SPHERE PACKINGS, LATTICES AND GROUPS

Material for Third Edition

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This document contains four sections.

- Preface to Third Edition
- Corrections to First Edition
- New version of Chapter 17
- Supplementary Bibliography for Third Edition

The third edition will be published by Springer-Verlag in late 1998 or early 1999. Comments on this material are welcomed, especially updates to the bibliography. Please send comments to N. J. A. Sloane, email address njas@research.att.com, mailing address AT&T Labs Research, Room C233, 180 Park Avenue, Florham Park, NJ 07932-0971, USA.

[The third edition went to press earlier this year, but we are still making updates to this file. Perhaps this should now be called Version 3.03. We have also added an index to the Preface to Third Edition — see page lii.

Preface to Third Edition

Interest in the subject matter of the book continues to grow. The Supplementary Bibliography has been enlarged to cover the period 1988 to 1998 and now contains over 800 items. Other changes from the second edition include a handful of small corrections and improvements to the main text, and this preface (an expanded version of the preface to the Second Edition) which contains a brief report on some of the developments since the appearance of the first edition.

We are grateful to a number of correspondents who have supplied corrections and comments on the first two editions, or who have sent us copies of manuscripts.[1] We thank in particular R. Bacher, R. E. Borcherds, P. Boyvalenkov, H. S. M. Coxeter, Y. Edel, N. D. Elkies, L. J. Gerstein, M. Harada, J. Leech, J. H. Lindsey, II, J. Martinet, J. McKay, G. Nebe, E. Pervin, E. M. Rains, R. Scharlau, F. Sigrist, H. M. Switkay, T. Urabe, A. Vardy, Z.-X. Wan and J. Wills. The new material was expertly typed by Susan K. Pope.

We are planning a sequel, tentatively entitled *The Geometry of Low-Dimensional Groups and Lattices*, which will include two earlier papers [Con36] and [Con37] not included in this book, as well as several recent papers dealing with groups and lattices in low dimensions ([CSLDL1]–[CSLDL8], [CoSl91a], [CoSl95a], etc.).

A Russian version of the first edition, translated by S. N. Litsyn, M. A. Tsfasman and G. B. Shabat, was published by Mir (Moscow) in 1990.

Recent developments, comments, and additional corrections. The following pages attempt to describe recent developments in some of the topics treated in the book. The arrangement roughly follows that of the chapters. Our coverage is necessarily highly selective, and we apologize if we have failed to mention some important results. In a few places we have also included additional comments on or corrections to the text.

Three books dealing with lattices have recently appeared: in order of publication, these are Ebeling, *Lattices and Codes* [Ebe94], Martinet, *Les Réseaux Parfaits des Espaces Euclidiens* [Mar96] and Conway and Fung, *The Sensual (Quadratic) Form* [CoFu97].

An extensive survey by Lagarias [Laga96] discusses lattices from many points of view not dealt with in our book, as does the Erdős-Gruber-Hammer [ErGH89] collection of **unsolved problems concerning lattices**. The encyclopedic work on **distance-regular graphs** by Brouwer, Cohen and Neumaier [BrCN89] discusses many mathematical structures that are related to topics in our book (see also Tonchev [Ton88]). See also the works by Aschbacher [Asch94] on **sporadic groups**, Engel [Eng86], [Eng93] on **geometric crystallography** [Eng93], Fejes Tóth and Kuperberg [FeK93], and Fejes Tóth [Fej97] on **packing and covering**, Gritzmann and Wills [GriW93b] on **finite packings and coverings**, and Pach and Agarwal [PacA95] on **combinatorial geometry**.

An electronic **data-base of lattices** is now available [NeSl]. This contains information about some 160,000 lattices in dimensions up to 64. The theta-series of

[1] We also thank the correspondent who reported hearing the first edition described during a talk as "the bible of the subject, and, like the bible, [it] contains no proofs". This is of course only half true.

the most important lattices can be found in [SloEIS]. The computer languages PARI [BatB91], KANT [Schm90], [Schm91] and especially MAGMA [BosC97], [BosCM94], [BosCP97] have extensive facilities for performing lattice calculations (among many other things).

Notes on Chapter 1: Sphere Packings and Kissing Numbers

Hales [Hal92], [Hal97], [Hal97a], [Hal97b] (see also Ferguson [Ferg97] and Ferguson and Hales [FeHa97]) has described a series of steps that may well succeed in proving the long-standing conjecture (the so-called "**Kepler conjecture**") that no packing of three-dimensional spheres can have a greater density than that of the face-centered cubic lattice. In fact, on August 9, 1998, just as the third editionn was going to press, Hales announced [Hal98] that the final step in the proof has been completed: the Kepler conjecture is now a theorem.

The previous best upper bound known on the density of a three-dimensional packing was due to Muder [Mude93], who showed that the density cannot exceed 0.773055... (compared with $\pi/\sqrt{18} = 0.74048\ldots$ for the f.c.c. lattice).

A paper by W.-Y. Hsiang [Hsi93] (see also [Hsi93a], [Hsi93b]) claiming to prove the Kepler conjecture contains serious flaws. G. Fejes Tóth, reviewing the paper for *Math. Reviews* [Fej95], states: "If I am asked whether the paper fulfills what it promises in its title, namely a proof of Kepler's conjecture, my answer is: no. I hope that Hsiang will fill in the details, but I feel that the greater part of the work has yet to be done." Hsiang [Hsi93b] also claims to have a proof that no more than 24 spheres can touch an equal sphere in four dimensions. For further discussion see [CoHMS], [Hal94], [Hsi95].

S. McLaughlin and T. C. Hales [McHa98] have announced a proof of the **dodecahedral conjecture**. This conjecture, weaker than the Kepler conjecture, states that the volume of any Voronoi cell in a packing of unit spheres in $\mathbb{R}^3$ is at least as large as the volume of a regular dodecahedron of inradius 1. See also K. Bezdek [Bez97] and Muder [Mude93].

A. Bezdek, W. Kuperberg and Makai [BezKM91] had established the Kepler conjecture for packings composed of parallel strings of spheres. See also Knill [Knill96].

There was no reason to doubt the truth of the Kepler conjecture. However, A. Bezdek and W. Kuperberg [BezKu91] show that there are packings of congruent ellipsoids with density $0.7533\ldots$, exceeding $\pi/\sqrt{18}$, and in [Wills91] this is improved to $0.7585\ldots$.

Using spheres of two radii $0 < r_1 < r_2$, one obviously obtains packings in 3-space with density $> 0.74048\ldots$, provided r_1/r_2 is sufficiently small. In [VaWi94] it is shown that that this is so even when $r_1/r_2 = 0.623\ldots$.

There are infinitely many three-dimensional nonlattice packings (the Barlow packings, see below) with the same density as the f.c.c. lattice packing. In [Schn98] it is shown that large finite subsets of the f.c.c. lattice are denser (in the sense of parametric density, see below) than subsets of any other Barlow packing.

What are all the best sphere packings in low dimensions? In [CoSl95a] we describe what may be *all* the best packings of nonoverlapping equal spheres in dimensions $n \leq 10$, where "best" means both having the highest density and not permitting any local improvement. For example, it appears that the best five-dimensional sphere packings are parameterized by the 4-colorings of the one-dimensional integer lattice. We also find what we believe to be the exact numbers of "uniform" packings among these, that is, those in which the automorphism group acts transitively. These assertions depend on certain plausible but as yet unproved postulates.

There are some surprises. We show that the Korkine-Zolotarev lattice Λ_9 (which continues to hold the density record it established in 1873) has the following astonishing property. Half the spheres can be moved bodily through arbitrarily large distances without overlapping the other half, only touching them at isolated instants, and yet the density of the packing remains the same at all times. A typical packing in this family consists of the points of

$$D_9^{\theta+} = D_9 \cup D_9 + \left(\left(\frac{1}{2} \right)^8, \frac{1}{2}\theta \right) ,$$

for any real number θ. We call this a "fluid diamond packing," since $D_9^{0+} = \Lambda_9$ and $D_9^{1+} = D_9^+$ (cf. Sect. 7.3 of Chap. 4). All these packings have the same density, the highest known in 9 dimensions. Agrell and Eriksson [AgEr98] show D_9^+ is a better 9-dimensional quantizer than any previously known. In [CoSl95a] we also discuss some new higher-dimensional packings, showing for example that there are extraordinarily many 16-dimensional packings that are just as dense as the Barnes-Wall lattice Λ_{16}.

Mordell-Weil Lattices. One of the most exciting developments has been Elkies' ([Elki], [Elki94], [Elki97]) and Shioda's [Shiod91d] construction of lattice packings from the Mordell-Weil groups of elliptic curves over function fields. These lattices have a greater density than any previously known in dimensions from about 80 to 4096, and provide the following new entries for Table 1.3 of Chap. 1:

n	54	64	80	104	128
$\log_2 \delta \geq$	15.88	24.71	40.14	67.01	97.40
reference	[Elki]	[Elki]	[Shi7]	[Shi7]	[Elki]

n	256	512	1024	2048	4096
$\log_2 \delta \geq$	294.80	797.12	2018.2	4891	11527
reference	[Elki]	[Elki]	[Elki]	[Elki]	[Elki]

In this Introduction we will use MW_n to denote an n-dimensional Mordell-Weil lattice. For further information about this construction see Shioda [Shiod88]–[Shiod91e], Oguiso and Shioda [OgS91], Dummigan [Dum94]–[Dum96], Gow [Gow89], [Gow89a], Gross [Gro90], [Gro96], Oesterlé [Oes90], Tiep [Tiep91]–[Tiep97b].

Several other new record packings will be mentioned later in this Introduction. Because of this, it seems worthwhile to include two new tables. (The latest versions of these two tables are also available electronically [NeSl].)

A new table of densest packings. Table I.1 gives the densest (lattice or nonlattice) packings and Table I.2 the highest kissing numbers presently known in dimensions up to 128. These tables update and extend Table 1.2 and part of Table 1.3 of Chapter 1, to which the reader is referred for more information about most of these packings. Others are described later in this Introduction. There are several instances in Table I.1 where the highest known density is achieved by a nonlattice packing: these entries are enclosed in parentheses.

At dimension 32, Q_{32} denotes the Quebbemann lattice constructed on page 220 (although the Mordell-Weil lattice MW_{32} or Bachoc's lattice $\mathcal{B}_{32}$ [Baco95], [Baco97] have the same density). The lattices $Q_{33}, \ldots, Q_{40}$ were constructed by Elkies [Elki] by laminating Q_{32} (or MW_{32} or $\mathcal{B}_{32}$) above certain half-lattice points. Each putative deep hole of norm 4 (two-thirds the minimal norm of the lattice) is surrounded by 576 lattice points. For $n \leq 8$ we obtain a lattice Q_{32+n} with center density $\delta = 2^{-24}3^{16+0.5n}/\sqrt{\lambda_n}$ and kissing number $261120 + 576\tau_n$, where λ_n and τ_n are respectively the determinant and kissing number of Λ_n (cf. Tables 6.1, 6.3). At the present time Q_{33} and Q_{34} are the densest packings known in those dimensions, and $Q_{32}, \ldots, Q_{40}$ have the highest kissing numbers presently known for *lattice* packings. The lattices Q_{30} and Q_{31} will be found on page 220. The lattice P_{48n} mentioned at dimension 48 is the extremal lattice found by Nebe [Nebe98].

A new table of kissing numbers. Table I.2 gives the highest kissing numbers presently known in dimensions $n \leq 128$. If a dimension n is not mentioned in the appropriate column, let m be the next lowest dimension that is mentioned, and use the sum of the entries for m and $n - m$. For more information about most of these packings see Tables 1.2 and 1.3. (Parenthesized entries indicate that higher kissing numbers can be obtained from nonlattice packings.)

The kissing number of the Mordell-Weil lattice MW_{44} was computed by G. Nebe (personal communication), and that of MW_{128} by Elkies [Elki]. However, a simple construction using binary codes yields higher kissing numbers [EdRS98].

A. Vardy has pointed out to us that by using the Nordstrom-Robinson code as inner code, and the Vladuts-Katsman-Tsfasman algebraic-geometry codes as outer codes, as in [TsV91], Theorem 3.4.16, one obtains a polynomial-time construction for a family of nonlinear binary codes with $d/n \geq 0.25$ and rate $R = k/n = 2/15\,(1 + o(1))$ as $n \to \infty$. Thus there is a polynomial-time construction for nonlattice packings with kissing number

$$\tau = 2^{0.1333n(1+o(1))} ,$$

a considerable improvement over Eq. (56) of Chap. 2. No polynomial-time construction is presently known however for a sequence of lattices in which the kissing number grows exponentially with dimension. See also [Alon97].

[CSLDL3] contains a simple and self-contained proof of the classification of **perfect lattices** in dimensions $n \leq 4$ and hence of the determination of the densest lattice packings in these dimensions (cf. Table 1.1 of Chap. 1). The main goal of [CSLDL3] is to study the perfect lattices in dimensions $n \leq 7$ found by Korkine and Zolotareff [Kor3], Voronoi [Vor1], Barnes [Bar6] – [Bar9], Scott [Sco1], [Sco2], Stacey [Sta1], [Sta2], and others, and to determine their automorphism groups, orbits of minimal vectors, and eutactic coefficients. It is shown that just 30 of the 33 seven-dimensional perfect lattices are extreme. Jaquet [Jaq93] has now shown that this list of 33 seven-dimensional perfect lattices is complete. See also [Anz91], [BatMa94] and especially Martinet [Mar96].

[CoSl95] describes an especially interesting imperfect 11-dimensional lattice, which we call **anabasic**: it is generated by its minimal vectors, but no set of 11 minimal vectors forms a basis.

Several important books have appeared that deal with the construction of very good codes and very dense lattices from algebraic curves and algebraic function fields (cf. §1.5 of Chap. 1 and §2.11 of Chap. 3): Goppa [Gop88], Tsfasman and Vladuts [TsV91] and Stichtenoth [Stich93], Lachaud et al. [Lac95]. Garcia and Stichtenoth

Table I.1(a) Densest packings presently known in dimensions $n \le 128$. The table gives the center density δ, defined on page 13.

n	δ (lattice)	δ (nonlattice)	Lattice (nonlattice)
0	0		Λ_0
1	$1/2 = 0.50000$		$\Lambda_1 \cong A_1 \cong \mathbb{Z}$
2	$1/2\sqrt{3} = 0.28868$		$\Lambda_2 \cong A_2$
3	$1/4\sqrt{2} = 0.17678$		$\Lambda_3 \cong A_3 \cong D_3$
4	$1/8 = 0.12500$		$\Lambda_4 \cong D_4$
5	$1/8\sqrt{2} = 0.08839$		$\Lambda_5 \cong D_5$
6	$1/8\sqrt{3} = 0.07217$		$\Lambda_6 \cong E_6$
7	$1/16 = 0.06250$		$\Lambda_7 \cong E_7$
8	$1/16 = 0.06250$		$\Lambda_8 \cong E_8$
9	$1/16\sqrt{2} = 0.04419$		Λ_9
10	$1/16\sqrt{3} = 0.03608$	$(5/128 = 0.03906)^*$	Λ_{10} $(P_{10c})^*$
11	$1/18\sqrt{3} = 0.03208$	$(9/256 = 0.03516)^*$	K_{11} $(P_{11a})^*$
12	$1/27 = 0.03704$		K_{12}
13	$1/18\sqrt{3} = 0.03208$	$(9/256 = 0.03516)^*$	K_{13} $(P_{13a})^*$
14	$1/16\sqrt{3} = 0.03608$		Λ_{14}
15	$1/16\sqrt{2} = 0.04419$		Λ_{15}
16	$1/16 = 0.06250$		Λ_{16}
17	$1/16 = 0.06250$		Λ_{17}
18	$1/8\sqrt{3} = 0.07217$	$(3^9/4^9 = 0.07508)^*$	Λ_{18} ($\mathcal{B}^*_{18}$ [BiEd98])*
19	$1/8\sqrt{2} = 0.08839$		Λ_{19}
20	$1/8 = 0.12500$	$(7^{10}/2^{31} = 0.13154)^*$	Λ_{20} ($\mathcal{B}^*_{20}$ [Vard95])*
21	$1/4\sqrt{2} = 0.17678$		Λ_{21}
22	$1/2\sqrt{3} = 0.28868$	$(0.33254)^*$	Λ_{22} ($\mathcal{A}^*_{22}$ [CoSl96])*
23	$1/2 = 0.50000$		Λ_{23}
24	1		Λ_{24}
25	$1/\sqrt{2} = 0.70711$		Λ_{25}
26	$1/\sqrt{3} = 0.57735$		Λ_{26}, T_{26} (see Notes on Chap. 18)

*Nonlattice packing.

[GaSt95] have given a fairly explicit construction for an infinite sequence of good codes over a fixed field $GF(q^2)$. Elkies [Elki97a] has worked on finding explicit equations for modular curves of various kinds that attain the Drinfeld-Vladuts bound. See also Manin and Vladuts [MaV85], Stichtenoth and Tsfasman [StTs92], Tsfasman [Tsf91], [Tsf91a]. The problem of decoding codes constructed from algebraic geometry is considered by Feng and Rao [FeRa94], Justesen et al. [JuL89], [JuL92], Pellikaan [Pel89], Skorobogatov and Vladuts [SkV90], Sudan [Suda96], [Suda97], Vladuts [Vlad90] (see also Lachaud et al. [Lac95]).

Quebbemann [Queb88] uses class field towers and Alon et al. [AlBN92], Sipser and Spielman [SipS96] and Spielman [Spiel96] use expander graphs to construct asymptotically good codes. The codes in [SipS96] and [Spiel96] can be decoded in linear time.

Several important papers have appeared dealing with the construction of dense lattices in high-dimensional space using algebraic number fields and global fields (cf. §1.5 of Chap. 1 and §7.4 of Chap. 8) — see Quebbemann [Queb89]–[Queb91a]

Table I.1(b) Densest packings presently known in dimensions $n \le 128$. The table gives the center density δ, defined on page 13.

n	δ (lattice)	δ (nonlattice)	Lattice (nonlattice)
27	$1/\sqrt{3} = 0.57735$	$(1/\sqrt{2} = 0.70711)^*$	$\mathcal{B}_{27}$ ($\mathcal{B}_{27}^*$ [Vard98])*
28	$2/3 = 0.66667$	$(1)^*$	$\mathcal{B}_{28}$ ($\mathcal{B}_{28}^*$ [Vard98])*
29	$1/\sqrt{3} = 0.57735$	$(1/\sqrt{2} = 0.70711)^*$	$\mathcal{B}_{29}$ ($\mathcal{B}_{29}^*$ [Vard98])*
30	$3^{13.5}/2^{22} = 0.65838$	$(1)^*$	Q_{30} ($\mathcal{B}_{30}^*$ [Vard98])*
31	$3^{15}/2^{23.5} = 1.20952$		Q_{31}
32	$3^{16}/2^{24} = 2.56578$		Q_{32} and others
33	$3^{16.5}/2^{25} = 2.22203$		Q_{33} [Elki94], [Elki]
34	$3^{16.5}/2^{25} = 2.22203$		Q_{34} [Elki94], [Elki]
35	$2\sqrt{2} = 2.82843$		B_{35} (p. 234)
36	$2^{18}/3^{10} = 4.43943$		Ks_{36} [KsP92]
37	$4/\sqrt{2} = 5.65685$		$\mathcal{D}_{37}$
38	8		$\mathcal{D}_{38}$
39	$3^{16}/2^{20}\sqrt{14} = 10.9718$		From P_{48p}, see p. 167
40	$3^{17}/2^{22.5} = 21.7714$		From P_{48p}, see p. 167
41	$3^{17}/2^{21.5} = 43.5428$		From P_{48p}
42	$3^{18}/2^{22} = 92.3682$		From P_{48p}
43	$3^{19}/2^{22.5} = 195.943$		From P_{48p}
44	$3^{20}/2^{23} = 415.657$	$(17^{22}/2^{43}3^{24} = 472.799)^*$	From P_{48p} ($\mathcal{A}_{44}$ [CoSl96])*
45	$3^{21}/2^{23.5} = 881.742$	$(17^{22.5}/2^{44}3^{24} = 974.700)^*$	From P_{48p} ($\mathcal{A}_{45}$ [CoSl96])*
46	$3^{21.5}/2^{23} = 2159.82$	$(13^{23}/3^{46.5} = 2719.94)^*$	From P_{48p} ($\mathcal{A}_{46}$ [CoSl96])*
47	$3^{23}/2^{24} = 5611.37$	$(35^{23.5}/2^{70}3^{24} = 5788.81)^*$	From P_{48p} ($\mathcal{A}_{47}$ [CoSl96])*
48	$3^{24}/2^{24} = 16834.1$		$P_{48n}, P_{48p}, P_{48q}$
54	$2^{15.88}$		MW_{54}
56	$1.5^{28} = 2^{16.38}$		$L_{56,2}(M)$, $\tilde{L}_{56,2}(M)$ [Nebe98]
64	$3^{16} = 2^{25.36}$		Ne_{64} [Nebe98], [Nebe98b]
80	$2^{40.14}$		MW_{80}
128	$2^{97.40}$		MW_{128}

*Nonlattice packing.

and Rosenbloom and Tsfasman [RoT90], as well as Tsfasman and Vladuts [TsV91].

The results in [Rus1] (see p. 19 of Chap. 1) have been generalized and extended in [ElkOR91], [Rus89]–[Rus92].

Yudin [Yud91] gives an upper bound for the number of disjoint spheres of radius r in the n-dimensional torus, and deduces from this a new proof of Levenshtein's bound (Eq. (42) of Chap. 1).

When discussing **Hermite's constant** γ_n on page 20 of Chap. 1, we should have mentioned that it can also be defined as the minimal norm of an n-dimensional lattice packing of maximal density, when that lattice is scaled so that its determinant is 1. See also Bergé and Martinet [BerM85].

Let $\mu(\Lambda)$ denote the minimal nonzero norm of a vector in a lattice Λ. Bergé and Martinet [BerM89] call a lattice **dual-critical** if the value of $\mu(\Lambda)\mu(\Lambda^*)$ is maximized (where Λ^* is the dual lattice). They prove that A_1, A_2, A_3, A_3^*, D_4, A_4, A_4^* are the only lattices in dimensions $n \le 4$ on which the product $\mu(\Lambda)\mu(\Lambda^*)$ attains

Table I.2(a) Highest kissing numbers τ presently known for packings in dimensions $n \leq 128$.

n	τ (lattice)	τ (nonlattice)	Lattice (nonlattice)
0	0		Λ_0
1	2		$\Lambda_1 \cong A_1 \cong \mathbb{Z}$
2	6		$\Lambda_2 \cong A_2$
3	12		$\Lambda_3 \cong A_3 \cong D_3$
4	24		$\Lambda_4 \cong D_4$
5	40		$\Lambda_5 \cong D_5$
6	72		$\Lambda_6 \cong E_6$
7	126		$\Lambda_7 \cong E_7$
8	240		$\Lambda_8 \cong E_8$
9	272	$(306)^*$	Λ_9 $(P_{9a})^*$
10	336	$(500)^*$	Λ_{10} $(P_{10b})^*$
11	438	$(582)^*$	$\Lambda_{11}^{\max}$ $(P_{11c})^*$
12	756	$(840)^*$	K_{12} $(P_{12a})^*$
13	918	$(1130)^*$	K_{13} $(P_{13a})^*$
14	1422	$(1582)^*$	Λ_{14} $(P_{14b})^*$
15	2340		Λ_{15}
16	4320		Λ_{16}
17	5346		Λ_{17}
18	7398		Λ_{18}
19	10668		Λ_{19}
20	17400		Λ_{20}
21	27720		Λ_{21}
22	49896		Λ_{22}

* The kissing number in a nonlattice packing may vary from sphere to sphere – we give the largest value (see Table 1.2)

a local maximum. That A_1, A_2, D_4 (and E_8) are the only dual-critical lattices in dimensions 1, 2, 4 and 8 respectively follows because these are the densest lattices, and they are unique and isodual. See also [CoSl94], [Mar96], [Mar97].

Fields [Fie2], [Fiel80] and Fields and Nicolich [FiNi80] have found the **least dense lattice packings** in dimensions $n \leq 4$ that minimize the density under local variations that preserve the minimal vectors.

For recent work on **quasilattices** and **quasicrystals** (cf. §2.1 of Chap. 1), see for example [BaaJK90], [Bru86], [Hen86], [MRW87], [OlK89], [RHM89], [RMW87], [RWM88], [Wills90a], [Wills90b]. Baake et al. [BaaJK90] show that many examples of quasiperiodic tilings of the plane arise from projections of root lattices.

Spherical codes. There has been considerable progress on the construction of spherical codes and the Tammes problem (cf. §2.3 of Chap. 1). In particular, Hardin, Smith and Sloane have computed extensive tables of spherical codes in up to 24 dimensions. For example, we have found conjecturally optimal packings of N spherical caps on a sphere in n dimensions for $N \leq 100$ and $n \leq 5$; and coverings and maximal volume codes in three dimensions for $N \leq 130$ and in four dimensions for $N \leq 24$. A book is in preparation [HSS]. Many of these tables are

Table I.2(b) Highest kissing numbers τ presently known for packings in dimensions $n \leq 128$.

n	τ (lattice)	τ (nonlattice)	Lattice (nonlattice)
23	93150		Λ_{23}
24	196560		Λ_{24}
25	196656		Λ_{25}
26	196848		Λ_{26}
27	197142		Λ_{27}
28	197736		Λ_{28}
29	198506		Λ_{29}
30	200046		Λ_{30}
31	202692		Λ_{31}
32	261120	(276032)*	Q_{32} and others ([EdRS98])
33	262272	(294592)*	Q_{33} ([EdRS98])
34	264576	(318020)*	Q_{34} ([EdRS98])
35	268032	(370892)*	Q_{35} ([EdRS98])
36	274944	(438872)*	Q_{36} ([EdRS98])
37	284160	(439016)*	Q_{37} ([EdRS98])
38	302592	(566652)*	Q_{38} ([EdRS98])
39	333696	(714184)*	Q_{39} ([EdRS98])
40	399360	(991792)*	Q_{40} ([EdRS98])
44	2708112	(2948552)*	MW_{44} ([EdRS98])
48	52416000		$P_{48n}, P_{48p}, P_{48q}$
64	138458880	(331737984)*	Ne_{64} [Nebe98b] ([EdRS98])
80	1250172000	(1368532064)*	L_{80} [BacoN98] ([EdRS98])
128	218044170240	(8863556495104)*	MW_{128} [Elki] ([EdRS98])

* The kissing number in a nonlattice packing may vary from sphere to sphere – we give the largest value (see Table 1.2)

also available electronically [SloHP]. The papers by Hamkins and Zeger [HaZe97], [HaZe97a] show how to construct good spherical codes by (a) "wrapping" a good lattice packing around a sphere or (b) "laminating" (cf. Chap. 6) a good spherical code in a lower dimension. Other recent papers dealing with the construction of spherical codes are Dodunekov, Ericson and Zinoviev [DEZ91], Kolushov and Yudin [KolY97], Kottwitz [Kott91], Lazić, Drajić and Senk [LDS86], [LDS87], Melissen [Mel97].

A series of papers by Boyvalenkov and coauthors [Boy93]–[BoyN95] has investigated (among other things) the best polynomials to use in the linear programming bounds for spherical codes (cf. Chaps. 9, 13). This has led to small improvements in the (rather weak) upper bounds on the kissing number in dimensions 19, 21 and 23 given in Table 1.5 [Boy94a]. Coverings of a sphere by equal spherical caps are also discussed in [Tar5], [Tar6].

Drisch and Sonneborn [DrS96] give a table of upper bounds on kissing numbers (cf. Table 1.5 of Chap. 1) that extends to dimension 49.

Concerning the **numbers of lattice points** in or on various regions, (cf. §2.4 of Chap. 1), see [ArJ79], [Barv90], [BoHW72], [DuS90], [Dye91], [ElkOR91], [GoF87], [GriW93a], [Kra88], [Levi87], [MaO90], [Sar90].

Gritzmann and Wills [GriW93b] give a survey of recent work on **finite packings** and **coverings** (cf. §1.5 of Chap. 1), with particular emphasis on "sausage problems" and "bin-packing." The term "sausage problem" arises from the "sausage catastrophe," first observed by Wills in 1983 [Wills83]: which arrangement of N equal three-dimensional spheres has the smallest convex hull? It appears that for N up to about 55 a sausage-like linear arrangement of spheres is optimal, but for all larger N except 57, 58, 63 and 64 a three-dimensional cluster is better. In [Wills93] the notion of **parametric density** was introduced, permitting a joint theory of finite and infinite packings, with numerous applications. There are many interesting papers on the best packings of N balls when N is large and the connection with the **Wulff shape**, etc.: see Arhelger et al. [ABB96], Betke and Böröczky [BetB97], Böröczky and Schnell [BorSch97], [BorSch98], [BorSch98a], Dinghas [Ding43], von Laue [Laue43], Schnell [Schn98], Wills [Wills90]–[Wills98a]. For the sausage catastrophe in dimension 4, see [GaZu92].

Several recent papers have studied the problems of packing N points in a (a) circle [Fej97], [GraLNO], [Mel94], [Mel97], (b) square [Fej97], [Golb70], [Mel97], [MolP90], [NuOs97], [PeWMG], [Scha65], [Scha71], [Vall89], (c) triangle [Fej97], [GraL95], [Mel97], and (d) torus (Hardin and Sloane, unpublished).

Other papers dealing with finite packings and coverings are [BetG84], [BetG86], [BetGW82], [Chow92], [DaZ87], [FGW90], [FGW91], [GrW85], [Wills90]–[Wills90b].

The techniques that we use in [HSS] to find spherical codes have also proved successful in constructing **experimental designs** for use in statistics, and have been implemented in a general-purpose experimental design program called *Gosset* [HaSl93], [HaSl96a]. The name honors both the amateur mathematician Thorold Gosset (1869 – 1962) (cf. p. 120 of Chap. 8) and the statistician William Seally Gosset (1876 – 1937).

We have also used the same optimization methods to find good (often optimal) packings of lines through the origin in $\mathbb{R}^n$ (that is, antipodal spherical codes), and more generally packings of m-dimensional subspaces of $\mathbb{R}^n$. These **Grassmannian packings** are described in [CoHS96], [ShS98], [CHRSS].

The material in the Appendix to Chapter 1 on **planetary perturbations** was obtained in conversations among E. Calabi, J. H. Conway and J. G. Propp about Conway's lectures on "Games, Groups, Lattices and Loops" at the University of Pennsylvania in 1987 — see [Prop88].

For the application of lattices in **string theory** (cf. §1.4 of Chap. 1), see for example Gannon and Lam [GaL90].

Finally, we cannot resist calling attention to the remark of Frenkel, Lepowsky and Meurman, that **vertex operator algebras** (or conformal field theories) are to lattices as lattices are to codes (cf. [DGM90]–[DGM90b], [Fre1]–[Fre5], [Godd89], [Hoehn95], [Miya96], [Miya98]).

Notes on Chapter 2: Coverings, Lattices and Quantizers

Miyake [Miya89, Section 4.9] gives an excellent discussion of the classical result that the **theta function** of an integral lattice is a **modular form** for an appropriate subgroup of $SL_2(\mathbb{Z})$.

The papers [CSLDL6], [CSLDL8] and Chap. 3 of [CoFu97] describe how the **Voronoi cell** of a lattice (cf. §1.2 of Chap. 2) changes as that lattice is continuously varied. We simplify the usual treatment by introducing new parameters which we call the vonorms and conorms of the lattice. [CSLDL6] studies lattices in one, two and three dimensions, ending with a proof of the theorem of Fedorov [Fed85], [Fed91] on the five types of three-dimensional lattices. The main result of [CSLDL6] (and Chap. 3 of [CoFu97]) is that each three-dimensional lattice is uniquely represented by a projective plane of order 2 labeled with seven numbers, the conorms of the lattice, whose minimum is 0 and whose support is not contained in a proper subspace. Two lattices are isomorphic if and only if the corresponding labelings differ only by an automorphism of the plane.

These seven "conorms" are just 0 and the six "Selling parameters" ([Sel74], [Bara80]). However, this apparently trivial replacement of six numbers by seven numbers whose minimum is zero leads to several valuable improvements in the theory:

(i) The conorms vary continuously with the lattice. (For the Selling parameters the variation is usually continuous but requires occasional readjustments.)

(ii) The definition of the conorms makes it apparent that they are invariants of the lattice. (The Selling parameters are almost but not quite invariant.)

(iii) All symmetries of the lattice arise from symmetries of the conorm function. (Again, this is false for the Selling parameters.)

[CSLDL8] (summarized in the Afterthoughts to Chap. 3 of [CoFu97]) uses the same machinery as [CSLDL6] to give a simple proof of the theorem of Delone (= Delaunay) [Del29], [Del37], as corrected by Stogrin [Sto73], that there are 52 types of **four-dimensional lattices**. We also give a detailed description of the 52 types of four-dimensional parallelotopes (these are also listed by Engels [Eng86]). Erdahl and Ryskov [ErR87], [RyE88] show that there are only 19 types of different Delaunay cells that occur in four-dimensional lattices. (See also [RyE89].)

We call a lattice that is geometrically congruent to its dual **isodual** ([CoSl94]). We have used the methods of [CSLDL6] to determine the densest three-dimensional isodual lattice [CoSl94]. This remarkable lattice, the **m.c.c.** (or **mean-centered cuboidal**) **lattice**, has Gram matrix

$$\frac{1}{2}\begin{bmatrix} 1+\sqrt{2} & 1 & 1 \\ 1 & 1+\sqrt{2} & 1-\sqrt{2} \\ 1 & 1-\sqrt{2} & 1+\sqrt{2} \end{bmatrix} . \tag{1}$$

In a sense this lattice is the geometric mean of the f.c.c. and b.c.c. lattices. (Consider the lattice generated by the vectors $(\pm u, \pm v, 0)$ and $(0, \pm u, \pm v)$ for real numbers u and v. If the ratio u/v is respectively 1, $2^{1/2}$ or $2^{1/4}$ we obtain the f.c.c., b.c.c. and m.c.c. lattices.) The m.c.c. lattice is also the thinnest isodual covering lattice. It is of course nonintegral. The m.c.c. lattice also recently appeared in a different context, as the lattice corresponding to the period matrix for the hyperelliptic Riemann surface $w^2 = z^8 - 1$ [BerSl97].

A **modular** lattice is an integral lattice that is geometrically similar to its dual (the term was introduced by Quebbemann [Queb95]; see also [Queb97]). In other words, an n-dimensional integral lattice Λ is *modular* if there exists a similarity σ of $\mathbb{R}^n$ such that $\sigma(\Lambda^*) = \Lambda$, where Λ^* is the dual lattice. If σ multiplies norms by N, Λ is said to be N-*modular*, and so has determinant $N^{n/2}$. A unimodular lattice is 1-modular. A modular lattice becomes isodual when rescaled so that its determinant is 1. For example, the sporadic root lattices E_8, F_4 ($\cong D_4$), G_2 ($\cong A_2$) are respectively

1-, 2- and 3-modular. In the last two cases the modularity maps short roots to long roots. The densest lattice packings presently known in dimensions 1, 2, 4, 8, 12, 16, 24, 48 and 56 are all modular.

Root lattices. In [CoSl91a] the Voronoi and Delaunay cells (cf. §1.2 of Chap. 2) of the lattices A_n, D_n, E_n and their duals are described in a simple geometrical way. The results for E_6^* and E_7^* simplify the work of Worley [Wor1], [Wor2], and also provide what may be new space-filling polytopes in dimensions 6 and 7. Pervin [Per90] and Baranovskii [Bara91] have also studied the Voronoi and Delaunay cells of E_6^*, E_7^*. Moody and Patera [MoP92], [MoP92a] have given a uniform treatment of the Voronoi and Delaunay cells of root lattices that also applies to the hyperbolic cases.

If a lattice Λ has covering radius R (cf. §1.2 of Chap. 2) then closed balls of radius R around the lattice points just cover the space. Sullivan [Sul90] defines the **covering multiplicity** $CM(\Lambda)$ to be the maximal number of times the interiors of these balls overlap. In [CoSl92] we show that the least possible covering multiplicity for an n-dimensional lattice is n if $n \leq 8$, and conjecture that it exceeds n in all other cases. We also determine the covering multiplicity of the Leech lattice and of the lattices I_n, A_n, D_n, E_n and their duals for small values of n. Although it appears that $CM(I_n) = 2^{n-1}$ if $n \leq 33$, it follows from the work of Mazo and Odlyzko [MaO90] that as $n \to \infty$ we have $CM(I_n) \sim c^n$, where $c = 2.089\ldots$. The results have applications to numerical integration.

The covering problem. Several better coverings of space by spheres have been found in low dimensions, giving improvements to Table 2.1 and Fig. 2.4 of Chap. 2. The lattice $A_n[s]$ mentioned on p. 116 of Chap. 4 is generated by the vectors of the translate $[s]+A_n$, where s is any divisor of $n+1$, and is the union of the r translates $[i]+A_n$ for $i = 0, s, 2s, \ldots, (r-1)s$, where $r = (n+1)/s$. This is the lattice A_n^{+r}, in the notation of [CSLDL1], or A_n^r in Coxeter's notation [Cox10]. It has determinant $(n+1)/r^2$ and minimal norm $rs/(n+1)$.

Baranovskii [Bara94] shows that A_9^{+5} has covering radius $\sqrt{24}/5$ and thickness

$$\Theta = (2^{13}3^{4.5}/5^{8.5})V_9 = 1.3158\ldots V_9 = 4.3402\ldots ,$$

whereas A_9^*, the old record-holder, has thickness

$$\Theta = (3^{4.5}11^{4.5}/2^{13}5^4)V_9 = 1.3306\ldots V_9 = 4.3889\ldots .$$

Furthermore, W. D. Smith [Smi88] has shown that Λ_{22}^* and Λ_{23}^* are better coverings than A_{22}^* and A_{23}^*. Smith finds that the thickness Θ of Λ_{22}^* is at most

$$2\sqrt{3}\left(\frac{\sqrt{17}}{3}\right)^{22} V_{22} = 27.8839\ldots$$

and the thickness of Λ_{23}^* is at most

$$2\left(\frac{\sqrt{31}}{4}\right)^{23} V_{23} = 15.3218\ldots .$$

For other work on the covering radius of lattices see [CaFR95].

Integer coordinates for integer lattices. [CSLDL5] is concerned with finding descriptions for integral lattices (cf. §2.4 of Chap. 2) using integer coordinates (possibly with a denominator). Let us say that an n-dimensional (classically) integral lattice Λ is **s-integrable**, for an integer s, if it can be described by vectors $s^{-1/2}(x_1, \ldots, x_k)$, with all $x_i \in \mathbb{Z}$, in a Euclidean space of dimension $k \geq n$. Equivalently, Λ is s-integrable if and only if any quadratic form $f(x)$ corresponding to Λ can be written as s^{-1} times a sum of k squares of linear forms with integral coefficients, or again, if and only if the dual lattice Λ^* contains a eutactic star of scale s. [CSLDL5] gives many techniques for s-integrating low-dimensional lattices (such as E_8 and the Leech lattice). A particular result is that any one-dimensional lattice can be 1-integrated with $k = 4$: this is Lagrange's four-squares theorem. Let $\phi(s)$ be the smallest dimension n in which there is an integral lattice that is not s-integrable. In 1937 Ko and Mordell showed that $\phi(1) = 6$. We prove that $\phi(2) = 12$, $\phi(3) = 14$, $21 \leq \phi(4) \leq 25$, $16 \leq \phi(5) \leq 22$, $\phi(s) \leq 4s + 2$ (s odd), $\phi(s) \leq 2\pi e s(1 + o(1))$ (s even) and $\phi(s) \geq 2 \ln \ln s / \ln \ln \ln s(1 + o(1))$.

Plesken [Plesk94] studies similar embedding questions for lattices from a totally different point of view. See also Cremona and Landau [CrL90].

Complexity. For recent results concerning the complexity of various lattice- and coding-theoretic calculations (cf. §1.4 of Chap. 2), see Ajtai [Ajt96], [Ajt97], Downey et al. [DowFV], Hastad [Has88], Jastad and Lagarias [HaL90], Lagarias [Laga96], Lagarias, Lenstra and Schnorr [Lag3], Paz and Schnorr [PaS87], Vardy [Vard97].

In particular, Vardy [Vard97] shows that computing the minimal distance of a binary linear code is NP-hard, and the corresponding decision problem is NP-complete. Ajtai [Ajt97] has made some progress towards establishing analogous results for lattices. Downey et al. [DowFV] show that computing (the nonzero terms in) the theta-series of a lattice is NP-hard.

For **lattice reduction algorithms** see also [Schn87], [Val90], [Zas3]. Most of these results assume the lattice in question is a sublattice of $\mathbb{Z}^n$. In this regard the results of [CSLDL5] mentioned above are especially relevant. Ivanyos and Szántó [IvSz96] give a version of the LLL algorithm that applies to indefinite quadratic forms.

Mayer [Maye93], [Maye95] shows that every Minkowski-reduced basis for a lattice of dimension $n \leq 6$ consists of strict Voronoi vectors (cf. [Rys8]). He also answers a question raised by Cassels ([Cas3], p. 279) by showing that in seven dimensions (for the first time) the Minkowski domains do not meet face to face.

Barvinok [Barv92a] has described a new procedure for finding the minimal norm of a lattice or the closest lattice vector to a given vector that makes use of theta functions (cf. [Barv90], [Barv91], [Barv92]).

The **isospectral problem** for planar domains was solved by Gordon, Webb and Wolpert in 1992 ([GWW92], [Kac66] see also [BuCD94]). A particularly simple solution appears in Chap. 2 of [CoFu97]. This has aroused new interest in other isospectrality problems, for instance that of finding the largest n such that any positive definite quadratic form of rank n is determined by its representation numbers. Equivalently, what is the smallest dimension in which there exist two inequivalent lattices with the same theta series? We shall call such lattices *isospectral*: they are the subject of Chap. 2 of [CoFu97]. As mentioned in §2.3 of Chap. 2, Witt [Wit4], Kneser [Kne5] and Kitaoka [Kit2] found isospectral lattices in dimensions 16, 12 and 8 respectively. Milnor [Mil64] pointed out the connection with the isospectral manifold problem.

In 1986 one of the present authors observed that pairs of isospectral lattices in 6 and 5 dimensions could be obtained from a pair of codes with the same weight enumerator given by the other author [Slo10]. These lattices are mentioned on p. 47, and have now been published in [CoSl92a]. One lattice of the six-dimensional pair is a scaled version of the cubic lattice I_6. The five-dimensional pair have Gram matrices

$$\begin{bmatrix} 2 & 0 & 0 & 2 & 2 \\ 0 & 2 & 0 & 2 & 0 \\ 0 & 0 & 2 & 0 & 2 \\ 2 & 2 & 0 & 8 & 4 \\ 2 & 0 & 2 & 4 & 8 \end{bmatrix} , \quad \begin{bmatrix} 2 & 1 & 0 & 2 & 2 \\ 1 & 2 & 0 & 2 & 2 \\ 0 & 0 & 6 & 4 & 4 \\ 2 & 2 & 4 & 8 & 4 \\ 2 & 2 & 4 & 4 & 8 \end{bmatrix} \tag{2}$$

and determinant 96 (and are in different genera).

The first pair of isospectral 4-dimensional lattices was found in 1990 by Schiemann [Schi90], by computer search, and we have been informed by Schulze-Pillot (personal communication) that Schiemann has since found at least a dozen such pairs. Another pair has been given by Earnest and Nipp [EaN91]. The main result of [CoSl92a] is to give a simple 4-parameter family of pairs of isospectral lattices, which includes many of the known examples (including Schiemann's first pair) as special cases. The typical pair of this family is

$$\langle 3w - x - y - z, \quad w + 3x + y - z, \quad w - x + 3y + z, \quad w + x - y + 3z \rangle$$

and

$$\langle -3w - x - y - z, \quad w - 3x + y - z, \quad w - x - 3y + z, \quad w + x - y - 3z \rangle \ ,$$

where w, x, y, z are orthogonal vectors of distinct lengths, and the pointed brackets mean "lattice spanned by."

Schiemann [Schi97] has now completed the solution to the original problem by showing that any three-dimensional lattice is determined by its theta series. Thus n-dimensional isospectral lattices exist if and only if n is at least 4. For more about these matters see [CoFu97].

Lattice quantizers. Coulson [Coul91] has found the mean squared error G for the perfect (and isodual) six-dimensional lattice $P_6^5 \cong A_6^{(2)}$ (defined in §6 of Chap. 8 and studied in [CSLDL3]). He finds $G = 0.075057$, giving an additional entry for Table 2.3 of Chap. 2. Viterbo and Biglieri [ViBi96] have computed G for the lattices of Eqs. (1) and (2), the Dickson lattices of page 36, and other lattices.

Agrell and Eriksson [AgEr98] have found 9- and 10-dimensional lattices with $G = 0.0716$ and 0.0708, respectively, and show that the nonlattice packings D_7^+ and D_9^+ have $G = 0.0727$ and 0.0711, respectively. These values are all lower (i.e. better) than the previous records.

Notes on Chapter 3: Codes, Designs and Groups

Lattice codes. Several authors have studied the error probability of codes for the Gaussian channel that make use of constellations of points from some lattice as the signal set – see for example Banihashemi and Khandani [BanKh96], de Buda [Bud2], [Bud89], Forney [Forn97], Linder et al. [LiSZ93], Loeliger [Loel97], Poltyrev [Polt94], Tarokh, Vardy and Zeger [TaVZ], Urbanke and Rimoldi [UrB98].

Urbanke and Rimoldi [UrB98], completing the work of several others, have shown that lattice codes bounded by a sphere can achieve the capacity $1/2 \log_2(1 + P/N)$ (where P is the signal power and N is the noise variance), using minimal-distance decoding. This is stronger than what can be deduced directly from the Minkowski-Hlawka theorem ([Bud2], [Cas2], [Gru1a], [Hla1], [Rog7]), which is that a rate of $1/2 \log_2(P/N)$ can be achieved with lattice codes.

There has been a great deal of activity on **trellis codes** (cf. §1.4 of Chap. 3) — see for example [BDMS], [Cal91], [CaO90], [Forn88], [Forn88a], [Forn89a], [Forn91], [FoCa89], [FoWe89], [LaVa95], [LaVa95a], [LaVa96], [TaVa97], [VaKs96].

Another very interesting question is that of finding **trellis representations** of the standard codes and lattices: see Forney [Forn94], [Forn94a], Feigenbaum et al. [FeFMMV], Vardy [Vard98a] and many related papers: [BanBl96], [BanKh97], [BlTa96], [FoTr93], [KhEs97], [TaBl96], [TaBl96a].

We have already mentioned recent work on Goppa codes and the construction of codes and lattices from algebraic geometry (cf. §2.11 of Chap. 3) under §1.5 of Chap. 1.

Tables of codes. Verhoeff's table [Ver1] of the best binary linear codes (cf. §2.1 of Chap. 3) has been greatly improved by Brouwer [BrVe93], [Bro98]. For other tables of codes see [BrHOS], [BrSSS], [Lits98], [SchW92].

For recent work on the **covering radius of codes** see the book by Cohen et al. [CHLL] as well as the papers [BrLP98], [CaFR95], [CLLM97], [DaDr94], [EtGr93], [EtGh93], [EtWZ95], [Habs94]–[Habs97], [LeLi96], [LiCh94], [Stru94], [Stru94a], [Tiet91], [Wee93], [Wille96].

Spherical t-designs. The work of Hardin and Sloane on constructing experimental designs mentioned under Chap. 1 has led to new results and conjectures on the existence of spherical 4-designs (cf. §3.2 of Chap. 3). In three dimensions, for example, we have shown that spherical 4-designs containing M points exist for $M = 12$, 14 and $M \geq 16$, and we conjecture that they do not exist for $M = 9, 10, 11, 13$ and 15 [HaSl92]. Similarly, we conjecture that in four dimensions they exist precisely for $M \geq 20$; in five dimensions for $M \geq 29$; in six dimensions for $M = 27$, 36 and $M \geq 39$; in seven dimensions for $M \geq 53$; and in eight dimensions for $M \geq 69$ [HaSl92]. The connections between experimental designs and spherical designs have become much clearer thanks to the work of Neumaier and Seidel [NeS92].

Other recent papers dealing with spherical designs and numerical integration on the sphere are [Atk82], [Baj91], [Baj91a], [Baj91b], [Baj92], [Baj98], [Boy95], [BoyDN], [BoyN94], [BoyN95], [KaNe90], [Kea87], [Kea1], [LySK91], [NeS88], [Neut83], [NeSJ85], [Rezn95], [Sei90], [Yud97]. Several of J. J. Seidel's papers (including in particular the joint papers [Del15], [Del16]) have been reprinted in [Sei91].

Finite matrix groups. For a given value of n there are only finitely many nonisomorphic finite groups of $n \times n$ integer matrices. This theorem has a long history and is associated with the names Jordan, Minkowski, Bieberbach and Zassenhaus (see [Mil5], [Bro10]). For $n = 2$ and 3 these groups were classified in the past century, because they are needed in crystallography (see also [AuC91], [John91]). The maximal finite subgroups of $GL(4, \mathbb{Z})$ were given by Dade [Dad1], and the complete list of finite subgroups of $GL(4, \mathbb{Z})$ by Bülow, Neubüser and Wondratschek [Bül1] and Brown et al. [Bro10]. The maximal irreducible finite subgroups of $GL(5, \mathbb{Z})$

were found independently by Ryskov [Rys4], [Rys5], and Bülow [Bül2]. That work was greatly extended by Plesken & Pohst [Ple5], who determined the maximal irreducible subgroups of $GL(n, \mathbb{Z})$ for $n = 6, 7, 8, 9$, and by Plesken [Ple3], who dealt with $n = 11, 13, 17, 19, 23$.

In these papers the subgroups are usually specified as the automorphism groups of certain quadratic forms. In [CSLDL2] we give a geometric description of the maximal irreducible subgroups of $GL(n, \mathbb{Z})$ for $n = 1, \ldots, 9,\ 11, 13, 17, 19, 23$, by exhibiting lattices corresponding to these quadratic forms (cf. §4.2(i) of Chap. 3): the automorphism groups of the lattices are the desired groups. By giving natural coordinates for these lattices and determining their minimal vectors, we are able to make their symmetry groups clearly visible. There are 176 lattices, many of which have not been studied before (although they are implicit in the above references and in [Con16]).

The book by Holt and Plesken [HoPl89] contains tables of perfect groups of order up to 10^6, and includes tables of crystallographic space groups in dimensions up to 10.

Nebe and Plesken [NePl95] and Nebe [Nebe96], [Nebe96a] (see also [Plesk96], [Nebe98a]) have recently completed the enumeration of the maximal finite irreducible subgroups of $GL(n, \mathbb{Q})$ for $n \leq 31$, together with the associated lattices. This is an impressive series of papers, which contains an enormous amount of information about lattices in dimensions below 32.

Notes on Chapter 4: Certain Important Lattices and Their Properties

Several recent papers have dealt with gluing theory (cf. §3 of Chap. 4) and related techniques for combining lattices: [GaL91]–[GaL92a], [Gers91], [Sig90], [Xu1]. Gannon and Lam [GaL92], [GaL92a] also give a number of new theta-function identities (cf. §4.1 of Chap. 4).

Scharlau and Blaschke [SchaB96] classify all lattices in dimensions $n \leq 6$ in which the root system has full rank.

Professor Coxeter has pointed out to us that, in the last line of the text on page 96, we should have mentioned the work of Bagnera [Bag05] along with that of Miller.

For recent work on quaternionic reflection groups (cf. §2 of Chap. 4) see Cohen [Coh91].

Hexagonal lattice A_2. The number of inequivalent sublattices of index N in A_2 is determined in [BerSl97a], and the problems of determining the best sublattices from the points of view of packing density, signal-to-noise ratio and energy are considered. These questions arise in cellular radio. See also [BaaPl95].

Kühnlein [Kuhn96] has made some progress towards establishing Schmutz's conjecture [Schmu95] that the distinct norms that occur in A_2 are strictly smaller than those in any other (appropriately scaled) two-dimensional lattice. See also Schmutz [Schmu93], Schmutz Schaller [Schmu95a], Kühnlein [Kuhn97]. A related idea (the Erdős number) is discussed in the Notes on Chapter 15.

Leech lattice. A very simple construction for the Leech lattice Λ_{24} was discovered by Bonnecaze and Solé ([BonS94], see also [BonCS95]): lift the binary Golay code to $\mathbb{Z}_4$ (the ring of integers mod 4), and apply "Construction A_4". The details are as follows.

The $[23,12,7]$ Golay code may be constructed as the cyclic code with generator polynomial $g_2(x) = x^{11} + x^9 + x^7 + x^6 + x^5 + x + 1$, a divisor of $x^{23} - 1 \pmod 2$. By Hensel-lifting this polynomial (using say Graeffe's root-squaring method, cf. [HaKCSS], p. 307) to $\mathbb{Z}_4$ we obtain

$$g_4(x) = x^{11} + 2x^{10} - x^9 - x^7 - x^6 - x^5 + 2x^4 + x - 1 ,$$

a divisor of $x^{23} - 1 \pmod 4$. By appending a zero-sum check symbol to the cyclic code generated by $g_4(x)$, we obtain a self-dual code of length 24 over $\mathbb{Z}_4$. Applying Construction A_4 (cf. Chapter 5), that is, taking all vectors in $\mathbb{Z}^{24}$ which when read mod 4 are in the code, we obtain the Leech lattice.

In this version of the Leech lattice the 196560 minimal vectors appear as 4.16.759 of shape $2^2 1^8 0^{14}$, 2.24.2576 of shape $2^1 1^{12} 0^{11}$, 32.759 of shape $1^{16} 0^8$ and 48 of shape $4^1 0^{23}$.

The general setting for this construction is the following ([BonCS95], Theorem 4.1). Define the Euclidean norms of the elements of $\mathbb{Z}_4$ by $N(0) = 0$, $N(\pm 1) = 1$, $N(2) = 4$, and define $N(u)$, $u = (u_1, \ldots, u_n) \in \mathbb{Z}_4^n$, by $N(u) = \sum N(u_i)$. Then if C is a self-dual code over $\mathbb{Z}_4$ in which the Euclidean norm of every codeword is divisible by 8, Construction A_4 produces an n-dimensional even unimodular lattice.

J. Young and N. J. A. Sloane showed (see [CaSl97]) that the other eight doubly-even binary self-dual codes of length 24 can also be lifted to codes over $\mathbb{Z}_4$ that give the Leech lattice (see also Huffman [Huff98a]).

[CaSl95] considers the codes obtained by lifting the Golay code (and others) from $\mathbb{Z}_2$ to $\mathbb{Z}_4$ to $\mathbb{Z}_8$ to ..., finally obtaining a code over the 2-adic integers $\mathcal{Z}_2$. For more about codes over $\mathbb{Z}_4$ see the Notes on Chapter 16.

For other recent results on the Leech lattice and attempts to generalize it see Bondal et al. [BKT87], Borcherds [Borch90], Conway and Sloane [CoSl94a], Deza and Grishukhin [DezG96], Elkies and Gross [ElkGr96], Harada and Lang [HaLa89], [HaLa90], Koike [Koik86], Kondo and Tasaka [Kon2], [KoTa87], Kostrikin and Tiep [KoTi94], Ozeki [Oze91], Seidel [Sei90b].

Lindsay [Lin88] describes a 24-dimensional 5-modular lattice associated with the proper central extension of the cyclic group of order 2 by the Hall-Janko group J_2 (cf. Chap. 10). The density of this lattice is about a quarter of that of the Leech lattice.

Napias [Napa94] has found some new lattices by investigating cross-sections of the Leech lattice, the 32-dimensional Quebbemann lattice and other lattices.

Shadows and parity (or characteristic) vectors. The notion of the "shadow" of a self-dual code or unimodular lattice, introduced in [CoSl90], [CoSl90a], has proved useful in several contexts, and if we were to rewrite Chapter 4 we would include the following discussion there. We will concentrate on lattices, the treatment for codes being analogous.

Let Λ be an n-dimensional odd unimodular (or Type I) lattice, and let Λ_0 be the even sublattice, of index 2. The dual lattice Λ_0^* is the union of four cosets of

Λ_0, say

$$\Lambda_0^* = \Lambda_0 \cup \Lambda_1 \cup \Lambda_2 \cup \Lambda_3 \ ,$$

where $\Lambda = \Lambda_0 \cup \Lambda_2$. Then we call $S := \Lambda_1 \cup \Lambda_3 = \Lambda_0^* \setminus \Lambda$ the *shadow* of Λ. If Λ is even (or Type II) we define its shadow S to be Λ itself. The following properties are easily established [CoSl90].

If $s \in S$ and $x \in \Lambda$, then $s \cdot x \in \mathbb{Z}$ if $x \in \Lambda_0$, $s \cdot x \in \frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}$ if $x \in \Lambda_1$. In fact the set $2S = \{2s : s \in S\}$ is precisely the set of *parity vectors* for Λ, that is, those vectors $u \in \Lambda$ such that

$$u \cdot x \equiv x \cdot x \ (\mathrm{mod}\ 2) \quad \text{for all} \quad x \in \Lambda \ .$$

Such vectors have been studied by many authors, going back at least as far as Braun [Brau40] (we thank H.-G. Quebbemann for this remark). They have been called *characteristic vectors* [Blij59], [Bor1], [Elki95], [Elki95a], [Mil7], *canonical elements* [Ser1], and *test vectors*. We recommend "parity vector" as the standard name for this concept.

The existence of a parity vector u also follows from the fact that the map $x \to x \cdot x$ (mod 2) is a linear functional from Λ to $\mathbb{F}_2$. The set $2S$ of all parity vectors forms a single class $u + 2\Lambda$ in $\Lambda/2\Lambda$. If Λ is even this is the zero class.

We also note that for any parity vector u, $u \cdot u \equiv n$ (mod 8).

Gerstein [Gers96] gives an explicit construction for a parity vector. Let $v_1, \ldots, v_n$ be a basis for Λ and $v_1', \ldots, v_n'$ the dual basis. Then $\sum c_i v_i$ is a parity vector if and only if $c_i \equiv v_i' \cdot v_i'$ (mod 2) for all i.

Elkies [Elki95], [Elki95a] shows that the minimal norm $p(\Lambda)$ of any parity vector for Λ satisfies $p(\Lambda) \le n$, and $p(\Lambda) = n$ if and only if $\Lambda = \mathbb{Z}^n$. Furthermore, if $p(\Lambda) = n - 8$ then $\Lambda = \mathbb{Z}^{n-r} \oplus M_r$, where M_r is one of the fourteen unimodular lattices whose components are E_8, D_{12}, E_7^2 A_{15}, D_8^2, $A_{11}E_6$, D_6^3, A_9^2, $A_7^2 D_5$, D_4^5, A_5^4, A_3^7, A_1^{22}, O_{23} (using the notation of Chapter 16).

The shadow may also be defined for a more general class of lattices. If Λ is a 2-*integral* lattice (i.e. $u \cdot v \in \mathcal{Z}_2$, the 2-adic integers, for all $u, v \in \Lambda$), and $\Lambda_0 = \{u \in \Lambda \ : \ u \cdot u \in 2\mathcal{Z}_2\}$ is the even sublattice, we define the shadow $S(\Lambda)$ of Λ as follows [RaSl98a]. If Λ is odd, $S(\Lambda) = (\Lambda_0)^* \setminus \Lambda^*$, otherwise $S(\Lambda) = \Lambda^*$. Then

$$S(\Lambda) = \{v \in \Lambda \otimes \mathbb{Q} : 2u \cdot v \equiv u \cdot u \ (\mathrm{mod}\ 2\mathcal{Z}_2) \quad \text{for all} \quad u \in \Lambda\} \ .$$

This includes the first definition of shadow as a special case. The theta series of the shadow (for both definitions) is related to the theta series of the lattice by

$$\Theta_{S(\Lambda)}(z) = (\det \Lambda)^{1/2} \left(\frac{e^{\pi i/4}}{\sqrt{z}}\right)^{\dim \Lambda} \Theta_\Lambda \left(1 - \frac{1}{z}\right) \ . \tag{3}$$

It is also shown in [RaSl98a] that if Λ has odd determinant, then for $u \in S(\Lambda)$,

$$u \cdot u \equiv \frac{1}{4} \text{ oddity } \Lambda \ (\mathrm{mod}\ 2\mathcal{Z}_2) \tag{4}$$

(compare Chap. 15). In particular, if Λ is an odd unimodular lattice with theta series

$$\Theta_\Lambda(z) = \sum_{r=0}^{[n/8]} a_r \theta_3(z)^{n-8r} \Delta_8(z)^r \tag{5}$$

(as in Eq. (36) of Chap. 7), then the theta series of the shadow is given by

$$\Theta_S(z) = \sum_{r=0}^{[n/8]} \frac{(-1)^r}{16^r} a_r \theta_4(2z)\theta_2(z)^{n-8r} \ . \tag{6}$$

For further information about the shadow theory of codes and lattices see [CoSl90], [CoSl90a], [CoSl98], [Rain98], [RaSl98], [RaSl98a]. See also [Dou95]–[DoHa97].

Coordination sequences. Crystallographers speak of "coordination number" rather than "kissing number." Several recent papers have investigated the following generalization of this notion ([BaaGr97], [BrLa71], [GrBS], [MeMo79], [O'Ke91], [O'Ke95]). Let Λ be a (possibly nonlattice) sphere packing, and form an infinite graph ? whose nodes are the centers of the spheres and which has an edge for every pair of touching spheres. The **coordination sequence** of ? with respect to a node $P \in$? is the sequence $S(0)$, $S(1)$, $S(2), \ldots$, where $S(n)$ is the number of nodes in ? at distance n from P (that is, such that the shortest path to P contains n edges).

If Λ is a lattice then the coordination sequence is independent of the choice of P. In [CSLDL7], extending the work of O'Keeffe [O'Ke91], [O'Ke95], we determine the coordination sequences for all the root lattices and their duals. Ehrhart's reciprocity law ([Ehr60]–[Ehr77], [Stan80], [Stan86]) is used, but there are unexpected complications. For example, there are points Q in the 11-dimensional "anabasic" lattice of [CoSl95], mentioned in the Notes to Chapter 1, with the property that $2Q$ is closer to the origin than Q (in graph distance).

We give two examples. For a d-dimensional lattice Λ it is convenient to write the generating function $S(x) = \sum_{n=0}^{\infty} S(n)x^n$ as $P_d(x)/(1-x)^d$, where we call $P_d(x)$ the *coordinator polynomial*. For the root lattice A_d it turns out that

$$P_d(x) = \sum_{k=0}^{d} \binom{d}{k}^2 x^k \ ,$$

and for E_8 we have

$$P_8(x) = 1 + 232x + 7228x^2 + 55384x^3 + 133510x^4 + 107224x^5 + 24508x^6 + 232x^7 + x^8 \ .$$

Thus the coordination sequence of E_8 begins 1, 240, 9120, 121680, 864960, ... For further examples see [BaHVe97], [BaHVe98], [CSLDL7], [GrBS], [SloEIS]. We do not know the coordination sequence of the Leech lattice.

In [CSLDL7] we also show that among all the Barlow packings in three dimensions (those obtained by stacking A_2 layers, cf. [CoSl95a]) the hexagonal close packing has the greatest coordination sequence, and the face-centered cubic lattice the smallest. More precisely, for any Barlow packing,

$$10n^2 + 2 \le S(n) \le [21n^2/2] + 2 \quad (n > 0) \ .$$

For any $n > 1$, the only Barlow packing that achieves either the left-hand value or the right-hand value for all choices of central sphere is the face-centered cubic lattice or hexagonal close-packing, respectively. This interesting result was conjectured by O'Keeffe [O'Ke95]; it had in fact already been established (Conway & Sloane 1993, unpublished notes). There is an assertion on p. 801 of [Hsi93] that is equivalent to saying that any Barlow packing has $S(2) = 44$, and so is plainly incorrect: as shown in [CoSl95a], there are Barlow packings with $S(2) = 42$, 43 and 44. [CSLDL7] concludes with a number of open problems related to coordination sequences.

Notes on Chapter 5: Sphere Packing and Error-Correcting Codes

The **Barnes-Wall lattices** ([Bar18], §6.5 of Chap. 5, §8.1 of Chap. 8) are the subject of a recent paper by Hahn [Hahn90].

On p. 152 of Chap. 5 we remarked that it would be nice to have a list of the best **cyclic codes** of length 127. Such a list has now been supplied by Schomaker and Wirtz [SchW92]. Unfortunately this does not improve the $n = 128$ entry of Table 8.5. Perhaps someone will now tackle the cyclic codes of length 255.

The paper by Ozeki mentioned in the postscript to Chap. 5 has now appeared [Oze87].

Construction B*. The following construction is due to A. Vardy [Vard95], [Vard98] (who gives a somewhat more general formulation). It generalizes the construction of the Leech lattice given in Eqs. (135), (136) of Chap. 4 and §4.4 of Chap. 5, and we refer to it as Construction B* since it can also be regarded as a generalization of Construction B of §3 of Chap.5

Let $\mathbf{0} = 0 \ldots 0$ and $\mathbf{1} = 1 \ldots 1$, and let $\mathcal{B}$ and $\mathcal{C}$ be (n, M, d) binary codes (in the notation of p. 75) such that $c \cdot (\mathbf{1} + b) = 0$ for all $b \in \mathcal{B}$, $c \in \mathcal{C}$. Let Λ be the sphere packing with centers

$$\mathbf{0} + 2b + 4x, \quad \mathbf{1} + 2c + 4y,$$

where x (resp. y) is any vector of integers with an even (resp. odd) sum, and $b \in \mathcal{B}$, $c \in \mathcal{C}$. (We regard the components of b and c as *real* 0's and 1's rather than elements of $\mathbb{F}_2$.) In general Λ is not a lattice.

The most interesting applications arise when d is 7 or 8, in which case it is easily verified that Λ has center density $M\, 7^{n/2}/4^n$ (if $d = 7$ and $n \geq 20$) or $M/2^{n/2}$ (if $d = 8$ and $n \geq 24$).

Vardy [Vard95], [Vard98] uses this construction to obtain the nonlattice packings $\mathcal{B}^*_{20}$ and $\mathcal{B}^*_{27} - \mathcal{B}^*_{30}$ shown in Table I.1. In dimension 20 he uses a pair of $(20, 2^9, 7)$ codes, but we will not describe them here since the same packing will be obtained more simply below. For dimensions 28 and 30 he takes $\mathcal{B} = \mathcal{C}^-$ to be the $[28, 14, 8]$ or $[30, 15, 8]$ double circulant codes constructed by Karlin (see [Mac6], p. 509). Both codes contain $\mathbf{1}$, are not self-dual, but are equivalent to their duals.

For $n = 27$ we shorten the length 28 code to obtain a $[27, 13, 8]$ code $\mathcal{A}$ and set $\mathcal{B} = \mathbf{1} + \mathcal{A}$, $\mathcal{C}$ = even weight subcode of $\mathcal{A}^-$. Similarly for $n = 29$.

Once n exceeds 31, we may use Construction D (see Chap. 8, §8) instead of Construction B*, obtaining a lattice packing from an $[n, k, 8]$ code. In particular, using codes with parameters $[37, 31, 8]$ and $[38, 22, 8]$ (Shearer [Shea88]) we obtain the lattices $\mathcal{D}_{37}$ and $\mathcal{D}_{38}$ mentioned in Table I.1.

As far as is known at the present time, codes with parameters $[32, 18, 8]$, $[33, 18, 8]$, $[34, 19, 8], \ldots, [38, 23, 8]$, $[39, 23, 8]$ might exist. If any one of these could be constructed, a new record for packing density in the corresponding dimension would be obtained for using Construction D.

Bierbrauer and Edel [BiEd98] pointed out that Construction B* also yields a new record 18-dimensional nonlattice packing, using the $[18, 9, 8]$ quadratic residue code and its dual. This packing has center density $(3/4)^9 = 0.07508\ldots$.

An alternative construction of Vardy's 20-dimension packing was given in [CoSl96].

This construction, which we call the **antipode construction**, also produces new records (denoted by $\mathcal{A}_n$ in Table I.1) in dimensions 22 and 44–47. It is an analogue of the "anticode" construction for codes ([Mac6], Chap. 17, Sect. 6). The common theme of the two constructions is that instead of looking for well-separated points, which is what most constructions do, now we look for points somewhere else that are close together and factor them out.

Let Λ be a unimodular lattice (the construction in [CoSl96] is slightly more general) of minimal norm μ in an n-dimension Euclidean space W. Let U, V be respectively k- and l-dimensional subspaces with $W = U \oplus V$, $n = k + l$, such that $\Lambda \cap U = K$ and $\Lambda \cap V = L$ are k- and l-dimensional lattices. Then the projections $\pi_U(\Lambda)$, $\pi_V(\Lambda)$ are the dual lattices K^*, L^*. Suppose we can find a subset $S = \{u_1, \ldots, u_s\} \subseteq K^*$ such that $\operatorname{dist}^2(u_i, u_j) \le \beta$ for all i, j. Then

$$\mathcal{A}(S) = \{\pi_V(w) : w \in \Lambda,\ \pi_U(w) \in S\}$$

is an l-dimensional packing of minimal norm $\mu - \beta$ and center density equal to $\delta = s(\mu - \beta)^{l/2} 2^{-l} / \sqrt{\det L}$.

In dimension 20 we take $\Lambda = \Lambda_{24}$, $L = \Lambda_{20}$, $K = \sqrt{2} D_4$, $K^* = 2^{-1/2} D_4^*$,

$$S = 2^{-1.5}\{0\ 0\ 0\ 0, \quad 1\ 1\ 1\ 1, \quad 2\ 0\ 0\ 0, \quad 1\ 1\ 1\ -1\}, \quad \beta = 1/2\ ,$$

which produces Vardy's packing $\mathcal{A}_{20} \cong \mathcal{B}_{20}^*$ of center density $7^{10}/2^{31} = 0.13154$ and kissing number 15360.

In dimension 22 we take $\Lambda = \Lambda_{24}$, $K = \sqrt{2} A_2$, $K^* = 2^{-1/2} A_2^*$, $S =$ three equally spaced vectors in K^* with $\beta = 1/3$, obtaining a packing $\mathcal{A}_{22}$ with center density $2^{-23} 3^{-10.5} 11^{11} = 0.33254$ and kissing number 41472. To obtain explicit coordinates for this packing, take those vectors of Λ_{24} in which the first three coordinates have the form $2^{-1.5}(a, a, a, \ldots)$, $2^{-1.5}(a+2, a, a, \ldots)$ or $2^{-1.5}(a, a-2, a, \ldots)$ and replace them by their respective projections $2^{-1.5}(a, a, a, \ldots)$, $2^{-1.5}(a+2/3, a+2/3, a+2/3, \ldots)$ and $2^{-1.5}(a-2/3, a-2/3, a-2/3, \ldots)$.

In dimensions 44–47, we take $\Lambda = P_{48p}$, for example, and as on p. 168 find subspaces U in $\mathbb{R}^{48}$ such that K^* is respectively $3^{-1/2} A_1^*$, $3^{-1/2} A_2^*$, $3^{-1/2} A_3^*$, $3^{-1/2} D_4^*$. In these four lattices we can find $s = 2, 3, 4, 4$ points, respectively, for which $\beta = 1/6$, $2/9$, $1/3$, $1/3$, obtaining the packings $\mathcal{A}_{44}$–$\mathcal{A}_{47}$ mentioned in Table I.1.

Notes on Chapter 6: Laminated Lattices

In 1963 Musès ([Cox18], p. 238; [Mus97], p. 7) discovered that the highest possible kissing number for a lattice packing in dimensions $n = 0$ through 8 (but presumably for no higher n) is given by the formula

$$n\left(n + \left\lceil \frac{2^n}{12} \right\rceil\right) \tag{7}$$

where $\lceil x \rceil$ is the smallest integer $\ge x$ (cf. Table 1.1).

All laminated lattices Λ_n in dimensions $n \le 25$ are known, and their kissing numbers are shown in Table 6.3. In dimensions 26 and above, as mentioned on pp. 178–179, the number of laminated lattices seems to be very large, and although they all have the same density, we do not at present know the range of kissing numbers that can be achieved.

In the mid-1980's the authors computed the kissing numbers of one particular sequence of laminated lattices in dimensions 26–32, obtaining values that can be seen in Table I.2. Because of an arithmetical error, the value we obtained in dimension 31 was incorrect. Musès [Mus97] independently studied the (presumed) maximal kissing numbers of laminated lattices (finding the correct value 202692 in dimension 31) and has discovered the formulae analogous to (7).

In the Appendix to Chapter 6, on page 179, third paragraph, it would have been clearer if we had said that, for $n \leq 12$, the integral laminated lattice $\Lambda_n\{3\}$ of minimal norm 3 consists of the projections onto v^- of the vectors of Λ_{n+1} having even inner product with v, where $v \in \Lambda_{n+1}$ is a suitable norm 4 vector. For $n \leq 10$, $K_n\{3\}$ is defined similarly, using K_{n+1} instead of Λ_{n+1}. Also $\Lambda_n\{3\}^-$, $K_n\{3\}^-$ denote the lattices orthogonal to these in $\Lambda_{23}\{3\}$.

A sequel to Plesken and Pohst [Ple6] has appeared — see [Plesk92].

Notes on Chapter 7: Further Connections Between Codes and Lattices

Upper bounds. The upper bounds on the minimal norm μ of a unimodular lattice and the minimal distance d of a binary self-dual code stated in Corollary 10 of Chapter 7 have been strengthened. In [RaSl98a] it is shown that an n-dimensional unimodular lattice has minimal norm

$$\mu \leq 2\left[\frac{n}{24}\right] + 2 , \tag{8}$$

unless $n = 23$ when $\mu \leq 3$. The analogous result for binary codes (Rains [Rain98]) is that minimal distance of a self-dual code satisfies

$$d \leq 4\left[\frac{n}{24}\right] + 4 , \tag{9}$$

unless $n \equiv 22 \pmod 4$ when the upper bound must be increased by 4.

These two bounds are obtained by studying the theta series (or weight enumerator) of the shadow of the lattice (or code) — see Notes on Chapter 4, especially equations (5), (6).

In [RaSl98] and [RaSl98a] it is proposed that a lattice or code meeting (8) or (9) be called **extremal**. This definition coincides with the historical usage for even lattices and doubly-even codes, but for odd lattices extremal has generally meant $\mu = [n/8] + 1$ and for singly-even codes that $d = 2[n/8] + 2$. In view of the new bounds in (8) and (9) the more uniform definition seems preferable. A lattice or code with the highest possible minimal norm or distance is called **optimal**. An extremal lattice or code is *a priori* optimal.

By using (5) and (6) it is often possible to determine the exact values of the highest minimal norm or minimal distance — see Table I.3, which is (essentially) taken from [CoSl90] and [CoSl90a]. The extremal code of length 62 mentioned in the table was recently found by Harada [Hara98a].

In the years since the manuscript of [CoSl90a] was first circulated, over 50 sequels have been written, supplying additional examples of codes in the range of Table I.3. In particular, codes with parameters $[70, 35, 12]$ (filling a gap in earlier versions of the table) were found independently by W. Scharlau and D. Schomaker [ScharS] and M. Harada [Hara97]. Other self-dual binary codes are constructed in [BrP91], [DoGH97a], [DoHa97], [Hara96], [Hara97], [KaT90], [PTL92], [Ton89], [ToYo96], [Tsa91], but these are just a sampling of the recent papers (see [RaSl98]).

Table I.3 Highest minimal norm (μ_n) of an n-dimensional integral unimodular lattice, and highest minimal distance (d_{2n}) of a binary self-dual code of length $2n$.

n	μ_n	d_{2n}	n	μ_n	d_{2n}
1	1	2	19	2	8
2	1	2	20	2	8
3	1	2	21	2	8
4	1	4	22	2	8
5	1	2	23	3	10
6	1	4	24	4	12
7	1	4	25	2	10
8	2	4	26	3	10
9	1	4	27	3	10
10	1	4	28	3	12
11	1	6	29	3	10
12	2	8	30	3	12
13	1	6	31	3	12
14	2	6	32	4	12
15	2	6	33	3	12
16	2	8	34	$3-4$	12
17	2	6	35	$3-4$	$12-14$
18	2	8	36	4	$12-16$

For ternary self-dual (and other) codes see [Hara98], [HiN88], [Huff91], [KsP92], [Oze87], [Oze89b], [VAL93].

The classification of Type I self-dual binary codes of lengths $n \leq 30$ given in [Ple12] (cf. p. 189 of Chap. 7) has been corrected in [CoPS92] (see also [Yor89]).

Lam and Pless [LmP90] have settled a question of long standing by showing that there is no $[24, 12, 10]$ self-dual code over $\mathbb{F}_4$. The proof was by computer search, but required only a few hours of computation time. Huffman [Huff90] has enumerated some of the extremal self-dual codes over $\mathbb{F}_4$ of lengths 18 to 28.

We also show in [CoSl90], [CoSl90a], [CoSl98] that there are precisely five Type I optimal (i.e. $\mu = \mu_n$) lattices in 32 dimensions, but more than 8×10^{22} optimal lattices in 33 dimensions; that unimodular lattices with $\mu = 3$ exist precisely for $n \geq 23$, $n \neq 25$; that there are precisely three Type I extremal self-dual codes of length 32; etc.

Nebe [Nebe98] has found an additional example of an extremal unimodular lattice (P_{48n}) in dimension 48, and Bachoc and Nebe [BacoN98] contruct two extremal unimodular lattices in dimension 80. One of these (L_{80}) has kissing number 1250172000 (see Table I.2). The existence of an extremal unimodular lattice in dimension 72 (or of an extremal doubly-even code of length 72) remains open.

Several other recent papers have studied extremal unimodular lattices, especially in dimensions 32, 40, 48, etc. Besides [CoSl90], [CoSl98], which we have already mentioned, see Bonnecaze et al. [BonCS95], [BonS94], [BonSBM], Chapman [Chap 96], Chapman and Solé [ChS96], Kitazume et al. [KiKM], Koch [Koch86], [Koch90], Koch and Nebe [KoNe93], Koch and Venkov [KoVe89], [KoVe91], etc. Other lattices are constructed in [JuL88].

For doubly-even binary self-dual codes, **Krasikov and Litsyn** [KrLi97] have recently shown that the minimal distance satisfies

$$d \leq 0.166315\ldots n + o(n), \quad n \to \infty \ . \tag{10}$$

No comparable bound is presently known for even unimodular lattices.

For a comprehensive survey of self-dual codes over all alphabets, see Rains and Sloane [RaSl98].

[RaSl98a] also gives bounds, analogous to (8), for certain classes of **modular lattices** (see Notes to Chapter 2), and again there is a notion of **extremal** lattice. (Scharlau and Schulze-Pillot [SchaS98] have proposed a somewhat different definition of extremality for modular lattices.) Bachoc [Baco95], [Baco97] has constructed a number of examples of extremal modular lattices with the help of self-dual codes over various rings. Further examples of good modular lattices can be found in Bachoc and Nebe [BacoN98], Martinet [Mar96], Nebe [Nebe96], Plesken [Plesk96], Tiep [Tiep97a], etc.

For generalizations of the theorems of Assmus-Mattson and Venkov (cf. §7 of Chap. 7), see [CaD92], [CaD92a], [CaDS91], [Koch86], [Koch90], [KoVe89].

Several papers are related to **multiple theta series** of lattices. Peters [Pet90], extending the work of Ozeki [Oze4], has investigated the second-order theta series of extremal Type II (or even) unimodular lattices (cf. §7 of Chap. 7). [Pet89] studies the Jacobi theta series of extremal lattices. See also Böcherer and Schulze-Pillot [BocS91], [BocS97].

The connections between multiple weight enumerators of self-dual codes and Siegel modular forms have been investigated by Duke [Duke93], Ozeki [Oze76], [Oze4], [Oze97] and Runge [Rung93]–[Rung96]. Borcherds, Freitag and Weissauer [BorchF98] study multiple theta series of the Niemeier lattices.

Ozeki [Oze97] has recently introduced another generalization of the weight enumerator of a code C, namely its *Jacobi polynomial*. For a fixed vector $v \in \mathbb{F}^n$, this is defined by

$$Jac_{\,C,v}(x,z) = \sum_{u \in C} x^{wt(u)} z^{wt(u \cap v)} \ .$$

These polynomials have been studied in [BanMO96], [BanO96], [BonMS97]. They have the same relationship to Jacobi forms [EiZa85] as weight enumerators do to modular forms.

P_{48q} **and** P_{48p}**.** G. Nebe informs us that the automorphism groups of P_{48q} and P_{48p} are in fact $SL_2(47)$ and $SL_2(23) \times S_3$. We have modified page 195 accordingly. In the first paragraph on page 195, it would have been clearer if we had said that the vectors of these two lattices have the same *coordinate* shapes.

Notes on Chapter 8: Algebraic Constructions for Lattices

As we discuss in §7 of Chap. 8, there are several constructions for lattices that are based on **algebraic number theory**. The article by Lenstra [Len92] and the books by Bach and Shallit [BacSh96], Cohen [CohCNT] and Pohst and Zassenhaus [PoZ89] describe algorithms for performing algebraic number theory computations. (See also Fieker and Pohst [FiP96].) The computer languages KANT, PARI and MAGMA (see the beginning of this Introduction) have extensive facilities for performing such calculations.

The papers [BoVRB, BoV98] give algebraic constructions for lattices that can be used to design signal sets for transmission over the Rayleigh fading channel.

Corrections to Table 8.1. There are four mistakes in Table 8.1. The entry headed $\omega_{IK} \to (GJH)$ should read

$$\begin{array}{rrrrr} -1 & -2 & 0 & 0 & 0 \\ \sigma & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ -\tau & -2 & 0 & 0 & 2 \end{array},$$

and the entry headed $\omega_{JI} \to (GKH)$ should read

$$\begin{array}{rrrrr} -1 & -2 & 0 & 0 & 0 \\ -\tau & -2 & 0 & 0 & 2 \\ \sigma & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}.$$

Further examples of new packings. **Dimensions 25 to 30.** As mentioned at the beginning of Chapter 17, the 25-dimensional unimodular lattices were classified by Borcherds [Bor1]. All 665 lattices (cf. Table 2.2) have minimal norm 1 or 2.

In dimension 26, Borcherds [Bor1] showed that there is a unique unimodular lattice with minimal norm 3. This lattice, which we will denote by S_{26}, was discovered by J. H. Conway in the 1970's.

The following construction of S_{26} is a modification of one found by Borcherds. We work inside a Lorentzian lattice P which is the direct sum of the unimodular Niemeier lattice A_4^6 and the Lorentzian lattice $I_{2,1}$ (cf. Chaps. 16 and 24). Thus $P \cong I_{26,1}$. Let $\rho = (-2,-1,0,1,2)$ denote the Weyl vector for A_4, so that $\rho' = \rho \oplus \rho \oplus \rho \oplus \rho$ is the Weyl vector for A_4^6, of norm 60, and let $v' = (4, 2 \mid 9) \in I_{2,1}$. Then S_{26} is the sublattice of P that is perpendicular to $v = \rho' \oplus v' \in P$. S_{26} can also be constructed as a complex 13-dimensional lattice over $Q[(1+\sqrt{5})/2]$ ([Con16], p. 62).

Here are the properties of S_{26}. It is a unimodular 26-dimensional lattice of minimal norm 3, center density $\delta = 3^{13}2^{-26} = .0237\ldots$ (not a record), kissing number 3120 (also not a record), with automorphism group of order $2^8.3^2.5^4.13 = 18720000$, isomorphic to $Sp_4(5)$ (cf. [Con16], p. 61; [Nebe96a]). The minimal norm of a parity vector is 10, and there are 624 such vectors. The group acts transitively on these vectors. The theta series begins

$$1 + 3120q^3 + 102180q^4 + 1482624q^5 + \cdots .$$

We do not know the covering radius.

There is a second interesting 26-dimensional lattice, T_{26}, an integral lattice of determinant 3, minimal norm 4 and center density $1/\sqrt{3}$. This is best obtained by forming the sublattice of T_{27} (see below) that is perpendicular to a norm 3 parity vector. T_{26} is of interest because it shares the record for the densest known packing in 26 dimensions with the (nonintegral) laminated lattices Λ_{26}. The kissing number is 117936 and the group is the same as the group of T_{27} below.

Bacher and Venkov [BaVe96] have classified all unimodular lattices in dimensions 27 and 28 that contain no roots, i.e. have minimal norm ≥ 3. In dimension 27 there are three such lattices. In two of them the minimal norm of a parity vector is 11. These two lattices have theta series

$$1 + 2664q^3 + 101142q^4 + 1645056q^5 + \cdots$$

and automorphism groups of orders 7680 and 3317760, respectively. The third, found in [Con16], we shall denote by T_{27}. It has a parity vector of norm 3, theta series

$$1 + 1640q^3 + 119574q^4 + 1497600q^5 + \cdots$$

and a group of order $2^{13}3^5 7^2 13 = 1268047872$, which is isomorphic to the twisted group $2 \times ({}^3D_4(2) : 3)$ ([Con16], p. 89; [Nebe96a]). That this is the unique lattice with a parity vector of norm 3 was established by Borcherds [Bor1].

The following construction of T_{27} is based on the descriptions in [Con16], p. 89 and [Bor1]. Let V be the vector space of 3×3 Hermitian matrices

$$y = \begin{bmatrix} a & C & \bar{B} \\ \bar{C} & b & A \\ B & \bar{A} & c \end{bmatrix} = (a, b, c \mid A, B, C), \quad a, b, c \ \text{ real} \ ,$$

over the real Cayley algebra with units $i_\infty = 1$, $i_0, \ldots, i_6$, in which i_∞, $i_{n+1} \to i$, $i_{n+2} \to j$, $i_{n+4} \to k$ generate a quaternion subalgebra (for $n = 0, \ldots, 6$). V has real dimension $3 + 8 \times 3 = 27$. We define an inner product on V by $Norm(y) = \sum Norm(y_{ij})$. The lattice T_{27} is generated by the 3×3 identity matrix and the 819 images of the norm 3 vectors

$$\underset{(3)}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}}, \quad \underset{(48)}{\begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}}, \quad \underset{(768)}{\begin{bmatrix} 0 & \bar{s} & \bar{s} \\ s & -1/2 & 1/2 \\ s & 1/2 & -1/2 \end{bmatrix}}$$

where $s = (i_\infty + i_0 + \cdots + i_6)/4$, under the group generated by the maps taking $(a, b, c \mid A, B, C)$ to $(a, b, c \mid iAi, iB, iC)$, $(b, c, a \mid B, C, A)$ and $(a, c, b \mid \bar{A}, -\bar{C}, -\bar{B})$, respectively, where $i \in \{\pm i_\infty, \ldots, \pm i_6\}$. These 820 norm 3 vectors and their negatives are all the minimal vectors in the lattice.

The identity matrix and its negative are the only parity vectors in T_{27} of norm 3. Taking the sublattice perpendicular to either vector gives T_{26}, which therefore has the same group as T_{27}.

In 28 dimensions Bacher and Venkov [BaVe96] show that there are precisely 38 unimodular lattices with no roots. Two of these have a parity vector of norm 4 and theta series

$$1 + 1728q^3 + 106472q^4 + \cdots \ ,$$

while for the other 36 the minimal norm of a parity vector is 12 and the theta series

$$1 + 2240q^3 + 98280q^4 + \cdots \ .$$

One of these 36 is the exterior square of E_8, which has group $2 \times G.2$, where $G = O_8^+(2)$ (whereas E_8 itself has group $2.G.2$). One of these 36 lattices also appears in Chapman [Chap97].

Bacher [Bace96] has also found lattices $\mathcal{B}_{27}$, $\mathcal{B}_{28}$, $\mathcal{B}_{29}$ in dimensions 27–29 which are denser than the laminated lattices Λ_{27}–Λ_{29}, and are the densest lattices presently known in these dimensions (although, as we have already mentioned in the Notes to Chapter 5, the densest packings currently known in dimensions 27 to 31 are all nonlattice packings).

$\mathcal{B}_{28}$ can be obtained by taking the even sublattice S_0 of S_{26}, which has determinant 4 and minimal norm 4, and finding translates $r_0 + S_0$, $r_1 + S_0$, $r_2 + S_0$ with $r_0 + r_1 + r_2 \in S_0$ and such that the minimal norm in each translate is 3. We then glue S_0 to a copy of A_2 scaled so that the minimal norm is 4, obtaining a lattice $\mathcal{B}_{28}$ with determinant 3, minimal norm 4, center density $1/\sqrt{3}$ and kissing number 112458. This is a nonintegral lattice since the r_i are not elements of the dual quotient S_0^*/S_0. $\mathcal{B}_{29}$ is obtained in the same way from T_{27}, and has determinant 3, minimal norm 4, center density $1/\sqrt{3}$ and kissing number 109884.

Dimensions 32, 48, 56. Nebe [Nebe98] studies lattices in dimension $2(p-1)$ on which $SL_2(p)$ acts faithfully. For $p \equiv 1 \pmod 4$ these are cyclotomic lattices over quaternion algebras. The three most interesting examples given in [Nebe98] are a 32-dimensional lattice with determinant 17^4, minimal norm $\mu = 6$, center density $\delta = 2^{-16}3^{16}17^{-2} = 2.2728\ldots$, kissing number $\tau = 233376$; a 56-dimensional lattice with det $= 1$, $\mu = 6$, $\delta = (3/2)^{28} = 85222.69\ldots$, $\tau = 15590400$; and a 48-dimensional even unimodular lattice with minimal norm 6 that is not isomorphic to either P_{48p} or P_{48q}, which we will denote by P_{48n}. Its automorphism group contains a subgroup $SL_2(13)$ whose normalizer in the full group is an absolutely irreducible group $(SL_2(13) \otimes SL_2(5)).2^2$.

Dimensions 36 and 60. Kschischang and Pasupathy [KsP92] combine codes over $\mathbb{F}_3$ and $\mathbb{F}_4$ to obtain lattice packings Ks_{36}, Ks_{60} with center densities δ given by

$$\log_2 \delta = 2.1504 \qquad \text{(in 36 dimensions)},$$

$$\log_2 \delta = 17.4346 \qquad \text{(in 60 dimensions)},$$

respectively, thus improving two entries in Table 1.3. Their construction is easily described using the terminology of §8 of Chap. 7. If $\mathcal{E}$ denotes the Eisenstein integers, there are maps

$$\begin{aligned} \pi_4 : &\qquad \mathcal{E} \to \mathcal{E}/2\mathcal{E} \to \mathbb{F}_4 \ , \\ \pi_3 : &\qquad 2\mathcal{E} \to 2\mathcal{E}/2\theta\mathcal{E} \to \mathbb{F}_3 \ , \end{aligned}$$

where $\theta = \sqrt{-3}$. If $\mathcal{C}_1$ is an $[n, k_1, d_1]$ code over $\mathbb{F}_4$, and $\mathcal{C}_2$ is an $[n, k_2, d_2]$ code over $\mathbb{F}_3$, we define Λ to be the complex n-dimensional Eisenstein lattice spanned by the vectors of $(2\theta\mathcal{E})^n$, $\pi_4^{-1}(\mathcal{C}_1)$ and $\pi_3^{-1}(\mathcal{C}_2)$. The real version of Λ (cf. §2.6 of Chap. 2) is then a $2n$-dimensional lattice with determinant $2^{2n-4k_1}3^{3n-2k_2}$ and minimal norm $= \min\{12, d_1, 4d_2\}$. In the case $n = 18$ Kschischang and Pasupathy take $\mathcal{C}_1$ to be the $[18, 9, 8]$ code S_{18} of [Mac4] and $\mathcal{C}_2$ to be the $[18, 17, 2]$ zero-sum code; and for $n = 30$ they take $\mathcal{C}_1$ to be the $[30, 15, 12]$ code Q_{30} of [Mac4] and $\mathcal{C}_2$ to be a $[30, 26, 3]$ negacyclic code. Christine Bachoc (personal communication) has found that Ks_{36} has kissing number 239598.

Elkies [Elki] points out that the densities of **Craig's lattices** (§6 of Chap. 8) can be improved by adjoining vectors with fractional coordinates. However, these improvements do not seem to be enough to produce new record packings, at least in dimensions up to 256. The Craig lattices (among others) have also been investigated by Bachoc and Batut [BacoB92].

Lattices from representations of groups. Of course nearly every lattice in the book could be described under this heading. We have already mentioned the lattices obtained by Nebe and Plesken [NePl95] and Nebe [Nebe96], [Nebe96a] from groups of rational matrices. See also Adler [Adle81].

Scharlau and Tiep [SchaT96], [SchaT96a] have studied lattices arising from representations of the symplectic group $Sp_{2n}(p)$. Among other things, [SchaT96] describes "p-analogues" of the Barnes-Wall lattices.

The concept of a "globally irreducible" lattice was first investigated by Thompson [Tho2], [Tho3], [Thomp76] in the course of his construction of the sporadic finite simple group Th. The construction involves a certain even unimodular lattice 248-dimensional lattice TS_{248} with minimal norm 16 (the **Thompson-Smith lattice**), with $Aut(TS_{248}) = 2 \times Th$. (For more about this lattice see also Smith [Smith76], Kostrikin and Tiep [KoTi94].) This lattice shares with Z, E_8 and the Leech lattice the property of being **globally irreducible**: $\Lambda/p\Lambda$ is irreducible for every prime p.

However, Gross [Gro90] remarks that over algebraic number rings such lattices are more common. He gives new descriptions of several familiar lattices as well as a number of new families of unimodular lattices. Further examples of globally irreducible lattices have been found by Gow [Gow89], [Gow89a]. See also Dummigan [Dum97], Tiep [Tiep91]–[Tiep97b].

Thompson and Smith actually constructed their lattice by decomposing the Lie algebra of type E_8 over $\mathbb{C}$ into a family of 31 mutually perpendicular Cartan subalgebras. Later authors have used other Lie algebras to obtain many further examples of lattices, including infinite families of even unimodular lattices. See Abdukhalikov [Abdu93], Bondal, Kostrikin and Tiep [BKT87], Kantor [Kant96], and especially the book by Kostrikin and Tiep [KoTi94].

Lattices from tensor products. Much of the final chapter of Kitaoka's book [Kita93] is concerned with the properties of tensor products of lattices. The minimal norm of a tensor product $L \otimes M$ clearly cannot exceed the product of the minimal norms of L and M, and may be less. Kitaoka says that a lattice L is of **E-type** if, for any lattice M, the minimal vectors of $L \otimes M$ have the form $u \otimes v$ for $u \in L$, $v \in M$. (This implies $\min(L \otimes M) = \min(L) \min(M)$.) Kitaoka elegantly proves that every lattice of dimension $n \leq 43$ is of E-type.

On the other hand the Thompson-Smith lattice TS_{248} is not of E-type. (Thompson's proof: Let $L = TS_{248}$, and consider $L \otimes L \cong Hom(L, L)$. The element of $L \otimes L$ corresponding to the identity element of $Hom(L, L)$ is easily seen to have norm 248, which is less than the square of the minimal norm of L.) Steinberg ([Mil7], p. 47) has shown that there are lattices in every dimension $n \geq 292$ that are not of E-type.

If an extremal unimodular lattice of dimension 96 (with minimal norm 10) could be found, or an extremal 3-modular lattice in dimension 84 (with minimal norm 16), etc., they would provide lower-dimensional examples of non-E-type lattices.

Coulangeon [Cogn98] has given a generalization of Kitaoka's theorem to lattices over imaginary quadratic fields or quaternion division algebras. Such tensor products provide several very good lattices. Bachoc and Nebe [BacoN98] take as their starting point the lattice L_{20} described on p. 39 of [Con16]. This provides a 10-dimensional representation over $\mathbb{Z}[\alpha]$, $\alpha = (1+\sqrt{-7})/2$ for the group $2.M_{22}.2$. (L_{20} is an extremal 7-modular lattice with minimal norm 8 and kissing number 6160.) Bachoc and Nebe form the tensor product of L_{20} with A_2^2 over $\mathbb{Z}[\alpha]$ and obtain a 40-dimensional extremal 3-modular lattice with minimal norm 8, and of L_{20} with E_8 to obtain an 80-dimensional extremal unimodular lattice L_{80} with minimal norm 8 and kissing number 1250172000 (see Notes on Chapter 1).

Lattices from Riemann surfaces. The period matrix of a compact Riemann surface of genus g determines a real $2g$-dimensional lattice. Buser and Sarnak [BuSa94] have shown that from a sphere packing point of view these lattices are somewhat disappointing: for large g their density is much worse than the Minkowski bound, neither the root lattices E_6, E_8 nor the Leech lattice can be obtained, and so on. Nevertheless, for small genus some interesting lattices occur [BerSl97], [Quin95], [QuZh95], [RiRo92], [Sar95], [TrTr84].

One example, the m.c.c. lattice, has already been mentioned in the Notes on Chapter 1. The period matrix of the Bring curve (the genus 4 surface with largest automorphism group) was computed by Riera and Rodríguez, and from this one can determine that the corresponding lattice is an 8-dimensional lattice with determinant 1, minimal norm 1.4934... and kissing number 20 (see [NeSl]).

Lattices and codes with no group. Etsuko Bannai [Bann90] showed that the fraction of n-dimensional unimodular lattices with trivial automorphism group approaches 1 as $n \to \infty$. Some explicit examples were given by Mimura [Mimu90]. Bacher [Bace94] has found a Type I lattice in dimension 29 and a Type II lattice in dimension 32 with trivial groups $\{\pm 1\}$. Both dimensions are the lowest possible.

Concerning codes, Orel and Phelps [OrPh92] proved that the fraction of binary self-dual codes of length n with trivial group approaches 1 as $n \to \infty$. A self-dual code with trivial group of length 34 (conjectured to be the smallest possible length) is constructed in [CoSl90a], and a doubly-even self-dual code of length 40 (the smallest possible) in [Ton89]. See also [BuTo90], [Hara96], [Huff98], [LePR93], and [Leo8] (for a ternary example).

Notes on Chapter 9: Bounds for Codes and Sphere Packings

Samorodnitsky [Samo98] shows that the Delsarte linear programming bound for binary codes is at least as large as the average of the Gilbert-Varshamov lower bound and the McEliece-Rodemich-Rumsey-Welch upper bound, and conjectures that this estimate is actually the true value of the pure linear programming bound.

Krasikov and Litsyn [KrLi97a] improve on Tietavainen's bound for codes with $n/2 - d = o(n^{1/3})$.

Laihonen and Litsyn [LaiL98] derive a straight-line upper bound on the minimal distance of nonbinary codes which improves on the Hamming, linear programming and Aaltonen bounds.

Levenshtein [Lev87], [Lev91], [Lev92] and Fazekas and Levenshtein [FaL95] have obtained new bounds for codes in finite and infinite polynomial association schemes (cf. p. 247 of Chap. 9).

There have been several recent improvements to the table of bounds on $A(n,d)$ given in Table 9.1 – a revised version appears below in the section "Patch for p. 248." A table of lower bounds on $A(n,d)$ extending to $n \leq 28$ (cf. Table 9.1 of Chap. 9) has been published by Brouwer et al. [BrSSS] (see also [Lits98]). The main purpose of [BrSSS], however, is to present a table of lower bounds on $A(n,d,w)$ for $n \leq 28$ (cf. §3.4 of Chap. 9).

Notes on Chapter 10: Three Lectures on Exceptional Groups

Curtis [Cur89a], [Cur90] discusses further ways to generate the Mathieu groups M_{12} and M_{24} (cf. Chaps. 10, 11).

Hasan [Has89] has determined the possible numbers of common octads in two Steiner systems $S(5,8,24)$ (cf. §2.1 of Chap. 10). The analogous results for $S(5,6,12)$ were determined by Kramer and Mesner in [KrM74].

Figure 10.1 of Chap. 10 classifies the binary vectors of length 24 into orbits under the Mathieu group M_{24}. [CoSl90b] generalizes this in the following way. Let $\mathcal{C}$ be a code of length n over a field $\mathbb{F}$, with automorphism group G, and let $\mathcal{C}_w$ denote the subset of codewords of $\mathcal{C}$ of weight w. Then we wish to classify the vectors of $\mathbb{F}^n$ into orbits under G, and to determine their distances from the various subcodes $\mathcal{C}_w$. [CoSl90b] does this for the first-order Reed-Muller, Nordstrom-Robinson and Hamming codes of length 16, the Golay and shortened Golay codes of lengths 22, 23, 24, and the ternary Golay code of length 12.

For recent work on the subgroup structure of various finite groups (cf. Postscript to Chap. 10) see Kleidman et al. [KlL88], [KlPW89], [KlW87], [KlW90], [KlW90a], Leibeck et al. [LPS90], Linton and Wilson [LiW91], Norton and Wilson [NoW89], Wilson [Wil88], [Wil89]. The "modular" version of the ATLAS of finite groups [Con16] has now appeared [JaLPW].

On page 289, in the proof of Theorem 20, change "$x \cdot y = y \cdot y = 64$" to "$x \cdot x = y \cdot y = 64$". On page 292, 8th line from the bottom, change "$\{i\}$" to "$\{j\}$"

Borcherds points out that in Table 10.4 on page 291 there is a third orbit of type 10 vectors, with group $M_{22}.2$. (Note that [Con16], p. 181, classifies the vectors up to type 16.)

Notes on Chapter 11: The Golay Codes and the Mathieu Groups

For more about the MOG (cf. §5 of Chap. 11) see Curtis [Cur89].

The cohomology of the groups M_{11}, M_{12} and J_1 has been studied in [BenCa87], [AdM91] and [Cha82], respectively.

Notes on Chapter 13: Bounds on Kissing Numbers

Drisch and Sonneborn [DrS96] have given an upper bound on the degree of the best polynomial to use in the main theorem of §1 of Chap. 13.

Notes on Chapter 15: On the Classification of Integral Quadratic Forms

A recent book by Buell [Bue89] is devoted to the study of binary quadratic forms (cf. §3 of Chap. 15). See also Kitaoka's book [Kita93] on the arithmetic theory of quadratic forms (mentioned already in the Notes on Chapter 8). Hsia and Icaza [HsIc97] give an effective version of Tartakovsky's theorem.

For an interpretation of the "oddity" of a lattice, see Eq. (4) of the Notes on

Tables. Nipp ([Nip91] has constructed a table of reduced positive-definite integer-valued four-dimensional quadratic forms of discriminant ≤ 1732. A sequel [Nip91a] tabulates five-dimensional forms of discriminant ≤ 256. These tables, together with a new version of the Brandt-Intrau [Bra1] tables of ternary forms computed by Schiemann can also be found on the electronic *Catalogue of Lattices* [NeSl].

Universal forms. The **15-theorem**. Conway and Schneeberger [Schnee97], [CoSch98] (see also [CoFu97]) have shown that for a positive-definite quadratic form with integer matrix entries to represent all positive integers it suffices that it represent the numbers 1, 2, 3, 5, 6, 7, 10, 14, 15. It is conjectured (the **290-conjecture**) that for a positive-definite quadratic form with integer values to represent all positive integers it suffices that it represent the numbers 1, 2, 3, 5, 6, 7, 10, 13, 14, 15, 17, 19, 21, 22, 23, 26, 29, 30, 31, 34, 35, 37, 42, 58, 93, 110, 145, 203, 290.

The 15-theorem is best-possible in the sense that for each of the nine critical numbers c there is a positive-definite diagonal form in four variables that misses only c. For example $2w^2 + 3x^2 + 4y^2 + 5z^2$ misses only 1, and $w^2 + 2x^2 + 5y^2 + 5z^2$ misses only 15. For the other c in the 290-conjecture the forms are not diagonal and sometimes involve five variables. For example a form that misses only 290 is

$$\begin{bmatrix} 2 & 1/2 & 0 & 0 & 0 \\ 1/2 & 4 & 1/2 & 0 & 0 \\ 0 & 1/2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 29 & 14\,1/2 \\ 0 & 0 & 0 & 14\,1/2 & 29 \end{bmatrix}.$$

For other work on universal forms see Chan, Kim and Raghavan [ChaKR], Earnest and Khosravani [EaK97], [EaK97b], Kaplansky [Kapl95].

M. Newman [Newm94] shows that any symmetric matrix A of determinant d over a principal ideal ring R is congruent to a tridigonal matrix

$$\begin{bmatrix} c_1 & d_1 & 0 & 0 & & & \\ d_1 & c_2 & d_2 & 0 & & & \\ 0 & d_2 & c_3 & d_3 & & & \\ & & & \cdots & & & \\ & & & & d_{n-2} & c_{n-1} & d_{n-1} \\ & & & & 0 & d_{n-1} & c_n \end{bmatrix}$$

in which d_i divides d for $1 \leq i \leq n-2$. In particular, the Gram matrix for a unimodular lattice can be put into tridiagonal form where all off-diagonal entries except perhaps the last one are equal to 1.

[CSLDL1] extends the classification of positive definite integral lattices of small determinant begun in Tables 15.8 and 15.9 of Chap. 15. Lattices of determinants 4 and 5 are classified in dimensions $n \leq 12$, of determinant 6 in dimensions $n \leq 11$, and of determinant up to 25 in dimensions $n \leq 7$.

The four 17-dimensional even lattices of determinant 2 (cf. Table 15.8) were independently enumerated by Urabe [Ura89], in connection with the classification of singular points on algebraic varieties. We note that in 1984 Borcherds [Bor1, Table -2] had already classified the 121 25-dimensional even lattices of determinant 2. Even lattices of dimension 16 and determinant 5 have been enumerated by Jin-Gen Yang [Yan94], and other lattice enumerations in connection with classification of singularities can be found in [Tan91], [Ura87], [Ura90], [Wan91]. For the connections between lattices and singularities, see Eberling [Ebe87], Kluitmann [Klu89], Slodowy [Slod80], Urabe [Ura93], Voigt [Voi85].

Kervaire [Kerv94] has completed work begun by Koch and Venkov and has shown that there are precisely 132 indecomposable even unimodular lattices in **dimension 32** which have a "complete" root system (i.e. the roots span the space). Only 119 distinct root systems occur.

Several recent papers have dealt with the construction and classifications of lattices, especially unimodular lattices, over rings of integers in number fields, etc. See for example Bayer-Fluckiger and Fainsilber [BayFa96], Benham et al. [BenEHH], Hoffman [Hof91], Hsia [Hsia89], Hsia and Hung [HsH89], Hung [Hun91], Takada [Tak85], Scharlau [Scha94], Zhu [Zhu91]–[Zhu95b].

Some related papers on class numbers of quadratic forms are Earnest [Earn88]–[Earn91], Earnest and Hsia [EaH91], Gerstein [Gers72], Hashimoto and Koseki [HaK86].

Hsia, Jochner and Shao [HJS], extending earlier work of Friedland [Fri89], have shown that for any two lattices Λ and M of dimension > 2 and in the same genus (cf. §7 of Chap. 15), there exist isometric primitive sublattices Λ' and M' of codimension 1.

Fröhlich and Thiran [FrTh94] use the classification of Type I lattices in studying the quantum Hall effect.

Erdős numbers. An old problem in combinatorial geometry asks how to place a given number of distinct points in n-dimensional Euclidean space so as to minimize the total number of distances they determine ([Chu84], [Erd46], [ErGH89], [SkSL]). In 1946 Erdős [Erd46] considered configurations formed by taking all the points of a suitable lattice that lie within a large region. The best lattices for this purpose are those that minimize what we shall call the *Erdős number* of the lattice, given by

$$E = Fd^{1/n},$$

where d is the determinant of the lattice and F, its *population fraction*, is given by

$$F = \lim_{x\to\infty} \frac{P(x)}{x}, \quad \text{if} \quad n \geq 3,$$

where $P(x)$ is the population function of the corresponding quadratic form, i.e. the number of values not exceeding x taken by the form.[2] The Erdős number is the population fraction when the lattice is normalized to have determinant 1. It turns out that minimizing E is an interesting problem in pure number theory.

In [CoSl91]] we prove all cases except $n = 2$ (handled by Smith [Smi91]) of the following proposition:

> The lattices with minimal Erdős number are (up to a scale factor) the even lattices of minimal determinant. For $n = 0, 1, 2, \ldots$ these determinants are
>
> $1, 2, 3, 4, 4, 4, 3, 2, 1, 2, 3, 4, 4, 4, \ldots,$
>
> this sequence continuing with period 8.

[2] For $n \leq 2$ these definitions must be modified. For $n = 0$ and 1 we set $E = 1$, while for $n = 2$ we define F by $F = \lim_{x\to\infty} x^{-1}P(x)\sqrt{\log x}$.

For $n \le 10$ these lattices are unique:

$$A_0, A_1, A_2, A_3 \simeq D_3, D_4, D_5, E_6, E_7, E_8, E_8 \oplus A_1, E_8 \oplus A_2,$$

with Erdős numbers

$$1,\ 1,\ 2^{-3/2}3^{1/4} \prod_{p \equiv 2(3)} \left(1 - \frac{1}{p^2}\right)^{-1/2} = 0.5533,$$

$$\frac{11}{24}4^{1/3} = 0.7276,\ \ \frac{4^{1/4}}{2} = 0.7071,\ \ \frac{4^{1/5}}{2} = 0.6598,\ \ \frac{3^{1/6}}{2} = 0.6005,$$

$$\frac{2^{1/7}}{2} = 0.5520,\ \ \frac{1}{2},\ \ \frac{2^{1/9}}{2} = 0.5400,\ \ \frac{3^{1/10}}{2} = 0.5581,$$

(rounded to 4 decimal places), while for each $n \ge 11$ there are two or more such lattices. The proof uses the p-adic structures of the lattices (cf. Chap. 15). The three-dimensional case is the most difficult. The crucial number-theoretic result needed for our proof was first established by Peters [Pet80] using the Generalized Riemann Hypothesis. The dependence on this hypothesis has been removed by Duke and Schulze-Pillot [DuS90].

Notes on Chapter 16: Enumeration of Unimodular Lattices

Recent work on the classification of lattices of various types has been described in the Notes on Chapter 15.

The **mass formula** of H. J. S. Smith, H. Minkowski and C. L. Siegel (cf. §2 of Chap. 16) expresses the sum of the reciprocals of the group orders of the lattices in a genus in terms of the properties of any one of them. In [CSLDL4] we discuss the history of the formula and restate it in a way that makes it easier to compute. In particular we give a simple and reliable way to evaluate the 2-adic contribution. Our version, unlike earlier ones, is visibly invariant under scale changes and dualizing. We then use the formula to check the enumeration of lattices of determinant $d \le 25$ given in [CSLDL1]. [CSLDL4] also contains tables of the "standard mass," values of the L-series $\Sigma(\frac{n}{m})m^{-s}$ (m odd), and genera of lattices of determinant $d \le 25$.

Eskin, Rudnick and Sarnak [ERS91] give a new proof of the mass formula using an "orbit-counting" method. Another proof is given by Mischler [Misch94].

The classification of the Niemeier lattices is rederived by Harada, Lang and Miyamoto [HaLaM94]. Montague [Mont94] constructs these lattices from ternary self-dual codes of length 24, and Bonnecaze et al. [BonGHKS] construct them from codes over $\mathbb{Z}_4$ using Construction A_4.

Codes over $\mathbb{Z}_4$. The glue code for the Niemeier lattice with components A_4^6 (cf. Table 16.1 of Chap. 16) is a certain linear code (the **octacode**) of length 8 over the ring $\mathbb{Z}_4$ of integers mod 4. This code may be obtained from the code generated by all cyclic shifts of the vector (3231000) by appending a zero-sum check symbol. The octacode contains 256 vectors and is self-dual with respect to the standard inner product $x \cdot y = \sum x_i y_i$ (mod 4). It is the unique self-dual code of length 8 and minimal Lee weight 6 [CoSl93].

In [FoST93] it is shown that if the octacode is mapped to a binary code of twice the length using the **Gray map**:

$$0 \to 00, \quad 1 \to 01, \quad 2 \to 11, \quad 3 \to 10, \tag{11}$$

then we obtain the (nonlinear) Nordstrom-Robinson code (mentioned in §2.12 of Chapter 2).

Furthermore, if we apply "Construction A_4" (see Notes on Chapter 4) to the octacode, we obtain the E_8 lattice [BonS94], [BonCS95].

The octacode may also be obtained by Hensel-lifting the Hamming code of length 7 from $GF(2)$ to $\mathbb{Z}_4$. If the same process is applied to an arbitrary binary Hamming code of length $2^{2m+1}-1$ ($m \geq 1$) we obtain the (nonlinear) Preparata and Kerdock codes [HaKCSS]. If this process is applied to the binary Golay code of length 23, then as already mentioned in the Notes on Chapter 4, we obtain a code over $\mathbb{Z}_4$ that lifts by Construction A_4 to the Leech lattice [BonS94], [BonCS95].

Many other nonlinear binary codes also have a simpler description as $\mathbb{Z}_4$ codes. Consider, for example, the $(10, 40, 4)$ binary nonlinear code found by **Best** [Bes1], which leads via Construction A to the densest 10-dimensional sphere packing presently known (Chapter 5, p. 140). This code now has the following simple description [CoSl94b]: take the 40 vectors of length 5 over $\mathbb{Z}_4$ obtained from

$$(c-d, b, c, d, b+c), \quad b, c, d \in \{+1, -1\} ,$$

and its cyclic shifts, and apply the Gray map (11). Litsyn and Vardy [LiV94] have shown that the Best code is unique.

In some cases the $\mathbb{Z}_4$-approach has also led to (usually nonlinear) binary codes that are better than any previously known: see Calderbank and McGuire [CaMc97], Pless and Qian [PlQ96], Shanbhag, Kumar and Helleseth [ShKH96], etc.

For other recent papers dealing with codes over $\mathbb{Z}_4$ see [DoHaS97], [Huff98a], [Rain98a], [RaSl98].

Bannai et al. [BanDHO97] investigate self-dual codes over $\mathbb{Z}_{2k}$ and their relationship with unimodular lattices.

Notes on Chapter 17: The 24-Dimensional Odd Unimodular Lattices

The chapter has been completely retyped for this edition, in order to describe the enumeration process more clearly, and to correct several errors in the tables.

Neighbours. Borcherds uses Kneser's notion of neighboring lattices [Kne4], [Ven2], defined as follows. Two lattices L and L' are *neighbors* if their intersection $L \cap L'$ has index 2 in each of them. We note the following properties.

(i) Let L be a unimodular lattice. Suppose $u \in L$, $1/2\ u \notin L$ and $u \cdot u \in 4\mathbb{Z}$. Let $L_u = \{x \in L : x \cdot u \equiv 0 \pmod 2\}$ and $L^u = \langle L_u, 1/2\ u \rangle$. Then L^u is a unimodular neighbor of L, all unimodular neighbors of L arise in this way, and u, u' produce the same neighbor if and only if $1/2\ u \equiv 1/2\ u' \bmod L_u$.

(ii) Let L be an even unimodular lattice. Suppose $u \in L$, $1/2\ u \notin L$ and $u \cdot u \in 4\mathbb{Z}$. Then L^u is an even unimodular lattice if and only if $u \cdot u \in 8\mathbb{Z}$. If $1/2\ u \equiv 1/2\ u' \bmod L$ then $L^u = L^{u'}$ and $1/2\ u \equiv 1/2\ u' \pmod{L_u}$.

(iii) Define the *integral part* of an arbitrary lattice M to be

$$[M] = M \cap M^* = \{l \in M : l \cdot M \subseteq \mathbb{Z}\} \ . \qquad (12)$$

If L is unimodular, and $u \in L$, $1/2\ u \notin L$, $u \cdot u \in 4\mathbb{Z}$, then the neighbor L^u is also equal to

$$[\ \langle\ L,\ 1/2\ u\ \rangle\ ]\ . \qquad (13)$$

The proofs are straightforward, and are written out in full in [Wan96]. (Note that a neighbor of a unimodular lattice need not be unimodular: it must have determinant 1, but need not be integral. Overlooking this point caused some inaccuracies in the statement of these properties in the second edition.)

For example, if $L = \mathbf{Z}^8$ and $x = (1/2)^8$, we obtain $B = E_8$.

As a second example, let L be the Niemeier lattice A_1^{24}, which we take in the form obtained by applying Construction A to $\mathcal{C}_{24}$ (see §1 of the previous chapter). Any $a \in A_1^{24}$ has the shape $8^{-1/2}\ (2c + 4z)$ for $c \in \mathcal{C}_{24}$, $z \in \mathbf{Z}^{24}$. There are just two inequivalent choices for x. If $x = 8^{-1/2}(2^{24})$ then L^u is the odd Leech lattice O_{24}, with minimal norm 3 (see the Appendix to Chap. 6). If $x = 8^{-1/2}(-6, 2^{23})$ then L^u is the Leech lattice Λ_{24} itself. In fact A_1^{24} is the unique even neighbor of Λ_{24} (see Fig. 17.1).

Eq. (5) can be used to construct new lattices even if x does not satisfy (1). For example many years ago Thompson [Tho7] showed that the Leech lattice can be obtained from D_{24} by

$$\Lambda_{24} = [\langle D_{24}, x\rangle] \qquad (14)$$

where

$$x = \left(\frac{0}{47}, \frac{1}{47}, \ldots, \frac{23}{47}\right)\ . \qquad (15)$$

But since the intersection $\Lambda_{24} \cap D_{24}$ has index 47 in each of them, Λ_{24} and D_{24} are not neighbors in our sense.

Notes on Chapter 18: Even Unimodular 24-Dimensional Lattices

An error in Venkov's proof was found and corrected by Wan [Wan96].

There are also two tiny typographical errors on page 435. In line 13, change "x_2" to "x_4", and in line 18 from the bottom, change "$x_1 = (0, 0, 1, 1, -2, -2)$" to "$x_1 = (0, 0, 1, 1, 2, -2)$".

Scharlau and Venkov [SchaV94] use the method of Chapter 17 to classify all lattices in the genus of BW_{16}.

Notes on Chapter 20: Finding Closest Lattice Point

The following papers deal with techniques for finding the closest lattice point to a given point (cf. Chap. 20), and with the closely related problem of "soft decision" decoding of various binary codes. Most of these are concerned with the Golay code and the Leech lattice. [AdB88], [Agr96], [Allc96], [Amr93], [Amr94], [BeeSh91], [BeeSh92], [Forn89], [FoVa96], [LaLo89], [RaSn93], [RaSn95], [RaSn98], [SnBe89], [Vard94], [Vard95a], [VaBe91], [VaBe93].

Notes on Chapter 21: Voronoi Cells of Lattices and Quantization Errors

Because of an unfortunate printer's error, the statement of one of the most important theorems on Voronoi cells (Theorem 10 of Chap. 21) was omitted from the first edition. This has now been restored at the top of page 474. Concerning the previous theorem (Theorem 9), Rajan and Shende [RaSh96] have shown that the conditions of the theorem are necessary and sufficient for the conclusion, and also that the only lattices whose relevant vectors are precisely the minimal vectors are the root lattices (possibly rescaled). Geometrically, this assertion is just that the root lattices are the only lattices whose Voronoi cells have inscribed spheres.

Viterbo and Biglieri [ViBi96] give an algorithm for computing the Voronoi cell of a lattice. A recent paper by Fortune [Fort97] surveys techniques for computing Voronoi and Delaunay regions for general sets of points.

Notes on Chapter 23: The Covering Radius of the Leech Lattice

Deza and Grishukhin [DezG96] claim to give a simpler derivation of the covering radius of the Leech lattice, obtained by a refinement of Norton's argument of Chapter 22. However, as the reviewer in *Math. Reviews* points out (MR98b: 11074), there is a gap in their argument. A correction will be published in *Mathematika*.

Notes on Chapter 25: The Cellular Structure of the Leech Lattice

On page 520, in Fig. 25.1(b) the shaded node closest to the top of the page should not be shaded.

Notes on Chapter 27: The Automorphism Group of the 26-Dimensional Lorentzian Lattice

Borcherds [Borch90] has generalized some of the results of Chap. 27, by showing that, besides the Leech lattice, several other well-known lattices, in particular the Coxeter-Todd lattice K_{12} and the Barnes-Wall lattice BW_{16}, are related to Coxeter diagrams of reflection groups of Lorentzian lattices.

Borcherds (personal communication) remarks that some of the questions on the last page of the chapter can now be answered.

(ii) There is a proof of the main result of this chapter that does not use the covering radius of the Leech lattice on page 199 of [Borch95a], but it is very long and indirect.

(iv) The group is transitive for $\mathrm{II}_{33,1}, \ldots$. This follows from Corollary 9.7 of [Borch87].

(v) The answer is yes: see Notes on Chapter 30.

Kondo [Kon97] has made use of the results in this chapter and in [Borch87] in determining the full automorphism group of the Kummer surface associated with a generic curve of degree 2.

Notes on Chapter 28: Leech Roots and Vinberg Groups

A lattice Λ in Lorentzian space $\mathbb{R}^{n,1}$ is called **reflexive** if the subgroup of $\mathrm{Aut}(\Lambda)$ generated by reflections has finite index in $\mathrm{Aut}(\Lambda)$. Esselmann [Ess90] (extending the work of Makarov [Mak65], [Mak66], Vinberg [Vin1]–[Vin13], Nikulin [Nik3], Prokhorov [Pro87] and the present authors, see Chap. 28) shows that for $n = 20$ and $n \geq 22$ reflexive lattices do not exist, and that for $n = 21$ the example found by Borcherds (the even sublattice of $I_{21,1}$ of determinant 4) is essentially the only one.

Allcock ([Allc97], [Allc97a]) found several new complex and quaternionic hyperbolic reflection groups, the largest of which behave rather like the reflection group of $\mathrm{II}_{25,1}$ described in Chapter 27.

Two other recent papers dealing with the Lorentzian lattices and related topics are Scharlau and Walhorn [SchaW92], [SchaW92a].

Notes on Chapter 30: A Monster Lie Algebra?

The Lie algebra of this chapter is indeed closely related to the Monster simple group. In order to get a well behaved Lie algebra it turns out to be necessary to add some imaginary simple roots to the "Leech roots." This gives the *fake* Monster Lie algebra, which contains the Lie algebra of this chapter as a large subalgebra. See [Borch90a] for details (but note that the fake Monster Lie algebra is called the Monster Lie algebra in this paper). The term "Monster Lie algebra" is now used to refer to a certain "Z/2Z-twisted" version of the fake Monster Lie algebra. The Monster Lie algebra is acted on by the Monster simple group, and can be used to show that the Monster module constructed by Frenkel, Lepowsky, and Meurman satisfies the moonshine conjectures; see [Borch92a].

For other recent work on the Monster simple group (Chap. 29) and related Lie algebras see [Borch86]–[BorchR96], [Con93], [Fre5], [Iva90], [Iva92], [Iva92a], [Lep88], [Lep91], [Nor90], [Nor92].

For further work on "Monstrous Moonshine" see Koike [Koik86], Miyamoto [Miya95], the conference proceedings edited by Dong and Mason [DoMa96], and the Cummins bibliography [Cumm].

Errata for the Low-Dimensional Lattices papers [CSLDL1] – [CSLDL7]

[CSLDL1]. On page 37, in line 8 from the bottom, the second occurrence of $p \equiv 1$ should be $p \equiv -1$.

[CSLDL2]. On page 41, in Table 1, the first occurrence of $Q_{23}(4)^{+4}$ should be $Q_{23}(4)^{+2}$.

[CSLDL3]. On page 56, in Table 1, the entry for P_5^3 should have $\mu = 2$ (not 4). On page 62, at about line 10, "$78k_c$" should be "$78c_k$". On page 68, in the entry for p_7^{12}, the third neighbor should be changed from "p_7^{28}" to "p_7^{4}". In the entry for

p_7^{13}, the six neighbors should read, in order, "$p_7^{11}, p_7^5, p_7^1, p_7^{17}, p_7^1, p_7^1$". On page 70, in the entry for p_7^{17}, the second neighbor should be changed from "p_7^{15}" to "p_7^7".

[CSLDL4]. In line 1 of page 273, change "7" to "3".

Index to Preface to Third Edition

Patches for Third Edition of SPLAG

Note the printer: these are to be pasted over the appropriate sections of the marked pages.

In case of difficulty, contact: N. J. A. Sloane, 973-360-8415 or 732-828-6098; fax: 973-368-8178.

Patch for page iv

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Patch for p. xii

We should like to express our thanks to E. S. Barnes, H. S. M. Coxeter, Susanna Cuyler, G. D. Forney, Jr., W. M. Kantor, J. J. Seidel, J.-P. Serre, P. N. de Souza, and above all John Leech, for their comments on the manuscript. Further acknowledgements appear at the end of the individual chapters. Any errors that remain are our own responsibility: please notify N. J. A. Sloane, Information Sciences Research, AT&T Labs - Research, 180 Park Avenue, Florham Park, NJ 07932-0971, USA (email: njas@research.att.com). We would also like to hear of any improvements to the tables.

Patch for p. 14

Figure 1.5. The densest sphere packings known in dimensions $n \le 48$. The vertical axis gives $\log_2 \delta + n(24-n)/96$, where δ is the center density. The Λ_n are laminated lattices, the K_n are described in Chap. 6, K_{12} is the Coxeter-Todd lattice, the crosses are nonlattice packings (Chap. 5, §§2.6,4.3), and Q_{32}, B_{36} and P_{48q} are described in Table 1.3a. The upper bound is Rogers' bound (39), (40). (See also Table I.1 of the Introduction.)

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Table 1.2. Sphere packings in up to 24 dimensions. (See Tables I.1, I.2 of the Introduction for recent improvements.)

Patch for p. 16

Table 1.3. Sphere packings in more than 24 dimensions. (See Tables I.1, I.2 of the Introduction for recent improvements.)

n	Name	$\log_2 \delta$	Bound	Kissing no.	Ch., §
32	Λ_{32}	0	5.52	208320	6,7
	BW_{32}	0		146880	8,8.2f
	C_{32}	1		249280	8,8.2h
	Q_{32}	1.359		261120	8,4
36	P_{36p}	-1	8.63	42840	5,5.5
	Λ_{36}	1		234456	5,5.3
	B_{36}	2			8,8.2d
48	Λ_{48}	12	15.27		6,7
	P_{48p}	14.039		52416000	5,5.7
	P_{48q}	14.039		52416000	5,5.7
60	P_{60p}	16.548	27.85	3908160	5,5.5
64	BW_{64}	16	31.14	9694080	8,8.2f
	Q_{64}	18.719		2611200	8,2
	P_{64c}	22			8,8.2e
80	$\eta(E_8)$	36	49.90		8,10c
96	$\eta(P_{48q})$	52.078	70.96		8,10g
104	$\eta(E_8)$	60	80.20		8,10c
128	BW_{128}	64	118.6	1260230400	8,8.2f
	P_{128b}	85			5,6.6
	$\eta(E_8)$	88			8,10c
136	$\eta(E_8)$	100	129.4		8,10c
150	$A_{150}^{(15)}$	113.06	153.2		8.6
180	$\eta(\Lambda_{20})$	133	206.7		8,10c
	$A_{180}^{(17)}$	154.12			8,6

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Table 1.3 (cont.)

n	Name	$\log_2 \delta$	Bound	Ch., §
192	$\eta(\Lambda_{24})$	156	230.0	8,10e
	$A_{192}^{(18)}$	171.44		8,6
256	BW_{256}	192	357.0	8,8.2f
	B_{256}	250		8,8.2g
	$A_{256}^{(23)}$	270.89		8,6
508	$A_{508}^{(41)}$	742.66	948.1	8,6
512	B_{512}	698	957.4	8,8.2g
520	$A_{520}^{(42)}$	767.46	980.1	8,6
1020	$A_{1020}^{(74)}$	1922	2406	8,6
1030	$A_{1030}^{(74)}$	1947	2439	8,6
2052	$A_{2052}^{(135)}$	4755	5871	8,6
4096	$\eta(\Lambda_{16})$	11344	13750	8,10c
4098	$A_{4098}^{(246)}$	11279	13758	8,6
8184	$\eta(\Lambda_{24})$	26712	31547	8,10e
8190	$A_{8190}^{(454)}$	26154	31573	8,6
8208	$\eta(\Lambda_{24})$	26808	31655	8,10e
16380	$A_{16380}^{(844)}$	59617	71325	8,6
16392	$\eta(\Lambda_{24})$	61608	71387	8,10e
32784	$\eta(\Lambda_{24})$	139488	159154	8,10e
65520	$\eta(\Lambda_{24})$	311364	350788	8,10e
65544	$\eta(\Lambda_{24})$	311496	350932	8,10e
131088	$\eta_3(\Lambda_{24})$	664962	767395	8,10f
262152	$\eta_3(\Lambda_{24})$	$1.475 \cdot 10^6$	$1.666 \cdot 10^6$	8,10f
524304	$\eta_3(\Lambda_{24})$	$3.178 \cdot 10^6$	$3.594 \cdot 10^7$	8,10f
1048584	$\eta_3(\Lambda_{24})$	$6.918 \cdot 10^6$	$7.711 \cdot 10^7$	8,10f

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Figure 2.4. The thickness of various lattice coverings in dimensions $n \leq 24$. The values for Λ_{13} to Λ_{15} and Λ_{17} to Λ_{23} are lower bounds. (They are computed using the *subcovering radius* of Chap. 6.) See Notes on Chapter 2 for recent improvements.

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Table 2.1. Coverings in up to 24 dimensions. (See Notes on Chapter 2 for recent improvements.)

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Table 2.3. Bounds for G_n, the mean squared error of the optimal n-dimensional quantizer. (92) is our conjectured lower bound. (See also Introduction to Third Edition.)

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where i, j, k, ℓ are integers with $i+j+k=0$ [Slo19]. With the first definition the Voronoi cells have volume $1/\sqrt{2}$, minimal norm $=1$, $\tau=12$, minimal vectors $=$ $(\pm 1,0,0)$, $(\pm 1/2, \pm\sqrt{3}/2, 0)$, $(0,-1/\sqrt{3}, \pm\sqrt{2/3})$ and $(\pm 1/2, 1/\sqrt{12}, \pm\sqrt{2/3})$, $\rho =$ $1/2$, $\Delta = \pi/3\sqrt{2} = 0.7405\ldots$, and $R = \rho\sqrt{2} = 1/\sqrt{2}$. There are two kinds of holes, deep or octahedral holes such as $(0, 1/\sqrt{3}, 1/\sqrt{6})$ and shallow or tetrahedral holes such as $(1/2, 1/\sqrt{12}, 1/\sqrt{24})$. Theta series:

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Then det $=4$, minimal norm $=2$, kissing number $\tau = 2n(n-1)$, minimal vectors $=$ all permutations of $(\pm 1, \pm 1, 0, \ldots, 0)$, $h = 2n-2$, packing radius $\rho = 1/\sqrt{2}$, center density $\delta = 2^{-(n+2)/2}$, and covering radius $R = \rho\sqrt{2}$ $(n=3)$ or $\rho\sqrt{n/2}$ $(n \geq 4)$. There are two kinds of hole, deep holes such as the glue vectors [2] (if $n \leq 4$) or [1] (if $n \geq 4$), and shallow holes such as the glue vectors [1] (if $n \leq 4$) or [2] (if $n \geq 4$). For $n=4$ there is only one kind of hole. The Voronoi cell is given in Chap. 21, and the Delaunay cells in §4.2 of Chap. 5. Glue vectors:

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maps the first version into the second. ((90) also defines the dual lattice D_4^* — see §7.4 — showing that $D_4 \cong D_4^*$.) Using the definition (86), det $=4$, minimal norm $=2$, $\tau = 24$, minimal vectors $=$ all permutations of $(\pm 1, \pm 1, 0, 0)$, $h=6$, $\rho = 1/\sqrt{2}$, $\Delta = \pi^2/16 = 0.6169\ldots$ $(\delta = 1/8)$, $R = \rho\sqrt{2} = 1$, typical deep holes $= (\pm 1/2, \pm 1/2, \pm 1/2, \pm 1/2)$ and $(0,0,0,\pm 1)$, $\Theta = \pi^2/4 = 2.4674\ldots$. The Voronoi cell is the regular 4-dimensional polytope known as the 24-cell or $\{3,4,3\}$ (Chap. 21, [Cox20, §8.2]). The three nonzero cosets $D_4 + [i]$, $i = 1,2,3$, are equivalent. Theta series

$$\Theta_{D_4}(z) = 1/2(\theta_3(z)^4 + \theta_4(z)^4) = \theta_2(2z)^4 + \theta_3(2z)^4 \tag{92}$$

(cf. Eq. (89) of Chap. 7). D_4 is the unique lattice with this density.

Using the second definition, (90), D_4 consists of the vectors (a, b, c, d) with a, b, c, d all in $\mathbf{Z}$ or all in $\mathbf{Z} + {}^1\!/\!_2$, and therefore may be regarded as the lattice of Hurwitz *integral quaternions* ([Cox8], [Cox18, p. 25], [Har5, §20.6], [Hur1]). This gives the lattice points the structure of a skew domain. As an Eisenstein lattice, D_4 is generated by (2,0) and $(1, \theta)$.

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holes such as $((1/6)^7, 5/6)$, surrounded by 9 lattice points (see Fig. 21.8). $E_8^* = E_8$, so there is no glue (or more precisely the only glue is $[0] = (0^8)$).

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and from Construction B$_3$, if d_e and d_o are respectively the minimal even and odd distances between codewords,

$$\rho = \min\{3/\sqrt{2}, {}^1\!/\!_2\sqrt{d_e}, {}^1\!/\!_2\sqrt{d_o + 3}\} , \tag{8}$$

$$\delta = M\rho^n 2^{-1} 3^{-n} . \tag{9}$$

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P_{48q} and P_{48p} have det = 1, minimal norm = 6, τ = 52416000, their minimal vectors are described in Table 5.3, $\rho = \sqrt{3}/2$, $\Delta = 0.00000002318\ldots$ ($\delta = (3/2)^{24} = 16834.112\ldots$), and the covering radii R are not known, but $R \geq 2$, corresponding to putatively deep holes $c(2^{12}, 0^{36})$. We have $\mathrm{Aut}(P_{48q}) = SL_2(47)$, and $\mathrm{Aut}(P_{48p}) = SL_2(23) \times \mathsf{S}_3$ [Tho7]. (It is because there is no suitable group containing both $L_2(23)$ and $L_2(47)$ that we know these lattices are inequivalent.) Theta series: see Eq. (68) and Eq. (57) of Chap. 2, Table 7.1. For cross sections see Corollary 8 of Chap. 6.

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(c) *The lattices* $A_n^{(m)}$. Craig's lattices $A_n^{(m)}$ described in §6 were originally constructed in [Cra5] as the lattices $\Lambda(\mathcal{A})$, where $\mathcal{A}$ is the ideal $((1 - \zeta_p)^m)$ in the cyclotomic field $\mathbf{Q}(\zeta_p)$ and $p = n + 1$ is a prime.

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Table 9.1. The best codes: bounds for $A(n,d)$ (a: [Bes1], b: [Pul82], c: [Rom1], d: [Ham88], e: [Kaik98], g: Etzion [CHLL], p. 58, k: [KiHa98], o: [Ost98]).

n	$d=4$	$d=6$	$d=8$	$d=10$
6	4	2	1	1
7	8	2	1	1
8	16	2	2	1
9	20	4	2	1
10	40^a	6	2	2
11	72^o	12	2	2
12	144^o	24	4	2
13	256	32	4	2
14	512	64	8	2
15	1024	128	16	4
16	2048	256	32	4
17	2720^c-3276	256-340	36-37	6
18	5312^g-6552	512-680	64-72^b	10
19	10496^d-13104	1024-1288	128-144	20
20	20480^a-26208	2048-2372	256-279	40
21	36864-43690	2560-4096	512	42^e-48^b
22	73728-87380	4096-6942	1024	50^e-88^b
23	147456-173784	8192-13774	2048	76^e-150
24	294912-344308^k	16384-24106	4096	128-280

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(12 13 14)(21 7 18)(17 1 20)(2 19 15)(6 3 11)(∞ 5 10)(16 0 9)(4 22 8),

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A pretty series of subgroups of $\cdot 0$ arises in the following way [Tho7]. The centralizer of a certain element x of order 3 in $\cdot 0$ has the form $\langle x\rangle \times 2\mathsf{A}_9$, the $2\mathsf{A}_9$ being the Schur double cover of A_9, and containing a natural sequence of subgroups $2\mathsf{A}_n\,(2 \le n \le 9)$. The centralizers B_n of $2\mathsf{A}_n$ are, for $n = 2, 3, \ldots, 9$, groups $\cdot 0$, $6Suz$, $2G_2(4)$, $2HJ$, $2U_3(3)$, $2L_3(2)$, $2\mathsf{A}_4$, C_6, where HJ (also called J_2) is the Hall-Janko simple group [Hal4] and Suz is Suzuki's sporadic simple group [Suz1]. It follows that HJ has a multiplier of order divisible by 2, and Suz a multiplier of order divisible by 6, and also that HJ has a 6-dimensional projective representation that can be written over $\mathbf{Q}(\sqrt{-3}, \sqrt{-5})$, while Suz has a 12-dimensional projective representation over $\mathbf{Q}(\sqrt{-3})$. The latter can be obtained as follows. Take an element ω of order 3 with no fixed point, and so satisfying (as a matrix) the equation $\omega^2 + \omega + 1 = 0$. Then, in the ring of 24×24 matrices, ω generates a copy of the complex numbers in which it is identified with $e^{2\pi i/3}$. If we define, for $x \in \Lambda$, $x(a + be^{2\pi i/3}) = ax + b(x\omega)$, the Leech lattice becomes the *complex Leech lattice* Λ_C (see Chap. 7, Example 12), a 12-dimensional lattice (or module) over the ring $\mathbf{Z}[e^{2\pi i/3}]$ of Eisenstein integers, and the automorphism group of Λ_C is the group $6Suz$. The complex Leech lattice has a natural coordinate system that displays a

Patch for p. 363 is on the next page

Table 15.2b. Reduced indefinite binary forms.

d	Forms
-51	$\{1^{7}2\}, \{3^{6}5^{4}7^{3}6\}$
-52	$\{1^{7}3^{5}9^{4}4\}, \{2^{6}8^{2}6^{4}\}, \{4^{6}\}$
-53	$\{1^{7}4^{5}7^{2}\}, \{2^{7}\}$
-54	$\{1^{7}5^{3}9^{6}2\}, \{3^{6}6\}$
-55	$\{1^{7}6^{5}5\}, \{2^{7}3^{5}10\}$
-56	$\{1^{7}7\}, \{2^{6}10^{4}4\}, \{4^{6}5^{4}8\}$
-57	$\{1^{7}8^{1}7^{6}3\}, \{2^{7}4^{5}8^{3}6\}$
-58	$\{1^{7}9^{2}6^{4}7^{3}\}, \{2^{6}11^{5}3^{7}\}$
-59	$\{1^{7}10^{3}5^{7}2\}$
-60	$\{1^{7}11^{4}4\}, \{2^{6}12\}, \{3^{6}8^{2}7^{5}5\}, \{4^{6}6\}$
-61	$\{1^{7}12^{5}3^{7}4^{5}9^{4}5^{6}\}, \{2^{7}6^{5}\}$
-62	$\{1^{7}13^{6}2\}$
-63	$\{1^{7}14\}, \{2^{7}7\}, \{3^{6}9^{3}6\}$
-64	$0^{8}\}, 0^{8}1\}, 0^{8}2\}, 0^{8}3^{7}5^{8}0, 0^{8}4\}, 0^{8}6^{4}8\}, 0^{8}7^{6}4\}, 0^{8}8\}$
-65	$\{1^{8}\}, \{2^{7}8^{1}\}, \{{}^{7}4\,{}^{5}10\}, \{5^{5}8^{3}7^{4}\}$
-66	$\{1^{8}2\}, \{3^{6}10^{4}5^{6}6\}$
-67	$\{1^{8}3^{7}6^{5}7^{2}9^{7}2\}$
-68	$\{1^{8}4\}, \{2^{8}\}, \{4^{6}8^{2}\}$
-69	$\{1^{8}5^{7}4^{5}11^{6}3\}, \{2^{7}10^{3}6\}$
-70	$\{1^{8}6^{4}9^{5}5\}, \{2^{8}3^{7}7\}$
-71	$\{1^{8}7^{6}5^{4}11^{7}2\}$
-72	$\{1^{8}8\}, \{2^{8}4\}, \{3^{6}12\}, \{4^{6}9^{3}7^{4}8\}, \{6^{6}\}$
-73	$\{1^{8}9^{1}8^{7}3^{8}\}, \{2^{7}12^{5}4^{7}6^{5}8^{3}\}$
-74	$\{1^{8}10^{2}7^{5}\}, \{2^{8}5^{7}\}$
-75	$\{1^{8}11^{3}6\}, \{2^{7}13^{6}3\}, \{5^{5}10\}$
-76	$\{1^{8}12^{4}5^{6}8^{2}9^{7}3^{8}4\}, \{2^{8}6^{4}10^{6}4\}$
-77	$\{1^{8}13^{5}4^{7}7\}, \{2^{7}14\}$
-78	$\{1^{8}14^{6}3\}, \{2^{8}7^{6}6\}$
-79	$\{1^{8}15^{7}2\}, (3^{8}5^{7}6^{5}9^{4}7^{3}10^{7})$
-80	$\{1^{8}16\}, \{2^{8}8\}, \{4^{6}11^{5}5\}, \{4^{8}\}, \{8^{4}\}$
-81	$0^{9}\}, 0^{9}1\}, 0^{9}2\}, 0^{9}3\}, 0^{9}4^{7}8^{9}0, 0^{9}5^{6}9^{3}8^{5}7^{9}0, 0^{9}6\}, 0^{9}9\}$
-82	$\{1^{9}\}, \{2^{8}9^{1}\}, (3^{7}11^{4}6^{8})$
-83	$\{1^{9}2\}$
-84	$\{1^{9}3\}, \{2^{8}10^{2}8^{6}\}, \{4^{6}12\}, \{4^{8}7^{5}\}$
-85	$\{1^{9}4^{7}9^{2}\}, \{2^{9}\}, \{5^{5}12^{7}3^{8}7^{6}\}, \{\text{””}{}^{7}6^{5}10\}$
-86	$\{1^{9}5^{6}10^{4}7^{3}11^{8}2\}$
-87	$\{1^{9}6\}, \{2^{9}3\}$
-88	$\{1^{9}7^{5}9^{4}8\}, \{2^{8}12^{4}6^{8}4\}, \{4^{6}13^{7}3^{8}8\}, \{4^{8}6\}$
-89	$\{1^{9}8^{7}5^{8}\}, \{2^{9}4^{7}10^{3}8^{5}\}$
-90	$\{1^{9}9\}, \{3^{9}\}, \{2^{8}13^{5}5\}, \{6^{9}9^{3}\}$
-91	$\{1^{9}10^{1}9^{8}3^{7}14\}, \{2^{9}5^{6}11^{5}6^{7}7\}$
-92	$\{1^{9}11^{2}8^{6}7^{8}4\}, \{2^{8}14^{6}4\}$
-93	$\{1^{9}12^{3}7^{4}11^{7}4^{9}3\}, \{2^{9}6\}$
-94	$\{1^{9}13^{4}6^{8}5^{7}9^{2}10^{8}3^{7}15^{8}2\}$
-95	$\{1^{9}14^{5}5\}, \{2^{9}7^{5}10\}$
-96	$\{1^{9}15^{6}4\}, \{3^{9}5^{6}12\}, \{2^{8}16\}, \{4^{8}8\}, \{6^{6}10^{4}8\}$
-97	$\{1^{9}16^{7}3^{8}11^{3}8^{5}9^{4}\}, \{2^{9}8^{7}6^{5}12^{7}4^{9}\}$
-98	$\{1^{9}17^{8}2\}, \{7^{7}\}$
-99	$\{1^{9}18\}, \{2^{9}9\}, \{3^{9}6\}, (5^{8}7^{6}9^{3}10^{7})$
-100	$0^{10}\}, 0^{10}1\}, 0^{10}2\}, 0^{10}3^{8}12^{4}7^{10}0, 0^{10}4\}, 0^{10}5\}, 0^{10}6^{8}\},$

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The glue codes for the Niemeier lattices E_6^4, D_4^6, A_2^{12}, A_1^{24} are respectively the tetracode $\mathcal{C}_4$, the hexacode $\mathcal{C}_6$, and the Golay codes $\mathcal{C}_{12}$, $\mathcal{C}_{24}$ (§§2.5.1, 2.5.2, 2.8.5, 2.8.2 of Chap. 3). The glue codes for A_{12}^2 and $A_{11}D_7E_6$ are given in full in Chap. 24. For A_4^6 the group $G_2(A_4^6)$ is isomorphic to $PGL_2(5)$ acting on $\{\infty, 0, 1, 2, 3, 4\}$. For A_3^8 the group $G_2(A_3^8)$ is isomorphic to $2^3.PSL_2(7)$ acting on the extended Hamming code of length 8 over the integers modulo 4. The glue codes are also described (often using different coordinates) by Venkov in Chap. 18. Other references dealing with the Niemeier lattices are [Ero1]-[Ero3].

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On the other hand, if $\mathbf{V}_1, \ldots, \mathbf{V}_\mu$ are a set of fundamental roots corresponding to an extended diagram, there are positive integers $c_1, \ldots, c_\mu$ such that $\mathbf{\Sigma}\ c_i\mathbf{V}_i = 0$ ([Bou1], [Cox20, p. 194], [Hum1, p. 58]). These integers are shown in Fig. 23.1. If this diagram occurs as a subgraph of a hole diagram, $\mathbf{v}_1, \ldots, \mathbf{v}_\mu$ are the corresponding vertices and $\mathbf{c}$ is the center of the hole, then $\mathbf{\Sigma}\ c_i(\mathbf{v}_i - \mathbf{c}) = 0$. Thus the center can be found from

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Table 25.1. A list of all 307 holes in the Leech lattice. The first 23 entries are the deep holes. The entries give the name of a hole P_i, the order $g(P_i)$ of its automorphism group, its scaled volume

$$\mathrm{svol}(P_i) = \mathrm{vol}(P_i) \cdot 24!\ ,$$

the norm $s(P_i)$ of its Weyl vector, and the determinant $d(P_i)$ of the Cartan matrix. The volume formula then becomes

$$\sum_i \mathrm{svol}(P_i)/g(P_i) = 24!/|Co_0| = 74613\ .$$

The name of a hole indicates the orbits of its automorphism group on the components of the diagram. Thus the hole $a_7^2a_3^2a_3a_1^2$ has two components of type a_7 that are equivalent under the automorphism group, also two equivalent components of type a_1, and three components of type a_3, only two of which are equivalent.

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Figure 27.3. A convenient set of 35 Leech roots, representing the Leech lattice points closest to a deep hole of type A_{24}. The coordinates of the points are as follows: $i : (0^i, +1, -1, 0^{23-i}|0)$ for $0 \le i \le 23$; $24 : (-1/2, (1/2)^{23}, 3/2|5/2)$; $25 : (-1^2, 0^{23}|0)$; $26 : (0^7, 1^{18}|4)$; $27 : ((1/2)^{12}, (3/2)^{13}|11/2)$; $28 : ((1/2)^{17}, (3/2)^8|9/2)$; $29 : (0^{22}, 1^3|1)$; $30 : (0^5, 1^{14}, 2^6|6)$; $31 : (0^{10}, 1^{14}, 2|4)$; $32 : (0^4, 1^{11}, 2^{10}|7)$; $33 : ((1/2)^9, (3/2)^{11}, (5/2)^5|15/2)$; $34 : (0^{14}, 1^{11}|3)$.

17

The 24-Dimensional Odd Unimodular Lattices

R. E. Borcherds

This chapter completes the classification of the 24-dimensional unimodular lattices by enumerating the odd lattices. These are (essentially) in one-to-one correspondence with neighboring pairs of Niemeier lattices.

1. Introduction

The even unimodular lattices in 24 dimensions were classified by Niemeier [Nie2] and the results are given in the previous chapter, together with the enumeration of the even and odd unimodular lattices in dimensions less than 24. There are twenty-four Niemeier lattices, and in the present chapter they will be referred to by their components D_{24}, $D_{16}E_8, \ldots$ (with the Leech lattice being denoted by Λ_{24}), and also by the Greek letters $\alpha, \beta, \ldots$ (see Table 16.1).

The *odd* unimodular lattices in 24 and 25 dimensions were classified in [Bor1]. In this chapter we list the odd 24-dimensional lattices. Only those with minimal norm at least 2 are given, i.e. those that are strictly 24-dimensional, since the others can easily be obtained from lower dimensional lattices (see the summary in Table 2.2 of Chapter 2).

Tables of all the 665 25-dimensional unimodular lattices and the 121 even 25-dimensional lattices of determinant 2 are available electronically from [BorchHP]. The 665 25-dimensional unimodular lattices are also available from the electronic *Catalogue of Lattices* [NeSl].

Two lattices are called *neighbors* if their intersection has index 2 in each of them [Kne4], [Ven2] (see Introduction to Third Edition for a discussion of this concept).

We now give a brief description of the algorithm used in [Bor1] to enumerate the 25-dimensional unimodular lattices.

The first step is to observe that there is a one-to-one correspondence between 25-dimensional unimodular lattices (up to isomorphism) and orbits of norm -4 vectors in the even Lorentzian lattice $\mathrm{II}_{25,1}$: the lattice Λ corresponds to the norm -4 vector v if and only if the sublattice of even vectors of Λ is isomorphic to the lattice v^{-}. So we can classify 25-dimensional unimodular lattices if we can classify negative norm vectors in $\mathrm{II}_{25,1}$.

We classify orbits of vectors of norm $-2n \leq 0$ in $\mathrm{II}_{25,1}$ by induction on n as follows. First of all the primitive norm 0 vectors correspond to the Niemeier lattices as in Section 1 of Chapter 26. So there are exactly 24 orbits of primitive norm 0 vectors, and any norm 0 vector can be obtained from a primitive one by multiplying it by some constant.

Suppose we have classified all orbits of vectors of norms $-2m$ with $0 \geq -2m > -2n$, and that we have a vector v of norm $-2n$. We fix a fundamental Weyl chamber for the reflection group of $\mathrm{II}_{25,1}$ containing v, as in Chapter 26. We look at the root system of the lattice v^-, and find that one of the following three things can happen:

1. There is a norm 0 vector z with $(z,v) = 1$. It turns out to be trivial to classify such norm $-2n$ vectors v: there is one orbit corresponding to each orbit of norm 0 vectors. They correspond to lattices v^- which are the sum of a Niemeier lattice and a 1-dimensional lattice generated by a vector of norm $2n$.
2. There is no norm 0 vector z with $(z,v) = 1$ and the root system of v^- is nonempty. In this case we choose a component of the root system of v^- and let r be its highest root. Then the vector $u = v + r$ has norm $-2(n-1)$, and the assumption about no norm 0 vectors z with $(z,v) = 1$ easily implies that u is still in the Weyl chamber of $\mathrm{II}_{25,1}$. Hence we have reduced v to some known vector u of norm $-2(n-1)$, and with a little effort it is possible to reverse this process and construct v from u.
3. Finally suppose that there are no roots in v^-. As v is in the Weyl chamber this implies that $(v,r) \leq -1$ for all simple roots r. By Theorem 1 of Chapter 27 there is a norm 0 (Weyl) vector w_{25} with the property that $(w_{25}, r) = -1$ for all simple roots r. Therefore the vector $u = v - w_{25}$ has the property that $(u,r) \leq 0$ for all simple roots r. So u is in the Weyl chamber, and has norm $-2n - (u, w_{25})$ which is larger than $-2n$ unless v is a multiple of w_{25}. So we can reconstruct v from the known vector u as $v = u + w_{25}$.

In every case we can reconstruct v from known vectors, so we get an algorithm for classifying the norm $-2n$ vectors in $\mathrm{II}_{25,1}$. (This algorithm breaks down in higher-dimensional Lorentzian lattices for two reasons: it is too difficult to classify the norm 0 vectors, and there is usually no analogue of the Weyl vector w_{25}.)

We now apply the algorithm above to find the 121 orbits of norm -2 vectors from the (known) norm 0 vectors, and then apply it again to find the 665 orbits of norm -4 vectors from the vectors of norm 0 and -2.

The neighbors of a strictly 24 dimensional odd unimodular lattice can be found as follows. If a norm -4 vector $v \in \mathrm{II}_{25,1}$ corresponds to the sum of a strictly 24 dimensional odd unimodular lattice Λ and a 1-dimensional lattice, then there are exactly two norm-0 vectors of $\mathrm{II}_{25,1}$ having inner product -2 with v, and these norm 0 vectors correspond to the two even neighbors of Λ.

The enumeration of the odd 24-dimensional lattices. Figure 17.1 shows the neighborhood graph for the Niemeier lattices, which has a node for each Niemeier lattice. If A and B are neighboring Niemeier lattices, there are three integral lattices containing $A \cap B$, namely A, B, and an odd unimodular lattice C (cf. [Kne4]). An edge is drawn between nodes A and B in Fig. 17.1 for each strictly 24-dimensional unimodular lattice arising in this way. Thus there is a one-to-one correspondence between the strictly 24-dimensional odd unimodular lattices and the edges of our neighborhood graph. The 156 lattices are shown in Table 17.1. Figure 17.1 also shows the corresponding graphs for dimensions 8 and 16.

For each lattice Λ in the table we give its components (in the notation of the previous chapter) and its even neighbors (represented by 2 Greek letters as in Table

16.1). The final column gives the orders $g_1 \cdot g_2$ of the groups $G_1(\Lambda)$, $G_2(\Lambda)$ defined as follows. We may write $Aut(\Lambda) = G_0(\Lambda).G_1(\Lambda).G_2(\Lambda)$ where G_0 is the reflection group. The group G_1 is the subgroup of $Aut(\Lambda)$ of elements fixing a fundamental chamber of the Weyl group and not interchanging the two neighbors. The group $G_2(\Lambda)$ has order 1 or 2 and interchanges the two neighbors of Λ if it has order 2. (It turns out that $G_2(\Lambda)$ has order 2 if and only if the two components of Λ are isomorphic.) The components are written as a union of orbits under $G_1(\Lambda)$, with parentheses around two orbits if they fuse under $G_2(\Lambda)$.

The first lattice in the table is the odd Leech lattice O_{24}, which is the only one with no norm 2 vectors. The number of norm 2 vectors is given by the formula

$$8h(A) + 8h(B) - 16$$

where $h(A)$ and $h(B)$ are the Coxeter numbers of the even neighbors of the lattice. These Coxeter numbers satisfy the inequality $h(B) \leq 2h(A) - 2$ and the lattices for which equality holds are indicated by a thick line in Figure 17.1. The Weyl vector $\rho(\Lambda)$ of the lattice Λ has norm given by the formula $\rho(\Lambda)^2 = h(A)h(B)$.

Table 17.1a

	Components		$g_1 g_2$
1	O_{24}	$\omega\ \psi$	$2^{12} M_{24}$
2	$A_1^8 O_{16}$	$\psi\ \psi$	$10321920 \cdot 2$
3	$A_1^{12} O_{12}$	$\psi\ \chi$	190080
4	$A_1^{16} O_8$	$\psi \phi$	43008
5	$A_2^2 A_1^{10} O_{10}$	$\chi\ \chi$	$2880 \cdot 2$
6	A_1^{24}	$\psi\ \tau$	138240
7	$A_2^4 A_1^8 O_8$	$\chi\ \phi$	384
8	$A_2^6 A_1^6 O_6$	$\chi\ \upsilon$	240
9	$A_3^2 A_1^{12} O_6$	$\phi\ \phi$	$384 \cdot 2$
10	$A_3 A_2^4 A_1^6 O_7$	$\phi\ \phi$	$48 \cdot 2$
11	$A_2^8 O_8$	$\phi\ \phi$	$336 \cdot 2$
12	$A_2^8 A_1^4 O_4$	$\chi\ \sigma$	384
13	$A_3^2 A_2^4 A_1^4 O_6$	$\phi\ \upsilon$	16
14	$A_3^4 A_1^8 O_4$	$\phi\ \tau$	384
15	$A_3^4 A_1^4 A_1^4 O_4$	$\phi\ \sigma$	48
16	$A_3^2 A_3 A_2^4 A_1^2 O_5$	$\phi\ \sigma$	16
17	$A_4 A_3 A_2^4 A_1^4 O_5$	$\upsilon\ \upsilon$	$8 \cdot 2$
18	$A_3^4 A_2^2 A_1^2 O_6$	$\upsilon\ \upsilon$	$16 \cdot 2$
19	$A_3^4 A_2^4 O_4$	$\phi\ \rho$	24
20	$A_3^6 O_6$	$\upsilon\ \tau$	240
21	$A_4^2 A_2^4 A_1^4 O_4$	$\upsilon\ \sigma$	16
22	$A_4 A_3^2 A_3 A_2^2 A_1^2 O_5$	$\upsilon\ \sigma$	4
23	$A_3^4 A_3^2 A_1^4 O_2$	$\phi\ \pi$	32
24	$A_4^2 A_3^2 A_2^2 A_1^2 O_4$	$\upsilon\ \rho$	4
25	$D_4^2 A_1^{16}$	$\tau\ \tau$	$576 \cdot 2$
26	$D_4 A_3^4 A_1^4 O_4$	$\tau\ \sigma$	48

Table 17.1b

	Components		$g_1 g_2$
27	$A_5(A_3A_3)A_2^4A_1O_4$	$\sigma\ \sigma$	$8 \cdot 2$
28	$A_4^2(A_3A_3)A_3(A_1A_1)O_5$	$\sigma\ \sigma$	$4 \cdot 2$
29	$A_5A_3^3(A_1^3A_1^3)A_1O_3$	$\sigma\ \sigma$	$12 \cdot 2$
30	$4A_3^4A_1^4O_4$	$\sigma\ \sigma$	$16 \cdot 2$
31	$D_4A_4A_2^6O_4$	$\sigma\ \sigma$	$12 \cdot 2$
32	A_3^8	$\phi\ \xi$	384
33	$A_4^2A_4A_3A_2^2A_1^2O_3$	$\upsilon\ \pi$	4
34	$A_4^2A_3^4O_4$	$\upsilon\ \pi$	16
35	$A_5A_4A_3A_3A_2A_2A_1O_4$	$\sigma\ \rho$	2
36	$A_4^4A_1^4O_4$	$\sigma\ \rho$	24
37	$A_4^3A_3^3O_3$	$\upsilon\ o$	12
38	$D_4^2A_3^4O_4$	$\tau\ \pi$	32
39	$A_5A_4^2A_3A_3A_1O_4$	$\sigma\ \pi$	4
40	$A_5^2A_3^2A_1^2A_1^2A_1^2O_2$	$\sigma\ \pi$	8
41	$A_5D_4A_3^2A_3A_1^2A_1O_3$	$\sigma\ \pi$	4
42	$D_4A_4^2A_4A_2^2O_4$	$\sigma\ \pi$	4
43	$A_6A_3^3A_2^3O_3$	$\rho\ \rho$	$6 \cdot 2$
44	$A_5^2A_3^2A_2^2O_4$	$\rho\ \rho$	$4 \cdot 2$
45	$A_5A_4^3A_1^3O_4$	$\rho\ \rho$	$6 \cdot 2$
46	$A_4^4A_3^2O_2$	$\upsilon\ \nu$	16
47	$A_5^2A_4A_3A_2^2O_3$	$\sigma\ o$	4
48	$A_6A_4A_4A_3A_2A_1A_1O_3$	$\rho\ \pi$	2
49	$A_5^2A_4^2A_1^2O_4$	$\rho\ \pi$	4
50	$D_4^4A_1^8$	$\tau\ \xi$	48
51	$A_5^2D_4A_3^2A_1^2O_2$	$\sigma\ \xi$	8
52	$A_5A_5A_4^2A_3O_3$	$\sigma\ \nu$	4

Table 17.1c

	Components		g_1g_2
53	$A_5^2D_4A_3^2A_1^2O_2$	σ ν	4
54	$A_6A_5A_4A_3A_2A_1O_3$	ρ o	2
55	$A_6D_4(A_4A_4)A_2O_4$	π π	$2\cdot 2$
56	$D_5A_4(A_4A_4)(A_2A_2)O_3$	π π	$2\cdot 2$
57	$A_7(A_3^2A_3^2)A_1^4O_1$	π π	$8\cdot 2$
58	$A_5^2D_4^2A_1^2O_4$	π π	$8\cdot 2$
59	$D_5A_5(A_3A_3)A_3(A_1A_1)A_1O_2$	π π	$2\cdot 2$
60	$D_5D_4A_3^4O_3$	π π	$8\cdot 2$
61	$A_5^4O_4$	ρ ξ	24
62	$A_6^2A_3^2A_2^2O_2$	ρ ν	4
63	$A_6A_5A_5A_3A_2O_3$	ρ ν	2
64	$A_5^4A_1^4$	σ μ	48
65	$A_5^2A_5D_4A_3A_1O_1$	σ λ	4
66	$A_6^2A_4A_3A_1^2O_3$	π o	4
67	$A_7A_4^2A_3^2O_3$	π o	4
68	$D_5A_5^2D_4A_1^2O_3$	π ξ	4
69	$D_5^2A_3^4O_2$	π ξ	16
70	$A_6D_5A_4A_4A_2O_3$	π ν	2
71	$A_7A_5D_4A_3A_1A_1A_1O_2$	π ν	2
72	$A_7A_5A_4^2A_1O_3$	π ν	4
73	$A_7A_5^2A_2^2O_3$	o o	$4\cdot 2$
74	D_4^6	τ ι	48
75	$A_6A_6A_5A_4A_1O_2$	ρ λ	2
76	$A_7A_6A_5A_2A_1O_3$	o ν	2
77	$A_8A_4^2A_3^2O_2$	o ν	4
78	$A_6^2A_5^2O_2$	ρ κ	4

Table 17.1d

	Components		$g_1 g_2$
79	$D_5^2 A_5^2 A_1^2 O_2$	$\pi\ \mu$	8
80	$A_6 A_6 D_5 A_4 O_3$	$\pi\ \lambda$	2
81	$A_7 D_5 D_4 A_3 A_3 O_2$	$\pi\ \lambda$	2
82	$D_5 D_5 A_5^2 A_1^2 O_2$	$\pi\ \lambda$	4
83	$A_7 D_5 A_5 A_3 A_1 A_1 A_1 O_1$	$\pi\ \lambda$	2
84	$D_6 D_4^3 A_1^6$	$\xi\ \xi$	$6 \cdot 2$
85	$D_6 A_5^2 A_3^2 O_2$	$\xi\ \nu$	4
86	$A_8 (A_5 A_5) A_3 O_3$	$\nu\ \nu$	$2 \cdot 2$
87	$D_6 A_5^2 A_3^2 O_2$	$\nu\ \nu$	$4 \cdot 2$
88	$A_7^2 D_4 (A_1^2 A_1^2) O_2$	$\nu\ \nu$	$4 \cdot 2$
89	$A_7^2 A_4^2 O_2$	$\pi\ \kappa$	4
90	$A_8 A_6 A_5 A_2 A_1 O_2$	$o\ \lambda$	2
91	$A_7^2 D_4^2 O_2$	$\pi\ \iota$	8
92	$A_7 D_5^2 A_3^2 O_1$	$\pi\ \iota$	4
93	$A_8 A_7 A_4 A_3 O_2$	$o\ \kappa$	2
94	$D_5^4 O_4$	$\xi\ \mu$	48
95	$D_6 D_5 A_5^2 O_3$	$\xi\ \lambda$	4
96	$A_8 A_6 D_5 A_2 O_3$	$\nu\ \lambda$	2
97	$A_7 D_6 A_5 A_3 A_1 O_2$	$\nu\ \lambda$	2
98	$A_9 A_5 D_4 A_3 A_1 A_1 O_1$	$\nu\ \lambda$	2
99	$A_7^3 O_3$	$o\ \iota$	12
100	$A_7^2 D_5 D_4 O_1$	$\pi\ \theta$	4
101	$A_9 A_6 A_5 A_2 O_2$	$\nu\ \kappa$	2
102	$A_8^2 A_3^2 O_2$	$\nu\ \kappa$	4
103	$A_7^2 D_6 A_1^2 O_2$	$\nu\ \iota$	4
104	$D_6^2 D_4^2 A_1^4$	$\xi\ \iota$	4

Table 17.1e

	Components		g_1g_2
105	$A_7^2D_5^2$	$\pi\ \eta$	8
106	$A_8A_7^2O_2$	$o\ \theta$	4
107	$E_6D_5A_5^2A_1^2O_1$	$\mu\ \lambda$	4
108	$E_6(A_6A_6)A_4O_2$	$\lambda\ \lambda$	$2\cdot 2$
109	$A_7E_6D_4(A_3A_3)O_1$	$\lambda\ \lambda$	$2\cdot 2$
110	$A_9(D_5D_5)(A_1A_1)A_1O_2$	$\lambda\ \lambda$	$2\cdot 2$
111	$D_7(A_5A_5)A_5A_1O_1$	$\lambda\ \lambda$	$2\cdot 2$
112	$A_{10}A_6A_5A_1O_2$	$\lambda\ \kappa$	2
113	$A_9A_8A_5O_2$	$\nu\ \theta$	2
114	$A_9D_6A_5A_3O_1$	$\nu\ \theta$	2
115	$A_8^2A_7O_1$	$o\ \zeta$	4
116	$D_7A_7D_5A_3O_2$	$\lambda\ \iota$	2
117	$A_9^2A_2^2O_2$	$\kappa\ \kappa$	$4\cdot 2$
118	$D_6^2D_6D_4A_1^2$	$\xi\ \eta$	2
119	$A_9A_7D_6A_1O_1$	$\nu\ \eta$	2
120	$A_9A_7D_6A_1O_1$	$\nu\ \zeta$	2
121	$A_{11}D_5D_4A_3O_1$	$\lambda\ \theta$	2
122	$A_9D_7A_5A_1A_1O_1$	$\lambda\ \theta$	2
123	$D_8D_4^4$	$\iota\ \iota$	$4\cdot 2$
124	$A_{11}A_8A_3O_2$	$\kappa\ \theta$	2
125	$E_6^2D_5^2O_2$	$\mu\ \eta$	8
126	$A_9E_6D_6A_1O_2$	$\lambda\ \eta$	2
127	$D_7A_7E_6A_3O_1$	$\lambda\ \eta$	2
128	$A_{10}E_6A_6O_2$	$\lambda\ \zeta$	2
129	$A_{11}D_6A_5A_1O_1$	$\lambda\ \zeta$	2
130	D_6^4	$\xi\ \epsilon$	8

Table 17.1f

	Components		$g_1 g_2$
131	$D_8 A_7^2 O_2$	$\iota\ \theta$	4
132	$A_{12} A_7 A_4 O_1$	$\kappa\ \zeta$	2
133	$D_8 D_6^2 A_1^2 A_1^2$	$\iota\ \eta$	2
134	$A_{11} D_7 D_5 O_1$	$\lambda\ \epsilon$	2
135	$A_{11}^2 O_2$	$\kappa\ \epsilon$	4
136	$D_9 A_7^2 O_1$	$\theta\ \eta$	4
137	$A_{13} D_6 A_3 A_1 O_1$	$\theta\ \zeta$	2
138	$D_8^2 D_4^2$	$\iota\ \epsilon$	2
139	$E_7 D_6 D_6 D_4 A_1$	$\eta\ \eta$	$1 \cdot 2$
140	$A_9 E_7 A_7 O_1$	$\eta\ \zeta$	2
141	$A_{12} A_{11} O_1$	$\kappa\ \delta$	2
142	$A_{11} D_9 A_3 O_1$	$\theta\ \epsilon$	2
143	$D_{10} D_6^2 A_1^2$	$\eta\ \epsilon$	2
144	$A_{15} A_8 O_1$	$\theta\ \delta$	2
145	D_8^3	$\iota\ \gamma$	6
146	$D_8^2 D_8$	$\iota\ \beta$	2
147	$A_{16} A_7 O_1$	$\zeta\ \delta$	2
148	$A_{15} D_8 O_1$	$\theta\ \beta$	2
149	$D_8 E_7^2 A_1^2$	$\eta\ \gamma$	2
150	$D_{10} E_7 D_6 A_1$	$\eta\ \beta$	1
151	$A_{15} E_7 A_1 O_1$	$\zeta\ \beta$	2
152	$D_{12} D_8 D_4$	$\epsilon\ \beta$	1
153	$E_8 D_8^2$	$\gamma\ \beta$	2
154	D_{12}^2	$\epsilon\ \alpha$	2
155	$A_{23} O_1$	$\delta\ \alpha$	2
156	$D_{16} D_8$	$\beta\ \alpha$	1

FIGURE 17.1 (copy from page 423) GOES SOMEWHERE HERE!

Supplementary Bibliography (1988-1998)

This supplementary bibliography covers the period 1988—1998, and also includes some earlier references that should have been included in the first edition. Besides the journal abbreviations used in the main bibliography, we also use

DCC	=	Designs, Codes and Cryptography
DCG	=	Discrete and Computational Geometry
EJC	=	European Journal of Combinatorics
JNB	=	Journal de Théorie des Nombres de Bordeaux (formerly Sém. Théor. Nombres Bordeaux)

Most (although not all) of these references are mentioned in the Preface to the Third Edition. (We apologize if we have overlooked any relevant papers, or, having listing them here, have failed to mention them in the Preface. No disrespect was intended.)

References

E. H. L. Aarts: see R. J. M. Vaessens.

[Abdu93] K. S. Abdukhalikov, *Integral invariant lattices in Lie algebras of type* A_{p^m-1} (in Russian), Mat. Sb. **184** (1993) = Russ. Acad. Sci. Sb. Math. **78** (1994), 447–478.

[AdM91] A. Adem, J. Maginnis and R. J. Milgram, *The geometry and cohomology of the Mathieu group* M_{12}, J. Alg. **139** (1991), 90–133.

[Adle81] A. Adler, *Some integral representations of* $PSL_2(\mathbb{F}_p)$ *and their applications*, J. Alg. **72** (1981), 115–145.

[AdB88] J.-P. Adoul and M. Barth, *Nearest neighbor decoding algorithms for spherical codes from the Leech lattice*, PGIT **34** (1988), 1188–1202.

D. Agrawal: see R. Urbanke.

[Agr96] E. Agrell, *Voronoi regions for binary linear block codes*, PGIT **42** (1996), 310–316.

[AgEr98] — and T. Eriksson, *Optimization of lattices for quantization*, PGIT **44** (1998), to appear.

[Ajt96] M. Ajtai, *Generating hard instances of lattice problems*, Electronic Colloq. Comp. Complexity, http://www.eccc.uni-trier.de/eccc/, Report TR96-007.

[Ajt97] — *The shortest vector problem in* L_2 *is NP-hard for randomized reductions*, Electronic Colloq. Comp. Complexity, http://www.eccc.uni-trier.de/eccc/, Report TR97-047.

[Allc96] D. Allcock, *Recognizing equivalence of vectors in the Leech lattice*, preprint, 1996.

[Allc97] — *New complex and quaternion-hyperbolic reflection groups*, preprint, 1997.

[Allc97a] — *The Leech lattice and complex hyperbolic reflections*, preprint, 1997.

[Alon97] N. Alon, *Packings with large minimum kissing numbers*, Discr. Math. **175** (1997), 249–251.

[AlBN92] — J. Bruck, J. Naor, M. Naor and R. M. Roth, *Construction of asymptotically good low-rate error-correcting codes through pseudo-random graphs*, PGIT **38** (1992), 509–516.

[Amr93] O. Amrani, Y. Be'ery and A. Vardy, *Bounded-distance decoding of the Leech lattice and the Golay code*, in *Algebraic Coding*, Lect. Notes Comp. Sci. **781** (1994), 236–248.

[Amr94] — — — F.-W. Sun and H. C. A. van Tilborg, *The Leech lattice and the Golay code: bounded-distance decoding and multilevel constructions*, PGIT **40** (1994), 1030–1043.

[Ang90] W. S. Anglin, *The square pyramid puzzle*, AMM **97** (1990), 102–124.

[Anz91] M. M. Anzin, *On variations of positive quadratic forms (with applications to the study of perfect forms)*, TMIS **196** (1991), 3–10.

[ArJ79] R. F. Arenstorf and D. Johnson, *Uniform distribution of integral points on 3–dimensional spheres via modular forms*, JNT **11** (1979), 218–238.

[ABB96] V. Arhelger, U. Betke and K. Böröczky, Jr., *Large finite lattice packings in E^3*, preprint, 1996.

[Asch94] M. Aschbacher, *Sporadic Groups*, Cambridge, 1994.

[AsL96] A. E. Ashikhmin and S. N. Litsyn, *Fast decoding algorithms for first order Reed-Muller and related codes*, DCC **7** (1996), 187–214.

[Atk82] K. Atkinson, *Numerical integration on the sphere*, JAMS **B 23** (1982), 332–347.

[AuC91] L. Auslander and M. Cook, *An algebraic classification of the three-dimensional crystallographic groups*, Advances Appl. Math. **12** (1991), 1–21.

[BaaGr97] M. Baake and U. Grimm, *Coordination sequences for root lattices and related graphs*, Zeit. Krist. **212** (1997), 253–256.

[BaaJK90] — D. Joseph, P. Kramer and M. Schlottmann, *Root lattices and quasicrystals*, J. Phys. **A 23** (1990), L1037–L1041.

[BaaPl95] — and P. A. B. Pleasants, *Algebraic solution of the coincidence problem in two and three dimensions*, Z. Naturforschung **50A** (1995) 711–717.

[BacSh96] E. Bach and J. O. Shallit, *Algorithmic Number Theory I: Efficient Algorithms*, MIT Press, Cambridge, MA, 1996.

[Bace94] R. Bacher, *Unimodular lattices without nontrivial automorphisms*, Int. Math. Res. Notes **2** (1994), 91–95.

[Bace97] — *Tables de réseaux entiers unimodulaires construits comme k-voisins de $\mathbb{Z}^n$*, JNB **9** (1997), 479–49.

[Bace96] — *Dense lattices in dimensions 27–29*, Invent. Math. **130** (1997), 153–158.

[BaHVe97] — P. de la Harpe and B. B. Venkov, *Séries de croissance et séries d'Ehrhart associées aux réseaux de racines*, CR **325** (1997) 1137–1142.

[BaHVe98] — — — *Séries de croissance et polynômes d'Ehrhart associés aux réseaux de racines*, preprint.

[BaVe95] — and B. B. Venkov, *Lattices and association schemes: a unimodular example without roots*, Annales Inst. Fourier **45** (1995), 1163–1176.

[BaVe96] — — *Réseaux entiers unimodulaires sans racine en dimension 27 et 28*, Preprint No. 332, Inst. Fourier, Grenoble, 1996.

C. Bachoc: see also A. Bonnecaze.

[Baco95] — *Voisinages au sens de Kneser pour les réseaux quaternioniens*, Comm. Math. Helv. **70** (1995), 350–374.

[Baco97] — *Applications of coding theory to the construction of modular lattices*, JCT **A** **78** (1997), 92–119.

[BacoB92] — and C. Batut, *Etude algorithmique de réseaux construits avec la forme trace*, Exper. Math. **1** (1992), 183–190.

[BacoN96] — and G. Nebe, *Classification of two genera of 32-lattices over the Hurwitz order*, Exper. Math. **6** (1997), 151–162.

[BacoN98] — — *Extremal lattices of minimum 8 related to the Mathieu group M_{22}*, JRAM **494** (1998), 155–171.

[Bag05] G. Bagnera, *I gruppi finiti di transforminazioni lineari dello spazio che contengono omologie*, Rend. Circ. Math. Palermo **19** (1905), 1–56.

T. Baicheva: see P. R. J. Östergård.

[Baj91] B. Bajnok, *Construction of spherical 4- and 5-designs*, Graphs Combin. **7** (1991), 219–233.

[Baj91a] — *Construction of designs on the 2-sphere*, EJC **12** (1991), 377–382.

[Baj91b] — *Chebyshev-type quadrature formulas on the sphere*, Congr. Numer. **85** (1991), 214–218.

[Baj92] — *Construction of spherical t-designs*, Geom. Dedicata **43** (1992), 167–179.

[Baj98] — *Constructions of spherical 3-designs*, Graphs Combin. **14** (1998), 97–107.

[BanBl96] A. H. Banihashemi and I. F. Blake, *Trellis complexity and minimal trellis diagrams of lattices*, preprint, 1996.

[BanKh96] — and A. K. Khandani, *An inequality on the coding gain of densest lattices in successive dimensions*, preprint.

[BanKh97] — — *On the complexity of decoding lattices using the Korkin-Zolotarev reduced basis*, PGIT, to appear.

[BanDHO97] Eiichi Bannai, S. T. Dougherty, M. Harada and M. Oura, *Type II codes, even unimodular lattices and invariant rings*, preprint, June 1997.

[BanMO96] — S. Minashima and M. Ozeki, *On Jacobi forms of weight 4*, Kyushu J. Math. **50** (1996), 335–370.

[BanO96] — and M. Ozeki, *Construction of Jacobi forms from certain combinatorial polynomials*, Proc. Japan Acad. **A 72** (1996), 359–363.

[Bann90] Etsuko Bannai, *Positive Definite Unimodular Lattices with Trivial Automorphism Group*, Memoirs Amer. Math. Soc. **429** (1990), 1–70.

[Bara80] E. P. Baranovskii, *The Selling reduction domain of positive quadratic forms in five variables* (in Russian), TMIS **152** (1980) 5–33 = PSIM **3** (1982), 5–35.

[Bara91] — *Partitioning of Euclidean space into L-polytopes of some perfect lattices*, TMIS **196** (1991) 27–26 = PSIM (1992, No. 4), 29–51.

[Bara94] — *The perfect lattices ?($\mathcal{A}^n$), and the covering density of ?($\mathcal{A}^9$)*, EJC **15** (1994), 317–323.

[Bart88] K. Bartels, *Zur Klassifikation quadratischer Gitter über diskreten Bewertungsringen*, Ph. Dissertation, Göttingen, 1988.

M. Barth: see J.-P. Adoul.

[Barv90] A. I. Barvinok, *The method of Newton sums in problems of combinatorial optimization* (in Russian), Disk. Mat. **2** (1990), 3–15.

[Barv91] — *Problems of combinatorial optimization, statistical sums and representations of the general linear group* (in Russian), Mat. Zamet. **49** (1991), 3–11 = Math. Notes **49** (1991), 3–9.

[Barv92] — *The method of statistical sums in combinatorial optimization problems*, Leningrad Math. J. **2** (1991), 987–1002.

[Barv92a] — *Optimization problems on lattices and theta functions*, preprint.

[Barv97] — *Lattice points and lattice polytopes*, Chap. 7 of [GoO'R97].

C. Batut: see also C. Bachoc.

[BatB91] — D. Bernardi, H. Cohen and M. Olivier, *User's Guide to PARI-GP*, 1991.

[Bat98] — *Classification of quintic eutactic forms*, Math. Comp., to appear.

[BatMa94] — and J. Martinet, *Radiographie des réseaux parfaits*, Exper. Math. **3** (1994), 39–49.

[BatQS95] — H.-G. Quebbemann and R. Scharlau, *Computations of cyclotomic lattices*, Exper. Math. **4** (1995), 175–179.

[Bay89] E. Bayer-Fluckiger, *Réseaux unimodulaires*, JNB **1** (1989), 535–589.

[Bay94] — *Galois cohomology and the trace form*, Jber. Deutsche Math.–Verein, **96** (1994), 35–55.

[Bay97] — *Lattices with automorphisms of given characteristic polynomial*, preprint.

[BayFa96] — and L. Fainsilber, *Non-unimodular Hermitian forms*, Invent. Math. **123** (1996), 233–240.

[BayM94] — and J. Martinet, *Formes quadratiques liées aux algèbres semi-simples*, JRAM **451** (1994), 51–69.

[BeaH89] S. Beale and D. K. Harrison, *A computation of the Witt index for rational quadratic forms*, Aeq. Math. **38** (1989), 86–98.

Y. Be'ery: see also O. Amrani, J. Snyders, A. Vardy.

[BeeSh91] — and B. Shahar, *VLSI architectures for soft decoding of the Leech lattice and the Golay codes*, Proc. IEEE Int. Workshop on Microelectronics in Comm., Interlaken, Switzerland, March 1991.

[BeeSh92] — — and J. Snyders, *Fast decoding of the Leech lattice*, IEEE J. Select. Areas Comm. **7** (1989), 959–967.

J. C. Belfiore: see J. Boutros.

[BenEHH] J. W. Benham, A. G. Earnest, J. S. Hsia and D. C. Hung, *Spinor regular positive definite ternary quadratic forms*, JLMS **42** (1990), 1–10.

[BenCa87] D. Benson and J. Carlson, *Diagrammatic methods for modular representations and cohomology*, Comm. Alg. **15** (1987), 53–121.

[Ber93] A.-M. Bergé, *Minimal vectors of a pair of dual lattices*, JNT **52** (1995), 284–298.

[BerM85] — and J. Martinet, *Sur la constante d'Hermite (étude historique)*, JNB **2** (No. 8, 1985), 1–15.

[BerM89] — — *Sur un problème de dualité lié aux sphères en géométrie des nombres*, JNT **32** (1989), 14–42.

[BerM89a] — — *Réseaux extrêmes pour un groupe d'automorphismes*, Astérisque **198**–**200** (1992), 41–46.

[BerM95] — — *Densité dans des familles de réseaux: application aux réseaux isoduaux*, L'Enseign. Math. **41** (1995), 335–365.

[BerM96] — — *Sur la classification des réseaux eutactiques*, JLMS **53** (1996), 417–432.

[BerMS92] — — and F. Sigrist, *Une généralisation de l'algorithme de Voronoi pour les formes quadratiques*, Astérisque **209** (1992), 137–158.

D. Bernardi: see C. Batut.

[BerSl97] M. Bernstein and N. J. A. Sloane, *Some lattices obtained from Riemann surfaces*, in *Extremal Riemann Surfaces (Contemporary Math. Vol. 201)*, J. R. Quine and P. Sarnak, Eds., Amer. Math. Soc., Providence, RI, 1997, pp. 29–32.

[BerSl97a] — — and P. E. Wright, *On sublattices of the hexagonal lattice*, Discr. Math. **170** (1997), 29–39.

D. Bessis: see H. J. Herrmann.

U. Betke: See also V. Arhelger.

[BetB96] — and K. Böröczky, Jr., *Asymptotic formulae for the lattice point enumerator*, preprint, 1996.

[BetB97] —— *Finite lattice packings and the Wulff-shape*, preprint, 1997.

[BetG84] — and P. Gritzmann, *Ueber L. Fejes Tóth's Wurstvermutung in kleinen Dimensionen*, AMAH **43** (1984), 299–307.

[BetG86] —— *An application of valuation theory to two problems in discrete geometry*, Discr. Math. **58** (1986), 81–85.

[BetGW82] —— and J. M. Wills, *Slices of L. Fejes Tóth's sausage conjecture*, Math. **29** (1982), 194–201.

[BetH98] — and M. Henk, *Finite packings of spheres*, DCG **19** (1998), 197–227.

[BetHW94] —— and J. M. Wills, *Finite and infinite packings*, JRAM **453** (1994), 165–191.

[BetHW95] ——— *Sausages are good packings*, DCG **13** (1995), 297–311.

[BetHW95a] ——— *A new approach to covering*, Math. **42** (1995), 251–263.

[BezBeCo] A. Bezdek, K. Bezdek and R. Connelly, *Finite and uniform stability of sphere packings*, preprint.

[BezK90] — and W. Kuperberg, *Maximum density space packings with congruent circular cylinders of infinite length*, Math. **37** (1990), 74–80.

[BezKu91] —— *Packing Euclidean space with congruent cylinders and with congruent ellipsoids*, in *Victor Klee Festschrift*, ed. P. Gritzmann and B. Sturmfels, DIMACS Series Vol. 4, Amer. Math. Soc., 1991, pp. 71–80.

[BezKM91] —— and E. Makai, Jr., *Maximum density space packing with parallel strings of spheres*, DCG **6** (1991), 277–283.

K. Bezdek: See also A. Bezdek.

[Bez97] — *Isoperimetric inequalities and the dodecahedral conjecture*, IJM **8** (1997), 759–780.

[BezCK85] — R. Connelly and G. Kertész, *On the average number of neighbors in a spherical packing of congruent circles*, in *Intuitive Geometry, Siofok 1985*, North-Holland, Amsterdam, 1987, pp. 37–52.

[BiEd98] J. Bierbrauer and Y. Edel, *Dense sphere packings from new codes*, preprint, 1998.

E. Biglieri: see also E. Viterbo.

[BDMS] — D. Divsalar, P. J. McLane and M. K. Simon, *Introduction to Trellis-Coded Modulation with Applications*, Macmillan, New York, 1991.

[Bla91] R. E. Blahut, *The Gleason-Prange theorem*, PGIT **37** (1991), 1269–1273.

I. F. Blake: see also A. H. Banihashemi, V. Tarokh.

[BlTa96] — and Tarokh, *On the trellis complexity of the densest lattices in* $\mathbb{R}^N$, SIAM J. Discr. Math. **9** (1996), 597–601.

[Blij59] F. van der Blij, *An invariant of quadratic forms mod 8*, Indag. Math. **21** (1959), 291–293.

[BocS91] S. Böcherer and R. Schulze-Pillot, *Siegel modular forms and theta series attached to quaternion algebras*, Nagoya Math. J. **121** (1991), 35–96.

[BocS97] —— *Siegel modular forms and theta series attached to quaternion algebras II*, Nagoya Math. J. **147** (1997), 71–106.

[BoHW72] J. Bokowski, H. Hadwiger and J. M. Wills, *Eine Ungleichung zwischen Volumen, Oberfläche und Gitterpunktanzahl konvexer Körper im n-dimensionalen euklidischen Raum*, Math. Zeit. **127** (1972), 363–364.

[BKT87] A. I. Bondal, A. I. Kostrikin and P. H. Tiep, *Invariant lattices, the Leech lattice and its even unimodular analogs in the Lie algebras A_{p-1}*, Mat. Sb. **130** (1986), 435–464 = Sb. **58** (1987), 435–465.

[Bonn90] P. G. Bonneau, *Weight distribution of translates of MDS codes*, Combinatorica **10** (1990), 103–105.

A. Bonnecaze: see also C. Bachoc.

[BonCS95] — A. R. Calderbank and P. Solé, *Quaternary quadratic residue codes and unimodular lattices*, PGIT **41** (1995), 366–377.

[BonGHKS] — P. Gaborit, M. Harada, M. Kitazume and P. Solé, *Niemeier lattices and Type II codes over $\mathbb{Z}_4$*, preprint.

[BonMS97] — B. Mourrain and P. Solé, *Jacobi polynomials, Type II codes, and designs*, DCC, submitted.

[BonS94] — and P. Solé, *Quaternary constructions of formally self-dual binary codes and unimodular lattices*, in *Algebraic Coding*, Lect. Notes Comp. Sci. **781** (1994), 194–205.

[BonSBM] —— C. Bachoc and B. Mourrain, *Type II codes over $\mathbb{Z}_4$*, *PGIT* **43** (1997), 969–976. PGIT **43** (1997), 969–976.

[Borch86] R. E. Borcherds, *Vertex algebras, Kac-Moody algebras, and the Monster*, PNAS **83** (1986), 3068–3071.

[Borch87] — *Automorphism groups of Lorentzian lattices*, J. Alg. **111** (1987), 133–153.

[Borch88] — *Generalized Kac-Moody algebras*, J. Alg. **115** (1988), 501–512.

[Borch90] — *Lattices like the Leech lattice*, J. Alg. **130** (1990), 219–234.

[Borch90a] — *The monster Lie algebra*, Adv. Math. **83** (1990), 30–47.

[Borch91] — *Central extensions of generalized Kac-Moody algebras*, J. Alg. **140** (1991), 330–335.

[Borch92] — *Introduction to the Monster Lie algebra*, in *Groups, Combinatorics and Geometry*, ed. M. W. Liebeck and J. Saxl, Camb. Univ. Press, 1992, pp. 99–107.

[Borch92a] — *Monstrous moonshine and monstrous Lie superalgebras*, Invent. Math. **109** (1992), 405–444.

[Borch94] — *Sporadic groups and string theory*, in *Proc. First European Congress of Math. (Paris, July 1992)*, ed. A. Joseph et al., Birkhauser, Vol. 1 (1994), pp. 411–421.

[Borch95] — *A characterization of generalized Kac-Moody algebras*, J. Alg. **174** (1995), 1073–1079.

[Borch95a] — *Automorphic forms on $O_{s+2,2}(\mathbb{R})$ and infinite products*, Invent. Math. **120** (1995), 161–213.

[Borch95b] — *Automorphic forms on $O_{s+2,2}(\mathbb{R})^+$ and generalized Kac-Moody algebras*, Proc. Internat. Congress Math. (Zürich, 1994), Vol II, pp. 744–742.

[Borch96] — *The moduli space of Enriques surfaces and the fake Monster Lie superalgebra*, Topology **35** (1996), 699–710.

[Borch96a] — *Modular moonshine III*, DMJ, to appear, 1998.

[Borch96b] — *Automorphic forms and Lie algebras*, in *Current Developments in Mathematics 1996*, to appear.

[BorchHP] — *Home page*: http://www.dpmms.cam.ac.uk/~reb/.

[BorchF98] — E. Freitag and R. Weissauer, *A Siegel cusp form of degree 12 and weight 12*, JRAM **494** (1998), 141-153.

[BorchR96] — and A. J. E. Ryba, *Modular moonshine II*, DMJ **83** (1996), 435–459.

K. Böröczky, Jr., see also V. Arhelger, U. Betke.

[BorSch97] — and U. Schnell, *Asymptotic shape of finite packings*, CJM, to appear.

[BorSch98] — — *Quasicrystals and the Wulff-shape*, DCG, to appear.

[BorSch98a] — — *Wulff-shape for non-peridic arrangements*, Lett. Math. Phys., to appear.

[BorWi96] — and J. M. Wills, *Finite sphere packing and critical radii*, Contributions to Algebra and Geometry **38** (1997), 193–211.

[BosC97] W. Bosma and J. Cannon, *Handbook of Magma Functions*, Sydney, May 22, 1995.

[BosCM94] — — and G. Mathews, *Programming with algebraic structures: Design of the Magma language*, in *Proceedings of the 1994 International Symposium on Symbolic and Algebraic Computation*, M. Giesbrecht, Ed., Association for Computing Machinery, 1994, 52–57.

[BosCP97] — — and C. Playoust, *The Magma algebra system I: The user language*, J. Symb. Comp. **24** (1997), 235–265.

[BoVRB] J. Boutros, E. Viterbo, C. Rastello and J. C. Belfiore, *Good lattice constellations for both Rayleigh fading and Gaussian channels*, PGIT **42** (1996), 502–518.

[BoV98] — — *A power and bandwidth efficient diversity technique for the Rayleigh fading channel*, PGIT to appear.

[Boy93] P. Boyvalenkov, *Nonexistence of certain symmetric spherical codes*, DCC **3** (1993), 69–74.

[Boy94] — *On the extremality of the polynomials used for obtaining the best known upper bounds for the kissing numbers*, J. Geom. **49** (1994), 67–71.

[Boy94a] — *Small improvements for the upper bounds for the kissing numbers in dimensions 19, 21 and 23*, Atti Sem. Mat. Fis. Univ. Modena **LXII** (1994), 159–163.

[Boy95] — *Computing distance distributions of spherical designs*, Lin. Alg. Applications **226–228** (1995), 277–286.

[Boy95a] — *Extremal polynomials for obtaining bounds for spherical codes and designs*, DCG **14** (1995), 167–183.

[BoyD95] — and D. Danev *Classification of spherical codes attaining the even Levenshtein bound*, in *Proc. Workshop on Optimal Codes and Related Topics*, Sozopol, Bulgaria, May 1995, pp. 21–24.

[BoyD95a] — — *On linear programming bounds for codes in polynomial metric spaces*, PPI, to appear.

[BoyD97] — — *On maximal codes in polynomial metric spaces*, Lect. Notes Comp. Sci. **1255** (1997), 29–38.

[BoyDB96] — — and S. P. Bumova, *Upper bounds on the minimum distance of spherical codes*, PGIT **42** (1996), 1576–1581.

[BoyDL97] — — and I. Landgev, *On maximal spherical codes II*, J. Combin. Designs, to appear.

[BoyDM] — — and M. Mitradjieva, *Linear programming bounds for codes in infinite projective spaces*, preprint.

[BoyDN] — — and S. Nikova, *Nonexistence of certain spherical designs of odd strengths and cardinalities*, DCG to appear.

[BoyD93] — and S. M. Dodunekov, *Some new bounds on the kissing numbers*, in *Proc. Sixth Joint Swedish-Russian Internat. Workshop on Information Theory*, Möele, 1993, pp. 389–393.

[BoyK94] — and P. Kazakov, *Nonexistence of certain spherical codes*, C. R. Bulgarian Acad. Sci. **47** (1994), 37–40.

[BoyK95] — — *New upper bounds for some spherical codes*, Serdica Math. J. **21** (1995), 231–238.

[BoyL95] — and I. Landgev, *On maximal spherical codes I*, in *Proc. XI AAECC Symposium*, Paris, 1995, Lect. Notes Comp. Sci. **948**, pp. 158–168.

[BoyN94] — and S. Nikova, *New lower bounds for some spherical designs*, in *Algebraic Coding*, Lect. Notes Comp. Sci. **781** (1994), 207–216.

[BoyN95] — — *Improvements of lower bounds for some spherical designs*, Math. Balcanika, to appear.

[BoyN97] — — *On lower bounds on the size of designs in compact symmetric spaces of rank 1*, Archiv. Math. **68** (1997), 81–88.

[Bro92] A. E. Brouwer, personal communication, 1992.

[Bro95] — *The linear programming bound for binary linear codes*, PGIT **39** (1995), 677–680.

[Bro98] — *Tables of bounds for linear codes*, to appear in [PHB].

[BrCN89] — A. M. Cohen and A. Neumaier, *Distance-Regular Graphs*, Springer-Verlag, NY, 1989.

[BrHOS] — H. O. Hämäläinen, P. R. J. Östergård, and N. J. A. Sloane, *Bounds on mixed binary ternary codes*, PGIT **44** (1998), 140–161.

[BrSSS] — J. B. Shearer, N. J. A. Sloane and W. D. Smith, *A new table of constant weight codes*, PGIT **36** (1990), 1334–1380.

[BrVe93] — and T. Verhoeff, *An updated table of minimum-distance bounds for binary linear codes*, PGIT **39** (1993), 662–677.

R. A. Brualdi: see also V. Pless.

[BrLP98] — S. Litsyn and V. S. Pless, *Covering radius*, in [PHB], to appear.

[BrP91] — and V. S. Pless, *Weight enumerators of self-dual codes*, PGIT **37** (1991), 1222–1225.

[Brau40] H. Braun, *Geschlecter quadratischer Formen*, JRAM **182** (1940), 32–49.

J. Bruck: see N. Alon.

[Bru86] N. D. de Bruijn, *Quasicrystals and their Fourier transform*, KNAW **48** (1986), 123–152.

G. O. Brunner: see also R. W. Grosse-Kunstleve.

[Brunn79] — *The properties of coordination sequences and conclusions regarding the lowest possible density of zeolites*, J. Solid State Chem. **29** (1979), 41–45.

[BrLa71] — and F. Laves, *Zum problem der koordinationszahl*, Wiss. Z. Techn. Univ. Dresden **20** (1971), 387–390.

[Bud89] R. de Buda, *Some optimal codes have structure*, IEEE J. Select. Areas Commun. **7** (1989), 893–899.

[Bue89] D. A. Buell, *Binary Quadratic Forms*, Springer-Verlag, NY, 1989.

[Bul73] R. Bülow, *Ueber Dadegruppen in $GL(5,\mathbb{Z})$*, Dissertation, Rheinisch-Westfälische Tech. Hochschule, Aachen, 1973.

S. P. Bumova: See P. G. Boyvalenkov.

[BuCD94] P. Buser, J. H. Conway, P. G. Doyle and K.-D. Semmler, *Some planar isospectral domains*, Inter. Math. Res. Notes **9** (1994), 391–400.

[BuSa94] — and P. Sarnak, *On the period matrix of a Riemann surface of large genus*, Invent. Math. **117** (1994), 27–56.

[BuTo89] F. C. Bussemaker and V. D. Tonchev, *New extremal doubly-even codes of length 56 derived from Hadamard matrices of order 28*, Discr. Math. **76** (1989), 45–49.

[BuTo90] — — *Extremal doubly-even codes of length 40 derived from Hadamard matrices*, Discr. Math. **82** (1990), 317–321.

A. R. Calderbank: see also A. Bonnecaze, G. D. Forney, Jr., A. R. Hammons, Jr.

[Cal91] — *The mathematics of modems*, Math. Intell. **13** (1991), 56–65.

[CaD92] — and P. Delsarte, *On error-correcting codes and invariant linear forms*, SIAM J. Discr. Math. **6** (1993), 1–23.

[CaD92a] — — *Extending the t-design concept*, TAMS, to appear.

[CaDS91] —— and N. J. A. Sloane, *A strengthening of the Assmus-Mattson theorem*, PGIT **37** (1991), 1261–1268.

[CaFR95] — P. C. Fishburn and A. Rabinovich, *Covering properties of convolutional codes and associated lattices*, PGIT **41** (1995), 732–746.

[CaFV98] — G. D. Forney, Jr. and A. Vardy, *Minimal tail-biting trellises: the Golay code and more*, PGIT, to appear.

[CHRSS] — R. H. Hardin, E. M. Rains, P. W. Shor and N. J. A. Sloane, *A group-theoretic framework for the construction of packings in Grassmannian space*, J. Algeb. Combin. (1998), to appear.

[CaMc97] — and G. McGuire, *Construction of a* $(64, 2^{37}, 12)$ *code via Galois rings*, DCC **10** (1997), 157–165.

[CaO90] — and L. H. Ozarow, *Nonequiprobable signaling on the Gaussian channel*, PGIT **36** (1990), 726–740.

[CaSl95] — and N. J. A. Sloane, *Modular and p-adic cyclic codes*, DCC **6** (1995), 21–35.

[CaSl97] —— *Double circulant codes over* $\mathbb{Z}_4$ *and even unimodular lattices*, J. Algeb. Combin. **6** (1997), 119–131.

J. Carlson: see D. Benson.

[ChaKR] W. Chan, M. H. Kim and S. Ragavan, *Ternary universal integral quadratic forms over real quadratic fields*, Indian J. Math., to appear.

[Cha82] G. R. Chapman, *Generators and relations for the cohomology ring of Janko's first group in the first twenty-one dimensions*, in *Groups – St. Andrews 1981*, London Math. Soc. Lect. Notes **71**, Cambridge Univ. Press, 1982.

[Chap96] R. Chapman, *Quadratic residue codes and lattices*, preprint.

[Chap97] — *Higher power residue codes*, Finite Flds. Applic. **3** (1997), 353–369.

[ChS96] — and P. Solé, *Universal codes and unimodular lattices*, JNB **8** (1996), 369–376.

[Chen91] C. L. Chen, *Construction of some binary linear codes of minimum distance five*, PGIT **37** (1991), 1429–1432.

W. Chen: see D. Li.

[Chow92] T. Y. Chow, *Penny-packings with minimal second moments*, Combinatorica **15** (1995), 151–158.

[Chu84] F. R. K. Chung, *The number of different distances determined by n points in the plane*, JCT **A 36** (1984), 342–354.

A. M. Cohen: see also A. E. Brouwer.

[Coh91] — *Presentations for certain finite quaternionic reflection groups*, in *Advances in Finite Geometries and Designs*, ed. J. W. P. Hirschfeld et al., Oxford, 1991, pp. 69–79.

[CNP96] — G. Nebe and W. Plesken, *Cayley orders*, Composit. Math. **103** (1996), 63–74.

[CHLL] G. D. Cohen, I. Honkala, S. N. Litsyn and A. C. Lobstein, *Covering Codes*, North-Holland, Amsterdam, 1997.

[CHLS] — — — and P. Solé, *Long packing and covering codes*, PGIT **43** (1997), 1617–1619.

[CLLM97] — S. N. Litsyn, A. C. Lobstein and H. F. Mattson, Jr., *Covering radius 1985–1994*, Applicable Algebra in Engineering, Communication and Computing, **8** (1997), 173–239.

H. Cohen: see also C. Batut.

[CohCNT] — *A Course in Computational Number Theory*, Springer-Verlag, NY, 1993.

[Coh92] S. D. Cohen, *The explicit construction of irreducible polynomials over finite fields*, DCC **2** (1992), 169–174.

R. Connelly: see A. Bezdek, K. Bezdek.

J. H. Conway: see also P. Buser, N. J. A. Sloane.

[Con92] — *The orbifold notation for surface groups*, in *Groups, Combinatorics and Geometry*, ed. M. W. Liebeck and J. Saxl, Camb. Univ. Press, 1992, pp. 438–447.

[Con93] — *From hyperbolic reflections to finite groups*, in *Groups and Computation*, DIMACS Series in Discrete Math., Vol. 11, Amer. Math. Soc., 1993, pp. 41–51.

[CoFu97] — assisted by P. Y. C. Fung, *The Sensual (Quadratic) Form*, Math. Assoc. Amer. 1997.

[CoHMS] — T. C. Hales, D. J. Muder and N. J. A. Sloane, *On the Kepler conjecture*, Math. Intelligencer **16** (No. 2, 1994), 5.

[CoHS96] — R. H. Hardin, and N. J. A. Sloane, *Packing lines, planes, etc.: packings in Grassmannian space*, Exper. Math. **5** (1996), 139–159.

[CoPS92] — V. Pless and N. J. A. Sloane, *The binary self-dual codes of length up to 32: A revised enumeration*, JCT **A 60** (1992), 183–195.

[CoSch98] — and W. A. Schneeberger, in preparation.

[CSLDL1] — and N. J. A. Sloane, *Low-dimensional lattices I: Quadratic forms of small determinant*, PRS **A 418** (1988), 17–41. (For errata, see end of Preface to 3rd Edition.)

[CSLDL2] — — *Low-dimensional lattices II: Subgroups of* $GL(n, \mathbb{Z})$, PRS **A 419** (1988), 29–68. (For errata, see end of Preface to 3rd Edition.)

[CSLDL3] — — *Low-dimensional lattices III: Perfect forms*, PRS **A 418** (1988), 43–80. (For errata, see end of Preface to 3rd Edition.)

[CSLDL4] — — *Low-dimensional lattices IV: The mass formula*, PRS **A 419** (1988), 259–286. (For errata, see end of Preface to 3rd Edition.)

[CSLDL5] — — *Low-dimensional lattices V: Integral coordinates for integral lattices*, PRS **A 426** (1989), 211–232.

[CSLDL6] — — *Low-dimensional lattices VI: Voronoi reduction of three-dimensional lattices*, PRS **A 436** (1991), 55–68.

[CSLDL7] —— *Low-dimensional lattices VII: Coordination sequences*, PRS **A 453** (1997), 2369–2389.

[CSLDL8] —— *Low-dimensional lattices VIII: The 52 four-dimensional parallelotopes*, in preparation.

[CoSl90] —— *A new upper bound for the minimum of an integral lattice of determinant one*, BAMS **23** (1990), 383–387; **24** (1991), 479.

[CoSl90a] —— *A new upper bound for the minimal distance of self-dual codes*, PGIT **36** (1990), 1319–1333.

[CoSl90b] —— *Orbit and coset analysis of the Golay and related codes*, PGIT **36** (1990), 1038–1050.

[CoSl91] —— *Lattices with few distances*, JNT **39** (1991), 75–90.

[CoSl91a] —— *The cell structures of certain lattices*, in *Miscellanea mathematica*, P. Hilton, F. Hirzebruch and R. Remmert, editors, Springer-Verlag, NY, 1991, pp. 71–107.

[CoSl92] —— *On the covering multiplicity of lattices*, DCG **8** (1992), 109–130.

[CoSl92a] —— *Four-dimensional lattices with the same theta series*, DMJ **6** (Inter. Math. Res. Notices 4) 1992, 93–96.

[CoSl93] —— *Self-dual codes over the integers modulo 4*, JCT **A 62** (1993), 30–45.

[CoSl94] —— *On lattices equivalent to their duals*, JNT **48** (1994), 373–382.

[CoSl94a] —— *D_4, E_8, Leech and certain other lattices are symplectic.* Appendix to P. Buser and P. Sarnak [BuSa94].

[CoSl94b] —— *Quaternary constructions for the binary single-error-correcting codes of Julin, Best and others*, DCC **4** (1994), 31–42.

[CoSl95] —— *A lattice without a basis of minimal vectors*, Math. **42** (1995), 175–177.

[CoSl95a] —— *What are all the best sphere packings in low dimensions?*, DCG **13** (1995), 383–403.

[CoSl96] —— *The antipode construction for sphere packings*, Invent. Math. **123** (1996), 309–313.

[CoSl98] —— *A note on unimodular lattices*, JNT (submitted).

[CoSl98a] —— *Codes and lattices*, PGIT, to appear.

M. Cook: see L. Auslander.

[CoRa93] R. Cools and P. Rabinowitz, *Monomial cubature rules since "Stroud": a compilation*, J. Comput. Appl. Math. **48** (1993), 309–326.

[Cogn94] R. Coulangeon, *Réseaux quaternioniens et invariant de Venkov*, Manuscripta Math. **82** (1994), 41–50.

[Cogn95] — *Réseaux unimodulaires quaternioniens en dimension inférieure ou égale à 32*, Acta Arith. **71** (1995), 9–24.

[Cogn96] — *Réseaux k-extrêmes*, PLMS **73** (1996), 555–574.

[Cogn97] — *Minimal vectors in the second exterior power of a lattice*, J. Alg. **194** (1997), 467–476.

[Cogn98] — *Tensor products of Hermitian lattices*, preprint.

[Coul91] D. Coulson, *The dual lattice of an extreme six-dimensional lattice*, JAMS **A 50** (1991), 373–383.

[Coxe95] H. S. M. Coxeter, *Kaleidoscopes: Selected Writings*, ed. F. A. Sherk et al., Wiley, NY, 1995.

[CrL90] J. Cremona and S. Landau, *Shrinking lattice polyhedra*, SIAM J. Discr. Math. **3** (1990), 338–348.

[Cumm] C. J. Cummins, *Moonshine bibliography*, published electronically at http://www-cicma.concordia.ca/faculty/cummins/moonshine.refs.html .

[Cur89] R. T. Curtis, *Further elementary techniques using the Miracle Octad Generator*, Proc. Edin. Math. Soc. **32** (1989), 345–353.

[Cur89a] — *Natural constructions of the Mathieu groups*, PCPS **106** (1989), 423–429.

[Cur90] — *Geometric interpretations of the 'natural' generators of the Mathieu groups*, PCPS **107** (1990), 19–26.

D. Danev: see P. Boyvalenkov.

[Danz89] L. Danzer, *Three-dimensional analogues of the planar Penrose tilings and quasicrystals*, Discr. Math. **76** (1989), 1–7.

[DaZ87] M. H. Dauenhauer and H. J. Zassenhaus, *Local optimality of the critical lattice sphere-packing of regular tetrahedra*, Discr. Math. **64** (1987), 129–146.

[DaDr94] A. A. Davydov and A. Yu. Brozhzhina-Labinskaya, *Constructions, families and tables of binary linear covering codes*, PGIT **40** (1994), 1270–1279.

[Del29] B. N. Delone, *Sur la partition régulière de l'espace à 4–dimensions*, Izv. Akad. Nauk SSSR Otdel. Fiz.-Mat. Nauk (1929) pp. 79–110 and 145–164.

[Del37] — *Geometry of positive quadratic forms*, Uspehi Mat. Nauk **3** (1937), 16–62; **4** (1938), 102–164.

P. Delsarte: see A. R. Calderbank.

[DezG96] M. Deza and V. Grishukhin, *Bounds on the covering radius of a lattice*, Math. **43** (1996), 159–164. [But see the review: *Math. Reviews* MR98b: 11074.]

[Ding43] A. Dinghas, *Uber einen geometrischen Satz von Wulff über die Gleichgewichtsform von Kristallen*, Z. Krist. **105** (1043), 304–414.

D. Divsalar: see E. Biglieri.

S. M. Dodunekov: see also P. Boyvalenkov.

[DEZ91] — T. Ericson and V. A. Zinoviev, *Concatenation methods for construction of spherical codes in n-dimensional Euclidean space*, PPI **27** (1991), 34–38.

[DGM90] L. Dolan, P. Goddard and P. S. Montague, *Twisted conformal field theory*, in *Trieste Conf. Recent Developments in Conformal Field Theories*, edited S. Randjbar-Daemi et al., World Scientific, Singapore, 1990, pp. 39–54.

[DGM90a] ——— *Conformal field theory of twisted vertex operations*, Nuclear Phys. **B 338** (1990), 529–601.

[DGM90b] ——— *Conformal field theory, triality and the Monster group*, Phys. Lett. **B 236** (1990), 165–72.

[DoMa96] C. Dong and G. Mason, editors, *Moonshine, the Monster, and Related Topics*, Contemp. Math **193**, Amer. Math. Soc., 1996.

[Dou95] S. T. Dougherty, *Shadow codes and weight enumerators*, PGIT **41** (1995), 762–768.

[DoGH97] — T. A. Gulliver and M. Harada, *Type II codes over finite rings and even unimodular lattices*, J. Alg. Combin., to appear.

[DoGH97a] ——— *Extremal binary self-dual codes*, PGIT **43** (1997), 2036–2046.

[DoHa96] — and M. Harada, *Shadow optimal self-dual codes*, Kyushu J. Math., to appear.

[DoHa97] —— *New extremal self-dual codes of length 68*, preprint.

[DoHa97a] —— *Self-dual codes constructed from Hadamard matrices and symmetric designs*, preprint.

[DoHa09] —— and M. Oura, *Formally self-dual codes*, preprint.

[DoHaS97] —— and P. Solé, *Shadow codes over* $\mathbb{Z}_4$, preprint, July 1997.

[DowFV] R. G. Downey, M. R. Fellows, A. Vardy and G. Whittle, *The parameterized complexity of some fundamental problems in coding theory*, SIAM J. Computing, to appear.

P. G. Doyle: see P. Buser.

D. B. Drajić: see D. E. Lazić.

[DrS96] T. Drisch and P. Sonneborn, *The optimal results from linear programming for kissing numbers*, preprint.

A. Yu Drozhzhina-Labinskaya: see A. A. Davydov.

E. Dubois: see A. K. Khandani.

T. S. Duff: see N. J. A. Sloane.

[Duke93] W. Duke, *On codes and Siegel modular forms*, Intern. Math. Res. Notices **5** (1993), 125–126.

[DuS90] — and R. Schulze-Pillot, *Representations of integers by positive ternary quadratic forms and equidistribution of lattice points on ellipsoids*, Invent. Math. **99** (1990), 49–57.

[Dum94] N. Dummigan, *Mordell-Weil lattices and Shafarevich-Tate groups*, preprint, 1994.

[Dum95] — *The determinants of certain Mordel-Weil lattices*, Amer. J. Math. **117** (995), 1409–1429.

[Dum96] — *Symplectic group lattices as Mordell-Weil sublattices*, JNT **61** (1996), 365–387.

[Dum97] — *Algebraic cycles and even unimodular lattices*, preprint.

[Dye91] M. Dyer, *On counting lattice points in polyhedra*, SIAM J. Comput. **20** (1991), 695–707.

[DyKa98] — and R. Kannan, *On Barvinok's algorithm for counting lattice points in fixed dimension*, preprint, 1998.

A. G. Earnest: see also J. W. Benham.

[Earn88] — *Minimal discriminants of indefinite ternary quadratic forms having specified class number*, Math. **35** (1988), 95–100.

[Earn89] — *Binary quadratic forms over rings of algebraic integers: A survey of recent results*, in *Théorie des Nombres*, ed. J.-M. de Koninck and C. Levesque, Gruyter, Berlin, 1989, pp. 133–159.

[Earn90] — *Discriminants and class numbers of indefinite integral quadratic forms*, in *Number Theory*, ed. R A. Mollin, Gruyter, Berlin, 1990, pp. 115–123.

[Earn91] — *Genera of rationally equivalent integral binary quadratic forms*, Proc. Roy. Soc. Ed. **A 119** (1991), 27–30.

[Earn94] — *The representation of binary quadratic forms by positive definite quaternary quadratic forms*, TAMS **345** (1994), 853–863.

[EaH91] — and J. S. Hsia, *One-class spinor genera of positive quadratic forms*, Acta Arith. **58** (1991), 133–139.

[EaK97] — and A. Khosravani, *Universal binary Hermitian forms*, MTAC **66** (1997), 1161–1168.

[EaK97a] — — *Representations of integers by positive definite binary Hermitian lattices over imaginary quadratic fields*, JNT **62** (1997), 368–374.

[EaK97b] — — *Universal positive quaternary quadratic lattices over totally real number fields*, Math., to appear.

[EaN91] — and G. Nipp, *On the theta series of positive quaternary quadratic forms*, C. R. Math. Rep. Acad. Sci. Canada **13** (No. 1, Feb. 1991), 33–38.

[EaN97] — — *On the classification of positive quaternary quadratic forms by theta series*, preprint.

[Ebe87] W. Eberling, *The Monodromy Groups of Isolated Singularities of Complete Intersections*, LNM **1293** (1987).

[Ebe94] — *Lattices and Codes*, Vieweg, Wiesbaden, 1994.

Y. Edel: see also J. Bierbrauer.

[EdRS98] — E. M. Rains and N. J. A. Sloane, *On kissing numbers in dimensions 32 to 128*, preprint 1998.

[Ehr60] E. Ehrhart, *Sur les polyèdres rationnels et les systèmes diophantiens linéaires*, CR **250** (1960), 959–961.

[Ehr67] — *Démonstration de la loi réciprocité du polyèdre rationnel*, CR **265** (1967), 91–94.

[Ehr73] — *Une extension de la loi de réciprocité des polyèdres rationnels*, CR **277** (1973), 575–577.

[Ehr77] — *Polynômes Arithmétiques et Méthode des Polyèdres en Combinatoire*, Internat. Series Numer. Math., Vol. **35**, Birkhauser, Basel, 1977.

[EiZa85] M. Eichler and D. Zagier, *The Theory of Jacobi Forms*, Birkhäuser, Boston, 1985.

[Elki] N. D. Elkies, personal communication.

[Elki94] — *Mordell-Weil lattices in characteristic 2: I. Construction and first properties*, Internat. Mathematics Research Notices (No. 8, 1994), 353–361.

[Elki95] — *A characterization of the $\mathbb{Z}^n$ lattice*, Math. Research Letters **2** (1995), 321–326.

[Elki95a] — *Lattices and codes with long shadows*, Math. Research Letters **2** (1995), 643–651.

[Elki96] — *Mock-laminated lattices*, preprint, 1996.

[Elki97] — *Mordell-Weil lattices in characteristic 2: II. The Leech lattice as a Mordell-Weil lattice*, Invent. Math. **128** (1996), 1–8.

[Elki97a] — *Explicit modular towers*, in *Proc. Allerton Conf.*, 1997.

[ElkGr96] — and B. H. Gross, *The exceptional cone and the Leech lattice*, Internat. Math. Research Notices **14** (1996), 665–698.

[ElkOR91] — A. M. Odlyzko and J. A. Rush, *On the packing density of superballs and other bodies*, Ivent. Math. **105** (1991), 613–639.

[Eng86] P. Engel, *Geometric Crystallography*, Reidel, Dordrecht, Holland, 1986.

[Eng93] — *Geometric crystallography*, Chap. 3.7 of [GruWi93].

R. M. Erdahl: see also S. S. Ryskov.

[ErR87] — and S. S. Ryskov, *The empty sphere*, CJM **39** (1987), 794–824.

[Erd46] P. Erdős, *On sets of distances of n points*, AMM **53** (1946), 248–250.

[ErGH89] — P. M. Gruber and J. Hammer, *Lattice Points*, Longman Scientific, 1989.

T. Ericson: see also S. M. Dodunekov.

[ErZi95] — and V. Zinoviev, *Spherical codes generated by binary partitions of symmetric pointsets*, PGIT **41** (1995), 107–129.

T. Eriksson: see E. Agrell.

[ERS91] A. Eskin, Z. Rudnick and P. Sarnak, *A proof of Siegel's weight formula*, Internat. Math. Research Notes (Duke Math. J.) **5** (1991), 65–69.

M. Esmaeili: see A. K. Khandani.

[Ess90] F. Esselmann, *Ueber die maximale Dimension von Lorentz-Gittern mit coendlicher Spiegelungsgruppe*, JNT **61** (1996), 103–144.

[EsPa70] D. Estes and G. Pall, *The definite octonary quadratic forms of determinant 1*, IJM **14** (1970), 159–163.

[EtGr93] T. Etzion and G. Greenberg, *Constructions for perfect mixed codes and other covering codes*, PGIT **39** (1993), 209–213.

[EtGh93] —— and I. S. Honkala, *Normal and abnormal codes*, PGIT **39** (1993), 1453–1456.

[EtWZ95] — V. Wei and Z. Zhang, *Bounds on the sizes of constant weight covering codes*, DCC **5** (1995), 217–239.

L. Fainsilber: see E. Bayer-Fluckiger.

[FaL95] G. Fazekas and V. I. Levenshtein, *On upper bounds for code distance and covering radius of designs in polynomial metric spaces*, JCT **A 70** (1995), 267–288.

M. Feder: see R. Zamir.

[Fed85] E. S. Fedorov, *Elements of the study of figures*, Zap. Mineralog. Obsc. (2) **2** (1885), 1–279. Reprinted by Izdat. Akad. Nauk SSSR, Moscow, 1953.

[Fed91] — *The symmetry of regular systems of figures*, Zap. Mineralog. Obsc. (2) **28** (1891), 1–146. English translation in *Symmetry of Crystals*, ACA Monograph no. 7, Amer. Crystallographic Assoc., New York, 1971, pp. 50–131.

[FeFMMV] J. Feigenbaum, G. D. Forney, Jr., B. H. Marcus, R. J. McEliece and A. Vardy, editors, *Special Issue on Codes and Complexity*, PGIT **42** (No. 6, 1996), 1649–2064.

[Fej95] G. Fejes Tóth, *Review of [Hsi93]*, Math. Review **95**g #52032, 1995.

[Fej97] — *Packing and covering*, Chap. 2 of [GoO'R97].

[FGW90] — P. Gritzmann and J. M. Wills, *On finite multiple packings*, Archiv. Math. **55** (1990), 407–411.

[FGW91] ——— *Finite sphere packing and sphere covering*, DCG **4** (1989), 19–40.

[FeK93] — and W. Kuperberg, *Packing and covering with convex sets*, Chap. 3.3 of [GruWi93]

[FeK93a] —— *Blichfeldt's density bound revisited*, Math. Ann. **295** (1993), 721–727.

[Fel92] A. vom Felde, *A new presentation of Cheng-Sloane's [32,17,8] code*, Archiv. Math. **60** (1993), 508–511.

M. R. Fellows: see R. G. Downey.

[FeRa94] G.-L. Feng and T. R. N. Rao, *Decoding algebraic-geometric codes up to the designed minimum distance*, PGIT **39** (1994), 37–45.

[Ferg97] S. P. Ferguson, *Sphere packings V*, Abstract 926-51-196, Abstracts Amer. Math. Soc. **18** (1997).

[FeHa97] — and T. C. Hales, *A formulation of the Kepler conjecture*, preprint.

[FiP96] C. Fieker and M. E. Pohst, *On lattices over number fields*, Lect. Notes Comp. Sci. **1122** (1996), 133-139.

[Fiel80] K. L. Fields, *The fragile lattice packings of spheres in three-dimensional space*, Acta Cryst. **A 36** (1980), 194–197.

[FiNi80] — and M. J. Nicolich, *The fragile lattice packings of spheres in four-dimensional space*, Acta Cryst. **A 36** (1980), 588–591.

P. C. Fishburn: see A. R. Calderbank.

O. M. Fomenko: see E. P. Golubeva.

G. D. Forney, Jr.: see also A. R. Calderbank, J. Feigenbaum.

[Forn88] — *Coset codes I: Introduction and geometrical classification*, PGIT **34** (1988), 1066–1070.

[Forn88a] — *Coset codes II: Binary lattices and related codes*, PGIT **34** (1988), 1152–1187.

[Forn89] — *A bounded-distance decoding algorithm for the Leech lattice, with generalizations*, PGIT **35** (1989), 906–909.

[Forn89a] — *Multidimensional constellations II: Voronoi constellations*, IEEE J. Select. Areas Commun. **7** (1989), 941–958.

[Forn91] — *Geometrically uniform codes*, PGIT **37** (1991), 1241–1260.

[Forn93] — *Progress in geometrically uniform codes*, in *Proc. Sixth Swedish-Russian Joint Workshop on Information Theory*, Mölle, Sweden, 1993, pp. 16–20.

[Forn94] — *Dimension/length profiles and trellis complexity of linear block codes*, PGIT **40** (1994), 1741–1752.

[Forn94a] — *Dimension/length profiles and trellis complexity of lattices*, loc. cit., 1753–1791.

[Forn97] — *Approaching AWGN channel capacity with coset codes and multilevel coset codes*, preprint, 1997.

[FoCa89] — and A. R. Calderbank, *Coset codes for partial response channels; or, coset codes with spectral nulls*, PGIT **35** (1989), 925–943.

[FoST93] — N. J. A. Sloane and M. D. Trott, *The Nordstrom-Robinson code is the binary image of the octacode*, in *Coding and Quantization: DIMACS/IEEE Workshop October 19–21, 1992*, R. Calderbank, G. D. Forney, Jr., and N. Moayeri, Eds., Amer. Math. Soc., 1993, pp. 19–26.

[FoTr93] — and M. D. Trott, *The dynamics of group codes: state spaces, trellis diagrams, and canonical encoders*, PGIT **39** (1993), 1491–1513.

[FoVa96] — and A. Vardy, *Generalized minimum distance decoding of Euclidean-space codes and lattices*, PGIT **42** (1996), 1992–2026.

[FoWe89] — and L. F. Wei, *Multidimensional signal constellations I: Introduction, figures of merit, and generalized cross constellations*, IEEE J. Select. Areas Commun. **7** (1989), 877–892.

[Fort97] S. Fortune, *Voronoi diagrams and Delaunay triangulations*, Chap. 20 of [GrO'R97].

E. Freitag: see R. E. Borcherds.

[Fri89] S. Friedland, *Normal forms for definite integer unimodular quadratic forms*, PAMS **106** (1989), 917–921.

[FrTh94] J. Fröhlich and E. Thiran, *Integral quadratic forms, Kac-Moody algebras, and fractional quantum Hall effects. An ADE-O classification*, J. Statist. Phys. **76** (1994), 209–283.

F. Y. C. Fung: see J. H. Conway.

P. Gaborit: see A. Bonnecaze.

[GaWi92] P. M. Gandini and J. M. Wills, *On finite sphere-packings*, Math. Pannonica **3** (1992), 19–29.

[GaZu92] — and A. Zucco, *On the sausage catastrophe in 4-space*, Math. **39** (1992), 274–278.

[GaL90] T. Gannon and C. S. Lam, *Construction of four-dimensional strings*, Phys. Rev. **D 41** (1990), 492–506.

[GaL91] — — *Gluing and shifting lattice constructions and rational equivalence*, Reviews in Mathematical Physics **3** (1991), 331–369.

[GaL92] — — *Lattices and theta-functions identities I: Theta constants*, J. Math. Phys. **33** (1992), 854–870.

[GaL92a] — — *Lattices and theta-function identities II: Theta series*, J. Math. Phys. **33** (1992), 871–887.

[GaSt95] A. Garcia and H. Stichtenoth, *A tower of Artin-Schreier extensions of function fields attaining the Drinfeld-Valdut bound*, Invent. Math. **121** (1995), 211–222.

Z. Gáspár: see T. Tarnai.

A. Gersho: see K. Zeger.

[Gers72] L. J. Gerstein, *The growth of class numbers of quadratic forms*, AJM **94** (1972), 221–236.

[Gers91] — *Stretching and welding indecomposable quadratic forms*, JNT **37** (1991), 146–151.

[Gers95] — *Nearly unimodular quadratic forms*, Annals Math. **142** (1995), 597–610.

[Gers96] — *Characteristic elements of unimodular lattices*, preprint, 1996.

A. Gervois: see also L. Oger.

[GLOG] — M. Lichtenberg, L. Oger and E. Guyon, *Coordination number of disordered packings of identical spheres*, J. Phys. **A 22** (1989), 2119–2131.

P. Goddard: see also L. Dolan.

[Godd89] — *Meromorphic conformal field theory*, in *Infinite-Dimensional Lie Algebras and Lie Groups*, edited by V. G. Kac, World Scientific, Singapore, 1989, pp. 556–587.

[Golb70] M. Goldberg, *The packing of equal circles in a square*, Math. Mag. **43** (1970), 24–30.

[GoF87] E. P. Golubeva and O. M. Fomenko, *Asymptotic equidistribution of integral points on the three-dimensional sphere*, LOMI **160** (1987), 54–71.

[GoO'R97] J. E. Goodman and J. O'Rourke, editors, *Handbook of Discrete and Computational Geometry*, CRC Press, Boca Raton, 1997.

[Gop88] V. D. Goppa, *Gometry and Codes*, Kluwer, Dordrecht, 1988.

[GWW92] C. S. Gordon, D. L. Webb and S. A. Wolpert, *One cannot hear the shape of a drum*, BAMS **27** (1992), 134–138.

[Gow89] R. Gow, *Even unimodular lattices associated with the Weil representations of the finite symplectic groups*, J. Alg. **22** (1989), 510–519.

[Gow89a] — *Unimodular integral lattices associated with the basic spin representations of $2A_n$ and $2S_n$*, BLMS **21** (1989), 257–262.

[GraL95] R. L. Graham and B. D. Lubachevsky, *Dense packings of equal disks in an equilateral triangle: from 22 to 34 and beyond*, Electron. J. Combin. **2** (1995), #A1, 33 pp.

[GraLNO] — — K. J. Nuermela and P. R. J. Östergård, *Dense packings of congruent circles in a circle*, Discr. Math. **181** (1998), 139–154.

G. Greenberg: see T. Etzion.

[Grie88] R. L. Griess, *Code loops and a large finite group containing triality for D_4*, Rend. Circ. Mat. Palermo **19** (1988), 79–98.

[Grie90] — *A Moufang loop, the exceptional Jordan algebra, and a cubic form in 27 variables*, J. Alg. **131** (1990), 281–293.

U. Grimm: see M. Baake.

V. Grishukhin: see M. Deza.

P. Gritzmann: see also U. Betke, G. Fejes Tóth.

[Gri87] — *Ueber die j-ten Ueberdeckungsdichten konvexer Körper*, Monat. Math. **103** (1987), 207–220.

[GrW85] — and J. M. Wills, *On two finite covering problems of Bambah, Rogers, Woods and Zassenhaus*, Monat. Math. **99** (1985), 279–296.

[GriW93a] — — *Lattice points*, Chap. 3.2 of [GruWi93].

[GriW93b] — — *Finite packing and covering*, Chap. 3.4 of [GruWi93].

[GrG94] B. Groneick and S. Grosse, *New binary codes*, PGIT **40** (1994), 510–512.

C. de Groot: see R. Piekert.

B. H. Gross: see also N. D. Elkies.

[Gro90] — *Group representations and lattices*, J. Amer. Math. Soc. **3** (1990), 929–960.

[Gro96] — *Groups over $\mathbb{Z}$*, Invent. Math. **124** (1996), 263–279.

S. Grosse: see B. Groneick.

[GrBS] R. W. Grosse-Kunstleve, G. O. Brunner and N. J. A. Sloane, *Algebraic description of coordination sequences and exact topological densities of zeollites*, Acta Cryst. **A 52** (1996), 879–889.

P. M. Gruber: see also P. Erdős.

[GruWi93] — and J. M. Wills, *Handbook of Convex Geometry*, North-Holland, Amsterdam, 2 vols., 1993.

[GruS85] B. Grünbaum and G. C. Shephard, *Patterns of circular disks on a 2–sphere*, in *Diskrete Geometrie 3 Kolloq., Salzburg 1985*, North-Holland, 1985, 243–251.

T. A. Gulliver: see also S. T. Dougherty.

[Gull95] — *New optimal ternary linear codes*, PGIT **41** (1995), 1182–185.

E. Guyon: see A. Gervois, L. Oger.

[Habd95] B. Habdank-Eichelsbacher, *Unimodulare Gitter über Reel-quadratischen Zahlkörpern*, Report 95-005, Ergänzungsreihe des Sonderforschungsbereichs 343, Bielefield, 1994.

[Habs94] L. Habsieger, *Lower bounds for q-ary covering codes*, JCT **A 67** (1994), 199–222.

[Habs96] — *Some new lower bounds for ternary covering codes*, Elect. J. Combin. **3** (1996), #23.

[Habs96a] — *A new lower bound for the football pool problem for 7 matches*, JNB **8** (1996), 481–484.

[Habs97] — *Binary codes with covering radius one: some new lower bounds*, Discr. Math. **176** (1997), 115–130.

H. Hadwiger: see J. Bokowski.

[Hahn90] A. J. Hahn, *The coset lattices of E. S. Barnes and G. E. Wall*, JAMS **A 49** (1990), 418–433.

S. G. Hahn: see J. K. Kim.

T. C. Hales: see also J. H. Conway, S. P. Ferguson, S. McLaughlin.

[Hal92] — *The sphere packing problem*, J. Computational Applied Math. **44** (1992), 41–76.

[Hal93] — *Remarks on the density of sphere packings in three dimensions*, Combinatorica **13** (1993), 181–187.

[Hal94] — *The status of the Kepler conjecture*, Math. Intelligencer **16** (No. 3, 1994), 47–58.

[Hal97] — *Sphere packings I*, DCG **17** (1997), 1–51.

[Hal97a] — *Sphere packings II*, DCG **18** (1997), 135–149.

[Hal97b] — *Sphere packings III*, preprint.

[Hal98] — *The Kepler conjecture*, preprint.

H. Hämäläinen: see also A. E. Brouwer.

[Ham88] — *Two new binary codes with minimum distance three*, PGIT **34** (1988), 875.

[HaZe97] J. Hamkins and K. Zeger, *Asymptotically dense spherical codes — Part I: wrapped spherical codes*, PGIT **43** (1997), 1774–1785.

[HaZe97a] — — *Asymptotically dense spherical codes — Part II: laminated spherical codes*, PGIT **43** (1997), 1786–1798.

J. Hammer: see P. Erdős.

[HaKCSS] A. R. Hammons, Jr., P. V. Kumar, A. R. Calderbank, N. J. A. Sloane and P. Solé, *The $\mathbb{Z}_4$-linearity of Kerdock, Preparata, Goethals and related codes*, PGIT **40** (1994), 301–319.

[HaLa89] K. Harada and M. L. Lang, *Some elliptic curves arising from the Leech lattice*, J. Alg. **125** (1989), 298–310.

[HaLa90] — *On some sublattices of the Leech lattice*, Hokkaido Math. J. **19** (1990), 435–446.

[HaLaM94] — — and M. Miyamoto, *Sequential construction of Niemeier lattices and their uniqueness*, JNT **47** (1994), 198–223.

M. Harada: see also Eiichi Bannai, A. Bonnecaze, S. T. Dougherty.

[Hara96] — *Existence of new extremal doubly-even codes and extremal singly-even codes*, DCC **8** (1996), 273–284.

[Hara97] — *The existence of a self-dual* $[70, 35, 12]$ *code and formally self-dual codes*, Finite Fields Applic. **3** (1997), 131–139.

[Hara98] — *New extremal ternary self-dual codes*, Australias. J. Combin., to appear.

[Hara98a] — *Construction of an extremal self-dual code of length 62*, preprint.

R. H. Hardin: see also A. R. Calderbank, J. H. Conway, N. J. A. Sloane.

[HaSl92] — and N. J. A. Sloane, *New spherical 4 designs*, Discr. Math. **106/107** (1992), 255–264.

[HaSl93] — — *A new approach to the construction of optimal designs*, J. Stat. Planning Inference **37** (1993), 339–369.

[HaSl95] — — *Codes (spherical) and designs (experimental)*, in *Different Aspects of Coding Theory*, ed. A. R. Calderbank, AMS Series Proceedings Symposia Applied Math., Vol. 50, 1995, pp. 179–206.

[HaSl96] — — *McLaren's improved snub cube and other new spherical designs in three dimensions*, DCG **15** (1996), 429–441.

[HaSl96a] — — *Operating Manual for Gosset: A General Purpose Program for Designing Experiments (Second Edition)*, Statistics Research Report No. 106, AT&T Bell Laboratories, Murray Hill, NJ, Oct. 1996.

[HSS] —— and W. D. Smith, *Spherical Codes*, book in preparation.

de la Harpe: see R. Bacher.

D. K. Harrison: see S. Beale.

[Has89] Z. Hasan, *Intersection of the Steiner systems of M_{24}*, Discr. Math. **78** (1989), 267–289.

[HaK86] K. Hashimoto and H. Koseki, *Class numbers of positive definite binary and ternary unimodular hermitian forms*, Proc. Japan Acad. **A 62** (1986), 323–326.

[Has88] J. Hastad, *Dual vectors and lower bounds for the nearest lattice point problem*, Combinatorica **8** (1988), 75–81.

[HaL90] — and J. C. Lagarias, *Simultaneously good bases of a lattice and its reciprocal lattice*, Math. Ann. **287** (1990), 163–174.

A. Havemose: see J. Justesen.

T. Helleseth: see also A. G. Shanbhag.

[HeK95] — and P. V. Kumar, *The algebraic decoding of the $\mathbb{Z}_4$-linear Goethals code*, PGIT **41** (1995), 2040–2048.

M. Henk: see U. Betke.

[Hen86] C. L. Henley, *Sphere packings and local environments in Penrose tilings*, Phys. Rev. **34** (1986), 797–816.

[HMB90] H. J. Herrmann, G. Mantica and D. Bessis, *Space-filling bearings*, Phys. Rev. Lett. **65** (1990), 3223–3226.

[HiHu93] R. J. Higgs and J. F. Humphreys, *Decoding the ternary Golay code*, PGIT **39** (1993), 1043–1046.

[Hig69] G. Higman and J. McKay, Errata to [Hig5] in following issue (p. 219).

[HiN88] R. Hill and D. E. Newton, *Some optimal ternary linear codes*, Ars Comb. **A 25** (1988), 61–72.

[HiN92] —— *Optimal ternary linear codes*, DCC **2** (1992), 137–157.

T.-L. Ho: see D. A. Rabson.

[Hof91] D. W. Hoffmann, *On positive definite Hermitian forms*, Manuscripta Math. **71** (1991), 399–429.

[Hoehn95] G. Höhn, *Selbstduale Vertexoperatorsuperalgebren und das Babymonster*, Ph.D. Dissertation, Univ. Bonn, 1995. Also Bonner Mathematische Schriften **286** (1996).

[Hoehn96] — *Self-dual codes over the Kleinian four group*, preprint, August 1996.

T. Høholdt: see J. Justesen.

[HoPl89] D. F. Holt and W. Plesken, *Perfect Groups*, Oxford, 1989.

I. S. Honkala: see G. D. Cohen, T. Etzion.

[Hou96] X.-D. Hou, On the covering radius of $R(1,m)$ in $R(3,m)$, PGIT **42** (1996), 1035–1037.

J. S. Hsia: see also J. W. Benham, A. G. Earnest.

[Hsia87] — *On primitive spinor exceptional representations*, JNT **26** (1987), 38–45.

[Hsia89] — *Even positive definite unimodular quadratic forms over real quadratic fields*, Rocky Mtn. J. Math. **19** (1989), 725–733.

[Hsia91] — *Representations by spinor genera II*, preprint.

[HsH89] — and D. C. Hung, *Even unimodular 8–dimensional quadratic forms over* $\mathbb{Q}(\sqrt{2})$, Math. Ann. **283** (1989), 367–374.

[HsIc97] — and M. I. Icaza, *Effective version of Tartakowsky's theorem*, preprint.

[HsJ97] — and M. Jöchner, *Almost strong approximations for definite quadratic spaces*, Ivent. Math. **129** (1997), 471–487.

[HsPC] — and J. P. Prieto-Cox, *Representations of positive definite Hermitian forms with approximation and primitive properties*, JNT **47** (1994), 175–189.

[HJS] — — and Y.-Y. Shao, *A structure theorem for a pair of quadratic forms*, PAMS **119** (1993), 731–734.

[Hsi92] W.-Y. Hsiang, *A simple proof of a theorem of Thue on the maximal density of circle packings in* E^2, L'Enseign. Math. **38** (1992), 125–131.

[Hsi93] — *On the sphere packing problem and the proof of Kepler's conjecture*, Internat. J. Math. **93** (1993), 739–831.

[Hsi93a] — *On the sphere packing problem and the proof of Kepler's conjecture*, in *Differential Geometry and Topology (Alghero, 1992)*, Word Scientific, River Edge, NJ, 1993, pp. 117–127.

[Hsi93b] — *The geometry of spheres*, in *Differential Geometry (Shanghai, 1991)*, Word Scientific, River Edge, NJ, 1993, pp. 92–107.

[Hsi95] — *A rejoinder to T. C. Hales's article "The states of the Kepler conjecture",* Math. Intelligencer **17** (No. 1, 1995), 35–42.

W. C. Huffman: see also V. Pless

[Huff90] — *On extremal self-dual quaternary codes of lengths 18 to 28*, PGIT **36** (1990), 651–660 and **37** (1991), 1206–1216.

[Huff91] — *On extremal self-dual ternary codes of lengths 28 to 40*, PGIT, submitted.

[Huff98] — *Codes and groups*, in [PHB].

[Huff98a] — *Decompositions and extremal Type II codes over* $\mathbb{Z}_4$, PGIT **44** (1998), to appear.

[HuTo95] — and V. D. Tonchev, *The existence of extremal self-dual* $[50, 25, 10]$ *codes and quasi-symmetric* $2-(49, 9, 6)$ *designs*, DCC **6** (1995), 97–105.

[Hump90] J. E. Humphreys, *Reflection Groups and Coxeter Groups*, Camb. Univ. Press, 1990.

J. F. Humphreys: see R. H. Higgs.

D. C. Hung: see also J. W. Benham, J. S. Hsia.

[Hun91] — *Even positive definite unimodular quadratic forms over* $\mathbb{Q}(\sqrt{3})$, MTAC **57** (1991), 351–368.

M. I. Icaza: see J. S. Hsia.

V. I. Iorgov: see V. Y. Yorgov.

[Iva90] A. A. Ivanov, *Geometric presentations of groups with an application to the Monster*, in Proc. Internat. Congress Math., Kyoto, 1990, Vol. II, pp. 1443–1453.

[Iva92] — *A geometric characterization of Fischer's Baby Monster*, J. Alg. Combin. **1** (1992), 45–69.

[Iva92a] — *A geometric characterization of the Monster*, in *Groups, Combinatorics and Geometry*, ed. M. W. Liebeck and J. Saxl, Camb. Univ. Press, 1992, pp. 46–62.

[IvSz96] G. Ivanyos and A. Szántó, *Lattice basis reduction for indefinite forms and an application*, Discr. Math. **153** (1996), 177–188.

[JKS97] W. C. Jagy, I. Kaplansky and A. Schiemann, *There are 913 regular ternary forms*, Math. **44** (1997), 332–341.

[Jame93] D. G. James, *Primitive representations by unimodular quadratic forms*, JNT **44** (1993), 356–366.

[JaLPW] C. Jansen, K. Lux, R. A. Parker and R. A. Wilson, *An Atlas of Brauer Characters*, Oxford, 1995.

[Jaq90] D.-O. Jaquet, *Domaines de Voronoi et algorithme de réduction des formes quadratiques définies positives*, JNB **2** (1990), 163–215.

[Jaq91] D.-O. Jaquet-Chiffelle, *Classification des réseaux dans* $\mathbb{R}^7$ *(à l'aide de la notion de formes parfaites)*, Astérisque **198**–**200** (1992), 7–8 and 177–185.

[Jaq92a] — *Description des voisines de* E_7, D_7, D_8 *et* D_9, JNB **4** (1992), 273–374.

[Jaq93] — *Enumération complète des classes de formes parfaites en dimension 7*, Ann. Inst. Fourier **43** (1993), 21–55.

[JaSi89] — and F. Sigrist, *Formes quadratiques contiguës à* D_7, CR **309** (1989), 641–644.

[JaSi94] — — *Classification des formes quadratiques réelles: un contre-example à la finitude*, Acta Arith. **68** (1994), 291–294.

H. E. Jensen: see J. Justesen.

[Jen93] J. N. Jensen, *On the construction of some very long cyclic codes*, PGIT **39** (1993), 1093–1094.

[Jen94] — *A class of constacyclic codes*, PGIT **40** (1994), 951–954.

A. Jessner: see W. Neutsch.

M. Jöchner: see J. S. Hsia.

[John91] R. W. Johnson, *Unimodularly invariant forms and the Bravais lattices*, Advances Appl. Math. **12** (1991), 22–56.

D. Joseph: see M. Baake.

[JuL88] D. Jungnickel and M. Leclerc, *A class of lattices*, Ars Comb. **26** (1988), 243–248.

J. Justesen: see also G. Lachaud.

[JuL89] — K. J. Larsen, A. Havemose, H. E. Jensen and T. Høholdt, *Construction and decoding of a class of algebraic geometry codes*, PGIT **35** (1989), 811–821.

[JuL92] — — H. E. Jensen and T. Høholdt, *Fast decoding of codes from algebraic curves*, PGIT **38** (1992), 111–119.

P. Kabal: see A. K. Khandani.

[Kac66] M. Kac, *Can one hear the shape of a drum?*, AMM **73** (1966), 1–23.

M. K. Kaikkonen: see also P. R. J. Östergård.

[Kaik98] — *Codes from affine permutation groups*, DCC , to appear.

[KaL95] G. Kalai and N. Linial, *On the distance distribution of codes*, PGIT **41** (1995), 1467–1472.

[KaNe90] J. Kallrath and W. Neutsch, *Lattice integration on the 7–sphere*, J. Comput. Appl. Math. **29** (1990), 9–14.

R. Kannan: see M. Dyer.

[Kant96] W. M. Kantor, *Note on Lie algebras, finite groups and finite geometries*, in *Groups, Difference Sets and the Monster*, eds. K. T. Arasu et al., de Gruyter, Berlin, 1996, pp. 73–81.

M. R. Kantorovitz: see K. Zeger.

I. Kaplansky: see also W. C. Jagy.

[Kapl95] — *Ternary positive quadratic forms that represent all odd positive integers*, Acta Arith. **70** (1995), 209–214.

[KaT90] S. N. Kapralov and V. D. Tonchev, *Extremal doubly-even codes of length 64 derived from symmetric designs*, Discr. Math. **83** (1990), 285–289.

[Kar93] J. K. Karlof, *Describing spherical codes for the Gaussian channel*, PGIT **39** (1993), 60–65.

P. Kazakov: see P. Boyvalenkov.

P. Keast: see also J. N. Lyness.

[Kea87] — *Cubature formulas for the surface of the sphere*, J. Comput. Appl. Math. **17** (1987), 151–172.

G. Kertész: see K. Bezdek.

[Kerv94] M. Kervaire, *Unimodular lattices with a complete root system*, L'Enseign. Math. **40** (1994), 59–104.

A. K. Khandani: see also A. H. Banihashemi.

[KhEs97] — and M. Esmaelili, *Successive minimization of the state complexity of self-dual lattices using Korkin-Zolotarev reduced bases*, preprint, 1997.

[KKD95] — P. Kabal and E. Dubois, *Computing the weight distribution of a set of points obtained by scaling, shifting and truncating a lattice*, PGIT **41** (1995), 1480–1482.

A. Khosravani: see A. G. Earnest.

[KiHa98] J. K. Kim and S. G. hahn, *A new upper bound for binary codes with minimum distance four*, DM **187** (1998), 291–295.

M. H. Kim: see W. Chan.

[Kita93] Y. Kitaoka, *Arithmetic of Quadratic Forms*, Canbridge, 1993.

M. Kitazume: see also A. Bonnecaze.

[KiKM] — T. Kondo and I. Miyamoto, *Even lattices and doubly even codes*, J. Math. Soc. Japan **43** (1991), 67–87.

[KlL88] P. B. Kleidman and M. W. Liebeck, *The Subgroup Structure of the Finite Classical Groups*, London Math. Soc. Lect. Note Series **129**, Camb. Univ. Press, 1988.

[KlPW89] — R. A. Parker and R. A. Wilson, *The maximal subgroups of the Fischer group* Fi_{23}, JLMS **39** (1989), 89–101.

[KlW87] — and R. A. Wilson, *The maximal subgroups of* Fi_{22}, PCPS **102** (1987), 17–23; **103** (1988), 383.

[KlW90] — — $J_3 \subset E_6(4)$ *and* $M_{12} \subset E_6(5)$, JLMS **42** (1990), 555–561.

[KlW90a] — — *The maximal subgroups of* $E_6(2)$ *and* $Aut(E_6(2))$, PLMS **60** (1990), 266–294.

[KLV95] Y. Klein, S. Litsyn and A. Vardy, *Two new bounds on the size of binary codes with a minimum distance of three*, DCG **6** (1995), 219–227.

M. Kléman: see Z. Olamy.

[Klu89] P. Kluitmann, *Addendum zu der Arbeit "Ausgezeichnete Basen von Milnorgittern einfacher Singularitäten" von E. Voigt*, AMSH **59** (1989), 123–124.

[Kne54] M. Kneser, *Zur Theorie der Kristallgitter*, Math. Ann. **127** (1954), 105–106.

[Kne74] — *Quadratische Formen*, Lecture notes, Göttingen, 1974.

[Knill96] O. Knill, *Maximizing the packing density on a class of almost periodic sphere packings*, Expos. Math. **14** (1996), 227–246.

[Koch86] H. Koch, *Unimodular lattices and self-dual codes*, in *Proc. Intern. Congress Math., Berkeley 1986*, Amer. Math. Soc. Providence RI **1** (1987), pp. 457–465.

[Koch90] — *On self-dual doubly-even extremal codes*, Discr. Math. **83** (1990), 291–300.

[KoNe93] — and G. Nebe, *Extremal unimodular lattices of rank 32 and related codes*, Math. Nach. **161** (1993), 309–319.

[KoVe89] — and B. B. Venkov, *Ueber ganzzahlige unimodulare euklidische Gitter*, JRAM **398** (1989), 144–168.

[KoVe91] — — (with an appendix by G. Nebe), *Ueber gerade unimodulare Gitter der Dimension 32, III*, Math. Nach. **152** (1991), 191–213.

[Koik86] M. Koike, *Moonshines of $PSL_2(\mathbb{F}_q)$ and the automorphism group of the Leech lattice*, Japan J. Math. **12** (1986), 283–323.

E. Kolev: see P. R. J. Östergård.

[KolY97] A. V. Kolushov and V. A. Yudin, *Extremal dispositions of points on the sphere*, Analysis Math. **23** (1997), 25–34.

[Kon97] S. Kondo, *The automorphism group of a generic Jacobian Kummer surface*, J. Algebraic Geometry **7** (1998), 589–609.

[Kon98] — *Niemeier lattices, Mathieu groups and finite groups of symplectic automorphisms of K3 surfaces*, Duke Math. J. **92** (1998), 593–598.

T. Kondo: see also M. Kitazume.

[KoTa87] — and T. Tasaka, *The theta functions of sublattices of the Leech lattice II*, J. Fac. Sci. Univ. Tokyo Sect. IA, Math. **34** (1987), 545–572

[KoMe93] J. Korevaar and J. L. H. Meyers, *Spherical Faraday cage for the case of equal point changes and Chebyshev-type quadrature on the sphere*, Integral Transforms and Special Functions **1** (1993), 105–117.

[KoMe94] — — *Chebyshev-type quadrature on multidimensional domains*, J. Approx. Theory **79** (1994), 144–164.

H. Koseki: see K. Hashimoto.

A. I. Kostrikin: see also A. I. Bondal.

[KoTi94] — and P. H. Tiep, *Orthogonal Decompositions and Integral Lattices*, De Gruyter, Berlin, 1994.

[Kott91] D. A. Kottwitz, *The densest packings of equal circles on a sphere*, Acta Cryst. **A 47** (1991), 158–165.

[KrM74] E. S. Kramer and D. M. Mesner, *Intersections among Steiner systems*, JCA **A 16** (1974), 273–285.

[Kram88] J. Kramer, *On the linear independence of certain theta-series*, Math. Ann. **281** (1988), 219–228.

P. Kramer: see M. Baake.

[KrLi97] I. Krasikov and S. Litsyn, *Linear programming bounds for doubly-even self-dual codes*, PGIT **43** (1997), 1238–1244.

[KrLi97a] — — *Linear programming bounds for codes of small size*, EJC **18** (1997), 647–656.

[Kra88] E. Krätzel, *Lattice Points*, Kluwer, Dordrecht, Holland, 1988.

F. R. Kschischang: see also A. Vardy.

[KsP92] — and S. Pasupathy, *Some ternary and quaternary codes and associated sphere packings*, PGIT **38** (1992) 227–246.

[Kuhn96] S. Kühnlein, *Partial solution of a conjecture of Schmutz*, Archiv. Math. **67** (1996), 164–172.

[Kuhn97] — *Multiplicities of non-arithmetic ternary quadratic forms and elliptic curves of positive rank*, preprint.

P. V. Kumar: see A. R. Hammons, Jr., T. Helleseth, O. Moreno, A. G. Shanbhag.

W. Kuperberg: see A. Bezdek, G. Fejes Tóth.

[Kuzj95] N. N. Kuzjurin, *On the difference between asymptotically good packings and coverings*, EJM **16** (1995), 35–40.

[LaS92] G. Lachaud and J. Stern, *Polynomial-time construction of codes I: linear codes with almost equal weights*, Proc. AAECC, vol. 3 (1992), pp. 151–161.

[LaS94] — — *Polynomial-time construction of codes II: spherical codes and the kissing number of spheres*, PGIT **40** (1994), 1140–1146.

[Lac95] — M. A. Tsfasman, J. Justesen and V. K.-W. Wei, editors, *Special Issue on Algebraic Geometry Codes*, PGIT **41** (No. 6, 1996), 1545–1772.

[LaVa95] A. Lafourcade and A. Vardy, *Asymptotically good codes have infinite trellis complexity*, PGIT **41** (1995), 555–559.

[LaVa95a] — — *Lower bounds on trellis complexity of block codes*, PGIT **41** (1995), 1938–1954.

[LaVa96] — — *Optimal sectionalization of a trellis*, PGIT **42** (1996), 689–703.

J. C. Lagarias: see also J. Hastad.

[Laga96] — *Point lattices*, in *Handbook of Combinatorics*, ed. R. L. Graham et al., North-Holland, Amsterdam, 1996, pp. 919–966.

J. Lahtonen: see T. Mittelholzer.

[LaiL98] T. Laihonen and S. Litsyn, *On upper bounds for minimum distance and covering radius of nonbinary codes*, DCC, to appear.

C. S. Lam: see T. Gannon.

[LmP90] C. W. H. Lam and V. Pless, *There is no (24,12,10) self-dual quaternary code*, PGIT **36** (1990), 1153–1156.

S. Landau: see J. Cremona.

I. Landgev: see P. Boyvalenkov.

[LaM92] P. Landrock and O. Manz, *Classical codes as ideals in group algebras*, DCC **2** (1992), 273–285.

[LaLo89] G. R. Lang and F. M. Longstaff, *A Leech lattice modem*, IEEE J. Select. Areas Comm. **7** (1989), 968–973.

M. L. Lang: see K. Harada.

K. J. Larsen: see J. Justesen.

[Laue43] M. v. Laue, *Der Wuffsche Satz für die Gleichgewichtsform von Kristallen*, Z. Krist. **105** (1943), 124–133.

F. Laves: see G. O. Brunner.

[LDS86] D. E. Lazić, D. B. Drajić and V. Senk, *A table of some small-size three-dimensional best spherical codes*, preprint.

[LDS87] — V. Senk and I. Seskar, *Arrangements of points on a sphere which maximize the least distance*, Bull. Appl. Math. **47** (No. 479, 1987), 5–21.

M. Leclerc: see D. Jungnickel.

[LePR93] H. Lefmann, K. T. Phelps and V. Rödl, *Rigid linear binary codes*, JCT **A 63** (1993), 110–128.

P. Lemke: see S. S. Skiena.

[Lemm1] P. W. H. Lemmens, *On the general structure of holes in integral lattices*, preprint.

[Lemm2] — *A note on lattices and linear subspaces in* $\mathbb{R}^n$, preprint.

[Lemm3] — *On intersections and sums of integral lattices in* $\mathbb{R}^n$, preprint.

[Len92] H. W. Lenstra, Jr., *Algorithms in algebraic number theory*, BAMS **26** (1992), 211–244.

J. Leon: see V. Pless.

[Lep88] J. Lepovsky, *Perspectives on vertex operators and the Monster*, PSPM **48** (1988), 181–197.

[Lep91] — *Remarks on vertex operator algebras and moonshine*, in *Proc 20th Internat. Conf. Differential Geometric Methods in Theoretical Physics, New York, 1991*, edited S. Catto and A. Rocha, World Scientific, Singapore, 1992.

V. I. Levenshtein: see also G. Fazekas.

[Lev87] — *Packing of polynomial metric spaces*, Proc. Third. Intern. Workshop on Information Theory, Sochi, 1987, pp. 271–274.

[Lev91] — *On perfect codes in the metric of deletions and insertions* (in Russian), Discr. Mat. **3** (1991), 3–20. English translation in Discrete Math. and Applic. **2** (No. 3, 1992).

[Lev92] — *Packing and decomposition problems for polynomial association schemes*, EJC **14** (1993), 461–477.

[Lev92a] — *Bounds for self-complementary codes and their applications*, in Proc. Eurocode '92.

[Lev92b] — *Designs as maximum codes in polynomial metric spaces*, Acta Applicandae Math. **29** (1992), 1–82.

[Lev93] — *Bounds for codes as solutions of extremum problems for systems of orthogonal polynomials*, in *Proc. 10th Int. Sympos. Applied Algebra etc.*, Lect. Notes Comp. Sci. **673**, 1993, 25–42.

[Lev95] — *Krawtchouk polynomials and universal bounds for codes and designs in Hamming space*, PGIT **41** (1995), 1303–1321.

[Levi87] B. M. Levitan, *Asymptotic formulae for the number of lattice points in Euclidean and Lobachevskii spaces*, Russian Math. Surveys **42** (1987), 13–42.

[LeLi96] F. Levy-dit-Vehel and S. Litsyn, *More on the covering radius of BCH codes*, PGIT **42** (1996), 1023–1028.

[LiCh94] D. Li and W. Chen, *New lower bounds for binary covering codes*, PGIT **40** (1994), 1122–1129.

M. Lichtenberg: see A. Gervois, L. Oger.

M. W. Liebeck: see also P. B. Kleidman.

[LPS90] — C. E. Praeger and J. Saxl, *The Maximal Factorizations of the Finite Simple Groups and Their Automorphism Groups*, Memoirs Amer. Math. Soc. **432** (1990).

[LiSZ93] T. Linder, C. Schlegel and K. Zeger, *Corrected proof of de Buda's theorem*, PGIT **39** (1993), 1735–1737.

[Lin88] J. H. Lindsey II, *A new lattice for the Hall-Janko group*, PAMS **103** (1988), 703–709.

N. Linial: see G. Kalai.

J. H. van Lint: see R. J. M. Vaessens.

[LiW91] S. A. Linton and R. A. Wilson, *The maximal subgroups of the Fischer groups Fi_{24} and Fi'_{24}*, PLMS **63** (1991), 113–164.

S. Litsyn: see also A. E. Ashikhmin, R. A. Brualdi, G. D. Cohen, Y. Klein, I. Krasikov, F. Levy-dit-Vehel, T. Laihonen.

[Lits98] — *An updated table of the best binary codes known*, to appear in [PHB].

[LiV93] — and A. Vardy, *Two new upper bounds for codes of distance 3*, in *Algebraic Coding*, Lect. Notes Comp. Sci. **781** (1994), 253–262.

[LiV94] — — *The uniqueness of the Best code*, PGIT **40** (1994), 1693–1698.

A. C. Lobstein: see G. D. Cohen.

[Loel97] H.-A. Loeliger, *Averaging bounds for lattices and linear codes*, PGIT **43** (1997), 1767–1773.

[Loma94] G. A. Lomadze, *On the number of ways of writing a number as the sum of nine squares* (in Russian), Acta Arith. **68** (1994), 245–253.

F. M. Longstaff: see G. R. Lang.

P. Loyer: see also P. Solé.

[LoSo93] — and P. Solé, *Quantizing and decoding for the usual lattices in the L_p-metric*, in *Algebraic Coding (Paris, 1993)*, Lect. Notes Comp. Sci **781** (1994), 225–235.

[LoSo94] — and P. Solé, *Les réseaux BW32 et U32 sont équivalents*, JNB **6** (1994) 359–362.

B. D. Lubachevsky: see R. L. Graham.

K. Lux: see C. Jansen.

[LySK91] J. N. Lyness, T. Sørevik and P. Keast, *Notes on integration and integer sublattices*, MTAC **56** (1991), 243–255.

J. Maginnis: see A. Adem.

[Mak65] V. S. Makarov, *A class of partitions of a Lobachevskii space*, SMD **6** (1965), 400–401.

[Mak66] — *A class of discrete groups in Lobachevskii space having an infinite fundamental domain of finite measure*, SMD **7** (1966), 328–331.

[MaV85] Y. I. Manin and S. G. Vladuts, *Linear codes and modular curves*, JSM **30** (1985), 2611–2643.

G. Mantica: see H. J. Herrmann.

O. Manz: see P. Landrock.

B. H. Marcus: see J. Feigenbaum.

J. Martinet: see also C. Batut, E. Bayer-Fluckiger, A.-M. Bergé.

[Mar95] — *Structure algébriques sur les réseaux*, in *Number Theory (Seminaire de Théorie des Nombres de Paris, 1992–93)*, Cambridge Univ. Press, 1995, pp. 167–186.

[Mar96] — *Les Réseaux Parfaits des Espaces Euclidiens*, Masson, Paris, 1996.

[Mar97] — *Une famille de réseaux dual-extrêmes*, JNB **9** (1997), 169–181.

[Mar98] — *Perfect and eutactic lattices: extensions of Voronoi's theory*, preprint.

G. Mason: see C. Dong.

H. F. Mattson, Jr.: see G. D. Cohen.

[Maye93] A. J. Mayer, *A Tiling Property of the Minkowski Fundamental Domain*, Ph. D. Disseration, Princeton Univ., 1993.

[Maye95] — *Low dimensional lattices have a strict Voronoi base*, Math. **42** (1995), 229–238.

[MaO90] J. E. Mazo and A. M. Odlyzko, *Lattice points in high-dimensional spheres*, Monat. Math. **110** (1990), 47–61.

R. J. McEliece: see J. Feigenbaum.

J. McKay: see G. Higman.

P. J. McLane: see E. Biglieri.

[McHa98] S. McLaughlin and T. C. Hales, *A proof of the dodecahedral conjecture*, preprint.

[MeMo79] W. M. Meier and H. J. Moeck, *The topology of three-dimensional 4-connected nets: classification of zeolite framework types using coordination sequences*, J. Solid State Chem. **27** (1979), 349–355.

[Mel94] J. B. M. Melissen, *Densest packing of eleven congruent circles in a circle*, Geomet. Dedicata **50** (1994), 15–25.

[Mel97] — *Packing and Covering with Circles*, Ph.D. Dissertation, Univ. Utrecht, Dec. 1997.

N. D. Mermin: see also D. A. Rabson, D. S. Rokhsar.

[MRW87] — D. A. Rabson and D. C. Wright, *Beware of 46-fold symmetry: the classification of two-dimensional crystallographic lattices*, Phys. Rev. Lett. **58** (1987), 2099–2101.

D. M. Mesner: see E. S. Kramer.

J. L. H. Meyers: see J. Korevaar.

[MiRS95] L. Michel, S. S. Ryshkov and M. Senechal, *An extension of Voronoi's theorem on primitive parallelotopes*, EJC **16** (1995), 59–63.

R. J. Milgram: see A. Adem.

[Mil64] J. Milnor, *Eigenvalues of the Laplace operator on certain manifolds*, PNAS **51** (1964), 542.

[Mimu90] Y. Mimura, *Explicit examples of unimodular lattices with the trivial automorphism group*, in *Algebra and Topology*, Korea Adv. Inst. Sci. Tech., Taejon, 1990, pp. 91–95.

S. Minashima: see Eichii Bannai.

[Misch94] M. Mischler, *La formule de Minkowski-Siegel pour les formes bilinéaires symmétriques dégénérées et définies positives*, Théorie des nombres 1992/93 – 1993/94, Univ. Franche-Comté, Besançon, 1994.

M. Mitradjieva: see P. Boyvalenkov.

[MiL96] T. Mittelholzer and J. Lahtonen, *Group codes generated by finite reflection groups*, PGIT **42** (1996), 519–528.

[Miya89] T. Miyake, *Modular Forms*, Springer-Verlag, NY, 1989.

I. Miyamoto: see M. Kitazume.

M. Miyamoto: see also K. Harada.

[Miya95] — 21 *involutions acting on the Moonshine module*, J. Alg. **175** (1995), 941–965.

[Miya96] — *Binary codes and vertex operator (super) algebras*, J. Alg. **181** (1996), 207–222.

[Miya98] — *Hamming code VOA and construction of VPOA's*, preprint.

H. J. Moeck: see W. M. Meier.

[MolP90] M. Mollard and C. Payan, *Some progress in the packing of equal circles in a square*, Discr. Math. **84** (1990), 303–307.

M. Monagan: see R. Peikert.

P. S. Montague: see also L. Dolan.

[Mont94] — *A new construction of lattices from codes over $GF(3)$*, Discr. Math. **135** (1994), 193–223.

[MoP92] R. V. Moody and J. Patera, *Voronoi and Delaunay cells of root lattices: classification of their faces and facets by Coxeter-Dynkin diagrams*, J. Phys. **A. 25** (1992), 5089–5134.

[MoP92a] — — *Voronoi domains and dual cells in the generalized kaleidescope with applications to root and weight lattices*, CJM **47** (1995), 573–605.

[MoKu93] O. Moreno and P. V. Kumar, *Minimum distance bounds for cyclic codes and Deligne's theorem*, PGIT **39** (1993), 1524–1534.

[MoPa93] W. O. J. Moser and J. Pach, *Research Problems in Discrete Geometry: Packing and Covering*, DIMACS Technical Report 93–32, DIMACS Center, Rutgers Univ., 1993.

B. Mourrain: see C. Bachoc, A. Bonnecaze.

D. J. Muder: see also J. H. Conway.

[Mud90] — *How big is an n-sided Voronoi polygon?*, PLMS **61** (1990), 91–108.

[Mude93] — *A new bound on the local density of sphere packings*, DCG **10** (1993), 351–375.

[Mus97] C. Musès, *The dimensional family approach in (hyper)sphere packing: a typological study of new patterns, structures, and interdimensional functions*, Applied Math. Computation **88** (1997), 1–26 (see Corrigenda in same issue).

J. Naor: see N. Alon.

M. Naor: see N. Alon.

[Napa94] H. Napias, *Sur quelques réseaux contenus dans les réseaux de Leech et de Quebbemann*, CR **319** (1994), 653–658.

[Napa96] — *A generalization of the LLL-algorithm over Euclidean rings or orders*, JNB **8** (1996), 387–396.

G. Nebe: see also C. Bachoc, A. Cohen, H. Koch.

[Nebe96] — *Finite subgroups of $GL_{24}(\mathbb{Q})$*, Exper. Math. **5** (1996), 163–195.

[Nebe96a] — *Finite subgroups of $GL_n(\mathbb{Q})$ for $25 \leq n \leq 31$*, Comm. Alg. **24** (1996), 2341–2397.

[Nebe97] — *The normaliser action and strongly modular lattices*, L'Enseign. Math. **43** (1997), 67–76.

[Nebe98] — *Some cyclo-quaternionic lattices*, J. Alg. **199** (1998), 472–498.

[Nebe98a] — *Finite quaternionic matrix groups*, Electron. J. Representation Theory, to appear.

[Nebe98b] — *A method of computing the minimum of certain lattices*, preprint.

[NePl95] — and W. Plesken, *Finite Rational Matrix Groups*, Memoirs A.M.S., vol. **116**, No. 556, 1995.

[NeSl] — and N. J. A. Sloane, *A Catalogue of Lattices*, published electronically at http://www.research.att.com/∼njas/lattices/.

[NeVe96] — and B. B. Venkov, *Nonexistence of extremal lattices in certain genera of modular lattices*, JNT **60** (1996), 310–317.

A. Neumaier: see also A. E. Brouwer.

[NeS88] — — *Discrete measures for spherical designs, eutactic stars and lattices*, KNAW **50** (1988), 321–334.

[NeS92] — and J. J. Seidel, *Measures of strength $2e$, and optimal designs of degree e*, Sankhyā **54** (1992), 299–309.

W. Neutsch: see also J. Kallrath.

[Neut83] — *Optimal spherical designs and numerical integration on the sphere*, J. Comput. Phys. **51** (1983), 313–325.

[NeSJ85] — E. Schrüfer and A. Jessner, *Efficient integration on the hypersphere*, J. Comput. Phys. **59** (1985), 167–175.

[Newm94] M. Newman, *Tridiagonal matrices*, Lin Alg. Appl. **201** (1994), 51–55.

D. E. Newton: see R. Hill.

M. J. Nicolich: see K. L. Fields.

S. Nikova: see P. Boyvalenkov.

G. Nipp: see also A. G. Earnest.

[Nip91] — *Quaternary Quadratic Forms*: *Computer Generated Tables*, Springer-Verlag, NY, 1991.

[Nip91a] — *Computer-Generated Tables of Quinary Quadratic Forms*, MTAC, Unpublished Mathematical Tables Collection, 1991.

[Nor90] S. P. Norton, *Presenting the Monster*?. Bull. Soc. Math. Belgium **62** (1990), 595–605.

[Nor92] — *Constructing the Monster*, in *Groups, Combinatorics and Geometry*, ed. M. W. Liebeck and J. Saxl, Camb. Univ. Press, 1992, pp. 63–76.

[NoW89] — and R. A. Wilson, *The maximal subgroups of $F_4(2)$ and its automorphism group*, Comm. Alg. **17** (1989), 2809–2824.

K. J. Nurmela: see also R. L. Graham.

[NuOs97] — and P. R. J. Östergård, *Packing up to 50 equal circles in a square*, DCG **18** (1997), 111–120.

A. M. Odlyzko: see N. D. Elkies, J. E. Mazo.

[Oes90] J. Oesterlé, *Empilements de sphères*, Sém. Bourbaki **42** (No. 727, 1990).

L. Oger: see also A. Gervois.

[OLGG] — M. Lichtenberg, A. Gervois and E. Guyon, *Determination of the coordination number in disordered packings of equal spheres*, J. Microscopy-Oxford **156** (1989), 65–78.

[OgS91] K. Oguiso and T. Shioda, *The Mordell-Weil lattice of a rational elliptic surface*, Comment. Math. Univ. St. Pauli **40** (1991), 83–99.

[O'Ke91] M. O'Keeffe, *N-dimensional diamond, sodalite and rare sphere packings*, Acta Cryst. **A 47** (1991), 748–753.

[O'Ke95] — *Coordination sequences for lattices*, Zeit. F. Krist. **210** (1995), 905–908.

[OlK89] Z. Olamy and M. Kléman, *A two-dimensional aperiodic dense tiling*, J. Phys. France **50** (1989), 19–37.

M. Olivier: see C. Batut.

[OrPh92] H. Oral and K. T. Phelps, *Almost all self-dual codes are rigid*, JCT **A 60** (1992), 264–276.

J. O'Rourke: see J. E. Goodman.

P. R. J. Östergård: see also A. E. Brouwer, R. L. Graham, K. J. Nurmela.

[Ost98] — T. Baicheva and E. Kolev, *Optimal binary one-error-correcting codes of length 10 have 72 codewords*, preprint.

[OsK96] — and M. K. Kaikkonen, *New single-error-correcting codes*, PGIT **42** (1996), 1261–1262.

M. Oura: see Eiichi Bannai, S. T. Dougherty.

L. H. Ozarow: see A. R. Calderbank.

M. Ozeki: see also Eiichi Bannai.

[Oze76] — *On the basis problem for Siegel modular forms of degree 2*, Acta Arith. **31** (1976), 17–30.

[Oze87] — *Ternary code construction of even unimodular lattices*, in *Theorie des Nombres, Quebec 1987*, Gruyter, Berlin, 1989, pp. 772–784.

[Oze89a] — *On the structure of even unimodular extremal lattices of rank 40*, Rocky Mtn. J. Math. **19** (1989), 847–862.

[Oze89b] — *On a class of self-dual ternary codes*, Science Reports Hirosaki Univ. **36** (1989), 184–191.

[Oze91] — *Quinary code construction of the Leech lattice*, Nihonkai Math. J. **2** (1991), 155–167.

[Oze97] — *On the notion of Jacobi polynomials for codes*, Math. Proc. Camb. Phil. Soc. **121** (1997), 15–30.

J. Pach: see W. O. J. Moser.

[PacA95] — and P. K.Agarwal, *Combinatorial Geometry*, Wiley, New York, 1995.

G. Pall: see D. Estes.

R. A. Parker: see also C. Jansen, P. B. Kleidman.

[PaW90] — and R. A. Wilson, *The computer construction of matrix representations of finite groups over finite fields*, J. Symb. Comp. **9** (1990), 583–590.

[Pas81] G. Pasquier, *A binary extremal doubly even self-dual code (64,32,12) obtained from an extended Reed-Solomon code over* F_{16}, PGIT **27** (1981), 807–808.

S. Pasupathy: see F. R. Kschischang.

J. Patera: see R. V. Moody.

C. Payan: see M. Mollard.

[PaS87] A. Paz and C. P. Schnorr, *Approximating integer lattices by lattices with cyclic factor groups*, in *Automata, Languages and Programming, Proc. 14th Internat. Colloq.*, edited T. Ottmann, Lect. Notes Comp. Sci. **267** (1987), 386–393.

[PeWMG] R. Peikert, D. Würtz, M. Monagan and C. de Groot, *Packing circles in a square: A review and new results*, Maple Technical Newsletter **6** (1991), 28–34.

[Pel89] R. Pellikaan, *On a decoding algorithm for codes on maximal curves*, PGIT **35** (1989), 1228–1232.

[Per90] E. Pervin, *The Voronoi cells of the* E_6^* *and* E_7^* *lattices*, in D. Jungnickel et al., editors, *Coding Theory, Design Theory, Group Theory*, Wiley, NY, 1993, pp. 243–248.

[Pet80] M. Peters, *Darstellungen durch definite ternäre quadratische Formen*, Acta Arith **34** (1977), 57–80.

[Pet89] __ *Jacobi theta series*, Rocky Mtn. J. Math. **19** (1989), 863–870.

[Pet90] __ *Siegel theta series of degree 2 of extremal lattices*, JNT **35** (1990), 58–61.

K. T. Phelps: see H. Lefmann, H. Oral.

P. A. B. Pleasants: see M. Baake.

W. Plesken: see also A. M. Cohen, D. F. Holt, G. Nebe.

[Plesk91] __ *Some applications of representation theory*, in *Representation Theory of Finite Groups and Finite-Dimensional Algebras (Bielefeld, 1991)*, Birkhauser, Basel, 1991, pp. 477–496.

[Plesk93] __ *Kristallographische Gruppen*, in *Group Theory, Algebra and Number Theory (Saarbrücken, 1993)*, De Gruyter, Berlin, 1996, pp. 75–96.

[Plesk94] __ *Additively indecomposable positive integral quadratic forms*, JNT **47** (1994), 273–283.

[Plesk96] __ *Finite rational matrix groups — a survey*, in *Proc. Conf. "The ATLAS: Ten Years After" (1995)*, to appear.

[Plesk92] __ and M. Pohst, *Constructing integral lattices with prescribed minimum II*, MTAC **60** (1993), 817–826.

[PlSo96] __ and B. Souvignier, *Computing rational representations of finite groups*, Exper. Math. **5** (1996), 39–47.

V. Pless: see also R. A. Brualdi, J. H. Conway, C. W. H. Lam.

[Pless89] — *Extremal codes are homogeneous*, PGIT **35** (1989), 1329–1330.

[PHB] — W. C. Huffman and R. A. Brualdi, editors, *Handbook of Coding Theory*, Elsevier, Amsterdam, 1998, to appear.

[PlQ96] — and Z. Qian, *Cyclic codes and quadratic residue codes over* $\mathbb{Z}_4$, PGIT **24** (1996), 1594–1600.

[PTL92] — V. Tonchev and J. Leon, *On the existence of a certain (64,32,12) extremal code*, PGIT **39** (1993), 214–215.

M. Pohst: see also C. Fieker, W. Plesken.

[PoZ89] — and H. Zassenhaus, *Algorithmic Number Theory*, Cambridge Univ. Press, 1989.

[Polt94] G. Poltyrev, *On coding without restrictions for the AWGN channel*, PGIT **40** (1994), 409–417.

C. E. Praeger: see also M. W. Liebeck.

[Pra89] — *Finite primitive permutation groups: A survey*, LNM **1456** (1989), 63–84.

J. P. Prieto-Cox: see J. S. Hsia.

[Pro87] M. N. Prokhorov, *The absence of discrete reflection groups with non-compact fundamental polyhedron of finite volume in Lobachevsky space of large dimension*, Izv. **28** (1987), 401–411.

[Prop88] J. G. Propp, *Kepler's spheres and Rubik's cube*, Math Mag. **61** (1988), 231–239.

[Pul82] C. L. M. van Pul, *On Bounds on Codes*, Master's Dissertation, Eindhoven University of Technology, Netherlands, 99 pages, 1982.

Z. Qian: see V. Pless.

H.-G. Quebbemann: see also C. Batut.

[Queb88] — *Cyclotomic Goppa codes*, PGIT **34** (1988), 1317–1320.

[Queb88a] — *On even codes*, Discr. Math. **98** (1991), 29–34.

[Queb89] — *Lattices from curves over finite fields*, preprint, 1989.

[Queb91] — *A note on lattices of rational functions*, Math. **38** (1991), 10–13.

[Queb91a] — *Unimodular lattices with isometries of large prime order II*, Math. Nach. **156** (1992), 219–224.

[Queb95] — *Modular lattices in Euclidean spaces*, JNT **54** (1995), 190–202.

[Queb97] — *Atkin-Lehner eigenforms and strongly modular lattices*, L'Enseign. Math. **43** (1997), 55–65.

[Queb98] — *A shadow identity and an application to isoduality*, preprint.

[Quin95] J. R. Quine, *Jacobian of the Picard curve*, Contemp. Math. **201** (1997), 33–41.

[QuZh95] — and P. L. Zhang, *Extremal symplectic lattices*, preprint, 1995.

[RaBa91] P. Rabau and B. Bajnok, *Bounds for the number of nodes in Chebyshev type quadrature formulas*, J. Approx. Theory **67** (1991), 199–214.

A. Rabinovich: see A. R. Calderbank.

P. Rabinowitz: see R. Cools.

[RHM89] D. A. Rabson, T.-L. Ho and N. D. Mermin, *Space groups of quasicrystallographic tilings*, Acta Cryst. **A** (1989), 538–547.

S. Raghavan: see W. Chan.

E. M. Rains: see also A. R. Calderbank, Y. Edel.

[Rain98] — *Shadow bounds for self-dual codes*, PGIT **44** (1998), 134–139.

[Rain98a] — *Optimal self-dual codes over* $\mathbb{Z}_4$, preprint.

[RaSl98] — and N. J. A. Sloane, *Self-dual codes*, to appear in [PHB].

[RaSl98a] — — *The shadow theory of modular and unimodular lattices*, JNT (submitted).

[RaSi94] B. S. Rajan and M. U. Siddiqi, *A generalized DFT for abelian codes over* $\mathbb{Z}_m$, PGIT **40** (1994), 2082–2090.

[RaSh96] D. S. Rajan and A. M. Shende, *A characterization of root lattices*, Discr. Math. **161** (1996), 309–314.

[RaSn93] M. Ran and J. Snyders, *On maximum likelihood soft decoding of binary self-dual codes*, IEEE Trans. Commun. **41** (1993), 439–443.

[RaSn95] — — *Constrained designs for maximum likelihood soft decoding of* $RM(2,m)$ *and the extended Golay codes*, IEEE Trans. Commun. **43** (1995), 812–820.

[RaSn96] — — *A cyclic* $[6,3,4]$ *group code and the hexacode over* $GF(4)$, PGIT **42** (1996), 1250–1253.

[RaSn98] — — *Efficient decoding of the Gosset, Coxeter-Todd and the Barnes-Wall lattices* , preprint, 1998.

T. R. N. Rao: see G.-L. Feng.

C. Rastello: see J. Boutros.

[Rezn95] B. Reznick, *Some constructions of spherical 5-designs*, Lin. Alg. Appl. **226**–**228** (1995), 163–196.

[RiRo92] G. Riera and R. E. Rodríguez, *The period matrix of Bring's curve*, PJM **154** (1992), 179–200.

B. Rimoldi: see R. Urbanke.

V. Rödl: see H. Lefmann.

R. E. Rodríguez: see G. Riera.

D. S. Rokhsar: see also N. D. Mermin.

[RMW87] — N. D. Mermin and D. C. Wright, *Rudimentary quasicrystallography: the icosahedral and decagonal reciprocal lattices*, Phys. Rev. **B 35** (1987), 5487–5495.

[RWM88] — D. C. Wright and N. D. Mermin, *The two-dimensional quasicrystallographic space groups with rotational symmetries less than 23-fold*, Acta Cryst. **A 44** (1988), 197–211.

[RoT90] M. Y. Rosenbloom and M. A. Tsfasman, *Multiplicative lattices in global fields*, Invent. Math. **101** (1990), 687–696.

R. M. Roth: see N. Alon.

Z. Rudnick: see A. Eskin.

[Rung93] B. Runge, *On Siegel modular forms I*, JRAM **436** (1993), 57–85.

[Rung95] — *On Siegel modular forms II*, Nagoya Math. J. **138** (1995), 179–197.

[Rung95a] — *Theta functions and Siegel-Jacobi forms*, Acta Math. **175** (1995), 165–196.

[Rung96] — *Codes and Siegel modular forms*, Discr. Math. **148** (1996), 175–204.

J. A. Rush: see also N. D. Elkies.

[Rus89] — *A lower bound on packing density*, Invent. Math. **98** (1989), 499–509.

[Rus91] — *Constructive packings of cross polytopes*, Math. **38** (1991), 376–380.

[Rus92] — *Thin lattice coverings*, JLMS **45** (1992), 193–200.

[Rus93] — *A bound, and a conjecture, on the maximum lattice-packing density of a superball*, Math. **40** (1993), 137–143.

[Rus94] — *An indexed set of density bounds on lattice packings*, Geom. Dedicata **53** (1994), 217–221.

[Rus96] — *Lattice packing of nearly Euclidean balls in spaces of even dimension*, Proc. Edin. Math. Soc. **39** (1996), 163–169.

A. J. E. Ryba: see R. E. Borcherds.

S. S. Ryskov: see also R. M. Erdahl, L. Michel.

[RyE88] — and R. M. Erdahl, *The empty sphere, Part II*, CJM **40** (1988), 1058–1073.

[RyE89] — — *Stepwise construction of L-bodies of lattices*, Uspekhi Mat. Nauk **44** (1989), 241–242 = Russian Math. Surveys **44** (1989), 293–294.

[Sal94] A. Sali, *On the rigidity of spherical t-designs that are orbits of finite reflection groups*, DCC **4** (1994), 157–170.

[Samo98] A. Samorodnitsky, *On the optimum of Delsarte's linear program*, JCT, to appear.

P. Sarnak: see also P. Buser, A. Eskin.

[Sar90] — *Some Applications of Modular Forms*, Camb. Univ. Press, 1990.

[Sar95] — *Extremal geometries*, Contemp. Math. **201** (1997), 1–7.

J. Saxl: see M. W. Liebeck.

[Scha65] J. Schaer, *The densest packing of 9 circles in a square*, CMB **8** (1965), 273–277.

[Scha71] — *On the packing of ten equal circles in a square*, Math. Mag. **44** (1971), 139–140.

R. Scharlau: see also C. Batut.

[Scha94] — *Unimodular lattices over real quadratic fields*, Math. Zeit. **216** (1994), 437–452.

[SchaB96] — and B. Blaschke, *Reflective integral lattices*, J. Alg. **181** (1996), 934–961.

[SchaS98] — and R. Schulze-Pillot, *Extremal lattices*, preprint, 1998.

[SchaT96] — and P. H. Tiep, *Symplectic groups, symplectic spreads, codes and unimodular lattices*, Inst. Exper. Math., Essen, preprint No. 16, 1996.

[SchaT96a] — — *Symplectic group lattices*, preprint, 1996.

[SchaV94] — and B. B. Venkov, *The genus of the Barnes-Wall lattice*, Comm. Math. Helv. **69** (1994), 322–333.

[SchaV96] — — *The genus of the Coxeter-Todd lattice*, preprint, 1995.

[SchaW92] — and C. Walhorn, *Integral lattices and hyperbolic reflection groups*, Astérisque **209** (1992), 279–291.

[SchaW92a] — — *Tables of hyperbolic reflection groups*, Ergänzungsreihe der Sonderforschungsbereichs **343** der Univ. Bielefeld, preprint 92–001, April 1992.

[ScharS] W. Scharlau and D. Schomaker, personal communication, April 1991.

A. Schiemann: see also W. C. Jagy.

[Schi90] — *Ein Beispiel positiv definiter quadratischer Formen der Dimension 4 mit gleichen Darstellungszahlen*, Arch. Math. **54** (1990), 372–375.

[Schi97] — *Ternary positive definite quadratic forms are determined by their theta series*, Math. Ann. **308** (1997), 507–517.

M. Schlottmann: see M. Baake.

C. Schlegel: see T. Linder.

[Schma] B. Schmalz, *The t-designs with prescribed automorphism group, new simple 6-designs*, J. Combin. Designs **1** (1993), 125–170.

[Schm90] J. Gf. von Schmettow, *A practical view of KANT*, DMV-Seminar "Konstruktive Zahlentheorie", Düsseldorf, August, 1990.

[Schm91] — *KANT, a tool for computations in algebraic number fields*, in *Computational Number Theory*, A. Pethö et al., editors, Gruyter, 1991.

[Schmu93] P. Schmutz, *Riemann surfaces with shortest geodesics of maximal length*, Geomet. Functional Anal. **3** (1993), 564–631.

[Schmu95] — *Arithmetic groups and the length spectrum of Riemann surfaces*, DMJ, to appear.

[Schmu95a] P. Schmutz Schaller, *The modular torus has maximal length spectrum*, preprint.

W. A. Schneeberger: see also J. H. Conway.

[Schnee97] — Ph. D. Dissertation, Princeton Univ., 1997.

U. Schnell: see also K. Böröczky, Jr.

[Schn96] — Review of [Hsi93a], Math. Review **96**f: 52028.

[Schn98] — *On the Wulff-shape of periodic sphere packings*, preprint.

C. P. Schnorr: see also A. Paz.

[Schn87] — *A hierarchy of polynomial time basis reduction algorithms*, Theoret. Comp. Sci. **53** (1987), 201–224.

D. Schomaker: see also W. Scharlau.

[SchW92] — and M. Wirtz, *On binary cyclic codes of odd lengths from 101 to 127*, PGIT **38** (1992), 516–518.

E. Schrüfer: see W. Neutsch.

R. Schulze-Pillot: see also S. Böcherer, W. Duke, R. Scharlau.

[ScP91] — *An algorithm for computing genera of ternary and quaternary quadratic forms*, in *Proc. Intern. Symp. Symbolic Algebraic Computation (Bonn, 1991)*, pp. 134–143.

[ScP93] — *Quadratic residue codes and cyclotomic lattices*, Archiv. Math. **60** (1993), 40–45.

J. J. Seidel: see also A. Neumaier.

[Sei90] — *Designs and approximation*, Contemp. Math. **111** (1990), 179–186.

[Sei90a] — *More about two-graphs*, in *Fourth Czechoslovak Symp. Combinatorics, Prachatice 1990*, Ann. Discr. Math. **51** (1992), 297–308.

[Sei90b] — *Integral lattices, in particular those of Witt and of Leech*, in *Proceedings Seminar 1986—1987, Mathematical Structures in Field Theories*, Math. Centrum, Amsterdam, 1990, pp. 37–53.

[Sei91] — *Geometry and Combinatorics Selected Works*, eds. D. G. Corneil and R. Mathon, Academic Press, San Diego, 1991.

[Sei95] — *Spherical designs and tensors*, in *Progress in Algebraic Combinatorics (Fukuoka 1993)*, Adv. Studies Pure Math., **24** (1996), 309–321.

[Sel74] E. Selling, *Ueber die binären und ternären quadratischen Formen*, JRAM **77** (1874), 143–229.

K.-D. Semmler: see P. Buser.

M. Senechal: see L. Michel.

V. Senk: see D. E. Lazić.

I. Seskar: see D. E. Lazić.

B. Shahar: see Y. Be'ery.

J. O. Shallit: see E. Bach.

[ShKH96] A. G. Shanbhag, P. V. Kumar and T. Helleseth, *Improved binary codes and sequence families from $\mathbb{Z}_4$-linear codes*, PGIT **42** (1996), 1582–1587.

Y.-Y. Shao: see J. S. Hsia, F. Z. Zhu.

J. B. Shearer: see also A. E. Brouwer.

[Shea88] — personal communication, June, 1988.

A. M. Shende: see D. S. Rajan.

G. C. Shephard: see B. Grünbaum.

T. Shioda: see also K. Oguiso.

[Shiod88] — *Arithmetic and geometry of Fermat curves*, in *Algebraic Geometry Seminar, Singapore 1987*, World Scientific, Singapore, 1988, pp. 95–102.

[Shiod89] — *The Galois representation of type E_8 arising from certain Mordell-Weil groups*, Proc. Japan Acad. **A 65** (1989) 195–197.

[Shiod89a] — *Mordell-Weil lattices and Galois representation*, Proc. Japan Acad. **A 65** (1989), 268–271, 296–299, 300–303.

[Shiod90] — *Construction of elliptic curves over $\mathbb{Q}(t)$ with high rank: A preview*, Proc. Japan Acad. **A 66** (1990), 57–60.

[Shiod90a] — *On the Mordell-Weil lattices*, Comment. Math. Univ. St. Pauli **39** (1990), 211–240.

[Shi90b] — *The theory of Mordell-Weil lattices*, Proc. Inter. Congress Math., Tokyo, 1990, Vol. I, pp. 473–489.

[Shiod91] — *Mordell-Weil lattices and sphere packings*, AJM **113** (1991), 931–948.

[Shiod91a] — *Mordell-Weil lattices of type E_8 and deformation of singularities*, in *Prospects in Complex Geometry*, LNM **1468** (1991), 177–202.

[Shiod91b] — *Construction of elliptic curves with high rank via the invariants of the Weyl groups*, J. Math. Soc. Japan **43** (1991), 673–719.

[Shiod91c] — *An infinite family of elliptic curves over $\mathbb{Q}$ with large rank via Néron's method*, Invent. Math. **106** (1991), 109–119.

[Shiod91d] — *A Collection: Mordell-Weil Lattices*, Max-Planck Inst. Math., Bonn, 1991.

[Shiod91e] — *Mordell-Weil lattice theory and its applications – a point of contact of algebra, geometry and computers* (in Japanese), Sugaku **43** (No. 2, 1991), 97–114.

P. W. Shor: see also A. R. Calderbank.

[ShS98] — and N. J. A. Sloane, *A family of optimal packings in Grassmannian manifolds*, J. Alg. Combin., 1998, to appear.

M. U. Siddiqi: see B. S. Rajan.

F. Sigrist: see also A.-M. Bergé, D.-O. Jaquet.

[Sig90] — *Formes quadratiques encapsulées*, JNB **2** (1990), 425–429.

M. K. Simon: see E. Biglieri.

[Sim94] J. Simonis, *Restrictions on the weight distribution of binary linear codes imposed by the structure of Reed-Muller codes*, PGIT **40** (1994), 194–196.

[SiV91] — and C. de Vroedt, *A simple proof of the Delsarte inequalities*, DCC **1** (1991), 77–82.

[SipS96] M. Sipser and D. A. Spielman, *Expander codes*, PGIT **42** (1996), 1710–1722.

[SkSL] S. S. Skiena, W. D. Smith and P. Lemke, *Reconstructing sets from interpoint distances*, 6th ACM Symp. Comp. Geom., June 1990, pp. 332–339.

[SkV90] A. N. Skorobogatov and S. G. Vladuts, *On the decoding of algebraic-geometric codes*, PGIT **36** (1990), 1051–1061.

N. J. A. Sloane: see also M. Bernstein, A. E. Brouwer, A. R. Calderbank, J. H. Conway, Y. Edel, G. D. Forney, Jr., R. W. Grosse-Kunstleve, A. R. Hammons, Jr., R. H. Hardin, G. Nebe, E. M. Rains, P. W. Shor.

[SloEIS] — *The On-Line Encyclopedia of Integer Sequences*, published electronically at http://www.research.att.com/∼njas/sequences/.

[SloHP] — *Home page*: http://www.research.att.com/∼njas/.

[SlHDC] — R. H. Hardin, T. S. Duff and J. H. Conway, *Minimal-energy clusters of hard spheres*, DCG **14** (1995), 237–259.

[Slod80] P. Slodowy, *Simple Singularities and Simple Algebraic Groups*, LNM **815** (1980).

[Smith76] P. E. Smith, *A simple subgroup of M? and* $E_8(3)$, BLMS **8** (1976), 161–165.

W. D. Smith: see also A. E. Brouwer, R. H. Hardin, S. S. Skiena.

[Smi88] — *Studies in Computational Geometry Motivated by Mesh Generation*, Ph. D. Dissertation, Dept. of Applied Mathematics, Princeton Univ., Sept. 1988.

[Smi91] — *Few-distance sets and the second Erdős number*, in preparation.

J. Snyders: see also Y. Be'ery, M. Ran.

[SnBe89] — and Y. Be'ery, *Maximum likelihood soft decoding of binary block codes and decoders for the Golay codes*, PGIT **35** (1989), 963–975.

[Soi87] L. H. Soicher, *Presentations of some finite groups with applications to the O'Nan simple group*, J. Alg. **108** (1987), 310–316.

[Soi90] — *A new existence and uniqueness proof for the O'Nan group*, BLMS **22** (1990), 148–152.

[Soi91] — *More on the group* Y_{555} *and the projective plane of order 3*, J. Alg. **36** (1991), 168–174.

P. Solé: see also C. Bachoc, A. Bonnecaze, R. Chapman, G. D. Cohen, S. T. Dougherty, A. R. Hammons, Jr., P. Loyer.

[Sole95] — D_4, E_6, E_8 *and the AGM*, Lect. Notes Comp. Sci. **948** (1995), 448–455.

[SolL98] — and P. Loyer, *U_n lattices, Construction B and AGM iterations*, EJC **19** (1998), 227–236.

P. Sonneborn: see T. Drisch.

T. Sørevik: see J. N. Lyness.

B. Souvignier: see also W. Plesken.

[Sou94] — *Irreducible finite integral matrix groups of degree 8 and 10*, MTAC **63** (1994), 335–350.

D. A. Spielman; See also M. Sipser.

[Spiel96] — *Linear-time encodable and decodable error-correcting codes*, PGIT **42** (1996), 1723–1731.

[Stan80] R. P. Stanley, *Decompositions of rational convex polytopes*, Ann. Discrete Math. **6** (1980), 333–342.

[Stan86] — *Enumerative Combinatorics*, Vol. 1. Wadsworth, Monterey, CA, 1986.

J. Stern: see G. Lachaud.

H. Stichtenoth: see also A. Garcia.

[Stich93] — *Algebraic Function Fields and Codes*, Springer-Verlag, NY, 1993.

[StTs92] — and M. A. Tsfasman, editors, *Coding Theory and Algebraic Geometry*, LNM **1518**, Springer-Verlag, NY, 1992.

[Sto73] M. I. Stogrin, *Regular Dirichlet-Voronoi partitions for the second triclinic group*, TMIS **123** (1973) = PSIM **123** (1973).

[Stru94] R. Struik, *An improvement of the Van Wee bound for binary linear covering codes*, PGIT **40** (1994), 1280–1284.

[Stru94a] — *On the structure of linear codes with covering radius two and three*, PGIT **40** (1994), 1417–1424.

[Suda96] M. Sudan, *Maximum likelihood decoding of Reed-Solomon codes*, Proc. 37th FOCS (1996), 164–172.

[Suda97] — *Decoding of Reed-Solomon codes beyond the error-correction bound*, J. Compl. **13** (1997), 180–193.

[SuW95] I. A. I. Suleiman and R. A. Wilson, *Standard generators for J_3*, Exper. Math. **5** (1995).

[Sul90] J. M. Sullivan, *A Crystalline Approximation Theorem for Hypersurfaces*, Ph. D. Dissertation, Princeton Univ., 1990.

F.-W. Sun: see also O. Amrani.

[SuTi95] — and H. C. A. van Tilborg, *The Leech lattice, the octacode and decoding algorithms*, PGIT **41** (1995), 1097–1106.

A. Szántó: see G. Ivanyos.

[Tak85] I. Takada, *On the classification of definite unimodular lattices over the ring of integers in $\mathbb{Q}(\sqrt{2})$*, Math. Japan. **30** (1985), 423–433.

[Tak90] — *On the genera and the class number of unimodular lattices over the ring of integers of real quadratic fields*, JNT **36** (1990), 373–379.

[Tan91] L.-Z. Tang, *Rational double points on a normal octic K3 surface*, Nagoya Math. J. **129** (1993), 147–177.

[TaGa91] T. Tarnai and Z. Gáspár, *Covering a sphere by equal circles, and the rigidity of its graph*, PCPS **110** (1991), 71–89.

[TaGa91a] — — *Arrangement of 23 points on a sphere (on a conjecture of R. M. Robinson)*, PRS **A 433** (1991), 257–267.

V. Tarokh: see also I. F. Blake.

[TaBl96] — and I. F. Blake, *Trellis complexity versus the coding gain of lattices I*, PGIT **42** (1996), 1796–1807.

[TaBl96a] — — *Trellis complexity versus the coding gain of lattices II*, PGIT **42** (1996), 1808–1816.

[TaVa97] — and A. Vardy, *Upper bounds on trellis complexity of lattices*, PGIT **43** (1997), 1294–1300.

[TaVZ] — — and K. Zeger, *Universal bound on the performance of lattice codes*, PGIT, to appear.

T. Tasaka: see T. Kondo.

E. Thiran: see J. Fröhlich.

[Thomp76] J. G. Thompson, *A conjugacy theorem for E_8*, J. Alg. **38** (1976), 525–530.

P. H. Tiep: see also A. I. Bondal, A. I. Kostrikin, R. Scharlau.

[Tiep91] — *Weil representations of finite symplectic groups, and Gow lattices* (in Russian), Mat. Sb. **182** (1991), 1177–1199 = Math. USSR-Sb. **73** (1992), 535–555.

[Tiep92] — *Automorphism groups of some Mordell-Weil lattices* (in Russian), Izv. Ross. Akad. Nauk Ser. Mat. **56** (1992), 509–537 = Russian Acad. Sci. Izv. Math. **40** (1993), 477–501.

[Tiep92a] — *Basic spin representations of alternating groups, Gow lattices, and Barnes-Wall lattices* (in Russian), Mat. Sb. **183** (1992), 99–116 = Russian Acad. Sci. Sb. Math. **77** (1994), 351–365.

[Tiep94] — *Globally irreducible representations of the finite symplectic group $Sp_4(q)$*, Commun. Alg. **22** (1994), 6439–6457.

[Tiep95] — *Basic spin representations of $2\mathbb{S}_n$ and $2\mathbb{A}_n$ as globally irreducible representations*, Archiv. Math. **64** (1995), 103–112.

[Tiep97] — *Weil representations as globally irreducible representations*, Math. Nach. **184** (1997), 313–327.

[Tiep97a] — *Globally irreducible representations of finite groups and integral lattices*, Geom. Dedicata **64** (1997), 85–123.

[Tiep97b] — *Globally irreducible representations of $SL_3(q)$ and $SU_3(q)$*, preprint.

[Tiet91] A. Tietäväinen, *Covering radius and dual distance*, DCG **1** (1991), 31–46.

H. C. A. van Tilborg: see O. Amrani, F.-W. Sun.

V. D. Tonchev: see also F. C. Bussemaker, W. C. Huffman, S. N. Kapralov, V. Pless.

[Ton88] — *Combinatorial Configurations*, Longman, London, 1988.

[Ton89] — *Self-orthogonal designs and extremal doubly even codes*, JCT **A 52** (1989), 197–205.

[ToYo96] — and V. Y. Yorgov, *The existence of certain extremal [54, 27, 10] self-dual codes*, PGIT **42** (1996), 1628–1631.

[TrTr84] C. L. Tretkoff and M. D. Tretkoff, *Combinatorial group theory, Riemann surfaces, and differential equations*, Contemp. Math. **33** (1984), 467–519.

M. D. Tretkoff: see C. L. Tretkoff.

M. D. Trott: see G. D. Forney, Jr.

[Tsa91] H.-P. Tsai, *Existence of certain extremal self-dual codes*, PGIT **38** (1992), 501–504.

M. A. Tsfasman: see also G. Lachaud, M. Y. Rosenbloom, H. Stichtenoth.

[Tsf91] — *Algebraic-geometric codes and asymptotic problems*, Discr. Appl. Math. **33** (1991), 241–256.

[Tsf91a] — *Global fields, codes, and sphere packings*, Astérisque **198–200** (1992), 373–396.

[Tsf96] — *Algebraic curves and sphere packing*, in *Arithmetic, Geometry and Coding Theory*, ed. R. Pellikaan et al., de Gruyter, Berlin, 1996, pp. 225–251.

[TsV91] — and S. G. Vladuts, *Algebraic-Geometric Codes*, Kluwer, Dordrecht, 1991.

[Ura87] T. Urabe, *Elementary transformations of Dynkin graphs and singularities on quartic surfaces*, Invent. Math. **87** (1987), 549–572.

[Ura89] — *Dynkin graphs and combinations of singularities on plane sextic curves*, Contemp. Math. **90** (1989), 295–316.

[Ura90] — *Tie transformations of Dynkin graphs and singularities on quartic surfaces*, Invent. Math. **100** (1990), 207–230.

[Ura93] — *Dynkin Graphs and Quadrilateral Singularities*, LNM **1548**, 1993.

[UrA95] R. Urbanke and D. Agrawal, *A counter example to a Voronoi region conjecture*, PGIT **41** (1995), 1195–1199.

[UrB98] — and B. Rimoldi, *Lattice codes can achieve capacity on the AWGN channel*, PGIT **44** (1998), 273–278.

[VAL93] R. J. M. Vaessens, F. H. L. Aarts and J. H. van Lint, *Genetic algorithms in coding theory – a table for $A_3(n,d)$*, Discr. Appl. Math. **45** (1993), 71-87.

[Val90] B. Vallée, *A central problem in the algorithmic geometry of numbers: Lattice reduction*, CWI Quarterly **3** (1990), 95–120.

[Vall89] G. Vallette, *A better packing of ten equal circles in a square*, Discr. Math. **76** (1989), 57–59.

A. Vardy: see also O. Amrani, A. R. Calderbank, R. G. Downey, J. Feigenbaum, G. D. Forney, Jr., Y. Klein, A. Lafourcade, S. Litsyn, V. Tarokh.

[Vard94] — *The Nordstrom-Robinson code: representation over $GF(4)$ and efficient decoding*, PGIT **40** (1994), 1686–1693.

[Vard95] — *A new sphere packing in 20 dimensions*, Invent. Math. **121** (1995), 119–133.

[Vard95a] — *Even more efficient bounded-distance decoding of the hexacode, the Golay code, and the Leech lattice*, PGIT **41** (1995), 1495–1499.

[Vard97] — *The intractability of computing the minimum distance of a code*, PGIT **43** (1997), 1757–1766.

[Vard98] — *Density doubling, double-circulants and new sphere packings*, TAMS, to appear.

[Vard98a] — *Trellis structure of codes*, to appear in [PHB].

[VaBe91] — and Y. Be'ery, *More efficient soft-decision decoding of the Golay codes*, PGIT **37** (1991), 667–672.

[VaBe93] — — *Maximum-likelihood decoding of the Leech lattice*, PGIT **39** (1993), 1435–1444.

[VaKs96] — and F.R. Kschischang, *Proof of a conjecture of McEliece regarding the expansion index of the minimal trellis*, PGIT **42** (1996), 2027–2034.

[VaWi94] S. Vassallo and J. M. Wills, *On mixed sphere packings*, Contributions to Algebra and Geometry **35** (1994), 67–71.

B. B. Venkov: see R. Bacher, H. Koch, G. Nebe.

T. Verhoeff: see A. E. Brouwer.

E. Viterbo: see also J. Boutros.

[ViBi96] — and E. Biglieri, *Computing the Voronoi cell of a lattice: the diamond-cutting algorithm*, PGIT **42** (1996), 161–171.

S. G. Vladuts: see also Y. U. Manin, A. N. Skorobogatov, M. A. Tsfasman.

[Vlad90] — *On the decoding of algebraic-geometric codes over $\mathbb{F}_q$ for $q \geq 16$*, PGIT **36** (1990), 1461–1463.

[Voi85] E. Voigt, *Ausgezeichnete Basen von Milnorgittern einfacher Singularitäten*, AMSH **55** (1985), 183–190.

C. de Vroedt: see J. Simonis.

[Wan96] Z. Wan, On the uniqueness of the Leech lattice, EJC **18** (1997), 455–459.

[Wan91] M. Wang, *Rational double points on sextic K3 surfaces*, preprint.

[War92] H. H. Ward, *A bound for divisible codes*, PGIT **38** (1992), 191–194.

D. L. Webb: see C. S. Gordon.

[Wee93] G. J. M. van Wee, *Some new lower bounds for binary and ternary covering codes*, PGIT **39** (1993), 1422–1424.

L. F. Wei: see G. D. Forney, Jr.

V. K.-W. Wei: see T. Etzion, G. Lachaud.

R. Weissauer: see R. E. Borcherds.

G. Whittle: see R. G. Downey.

[Wille96] L. T. Wille, *New binary covering codes obtained by simulated annealing*, PGIT **46** (1996), 300–302.

[Will87] D. E. G. Williams, *Close packing of spheres*, J. Chem. Phys. **87** (1987), 4207–4210.

J. M. Wills: see also U. Betke, J. Bokowski, K. Böröczky, Jr., G. Fejes Tóth, P. M. Gandini, P. Gritzmann, P. M. Gruber, S. Vassallo.

[Wills83] — *Research Problem 35*, Period. Math. Hungar. **14** (1983), 312–314.

[Wills90] — *Kugellagerungen und Konvexgeometrie*, Jber. Dt. Math.-Verein. **92** (1990), 21–46.

[Wills90a] — *Dense sphere packings in cylindrical Penrose tilings*, Phys. Rev. **B 42** (1990), 4610–4612.

[Wills90b] — *A quasi-crystalline sphere-packing with unexpected high density*, J. Phys. France **51** (1990), 1061–1064.

[Wills91] — *An ellipsoid packing in E^3 of unexpected high density*, Math. **38** (1991), 318–320.

[Wills93] — *Finite sphere packings and sphere coverings*, Rend. Semin. Mat. Messina (II) **2** (1993), 91–97.

[Wills95] — *Sphere packings, densities and crystals*, in *Intuitive Geometry (Budapest 1995)*, Bolyai Society Math. Studies **6** (1997), pp. 223–234.

[Wills96] — *Sphere packings and the concept of density*, Rend. Circ. Mat. Palermo **41** (1996), 245–252.

[Wills96a] — *Lattice packings of spheres and the Wulff-shape*, Math. **43** (1996), 229–236.

[Wills97] — *On large lattice packings of spheres*, Geom. Dedicata **65** (1997), 117–126.

[Wills97a] — *Finite sphere packings and the methods of Blichfeldt and Rankin*, Acta Math. Hungar. **75** (1997), 337–342.

[Wills97b] — *Parametric density, online-packings and crystal growth*, Rend. Circ. Mat. Palermo **50** (1997), 413–424.

[Wills98] — *Crystals and quasicrystals, sphere packings and Wulff shape*, preprint.

[Wills98a] — *Spheres and sausages, crystals and catastrophes, and a joint packing theory*, Math. Intelligencer **20** (1998), 16–21.

R. A. Wilson: see also C. Jansen, P. B. Kleidman, S. A. Linton, S. P. Norton, R. A. Parker, A. I. A. Suleiman.

[Wil88] — *On the 3–local subgroups of Conway's group Co_1*, J. Alg. **113** (1988), 261–262.

[Wil89] — *Vector stabilizers and subgroups of Leech lattice groups*, J. Alg. **127** (1989), 387–408.

[Wil90] — *The 2– and 3-modular characters of J_3, its covering group and automorphism group*, J. Symb. Comp. **10** (1990), 647–656.

M. Wirtz: see D. Schomaker.

S. A. Wolpert: see C. S. Gordon.

D. C. Wright: see N. D. Mermin, D. S. Rokhsar.

P. E. Wright: see M. Bernstein.

D. Würtz: see R. Peikert.

[Xu1] X. Xu, *Untwisted and twisted gluing techniques for constructing self-dual lattices*, preprint.

[Yan94] J.-G. Yang, *Rational double points on a normal quintic K3 surface*, Acta Math. Sinica **10** (1994), 348–361.

V. Y. Yorgov: see also V. Tonchev.

[Yor89] — *On the extremal doubly-even codes of length 32*, Proc. Fourth Joint Swedish-Soviet International Workshop on Information Theory, Gotland, Sweden, 1989, pp. 275–279.

V. A. Yudin: see also A. V. Kolushov.

[Yud91] — *Sphere-packing in Euclidean space and extremal problems for trigonometrical polynomials*, Discr. Appl. Math. **1** (1991), 69–72.

[Yud97] — *Lower bounds for spherical designs*, Izv. Math. **61** (1997), 673–683.

D. Zagier: see M. Eichler.

[ZaF96] R. Zamir and M. Feder, *On lattice quantization noise*, PGIT **42** (1996), 1152–1159.

H. J. Zassenhaus: see also M. H. Dauenhauer, M. Pohst.

[Zas3] — *A new reduction method for positive definite quadratic forms*, preprint.

K. Zeger: see also J. Hamkins, C. Linder, V. Tarokh.

[ZeG94] — and A. Gersho, *Number of nearest neighbors in a Euclidean code*, PGIT **40** (1994), 1647–1649.

[ZeK93] — and M. R. Kantorovitz, *Average number of facets per cell in tree-structured vector quantizer partitions*, PGIT **39** (1993), 1053–1056.

P. L. Zhang: see J. R. Quine.

Z. Zhang: see T. Etzion.

[Zhu88] F. Z. Zhu, *On nondecomposability and indecomposability of quadratic forms*, Sci. Sinica **A 31** (1988), 265–273.

[Zhu91] — *Classification of positive definite Hermitian forms over $Q(\sqrt{-11})$*, J. Math (P.R.C.) **11** (1991), 188–195.

[Zhu91a] — *On the classification of positive definite unimodular Hermitian forms*, Chinese Sci. Bull. **36** (1991), 1506–1511.

[Zhu93] — *Classification of positive definite Hermitian forms over $Q(\sqrt{-19})$*, Northeastern Math. J. **9** (1993), 89–97.

[Zhu94] — *Construction of indecomposable definite Hermitian forms*, Chin. Ann. Math. **15 B** (1994), 349–360.

[Zhu95] — *On the construction of positive definite indecomposable unimodular even Hermitian forms*, JNT **50** (1995), 318–322.

[Zhu95a] — *On the construction of indecomposable positive definite Hermitian forms over $\mathbb{Z}[\sqrt{-m}]$*, Algebra Colloq. **2** (1995), 221–230.

[Zhu95b] — *Construction of indecomposable definite Hermitian forms*, Chinese Science Bull. **40** (1995), 177–180.

[Zhu88a] — and Y.-Y. Shao, *On the construction of indecomposable positive definite forms over $\mathbb{Z}$*, Chinese Ann. Math. **B 9** (1988), 79–94.

V. A. Zinoviev: see S. M. Dodunekov.

[Zo96] Ch. Zong, *Strange Phenomena in Convex and Discrete Geometry*, Springer-Verlag, NY, 1996.

A. Zurco: see P. M. Gandini.