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# Quantum correlation exists in any non-product state

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Simultaneous existence of correlation in complementary bases is a fundamental feature of quantum correlation, and we show that this characteristic is present in any non-product bipartite state. We propose a measure via mutually unbiased bases to study this feature of quantum correlation, and compare it with other measures of quantum correlation for several families of bipartite states.

uantum systems can be correlated in ways inaccessible to classical objects. This quantum feature of correlations not only is the key to our understanding of quantum world, but also is essential for the powerful applications of quantum information and quantum computation<sup>1-19</sup>. In order to characterize the correlation in quantum state, many approaches have been proposed to reveal different aspects of quantum correlation, such as the various measures of entanglement<sup>6-10</sup> and the various measures of discord and related measures<sup>17-21</sup>, etc. It is believed that some aspects of quantum correlation could still exist without the presence of entanglement and these aspects could be revealed via local measurements with respect to some basis of a local system.

The simultaneous existence of complementary correlations in different bases is revealed very early by the Bell's inequalities <sup>22</sup>. Bell's inequalities quantify quantum correlation via expectation values of local complementary observables. In Ref. 23, the feature of genuine quantum correlation is revealed by defining measures based on invariance under a basis change: for a bipartite quantum state, the classical correlation is the maximal correlation present in a certain optimum basis, while the quantum correlation is characterized as a series of residual correlations in the bases mutually unbiased (MU) to the optimum basis. In this paper, we use the fact that the essential feature of the quantum correlation is that it can be present in any two mutually unbiased bases (MUBs) simultaneously. Thus, one of the two bases is not necessarily the optimum basis to reveal the maximal classical correlation in this paper. With respect to the measure proposed here, we shall show that only the product states do not contain quantum correlation. A product state contains neither any quantum correlation nor any classical correlation; while any non-product bipartite state contains correlation that is fundamentally quantum! We shall also reveal interesting properties of this measure by comparing this measure to other measures of quantum correlation for several families of bipartite states.

The MUBs constitute now a basic ingredient in many applications of quantum information processing: quantum state tomography<sup>24</sup>, quantum cryptography<sup>25</sup>, discrete Wigner function<sup>26</sup>, quantum teleportation<sup>27</sup>, quantum error correction codes<sup>28</sup>, and the mean king's problem<sup>29</sup>. Two orthonormal bases  $\{|\psi_i\rangle\}$  and  $\{|\phi_j\rangle\}$  of a d-dimensional Hilbert space H are said to be mutually unbiased if and only if

$$\left|\left\langle \psi_i \middle| \phi_j \right\rangle \right| = \frac{1}{\sqrt{d}}, \quad \forall \ 1 \le i, j \le d.$$
 (1)

In a d-dimensional Hilbert space, there exist at least 3 MUBs (when d is a power of a prime number, a full set of d + 1 MUBs exists, more details can be found in Ref. 30).

We recall the quantity defined in Ref. 23. Let  $H_{ab} = H_a \otimes H_b$  with dim  $H_a = d_a$  and dim  $H_b = d_b$  be the state space of the bipartite system A+B shared by Alice and Bob. Let  $\{|i\rangle\}$  and  $|j'\rangle$  be the orthonormal bases of  $H_a$  and  $H_b$  respectively. Alice selects a basis  $\{|i\rangle\}$  of  $H_a$  and performs a measurement projecting her system onto the basis states. The Holevo quantity  $\chi\{\rho_{ab}|\{|i\rangle\}\}$  of  $\rho_{ab}$  with respect to Alice's local projective measurement onto the basis  $\{|i\rangle\langle i|\}$ , is defined as  $\chi\{\rho_{ab}|\{|i\rangle\}\}=\chi\{p_i;\rho_i^b\}\equiv S\left(\sum_i p_i\rho_i^b\right)-\sum_i p_i S\left(\rho_i^b\right)$ . A basis  $\{|i\rangle\}$  that achieves the maximum (denoted as  $C_1(\rho_{ab})$ ) of the Holevo quantity is called a  $C_1$ -basis of  $\rho_{ab}$ . There could exist many  $C_1$ -

bases for a state  $\rho_{ab}$ , and the set of these bases is denoted as  $\Gamma_{\rho_{ab}}$ . Let  $\Omega_{\Pi^a}$  be the set of all bases that are mutually unbiased to  $\Pi^a$ ,  $\Pi^a \in \Gamma_{\rho_{ab}}$ . The quantity of quantum correlation in Ref. 23, denoted by  $Q_2(\rho_{ab})$ , is defined as

$$Q_2(\rho_{ab}) \equiv \max_{\Pi^a \in \Gamma_{\rho_{ab}}} \max_{\tilde{\Pi}^a \in \Omega_{\Pi^a}} \chi \left\{ \rho_{ab} \middle| \tilde{\Pi}^a \right\}. \tag{2}$$

In other words,  $Q_2$  is defined as the Holevo quantity of Bob's accessible information about Alice's results, maximized over Alice's projective measurements in the bases that are mutually unbiased to a  $C_1$ -basis  $\Gamma_{\rho_{ab}}$ , and further maximized over all possible  $C_1$ -bases (if not unique). Thus,  $Q_2$  is actually the maximum correlation present simultaneously in any set of two MUBs of which one is a  $C_1$ -basis.

### Results

Our approach - Correlation measure based on MUBs. We now present our approach in a more general way.

**Definition.** Let  $\Delta$  denote the set of all two-MUB sets, i.e.,

$$\Delta = \{\{\{|i_1\rangle\}, \{|j_2\rangle\}\} : \{|i_1\rangle\} \text{ is MU to } \{|j_2\rangle\}\}. \tag{3}$$

We define

$$\mathcal{C}_{2}(\rho_{ab}) \equiv \max_{\left(\Pi_{1}^{a}, \Pi_{2}^{a}\right) \in \Delta} \min\left\{\chi\left\{\rho_{ab} \left| \Pi_{1}^{a}\right\}, \chi\left\{\rho_{ab} \left| \Pi_{2}^{a}\right\}\right\}\right\}. \tag{4}$$

The quantity  $C_2$  represents the maximal amount of correlation that is present simultaneously in any two MUBs. Similar to the other usual measures of quantum correlation,  $C_2$  is local unitary invariant, that is,  $C_2(\rho_{ab}) = C_2\left(U_a \otimes U_b \rho_{ab} U_a^{\dagger} \otimes U_b^{\dagger}\right)$  for any unitary operators  $U_a$  and  $U_b$  acting on  $H_a$  and  $H_b$  respectively.

 $C_2$  **versus**  $Q_2$ . Although both  $C_2$  and  $Q_2$  represent quantum correlation (here the symbol  $C_2$  is associated with *Correlation* in two MUBs) instead of classical correlation (represented by  $C_1$ ), they are actually quite different. As  $C_2$  is defined as the maximum correlation present simultaneously in any two MUBs while  $Q_2$  is the maximum correlation present simultaneously in two MUBs of which one is a  $C_1$ -basis, it is obvious that

$$C_2(\rho_{ab}) \ge Q_2(\rho_{ab}) \tag{5}$$

for any  $\rho_{ab}$ . In a sense,  $C_2$  is more essential than  $Q_2$  since the maximum in the former one is taken over arbitrarily two MUBs. Thus,  $C_2$  may reveal more quantum correlation than  $Q_2$ .

From the following Theorem and Examples, one knows that there do exist states such that  $C_2 > Q_2$  (Example 4) and  $C_2 = Q_2$  (Examples 1,2,3). A clear illustration of the difference between  $C_2$  and  $C_2$  is also given in Fig. 3. We know that  $C_2$  does not exceed quantum discord  $C_2$  for all the known examples<sup>23</sup>. However, one can easily find states such that  $C_2$  exceeds quantum discord  $C_2$  (See Fig. 3).

The nullity of  $C_2$ . Now we show that any bipartite quantum state contains nonzero correlation simultaneously in two mutually unbiased bases unless it is a product state, this result is stated as the following theorem, while the proof is left to the Method.

**Theorem.**  $C_2(\rho_{ab}) = 0$  if and only if  $\rho_{ab}$  is a product state.

In a sense, this theorem implies that, any non-product bipartite state contains genuine quantum correlation, and  $\mathcal{C}_2$  reveals the amount of quantum correlation in the state. In addition, we know that  $\mathcal{C}_2$  is different from the quantity  $Q_2$  in Ref. 23 since  $Q_2(\rho_{cq})=0$  for any classical-quantum state  $\rho_{cq}$  while  $\mathcal{C}_2=0$  only for product states. The difference between the measure  $\mathcal{C}_2$  and other measures of quantum correlation shall be discussed below for several families of bipartite states in more details.

Now, we shall calculate the quantity for several families of bipartite states, and see how our measure in terms of MUBs is well justified as a measure of quantum correlation.

**Example 1 - Pure states.** For a bipartite pure state with the Schmidt decomposition  $|\psi\rangle = \sum_i \sqrt{\lambda_i} |a_i\rangle |b_i\rangle$ ,  $\mathcal{C}_2 = Q_2 = S(\rho_a) = S(\rho_b) = \sum_i -\lambda_i \log_2 \lambda_i$ . It can be easily checked that  $\mathcal{C}_2$  coincides with the entropy of either reduced state for any pure state, which is also the usual measure of entanglement in a pure state.

**Example 2 - Werner states.** Next, we consider the Werner states of a  $d \otimes d$  dimensional system<sup>5</sup>,

$$\rho_{w} = \frac{1}{d(d-\alpha)}(I - \alpha P), \tag{6}$$

where  $-1 \leq \alpha \leq 1$ , I is the identity operator in the  $d^2$ -dimensional Hilbert space, and  $P = \sum_{i,j=1}^d |i\rangle\langle j| \otimes |j\rangle\langle i|$  is the operator that exchanges A and B. For a local measurement with respect to basis states  $\{|e_i\rangle\}$  of  $H_a$ , with probability  $p_i = \frac{1}{d}$ , Alice will obtain the k-th basis state  $|e_k\rangle$ , and Bob will be left with the state  $\rho_k^b = \frac{1}{d-\alpha}(I-\alpha|e'_k\rangle\langle e'_k|)$ , where  $|e'_k\rangle = \sum_j \alpha_{kj}|j'\rangle$  with  $\alpha_{kj} = \langle e_k|j\rangle$ . It is straightforward to show that

$$C_2(\rho_w) = \chi\{p_i; \rho_i^B\} = \log_2\left(\frac{d}{d-\alpha}\right) + \frac{1-\alpha}{d-\alpha}\log_2(1-\alpha) = Q_2(\rho_w).$$
 (7)

The entanglement of formation  $E_f$  for the Werner states is given as

$$E_f(\rho_w) = h\left(\frac{1}{2}\left(1 + \sqrt{1 - \left[\max\left(0, \frac{d\alpha - 1}{d - \alpha}\right)\right]^2}\right)\right), \text{ with } h(x) \equiv -x$$

 $\log_2 x - (1-x)\log_2(1-x)^{31}$ . The three different measures of quantum correlation, i.e.,  $C_2$ , the quantum discord D and the entanglement of formation  $E_f$ , are illustrated in Fig. 1 for comparison. From this figure, we see that the curve for

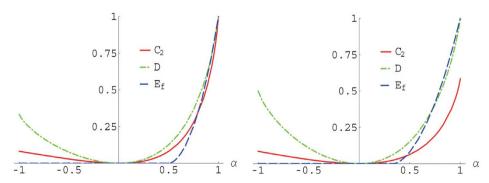
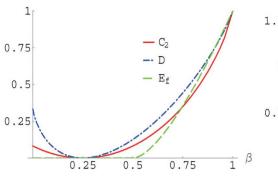


Figure 1 | Measures of quantum correlation for the Werner states as functions of  $\alpha$  when d = 2 (left) and d = 3 (right). The red curve represents our measure  $C_2$ , the green curve represents the quantum discord D and the blue curve represents the entanglement of formation  $E_f$ .





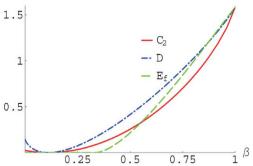


Figure 2 | Measures of quantum correlation for the isotropic states as functions of  $\beta$  when d = 2 (left) and d = 3 (right). The red curve represents our measure  $C_2$ , the blue curve represents the quantum discord D and the green curve represents the entanglement of formation  $E_f$ .

entanglement of formation intersects the other two curves; thus,  $E_f$  can be larger or smaller than  $C_2$ .

**Example 3 - Isotropic states.** For the  $d \otimes d$  isotropic states

$$\rho = \frac{1}{d^2 - 1} \left( (1 - \beta)I + \left( d^2 \beta - 1 \right) P^+ \right), \quad \beta \in [0, 1], \tag{8}$$

where  $P^+ = |\Phi^+\rangle \langle \Phi^+|$ ,  $|\Phi^+| = \frac{1}{\sqrt{d}} \sum_i |i\rangle |i'\rangle$  is the maximally entangled pure state in  $\mathbb{C}^d \otimes \mathbb{C}^d$ . Let  $\{|e_k\rangle \langle e_k|\}$  be an arbitrarily given projective measurement on Alice's part. Bob's state after after Alice gets the k-th measurement result is

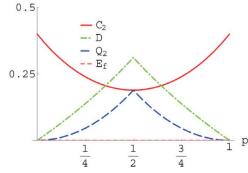
$$\rho_{k}^{b} = \frac{1}{d^{2} - 1} \left( d(1 - \beta)I + \left( d^{2}\beta - 1 \right) |e'_{k}\rangle\langle e'_{k}| \right), \tag{9}$$

where  $|e'_k\rangle = \sum_j \alpha_{kj} |j'\rangle$  with  $\alpha_{kj} = \langle e_k | j \rangle$ . As the eigenvalues of  $\rho_k^b$  does not depend on the basis for Alice's measurement, one can easily show that

$$C_{2}(\rho) = Q_{2}(\rho) = \log_{2} d + \frac{d\beta + 1}{d+1} \log_{2} \frac{d\beta + 1}{d+1} + \frac{d - d\beta}{d+1} \log_{2} \frac{d - d\beta}{d^{2} - 1}.$$
(10)

The entanglement of formation  $E_f$  for the isotropic states is given as  $^{32,33}$ 

$$E_{f}(\rho) = \begin{cases} 0, & \beta \leq \frac{1}{d}, \\ h(\gamma) + (1 - \gamma)\log_{2}(d - 1), & \frac{1}{d} < \beta < \frac{4(d - 1)}{d^{2}}, \\ \frac{(\beta - 1)d\log_{2}(d - 1)}{d - 2} + \log_{2}d, & \frac{4(d - 1)}{d^{2}} \leq \beta \leq 1, \end{cases}$$
(11)



where  $\gamma = \frac{1}{d} \left( \sqrt{\beta} + \sqrt{(d-1)(1-\beta)} \right)^2$ . The quantum discord of the isotropic state is<sup>34</sup>

$$D(\rho) = \beta \log_2 \beta + \frac{1 - \beta}{d + 1} \log_2 \frac{1 - \beta}{d^2 - 1} - \frac{1 + d\beta}{d + 1} \log_2 \frac{1 - \beta - \frac{1}{d} + d\beta}{d^2 - 1}.$$
 (12)

The three different measures of quantum correlation, i.e.,  $C_2$ , the quantum discord D and the entanglement of formation  $E_f$  are illustrated in Fig. 2 for comparison. From this figure, we see that the curve for entanglement of formation intersects the other two curves; thus,  $E_f$  can be larger or smaller than  $C_2$ .

**Example 4 - A family of two-qubit states.** As the last example, we consider a family of two-qubit states that are Bell-diagonal states. This family of states admit the form

$$\sigma_{ab} = \frac{1}{4} \left( I_2 \otimes I_2 + \sum_{j=1}^3 r_j \sigma_j \otimes \sigma_j \right). \tag{13}$$

We rearrange the three numbers  $\{r_1, r_2, r_3\}$  according to their absolute values and denote the rearranged set as  $\{\bar{r}_1, \bar{r}_2, \bar{r}_3\}$  such that  $|\bar{r}_1| \ge |\bar{r}_2| \ge |\bar{r}_3|$ . Next we show that

$$C_{2}(\sigma_{ab}) = 1 - h \left( \frac{1 + \sqrt{(\overline{r}_{1}^{2} + \overline{r}_{2}^{2})/2}}{2} \right). \tag{14}$$

Without loss of generality, we prove (14) only for the case  $r_1 \ge r_2 \ge r_3$ . A projective measurement performed on qubit A can be written as  $P^a_{\pm} = \frac{1}{2}(I_2 \pm \vec{n} \cdot \vec{\sigma})$ , parameterized by the unit vector  $\vec{n}$ . When Alice

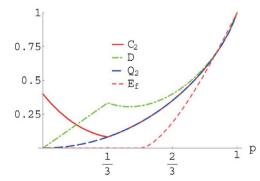


Figure 3 | Different measures of quantum correlation for two special classes of states:  $\rho_1 = \frac{1}{2} |\psi^+\rangle \langle \psi^+| + \frac{p}{2} |\phi^+\rangle \langle \phi^+| + \frac{1-p}{2} |\phi^-\rangle \langle \phi^-|$  (left) and  $\rho_2 = p|\psi^-\rangle \langle \psi^-| + \frac{1-p}{2} (|\psi^+\rangle \langle \psi^+| + |\phi^+\rangle \langle \phi^+|)$  (right). In each figure, the red curve represents our measure  $C_2$ , the green curve represents the quantum discord D, the blue curve represents the measure  $Q_2$ , and the dashed orange curve represents the entanglement of formation  $E_f$ .

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obtains  $p_{\pm}$ , Bob will be in the corresponding states  $\rho_{\pm}^{b}=\frac{1}{2}(I_{2}\pm\sum_{j}n_{j}r_{j}\sigma_{j})$ , each occurring with probability  $\frac{1}{2}$ . The entropy  $S(\rho_{\pm}^{b})$  reaches its minimum value  $h\left(\frac{1+|r_{1}|}{2}\right)$  when  $\vec{n}=(1,0,0)$ . Let  $\vec{n}_{1}=(x,y,0)$  and  $\vec{n}_{2}=(a,b,0)$  with ax+by=0, then  $P_{\pm}^{(1)}$  is mutually unbiased to  $P_{\pm}^{(2)}$ , where  $P_{\pm}^{(1)}=\frac{1}{2}(I_{2}\pm\vec{n}_{1}\cdot\vec{\sigma})$ ,  $P_{\pm}^{(2)}=\frac{1}{2}(I_{2}\pm\vec{n}_{2}\cdot\vec{\sigma})$ . It is immediate that  $\chi\left\{\sigma_{ab}\left|P_{\pm}^{(1)}\right.\right\}=1-h\left(\frac{1+\sqrt{x^{2}r_{1}^{2}+y^{2}r_{2}^{2}}}{2}\right)$  and  $\chi\left\{\sigma_{ab}\left|P_{\pm}^{(2)}\right.\right\}=1-h\left(\frac{1+\sqrt{a^{2}r_{1}^{2}+b^{2}r_{2}^{2}}}{2}\right)$ . Thus  $\mathcal{C}_{2}\{\sigma_{ab}\}=1-h\left(\frac{1+\sqrt{(r_{1}^{2}+r_{2}^{2})/2}}{2}\right)$  as desired since h(c) is a monotonic

decreasing function when  $c \ge \frac{1}{2}$ . Our quantity  $C_2$  is compared with the quantum discord D and the entanglement of formation  $E_f$  for  $\rho_1$  and  $\rho_2$  in Fig. 3.

From the left figure of Fig. 3, it is clear that  $\mathcal{C}_2$  is quite different from both D and  $Q_2$ . Unlike  $Q_2$  that does not exceed D for all known examples,  $\mathcal{C}_2$  can exceed D. We have  $\mathcal{C}_2(\rho_1) < D(\rho_1)$  when p is closed to  $\frac{1}{2}$ , while  $\mathcal{C}_2(\rho_1) > D(\rho_1)$  when p is closed to 0 or 1; we also have  $\mathcal{C}_2(\rho_1) = Q_2(\rho_1)$  when  $p = \frac{1}{2}$ , and  $\mathcal{C}_2(\rho_1)$  increases monotonously while  $Q_2(\rho_1)$  decreases monotonously when p deviates from  $\frac{1}{2}$ . In Fig. 3, the difference between our measure  $\mathcal{C}_2$  and the other measures is well illustrated by the extreme cases when p = 0 or 1 in the left figure and when p = 0 in the right figure. For example, for  $\sigma = \frac{1}{2} \left| \psi^+ \right\rangle \left\langle \psi^+ \right| + \frac{1}{2} \left| \phi^+ \right\rangle \left\langle \phi^+ \right|$ , our measure has a finite value while the other measures vanish.

**Correlation revealed via more MUBs.** In addition, we can define a quantity based on m MUBs ( $3 \le m \le \dim H_a + 1$ ), namely,

$$\begin{split} \mathcal{C}_{m}(\rho_{ab}) &\equiv \max_{\left(\Pi_{1}^{a}, \Pi_{2}^{a}, \cdots, \Pi_{m}^{a}\right) \in \Delta_{m}} \min \left\{ \chi \left\{ \rho_{ab} \middle| \Pi_{1}^{a} \right\}, \right. \\ &\left. \chi \left\{ \rho_{ab} \middle| \Pi_{2}^{a} \right\}, \cdots, \chi \left\{ \rho_{ab} \middle| \Pi_{m}^{a} \right\} \right\}. \end{split} \tag{15} \end{split}$$

where

$$\Delta_m = \left\{ \left( \Pi_1^a, \Pi_2^a, \cdots, \Pi_m^a \right) : \right.$$

$$\Pi_{\iota}^a \text{ is MU to } \Pi_{\iota}^a \text{ for any } k \neq l \right\}.$$
(16)

It is clear that  $C_{k+1} \le C_k \le C_2$ . The following are obvious from the arguments in the previous examples: i)  $C_m(\rho) = 0$  if and only if  $\rho$  is a product state, ii)  $C_m = C_2$  for both the Werner states and the isotropic

states, and iii) 
$$C_3(\sigma_{ab}) = 1 - h\left(\frac{1 + \sqrt{(r_1^2 + r_2^2 + r_3^2)/3}}{2}\right)$$
 for the family of two-qubit states in Eq. (13).

# **Discussion**

We have provided a very different approach to quantify quantum correlation in a bipartite quantum state. Our approach captures the essential feature of quantum correlation: the simultaneous existence of correlations in complementary bases. We have proved that the only states that don't have this feature are the product states, which contains no correlation (classical or quantum) at all. Thus, any non-product state contains correlation that is fundamentally quantum. This feature of quantum correlation characterized here could be the key feature that enables quantum key distribution (QKD) with

entangled states, since the quantum correlation that exists simultaneously in MUBs, which can be quantified by  $C_m$ , is the resource for entanglement-based QKD via MUBs.

### Method

**Proof of the Theorem in the Main Text.** The 'if' part is obvious, and we only need to show the 'only if' part. In other words, we only need to prove that  $\rho_{ab} = \rho_a \otimes \rho_b$  if either  $\chi(\rho_{ab} | \Pi_1^a) = 0$  or  $\chi(\rho_{ab} | \Pi_2^a) = 0$  for any MUB pair  $(\Pi_1^a, \Pi_2^a) \in \Delta$ . It is equivalent to show that both  $\chi(\rho_{ab} | \Pi_1^a) \neq 0$  and  $\chi(\rho_{ab} | \Pi_2^a) \neq 0$  for a certain MUB pair  $(\Pi_1^a, \Pi_2^a) \in \Delta$  if  $\rho_{ab}$  is not a product state.

We assume that  $\rho_{ab}$  is not a product state, then the maximal classical correlation is nonzero, i.e.,  $C_1(\rho_{ab}) \neq 0$ . Let  $\{|e_i\rangle\} \in \Gamma_{\rho_{ab}}$ , we have  $\chi(\rho_{ab}|\{|e_i\rangle\}) \neq 0$ . Therefore, we only need to find a second basis (MU to  $\{|e_i\rangle\}$ ) such that the corresponding Holevo quantity is nonzero. We denote the projective measurement corresponding to  $\{|e_i\rangle\}$  by  $\Pi = \{\Pi_k = |e_k\rangle\langle e_k|\}$ . Then

$$\Pi\left(\rho_{ab}\right) = \sum_{k} \Pi_{k} \otimes I_{b} \rho_{ab} \Pi_{k} \otimes I_{b} = \sum_{k} p_{k}^{(1)} |e_{k}\rangle \langle e_{k}| \otimes \rho_{k}^{b(1)}$$
. As  $C_{1}(\rho_{ab}) \neq 0$ , we know that  $\rho_{k_{0}}^{b(1)} \neq \rho_{b}$  and  $\rho_{l_{0}}^{b(1)} \neq \rho_{b}$  at least for some  $k_{0}$  and  $l_{0}$ . We arbitrarily choose a basis  $\{|f_{i}\rangle\}$  that is MU to  $\{|e_{i}\rangle\}$ . If  $\chi\{\rho_{ab}|\{|f_{i}\rangle\}\} \neq 0$ , then we already obtain the second basis and the theorem is true.

If  $\chi\{\rho_{ab}|\{|f_i\rangle\}\}=0$ , we can construct the MUB pair as follows. As in this case, the measurement corresponding to  $\{|f_i\rangle\}$  yields the following output state

$$\begin{pmatrix} p_1^{(2)} \rho_b & 0 & \dots & 0 \\ 0 & p_2^{(2)} \rho_b & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & p_A^{(2)} \rho_b \end{pmatrix}. \tag{17}$$

Thus,  $\rho_{ab}$  can be represented as

$$\begin{pmatrix} p_1^{(2)}\rho_b & * & \dots & * \\ * & p_2^{(2)}\rho_b & \dots & * \\ \vdots & \vdots & \ddots & \vdots \\ * & * & \dots & p_2^{(2)}\rho_b \end{pmatrix}$$
(18)

with respect to the local basis  $\{|f_i\rangle\}$ , and at least one of the off-diagonal blocks is not zero (otherwise,  $\rho_{ab}$  is a product state). Without loss of generality we assume that the (1,2)-block-entry of the above matrix is nonzero. It follows that there exists a 2 by 2 unitary matrix  $U_2$ , such that, under the local basis  $\{U_2 \oplus I_{d-2}[f_i\rangle\}$ , the state admits the

$$\begin{pmatrix} q_1^{(2)}\varrho_b & * & \cdots & * \\ * & q_2^{(2)}\sigma_b & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ * & * & \cdots & p_s^{(2)}\varrho_b \end{pmatrix}$$
(19)

with  $\varrho_b \neq \rho_b$  and  $\sigma_b \neq \rho_b$ . That is  $\chi\{\rho_{ab}|\{U_2 \oplus I_{d-2}|f_i\}\}\} \neq 0$ . This unitary matrix  $U_2$  can be chosen as

$$U_2 = \begin{pmatrix} \sqrt{1 - \epsilon^2} & \epsilon \\ -\epsilon & \sqrt{1 - \epsilon^2} \end{pmatrix}. \tag{20}$$

with  $\epsilon$  a very small positive number. Even though  $\chi\{\rho_{ab}|\{U_2\oplus I_{d-2}|f_i\}\}\}$  could be very small, it is nonzero. As  $\epsilon$  is a very small and  $\chi\{\rho_{ab}|\{|e_i\rangle\}\}\neq 0$ , we also have  $\chi\{\rho_{ab}|\{U_2\oplus I_{d-2}|e_i\rangle\}\}\neq 0$ . Thus, the Holevo quantity is nonzero at least for a certain MUB pair (i.e.,  $\{U_2\oplus I_{d-2}|e_i\rangle\}$  and  $\{U_2\oplus I_{d-2}|f_i\rangle\}$ ), and therefore  $\mathcal{C}_2(\rho_{ab})\neq 0$ .

Thus,  $C_2(\rho_{ab}) \neq 0$  for any  $\rho_{ab}$  that is not a product state. The proof is completed.

- Einstein, A., Podolsky, B. & Rosen, N. Can quantum-mechanical description of physical reality be considered complete? *Phys. Rev.* 47, 777–780 (1935).
- Schrödinger, E. Die gegenwärtige Situation in der Quantenmechanik. Naturwissenschaften 23, 807–812 (1935).
- Einstein, A., Born, M. & Born, H. The Born-Einstein Letters: correspondence between Albert Einstein and Max and Hedwig Born from 1916 to 1955 (Walker, New York, 1971).
- Bennett, C. H., Bernstein, H. J., Popescu, S. & Schumacher, B. Concentrating partial entanglement by local operations. *Phys. Rev. A* 53, 2046–2052 (1996).
- Werner, R. F. Quantum states with Einstein-Podolsky-Rosen correlations admitting a hidden-variable model. *Phys. Rev. A* 40, 4277–4281 (1989).
- Horodecki, R., Horodecki, P., Horodecki, M. & Horodecki, K. Quantum entanglement. Rev. Mod. Phys. 81, 65–942 (2009).
- 7. Gühne, O. & Tóth, G. Entanglement detection. Phys. Rep. 474, 1 (2009).
- 8. Usha Devi, A. R. & Rajagopal, A. K. Generalized Information Theoretic Measure to Discern the Quantumness of Correlations. *Phys. Rev. Lett.* **100**, 140502 (2008).



- Eltschka, C. & Siewert, J. Negativity as an Estimator of Entanglement Dimension. Phys. Rev. Lett. 111, 100503 (2013).
- Oppenheim, J., Horodecki, M., Horodecki, P. & Horodecki, R. Thermodynamical Approach to Quantifying Quantum Correlations. *Phys. Rev. Lett.* 89, 180402 (2002).
- 11. Skrzypczyk, P., Navascués, M. & Cavalcanti, D. Quantifying Einstein-Podolsky-Rosen Steering. *Phys. Rev. Lett.* **112**, 180404 (2014).
- 12. Vedral, V. Quantum entanglement. Nature Phys. 10, 256 (2014).
- 13. Georgescu, I. Quantum entanglement: Now you see it. *Nature Phys.* **9**, 394 (2013).
- 14. Strobel, H. et al. Fisher information and entanglement of non-Gaussian spin states. *Science* **345**, 424 (2014).
- 15. Yu, T. & Eberly, J. H. Sudden Death of Entanglement. Science 323, 598 (2009).
- 16. Dakić, B. *et al.* (2012) Quantum discord as resource for remote state preparation. *Nature Phys.* **8**, 666.
- Wu, S., Poulsen, U. V. & Mølmer, K. Correlations in local measurements on a quantum state, and complementarity as an explanation of nonclassicality. *Phys. Rev. A* 80, 032319 (2009).
- 18. Luo, S. Using measurement-induced disturbance to characterize correlations as classical or quantum. *Phys. Rev. A* 77, 022301 (2008).
- 19. Modi, K., Brodutch, A., Ćable, H., Paterek, T. & Vedral, V. The classical-quantum boundary for correlations: Discord and related measures. *Rev. Mod. Phys.* **84**, 1655 (2012).
- Ollivier, H. & Zurek, W. H. Quantum Discord: A Measure of the Quantumness of Correlations. *Phys. Rev. Lett.* 88, 017901 (2001).
- 21. Luo, S. & Fu, S. Measurment-induced nonlocality. *Phys. Rev. Lett.* **106**, 120401 (2011)
- 22. Bell, J. S. On the Einstein Podolsky Rosen paradox. Physics 1, 195-200 (1964).
- 23. Wu, S., Ma, Z., Chen, Z. & Yu, S. Reveal quantum correlation in complementary bases. *Sci. Rep.* 4, 4036 (2014).
- Wootters, W. K. & Fields, B. D. Optimal state-determination by mutually unbiased measurements. Ann. Phys. (N.Y.) 191, 363 (1989).
- 25. Bechmann-Pasquinucci, H. & Peres, A. Quantum cryptography with 3-state systems. *Phys. Rev. Lett.* **85**, 3313 (2000).
- 26. Wootters, W. K. A Wigner-function formulation of finite-state quantum mechanics. *Ann. Phys. (N.Y.)* **176**, 1 (1987).
- Koniorczyk, M., Buzek, V. & Janszky, J. Wigner-function description of quantum teleportation in arbitrary dimensions and a continuous limit. *Phys. Rev. A* 64, 034301 (2001)
- Paz, J. P., Roncaglia, A. J. & Saraceno, M. Qubits in phase space: Wigner-function approach to quantum-error correction and the mean-king problem. *Phys. Rev. A* 72, 012309 (2005).

- Englert, B. G. & Aharonov, Y. The mean king's problem: prime degrees of freedom. *Phys. Lett. A* 284, 1 (2001).
- 30. Durt, T., Englert, B. G., Bengtsson, I. & Zyczkowski, K. On mutually unbiased bases. *Int. J. Quant. Inf.* **8**, 535 (2010).
- Wootters, W. K. Entanglement of formation and concurrence. Quant. Inf. & Comp. 1, 27–34 (2001).
- Terhal, B. M. & Vollbrecht, K. G. H. Entanglement of Formation for Isotropic States. *Phys. Rev. Lett.* 85, 2625 (2000).
- 33. Fei, S. M. & Li-Jost, X. R function related to entanglement of formation. *Phys. Rev.* A 73, 024302 (2006).
- Chitambar, E. Quantum correlation in high-dimensional states of high symmetry. Phys. Rev. A 86, 032110 (2012).

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### **Author contributions**

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### Additional information

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