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Fault-tolerant quantum computation with a soft-decision decoder for error correction and detection by teleportation

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Fault-tolerant quantum computation with quantum error-correcting codes has been considerably developed over the past decade. However, there are still difficult issues, particularly on the resource requirement. For further improvement of fault-tolerant quantum computation, here we propose a *soft-decision* decoder for quantum error correction and detection by teleportation. This decoder can achieve almost optimal performance for the depolarizing channel. Applying this decoder to Knill's C_4/C_6 scheme for fault-tolerant quantum computation, which is one of the best schemes so far and relies heavily on error correction and detection by teleportation, we dramatically improve its performance. This leads to substantial reduction of resources.

Quantum computers[1,2] are expected to outperform current classical computers. Many problems intractable for classical computers are believed to be solved by quantum computers more efficiently[1,3–11]. The most famous one is the prime number factoring problem[3], the difficulty of which ensures today's internet security.

The origin of the speed of quantum computation is quantum superposition of physical states. This enables us to perform a great number of calculations in parallel (*quantum parallelism*). Unfortunately, the quantum superposition is very fragile. The destruction of the superposition is called *decoherence*. The decoherence induces errors in quantum computation[12] and makes quantum computers difficult to be realized.

The standard approaches to this problem are based on quantum error correction. Using quantum error-correcting codes, we can make quantum computation fault-tolerant[1,13]. If the error probabilities of elementary operations are lower than a *threshold*, we can, in principle, perform arbitrarily long quantum computation reliably. This fact is known as the threshold theorem.

The threshold has gone up to about 1%[14–19] as a result of theoretical advances over the past decade. Although this value is comparable to error probabilities in state-of-the-art experiments[2,20,21], this does not mean that the realization of quantum computers is within reach. There are still difficult issues, particularly on resource requirement. First, the threshold is the value at which necessary resources become infinite. Therefore, the error probabilities should be much lower than the threshold. Second, even if the error probabilities become as low as 0.1%, the resources required for practical quantum computation will still be enormous[22,23]. Thus, further improvement of fault-tolerant quantum computation has been desired.

Towards more efficient fault-tolerant quantum computation, here we propose a new decoder using *soft-decision* decoding. Decoding is a crucial part of error correction in both quantum and classical situations. In the history of classical error correction, the use of soft-decision decoding based on probabilistic inference, instead of conventional hard-decision decoding based on algebraic techniques, was a key step to achieve the theoretical limit[24]. This is natural because decoding is, in essence, a problem of probabilistic inference. In general, such a problem is computationally hard. In the case of classical error correction, clever algorithms and approximations with appropriate error-correcting codes have enabled efficient soft-decision decoding. In the case of quantum error correction, an efficient soft-decision (optimal) decoding is possible for quantum concatenated codes, which has been shown by Poulin[25]. The decoding has displayed high performance on a simple quantum channel called the depolarizing channel. To the best of our knowledge, however, this has not been applied to fault-tolerant quantum computation. The reason for this is probably as follows: this algorithm is based on conventional syndrome measurements, which require many iterative fault-tolerant measurements[13,26] and consequently may not be able to achieve high performance in fault-tolerant quantum computation; probabilistic inference seems difficult in the case of fault-tolerant quantum computation because the estimation of error probabilities will be difficult.

Instead of syndrome measurements, here we focus on quantum error correction by teleportation proposed by Knill[16,27], which is more efficient and therefore more suitable for fault-tolerant quantum computation. We propose a soft-decision decoder for it. Using the depolarizing channel, we found that the performance of this decoder is very insensitive to the difference between the actual error probability and that assumed for the decoding. This means that it is unnecessary to estimate actual error probabilities accurately, and consequently opens the possibility of applying soft-decision decoding to fault-tolerant quantum computation. Applying this decoder to Knill's C_4/C_6 scheme for fault-tolerant quantum computation[16], which is one of the best schemes so far and relies heavily on error correction and detection by teleportation, we improve its performance dramatically. This leads to substantial reduction of resources and will open a new way to large-scale quantum computers.

Results

Performance for the depolarizing channel. To evaluate the performance of our soft-decision decoder, we first investigated the performance for the *depolarizing channel*[1,25], which is the standard model for noisy quantum channels. On this channel, three Pauli errors, X, Y, and Z, occur with equal probability $p_{dep}/3$ on each physical qubit, where p_{dep} denotes the error probability for the depolarizing channel. (Here, three Pauli operators are denoted by X, Y, and Z, and an identity operator is denoted by I).

We estimated the decoding error probability for the depolarizing channel by numerical simulation. In this simulation, it is assumed that errors occur only on the channel and the other operations (encoding and decoding) are performed perfectly (see Methods and Supplementary Information for the details of the simulation). The error-correcting code used in the present work is the C_4/C_6 code[16] (see Supplementary Information for the details of the C_4/C_6 code).

In this case, we can design an optimal decoding if we know p_{dep} (see Supplementary Information). In actual channels, however, p_{dep} may be unknown, and therefore we must estimate p_{dep} and use the estimated value for the decoding. Here this value used for the decoding is denoted by p_0. If the performance of the decoding is sensitive to the difference between p_{dep} and p_0, the decoding will be not useful practically. Thus, we first examined the p_0 dependence of the performance of the decoding. The result is shown in Fig. 1, where p_{dep} = 10%. The result clearly shows that the performance of the decoding is very insensitive to the difference between p_{dep} and p_0. (This is the case for the other values of p_{dep}.) This property is very significant for fault-tolerant quantum computation because the accurate estimation of error probabilities in fault-tolerant quantum computation will be difficult.

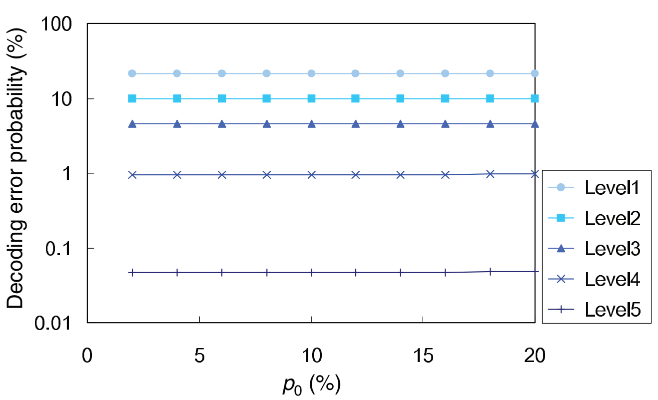

Figure 1 | Dependence of decoding error probability for the depolarizing channel on p_0. The depolarizing error probability, p_{dep}, is 10%. The decoder is designed such that it is optimal for the depolarizing channel with error probability p_0. Different symbols (colors) correspond to different concatenation levels, as shown in the figures.

Encouraged by the above result, we design our soft-decision decoder such that it is optimal for the depolarizing channel with error probability of 19%, which is associated with the threshold for the depolarizing channel (see below). This decoder can achieve high performance not only for the depolarizing channel but also for fault-tolerant quantum computation, as expected. (See Supplementary Information for the details of the decoder design).

The simulation results for the depolarizing channel are shown in Fig. 2. Figures 2(a) and 2(b) correspond to Knill's hard-decision decoder[16] and our soft-decision decoder, respectively (see Methods and Supplementary Information for the two decoding algorithms). The thresholds for them are 13.6% and 18.8%, respectively. When p_{dep} is much smaller than the threshold, power laws hold as shown in Fig. 2.

Logical controlled-NOT gate. It is known that the error threshold for fault-tolerant quantum computation is usually determined by that for the logical controlled-NOT (CNOT) gate because it is the noisiest elementary gate. In this sense, the logical CNOT gate is the most important gate for fault-tolerant quantum computation.

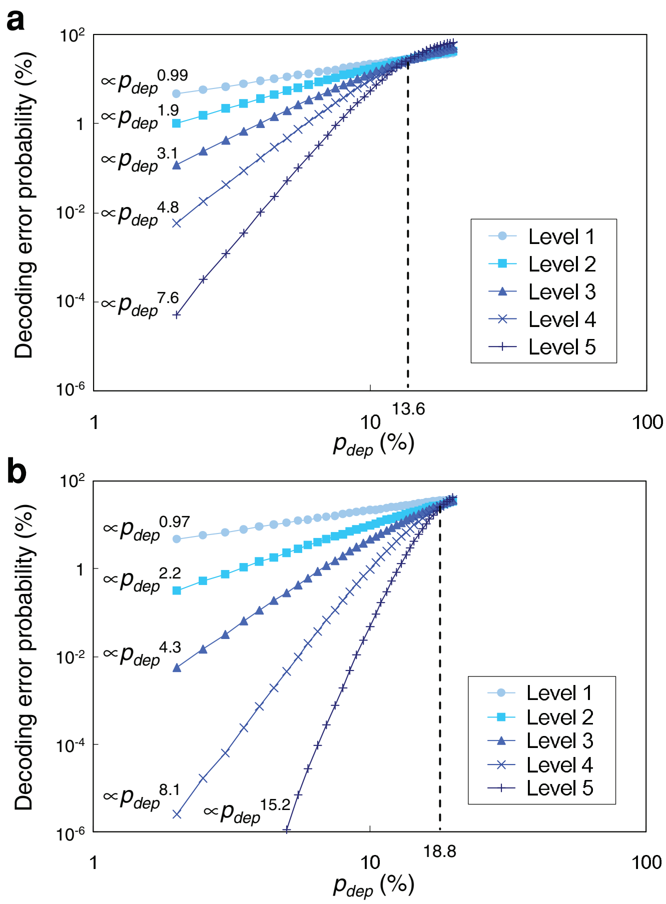

Figure 2 | Decoding error probability for the depolarizing channel. (a) and (b) correspond to the hard-decision and soft-decision decoders, respectively. Different symbols (colors) correspond to different concatenation levels, as shown in the figures. The thresholds are indicated by the vertical dashed lines. When p_{dep} is much smaller than the threshold, power laws hold as shown in the figures.

We numerically simulated logical CNOT gates to evaluate the performance of our soft-decision decoder for fault-tolerant quantum computation. In this simulation, we have assumed that errors occur only on physical CNOT gates with probability p_{CNOT} and the other operations are perfect. This assumption is valid and useful in the following sense: physical CNOT gates are usually the noisiest physical elementary gate; physical CNOT gates are used most frequently in fault-tolerant quantum computation, and consequently their effects are dominant[23]; if the other errors should be taken into account, we can effectively model such a case by assuming noisier physical CNOT gates and can use the present results. (The effect of latency is beyond the scope of the present paper.) The model of a noisy physical CNOT gate used here is the standard one[16], where 15 two-qubit Pauli errors occur with equal probability $p_{CNOT}/15$. (See Methods and Supplementary Information for the details of the simulation).

The symbols (circle, square, triangle, and cross) in Fig. 3 were obtained by the simulation. Since power laws hold again and the exponents are nearly equal to those for the depolarizing channel, we have assumed that the error probabilities of logical CNOT gates can be modeled by the depolarizing channel. Thus, the curves in Fig. 3 were estimated with the results for the depolarizing channel (see Supplementary Information for the detailed estimation).

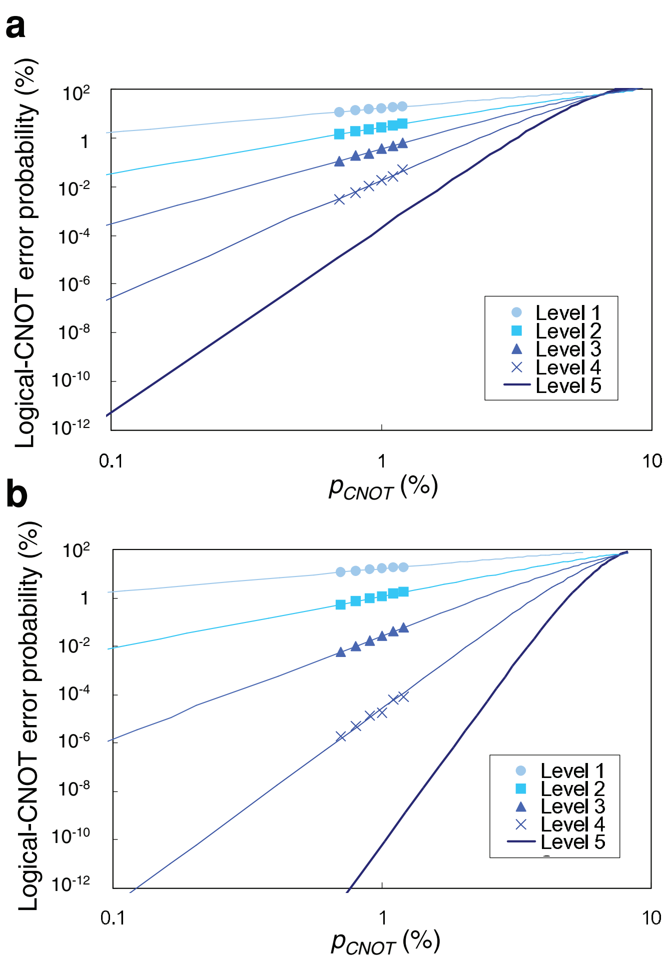

Figure 3 | Logical-CNOT error probability. (a) and (b) correspond to the hard-decision and soft-decision decoders, respectively. The symbols (circle, square, triangle, and cross) are the simulation results. The curves were estimated with the results for the depolarizing channel (see the text). Different symbols (colors) correspond to different concatenation levels and the lowest bold lines correspond to level 5, as shown in the figures.

Discussion

First, we discuss the results for the depolarizing channel (Fig. 2). The threshold for the soft-decision decoder is very close to a theoretical limit known as the hashing bound (18.9%)[25,28]. This shows the high performance of the soft-decision decoder. More importantly, one should pay attention to the exponents for the power laws which hold when p_{dep} is much smaller than the threshold. The exponents represent the minimum number of physical-qubit errors inducing decoding errors. The exponents for the hard-decision decoder are approximately a Fibonacci sequence (1, 2, 3, 5, 8, …). This fact has been pointed out by Knill[16]. On the other hand, the exponents for the soft-decision decoder are approximately the geometric sequence, 2^{l-1}, where l is the concatenation level. Since the code distance of the C_4/C_6 code is given by 2^l, each exponent is approximately equal to a half of the corresponding code distance. This indicates that the soft-decision decoding is almost optimal. (A quantum code with distance d has the potential to correct $(d - 1)/2$ qubit errors[1].) Since the geometric sequence is much greater than the Fibonacci sequence for high concatenation levels, the decoding error probability for the soft-decision decoder becomes lower much faster than that for the hard-decision one as p_{dep} becomes smaller. This also shows the high performance of the soft-decision decoder. Here it should be noted that these high performances can be achieved by computationally efficient decoding calculations (see Supplementary Information for the details of the calculations).

Next, we discuss the results for logical CNOT gates (Fig. 3). The error probability for level-4 encoding with the soft-decision decoder is a little lower than that for level-5 encoding with the hard-decision decoder. Since the total number of physical qubits required for the preparation of a level-l encoded qubit ($l \geq 2$) is given by $4 \times 12^{l-1}$ (see Supplementary Information for the derivation and validity of this formula), where it is assumed that necessary and sufficient auxiliary qubits for fully parallel computation are used, this result concludes that the qubit resource for the C_4/C_6 scheme is one order of magnitude reduced by using the soft-decision decoder, as expected. On the other hand, if level-5 encoding is used, the soft-decision decoder allows one to use much noisier physical CNOT gates to achieve the same value of the logical-CNOT error probability. These dramatic improvements are the consequence of the almost optimal performance of the present decoder.

Finally, we discuss the resource requirement for factoring a 1000-bit integer by Shor's algorithm[3,22,23]. From our estimation, this application requires about 10^{14} logical CNOT gates (see Supplementary Information for the detailed estimation). Thus, the error probability of a logical CNOT gate should be lower than 10^{-12}%. Using the hard-decision decoder, we can achieve this value by level-5 (324-qubit) encoding if the error probability of a physical CNOT gate is lower than 0.1%. On the other hand, the soft-decision decoder enables one to achieve the same value by level-4 (108-qubit) encoding under the same condition. These results are surprisingly good in comparison with the recent results for surface codes[22,23], where a logical qubit is encoded into several thousands of physical qubits under similar conditions. If we count auxiliary physical qubits, then the total number of physical qubits for an encoded qubit is given by the above formula. That is, a level-4 logical qubit requires 6912 physical qubits, which is comparable to the cases of surface codes. While surface codes have a remarkable advantage that they require only nearest-neighbor interactions (the C_4/C_6 scheme requires more complicated interactions), the C_4/C_6 scheme has the potential for further reduction of the number of physical qubits because 6804 of the 6912 qubits are auxiliary ones. Thus, the soft-decision decoder will open a new way to practical quantum computers.

Methods

Soft-decision decoding. The goal of decoding for quantum error correction by teleportation is to decide a reliable result of the encoded Bell measurement, $\{b_x, b_z\}$, with the data of the physical measurements (see Supplementary Information for details). In Knill's hard-decision decoding for the C_4/C_6 code[16], at each level of concatenation, the value of each encoded qubit is decided as 0, 1, or E, where E is a symbol indicating 'error detected'. (Since both C_4 and C_6 are error-detecting codes, the decoding result includes 'error detected'.) We call this decoding 'hard-decision' because only the three values are used in each step of the decoding. Also note that b_x and b_z are decided *independently*, that is, their correlation is ignored.

Our soft-decision decoding is as follows. In this decoding, we calculate the conditional probability, $P(b_x, b_z)$, that $\{b_x, b_z\}$ becomes $\{0, 0\}$, $\{0, 1\}$, $\{1, 0\}$, or $\{1, 1\}$ on the condition that the data of the physical measurements are given. (The detailed algorithm is presented in Supplementary Information.) We call this decoding 'soft-decision' because real-valued quantities (probabilities) are used for the decoding. Furthermore, the correlation between b_x and b_z is taken into account as the joint probability $P(b_x, b_z)$, unlike the hard-decision decoding.

For this calculation, we must know the error probabilities for the physical measurements. Instead of estimating the error probabilities in each case, the decoder is designed such that it is optimal for the depolarizing channel with error probability of 19%, as mentioned in the Results section. Note that this calculation is efficiently performed (see Supplementary Information).

Obtaining $P(b_x, b_z)$, we decide the result of the Bell measurement as the value maximizing $P(b_x, b_z)$. Thus, the error correction by teleportation with the soft-decision decoding is achieved.

This decoding can easily be modified for error detection, as suggested by Poulin[25]. If the maximum probability obtained in the decoding is lower than a specific value set appropriately in advance, then the decoder outputs E ('error detected'). This error detection is useful for preparing encoded states by postselection. In fact, we have used this decoder in the state preparation and achieved the lower error probabilities of logical CNOT gates (see Supplementary Information for details).

Simulation methods. The simulation for the depolarizing channel is done as follows (see Supplementary Information for details). In this case, errors occur only on the channel (the other operations are perfect). First, a logical Bell pair is prepared. Next, the first logical qubit of the Bell pair is transmitted through the depolarizing channel, where depolarizing errors occur. After that, we correct the errors by teleportation. Then, the Bell pair is disentangled by a transversal CNOT gate. Finally, the two logical qubits are measured and decoded in an appropriate manner. If both the measurement results are 0, the decoding has succeeded. Otherwise, the decoding has failed.

The simulation for the logical CNOT gate is done as follows (see Supplementary Information for details). In this case, errors occur only on physical CNOT gates used in the logical CNOT gate the error probability of which is to be estimated (the other operations are perfect). First, two error-free logical Bell pairs are prepared. Next, an error-free transversal CNOT gate is performed on the first logical qubits of the two Bell pairs, which is followed by the noisy logical CNOT gate on the first logical qubits. Here the noisy logical CNOT gate is implemented by a noisy transversal CNOT gate followed by error correction by teleportation with noisy physical CNOT gates, as Knill did in Ref. 16. Finally, after the two Bell pairs are disentangled with two error-free transversal CNOT gates, the four logical qubits are measured and decoded in an appropriate manner. If all the measurement results are 0, the logical CNOT gate has succeeded. Otherwise, the logical CNOT gate has failed.

The simulators used in the present work are described in Supplementary Information.

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Author contributions

H. G. proposed and devised the soft-decision decoding for quantum error correction and detection by teleportation, performed all the simulations, and wrote the paper. H. U. suggested the application of soft-decision decoding to quantum error correction and devised the soft-decision decoding.

Additional information

Supplementary information accompanies this paper at http://www.nature.com/scientificreports

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CORRIGENDUM: Fault-tolerant quantum computation with a soft-decision decoder for error correction and detection by teleportation

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The Supplementary Information of this Article contains errors. The physical-qubit interchanges are unnecessary in Figures S9 and S10. The correct Figures S9 and S10 appear below as Figures 1 and 2 respectively. In addition the figure legends of S9 and S10 should not contain the sentence ``The qubit interchanges after the Hadamard gates come from the definition of the encoded Hadamard gate (see Fig. S1a)". Corresponding to this correction, in both the hard- and soft-decision decoding algorithms, the definition of the logical X gate, instead of the logical Z gate, is used for $\{m_{zj}: j=1, 2, ¼, 12\}$.

In addition the equations for the relative probabilities for $R_1^{(1)}(d_{x1}, d_{x2}, d_{z1}, d_{z2})$ and $R^{(2)}(d'_{x1}, d'_{x2}, d'_{z1}, d'_{z2})$ in Section IIIC (``Soft-decision decoding algorithm") contain errors.

In $R_1^{(1)}(d_{x1}, d_{x2}, d_{z1}, d_{z2})$:

$$\delta[l_{z1}+l_{z3}=d_{z1}]\delta[l_{z3}+l_{z4}=d_{z2}],$$

should read

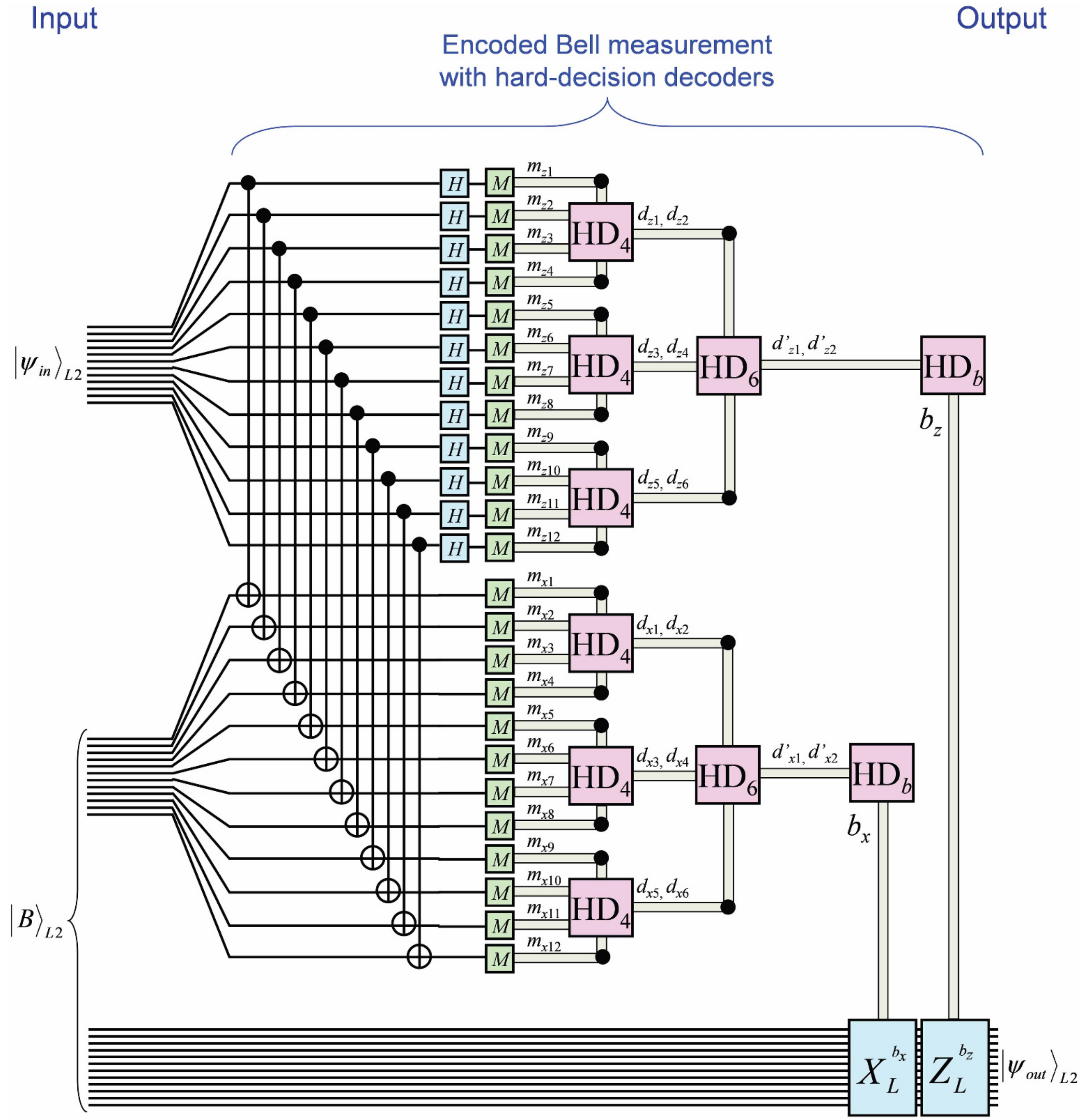
$$\delta[l_{z1}+l_{z2}=d_{z1}]\delta[l_{z2}+l_{z4}=d_{z2}],$$

And in $R^{(2)}(d'_{x1}, d'_{x2}, d'_{z1}, d'_{z2})$:

$$\times\,\delta[d_{x1}+d_{x3}+d_{x4}=d'_{x1}]\delta[d_{x4}+d_{x5}=d'_{x2}]$$
$$\times\,\delta[d_{z1}+d_{z3}+d_{z4}=d'_{z1}]\delta[d_{z4}+d_{z5}=d'_{z2}].$$

should read

$$\times\,\delta[d_{x3}+d_{x4}+d_{x6}=d'_{x1}]\delta[d_{x4}+d_{x5}=d'_{x2}]$$
$$\times\,\delta[d_{z2}+d_{z3}=d'_{z1}]\delta[d_{z1}+d_{z3}+d_{z4}=d'_{z2}],$$



Input
Output
Encoded Bell measurement
with hard-decision decoders
$|\psi_{in}\rangle_{L2}$
$|B\rangle_{L2}$
H
M
m_{z1} … m_{z12}
m_{x1} … m_{x12}
HD_4
HD_6
HD_b
d_{z1}, d_{z2}
d_{z3}, d_{z4}
d_{z5}, d_{z6}
d'_{z1}, d'_{z2}
d_{x1}, d_{x2}
d_{x3}, d_{x4}
d_{x5}, d_{x6}
d'_{x1}, d'_{x2}
b_z
b_x
$X_L^{b_x}$
$Z_L^{b_z}$
$|\psi_{out}\rangle_{L2}$

Figure 1 |

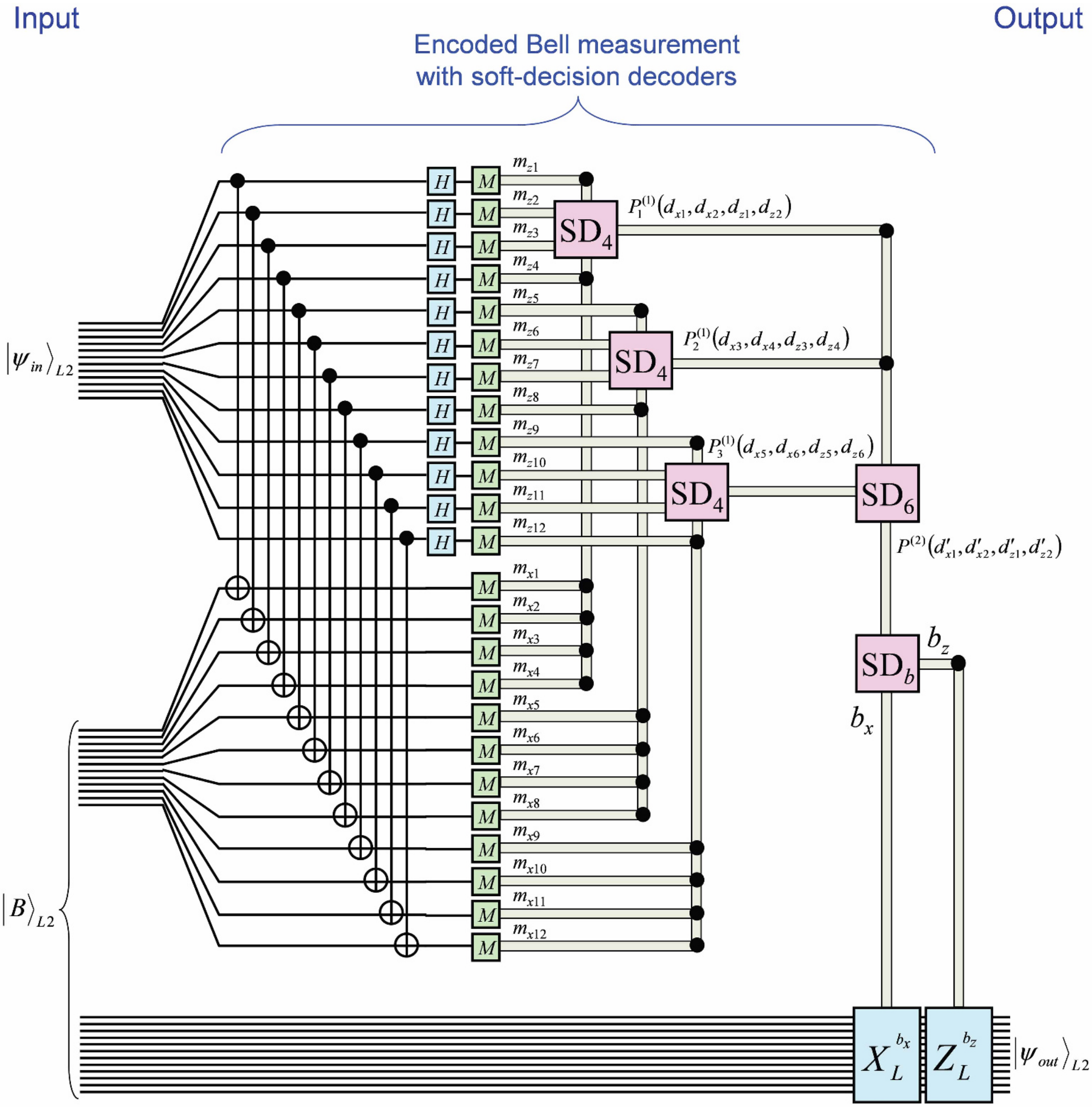

Input
Output
Encoded Bell measurement
with soft-decision decoders
$|\psi_{in}\rangle_{L2}$
$|B\rangle_{L2}$
H
M
m_{z1}
m_{z2}
m_{z3}
m_{z4}
m_{z5}
m_{z6}
m_{z7}
m_{z8}
m_{z9}
m_{z10}
m_{z11}
m_{z12}
m_{x1}
m_{x2}
m_{x3}
m_{x4}
m_{x5}
m_{x6}
m_{x7}
m_{x8}
m_{x9}
m_{x10}
m_{x11}
m_{x12}
SD_4
$P_1^{(1)}(d_{x1}, d_{x2}, d_{z1}, d_{z2})$
SD_4
$P_2^{(1)}(d_{x3}, d_{x4}, d_{z3}, d_{z4})$
$P_3^{(1)}(d_{x5}, d_{x6}, d_{z5}, d_{z6})$
SD_4
SD_6
$P^{(2)}(d'_{x1}, d'_{x2}, d'_{z1}, d'_{z2})$
SD_b
b_z
b_x
$X_L^{b_x}$
$Z_L^{b_z}$
$|\psi_{out}\rangle_{L2}$

Figure 2 |