UTTG-22-15

Necessary and Sufficient Condition on the Lindblad Equation to Prevent Entropy Decrease

Steven Weinberg*
Theory Group, Department of Physics, University of Texas
Austin, TX, 78712

Abstract

It is shown that in order for the solutions of the Lindblad equation never to give a decreasing von Neumann entropy, it is necessary and sufficient that the operators appearing in this equation should be unitary linear combinations of their adjoints. In this case, these operators may be replaced with Hermitian operators, without changing the evolution of density matrices.

*Electronic address: weinberg@physics.utexas.edu

It is well known in the theory of open systems that, under the condition of time-translation invariance, if general density matrices evolve linearly so as to remain Hermitian and with unit trace, then they must obey a differential equation of the form[1]

$$\frac{d}{dt}\rho(t) = -i[\mathcal{H},\rho(t)] + \sum_a \Delta_a \left[N_a\,\rho(t)\,N_a^\dagger - \frac{1}{2}N_a^\dagger N_a\,\rho(t) - \frac{1}{2}\rho(t)\,N_a^\dagger N_a\right] \tag{1}$$

where $\mathcal{H}$ is a Hermitian operator, N_a are various other linearly independent operators, not necessarily Hermitian, and Δ_a are non-zero real numbers, all of these operators and numbers independent of ρ and t. If the density matrix is positive at some time then it will remain positive at least for a finite range of future times if all Δ_a are positive, and under reasonable additional conditions this sufficient condition is also necessary. (Specifically, if any entangled density matrix for a compound system $\mathcal{S}\otimes\mathcal{S}$ consisting of two isolated copies of a system $\mathcal{S}$, each of which evolves according to Eq. (1), remains positive for a finite range of future times if it is positive at an initial time, then the linear mapping $\rho(t)\to\rho(t')$ of the density matrix of $\mathcal{S}$ for $t'>t$ in this range is completely continuous[2], which implies[3] that $\Delta_a>0$.) With $\Delta_a>0$, we can define operators $L_a\equiv\sqrt{\Delta_a}N_a$, and write Eq.(1) as the *Lindblad equation*[4]:

$$\frac{d}{dt}\rho(t) = -i[\mathcal{H},\rho(t)] + \sum_a \left[L_a\,\rho(t)\,L_a^\dagger - \frac{1}{2}L_a^\dagger L_a\,\rho(t) - \frac{1}{2}\rho(t)\,L_a^\dagger L_a\right] \tag{2}$$

With a suitable re-definition of $\mathcal{H}$, the L_a can be given any traces we like.** We will assume here that this has been done so that the L_a are traceless, which in the derivation of Eq. (1) seems the most natural choice.

The Lindblad equation applies in such general circumstances, that it is used not only in the theory of open systems, but also for closed systems in theories that attempt to go beyond ordinary quantum mechanics.[5] In the present paper we will not need to specify whether we are dealing with open systems in ordinary quantum mechanics or with general systems in extended versions of quantum mechanics.

I wish here to address the question whether there is any physical reason why the operators L_a must in general be Hermitian. As described in ref. [6], the Lindblad equation with L_a Hermitian exhibits interesting late-time

**The effect on the evolution of the density matrix of adding a term $\xi_a\mathbf{1}$ to each L_a (with arbitrary complex numbers ξ_a) is cancelled if we add a Hermitian term $-(1/2i)\sum_a(\xi_a L_a^\dagger - \xi_a^* L_a)$ to $\mathcal{H}$.

behavior, reminiscent of what is supposed to happen in a measurement, but that does not in itself provide a reason why the L_a must be Hermitian.

A reason may be found in the requirement the the von Neumann entropy must never decrease. It has already been shown by Banks, Peskin, and Susskind that requiring the L_a to be Hermitian is sufficient to give a non-decreasing entropy,[7] but they mentioned that they did not know what condition is necessary. It is shown here that a necessary and sufficient condition for the entropy never to decrease for an arbitrary initial density matrix is that the L_a must be unitary linear combinations of their adjoints. This has the consequence that the L_a may be replaced by other matrices that are Hermitian, without affecting the evolution of any density matrix.

The von Neumann entropy for a given density matrix ρ is defined by

$$S[\rho] \equiv -\mathrm{Tr}\Big(\rho \ln \rho\Big) \ . \tag{3}$$

Its rate of change is

$$\frac{d}{dt} S[\rho(t)] = -\mathrm{Tr}\left(\frac{d\rho(t)}{dt}\left[1 + \ln \rho(t)\right]\right) \ .$$

or, using the constancy of the trace,

$$\frac{d}{dt} S[\rho(t)] = -\mathrm{Tr}\left(\frac{d\rho(t)}{dt} \ln \rho(t)\right) \ . \tag{4}$$

Using Eq. (2), and from now on dropping the time argument, this is

$$\dot{S}[\rho] = -\mathrm{Tr}\left(\sum_a \Big[L_a \rho L_a^\dagger - \rho L_a^\dagger L_a\Big] \ln \rho\right) \ . \tag{5}$$

(The first term on the right in Eq. (2) does not contribute here, because $\mathrm{Tr}\Big([\mathcal{H}, \rho] \ln \rho\Big) = \mathrm{Tr}\Big(\mathcal{H}[\rho, \ln \rho]\Big) = 0$.) Because any density matrix is Hermitian, it can be put in the form

$$\rho = UPU^\dagger \tag{6}$$

where U is unitary, and P is diagonal, and like ρ Hermitian and positive with unit trace:†

$$P_{ij} = p_i \delta_{ij} \ , \qquad 0 \le p_i \le 1 \ , \qquad \sum_i p_i = 1 \tag{7}$$

†We take the Hilbert space to have a finite dimensionality d, so that indices i, j, k, l, m, n run over d values. It is expected that the considerations of this paper can be extended to infinite dimensional Hilbert spaces, as long as traces are well-defined.

Eq. (5) may then be written

$$\dot{S}[\rho] = -\sum_{ij} M^U_{ij} p_j [\ln p_i - \ln p_j] \ , \tag{8}$$

where

$$M^U_{ij} \equiv \sum_a \left| \left(U^\dagger L_a U \right)_{ij} \right|^2 \tag{9}$$

If M^U_{ij} is symmetric for all U, then for any density matrix we can rewrite Eq. (8) as

$$\dot{S}[\rho] = \frac{1}{2} \sum_{ij} M^U_{ij} \left(p_i - p_j \right) \left(\ln p_i - \ln p_j \right) \ , \ . \tag{10}$$

The function $\ln p$ increases monotonically, so $\left(p_i - p_j\right)(\ln p_i - \ln p_j)$ is positive for all p_i and p_j, and so $\dot{S}$ is positive for all U and all p_i — that is, for all ρ. Thus taking M^U_{ij} to be symmetric for all U is a sufficient condition for $\dot{S}[\rho]$ to be positive for any density matrix ρ. In particular, if the L_a are Hermitian, then $(U^\dagger L_a U\Big)^*_{ij} = (U^\dagger L_a U\Big)_{ji}$, so M^U_{ij} is symmetric for all U, and the entropy does not decrease. But although taking the L_a to be Hermitian is a sufficient condition for non-decreasing entropy, as we shall see, it is not a necessary condition.

Now let us find a *necessary* condition for $\dot{S}[\rho]$ to be positive for any possible density matrix. We first show that it is necessary as well as sufficient that M^U_{ij} should be symmetric for all U. Suppose it were not symmetric for some U. Then there would be at least one pair of indices k and l for which $M^U_{kl} > M^U_{lk}$. Consider a density matrix of the form $UPU^\dagger$, with P diagonal and with the only non-zero diagonal elements p_k and p_l. Then Eq. (8) gives

$$\dot{S}[\rho] = -M^U_{kl}\, p_l \left(\ln p_k - \ln p_l \right) - M^U_{lk}\, p_k \left(\ln p_l - \ln p_k \right) .$$

We can rewrite this as

$$\dot{S}[\rho] = \frac{1}{2} \left(M^U_{kl} + M^U_{lk} \right) (p_k - p_l) \ \ln \left(\frac{p_k}{p_l} \right) - \frac{1}{2} \left(M^U_{kl} - M^U_{lk} \right) (p_k + p_l) \ \ln \left(\frac{p_k}{p_l} \right) \ .$$

As long as $M^U_{kl} > M^U_{lk}$, it is always possible to find positive p_k and p_l with $p_k > p_l$ and $p_k + p_l = 1$ that are close enough together so that

$$\frac{p_k + p_l}{p_k - p_l} > \frac{M^U_{kl} + M^U_{lk}}{M^U_{kl} - M^U_{lk}}$$

in which case $\dot{S}[UPU^\dagger] < 0$. Hence the symmetry of M^U_{ij} for all i, j, and U is indeed necessary to have non-decreasing entropy for all density matrices, as was to be shown.

We still need to translate this into a necessary and sufficient condition on the L_a themselves. The symmetry condition for M^U_{ij} may be written

$$\sum_{klmn} U^\dagger_{ik} U_{lj} U^\dagger_{jm} U_{ni} X_{klmn} = 0 \;, \tag{11}$$

where

$$X_{klmn} \equiv \sum_a \Big[(L_a)_{kl} (L^\dagger_a)_{mn} - (L^\dagger_a)_{kl} (L_a)_{mn} \Big] \;. \tag{12}$$

But it would be premature to conclude from this that X_{klmn} must vanish, because even though the unitary U_{ij} are completely unconstrained, the coefficient of X_{klmn} in Eq. (11) is not. To deal with this, suppose we introduce two Hermitian operators A and B that commute with each other but are otherwise unconstrained. They can be simultaneously diagonalized — that is, we can find a unitary U_{ij} and sets of numbers a_i and b_j, such that

$$A_{nk} = \sum_i U_{ni} a_i U^\dagger_{ik} \;, \qquad B_{lm} = \sum_j U_{lj} b_j U^\dagger_{jm} \;.$$

Using this U in Eq. (11) and contracting with $a_i b_j$ then gives

$$\sum_{klmn} A_{nk} B_{lm} X_{klmn} = 0 \;. \tag{13}$$

Now, A and B are subject to no linear constraints except that they commute, so if we put no constraints at all on A and B then the left-hand side of Eq. (13) would have to be a linear combination of components of the commutator. That is, for unconstrained A and B, we must have

$$\sum_{klmn} A_{nk} B_{lm} X_{klmn} = \mathrm{Tr}\Big([A, B] C \Big) \;,$$

for some operator C. Equating coefficients of A_{nk} and B_{lm} on both sides of this equation then gives

$$X_{klmn} = \delta_{kl} C_{mn} - \delta_{mn} C_{kl} \;. \tag{14}$$

Recall that we have defined the Hamiltonian $\mathcal{H}$ (as we can) so that the L_a are all traceless. Then, contracting the indices k and l in Eq. (14), we have

$$0 = \sum_k X_{kkmn} = C_{mn} d - \delta_{mn} \sum_k C_{kk} \;, \tag{15}$$

so $C_{mn} \propto \delta_{mn}$. But then using this in Eq. (14), we have $X_{klmn} = 0$, or explicitly

$$\sum_a (L_a)_{kl}(L_a^\dagger)_{mn} = \sum_a (L_a^\dagger)_{kl}(L_a)_{mn} \ . \tag{16}$$

The L_a are linearly independent, so there exist an equal number of dual operators D_a, such that

$$\mathrm{Tr}\Big(L_a^\dagger D_b\Big) = \delta_{ab} \tag{17}$$

Contracting Eq. (16) with $(D_b)_{nm}$ then gives

$$L_b = \sum_a V_{ba} L_a^\dagger \ , \tag{18}$$

where $V_{ba} = \mathrm{Tr}(L_a D_b)$. Inserting Eq. (18) and its adjoint on the left-hand side of the necessary condition Eq. (16) gives

$$\sum_{abc} V_{ab} V_{ac}^* (L_b^\dagger)_{kl}(L_c)_{mn} = \sum_a (L_a^\dagger)_{kl}(L_a)_{mn} \ .$$

Since the L_a are independent, this shows that $\sum_a V_{ab} V_{ac}^* = \delta_{bc}$, or in other words

$$V^\dagger V = 1 \ . \tag{19}$$

Thus it is necessary for the non-decrease of entropy that the L_a should be linearly related to their adjoints, as in Eq. (18), with unitary matrix V_{ba}.

We can easily see that this condition is also sufficient for the the non-decrease of entropy. Using Eq. (18) and then Eq. (19) in the definition (9), we see that

$$\begin{aligned} M_{ij}^U &= \sum_{abc} V_{ab} V_{ac}^* \, (U^\dagger L_b^\dagger U)_{ij} (U^\dagger L_c^\dagger U)_{ij}^* \\ &= \sum_b (U^\dagger L_b^\dagger U)_{ij} (U^\dagger L_b^\dagger U)_{ij}^* = \sum_b (U^\dagger L_b U)_{ji}^* (U^\dagger L_b U)_{ji} = M_{ji}^U \end{aligned}$$

so Eqs. (18) and (19) imply that M_{ij}^U is symmetric for arbitrary unitary U, which as we have seen is a sufficient condition for the non-decrease of entropy.

Where Eqs. (18) and (19) are satisfied, we can replace the L_a with matrices L'_a that are Hermitian, without changing the evolution of the density matrix. First note that the adjoint of Eq. (18) gives $L_a^\dagger = \sum_c V_{ac}^* L_c$, and inserting this back in Eq. (18) then gives $L_b = \sum_c (VV^*)_{bc} L_c$, so since the L_a are independent, we conclude that

$$VV^* = 1 \ . \tag{20}$$

Now, define [††]

$$L'_a \equiv \sum_b (V^{1/2})^*_{ab} L_b \ . \tag{21}$$

According to Eqs. (18) and (19),

$$L'_a = \sum (V^{*\,1/2} V)_{ab} L^\dagger_b = \sum_b (V^{1/2})_{ab} L^\dagger_b = \left[\sum_b (V^{1/2})^*_{ab} L_b\right]^\dagger = L'^\dagger_a$$

so the L'_a are Hermitian. Also, in consequence of the unitarity of V, we have

$$\sum_a (L'_a)_{kl} (L'^\dagger_a)_{mn} = \sum_{abc} V^{*\,1/2}_{ab} V^{1/2}_{ac} (L_b)_{kl} (L^\dagger_c)_{mn} = \sum_b (L_b)_{kl} (L^\dagger_b)_{mn}$$

so the L'_a are equivalent to the L_a, in the sense that the evolution equation can be written in terms of the L'_a in the same way as in terms of the L_a:

$$\frac{d}{dt}\rho(t) = -i[\mathcal{H}, \rho(t)] + \sum_a \left[L'_a\,\rho(t)\,L'^\dagger{}_a - \frac{1}{2} L'^\dagger{}_a L'_a\,\rho(t) - \frac{1}{2}\rho(t)\,L'^\dagger_a L'_a\right] \tag{22}$$

Hence if we require that the von Neumann entropy must never decrease, then we can choose the matrices in the Lindblad equation to be Hermitian with no loss of generality,

Incidentally, if we had not defined the Hamiltonian $\mathcal{H}$ so that the L_a are traceless, then we could not have concluded that the L_a must be related to their adjoints by a linear transformation, as in Eq. (18). We can see this directly in Eq. (5); a shift in L_a by a term ξ_a times the unit matrix does not affect the rate of change of entropy, even if the ξ_a are complex. (This remark is obvious, because Eq. (5) shows that the rate of change of entropy does not depend on $\mathcal{H}$, so since changing the L_a by terms proportional to the unit matrix can be compensated by a change in $\mathcal{H}$. it can have no effect on the evolution of entropy.) It is still true that we can choose the matrices in the Lindblad equation to be Hermitian without loss of generality, but in general we must define $\mathcal{H}$ so that $\mathrm{Tr} L_a$ vanishes (or is real), and then make the linear transformation (21), .

[††]The definition of $V^{1/2}$ is of course not unique. Any matrix for which $V^{1/2}V^{1/2} = V$ will do.

Acknowledgments

This material is based upon work supported by the National Science Foundation under Grant Number PHY-1316033 and with support from The Robert A. Welch Foundation, Grant No. F-0014.

---

1. For a straightforward derivation, see P. Pearle, Eur. J. Phys. **33**, 805 (2012).

2. F. Benatti, R. Floreanini, and R. Romano, J. Phys. A Math. Gen. **35**, L351 (2002), For complete continuity, see W. F. Stinnespring, Proc. Am. Math. Soc. **6**, 211 (1955); M. D. Choi, J. Canada Math. **24**, 520 91972).

3. M. D. Choi, it Linear Algebra and its Applications **10**285 (1975).

4. G. Lindblad, Commun. Math. Phys. **48**, 119 (1976); V. Gorini, A. Kossakowski and E. C. G. Sudarshan, J. Math. Phys. **17**, 821 (1976).

5. G. C. Ghirardi, A. Rimini, and T. Weber, Phys. Rev. D **34**, 470 (1986); P. Pearle, Phys. Rev. A **39**, 2277 (1989), and in *Quantum Theory: A Two-Time Success Story* (Yakir Aharonov Festschrift), eds. D. C. Struppa & J. M. Tollakson (Springer, 2013), Chapter 9. [arXiv:1209.5082]

6. This was added in the second edition of S. Weinberg, *Lectures on Quantum Mechanics* (Cambridge University Press, Cambridge, UK, 2015), Sec. 6.9.

7. T. Banks, M. Peskin, and L. Susskind, Nuclear Phys. B **244**, 125 (1984). A modified proof is given below.