THE VERLINDE ALGEBRA AND THE COHOMOLOGY OF THE GRASSMANNIAN

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ABSTRACT

The article is devoted to a quantum field theory explanation of the relationship between the Verlinde algebra of the group U(k) at level N-k and the "quantum" cohomology of the Grassmannian of complex k planes in N space. In §2, I explain the relation between the Verlinde algebra and the gauged WZW model of G/G; in §3, I describe the quantum cohomology and its origin in a quantum field theory; and in §4, I present a path integral argument for mapping between them.

[★] Research supported in part by NSF Grant PHY92-45317.

1. Introduction

My main goal in these lecture notes will be to elucidate a formula of Doron Gepner [1], which relates two mathematical objects, one rather old and one rather new. Along the way we will consider a few other matters as well.

The old structure is the cohomology ring of the Grassmannian G(k, N) of complex k planes in N space – except that one considers the quantum cohomology (or Floer instanton homology) rather than the classical cohomology. The new structure is the Verlinde algebra, which computes the Hilbert polynomial of the moduli space of vector bundles on a curve. Gepner's formula, as we will consider it here, says that the quantum cohomology ring of G(k, N) coincides with the Verlinde algebra of the group U(k) essentially at level N - k.

Gepner discovered his formula by computing the left and right hand side and observing that they were equal. We will seek a more conceptual explanation, by representing the quantum cohomology ring of the Grassmannian in a quantum field theory and reducing that quantum field theory at low energies to another quantum field theory which is known to compute the Verlinde algebra.

The Verlinde formula appears in several quantum field theories. Of these, the one that is relevant here is the gauged WZW model, of G/G. The shortest and most complete explanation of its relation to the Verlinde formula is due to Gerasimov [4], and I will explain his argument in §2. Part of the charm of the G/G model is that it can be abelianized, that is, reduced to a theory in which the gauge group is the maximal torus of G, extended by the Weyl group [5]. The argument is simple in concept and will be summarized in §2.6.

In §3, I explain at a qualitative level how the quantum cohomology of the Grass-

[†] Gepner actually discusses the classical cohomology of the Grassmannian and identifies it with a close cousin of the Verlinde algebra. The refinement of Gepner's formula that we will consider was conjectured by Vafa [2] and Intriligator [3].

[‡] Actually if one decomposes the Lie algebra of U(k) as $su(k) \times u(1)$, then the level is (N - k, N), that is level N - k for the su(k) factor and level N for the u(1) factor. The source of this subtlety will become clear in §4.6.

mannian is represented in a quantum field theory, and some general techniques for studying this field theory and reducing it to a problem in gauge theory. In §4, I describe the arguments in more technical detail. The analysis actually should be adaptable to other manifolds that can be realized as symplectic quotients of linear spaces, such as flag manifolds and toric varieties. (The quantum cohomology of a toric variety has been studied by Batyrev [6]; that of a general flag manifold has apparently not yet been studied.)

This paper, despite its length, is based on an idea that can be described very simply. The two dimensional supersymmetric sigma model with target G(k, N)can be described as a U(k) gauge theory (in $\mathcal{N}=2$ superspace) with N multiplets of chiral superfields in the fundamental representation of U(k). It was studied from this point of view in the case of k=1 (that is \mathbb{CP}^{N-1}) many years ago [7,8], and the generalization to arbitrary k is also familiar [9,10]. At low energy, a suitable U(k) gauge theory with the matter content just stated reduces to the supersymmetric sigma model of the Grassmannian. On the other hand, integrating out the N matter multiplets, one gets an effective action for the U(k) gauge multiplet. Because of a sort of mixing between scalars and vectors, this low energy effective action has no massless particles; this is how the presence of a mass gap has been shown in the past. The novelty in the present paper is simply the observation that the low energy effective action is in fact a gauged WZW model of U(k)/U(k). Under this low energy reduction, the topological correlation functions of the sigma model – which compute the quantum cohomology of G(k, N) – are mapped into correlation functions of the U(k)/U(k) model that can be computed (as we recall in §2) in terms of the Verlinde algebra. This gives the map between the two theories.

The quantum cohomology of the Grassmannian has also been studied – using, more or less, a classical version of the same setup we will follow – by Bertram, Daskapoulos, and Wentworth [11]. And there is a forthcoming mathematical approach to Gepner's formula in work of Braam and Agnihorti. The cohomology of the Grassmannian is closely related to the chiral ring of a certain $\mathcal{N}=2$ superconformal field theory [12] (somewhat misleadingly called a $U(N)/U(k) \times U(N-k)$

coset model); this model probably should be included in the story, but that will not be done here. Some of the phenomena we will study have analogs for real and symplectic Grassmannians, as in [13] and the second paper cited in [1]; it would be interesting to try to extend the analysis for those cases.

§2 and §3 can be read independently of one another. §4 requires more familiarity with methods of physics than either §2 or §3. Physicists may want to start with §4.

2. The Verlinde Formula And The G/G Model

First of all, the Verlinde algebra counts theta functions, such as the classical theta functions of Jacobi and their generalizations. In modern language, the classical theta functions can be described as follows. Let \mathcal{T} be a complex torus of dimensions g, \mathcal{L} a line bundle defining a principal polarization, and s a positive integer. Then the space of level s theta functions is $H^0(\mathcal{T}, \mathcal{L}^{\otimes s})$. The dimension of this space can be readily determined from the Riemann-Roch theorem.

For example, \mathcal{T} might be the Jacobian \mathcal{J} of a complex Riemann surface Σ , that is, the moduli space of holomorphic line bundles over Σ of some given degree. This example suggests the generalization to the "non-abelian theta functions" of A. Weil. Here one replaces the Jacobian of Σ by the moduli space \mathcal{R} of rank k (stable) holomorphic vector bundles over Σ ; now a "non-abelian theta function" at level s is an element of $H^0(\mathcal{R}, \mathcal{L}^{\otimes s})$. Though the Riemann-Roch theorem gives a formula for the dimension of this space, this formula is difficult to use in the non-abelian case as it involves invariants of \mathcal{R} that are not easy to determine directly.

The Verlinde algebra gives on the other hand a practical formula for the dimension of $H^0(\mathcal{R}, \mathcal{L}^{\otimes s})$; the formula was described very explicitly by Raoul Bott in [14]. Roughly speaking, the origin of the Verlinde formula in differential geometry is as follows. Via Hodge theory, \mathcal{R} is endowed with a natural Kahler metric. As the dimension of the space of non-abelian theta functions is independent of

the complex structure of Σ , one can choose the complex structure to simplify the problem. It is convenient to take Σ to be a nearly degenerate surface consisting of three-holed spheres joined by long tubes. Using the behavior of the differential geometry of \mathcal{R} in this limit, one can write $H^0(\mathcal{R}, \mathcal{L}^{\otimes s})$ as a sum of tensor products of similar spaces for a three-holed sphere with some branching around the holes. The Verlinde algebra encodes the details of this.

The Verlinde algebra arises in several quantum field theories:

- (1) It originally arose [15] in the WZW model, a conformal field theory (whose Lagrangian we will recall later) that governs maps from a Riemann surface Σ to a compact Lie group G. Mathematically, this model is related to representations of affine Lie algebras, the unitary action of the modular group $SL(2, \mathbf{Z})$ on their characters, etc.
- (2) The Verlinde formula is an important ingredient in understanding Chern-Simons gauge theory on a three-manifold. Thus it is relevant to the knot and three-manifold invariants constructed from quantum field theory.
- (3) The Verlinde formula also enters in the gauged WZW model, governing a pair (g, A), where A is a connection on a principal G bundle P over a Riemann surface, and g is a section of $P \times_G G$ (where G acts on itself via the adjoint action).

Of these it is the third – which was discovered most recently – that will enter our story. The present section is therefore mainly devoted to an explanation – following Gerasimov [4], who reinterpreted earlier formulas [16,17] – of the gauged WZW model and its relation to nonabelian theta functions.

2.1. Gauge Theory And The Prequantum Line Bundle In Two Dimensions

Let G be a compact Lie group, Σ a closed oriented two-manifold without boundary, and P a principal G bundle over Σ . To achieve some minor simplifications in the exposition, I will suppose G simple, connected, and simply connected. (Notation aside, the only novelty required to treat a general compact Lie group is that more care is required in defining the functional $\Gamma(g, A)$ that appears below; see [18, §4].) One consequence of the assumption about G is that P is trivial.

Let \mathcal{A} be the space of connections on P. \mathcal{A} has a natural symplectic structure ω that can be defined with no choice of metric or complex structure on Σ . (This and some other facts that I summarize presently are originally due to Atiyah and Bott [20].) The symplectic structure can be defined by the formula

$$\omega(a_1, a_2) = \frac{1}{2\pi} \int_{\Sigma} \operatorname{Tr} a_1 \wedge a_2, \qquad (2.1)$$

where a_1 and a_2 are adjoint-valued one-forms representing tangent vectors to \mathcal{A} . Here Tr is an invariant quadratic form on the Lie algebra of G, defined for G = SU(k) to be the trace in the k dimensional representation; in general one can take Tr to be the smallest positive multiple of the trace in the adjoint representation such that the differential form Θ introduced below has periods that are multiples of 2π .

A prequantum line bundle \mathcal{L} over \mathcal{A} is a unitary line bundle with a connection of curvature $-i\omega$. \mathcal{L} exists and is unique up to isomorphism since \mathcal{A} is an affine space. We can take \mathcal{L} to be the trivial bundle with a connection defined by the following formula:

$$\frac{D}{DA_i} = \frac{\delta}{\delta A_i} + \frac{i}{4\pi} \epsilon^{ij} A_j. \tag{2.2}$$

 (ϵ^{ij}) is the Levi-Civita antisymmetric tensor; when local complex coordinates are introduced, we will take $\epsilon^{z\overline{z}} = -\epsilon^{\overline{z}z} = i$.) The k^{th} power $\mathcal{L}^{\otimes k}$ is therefore the trivial bundle endowed with the connection

$$\frac{D}{DA_i} = \frac{\delta}{\delta A_i} + \frac{ik}{4\pi} \epsilon^{ij} A_j. \tag{2.3}$$

Let \widehat{G} be the group of gauge transformations. If P is trivialized, a gauge

transformation is a map $g: \Sigma \to G$, and acts on the connection A by

$$A \to A^g = gAg^{-1} - dg \cdot g^{-1}.$$
 (2.4)

More invariantly, g is a section of $P \times_G G$, where G acts on itself in the adjoint representation, and the action of g on \mathcal{A} should be written as

$$d_A \to g d_A g^{-1}, \tag{2.5}$$

with d_A the gauge-covariant extension of the exterior derivative. At the Lie algebra level this is

$$A \to A - d_A \alpha,$$
 (2.6)

where α is a section of $P \times_G \mathbf{g}$, with \mathbf{g} being the Lie algebra of G, on which G acts by the adjoint action.

The action of the gauge group on the space \mathcal{A} of connections lifts to an action on the prequantum line bundle. At the Lie algebra level, the lift is generated by the operators

$$D_i \frac{D}{DA_i} - \frac{ik}{4\pi} \epsilon^{ij} F_{ij}, \tag{2.7}$$

with $F = dA + A \wedge A$ the curvature form. (2.7) means very concretely that the infinitesimal gauge transformation (2.6) is represented on sections of \mathcal{L} by the operator

$$\int_{\Sigma} \operatorname{Tr} \alpha \left(D_i \frac{D}{DA_i} - \frac{ik}{4\pi} F \right). \tag{2.8}$$

Even globally, at the group level, the \widehat{G} action on \mathcal{L} can be described rather explicitly [21]. Pick a three manifold B with $\partial B = \Sigma$. Given $g: \Sigma \to G$, extend g to a map (which I will also call g) from B to G. (The extension exists because of

our assumption that $\pi_0(G) = \pi_1(G) = 0$. See [19,18] for the definition of Γ without this simplifying assumption.) Define as in [22]

$$\Gamma(g) = \frac{1}{12\pi} \int_{B} \operatorname{Tr} g^{-1} dg \wedge g^{-1} dg \wedge g^{-1} dg, \qquad (2.9)$$

which is known as the Wess-Zumino anomaly functional [23]. This is equivalent to

$$\Gamma(g) = \int_{B} g^{*}(\Theta), \qquad (2.10)$$

where

$$\Theta = \frac{1}{12\pi} \operatorname{Tr} g^{-1} dg \wedge g^{-1} dg \wedge g^{-1} dg \qquad (2.11)$$

is a left- and right-invariant closed three-form on G. The periods of Θ are multiples of 2π . This ensures that, regarded as a map to $\mathbf{R}/2\pi\mathbf{Z}$, $\Gamma(g)$ depends only on the restriction of g to Σ . Note that Γ is defined purely in differential topology; no metric or complex structure on Σ is required.

It follows rather directly from the definition of Γ that for $g, h : \Sigma \to G$,

$$\Gamma(gh) = \Gamma(g) + \Gamma(h) - \frac{1}{4\pi} \int_{\Sigma} \operatorname{Tr} g^{-1} dg \wedge dh \cdot h^{-1}.$$
 (2.12)

A variant of this equation is called the Polyakov-Wiegmann formula [24].

Now, given a connection A on P, set

$$W(g,A) = \Gamma(g) - \frac{1}{4\pi} \int_{\Sigma} \operatorname{Tr} A \wedge g^{-1} dg.$$
 (2.13)

From (2.12), it follows almost immediately that

$$W(gh, A) = W(g, A^h) + W(h, A).$$
(2.14)

From this we can define an action of the gauge group \widehat{G} on the space of functions

of A. In fact, setting

$$g^*\chi(A) = \exp(ikW(g, A)) \cdot \chi(A^g), \tag{2.15}$$

we have $(gh)^* = h^*g^*$. Differentiating (2.15) with respect to g at g = 1, one sees that this particular lift induces (2.7) at the Lie algebra level; so (2.15) is the desired lifting of the action of the gauge group to an action on the prequantum line bundle \mathcal{L} .

2.2. Non-Abelian Theta Functions

So far we have considered Σ simply as a closed, oriented two-manifold without boundary. If one picks a complex structure on Σ , some additional interesting constructions can be made [20]. A complex structure on Σ induces a complex structure on the space \mathcal{A} of connections. One simply declares that the (0,1) part of A is holomorphic and the (1,0) part is antiholomorphic. If z, \overline{z} are local complex coordinates on Σ , then the connection (2.3) characterizing $\mathcal{L}^{\otimes k}$ can be written

$$\frac{D}{DA_z} = \frac{\delta}{\delta A_z} - \frac{k}{4\pi} A_{\overline{z}}$$

$$\frac{D}{DA_{\overline{z}}} = \frac{\delta}{\delta A_{\overline{z}}} + \frac{k}{4\pi} A_z.$$
(2.16)

The complex structure on \mathcal{A} can be described in very down-to-earth terms by saying that a holomorphic function on \mathcal{A} is a function annihilated by $\delta/\delta A_z$. Correspondingly, a holomorphic section of $\mathcal{L}^{\otimes k}$ is a section annihilated by D/DA_z . Even more explicitly, a holomorphic section of $\mathcal{L}^{\otimes k}$ is a function $\chi(A_z, A_{\overline{z}})$ which can be written

$$\chi(A_z, A_{\overline{z}}) = \exp\left(\frac{k}{4\pi} \int_{\Sigma} \operatorname{Tr} A_z A_{\overline{z}}\right) \cdot \widehat{\chi}(A_{\overline{z}}),$$
(2.17)

with $\widehat{\chi}(A_{\overline{z}})$ an ordinary holomorphic function on \mathcal{A} .

Once a complex structure is picked on Σ , the connection A determines operators $\overline{\partial}_A$ giving complex structures to vector bundles $P \times_G \mathbf{r}$, with \mathbf{r} a representation of G. The action of gauge transformations on A can be described by the action on the $\overline{\partial}_A$ operators:

$$\overline{\partial}_A \to g \cdot \overline{\partial}_A \cdot g^{-1}.$$
 (2.18)

Since this formula makes sense for complex g, the \widehat{G} action on \mathcal{A} extends to an action of the complexified gauge group $\widehat{G}_{\mathbf{C}}$ (consisting of maps of Σ to the complexification $G_{\mathbf{C}}$ of G).

Two $\overline{\partial}_A$ operators define equivalent holomorphic bundles if and only if they are related as in (2.18). So the quotient $\mathcal{A}/G_{\mathbf{C}}$ (in case the $G_{\mathbf{C}}$ action is not free, the quotient must be taken in the sense of geometric invariant theory) is the same as the moduli space \mathcal{R} of (stable) holomorphic principal G bundles over Σ .

The formulas used to describe the lift of the \widehat{G} action to \mathcal{L} make sense when g is complex, so we get a lift of the $\widehat{G}_{\mathbf{C}}$ action to \mathcal{L} . One defines a line bundle over \mathcal{R} — which we will also call \mathcal{L} — whose sections over an open set $U \subset \mathcal{R}$ are the same as the $\widehat{G}_{\mathbf{C}}$ -invariant sections of \mathcal{L} over the inverse image of U in \mathcal{A} .

Non-Abelian Theta Functions

The space of non-abelian theta functions, at level k, is $H^0(\mathcal{R}, \mathcal{L}^{\otimes k})$. From what has just been said, this is the same as the \widehat{G} -invariant (or equivalently, $\widehat{G}_{\mathbf{C}}$ -invariant) subspace $\mathcal{H}^{\widehat{G}}$ of $\mathcal{H} = H^0(\mathcal{A}, \mathcal{L}^{\otimes k})$. We want to determine the dimension of $\mathcal{H}^{\widehat{G}}$.

The strategy, as in [4], will be as follows. We will find a very convenient description of the action of \widehat{G} on \mathcal{H} . In fact, for $g \in \widehat{G}$, we will find an explicit integral kernel K(A, B; g) (with $A, B \in \mathcal{A}$) such that for $\chi \in \mathcal{H}$,

$$g^*\chi(A) = \int DB \ K(A, B; g)\chi(B).$$
 (2.19)

(For fixed g, K(A, B; g) is a section of $p_1^*(\mathcal{L}^{\otimes k}) \otimes p_2^*(\mathcal{L}^{\otimes (-k)})$ over $\mathcal{A} \times \mathcal{A}$, with p_1 and

 p_2 being the two projections to \mathcal{A} .) Here DB is the natural symplectic measure, normalized in a way that will be specified in §2.4, on the symplectic manifold \mathcal{A} .

Now the projection operator $\Pi: \mathcal{H} \to \mathcal{H}^{\widehat{G}}$ can be written

$$\Pi = \frac{1}{\operatorname{vol}(\widehat{G})} \int_{\widehat{G}} Dg \ g^*. \tag{2.20}$$

Here formally Dg is a Haar measure on \widehat{G} and $\operatorname{vol}(\widehat{G})$ is the volume of \widehat{G} computed with the same measure. The dimension of $\mathcal{H}^G = H^0(\mathcal{R}, \mathcal{L}^{\otimes k})$ is the same as $\operatorname{Tr} \Pi$, and so can evidently be written

$$\dim H^0(\mathcal{R}, \mathcal{L}^{\otimes k}) = \frac{1}{\operatorname{vol}(\widehat{G})} \int Dg \ DA \quad K(A, A; g). \tag{2.21}$$

It remains to construct a suitable kernel K. This will be done using gauged WZW models.

2.3. Gauged WZW Models

Let Σ be a complex Riemann surface; the complex structure determines the Hodge duality operator * on one-forms. For a map $g: \Sigma \to G$, the WZW functional is

$$I(g) = -\frac{1}{8\pi} \int_{\Sigma} \operatorname{Tr} g^{-1} dg \wedge *g^{-1} dg - i\Gamma(g), \qquad (2.22)$$

where $\Gamma(g)$ was defined in (2.9). While $\Gamma(g)$ is defined purely in differential topology, the first term in the definition of I(g) depends on the complex structure of Σ through the * operator. The quantum field theory with Lagrangian L(g) = kI(g) is conformally invariant and describes level k highest weight representations of the loop group of G and the action of the modular group on their characters.

 $G \times G$ acts on G by left and right multiplication $(g \to agb^{-1})$. Let us denote this copy of $G \times G$ as $G_L \times G_R$ with G_L and G_R acting on the left and right

respectively. I(g) is invariant under $G_L \times G_R$. We want to pick a subgroup $H \subset G_L \times G_R$ and construct a gauge invariant extension of I(g) with gauge group H. What this means is that we introduce a principal H bundle P, with connection A, and we replace the map $g: \Sigma \to G$ by a section of the bundle $P \times_H G$; here G is understood as the trivial principal G bundle over Σ , and H acts on G via its chosen embedding in $G_L \times G_R$. We want to construct a natural, gauge invariant functional I(g,A) that reduces at A=0 to I(g).

There is no problem in constructing a gauge invariant extension of the first term in (2.22). One simply replaces the exterior derivative by its gauge-covariant extension:

$$-\frac{1}{8\pi} \int_{\Sigma} \operatorname{Tr} g^{-1} d_A g \wedge *g^{-1} d_A g. \tag{2.23}$$

On the other hand, there is a topological obstruction to constructing a gauge invariant extension of $\Gamma(g)$. The requirement is that the class in $H^3(G, \mathbf{Z})$ that determines the functional Γ (and which in real cohomology is represented by the differential form Θ) should have an extension to the equivariant cohomology group $H^3_H(G, \mathbf{Z})$. This is explained in [18,§4]; a quick explanation at the level of de Rham theory (ignoring the torsion in $H^3_H(G, \mathbf{Z})$) is in the appendix of [17].

As explained, for instance, in that appendix, the condition for existence of a gauge invariant extension of $\Gamma(g)$ can be put in the following very explicit form. If T_a , $a = 1 \dots \dim(H)$ are a basis of the Lie algebra of H, and if the embedding $H \subset G_L \times G_R$ is described at the Lie algebra level by $T_a \to (T_{a,L}, T_{a,R})$, then the requirement is

$$\operatorname{Tr} T_{a,L} T_{b,L} = \operatorname{Tr} T_{a,R} T_{b,R}, \quad \text{for all } a, b.$$
 (2.24)

A subgroup $H \subset G_L \times G_R$ obeying this condition is said to be anomaly-free. For

such an H, the gauge invariant extension of Γ exists and is explicitly

$$\Gamma(g, A) = \Gamma(g) - \frac{1}{4\pi} \sum_{a} \int_{\Sigma} A^{a} \operatorname{Tr} \left(T_{a,L} dg \cdot g^{-1} + T_{a,R} g^{-1} dg \right)$$

$$- \frac{1}{8\pi} \sum_{a,b} \int A^{a} \wedge A^{b} \operatorname{Tr} \left(T_{a,R} g^{-1} T_{b,L} g - T_{b,R} g^{-1} T_{a,L} g \right).$$
(2.25)

Combining these formulas, one gets for anomaly-free H a gauge invariant extension of the WZW functional,

$$I(g,A) = -\frac{1}{8\pi} \int_{\Sigma} \text{Tr } g^{-1} d_A g \wedge *g^{-1} d_A g - i\Gamma(g,A).$$
 (2.26)

The quantum field theories with Lagrangians L(g, A) = kI(g, A), k a positive integer, are called G/H models.* Note that for given G and H, there may be several G/H models, since there may be several anomaly-free embeddings of H in $G_L \times G_R$.

If H is any subgroup of G, then the diagonal embedding of H in $G_L \times G_R$ is always anomaly free. The model determined by such a diagonal embedding is often called "the" G/H model. If we pick local complex coordinates z, \overline{z} on Σ (which will facilitate a small calculation needed presently) and write the measure $|dz \wedge d\overline{z}|$ as d^2z , then the Lagrangian of the diagonal G/H model is explicitly k times

$$I(g,A) = I(g) - \frac{1}{2\pi} \int_{\Sigma} d^2 z \operatorname{Tr} A_z \partial_{\overline{z}} g \cdot g^{-1}$$

$$+ \frac{1}{2\pi} \int_{\Sigma} d^2 z \operatorname{Tr} A_{\overline{z}} g^{-1} \partial_z g - \frac{1}{2\pi} \int_{\Sigma} d^2 z \operatorname{Tr} \left(A_z A_{\overline{z}} - A_z g A_{\overline{z}} g^{-1} \right).$$

$$(2.27)$$

Our interest will center on the special case of the diagonal G/H model for

 $[\]star$ The terminology is somewhat misleading since these models are not the most obvious sigma models with target space G/H; and one is not allowed to use the most obvious H actions on G, such as the left or right actions, which are anomalous. The terminology is used because the models are believed [26,27, 16,17] to be equivalent to GKO models [25], which were originally described algebraically, and are conventionally called G/H models or coset models. The claimed equivalence to the GKO models implies in particular that the models are conformally invariant at the quantum level.

H = G. This is then the G/G model with adjoint action of G on itself. The Lagrangian is k times (2.27), and the partition function at level k is

$$Z_k(G,\Sigma) = \frac{1}{\operatorname{vol}(\widehat{G})} \int Dg \ DA \ \exp\left(-kI(g,A)\right). \tag{2.28}$$

Our goal is to use (2.21) to show that $Z_k(G, \Sigma)$ coincides with the dimension of the space of non-abelian theta functions at level k.

2.4. The Kernel

One more special case is important: $H = G_L \times G_R$. This is an anomalous subgroup, so there is no gauge invariant G/H Lagrangian and no G/H quantum field theory for this H. We will do something else instead.

Denote the G_L and G_R components of an H connection as A and B. Set

$$I(g, A, B) = I(g) + \frac{1}{2\pi} \int d^2z \operatorname{Tr} \left(A_{\overline{z}} g^{-1} \partial_z g - B_z \partial_{\overline{z}} g \cdot g^{-1} + B_z g A_{\overline{z}} g^{-1} - \frac{1}{2} A_z A_{\overline{z}} - \frac{1}{2} B_z B_{\overline{z}} \right).$$

$$(2.29)$$

This functional is determined by the following: it is not gauge invariant, but its change under a gauge transformation is independent of g and related in a useful way to the geometry of the prequantum line bundle. In fact, under an infinitesimal gauge transformation

$$\delta g = vg - gu, \quad \delta A = -d_A u, \quad \delta B = -d_A v,$$
 (2.30)

we have

$$\delta I(g, A, B) = \frac{i}{4\pi} \int_{\Sigma} \operatorname{Tr} \left(u \, dA - v \, dB \right). \tag{2.31}$$

(The fact that an extension I(g, A, B) of I(G) exists with these properties has a conceptual explanation noted in the appendix to [17].)

Now, set

$$K(A, B; g) = \exp(-kI(g, A, B)).$$
 (2.32)

In its dependence on A, K can be interpreted as a holomorphic section of $\mathcal{L}^{\otimes k}$; this just means that K is independent of A_z except for the exponential factor prescribed in (2.17). Likewise, in its dependence on B, K is an anti-holomorphic section of $\mathcal{L}^{\otimes (-k)}$; this means that it is independent of $B_{\overline{z}}$ except for a similar exponential. In [17], the above facts were used to describe holomorphic factorization of WZW and coset models. Gerasimov's insight [4] was that K is actually the kernel representing the action of the gauge group on $\mathcal{H} = H^0(\mathcal{A}, \mathcal{L}^{\otimes k})$. This means that for $\chi \in \mathcal{H}$ and $g \in \widehat{G}$,

$$g^*\chi(A) = \int DB \quad K(A, B; g)\chi(B). \tag{2.33}$$

To show this, we first as in (2.17) write the holomorphic section χ as

$$\chi(B) = \exp\left(\frac{k}{4\pi} \int_{\Sigma} d^2 z \operatorname{Tr} B_z B_{\overline{z}}\right) \widehat{\chi}(B_{\overline{z}})$$
(2.34)

with $\hat{\chi}$ an ordinary holomorphic function. The *B*-dependent factors in the integral on the right hand side of (2.33) are

$$\int DB \exp\left(\frac{k}{2\pi} \int_{\Sigma} d^2z \operatorname{Tr}\left(B_z B_{\overline{z}} - B_z A_{\overline{z}}^g\right)\right) \cdot \widehat{\chi}(B_{\overline{z}}). \tag{2.35}$$

To perform such an integral, the basic fact is that if $f(\phi)$ is a holomorphic function that grows at infinity more slowly than $\exp(|\phi|^2)$, then

$$\frac{1}{\pi} \int_{\mathbf{C}} |d\phi \wedge d\overline{\phi}| \exp(-\overline{\phi}\phi + a\overline{\phi}) f(\phi) = f(a). \tag{2.36}$$

Using this fact and normalizing the symplectic measure on A so that

$$\int DB \exp\left(\frac{k}{2\pi} \int_{\Sigma} d^2 z \operatorname{Tr} B_z B_{\overline{z}}\right) = 1$$
 (2.37)

(to avoid a determinant that would otherwise arise in using (2.36)), we get simply

$$\int DB \exp\left(\frac{k}{2\pi} \int_{\Sigma} d^2 z \operatorname{Tr}\left(B_z B_{\overline{z}} - B_z A_{\overline{z}}^g\right)\right) \cdot \widehat{\chi}(B_{\overline{z}}) = \widehat{\chi}(A_{\overline{z}}^g). \tag{2.38}$$

The integral in (2.33) thereby becomes

$$\int DB \ K(A,B;g)\chi(B) = \exp\left(-k\left(I(g) + \frac{1}{2\pi}\int d^2z \operatorname{Tr} A_{\overline{z}}g^{-1}dg - \frac{1}{4\pi}\int d^2z \operatorname{Tr} A_z A_{\overline{z}}\right)\right)$$
$$\cdot \widehat{\chi}(A_{\overline{z}}^g). \tag{2.39}$$

Using (2.34) to reexpress $\widehat{\chi}$ in terms of χ , and using the explicit forms of I(g) and A^g , we get

$$\int DB \ K(A, B; g) \chi(B) = \exp\left(ik\left(\Gamma(g) - \frac{1}{4\pi} \int_{\Sigma} \operatorname{Tr} A \wedge g^{-1} dg\right)\right) \cdot \chi(A^g). \tag{2.40}$$

Using the definition of g^* in (2.15), this indeed coincides with the desired formula (2.33).

As we saw in arriving at (2.21), it follows that the dimension of the space of non-abelian theta functions is

$$\dim H^0(\mathcal{R}, \mathcal{L}^{\otimes k}) = \frac{1}{\operatorname{vol}(\widehat{G})} \int Dg \ DA \ K(A, A; g). \tag{2.41}$$

But the integral on the right is precisely the partition function (2.28) of the G/G model (since $K(A, A; g) = \exp(-kI(g, A))$). So we have arrived at the main goal of this section: identifying the dimension of the space of non-abelian theta functions with the partition function of the G/G model.

<u>Inclusion Of Marked Points</u>

Now we would like to extend the analysis slightly to the case of a Riemann surface Σ with marked points labeled by representations of G. The G/G model in this situation will give a path integral representation of the Verlinde algebra. (This generalization might be omitted on a first reading.)

Suppose one has a represention ρ of a compact Lie group G in a Hilbert space \mathcal{H} . Then as in (2.20), the projection operator onto the invariant subspace of \mathcal{H} is

$$\Pi = \frac{1}{\text{vol}(G)} \int_{G} Dg \ \rho(g), \tag{2.42}$$

with Dg an invariant measure on G and vol(G) the volume of G computed with that measure. The trace of Π is the multiplicity with which the trivial representation of G appears in \mathcal{H} .

Now pick an irreducible representation V of G, that is a vector space V in which G acts irreducibly by $g \to \rho_V(g) \in \operatorname{Aut}(V)$. We want a formula for the multiplicity with which V appears in G. We can reduce to the previous case as follows. Let \overline{V} be the dual or complex conjugate representation of G. The multiplicity with which V appears in \mathcal{H} is the same as the multiplicity with which the trivial representation appears in $\mathcal{H} \otimes \overline{V}$. So we define the projection operator Π_V onto the G-invariant subspace of $\mathcal{H} \otimes \overline{V}$:

$$\Pi_V = \frac{1}{\text{vol}(G)} \int Dg \ \rho(g) \otimes \rho_{\overline{V}}(g). \tag{2.43}$$

The multiplicity with which V appears in \mathcal{H} is

$$\operatorname{mult}(V) = \operatorname{Tr} \Pi_V. \tag{2.44}$$

We want to apply this to the case in which G is replaced by the group \widehat{G} of gauge transformations of a principal G bundle $P \to \Sigma$; and \mathcal{H} will be, as above,

 $H^0(\mathcal{A}, \mathcal{L}^{\otimes k})$. The representations we will use will be the following simple ones. For a point $x \in \Sigma$, let $r_x : \widehat{G} \to G$ be the map of evaluation at x. For any representation $\rho_V : G \to \operatorname{Aut}(V)$ of G, we have the corresponding representation $\rho_{x,V} = \rho_V \circ r_x$ of \widehat{G} . Pick now points $x_i \in \Sigma$, labeled by representations V_i , and let $V = \otimes_i V_i$ with \widehat{G} acting by

$$\rho_V = \otimes_i \rho_{x_i, V_i}. \tag{2.45}$$

The conjugate representation is $\rho_{\overline{V}} = \bigotimes_i \rho_{x_i, \overline{V}_i}$.

We want to find a path integral representation of the multiplicity with which V appears in \mathcal{H} , along the lines of (2.44). To this aim we must calculate

$$\operatorname{Tr}\left(\rho(g)\otimes\rho_{\overline{V}}(g)\right) = \operatorname{Tr}\rho(g)\cdot\operatorname{Tr}\rho_{\overline{V}}(g). \tag{2.46}$$

Here the first factor has a path integral expression; in fact,

$$\operatorname{Tr} \rho(g) = \int DA \ K(A, A; g), \tag{2.47}$$

with K(A, B; g) the kernel introduced in (2.32). The second factor is simply

$$\operatorname{Tr} \rho_{\overline{V}}(g) = \prod_{i} \operatorname{Tr}_{\overline{V}_{i}} g(x_{i}). \tag{2.48}$$

So we get

$$\operatorname{mult}(V) = \dim \left(\mathcal{H} \otimes \overline{V} \right)^{\widehat{G}} = \frac{1}{\operatorname{vol}(\widehat{G})} \int Dg \ DA \ \exp(-kI(g, A)) \cdot \prod_{i} \operatorname{Tr}_{\overline{V}_{i}} g(x_{i}).$$
(2.49)

The right hand side is usually called the (unnormalized) correlation function,

$$\left\langle \prod_{i} \operatorname{Tr}_{\overline{V}_{i}} g(x_{i}) \right\rangle$$
 (2.50)

in the gauged WZW model. (2.50) would be unchanged if all \overline{V}_i are replaced by V_i ; the gauged WZW action has a symmetry (coming from an involution of G that exchanges all representations with their complex conjugates) that ensures this.

Relation To The Verlinde Algebra

Now let us relate this to the Verlinde algebra. Let T be the maximal torus of G and G/T the quotient of G by the right action of T. For any irreducible representation V of G, there is a homogeneous line bundle S over G/T such that $H^0(G/T, S)$ is isomorphic to V.

Given marked points x_1, \ldots, x_s on Σ , let $\widehat{\mathcal{A}}$ be the symplectic manifold

$$\widehat{\mathcal{A}} = \mathcal{A} \times \prod_{i=1}^{s} (G/T)_{i}$$
(2.51)

where $(G/T)_i$ is a copy of G/T "sitting" at x_i . This is an informal way to say that the gauge group \widehat{G} (and its complexification $\widehat{G}_{\mathbf{C}}$) acts on $(G/T)_i$ by composition of the evaluation map r_{x_i} with the natural action of G (or $G_{\mathbf{C}}$) on G/T.

If we are given irreducible representations V_i of G, let for each i S_i be a homogeneous line bundle over $(G/T)_i$ such that $H^0((G/T)_i, S_i) \cong \overline{V}_i$. Define a homogeneous line bundle $\widehat{\mathcal{L}}$ over $\widehat{\mathcal{A}}$ by

$$\widehat{\mathcal{L}} = \mathcal{L}^{\otimes k} \otimes (\otimes_i \mathcal{S}_i) \,. \tag{2.52}$$

(In an obvious way, I have identified the line bundles \mathcal{L} and \mathcal{S}_i with their pullbacks to $\widehat{\mathcal{A}}$.) Then

$$H^{0}(\widehat{\mathcal{A}}, \widehat{\mathcal{L}}) = H^{0}(\mathcal{A}, \mathcal{L}^{\otimes k}) \otimes (\otimes_{i} \overline{V}_{i}).$$
(2.53)

The multiplicity $\operatorname{mult}(V)$ of (2.49) is therefore the same as the dimension of the \widehat{G} -invariant subspace of $H^0(\widehat{\mathcal{A}}, \widehat{\mathcal{L}})$:

$$\operatorname{mult}(V) = \dim \left(H^0(\widehat{\mathcal{A}}, \widehat{\mathcal{L}})^{\widehat{G}} \right).$$
 (2.54)

On the other hand, let $\widehat{\mathcal{R}}$ be the quotient of $\widehat{\mathcal{A}}$ by $\widehat{G}_{\mathbf{C}}$ (the quotient being taken in the sense of geometric invariant theory, using the ample line bundle $\widehat{\mathcal{L}}$). $\widehat{\mathcal{R}}$ is

called the moduli space of holomorphic bundles over Σ with parabolic structure, the parabolic structure being a reduction of the structure group to T at the marked points x_i . (By a theorem of Mehta and Seshadri [28], $\widehat{\mathcal{R}}$ coincides with the moduli space of flat connections on $P \to \Sigma - \{x_i\}$ with certain branching about the x_i , up to gauge transformation.) The $\widehat{G}_{\mathbf{C}}$ -invariant line bundle $\widehat{\mathcal{L}} \to \widehat{\mathcal{A}}$ descends to a line bundle over $\widehat{\mathcal{R}}$, which we will also call $\widehat{\mathcal{L}}$, whose sections over an open set $U \subset \mathcal{R}$ are \widehat{G} -invariant sections of $\widehat{\mathcal{L}}$ over the inverse image of U in \widehat{A} . So in particular

$$H^{0}(\widehat{\mathcal{R}},\widehat{\mathcal{L}}) = H^{0}(\widehat{\mathcal{A}},\widehat{\mathcal{L}})^{\widehat{G}}.$$
(2.55)

Both $\widehat{\mathcal{R}}$ and $\widehat{\mathcal{L}}$ depend on the V_i , but I will not indicate this in the notation.

The left hand side of (2.55) is the space of non-abelian theta functions with parabolic structure. If we combine (2.49), (2.50), (2.54), and (2.55), we find that the dimension of this space is naturally written as a correlation function in the gauged WZW model:

$$\dim H^0(\widehat{\mathcal{R}}, \widehat{\mathcal{L}}) = \left\langle \prod_{i=1}^s \operatorname{Tr}_{V_i} g(x_i) \right\rangle. \tag{2.56}$$

The Verlinde Algebra

As a special case of this, the Verlinde algebra is defined as follows. For given "level" k, the loop group of the compact Lie group G has a finite number of isomorphism classes of unitary, integrable representations; their highest weights are a distinguished list of isomorphism classes V_{α} , $\alpha \in W$ of representations of G. Let X be the \mathbf{Z} module freely generated by the V_{α} . X has a natural metric given by $g(V_{\alpha}, V_{\beta}) = 1$ if $V_{\alpha} = \overline{V}_{\beta}$ and otherwise $g(V_{\alpha}, V_{\beta}) = 0$. It also has a natural multiplication structure that we will describe presently. X endowed with this structure is called the Verlinde algebra.

Using the metric on X, a multiplication law $V_{\alpha} \cdot V_{\beta} = \sum_{\gamma} N_{\alpha\beta}{}^{\gamma}V_{\gamma}$ can be defined by giving a cubic form $N_{\alpha\beta\gamma}$ which is interpreted as $\sum_{\delta} g_{\gamma\delta} N_{\alpha\beta}{}^{\delta}$. Such a cubic form is defined as follows.

Take Σ to be a curve of genus zero with three marked points x_i , i = 1...3, labeled by integrable representations V_{α_i} , $\alpha_i \in W$. The choice of the α_i and of a level k determines a moduli space $\widehat{\mathcal{R}}$ of parabolic bundles with a line bundle $\widehat{\mathcal{L}}$. The structure constants of the Verlinde algebra are

$$N_{\alpha_1,\alpha_2,\alpha_3} = \dim H^0(\widehat{\mathcal{R}},\widehat{\mathcal{L}}).$$
 (2.57)

So in other words, from (2.56), the Verlinde structure functions are the genus zero three point functions of the G/G model:

$$N_{\alpha_1,\alpha_2,\alpha_3} = \left\langle \prod_{i=1}^3 \operatorname{Tr}_{V_i} g(x_i) \right\rangle. \tag{2.58}$$

The basic phenomenon under study in the present paper is a relation between the quantum cohomology of the Grassmannian and the G/G model; the result can be applied to the Verlinde algebra because of (2.58). The special case of a genus zero surface with three marked points is fundamental because the general case can be reduced to this by standard sewing and gluing arguments. In fact, such sewing and gluing arguments, applied to a genus zero curve with four marked points, yield the associativity of the Verlinde algebra.

Higher Cohomology

Obviously, the above discussion has only a physical level of rigor. Among many points that should be clarified I will single out one.

If the V_i are integrable representations at level k, then the higher cohomology $H^i(\widehat{\mathcal{R}},\widehat{\mathcal{L}})$, i>0 vanishes, and $\dim H^0(\widehat{\mathcal{R}},\widehat{\mathcal{L}})$ coincides with the Euler characteristic $\chi(\widehat{\mathcal{R}},\widehat{\mathcal{L}}) = \sum_i (-1)^i \dim H^i(\widehat{\mathcal{R}},\widehat{\mathcal{L}})$. From comments made to me by R. Bott and G. Segal, it appears that for (2.56) to hold for arbitrary representations V_i (perhaps not integrable), one must replace $\dim H^0(\widehat{\mathcal{R}},\widehat{\mathcal{L}})$ by $\chi(\widehat{\mathcal{R}},\widehat{\mathcal{L}})$. A rigorous treatment of the G/G model should show the restriction to integrable representations in deriving (2.56); there may also be a supersymmetric version of the derivation that naturally gives the Euler characteristic and holds for all representations.

2.5. Some Additional Properties

The reader may wish at this stage to turn to §3. However, I will pause here and in §2.6 below to explain a few additional facts that have their own interest and will be needed at a few points in §4.

Topological Field Theory

First of all, the gauged WZW theory of G/H is in general conformally invariant but not topologically invariant. A conformal structure appears in the definition of the Lagrangian. However, for H = G we have evaluated the partition function of the G/H model, and found it to be an integer, independent of the conformal structure of Σ , and equal to the dimension of the space of non-abelian theta functions. This strongly suggests that the G/G model is actually a topological field theory. Let us try to demonstrate that directly.

A conformal structure on Σ can be specified by giving a metric h, uniquely determined up to Weyl scaling. Under a change in h, the change in the G/G Lagrangian is

$$\delta L = \frac{k}{8\pi} \int_{\Sigma} d^2 z \sqrt{h} (h^{z\overline{z}})^2 \left(\delta h_{\overline{z}\overline{z}} \operatorname{Tr}(g^{-1} D_z g)^2 + \delta h_{zz} \operatorname{Tr}(D_{\overline{z}} g \cdot g^{-1})^2 \right). \tag{2.59}$$

Though this expression does not vanish identically, it vanishes when the classical equations of motion are obeyed. In fact, under a variation of the connection A, the Lagrangian changes by

$$\delta' L = \frac{k}{2\pi} \int_{\Sigma} d^2 z \sqrt{h} h^{z\overline{z}} \operatorname{Tr} \left(\delta A_{\overline{z}} g^{-1} D_z g - \delta A_z D_{\overline{z}} g \cdot g^{-1} \right). \tag{2.60}$$

So the classical Euler-Lagrange equations, asserting the vanishing of $\delta'L$, are

$$0 = g^{-1}D_z g = D_{\overline{z}}g \cdot g^{-1}. \tag{2.61}$$

Since (2.59) vanishes when (2.61) does, the G/G model is classically a topological field theory. Quantum mechanically the analog of using the equations of motion

is to make a suitable change of variables in the path integral. In this case, we consider the infinitesimal redefinition of A

$$\delta A_z = \frac{1}{4} \delta h_{zz} h^{z\overline{z}} D_{\overline{z}} g \cdot g^{-1}$$

$$\delta A_{\overline{z}} = -\frac{1}{4} \delta h_{\overline{z}\overline{z}} h^{z\overline{z}} g^{-1} D_z g.$$
(2.62)

(This is a complex change of coordinates that entails an infinitesimal displacement of the integration contour in the complex plane, or more exactly a displacement of the cycle of integration in the complexification of \mathcal{A} .) Substituting in (2.61), we see that the Lagrangian L(g,A) is invariant under a change of metric on Σ compensated by the transformation (2.62) of the field variables. The path integral for the partition function

$$\int Dg \ DA \ \exp(-L(A,g)) \tag{2.63}$$

is therefore invariant under the combined change of metric and integration variable, provided the measure DA is invariant. To this effect, we must compute a Jacobian or, at the infinitesimal level, the divergence of the vector field that generates the change of variables (2.62). This is formally

$$\int_{\Sigma} \left(\frac{\delta}{\delta A_z(x)} \delta A_z(x) + \frac{\delta}{\delta A_{\overline{z}}(x)} \delta A_{\overline{z}}(x). \right)$$
 (2.64)

This vanishes, as δA_z is independent of A_z and $\delta A_{\overline{z}}$ is independent of $A_{\overline{z}}$. This completes the explanation of why the G/G model is a topological field theory.

Let us note now that the other Euler-Lagrange equation of motion, obtained by varying with respect to g, is

$$D_{\overline{z}}(q^{-1}D_z q) + F_{\overline{z}z} = 0, (2.65)$$

with F the curvature of the connection A. So given (2.61), this implies that

$$F = 0. (2.66)$$

Comparison To The Obvious Topological Field Theory

If the goal were to construct a topological field theory using the fields g, A, the more obvious way to do it would be to take the Lagrangian to be simply

$$L'(g,A) = -ik\Gamma(g,A), \tag{2.67}$$

which manifestly corresponds to a topological field theory, since it is defined without use of any metric or conformal structure. How does this theory compare to the G/G WZW model?

More generally, let us consider the family of theories

$$L_{k'}(g,A) = -\frac{k'}{8\pi} \int_{\Sigma} \text{Tr } g^{-1} d_A g \wedge *g^{-1} d_A g - ik\Gamma(g,A), \qquad (2.68)$$

with positive k'. This coincides with the G/G model at k' = k, and with the manifestly topologically invariant model at k' = 0. It is straightforward to work out that the classical equations of motion are

$$0 = q^{-1}D_z q - \lambda D_z q \cdot q^{-1} = D_{\overline{z}} q \cdot q^{-1} - \lambda q^{-1} D_{\overline{z}} q, \tag{2.69}$$

with

$$\lambda = \frac{k' - k}{k' + k}.\tag{2.70}$$

For $0 < k' < \infty$, one has

$$-1 < \lambda < 1. \tag{2.71}$$

(2.69) implies

$$d_A g = 0. (2.72)$$

For instance, the first equation in (2.69) is equivalent to

$$(1 - \lambda \operatorname{Ad}(g)) (D_z g) = 0, \tag{2.73}$$

with $Ad(g)(x) = gxg^{-1}$. Since $|Ad(g)| \le 1$ and $|\lambda| < 1$, (2.73) implies $D_zg = 0$, and similarly (2.69) implies $D_{\overline{z}}g = 0$.

Given that (2.72) follows from the classical equations of motion, the same sort of reasoning as above shows that the Lagrangians $L_{k'}$ describe a family of topological field theories: a change of metric can be compensated by a change of integration variable with trivial Jacobian.

Now, to study the k' dependence, look at

$$\frac{\partial L_{k'}}{\partial k'} = -\frac{1}{8\pi} \int_{\Sigma} \text{Tr}(g^{-1}d_A g \wedge *g^{-1}d_A g). \tag{2.74}$$

By virtue of (2.72), this expression vanishes by the classical equations of motion, so classically the family of theories governed by $L_{k'}$ is constant.

From the above discussion, we know how we should proceed quantum mechanically: we should find a change of integration variable that compensates for the k' dependence of the Lagrangian. Such a change of variable exists because of (2.72); one can take explicitly

$$\delta A_z = -\frac{\delta k'}{k + k'} \left(1 - \lambda \text{Ad}(g^{-1}) \right)^{-1} (D_z g \cdot g^{-1}). \tag{2.75}$$

Now, however, a difference arises from our earlier discussion. Because δA_z is a function of A_z , the Jacobian of the transformation in (2.75) is not necessarily 1; the integration measure in the path integral may not be invariant. The change of the integration measure is formally

$$\int_{\Sigma} \operatorname{Tr} \frac{\delta}{\delta A_z(x)} \delta A_z(x). \tag{2.76}$$

Since

$$\frac{\delta}{\delta A_z(x)} \delta A_z(y) \sim \delta^2(x, y), \tag{2.77}$$

this is ill-defined, proportional to $\delta^2(0)$. In any event, since (2.76) is the integral over Σ of a local quantity, any regularization should be of that form. Quantities

analogous to (2.76) are regularized (albeit in a slightly *ad hoc* fashion) in [5], in deriving eqn. (6.22). I will not repeat such a calculation here, but I will just explain what general form the answer must have, by asking what is the most general possible perturbation of the G/G model.

The Complete Family Of Theories

Let us simply go back to the gauged WZW model of G/G, and ask what kind of perturbations it has (see also [29]). We permit the perturbation of the Lagrangian to be the integral of an arbitrary local functional of g, A, and a metric h on Σ . In this way we will obtain continuous perturbations of the G/G model, but forbid discrete perturbations (notably changes in k) that cannot be described via the addition of a local functional to the Lagrangian.

Perturbations that vanish by the classical equations of motion are irrelevant, since they can be eliminated by a change of integration variables as described above. (Even if the integration measure is not invariant under the change of variables, changes of variables can be used to eliminate the perturbations that vanish by the equations of motion in favor of other perturbations that do not so vanish.) In classifying perturbations, we therefore can work modulo operators that vanish by the classical equations of motion. Given (2.61) and (2.66), this means that we can discard anything proportional to $d_A g$ or F.

The gauge invariant local operators, modulo operators that vanish by the equations of motion, are generated by operators of the form U(g), with U some function on G that is invariant under conjugation. Since U(g) is a zero-form, to construct from it a perturbation of the Lagrangian, we need also a metric h on Σ , or at least a measure μ , such as the Riemannian measure. The curvature scalar of h will be called R. The most interesting perturbations are

$$Q_U = \int_{\Sigma} d\mu \ U(g) \tag{2.78}$$

and

$$S_U = \int_{\Sigma} d^2z \sqrt{h} R \ U(g). \tag{2.79}$$

(2.78) breaks the diffeomorphism invariance of the G/G model down to invariance under the group of diffeomorphisms that preserve the measure μ . The G/G model perturbed as in (2.78) is an interesting family of theories invariant under area-preserving diffeomorphisms (and reducing for $k \to \infty$ to two dimensional Yang-Mills theory, which has the same invariance).

Slightly less obviously, the G/G model perturbed by (2.79) is still a topological field theory. In fact, under an infinitesimal change in h, $\sqrt{h}R$ changes by a total derivative (so that $\int_{\Sigma} d^2z\sqrt{h}R$ is a topological invariant, a multiple of the Euler characteristic). After integrating by parts, the change in (2.79) under a change in h is

$$\delta S_U \sim \int_{\Sigma} d^2 z \sqrt{h} \left(\delta h_{i'j'} - h_{i'j'} h^{kl} \delta h_{kl} \right) h^{i'i} h^{j'j} D_i D_j U(g). \tag{2.80}$$

This vanishes by the equations of motion, since $d_A(g) = 0$ implies dU = 0. Hence one can compensate for δS_U with a redefinition of A (and the Jacobian for the transformation is trivial, since the requisite δA is independent of A).

Other perturbations, such as $\int_{\Sigma} d^2z \sqrt{h}R^2U(g)$, are less interesting, since (i) they do not possess the large invariances of the theories perturbed by Q_U or S_U ; (ii) they vanish as a negative power of t if the metric of Σ is scaled up by $h \to th$, t >> 1. The latter property means that in most applications of these systems, such perturbations (if not prevented by (i)) can be conveniently eliminated.

Since the most general perturbation of the G/G model that preserves the diffeomorphism invariance is of the form of S_U , the regularized version of (2.76) must be equivalent to S_U for some U. By the same token, for any k', the G/G model must be equivalent to the $L_{k'}$ model perturbed by some S_U (with a k'-dependent U), and vice-versa. In particular, setting k' = 0, the G/G model is equivalent to the manifestly topologically invariant model with Lagrangian $-ik\Gamma(g, A)$, perturbed by some S_U . The requisite U's in these statements can in fact be computed at least heuristically along the lines of the derivation of eqn. (6.22) of [5], but I will not do so here.

Interpretation

Note that the conjugation-invariant function U(g) that entered above can be expressed as a linear combination of the characters $\text{Tr}_V g$, as V runs over irreducible representations of G. These are precisely the operators whose correlation functions were interpreted algebro-geometrically in (2.56), so the theories obtained by perturbing the G/G model are all computable in terms of the Verlinde algebra.

2.6. Abelianization

I will now briefly describe another interesting facet of the G/G model, introduced in [5], which apart from its beauty will enter at a judicious moment in §4.*

A recurring and significant theme in the theory of compact Lie groups is the reduction to the maximal torus T, extended by the Weyl group W. As explained in [5], the G/G model admits such an reduction to the maximal torus. It is equivalent to the T/T model (that is, the G/H model with both G and H set equal to T) perturbed by S_U , where U is a certain Weyl-invariant function on T and S_U is defined in (2.79).

At the level of precision explained in [5], the abelianization of the model proceeds as follows. Pick a maximal torus $T \subset G$, with Lie algebra \mathbf{t} . Impose the "gauge condition" $g \in T$. Decompose the connection as $A = A_0 + A_{\perp}$, where A_0

^{*} A computation reaching a rather similar conclusion is sketched in [4], but unfortunately the fermionic symmetry δ introduced in equations (71)-(74) of that paper does not obey $\delta^2 = 0$, which would be needed to justify the computation. I will therefore concentrate on sketching the argument of [5].

[†] This is not really valid globally as a gauge condition. One must think in terms of integrating over the fibers of the map $G \to T/W$ that maps a group element to its conjugacy class.

is the part of the connection valued in \mathbf{t} , and A_{\perp} is valued in the orthocomplement \mathbf{t}_{\perp} of \mathbf{t} . In this gauge the G/G Lagrangian takes the form

$$L_{G/G}(g,A) = L_{T/T}(g,A_0) - \frac{k}{2\pi} \int_{\Sigma} d^2z \operatorname{Tr} \left(A_{\perp,z} A_{\perp,\overline{z}} - A_{\perp,z} g A_{\perp,\overline{z}} g^{-1} \right). \quad (2.81)$$

Here

$$L_{T/T}(g, A_0) = kI_{T/T}(g, A_0)$$
(2.82)

is the Lagrangian of the T/T model, at level k. The G/G model, in this gauge, differs from the T/T model by the last term in (2.81), which involves A_{\perp} . To reduce the G/G model to something like the T/T model, one must "integrate out" A_{\perp} to reduce to a description involving g and A_0 only. Happily, the A_{\perp} integral is Gaussian:

$$\int DA_{\perp} \exp\left(\frac{k}{2\pi} \int_{\Sigma} d^2z \operatorname{Tr}\left(A_{\perp,z} A_{\perp,\overline{z}} - A_{\perp,z} g A_{\perp,\overline{z}} g^{-1}\right)\right). \tag{2.83}$$

Such a Gaussian integral formally gives rise to a determinant (as we briefly explain in §3.5 below). In comparing the G/G model to the T/T model, another determinant arises: the Fadde'ev-Popov determinant comparing the volume of \widehat{G} to the volume of \widehat{T} . These two determinants are rather singular but at the same time extremely simple, because the exponent in (2.83) (like the corresponding expression in the Fadde'ev-Popov determinant) is a local functional without derivatives. In [5], Blau and Thompson calculate these determinants, with a plausible regularization, and argue that the G/G model is equivalent to a T/T model with Lagrangian

$$\widehat{L}_{T/T}(g, A_0) = (k + \rho)I(g, A_0) - \frac{1}{4\pi} \int_{\Sigma} d^2x \sqrt{h}R \log \det_{\mathbf{t}_{\perp}} (1 - \mathrm{Ad}(g)). \tag{2.84}$$

Here ρ is the dual Coxeter number of G, and $\det_{\mathbf{t}_{\perp}}(1-\mathrm{Ad}(g))$ is the determinant of $1-\mathrm{Ad}(g)$, regarded as an operator on \mathbf{t}_{\perp} . (This well-known Weyl-invariant function

enters in the Weyl character formula, where it has a somewhat similar origin, involving a comparison of the volumes of G and T.) In §4.6, we will have occasion to use (2.84) for the case that G = U(k). For that case, if $g = \text{diag}(\sigma_1, \ldots, \sigma_k)$, the eigenvalues of 1 - Ad(g) acting on \mathbf{t}_{\perp} are the numbers $1 - \sigma_i \sigma_j^{-1}$, for $1 \leq i, j \leq k$, $i \neq j$. Hence the correction term in (2.84) becomes in this case

$$\Delta L = -\frac{1}{4\pi} \int_{\Sigma} d^2x \sqrt{h} R \left(\sum_{i \neq j} \ln(\sigma_i - \sigma_j) - (k - 1) \sum_i \ln \sigma_i \right). \tag{2.85}$$

Given the role of the G/G model in counting non-abelian theta functions, its reduction to a T/T model is a kind of abelianization of the problem of counting such functions. In §7.3 of [5], this is pursued further to obtain a completely explicit count of non-abelian theta functions for G = SU(2). The role of the endpoint contributions in equation (7.14) of that paper still deserves closer study.

3. The Quantum Cohomology Of The Grassmannian

The Grassmannian G(k, N) is the space of all k dimensional subspaces of a fixed N dimensional complex vector space $V \cong \mathbb{C}^N$. If we want to make the dependence on V explicit, we write $G_V(k, N)$.

By associating with a k dimensional subspace of V the N-k dimensional orthogonal subspace of the dual space V^* , we see that $G_V(k,N) \cong G_{V^*}(N-k,N)$. The relation that we will explain here and in §4 between the Verlinde algebra of U(k) at level $(N-k,N)^*$ and the quantum cohomology of G(k,N) therefore implies that the Verlinde algebra of U(k) at level (N-k,N) coincides with that of U(N-k) at level (k,N). This is a surprising fact that had been noted earlier. (For instance, see [30] and [31, p. 212, Proposition (10.6.4)] for $k \leftrightarrow N-k$ symmetry of loop group representations and [32,33] for such symmetry of the Verlinde algebra.)

^{*} That is, at levels N-k and N for the su(k) and u(1) factors in the Lie algebra of U(k).

One way to describe G(k, N) is as follows. Let B be the space of all linearly independent k-plets $e_1, \ldots, e_k \subset V$. A point in B labels a k-plane V with a basis. The group $GL(k, \mathbf{C})$ acts on B by change of basis, $e_i \to \sum_j W_i{}^j e_j$, $W \in GL(k, \mathbf{C})$. Since $GL(k, \mathbf{C})$ acts simply transitively on the space of bases of V, upon dividing by $GL(k, \mathbf{C})$ we precisely forget the basis and therefore

$$G(k, N) = B/GL(k, \mathbf{C}). \tag{3.1}$$

B is dense and open in the k-fold product $\mathbf{C}^{kN} = V \times V \times \ldots \times V$ (since the generic k-plet $e_1, \ldots, e_k \subset V$ is a basis of V), so G(k, N) is a quotient of a dense open subset of \mathbf{C}^{kN} by $GL(k, \mathbf{C})$. In fact, G(k, N) is the good quotient of \mathbf{C}^{kN} by $GL(k, \mathbf{C})$ that would be constructed in geometric invariant theory.

There is also a symplectic version of this, which will be more relevant in what follows. Pick a Hermitian metric on V so that $V^k = \mathbf{C}^{kN}$ gets a metric and a symplectic structure. In linear coordinates ϕ^{is} , $i = 1 \dots k$, $s = 1 \dots N$ on \mathbf{C}^{kN} , the symplectic form is

$$\omega = i \sum_{i,s} d\phi^{is} \wedge d\overline{\phi}_{is}. \tag{3.2}$$

 ω is not invariant under $GL(k, \mathbf{C})$, but it is invariant under a maximal compact subgroup $U(k) \subset GL(k, \mathbf{C})$.

To this symplectic action is associated a "moment map" μ from \mathbf{C}^{kN} to the dual of the Lie algebra of U(k), given by the angular momentum functions that generate U(k) via Poisson brackets. In this case we can take the moment map to be

$$\mu: (e_1, \dots, e_k) \to \{(e_i, e_j) - \delta_{ij}\}.$$
 (3.3)

In other words, $\mu = 0$ precisely if the vectors e_1, \ldots, e_k are orthonormal.

Every k-plane has an orthonormal basis, unique up to the action of U(k), so

$$G(k, N) = \mu^{-1}(0)/U(k). \tag{3.4}$$

This is the description of G(k, N) that we will actually use. We will also want to remember one fact: μ is a quadratic function on the real vector space underlying \mathbf{C}^{kN} . In components,

$$\mu^{i}{}_{j} = \sum_{s} \phi^{is} \overline{\phi}_{js} - \delta^{i}{}_{j}. \tag{3.5}$$

3.1. Cohomology

Now we need to discuss the cohomology of G(k, N). We begin with the classical cohomology. Over G(k, N) there is a "tautological" k-plane bundle E (whose fiber over $x \in G(k, N)$ is the k plane in V labeled by x) and a complementary bundle F (of rank N - k):

$$0 \to E \to V \cong \mathbf{C}^N \to F \to 0. \tag{3.6}$$

Obvious cohomology classes of G(k, N) come from Chern classes. We set

$$x_i = c_i(E^*), (3.7)$$

where * denotes the dual. (It is conventional to use E^* rather than E, because $\det E^*$ is ample.) This is practically where Chern classes come from, as G(k, N) for $N \to \infty$ is the classifying space of the group U(k). It is known that the x_i generate $H^*(G(k, N))$ with certain relations. The relations come naturally from the existence of the complementary bundle F in (3.6). Let $y_j = c_j(F^*)$, and let $c_t(\cdot) = 1 + tc_1(\cdot) + t^2c_2(\cdot) + \ldots$ Then as a consequence of (3.6),

$$c_t(E^*)c_t(F^*) = 1,$$
 (3.8)

and $H^*(G(k, N))$ is generated by the x_i, y_j with relations (3.8). If one wishes, these relations can be partially solved to express the y_j in terms of the x_i (or vice-versa).

Now we come to the quantum cohomology, which originally entered in string theory, where [34] it enters the theory of the Yukawa couplings (which are related to quark and lepton masses), and in Floer/Gromov theory of symplectic manifolds [35]. Additively, the quantum cohomology is the same as the classical one, but the ring structure is different.

Giving a ring structure on $W = H^*(G(k, N))$ is the same as giving the identity $1 \in W$ and a cubic form

$$(\alpha, \beta, \gamma) = \int_{G(k,N)} \alpha \cup \beta \cup \gamma.$$
 (3.9)

The cubic form determines a metric

$$g(\alpha, \beta) = (\alpha, \beta, 1), \tag{3.10}$$

and given a metric the cubic form $W \times W \times W \to \mathbf{C}$ determines a ring structure $W \times W \to W$.

So I will explain the quantum cohomology ring by describing the quantum cubic form. To this aim, let Σ be a closed oriented two-manifold (which in string theory would be the "world-sheet," analogous to the world-line of a particle). Let $P \in \Sigma$. Let $\mathcal{W} = \operatorname{Maps}(\Sigma, G(k, N))$. Evaluation at P gives a map

$$\mathcal{W} \xrightarrow{\text{ev}(P)} G(k, N),$$
 (3.11)

by which $\alpha \in H^*(G(k, N))$ pulls back to $\widehat{\alpha}(P) = \text{ev}(P)^*(\alpha) \in H^*(\mathcal{W})$.

Now pick a complex structure on Σ , and let $\mathcal{M} \subset \mathcal{W}$ be the space of holomorphic maps of Σ to G(k, N). We have $\mathcal{M} = \cup_{\lambda} \mathcal{M}_{\lambda}$, with \mathcal{M}_{λ} being the connected components of \mathcal{M} . In the case of the Grassmannian, the components \mathcal{M}_{λ} are determined by the degree, defined as follows. If $\eta = c_1(E^*)$, which generates $H^2(G(k, N), \mathbf{Z})$, and $\Phi : \Sigma \to G(k, N)$ is such that $\int_{\Sigma} \Phi^*(\eta) = d$, then Φ is said to be of degree d. Since det E^* is ample, holomorphic curves only exist for $d \geq 0$.

The quantum cubic form is defined as follows (ignoring analytical details and tacitly assuming that the \mathcal{M}_{λ} are smooth and compact). Let Σ be of genus zero. Let P, Q, R be three points in Σ . Then for $\alpha, \beta, \gamma \in H^*(G(k, N))$, we set

$$\langle \alpha, \beta, \gamma \rangle = \sum_{d} e^{-dr} \cdot \int_{\mathcal{M}_d} \widehat{\alpha}(P) \cup \widehat{\beta}(Q) \cup \widehat{\gamma}(R),$$
 (3.12)

with r a real parameter.

In what sense does $\langle \alpha, \beta, \gamma \rangle$ generalize the classical cubic form? One component of \mathcal{M} , namely \mathcal{M}_0 , consists of constant maps $\Sigma \to G(k, N)$. This component is a copy of G(k, N) itself. Under that identification the evaluation maps at P, Q, and R all coincide with the identity, so the contribution of \mathcal{M}_0 to $\langle \alpha, \beta, \gamma \rangle$ coincides with the classical cubic form defined as in (3.9). The quantum cubic form differs from the classical one by contributions of the rational curves of higher degree. These contributions are small for r >> 0. In practice, for dimensional reasons, for every given α, β, γ of definite dimension, the sum in (3.12) receives a contribution from at most one value of d. (This is in marked contrast to the much-studied case of a Kahler manifold of $c_1 = 0$, where every positive d can contribute to the same correlation function.) Therefore, no information is lost if we set r = 0, and that is what we will do in the rest of this section.

It follows from the definition (for any Kahler manifold, not just the Grassmannian) that

$$\langle \alpha, \beta, 1 \rangle = (\alpha, \beta, 1)$$
 (3.13)

and thus that the classical and quantum metrics coincide. This is equivalent to the statement that rational maps of positive degree do not contribute to $\langle \alpha, \beta, 1 \rangle$. In fact (as $\widehat{1}(R) = 1$), the contribution of a component \mathcal{M}_{λ} of positive degree is

$$\int_{\mathcal{M}_{\lambda}} \widehat{\alpha}(P) \cup \widehat{\beta}(Q). \tag{3.14}$$

A group $F \cong \mathbb{C}^*$ acts on \mathbb{CP}^1 leaving fixed the points P and Q. F acts freely on \mathcal{M}_{λ} , if \mathcal{M}_{λ} is a component of rational maps of positive degree. The classes

 $\widehat{\alpha}(P)$ and $\widehat{\beta}(Q)$ in the cohomology of \mathcal{M}_{λ} are pullbacks from \mathcal{M}_{λ}/F . Therefore, on dimensional grounds (3.14) vanishes.

3.2. The Grassmannian

Let us now work out the quantum cohomology ring of the Grassmannian. As a preliminary, we note that the contribution of a moduli space \mathcal{M}_d to the quantum cubic form obeys an obvious dimensional condition: it vanishes unless the sum of the dimensions of α , β , γ equals the (real) dimension of \mathcal{M}_d . The component \mathcal{M}_d of genus zero holomorphic curves of degree d in G(k, N) has (according to the Riemann-Roch theorem) complex dimension $\dim_{\mathbf{C}} G(k, N) + dN$. The fact that this depends on d means that the dimensional condition depends on d and therefore that the quantum cohomology ring is not \mathbf{Z} -graded. However, the fact that the real dimensions are all equal modulo 2N means that the cohomology is $\mathbf{Z}/2N\mathbf{Z}$ -graded.

Returning to the relations $c_t(E^*)c_t(F^*) = 1$ that define the cohomology of the Grassmannian, we see that (as the left hand side is a priori a polynomial in t of degree N) the classical relations are of dimension 0, 2, 4, ..., 2N. To a classical relation of degree 2k, the rational curves of degree d > 0 will add a correction of degree 2k - 2dN; this therefore must vanish unless k = N and d = 1. Therefore, of the defining relations of the cohomology, the only one subject to a quantum correction is the "top" relation $c_k(E^*)c_{N-k}(F^*) = 0$, and the correction is an element of $H^*(G(k, N))$ of degree 0 and hence simply an integer. So the non-trivial effect of the quantum corrections will be simply to generate a relation of the form

$$c_k(E^*)c_{N-k}(F^*) = a,$$
 (3.15)

for some $a \in \mathbf{Z}$. Moreover, a is to be computed by examining rational curves in the Grassmannian of degree 1. We will find that $a = (-1)^{N-k}$, so the quantum

cohomology ring can be described by the relations

$$c_t(E^*)c_t(F^*) = 1 + (-1)^{N-k}t^N.$$
 (3.16)

This correction has been described previously [36] in the special case of k = 1 (complex projective space). Despite its simple form, the correction has a dramatic effect: while the classical cohomology ring is nilpotent (in the sense that every element of positive degree is nilpotent), the quantum cohomology ring is semi-simple. This is evident in its Landau-Ginzburg description [12,3,1] which we consider presently.

Computation Of a

For X a submanifold of G(k, N), let [X] be its Poincaré dual cohomology class. For instance, for p a point in the Grassmannian, [p] is a top dimensional class, obeying g(1, [p]) = 1. (It does not matter here if the metric $g(\cdot, \cdot)$ is defined using the classical or quantum cubic form, since we have seen that these determine the same metric.) The definition of the quantum ring structure from the quantum cubic form is such that $a = c_k(E^*)c_{N-k}(F^*)$ can be computed as

$$a = \langle c_k(E^*), c_{N-k}(F^*), [p] \rangle.$$
 (3.17)

 $c_k(E^*)$ equals the Poincaré dual of the zero locus of a generic section of E^* . The dual of the exact sequence (3.6) reads

$$0 \to F^* \to V^* \to E^* \to 0,$$
 (3.18)

with V^* a fixed N dimensional complex vector space. The image in E^* of any fixed vector $w \in V^*$ gives a holomorphic section \overline{w} of E^* . If as before e_1, \ldots, e_N is a basis of V, and w is the linear form that maps $\sum_{i=1}^N r^i e_i$ to r^1 , then the restriction of w to $E \subset V$ vanishes precisely if E consists only of vectors with $r^1 = 0$. This is

a copy of G(k, N-1) which we will call X_w . Since \overline{w} has only a simple zero along X_w (any E can be perturbed in first order to get one for which $\overline{w} \neq 0$), we have

$$c_k(E^*) = [X_w]. (3.19)$$

For future use, let us note that

$$\int_{G(k,N)} c_k(E^*)^{N-k} = 1.$$
 (3.20)

Indeed, we can pick N-k holomorphic sections of E^* whose zero sets intersect transversely at a single point. To do so, let w_i for $i=1,\ldots,N-k$ be the linear form on V that maps $\sum_{i=1}^{N} r^i e_i$ to r^i . Then the \overline{w}_i have the required properties, vanishing precisely for E the k-plane spanned by e_{N-k+1},\ldots,e_N .

Now let us compute $c_{N-k}(F^*) = (-1)^{N-k}c_{N-k}(F)$. Under the holomorphic surjection $V \to F$, any vector $v \in V$ projects to a holomorphic section \overline{v} of F. \overline{v} vanishes precisely if $v \in E$; let $Y_v = \{E \in G(k,N) | v \in E\}$. Then \overline{v} has a simple zero along Y_v , so $c_{N-k}(F) = [Y_v]$ and therefore

$$c_{N-k}(F^*) = (-1)^{N-k}[Y_v]. (3.21)$$

Rational curves of degree one in G(k, N) can all be described as follows. Let (s,t) be homogeneous coordinates for \mathbf{CP}^1 . For r_1, \ldots, r_k a set of k linearly independent vectors in the N dimensional vector space V, let $\{r_1, \ldots, r_k\}$ be the k-plane that they span. Then a rational curve of degree one in G(k, N) is of the form

$$(s,t) \to \{sr_0 + tr_1, r_2, r_3, \dots, r_k\},$$
 (3.22)

with r_0, \ldots, r_k being linearly independent vectors in V.

We have to calculate

$$a = (-1)^{N-k} \int_{\mathcal{M}_1} \widehat{[X_w]}(P) \cup \widehat{[Y_v]}(Q) \cup \widehat{[p]}(R). \tag{3.23}$$

Here \mathcal{M}_1 is the space of degree 1 rational curves, $w \in V^*$, $v \in V$, and P, Q, R are points in \mathbb{CP}^1 . If everything is sufficiently generic, a is simply the number of degree one curves that pass through X_e at P, through Y_f at Q, and through p at R.

We choose p to be an arbitrary point in G(k, N) corresponding to a k-plane spanned by vectors v_1, \ldots, v_k . We take $v = v_0$ to be linearly independent of these, and we pick w to be any linear form that maps v_0 to 1, v_1 to -1, and the v_j of j > 1 to 0.

From the explicit description of degree one curves in (3.22), we see that the k-planes represented by points in the image of such a curve are subspaces of a common k + 1-plane. For a curve that passes through Y_v at Q and through p at R, this is clearly the k + 1-plane W spanned by v_0, v_1, \ldots, v_k . Requiring that the curve pass also through X_w at P determines the curve uniquely. For instance, if Q = (1,0), R = (0,1), and P = (1,1), then the degree 1 curve must be

$$(s,t) \to \{v_0 s + v_1 t, v_2, \dots, v_k\}.$$
 (3.24)

The subvarieties $[\widehat{X}_w](P)$, $[\widehat{Y}_f](Q)$, and $[\widehat{p}](R)$ of \mathcal{M}_{∞} meet transversely at that point, so we get finally

$$a = (-1)^{N-k} (3.25)$$

as claimed above.

Landau-Ginzburg Formulation

Write

$$c_t(E^*) = \sum_{i=0}^k x_i t^i, \tag{3.26}$$

with $x_i = c_i(E^*)$. Define functions $y_j(x_i), j \ge 0$ by

$$\frac{1}{c_t(E^*)} = \sum_{j \ge 0} y_j t^j. \tag{3.27}$$

Classically, the cohomology ring of G(k, N) is described by the relations

$$y_j = 0$$
, for $N - k + 1 \le j \le N$. (3.28)

Let

$$-\log c_t(E^*) = \sum_{r>0} U_r(x_1, \dots, x_k) t^r.$$
 (3.29)

So

$$-t^{j}c_{t}(E^{*})^{-1} = -\frac{\partial}{\partial x_{j}}\log c_{t}(E^{*}) = \sum_{r>0} \frac{\partial U_{r}}{\partial x_{j}}t^{r}.$$
(3.30)

Hence if

$$W_0 = (-1)^{N+1} U_{N+1} (3.31)$$

then

$$\frac{\partial W_0}{\partial x_i} = (-1)^N y_{N+1-i}, \text{ for } 1 \le i \le k.$$
(3.32)

So the defining relations of the classical cohomology take the form

$$dW_0 = 0. (3.33)$$

To obtain in a similar way the quantum cohomology ring, set

$$W = W_0 + (-1)^k x_1. (3.34)$$

The relations dW = 0 now give

$$y_{N+1-i} + (-1)^{N-k} \delta_{i,1} = 0. (3.35)$$

Therefore the relation $c_t(E^*) \cdot (\sum_j y_j t^j) = 1$ becomes

$$\left(\sum_{i=0}^{k} x_i t^i\right) \cdot \left(\sum_{j=0}^{N-k} y_j t^j - (-1)^{N-k} t^N + O(t^{N+1})\right) = 1.$$
 (3.36)

Keeping only the terms of order at most t^N , this becomes

$$\left(\sum_{i=0}^{k} x_i t^i\right) \cdot \left(\sum_{j=0}^{N-k} y_j t^j\right) = 1 + (-1)^{N-k} t^N.$$
 (3.37)

This coincides with the quantum cohomology ring as described in (3.16). The function W is called the Landau-Ginzburg potential.

If we introduce the roots of the Chern polynomial

$$c_t(E^*) = \prod_{i=1}^k (1 + \lambda_i t),$$
 (3.38)

then W can be written

$$W(\lambda_1, \dots, \lambda_k) = \frac{1}{k+1} \sum_{j=1}^k \left(\lambda_j^{N+1} + (-1)^k \lambda_j \right).$$
 (3.39)

Now let us discuss integration. Integration defines a linear functional on the top dimensional cohomology of G(k, N), which is the cohomology in real dimension

2k(N-k):

$$f \to I(f) = \int_{G(k,N)} f. \tag{3.40}$$

Since $H^{2k(N-k)}(G(k,N))$ is one dimensional, any two linear functionals on that space are proportional. Such a linear functional can be obtained as follows in the Landau-Ginzburg description. We examine the classical case first. If we consider $x_i = c_i(E^*)$ to be of degree i, then the top dimensional cohomology consists of polynomials f of degree k(N-k) modulo the ideal generated by $\partial W_0/\partial x_i$, $i=1\ldots k$. Consider the linear form on homogeneous polynomials of degree k(N-k) defined by

$$J(f) = (-1)^{k(k-1)/2} \left(\frac{1}{2\pi i}\right)^k \oint dx_1 \dots dx_k \frac{f}{\prod_{i=1}^k \partial W_0 / \partial x_i}.$$
 (3.41)

The integration contour is a product of circles enclosing the poles in the denominator. J(f) annihilates the ideal generated by dW_0 , since if f is divisible by, say, $\partial W/\partial x_i$, then one of the denominators in (3.41) is canceled and one of the contour integrals vanishes.

For f of degree k(N-k), the integral in (3.41) is unaffected if W_0 is replaced by W; this follows from taking the contour integral on (3.41) to be a large contour. The integral can then be evaluated as a simple sum of residues:

$$J(f) = (-1)^{k(k-1)/2} \sum_{dW=0} \frac{f}{\det\left(\frac{\partial^2 W}{\partial x_i \partial x_j}\right)}.$$
 (3.42)

It is convenient to change variables from the x_i to the λ_a . One has

$$\det\left(\frac{\partial^2 W}{\partial x_i \partial x_j}\right)\Big|_{dW=0} = \det\left(\frac{\partial^2 W}{\partial \lambda_a \partial \lambda_b}\right) \cdot \det\left(\frac{\partial \lambda_a}{\partial x_i}\right)^2. \tag{3.43}$$

The Jacobian in the change of variables from x_i to λ_a is the Vandermonde deter-

minant:

$$\det\left(\frac{\partial x_i}{\partial \lambda_a}\right)^2 = \prod_{a \le b} (\lambda_a - \lambda_b)^2. \tag{3.44}$$

So

$$J(f) = \frac{(-1)^{k(k-1)/2}}{k!} \sum_{dW(\lambda_a)=0} \frac{f \cdot \prod_{a < b} (\lambda_a - \lambda_b)^2}{\prod_a dW/d\lambda_a} = \frac{(-1)^{k(k-1)/2}}{k!(2\pi i)^k} \oint d\lambda_a \frac{f \cdot \prod_{a < b} (\lambda_a - \lambda_b)^2}{\prod_a dW/d\lambda_a}.$$
(3.45)

The integration contour in each λ_a integral is a circle running counterclockwise around the origin. A factor of k! comes here because, as the map from the λ_a to the x_i is of degree k!, each critical point of $W(x_i)$ corresponds to k! critical points of $W(\lambda_a)$.

To verify that J(f) is correctly normalized to coincide with I(f), we set $f = c_k(E^*)^{N-k} = \prod_a \lambda_a^{N-k}$. According to (3.20), I(f) = 1. To verify that J(f) = 1, we use the contour integral version of (3.45). In the denominator we can replace $dW/d\lambda_a = \lambda_a^N + (-1)^k$ by $\lambda_a^N - \lambda_a^{N-k}$ without changing the behavior on a large contour enough to affect the integral. Then

$$J(f) = \frac{(-1)^{k(k-1)/2}}{k!(2\pi i)^k} \oint d\lambda_1 \dots d\lambda_k \frac{\prod_{a < b} (\lambda_a - \lambda_b)^2}{\prod_c (\lambda_c^k - 1)}.$$
 (3.46)

The integral is easily done as a sum of residues. The poles are at $\lambda_a^k = 1$, for $1 \leq a \leq k$. Because of the Vandermonde determinant in the numerator, the λ_a must be distinct. Up to a permutation, one must have $\lambda_a = \exp(2\pi i a/k)$; evaluating the residue at this value of the λ_a and including a factor of k! from the sum over permutations, one gets J(f) = 1.

3.3. Quantum Field Theory Interpretation

Physicists would never actually begin with the definition that I have given above for the quantum cubic form. Rather, everything begins with considerations on the function space $W = \text{Maps}(\Sigma, G(k, N))$. Physicists are mainly interested in quantum field theory, which is conveniently formulated in terms of integration over spaces such as W.

For instance, let Σ be a complex Riemann surface with Hodge duality operator *, pick a Hermitian metric on G(k,N) (such as the natural U(k)-invariant metric), and for a map $\Phi: \Sigma \to G(k,N)$, set

$$L(\Phi) = \int_{\Sigma} (d\Phi, *d\Phi). \tag{3.47}$$

Then in the "bosonic sigma model with target space G(k, N)" we consider integrals such as

$$\int_{\mathcal{W}} D\Phi \exp\left(-\frac{L(\Phi)}{\lambda}\right),\tag{3.48}$$

with λ a positive real number. This is not complete pie in the sky. For instance, to make the definition more concrete, one can triangulate Σ and make a finite dimensional approximation to the integral. Then the problem is to adjust λ , while refining the triangulation, so that the given integral (and related ones) converges as the triangulation is infinitely refined.

For a homogeneous space of positive curvature such as the Grassmannian, one knows at a physical level of rigor precisely how to do this: λ must be taken to vanish in inverse proportion to the logarithm of the number of vertices in the triangulation. This is a consequence of a phenomenon known as "asymptotic freedom," which plays a crucial role in the theory of the strong interactions in four dimensions; sigma models with targets such as the Grassmannian were intensively studied in the late 1970's and early 1980's as simple cases of asymptotically free quantum

field theories. Asymptotic freedom actually plays an important role in our story, since it leads to the mass gap that will be essential in §4.

Supersymmetric Sigma Models

What we actually want to do is to transfer the integral over the space of holomorphic maps that defined the quantum cohomology ring,

$$\langle \alpha, \beta, \gamma \rangle = \sum_{\lambda} \int_{\mathcal{M}_{\lambda}} \widehat{\alpha}(P) \cup \widehat{\beta}(Q) \cup \widehat{\gamma}(R),$$
 (3.49)

to an integral over the space W of all maps of Σ to G(k, N). Reversing the usual logic, this is done as follows. The condition that $\Phi: \Sigma \to G(k, N)$ is holomorphic is an equation

$$0 = \left(\overline{\partial}\Phi\right)^{1,0} \tag{3.50}$$

which asserts the vanishing of a section

$$s: \Phi \to (\overline{\partial}\Phi)^{1,0}$$
 (3.51)

of an infinite dimensional vector bundle Y over \mathcal{W} . (Y is the bundle whose fiber at $\Phi \in \mathcal{W}$ is the space of (0,1) forms on Σ with values in $\Phi^*(T^{1,0}G(k,N))$, with $T^{1,0}G(k,N)$ being the (1,0) part of the complexified tangent bundle of G(k,N). The point of the definition is just that $(\overline{\partial}\phi)^{1,0}$ is a vector in Y.)

The space \mathcal{M} of holomorphic maps, being defined by the vanishing of a section $s: \mathcal{W} \to Y$, is Poincaré dual to the Euler class $\chi(Y)$ of Y. So formally we can write

$$\langle \alpha, \beta, \gamma \rangle = \int_{\mathcal{M}} \widehat{\alpha} \cup \widehat{\beta} \cup \widehat{\gamma} = \int_{\mathcal{W}} \widehat{\alpha} \cup \widehat{\beta} \cup \widehat{\gamma} \cup \chi(Y). \tag{3.52}$$

Now, there are any number of ways to write a differential form representing $\chi(Y)$, but one nice way (formulated mathematically by Mathai and Quillen [37]) uses a

section s and has a nice exponential factor $\exp(-|s|^2/\lambda)$, with $|s|^2$ the norm of s with respect to a metric on Y, and λ a positive real number. For the section indicated in (3.51), the norm with respect to the natural metric is

$$|s|^2 = \int_{\Sigma} (d\Phi, *d\Phi), \tag{3.53}$$

which is precisely the Lagrangian introduced above for the bosonic sigma model with target space the Grassmannian.

So the long and short of it is that we get a representation

$$\langle \alpha, \beta, \gamma \rangle = \int_{\mathcal{W} \times \dots} \int D\Phi \ \dots \ \exp\left(-\frac{1}{\lambda} \int_{\Sigma} |d\Phi|^2 + \dots\right) \widehat{\alpha}(P) \widehat{\beta}(Q) \widehat{\gamma}(R), \quad (3.54)$$

much like the bosonic sigma model, but with "fermions," represented by "..." The quantum field theory that appears here is in fact a twisted form of the usual supersymmetric nonlinear sigma model, as I explained in [38]; in the twisted model, the fermions can be interpreted in terms of differential forms on the function space W. The relation to the Mathai-Quillen formula was explained by Atiyah and Jeffrey [39] in the analogous case of four dimensional Donaldson theory.

It should be fairly obvious that instead of G(k, N) we could use a general Kahler manifold X in the above discussions, at least at the classical level. (If we are willing to give up the interpretation in terms of twisting of a unitary supersymmetric model, we can even consider almost complex manifolds that are not Kahler.) At the quantum level, the situation is more subtle. There are two main branches in the subject. If $c_1(X) = 0$, the supersymmetric sigma model (with a suitable choice of the Kahler metric of X) is conformally invariant; such models provide classical solutions of string theory. On the other hand, if $c_1 > 0$, as in the case of the Grassmannian, one is in a quite different world, with asymptotic freedom and analogs of the mass generation and chiral symmetry breaking seen in the strong interactions.

3.4. Strategy

To try to say something of substance in this situation, we use the realization of G(k,N) as $\mu^{-1}(0)/U(k)$, where μ is the moment map from \mathbf{C}^{kN} to the Lie algebra of U(k). One is tempted to try to lift a map $\Phi: \Sigma \to G(k,N)$ to a map $\widehat{\Phi}: \Sigma \to \mathbf{C}^{kN}$. There is not a natural way to do this, and there may even be a topological obstruction.

So instead we proceed as follows. Let P be a principal U(k) bundle over Σ , A a connection on P, $\widehat{\Phi}$ a section of $P \times_{U(k)} \mathbf{C}^{kN}$, and S a two-form on Σ with values in the adjoint bundle ad(P). Take

$$\widehat{L}(\widehat{\Phi}, A, S) = \int_{\Sigma} \left((d_A \widehat{\Phi}, *d_A \widehat{\Phi}) + i(S, \mu \circ \widehat{\Phi}) \right). \tag{3.55}$$

Then classically the theory described by $\widehat{L}(\widehat{\Phi},A,S)$ is equivalent to the bosonic sigma model with target G(k,N). This can be seen as follows. The Euler-Lagrange equation of S is $\mu \circ \widehat{\Phi} = 0$, so, under the natural projection $P \times_{U(k)} \mathbf{C}^{kN} \to \mathbf{C}^{kN}/U(k)$, $\widehat{\Phi}$ maps to $\Phi : \Sigma \to \mu^{-1}(0)/U(k) = G(k,N)$. The Euler-Lagrange equation for A identifies P and A with the pull-back by $\widehat{\Phi}$ of the tautological principal U(k) bundle and connection over G(k,N). Once these restrictions and identifications are made, $\widehat{L}(\widehat{\Phi},A,S)$ reduces to the Lagrangian $L(\Phi)$ of the bosonic sigma model of the Grassmannian.

This sort of reasoning is still valid quantum mechanically. For instance, using

$$\int_{-\infty}^{\infty} \frac{dx}{2\pi} e^{ixy} = \delta(y) \tag{3.56}$$

(and the obvious generalization of that formula to several variables) we get the path integral formula

$$\int DS \exp\left(-i\int_{\Sigma} (S, \mu \circ \widehat{\Phi})\right) = \delta(\mu \circ \widehat{\Phi}). \tag{3.57}$$

So the S integral places on $\widehat{\Phi}$ precisely the restriction that one would guess from the classical Euler-Lagrange equations. From simple properties of Gaussian integrals (which are introduced below), one similarly deduces that, quantum mechanically as classically, the A integral has the effect of identifying P, A with the pull-backs of the tautological objects over the Grassmannian.

Similar reasoning holds after including fermions, so we get for the quantum cubic form a representation of the general kind

$$\langle \alpha, \beta, \gamma \rangle = \int D\widehat{\Phi} \, DA \, DS \, \dots \exp\left(-\int_{\Sigma} \left((d_A \widehat{\Phi}, *d_A \widehat{\Phi}) + i(S, \mu \circ \widehat{\Phi}) + \dots \right) \right) \cdot \widehat{\alpha}(P) \widehat{\beta}(Q) \widehat{\gamma}(R).$$
(3.58)

As before, "..." represents terms involving fermions that are not indicated explicitly.

3.5. Reversing the Order Of Integration

In sum, (3.58) will reduce to (3.54) if we integrate over A and S first. To get something interesting, we instead integrate first over $\widehat{\Phi}$. The key point is that $\widehat{\Phi}$ is a section of a bundle over Σ with linear fibers (a \mathbf{C}^{kN} bundle) and that \widehat{L} is quadratic in $\widehat{\Phi}$. Consequently, the $\widehat{\Phi}$ integral is a Gaussian integral.

The basic one dimensional formula

$$\int_{-\infty}^{\infty} \frac{dx}{\sqrt{2\pi}} \exp(-\lambda x^2/2) = \frac{1}{\sqrt{\lambda}}$$
 (3.59)

has the n dimensional generalization

$$\int_{-\infty}^{\infty} \frac{dx_1 \dots dx_n}{(2\pi)^{n/2}} \exp\left(-\frac{1}{2} \sum_{i,j} M_{ij} x_i x_j\right) = \frac{1}{\sqrt{\det M}},\tag{3.60}$$

for any quadratic form M with positive real part; this is demonstrated by picking a coordinate system in which $M = \text{diag}(m_1, \ldots, m_n)$.

In our case the $\widehat{\Phi}$ integral is (apart from terms involving fermions)

$$\int D\Phi \exp\left(-\int_{\Sigma} \left((d_A \widehat{\Phi}, *d_A \widehat{\Phi}) + i(S, \mu \circ \widehat{\Phi}) \right) \right). \tag{3.61}$$

This is an infinite dimensional Gaussian integral with M the quadratic form associated with the elliptic differential operator

$$M' = (d_A^* d_A + iS) \otimes 1_N. (3.62)$$

The notation reflects the fact that the \mathbf{C}^{kN} bundle of which $\widehat{\Phi}$ is a section is actually a sum of N copies of a \mathbf{C}^k bundle, and M is the sum of N copies of a quadratic form derived from an operator (namely $d_A^*d_A + iS$) on sections of that \mathbf{C}^k bundle. So the integral over $\widehat{\Phi}$ gives

$$\frac{1}{\sqrt{\det M'}} = \left(\frac{1}{\sqrt{\det(d_A^* d_A + iS)}}\right)^{N/2}.$$
(3.63)

The determinant of the elliptic differential operator $d_A^*d_A + iS$ can be conveniently defined using the ζ -function regularization of Ray and Singer.

So modulo fermions we get

$$\langle \alpha, \beta, \gamma \rangle = \int DA \, DS \dots \left(\frac{1}{\sqrt{\det(d_A^* d_A + iS)}} \right)^{N/2} \cdot \widehat{\alpha} \widehat{\beta} \widehat{\gamma}.$$
 (3.64)

So we have transformed the problem of computing the quantum cohomology of the Grassmannian to a problem involving integration over the connection A and over S – a problem in quantum gauge theory. This brings us into an entirely different world, that of §2 of this paper.

As this stage we can see why – as topologists might expect – the sigma model of G(k, N) simplifies in the limit of k fixed, $N \to \infty$. The integrand in (3.64)

has a sharp peak at the minimum of the determinant, and "everything" can be calculated in an asymptotic expansion in powers of 1/N, by expanding around this peak.

For fixed N, it is not true that "everything" can be calculated, but the topological quantities can be, reducing to a saddle point by a more elaborate argument. The essence of the matter is that although the classical Lagrangian \widehat{L} is conformally invariant, the quantum theory is not (because, for instance, with Ray-Singer or any other regularization, the determinant introduced above is not conformally invariant). The topological quantities are however not just conformally invariant but completely independent of the metric of Σ . Scaling up the metric of Σ by a very large real factor, life simplifies because of the basic physical properties of the model – asymptotic freedom and the dynamically generated mass gap. At very large distances (that is, if the metric on Σ is scaled up by a very big factor), the complicated integral over A, S, and fermions in (3.64) reduces to a local and tractable quantum field theory – in fact it reduces to the gauged WZW model (of U(k)/U(k)) that was analyzed in §2.

There is a basic principle here: every quantum field theory with a mass gap reduces at very big distances to a topological field theory. Often the topological field theory that so arises is more or less trivial, but in the case of the supersymmetric sigma model of the Grassmannian, it is the gauged WZW model. This large distance reduction of the Grassmannian sigma model to a gauged WZW model, plus the relation explained in §2 between the gauged WZW model and the Verlinde algebra, give the relation between the quantum cohomology of the Grassmannian and the Verlinde algebra.

It is well known that at large distances, massive particles can be neglected and massless particles dominate. Less fully appreciated is that beyond the reach of the propagating fields, a non-trivial dynamics of the vacuum or topological field theory may prevail.

Differential Geometry Of The Moduli Space Of Bundles

A detailed discussion of the reduction of the sigma model to the gauged WZW model will be the subject of §4, but here I will make a few naive remarks. The integrand in (3.64) actually has its maximum for flat connections – with some branching at the points $P, Q, R \in \Sigma$ at with $\widehat{\alpha}, \widehat{\beta}$, and $\widehat{\gamma}$ are inserted. The moduli space of such flat connections is (by a theorem of Mehta and Seshadri [28]) the same as the moduli space \mathcal{R} of rank k stable holomorphic vector bundles over Σ , with some parabolic structure at P, Q, R determined by the branching. So the integral gives some differential geometry of \mathcal{R} . (In view of (2.66), \mathcal{R} can be interpreted as a space of classical solutions of the gauged WZW model.) In the large N limit, direct analysis of the determinant in (3.64) shows that the differential geometric quantity that appears is the volume of the symplectic manifold \mathcal{R} , times $N^{\text{dimc }\mathcal{R}}$. This is the leading large N behavior of the Riemann-Roch formula for the dimension of the space $H^0(\mathcal{R}, \mathcal{L}^{N-k})$ of non-abelian theta functions. This simple direct argument relates the quantum cohomology of G(k, N) to the dimension of the space of non-abelian theta functions for large N.

The only way I know to establish this as an exact relation, not just as asymptotic one for large N, is to reduce the sigma model of G(k, N) to the gauged WZW model as we will do in §4, and then study that model as in §2.

4. From The Grassmannian To The G/G Model

This section is organized as follows. After recalling some background about $\mathcal{N}=2$ models in two dimensions in §4.1, we construct in §4.2 the sigma model whose target space is the Grassmannian G(k,N). Then we analyze its behavior at long distances in §4.3-6. In §4.7-8, we enter the computational stage and work things out in detail in the simplest non-trivial case.

In the past, the long distance behavior of the Grassmannian sigma model has been analyzed on \mathbb{R}^2 [7–10]; the main results were spontaneously broken chiral

symmetry, the existence of a mass gap, and a determination for large N of the spectrum of low-lying states. The novelty here is to examine the long distance behavior more globally, uncovering the relation to the gauged WZW model and thereby (in view of $\S 2$) the Verlinde algebra.

I will make a small change in notation in this section. In §2, we considered general compact Lie groups, and (as is conventional in mathematics) we took the Lie algebra to consist of anti-hermitian matrices (so the quadratic form (a, b) = Tr ab is negative definite). The reason that this convention is standard for general Lie groups is that in the case of a real group, whose representations may also all be real, it is unnatural to introduce factors of i and therefore the group generators are naturally anti-hermitian. In this section, the gauge group will be the unitary group U(k), which will arise in a natural complex representation, and I will follow the standard physics convention that the group generators are hermitian matrices; thus (a,b) = Tr ab will be positive definite. The complexification of the Lie algebra of U(k) consists of all $k \times k$ complex matrices; if σ is such a matrix, then $\overline{\sigma}$ will denote its hermitian adjoint.

4.1. Background

We will work in $\mathcal{N}=2$ superspace in two dimensions, conventions and the basic setup being as explained in [40,41]. The detailed formulas of this subsection are presented mainly for reference, and most readers will want to skim them.

We consider first flat superspace with bosonic coordinates x^m , m = 0, 1 (and Lorentz signature -+) and fermionic coordinates θ^{α} , $\overline{\theta}^{\alpha}$. In a light-cone basis, supersymmetry is realized geometrically by the operators

$$Q_{\pm} = \frac{\partial}{\partial \theta^{\pm}} + i\overline{\theta}^{\pm} \left(\frac{\partial}{\partial x^{0}} \pm \frac{\partial}{\partial x^{1}} \right)$$

$$\overline{Q}_{\pm} = -\frac{\partial}{\partial \overline{\theta}^{\pm}} - i\theta^{\pm} \left(\frac{\partial}{\partial x^{0}} \pm \frac{\partial}{\partial x^{1}} \right).$$
(4.1)

These operators commute with the superspace covariant derivatives

$$D_{\pm} = \frac{\partial}{\partial \theta^{\pm}} - i\overline{\theta}^{\pm} \left(\frac{\partial}{\partial x^{0}} \pm \frac{\partial}{\partial x^{1}} \right)$$

$$\overline{D}_{\pm} = -\frac{\partial}{\partial \overline{\theta}^{\pm}} + i\theta^{\pm} \left(\frac{\partial}{\partial x^{0}} \pm \frac{\partial}{\partial x^{1}} \right)$$
(4.2)

which are used in constructing Lagrangians.

To formulate gauge theory, one introduces a gauge field in superspace, replacing the differential operators D_{α} , \overline{D}_{α} , and $\partial_{m} = \partial/\partial x^{m}$ by gauge covariant derivatives \mathcal{D}_{α} , $\overline{\mathcal{D}}_{\alpha}$, and \mathcal{D}_{m} . On the superspace gauge fields one imposes the very strong constraints

$$0 = \{ \overline{\mathcal{D}}_{\alpha}, \overline{\mathcal{D}}_{\beta} \} = \{ \mathcal{D}_{\alpha}, \mathcal{D}_{\beta} \}$$

$$\{ \mathcal{D}_{\pm}, \overline{\mathcal{D}}_{\pm} \} = 2i \left(\mathcal{D}_{0} \pm \mathcal{D}_{1} \right).$$
 (4.3)

Among other things, these conditions permit the existence of "chiral superfields," superspace fields Φ obeying

$$\overline{\mathcal{D}}_{\alpha}\Phi = 0. \tag{4.4}$$

With the aid of the constraints one can take locally

$$\mathcal{D}_{\alpha} = e^{-V} D_{\alpha} e^{V}$$

$$\overline{\mathcal{D}}_{\alpha} = e^{V} \overline{D}_{\alpha} e^{-V}$$
(4.5)

where V is a real Lie algebra-valued function on superspace, called a vector superfield. After also fixing some residual gauge invariance, one can go to a "Wess-Zumino gauge," in which

$$V = \theta^{-}\overline{\theta}^{-}(v_{0} - v_{1}) + \theta^{+}\overline{\theta}^{+}(v_{0} + v_{1}) - \sqrt{2}\sigma\theta^{-}\overline{\theta}^{+} - \sqrt{2}\overline{\sigma}\theta^{+}\overline{\theta}^{-} + 2i\theta^{-}\theta^{+}\left(\overline{\theta}^{-}\overline{\lambda}_{-} + \overline{\theta}^{+}\overline{\lambda}_{+}\right) + 2i\overline{\theta}^{+}\overline{\theta}^{-}(\theta^{+}\lambda_{+} + \theta^{-}\lambda_{-}) + 2\theta^{-}\theta^{+}\overline{\theta}^{+}\overline{\theta}^{-}D.$$

$$(4.6)$$

Here v_m is an ordinary two-dimensional gauge field, and the other fields are bose and fermi matter fields. σ is a complex $k \times k$ matrix, and – as V is hermitian –

 $\overline{s}igma$ is its hermitian adjoint. We write $F = F_{01} = \partial_0 v_1 - \partial_1 v_0 + [v_0, v_1]$ for the curvature of v. The supersymmetry transformation laws for this multiplet are

$$\delta v_{m} = i\overline{\epsilon}\sigma_{m}\lambda + i\epsilon\sigma_{m}\overline{\lambda}$$

$$\delta \sigma = -i\sqrt{2}\overline{\epsilon}_{+}\lambda_{-} - i\sqrt{2}\epsilon_{-}\overline{\lambda}_{+}$$

$$\delta \overline{\sigma} = -i\sqrt{2}\epsilon_{+}\overline{\lambda}_{-} - i\sqrt{2}\overline{\epsilon}_{-}\lambda_{+}$$

$$\delta D = -\overline{\epsilon}_{+}(D_{0} - D_{1})\lambda_{+} - \overline{\epsilon}_{-}(D_{0} + D_{1})\lambda_{-} + \epsilon_{+}(D_{0} - D_{1})\overline{\lambda}_{+} + \epsilon_{-}(D_{0} + D_{1})\overline{\lambda}_{-}$$

$$+ \sqrt{2}\epsilon_{+}[\sigma, \overline{\lambda}_{-}] + \sqrt{2}\epsilon_{-}[\overline{\sigma}, \overline{\lambda}_{+}] + \sqrt{2}[\sigma, \lambda_{+}]\overline{\epsilon}_{-} + \sqrt{2}[\overline{\sigma}, \lambda_{-}]\overline{\epsilon}_{+}$$

$$\delta \lambda_{+} = i\epsilon_{+}D + \sqrt{2}(D_{0} + D_{1})\overline{\sigma}\epsilon_{-} - F_{01}\epsilon_{+} - [\sigma, \overline{\sigma}]\epsilon_{+}$$

$$\delta \lambda_{-} = i\epsilon_{-}D + \sqrt{2}(D_{0} - D_{1})\sigma\epsilon_{+} + F_{01}\epsilon_{-} + [\sigma, \overline{\sigma}]\epsilon_{-}$$

$$\delta \overline{\lambda}_{+} = -i\overline{\epsilon}_{+}D + \sqrt{2}(D_{0} + D_{1})\sigma\overline{\epsilon}_{-} - F_{01}\overline{\epsilon}_{+} + [\sigma, \overline{\sigma}]\overline{\epsilon}_{-}$$

$$\delta \overline{\lambda}_{-} = -i\overline{\epsilon}_{-}D + \sqrt{2}(D_{0} - D_{1})\overline{\sigma}\epsilon_{+} + F_{01}\overline{\epsilon}_{-} - [\sigma, \overline{\sigma}]\overline{\epsilon}_{-}.$$

$$(4.7)$$

The basic gauge invariant field strength is

$$\Sigma = \frac{1}{2\sqrt{2}} \{ \overline{\mathcal{D}}_{+}, \mathcal{D}_{-} \} = \sigma + i\sqrt{2}\theta^{+}\overline{\lambda}_{+} - i\sqrt{2}\overline{\theta}^{-}\lambda_{-} + \sqrt{2}\theta^{+}\overline{\theta}^{-}D$$

$$- i\overline{\theta}^{-}\theta^{-}(D_{0} - D_{1})\sigma - i\theta^{+}\overline{\theta}^{+}(D_{0} + D_{1})\sigma$$

$$+ \sqrt{2}\overline{\theta}^{-}\theta^{-}\theta^{+}(D_{0} - D_{1})\overline{\lambda}_{+} - \sqrt{2}\theta^{+}\overline{\theta}^{+}\overline{\theta}^{-}(D_{0} + D_{1})\lambda_{-} - i\sqrt{2}\theta^{+}\overline{\theta}^{-}F_{01}$$

$$- 2i\theta^{-}\overline{\theta}^{-}\theta^{+}[\sigma, \overline{\lambda}_{-}] - 2i\overline{\theta}^{-}\theta^{+}\overline{\theta}^{+}[\sigma, \lambda_{+}]$$

$$- \overline{\theta}^{-}\theta^{-}\theta^{+}\overline{\theta}^{+}((D_{0}^{2} - D_{1}^{2})\sigma - [\sigma, [\sigma, \overline{\sigma}]]) + i\theta^{-}\overline{\theta}^{-}\theta^{+}\overline{\theta}^{+}[\sigma, \partial_{m}v^{m}].$$

$$(4.8)$$

(The last term does not really spoil gauge invariance: the gauge transformations that preserve Wess-Zumino gauge have a certain θ dependence which requires this term to be present.) Σ is a twisted chiral superfield; this means that (by the Bianchi identity together with the constraints)

$$\overline{\mathcal{D}}_{+}\Sigma = \mathcal{D}_{-}\Sigma = 0. \tag{4.9}$$

With the aid of Σ , it is straightforward to construct gauge invariant La-

grangians. The standard gauge kinetic energy is

$$L_{g} = -\frac{1}{4e^{2}} \int d^{2}x \ d^{4}\theta \operatorname{Tr} \overline{\Sigma} \Sigma$$

$$= \frac{1}{e^{2}} \int d^{2}x \ \operatorname{Tr} \left(\frac{1}{2} F_{01}^{2} + |D_{0}\sigma|^{2} - |D_{1}\sigma|^{2} + i\overline{\lambda}_{-}(D_{0} + D_{1})\lambda_{-} + i\overline{\lambda}_{+}(D_{0} - D_{1})\lambda_{+} + \frac{1}{2} D^{2} - \frac{1}{2} [\sigma, \overline{\sigma}]^{2} - \sqrt{2}\lambda_{+} [\sigma, \overline{\lambda}_{-}] + \sqrt{2} [\overline{\sigma}, \lambda_{-}] \overline{\lambda}_{+} \right).$$

$$(4.10)$$

One more term constructed from gauge fields only is important. Using the fact that Σ is a twisted chiral superfield, there is an invariant interaction of the form

$$L_{D,\theta} = \frac{it}{2\sqrt{2}} \int d^2x \, d\theta^+ \, d\overline{\theta}^- \operatorname{Tr} \Sigma|_{\theta^- = \overline{\theta}^+ = 0} + c.c. = \int d^2x \left(-r \operatorname{Tr} D + \frac{\theta}{2\pi} \operatorname{Tr} F_{01} \right), \tag{4.11}$$

with

$$t = ir + \frac{\theta}{2\pi}. (4.12)$$

Matter Fields

Chiral superfields are functions Φ on superspace, transforming in some given unitary representation V of the gauge group, and obeying

$$\overline{\mathcal{D}}_{\pm}\Phi = 0. \tag{4.13}$$

Such a field has an expansion in components

$$\Phi = \phi + \sqrt{2}\theta^{\alpha}\psi_{\alpha} + \theta^{\alpha}\theta_{\alpha}F. \tag{4.14}$$

The supersymmetry transformation laws for this multiplet are (by dimensional

reduction from [40, p. 50])

$$\delta\phi = \sqrt{2} \left(\epsilon_{+} \psi_{-} - \epsilon_{-} \psi_{+} \right)$$

$$\delta\psi_{+} = i\sqrt{2} \left(D_{0} + D_{1} \right) \phi \overline{\epsilon}_{-} + \sqrt{2} \epsilon_{+} F - 2 \overline{\sigma} \phi \overline{\epsilon}_{+}$$

$$\delta\psi_{-} = -i\sqrt{2} \left(D_{0} - D_{1} \right) \phi \overline{\epsilon}_{+} + \sqrt{2} \epsilon_{-} F + 2 \sigma \phi \overline{\epsilon}_{-}$$

$$\delta F = -i\sqrt{2} \overline{\epsilon}_{+} \left(D_{0} - D_{1} \right) \psi_{+} - i\sqrt{2} \overline{\epsilon}_{-} \left(D_{0} + D_{1} \right) \psi_{-}$$

$$+ 2 \left(\overline{\epsilon}_{+} \overline{\sigma} \psi_{-} + \overline{\epsilon}_{-} \sigma \psi_{+} \right) + 2i \left(\overline{\epsilon}_{-} \overline{\lambda}_{+} - \overline{\epsilon}_{+} \overline{\lambda}_{-} \right) \phi.$$

$$(4.15)$$

The usual kinetic energy for a multiplet of such chiral superfields is

$$L_{ch} = \frac{1}{4} \int d^2x \, d^4\theta \, \overline{\Phi} \Phi = \int d^2x \, \left(|D_0\phi|^2 - |D_1\phi|^2 + |F|^2 + i\overline{\psi}_+(D_0 - D_1)\psi_+ + i\overline{\psi}_-(D_0 + D_1)\psi_- + \overline{\phi}D\phi - \overline{\phi}\{\sigma, \overline{\sigma}\}\phi - \sqrt{2}\overline{\psi}_+\overline{\sigma}\psi_- - \sqrt{2}\overline{\psi}_-\sigma\psi_+ + i\sqrt{2}\overline{\psi}_+\overline{\lambda}_-\phi - i\sqrt{2}\overline{\psi}_-\overline{\lambda}_+\phi + i\sqrt{2}\overline{\phi}\lambda_+\psi_- - i\sqrt{2}\overline{\phi}\lambda_-\psi_+ \right).$$

$$(4.16)$$

The $|D_{\alpha}\phi|^2$ term in (4.16) is the conventional free kinetic energy corresponding to a sigma model with a flat metric on $V \cong \mathbb{C}^r$. The $\overline{\phi}D\phi$ term is the coupling of D to the moment map, in the sense that if we pick a basis T_a , $a = 1 \dots \dim G$ for the Lie algebra of G, then this term is

$$\int d^2x \sum_a D^a(\overline{\phi}, T_a \phi), \tag{4.17}$$

and the functions $(\overline{\phi}, T_a \phi)$ are the components of the moment map.

A more general Kahler metric on V (and accordingly, a more general form of the moment map) could be obtained by replacing the function $\overline{\Phi}\Phi$ on the left hand side of (4.16) with a more general Kahler potential $K(\Phi, \overline{\Phi})$. These matters are explained in some detail in [42].

4.2. The Model

Now we can construct the actual model of interest. We take the gauge group to be G = U(k). We take kN chiral superfields Φ^{is} , $i = 1 \dots k$, $s = 1 \dots N$, regarded as N copies of the defining k dimensional representation of G. The action of G commutes with a global symmetry group $H \cong U(N)$ which one can think of as the unitary transformations of \mathbb{C}^N .

The Lagrangian that we actually wish to study is simply

$$L = L_{gauge} + L_{D,\theta} + L_{ch}. \tag{4.18}$$

The potential energy is determined by the following terms in L:

$$L_{pot} = \frac{1}{2e^2} \operatorname{Tr} D^2 - r \operatorname{Tr} D + \overline{\phi} D\phi - \frac{1}{2e^2} \operatorname{Tr} [\sigma, \overline{\sigma}]^2 - \overline{\phi} \{\sigma, \overline{\sigma}\} \phi. \tag{4.19}$$

Upon integrating out D, the potential energy is

$$V = \frac{e^2}{2} \sum_{i,j=1}^k \left(\sum_s \overline{\phi}_{is} \phi^{js} - \delta_i^{j} r \right)^2 + \frac{1}{2e^2} \operatorname{Tr}[\sigma, \overline{\sigma}]^2 + \overline{\phi} \{\sigma, \overline{\sigma}\} \phi. \tag{4.20}$$

The space of classical vacua is the space of zeroes of V up to gauge transformation. For V to vanish, ϕ must be non-zero, and therefore σ must vanish. As anticipated in §3, the first term in the potential is the square of the moment map for the action of U(k) on \mathbb{C}^{kN} . This term vanishes precisely if the vectors in \mathbb{C}^N represented by the rows of ϕ , divided by \sqrt{r} , are orthonormal. The k dimensional subspace $V \subset \mathbb{C}^N$ spanned by the rows of ϕ is gauge invariant, and is the only gauge invariant data determined by ϕ (since any two orthonormal bases of V are related by the action of U(k)). Moreover, every k dimensional subspace $V \subset \mathbb{C}^N$ has such an orthonormal basis. So the space of classical vacua is the Grassmannian G(k, N) of k dimensional subspaces of \mathbb{C}^N .

Since the condition for vanishing energy is

$$\sum_{s} \overline{\phi}_{is} \phi^{js} = \delta_i{}^j r, \tag{4.21}$$

the radius of the space of vacua is \sqrt{r} and the Kahler class is proportional to r. Classically, the space of vacua shrinks to a point for r=0; for r<0 the classical energy can no longer vanish and it appears that supersymmetry is spontaneously broken. Quantum mechanically, the situation is rather different and there is a smooth continuation to negative r with unbroken supersymmetry, as discussed (for k=1) in [41,§3.2]; the existence of this continuation will be exploited below.

The choice of a classical vacuum spontaneously breaks the symmetry group U(N) to $U(k) \times U(N-k)$, while leaving supersymmetry unbroken. The oscillations in the vacuum are massless Goldstone bosons at the classical level. Their supersymmetric partners are, of course, also massless classically. Other modes are readily seen to have masses proportional to e. The model therefore reduces at long distances (or equivalently for $e \to \infty$) to the supersymmetric nonlinear sigma model with target space the Grassmannian; we will more briefly call this the Grassmannian sigma model.

At the quantum level, spontaneous breaking of a continuous symmetry such as the U(N) symmetry of this model is not possible in two dimensions. The symmetry must therefore be restored by quantum corrections. Exhibiting this symmetry restoration, and the associated mass gap, was a primary goal of early investigations of the model.

R Symmetries

A right-moving R-symmetry in an $\mathcal{N}=2$ model in two dimensions is a symmetry under which $\theta^+ \to e^{i\alpha}\theta^+$, $\overline{\theta}^+ \to e^{-i\alpha}\overline{\theta}^+$, while $\theta^-, \overline{\theta}^-$ are invariant. A left-moving R-symmetry obeys the analogous condition with θ^+ and θ^- exchanged.

The Grassmannian sigma model as constructed above has at the classical level

a right-moving R-symmetry J_R^* under which the charges of the various fields are as follows: $(\psi_+, F, \sigma, \lambda_-)$ have charges (-1, -1, 1, 1), their complex conjugates have opposite charge, and other fields have charge zero. Similarly there is classically a left-moving R-symmetry J_L under which $(\psi_-, F_i, \sigma, \lambda_+)$ have charges (-1, -1, -1, 1), their complex conjugates have opposite charges, and other fields are neutral.

At the quantum level, the sum $J_V = J_R + J_L$ is a "vector" symmetry, that is, it transforms left- and right-moving fermions the same way, so it is free of anomaly and generates a U(1) symmetry.

However, the "axial" combination $J_A = J_R - J_L$ is anomalous. The anomaly can be described as follows. Let a be a U(1) connection with first Chern class 1, and embed this in G = U(K) so that the U(k) gauge field is v = diag(a, 0, 0, ..., 0). In such an instanton field, the index of the Dirac operator acting on ψ_+ is N^{\dagger} . Similarly the ψ_- index is -N. The total anomaly in $J_A = J_R - J_L$ is the difference of these or 2N. The anomaly in any instanton field would be an integer multiple of this.

So J_A is conserved only modulo 2N. The only symmetries we can construct from J_A are the discrete transformations $\exp(2\pi i t J_A/2N)$, with $t \in \mathbf{Z}$. This gives a discrete group, isomorphic to \mathbf{Z}_{2N} , of chiral symmetries. If unbroken, these symmetries would prevent ψ and λ from gaining a mass. One of the main results of the old literature on this model was that this \mathbf{Z}_{2N} is spontaneously broken down to \mathbf{Z}_2 , making a mass gap possible; the surviving \mathbf{Z}_2 is just the operation $(-1)^F$ that counts fermions modulo two.

The Twisted Model

Any $\mathcal{N}=2$ supersymmetric theory in two dimensions with an R symmetry

 $[\]star$ We will somewhat imprecisely use the symbol J_R to denote either the current or the corresponding charge; and similarly for other currents introduced momentarily.

[†] The index is defined as the number of ψ_+ zero modes minus the number of $\overline{\psi}_+$ zero modes. Recall that ψ_+ transforms as a sum of N copies of the defining k-dimensional representation of U(k); each of these contributes 1 to the index.

can be twisted to obtain a topological field theory. The construction, as explained in [43], which the reader can consult for details, involves adding to the usual stress tensor the derivative of the R-current. As we have just seen, in the case of the Grassmannian there is only one anomaly-free R-symmetry. Consequently, only one twisted topological field theory can be constructed; it is related to the quantum cohomology of the Grassmannian, which was introduced in §3.

In going from the untwisted to the twisted model, the spin of every field decreases (in the convention of [41]) by $J_V/2$. For instance, in the untwisted model, ψ_+ and $\overline{\psi}_+$ have spin 1/2 and $J_V = \mp 1$, so in the twisted model they have respectively spin 1 and 0. More generally the fermi fields that have spin zero in the twisted model are $\overline{\psi}_+, \psi_-, \overline{\lambda}_-, \lambda_+$. If the twisted model is formulated on \mathbf{S}^2 , the spin zero fields each have one zero mode and the spin one fields have none. Since $\overline{\psi}_+, \psi_-, \overline{\lambda}_-, \lambda_+$ have kN, kN, k^2, k^2 components respectively and have $J_A = 1, 1, -1, -1$, the total J_A value of the zero modes is $kN + kN - k^2 - k^2 = 2k(N - k)$, and this is the anomaly in J_A conservation due to coupling to the curvature of S^2 . (Not coincidentally, 2k(N - k) is the dimension of G(k, N).) More generally, on a surface of genus g, the spin one fields would have g zero modes, so the violation of J_A is

$$\Delta J_A = 2k(N-k)(1-g) = k(N-k) \int_{\Sigma} d^2x \sqrt{h} \frac{R}{2\pi}.$$
 (4.22)

Here I have written the Euler characteristic of Σ , which of course equals 2(1-g), as the familiar curvature integral.

Fermionic Symmetry Of The Twisted Theory

The untwisted theory, formulated on a flat world-sheet, possesses fermionic symmetries, that were described in detail in equations (4.7), (4.15). After twisting, the fermionic parameters ϵ_+ and $\overline{\epsilon}_-$ in the transformation laws have spin zero; let Q_- and \overline{Q}_+ be the symmetries generated by those transformations, and let $Q = Q_- + \overline{Q}_+$. By a standard calculation, $Q_-^2 = \overline{Q}_+^2 = \{Q_-, \overline{Q}_+\} = 0$ and in

particular $Q^2 = 0$. Moreover, the stress tensor can be written as $T = \{Q, \Lambda\}$ for some Λ . It follows that if we restrict ourselves to operators that are annihilated by Q (or more exactly to cohomology classes of such operators), the theory can be interpreted as a topological field theory. Each cohomology class of Q-invariant operators has representatives annihilated by both Q_- and \overline{Q}_+ .

The relevant observables are easily found. The transformation laws of the topological theory are found from the microscopic transformation laws (4.7), (4.15) by setting $\epsilon_{-} = \overline{\epsilon}_{+} = 0$ and keeping $\epsilon_{+}, \overline{\epsilon}_{-}$. By inspection of the transformation laws, σ (but not $\overline{\sigma}$) is invariant, so that any gauge invariant holomorphic function of σ is a suitable vertex operator in the topological theory. Such functions are linear combinations of characters, so the basic operators constructed this way are

$$O_V(x) = \text{Tr}_V \,\sigma(x),\tag{4.23}$$

with V an irreducible representation of G = U(k) and Tr_V the trace in that representation.

Actually, these are the only relevant operators. In fact, even before twisting, the model (for r >> 0) is equivalent at long distances, as we saw above, to a sigma model with target the Grassmannian G(k, N). Consequently, the twisted model is simply the standard A model of G(k, N) (the A model for any Kahler target is explained in detail in [43]) so the cohomology classes of observables are in one-to-one correspondence with the de Rham cohomology of G(k, N).

Indeed, upon integrating out the massive fields, $\sigma(x)$ turns into a bilinear expression in massless fermions tangent to G(k, N), with values in the adjoint representation of U(k). It is easy to calculate this explicitly in the weak coupling, low energy limit. To this aim, we need only evaluate a tree diagram, and we can ignore the kinetic energy of the massive σ field. The relevant part of the Lagrangian is simply

$$-\int d^2x \left(\sum_{ijs} \overline{\phi}_{is} \{\sigma, \overline{\sigma}\}^i{}_j \phi^{js} + \sqrt{2} \sum_{ijs} \overline{\psi}_{+is} \overline{\sigma}^i{}_j \psi_-{}^{js} \right). \tag{4.24}$$

Because of the overall U(N) invariance, it suffices to work out the effective operator representing σ in the low energy theory at one particular point on G(k, N). We take this to be the point represented by $\phi^{is} = \sqrt{r}\delta^{is}$ for $1 \leq s \leq k$, $\phi^{is} = 0$ for s > k. With this choice, (4.24) becomes

$$-\int d^2x \left(2r \operatorname{Tr} \overline{\sigma}\sigma + \sqrt{2} \sum_{ijs} \overline{\psi}_{+is} \overline{\sigma}^i{}_j \psi_{-}{}^{js}\right). \tag{4.25}$$

The quickest way to evaluate the tree diagram is simply to impose the equation of motion of $\overline{\sigma}$; this gives

$$\sigma^{j}{}_{i} = -\frac{1}{r\sqrt{2}} \sum_{s} \overline{\psi}_{+is} \psi_{-}^{js}. \tag{4.26}$$

In the interpretation of the low energy theory in terms of differential forms on G(k, N), ψ_-/\sqrt{r} and $\overline{\psi}_+/\sqrt{r}$ are (1, 0) and (0, 1) forms. $(\psi_- \text{ and } \overline{\psi}_+ \text{ have been normalized to have canonical kinetic energies; their natural normalization as differential forms involves dividing by <math>\sqrt{r}$.) So σ is represented in the low energy theory by a (1, 1) form or more exactly by the operator in the G(k, N) model determined by this (1, 1) form.

The chosen vacuum $\phi^{is} = \sqrt{r}\delta^{is}$ is invariant up to a gauge transformation under a subgroup $U(k) \times U(N-k)$ of U(N). For σ to be a U(N)-invariant form in the adjoint representation of the gauge group, it must transform in the adjoint representation of the unbroken U(k) (since the unbroken symmetry is a mixture of this with a gauge transformation) and be invariant under U(N-k). The U(N) action can then be used to extend σ in a unique way to an invariant (1,1) form on G(k,N). It is evident that the right hand side of (4.26) has the required properties.

Conversely, the right hand side of (4.26) is the only bilinear expression in $\overline{\psi}_+$ and ψ_- with the claimed properties, so any adjoint-valued U(N)-invariant (1,1) form would be a multiple of σ . Such a form is the curvature of the tautological U(k) bundle E^* with its natural connection. So up to a constant, which I will not verify directly (it can be absorbed in the constant later called c), σ coincides in

the low energy theory with the tautological curvature. Hence classical expressions $O_V = \text{Tr}_V \sigma$ coincide with the corresponding polynomials in Chern classes on G(k, N), and as quantum operators in the twisted theory, the O_V coincide with the elements of the quantum cohomology determined by those classes. The fact that the tautological classes generate the cohomology of G(k, N) ensures that the O_V span the space of observables of the twisted theory.

There is a more conceptual approach to identifying σ with the tautological curvature which I will indicate very briefly. Let $\lambda_{-} = \eta_{0} - \eta_{1}$, $\overline{\lambda}_{+} = \eta_{0} + \eta_{1}$. Restrict to the diagonal fermionic symmetry with $\epsilon_{+} = \overline{\epsilon}_{-} = \epsilon$. Then a key part of the symmetry algebra is

$$\delta v_m = 2i\epsilon \eta_m$$

$$\delta \eta_m = \sqrt{2}\epsilon D_m \sigma$$

$$\delta \sigma = 0.$$
(4.27)

This multiplet describes the equivariant cohomology of the gauge group acting on the space \mathcal{A} of connections. The interpretation of σ as the curvature of the tautological bundle over the quotient is standard in equivariant cohomology. This interpretation holds independent of any specific Lagrangian model; the salient feature of the particular model we are considering is that the connection v_m is identified via the low energy equations of motion with the pullback of the tautological connection on $E^* \to G(k, N)$.

<u>Instantons</u>

A correlation function

$$\left\langle \prod_{i} O_{V_i}(x_i) \right\rangle \tag{4.28}$$

on a Riemann surface Σ can be computed as follows. The O_{V_i} determine cohomology classes of G(k, N) as we have just seen; pick Poincaré dual cycles H_i . Let d be the non-negative integer, if any, such that the moduli space of holomorphic maps

 $\Phi: \Sigma \to G(k,N)$ obeying

$$\Phi(x_i) \in H_i \tag{4.29}$$

has virtual dimension zero. The correlation function (4.28) is zero if such a d does not exist; otherwise it is

$$\left\langle \prod_{i} O_{V_i}(x_i) \right\rangle = \exp(-dr) \cdot N_{\{H_i\}} \tag{4.30}$$

with $N_{\{H_i\}}$ the "number" of holomorphic maps $\Phi: \Sigma \to G(k, N)$ that obey (4.29). (In general, in defining this number, one must make a suitable perturbation of the equation to avoid possible degeneracies; that is why I have put the word "number" in quotes.) (4.30) follows from the standard description of the A model, as explained in [43].

More microscopically, to see the appearance of instantons, one can begin with the transformation laws (4.7), (4.15). The calculation of the correlation function in (4.28) can be localized, by a standard argument, on the fixed points of Q_-, \overline{Q}_+ . An analysis as in [41], pp. 184-8, identifies those fixed points (for r >> 0) with the holomorphic maps of Σ to the Grassmannian. Those holomorphic maps appear in precisely the form in which they were studied by Bertram, Daskapoulos, and Wentworth [11].

4.3. Some Renormalization Factors

Before analyzing the quantum theory, I want to first point out a few details involving renormalization.

Any topological field theory in two dimensions could be modified by the addition of a term

$$\Delta L = a \int_{\Sigma} d^2x \sqrt{h} \frac{R}{2\pi} \tag{4.31}$$

without affecting the topological invariance. The affect of this is merely to multiply a genus g amplitude by a factor of $\exp(a(2-2g))$.

An important role in the analysis will be played by the Kahler parameter r. For instance, r enters in the basic formula (4.30) expressing correlation functions in terms of instantons. However, as we see in (4.30), the r dependence of a degree d instanton contribution is known a priori. Moreover, because of the \mathbf{Z} grading of the classical cohomology, every given correlation function in genus g receives a contribution at most only from one known value of d. Therefore, there is no material loss in setting r to 0, and we will do that eventually in §4.7.

Another normalization question involves the possibility of multiplying an operator of degree w by a factor $\exp(uw)$ with some constant u. One can show by keeping track of the classical \mathbf{Z} grading that this can be absorbed in adding constants to r and a. This normalization question will arise below because we will find that the field σ of the Grassmannian sigma model has a macroscopic interpretation as

$$\sigma = cg, \tag{4.32}$$

where c is a constant that we will determine only approximately and g is the elementary field of a gauged WZW model.

In practice, in our computations we will not try to determine the precise values of a and c. At the end, when we enter the computational stage, we will identify the values of these parameters by checking a couple of special cases of the formulas.

4.4. Quantum Properties Of The Model

We come finally to the point of the present paper – the calculation mapping the Grassmannian sigma model onto the G/G model, and thence the Verlinde algebra. The calculations themselves are not new [7–10], and I will therefore present them rather briefly. What is new is the result that we will get by considering these computations in a global context.

We begin with the expression for D in terms of matter fields that is obtained

by varying the potential energy term (4.19) with respect to D:

$$-\frac{1}{e^2}D^i{}_j = \sum_{s=1}^N \phi^{is}\overline{\phi}_{js} - r\delta^i_s. \tag{4.33}$$

At the classical level, for r >> 0, vanishing of the $D^i{}_j$ – which is needed to set the energy to zero – requires that the ϕ^{is} should have non-zero vacuum expectation values. This in turn spontaneously breaks the global U(N) symmetry (and ensures the existence of massless Goldstone bosons and the absence of a mass gap). Such spontaneous breaking of a continuous symmetry is, however, impossible in two dimensions.

The resolution of this conundrum has long been known. Quantum mechanically the operator $O^i{}_j = \sum_{s=1}^N \phi^{is} \overline{\phi}_{js}$ can have an expectation value even if the ϕ^{is} do not. If this expectation value can equal $r\delta^i{}_j$, then the $D^i{}_j$ can vanish without spontaneous breaking of the U(N) symmetry.

To investigate this phenomenon, let us compute the expectation value of O^{i}_{j} . We will first do this in a naive approximation, treating the ϕ 's as free fields with the mass term that we can read off from the classical Lagrangian. Then we will discuss the conditions for validity of the approximation. We will do the calculation on Euclidean \mathbb{R}^{2} , making the standard Wick rotations from the Lorentz signature Lagrangian given above.

The mass term for the ϕ field in the Lagrangian is $\sum_{i,j,s} \overline{\phi}_{is} \{\sigma, \overline{\sigma}\}^i{}_j \phi^{is}$, with $\{\cdot, \cdot\}$ the anticommutator. Treating the ϕ 's as free fields with that mass term, the expectation value of $O^i{}_j$ is simply

$$\langle O \rangle = N \int \frac{d^2k}{(2\pi)^2} \frac{1}{k^2 + \{\sigma, \overline{\sigma}\}}.$$
 (4.34)

The factor of N comes from summing over s.

The integral in (4.34) is logarithmically divergent. The divergence can be regularized by subtracting a similar integral with $\{\sigma, \overline{\sigma}\}$ replaced by a multiple of

the identity, say $2\mu^2$, with μ an arbitrary "subtraction point." The subtraction can be interpreted as an additive renormalization of r. After this regularization, the integral can be evaluated, and one gets

$$\langle O \rangle = -\frac{N}{4\pi} \ln \left(\{ \sigma, \overline{\sigma} \} / 2\mu^2 \right).$$
 (4.35)

The condition for D to vanish in this approximation is hence that

$$-\frac{N}{4\pi}\ln\left(\{\sigma,\overline{\sigma}\}/2\mu^2\right) - r = 0,\tag{4.36}$$

or

$$\{\sigma, \overline{\sigma}\} = 2\mu^2 \exp\left(-4\pi r/N\right). \tag{4.37}$$

This is however only a necessary condition for vanishing of the energy. Another condition comes from the presence in the classical Lagrangian of a term proportional to $\text{Tr}[\sigma, \overline{\sigma}]^2$. This term gives a contribution to the energy that vanishes precisely when $[\sigma, \overline{\sigma}] = 0$, so in seeking to describe the vacuum, we may assume that σ and $\overline{\sigma}$ commute and therefore rewrite (4.37) in the form

$$\sigma \overline{\sigma} = \mu^2 \exp\left(-4\pi r/N\right). \tag{4.38}$$

This means that

$$\sigma = cg \tag{4.39}$$

with g a unitary matrix -g = 1 – and c the constant

$$c = \mu \exp\left(-2\pi r/N\right). \tag{4.40}$$

Thus, we have obtained a kind of sigma model with a field g taking values in the unitary group.

The vacuum expectation value of σ that we have just found gives a positive mass squared to the ϕ^{is} , so that they will have zero vacuum expectation value, restoring the U(N) symmetry. However, the discrete chiral symmetry (conservation of J_A modulo 2N) is spontaneously broken in this process. Indeed, since σ has $J_A = 2$, the vacuum expectation value of σ breaks \mathbf{Z}_{2N} down to \mathbf{Z}_2 . (For instance, this is discussed in detail for k = 1 on p. 310 of [8].) As the broken symmetry is discrete, this does not produce Goldstone bosons and is compatible with the existence of a mass gap. In fact, the broken symmetry helps in getting a mass gap, since most of the fermions obtain masses at tree level from the vacuum expectation value of σ .

Validity Of The Approximation

Before proceeding to unravel further subtleties, let us discuss the conditions for validity of the approximation.

The traditional region of validity of the above approximation, as in [7,8], is k fixed, $N \to \infty$, with r and $1/e^2$ of order N. In this limit, the corrections to the approximation (of treating the ϕ 's as free fields with a σ -dependent mass) are of order 1/N. The above computation is part of the beginning of a systematic expansion of all physical observables in powers of 1/N. Many important features of the theory involve properties that are stable under perturbation – like whether there is a mass gap, what symmetries are spontaneously broken, and certain aspects of the topological sector. For addressing such questions, the 1/N expansion is good enough for fixed k and sufficiently big N.

That is not enough for us, because we wish to relate the Verlinde algebra to the cohomology of the Grassmannian for all k and N. Happily, there is another region of validity of the approximation. At the classical level, the matrix O is positive definite, and accordingly for r < 0 it would be impossible for the energy to vanish. Quantum mechanically, because of the subtraction that was needed in the above computation, O is not positive definite. Accordingly, zero energy is possible also for negative r at the quantum level; indeed, the solution (4.38) makes sense for

either sign of r. (The continuation of the model to negative r was discussed in [41,§3.2] for the case k = 1.)

I claim that for any k and N, the computation leading to (4.38) is a valid approximation for r << 0. The reason for this is that the approximate vacuum state given by this computation has exponentially large σ for $r \to \infty$. This gives an exponentially large mass to the Φ multiplet, so ϕ and ψ loops can be ignored except perhaps for renormalization effects involving diagrams with poor ultraviolet convergence. In this super-renormalizable theory, the only such diagram is the one loop diagram whose evaluation leads to (4.38).

The quantum cohomology of the Grassmannian involves, naively, the behavior for r >> 0. However, because the first Chern class of G(k, N) is positive, every topological correlation function of the twisted theory (of operators of definite dimension or J_A) receives a contribution only from one value of the instanton number and hence depends on r as $\exp(-dr)$ with a known constant d that appeared in (4.30). The behavior for r << 0 therefore determines the behavior for r >> 0. Consequently, the fact that our approximation is valid for the theory continued to r << 0 means that it is good enough for studying the topological sector of the twisted theory. We will now explore the implications.

4.5. THE MASS GAP AND THE WZW ACTION

Because σ was determined to be an arbitrary unitary matrix (times a fixed constant), it appears at first sight that the model has a continuous vacuum degeneracy and therefore massless particles, at least in this approximation. This can hardly be correct because the massless σ particles, subject to the constraint (4.38), do not furnish a representation of $\mathcal{N}=2$ supersymmetry. (The ϕ fields are massive in this approximation, as we have noted, and so cannot help.) This puzzle was resolved in the old literature in the context of the 1/N expansion; the resolution involves giving a mass to σ by mixing with the U(k) gauge field v. (See, for instance, p. 308 of [8] for k=1 and the discussion of the $\phi_5 - \lambda$ propagator on pp. 165-6 of

[9] for general k.) I will not present the detailed computations here, as they are standard; I will merely summarize them and focus on the interpretation, which is all that is new.

The 1/N expansion amounts to integrating out the chiral superfields Φ^{is} to obtain an effective action for the gauge multiplet. The $\sigma - v$ mixing comes from the one loop diagram of figure (1). The non-vanishing contribution is the one in which the particles running around the loop are fermions. However, perhaps even more fundamental is the one-loop diagram with external sigma fields only and internal fermions, shown in figure (2). The fermions ψ^{is} , for $s = 1 \dots N$, form N copies of the fundamental representation of U(k). Let us suppress the s index and look at a single multiplet ψ^i . The key point is that, looking back to the Lagrangian (4.16), the fermions receive their mass from a coupling $-\sqrt{2}\,\overline{\psi}_{-i}\sigma^{i}{}_{j}\psi_{+}{}^{j}-c.c.$ This coupling breaks the $U(k)_L \times U(k)_R$ chiral symmetry of the fermion kinetic energy down to a diagonal U(k). Therefore, when we integrate out the fermions to get an effective action for σ , we are dealing with the standard problem of integrating out massive fermions that receive their mass from spontaneous chiral symmetry breaking. It is precisely in connection with this problem that the anomalous Wess-Zumino interaction, defined in (2.9), was originally discovered. The long wavelength limit of the effective interaction obtained by integrating out σ is therefore – allowing for the N multiplets – precisely

$$L_{eff}(\sigma) = N\Gamma(\sigma). \tag{4.41}$$

The form of this interaction is completely determined by $U(k)_L \times U(k)_R$ invariance and the chiral anomaly. (Let me warn that reader that the $U(k)_L \times U(k)_R$ symmetry just invoked is explicitly broken by interactions, such as the gauge couplings, that do not contribute to $L_{eff}(\sigma)$ in leading order in 1/N. The corrections to the leading large N behavior are the subject of the next sub-section.)

Now we include the gauge fields and Feynman diagrams such as that of figure (1). Such diagrams must extend (4.41) to a gauge invariant effective action $L_{eff}(\sigma, v)$. The minimal choice, in some sense, is the gauge invariant extension of the Wess-Zumino action that was defined in (2.25):

$$L_{eff}(\sigma, v) = N\Gamma(\sigma, v). \tag{4.42}$$

Is this minimal form correct? Apart from terms that vanish by the equations of motion and terms of higher dimension that can be ignored at long distances, a non-minimal gauge invariant term (on a flat world sheet) would have to be of the form

$$\int_{\Sigma} \operatorname{Tr} FW(\sigma), \tag{4.43}$$

with $F = dv + v \wedge v$ and W a function of σ that transforms in the adjoint representation. In the next sub-section, we will show that such a term is not generated, even by corrections to the 1/N expansion. We will also discuss the role of the terms that vanish by the equations of motion, and some curvature-dependent terms.

Synthesis

If we take the Lagrangian $N\Gamma(\sigma, v)$ by itself, it describes a level N gauged WZW model of U(k)/U(k). This sort of model was analyzed in §2, and as we know from §2.5, it describes a topological field theory. There are no propagating modes at all, massless or massive. If one adds conventional kinetic energy for σ and v (such terms are certainly present in our underlying Lagrangian), one has propagating modes but massive ones. Indeed the conventional kinetic energy is irrelevant in the infrared and the large distance behavior is that of the gauged WZW model.

Thus, the Grassmannian sigma model – even if one does not restrict a priori to its topological sector – reduces at long distances to a topological field theory. In fact, any theory with a mass gap will do this, since at distances at which the massive particles can be neglected, all that survives is dynamics of the vacuum or topological field theory.

In the case of the Grassmannian sigma model, there was an underlying topological sector, described in §4.2, and visible from the classical Lagrangian before any analysis of its quantum properties. The basic observable in this topological sector was the σ field that appears in (4.42) (but now restricted to $\overline{\sigma}\sigma$ = constant). Thus the topological sector, defined microscopically, passes over at large distances to the gauged WZW model governing the σ field. This is the mapping from the quantum cohomology of the Grassmannian to the gauged WZW model (and thence the Verlinde algebra) that is the main goal of this paper.

In what follows, we will analyze the corrections to the 1/N expansion and eventually pin down the details of the mapping from the quantum cohomology to the gauged WZW model.

Search For Manifest Supersymmetry

It would be attractive to find an extension of (4.42), including the fermi partners of σ and v, with manifest $\mathcal{N}=2$ supersymmetry. Of course, the one-loop effective action from which (4.42) was defined is such an extension, but it would be nice to find, for instance, a compact description in $\mathcal{N}=2$ superspace of a local interaction describing the long-wavelength part of the one-loop effective action. I have been unable to do this and leave it as an interesting question.

However, let us truncate to the abelian case in which σ , v, and their fermionic partners are diagonal matrices (for instance $\sigma = \operatorname{diag}(\sigma_1, \ldots, \sigma_k)$). In this case, the field strength is similarly diagonal (say $\Sigma = \operatorname{diag}(\Sigma_1, \ldots, \Sigma_k)$). With this truncation, it is possible to find an explicit, local superspace interaction that describes all of the anomalous interactions. This interaction (which in the abelian case was discussed in [41], §3.2), is

$$L_{\mathcal{N}=2} = \frac{1}{\sqrt{2}} \sum_{i=1}^{k} \int d^2x \, d\theta^+ \, d\overline{\theta}^- \left(\frac{it\Sigma_i}{2} - \frac{N}{2\pi} \Sigma_i \ln(\Sigma_i/\mu) \right) + c.c. \tag{4.44}$$

The ease of writing this interaction in the diagonal case and the difficulty of describing its full non-abelian generalization may be related to the utility of abelianization

in the next subsection.

4.6. Corrections

Now we turn to analyzing the corrections to this approximation. We can ignore operators of dimension higher than two, which are irrelevant at long distances. Terms of dimension less than two, such as (2.78), cannot arise as they would violate the underlying $\mathcal{N}=2$ supersymmetry or (as a consequence) the topological invariance of the twisted sector. Also, we can ignore terms that vanish by the v equations of motion and so can be eliminated, as in §2.5, by a redefinition of v. Such terms (which are of the form $\int_{\Sigma} \sigma^*(B)$, with B an adjoint-invariant two-form on U(k)) would play no role in our subsequent analysis.

We are left with three issues to consider:

- (1) First, there might be corrections to the discrete data, the "level" of the effective WZW model. In the one loop approximation, we found the level to be N; however, corrections of relative order 1/N could shift this by a constant. Actually, this has to be formulated more precisely because the Lie algebra of U(k), which we will call u(k), is not simple; it can be split as $su(k) \oplus u(1)$, where su(k) consists of the traceless $k \times k$ hermitian matrices and u(1) is the center of u(k). In general, one could have a gauged WZW model for U(k) of level (N_1, N_2) , by which I mean that the Lagrangian would be determined by the quadratic form (\cdot, \cdot) on u(k) such that $(a, b) = N_1 \operatorname{Tr} ab$ for $a, b \in su(k)$, and $(a, b) = N_2 \operatorname{Tr} ab$ for $a, b \in u(1)$. Thus, the first correction to the 1/N approximation might lead to $(N_1, N_2) = (N + u, N + v)$ where u, v are integers (perhaps depending on k); higher order corrections in 1/N must vanish as they could not be integral for all N.
 - (2) Second, the low energy effective action might contain a term

$$\int_{\Sigma} \operatorname{Tr} FW(\sigma), \tag{4.45}$$

with as above F the u(k) curvature, and W a function of σ that transforms in the adjoint representation. Though this term vanishes by the equations of motion

of the low-energy gauged WZW model and so could be eliminated even from the quantum theory by a field redefinition (as described in §2.5), it could still play a role that will be explained later.

Note that a constant term in W (that is, a multiple of the identity) could be absorbed in an additive renormalization of t; we will not try to determine such a renormalization, and all of our statements about W will hold modulo an additive constant.

(3) Finally, we need to know whether, when the twisted theory is formulated on a curved world-sheet, the effective Lagrangian contains a term

$$\Delta L = \int_{\Sigma} d^2x \sqrt{h} \frac{R}{4\pi} U(\sigma), \qquad (4.46)$$

with R the world-sheet curvature and $U(\sigma)$ a function invariant under conjugation. As we have discussed in §2.5, any continuous deformation of the gauged WZW model that preserves the topological invariance and cannot be eliminated by a change of variables is of this form. (By contrast, the deformations considered above in (1) are discrete, not continuous, and the deformations in (2) can be removed by a change of variable.)

Now, here are the answers that I will claim for these three questions:

- (A1) I will claim that the level of the effective gauged WZW model is really (N-k, N). The correction can be thought of as a 1/N correction that comes from integrating out the U(k) gauge multiplet (the u(1) level is not shifted since the gauge multiplet is neutral under u(1)).
- (A2) I will claim that W=0, in other words that no term of the form (4.45) is generated.
 - (A3) I will claim that a term of the form (4.46) is generated, with

$$U = (N - k) \ln \det \sigma + \text{constant.}$$
 (4.47)

This might be regarded as the minimal possibility compatible with the anomaly

formula (4.22).

Abelianization

Now I will explain how I will do the calculation. A 1/N expansion will not suffice, since we do not want to be limited to sufficiently large N. Instead, we will study the theory in the alternative regime of $r \ll 0$.

To identify the corrections to the effective action of the three types discussed above, it suffices to work in the region of field space in which σ is a diagonal matrix with distinct eigenvalues $\sigma_1, \ldots, \sigma_k$. Moreover, we impose the condition of unbroken supersymmetry (or vanishing vacuum energy); in the approximation of (4.38) – which is valid for r << 0 – the condition is

$$\overline{\sigma}_i \sigma_i = \mu^2 \exp(-4\pi r/N), \text{ for } i = 1, \dots, k.$$
 (4.48)

The distinct values of the σ_i break U(k) to a diagonal subgroup $U(1)^k$. Calculations are relatively easy because the chiral superfields and the "off-diagonal" part of the gauge multiplet have large masses, of order $\overline{\sigma}\sigma$, which can be read off from the classical Lagrangian. The fields which remain massless in this approximation (and actually get masses at one loop, smaller by a factor of e^2) are the diagonal part of the gauge multiplet. The effective action for the massless modes, including the one loop correction, has already been written with manifest $\mathcal{N}=2$ supersymmetry in (4.44). This is a kind of gauged WZW model of $U(1)^k$. So in this regime, we get a kind of abelianization of the Grassmannian sigma model.

This should not come as a complete surprise, since as we have recalled in §2.6, the gauged WZW model has a precisely analogous abelianization. Now, we will have to be careful in using abelianization to compute the effects of types (1), (2), and (3), because in going from the gauged WZW model to its abelianization, precisely analogous terms are generated. These were computed in [5] and described in §2.6, and are as follows.

- (B1) The shift in level in going from the gauged WZW model of U(k) to its abelianization is (k,0). (There is obviously no shift of the u(1) level under abelianization since u(1) is already abelian.)
 - (B2) No term of the form (4.45) is generated.
- (B3) The term of the form (4.46) that is generated in abelianizing the gauged WZW model was presented in equation (2.85).

Now in verifying claims (A1), (A2), and (A3), we will integrate out from the Grassmannian sigma model the fields that, in the abelianized regime, have tree level masses; thus we will get the precise abelianized theory that is equivalent to the Grassmannian sigma model. Then we will interpret the result as a sum of two contributions: the terms claimed in (A1), (A2), and (A3) which describe how to go from the topological sector of the Grassmannian sigma model to an equivalent gauged WZW model; and the terms (B1), (B2), and (B3), which arise in abelianization of the gauged WZW model.

So claims (A1), (A2), and (A3) are equivalent to the following claims, which are the ones that we will actually check:

- (C1) After abelianization, there is no shift in the level of the Grassmannian sigma model from the naive result (N, N). We interpret this to mean that the topological sector of the Grassmannian sigma model is equivalent to a gauged WZW model of U(k)/U(k) at level (N k, N), and the level of that model is shifted by (k, 0) upon abelianization.
- (C2) There will be no induced term of the type (4.45), in abelianizing either of the two models or in comparing them.
- (C3) The induced term of type (4.46) in abelianization of the Grassmannian model will be the sum of (4.47) and the contribution (2.85) that arises in abelianizing the gauged WZW model. The sum of these is simply

$$\widetilde{U}(\sigma) = (N-1) \ln \det \sigma - \sum_{i \neq j} \ln(\sigma_i - \sigma_j).$$
 (4.49)

The Calculation

Now I will explain the calculation justifying (C1), (C2), and (C3). In discussing (C1) and (C2), world-sheet curvature is irrelevant, and we can work on a flat \mathbb{R}^2 .

(C1) and (C2) can be taken together and deduced from the following principle. Suppose that a u(1) gauge field v, with field strength f = dv, is coupled to a Dirac fermion χ , of charge q. Let χ have a mass term

$$L_{mass} = -\int_{\Sigma} d^2x \left(\overline{\chi}_- m \chi_+ + \overline{\chi}_+ \overline{m} \chi_- \right). \tag{4.50}$$

We want to integrate out χ to get an effective action for v. The dependence of the effective action on the phase of m comes only from the chiral anomaly and is

$$L_{\text{eff}} = \dots + q \int_{\Sigma} \frac{f}{(2\pi)} \text{Im} \log m.$$
 (4.51)

Now we look at the Grassmannian sigma model in the abelianized regime of r << 0, σ_i large (obeying (4.48)) and distinct. The chiral superfields Φ^{js} and the off-diagonal part of the gauge multiplet have bare masses of order $|\sigma_i|$. They can be integrated out in a one loop approximation; higher order corrections would be of order e^2 and irrelevant. Integrating out massive bosons does not give terms relevant to (C1) or (C2), while the contributions of fermions can be deduced from (4.51).

To do so explicitly, let v_i , i = 1 ... k, be the diagonal components of the gauge field. First we work out the contributions of chiral superfields. Each v_i is coupled to N chiral superfields Φ^{is} , s = 1 ... N, of charge 1. The fermi elements of these superfields have mass $\sqrt{2}\sigma_i$ (by inspection of (4.16)), so their contribution is

$$N\sum_{i=1}^{k} \int_{\Sigma} \frac{dv_i}{2\pi} \operatorname{Im} \ln \sigma_i. \tag{4.52}$$

This is the level N gauged WZW action computed of equation (4.42), specialized to the case that only the diagonal components of v and σ are non-zero.

Now, we come to the off-diagonal part of the gauge fields. Again, the relevant contribution comes from the phases of the masses of the off-diagonal fermions λ^{i}_{j} , $i \neq j$. Since (4.52) coincides with the level N gauged WZW action, the claims (C1) and (C2) amount to the assertion that no additional contribution will come from integrating out the λ^{i}_{j} .

By inspection of (4.10), the mass of $\lambda^i{}_j$ is $\sqrt{2}(\sigma_i - \sigma_j)$. The gauge field v_i interacts with the $\lambda^i{}_j$, $j \neq i$, of charge 1, and with the $\lambda^m{}_i$, $m \neq i$, of charge -1. Their contribution adds up to

$$\sum_{i,j} \int_{\Sigma} \frac{dv_i}{2\pi} \left(\operatorname{Im} \ln(\sigma_i - \sigma_j) - \operatorname{Im} \ln(\sigma_j - \sigma_i) \right). \tag{4.53}$$

This is zero, or more exactly, it is independent of the σ_i . Consequently it can be interpreted as a constant term in W or an additive renormalization of t; as noted in the paragraph following (4.45), we will not keep track of such effects. As for the diagonal components of λ , they are neutral and do not couple to the v_i .

It remains to discuss (C3). To this aim, we can take Σ to be a Riemann surface of genus zero and take the σ_i to be constants. As $\int_{\Sigma} d^2x \sqrt{h}R/4\pi = 1$ in genus zero, the claim (C3) amounts to the assertion that the path integral $\int D\Phi_i \dots e^{-L}$ is a constant multiple of

$$(\det \sigma)^{-(N-1)} \prod_{i \neq j} (\sigma_i - \sigma_j). \tag{4.54}$$

To verify this, we first integrate out the massive fields in the same one loop approximation as above. As before, the boson determinant is real and depends only on $|\sigma_i|$, while the fermion determinant has a phase that can be extracted from the chiral anomaly. Ordinarily, there is no chiral anomaly for fermions in a gravitational field in two dimensions, but the twisting to produce the topological theory involves a modification of the fermion kinetic energy that introduces such an anomaly.

We could proceed as above, starting with the anomaly formula analogous to (4.51). For the sake of variety, however, let us note that the anomaly can be captured by the path integral over the zero modes of the fermion kinetic energy. For instance, the fermions ψ^{is} from the chiral multiplets have components $\overline{\psi}_+, \psi_-$ of spin zero and other components of spin one. The zero modes of the fermion kinetic energy are the constant modes of $\overline{\psi}_+, \psi_-$, and the path integral over those modes is

$$\int d\overline{\psi}_{+is} d\psi_{-}^{jt} \exp\left(\sum_{is} \overline{\psi}_{+is} \overline{\sigma}_{i} \psi_{-}^{is}\right) = \det \overline{\sigma}^{N} = \operatorname{constant} \cdot \det \sigma^{-N}, \quad (4.55)$$

where we have used the fact that $\sigma \overline{\sigma} = \text{constant}$. Similarly, for the off-diagonal λ fields, the zero modes of the kinetic energy are the constant modes of $\overline{\lambda}_{-}$, λ_{+} , and the path integral over those modes is

$$\prod_{i \neq j} \int d\overline{\lambda}_{-j}^{i} d\lambda_{+}^{j} \exp\left(\sqrt{2}(\sigma_{i} - \sigma_{j})\overline{\lambda}_{-j}^{i} \lambda_{+j}^{j}\right) = \operatorname{const} \cdot \prod_{i \neq j} (\sigma_{i} - \sigma_{j}). \tag{4.56}$$

Comparing (4.55) and (4.56) with the claim made in (4.49) concerning (C3), we see that we are missing precisely one factor of $\det \sigma$. This must come from the remaining integral over the diagonal components of the gauge field. Indeed, though the diagonal fermions $\lambda^i{}_i$ are massless at tree level, they receive at the one loop level a mass term with the form

$$\sum_{i=1}^{k} \left(\lambda_{+}^{i} i \overline{\sigma}_{i}^{-1} \overline{\lambda}_{-}^{i} i + c.c. \right) \tag{4.57}$$

This term can be straightforwardly calculated or can be read off from the $\mathcal{N}=2$ extension (4.44) of the bosonic anomalous interactions. Integrating out the diagonal fermions therefore gives (up to a constant) a factor of

$$(\det \overline{\sigma})^{-1} = \operatorname{constant} \cdot (\det \sigma).$$
 (4.58)

This is the last factor needed for (C3).

This factor could in a more general way be predicted as follows. Just because the diagonal theory (including the λ^{i}_{i}) is a product of k sub-theories, the phase it produces must be of the form

$$\prod_{i=1}^{k} F(\sigma_i) \tag{4.59}$$

for some function F. Given this factorized form, to agree with the anomaly formula (4.22) it must be that $F(\sigma) = \sigma$.

4.7. THE VERLINDE ALGEBRA AND THE GRASSMANNIAN

In this subsection, we will put the pieces together and write down the precise connection between the Verlinde algebra and the quantum cohomology of the Grassmannian.

First of all, we consider the map from the cohomology ring of the Grassmannian to the Verlinde algebra. The quantum cohomology ring of G(k, N) is generated by operators of the form

$$O_V = \text{Tr}_V \, \sigma, \tag{4.60}$$

with V an irreducible representation of U(k). Thinking of σ is the curvature of the natural connection on the tautological rank k bundle over G(k, N), $\text{Tr}_V \sigma$ can be interpreted as a characteristic class of that bundle and hence a cohomology class of G(k, N).

On the other hand, working at long distances, σ becomes a unitary matrix (up to a constant that we will eventually pin down) and then $\text{Tr}_V \sigma$ can be interpreted as an operator in the effective gauged WZW model of U(k)/U(k). We worked out in (2.56),(2.58) the interpretation of this operator: it is the element of the Verlinde algebra determined by the representation V. Of course, from what we have said in §4.6, the Verlinde algebra in question is the one for the group U(k) at level (N-k,N).

So we have gotten the precise mapping from the cohomology of the Grassmannian to the Verlinde algebra. A couple of points should be clarified:

(1) It is essential that in mapping from the Grassmannian sigma model to the gauged WZW model, there is no correction of type (4.45). Such a term, since it vanishes by the equations of motion in the gauged WZW model, could be transformed away by a redefinition of σ and v. But the resulting redefinition of σ would cause the operator O_V to mix with similar operators for other representations. Thus, were terms with the structure (4.45) to appear, their precise form would enter in determining the map from the cohomology of G(k, N) to the Verlinde algebra.

By contrast, corrections to the gauged WZW action that vanish by the v equations of motion and so can be removed by redefinition of v are immaterial, since $O_V(\sigma)$ is independent of v. As noted at the beginning of §4.3, we have made no claim that corrections that vanish by the v equations of motion are not generated or have any particular structure.

(2) In the rest of this paper, we will set the Kahler parameter r to zero; as explained in §4.3, this involves no essential loss of information. Two other constants discussed in §4.3 also enter. One is the constant c in the relation $\sigma = c \cdot \text{unitary matrix}$, which we evaluated only approximately in (4.40). The other is the additive renormalization constant called a in (4.31), which is unknown since we did not attempt to determine the constant in (4.47). For the time being, we will set c and a to 1 and 0; eventually we will verify that this is correct (for r = 0) by checking special cases of the formulas.

Correlation Functions And The Metric

Now, let us determine precisely how the correlation functions and the metric in the Grassmannian sigma model compare to those in the gauged WZW model. The essential point that goes beyond what we have just said above is that one must include the correction term of (4.46), (4.47). By topological invariance, σ can be

treated as a constant, so the correction factor in the path integral is

$$\exp(-\Delta L) = \exp\left(-\int_{\Sigma} d^2x \sqrt{h} \frac{R}{4\pi} (N-k) \ln \det \sigma\right) = (\det \sigma)^{(g-1)(N-k)}. \quad (4.61)$$

The point or points at which det σ is inserted are immaterial. (It will turn out that det σ is an invertible element of the Verlinde algebra or quantum cohomology.)

So if $\langle \ \rangle_{G(k,N)}$ denotes an expectation value of the path integral of the Grassmannian sigma model, and $\langle \ \rangle_{WZW}$ denotes a path integral in the gauged WZW model, then the relation between these symbols in genus g is

$$\left\langle \prod_{i=1}^{s} \operatorname{Tr}_{V_i}(\sigma_i) \right\rangle_{G(k,N)} = \left\langle \prod_{i=1}^{s} \operatorname{Tr}_{V_i}(\sigma_i) \cdot (\det \sigma)^{(g-1)(N-k)} \right\rangle_{WZW}. \tag{4.62}$$

Henceforth we will abbreviate $\operatorname{Tr}_{V_i}(\sigma_i)$ as V_i . From (4.62) one can see that the natural metric on the cohomology of G(k, N) (given by Poincaré duality) does not coincide with the natural metric on the Verlinde algebra. Let us call these metrics (the sigma model and Verlinde metrics) g_{σ} and g_V , respectively. We recall that the metric is defined by a two point function in genus 0, so

$$g_{\sigma}(V_1, V_2) = \langle V_1 V_2 \rangle_{G(k,n)}$$

$$g_V(V_1, V_2) = \langle V_1 V_2 \rangle_{WZW} = \langle V_1 V_2 \cdot (\det \sigma)^{(N-k)} \rangle_{G(k,N)},$$

$$(4.63)$$

with the correlation functions being in genus zero.

Now let us compare the ring structure on the cohomology of the Grassmannian to the ring structure of the gauged WZW model. We recall that in either of the two theories, the ring structure is introduced by interpreting the genus zero three point function in terms of a binary operation, say $V_1, V_2 \rightarrow V_1 \cdot V_2$, according to the following formula:

$$\langle V_1 V_2 V_3 \rangle = g(V_1 \cdot V_2, V_3).$$
 (4.64)

The relation between the genus zero three point functions of the two theories is

from (4.62)

$$\langle V_1 V_2 V_3 \rangle_{WZW} = \langle V_1 V_2 V_3 (\det \sigma)^{(N-k)} \rangle_{G(k,N)}. \tag{4.65}$$

In particular the three point functions of the G(k, N) and gauged WZW models do not coincide. However, they differ by the same factor of $(\det \sigma)^{(N-k)}$ that enters in the relation between the metrics. This means in fact, upon putting together the last few formulas, that the multiplication laws are the same in the two theories.

So finally, our natural map from the quantum cohomology of the Grassmannian to the Verlinde algebra is a ring homomorphism – justifying terminology that was used above.

Non-Abelian Theta Functions

In the title of this paper and in much of the writing of it, I have emphasized the Verlinde algebra, which determines the dimension of the space of non-abelian theta functions. However, the above gives directly a formula for the dimension of the space \mathcal{H} of non-abelian theta functions without having to detour via the Verlinde algebra. Of course, we will count non-abelian theta functions for the group U(k) at level (N - k, N); because the U(1) theory is well understood, there is no essential difficulty in generalizing to other levels.

Let $\langle 1 \rangle^g$ denote the partition function in genus g. Then the dimension of \mathcal{H} on a Riemann surface Σ of genus g is

$$\dim \mathcal{H} = \langle 1 \rangle_{WZW}^g = \left\langle (\det \sigma)^{-(g-1)(N-k)} \right\rangle_{G(k,N)} = \left\langle (\det \sigma)^{k(g-1)} \right\rangle_{G(k,N)} \tag{4.66}$$

In the last expression, I have used the fact that $(\det \sigma)^N = 1$, as one can deduce from the Landau-Ginzburg description of the quantum cohomology (we do this below for k = 2). The right hand side of (4.66) can be evaluated by counting holomorphic maps of Σ to G(k, N) obeying certain conditions.

4.8. Getting Down To Earth

In this section, we will make everything completely explicit in the cases of k = 1 and k = 2. (Some of the issues are discussed by Gepner for general k in the last paper in [1].)

First we dispose of G(1, N), that is, \mathbb{CP}^{N-1} . Over \mathbb{CP}^{N-1} there is a tautological principal U(1) bundle P. Let W be the standard representation of U(1), of "charge one." Associated to P in the representation W is a line bundle \mathcal{L} . The representation W determines an operator $\mathrm{Tr}_W \sigma = \sigma$ which we will call x. In the sigma model, interpreting σ as the curvature of the natural connection on \mathcal{L} , $x = c_1(\mathcal{L})$.

The cohomology ring of \mathbb{CP}^{N-1} is generated by x and classically is $\mathbb{C}[x]/x^{N-1}$. But from [36] or the k=1 case of (3.16), the quantum cohomology ring is

$$R = \mathbf{C}[x]/(x^N - 1). \tag{4.67}$$

The metric on the cohomology determined by Poincaré duality is

$$q_{\sigma}(x^k, x^l) = \delta_{k+l} N_{-1},$$
 (4.68)

where in view of (4.67), k and l are evaluated modulo N.

On the other hand, in the gauged WZW model, $x = \text{Tr}_W \sigma$ should be identified with the element of the Verlinde algebra for U(1) at level N determined by the representation W. The structure of this algebra is well known. It is generated by W with the relation $W^N = 1$, just as in (4.67), and the metric is

$$g_V(W^k, W^l) = \delta_{k+l,0},$$
 (4.69)

with again k and l taken modulo N. (4.68) and (4.67) are related in the fashion predicted by (4.63). Indeed since for either metric g(a,b) = g(ab,1), (4.63) is

equivalent to

$$g_V(x^k, 1) = g_\sigma(x^k, x^{N-1}).$$
 (4.70)

That disposes of k = 1. Obviously, we cannot expect the non-abelian case k = 2 to be as trivial as that.

The Verlinde Algebra For k=2

First we describe explicitly (but not in a fully self-contained fashion) the Verlinde algebra of the group U(2) at the desired level (N-2, N).

We have an exact sequence

$$1 \to Z_2 \to SU(2) \times U(1) \xrightarrow{f} U(2) \to 1. \tag{4.71}$$

Here the map f is as follows: we identify SU(2) with the 2×2 unitary matrices of determinant 1, U(1) with 2×2 unitary matrices that are multiples of the identity, and for $x \in SU(2)$, $y \in U(1)$, let f(x,y) = xy.

The gauged WZW action of U(2) at level (N-2,N) restricts, if one takes the fields to lie in SU(2), to the SU(2) action at level N-2; but if one takes the fields to be in U(1), it restricts to the U(1) gauged WZW action at level 2N. (A factor of two arises simply because the trace of the identity matrix in the fundamental representation of U(2) is 2.) Therefore, we will proceed by comparing the Verlinde algebra of U(2) at level (N-2,N) to that of $SU(2) \times U(1)$ at level (N-2,2N).

The SU(2) Verlinde algebra was described explicitly in [14]. If V_1 is the two dimensional representation of SU(2), and V_n is its n^{th} symmetric tensor power, then the Verlinde algebra of SU(2) is the usual representation ring of SU(2), subject to the relation

$$V_{N-1} = 0. (4.72)$$

The representation ring of SU(2), subject to this relation, is spanned additively

by V_0, \ldots, V_{N-2} . The multiplication law can be described explicitly as

$$V_i \times V_j = \sum_t N_{ijt} V_t, \tag{4.73}$$

where N_{ijt} is 1 if the following relations and their cyclic permutations are obeyed:

$$i+j \ge t$$
, $N-2-i+j \ge N-2-t$, $2(N-2)-i-j \ge t$. (4.74)

Otherwise $N_{ijt} = 0$. The metric on the Verlinde algebra is

$$g(V_s, V_t) = \delta_{s,t}. (4.75)$$

The U(1) Verlinde algebra has already been introduced above. It is generated by the charge one representation W, and the defining relation at level 2N is

$$W^{2N} = 1. (4.76)$$

The metric is

$$g(W^u, W^v) = \delta_{u+v,0}. (4.77)$$

The $SU(2) \times U(1)$ Verlinde algebra at level (N-2,2N), is therefore spanned additively by the elements V_iW^j , for $i=0,\ldots,N-2,\ j=0,\ldots,2N-1$, corresponding to the representation $V_i \otimes W^{\otimes j}$. The multiplication law and metric are products of the multiplication law and metric of SU(2) and U(1).

Now we want to proceed to $U(2) = (SU(2) \times U(1))/\mathbb{Z}_2$. Dividing by \mathbb{Z}_2 halves the volume of the group manifold (if one uses a fixed Haar measure in an obvious sense). The Verlinde algebra is defined on a certain space of conformal blocks which can be constructed by quantizing an appropriate phase space \mathcal{M} – for instance, the phase space of the gauged WZW model. When the volume of the group is halved, the volume of \mathcal{M} is divided by $2^2 = 4^*$; therefore, in the semiclassical limit of large N, the Verlinde algebra of U(2) at level (N-2,N) will have one fourth the dimension of that of $SU(2) \times U(1)$ at level (N-2,2N).

One factor of two is obvious. Among all $SU(2) \times U(1)$ representations, we must restrict to those that are representations of U(2). This means keeping only V_iW^j with i+j even. The second factor of two is less obvious. One must impose the equivalence relation

$$V_i W^j = V_{N-2-i} W^{j+N}. (4.78)$$

I refer the interested reader to [44] for an explanation (in the analogous case of $SO(3) = SU(2)/\mathbf{Z_2}$) of such matters.

Note that if we set $\tau(V_iW^j) = V_{N-2-i}W^{j+N}$, then the Verlinde algebra of $SU(2) \times U(1)$ at level (N-2,N) obeys $\tau(a)b = a\tau(b) = \tau(ab)$. This ensures that the Verlinde algebra of $SU(2) \times U(1)$ induces a natural algebra structure on the quotient by the relations (4.78). This quotient algebra, restricted to i+j even (a τ -invariant condition) is the Verlinde algebra of U(2) at level (N-2,N). The metric is

$$g_V(V_i W^s, V_j W^t) = \delta_{i,j} \delta_{s+t,0} + \delta_{i,N-2-j} \delta_{s+t-N,0}.$$
 (4.79)

A complete but redundant set of relations for the U(2) Verlinde algebra would be the relations

$$V_{N-1} = 0, \quad W^{2N} = 1 \tag{4.80}$$

inherited from the SU(2) and U(1) algebras, along with (4.78). A special case of

^{*} \mathcal{M} is the moduli space of flat connections in genus one and consists of pairs of commuting elements of the gauge group G, divided by the Weyl group. Because one has a pair of elements of G, the volume of \mathcal{M} is decreased by a factor of n^2 if one divides G by a group of order n.

(4.78) is

$$V_{N-2}W^N = 1. (4.81)$$

It will become clear presently that (4.80) and (4.81) suffice to characterize the Verlinde algebra.

Representations And Characters

Two representations of the group U(2) will play a distinguished role. The first is the standard two dimensional representation \mathcal{V}_1 . Under restriction to $SU(2) \subset$ U(2), \mathcal{V}_1 restricts to the standard two dimensional representation V_1 of SU(2), and the scalars in U(2) act with charge 1. So \mathcal{V}_1 pulls back to the representation $V_1 \otimes W$ of $SU(2) \times U(1)$.

The other important representation is $\eta = \wedge^2 \mathcal{V}_1$. SU(2) acts trivially on η , and the scalars in U(2) act with charge 2, so η pulls back to the representation W^2 of $SU(2) \times U(1)$.

So the elements of the U(2) Verlinde algebra determined by \mathcal{V}_1 and η are just V_1W and W^2 . What elements in the quantum cohomology do these same representations determine? Over G(2, N), there is a tautological complex two-plane bundle E^* with curvature matrix represented by the quantum field σ . The Chern classes of E^* are

$$c_1(E^*) = \operatorname{Tr}_{\mathcal{V}_1} \sigma$$

$$c_2(E^*) = \operatorname{Tr}_{\eta} \sigma.$$
(4.82)

So under the natural mapping from the quantum cohomology of G(2, N) to the Verlinde algebra of U(2), $c_1(E^*)$ and $c_2(E^*)$ correspond to the representations \mathcal{V}_1 and η .

Let us now briefly discuss the classical representation ring of U(2); the Verlinde algebra is a quotient of this, as we have described. Consider the maximal abelian

subgroup of U(2) of matrices of the form

$$\sigma = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix},\tag{4.83}$$

with $|\lambda_1| = |\lambda_2| = 1$. The character of a representation R of U(2) is $\text{Tr}_R \sigma$ regarded as a function of the λ_i . For instance, the characters of \mathcal{V}_1 and η are

$$\operatorname{Tr}_{\mathcal{V}_1} \sigma = \lambda_1 + \lambda_2$$

$$\operatorname{Tr}_{\eta} \sigma = \lambda_1 \lambda_2.$$
(4.84)

If \mathcal{V}_n is the n^{th} symmetric tensor power of \mathcal{V}_1 , then its character is

$$\operatorname{Tr}_{\mathcal{V}_n} \sigma = \frac{\lambda_1^{n+1} - \lambda_2^{n+1}}{\lambda_1 - \lambda_2}.$$
 (4.85)

Any irreducible representation of U(2) is of the form $\mathcal{V}_s \eta^t$ for some integers s, t (with $s \geq 0$). The equivalence relation (4.78) becomes

$$\mathcal{V}_s \eta^t \leftrightarrow \mathcal{V}_{N-2-s} \eta^{s+t+1}. \tag{4.86}$$

In general, the map from a representation of U(2) to its character is an isomorphism between the ring of representations of U(2) and the ring of Laurent polynomials in λ_1 and λ_2 that are invariant under the Weyl group, which acts by $\lambda_1 \leftrightarrow \lambda_2$.

As we have sketched above, the Verlinde algebra of U(2) is a quotient of the classical representation ring of U(2) by a certain ideal. Under the isomorphism between the representation ring and the character ring, the generators of this ideal can be identified with certain Laurent polynomials in the λ 's. For instance, the

relations in (4.80) and (4.81) become

$$\frac{\lambda_1^N - \lambda_2^N}{\lambda_1 - \lambda_2} = 0$$

$$(\lambda_1 \lambda_2)^N = 1$$

$$\lambda_1 \lambda_2 \frac{\lambda_1^{N-1} - \lambda_2^{N-1}}{\lambda_1 - \lambda_2} = 1.$$

$$(4.87)$$

The second relation in (4.87) has the following implication. The classical representation ring of U(2) is a ring of Laurent polynomials in the λ 's, including negative powers. But by multiplying by a suitable power of $1 = (\lambda_1 \lambda_2)^N$, one can clear the denominators and regard the Verlinde algebra as the quotient of the ring of Weylinvariant polynomials (not Laurent polynomials) in the λ 's by a certain ideal \mathcal{I} . We will learn that \mathcal{I} is in fact generated by the first and third relations in (4.87).

If we multiply the first relation in (4.87) by $\lambda_1 + \lambda_2$ and subtract the third, we learn that

$$\frac{\lambda_1^{N+1} - \lambda_2^{N+1}}{\lambda_1 - \lambda_2} + 1 = 0. \tag{4.88}$$

Cohomology Ring Of G(2, N)

Now return to the sigma model interpretation of σ as the curvature of the tautological two-plane bundle E over G(2, N). If we introduce the roots $\widetilde{\lambda}_1, \widetilde{\lambda}_2$ of the Chern polynomial, then the Chern classes of E are

$$c_1(E^*) = \widetilde{\lambda}_1 + \widetilde{\lambda}_2$$

$$c_2(E^*) = \widetilde{\lambda}_1 \widetilde{\lambda}_2.$$
(4.89)

We observed earlier that under the map from cohomology of G(2, N) to the Verlinde algebra, $c_1(E^*)$ and $c_2(E^*)$ correspond to \mathcal{V}_1 and η . If in turn we identify the Verlinde algebra as a quotient of the character ring, \mathcal{V}_1 and η are identified with their characters, which were given in (4.84). As (4.84) and (4.89) coincide, the identification between these rings can be interpreted as $\lambda_i \leftrightarrow \tilde{\lambda}_i$. Henceforth, we make this identification and drop the tildes.

The quantum comology ring of G(2, N) is the ring of polynomials in $c_1(E^*)$ and $c_2(E^*)$ modulo an ideal \mathcal{J} . As explained in §3.2, \mathcal{J} can be described as follows. Let

$$W(\lambda_1, \lambda_2) = \frac{1}{N+1} \left(\lambda_1^{N+1} + \lambda_2^{N+1} \right) + (\lambda_1 + \lambda_2). \tag{4.90}$$

Since it is Weyl-invariant, W can be regarded as a polynomial in $c_1 = \lambda_1 + \lambda_2$ and $c_2 = \lambda_1 \lambda_2$. The ideal \mathcal{J} is generated by the relations

$$0 = dW = \frac{\partial W}{\partial c_1} dc_1 + \frac{\partial W}{\partial c_2} dc_2. \tag{4.91}$$

Since

$$d\lambda_1 = \frac{\lambda_1 dc_1 - dc_2}{\lambda_1 - \lambda_2}$$

$$d\lambda_2 = \frac{-\lambda_2 dc_1 + dc_2}{\lambda_1 - \lambda_2},$$
(4.92)

we have

$$dW = \left(\lambda_1^{N} + 1\right) d\lambda_1 + \left(\lambda_2^{N} + 1\right) d\lambda_2 = dc_1 \left(\frac{\lambda_1^{N+1} - \lambda_2^{N+1}}{\lambda_1 - \lambda_2} + 1\right) - dc_2 \left(\frac{\lambda_1^{N} - \lambda_2^{N}}{\lambda_1 - \lambda_2}\right).$$
(4.93)

The quantum cohomology of G(2, N) is therefore defined by the relations

$$0 = \frac{\lambda_1^N - \lambda_2^N}{\lambda_1 - \lambda_2} = \frac{\lambda_1^{N+1} - \lambda_2^{N+1}}{\lambda_1 - \lambda_2} + 1.$$
 (4.94)

If we compare this to (4.88) and to the first equation in (4.87), we see that these relations hold in the Verlinde algebra, and consequently the Verlinde algebra is a quotient of the quantum cohomology of G(2, N).

To show that the two algebras coincide (and that all additional relations we found earlier for the Verlinde algebra are consequences of (4.93)), it suffices to compare the dimensions of the two algebras. From the description of the Verlinde algebra as being spanned by the elements V_iW^j , with $0 \le i \le N-2$, $0 \le j \le N-1$

2N-1, with a two-fold restriction and a two-fold identification, its dimension is N(N-1)/2. On the other hand, with c_1 and c_2 considered to be of degree 1 and 2, respectively, the potential $W(c_1, c_2)$ is homogeneous of degree N+1; it follows by a simple counting that the polynomial ring in the c_j modulo the ideal dW=0 has dimension N(N-1)/2. This completes the explicit verification of the equivalence between these rings.

Moreover, we can now dispose of the constant c in the relation $\sigma = cg$. This constant corresponds to a possible constant in the relation $\lambda_i \leftrightarrow \widetilde{\lambda}_i$. The relations above such as (4.94) are not invariant under rescaling of the λ 's, and so the agreement with the Verlinde algebra would be ruined if we modified the value of c. A similar argument holds for k > 2.

The Metric

It remains to show that the metric on the Verlinde algebra and the metric on the quantum cohomology of G(2, N) are related in the expected fashion.

Since either metric obeys g(a,b) = g(ab,1), to verify (4.63), it suffices to show that

$$g_V(a,1) = g_\sigma(a, (\det \sigma)^{N-2}).$$
 (4.95)

We already know the Verlinde metric:

$$g_V(\mathcal{V}_s \eta^t, 1) = \delta_{s,0} \delta_{t,0} + \delta_{N-2-s,0} \delta_{t-1,0}. \tag{4.96}$$

Now let us compute the metric on the cohomology of G(2, N). Since there are no instanton corrections to the metric, we need only compute the classical metric on the cohomology. According to (3.45), that metric is

$$g_{\sigma}(a,b) = -\frac{1}{2} \sum_{dW(\lambda)=0} \frac{ab(\lambda_1 - \lambda_2)^2}{N^2 \lambda_1^{N-1} \lambda_2^{N-1}} = -\frac{1}{2} \sum_{\lambda_1^N = \lambda_2^N = -1} \frac{ab(\lambda_1 - \lambda_2)^2}{N^2 \lambda_1^{N-1} \lambda_2^{N-1}}.$$
(4.97)

The symmetric polynomials in the λ 's of degree 2(N-2) (corresponding to

the top dimensional cohomology of G(2, N) are of the form

$$f_r = \frac{\lambda_1^{2r+1} - \lambda_2^{2r+1}}{\lambda_1 - \lambda_2} (\lambda_1 \lambda_2)^{N-2-r}, \text{ with } 0 \le r \le N - 2.$$
 (4.98)

A simple calculation gives

$$g_{\sigma}(f_r, 1) = -\frac{1}{2N^2} \sum_{\lambda_1^N = \lambda_2^N = -1} \left(\left(\frac{\lambda_1}{\lambda_2} \right)^{1+r} - \left(\frac{\lambda_1}{\lambda_2} \right)^r - \left(\frac{\lambda_1}{\lambda_2} \right)^{-r} + \left(\frac{\lambda_1}{\lambda_2} \right)^{-1-r} \right) = \delta_{r,0}.$$

$$(4.99)$$

This reproduces the first term on the right of (4.96) up to the shift predicted in (4.95). To interpret the second term on the right of (4.96), note that while classically for homogeneous f, $g_{\sigma}(f,1)$ is non-zero unless f is of degree 2(N-2), the quantum cohomology is only graded modulo N (in complex dimension), so we can also consider the case that f is of degree N-4; by a calculation similar to the above, this reproduces the second term in (4.96).

Moreover, we can now dispose of the renormalization constant a of equation (4.31). Inclusion of this term would rescale the metric by a factor of e^{2a} ; the agreement between the two metrics means that the above formulas are normalized correctly, at r = 0. Though we have made this check on the value of a (and a similar, earlier check for c) only for k = 2, the arguments are similar for any k.

Acknowledgements: I benefited from discussions with G. Segal at an early stage of this work.

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FIGURE CAPTIONS

- 1) The one-loop diagram describing σv mixing
- 2) The one-loop diagram generating the Wess-Zumino coupling for σ .

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