# Superconformal Field Theory on Threebranes at a Calabi-Yau Singularity

Igor R. Klebanov\* and Edward Witten
Institute for Advanced Study, Olden Lane, Princeton, New Jersey 08540, USA

#### Abstract

Just as parallel threebranes on a smooth manifold are related to string theory on  $AdS_5 \times \mathbf{S}^5$ , parallel threebranes near a conical singularity are related to string theory on  $AdS_5 \times X_5$ , for a suitable  $X_5$ . For the example of the conifold singularity, for which  $X_5 = (SU(2) \times SU(2))/U(1)$ , we argue that string theory on  $AdS_5 \times X_5$  can be described by a certain  $\mathcal{N} = 1$  supersymmetric gauge theory which we describe in detail.

July 1998

<sup>\*</sup>On leave from Joseph Henry Laboratories, Princeton University.

### 1 Introduction

Recently, Maldacena argued that the 't Hooft large N limit [1] of  $\mathcal{N}=4$  SU(N) gauge theory is related to Type IIB strings on  $AdS_5 \times \mathbf{S}^5$  [2]. (For earlier work on relations between large N gauge theories and strings or supergravity, see [3, 4, 5, 6].) This correspondence was sharpened in [7, 8], where it was shown how to calculate the correlation functions of gauge theory operators from the response of the Type IIB theory on  $AdS_5 \times \mathbf{S}^5$  to boundary conditions.

An interesting generalization of this duality between gauge theory and strings is to consider other backgrounds of Type IIB theory of the form  $AdS_5 \times X_5$  where  $X_5$  is a five-dimensional Einstein manifold bearing five-form flux. The arguments given in [2, 8] indicate that these backgrounds are related to four-dimensional conformal field theories. In general, these theories are different from the  $\mathcal{N}=4$  SU(N) gauge theory. This is obvious from the fact that only  $\mathbf{S}^5$  preserves the maximal number of supersymmetries (namely 32) while other Einstein manifolds lead to reduced supersymmetry. In the early days of Kaluza-Klein supergravity, Romans gave a partial list of five-dimensional Einstein manifolds together with their isometries and the degree of supersymmetry [9]. It is of obvious interest to find a field theoretic interpretation of the Romans compactifications, and in this paper we report on some progress in this direction.

Most of the Einstein manifolds considered by Romans preserve no supersymmetry, which makes construction of the field theory difficult. Instead, we will focus on the cases with some unbroken supersymmetry. In one particular case, we will identify string theory on  $AdS_5 \times X_5$  with a field theory. This is the case that  $X_5$  is a homogeneous space  $T^{1,1} = (SU(2) \times SU(2))/U(1)$ , with the U(1) being a diagonal subgroup of the maximal torus of  $SU(2) \times SU(2)$ . (If  $\sigma_i^{L,R}$  are the generators of the left and right SU(2)'s, then the U(1) is generated by  $\sigma_3^L + \sigma_3^R$ .) According to the counting of supersymmetries in [9], this compactification should be dual to an  $\mathcal{N}=1$  superconformal field theory in four dimensions.

We will find that the construction of the field theory is greatly facilitated by the observation that it is the infrared limit of the world volume theory on coincident Dirichlet threebranes [10, 11] placed at a conical singularity of a non-compact Calabi-Yau threefold. Thus, we are dealing with a special case of a connection between compactification on Einstein manifolds and the metric of threebranes placed at Calabi-Yau singularities, which was recently pointed out in [12]. The subject has been investigated independently of the present work in [13].

We will also find some interesting relations with theories on D3-branes placed at orbifold singularities, which have been extensively studied [14, 16, 17] and in particular related to AdS compactifications [15]. In general, these theories are related to Type IIB on  $AdS_5 \times \mathbf{S}^5/\Gamma$  where  $\Gamma$  is a discrete subgroup of SU(4). It turns out that the coset

space  $T^{1,1} = (SU(2) \times SU(2))/U(1)$  may be obtained by taking  $\Gamma = \mathbf{Z}_2$ , embedded in SU(4) so as to preserve  $\mathcal{N} = 2$  supersymmetry, and blowing up the fixed circle of  $\mathbf{S}^5/\mathbf{Z}_2$  (an operation that breaks  $\mathcal{N} = 2$  to  $\mathcal{N} = 1$ ). On the field theory side, the  $\mathcal{N} = 2$  superconformal theory corresponding to  $\mathbf{S}^5/\mathbf{Z}_2$  flows to the  $\mathcal{N} = 1$  IR fixed point corresponding to  $T^{1,1}$ . The necessary relevant perturbation of the superpotential is odd under the  $\mathbf{Z}_2$  and, therefore, corresponds to a blow-up mode of the orbifold [18, 19].

# 2 Branes at Conical Singularities and Einstein Spaces

We will consider parallel branes near a conical singularity. By a conical singularity on an n-dimensional manifold  $Y_n$ , we mean a point (which we will label as r = 0) near which the metric can locally be put in the form

$$h_{mn}dx^m dx^n = dr^2 + r^2 g_{ij} dx^i dx^j$$
,  $(i, j = 1, ..., n - 1)$ . (1)

Here  $g_{ij}$  is a metric on an n-1-dimensional manifold  $X_{n-1}$ ; the point r=0 is a singularity unless  $X_{n-1}$  is a round sphere. The basic property of this metric, which makes it "conelike," is that there is a group of diffeomorphisms of  $Y_n$  that rescale the metric. This group is  $r \to tr$  with t > 0; the group is thus isomorphic to  $\mathbf{R}_+^*$  (the multiplicative group of positive real numbers). We call  $Y_n$  a cone over  $X_{n-1}$ ;  $X_{n-1}$  is obtained if one deletes the singularity at r=0 and divides by  $\mathbf{R}_+^*$ .

For n > 2, the condition that  $Y_n$  is Ricci-flat is that that  $X_{n-1}$  is an Einstein manifold of positive curvature. (For example, if  $X_{n-1}$  is a round sphere, then  $Y_n$  is flat, not just Ricci-flat.) In fact, by a conformal transformation the metric on  $Y_n$  can be brought to the form

$$\hat{h}_{mn}dx^m dx^n = d\phi^2 + g_{ij}dx^i dx^j , \qquad \phi = \ln r .$$
 (2)

If  $h_{mn}$  is a Ricci flat metric, then by applying the conformal transformation to the Ricci tensor we find that  $g_{ij}$  is an Einstein metric,

$$R_{ij} = (n-2)g_{ij} . (3)$$

*Threebranes* 

We will mainly consider the case that n = 6. We take space-time to be  $M_4 \times Y_6$ , with  $M_4$  being four-dimensional Minkowski space and  $Y_6$  as above. We consider N parallel D3-branes on  $M_4 \times P$ , with P the singularity of  $Y_6$ . The resulting ten-dimensional space-time has the metric [12, 20]

$$ds^{2} = H^{-1/2}(r) \left[ -dt^{2} + d\vec{x}^{2} \right] + H^{1/2}(r) \left[ dr^{2} + r^{2} g_{ij} dx^{i} dx^{j} \right] , \qquad (4)$$

where

$$H(r) = 1 + \frac{L^4}{r^4}$$
,  $L^4 = 4\pi g_s N(\alpha')^2$ .

The near-horizon  $(r \to 0)$  limit of the geometry is  $AdS_5 \times X_5$  and, as we have shown,  $X_5$  is an Einstein manifold. Thus, in the spirit of [2], we may identify the field theory on the D3-branes at a conical singularity as the dual of Type IIB string theory on  $AdS_5 \times X_5$ .

The above considerations facilitate the counting of unbroken supersymmetries. The Killing spinor equation in the metric  $h_{mn}$  is

$$(\partial_m + \frac{1}{4}\omega_{mab}\Gamma^{ab})\eta = 0. (5)$$

Evaluating this in the metric (1), we find [12]

$$(\partial_i + \frac{1}{4}\omega_{ijk}\Gamma^{jk} + \frac{1}{2}\Gamma_i^r)\eta = 0 , \qquad (6)$$

which is equivalent to the  $X_5$  part of the Killing spinor equation in Type IIB compactification on  $AdS_5 \times X_5$ , including the effects of the five-form field strength (which contributes the  $\Gamma_i^r$  term). Here  $\Gamma_i^r = \Gamma_{is} n^s$  where  $n^s$  is the unit vector in the radial direction. Thus, the number of unbroken supersymmetries on  $X_5$  is the same as the number of unbroken supersymmetries on the six-dimensional cone. If the cone is a manifold of SU(3) holonomy (a Calabi-Yau threefold), then there are eight unbroken supersymmetries – four for left-movers and four for right-movers. These cases correspond to  $\mathcal{N}=1$  superconformal field theories in 4 dimensions, i.e. we may construct such field theories as the infrared limits of the theories on D3-branes placed at Calabi-Yau singularities. If, however, the cone has SU(2) holonomy, then by placing D3-branes at the singularity we obtain an  $\mathcal{N}=2$  superconformal field theory.

This point of view also helps in identifying which symmetries will be R-symmetries of the superconformal field theory. For example, to identify the R symmetries of string theory on  $AdS_5 \times X_5$  we work on a Calabi-Yau threefold  $Y_6$  which is a cone over  $X_5$ . Let  $\Omega$  be a holomorphic three-form such that  $-i\Omega \wedge \overline{\Omega}$  is the volume form of  $Y_6$ . Then the chiral superspace volume form  $d^2\theta$  (the  $\theta$ 's being fermionic coordinates of positive chirality) transforms like  $\Omega$ , so symmetries are R-symmetries precisely if they act non-trivially on  $\Omega$ .

#### An Example

For the special case where the five-dimensional Einstein manifold  $X_5$  is  $T^{1,1} = (SU(2) \times SU(2))/U(1)$ , the Calabi-Yau threefold turns out to be particularly simple. Since  $T^{1,1}$  has  $SU(2) \times SU(2) \times U(1) = SO(4) \times U(1)$  symmetry, we look for an

<sup>&</sup>lt;sup>1</sup>Recall that an  $\mathcal{N}=1$  superconformal field theory has eight fermionic symmetries – four ordinary supersymmetries and four superconformal symmetries.

isolated Calabi-Yau singularity with that symmetry. A fairly obvious candidate is the "conifold" [21, 22] which for our purposes is the complex manifold C

$$z_1^2 + z_2^2 + z_3^2 + z_4^2 = 0 (7)$$

with an "ordinary double point" singularity at  $z_i = 0$ . The  $z_i$  transform in the fourdimensional representation of SO(4), and have "charge one" relative to the U(1). The holomorphic three-form is

$$\Omega = \frac{dz_2 \wedge dz_3 \wedge dz_4}{z_1},\tag{8}$$

and has "charge two" under the U(1), which will therefore be an R-symmetry group.

This manifold C is a cone because the equation defining it transforms with definite weight under  $z_i \to tz_i$ , with  $t \in \mathbb{C}^*$ . This  $\mathbb{C}^*$  action is in fact the complexification of the U(1) R-symmetry group noted above. If we restrict t to be real and positive, we get a group  $\mathbb{R}_+^*$  of scalings under which the Calabi-Yau metric that we will find momentarily transforms homogeneously, just like the holomorphic three-form.

To identify  $X_5$  topologically, we note that after omitting the singularity at the origin, dividing C by  $\mathbf{R}_+^*$  is equivalent to intersecting it with the unit sphere

$$|z_1|^2 + |z_2|^2 + |z_3|^2 + |z_4|^2 = 1. (9)$$

The group SO(4) acts transitively on this intersection. Any given point on the intersection, such as  $Q:(z_1,z_2,z_3,z_4)=(1/\sqrt{2},i/\sqrt{2},0,0)$ , is invariant under only a single  $U(1) \subset SO(4)$  (for instance, Q is invariant under the subgroup U(1)=SO(2) that rotates  $z_3$  and  $z_4$ ), so  $X_5=SO(4)/U(1)=(SU(2)\times SU(2))/U(1)$  [22].

A Calabi-Yau metric on C can be written quite explicitly [22]. One can describe any SO(4)-invariant Kahler metric on C by an SO(4)-invariant Kahler potential K. The most general possibility is that  $K(z_1, z_2, z_3, z_4)$  must be a function only of  $\sum_i \bar{z}_i z_i$ . To get a conical metric – which scales homogeneously under  $z_i \to tz_i$  for  $t \in \mathbf{R}_+^*$  – we must take  $K = (\sum_i \bar{z}_i z_i)^{\gamma}$  for some exponent  $\gamma$ . The Kahler form  $\omega = -idz_i d\bar{z}_j (\partial^2 K/\partial z_i \partial \bar{z}_j)$  scales as  $\omega \to t^{2\gamma}\omega$ . We determine  $\gamma$  by asking that the metric should be a Calabi-Yau metric; this is equivalent to the requirement that  $\omega \wedge \omega \wedge \omega = -i\Omega \wedge \overline{\Omega}$ . Since  $\Omega$  transforms as  $\Omega \to t^2\Omega$ , we get  $\gamma = 2/3$ .

Starting from this description, to exhibit the metric in the standard conical form of (1), one reparametrizes  $\mathbf{R}_+^*$  by a new scaling variable  $\tilde{t}=t^{2/3}$ , so that the Kahler form  $\omega$  (or equivalently the Kahler metric) scales as  $\omega \to \tilde{t}^2 \omega$ . Then we introduce a radial variable  $r=(\sum_i \bar{z}_i z_i)^{1/3}$ , which transforms as  $r\to \tilde{t}r$ . In terms of r plus a set of angular variables (invariant under scaling) the metric takes the form (1).

The angular part of the metric has been described as follows [23, 22].  $X_5 = T^{1,1}$  is a U(1) bundle over  $\mathbf{S}^2 \times \mathbf{S}^2$ . We choose the coordinates  $(\theta_1, \phi_1)$  and  $(\theta_2, \phi_2)$  to parametrize

the two spheres in a conventional way, while the angle  $\psi \in [0, 4\pi)$  parametrizes the U(1) fiber. Then the metric may be written as

$$ds^{2} = a(d\psi + \cos\theta_{1}d\phi_{1} + \cos\theta_{2}d\phi_{2})^{2} + b\sum_{i=1}^{2} \left[ d\theta_{i}^{2} + \sin^{2}\theta_{i}d\phi_{i}^{2} \right] . \tag{10}$$

If we choose a = 1/9 and b = 1/6, we obtain an Einstein metric with  $R_{ij} = 4g_{ij}$ . It can be obtained as the angular part of the conical Calabi-Yau metric described above [22].

#### Description As A Quotient

To construct a field theory that describes threebranes at a conifold singularity, it is very helpful to have a description of the conifold as a quotient. (This description has been used [24] in analyzing the structure of Kahler moduli space in Calabi-Yau compactification.) After an obvious linear change of variables, we can describe C by the equation

$$z_1 z_2 - z_3 z_4 = 0 . (11)$$

This equation can be "solved" by writing

$$z_1 = A_1 B_1 , z_2 = A_2 B_2 , z_3 = A_1 B_2 , z_4 = A_2 B_1 . (12)$$

Note that we obtain the same  $z_i$  if we transform the A's and B's by

$$A_k \to \lambda A_k , \qquad B_l \to \lambda^{-1} B_l , \qquad k, l = 1, 2$$
 (13)

with  $\lambda \in \mathbf{C}^*$ . The  $SO(4) = SU(2) \times SU(2)$  symmetry of the conifold is easy to describe in this formulation: one SU(2) acts on the  $A_i$ , and one on the  $B_j$ . If we write  $\lambda = se^{i\alpha}$ , with s real and positive and  $\alpha$  real, then, away from the singular point  $z_i = 0$ , s can be selected to set

$$|A_1|^2 + |A_2|^2 = |B_2|^2 + |B_2|^2. (14)$$

The conifold is obtained by further dividing by U(1):

$$A_k \to e^{i\alpha} A_k \ , \qquad B_l \to e^{-i\alpha} B_l \ .$$
 (15)

To identify the angular manifold  $X_5$  from this point of view, note that we can divide by the scaling  $z_i \to sz_i$  (with real positive s) by setting  $|A_1|^2 + |A_2|^2 = B_1|^2 + |B_2|^2 = 1$ . At this point we are on  $\mathbf{S}^3 \times \mathbf{S}^3 = SU(2) \times SU(2)$ . Then dividing by (15) gives us back  $X_5 = (SU(2) \times SU(2))/U(1)$ .

Our goal in the next section will be to find the  $\mathcal{N}=1$  superconformal field theory which is dual to the Type IIB theory compactified on  $AdS_5 \times T^{1,1}$ . We will think of this theory as the infrared limit of the theory on N coincident D3 branes placed at the conical singularity of  $M_4 \times C$ .

# 3 Construction of the Field Theory

Our construction of the  $\mathcal{N}=1$  superconformal field theory on the D3-branes at the conical singularity of  $M_4 \times C$  will be guided by the parametrization of the conifold C in terms of  $A_1$ ,  $A_2$ ,  $B_1$ ,  $B_2$  given in (12). We consider a U(1) gauge theory with  $\mathcal{N}=1$  supersymmetry, and introduce  $A_k$  and  $B_l$ , k, l=1, 2 as chiral superfields of charges 1 and -1 respectively. The D auxiliary field of the U(1) vector multiplet is given by

$$D = |A_1|^2 + |A_2|^2 - |B_1|^2 - |B_2|^2 - \zeta, \tag{16}$$

with  $\zeta$  the coefficient of the Fayet-Iliopoulos term in the Lagrangian. The moduli space of vacua is found by setting D to zero and dividing by the gauge group U(1). If we set  $\zeta = 0$ , then the condition D = 0 is the condition (14), and dividing by the gauge group is the equivalence relation (15). So the moduli space of vacua is the conifold C. For  $\zeta \neq 0$ , one gets instead a resolution of the conifold singularity, in fact two different "small resolutions" depending on the sign of  $\zeta$ . (The "flop" between them at  $\zeta = 0$  is a prototype of topology change in Calabi-Yau sigma models.) In the present paper, we wish to study threebranes on the conifold; so we set  $\zeta = 0$ .

To describe parallel threebranes on  $M_4 \times C$ , we really should introduce a second U(1), which will be the unbroken U(1) on the threebrane worldvolume. The gauge group is then  $U(1) \times U(1)$ , and the chiral multiplets  $A_i$  and  $B_j$  have respective charges (1,-1) and (-1,1). All chiral multiplets are neutral under the diagonal U(1), which thus decouples and is the expected free U(1) gauge multiplet on the threebrane worldvolume. Modulo this free U(1), the model is equivalent to the one analyzed in the last paragraph. The chiral multiplets describe threebrane motion on C. The model can thus be considered to describe the low energy behavior of a threebrane on  $M_4 \times C$  whose worldvolume in a configuration of minimum energy is of the form  $M_4 \times Q$ , where Q is a point in C determined by a choice of vacuum of the field theory.

Now we want to generalize to the case of N parallel threebranes. The natural guess is that the gauge group should be  $U(N) \times U(N)$  gauge theory with chiral fields  $A_k$ , k = 1, 2 transforming now in the  $(N, \overline{N})$  representation and  $B_l$ , l = 1, 2 transforming in the  $(\overline{N}, N)$  representation. A renormalizable superpotential is not possible, so as a first guess we suppose that the superpotential vanishes. We will think of  $A_k$  and  $B_l$  as  $N \times N$  matrices. By assuming that the matrices  $A_k$ ,  $B_l$  are (in some basis) diagonal, with distinct eigenvalues, one finds a family of vacua parametrized by the positions of N threebranes at distinct points on the conifold. The gauge group is broken down to  $U(1)^N$ , one factor of U(1) for each threebrane. This is an encouraging sign, but it cannot be the whole story, since the vacua just described have massless charged chiral multiplets that should not be present in a theory describing N threebranes at generic smooth points.

To proceed, we must introduce a superpotential that will give mass to the unwanted massless multiplets. The model without the superpotential has an  $SU(2) \times SU(2)$  symmetry, with one SU(2) acting on the  $A_k$  and one on the  $B_l$ . There is also an anomaly-free U(1) R-symmetry, under which  $A_k$  and  $B_l$  both have charge 1/2. This  $SU(2) \times SU(2) \times U(1)_R$  symmetry is a symmetry of the conifold C. So the superpotential must preserve the symmetry. The most general superpotential that does so is

$$W = \frac{\lambda}{2} \epsilon^{ij} \epsilon^{kl} \text{Tr} A_i B_k A_j B_l \tag{17}$$

for some constant  $\lambda$ . Note that W has  $U(1)_R$  charge two, the correct value for a superpotential.

It is not hard to see that this superpotential does the right job at the classical level. The model has a family of vacua in which the matrices  $A_k$ ,  $B_l$  are (in some basis) diagonal and otherwise generic. The gauge group is broken to  $U(1)^N$ . The diagonal components of  $A_k$ ,  $B_l$  describe the motion of N threebranes on C. The off-diagonal fields all receive mass from the superpotential (plus Higgs mechanism). So the low energy theory is as desired.

Another way to reach this conclusion is the following. The  $U(N) \times U(N)$  gauge symmetry can be broken down to a diagonal U(N) by letting  $A_k$  and  $B_l$  be multiples of the identity matrix. In such a vacuum, one of the matrices  $A_k$ ,  $B_l$  can be Higgsed away. The other three give three chiral superfields, say X, Y, and Z, transforming in the adjoint representation of the unbroken U(N). The superpotential (17), after allowing for the expectation value of one linear combination of the fields, reduces to a cubic expression TrX[Y,Z]. The U(N) gauge theory of three chiral superfields with that superpotential flows in the infrared to the  $\mathcal{N}=4$  super Yang-Mills theory that describes N threebranes near a smooth point in C. This can be further Higgsed, if desired, to the  $U(1)^N$  theory described in the last paragraph.

#### Interpretation Of The Superpotential

To formulate a precise conjecture, it remains to discuss the meaning of the superpotential (17), which as a perturbation of free field theory is unrenormalizable. As in [25],[26],[28], the basic idea is to first consider the theory with  $\lambda=0$ . We want to understand its infrared dynamics. The U(1) factors in the gauge group have positive beta function and will decouple in the infrared. So in analyzing the infrared behavior, we replace the gauge group by  $SU(N) \times SU(N)$ .

<sup>&</sup>lt;sup>2</sup>We define R-symmetries so that chiral superspace coordinates have charge 1. Gluino fields  $\lambda$  hence have charge 1, so that the superspace field strength  $\mathcal{W} = \lambda + \theta F + \ldots$  transforms homogeneously. For  $A_k = a_k + \theta \psi_k + \ldots$ ,  $B_l = b_l + \theta \chi_l + \ldots$  to have charge 1/2, the fermion components  $\psi$ ,  $\chi$  have charge -1/2. The adjoint representation of U(N) has  $C_2 = 2N$ , while the fundamental has  $C_2 = 1$ . With these values of the Casimirs and R-charges, the anomalies in the R-symmetry cancel.

From the point of view of either of the SU(N) factors, this is an SU(N) gauge theory with chiral multiplets transforming in 2N copies of  $N \oplus \overline{N}$ . That theory flows [27] to a nontrivial infrared fixed point at which the anomaly-free R-symmetry of the Lagrangian becomes the R-symmetry in the superconformal algebra. We assume that the same is true for an  $SU(N) \times SU(N)$  theory with two copies of  $(N, \overline{N}) \oplus (\overline{N}, N)$ . This theory has, microscopically, a unique anomaly-free R-symmetry, analyzed above, under which the chiral superfields all have charge 1/2. We expect this (as in the examples in [27]) to become the R-symmetry in the superconformal algebra of the infrared fixed point theory. If so, dimensions of chiral superfields at the nontrivial fixed point are determined from their R-charges. In particular, the superpotential W of (17) has R-charge 2, and hence is a marginal perturbation of the infrared fixed point.

One can be more precise here; using techniques in [26], one can argue that the superpotential perturbation of this theory is an exactly marginal operator and gives a line of fixed points. As usual, this has to do with the high degree of symmetry of the operator: it is invariant under  $SU(2) \times SU(2)$ .

Let us impose the conditions of conformal invariance on the theory with the superpotential (17). From the vanishing of the exact beta function [29] we get the equation

$$3C_2(G) - \sum_{i=A_k, B_l} T(R_i)(1 - 2\gamma_i) = 0.$$
 (18)

Due to the  $SU(2) \times SU(2)$  symmetry, the anomalous dimensions satisfy

$$\gamma_{A_1} = \gamma_{A_2} \; , \qquad \gamma_{B_1} = \gamma_{B_2} \; . \tag{19}$$

The two anomalous dimensions,  $\gamma_A$  and  $\gamma_B$ , are functions of the gauge couplings  $g_1$  and  $g_2$ , and the superpotential strength  $\lambda$ . By applying (18) to either of the two SU(N)'s we find the condition

$$\gamma_A(g_1, g_2, \lambda) + \gamma_B(g_1, g_2, \lambda) + \frac{1}{2} = 0$$
 (20)

Requiring the scale invariance of the superpotential leads to exactly the same condition. So, we have one equation for three coupling constants, and this gives a critical surface. The surface of fixed points is generated by two exactly marginal operators, the superpotential (17) and the difference between the kinetic energies of the two SU(N)'s. If we further impose a symmetry under the interchange of the two gauge groups, then the fixed surface degenerates into a fixed line.

The argument for exact marginality does not go through if we consider a less symmetric superpotential. For instance, we could consider a superpotential

$$hTr(A_1B_1)^2. (21)$$

This preserves a  $\mathbb{Z}_2$  symmetry under the interchange of the two gauge groups, and we will keep the gauge couplings equal,  $g_1 = g_2 = g$ . Since the SU(2)'s are broken, there are two different anomalous dimensions: for  $A_1(B_1)$  and for  $A_2(B_2)$ . Now the vanishing of the beta functions gives

$$\gamma_1(g,h) + \gamma_2(g,h) + \frac{1}{2} = 0$$
,

while the scale invariance of the superpotential requires

$$\gamma_1(g,h) + \frac{1}{4} = 0$$
.

Now we find two equations for two different functions of g and h, and we do not expect a critical line to exist.

The conclusion that there are two and only two exactly marginal operators, the superpotential (17) and the difference between the two kinetic energies, is confirmed by considering the special case of N=2, where our model is an  $SU(2) \times SU(2) \sim SO(4)$  gauge theory coupled to four vectors of SO(4). This theory has been studied by Intriligator and Seiberg in [25]. Here the superpotential (17) reduces to the baryon operator introduced in [25], which was found to be exactly marginal: under duality it goes into the difference between the kinetic energies of the two SU(2)'s, which is perturbatively marginal. This case was also emphasized by Leigh and Strassler [26] as an example of an operator that is irrelevant as a perturbation of the free fixed point but exactly marginal as a perturbation of a nontrivial infrared fixed point.

We can now state a precise conjecture. Type IIB string theory on  $AdS_5 \times T^{1,1}$ , with N units of Ramond-Ramond flux on  $T^{1,1}$ , should be equivalent to the theory obtained by starting with  $SU(N) \times SU(N)$  gauge theory, with two copies of  $(N, \overline{N}) \oplus (\overline{N}, N)$ , flowing to an infrared fixed point, and then perturbing by the superpotential (17).

#### Comparison Of R-Symmetries

We will make several additional checks of this conjecture.

We begin with a more careful comparison of the U(1) R-symmetries of the conifold model and the field theory. Consider the transformation  $z_i \to e^{i\phi}z_i$  of the conifold. This transformation acts on the holomorphic three-form by  $\Omega \to e^{2i\phi}\Omega$ . The chiral superspace coordinates transform as  $\Omega^{1/2}$  and thus as  $e^{i\phi}$ .

Now set  $\phi = \pi$ . This gives an element of the R-symmetry group that acts on the conifold as  $z_i \to -z_i$  and on the chiral superspace coordinates  $\theta$  by  $\theta \to -\theta$ . Let us identify this transformation in the gauge theory. Since  $A_k$  and  $B_l$  have R-charge 1/2,

<sup>&</sup>lt;sup>3</sup>The reason for this is that  $\Omega$  can be written in terms of a covariantly constant spinor  $\eta$  of definite chirality as  $\Omega_{ijk} = \eta^T \Gamma_{ijk} \eta$ , so  $\Omega$  transforms like  $\eta^2$ . But supersymmetries are generated by covariantly constant spinors, and so the supersymmetry generators transform like  $\eta$ .

they transform under  $\theta \to -\theta$  as  $A_k \to iA_k$ ,  $B_l \to iB_l$ . This agrees with expectations from the conifold since the conifold coordinates  $z_i$  are represented in the gauge theory as  $\operatorname{Tr} A_k B_l$  and thus indeed transform as  $z_i \to -z_i$ . This is an interesting check because it depends on the fact that  $A_k$  and  $B_l$  have R-charge 1/2, a fact that was deduced from considerations of anomaly cancellation that are seemingly unrelated to the geometry of the conifold.

#### Global Structure Of The Symmetry Group

We identified earlier an  $SU(2) \times SU(2)$  symmetry group of the gauge theory, with one SU(2) acting on  $A_k$  and one on  $B_l$ . To be more precise, the group that acts faithfully modulo gauge transformations is  $(SU(2) \times SU(2))/\mathbb{Z}_2$  where  $\mathbb{Z}_2$  is the diagonal subgroup of the product of the centers of the two SU(2)'s. In other words, the transformation  $A_k \to -A_k$ ,  $B_l \to B_l$  is equivalent to  $A_k \to A_k$ ,  $B_l \to -B_l$ . The reason for this is that the gauge group  $U(N) \times U(N)$  contains a U(1) subgroup that acts by  $A_k \to e^{i\alpha}A_k$ ,  $B_l \to e^{-i\alpha}B_l$ . Setting  $\alpha = \pi$ , we get the gauge transformation  $A_k \to -A_k$ ,  $B_l \to -B_l$ , so that the transformations  $A \to -A$ ,  $B \to B$  and  $A \to A$ ,  $B \to -B$  are indeed gauge equivalent to each other.

The group acting faithfully is thus  $(SU(2) \times SU(2))/\mathbb{Z}_2 = SO(4)$ . But this is the global form of the corresponding symmetry group of the conifold. For indeed, the conifold coordinates  $z_i$ , obeying  $\sum_i z_i^2 = 0$ , transform in the vector representation of SO(4).

#### Reflection

Now we will analyze the discrete symmetries on the two sides. First of all, on the conifold side, the SO(4) symmetry group extends to O(4), as the conifold is invariant under  $z_4 \to -z_4$  with other coordinates invariant. This transformation changes the sign of the holomorphic three-form  $\Omega$ , so it is an R-symmetry, acting on the chiral superspace coordinates as  $\theta \to i\theta$ .

To compare to the gauge theory, we note that the reflection exchanges the two factors in  $SO(4) = (SU(2) \times SU(2))/\mathbb{Z}_2$ . So it must exchange the A's and B's. As A transforms under  $U(N) \times U(N)$  as  $(N, \overline{N})$  while B transforms as  $(\overline{N}, N)$ , a transformation that exchanges A and B must be accompanied by either (1) exchange of the two factors of the gauge group, or (2) charge conjugation, that is an outer automorphism of each U(N) that exchanges the N and  $\overline{N}$  representations.

The reason that the two options exist is that, as we will see shortly, the theory actually has global symmetries of each kind. But for the moment, we specify further that we want to examine a symmetry of the conifold that acts by  $z_4 \rightarrow -z_4$  without reversal of the orientation of the string worldsheet. Charge conjugation is associated in D-brane theory with such orientation reversal. So the "geometrical" symmetry

 $z_4 \to -z_4$  with no worldsheet orientation reversal corresponds to  $A \leftrightarrow B$  together with exchange of the two factors in the gauge group.

It remains to understand, from the point of view of field theory, why such a symmetry is an R-symmetry. The point is that under  $A_k \leftrightarrow B_k$ , the superpotential  $W = \lambda \text{Tr}(A_1B_1A_2B_2 - A_1B_2A_2B_1)$  is odd. So we must accompany the transformation described so far by an additional transformation under which W is odd. Such a transformation, if it exists, is not unique since one could always multiply by an ordinary symmetry of the theory. The missing transformation is uniquely determined if one asks that it should leave fixed the lowest components of the superfields  $A_i$  and  $B_j$ . The required transformation is the naive R-symmetry that acts on chiral superspace coordinates by  $\theta \to i\theta$ , acts on gluinos by  $\lambda \to i\lambda$ , leaves invariant the superfields A and B, and (therefore) acts on fermionic components  $\psi$  of A or B by  $\psi \to -i\psi$ . We will call this transformation  $\Upsilon$ . We note in particular that the fact that  $\theta \to i\theta$  under  $\Upsilon$  is in agreement with the  $z_4 \to -z_4$  symmetry of the conifold, and that under such an R-transformation the superpotential is odd.

What remains is to show that the combined operation of exchanging A and B, exchanging the two factors in the gauge group, and acting with  $\Upsilon$  has no anomaly. There is a subtlety here that is relatively unfamiliar since anomalies under global symmetries that act by outer automorphisms of the gauge group are not often studied. By itself (in addition to being explicitly violated by the superpotential),  $\Upsilon$  can sometimes have an anomaly in an instanton field. In fact, in an instanton field of the first U(N) with instanton number k, the gluinos of the first U(N) have 2Nk zero modes, while A and B have 2Nk each, so the path integral measure transforms under  $\Upsilon$  as  $i^{2Nk}(-i)^{4Nk}=(-1)^{Nk}$ , and is invariant if and only if Nk is even. Since  $\Upsilon$  is in any case not a symmetry (being violated by the superpotential), whether  $\Upsilon$  by itself has an anomaly is not of great physical relevance. What we really want to know is whether there is an anomaly in  $\Upsilon$  combined with the exchange of the two gauge group factors (and  $A \leftrightarrow B$ ). For this, we look at a classical field configuration that is invariant under exchange of the U(N)'s and the superfields, and study how the fermion determinant in such a configuration transforms under  $\Upsilon$ . For a classical configuration to be invariant under exchange of the U(N)'s and the superfields, there must be equal instanton numbers in the two U(N)'s. The computation of the transformation of the measure proceeds just as before, but now there is a factor of  $(-1)^{Nk}$  from each U(N), so overall the measure is invariant.

### Center Of $SL(2, \mathbf{Z})$

One additional discrete symmetry of string theory on the conifold should be compared to the gauge theory.

This is the element

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \tag{22}$$

which generates the center of  $SL(2, \mathbf{Z})$ . This symmetry, which we will call w, acts trivially on the coupling (and theta angle) of Type IIB superstring theory, and is a symmetry of Type IIB on the conifold as long as the B-fields vanish, as we have so far assumed.

The transformation w is equivalent to  $\Omega(-1)^{F_L}$ , where  $\Omega$  is the exchange of left and right movers on the string worldsheet, and  $(-1)^{F_L}$  multiplies left-moving worldsheet fermions by -1. If we write  $Q_L$  and  $Q_R$  for supersymmetries that come from left and right movers, then  $\Omega$  acts by  $Q_L \leftrightarrow Q_R$ , and  $(-1)^{F_L}$  by  $Q_L \to -Q_L$ ,  $Q_R \to Q_R$ . Hence w acts by  $Q_L \to Q_R$ ,  $Q_R \to -Q_L$ . Now let us consider what happens in the presence of parallel threebranes whose world-volume is spanned by Minkowski coordinates  $x^0, \ldots, x^3$ . The unbroken supersymmetries are linear combinations of the form  $\epsilon_L Q_L + \epsilon_R Q_R$ , where  $\epsilon_R = \Gamma_{0123} \epsilon_L$ . Hence w, which acts by  $\epsilon_L \to \epsilon_R$ ,  $\epsilon_R \to -\epsilon_L$ , is equivalent to

$$w: \epsilon_L \to \Gamma_{0123} \epsilon_L.$$
 (23)

 $\Gamma_{0123}$  acts on spinors of positive chirality as i and on spinors of negative chirality as -i. So w (like the reflection  $z_4 \to -z_4$  that we just analyzed) is an R-symmetry, acting on chiral superspace coordinates by  $\theta \to i\theta$ .

Now let us look at what happens in field theory. We expect  $w = \Omega(-1)^{F_L}$  to act on the gauge group by charge conjugation. The reason for this is that  $\Omega$ , the reversal of worldsheet orientation, is the basic charge conjugation operation for open strings. On the other hand, w commutes with  $SU(2) \times SU(2)$  and so will not exchange A and B. Since A transforms as  $(N, \overline{N})$  and B as  $(\overline{N}, N)$ , a charge conjugation operation that does not exchange A and B must exchange the two factors in the gauge group. So we identify w with a transformation that exchanges the two factors of the gauge group, in such a way that the N of the first U(N) is exchanged with the  $\overline{N}$  of the second (and  $\overline{N}$  of the first is exchanged with N of the second) while mapping A to A and B to B.

Let us suppress the SU(2) internal symmetry index of A and make explicit the gauge indices. Thus we write A as a matrix  $A^a{}_b$ , where a labels the N of the first U(N) and b labels the  $\overline{N}$  of the second U(N). The symmetry w exchanges the two types of indices, so we can regard it as an operation that maps A to its "transpose"  $A^T$ . Under this operation of transposing all  $A_k$  and  $B_l$ , the superpotential  $W = \lambda \text{Tr} (A_1B_1A_2B_2 - A_1B_2A_2B_1)$  changes sign. To compensate for this, we must include the action of our friend  $\Upsilon$ , which we encountered in analyzing the reflection  $z_4 \to -z_4$ . Since  $\Upsilon$  acts on chiral superspace coordinates as  $\theta \to i\theta$ , w acts in this way in the gauge theory, just as we found in the underlying string theory.

#### Counting Of Moduli

We will now make perhaps the most obvious comparison of all: counting moduli on the two sides and discussing how they transform under symmetries.

On the  $AdS_5 \times T^{1,1}$  side, one modulus is the  $\tau$  parameter of Type IIB. An additional modulus arises because topologically  $T^{1,1} = \mathbf{S}^2 \times \mathbf{S}^3$ . (We give an explanation of this statement later.) In particular, the second Betti number of  $T^{1,1}$  is 1. This means that, in Type IIB compactification on  $T^{1,1}$ , the Ramond and NS *B*-fields each have a zero mode, leading to two "theta angles" that appear in labeling a choice of vacuum. They combine under  $\mathcal{N}=1$  supersymmetry into a chiral superfield  $\phi$ , which parametrizes a complex torus.

On the gauge theory side, after decoupling the U(1)'s, one has the renormalization scales  $\Lambda_1$  and  $\Lambda_2$  of the two SU(N)'s (as is standard in supersymmetric theories, we include the theta angles in the definition of the  $\Lambda_i$ , so that they are complex-valued). In addition, we have the coupling constant  $\lambda$  that appears in the superpotential (17). From  $\Lambda_1$ ,  $\Lambda_2$  and  $\lambda$ , we can form two dimensionless combinations  $u_i = \lambda \Lambda_i$ . As we have explained earlier, the gauge theory has a surface of fixed points parametrized by the  $u_i$ .

We propose that the two moduli  $u_i$  of the gauge theory correspond to the two moduli  $\tau$  and  $\phi$  in Type IIB on  $AdS_5 \times \mathbf{S}^5$ . As a check, let us look at the behavior under the symmetry w that generates the center of  $SL(2,\mathbf{Z})$ . Under w,  $\tau$  is even but (as w acts by -1 on both the Ramond and NS B-fields)  $\phi$  is odd. Now on the gauge theory side, we identified w with an operation that among other things exchanges the two SU(N) factors of the gauge group. Under this transformation, clearly  $u_1 + u_2$  is even, but  $u_1 - u_2$  is odd. So we propose to identify the even variable with the Type IIB coupling  $\tau$  and the odd variable with the theta angles  $\phi$ .

It is rather odd to associate a difference of gauge couplings  $u_1-u_2$  with a torus-valued superfield  $\phi$ . Such an identification has, however [16], already been made in the case of a certain  $AdS_5 \times \mathbf{S}^5/\mathbf{Z}_2$  orbifold. As we will see next, this orbifold theory can "flow" to the  $AdS_5 \times T^{1,1}$  vacuum considered in the present paper. Thus the identification made in [16] implies that the difference of the coupling parameters of the two SU(N)'s should indeed be identified with the torus-valued parameter  $\phi$ .

#### Comparison To Orbifold Theory

The duality that we have proposed can be checked in an interesting way by comparing to a certain  $AdS_5 \times \mathbf{S}^5/\mathbf{Z}_2$  background. If  $\mathbf{S}^5$  is described by an equation

$$\sum_{i=1}^{6} x_i^2 = 1,\tag{24}$$

with real variables  $x_1, \ldots, x_6$ , then the  $\mathbb{Z}_2$  in question acts as -1 on four of the  $x_i$  and

as +1 on the other two. The importance of this choice is that this particular  $\mathbb{Z}_2$  orbifold of  $AdS_5 \times \mathbb{S}^5$  has  $\mathcal{N}=2$  superconformal symmetry. Using orbifold results for branes [14], this model has been identified [15] as an AdS dual of a  $U(N) \times U(N)$  theory with hypermultiplets transforming in  $(N, \overline{N}) \oplus (\overline{N}, N)$ . From an  $\mathcal{N}=1$  point of view, the hypermultiplets correspond to chiral multiplets  $A_k, B_l, k, l=1, 2$  in the  $(N, \overline{N})$  and  $(\overline{N}, N)$  representations respectively. The model also contains, from an  $\mathcal{N}=1$  point of view, chiral multiplets  $\Phi$  and  $\tilde{\Phi}$  in the adjoint representations of the two U(N)'s. The superpotential is

$$g \text{Tr} \Phi(A_1 B_1 + A_2 B_2) + g \text{Tr} \tilde{\Phi}(B_1 A_1 + B_2 A_2)$$
.

Now, let us add to the superpotential of this  $\mathbb{Z}_2$  orbifold a relevant term,

$$\frac{m}{2}(\mathrm{Tr}\Phi^2 - \mathrm{Tr}\tilde{\Phi}^2) \ . \tag{25}$$

It is straightforward to see what this does to the field theory. We simply integrate out  $\Phi$  and  $\tilde{\Phi}$ , to find the superpotential

$$\frac{g^2}{2m} \left[ \text{Tr}(A_1 B_1 A_2 B_2) - \text{Tr}(B_1 A_1 B_2 A_2) \right] .$$

This expression is familiar from (17), so the  $\mathbb{Z}_2$  orbifold with relevant perturbation (25) apparently flows to the  $T^{1,1}$  model associated with the conifold. (However, this way of obtaining the  $T^{1,1}$  model does not explain all of its symmetries; the  $T^{1,1}$  model has symmetries that arise only at the endpoint of the renormalization group flow from  $\mathbb{S}^5/\mathbb{Z}_2$ .)

Let us try to understand why this works from the point of view of the geometry of  $\mathbf{S}^5/\mathbf{Z}_2$ . The perturbation in (25) is odd under exchange of the two U(N)'s. The exchange of the two U(N)'s is the quantum symmetry of the  $AdS_5 \times \mathbf{S}^5/\mathbf{Z}_2$  orbifold – the symmetry that acts as -1 on string states in the twisted sector and +1 in the untwisted sector. So we must, as in [19], associate this perturbation with a twisted sector mode of string theory on  $AdS_5 \times \mathbf{S}^5/\mathbf{Z}_2$ . The twisted sector mode which is a relevant perturbation of the field theory is the blowup of the orbifold singularity of  $\mathbf{S}^5/\mathbf{Z}_2$ .

Let us consider the geometry produced by this blowup. First we recall the blowup (in the complex sense) of a codimension four  $\mathbb{Z}_2$  orbifold singularity  $\mathbb{R}^4/\mathbb{Z}_2$ . Let  $\mathbb{Z}_2$  act by sign change on four coordinates  $x_1, \ldots, x_4$  of  $\mathbb{R}^4$ . Blowup of such a singularity replaces it by a copy of  $\mathbb{S}^2$ . One way to describe the  $\mathbb{S}^2$  is as follows. The smooth

<sup>&</sup>lt;sup>4</sup>Note that in string theory, blowup of a codimension four  $\mathbf{R}^4/\mathbf{Z}_2$  singularity is usually a marginal deformation, but in the present context, blowup of the  $\mathbf{S}^5/\mathbf{Z}_2$  singularity is a relevant deformation. This statement follows from a knowledge of the  $\mathcal{N}=2$  super Yang-Mills theory dimensions but has not yet been explained from supergravity.

part of  $\mathbf{R}^4/\mathbf{Z}_2$ , where the  $x_i$  are not all zero, maps to  $\mathbf{RP}^3$  by mapping  $x_1, \ldots, x_4$  to their image in real projective space. One can identify  $\mathbf{RP}^3$  with the SO(3) group manifold. SO(3) can be projected to the two-sphere  $\mathbf{S}^2 = SO(3)/U(1)$ , where the U(1) is a maximal torus of SO(3). The resulting map of the smooth part of  $\mathbf{R}^4/\mathbf{Z}_2$  to  $\mathbf{S}^2$  can be made completely explicit as follows. Let  $n_1 = x_1 + ix_2$ ,  $n_2 = x_3 + ix_4$ , and map a point with coordinates  $x_1, \ldots, x_4$  to

$$\vec{b} = \frac{(n, \vec{\sigma}n)}{(n, n)} \tag{26}$$

where  $\vec{\sigma}$  are the Pauli matrices and the inner product is such that  $(n, n) = |n_1|^2 + |n_2|^2$ . The map of  $\mathbf{R}^4/\mathbf{Z}_2$  to  $\mathbf{S}^2$  given by this formula is ill-defined at the origin, but the blowup precisely replaces the singular point at the origin by a copy of  $\mathbf{S}^2$  in such a way that the map is well-defined everywhere.

Now return to  $\mathbf{S}^5/\mathbf{Z}_2$ , which we describe as in (24).  $\mathbf{S}^5/\mathbf{Z}_2$  is fibered over  $\mathbf{Z}_2$ , with a map that is given by precisely the same formula as in (26). Just as in the  $\mathbf{R}^4/\mathbf{Z}_2$  case, the blowup renders this map well-defined even where  $x_1, \ldots, x_4$  all vanish. The fiber of the map from  $\mathbf{S}^5/\mathbf{Z}_2$  to  $\mathbf{S}^2$  can be determined by looking at the inverse image of any point, say the point with  $\vec{b} = (0, 0, 1)$ . The inverse image of this point is given by  $x_3 = x_4 = 0$  and is a copy of  $\mathbf{S}^3$ . Hence the blowup of  $\mathbf{S}^5/\mathbf{Z}_2$  is an  $\mathbf{S}^3$  bundle over  $\mathbf{S}^2$ .

What about  $T^{1,1} = (SU(2) \times SU(2))/U(1)$ ? By "forgetting," say, the second SU(2),  $T^{1,1}$  maps to  $SU(2)/U(1) = \mathbf{S}^2$ . So  $T^{1,1}$  is also a fiber bundle over  $\mathbf{S}^2$ . The fiber is what was forgotten in the map, namely the second SU(2). Since SU(2) is isomorphic topologically to  $\mathbf{S}^3$ ,  $T^{1,1}$  is also an  $\mathbf{S}^3$  bundle over  $\mathbf{S}^2$ .

If we can show that these two  $S^3$  bundles are equivalent topologically, we will get a new understanding of the relation between the  $T^{1,1}$  and  $S^5/\mathbb{Z}_2$  models: blowup of  $S^5/\mathbb{Z}_2$  has simply produced  $T^{1,1}$ . In fact,  $S^3$  bundles over  $S^2$  are classified by  $\pi_1(SO(4)) = \mathbb{Z}_2$ , so there are only two possibilities for what our bundles might be.

Both  $S^3$  bundles we have met above are special cases of the following more general construction. Any circle bundle S over  $S^2$  can be built by starting with trivial bundles over the upper and lower hemispheres of  $S^2$ , and gluing them together over the equator with the gluing function

$$\begin{pmatrix} \cos n\theta & \sin n\theta \\ -\sin n\theta & \cos n\theta \end{pmatrix} \tag{27}$$

There is a subtlety hidden in this statement. After setting  $x_3 = x_4 = 0$ , we are left with  $x_1, x_2, x_5, x_6$ , with  $x_1^2 + x_2^2 + x_5^2 + x_6^2 = 1$ ; this looks like  $\mathbf{S}^3$ , but we are to divide by a sign reversal of  $x_1, x_2$ . The quotient by that sign reversal is a manifold, because the fixed points have codimension two, and is a copy of  $\mathbf{S}^3$ . One can prove this explicitly by writing  $x_1 = \rho \cos \theta$ ,  $x_2 = \rho \sin \theta$  with  $0 \le \theta \le 2\pi$ . The sign change of  $x_1, x_2$  amounts to  $\theta \to \theta + \pi$ ; dividing by this operation means that  $\theta$  ranges from 0 to  $\pi$ . The effect of this can be undone topologically by just replacing  $\theta$  by  $\theta' = 2\theta$ , which ranges from 0 to  $2\pi$ .

for some integer n. Here  $\theta$  is an angular variable on the equator of  $\mathbf{S}^2$ , and we regard S as an SO(2) bundle. It can be shown that the integer n equals the Euler class, or first Chern class, of the circle bundle S. Now let S and S' be circle bundles of first Chern class n and n' respectively. One can canonically make a three-sphere bundle W(S,S') whose fiber is the "join" of S and S'; concretely, if the fiber of S is a circle  $u^2 + v^2 = 1$  and the fiber of S' is a circle  $u^2 + v^2 = 1$ , then the fiber of W(S,S') is the three-sphere  $u^2 + v^2 + w^2 + x^2 = 1$ . W(S,S') can be built starting with trivial bundles over the upper and lower hemispheres of  $\mathbf{S}^2$ , and gluing them on the equator with the gluing function

$$\begin{pmatrix}
\cos n\theta & \sin n\theta & 0 & 0 \\
-\sin n\theta & \cos n\theta & 0 & 0 \\
0 & 0 & \cos n'\theta & \sin n'\theta \\
0 & 0 & -\sin n'\theta & \cos n'\theta
\end{pmatrix}.$$
(28)

This describes a trivial or non-trivial element of  $\pi_1(SO(4))$  depending on whether n+n' is even or odd. So the three-sphere bundle W(S, S') is non-trivial if and only if n+n' is odd.

As we will see, the blowup of  $\mathbf{S}^5/\mathbf{Z}_2$  can be identified naturally as W(S, S') with n=2, n'=0, while  $T^{1,1}$  can be identified naturally as W(S, S') with n=n'=1. So these bundles are both topologically trivial, isomorphic to the product  $\mathbf{S}^2 \times \mathbf{S}^3$ .

To verify the claim about  $\mathbf{S}^5/\mathbf{Z}_2$ , we note that when  $x_1,\ldots,x_4$  are not all zero, they determine (after dividing by  $\mathbf{Z}_2$ ) an element of  $\mathbf{RP}^3$ .  $\mathbf{RP}^3$ , which can be regarded as the group manifold of SO(3), is a circle bundle over  $\mathbf{S}^2 = SO(3)/U(1)$  with Euler class 2. We let S be this circle bundle over  $\mathbf{S}^2$ . We let S' be the trivial circle bundle, of n'=0, whose fiber is parametrized by  $x_5, x_6$  with  $x_5^2 + x_6^2 = 1$ . By reexamining the argument by which we showed that  $\mathbf{S}^5/\mathbf{Z}_2$  is a three-sphere bundle over  $\mathbf{S}^2$ , it can be seen that the fiber is precisely W(S,S') with the stated S and S'. The identification of  $T^{1,1}$  with W(S,S') where n=n'=1 is more elementary. It follows from noting that the second SU(2), which we "forget" to map  $T^{1,1}=(SU(2)\times SU(2))/U(1)$  to  $\mathbf{S}^2$ , can be written as a sphere  $u^2+v^2+w^2+x^2=1$ , where U(1) acts by rotation of the u-v plane together with rotation of the w-x plane.

There is a somewhat shorter and, perhaps, more powerful argument for why the blowup of the fixed circle of  $\mathbf{S}^5/\mathbf{Z}_2$  gives  $T^{1,1}$ . This argument relies on the fact that both sides have a U(1) R-symmetry.  $\mathbf{S}^5/\mathbf{Z}_2$  and  $T^{1,1}$  are  $\mathbf{S}^1$  bundles over complex surfaces, say B and B'.  $\mathbf{S}^5/\mathbf{Z}_2$  and  $T^{1,1}$  can be reconstructed from B and B' using the Calabi-Yau condition, so it suffices to compare B and B'.

Dividing  $S^5$  by U(1) gives  $\mathbb{CP}^2$ ; dividing  $S^5/\mathbb{Z}_2$  by U(1) gives  $\mathbb{CP}^2/\mathbb{Z}_2$ , where  $\mathbb{CP}^2$  has homogeneous coordinates  $a_1, a_2, a_3$ , and  $\mathbb{Z}_2$  acts by sign reversal on  $a_1, a_2$ .  $\mathbb{CP}^2/\mathbb{Z}_2$  has an orbifold singularity at  $a_1 = a_2 = 0$ ; if this is deformed or blown up, we should get the quotient of  $T^{1,1}$  by U(1), which is  $S^2 \times S^2$  or  $\mathbb{CP}^1 \times \mathbb{CP}^1$ . We can describe

 $\mathbb{CP}^2/\mathbb{Z}_2$  in terms of the  $\mathbb{Z}_2$ -invariant polynomials in  $a_1, a_2, a_3$ , which are generated by

$$u_1 = a_1^2 , \quad u_2 = a_2^2 , \quad u_3 = a_1 a_2 ,$$
 (29)

and  $a_3$ . They are homogeneous coordinates for a weighted projective space  $\mathbf{WCP}^3_{2,2,2,1}$ , and obey

$$u_1 u_2 - u_3^2 = 0 (30)$$

The weighted projective space is actually equivalent to an ordinary projective space, for the following reason. In the scaling  $(u_1, u_2, u_3, a_3) \rightarrow (\lambda^2 u_1, \lambda^2 u_2, \lambda^2 u_3, \lambda a_3)$  by which  $\mathbf{WCP}_{2,2,2,1}^3$  is defined, if we set  $\lambda = -1$ , we have simply  $a_3 \rightarrow -a_3$ . So in dividing by this scaling to obtain the weighted projective space  $\mathbf{WCP}_{2,2,2,1}^3$ , we among other things divide by  $a_3 \rightarrow -a_3$ . That step can be accomplished by restricting to the invariant functions, which are generated by  $u_1, u_2, u_3$ , and  $u_4 = a_3^2$ . The  $u_i$  all have the same weight (two), so are homogeneous coordinates for an ordinary projective space  $\mathbf{CP}^3$ . In this projective space,  $\mathbf{CP}^2/\mathbf{Z}_2$  is defined by the equation (30). To deform the singularity (which has the same topological effect as blowing it up), we deform the equation to

$$u_1 u_2 - u_3^2 + \epsilon u_4^2 = 0 , (31)$$

which by an obvious linear change of variables can be brought to the form  $\sum_{i=1}^4 z_i^2 = 0$ . This equation in  $\mathbb{CP}^3$  defines a copy of  $\mathbb{CP}^1 \times \mathbb{CP}^1$ ; this is proved by "solving" the equation in terms of A's and B's as in (12). Thus, as claimed, the blowup of the singularity of  $\mathbb{CP}^2/\mathbb{Z}_2$  gives  $\mathbb{S}^2 \times \mathbb{S}^2$ .

#### Chiral Operators

Let us now discuss the chiral operators. These operators have the lowest possible dimension for a given R-charge. We have assigned the R-charge 1/2 to each of the A's and B's. Thus, the lowest possible R-charge for a gauge invariant operator is 1. The corresponding chiral operators, namely

$$\operatorname{Tr} A_k B_l$$
, (32)

have dimension 3/2 and transform as (2,2) under the  $SU(2) \times SU(2)$  global symmetry. In general, we find chiral operators of positive integer R-charge n and dimension 3n/2,

$$C_L^{k_1 k_2 \dots k_n} C_R^{l_1 l_2 \dots l_n} \operatorname{Tr} A_{k_1} B_{l_1} A_{k_2} B_{l_2} \dots A_{k_n} B_{l_n}$$
 (33)

The equations for a critical point of the superpotential

$$B_1 A_k B_2 = B_2 A_k B_1 , \qquad A_1 B_l A_2 = A_2 B_l A_1 , \qquad (34)$$

tell us that (modulo descendants) we can freely permute all A's and all B's in (33). Thus,  $C_L$  and  $C_R$  must be completely symmetric, and for R-charge n we find chiral

operators in the (n+1, n+1) of  $SU(2) \times SU(2)$ . If we think of  $\operatorname{Tr} A_k B_l$  as a vector of SO(4), which we write  $z_i$ , then these representations are precisely the symmetric traceless polynomials in  $z_i$  of order n. Note that  $z_i$  is the original conifold coordinate, and that we have obtained the expected wave functions on the conifold.

The chiral operators we have found are rather analogous to the traceless symmetric polynomials  $\operatorname{Tr} X^{i_1} X^{i_2} \dots X^{i_n}$  in the  $\mathcal{N}=4$  SYM theory. Let us recall that, on the supergravity side, such operators were identified with modes of  $h^{\alpha}_{\alpha}$  (the trace of the metric on the compact manifold) and the four-form gauge potential on  $\mathbf{S}^5$ ; these are described by scalar spherical harmonics [30]. Thus, we expect that the spectrum of the chiral operators (33) should coincide with the spectrum of scalar spherical harmonics on  $SU(2) \times SU(2)/U(1)$ . Geometrically, this space is a product of two three-spheres with combined rotations around one of the axes modded out. Spherical harmonics on the first three-sphere are labeled by  $(J, J_3, I_3)$ , where  $J_3$  and  $I_3$  are two magnetic quantum numbers. Similarly, the quantum numbers on the second three-sphere are  $(\tilde{J}, \tilde{J}_3, \tilde{I}_3)$ . To mod out the U(1) we impose the constraint

$$I_3 + \tilde{I}_3 = 0$$
 . (35)

Then the R-charge is identified with

$$I_3 - \tilde{I}_3 = 2I_3 {,} {(36)}$$

and the remaining quantum numbers  $(J, J_3, \tilde{J}, \tilde{J}_3)$  label the  $SU(2) \times SU(2)$  representations. Chiral multiplets are obtained from the smallest representation (with the smallest quadratic Casimir operator) of given R-charge. For R-charge n, this representation is (n+1, n+1). This agrees with the spectrum of operators (33) that we found on the field theory side.

## 4 Extensions to M theory

In the preceding sections we presented a duality between Einstein space compactifications of Type IIB theory and large N field theories on D3-branes placed at conical singularities. There are obvious extensions of these results to M-theory.

For instance, the geometry  $AdS_7 \times X_4$ , where  $X_4$  is a four-dimensional compact Einstein manifold, is created by placing a large number of M5-branes at a conical singularity of  $M_6 \times Y_5$  where  $Y_5$  is a five-dimensional manifold that is a cone over  $X_4$ .

Another class of theories, which has more connections with old supergravity literature, concerns compactifications of eleven-dimensional supergravity on  $AdS_4 \times X_7$ , where  $X_7$  is a seven-dimensional compact Einstein manifold. Such backgrounds have been investigated in some detail (for a classic review, see [31]). In the context of

the AdS/CFT correspondence, each of these backgrounds corresponds to a threedimensional conformal field theory. This dual theory may be defined as the infrared limit of the world volume theory on N coincident M2-branes placed at a conical singularity of  $M_3 \times Y_8$ .  $Y_8$  is a Ricci-flat manifold, a cone over  $X_7$ . Its metric near the singularity takes the form

$$h_{mn}dx^m dx^n = dr^2 + r^2 g_{ij} dx^i dx^j$$
,  $(i, j = 1, ..., 7)$ , (37)

where r is the radial coordinate which vanishes at the singularity, and  $g_{ij}$  is the metric on  $X_7$ . The supersymmetry of the resulting three-dimensional theory is determined by the holonomy of  $Y_8$ . In fact, manifolds  $Y_8$  of Spin(7), SU(4), Sp(2) or smaller holonomy correspond to superconformal theories with  $\mathcal{N} = 1, 2, 3$  or higher supersymmetry, respectively. This has been recently pointed out also in [32].

The cases most analogous to our D3-brane construction correspond to three-dimensional  $\mathcal{N}=2$  theories and are found by placing M2-branes at conical singularities of Calabi-Yau four-folds. Consider, for instance, the non-compact four-fold defined by

$$\sum_{i=1}^{5} z_i^2 = 0 \ . \tag{38}$$

The singularity is at  $z_i = 0$ . The set of points at unit distance from the singularity,

$$\sum_{i=1}^{5} |z_i|^2 = 1 , (39)$$

is a coset SO(5)/SO(3). Indeed, the solutions to (38), (39) are obtained by SO(5) rotations of

$$z_1 = 1/\sqrt{2}$$
,  $z_2 = i/\sqrt{2}$ ,  $z_3 = z_4 = z_5 = 0$ . (40)

The subgroup of SO(5) that leaves this solution fixed is SO(3) (acting on  $z_3, z_4, z_5$ ), so the space of solutions is SO(5)/SO(3).

The conifold (38) is thus a cone over the homogeneous space SO(5)/SO(3). This homogeneous space is the example called  $V_{5,2}$  in [33] and in Table 6 of [31]. The isometries of the conifold are  $SO(5) \times U(1)$ ; the U(1) acts by  $z_i \to \gamma z_i$ , and is a group of R-symmetries (just as for the example we studied in section 3) because it acts nontrivially on the canonical line bundle of the conifold. We hope it will be possible to construct a three-dimensional field theory corresponding to M2-branes on (38).

## Acknowledgements

We are grateful to J. Figueroa-O'Farrill, S. Gubser, A. Hanany, N. Seiberg, and M. Strassler for discussions. The work of I.R.K. was supported in part by US Department of Energy grant DE-FG02-91ER40671 and by the James S. McDonnell Foundation Grant No. 91-48. The work of E.W. is supported in part by NSF grant PHY-9513835.

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