PHASE TRANSITIONS
IN M-THEORY AND F-THEORY

Edward Witten[1]

School of Natural Sciences, Institute for Advanced Study
Olden Lane, Princeton, NJ 08540, USA

Phase transitions are studied in M-theory and F-theory. In M-theory compactification to five dimensions on a Calabi-Yau, there are topology-changing transitions similar to those seen in conformal field theory, but the non-geometrical phases known in conformal field theory are absent. At boundaries of moduli space where such phases might have been expected, the moduli space ends, by a conventional or unconventional physical mechanism. The unconventional mechanisms, which roughly involve the appearance of tensionless strings, can sometimes be better understood in F-theory.

March, 1996

---

1. Introduction

In recent investigations of non-perturbative behavior of string theory, many surprising phenomena have been found. Such phenomena can very crudely be separated into two kinds. First are cases in which the phenomenon itself (such as occurrence of enhanced gauge theory or extra massless particles at a particular value of a scalar field) is not surprising, but its occurrence in a particular string theory, under particular conditions, came as a surprise. Extended gauge symmetry or extra massless matter resulting from a Calabi-Yau singularity or a small heterotic string instanton or strong heterotic string coupling are examples of this kind. Associated with all these surprises is generally a meta-surprise which is simply our ability to understand the phenomenon!

The second kind of surprises are those in which the phenomenon that is found to occur in string theory was not previously known to be possible in any physics model under any conditions. These include critical points and phase transitions that cannot be described by weakly coupled field theory of any sort, roughly because the objects that are becoming light are strings instead of particles [1-5]. For example, in six dimensions, practically any dynamics involving tensor multiplets is exotic in this sense, and much can be learned from rather general arguments [4].

The main goal of the present paper is a more microscopic study of some such phenomena in five and six dimensions, using M-theory and F-theory. In the M-theory case, we begin by studying in general the vector moduli space of M-theory compactified to five dimensions on a Calabi-Yau manifold X. Because of the usual relation of M-theory to Type IIA, this compactification is a certain strong coupling limit of the Type IIA superstring compactified to *four* dimensions on X. The vector moduli space of Type IIA superstring theory is extremely rich [6,7], with abundant "phase transitions" between a variety of geometrical and non-geometrical phases (that is, phases describable as sigma models with Calabi-Yau target and phases that must be described as more abstract conformal field theories). We put the phrase "phase transitions" in quotes because in the four-dimensional context, these are not true phase transitions, though they look like sharp phase transitions in a mean field theory approximation [7].

As we will see in section two, in M-theory compactified to five dimensions on X, there is a somewhat similar story, in developing which we draw on and extend certain results obtained recently   [8-10]. The five-dimensional case, however, has some crucial novelties. First of all, in the five-dimensional case, one does get sharp phase transitions between the

different geometrical phases. Second (as one might suspect from the fact that abstract conformal field theory has no evident role in M-theory), the non-geometrical phases do not appear in the five-dimensional story. Third and especially, at the overall boundaries of the moduli space (beyond which in four dimensions the non-geometrical phases appear), one gets a "surprise of the second kind," outside the scope of conventional low energy effective field theory. This involves a critical point at which an infinite number of particles of arbitrarily high spin go to zero mass, including electric charges and states of a dual magnetic string.

The earlier examples of "phenomena of the second kind" [1-5] have generally been in six dimensions and involve light strings that have both electric and magnetic couplings. Six dimensions is a very natural dimension for non-critical strings, because the interesting six-dimensional non-critical strings (whose tension vanishes at some point in moduli space) couple to a tensor multiplet, which is special to six dimensions. One may therefore wonder if the tensionless strings seen in M-theory could be better understood by lifting the picture to six dimensions (just as one has at this point already lifted the traditional problem of the vector moduli space from four dimensions to five in going from Type IIA to M-theory). Fortuitously, a remarkable construction known as F-theory [11] does make it possible to lift the whole discussion to six dimensions, at least for a suitable class of Calabi-Yau manifolds.

In [12], the heterotic string compactified to six dimensions on K3 was related to F-theory compactified to six dimensions on certain Calabi-Yau manifolds X. This is a very interesting case for the study of non-critical strings, because the heterotic string on K3 has (for most values of the instanton numbers) a strong coupling singularity [13] at which apparently [4,5] a string goes to zero tension. It also apparently has a transition in which an instanton turns into an M-theory five-brane, again via appearance of a tensionless string [3,4]. The general arguments used previously do not give very precise information about what kind of string goes to zero tension at these singularities, but such information can be extracted from F-theory. We will see, for instance, that in certain cases the non-critical strings that go to zero tension at the strong coupling singularity are objects that have been encountered before, and in other cases they are new.

In section four, we go on to consider the occurrence of non-critical strings of vanishing tension in string compactification to four dimensions. One easy example (in view of [14] and [1]) is the Type IIB superstring on a Calabi-Yau manifold, which develops a tensionless non-critical string when one approaches a conifold singularity from the Kahler side. The same tensionless string arises in heterotic string compactification on a certain Calabi-Yau manifold, as we show by considering an F-theory dual.

2. The Vector Moduli Space Of M-Theory

2.1. Generalities

The Kahler metric of a Calabi-Yau manifold X (for given complex structure) depends on $b_2 = \dim H^2(X)$ parameters, which determine the cohomology class of the Kahler form in the cohomology group $H^2(X, \mathbf{R})$. One function of these parameters is the volume of X, and is associated with a hypermultiplet, while the other $b_2 - 1$ parameters, which control the "shape" of X, are associated with the vector multiplets [8]. To study the vector moduli space, we are thus mainly interested in varying the shape, without letting the overall volume go to zero or infinity.

In Calabi-Yau compactification of Type IIA superstring theory to four dimensions, there are theta angles that are related by supersymmetry to the Kahler class of the metric; upon including them, one describes the vector moduli space in terms of the complexification of $H^2(X, \mathbf{R})$. In Calabi-Yau compactification of M-theory to five dimensions, the theta angles are absent, so the real parameters of $H^2(X, \mathbf{R})$ (with the volume scaled out) are the relevant ones.

For a given Calabi-Yau manifold X, the possible Kahler metrics fill out a "cone" in $H^2(X, \mathbf{R})$. As one approaches the boundary of the cone, X develops a singularity. In Type IIA compactification on X, the sigma model of X is singular only when the classical manifold X is singular and in addition a certain theta angle vanishes. By using a generic value of the theta angle, one can go smoothly past the singularity to get to another "phase" of the conformal field theory. This phase is defined in a region that is outside the Kahler cone of X; it might be the Kahler cone of another Calabi-Yau manifold Y (in which case this transition is a topology-changing process $X \to Y$), or it might be associated with a more abstract conformal field theory (such as a Landau-Ginzburg model).

The five-dimensional vector multiplet contains only one scalar; upon compactification to four dimensions, it gains a second scalar, related to the world-sheet theta angle of the Type IIA sigma model. This means that effectively in five dimensions, the theta angles are frozen to zero, so there is no way to go around the singularities. Any continuation from one phase to another will necessarily involve going through the singularities. That is why in five dimensions we will get sharply defined phases and phases transitions, which are smoothed out if one compactifies to four dimensions (roughly as, for instance, a standard ferromagnetic phase transition in three dimensions in a system with continuous symmetry is smoothed out if one compactifies to two dimensions).

3

There are various possibilities for how X may behave as one approaches the boundary of the Kahler cone. We will in this paper consider only the case that in the limit X is a complex three-manifold with singularities (as opposed to the possibility that X collapses to a space of complex dimension less than three on the boundary of the Kahler cone). The possible singularities can be very crudely classified as follows:

(1) It may be that a complex curve E is collapsing to a point as one approaches the boundary of the Kahler cone.

(2) It may be that a complex divisor D is collapsing, either to (a) a curve, or (b) a point.

Case (1) is the case of topology change - on the other side of the boundary of the Kahler cone, one has the Kahler cone of a different (but birationally equivalent) Calabi-Yau manifold Y. In section 2.2, we will see how this topology change can be described in M-theory.

Let us call the union of the Kahler cones of all Calabi-Yau's that are birationally equivalent to X the "extended Kahler cone" of X. When one approaches the boundary of the extended Kahler cone, one gets a singularity of type (2). In Type IIA compactification on X, one can continue past type (2) singularities, and then one sees phases based on more abstract conformal field theories rather than sigma models. We will explain in section 2.3 that in M-theory most or possibly all of the more abstract phases are absent. So the vector moduli space of X is just the extended Kahler cone (possibly with a few but not all of the non-geometrical phases added).

This means in particular that the vector moduli space is, with its natural metric, an *incomplete* manifold. One can reach the boundary in a finite distance. What physics can be associated with this? One way to get a boundary in the moduli space is to find an enhanced $SU(2)$ gauge symmetry. In supersymmetric $SU(2)$ gauge theory in five dimensions, the only scalar field is a field ϕ in the adjoint representation of $SU(2)$; the moduli space of classical vacua is thus parametrized by the order parameter $u = \mathrm{tr}\phi^2$, which is real and non-negative, so the moduli space is the half-line $u \geq 0$. An $SU(2)$ gauge symmetry is restored at the boundary point $u = 0$. It has indeed been shown that a boundary of the Kahler cone of type (2a) is associated with enhanced $SU(2)$ gauge symmetry [15]. (See also [16] for the case of collapse of a divisor to a curve of genus zero, and [17] for related issues.)

Note that in five-dimensional supersymmetric field theory, to restore a gauge symmetry of rank greater than one requires adjusting more than one real parameter; thus $SU(2)$

4

is the only extended gauge symmetry that can appear on going to the boundary of the extended Kahler cone in a generic fashion. More generally, as long as the physics is free in the infrared, and so describable by an effective classical field theory, restoration of an $SU(2)$ gauge symmetry or a discrete $\mathbf{Z}_2$ symmetry is the only way to produce a boundary of the moduli space. Thus the fact that a singularity of type (2a) gives precisely an $SU(2)$ gauge symmetry is no accident.

There remains the case (2b), which is possibly more typical, and brings us finally to a "surprise of the second kind" as promised in the introduction. As we will explain in section 2.4, when one approaches a singularity of type (2b), one gets a novel kind of low energy physics with infinitely many particles of arbitrarily high spin all going to zero mass. There is also a tensionless non-critical string which is "magnetically" charged with respect to this infinity of light "electric" charges. The limiting theory where all these particles reach zero mass is (as in like examples mentioned in the introduction) not free in the infrared and so beyond the reach of the comments in the last paragraph.

2.2. Collapse Of A Curve

In compactification of M-theory on a Calabi-Yau manifold X to a five-manifold W, vector fields A^a, $a = 1, \ldots, \dim H^2(X)$ arise from the five-dimensional reduction of the underlying three-form field C. These vector fields interact, among other things, through Chern-Simons couplings

$$L_{CS} = \frac{1}{24\pi^2} \int_W d^5x \, \epsilon^{\mu\alpha\beta\gamma\delta} A_\mu^a \partial_\alpha A_\beta^b \partial_\gamma A_\delta^c \lambda_{abc}. \tag{2.1}$$

Here the λ_{abc} are constants (integers as discussed later) determined by the intersection ring of X; they in turn determine the metric on the vector moduli space.

Now consider the behavior as one approaches the boundary of the Kahler cone and a complex curve E collapses. A BPS-saturated hypermultiplet, which arises by wrapping a two-brane over E, goes to zero mass in this limit. The effective Lagrangian of a five-dimensional hypermultiplet H can depend on a real mass parameter m, which enters by terms $m^2|\phi|^2 + m\overline{\psi}\psi$, where ϕ and ψ are the bosons and fermions in H. The particles obtained by quantizing ϕ and ψ have mass $|m|$. Thus, continuation to negative m makes sense in the low energy effective action.

What would a continuation to negative m mean in terms of M-theory, in the present context? For the hypermultiplet H that arises by wrapping a two-brane over E, m is

the area of E. Thus the continuation to negative m is a kind of continuation to negative area. This has been encountered in studying the vector moduli space in Type IIA superstring theory: the continuation to negative area is a "flop" to a different (but birationally equivalent) Calabi-Yau manifold Y. In this transition, the curve E, whose area should superficially become negative, disappears and is replaced by a curve F on Y of positive area but opposite cohomology class to E. [2]

To clarify the physical meaning of the sign of m, recall that in five dimensions the Lorentz group $SO(1,4)$ has only one spinor representation, which is pseudo-real. The Clifford algebra $\{\Gamma_\mu, \Gamma_\nu\} = 2\eta_{\mu\nu}$, has, however, two representations, one with $\Gamma_0\Gamma_1\Gamma_2\Gamma_3\Gamma_4 = 1$, and the other with $\Gamma_0\Gamma_1\Gamma_2\Gamma_3\Gamma_4 = -1$. The two representations differ by $\Gamma_\mu \to -\Gamma_\mu$, which preserves the anticommutation relations. While $SO(1,4)$ has only one spinor representation, $SO(4)$, which is the little group of a massive particle, has two. Once a representation of the Clifford algebra is picked, the Dirac equation with mass reads $(i\Gamma^\mu D_\mu - m)\psi = 0$, and the sign of m determines under which of the two spinor representations of $SO(4)$ the massive particle transforms. Thus, the two signs of m are physically inequivalent, but which sign of m goes with which representation of $SO(4)$ depends on which representation of the Clifford algebra one uses – since $\Gamma^\mu \to -\Gamma^\mu$ has the same effect on the Dirac equation as $m \to -m$.

Now, the manifolds X and Y – with Y being obtained from X by "analytic continuation to negative area" – have different intersection forms, and therefore different values of the coefficients λ_{abc} of equation (2.1). If the M-theory on X looks at low energies like a theory with a hypermultiplet of mass m interacting with gauge fields, and if continuation past $m = 0$ corresponds to jumping from X to Y, then we must see that coupling of gauge fields to a charged hypermultiplet results in a jump in the Chern-Simons coefficients when one passes through $m = 0$.

This is implicit in computations in [9] and is not hard to verify directly. Suppose that our hypermultiplet H couples to a linear combination $A = \sum_a c_a A^a$ of the vectors, with the c_a being coefficients. Consider the one-loop AAA amplitude due to the charged fermion loop. [3] If the three external photons have momenta p, q, and $-p - q$, and polarizations α,

---

[2] An example of this phenomenon is given at the end of the present paper.

[3] On dimensional grounds in this unrenormalizable theory, or by arguments involving locality or the quantization of the Chern-Simons coupling, diagrams with more loops cannot renormalize the Chern-Simons interaction.

β, and γ, then the diagram in which they are attached to the fermion loop in that cyclic order gives an amplitude

$$\frac{1}{(2\pi)^5} \int d^5k \, \mathrm{tr}\Gamma_\mu \frac{1}{\Gamma \cdot (k+p) - m} \Gamma_\alpha \frac{1}{\Gamma \cdot k - m} \Gamma_\beta \frac{1}{\Gamma \cdot (k-q) - m}. \qquad (2.2)$$

After doing the Dirac algebra, the parity-violating part of this is

$$-\frac{m\epsilon_{\mu\alpha\beta\gamma\delta}p^\gamma q^\delta}{8\pi^5} \int \frac{d^5k}{(k^2 - m^2)^3} \qquad (2.3)$$

plus terms of higher order in external momenta. Notice that an explicit factor of m appears in the numerator; on the other hand, after Wick rotation, the integral in (2.3) can be evaluated to give $\pi^3/2|m|$. Thus, the amplitude is proportional to $m/|m| = \mathrm{sign}\, m$ and is in fact

$$-\frac{i\,\mathrm{sign}\,m}{16\pi^2}\epsilon_{\mu\alpha\beta\gamma\delta}p^\gamma q^\delta. \qquad (2.4)$$

After adding the crossed diagram, this corresponds to an A^3 interaction vertex

$$-\frac{\mathrm{sign}\,m}{48\pi^2} \int_W d^5x \; \epsilon^{\mu\alpha\beta\gamma\delta}A_\mu\partial_\alpha A_\beta\partial_\gamma A_\delta. \qquad (2.5)$$

The jump when m changes sign is thus

$$\frac{1}{24\pi^2} \int_W d^5x \; \epsilon^{\mu\alpha\beta\gamma\delta}A_\mu\partial_\alpha A_\beta\partial_\gamma A_\delta. \qquad (2.6)$$

This should be compared with the change in the intersection form under birational transformation from X to Y. The change is that, for every curve E that collapses, the Yukawa coupling of the multiplet containing the corresponding A changes by 1. (See, for example, pp. 209-211 of [7].) This agrees with (2.6) provided that the expression (2.6) is the correctly normalized five-dimensional Chern-Simons action with coefficient 1. In verifying this, there are some subtleties. Recall that the Chern-Simons integral on a five-manifold W can be usefully defined in terms of the integral of $F \wedge F \wedge F$ over a six-manifold Z with boundary W; the normalization of the Chern-Simons action is conventionally chosen so that this integral is independent of Z (and of the extension of the gauge field over Z) precisely modulo 2π. The expression

$$\frac{1}{4\pi^2} \int_Z d^6x \; \epsilon^{\mu\nu\alpha\beta\gamma\delta}\partial_\mu A_\nu\partial_\alpha A_\beta\partial_\gamma A_\delta \qquad (2.7)$$

is correctly normalized so that, for closed oriented six-manifolds Z and $U(1)$ gauge fields A, its possible values are $2\pi n$ for arbitrary integer n. So, because the denominator is $24\pi^2$

instead of $4\pi^2$, (2.6) is 1/6 of the Chern-Simons interaction as it would conventionally be defined.

This seems to have the following interpretation. Suppose that the closed oriented six-manifold Z is actually a spin manifold with $p_1 = 0$. [4] Then one can show that (2.7) is divisible by 12π, and not just 2π. This amounts to the following topological fact. If L is a complex line bundle over an oriented six-manifold Z, then in general $c_1(L)^3$ can be an arbitrary integer; but if Z is spin and has $p_1 = 0$, then $c_1(L)^3$ is divisible by six. Indeed, on a six-manifold Z that is spin with $p_1 = 0$, the index of the Dirac operator, for spinors with values in L, is $c_1(L)^3/6$, showing that $c_1(L)^3$ is divisible by six.

Suppose, then, that one wants to define the Chern-Simons interaction not on arbitrary oriented five-manifolds W but only for those that are spin manifolds with $p_1 = 0$. If W has the stated properties, it can be shown using cobordism theory to be the boundary of a six-manifold Z that is likewise spin, with $p_1 = 0$, and by using only such Z's in defining the Chern-Simons interaction on W, one can ensure that (2.6) is uniquely defined modulo 2π, even though the denominator $24\pi^2 = 6 \cdot 4\pi^2$ in (2.6) is six times the denominator in (2.7).

It remains, then, to explain why in M-theory on $W \times X$, with W a five-manifold and X a Calabi-Yau manifold, we are only interested in the case that W is spin and has $p_1 = 0$. W must of course be spin because the theory has fermions. The restriction to $p_1 = 0$ comes because [18,19] in M-theory there is an interaction $C \wedge I(R)$ where C is the three-form of M-theory and $I(R)$ is an eight-form constructed as a quartic polynomial in the Riemann tensor. Because of this interaction, the equations of motion can only be obeyed if $I(R)$ vanishes cohomologically. In compactification on $W \times X$, $I(R)$ is a multiple of $p_1(W) \cdot p_1(X)$, and as $p_1(X) \neq 0$ for an arbitrary Calabi-Yau manifold X, it follows that one must require $p_1(W) = 0$.

---

[4] In what follows, we will not analyze the torsion and so will not get a complete result. One should really use the characteristic class $p_1/2$ (which is well-defined for spin manifolds) rather than p_1. The argument given momentarily that $c_1(L)^3$ is divisible by six only requires that $p_1/2$ vanish modulo torsion. For the cobordism assertion of the next paragraph, $p_1/2$ should vanish; I do not know if the statement still holds if $p_1/2$ is a torsion class. The M-theory argument at the end of this sub-section only shows that $p_1(W)/2$ can be assumed to vanish mod torsion; I do not know if M-theory allows torsion in $p_1(W)/2$.

2.3. Absence Of The Non-Geometrical Phases

In Type IIA compactification on a Calabi-Yau X, there are many geometrical and non-geometrical phases. We have just seen that the geometrical phases are connected in M-theory (though, unlike the Type IIA case, one must go through true phase transitions to connect them). About the non-geometrical phases, one faces a puzzle: they are described in string theory by relatively abstract conformal field theories (rather than sigma models), and it is hard to see what this could correspond to in M-theory. We will now argue that the non-geometrical phases are absent in M-theory. (It follows that the non-geometrical phases are also absent in F-theory, which turns into M-theory upon compactification on a circle.)

Before getting into a general discussion, let us first mention an important special case of how to see the absence in M-theory compactification to five dimensions of a continuation to a non-geometrical phase. At certain kinds of boundaries of the generalized Kahler cone [15], one gets an enhanced $SU(2)$ gauge symmetry. (A derivation of this result is given in the next subsection.) As explained in section 2.1, the order parameter for such a symmetry enhancement is a real, positive semi-definite field $u = \text{tr}\phi^2$, with ϕ a *real* scalar in the adjoint representation. Thus, one sees in the low energy field theory that $u = 0$ is a boundary of the moduli space, with no way to continue beyond it. After compactification to four dimensions, ϕ and u become complex and one can continue past $u = 0$ to a non-geometrical phase.

Now let us analyze why such boundaries appear. In Type IIA superstring theory on $\mathbf{R}^4 \times X$, the number of vector multiplets is $b_2(X)$, and the vector moduli space is modeled on $H^2(X, \mathbf{C})$. In M-theory on $\mathbf{R}^5 \times X$, there are only $b_2(X) - 1$ vector multiplets; the overall volume of X transforms in a hypermultiplet [8], while the scalars in vector multiplets are the "shape" parameters in $H^2(X, \mathbf{R})$. The vector moduli space is thus roughly the projectivization (in the real sense) of $H^2(X, \mathbf{R})$.

In a given geometrical phase, in taking a large volume on X, one can read off from eleven-dimensional supergravity the metric on the M-theory vector moduli space [8]. The result so obtained is exact, in that phase, since the unbroken five-dimensional supersymmetry permits no corrections. Indeed, as the volume of X transforms in a hypermultiplet, the metric on the vector moduli space is independent of the volume, so can be computed in the large volume, field theory limit. This metric is completely determined by the Chern-Simons couplings that we discussed in section 2.2 [20] and thus by the intersection form of X.

In another geometrical phase based on a different birational model Y, the metric on vector moduli space is determined by the intersection form of Y. This is different from that of X, so the metric and other couplings are non-analytic in crossing the phase boundary, as befits a *bona fide* phase transition.

Now to compare M-theory on $\mathbf{R}^5 \times X$ to Type IIA superstring theory on $\mathbf{R}^4 \times X$, we begin by looking at M-theory on $\mathbf{R}^4 \times \mathbf{S}^1 \times X$. As the radius, R, of the $\mathbf{S}^1$ goes to infinity, this goes over to M-theory on $\mathbf{R}^5 \times X$, while as it goes to zero, one gets Type IIA on $\mathbf{R}^4 \times X$. If g_M is the M-theory metric on $\mathbf{R}^4 \times X$, and g_{II} is the Type IIA metric on $\mathbf{R}^4 \times X$, then the relation between them is (see p. 93 of [21]) $g_M = g_{II}/T^{1/3}R$. (T is the two-brane tension, included here for dimensional reasons.) If then K_M and K_{II} are the Kahler classes of X as measured in M-theory or in the Type IIA theory, one has likewise

$$K_M = \frac{1}{T^{1/3}R} K_{II}. \tag{2.8}$$

This means that if one keeps K_M fixed while taking R to infinity, then K_{II} must go to infinity. Thus, M-theory in five dimensions only "sees" what in conformal field theory would be understood as the region at infinity in the moduli space.

In going from the Type II vector moduli space to the M-theory vector moduli space, the overall scale of K_{II} should be eliminated (since the volume is part of a hypermultiplet in M-theory). It is clear from (2.8) that the scale of K_{II} is to be removed by scaling $K_{II} \to \infty$ as $R \to \infty$. Note that since the metric on the Type IIA vector moduli space depends only on K_{II}/α', we could alternatively take $\alpha' \to 0$ instead of $K_{II} \to \infty$.

(2.8) has only been derived so far in a semi-classical sense, for long wavelengths. To make an exact statement, one must be more precise about what K_{II} means. There are at least two important sets of natural coordinate systems on vector moduli space: one can use the coupling constants of the linear sigma model [7], which I will call linear sigma model coordinates but which in [22] are called algebraic coordinates; or one can use the special coordinates of $N = 2$ special geometry, which in that paper are called σ-model coordinates. Linear sigma model coordinates have no natural meaning in M-theory, so to make the most natural comparison between Type IIA and M-theory, we should understand K_{II} to be the Kahler class as defined in special geometry. Indeed, the components of RK_M are the special coordinates of M-theory on $\mathbf{S}^1 \times X$.

In linear sigma model coordinates, the geometrical and non-geometrical phases appear to have a co-equal status: each occupies a cone in $H^2(X)$. In special coordinates, the

10

story is quite different. To within stringy corrections that move the phase boundaries by an amount of order α', the geometrical phases occupy in special coordinates the same cones that they occupy in linear sigma model coordinates, but the non-geometrical phases are squashed to small regions, with a thickness of order α', along the boundaries of the extended Kahler cone. This result is derived in [22], and is depicted clearly in figure 6 of that paper. (For further related discussion see [23].) We will call the squashed cones occupied by the non-geometrical phases "plates."

Now we can easily see what happens in going from Type IIA to M-theory. This is done, as explained above, by scaling K_{II} to infinity, or equivalently by taking the limit as $\alpha' \to 0$ with K_{II} fixed. When we do so, the plates (whose thickness in at least one direction is of order α', at least in the cases that have been analyzed) are flattened to nothing. But for $\alpha' \to 0$, the geometrical phases occupy precisely the cones of the linear sigma model – the corrections come from world-sheet instantons whose contributions vanish as $\alpha' \to 0$.

The upshot is that in M-theory, the vector moduli space for any geometrical phase is simply the corresponding cone in $H^2(X)$ (divided by overall scaling), with the metric obtained from eleven-dimensional supergravity. All geometrical phases must be included, as we saw in section 2.2, and the overall vector moduli space is the union of all the geometrical phases, making up the extended Kahler cone. When one reaches the boundary of the extended Kahler cone, the moduli space ends, either with extended $SU(2)$ gauge symmetry [15] or via more exotic physics that we come to next.

2.4. Collapse Of A Divisor

Now we will consider what happens in M-theory when a divisor D collapses as one approaches a boundary of the Kahler cone. There are two cases to consider: (a) D collapses to a curve E; (b) D collapses to a point. The former case was studied in [15].

Simple examples of the two cases are as follows:

(a) D could consist of $\mathbf{P}^1 \times F$, with F a curve. Then one could consider a limit in which $\mathbf{P}^1$ collapses to a point, so that D collapses to F. This can also be generalized to a fiber bundle, as analyzed in [15]. When D collapses in this way, one gets a curve of A_1 singularities (parametrized by F).

(b) Consider an isolated $\mathbf{Z}_3$ orbifold point, say the singularity at the origin in $X' = \mathbf{C}^3/\mathbf{Z}_3$, where $\mathbf{Z}_3$ acts by $(z_1, z_2, z_3) \to (\omega z_1, \omega z_2, \omega z_3)$, with $\omega^3 = 1$. By blowing up the origin, replacing the origin in X' by a copy of $\mathbf{P}^2$, one gets a smooth (non-compact) Calabi-Yau manifold X, which looks near the $\mathbf{P}^2$ like a piece of a global Calabi-Yau manifld X'.

One can approach a boundary of the Kahler cone of X' by letting the volume of $D = \mathbf{P}^2$ go to zero, so that D collapses to a point.

Now, in either example, let us look for BPS-saturated states that can be made by wrapping two-branes or five-branes on D and that go to zero mass as one approaches the singularity. (See [10] for some background.) In case (a), the lightest states come from wrapping a two-brane over $\mathbf{P}^1 \times x$, where x is an arbitrary point in F. The moduli space $\mathcal{M}$ of such two-branes is a copy of F. We will later quantize the collective coordinates corresponding to this moduli space and recover the spectrum claimed in [15]. If r is the area of the $\mathbf{P}^1$, such states have mass proportional to r. One can consider a two-brane wrapped over a more general complex curve in D; this gives a BPS-saturated state whose mass does not vanish as $r \to 0$. Looking for other light states, one can make a low-tension string by wrapping a five-brane over D. Such a string has tension of order r, so (if the usual sort of dimensional analysis can be applied) states obtained as excitations of such a string have masses of order $r^{1/2}$. Thus the lightest states, for $r \to 0$, come from the two-brane wrapped on a copy of $\mathbf{P}^1$.

Now we come to case (b). First of all, the opportunities for two-brane wrapping are much richer. Let r be the Kahler class of $D = \mathbf{P}^2$, and let (y_1, y_2, y_3) be homogeneous coordinates for D. Then one can in a supersymmetric fashion wrap a two-brane over any complex curve in D given by a degree n homogenous equation $f_n(y_1, y_2, y_3) = 0$. Let $\mathcal{M}_n$ be the moduli space of such curves; thus the dimension of $\mathcal{M}_n$ is

$$\dim \mathcal{M}_n = \frac{n(n+3)}{2}. \tag{2.9}$$

A two-brane wrapped over such a degree n curve has area nr, so quantization of $\mathcal{M}_n$ will give states whose mass is $nr \cdot T_2$, with T_2 the two-brane tension. The fundamental difference from the previous case is that all values of n arise, and not only $n = 1$. [5] In case (b), since the volume of D is $r^2/2$, one can make a string with tension of order r^2 by wrapping a five-brane over D. If the usual dimensional analysis holds, states obtained by quantizing such a string have masses of order $(r^2)^{1/2} = r$, just like the two-brane states.

---

[5] In case (a), one can consider several parallel two-branes, each wrapped over a copy of $\mathbf{P}^1$, and look for a bound state – which would give $n > 1$ – but it is believed that such bound states do not exist. In case (b) one gets bound states for free simply because the generic f_n is irreducible, so that the wrapped two-brane is not a combination of objects of lower degree.

Because of duality between two-branes and five-branes (and the fact that the complex curve $f_n = 0$ has a non-zero intersection number with $D = \mathbf{P}^2$), the string obtained from the five-brane is "magnetically" charged with respect to the "electric" charges that come from the two-branes. Thus, while in case (a) one gets eventually near the boundary of the Kahler cone only finitely many light states from quantization of $\mathcal{M}$, in case (b) we will get infinitely many "electric" states from quantizing the $\mathcal{M}_n$'s, together with all the modes of the "magnetic" string.

Quantum Numbers Of Electric States

We will now try to learn a little more by analyzing the quantum numbers of the "electric" states, that is the two-brane wrapping modes. (We cannot make a similar analysis for "magnetic" states, coming from the light string, since we do not know how to quantize it; as it is strongly coupled, it is not clear that it can be quantized in the conventional sense.)

In Calabi-Yau compactification of M-theory, there are eight unbroken real supersymmetries, transforming as a spinor of $SO(1,4)$. Under the little group $SO(4) = SU(2)_1 \times SU(2)_2$ of a massless particle, the supercharges transform as $2(1/2, 0) \oplus 2(0, 1/2)$, that is two copies each of $(1/2, 0)$ and $(0, 1/2)$. The presence of a two-brane breaks half of the supersymmetries, in an $SO(4)$ invariant fashion; with a suitable choice of orientation we can suppose that the $2(0, 1/2)$ supercharges are broken and the $2(1/2, 0)$ supercharges annihilate the classical two-brane configuration.

The breaking of the $2(0, 1/2)$ supercharges gives four fermion zero modes (related by the unbroken supersymmetries to spatial translations), whose quantization gives four states that transform as $2(0, 0) \oplus (0, 1/2)$, which is the content of a "half-hypermultiplet" H_0. The rest of the story involves quantization of the collective coordinates of the two-branes, that is, quantization of the moduli spaces $\mathcal{M}$ (or $\mathcal{M}_n$) of complex curves.

There are four unbroken supersymmetries, and they must be realized in the quantum mechanics of the collective coordinates. This occurs in a fairly standard way. The quantum states are differential forms on $\mathcal{M}$. Because $\mathcal{M}$ is a Kahler manifold, on the differential forms there act four natural operators $Q_i = \partial$, $\overline{\partial}$, ∂^*, and $\overline{\partial}^*$, which generate a $0+1$-dimensional supersymmetry algebra and represent the unbroken $2(1/2, 0)$ supercharges in acting on the differential forms on $\mathcal{M}$. The BPS-saturated states are the states annihilated by the Q_i, that is, the harmonic forms on $\mathcal{M}$.

Now we would like to know the spins of these BPS-saturated states. In the rotation group $SO(4) = SU(2)_1 \times SU(2)_2$, both $SU(2)_1$ and $SU(2)_2$ act trivially on the bosonic collective coordinates of the two-branes – the modes tangent to $\mathcal{M}$. The story is different for the fermionic collective coordinates. As they are generated from the bosonic ones by the Q_i, which transform as $2(1/2, 0)$, the fermionic collective coordinates transform trivially under $SU(2)_2$, which therefore acts trivially on the harmonic forms on $\mathcal{M}$. But by the same token, the fermionic collective coordinates transform non-trivially under $SU(2)_1$, which therefore acts non-trivially on the harmonic forms.

In fact, the action of $SU(2)_1$ on the harmonic forms on $\mathcal{M}$ is just the standard action of $SU(2)$ on the cohomology of a Kahler manifold. One can pick a standard basis J_3, J_+, J_- of the Lie algebra of $SU(2)_1$ so that J_3 acts on a (p, q) form on $\mathcal{M}$ by multiplication by $((p + q) - \dim_{\mathbf{C}} \mathcal{M})/2$, while J_+ and J_- act by wedge product or contraction with the Kahler form. In particular, the $SU(2)_1$ multiplet of highest spin (detected by the largest J_3 eigenvalue) consists of the powers of the Kahler form and has spin $\frac{1}{2}\dim_{\mathbf{C}} \mathcal{M}$.

Now we can analyze the examples (a) and (b) above. In example (a), with a divisor collapsing to a curve, the relevant moduli space is $\mathcal{M} = F$, which is a Riemann surface of genus, say, g. The cohomology of $\mathcal{M}$ consists of a $(0, 0)$-form, of $J_3 = -1/2$, a $(1, 1)$-form, of $J_3 = 1/2$, and $2g$ additional $(1, 0)$ or $(0, 1)$-forms, of $J_3 = 0$. The combined spectrum from quantization of $\mathcal{M}$ is thus $(1/2, 0) \oplus 2g(0, 0)$. When one tensors this with the half-hypermultiplet H_0 related to the translations, one gets a vector multiplet, consisting of $(1/2, 1/2) \oplus 2(1/2, 0)$, and g hypermultiplets, that is, g copies of $4(0, 0) \oplus 2(0, 1/2)$, This is the spectrum obtained in [15], by a different but not totally unrelated method. The one charged vector multiplet just found, together with its partner of opposite charge, generates an enhanced $SU(2)$ gauge symmetry. The g hypermultiplets, together with their charge conjugates and some neutral modes that come from complex structure deformations, make g hypermultiplets in the adjoint representation of $SU(2)$.

In example (b), with a divisor collapsing to a point, we see using (2.9) that the maximum value of J_3 in quantizing $\mathcal{M}_n$ is $n(n+3)/4$. Since the mass is $M \sim n$, this gives the relation – familiar from quantization of strings – $J \sim M^2$ for large M.

Along with the "electric" states that we have just analyzed, the same model also has "magnetic" states associated with the light string. How could one hope to find a common origin for the electric and magnetic states together? The only obvious hope is to use F-theory, replacing M-theory on $\mathbf{R}^5 \times X$ with F-theory on $\mathbf{R}^5 \times \mathbf{S}^1 \times X$. Here, as explained in [11], X must be a Calabi-Yau that admits an elliptic fibration. If one obtains a light

anti-self-dual non-critical string in six dimensions, then in the reduction to five dimensions, the "electric" modes in five dimensions would come by wrapping the string on $\mathbf{S}^1$, while the "magnetic" states in five dimensions would be unwrapped states of the same string. In the next section we study non-critical strings in F-theory.

3. Non-Critical Strings In F-Theory

A natural class of supersymmetric models in six dimensions is obtained by compact-ifying the $E_8 \times E_8$ heterotic string on K3, with $12 + n$ instantons in one E_8 and $12 - n$ in the second. (There is of course no essential loss in limiting to $n \geq 0$.) According to [12], this model is equivalent to F-theory on the Hirzebruch surface $\mathbf{F}_n$. That surface can be described roughly as the quotient $\mathbf{C}^4/\mathbf{C}^* \times \mathbf{C}^*$, where $\mathbf{C}^4$ has complex coordinates x, y, u, v, and $\mathbf{C}^* \times \mathbf{C}^*$ acts by

$$(x, y, u, v) \to (\lambda x, \lambda y, \mu u, \lambda^n \mu v), \tag{3.1}$$

with $\lambda, \mu \in \mathbf{C}^*$. For the properties that we want to see, it is helpful to give a "symplectic" description of $\mathbf{F}_n$, for which we introduce the D-functions

$$\begin{aligned} D_1 &= |x|^2 + |y|^2 + n|v|^2 - r_1 \\ D_2 &= |u|^2 + |v|^2 - r_2. \end{aligned} \tag{3.2}$$

Then one defines the $\mathbf{F}_n$ as the space of solutions of $D_1 = D_2 = 0$, divided by $U(1) \times U(1)$; the $U(1) \times U(1)$ action is given in (3.1), with now $\lambda, \mu \in U(1)$. This exhibits $\mathbf{F}_n$ as a Kahler manifold whose Kahler class depends on the two real parameters r_1 and r_2.

The Hirzebruch surface is fibered over $\mathbf{P}^1$ by the map that simply "forgets" u and v; the fibers (obtained by projectivizing $u - v$ space) are again copies of $\mathbf{P}^1$. The generic section of the fibration $\mathbf{F}_n \to \mathbf{P}^1$ is given by the equation

$$v = u g_n(x, y), \tag{3.3}$$

where g_n is an arbitrary homogeneous polynomial of degree n. There is also an "exceptional section" E given by

$$u = 0. \tag{3.4}$$

It is a copy of $\mathbf{P}^1$, embedded in $\mathbf{F}_n$ with self-intersection number $-n$.

The heterotic string compactified on K3 has for $n \neq 0$ a rather mysterious strong coupling singularity [13]. According to [12], this singularity is associated in F-theory with the collapse of the exceptional section E. Notice that on E, that is at $u = 0$, the equations $D_1 = D_2 = 0$ reduce to $|v|^2 = r_2$ and $|x|^2 + |y|^2 = r_1 - nr_2$. Thus, the area of E vanishes at $r_1 - nr_2 = 0$. When this happens, an F-theory string obtained by wrapping a Type IIB threebrane around E will go to zero tension. Thus, as has been suspected on more generic grounds [4,5], a tensionless string appears at the point of the strong coupling singularity.

To understand more, we should know something about the singularity that $\mathbf{F}_n$ develops when one reaches $r_1 - nr_2 = 0$, with r_1 and r_2 positive. With this condition on the parameters, the equations (3.2) imply

$$|u|^2 = \frac{1}{n} \left(|x|^2 + |y|^2 \right). \tag{3.5}$$

Thus, $|u|$ is completely determined in terms of x, y. Moreover, the $U(1)$ symmetry associated with the parameter μ in (3.1) can be uniquely fixed (except at $x = y = u = 0$, where our subsequent assertions can be seen to hold anyway) by asking for u to be, say, positive. Thus one can eliminate and forget about u and μ, and describe this degeneration $\widetilde{\mathbf{F}}_n$ of $\mathbf{F}_n$ in terms of three variables x, y, v, with one equation

$$|x|^2 + |y|^2 + n|v|^2 = r_1 \tag{3.6}$$

and one $U(1)$ symmetry to divide by, namely

$$(x, y, v) \rightarrow (\lambda x, \lambda y, \lambda^n v). \tag{3.7}$$

$\widetilde{\mathbf{F}}_n$ has (for $n > 1$) a singularity at the point P with $x = y = 0$. This singularity is simply a $\mathbf{Z}_n$ orbifold singularity; near $x = y = 0$, $\widetilde{\mathbf{F}}_n$ looks like the quotient of the $x - y$ plane by the $\mathbf{Z}_n$ symmetry

$$(x, y) \rightarrow (\zeta x, \zeta y), \quad \text{with } \zeta^n = 1. \tag{3.8}$$

In fact, $\widetilde{\mathbf{F}}_n$ is simply the weighted projective space $\mathbf{P}^2_{1,1,n}$, the subscripts being the weights. To see this, one simply goes back from the symplectic description of $\widetilde{\mathbf{F}}_n$ by (3.6) and (3.7) to the complex description of the same space, in which one replaces (3.6) by a condition that x, y, and v are not all zero and permits λ in (3.7) to range over $\mathbf{C}^*$.

It is now possible to make some interesting statements about the nature of the string that arises at the strong coupling singularity, for various values of n.

16

$n = 1$

$n = 1$ is the unique case in which there is no orbifold singularity; in fact, $\widetilde{\mathbf{F}}_1$ is the ordinary projective space $\mathbf{P}^2$, all weights being one. The point P is a smooth point in $\mathbf{P}^2$, which is replaced by a two-sphere E of self-intersection -1 in going from $\mathbf{P}^2$ to $\mathbf{F}_1$. The operation of replacing a smooth point P by a two-sphere E of self-intersection -1 is known as "blowing-up P," and is possible for any smooth point on a complex surface. We have just recovered the standard fact that $\mathbf{F}_1$ is equivalent to the surface made by blowing-up $\mathbf{P}^2$ at a point, to give an "exceptional curve" E.

To understand the strong coupling singularity for $n = 1$, we should study the string made by wrapping a Type IIB threebrane over E. Analysis of this string only depends on the behavior of $\mathbf{F}_1$ in a neighborhood of E. But this behavior is universal – locally one would get the same picture after blowing up any smooth point on any complex surface. Thus, the strong coupling singularity for $n = 1$ has nothing really to do with the details of $\mathbf{F}_1$, and just involves the behavior of F-theory under blow-down of a two-sphere of self-intersection number -1 to make a smooth point.

One of the things one most wants to know about the strong coupling singularities of the heterotic string is whether, upon adjusting a tensor multiplet to reach the strong coupling singularity, one can make a transition to a "Higgs banch" with a different number n_T of tensor multiplets. In [4], it was shown using anomalies that this could possibly occur only for $n = 1$ and $n = 4$. (In this paper, the question will be addressed only for $n = 1$, but I understand that the $n = 4$ case will be discussed elsewhere [24].) As background, let us recall [11] that in F-theory on a complex surface B, $n_T = b_2(B) - 1$. For instance, the Hirzebruch surfaces have $b_2 = 2$ (corresponding to the two Kahler parameters r_1 and r_2 introduced above), so $n_T = 1$, as expected for a perturbative heterotic string. Under blow-up of a point, b_2 increases by one, while under blow-down, b_2 decreases by one.

In particular $\mathbf{P}^2$ has $b_2 = 1$, so F-theory on $\mathbf{P}^2$ would have $n_T = 0$, no tensor multiplets at all. This strongly hints that the Higgs branch for $n = 1$ is simply F-theory on $\mathbf{P}^2$. We give some checks on this below, but first we point out some general consequences of assuming that the Higgs branch exists in this situation. Since the whole analysis is local, if the transition between $\mathbf{P}^2$ and $\mathbf{F}_1$ is possible in F-theory, analogous transitions are possible for arbitrary blow-ups and blow-downs (involving smooth points).

For instance, one could start with $\mathbf{F}_n$ for any n, and blow up a point $Q \in \mathbf{F}_n$, to get a surface S with $b_2 = 3$, corresponding to a model with $n_T = 2$. Then, if one finds a two-sphere $J \subset S$ with self-intersection number -1, one can blow down J to get another

but perhaps different model with $b_2 = 2$ and $n_T = 1$. For instance, the fibers of $\mathbf{F}_1 \to \mathbf{P}^1$ are two-spheres with self-intersection number zero and so cannot be blown down; but the fiber J containing Q acquires self-intersection number -1 when Q is blown up. Thus, after blowing up Q, one can blow down J, getting back to $n_T = 1$. In fact, the result of blowing up Q and then blowing down J is to make a transition from $\mathbf{F}_n$ to $\mathbf{F}_{n-1}$ if Q is generic, or to $\mathbf{F}_{n+1}$ if Q lies in E.

This should be compared to the situation seen in M-theory on K3 $\times$ $\mathbf{S}^1/\mathbf{Z}_2$, with instanton numbers $(12 + n, 12 - n)$ at the two ends. Of course, this model is also believed to be equivalent to the heterotic string on K3 and thus to F-theory on $\mathbf{F}_n$. In M-theory, one can apparently bring about a change in n by a process in which a small instanton is emitted from one of the ends, turning into a five-brane which then travels to the other end of $\mathbf{S}^1/\mathbf{Z}_2$ and is reabsorbed. Notice that this is a two-step process and that the intermediate stage has $n_T = 2$ (with one tensor multiplet carried by the five-brane), just as in the F-theory process for changing n. It is very natural to suspect that these processes coincide, and thus, that the transition to and from the Higgs branch for $n = 1$ is the same as the M-theory transition involving a small E_8 instanton that is emitted from the boundary.

We will give further evidence for this below, but first we must finally look a little more microscopically at F-theory in a neighborhood of the exceptional curve E. According to [12], to do F-theory on $\mathbf{F}_1$, we introduce two more variables X, Y, transforming as

$$(X, Y) \to (\lambda^6 \mu^4 X, \lambda^9 \mu^6 Y), \tag{3.9}$$

and write an equation

$$Y^2 = X^3 + f(x, y, u, v)X + g(x, y, u, v), \tag{3.10}$$

where f is of degree $(12, 8)$ in λ and μ, and g is of degree $(18, 12)$.

Let us compare this to F-theory on $\mathbf{P}^2$. For this, we use homogeneous coordinates x, y, v, scaling by $(x, y, v) \to (\lambda x, \lambda y, \lambda v)$, and introduce two more variables X, Y scaling as $(X, Y) \to (\lambda^6 X, \lambda^9 Y)$, and we write an equation

$$Y^2 = X^3 + \widetilde{f}(x, y, v)X + \widetilde{g}(x, y, v) \tag{3.11}$$

where $\widetilde{f}$ is of degree 12 and $\widetilde{g}$ is of degree 18.

For instance, a typical monomial in f is $x^{n_x} y^{n_y} u^{n_u} v^{n_v}$ with

$$n_x + n_y + n_v = 12$$
$$n_u + n_v = 8.$$

(3.12)

By contrast, a typical monomial in $\widetilde{f}$ is $x^{\widetilde{n}_x} y^{\widetilde{n}_y} v^{\widetilde{n}_v}$ with

$$\widetilde{n}_x + \widetilde{n}_y + \widetilde{n}_v = 12.$$

(3.13)

We see that for every possible monomial in f, there is a corresponding possible monomial in $\widetilde{f}$, with $\widetilde{n}_x = n_x$, $\widetilde{n}_y = n_y$, and $\widetilde{n}_v = n_v$. But some monomials in $\widetilde{f}$ are not associated to any possible monomials in f – the missing ones are those with $\widetilde{n}_v > 8$ (they would correspond to monomials in f with $n_u < 0$). A simple count shows that there are $4 + 3 + 2 + 1 = 10$ monomials present in $\widetilde{f}$ and not in f. Similarly, there are $6 + 5 + \ldots + 1 = 21$ monomials present in $\widetilde{g}$ and not in g. Altogether, in going from $\mathbf{F}_1$ to $\mathbf{P}^2$, we gain $10 + 21 = 31$ monomials.

On the other hand, some coefficients in $f, g, \widetilde{f}$, and $\widetilde{g}$ can be eliminated by reparametrizations of the variables. In $\mathbf{P}^2$, one has a nine-dimensional group $GL(3)$ of linear transformations of x, y, v. For $\mathbf{F}_1$, the corresponding counting is a little trickier. There is a $GL(2)$ that acts on x, y (four dimensional); one also has $\delta u = \epsilon u$, $\delta v = \epsilon' v + \epsilon'' u x + \epsilon''' u y$, making 8 reparametrizations in all; however, one combination of these is a symmetry of f and g, so only 7 parameters can be removed by redefinition of x, y, u, and v.

Thus, F-theory on $\mathbf{P}^2$ has 31 more monomials than F-theory on $\mathbf{F}_1$, but two $(= 9 - 7)$ more of them can be removed by redefinitions of the variables; so altogether the moduli space of F-theory on $\mathbf{P}^2$ has $31 - 2 = 29$ more hypermultiplets than that of F-theory on $\mathbf{F}_1$. The number 29 is the expected amount by which the number of hypermultiplets must increase (to cancel anomalies) in a transition in which the number of tensor multiplets decreases by one. This counting thus lends support to the idea that a transition is possible from the "Coulomb branch," F-theory on $\mathbf{F}_1$, to a "Higgs branch," F-theory on $\mathbf{P}^2$.

Now let us try to check the idea that the phase transition in which a point is blown up (passing for instance from $\mathbf{P}^2$ to $\mathbf{F}_1$) is the same as the transition in which a small E_8 instanton is emitted from the boundary in M-theory. The light string in that transition carries a rank eight current algebra, as explained in [3]. Let us look for such a current algebra in the string obtained in F-theory by wrapping a Type IIB three-brane over a two-sphere E of self-intersection number -1.

E is pierced by a certain number of Type IIB seven-branes. The current algebra will arise from intersections of Type IIB seven-branes and three-branes. So the first task is to count how many seven-branes intersect E. As in [11], we do this by counting parameters; we ask how many parameters there are in the equation $Y^2 = X^3 + fX + g$ when restricted to E, and interpret these parameters as the positions of the seven-branes. Since we now only want to look at the structure on E, we set $u = 0$, so we only care about the monomials with $n_u = 0$. According to (3.12) the monomials in f with $n_u = 0$ have $n_v = 8$, $n_x + n_y = 4$, giving five monomials in all; likewise there are seven relevant monomials in g, and so $5+7 = 12$ in all. After removing four parameters that can be absorbed in $GL(2)$ transformations of x, y, we have $12 - 4 = 8$ relevant parameters that we interpret as positions of seven-branes, so there are eight such seven-branes.

What remains is a perturbative string computation (in Minkowski space with coordinates $x^0, \ldots, x^9$) involving the transverse intersection of a Type IIB three-brane at, say, $x^4 = \ldots = x^9 = 0$ with a Type IIB seven-brane at, say, $x^2 = x^3 = 0$. Obviously, their intersection is the cosmic string given by $x^2 = \ldots = x^9 = 0$. (We call it a cosmic string to avoid confusion with the elementary Type IIB strings that will enter momentarily.) In working out the excitation spectrum along this cosmic string, one must quantize elementary Type IIB strings that start on the three-brane and end on the seven-brane, or vice-versa. For either of these orientations, a standard free field calculation shows that there are no massless bosons, and a single massless fermion that is left-moving along the cosmic string. (The calculation is actually isomorphic to the analysis of the DN sector for Type I Dirichlet one-branes; see the concluding paragraphs of [25].) Allowing for both orientations, one gets two left-moving fermions, that is a rank one current algebra, on the intersection of a seven-brane with a three-brane.

In our F-theory problem, the non-critical string comes from a three-brane that has, as we saw above, transverse intersections with *eight* seven-branes, so it carries altogether a rank eight current algebra. This gives strong support to the idea that this string is the same as the one that is associated to small E_8 instantons.

$n = 2$

Now we move on to $n = 2$. The main difference is that the exceptional curve E now has self-intersection number -2, so that blowing it down gives a $\mathbf{Z}_2$ orbifold singularity, the quotient of the $x - y$ plane by $(x, y) \to (-x, -y)$.

The crucial property of this singularity is that it looks locally like a singularity of a hyper-Kahler manifold, simply because the holomorphic two-form $dx \wedge dy$ is invariant under the $\mathbf{Z}_2$. (This property will not recur for any $n > 2$.) This particular singularity is called an A_1 singularity.

Now, recall that F-theory is simply Type IIB superstring theory with a variable complex coupling "constant." The coupling varies in such a way as to compensate for the curvature of the space-time. If one takes the ten-dimensional space-time to be $\mathbf{R}^6 \times B$, with B a hyper-Kahler manifold, then there is no reason for the coupling to vary on B, so in this situation F-theory reduces to ordinary Type IIB theory.

$\mathbf{F}_2$ is not hyper-Kahler, so F-theory on $\mathbf{F}_2$ is not equivalent (in any evident way) to an ordinary Type IIB theory. However, in a neighborhood of E, $\mathbf{F}_2$ does look like a hyper-Kahler manifold. In particular, in F-theory on E, the Type IIB coupling is *constant* in a neighborhood of E. This constant is arbitrary, and can be taken to be small. Thus, the strong coupling singularity for $n = 2$ is equivalent to a phenomenon that can occur in weakly coupled Type IIB superstring theory. It is simply the behavior of the Type IIB theory as one approaches an A_1 singularity.

The strong coupling singularity of the heterotic string for $n = 2$, that is, for instanton numbers $(14, 10)$, thus involves the appearance of the same non-critical string that has been studied for Type IIB at an A_1 singularity [1,2]. Because Type IIB has twice as much supersymmetry as the heterotic string, this string can carry $(0, 2)$ spacetime supersymmetry in six dimensions, even though we are finding it in an F-theory model that has only $(0, 1)$ spacetime supersymmetry. Some consequences of the extra supersymmetry were discussed in [4]. This string is controlled by five relevant parameters (the scalars in a $(0, 2)$ tensor multiplet) rather than one for the non-critical strings with $n \neq 1$. As a result, one can "go around" the singularity, unlike the other cases. The extra parameters correspond to the non-polynomial deformation of F-theory on $\mathbf{F}_2$ which was used in [12] to show that the $n = 2$ and $n = 0$ models are the same. At the critical point of this string, there is a $\mathbf{Z}_2$ symmetry, familiar in the Type IIB description, which in the present context is the strong-weak coupling symmetry of the heterotic string for instanton numbers $(12, 12)$ or $(14, 10)$. In the $(14, 10)$ case, the existence of this symmetry was first suggested in [26].

$n > 2$

Now we consider the case of $n > 2$. The new ingredient is that $\widetilde{\mathbf{F}}_n$ does not look like a Calabi-Yau manifold near its singularity. The singularity of $\widetilde{\mathbf{F}}_n$ is now the orbifold

singularity obtained by dividing the $x - y$ plane – which we will call W – by $(x, y) \to$ $(\zeta x, \zeta y)$, with $\zeta^n = 1$. The holomorphic two-form $dx \wedge dy$ on W is multiplied by ζ^{2n} under this operation.

This is the situation where, in F-theory, one restores the Calabi-Yau property by letting the coupling "constant" vary with position. Instead of $\widetilde{\mathbf{F}}_n$, one considers a Calabi-Yau manifold Z that maps to $\widetilde{\mathbf{F}}_n$, with the generic fibers being two-tori, whose complex structure is determined by the expectation value of the scalars of the Type IIB theory. In our case, we only need to know the local behavior near the singularity, so it is enough to find a Calabi-Yau manifold fibered by two-tori over $W/\mathbf{Z}_n$.

The most obvious thing to do is to take a *constant* two-torus A, with $\mathbf{Z}_n$ action, and look at $(W \times A)/\mathbf{Z}_n$, which maps to $W/\mathbf{Z}_n$ (by the map that forgets A) with the generic fiber being a two-torus, that is, a copy of A. For $(W \times A)/\mathbf{Z}_n$ to be a Calabi-Yau manifold, the $\mathbf{Z}_n$ action on A should be such that a holomorphic differential λ transforms under $(x, y) \to (\zeta x, \zeta y)$ as $\lambda \to \zeta^{-2}\lambda$. This is possible if and only if ζ^2 is of order $2, 3, 4$, or 6. Thus, n must be 3,4,6,8, or 12. These are the values of n studied in [12], and the only ones that will be considered here. [6]

The conclusion, then, is that for $n = 3, 4, 6, 8$, or 12, what happens at the strong coupling point is the appearance in space-time of what at least macroscopically looks like an orbifold singularity. In constructing this orbifold, one must use the $SL(2, \mathbf{Z})$ symmetry of the Type IIB theory. One finds this singularity, at least macroscopically, by dividing the theory by an operation that acts on the $x - y$ plane by $(x, y) \to (\zeta x, \zeta y)$ together with an $SL(2, \mathbf{Z})$ transformation which, for $n = 3, 4, 6, 8$, or 12, is of order $a_n = 3, 2, 3, 4$, or 6.

To be more precise about what one means here by an "orbifold," recall how orbifolds enter in perturbative string theory: one can make a non-singular orbifold conformal field theory; or by shifting a theta angle (as in [27]), one can get a conformal field theory singularity where non-perturbative states may become massless. Here our "orbifold" (whose theta angle is frozen at the critical value) corresponds to a truly singular configuration, since it supports a tensionless string. In any event, the two types of orbifold differ only in the structure near the space-time singularity, so macroscopically when the heterotic string gets a strong coupling singularity the F-theory description really does develop an orbifold singularity.

---

[6] The other possible values of n for $E_8 \times E_8$ heterotic strings on K3 with instanton numbers $(12 + n, 12 - n)$ are 5 and 7; I understand that results have been obtained for those values of n [24].

We can now see that the case $n = 4$ is exceptional. Only special two-tori have symmetries that act with order 3, 4, or 6 on a holomorphic differential; only for unique values of the coupling constants does the Type IIB theory have such symmetries. Thus, for $n = 3, 6, 8$, or 12, the behavior near the strong coupling singularity of the heterotic string involves the Type IIB theory at a special strong coupling point, one of the orbifold points in the ten-dimensional Type IIB moduli space. It may be that at one of those special strong coupling points, the Type IIB theory has some unusual dynamics. No such dynamics is possible in conventional field theory, since the ten-dimensional supersymmetry does not permit any extra particles to become massless at a special value of the coupling, but it is conceivable that something novel happens in string theory. If that is the case, one may need to first understand this behavior to get a real understanding of the heterotic string strong coupling singularity for $n = 3, 6, 8$, or 12.

On the other hand, for $n = 4$, $a_n = 2$, and since a generic two-torus has a symmetry ("multiplication by -1") that acts with order two on a holomorphic differential, the effective Type IIB coupling can have an arbitrary value in the region of space that is important for understanding the heterotic string strong coupling singularity. It should thus be possible to come relatively close to understanding in weak coupling the non-critical string relevant to $n = 4$. In fact, the element -1 of $SL(2, \mathbf{Z})$ which we need to use acts as -1 on, for instance, the Neveu-Schwarz two-form of the theory; the symmetry that does this is the reversal of world-sheet orientation, the exchange of the left- and right-movers of the theory. The singular space-time at the $n = 4$ strong coupling singularity is thus at least macroscopically a kind of orientifold made by dividing the (x, y) plane by $(x, y) \to (ix, iy)$ while also reversing the Type IIB world-sheet orientation. The relevant operation reversing the world-sheet orientation is actually of order four in acting on the fermions.

Note that the $n = 3$ example involves a sort of $\mathbf{Z}_3$ orbifold closely related to an M-theory example studied in section 2.4.

4. Some Examples In Four Dimensions

This paper has been mainly concerned with phase transitions in five and six dimensions, but I will here briefly point out that critical points with tensionless strings will also be common in four dimensional models with $N = 1$ and $N = 2$ spacetime supersymmetry.

For Calabi-Yau compactification of Type II, the most obvious example is to consider the Type IIB theory at a point where the Type IIA theory gets a massless charged hypermultiplet, or vice-versa. It is then obvious by an argument as in [1], involving the

equivalence of the two theories after further compactification on a circle, that Type IIB gets a tensionless string at such a point. More concretely, Type IIA gets a massless hypermultiplet by wrapping a two-brane around a collapsing two-cycle, while Type IIB gets a tensionless string by wrapping a three-brane over the same collapsing two-cycle.

An important example of a collapsing two-cycle is the one that arises at a conifold singularity reached by varying Kahler parameters. The collapsing two-cycle is then a two-sphere of normal bundle $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$. (This is the "generic" normal bundle for a holomorphic two-sphere in a Calabi-Yau three-fold.) The string that appears here of course carries $N = 2$ supersymmetry, since we have found it in Calabi-Yau compactification of Type II. It is a fairly close analog of the tensionless string that one gets in Type IIB on K3 when a two-cycle collapses.

Now we want to get the *same* string in a Calabi-Yau compactification of the heterotic string. (Thus, we will be finding a string that carries $N = 2$ supersymmetry in an $N = 1$ model, as happened for $n = 2$ in section 3. Many variations that will not be explored here give strings that only carry $N = 1$ supersymmetry.) First of all, to apply Type IIB techniques to the heterotic string, we will, of course, use F-theory as in [11,12]. We need a complex threefold W with the following properties:

(1) W has sufficiently positive curvature that one can do F-theory with W as base; there should be a Calabi-Yau fourfold that maps to W with generic fiber a two-torus.

(2) There should be a map $W \to B$ with B a complex surface and the generic fiber being a $\mathbf{P}^1$. In that case, one can fiber-wise use the relation [11] between Type IIB on $\mathbf{P}^1$ (with seven-branes and a variable coupling) and the heterotic string on $\mathbf{T}^2$. (In our example, the fibers of $W \to B$ will *all* be two-tori, so this fiber-wise transformation is justified simply by an adiabatic argument.) Thus, on replacing W with a Calabi-Yau three-fold Z that maps to W with generic fiber a two-torus, Type IIB on W will be equivalent to the heterotic string on Z.

(3) W will have a holomorphic two-sphere E, with normal bundle $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$, which collapses as a Kahler parameter is varied.

Condition (3) puts us in the situation we encountered in section 3 for $n = 2$. Though W is not a Calabi-Yau manifold, a neighborhood of E looks like one, so near E the Type IIB coupling is *constant*, and one can study the collapse of E using weakly coupled Type IIB string theory. Therefore, one will get the same tensionless string as in collapse of a two-sphere with local structure of E in Calabi-Yau compactification of Type IIB.

To obey condition (1), we start with $\mathbf{C}^6$ with coordinates $x_1, \ldots, x_6$ and the scalings

$$(x_1, x_2, x_3, x_4, x_5, x_6) \to (\lambda x_1, \lambda x_2, \mu x_3, \mu x_4, \nu x_5, \nu \lambda \mu x_6). \qquad (4.1)$$

Either one divides by these scalings with $\lambda, \mu, \nu \in (\mathbf{C}^*)^3$ and omits certain linear subspaces of $\mathbf{C}^6$ or – more helpful for exhibiting the Kahler parameters – one takes $\lambda, \mu, \nu \in U(1)^3$ and imposes the D-field equations:

$$|x_1|^2 + |x_2|^2 + |x_6|^2 = r_1$$
$$|x_3|^2 + |x_4|^2 + |x_6|^2 = r_2 \qquad (4.2)$$
$$|x_5|^2 + |x_6|^2 = r_3.$$

Our space W is just the space of solutions of (4.2) (with suitable r_i) divided by $U(1)^3$. To do F-theory with W as base, we introduce two new variables X, Y and an equation

$$Y^2 = X^3 + f(x_1, \ldots, x_6)X + g(x_1, \ldots, x_6), \qquad (4.3)$$

where (to get a Calabi-Yau four-fold) f is of degree $(6, 6, 4)$ and y of degree $(9, 9, 6)$ in λ, μ, ν. I will leave it to the reader to verify that generic such f and g give a smooth Calabi-Yau four-fold corresponding to a model that generically has the gauge symmetry completely broken.

To verify condition (2), note that forgetting x_5, x_6 gives a map from W to $B = \mathbf{P}^1 \times \mathbf{P}^1$, the fibers being (all) $\mathbf{P}^1$'s obtained by projectivizing x_5, x_6. Thus, Type IIB on W is equivalent to the heterotic string on a Calabi-Yau Z that is elliptically fibered over $\mathbf{P}^1 \times \mathbf{P}^1$. This particular Calabi-Yau has played a prominent role in the last year [28,12,29].

To verify condition (3), we take $r_2 > 0$, $r_3 > r_2$, $r_1 > r_2$, and let E be the two-sphere defined by $x_3 = x_4 = 0$. We can uniquely solve the last two equations in (4.2) and fix the $U(1)^2$ associated with μ and ν by taking $x_6 = \sqrt{r_2}$, $x_5 = \sqrt{r_3 - r_2}$. Then x_1 and x_2, modulo scaling by λ, parametrize a two-sphere E of Kahler class proportional to $r_1 - r_2$. E can be seen to have normal bundle $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$. Now note that in the limit $r_1 \to r_2$ (keeping $r_3 > r_2 > 0$), E collapses. In fact, if we continue to $r_1 < r_2$, E disappears and is replaced by a two-sphere F defined by $x_1 = x_2 = 0$; F is also of normal bundle $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$, with Kahler class proportional to $r_2 - r_1$. This situation is a typical Type II "flop" (note the change in sign of the area). At $r_1 = r_2$, Type IIB on W and therefore also the heterotic string on Z has the tensionless string discussed at the beginning of this section.

References

[1] E. Witten, "Some Comments on String Dynamics," hepth/9507121.

[2] A. Strominger, "Open p-Branes," hepth/9512059.

[3] O. Ganor and A. Hanany, "Small Instantons And Tensionless Strings," hepth/9602120.

[4] N. Seiberg and E. Witten, "Comments On String Dynamics In Six Dimensions," hepth/9603003.

[5] M. Duff, H. Lu, and C. N. Pope, "Heterotic Phase Transitions And Singularities Of The Gauge Dyonic String," hepth/9603037.

[6] P. Aspinwall, D. Morrison, and B. Greene , "Calabi-Yau Moduli Space, Mirror Manifolds, And Space-Time Topology Change In String Theory," Nucl. Phys. **B416** (1994) 414.

[7] E. Witten, "Phases Of $N = 2$ Theories In Two Dimensions," Nucl. Phys. **B403** (1993) 159,

[8] A.C. Cadavid , A. Ceresole , R. D'Auria , S. Ferrara, "Eleven-Dimensional Supergravity Compactified on Calabi-Yau Threefold," hepth/9506144.

[9] I. Antoniadia, S. Ferrara, and T. R. Taylor, "$N = 2$ Heterotic Superstring and its Dual Theory in Five Dimensions," hepth/9511108.

[10] S. Ferrara, R. R. Khuri, and R. Minasian, "M-Theory on a Calabi-Yau Manifold," hepth/9602102.

[11] C. Vafa, "Evidence For F-Theory," hepth/9602022.

[12] D. Morrison and C. Vafa, "Compactification Of F-Theory On Calabi-Yau Threefolds, I," hepth/9602114.

[13] M. Duff, R. Minasian, and E. Witten, "Evidence For Heterotic/Heterotic Duality, hepth/9601036.

[14] K. Becker, M. Becker, and A. Strominger, "Fivebranes, Membranes, And Nonperturbative String Theory," hepth/9507158.

[15] S. Katz, D. Morrison, and R. Plesser, "Enhanced Gauge Symmetry In Type II String Theory," hepth/9601108.

[16] P. Aspinwall, "Enhanced Gauge Symmetry And Calabi-Yau Three-Folds," hepth/9511171.

[17] A. Klemm and P. Mayr, "Strong Coupling Singularities And Non-Abelian Gauge Symmetries In $N = 2$ String Theory," hepth/9601014.

[18] C. Vafa and E. Witten, "A One-Loop Test Of String Duality," Nucl.Phys. **B447** (1995) 261.

[19] M. J. Duff, J. T. Liu, and R. Minasian, "Eleven-Dimensional Origin Of String-String Duality: A One Loop Test" Nucl. Phys. **B452** (1995) 261.

[20] M. Gunaydin, G. Sierra, and P. K. Townsend, "The Geometry Of $N = 2$ Maxwell-Einstein Supergravity And Jordan Algebras," Nucl. Phys. **B242** (1984) 244, "Gauging

Of The $D = 5$ Maxwell-Einstein Supergravity: More On Jordan Algebras," Nucl. Phys. **B253** (1985) 573.

[21] E. Witten, "Dynamics Of String Theory In Various Dimensions," Nucl. Phys. **B443** (1995) 85.

[22] P. S. Aspinwall, B. R. Greene, and D. R. Morrison, "Measuring Small Distances In $N = 2$ Sigma Models."

[23] P. S. Aspinwall, "Minimum Distances In Non-Trivial String Target Spaces," Nucl. Phys. **B431** (1994) 78.

[24] D. Morrison and C. Vafa, to appear.

[25] J. Polchinski and E. Witten, "Evidence For Heterotic - Type I String Duality," hepth/9511157.

[26] G. Aldazabal, A. Font, L. E. Ibanez, and F. Quevedo, "Heterotic/Heterotic Duality In $D = 6$, $D = 4$," hepth/9602097.

[27] P. Aspinwall, "Enhanced Gauge Symmetries And K3 Surfaces," hepth/9507012.

[28] S. Kachru and C. Vafa, "Exact Results For $N = 2$ Compactifications Of Heterotic Strings," Nucl. Phys. **B450** (1995) 69.

[29] P. S. Aspinwall and M. Gross, "Heterotic-Heterotic String Duality And Multiplet K3 Fibrations," hepth/9602118.