Quantum Gravity In De Sitter Space

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We discuss some general properties of quantum gravity in de Sitter space. It has been argued that the Hilbert space is of finite dimension. This suggests a macroscopic argument that General Relativity cannot be quantized – unless it is embedded in a more complete theory that determines the value of the cosmological constant. We give a definition of the quantum Hilbert space using the asymptotic behavior in the past and future, without requiring detailed microscopic knowledge. We discuss the difficulties in defining any precisely calculable or measurable observables in an asymptotically de Sitter spacetime, and explore some meta-observables that appear to make mathematical sense but cannot be measured by an observer who lives in the spacetime. This article is an expanded version of a lecture at Strings 2001 in Mumbai.

De Sitter *n*-space or dS_n is the maximally symmetric *n*-dimensional spacetime with positive cosmological constant Λ . Its symmetry group is SO(1, n). If we introduce variables x_0, x_1, \ldots, x_n obeying $x_0^2 - \sum_{i=1}^n x_i^2$, the de Sitter metric is simply (up to a constant factor)

$$ds^{2} = -dx_{0}^{2} + \sum_{i=1}^{n} dx_{i}^{2}.$$
 (1)

Alternatively, one can write the metric as

$$ds^2 = -dt^2 + \cosh^2 t \, d\Omega^2,\tag{2}$$

where $d\Omega^2$ is the metric on a unit round (n-1)-sphere. This spacetime has compact spatial sections (such as t=0), so when we speak of asymptotically de Sitter space – as we should in the presence of gravity, since the metric fluctuates – the asymptopia in question is in the past and future. There is no notion of spatial infinity. This is in sharp contrast with Anti de Sitter space, the maximally symmetric spacetime of negative cosmological constant, where, as we have come to know well in the last few years, asymptopia is at spatial infinity. It also contrasts with Minkowski space, which from a conformal point of view has a natural null infinity.

In de Sitter space, there is no positive conserved energy. In fact, no matter what generator we pick for SO(1,n), the corresponding Killing vector field, though perhaps timelike in some region of de Sitter space, is spacelike in some other region. For example, a typical Lorentz generator in de Sitter space is

$$K = x_1 \frac{\partial}{\partial x_0} + x_0 \frac{\partial}{\partial x_1}.$$
 (3)

Whether this generator moves us forwards or backwards in time (towards increasing or decreasing x_0) depends on the sign of x_1 . The conserved charge associated with K is positive for excitations supported at positive x_1 and negative for those at negative x_1 . This is the best we can do: there is no positive conserved energy in de Sitter space.

Consequently, there cannot be unbroken supersymmetry in de Sitter space. If there is a nonzero supercharge Q, we can (possibly after replacing Q by $Q+Q^{\dagger}$ or $i(Q-Q^{\dagger})$) assume that Q is Hermitian. Then Q^2 cannot be zero, and is a nonnegative bosonic conserved quantity; but there is no such object.

We can rotate de Sitter space to Euclidean signature by setting $x_0 \to ix_0$ (or equivalently, set $t = i\tau$ and take $\tau = \pi/2 - \theta$). The Euclidean continuation is a standard n-sphere

 \mathbf{S}^n , with symmetry group SO(n+1). After the continuation, the operator K becomes the generator of a rotation, and obeys

$$\exp(2\pi K) = 1. \tag{4}$$

Because of this, the Euclidean de Sitter path integral can be interepreted in terms of a thermal ensemble. This leads to the notion of a de Sitter temperature [1] and the associated entropy [2]. Like the Bekenstein-Hawking entropy of a black hole, the de Sitter entropy can be written

$$S = \frac{A}{4G},\tag{5}$$

where G is Newton's constant, and A is the area of a horizon. In this case, however, the horizon is observer-dependent, and because of this it is not entirely clear which concepts about black holes carry over to de Sitter space.

An observer in de Sitter space can only see part of the space. This is because of the exponential inflation that occurs in the future: the space expands so fast that light rays do not manage to propagate all the way around it. To make the causal structure of de Sitter space clear, one can introduce a new "time" coordinate u by

$$u = 2\tan^{-1}e^t, (6)$$

so that for $-\infty < t < \infty$, u ranges over $0 < u < \pi$. The metric becomes

$$ds^2 = \frac{1}{\sin^2 u} \left(-du^2 + d\Omega^2 \right). \tag{7}$$

The asymptotic past \mathcal{I}_{-} consists of a copy of \mathbf{S}^{n-1} at u=0, and the asymptotic future \mathcal{I}_{+} consists of a copy of \mathbf{S}^{n-1} at $u=\pi$. Any trajectory in de Sitter space begins at some point P in \mathcal{I}_{-} and ends at some point Q in \mathcal{I}_{+} . From a causal point of view, in a sense considered by Bousso [3], any observer can be identified with the pair (P,Q). The region of de Sitter space that one can influence, and likewise the region that one can see, depend only on P and Q, and not on the details of one's trajectory in spacetime. What one can see is determined only by Q, and the region that one can influence depends only on P.

To describe in detail the horizon of an observer, let us write $d\Omega^2 = d\chi^2 + \sin^2\chi d\widetilde{\Omega}^2$, where χ is a polar angle, ranging over $0 \le \chi \le \pi$, and $d\widetilde{\Omega}^2$ is the round metric on an (n-2)-sphere. The de Sitter metric then becomes

$$ds^{2} = \frac{1}{\sin^{2} u} \left(-du^{2} + d\chi^{2} + \sin^{2} \chi \, d\widetilde{\Omega}^{2} \right). \tag{8}$$

Consider now an observer who sits at the "north pole" of the sphere, that is, at $\chi=0$. (In fact, any geodesic in de Sitter space is equivalent to $\chi=0$ by the action of the de Sitter group.) From the form of the metric, we see that the propagation of light rays is bounded by $|d\chi/du| \leq 1$. Since the spacetime "ends" in this coordinate system at $u=\pi$, a light ray emitted at $\chi > \pi - u$ will never reach the observer at $\chi=0$. So the boundary of the region that this observer can see is given by

$$\chi = \pi - u. \tag{9}$$

This is the horizon. In general, the (n-2)-sphere of given χ and u has metric $(\sin \chi/\sin u)^2 d\widetilde{\Omega}^2$, and its area is proportional to $(\sin \chi/\sin u)^{n-2}$. Relating χ to u by (9), we see that the horizon area is time-independent. This is in keeping with general theorems saying that the area of the past horizon of an observer cannot decrease in time. For de Sitter space, this horizon area is precisely constant, and for a generic perturbation of de Sitter space, it is an increasing function of time.

By studies of D-branes and in a variety of other ways, we have learned in the last few years to interpret the Bekenstein-Hawking entropy of a black hole like every other entropy in statistical mechanics: it is the logarithm of the number of quantum states of the black hole. It has been argued [4] that the same holds for de Sitter space, more precisely that the Hilbert space of quantum gravity in asymptotically de Sitter space time has a finite dimension N, and that the entropy of de Sitter space is

$$S = \ln N. \tag{10}$$

This is a very interesting answer for many reasons, including the fact that the de Sitter Hilbert space has infinite dimension perturbatively. There is no contradiction here, since perturbation theory is an expansion in powers of $G\Lambda^{(n-2)/n}$. If the above formula for N is correct, then N diverges exponentially as $G\Lambda^{(n-2)/n} \to 0$, so perturbation theory, in exhibiting an infinite-dimensional Hilbert space, gives the right answer for the weak coupling limit. Moreover, one can argue to a reasonable extent that perturbation theory may be breaking down in this situation as one moves away from the weak coupling limit. (The arguments have been explained to me by T. Banks, R. Bousso, and G. Horowitz, and I will briefly allude to them later.) However, there is no known controlled calculation incorporating the breakdown of perturbation theory and exhibiting the claimed behavior of N.

If the quantum Hilbert space in de Sitter space really has a finite dimension N, this gives a strong hint that Einstein's theory with Lagrangian

$$I = -\frac{1}{8\pi G} \int d^n x \sqrt{g} R - \Lambda \int d^n x \sqrt{g}$$
 (11)

cannot be quantized and must be derived from a more fundamental theory that determines the possible values of $G\Lambda^{(n-2)/n}$. Indeed, N is claimed to be a nontrivial function of $G\Lambda^{(n-2)/n}$ (since it diverges exponentially as $G\Lambda^{(n-2)/n} \to 0$) yet it obviously takes integer values. Clearly, this implies that N cannot be a continuous function of $G\Lambda^{(n-2)/n}$ if the latter can vary continuously. This suggests that Einstein's theory cannot be quantized for general values of G and Λ but must be derived from a more fundamental theory that determines the possible values of $G\Lambda^{(n-2)/n}$.

This conclusion should not seem too surprising, since a similar argument – at least for some values of n – can be made for negative cosmological constant. In that case, quantum gravity in asymptotically Anti de Sitter space is related to a conformal field theory on the boundary. For n=3, the boundary theory has central charge c proportional to $G\Lambda^{1/3}$ (as was seen from a canonical point of view in [5] and later understood as a special case of the AdS/CFT correspondence). The Zamolodchikov c-theorem implies that c is constant in a family of conformal field theories in two dimensions, c so c of c annot be continuously varied. The same argument holds in any dimension c of c of which there is a suitable c-theorem.

Going back to de Sitter space, what values of N are in fact possible? If classically there existed compactifications to de Sitter space (perhaps depending on discrete fluxes to introduce an integer), then in the classical limit we would have $N \to \infty$. However, an important no go theorem [6,7] says that there is no classical compactification of ten- or eleven-dimensional supergravity to de Sitter space of any dimension. This means that there is no classical way to get de Sitter space from string theory or M-theory. By a "classical" compactification, I would mean a family of compactifications in which $G\Lambda^{(n-2)/n}$ becomes arbitrarily small and a supergravity or string theory description becomes arbitrarily good. The no go theorem means that this does not exist.

In fact, classical or not, I don't know any clear-cut way to get de Sitter space from string theory or M-theory. This last statement is not very surprising given the classical

¹ To be more precise, c is constant in a family of unitary theories with normalizable vacuum. These properties are expected to hold for CFT's arising in the AdS/CFT correspondence.

no go theorem. For, in view of the usual problems in stabilizing moduli, it is hard to get de Sitter space in a reliable fashion at the quantum level given that it does not arise classically. (For an analysis of a situation in which most moduli can be stabilized, leading in the large volume limit to a nonsupersymmetric vacuum with $\Lambda = 0$, see [8].)

The absence of a classical de Sitter limit suggests that the possible values of N in string/M-theory are sporadic, rather than arising from infinite families, and that there might be only finitely many choices. If the number of choices is finite, I would not personally expect it to be possible to get $N > 10^{10^{100}}$. But de Sitter space with such large N is needed to agree with the most obvious interpretation of recent astronomical data!²

The fact that N, the dimension of the Hilbert space \mathcal{H} , is finite means that the de Sitter symmetry group SO(1, n-1) cannot act on \mathcal{H} . Indeed, SO(1, n-1) has no (non-trivial) finite-dimensional unitary representations!

This may sound like a problem, but in fact it is not. The de Sitter group does not act on \mathcal{H} because the spatial sections of de Sitter space are compact. Always, in General Relativity, the spacetime symmetry generators (being gauge charges) can be expressed as surface terms at infinity. In the case of de Sitter space there is no (spatial) infinity and hence the de Sitter generators are zero. Thus, what I have informally called "quantum gravity in de Sitter space" does not have the invariance of classical de Sitter space. This is one consequence of the fact that the only asymptopia is in the past and future.

Absence of de Sitter symmetries should lead one to ask in what precise sense one might refer to some hypothetical system as "quantum gravity in de Sitter space." Let us think about the analogous question for Λ vanishing or negative. It has been clear since very early studies of quantum gravity [9] that one cannot define local field operators in a theory with gravity, but that in an asymptotically flat spacetime of dimension n > 3 one

² I should note that the point of view of Banks [4] is rather different. He argues that string/M-theory as we know it is the Hilbert space version of a theory that can also be formulated with matrices of arbitrary finite size N. He takes N as an input and proposes that, taking into account some conjectured "large graviton" effects that cannot be seen in classical field theory, some form of M-theory dynamics would be possible for any arbitrarily big N and would lead to a scale of supersymmetry breaking that is very low compared to the Planck mass, but very large compared to the scale set by the cosmological constant.

can define an S-matrix that apparently has all the usual properties.³ It is reasonable to insist that a quantum gravity theory in n dimensions for n > 3 is characterized by having a unitary, analytic S-matrix with a massless spin two particle that has the expected sort of low energy interations. This is a satisfactory description of the "output" that a quantum gravity theory in asymptotically flat spacetime should produce, but leaves completely open the big question of how the S-matrix is supposed to be computed at least in principle, that is, what kind of theory produces this output. (Matrix theory [10] supplies a possible answer in certain situations, though it has not yet been put in a systematic framework.)

For $\Lambda < 0$, the situation is much better. There is no notion of an S-matrix in asymptotically Anti de Sitter spacetime, but instead one has the correlation functions of the boundary conformal field theory. The local operator product relations of CFT makes the boundary CFT for $\Lambda < 0$ a much richer structure than the S-matrix for $\Lambda = 0$. As Λ approaches zero from below, the boundary CFT degenerates to the S-matrix, whose structure is much less rich. The degeneration occurs because for a connected correlation function $\langle \phi_1(x_1)\phi_2(x_2)\dots\phi_n(x_n)\rangle$ of the boundary CFT to have a limit as Λ approaches zero from below, one must take some of the x_i to the past and some to the future; what survive are the S-matrix elements of the $\Lambda = 0$ theory. In this limit, the local operator product relations of the CFT are lost; operators creating in states (or out states) just commute or anticommute with each other. To my thinking, boundary CFT for $\Lambda < 0$ perhaps can be regarded as not just an output but a dynamical principle that defines what we mean by quantum gravity for $\Lambda < 0$. From this point of view, what is missing is an understanding of what sort of limit corresponds to having a macroscopic spacetime in the interior.

What are we to make of the "holography" of 't Hooft and Susskind? For $\Lambda < 0$, this can be the assertion that quantum gravity is holographically dual to a conformal field theory on the boundary. For $\Lambda = 0$, the situation is much less satisfactory. Hopefully, for $\Lambda = 0$, holography means more than a mere assertion that quantum gravity in asymptotically flat spacetime does not have local field operators but only an S-matrix. This degree of understanding predates even the early beginnings of string theory [9]. Hopefully,

³ For $n \leq 3$, the presence of any mass causes the universe not to be asymptotically flat, and thwarts the existence of an S-matrix in any standard sense. Above four dimensions, the S-matrix acts between Fock states with finitely many initial and final particles; in four dimensions, because of the usual infrared problems with massless gauge particles, one must consider more subtle initial and final states.

holography means that, in some yet unclear way, the theory can be described (and the S-matrix computed) in terms of degrees of freedom that "live" at infinity. Regrettably, however, for $\Lambda = 0$ it is not very clear how even to begin in that direction, as Minkowski space has a natural null infinity (leading back to the notion of an S-matrix, perhaps), but not a natural spatial infinity.⁴

At any rate, getting back to the case of $\Lambda > 0$, we would ideally like to understand a dynamical principle generating quantum gravity theories in asymptotically de Sitter spacetime, but at a very minimum we would like a reasonable description of what sort of output characterizes them. It is not enough to merely have a finite dimensional Hilbert space! I will not really have an answer to propose, but it does seem that any answer, if there is one, would have to make use of the behavior in the asymptotic regions in the far past and future since that is the only asymptopia we have.

The De Sitter Hilbert Space

Now we will make a few remarks about the Hilbert space of de Sitter space. We start with some standard observations about perturbation theory, ⁵ and then we will try to suggest a nonperturbative definition of a Hilbert space.

In perturbation theory, the starting point is a free field in de Sitter space. Such a free field can be quantized to obtain a quantum Hilbert space. Though there is no notion of a state of minimum energy, the Hilbert space of a free field does contain a distinguished de Sitter-invariant state, which one might call the vacuum, $|\Psi\rangle$. For a free field, $|\Psi\rangle$ is the unique quantum state which is de Sitter invariant and Gaussian.

Concretely, let ϕ be a scalar field, and let T be the spatial section of de Sitter space obtained by setting $x_0 = 0$ in (1). Thus, T is the "equator" in the Euclidean form \mathbf{S}^n of de Sitter space. We denote a configuration of the ϕ -field on T as $\phi(x)$. A quantum wavefunction, in the "coordinate space" representation, is a functional $\Psi(\phi(x))$. A Gaussian wavefunction has the form

$$\Psi(\phi) = \exp\left(-\frac{1}{2} \int_{T \times T} d^{n-1}x \, d^{n-1}y \, D(x, y) \phi(x) \phi(y)\right) \tag{12}$$

⁴ Or it has a spatial infinity consisting of only one point, which is not very helpful. For a little more detail on some issues discussed in the last few paragraphs, see my talk at Strings '98: http://online.itp.ucsb.edu/online/strings98/witten/.

⁵ For more on perturbation theory in de Sitter space from different points of view, see [11] and [12].

with some kernel D(x,y). (Here $d^n x$ is short for the standard Riemannian measure $dx_1 \dots dx_n \sqrt{g}$.) There is a unique D(x,y) for which a wavefunction of this form is de Sitter invariant.

In fact, a Gaussian state can be usefully characterized by the two-point function $G(x,y) = \langle \Psi | \phi(x) \phi(y) | \Psi \rangle$. As usual for Gaussian integrals, G can be obtained by inverting the operator corresponding to D. The appropriate de Sitter-invariant G can be readily found by working in Euclidean signature and regarding G as a function on $\mathbf{S}^n \times \mathbf{S}^n$ which is then restricted to $T \times T$. In fact, G is determined by the standard equation for the propagator

$$\left(-\Delta^2 + m^2 + \alpha R\right)G(X, Y) = \delta^n(X, Y). \tag{13}$$

Here X, Y denote points in \mathbf{S}^n , and we have included a coupling to the Ricci scalar R as well as a mass term. Eqn. (13) uniquely determines G and hence (upon taking an operator inverse) it uniquely determines a de Sitter-invariant Gaussian state $|\Psi\rangle$.

In fact, $|\Psi\rangle$ can conveniently be computed by a path integral on a hemisphere H of boundary T, say the hemisphere $x_0 < 0$. One simply carries out a path integral over fields on H whose boundary values are given by ϕ . We let \mathcal{A}_{ϕ} be the space of fields Φ on H whose restriction to T is equal to ϕ . Then we define

$$\Psi(\phi) = \int_{\mathcal{A}_{\phi}} D\Phi \, e^{-I(\Phi)}. \tag{14}$$

With this definition, the two point function $\langle \Psi | \phi(x) \phi(y) | \Psi \rangle$ can be computed by a path integral on the full sphere \mathbf{S}^n – the path integral on one hemisphere computes $|\Psi\rangle$ and the path integral on the other hemisphere computes $\langle \Psi |$. The correlation $\langle \Psi | \phi(x) \phi(y) | \Psi \rangle$ is thus computed, from this point of view, by a path integral on the full sphere (with $\phi(x)$ and $\phi(y)$ inserted to the equator), and this makes it clear that it obeys the usual covariant equation (13).

In perturbation theory, other quantum states are polynomials times the "vacuum" state Ψ . Thus, they take the general form

$$\Psi_f(\phi) = \int dx^{(1)} \dots dx^{(s)} \ f(x^{(1)}, \dots, x^{(s)}) \phi(x^{(1)}) \dots \phi(x^{(s)}) \Psi(\phi). \tag{15}$$

(Here $x^{(1)}, \ldots, x^{(s)}$ are s points in T.) If we are doing quantum gravity, we must discard most of these states: physically acceptable states are invariant under the de Sitter group. De Sitter invariance gives a severe restriction on f. But as f depends on an arbitrarily large

number of points in T, and the de Sitter group has finite dimension, de Sitter invariant f's do exist for every sufficiently large s. Moreover, for sufficiently large s, the space of such f's is infinite-dimensional. Thus, the perturbative Hilbert space has infinite dimension.

So far we have just given a free field description, which could be used in principle (modulo the usual problems with renormalizability) as the starting point in constructing perturbation theory. It is clear that if s is very large, or for fixed s if f is very rapidly varying, perturbation theory may break down because of a large gravitational back-reaction. This is why the fact that the Hilbert space is infinite-dimensional in perturbation theory does not guarantee that it is really infinite-dimensional. But it is not at all clear how to actually do a calculation exhibiting the alleged finite-dimensionality of the quantum Hilbert space \mathcal{H} .

Nonperturbative Definition of \mathcal{H}

In fact, our goal here will be far more modest. We will just try to give a natural definition of \mathcal{H} and thus give a precise framework in which, in principle, it might be possible to eventually address the issue of its finite-dimensionality.

To give a nonperturbative definition of \mathcal{H} along the general lines that we used above—in terms of functionals of fields on a spatial slice obeying suitable conditions—would require a great deal of microscopic knowledge. (What kinds of fields are considered? What are the proper degrees of freedom to use at short distances? What kinds of topologies are allowed? What are the boundary conditions near places where the fields develop singularities?) Such a definition may not exist at all, even in principle. I believe that instead, by using a de Sitter analog of the familiar holographic construction for Anti de Sitter space, we can give a nonperturbative definition of the Hilbert space that does not depend on any detailed microscopic knowledge (and does not make it clear if \mathcal{H} is finite-dimensional). We will avoid needing microscopic knowledge in the definitions because we will only need to know what the fields can look like in the far past and the far future where things are tame; we will not need to know what sort of microscopic fluctuations are possible.

Before entering into detailed discussion, I want to point out that this sort of approach seems to give increasingly less information as the cosmological constant is increased. For negative cosmological constant, the sort of reasoning that we will use gives the boundary conformal field theory, which, as discussed above, might actually be regarded as supplying a dynamical principle. For zero cosmological constant, we get the S-matrix, which at least is a reasonable output for a theory to produce. For positive cosmological constant, all

we will really get is a definition of a Hilbert space (unless some further meaning can be assigned to the "meta-observables" that we discuss later).

Let us first recall the familiar procedure for negative cosmological constant. We start with the Anti de Sitter metric, which near the boundary (which we will take to be at $r = \infty$) looks like $ds^2 = dr^2 + \frac{1}{4}e^{2r}d\vec{x}^2$. We introduce an arbitrary conformal metric $g_{ij}dx^i dx^j$ on the boundary and generalize the AdS metric to one that looks like $dr^2 + \frac{1}{4}e^{2r}g_{ij}dx^i dx^j$ for $r \to \infty$. By considering the dependence on g_{ij} , we get the correlation functions of the stress tensor in the boundary conformal field theory.

De Sitter space is somewhat similar but with different signature. For $t \to \pm \infty$, the de Sitter metric of eqn. (2) looks like $ds^2 = -dt^2 + \frac{1}{4}e^{\pm 2t}d\Omega^2$, with $d\Omega^2$ the round metric on \mathbf{S}^{n-1} . Now to prepare an initial or final state $|i\rangle$ or $\langle f|$, we pick a conformal metric $g^{(i)}$ or $g^{(f)}$ on the sphere and ask that the spacetime metric should be asymptotic in the far past to $-dt^2 + \frac{1}{4}e^{-2t}g_{ab}^{(i)}dx^adx^b$ or in the far future to $-dt^2 + \frac{1}{4}e^{2t}g_{ab}^{(f)}dx^adx^b$.

The path integral for metrics with this asymptopia in the past and future gives an observable that we may call $\langle f|i\rangle$. These (and their generalizations to include asymptotic fields other than the metric) are the only observables that I can see in asymptotically de Sitter space time. Actually, it might be better to call this kind of object a "calculable" rather than an "observable," since formulating it requires a global view of \mathcal{I}_- and \mathcal{I}_+ (the infinite past and future, where $g^{(i)}$ and $g^{(f)}$ are defined), and this is not available to any observer living in this spacetime. So it may be calculable, but it is not observable in the usual sense. Since I consider the term "calculable" (used as a noun) to be rather clumsy, I will instead refer to these objects as meta-observables.

For the moment, let us just try to converge on a definition of a Hilbert space. The rough idea is that for any function $\Psi(g^{(i)})$ (of a suitable class), we would regard

$$|\Psi\rangle = \int Dg^{(i)}\Psi(g^{(i)})|i\rangle \tag{16}$$

as a quantum state. However, we want to take the quotient by null vectors. It may be the case that for some $g^{(i)}$, the matrix element $\langle f|i\rangle$ is zero for all $g^{(f)}$. If so, we want to regard $|i\rangle$ as a null vector and set it to zero.

More generally, the matrix

$$M(f,i) = \langle f|i\rangle \tag{17}$$

is an $\infty \times \infty$ matrix, but it may have a finite rank. If its rank is finite, we want this rank to be the dimension of the quantum Hilbert space. We regard any linear combination $|\Psi\rangle$

of the $|i\rangle$'s such that $\langle f|\Psi\rangle = 0$ for all $\langle f|$ as a null vector. Taking the quotient of the space of $|\Psi\rangle$'s by the null vectors, we get a vector space \mathcal{H}_i of initial states. Likewise, taking the quotient of the final spaces by the null vectors (such that $\langle \Psi|i\rangle = 0$ for all $|i\rangle$), we get a vector space \mathcal{H}_f of final states.

I have called these initial and final spaces "vector spaces" rather than "Hilbert spaces" because as of yet, we have given no definition of an inner product on either \mathcal{H}_i or \mathcal{H}_f . All we have so far is a pairing $\mathcal{H}_f \otimes \mathcal{H}_i \to \mathbf{C}$, computed from a path integral for fields with a specified asymptotic behavior in the past and future. This pairing is bilinear (rather than being complex linear in one argument and antilinear in the other) since no complex conjugation is involved in computing the path integral.

By considering only the fields in the past or only the fields in the future, there seems to be no way, without drawing on a great deal of hypothetical microscopic knowledge, to define an inner product on \mathcal{H}_i or on \mathcal{H}_f . However, we can extract a Hilbert space structure from the bilinear pairings $\langle f|i\rangle$ if we take into account that CPT symmetry (which as far as we know is valid in the presence of quantum gravity) gives an antilinear map from the past to the future, that is, from \mathcal{H}_i to \mathcal{H}_f . If we denote the CPT transformation as Θ , then we can define a Hermitian pairing (,) on \mathcal{H}_i as follows: for $|i\rangle$, $|j\rangle \in \mathcal{H}_i$, we set

$$(j,i) = \langle \Theta j | i \rangle. \tag{18}$$

Since we have divided by null vectors, this hermitian product is nondegenerate. Now we can formulate a *unitarity conjecture*: it actually is positive definite. This should be added to the *entropy conjecture*: the Hilbert space defined in this fashion is finite-dimensional, and for small cosmological constant, its dimension is approximately given by the semiclassical entropy formula.

Note that, if our reasoning is correct, we have defined a Hilbert space but not an Smatrix. Just in order to define an inner product without detailed microscopic knowledge,
we had to use the path integral over all of spacetime to get a pairing $\langle \mid \rangle$ between initial
and final states. We have no other such pairing at our disposal, so we get a Hilbert space
structure but not anything that one could characterize as an S-like matrix.

One might have been tempted to argue that the de Sitter Hilbert space cannot be finite-dimensional, because one could in the far past divide the space into an arbitrarily large number of causally disconnected regions and place zero or one elementary particle in each region in an arbitrary fashion.⁶ It has been argued ([4] and private communications by T. Banks, R. Bousso, and G. Horowitz) that this attempt to prove the infinite-dimensionality of the de Sitter Hilbert space fails because (given black hole formation and the like) such a generic initial state does not really lead to a de Sitter-like evolution. The idea that the matrix M(f,i) might have finite rank is suggested by this argument. Notice that from this point of view, we do not get any insight about what would be meant by the evolution starting from an arbitrary initial state $|i\rangle$. All we can determine is whether it has a nonzero pairing with final de Sitter states.

Big Bang and Oscillatory Universes

Defining a Hilbert space without detailed microscopic knowledge may not seem like much, but things could have been worse. Let us consider a Big Bang universe that is asymptotic in the future to a de Sitter spacetime. (For a recent discussion of related issues see [13].) We suppose that the initial conditions are somehow fixed quantum mechanically. The real Universe may be of this type.

In this case, all that we can specify is the conformal structure $g^{(f)}$ in the far future. The path integral with such final conditions gives a function of $g^{(f)}$ which we might think of as $\langle f|\chi\rangle$, where $|\chi\rangle$ is a distinguished state of the world, determined by the quantum initial conditions. Note that $|\chi\rangle$ is in some sense a state vector of the world, but it is not a vector in the physical de Sitter Hilbert space \mathcal{H} that we defined earlier. That is because it was not produced by specifying some incoming conformal structure $g^{(i)}$ in a world with de Sitter-like initial conditions, but rather was generated in some more general quantum mechanical fashion. Since the initial conditions do not correspond to a state of the form we considered before, we are not entitled to divide by null vectors in the above fashion. The best one can say along these lines is that possibly the function of $g^{(f)}$ given by the path integral with final conditions $g^{(f)}$ defines a vector in an infinite-dimensional space associated with de Sitter space. (It is hard to give a precise definition of this infinite-dimensional space, since we do not know what class of functionals Ψ should be used in eqn. (16).) Of course the function $\langle f|\chi\rangle$ is a meta-observable, beyond the control of any observer in such a universe.

⁶ The notion of a particle is murky in de Sitter space in general, but is better-defined in the far past and the far future. Anyway, all we really need here is a modest, localized disturbance of some kind; it is not important to precisely interpret it in terms of particles.

Worse from this point of view than a Big Bang (or Big Crunch) universe would be an oscillatory universe, by which I mean simply a universe in which we are given no statement at all about any asymptotic region in which a simplification occurs, for example because the universe is oscillating so that this never happens. In this case, one could say nothing at all without detailed microscopic knowledge.

More On The Meta-Observables

Now we will explore the meta-observables in somewhat more detail, despite the fact that their physical interpretation is unclear. The basic idea is to convert the meta-observable $\langle f|i\rangle$ into a series of correlation (or meta-correlation) functions by expanding the conformal metrics $g^{(i)}$ and $g^{(f)}$ near the standard round one.

To keep things simple, we will only consider the analogous expansion for a scalar field. Moreover, we will set m = 0 in (13), and we will set α to the conformally invariant value (n-2)/4(n-1). The purpose of this is just to keep the formulas simple; I do not believe that this specialization affects any qualitative conclusion.

In the conformally invariant case, the propagator can be described in a particularly simple form. In Euclidean signature, the propagator between two points $X, Y \in \mathbf{S}^n$ is a multiple of $1/(1 - X \cdot Y)^{(n-2)/2}$. To go to Lorentz signature, we write $X = (x_0, \vec{x})$, $Y = (y_0, \vec{y})$, and then we make a Wick rotation $x_0 \to ix_0$, $y_0 \to iy_0$. The propagator becomes a multiple of

$$\frac{1}{(1-\vec{x}\cdot\vec{y}+x_0y_0)^{(n-2)/2}}. (19)$$

To define the meta-observables, we want to take $x_0, y_0 \to \pm \infty$ and imitate the usual definition of boundary correlation functions in Anti de Sitter space.

Since $\vec{x}^2 - x_0^2 = \vec{y}^2 - y_0^2 = 1$, when $x_0, y_0 \to \pm \infty$, \vec{x} and \vec{y} must also diverge. We can take $\vec{x} = |x_0|\vec{a}$, $\vec{y} = |y_0|\vec{b}$, where \vec{a} , \vec{b} are points in \mathbf{S}^{n-1} . In fact, \vec{a} and \vec{b} are (depending on the signs of x_0 and y_0) points in past infinity \mathcal{I}_- or future infinity \mathcal{I}_+ ; we saw earlier that these are copies of \mathbf{S}^{n-1} .

The propagator is now for $x_0, y_0 \to \pm \infty$ a multiple of

$$\frac{1}{(x_0 y_0)^{(n-2)/2}} \frac{1}{(1 - \vec{a} \cdot \vec{b} \operatorname{sign}(x_0 y_0))^{(n-2)/2}}.$$
 (20)

The overall power of x_0y_0 means, if we imitate the familiar logic of the Anti de Sitter case, that we are dealing with a conformal field on the boundary of conformal dimension (n-2)/2. After removing this prefactor, what remains is the function that in the AdS

case we would interpret as the two point function of a conformal field on the boundary. It is simply

$$\frac{1}{(1 - \vec{a} \cdot \vec{b} \operatorname{sign}(x_0 y_0))^{(n-2)/2}}.$$
(21)

One thing which is unusual compared to the AdS case is that infinity has two components, \mathcal{I}_{-} and \mathcal{I}_{+} , and that in (21) we may have a correlator between two operators both inserted on the same component, in which case $sign(x_0y_0) = 1$, or inserted on opposite components, in which case $sign(x_0y_0) = -1$.

If the two fields are inserted on the same component, we get a singularity at $\vec{a} = \vec{b}$. This is not a surprise. More surprising is that for two fields inserted on opposite components, there is a singularity at $\vec{a} = -\vec{b}$. Though perhaps unexpected at first sight, this has a simple explanation. As we saw in analyzing the null geodesics using the form (8) of the metric, light rays emitted in the far past at a point \vec{a} all arrive (independent of their direction of propagation) in the far future at the antipodal point $-\vec{a}$. This convergence of the light rays produces the singularity in the "past-future" correlator.

This past-future singularity is presumably related to the fact that, classically, any initial state in de Sitter space can propagate to a state in the far future. If the past-future propagator were smooth and singularity free, propagation from the past to the future might project onto a finite-dimensional space of states (with the other states becoming null vectors), as one desires to make the Hilbert space \mathcal{H} finite-dimensional. Thus, one might suspect that the past-future correlator, after averaging over quantum fluctuations, would actually be singularity-free. There is actually a result in classical relativity (pointed out in this context by R. Bousso) that goes in this direction. In a generic perturbation of de Sitter space, the area of the horizon of an observer at future infinity goes to zero at a finite point in the past because the light rays going back into the past have time to converge before the Big Bang [14]. This contrasts with de Sitter space where, as we saw in discussing (8), the horizon area is independent of time and the backward-going geodesics only meet at past infinity, where they produce a singularity in the past-future correlator.

There is one other important difference between the correlation functions in the AdS case and their de Sitter cousins. This arises because in the de Sitter case the spatial sections are compact and one wants to project onto the invariants of the de Sitter symmetry group SO(1, n).

To try to see what this means concretely, we first write down a correlation function between s fields at points $\vec{a}^{(1)}, \dots \vec{a}^{(s)}$ in the past and s fields at points $\vec{b}^{(1)}, \dots \vec{b}^{(s)}$ in the

future. In writing the correlation function, we will for illustrative purposes write only the terms in which all propagators connect past and future points. (I focus on these terms because they are the ones that seem surprising.) We get

$$\sum_{\Pi} \prod_{j=1}^{s} \frac{1}{(1 + \vec{a}^{(j)} \cdot \vec{b}^{\Pi(j)})^{(n-2)/2}}.$$
 (22)

Here Π is a permutation of the set of s elements.

We have evaluated this correlator in free field theory, and of course perturbation theory (and nonperturbative physics) will generate a variety of corrections to such a formula. But should we really expect to get an answer of this general form, that is depending on an arbitrary set of points $\vec{a}^{(j)}$, $\vec{b}^{(j)}$? In a fixed de Sitter spacetime, this would be reasonable, but in a theory that includes gravitational fluctuations, it is not. One has no natural way to match up past and future infinity precisely, so one must allow for an SO(1,n) rotation of the \vec{b} 's relative to the \vec{a} 's. What this means in practice is that it makes sense to fix the \vec{a} 's to arbitrary points in $\mathcal{I}_- = \mathbf{S}^{n-1}$, but then the \vec{b} 's should be fixed only up to an SO(1,n) rotation. One way to implement this idea is to integrate over the \vec{b} 's with a weight function $f(\vec{b}^{(1)}, \ldots, \vec{b}^{(s)})$ that is of conformal dimension (n+2)/2 in each variable, so as to achieve SO(1,n) invariance. Thus, quantities that make more sense in the presence of quantum gravity than the correlators written in (22) are an integrated version

$$\int d\vec{b}^{(1)} \dots d\vec{b}^{(s)} f(\vec{b}^{(1)}, \dots, \vec{b}^{(s)}) \sum_{\Pi} \prod_{j=1}^{s} \frac{1}{(1 + \vec{a}^{(j)} \cdot \vec{b}^{\Pi(j)})^{(n-2)/2}}.$$
 (23)

It is possible to choose f so that the integral converges. Of course, in (23) we have written a free-field approximation to the correlator.

Note that a suitable f only exists for $s \geq 2$. So in particular, such a correlator, in the presence of quantum gravity, requires looking at the behavior of the theory at distinct points in the infinite past and future. So measuring such an object is beyond the scope of any one observer in de Sitter space, who experiences precisely one point in the asymptotic future. In that sense, the object that we have described is perhaps better characterized as a meta-correlator, computable and interpretable only by an observer external to the spacetime. This makes its interpretation obscure. We turn next to the question of what, if anything, an observer in the spacetime can measure in a precise way.

Precision Of Physics

We are accustomed to physical theories that make, within the rules set by quantum mechanics, predictions of arbitrary precision that can be tested experimentally, in principle, with any required accuracy. For example, we customarily assume that the g-factor of the electron is a well-defined real number. It is true that any given experiment only measures g with some limited (but perhaps very good) precision. But one customarily assumes that there is no bound to the precision with which g could be measured, in principle, given the necessary time, resources, and skill.

Likewise, any given theoretical calculation in a complicated theory such as QED is only an approximation. But conventionally, in flat space quantum field theory, to the extent that our theories are correct and thus in particular well-defined⁷ there is no limit in principle to the accuracy in which the computations can be done, again given sufficient resources, skill, and patience. We do not assume that the theories that we have now are valid to arbitrary precision, but we usually assume that these theories are approximations to a better theory that does make absolutely precise predictions, in principle, for the electron g-factor, the ratios of hadron masses, and so on.

In an eternal universe, in the absence of gravity, with a constant free energy supply generated by stars, this makes perfect sense. In a more realistic description of nature, taking the expansion of the universe into account, there are many pitfalls.

De Sitter space (or a cosmology asymptotic to it in the far future) is a particularly unfavorable case for achieving the usually assumed degree of precision. For example, if it is true that the dimension of the quantum Hilbert space is finite, this puts a limit on the conceivable complexity of any experimental apparatus or computational machinery. The inflation that will occur in the future in de Sitter space puts a limit on the time in which the experiment must be conducted (or the computation performed) before the free energy supply runs down.

Even the concept of an observer in de Sitter space as a living creature making an observation has only limited validity. For life itself is only an approximation, valid in the limit of a complex organism or civilization. There might be a cosmology in which the approximation we call life is better and better in the future, but this requires a process of adaptation to longer and longer time scales and lower and lower temperatures [15], neither of which is possible in de Sitter space (where inflation sets a maximum time scale, and

⁷ There may be a limit to the precision with which QED can be defined because of the ultraviolet behavior, so QCD, which is asymptotically free, might give a better example.

the de Sitter temperature is a minimum temperature). The approximation we know as life thus breaks down in the far future in an asymptotically de Sitter world, and this will put an end to any measurement (or computation) performed by an observer or civilization in such a spacetime, and hence an upper bound to its precision.⁸

It is thus just as well that the only candidates we can see for quantities that might be calculable with arbitrary precision are the meta-observables, which extend beyond any one horizon. Since the horizon volume (after a limited number of inflationary e-foldings) does not contain a living observer anyway, any precisely calculable quantities associated with the interior of a horizon would be wasted.

Where does this leave string theory? Like physics as we know it, string theory as we know it deals in precisely defined quantities, such as the S-matrix in an asymptotically flat spacetime, or the correlation functions of the boundary conformal field theory for the case of negative cosmological constant. If quantities with this degree of precision do not exist—which seems to be the case in de Sitter space if one rejects the meta-observables—then it is not clear just what one should aim to compute. This question has nothing specifically to do with string theory, and any answer to it that makes sense might make sense in string theory.

The problem with de Sitter space can actually be divided into two parts. One aspect is that because of the horizon experienced by an observer, one cannot hope to witness the final state of the whole universe. The other side of the problem, which seems more acute to me, is that, as indicated above, one also cannot in de Sitter space make sense in a precise way of what we usually regard as local particle physics quantities. In this context, let us consider cosmologies that accelerate more slowly than de Sitter space, as recently considered in [16,17]. In the models considered in those papers, the scale factor R of the universe varies with cosmic time t as $R \sim t^{1+\alpha}$ for some $\alpha > 0$. There is still an observer-dependent horizon, just as for de Sitter space, and the final state of the universe

⁸ I believe that for a civilization of ever-increasing complexity to exist in the far future also requires that the universe should have $\Omega=1$, that is, flat spatial sections. For $\Omega<1$, the galaxies or clusters of galaxies, if not gravitationally bound, recede from each other with constant asymptotic velocities, and any civilization has only a fixed number of atoms (or stable elementary particles) at its disposal, namely those that are bound to the local galaxy or cluster. This presumably puts an upper bound on the possible complexity (though this last assertion has been questioned by Dyson). With $\Omega=1$, it may be possible to keep absorbing matter from the surroundings and to grow in complexity.

as a whole is not observable. But the curvature of the universe vanishes for $t \to \infty$; does this mean that local particle physics observables such as the g-factor of the electron become well-defined and measurable in the far future? Some necessary conditions are obeyed; for example, the time scale of the cosmic expansion gets longer and the temperature goes to zero for $t \to \infty$. So from this point of view there seems to be no obstruction to a precise measurement. However, an observer in such a universe would have to perform all experiments with a finite supply of elementary particles and free energy stored up before the acceleration progresses too far. ⁹ Under these conditions, it seems doubtful that one could perform asymptotically precise measurements.

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⁹ The prospects for gathering additional matter are less favorable than in an open but non-accelerating universe, which was discussed in the previous footnote, and corresponds to $\alpha = 0$.

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