Quantum Gravity Partition Functions In Three Dimensions

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Abstract

We consider pure three-dimensional quantum gravity with a negative cosmological constant. The sum of known contributions to the partition function from classical geometries can be computed exactly, including quantum corrections. However, the result is not physically sensible, and if the model does exist, there are some additional contributions. One possibility is that the theory may have long strings and a continuous spectrum. Another possibility is that complex geometries need to be included, possibly leading to a holomorphically factorized partition function. We analyze the subleading corrections to the Bekenstein-Hawking entropy and show that these can be correctly reproduced in such a holomorphically factorized theory. We also consider the Hawking-Page phase transition between a thermal gas and a black hole and show that it is a phase transition of Lee-Yang type, associated with a condensation of zeros in the complex temperature plane. Finally, we analyze pure three-dimensional supergravity, with similar results.

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1. Introduction

This paper is devoted to describing some explicit computations relevant to three-dimensional pure quantum gravity with negative cosmological constant. The classical action can be written

$$I = \frac{1}{16\pi G} \int d^3 x \sqrt{g} \left(R + \frac{2}{\ell^2} \right). \qquad (1.1)$$

Three-dimensional gravity with a cosmological constant was first studied in [1]. Some early milestones in the study of this theory, both of them special to the case of negative cosmological constant, have been the construction of an asymptotic Virasoro symmetry [2] and the recognition that the theory admits black holes [3,4]. The asymptotic Virasoro

symmetry is now understood as part of the structure of a dual two-dimensional conformal field theory [5] and this structure together with modular invariance can be used to determine the entropy of a black hole of asymptotically large mass [6]. For a recent review of the extensive work on this subject, with references, see [7].

None of the results mentioned in the last paragraph are special to the case of *pure* three-dimensional gravity, without additional fields. Our intent here, however, is to study this minimal theory, with the goal of computing its exact energy levels in a spacetime that is asymptotic at spatial infinity to the classical "vacuum" of three-dimensional Anti de Sitter space, AdS$_3$. As always in General Relativity, a well-posed problem is obtained by specifying what the world should look like at spatial infinity – in this case we ask that it should coincide with AdS$_3$ at infinity – and then analyzing all possible "interiors." Our problem in this paper is to compute the precise quantum energy levels that arise. We perform a computation based on known concepts, and in a sense that we will explain, this computation is not successful. The reasons for this failure are not clear and we consider several hypotheses.

It is convenient to summarize the spectrum of energy levels in the form of a trace Tr $\exp(-\beta H)$, where H is the Hamiltonian and β is a positive real number (or more generally a complex number with positive real part). As usual in General Relativity, the Hamiltonian is defined via the ADM procedure in terms of the leading behavior at spatial infinity of the correction to the pure AdS$_3$ metric. It was indeed by carefully examining this procedure that the asymptotic Virasoro symmetry (which generalizes the obvious conservation laws such as conservation of energy) was discovered [2].

There is also a conserved angular momentum J which generates a rotation at infinity of the asymptotically AdS$_3$ spacetime and commutes with H. Consequently, one can introduce an additional parameter θ and try to compute a more general partition function:

$$Z(\beta, \theta) = \text{Tr} \, \exp(-\beta H - i\theta J). \tag{1.2}$$

This partition function is naturally computed via a Euclidean path integral. According to the standard recipe, the integral is taken over Euclidean three-geometries that are conformal at infinity to a two-torus Σ with modular parameter $\tau = \theta/2\pi + i\beta$. We write $|dz|^2$ for a flat metric on this torus, where z is a complex coordinate subject to the identifications $z \to z+1$, $z \to z+\tau$, and introduce another coordinate $u > 0$, such that conformal infinity will be the region $u \to 0$. Then the metric should behave for $u \to 0$ like

$$ds^2 = \frac{|dz|^2 + du^2}{u^2} \tag{1.3}$$

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plus subleading terms.

The Euclidean path integral also has a natural interpretation in the dual two-dimensional CFT [5]: it is the genus one partition function of this dual theory, on the two-dimensional surface Σ. In fact, by definition, the Hamiltonian and momentum H and P of the dual CFT coincide with H and J, the Hamiltonian and angular momentum of the original theory in AdS$_3$. This dual interpretation is informative, but is not really needed to motivate the computation that we perform.

This Euclidean path integral is a formal recipe, for various reasons. One problem is that, in general, the Euclidean quantum gravity path integral is not convergent because the action is not bounded below [8]. The only known way to deal with this problem is to expand around a classical solution; in doing so, one can obtain a meaningful result at least in perturbation theory. There is no clearly established claim in the literature that topologies that do not admit classical solutions do not contribute to the Euclidean path integral. But there is also no known method to evaluate their contributions.

It turns out that in the present case, one can describe the classical solutions completely. This is explained in Sec. 2.1. The classical solutions are precisely the ones considered in [9] and in [10] (in studying the elliptic genus of certain AdS$_3$ models derived from string theory). Moreover, perturbation theory around a classical solution terminates with the one-loop term, and that term can be easily evaluated by adapting the arguments of [2]. This is the content of Sec. 2.2.

We are therefore in a position to write down the complete sum of *known* contributions to the path integral. This is done in Sec. 3. The sum turns out to require some regularization, and we use what we believe is a natural regularization, analogous to zeta function regularization [11]. We ultimately obtain an explicit, though complicated, formula for the sum of known contributions to $Z(\beta, \theta)$. However – and this is our main result – the sum is not physically sensible: it cannot be written as $\mathrm{Tr}\exp(-\beta H - i\theta J)$ for any commuting operators H, J in a Hilbert space. (According to [12], this possibility was also conjectured by S. Minwalla.)

We do not know the correct interpretation of this result. In Sec. 4, we discuss some possibilities. One possibility is that the *minimal* theory that we postulate here actually does not exist. There are many quantum theories that do exist that look semiclassically like three-dimensional gravity coupled to additional fields; indeed, there are a plethora of

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known string-derived models, such as the ones studied in [10]. However, it is not clear[1] that there exists nonperturbatively a minimal theory that one would want to call pure gravity. One possible interpretation of our result is to indicate that this minimal theory does not exist.

The other possibility, broadly speaking, is that there are additional contributions to the path integral beyond the known ones that we evaluate in Secs. 2 and 3. To be concrete, we consider in Sec. 4 two scenarios. One is that the minimal theory of three-dimensional gravity, in addition to the known BTZ black holes and Brown-Henneaux boundary excitations, also describes cosmic strings. The motivation for this proposal is that known models of three-dimensional quantum gravity, such as the string-based models considered in [10], do always have cosmic strings. Perhaps this is also true for the "minimal" theory, if it exists.

The second scenario is that in addition to the real saddle points that are classified and evaluated in Secs. 2 and 3, an exact description of the theory should also include complex saddle points. We describe a specific scenario in which the inclusion of complex saddle points leads to the holomorphically factorized partition function that was proposed (based on highly speculative reasoning) in [13]. This involves a doubled sum over saddle points similar to what is assumed in [12,14]. The resulting partition function is consistent with an interpretation as $\mathrm{Tr}\,\exp(-\beta H - i\theta J)$ with some Hilbert space operators H, J.

Though we are not able to put this proposal in a convincing form, we do uncover one interesting fact, which concerns the semiclassical behavior of the partition function (assumed to be holomorphically factorized and extremal) as the gravitational coupling G goes to zero with fixed AdS radius ℓ. In this semiclassical limit, the partition function $Z = \mathrm{Tr}\,\exp(-\beta H - i\theta J)$ is always dominated, as long as β and θ are real, by a real saddle point corresponding to a real geometry. The partition function can be dominated by a classical saddle point, but only if one asks a more exotic question with complex values of β and θ.

In Secs. 5 and 6 we discuss questions of black hole physics in three dimensions. In section 5 we discuss black hole entropy: a theory of pure quantum gravity, assuming it

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[1] The problem may be analogous to trying to define a minimal string theory in four dimensions. There are many theories that macroscopically are four-dimensional string theories, such as theories obtained by compactification to four dimensions from the critical dimension, or gauge theories with flux tubes or vortex lines, but it is plausible that there is nothing that should sensibly be called a minimal four-dimensional quantum string theory.

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exists, should give a proper microscopic accounting of the Bekenstein-Hawking entropy of the BTZ black hole. In the semi-classical limit this entropy is just the horizon area. The computations of section 2, however, allow us to determine the perturbative corrections to this semi-classical result, which are described in section 5.1. In section 5.2 we show that an infinite series of corrections can be reproduced in a holomorphically factorized theory.

In investigating these holomorphically factorized theories, we noticed an additional interesting phenomenon that is the subject of Sec. 6. Three-dimensional quantum gravity, as a function of β, exhibits in the semi-classical limit the Hawking-Page phase transition [15] between a thermal gas (in this case, a gas of Brown-Henneaux boundary excitations) and a black hole. At first sight, it is puzzling how such a phase transition can be compatible with holomorphic factorization. We show, however, that this question has a natural answer, in terms of a condensation on the phase boundary of Lee-Yang zeroes of the partition function.

Finally, in Sec. 7, we extend our analysis to pure $\mathcal{N} = 1$ supergravity, obtaining results similar to what we find in the bosonic case.

2. Known Contributions To The Path Integral

Our task in this section is two-fold. The first step is to classify Euclidean solutions of Einstein's equations, with negative cosmological constant, with the asymptotic behavior described in the introduction. We write M for the three-dimensional spacetime, and Σ for its conformal boundary; as in the introduction, Σ is a Riemann surface of genus 1. We assume that M is smooth, that the metric on M is complete, and that Σ is the only "end" of M. According to the standard logic of Euclidean quantum gravity, known contributions to the path integral have these properties; it may be appropriate physically to relax them, but we do not know how.

We proceed in two steps. In Sec. 2.1, we describe the possible choices of M. The result is actually well-known in the theory of hyperbolic three-manifolds. In Sec. 2.2, we evaluate the contribution to the partition function of a particular M. The sum over different choices of M is postponed to Sec. 3. In Sec. 2.3, we describe the general form of the partition function in a theory with finite entropy.

2.1. Classification Of Solutions

The automorphism group of AdS_3 is $SO(3,1)$, which is the same as $SL(2,\mathbb{C})/\mathbb{Z}_2$. We may write the metric on a dense open subset of AdS_3 as[2]

$$ds^2 = \frac{|dz|^2 + du^2}{u^2}, \qquad u > 0, \qquad z \in \mathbb{C}. \tag{2.1}$$

If we combine the (z,u) coordinates into a single quaternion $y = z + ju$, the action of an element $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{C})$ can be written succinctly as

$$y \to (ay + b)\,(cy + d)^{-1}. \tag{2.2}$$

In this expression the element $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \in SL(2,\mathbb{C})$ acts trivially, so (2.2) actually describes the action of $SL(2,\mathbb{C})/\mathbb{Z}_2$ on AdS_3. In general, any classical solution M of three-dimensional gravity with negative cosmological constant takes the form AdS'_3/Γ, where Γ is a discrete subgroup of $SO(3,1)$ and AdS'_3 is the part of AdS_3 on which Γ acts discretely.

The conformal boundary of M can be constructed as follows. First of all, the conformal boundary of AdS_3 is a two-sphere, which one can think of as $\mathbb{CP}^1$, acted on by $SL(2,\mathbb{C})$ in the usual way. This $\mathbb{CP}^1$ may be regarded as the complex z-plane in (2.1) at $u \to 0$ plus a point at infinity. From (2.2), one can see that $SL(2,\mathbb{C})$ acts on this $\mathbb{CP}^1$ in the familiar fashion

$$z \to \frac{az + b}{cz + d}. \tag{2.3}$$

To construct the conformal boundary of $M = \mathrm{AdS}'_3/\Gamma$, one first throws away a certain subset of $\mathbb{CP}^1$ in a neighborhood of which the discrete group Γ acts badly. This set is closed, so its complement is an open subset $U \subset \mathbb{CP}^1$.

The discrete group Γ acts freely on U, and the conformal boundary of M is the quotient

$$\Sigma = U/\Gamma. \tag{2.4}$$

Since the action of $SL(2,\mathbb{C})$ preserves the holomorphic structure of $\mathbb{CP}^1$, U/Γ carries a natural holomorphic structure and the isomorphism (2.4) is valid holomorphically, not just topologically.

Given this, let us investigate the condition for Σ to be of genus 1. Σ is topologically a two-torus, so its fundamental group is $\pi_1(\Sigma) = \mathbb{Z} \oplus \mathbb{Z}$. It follows from eqn. (2.4) that the

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[2] We set $\ell = 1$ unless otherwise indicated.

fundamental group of U is a subgroup of the fundamental group of Σ. Possible subgroups $\pi_1(U) \subset \mathbb{Z} \times \mathbb{Z}$ are of three types:

(i) $\pi_1(U)$ may be a subgroup of $\pi_1(\Sigma)$ of finite index, isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$. (A special case is $\pi_1(U) = \pi_1(\Sigma)$.)

(ii) $\pi_1(U)$ may consist of multiples of a single non-zero vector $x \in \pi_1(\Sigma)$, in which case $\pi_1(U) \cong \mathbb{Z}$.

(iii) $\pi_1(U)$ may be trivial.

In case (i), U is a finite cover of Σ, and therefore is itself a Riemann surface of genus 1. However, a Riemann surface of genus 1 is not isomorphic to an open subset of $\mathbb{CP}^1$. So case (i) cannot arise.

Cusp Geometry

In case (iii), U is the universal cover of Σ, and so is isomorphic to $\mathbb{C}$ (or $\mathbb{R}^2$). The holomorphic structure of $\mathbb{C}$ is unique up to isomorphism. $\mathbb{C}$ is isomorphic holomorphically to an open subset of $\mathbb{CP}^1$ in essentially only one way: it is the complement of one point, say the point at $z = \infty$. The subgroup of $SL(2, \mathbb{C})$ that leaves fixed the point at infinity consists of the triangular matrices

$$\begin{pmatrix} \lambda & w \\ 0 & \lambda^{-1} \end{pmatrix}. \tag{2.5}$$

The point $z \in \mathbb{C}$ corresponds to $\begin{pmatrix} z \\ 1 \end{pmatrix} \in \mathbb{CP}^1$, and a triangular matrix acts by $z \to \lambda^2 z + \lambda w$.

Since U is simply-connected, Γ must be isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$ (in order to get the right fundamental group for Σ). Any discrete group of triangular matrices that is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$ is generated by two strictly triangular matrices

$$\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}, \tag{2.6}$$

where a and b are complex numbers that are linearly independent over $\mathbb{R}$. (If a and b are linearly dependent over $\mathbb{R}$, then the group generated by these matrices is not discrete, since a suitable linear combination $ma + nb$, for $m, n \in \mathbb{Z}$, can be arbitrarily small.) Up to conjugacy by a diagonal matrix, the only invariant of such a group is the ratio b/a. Therefore, we can reduce to the case $a = 1$, $b = \tau$, and moreover by taking $b \to -b$ (which does not affect the group generated by the two matrices) we can assume that $\operatorname{Im} \tau > 0$.

We have therefore arrived precisely at the group of symmetries

$$z \to z + m + n\tau, \ m, n \in \mathbb{Z} \tag{2.7}$$

of the complex z-plane. The quotient is a genus 1 surface Σ with an arbitrary τ-parameter. The three-manifold $M = \mathrm{AdS}_3/\Gamma$ can be described very concretely and is a standard example. The (u, z) coordinates in (2.1) cover precisely the subspace AdS_3' on which Γ acts "nicely." So $M = \mathrm{AdS}_3'/\Gamma$ is given by the metric (2.1), now subject to the identification (2.7).

But M does not obey the conditions described at the beginning of this section. M is smooth and has a complete Einstein metric. However, in addition to the "end" at $u = 0$, which is the one required by our boundary condition, M also has a second "end" at $u = \infty$. Our problem was to classify Einstein manifolds with only one end; M does not qualify.

The second end of M is one at which the metric of Σ collapses to zero (rather than blowing up, as it does for $u \to 0$). An end of this kind is known as a "cusp." In the theory of hyperbolic three-manifolds, one often considers hyperbolic metrics with such cusps. One of the main results about them is that, under suitable conditions, such a metric can be slightly perturbed so as to eliminate the cusp. In the present context, this gives the sort of metric that we will describe momentarily.

Physically, we cannot be certain that omitting the case of the "cusp" is the correct thing to do. However, restricting to cuspless metrics appears to be the right thing to do in known examples of AdS_3 theories. And pragmatically, we believe that from what is known of three-dimensional gravity, it would be difficult to give a sensible procedure for evaluating the contribution of the spacetime with the cusp. The reason for the last statement is that non-trivial one-cycles (loops in Σ) become sub-Planckian in length near a cusp, so a semiclassical treatment is not valid.

Semiclassical Geometries

The last case to consider – case *(ii)* – is that the fundamental group of U is $\mathbb{Z}$. This means that topologically U is $\mathbb{R} \times S^1$ (S^1 is a circle). The holomorphic structure of U is then uniquely determined to be that of the z-plane minus a point, which we may as well take to be the point at $z = 0$. The subgroup of triangular matrices that preserve the point $z = 0$ is simply the group of diagonal matrices.

Therefore Γ is a discrete subgroup of the group of diagonal matrices. It cannot be a finite group (or the first Betti number of $\Sigma = U/\Gamma$ would be 1, while the desired value is 2). This being so, there are essentially two cases to consider.

First, Γ may be isomorphic to $\mathbb{Z}$, generated by a matrix of the form

$$W = \begin{pmatrix} q & 0 \\ 0 & q^{-1} \end{pmatrix} \in SL(2,\mathbb{C}). \tag{2.8}$$

By exchanging the two eigenvalues we can assume that $|q| < 1$. (If $|q| = 1$, then either q is a root of unity and the group generated by W is a finite group, or q is not a root of unity and the subgroup of $SL(2,\mathbb{C})$ generated by W is not discrete.) Alternatively, Γ may be isomorphic to $\mathbb{Z} \times \mathbb{Z}_n$, generated by W together with

$$Y = \begin{pmatrix} \exp(2\pi i/n) & 0 \\ 0 & \exp(-2\pi i/n) \end{pmatrix}, \tag{2.9}$$

with some integer n.

Let us first consider the case that $\Gamma = \mathbb{Z}$. Then $\Sigma = U/\Gamma$ is obtained from the complex z-plane by throwing away the point $z = 0$ and dividing by the group generated by W. It is convenient to write $z = \exp(2\pi i w)$, so that w is defined modulo

$$w \to w + 1 \tag{2.10}$$

and W acts by

$$w \to w + \frac{\log q}{2\pi i}. \tag{2.11}$$

The quotient of the w-plane by (2.10) and (2.11) is a Riemann surface of genus 1, as required. The complex modulus of this surface is $\tau = \frac{\log q}{2\pi i}$, i.e. it is given by $q = e^{2\pi i \tau}$. More generally, however, the modulus of this Riemann surface is defined only up to $\tau \to (a\tau + b)/(c\tau + d)$ with integers a, b, c, d obeying $ad - bc = 1$. Therefore, we will get an equivalent Riemann surface if

$$q = \exp(2\pi i(a\tau + b)/(c\tau + d)) \tag{2.12}$$

for such a, b, c, d.

One might conclude, therefore, that we get a three-manifold obeying the required conditions for every choice of a, b, c, d. This is not quite the case, for two reasons. First, an overall sign change of a, b, c, d does not affect q or the associated three-manifold. Second, once c and d are given, a and b are uniquely determined by $ad - bc = 1$ up to shifts of

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the form $(a, b) \rightarrow (a, b) + t(c, d)$, $t \in \mathbb{Z}$. Under this transformation, q as defined in eqn. (2.12) is invariant. So the possible three-manifolds really only depend on the choice of the pair c, d of relatively prime integers, up to sign. For each such pair, we find integers a, b such that $ad - bc = 1$, and identify q via eqn. (2.12). This gives a manifold that we will call $M_{c,d}$. This family of manifolds were first discussed in the context of three-dimensional gravity in [9].
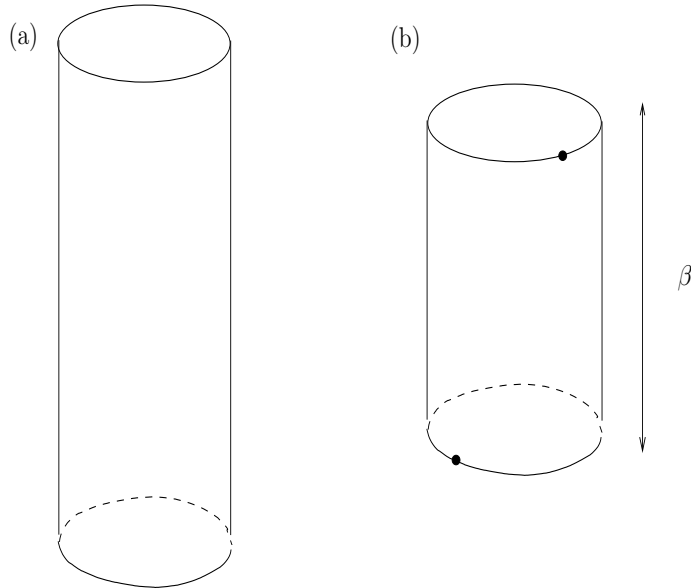
The Geometry of $M_{c,d}$

Figure 1:   a) An infinite cylinder representing AdS_3. The boundary of the cylinder represents conformal infinity; time translations act by vertical shifts. b) A slice of height determined by β. The manifold $M_{0,1}$ is built by gluing together the top and bottom, after a rotation that identifies the boundary points marked by solid dots.

The simplest such manifold is $M_{0,1}$, which we will now describe in more detail. If we identify the real one-parameter subgroup $\text{diag}(e^b, e^{-b})$ of $SL(2, \mathbb{C})$ as the group of time translations, the AdS_3 metric (2.1) can be put in the form

$$ds^2 = \cosh^2 r \; dt^2 + dr^2 + \sinh^2 r \; d\phi^2, \tag{2.13}$$

with $-\infty < t < \infty$, $0 \leq r < \infty$, and $0 \leq \phi \leq 2\pi$. Described in this coordinate system is the subset $\text{AdS}_3' \subset \text{AdS}_3$ on which Γ acts nicely; its topology is $D \times \mathbb{R}$, where D is a

two-dimensional open disc parameterized by r and ϕ. Conformal infinity is at $r = \infty$. (See Fig. 1.) The element $\mathrm{diag}(e^b, e^{-b})$ acts by $t \to t + b$. The group of spatial rotations is the one-parameter group $\mathrm{diag}(e^{i\theta}, e^{-i\theta})$, acting by $\phi \to \phi + \theta$.

The group element W therefore generates a combined time-translation and spatial rotation. To explicitly divide by W, we "cut" AdS_3' at times $t = 0$ and $t = 2\pi \operatorname{Im} \tau$. Then we glue together the top and bottom of the region $0 \le t \le 2\pi \operatorname{Im} \tau$ after making a spatial rotation by an angle $2\pi \operatorname{Re} \tau$. This is sketched in Fig. 1b. The resulting spacetime $M_{0,1}$ is topologically $D \times S^1$, so its fundamental group is indeed $\mathbb{Z}$.

The path integral in this spacetime has a simple semiclassical meaning, since it may be interpreted in terms of Hamiltonian time evolution. A state is prepared at time zero and propagates a distance $\beta = 2\pi \operatorname{Im} \tau$ forward in Euclidean time. In this process, the state vector is multiplied by the time evolution operator $\exp(-\beta H)$, where H is the Hamiltonian. Then, after a spatial rotation by an angle $\theta = 2\pi \operatorname{Re} \tau$, which acts on the state by $\exp(-i\theta J)$, we glue the top and bottom of the figure, which results in taking the inner product of the final state with the initial state. The whole operation gives the trace $\operatorname{Tr} \exp(-\beta H - i\theta J)$ defined in the Hilbert space of perturbative fluctuations around AdS_3. This Hamiltonian interpretation of the path integral of $M_{0,1}$ will be the basis for evaluating it in Sec. 2.2.

The other manifolds $M_{c,d}$ are obtained from $M_{0,1}$ by modular transformations, that is, by diffeomorphisms that act non-trivially on the homology of Σ. This fact will allow us to evaluate their contributions to the path integral. These other manifolds may be thought of as Euclidean black holes.

For example, in the (t, ϕ) coordinates introduced above, $M_{0,1}$ involves the identifications

$$\phi + it \sim \phi + it + 2\pi \sim \phi + it + 2\pi\tau. \tag{2.14}$$

From the metric (2.13), note that the ϕ circle is contractible in the bulk, since the coefficient of $d\phi^2$ becomes zero at the origin $r = 0$. The manifold $M_{1,0}$ is described by the same coordinate system (2.13), but with new identifications

$$\phi + it \sim \phi + it + 2\pi \sim \phi + it - \frac{2\pi}{\tau}. \tag{2.15}$$

These identifications (2.15) may be written in the form of (2.14) by taking $\phi + it \to \frac{1}{\tau}(\phi + it)$. After this transformation, the coordinate that is contractible in the bulk now involves a combination of ϕ and t.

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If we take τ to be imaginary, the scaling by $1/\tau$ has the effect of exchanging (and rescaling) ϕ and t. $M_{1,0}$ is hence a Euclidean black hole, in which the "time" circle is contractible rather than the "space" circle. If we rotate to Lorentzian signature by $t \to it$, then the locus $r = 0$ where the coefficient of dt^2 vanishes is the horizon of the black hole. In fact, $M_{1,0}$ is just the Euclidean version of the BTZ black hole [16]. The more general manifolds $M_{c,d}$ are often referred to as the $SL(2,\mathbb{Z})$ family of black holes.

Orbifolds

Finally, let us consider the extension in which Γ has a second generator given by (2.9). We may assume that $n = 2m$ is even, since the element $\mathrm{diag}(-1,-1)$ (the non-trivial element of the center of $SL(2,\mathbb{C})$) acts trivially on $\mathbb{CP}^1$. We also may as well assume that $m > 1$, since the case $m = 1$ leads to nothing new.

What we get when Γ has an additional generator with $m > 1$ is simply a three-dimensional space of the form $M_{c,d}/\mathbb{Z}_m$. The conformal boundary is still a Riemann surface of genus 1, and by changing q we can adjust its modular parameter as we wish. However, for $m > 1$, the group element (2.9) acts on AdS_3' with fixed points, meaning that $M_{c,d}/\mathbb{Z}_m$ has orbifold singularities. The fixed points are of codimension 2 and the singularities look locally like $\mathbb{R}^2/\mathbb{Z}_m$, where $\mathbb{Z}_m$ acts as a rotation by an angle $2\pi/m$. This produces a deficit angle

$$\theta = 2\pi(1 - 1/m). \tag{2.16}$$

The picture is sketched in Fig. 2.

At the classical level, the physical meaning in three-dimensional gravity of a codimension two singularity characterized by a deficit angle is usually that it represents the orbit of a massive particle. The mass of the particle is related to the deficit angle [17]. The particular values of the deficit angle in eqn. (2.16) are special because they are related to orbifolds.

Of course, there are many consistent theories of three-dimensional quantum gravity plus matter, obtained from various widely studied string theory and M-theory constructions. They each have massive particles of various sorts, the precise masses and spins being model-dependent. It would be hopeless to try to completely solve such a general theory.

The question of interest in the present paper is whether there exists a theory that is in some sense minimal and solvable. Our hypothesis is that the minimal theory should be described by smooth geometries without orbifold or deficit angle singularities. Pragmatically, it appears difficult to avoid the problem found in Sec. 3 simply by including a small
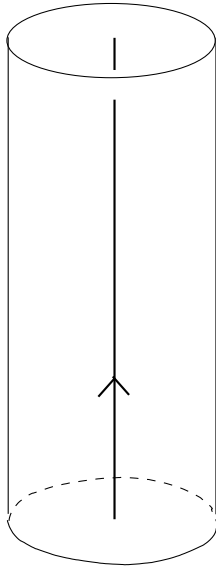
Figure 2: The black line represents an orbifold singularity in the interior of a spacetime that at infinity looks like AdS$_3$.

set of such singularities. In any event, were we to allow singularities, we would not know which ones to allow.

In the dual CFT, massive particles correspond to primary operators of positive dimension. Any two-dimensional CFT – and they are abundant, of course – can be interpreted as a dual AdS$_3$ theory. If this dual theory has a macroscopic, semiclassical interpretation, it can be interpreted in terms of AdS$_3$ with particles and black holes. From the holographic point of view, it is difficult to understand which CFTs have a semiclassical interpretation, but it is clear that, whether or not there is what one might call a minimal theory, there are many consistent theories with massive particles.

2.2. Evaluation Of The Partition Function

We write $Z_{c,d}(\tau)$ for the contribution to the partition function of the manifold $M_{c,d}$. Because the manifolds $M_{c,d}$ are all diffeomorphic to each other, the functions $Z_{c,d}(\tau)$ can all be expressed in terms of any one of them, say $Z_{0,1}(\tau)$, by a modular transformation. The formula is simply

$$Z_{c,d}(\tau) = Z_{0,1}((a\tau + b)/(c\tau + d)), \qquad (2.17)$$

where a and b are any integers such that $ad - bc = 1$. The partition function, or rather the sum of known contributions to it, is

$$Z(\tau) = \sum_{c,d} Z_{c,d}(\tau) = \sum_{c,d} Z_{0,1}((a\tau + b)/(c\tau + d)). \qquad (2.18)$$

The summation here is over all integers c and d which are relatively prime and have $c \geq 0$.

This formula shows that the key point is to evaluate $Z_{0,1}(\tau)$. We recall that this contribution is simply Tr $\exp(-\beta H - i\theta J)$, computed in the Hilbert space that describes small fluctuations about AdS$_3$ (as opposed to black holes). If we know the eigenvalues of the commuting operators H and J in the Hilbert space of small fluctuations, then we can compute the trace.

In the most naive semiclassical approximation, $Z_{0,1}(\tau)$ is just $\exp(-I)$, where I is the classical action. In computing this action, one can not just evaluate the action (1.1) for the solution (2.13); such a computation would give an infinite answer, coming from the boundary at $r \to \infty$. The full action includes the Gibbons-Hawking boundary term, which has the opposite sign of the Einstein-Hilbert term (1.1). This extra term removes the divergence, and one arrives at a finite (negative) answer for the action of $M_{0,1}$ [16]:

$$I = -4\pi k \mathrm{Im}\,\tau \qquad (2.19)$$

where $k = \ell/16G$. Therefore, in this approximation, we have

$$Z_{0,1}(\tau) \simeq |\bar{q}q|^{-k}. \qquad (2.20)$$

The result (2.20) has a simple interpretation, in terms of facts explained in [6]. Three-dimensional pure gravity with the Einstein-Hilbert action (1.1) is dual to a conformal field theory with central charge $c_L = c_R = 3\ell/2G = 24k$. The formula $c_L = c_R = 3\ell/2G$ is actually the semiclassical approximation of [2]. Depending on how the theory is regularized, there may be quantum corrections to this formula, but they preserve $c_L = c_R$ because the Einstein-Hilbert theory is parity symmetric. (By adding to the action a gravitational Chern-Simons term, we could generalize to $c_L \neq c_R$. We omit this here.) For our purposes, we simply parametrize the theory in terms of $k = c_L/24 = c_R/24$. Since c_L and c_R are physical observables (defined in terms of the two-point function of the stress tensor in the boundary CFT, or an equivalent bulk computation), the theory when parametrized in this way does not depend on any choice of formalism. (By contrast, if we describe

the results in terms of the microscopic variable ℓ/G, then it is also necessary to describe the regularization.) It is conjectured in [13] that pure three-dimensional quantum gravity exists, if at all, only for integer k. However, for our present purposes, we do not need to know if this is correct.

Let L_0 and $\widetilde{L}_0$ be the Hamiltonians for left- and right-moving modes of the CFT. They are related to what we have called H and J by

$$
\begin{aligned}
H &= L_0 + \widetilde{L}_0 \\
J &= L_0 - \widetilde{L}_0.
\end{aligned}
\tag{2.21}
$$

The CFT ground state has $L_0 = -c_L/24$, $\widetilde{L}_0 = -c_R/24$, or in the present context $L_0 = \widetilde{L}_0 = -k$. Equivalently, this state has $H = -2k$, $J = 0$. Its contribution to $\mathrm{Tr}\,\exp\left(-2\pi(\mathrm{Im}\,\tau)H + 2\pi i(\mathrm{Re}\,\tau)J\right)$ is $\exp(4\pi k\mathrm{Im}\,\tau) = |\bar{q}q|^{-k}$, as in eqn. (2.20).

Thus, although seemingly only a naive approximation, eqn. (2.20) actually gives the exact result for the contribution of the ground state to the partition function (if the theory is parametrized in terms of the central charges). As a result, it is also exact for $\mathrm{Im}\,\tau \to \infty$.

To get a complete answer, we also need to know the energies of excited states. In more than $2+1$ dimensions, the trace would receive contributions from a gas of gravitons as well as from other particles, if present. Their energies would in general receive complicated perturbative (and perhaps nonperturbative) corrections. Hence, although the function $Z_{0,1}(\tau)$ has a natural analog in any dimension, above $2+1$ dimensions one would not expect to be able to compute it precisely.

In $2+1$ dimensions, there are no gravitational waves. Naively, there are no perturbative excitations at all above the AdS_3 vacuum, and hence one might at first expect the formula (2.20) to be exact. However, this is mistaken, because of the insight of Brown and Henneaux [2]. There must at least be states that correspond to Virasoro descendants of the identity, or in other words states obtained by repeatedly acting on the CFT vacuum $|\Omega\rangle$ with the stress tensor. If L_n and $\widetilde{L}_n$ are the left- and right-moving modes of the Virasoro algebra, then a general such state is

$$
\prod_{n=2}^{\infty} L_{-n}^{u_n} \prod_{m=2}^{\infty} \widetilde{L}_{-m}^{v_m} |\Omega\rangle,
\tag{2.22}
$$

with non-negative integers u_n, v_m. (For a state of finite energy, almost all of these integers must vanish. The products begin with $n, m = 2$ since L_{-1} and $\widetilde{L}_{-1}$ annihilate the CFT

ground state.) A state of this form is an eigenstate of L_0 and $\widetilde{L}_0$ with $L_0 = -k + \sum_{n=2}^{\infty} n u_n$, $\widetilde{L}_0 = -k + \sum_{m=2}^{\infty} m v_m$. The contribution of these states to the partition function is then

$$Z_{0,1}(\tau) = |\bar{q}q|^{-k} \frac{1}{\prod_{n=2}^{\infty} |1 - q^n|^2}. \tag{2.23}$$

It is convenient to introduce the Dedekind η function, defined by

$$\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n). \tag{2.24}$$

Eqn. (2.23) can then be rewritten

$$Z_{0,1}(\tau) = \frac{1}{|\eta(\tau)|^2} |\bar{q}q|^{-(k-1/24)} |1 - q|^2. \tag{2.25}$$

This formula will be useful in Sec. 3 because $(\operatorname{Im}\tau)^{1/2}|\eta(\tau)|^2$ is modular-invariant.

We will show shortly how the Virasoro descendants of eqn. (2.22) arise in an analysis along the lines of Brown and Henneaux. The argument will also show that the formula (2.23) is exact to all order of perturbation theory. As for whether there are non-perturbative corrections to eqn. (2.23), it is not clear whether this question is well-defined, since it may be impossible to separate the question of unknown nonperturbative corrections to $Z_{0,1}$ from the more general question of unknown nonperturbative corrections to the exact partition function Z. At any rate, it will be easier to discuss what nonperturbative corrections would mean after explaining why eqn. (2.23) agrees with perturbation theory.

Before presenting this argument, let us discuss what the answer means from the general point of view of quantum mechanical perturbation theory. Usually, one defines an effective action I_{eff} such that the partition function is $Z = \exp(-k I_{\mathrm{eff}})$. I_{eff} is equal to the classical action I plus contributions generated from r-loop diagrams:

$$I_{\mathrm{eff}} = I + \sum_{r=1}^{\infty} k^{-r} I_r. \tag{2.26}$$

Here I_r is a function generated from Feynman diagrams with r loops. In a general theory, the expansion (2.26) is only asymptotic in $1/k$; there may be further contributions that are non-perturbatively small for $k \to \infty$. However, in the present context, the formula (2.23) implies that (for perturbation theory around the manifold $M_{0,1}$) there are no corrections beyond one-loop order; I_{eff} is simply the classical action plus a one-loop correction. In other

words, the formula implies that if $Z_{0,1}$ is directly computed in perturbation theory, then the perturbation series terminates with the one-loop term. (Such a direct computation is also briefly discussed below.)

In a sense, this should not come as a surprise. The gauge theory interpretation of three-dimensional gravity [18,19] (or even more naively the absence of any local excitations) suggests that in some sense it is an integrable system. It is often true for quantum integrable systems that properly chosen quantities are one-loop exact.

Derivation Of The Formula

In most of physics, the quantum Hilbert space of a theory is constructed by quantizing an appropriate classical phase space. This framework may ultimately be inadequate for quantum gravity, but it is certainly adequate for constructing the perturbative Hilbert space that is needed to compute $Z_{0,1}$ in perturbation theory.

The phase space of any physical theory is simply the space of its classical solutions. In $2 + 1$-dimensional gravity with AdS boundary conditions at infinity, naively speaking the phase space relevant to perturbation theory consists of only a single point, since any solution that is close to the AdS$_3$ solution (close enough to exclude black holes) actually is diffeomorphic to AdS$_3$.

However, we must be more careful here. In General Relativity, one should divide only by diffeomorphisms that approach the identity fast enough at infinity. After doing so, one constructs the classical phase space $\mathcal{M}$; it parametrizes classical solutions that obey the boundary conditions modulo diffeomorphisms that vanish fast enough at infinity. One is then left with an action of a group G that consists of those diffeomorphisms that preserve the boundary conditions, modulo those that vanish fast enough at infinity that they are required to act trivially on physical states.

In General Relativity in $3+1$ or more dimensions, in a spacetime that is asymptotically Lorentzian, the group G is the Poincaré group. This is why quantum gravity, in an asymptotically Lorentzian spacetime, is Poincaré invariant. In $2 + 1$-dimensional gravity with negative cosmological constant, the obvious analog of this answer would be $SL(2, \mathbb{R}) \times SL(2, \mathbb{R})$ (or a group locally isomorphic to that one), the group of symmetries of AdS$_3$. However, Brown and Henneaux showed [2] that the actual answer turns out to be the infinite-dimensional group $G = \text{diff}\, S^1 \times \text{diff}\, S^1$ (which contains $SL(2, \mathbb{R}) \times SL(2, \mathbb{R})$ as a subgroup). By computing Poisson brackets, they also showed that the quantum symmetry

group will be not G but a central extension with (in a semiclassical approximation) $c_L = c_R = 3\ell/2G$. We denote this central extension as $\widehat{\mathrm{diff}}\,S^1 \times \widehat{\mathrm{diff}}\,S^1$.

We will carry the analysis of [2] just slightly farther to describe the phase space $\mathcal{M}$ and the resulting energy levels. First of all, $\mathcal{M}$ is a homogeneous space for G, since if we divide by all diffeomorphisms, then the classical solutions that obey the boundary conditions (and are close enough to AdS_3) are equivalent to AdS_3. So $\mathcal{M}$ is G/H, where H is some subgroup of G. In fact, H is the subgroup of G that leaves fixed a given point on $\mathcal{M}$. Differently put, the symmetry group of a point on $\mathcal{M}$ is isomorphic to H. In the present context, we simply pick the point on $\mathcal{M}$ corresponding to AdS_3 and observe that its symmetry group is $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$. So this is H, and the phase space is $\mathcal{M} = (\mathrm{diff}\,S^1 \times \mathrm{diff}\,S^1)/(SL(2,\mathbb{R}) \times SL(2,\mathbb{R})) = (\mathrm{diff}\,S^1/SL(2,\mathbb{R}))^2$.

In general, the quantization of a homogeneous space G/H should be expected to give a Hilbert space that is an irreducible representation R of G (or more generally of a central extension of G). Moreover, as G/H contains an H-invariant point, the representation R will contain a vector that is an eigenvector for the action of H (and hence is H-invariant if H is a simple non-abelian group).

In the present problem, a representation of the Virasoro group $\widehat{\mathrm{diff}}\,S^1$ in which L_0 is bounded below and in which there is an $SL(2,\mathbb{R})$-invariant vector is uniquely determined once the central charge $c = 24k$ is given. It is the "vacuum" representation, containing a vector $|\Omega\rangle$ with $(L_n + k\delta_{n,0})|\Omega\rangle = 0$, $n \geq -1$. This representation is spanned by states of the form

$$\prod_{n=2}^{\infty} L_{-n}^{a_n}|\Omega\rangle, \tag{2.27}$$

with energy

$$\epsilon = -k + \sum_{n=2}^{\infty} na_n. \tag{2.28}$$

Quantizing $(\mathrm{diff}\,S^1/SL(2,\mathbb{R}))^2$ gives a tensor product of two such representations for the two factors of $G = (\mathrm{diff}\,S^1)^2$, and in this way we arrive at the spectrum claimed in eqn. (2.22), and hence at the formula (2.23) for the partition function.

Quantization of homogeneous spaces of $\mathrm{diff}\,S^1$ is described in [20]. The fact that quantization of the quotient $\mathrm{diff}\,S^1/SL(2,\mathbb{R})$ gives the vacuum representation of $\widehat{\mathrm{diff}}\,S^1$ is explained from several points of view in eqns. (170), (172), and (174) of that paper.

The states that we have described can be regarded as boundary excitations, supported near the boundary of AdS_3, since $\mathcal{M}$ collapses to a point if we are allowed to make general

changes of coordinate near the boundary. We call these boundary excitations the Brown-Henneaux (BH) states, since they are so closely related to the analysis in [2].

Corrections?

Now let us ask to what extent there may be corrections to the spectrum just described.

First we consider the question of quantum corrections to the formula (2.28) for this family of states. We can dispose of this question immediately: this particular family of states transforms in an irreducible representation of the symmetry group, and this representation has no possible deformations once the central charge is fixed.

So there are no quantum corrections to the energies of the states just described. Could there be additional states contributing to $Z_{0,1}$?

In general, in three-dimensional gravity, there may be additional states of various kinds, corresponding for instance to massive or even massless particles. What do we expect in "pure" three-dimensional gravity, if it exists? It is important to distinguish two length scales: the AdS$_3$ length scale ℓ, and the Planck length G. In the semiclassical regime of $k \to \infty$, we have $G << \ell$.

In eqn. (2.28), energies are measured in units of $1/\ell$. A reasonable minimum condition for anything that one might call a theory of pure gravity is that the energy of any state that is not part of the BH spectrum goes to infinity in the semiclassical limit $k \to \infty$ (when measured relative to the energy of the ground state). States that are not part of the BH spectrum and whose excitation energies remain fixed for $k \to \infty$ would be interpreted in terms of non-gravitational fields propagating in AdS$_3$.

As for what sort of states might have energies that go to infinity for $k \to \infty$, we certainly expect BTZ black holes, with energy bounded below by k. As for what else there may be, the sky is the limit in terms of conceivable speculations. If so inclined, one can postulate solitons with excitation energies proportional to k, states similar to D-branes with energies proportional to $k^{1/2}$, etc. Orbifold singularities with fixed deficit angle, as described in relation to eqn. (2.16), would also have excitation energy of order k.

Our point of view is that the question of whether there are nonperturbative corrections to $Z_{0,1}$ from states whose excitation energy diverges as $k \to \infty$ is hard to separate from the more general question of unknown nonperturbative corrections to the exact partition function Z. Since the perturbative evaluation of $Z_{0,1}$ gives a convergent and physically sensible result (which is even one-loop exact), we may as well regard hypothetical contributions in which $M_{0,1}$ is enriched with a soliton, a D-brane, an orbifold singularity, or some

other unknown type of excitation as representing different, presently unknown sectors of the path integral. In this sense, the formula for $Z_{0,1}$ is exact.

Comparison To Perturbation Theory

Since we claim that the formula for $Z_{0,1}$ is one-loop exact, the question arises of why not to simply calculate it by evaluating the relevant one-loop determinants.

We know of two efforts to do so. In [16], a formula much simpler than our result (2.23) is claimed. The analysis relies on the relation of the relevant product of determinants to Ray-Singer and Reidemeister torsion [11]. For an irreducible flat connection on a compact manifold without boundary, the torsion is simply a number, but for a manifold with boundary (such as $M_{0,1}$ effectively is), the torsion must be understood as measure on a certain moduli space associated with the boundary. A rather subtle treatment of this is needed, we suspect, to compute the one-loop correction using its interpretation via torsion.

On the other hand, in [21], the one-loop correction was studied by expressing the determinants in terms of the appropriate heat kernels, which were evaluated using a method of images (analogous to the derivation of the Selberg trace formula). We believe that this method is conceptually completely correct. The claimed result is qualitatively similar to our formula (2.23) (for example, it has an expansion in integer powers of q and $\bar{q}$), and we hope that it will prove possible to resolve any discrepancies between the result in [21] and the one claimed here.

As for why it is much simpler to compute the one-loop correction via the Hamiltonian route that we have followed, this should not really come as a surprise. In general, path integrals on a product $S^1 \times Y$ are often most easily evaluated by constructing an appropriate Hilbert space in quantization on Y and then taking a trace.[3] That is especially likely to be true in the present situation, in which the relevant excitations are rather subtle boundary excitations, already known but difficult to rediscover.

---

[3] An excellent example of this statement is given by Chern-Simons gauge theory with a compact gauge group for the case that Y is a compact Riemann surface without boundary. The partition function on $S^1 \times Y$ is an integer, the dimension of the physical Hilbert space $\mathcal{H}$ associated with Y. It is not too hard to describe $\mathcal{H}$, compute its dimension, and thereby learn the value of the path integral on $S^1 \times Y$. But direct evaluation of this path integral by Lorentz-covariant methods is difficult, even in perturbation theory.

2.3. The General Form Of The Partition Function

We conclude with an analysis of the following question: in general, what sort of function can be written as $\mathrm{Tr}\exp(-\beta H)$, where H is a hermitian operator on a Hilbert space $\mathcal{H}$? We assume that H is constrained so that $\mathrm{Tr}\exp(-\beta H)$ is convergent whenever $\mathrm{Re}\,\beta > 0$. (According to standard assumptions about pure three-dimensional gravity, this condition is satisfied, but we discuss in Sec. 4.1 one way that it might fail.)

For $\mathrm{Tr}\exp(-\beta H)$ to be convergent, the number n_E of states of energy less than E must be finite for each E. Indeed, the trace is bounded below by $n_E\exp(-\beta E)$. Finiteness of n_E for all E implies in particular that H must have a discrete spectrum.

Let E_* be any value of the energy and let $E_1, \ldots, E_n$ be the energy eigenvalues that are no greater than E_*. Their contribution to the partition function is $\sum_{i=1}^{n}\exp(-\beta E_i)$, and the full partition function is therefore

$$\mathrm{Tr}\exp(-\beta H) = \sum_{i=1}^{n}\exp(-\beta E_i) + \mathcal{O}(\exp(-\beta E_*)). \tag{2.29}$$

Here $\mathcal{O}(\exp(-\beta E_*))$ is a function that is bounded by a multiple of $\exp(-\beta E_*)$.

We have to be careful here with one point: any finite sum of states of energy greater than E_* makes a contribution of the form $\mathcal{O}(\exp(-\beta E_*))$, but in general an infinite set of states with energy $\geq E_*$ may make a contribution that is not bounded in this way. However, our hypothesis that $\mathrm{Tr}\exp(-\beta H)$ converges whenever $\mathrm{Re}\,\beta > 0$ ensures that the number n_E of states of energy less than E grows with E more slowly than any exponential. This is enough to justify the error estimate in eqn. (2.29).

Our problem is slightly more general. We have a pair of commuting hermitian operators H and J, and we are computing the trace $\mathrm{Tr}\exp(-\beta H - i\theta J)$. However, the eigenvalues of J are integers, so

$$\mathrm{Tr}\exp(-\beta H - i\theta J) = \sum_{j\in\mathbb{Z}} e^{-ij\theta}\,\mathrm{Tr}_j\,\exp(-\beta H), \tag{2.30}$$

where Tr_j is a trace in the subspace $\mathcal{H}_j$ in which J acts with eigenvalue j. The functions $\mathrm{Tr}_j\exp(-\beta H)$ should be constrained exactly as in eqn. (2.29).

In Sec. 3, we will find instead that the evaluation of known contributions to $\mathrm{Tr}_0\exp(-\beta H)$ take the form just described up to a certain energy E_*, beyond which this form breaks down. The result looks like

$$\sum_{i=1}^{n}\exp(-\beta E_i) + \exp(-\beta E_*)\Big(f + \sum_{s=1}^{\infty} f_s\beta^{-s}\Big) \tag{2.31}$$

where f is a negative integer, rather than being positive as it should be, and the corrections decay as power laws, rather than exponentials. (The series in (2.31) may be only asymptotic.) Both the fact that f is negative and the fact that there are power-law corrections mean that this function cannot be interpreted as Tr $\exp(-\beta H)$.

Finally, let us compare the assertion that Tr $\exp(-\beta H)$ can only converge if H has a discrete spectrum with standard results about the thermodynamics of physically sensible systems with continuous spectrum, such as a free gas of particles on the real line. What is usually computed in such a case is the free energy per unit volume; the total free energy is infinite simply because the volume is infinite. To make the total free energy finite, one can place the gas in a finite volume; then the spectrum is discrete and Tr $\exp(-\beta H)$ converges. A gas of particles in Anti-de Sitter space will have a discrete spectrum and finite partition function Tr $\exp(-\beta H)$ as well. This is because the motion of any finite energy particle in Anti-de Sitter space is restricted to a finite volume region in the interior of AdS.

3. Computing the Sum over Geometries

As explained in Secs. 1 and 2, the known contributions to the partition function of pure gravity in a spacetime asymptotic to AdS_3 come from smooth geometries $M_{c,d}$, where c and d are a pair of relatively prime integers (with a pair c, d identified with $-c, -d$). Their contribution to the partition function, including the contribution from the Brown-Henneaux excitations, is

$$Z(\tau) = \sum_{c,d} Z_{0,1}(\gamma\tau), \tag{3.1}$$

where

$$\gamma\tau = \frac{a\tau + b}{c\tau + d}, \qquad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}) \tag{3.2}$$

and

$$Z_{0,1}(\tau) = \left| q^{-k} \prod_{n=2}^{\infty} (1 - q^n)^{-1} \right|^2 = \frac{|\bar{q}q|^{-k+1/24}|1 - q|^2}{|\eta(\tau)|^2}. \tag{3.3}$$

The summation in (3.1) is over all relatively prime c and d with $c \geq 0$. Since $Z_{0,1}(\tau)$ is invariant under $\tau \to \tau + 1$, the summand in (3.1) is independent of the choice of a and b in (3.2). This sum over c and d in (3.1) should be thought of as a sum over the coset $PSL(2,\mathbb{Z})/\mathbb{Z}$, where $\mathbb{Z}$ is the subgroup of $PSL(2,\mathbb{Z}) = SL(2,\mathbb{Z})/\{\pm 1\}$ that acts by

$\tau \rightarrow \tau + n$, $n \in \mathbb{Z}$. Given any function of τ, such as $Z_{0,1}(\tau)$, that is invariant under $\tau \rightarrow \tau + 1$, one may form a sum such as (3.1), known as a Poincaré series.

The function $\sqrt{\operatorname{Im} \tau}|\eta(\tau)|^2$ is modular-invariant. We can therefore write $Z(\tau)$ as a much simpler-looking Poincaré series,

$$Z(\tau) = \frac{1}{\sqrt{\operatorname{Im} \tau}|\eta(\tau)|^2} \sum_{c,d} \left(\sqrt{\operatorname{Im} \tau}|\bar{q}q|^{-k+1/24}|1-q|^2 \right)\Big|_\gamma , \tag{3.4}$$

where $(\ldots)|_\gamma$ is the transform of an expression $(\ldots)$ by γ. Writing out explicitly $|1-q|^2 = 1 - q - \bar{q} + \bar{q}q$, we see that we really need a sum of four Poincaré series, each of the form

$$E(\tau; n, m) = \sum_{c,d} \left(\sqrt{\operatorname{Im} \tau} q^{-n} \bar{q}^{-m} \right)\Big|_\gamma , \tag{3.5}$$

with $n - m$ equal to 0 or ± 1. Precisely this sum, or rather its s-dependent generalization introduced below, has been studied in Sec. 3.4 of [22], as was pointed out by P. Sarnak.[4] If we set $\kappa = n + m$, $\mu = m - n$, and use the fact that $\operatorname{Im}(\gamma\tau) = \operatorname{Im}\tau/|c\tau + d|^2$, then the basic Poincaré series can be written

$$E(\tau; \kappa, \mu) = \sqrt{\operatorname{Im} \tau} \sum_{c,d} |c\tau + d|^{-1} \exp\{2\pi\kappa \operatorname{Im} \gamma\tau + 2\pi i\mu \operatorname{Re} \gamma\tau\}. \tag{3.6}$$

When $\kappa = 0$ and $\mu = 0$, this sum is a non-holomorphic Eiseinstein series of weight $1/2$. Sometimes we omit τ and write just $E(\kappa, \mu)$. In terms of this function, the partition function is

$$Z(\tau) = \frac{1}{\sqrt{\operatorname{Im} \tau}|\eta(\tau)|^2} \big(E(2k - 1/12, 0) + E(2k + 2 - 1/12, 0) \\ - E(2k + 1 - 1/12, 1) - E(2k + 1 - 1/12, -1)\big). \tag{3.7}$$

3.1. The Regularized Sum

The sums (3.6) that define the Poincaré series we need are divergent. (Such divergences have been encountered before in similar sums related to three-dimensional gravity [10,23,14].) Using

$$\frac{a\tau + b}{c\tau + d} = \frac{a}{c} - \frac{1}{c(c\tau + d)}, \tag{3.8}$$

---

[4] Actually, the generalization to arbitrary integer values of $n - m$ is considered in [22]. We would encounter the same generalization if we modify the original Einstein-Hilbert action (1.1) to include the gravitational Chern-Simons term.

one may show that if $\tau = x + iy$ then

$$\begin{aligned}
\mathrm{Im}\,(\gamma\tau) &= \frac{y}{(cx+d)^2 + c^2 y^2} \\
\mathrm{Re}\,(\gamma\tau) &= \frac{a}{c} - \frac{cx+d}{c((cx+d)^2 + c^2 y^2)}.
\end{aligned} \tag{3.9}$$

For $\mu = 0$, the exponential factor in the definition of E is

$$\exp\left\{2\pi\kappa\;\mathrm{Im}\;\gamma\tau\right\} = \exp\left\{2\pi\kappa\;\frac{y}{(cx+d)^2 + c^2 y^2}\right\}, \tag{3.10}$$

and this goes to 1 for $c, d \to \infty$. So the first two terms in (3.7) diverge linearly as $\sum_{c,d} |c\tau + d|^{-1}$ at large c and d. The other terms also diverge, though more slowly.

We claim that this divergence has a natural regularization. On the upper half plane, which we call $\mathcal{H}$, there is a natural $SL(2, \mathbb{R})$-invariant Laplacian:

$$\Delta = -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right). \tag{3.11}$$

A short calculation shows that the function $y^{1/2}$ is an eigenfunction of Δ: $\Delta(y^{1/2}) = (1/4)y^{1/2}$. Since Δ is $SL(2, \mathbb{R})$-invariant, the same is true of $(\mathrm{Im}\,\gamma\tau)^{1/2}$ for any $\gamma \in SL(2, \mathbb{Z})$ (or even $SL(2, \mathbb{R})$):

$$\Delta\,\sqrt{\mathrm{Im}\,(\gamma\tau)} = \frac{1}{4}\sqrt{\mathrm{Im}\,(\gamma\tau)}. \tag{3.12}$$

Using this, one may verify that although the Poincaré series for $E(\tau; \kappa, \mu)$ is divergent, the corresponding series for $(\Delta - 1/4)E(\tau; \kappa, \mu)$ actually converges. (This series is obtained by acting termwise with $(\Delta - 1/4)$ on the Poincaré series for E.) To see this, we just observe, using (3.10), that contributions in which derivatives appearing in Δ act on the exponential factor in (3.6) get an extra convergence factor of order $1/(\max(c,d))^2$ and lead to a convergent sum over c and d. On the other hand, contributions in which none of the derivatives act on the exponential actually vanish, because of (3.12).

So $(\Delta - 1/4)E$ requires no regularization. Similarly, if we set $F = \sqrt{\mathrm{Im}\,\tau}|\eta|^2$, then $(\Delta - 1/4)(FZ)$ requires no regularization, where Z is the partition function. Since no regularization is required, we assume that the naive sum gives $(\Delta - 1/4)(FZ)$ correctly.

The Laplacian Δ, acting on the Hilbert space of square-integrable $SL(2, \mathbb{Z})$-invariant functions on $\mathcal{H}$, has a continuous spectrum starting at $1/4$. It also has a discrete spectrum, but there are no discrete modes with eigenvalue $1/4$. So the operator $\Delta - 1/4$ is invertible

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acting on square-integrable functions, and roughly speaking, we now want to argue that once $(\Delta - 1/4)(FZ)$ is known, FZ is uniquely determined.

There is a subtlety in making this argument, because actually Z and FZ are not themselves square-integrable. Rather, we expect that Z grows exponentially for $y \to \infty$, because the AdS$_3$ vacuum has negative energy $-2k$. There are also Brown-Henneaux excitations of this vacuum, again with negative energy, and they contribute additional exponentially growing terms for $y \to \infty$. If we assume that all exponentially growing contributions to Z come from the known Brown-Henneaux states, then this, together with a knowledge of $(\Delta - 1/4)(FZ)$, is enough to determine Z uniquely.

But for our main application, we do not really need this assumption (which appears to be not quite correct according to the analysis of Sec. 3.3). The explicit calculation of Secs. 3.2 and 3.3 will give us a function Z, compatible with the Poincaré series for $(\Delta - 1/4)(FZ)$, such that the fastest growing exponentials agree with the Brown-Henneaux spectrum. The leading departure from the contributions of that spectrum are determined at the end of Sec. 3.3 and take the form

$$Z' = \frac{1}{|\eta|^2}\Big(-6 + \frac{(\pi^3 - 6\pi)(11 + 24k)}{9\zeta(3)}y^{-1}$$
$$+ \frac{5(53\pi^6 - 882\pi^2) + 528(\pi^6 - 90\pi^2)k + 576(\pi^6 - 90\pi^2)k^2}{2430\zeta(5)}y^{-2} + \mathcal{O}(y^{-3})\Big).$$

(3.13)

This function cannot be written as a sum of exponentials with positive integer coefficients, since the leading coefficient is negative, and the corrections do not have the right form. Any other candidate for Z would be obtained by adding a correction $\widetilde{Z}$ such that $(\Delta - 1/4)(F\widetilde{Z}) = 0$. But that equation is not obeyed by any function $\widetilde{Z}$ that would solve our problem; the equation $(\Delta - 1/4)(F\widetilde{Z}) = 0$ is not obeyed by $-Z'$ or by any function that differs from it for $y \to \infty$ by a sum of exponentials with positive integer coefficients. So the problem that we will find is not affected by adding to the function Z that we compute an additional contribution $\widetilde{Z}$ that obeys $(\Delta - 1/4)(F\widetilde{Z}) = 0$.

More Convenient Alternative

Although this argument is satisfying conceptually, it does not give a convenient way to determine Z in practice. A much more convenient method is to adapt ζ-function regularization [11] to this problem.

In the present context, the analog of ζ-function regularization is to replace the Poincaré series (3.5) by a more general one depending on a parameter s:

$$E(\tau; s, n, m) = \sum_{c,d} \left((\operatorname{Im}\tau)^s q^{-n} \bar{q}^{-m} \right)\big|_\gamma . \tag{3.14}$$

Since $(\operatorname{Im}\gamma\tau)^s = y^s/|c\tau + d|^{2s}$, the series $E(\tau; s, n, m)$ converges for $\operatorname{Re} s > 1$. Our original problem concerns the case $s = 1/2$. As we will see shortly, $E(\tau; s, n, m)$, defined initially for $\operatorname{Re} s > 1$, can be analytically continued to $s = 1/2$ without any problem. This analytic continuation gives a natural way to define the original function $E(\tau; n, m)$ and therefore the partition function Z. This turns out to be a very practical and useful method to study $E(\tau; n, m)$.

The only problem with this approach is that the physical meaning of the parameter s in three-dimensional gravity is unclear; hence, it is not clear *a priori* that the analytic continuation will give the right answer. The argument involving $(\Delta - 1/4)(FZ)$ is clearer conceptually, because $(\Delta - 1/4)(FZ)$ is a physical observable and we simply use the fact that the formal path integral expression for it converges.

However, it is not difficult to show that the two methods give the same result. For this, we observe that the function $H(\tau; s, m, n) = (\Delta + s(s - 1))E(\tau; s, m, n)$ can also be represented by a Poincaré series, convergent when $\operatorname{Re} s > 0$. For $\operatorname{Re} s > 1$, where both Poincaré series converge, the two functions E and H, both defined by their Poincaré series, obey $H = (\Delta + s(s - 1))E$. This automatically remains true after analytic continuation to $s = 1/2$. So the function $E(\tau; 1/2, m, n)$ defined by analytic continuation from $\operatorname{Re} s > 1$ has the property that $(\Delta - 1/4)E(\tau; 1/2, m, n)$ is given by the obvious Poincaré series.

In terms of $\kappa = m + n$, $\mu = m - n$, the regularized Poincaré series is

$$E(s, \kappa, \mu) = \sum_{c,d} \frac{y^s}{|c\tau + d|^{2s}} \exp\left\{ 2\pi\kappa \operatorname{Im} \gamma\tau + 2\pi i\mu \operatorname{Re} \gamma\tau \right\} . \tag{3.15}$$

When $\kappa = 0$ and $\mu = 0$, this sum is the non-holomorphic Eisenstein series of weight s.

3.2. Poisson Resummation

Sec. 3.4 of [22] contains precisely what we need to analyze the sum (3.15), make the analytic continuation, and determine if the partition function Z is physically sensible.

First, define $d = d' + nc$, where n is an integer and d' runs from 0 and $c - 1$. We may separate out the sum over n in (3.15) to get

$$E(s, \kappa, \mu) = y^s e^{2\pi(\kappa y + i\mu x)} + \sum_{c>0} \sum_{d' \in \mathbb{Z}/c\mathbb{Z}} \sum_{n \in \mathbb{Z}} f(c, d', n) \qquad (3.16)$$

where

$$f(c, d', n) = \frac{y^s}{|c(\tau + n) + d'|^{2s}} \exp\left\{ \frac{2\pi\kappa y}{|c(\tau + n) + d'|^2} + 2\pi i\mu \left(\frac{a}{c} - \frac{cx + d}{c|c(\tau + n) + d'|^2} \right) \right\}. \qquad (3.17)$$

The first term in eqn. (3.16) comes from $c = 0, d = 1$.

The Poisson summation formula allows us to turn the sum over n into a sum over a Fourier conjugate variable $\hat{n}$

$$\sum_{n \in \mathbb{Z}} f(c, d', n) = \sum_{\hat{n} \in \mathbb{Z}} \hat{f}(c, d', \hat{n}) \qquad (3.18)$$

where $\hat{f}(c, d', \hat{n})$ is the Fourier transform

$$\hat{f}(c, d', \hat{n}) = \int_{-\infty}^{\infty} dn \ e^{2\pi i n \hat{n}} f(c, d', n)$$

$$= \exp\left(2\pi i \left(\frac{\mu a - \hat{n} d'}{c} - \hat{n} x \right) \right) \int_{-\infty}^{\infty} dt \ e^{2\pi i \hat{n} t} \left(\frac{y}{c^2(t^2 + y^2)} \right)^s \exp\left\{ \frac{2\pi(\kappa y - i\mu t)}{c^2(t^2 + y^2)} \right\}. \qquad (3.19)$$

We have written the integral in terms of a shifted integration variable $t = n + x + \frac{d'}{c}$. Upon Taylor expanding the exponential that appears in the integral and introducing $T = t/y$, we get

$$\hat{f}(c, d', \hat{n}) = \sum_{m=0}^{\infty} c^{-2(s+m)} e^{2\pi i \left(\frac{\mu a - \hat{n} d'}{c} - \hat{n} x \right)} \frac{(2\pi)^m}{m!} \int_{-\infty}^{\infty} dt \ e^{2\pi i \hat{n} t} \left(\frac{y}{t^2 + y^2} \right)^{m+s} \left(\kappa - i\mu \frac{t}{y} \right)^m$$

$$= \sum_{m=0}^{\infty} c^{-2(s+m)} e^{2\pi i \left(\frac{\mu a - \hat{n} d'}{c} - \hat{n} x \right)} \frac{(2\pi)^m}{m!} y^{1-m-s} \int_{-\infty}^{\infty} dT \ e^{2\pi i \hat{n} T y} \left(1 + T^2 \right)^{-m-s} \left(\kappa - i\mu T \right)^m. \qquad (3.20)$$

In (3.20), c and d' do not appear in the integrals but only in the elementary prefactors, so we can study the sums over c and d' explicitly. Note that, as μ is an integer, (3.20) depends on a only modulo c. The value of a mod c is determined by d', given that $ad' = 1$ mod c. For a given d', such an a exists if and only if d' lies in the set $(\mathbb{Z}/c\mathbb{Z})^*$ of residue

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classes mod c that are invertible multiplicatively. So, dropping the prime from d', we may write the sum over that variable as

$$S(-\hat{n}, \mu; c) = \sum_{d \in (\mathbb{Z}/c\mathbb{Z})^*} \exp\left\{ 2\pi i \left(\frac{-\hat{n}d + \mu d^{-1}}{c} \right) \right\}, \qquad (3.21)$$

where $d^{-1} \in (\mathbb{Z}/c\mathbb{Z})^*$ is the multiplicative inverse of d. This sum is known as a Kloosterman sum.

Rearranging the sums in (3.16), we now have

$$E(s, \kappa, \mu) = y^s e^{2\pi(\kappa y + i\mu x)} + \sum_{\hat{n}} e^{-2\pi i \hat{n}x} E_{\hat{n}}(s, \kappa, \mu) \qquad (3.22)$$

where

$$E_{\hat{n}}(s, \kappa, \mu) = \sum_{m=0}^{\infty} I_{m,\hat{n}}(s, \kappa, \mu)\, y^{1-m-s} \left(\sum_{c=1}^{\infty} c^{-2(m+s)} S(-\hat{n}, \mu; c) \right). \qquad (3.23)$$

Here we have defined the integral

$$I_{m,\hat{n}}(s, \kappa, \mu) = \frac{(2\pi)^m}{m!} \int_{-\infty}^{\infty} dT\, e^{2\pi i \hat{n}Ty} \left(1 + T^2 \right)^{-m-s} \left(\kappa - i\mu T \right)^m. \qquad (3.24)$$

Note that (3.23) is independent of x, so that (3.22) has the form of a Fourier expansion in x with Fourier coefficients $E_{\hat{n}}(s, \kappa, \mu)$ given by (3.23). These Fourier coefficients are typically complicated functions of y, since the integral (3.24) depends on y.

These Fourier coefficients are precisely what we want. In view of (3.7), the Fourier expansion of the function E with respect to x will give a similar expansion for the partition function Z. The Fourier coefficients of Z are the functions $\mathrm{Tr}_j \exp(-\beta H)$, the partition function restricted to states with angular momentum $J = j$. These are the functions that we want to understand.

We are almost ready to analyze what happens when we continue the formulas to $s = 1/2$. For $m > 0$, the integral in (3.24) is convergent for $\mathrm{Re}\, s > 0$, and likewise the sum

$$\sum_{c=1}^{\infty} c^{-2(m+s)} S(-\hat{n}, \mu; c) \qquad (3.25)$$

converges for $\mathrm{Re}\, s > 0$. A problem does occur for $m = 0$, since then neither the integral nor the sum is convergent at $s = 1/2$. We return to this shortly.

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3.3. The $\hat{n} = 0$ Mode

Let us first consider the Fourier mode which is constant in x, i.e. the $\hat{n} = 0$ term in (3.22). In this case the integral (3.24) is independent of y, and may be evaluated explicitly. For $\mu = 0$, the result can be expressed in terms of Γ functions

$$I_{m,0}(s, \kappa, 0) = \kappa^m \frac{2^m \pi^{m+1/2} \Gamma(s + m - 1/2)}{m! \Gamma(s + m)}, \tag{3.26}$$

while for $\mu = \pm 1$, we require also hypergeometric functions

$$
\begin{aligned}
I_{m,0}(s, \kappa, \pm 1) = {} & \cos\left(\frac{m\pi}{2}\right) \frac{(2\pi)^m \Gamma\left(\frac{1 \pm m}{2}\right) \Gamma\left(\frac{m-1}{2} + s\right)}{m! \Gamma(m+s)} {}_2F_1\left(\frac{m-1}{2} + s, -\frac{m}{2} + s; \frac{1}{2}; \kappa^2\right) \\
& + m\kappa \sin\left(\frac{m\pi}{2}\right) \frac{(2\pi)^m \Gamma\left(\frac{m}{2}\right) \Gamma\left(\frac{m}{2} + s\right)}{m! \Gamma(m+s)} {}_2F_1\left(\frac{1-m}{2}, \frac{m}{2} + s; \frac{3}{2}; \kappa^2\right).
\end{aligned}
\tag{3.27}
$$

When $m = 0$, this formula simplifies to

$$I_{0,0}(s, \kappa, \pm 1) = \sqrt{\pi} \frac{\Gamma(s - 1/2)}{\Gamma(s)}. \tag{3.28}$$

The sum over c may also be evaluated exactly (see Sec. 2.5 of [22]). For $\mu = 0$ this evaluation involves the Kloosterman sum $S(0, 0; c)$; from the definition (3.21), one can see that $S(0, 0; c)$ is equal to the Euler totient function $\phi(c)$ (which is defined as the number of positive integers less than c that are relatively prime to c). The sum over c is a standard one

$$\sum_{c=1}^{\infty} c^{-2(m+s)} S(0, 0; c) = \sum_{c>0} c^{-2(m+s)} \phi(c) = \frac{\zeta(2(m+s) - 1)}{\zeta(2(m+s))}. \tag{3.29}$$

This formula can be obtained as follows. We start by noting a basic property of the totient function: for any n, $\sum_{d|n} \phi(d) = n$. In order to evaluate the sum $\sum_c c^{-\sigma} \phi(c)$, let us multiply this sum by $\zeta(\sigma) = \sum_n n^{-\sigma}$. This gives

$$
\begin{aligned}
\zeta(\sigma) \sum_{c=1}^{\infty} c^{-\sigma} \phi(c) &= \sum_{n,c=1}^{\infty} (nc)^{-\sigma} \phi(c) \\
&= \sum_{m=1}^{\infty} m^{-\sigma} \sum_{c|m} \phi(c) \\
&= \sum_{m=1}^{\infty} m^{1-\sigma} = \zeta(\sigma - 1).
\end{aligned}
\tag{3.30}
$$

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Setting $\sigma = 2(s+m)$ gives (3.29).

For $\mu = \pm 1$, the Kloosterman sum becomes a special case of what is known as Ramanujan's sum (see Sec. 2.5 of [22]):

$$S(\hat{n}, 0; c) = \sum_{d \in (\mathbb{Z}/c\mathbb{Z})^*} e^{2\pi i d/c} = \mu(c). \tag{3.31}$$

Here $\mu(c)$ is the Möbius function, which is defined as follows: $\mu(c) = 0$ if c is not square-free, while if c is the product of k distinct prime numbers, then $\mu(c) = (-1)^k$. The sum over c is given by

$$\sum_{c=1}^{\infty} c^{-2(s+m)} S(0, \pm 1; c) = \sum_{c=1}^{\infty} c^{-2(s+m)} \mu(c) = \frac{1}{\zeta(2(s+m))}. \tag{3.32}$$

To prove this, one may use the basic property of the Möbius function: for any n, $\sum_{d|n} \mu(d) = \delta_{n,1}$. To compute $\sum_c c^{-\sigma} \mu(c)$, we multiply this sum by $\zeta(\sigma)$ to get

$$
\begin{aligned}
\zeta(\sigma) \sum_{c=1}^{\infty} c^{-\sigma} \mu(c) &= \sum_{n,c=1}^{\infty} (cn)^{-\sigma} \mu(c) \\
&= \sum_{m=1}^{\infty} m^{-\sigma} \sum_{c|m} \mu(c) \\
&= \sum_{m=1}^{\infty} m^{-\sigma} \delta_{m,1} = 1.
\end{aligned}
\tag{3.33}
$$

Setting $\sigma = 2(m+s)$ gives (3.32).

Putting these formulae together gives an exact expression for the x independent part of our Poincaré series (3.16):

$$E_0(\hat{s}, \kappa, \mu) = \sum_{m=0}^{\infty} w_m(s, \kappa, \mu) y^{1-m-s}. \tag{3.34}$$

The constants $w_m(s, \kappa, \mu)$ in this expansion are independent of x and y, and are given explicitly by equations (3.26), (3.27), (3.29) and (3.32) – we will write them out in more detail for the cases of interest below. The expansion (3.34) is one of our key results. It is an explicit series expansion of our Poincaré series in powers of y^{-1}

Now we can study the behavior at $s = 1/2$ of the delicate contribution with $m = 0$. For $\mu = 0$, this term is finite because the factor of $\Gamma(s - 1/2)$ appearing in (3.26) and

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the factor of $\zeta(2s)$ appearing in (3.29) both have simple poles at $s = 1/2$; as $s \to 1/2$, $\Gamma(s - 1/2)/\zeta(2s) \to 2$. In fact, all of the complicated factors cancel to give

$$E_0(1/2, \kappa, 0) = -y^{1/2} + \mathcal{O}(y^{-1/2}). \tag{3.35}$$

The coefficient of the leading term is negative because $\zeta(0) = -1/2$. For $\mu = \pm 1$, the $m = 0$ term is finite because the factors of $\Gamma(s - 1/2)$ in (3.28) and $\zeta(2s)$ in (3.32) both have simple poles at $s = 1/2$. We find

$$E_0(1/2, \kappa, \pm 1) = 2y^{1/2} + \mathcal{O}(y^{-1/2}). \tag{3.36}$$

Going back to (3.22), and writing simply $E(\kappa, \mu)$ for $E(1/2, \kappa, \mu)$, we now have

$$E(\kappa, 0) = y^{1/2} \exp(2\pi\kappa y) - y^{1/2} + \mathcal{O}(y^{-1/2}). \tag{3.37}$$

Actually, we have not yet discussed the Fourier modes with $\hat{n} \neq 0$; however, the integrals (3.24) vanish exponentially for $y \to \infty$ and so do not affect the assertion in eqn. (3.37). Likewise, we have

$$E(\kappa, \pm 1) = y^{1/2} \exp(2\pi(\kappa y \pm ix)) + 2y^{1/2} + \mathcal{O}(y^{-1/2}). \tag{3.38}$$

If we evaluate eqn. (3.7) for the partition function Z keeping only the first terms in eqns. (3.37) and (3.38) – the exponentially growing terms – then the formula for Z simply reduces to $Z_{0,1}$, the contribution of the Brown-Henneaux states. What we have gained from all the work that we have done is that we can now calculate corrections to $Z_{0,1}$. The leading corrections come from the corrections to the exponential terms in (3.37) and (3.38). Adding them up and evaluating (3.7), we get

$$Z = Z_{0,1} + \frac{1}{|\eta|^2} \left(-6 + \mathcal{O}(y^{-1})\right). \tag{3.39}$$

Since $1/|\eta|^2 \sim |q|^{-1/12}$, Z is governed by the Brown-Henneaux spectrum up to energy $-1/12$ (slightly below the classical black hole threshold, which is at zero energy). The number of states at that energy is not a positive integer, as one would hope, but rather -6.

Moreover, the derivation shows that the corrections are given by a power series in $1/y$, not a sum of exponentials. As was explained in Sec. 2.3, for Z to have an interpretation as $\text{Tr} \exp(-\beta H)$, the corrections must be given by exponentials.

Thus we have arrived at the main conclusion of this paper: the sum of known contributions to the partition function of pure three-dimensional gravity is not physically sensible.

It is illustrative to compute the next few terms in the expansion of the partition function (3.39) in powers of y. This means taking $m \neq 0$ in equations (3.26), (3.27), (3.29) and (3.32). Unlike the $m = 0$ case, these expressions do not have poles at $s = 1/2$, so the computation is straightforward. Let us start with the $\mu = 0$ case. We find that the coefficients $w_m(s, \kappa, \mu)$ are given by

$$w_m(1/2, \kappa, 0) = \frac{2^m \pi^{m+1/2} \zeta(2m)}{m\Gamma(m+1/2)\zeta(2m+1)} \kappa^m. \tag{3.40}$$

So the next few terms in the Poincaré series (3.34) are

$$E_0(1/2, \kappa, 0) = -y^{1/2} + \left(\frac{2\pi^3}{3\zeta(3)}\kappa\right) y^{-1/2} + \left(\frac{4\pi^6}{135\zeta(5)}\kappa^2\right) y^{-3/2} + \mathcal{O}(y^{-5/2}) \tag{3.41}$$

To evaluate the $\mu = \pm 1$ terms, note that at $s = 1/2$ the first two arguments of the relevant hypergeometric function appearing in (3.27) are integers. This means that the formula (3.27) simplifies considerably at $s = 1/2$ – it is just a polynomial in κ. It is

$$I_{m,0}(\frac{1}{2}, \kappa, \pm 1) = \frac{2\pi^{m+1/2}}{m\Gamma(m+1/2)} T_m(\kappa) \tag{3.42}$$

where $T_m(\kappa)$ denotes a Chebyshev polynomial of the first kind. So the coefficients appearing in (3.34) are

$$w_m(1/2, \kappa, \pm 1/2) = \frac{2\pi^{m+1/2}}{m\Gamma(m+1/2)\zeta(2m+1)} T_m(\kappa). \tag{3.43}$$

This allows us to write down the next few terms in the series:

$$E_0(1/2, \kappa, \pm 1) = 2y^{1/2} + \left(\frac{4\pi}{\zeta(3)}\kappa\right) y^{-1/2} + \left(\frac{4\pi^2}{3\zeta(5)}(2\kappa^2 - 1)\right) y^{-3/2} + \mathcal{O}(y^{-5/2}). \tag{3.44}$$

Going back to (3.22), and writing $E(\kappa, \mu)$ for $E(1/2, \kappa, \mu)$, we have

$$E(\kappa, 0) = y^{1/2} \exp(2\pi\kappa y) - y^{1/2} + \left(\frac{2\pi^3}{3\zeta(3)}\kappa\right) y^{-1/2} + \left(\frac{4\pi^6}{135\zeta(5)}\kappa^2\right) y^{-3/2} + \mathcal{O}(y^{-5/2}) \tag{3.45}$$

and

$$E(\kappa, \pm 1) = y^{1/2} \exp(2\pi(\kappa y \pm ix)) + 2y^{1/2} + \left(\frac{4\pi}{\zeta(3)}\kappa\right) y^{-1/2} + \left(\frac{4\pi^2}{3\zeta(5)}(2\kappa^2 - 1)\right) y^{-3/2} + \mathcal{O}(y^{-5/2}). \tag{3.46}$$

Plugging this into (3.7) gives the expansion of the partition function

$$Z = Z_{0,1} + \frac{1}{|\eta|^2}\bigg(-6 + \frac{(\pi^3 - 6\pi)(11 + 24k)}{9\zeta(3)}y^{-1}$$
$$+ \frac{5(53\pi^6 - 882\pi^2) + 528(\pi^6 - 90\pi^2)k + 576(\pi^6 - 90\pi^2)k^2}{2430\zeta(5)}y^{-2} + \mathcal{O}(y^{-3})\bigg).$$
$$(3.47)$$

The additional contributions to the partition function in this expression have two notable features. First, and most importantly, they are not zero. Thus, as described above, the partition function truly cannot be represented as a sum of exponentials. Second, they differ qualitatively from the leading y^0 term: the additional coefficients appearing here are positive and irrational, rather than negative and integer.

3.4. $\hat{n} \neq 0$ Modes

Now we will consider the $\hat{n} \neq 0$ terms. For $\mu = 0$ and $\hat{n} \neq 0$, the integral (3.24) is a K-Bessel function

$$I_{m,\hat{n}}(s, \kappa, 0) = \frac{2^{s+1}\pi^{2s+m}|\hat{n}|^{s+m-1/2}}{m!\Gamma(s + m)}y^{s+m-1/2}K_{s+m-1/2}(2\pi|\hat{n}|y). \qquad (3.48)$$

The Kloosterman sum is now the general case of Ramanujan's sum

$$S(\hat{n}, 0; c) = \sum_{d \in (\mathbb{Z}/c\mathbb{Z})^*} e^{2\pi i\hat{n}d/c} = \sum_{\delta|\hat{n}} \mu(\delta). \qquad (3.49)$$

We will simply quote the answer for the sum over c (see Sec. 2.5 of [22])

$$\sum_{c=1}^{\infty} c^{-2(s+m)}S(\hat{n}, 0; c) = \frac{1}{\zeta(2(s+m))}\sum_{\delta|\hat{n}} \delta^{1-2(s+m)}. \qquad (3.50)$$

Taking $s = 1/2$ gives a Fourier coefficient of the regularized partition function. Consider first the $m = 0$ term. For $\hat{n} = 0$, this was the dangerous term in the analytic continuation, but for $\hat{n} \neq 0$, it simply vanishes, because the $\zeta(2s)$ in (3.50) has a pole at $c = 1/2$, causing the Kloosterman sum to vanish, and – unlike the $\hat{n} = 0$ case – the integral (3.48) is finite at $s = 1/2$. The other terms are non-zero, and give

$$E_{\hat{n}}(1/2, k, 0) = \sum_{m=1}^{\infty} \frac{2^{3/2}\pi^{m+1}|\hat{n}|^m}{m!\Gamma(m + 1/2)\zeta(2m + 1)}\bigg(\sum_{\delta|\hat{n}} \delta^{-2m}\bigg)\sqrt{y}K_m(2\pi|\hat{n}|y). \qquad (3.51)$$

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Each term in the sum over m vanishes for large y as $e^{-2\pi|\hat{n}|y}$.

Unfortunately, when $\mu \neq 0$ the coefficients in the expansion of the partition function are more difficult to compute. The problem is not the integral (3.24); these integrals are finite and can be evaluated analytically, although we will not write the answer here. The answer is a finite sum of Bessel functions of the form appearing in (3.48), each of which is multiplied by a polynomial in y. As $y \to \infty$, these Bessel functions cause the integrals vanish as $e^{-2\pi|\hat{n}|y}$, just as in the $\mu = 0$ case described above.

However, when $\mu \neq 0$, the sum over c

$$\sum_{c=1}^{\infty} c^{-2(m+s)} S(\hat{n}, \mu, c), \tag{3.52}$$

though of considerable number-theoretic interest, cannot be expressed in terms of familiar number-theoretic functions such as the Riemann zeta function. Analytic properties of these sums have been extensively studied. We will simply quote the relevant results. It has been shown that the sum (3.52) defines a meromorphic function on the complex s plane. This function is essentially the Selberg zeta function associated to the modular domain $D = \mathcal{H}/SL(2,\mathbb{Z})$. When we take $s = 1/2$, the function (3.52) remains regular for elementary reasons if $m > 0$. Indeed, one can see directly that the sum converges for these values, by noting from the definition of the Kloosterman sum (3.21) that $|S(\hat{n}, \mu, c)| \leq c$. To understand the case $m = 0$, one needs deeper results that can be found in Chapter 9 of [22] (and were described to us by P. Sarnak). The key result is that the only poles of this sum, or of its analogs for other congruence subgroups of $SL(2,\mathbb{Z})$, are at the points $s + m = 1/2 + it_j$, where $t_j^2 - 1/4$ is one of the discrete eigenvalues of the hyperbolic Laplacian $\Delta = -y^2(\partial_x^2 + \partial_y^2)$ on D. A pole at $s = 1/2$, $m = 0$, will therefore arise precisely if Δ has a discrete eigenvalue at $\lambda = 1/4$. It is a happy fact that no such eigenvalue exists – the smallest discrete eigenvalue of the Laplacian on D is of order $\lambda_1 \approx 90$. Therefore, the sum (3.52) may be analytically continued to give a finite value at $s = 1/2$ for all values of m, including the dangerous case $m = 0$. In addition, simple bounds suffice to show that there is no problem with the sum over m.

This argument shows that all $\hat{n} \neq 0$ Fourier coefficients of the partition function (3.1) are finite. Together with the results of the previous subsection for $\hat{n} = 0$, this allows us to conclude that the regularization scheme described in Sec. 3.1 provides a finite answer for the partition function (3.1). We conjecture that, just as for $\hat{n} = 0$, the results are not compatible with a Hilbert space interpretation.

3.5. Aside: the Tree Level Sum over Geometries

For comparison, we will now consider the sum which arises if we neglect the one-loop contribution to the path integral described in Sec. 2.2. As we will see, in this case the answer is if anything even worse.

The sum over geometries, if we take account of only the classical action and not the one-loop correction, is

$$Z^*(\tau) = \sum_{c,d} \exp\{2\pi k \operatorname{Im} \gamma\tau\}. \tag{3.53}$$

This sum is quadratically divergent, since at large c and d the summand approaches one. Before attempting to regularize the sum, let us compare it to the "correct" sum (3.4) that does include the one-loop correction. The "correct" formula has a factor of $1/\sqrt{\operatorname{Im}\tau}|\eta|^2$ that is outside the summation. This factor is certainly important, but it does not affect whether the sum converges. There is also a factor of $|1 - q|^2$ inside the sum. This factor also turned out in the above analysis to be less important than it may have appeared; we simply expanded it as $1 - q - \bar{q} + q\bar{q}$, and wrote the partition function as a sum of four terms. All four terms were similar, and nothing particularly nice happened in adding them up.

The factor in (3.4) that actually is important, resulted from the one-loop correction, and has no analog in the "naive" sum (3.53) is the innocent-looking factor of $\sqrt{\operatorname{Im}\tau}|_\gamma$ that is *inside* the summation. Because of this factor, we had to evaluate our Poincaré series at $s = 1/2$.

This factor is absent in eqn. (3.53), so now, if we try to define the naive sum Z^* by introducing a parameter s as before, we will have to evaluate the resulting function at $s = 0$. In fact, the necessary s-dependent function was already introduced in eqn. (3.15); $Z^*(\tau)$ is formally given by the function $E(s, k, 0)$ at $s = 0$:

$$Z^*(\tau) = \lim_{s \to 0} E(s, k, 0). \tag{3.54}$$

However, as an analytic function in s, $E(s, k, 0)$ has a pole at $s = 0$. To see this, consider the expansion (3.34) of the part of $E(s, k, 0)$ which is constant in x. The $m = 0$ term in this sum vanishes, because of the pole in $\Gamma(s)$ at $s = 0$. The $m = 1$ term gives

$$E(s, k, 0) = \sqrt{\pi}\frac{\zeta(1 + 2s)\Gamma(s + 1/2)}{\zeta(2 + 2s)\Gamma(1 + s)} + \mathcal{O}(y^{-1}) \tag{3.55}$$

which has a pole at $s = 0$ coming from the harmonic series $\zeta(1) = \infty$.

Hence, without the one-loop correction, the divergence of the sum over geometries becomes more serious. Of course, even if we could make sense of the function Z^*, we still might have trouble giving it a Hilbert space interpretation.

One might wonder what happens if we modify the definition of Z^* to include a finite subset of the Brown-Henneaux states – for example, the states of negative energy. This means that before summing over geometries, we multiply the exponential of the classical action by a polynomial $\sum_{n,m=0}^{t} a_{n,m} q^n \bar{q}^m$. The sum over geometries is a sum of terms each of which is similar to what was just described, with one such term for each monomial $q^n \bar{q}^m$. Each individual monomial will contribute a pole at $s = 0$, and generically this will survive in the sum.

4. Possible Interpretations

So far, we have analyzed the sum of known contributions to the partition function of pure three-dimensional gravity. As we learned in Sec. 3, the resulting function cannot be interpreted as $\mathrm{Tr} \exp(-\beta H)$ for any Hilbert space operator H.

We will now address the question of how to interpret this result. The most straight-forward interpretation is to take the result at face value. Three-dimensional pure gravity may not exist as a quantum theory; to get a consistent theory, it may be necessary to complete it by adding additional degrees of freedom, and there may be no canonical way to do this.

The other possibility is that some unknown contributions to the partition function should be added to the terms that we have evaluated. Here all sorts of speculations are possible. We will consider two quite different possibilities.

4.1. Cosmic Strings

Known consistent models of $2 + 1$-dimensional gravity with negative cosmological constant arise from string theory. For example, a famous class of models comes from Type IIB superstring theory on $\mathrm{AdS}_3 \times S^3 \times X$, where X is either a K3 surface or a four-torus.

In these models, the dimensionless ratio $k = \ell/16G$ is never a variable parameter, but always takes quantized values determined by fluxes that are chosen in the compact-ification. (The fact that ℓ/G is not continuously variable is actually [13] a more general consequence of the Zamolodchikov c-theorem applied to the boundary CFT.) Moreover, it

is always possible to have domain walls across which the fluxes jump. The domain walls are constructed from suitably wrapped branes.

In $2 + 1$ dimensions, a domain wall has a $1 + 1$-dimensional world-volume and so can be viewed as a cosmic string. The existence of these cosmic strings makes the models much more unified, as regions with different fluxes can appear as different domains in a single spacetime.

The usual $\text{AdS}_3 \times S^3 \times X$ models have supersymmetric moduli, and the values of the string tension depend on these moduli. There is a particularly interesting supersymmetric value of the string tension at which "long strings" become possible. These are strings that can expand to an arbitrarily large size at only a finite cost of energy [24,25]. When long strings exist, the energy spectrum is continuous above a certain minimum energy, and the partition function $\text{Tr} \exp(-\beta H)$ therefore diverges for all β.

The numerical value of the long string tension and of the excitation energy above the ground state beyond which the spectrum is continuous are proportional to the jump in $k = \ell/16G$ in crossing the string. The threshold excitation energy is of order 1 (above the ground state at energy $-2k$) if the jump in k is of order 1.

Since well-established models of three-dimensional gravity have such cosmic strings, perhaps they also present in minimal three-dimensional gravity, if it exists. In anything that one would want to call pure gravity, the string tension T measured in units of $1/\ell^2$ must go to infinity as $k \to \infty$. Otherwise, the cosmic strings would contribute excitations at the AdS_3 scale, and one would describe the model as a theory of three-dimensional gravity plus matter. We have no idea if the string tension should be proportional to k (as one might expect for solitons), to $k^{1/2}$ (as for D-branes), etc. If the jumps in k are correctly matched with T, then the strings are long strings and the partition function $\text{Tr} \exp(-\beta H)$ that we have been trying to compute in this paper is actually divergent. If the jumps in k are smaller than this, then the partition function converges, but to compute it might involve corrections from the strings.

If one thinks that the strings should be long strings, then the requirement $T >> \ell^2$ means that the jumps in k in crossing strings are much greater than 1. There is some tension between this and the proposal in [13] that k can take any integer value. If all allowed values of k are connected by strings or domain walls (as in the known string/M-theory models), then the fact that the jumps in k are large means that the allowed values of k are sparse.

4.2. Doubled Sum Over Geometries

The scenario just described is obviously rather speculative, but at least it has the virtue of underscoring our point that the range of conceivable unknown contributions to the AdS$_3$ partition function is quite large.

We will now describe a quite different scenario. To motivate it, we return to the formula (2.19) for the classical action of the basic spacetime $M_{0,1}$:

$$I = -4\pi k \mathrm{Im}\,\tau. \tag{4.1}$$

We write more explicitly

$$I = 2\pi i k(\tau - \bar{\tau}). \tag{4.2}$$

Hence the classical approximation for the contribution of this spacetime to the partition function is $\exp(-I) = \exp(-2\pi i k(\tau - \bar{\tau})) = q^{-k}\bar{q}^{-k}$. We notice that this is locally the product of a holomorphic function of k and an antiholomorphic function, and is globally such a product if k is an integer (ensuring that q^{-k} is single-valued). The one-loop correction preserves this factorized form, and therefore the formula (2.25) for the exact partition function $Z_{0,1}$ asociated with $M_{0,1}$ has the same properties; in fact, $Z_{0,1} = F_k(q)F_k(\bar{q})$, with

$$F_k(q) = q^{-k} \prod_{n=2}^{\infty} (1 - q^n)^{-1}. \tag{4.3}$$

To the extent that known formulations of three-dimensional gravity are valid, this sort of factorization holds for the contribution to the partition function of any classical geometry. See [26] for a detailed example. The gauge theory description [18,19] of three-dimensional gravity gives a natural explanation of this. With negative cosmological constant, in Lorentz signature, the gauge group is $SL(2, \mathbb{R}) \times SL(2, \mathbb{R})$; the theory is a product of two decoupled $SL(2, \mathbb{R})$ theories, associated respectively with left- and right-moving modes in the boundary CFT, and this corresponds to holomorphic factorization in the Euclidean form of the theory.

In [13], it was suggested that the *exact* partition function of pure three-dimensional gravity is holomorphically factorized. It was observed that if this is the case, and the Brown-Henneaux spectrum is exact until one gets above the classical black hole threshold, then the partition function can be determined uniquely. Dual CFT's consistent with the necessary spectrum have been called extremal CFT's [27]. In subsequent work, it has been shown that the genus 2 partition function of such a CFT can be uniquely and consistently

determined [28], and there has been some work comparing it to what would be expected from three dimensions [29], but on the other hand an interesting but slightly technical argument has been given which may show that extremal CFT's do not exist [30]. Also, it has been argued [31] that extremal CFT's, if they exist, generally do not have monster symmetry for $k > 1$, in contrast to what happens [32] for $k = 1$.

Now let us discuss holomorphic factorization in view of the sum over geometries. Associated to an element

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{4.4}$$

of $SL(2, \mathbb{Z})$ is a classical spacetime $M_{c,d}$. Its action is obtained by applying a modular transformation to (4.2):

$$I_\gamma(\tau) = 2\pi i k \left(\gamma\tau - \gamma\bar{\tau} \right). \tag{4.5}$$

As usual, $\gamma\tau = (a\tau + b)/(c\tau + d)$, $\gamma\bar{\tau} = (a\bar{\tau} + b)/(c\bar{\tau} + d)$. The partition function of the manifold $M_{c,d}$ is

$$Z_{c,d} = F_k(q)\big|_\gamma \, F_k(\bar{q})\big|_\gamma , \tag{4.6}$$

and is holomorphically factorized just like $Z_{0,1}$.

However, when we sum over geometries to evaluate the partition function

$$Z = \sum_{\gamma \in W} F_k(q)\big|_\gamma F_k(\bar{q})\big|_\gamma, \tag{4.7}$$

holomorphic factorization is lost. (Here W is set of classical geometries $M_{c,d}$, isomorphic to the coset space $PSL(2, \mathbb{Z})/\mathbb{Z}$, with $\mathbb{Z}$ being the upper triangular subgroup of $SL(2, \mathbb{Z})$.) In fact, this formula is not simply a product of holomorphic and antiholomorphic functions, but a sum of such products, somewhat like the partition function of a rational conformal field theory. (However, in that case, each term appearing in the sum is separately invariant under $T : \tau \rightarrow \tau + 1$.)

Holomorphic factorization has been lost because the sum over topologies is a common sum for left- and right-movers. What could be added to restore holomorphic factorization? This question has a simple answer, though whether the answer is really relevant to three-dimensional gravity remains to be seen.

If we formally introduce separate topological sums for holomorphic and antiholomorphic variables, defining an extended partition function

$$\widehat{Z} = \sum_{\gamma, \gamma' \in W} F_k(q)\big|_\gamma \, F_k(\bar{q})\big|_{\gamma'} , \tag{4.8}$$

then holomorphic factorization is restored (if the sum converges or can be regularized in a satisfactory way). After all, (4.8) can be written in the manifestly holomorphically factorized form

$$\widehat{Z} = \left(\sum_{\gamma \in W} F_k(q)\big|_\gamma \right) \left(\sum_{\gamma' \in W} F_k(\bar{q})\big|_{\gamma'} \right). \tag{4.9}$$

The question (apart from regularizing these sums) is to interpret the double sum over topologies.

What is the classical action corresponding to a given term in the sum in (4.9)? We can answer this question by simply applying separate modular transformations to holomorphic and antiholomorphic variables on the right hand side of eqn. (4.2). The "action" of a "classical solution" that would lead to the term in (4.8) labeled by a pair γ, γ' must be

$$I_{\gamma,\gamma'} = 2\pi i \left(\gamma\tau - \gamma'\bar{\tau} \right). \tag{4.10}$$

For $\gamma \neq \gamma'$, this formula is not real, so it is not the action of a real solution of the Einstein equations.

The most obvious way to try to interpret the formula is to interpret it as the action of a complex-valued solution of the Einstein equations. As we noted in the introduction, naively speaking the Euclidean path integral is a sum over three-manifolds M that obey the boundary conditions. But there is no established way to evaluate the contribution to the Euclidean functional integral of a given M unless M admits a classical solution of the equations of motion that one can expand around. It is because of this that our first step in Sec. 2.1 was to classify the classical solutions; what we classified were the solutions of the usual real Einstein equations with Euclidean signature.

It seems strange to simply ignore three-manifolds that do not admit classical solutions. Are their contributions meaningless or zero for some reason?

One obvious way to slightly generalize the usual framework is to consider the complexified equations of motion – the Einstein equations $R_{\mu\nu} = -\Lambda g_{\mu\nu}$, defined in the usual way except with a complex-valued but nondegenerate metric tensor $g_{\mu\nu}$. If such a complex-valued solution is given, one can hope that perturbation theory around it would make sense. In the present theory, in which perturbation theory about a classical solution is one-loop exact, one could hope to make sense of the contribution of a complex saddle point (that is, a complex-valued solution of the equations of motion) to the partition function.

Unfortunately, we have been unable to find a convincing family of solutions of the complexified Einstein equations depending on the pair $\gamma, \gamma' \in W$. Since this is the case,

we think another possibility is that the nonperturbative framework of quantum gravity really involves a sum not over ordinary geometries in the usual sense, but over some more abstract structures that can be defined independently for holomorphic and antiholomorphic variables. (A similar idea is expressed in Sec. 3.1 of [29].) Only when the two structures coincide can the result be interpreted in terms of a classical geometry.

"Seeing" The Non-Classical Geometries

None of this is terribly convincing – no more so, certainly, than the discussion of cosmic strings in Sec. 4.1. However, we can describe one fact that offers some slight encouragement.

Let us examine the classical limit of the partition function, by which we mean the limit of $k \to \infty$ with fixed τ. First we will consider the ordinary partition function Z, defined by summing over ordinary geometries, and then we will consider the extended partition function $\widehat{Z}$, defined in (4.8) by a double sum whose meaning is unclear.

In the classical limit, the sum over geometries is dominated by the geometry that has the most negative classical action. In other words, we must pick γ to minimize the quantity

$$I_\gamma = 2\pi i(\gamma\tau - \gamma\bar{\tau}). \tag{4.11}$$

We can also write

$$I_\gamma = \mathrm{Re}\,(4\pi i\gamma\tau) = \mathrm{Re}\,(-4\pi i\gamma\bar{\tau}). \tag{4.12}$$

For generic τ, the maximum occurs for a unique γ.

There actually is a complete democracy between the different choices of γ, because they are all obtained from each other by $SL(2,\mathbb{Z})$ transformations. Each γ dominates the partition function (in the large k limit) for τ in a suitable region of the upper half plane. The thermal AdS manifold $M_{0,1}$ dominates in the usual fundamental domain $|\tau| > 1$, $|\mathrm{Re}\,\tau| < 1/2$. Across the arc $|\tau| = 1$, $\mathrm{Re}\,\tau \leq 1/2$, there is a Hawking-Page phase transition to a region dominated by $M_{1,0}$, which is the Euclidean black hole. The upper half plane is tesselated in phases dominated by different classical solutions. Details are further discussed in Sec. 6.

Now let us consider the extended partition function $\widehat{Z}$. The action, given in eqn. (4.10) now depends on the pair γ, γ', and is complex. What we want to minimize is the real part of the action:

$$\mathrm{Re}\,I_{\gamma,\gamma'} = \mathrm{Re}\,(2\pi i(\gamma\tau - \gamma'\bar{\tau})) = \mathrm{Re}\,(2\pi i\gamma\tau) + \mathrm{Re}\,(-2\pi i\gamma'\bar{\tau}) = \mathrm{Re}\,(2\pi i\gamma\tau) + \mathrm{Re}\,(2\pi i\gamma'\tau). \tag{4.13}$$

The minimum is always at $\gamma = \gamma'$. In fact, we can use eqn. (4.12) to show that

$$\text{Re}\, I_{\gamma,\gamma'} = \frac{1}{2}\left(I_\gamma(\tau) + I_{\gamma'}(\tau)\right). \qquad (4.14)$$

Whatever is the minimum with respect to x of $I_x(\tau)$, the minimum of $\text{Re}\, I_{\gamma,\gamma'}(\tau)$ is at $\gamma = \gamma' = x$.

The conclusion is that, even if non-classical geometries exist and the correct partition function is a sum over non-classical geometries as well as classical ones, the semiclassical limit of large k is always dominated by a classical geometry. Non-classical geometries exist, but they never dominate the semiclassical limit.

What do we have to do to "see" a non-classical geometry in the semiclassical limit? Naturally, we have to ask a non-classical question. The partition function as we have defined it so far is $Z(\tau) = \text{Tr}\, \exp(2\pi i \tau L_0 - 2\pi i \bar\tau \widetilde{L}_0)$, where τ is a point in the upper half plane $\mathcal{H}$ and $\bar\tau$ is its complex conjugate. In terms of $\widetilde\tau = -\bar\tau$, which also takes values in $\mathcal{H}$, we have $Z(\tau) = \text{Tr}\, \exp(2\pi i \tau L_0 + 2\pi i \widetilde\tau \widetilde{L}_0)$. Now instead of defining $\widetilde\tau$ as $-\bar\tau$, let us relax this condition and simply think of $\widetilde\tau$ as a second point in $\mathcal{H}$.[5] Then we should rewrite (4.14) in the form

$$\text{Re}\, I_{\gamma,\gamma'} = \frac{1}{2}\left(I_\gamma(\tau) + I_{\gamma'}(\widetilde\tau)\right). \qquad (4.15)$$

Now it is clear that given any pair γ, γ', we can pick the pair $\tau, \widetilde\tau$ so that the extended partition function $\widehat{Z}$ is dominated in the semiclassical limit by γ, γ'. We simply pick τ so that $I_x(\tau)$ is minimized for $x = \gamma$, and $\widetilde\tau$ so that $I_x(\widetilde\tau)$ is minimized for $x = \gamma'$.

Thus, for each non-classical geometry defined by a pair γ, γ', we can find a question that that geometry dominates in the semiclassical limit. But to do this, we have to ask a rather exotic question that itself depends on a rather unusual analytic continuation.

5. Black Hole Entropy and its Corrections

In this and the following section we will discuss a few implications of the assumption of holomorphic factorization. In this section we will discuss black hole entropy, and how it can be reproduced in holomorphically factorized theories.

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[5] The partition function of a theory in which the high energy density of states is given by the entropy of a BTZ black hole remains convergent after this analytic continuation to general $\tau, \widetilde\tau \in \mathcal{H}$.

In Sec. 2, we computed the perturbative corrections to the saddle point action for the geometry $M_{0,1}$. Using a modular transformation, this leads to a new formula for the subleading corrections to the entropy of the BTZ black hole. We will compare these new subleading corrections with those that occur in the extremal holomorphic partition functions of [13].

5.1. Subleading Corrections to the Bekenstein-Hawking Formula

As we noted in Sec. 2, the geometry we called $M_{1,0}$ is just the Euclidean continuation of the BTZ black hole. The modular parameter τ is related to the Hawking temperature β^{-1} and angular potential θ of the black hole, by $\tau = \theta + i\beta$. The black hole partition function is found by applying the modular transformation $\tau \to -1/\tau$ to the expression (2.23) for the partition function $Z_{0,1}$ of thermal AdS. This gives the one-loop corrected partition function for the Euclidean BTZ black hole

$$Z_{1,0} = \mathcal{Z}(\tau)\bar{\mathcal{Z}}(\bar{\tau}) \tag{5.1}$$

where we have defined the holomorphic piece of the BTZ partition function

$$\mathcal{Z}(\tau) = \frac{q_-^{-(k-1/24)}(1 - q_-)}{\eta(-1/\tau)}. \tag{5.2}$$

In this formula we have defined $q_- = e^{-2\pi i/\tau}$. It will also be useful to write $\mathcal{Z}(\tau)$ as

$$\mathcal{Z}(\tau) = \sum_{\Delta=-k}^{\infty} C_\Delta q_-^\Delta \tag{5.3}$$

where the coefficients C_Δ are

$$C_\Delta = p(\Delta' + k) - p(\Delta' + k - 1). \tag{5.4}$$

Here $p(N)$ denotes the number of partitions of the integer N.

Equation (5.2) encodes various quantum corrections to the thermodynamic properties of the BTZ black hole. Equation (5.2) is a canonical ensemble partition function, so one may, for example, compute the black hole entropy using the usual formula

$$S(\beta, \theta) = \log Z_{1,0} - \beta \ Z_{1,0}^{-1} \ \frac{\partial Z_{1,0}}{\partial \beta} \tag{5.5}$$

where $\tau = \theta + i\beta$. This is a canonical ensemble entropy, evaluated at fixed temperature β^{-1} and angular potential θ.

For the purpose of comparing with CFT predictions, however, it is more useful to compute the microcanonical entropy, which counts the number of states $N(M, J)$ at fixed energy M and angular momentum J. The energy M and angular momentum J of a state are related to left and right-moving dimensions of the state by[6]

$$M = \Delta + \bar{\Delta}$$
$$J = \Delta - \bar{\Delta}. \tag{5.6}$$

So we may write this density of states as $N(\Delta, \bar{\Delta})$. This density of states is computed from the partition function $Z_{1,0}(\tau, \bar{\tau})$ by a pair of Laplace transforms

$$N(\Delta, \bar{\Delta}) = \left(\int_{i\epsilon-\infty}^{i\epsilon+\infty} d\tau \right) \left(\int_{i\epsilon-\infty}^{i\epsilon+\infty} d\bar{\tau} \right) q^{-\Delta} \bar{q}^{-\bar{\Delta}} Z_{1,0}(\tau, \bar{\tau}). \tag{5.7}$$

These are the usual Laplace transforms that appear when going from canonical to microcanonical ensemble. We should emphasize that in this expression τ and $\bar{\tau}$ should be regarded as independent variables.

We should note that, since (5.2) is a semiclassical partition function around a black hole background, it will not have an exact interpretation as a quantum mechanical trace. Moreover, the quantization of Δ and $\bar{\Delta}$ will not be visible in this approximation.[7] So equation (5.7) should be thought of as only a semi-classical approximation to the number of states with dimension $(\Delta, \bar{\Delta})$ in the exact quantum theory. Still, we want to explore this semiclassical formula.

The microcanonical entropy $S(\Delta, \bar{\Delta})$ is just

$$S(\Delta, \bar{\Delta}) = \log N(\Delta, \bar{\Delta}). \tag{5.8}$$

Of course, since $Z_{1,0}$ is holomorphicaly factorized, we may write

$$N(\Delta, \bar{\Delta}) = \mathcal{N}(\Delta)\mathcal{N}(\bar{\Delta}), \qquad S(\Delta, \bar{\Delta}) = \mathcal{S}(\Delta) + \mathcal{S}(\bar{\Delta}) \tag{5.9}$$

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[6] We are working in AdS units, with $\ell = 1$.

[7] To be more precise, in an exact theory with finite entropy, $N(\Delta, \bar{\Delta})$ will be a sum of delta functions of $\Delta, \bar{\Delta}$ multiplying positive integers; but the semiclassical approximation does not have this form.

where

$$\mathcal{N}(\Delta) = \int_{i\epsilon-\infty}^{i\epsilon+\infty} d\tau \, q^{-\Delta} \mathcal{Z}(\tau). \tag{5.10}$$

We may use the expansion (5.3) to get

$$\begin{aligned}
\mathcal{N}(\Delta) &= \sum_{\Delta'=-k}^{\infty} C_\Delta \int_{i\epsilon-\infty}^{i\epsilon+\infty} d\tau q^{-\Delta} q_-^{\Delta'} \\
&= \sum_{\Delta'=-k}^{\infty} C_\Delta \int_{i\epsilon-\infty}^{i\epsilon+\infty} d\tau \exp\left\{-2\pi i \left(\Delta\tau + \frac{\Delta'}{\tau}\right)\right\}.
\end{aligned} \tag{5.11}$$

To do this integral, recall the following representation of the Bessel function as a contour integral:

$$I_1(z) = \frac{1}{2\pi i} \oint t^{-2} e^{(z/2)(t+t^{-1})} dt \tag{5.12}$$

where the contour encloses the origin in a counterclockwise direction. Taking $t \to 1/t$ and letting $z = 4\pi\sqrt{-\Delta\Delta'}$, this becomes the integral appearing in (5.11). We end up with the following formula for the microcanonical entropy:

$$\mathcal{N}(\Delta) = e^{\mathcal{S}(\Delta)} = 2\pi \sum_{\Delta'=-k}^{\infty} C_\Delta \sqrt{\frac{-\Delta'}{\Delta}} I_1(4\pi\sqrt{-\Delta\Delta'}) \tag{5.13}$$

where C_Δ was defined in (5.4). Formula (5.13) is the main result of this section.

In the semiclassical approximation, this formula will be dominated by the term with $\Delta' = -k$. We may then use the asymptotic formula for the Bessel function

$$I_1(z) = \frac{1}{\sqrt{2\pi z}} e^z \left(1 - \frac{3}{8} z^{-1} + \ldots\right), \qquad \text{at } z \to \infty \tag{5.14}$$

to get

$$\mathcal{S}(\Delta) = \log \mathcal{N}(\Delta) = 4\pi\sqrt{k\Delta} + \frac{1}{4}\log k - \frac{3}{4}\log \Delta - \frac{1}{2}\log 2 + \ldots \tag{5.15}$$

The first term is the usual Bekenstein-Hawking term, proportional to the area of the BTZ black hole. The other terms are logarithmic corrections that typically appear when the entropy is computed in microcanonical, as opposed to a canonical, ensemble. They were computed for the BTZ black hole in [33].

Of course, equation (5.13) contains many interesting subleading terms in addition to these logarithmic terms. Let us consider the terms in (5.13) with $\Delta > -k$. For large values of Δ/k, which is to say for black holes whose radius is large in AdS units, these terms are

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exponentially subleading. To see this, let us compare the $\Delta = -k$ and $\Delta = -k + 1$ terms in the sum using the asymptotic behavior of the Bessel function (5.14). We find

$$\frac{I_1(4\pi\sqrt{k\Delta})}{I_1(4\pi\sqrt{(k-1)\Delta})} \sim e^{2\pi\sqrt{\Delta/k}}, \qquad \text{as } \Delta k \to \infty. \qquad (5.16)$$

Thus for Δ/k large the sum is dominated by the $\Delta = -k$ term plus terms which are exponentially small in $\sqrt{\Delta/k}$. For black holes whose size is of order the AdS scale, these additional terms can become relevant.

We should make one more important comment about the structure of the subleading terms in (5.13). The terms with $\Delta' < 0$ are qualitatively of the same form as the leading contribution described above. The terms with $\Delta' > 0$ are qualitatively different: for these terms the factors of $\sqrt{-\Delta'}$ appearing in (5.13) are imaginary, and we must worry, for example, about which branch of the solution (5.13) we should take. At this point one might wonder about the physical interpretation and implications of these funny terms. However, as we will see below, it is precisely these unusual-looking terms which are absent in the correct microscopic computation of the black hole entropy. So we will not worry too much about their appearance here.

Our goal is to understand to what extent the microcanonical entropy formula (5.13) can be reproduced in a holomorphically factorized partition function. Of course, we could attempt to reproduce the canonical formula (5.5) instead, as these two expressions contain precisely the same information. But we will find the problem of computing (5.13) to be technically simpler. Our conclusion will be that the terms in (5.13) with $-k \leq \Delta' < 0$ are reproduced in the full partition function.

Before proceeding, let us ask under what circumstances we expect the formula (5.13) for the entropy to be a good approximation to the number of states of the exact quantum theory. Of course, we must take k to be large, in order for the semi-classical approximation to be valid. In addition, we must take Δ to be large so that the corresponding black hole is large in Planck units and dominates the entropy of the system.

5.2. The Rademacher Expansion for Holomorphic Partition Functions

Let us now consider a modular-invariant, holomorphic CFT with central charge $c = 24k$. For such CFTs, k must be an integer, so the holomorphic part of the partition function can be expanded as

$$Z(\tau) = \sum_{\Delta=-k}^{\infty} F_\Delta q^\Delta \qquad (5.17)$$

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where the F_Δ are some positive integers. Modular invariance fixes $Z(\tau)$ in terms of the finite set of coefficients $\{F_\Delta | - k \leq \Delta \leq 0\}$. Once these polar coefficients are fixed, an explicit formula can then be given for $Z(\tau)$ as a polynomial in the j-invariant, although we will not write this formula here. The important point is that all of the coefficients F_Δ with $\Delta > 0$ are fixed in terms of the coefficients F_Δ with $\Delta \leq 0$. The formula for the F_Δ with $\Delta > 0$ in terms of these polar terms is

$$F_\Delta = 2\pi \sum_{-k \leq \Delta' < 0} \sqrt{\frac{|\Delta'|}{\Delta}} F_{\Delta'} \sum_{n=1}^{\infty} \frac{1}{n} S(\Delta, |\Delta'|, n) I_1\left(\frac{4\pi}{n}\sqrt{\Delta|\Delta'|}\right) \qquad (5.18)$$

where $S(\Delta, \Delta', n)$ is the Kloosterman sum defined in (3.21). This is a convergent series expansion of F_Δ. Expansions of the form (5.18) are known as Rademacher expansions. Rademacher expansions were first applied to three-dimensional gravity in [10].

In [13], a conjecture was made regarding the form of the partition function (5.17). The conjecture is that the theory contains no primary fields with $-k < \Delta \leq 0$. CFTs with this property are known as extremal CFTs, since in a sense this is the most extreme conjecture possible: modular invariance forces one to include additional primaries with $\Delta > 0$. In terms of the coefficients F_Δ, the conjecture of [13] is that

$$F_\Delta = C_\Delta \qquad \text{for } -k < \Delta \leq 0. \qquad (5.19)$$

Here C_Δ are the perturbative coefficients defined in (5.4).

The extremal partition function (or any similar function in which the number of primary fields of dimension $\leq k$ is not too large) is consistent with the leading Bekenstein-Hawking term in the black hole entropy for Δ, k large. We are now in a position to ask whether subleading terms given by (5.13) are reproduced as well. First, we note that the sums (5.18) and (5.13) are quite similar. In particular, when $n = 1$ the Kloosterman sum is 1. In this case the summand in (5.18) coincides precisely with that in (5.13), at least when $\Delta' < 0$. Thus the Rademacher expansion (5.18) contains the first k terms in the one-loop corrected black hole entropy (5.13). This is the main result of this section.

There are two important differences between (5.18) and (5.13). The first is that the sum in (5.13) is over $-k < \Delta' < \infty$, while the sum (5.18) is only over $-k < \Delta' < 0$. So the funny terms in (5.13) with $\Delta' > 0$ are absent in the full microscopic entropy. It would be interesting to understand the physical interpretation of these terms, but we will not attempt to do so here.

The second important difference is that the Rademacher expansion (5.18) includes terms with $n > 1$. These terms provide exponentially small corrections to the black hole entropy. To see this, we may again use the asymptotic behavior of the Bessel function (5.14). For black holes with large Δ the ratio of the $n = 1$ and $n = 2$ terms in the sum (5.18) goes like

$$\frac{I_1(2\pi\sqrt{\Delta|\Delta'|})}{I_1(4\pi\sqrt{\Delta|\Delta'|})} \sim e^{-2\pi\sqrt{\Delta|\Delta'|}}. \tag{5.20}$$

So at large Δ, the $n > 1$ terms in the Rademacher expansion are exponentially small in Δ. Again, it would be interesting to study the physical interpretation of these terms.

6. The Hawking-Page Phase Transition

In this section we address an apparent puzzle posed by holomorphic factorization and the Hawking-Page transition [15]. As we will see, the resolution to this puzzle is that if holomorphic factorization is valid, then the Hawking-Page phase transition is described by a condensation of Lee-Yang zeroes of the partition function [34,35]. We examined the zeroes of the partition function as a result of a question from M. Kaneko, and the zeroes we find are similar to what have been obtained in investigations [36,37] of certain somewhat similar modular functions.

The partition function $Z(\tau)$ computes a canonical ensemble partition function at fixed temperature Im τ and angular potential Re τ. As described in Sec. 2, for each value of τ, an infinite family of classical geometries $M_{c,d}$ will contribute to $Z(\tau)$. The pair of integers (c, d) labels an element of the coset $SL(2, \mathbb{Z})/\mathbb{Z}$, where $\mathbb{Z}$ denotes the shifts $\tau \to \tau + n$. In Sec. 2.2, we computed the contribution of $M_{c,d}$ to the partition sum $Z(\tau)$. If we choose an element $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ of $SL(2, \mathbb{Z})$ corresponding to $M_{c,d}$, the contribution to the partition function was given by $Z_{0,1}(\frac{a\tau+b}{c\tau+d})$.

For a given value of τ, one can ask which geometry $M_{c,d}$ has the largest contribution to $Z(\tau)$ in the semiclassical limit $k \to \infty$. As explained in [9,10] and in Sec. 4.2, this question can be answered by maximizing the classical action $-4\pi\text{Im}(\gamma\tau)$ as a function of γ. If τ is in the usual fundamental domain $|\tau| > 1$, $|\text{Re }\tau| \leq 1/2$, or any of its translates by $\tau \to \tau + n$, then the answer is that the dominant classical solution is $M_{0,1}$, which describes thermal AdS space. For any τ, the dominant classical solution can be found by asking which value of $\gamma \in SL(2, \mathbb{Z})$ will take τ into the fundamental domain or one of its translates under $\tau \to \tau + n$. For example, $M_{1,0}$, whose Lorentzian continuation describes a black

hole in equilibrium with its Hawking radiation, dominates whenever there exists an n such that $-1/(\tau + n)$ lies in the fundamental domain. Roughly speaking, this is a regime of low temperature and angular potential. More complicated geometries $M_{c,d}$, which do not have simple Lorentzian interpretations (as they do not admit an analytic continuation to a real solution with Lorentz signature) dominate in other regions of the upper half plane $\mathcal{H}$. Putting this together, we find that the phase diagram of three-dimensional gravity, in the semiclassical limit, is given by a sort of subtesselation of the usual tesselation of $\mathcal{H}$ by fundamental domains of $SL(2, \mathbb{Z})$.

We get a subtesselation because certain boundaries between fundamental domains can be crossed without any jump in the dominant geometry (an example is that, for $\mathrm{Im}\,\tau \gg 1$, there is no jump in crossing between domains related by $\tau \to \tau + n$). The usual tesselation of $\mathcal{H}$ by fundamental domains is sketched in Fig. 3a, and the subtesselation relevant to three-dimensional gravity is sketched in Fig. 3b.
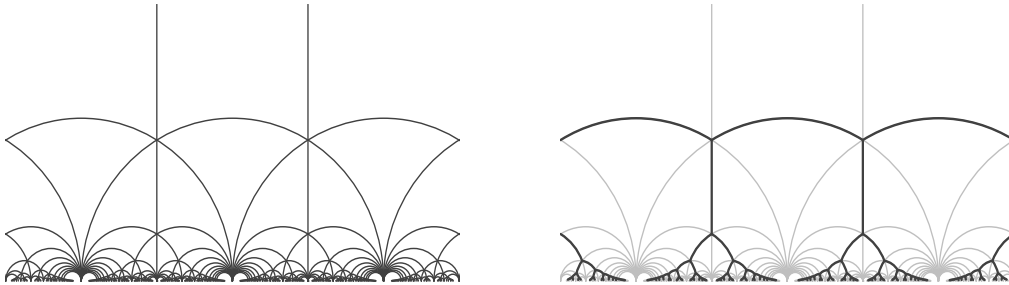

Figure 3:   a) The standard tesselation of the upper half τ-plane by $SL(2, \mathbb{Z})$ fundamental regions. b) The subtesselation that represents the phase diagram of three dimensional gravity. The phase boundaries, represented by solid black arcs, connect fixed points of $SL(2, \mathbb{Z})$ of order 3. This characterization was found by considering special cases and then using modular invariance.

It is believed that the phase structure as a function of τ of a wide range of three-dimensional gravity theories (with different sets of fields) is as we have just described. For any finite value of the dimensionless ratio $k = \ell/16G$, the partition function is smooth as a function of τ, but for $k \to \infty$ or $G \to 0$, it becomes non-smooth along the curves drawn in Fig. 3b.

We will address here a puzzle that arises if one assumes holomorphic factorization. In this case, we take k to be a positive integer. The partition function factorizes as

$\widehat{Z} = Z_k(q)Z_k(\bar{q})$, where the function Z_k is a holomorphic and modular-invariant function. Z_k has an expansion near $q = 0$ involving the negative energy Brown-Henneaux states,

$$Z_k = q^{-k} + \dots, \tag{6.1}$$

and has no other singularities. In order for $\widehat{Z}$ to exhibit the phase structure sketched in Fig. 3b in the semiclassical limit, Z_k must do the same.

The poses a puzzle, since Z_k is holomorphic. There is no problem for a sequence of smooth but not holomorphic functions depending on a parameter to become non-smooth along a phase boundary in some limit. But how can this happen to a sequence of holomorphic functions?

This question was actually answered by Lee and Yang [34,35]. The original idea of Lee and Yang is that although a system in finite volume can have no phase transition, its partition function, in its dependence on the complexified thermodynamic variables, can have zeroes. Then, in the infinite volume limit, the zeroes become more numerous and may become dense or "condense" along a certain arc, and a true phase transition can emerge.

In our problem, the limit $k \to \infty$ is analogous to a thermodynamic limit, since the effective size of AdS space (accessible at a given temperature) grows with k. In a fundamental domain compactified by adding its "cusp," the function Z_k has precisely k poles, exhibited in (6.1). Hence the number of zeroes in a fundamental domain is also equal to k. Here in the case of zeroes that are on the boundary of the fundamental domain, one must count only half the zeroes (or one half of the boundary) to avoid double-counting.

Before going into details, we will give an informal explanation in which we ignore the one-loop correction. We consider the phase boundary along the arc $|\tau| = 1$, $|\mathrm{Re}\,\tau| \leq 1/2$ separating the thermal AdS phase from the Euclidean black hole. We will denote the portion of this arc with $\mathrm{Re}\,\tau \geq 0$ by C:

$$C = \{\tau = x + iy \big| |\tau| = 1, 0 \leq x \leq 1/2\}. \tag{6.2}$$

The other portion of the arc, with $\mathrm{Re}\,\tau \leq 0$, is the image of C under the modular transformation $\tau \to -1/\tau$. Now, the contributions of $M_{0,1}$ and $M_{1,0}$ to the holomorphic Poincaré series (ignoring the one-loop correction) are $\exp(-2\pi ik\tau)$ and $\exp(-2\pi ik(-1/\tau))$. On the arc C, we have $1/\tau = \bar{\tau}$ and the sum of these two contributions is

$$\exp(-2\pi ik\tau) + \exp(2\pi ik\bar{\tau}). \tag{6.3}$$

The sum vanishes if $\exp(2\pi i k(\tau + \bar{\tau})) = -1$. Setting $\tau = \exp(i\theta)$, $\pi/3 \leq \theta \leq 2\pi/3$, we need $\exp(4\pi i k \cos\theta) = -1$ or $4\pi k \cos\theta = (2n+1)\pi$, $n \in \mathbb{Z}$. On the arc C, we have $0 \leq \cos\theta \leq 1/2$. So this equation has k roots. In the limit $k \to \infty$, these roots become dense everywhere along the arc. This is the condensation of Lee-Yang zeroes of the partition function.

Now we give a more precise account including the one-loop correction. However, the result does not depend on all the details of the function Z_k. We will show, following [36,37], that for a certain class of holomorphic functions, all zeroes lie on the arc C or one of its images under $SL(2,\mathbb{Z})$, as shown in Fig. 3. In the large k limit, the zeroes become dense on C.

We consider a modular-invariant function Z_k that is regular except for a pole of order k at $q = 0$. Such a function has a Laurent expansion

$$Z_k(\tau) = \sum_{\Delta=-k}^{\infty} F_\Delta q^\Delta . \tag{6.4}$$

We require that $F_{-k} = 1$ and that the coefficients F_Δ, $\Delta \leq 0$, do not increase too quickly. More precisely, we will show that if

$$F_\Delta < e^{2\pi(0.61)(\Delta+k)}, \qquad \text{for } \Delta \leq 0 \tag{6.5}$$

then the zeroes are all as just described. The origin of the numerical coefficient will be described below. This bound has a simple physical explanation: when there are too many primary fields of low dimension, i.e. too much light matter in the theory, the formation of stable black holes is not possible. Problems with black hole physics when there are many light species of matter have been considered in, for example, [38].

We should emphasize that the extremal partition functions proposed in [13] obey the bound (6.5). So in the large k limit the extremal partition functions exhibit the condensation of zeroes described above. In fact, for $\Delta < 0$ and large k the coefficients of the extremal partition function grow much more slowly than (6.4). For large $\Delta + k$, they grow like

$$F_\Delta \sim e^{\frac{\pi}{6}\sqrt{\Delta+k}} << e^{2\pi(0.61)(\Delta+k)}. \tag{6.6}$$

So one could add many more additional primaries with dimension $\Delta < 0$ without spoiling the phase transition described above.

Our proof of the foregoing assertions relies on a few specific facts about modular functions, which we now review.

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6.1. Review: Properties of J

First, we recall a few properties of the modular invariant $J(\tau)$. Proofs of many of our assertions in this and the next section can be found in [39]. The J function has a $q-$expansion

$$J(\tau) = \sum_{m \geq -1} c(m)q^m = \frac{1}{q} + 196884q + \dots \tag{6.7}$$

where the coefficients $c(m)$ are positive integers. Our choices of overall normalization and constant term $c(0) = 0$ agree with [32] but differ from the most classical convention. As the $c(m)$ are positive, it follows that J obeys

$$J(\tau) = \bar{J}(-\bar{\tau}) \tag{6.8}$$

and so is real along the imaginary τ axis. In particular, it can be shown that J decreases monotonically along the imaginary axis from $J(i\infty) = \infty$ to a minimum at $\tau = i$ with $J(i) = 984$. (Continuing past i on the imaginary axis, J grows again in view of its invariance under $\tau \to -1/\tau$.)

The modular domain

$$D = \{\tau = x + iy \big| |\tau| \geq 1, -1/2 \leq x \leq 1/2\} \tag{6.9}$$

is a fundamental region for $SL(2, \mathbb{Z})$. On the interior of D, J takes every complex value exactly once. (This follows from the fact that in the compactified fundamental domain, J has precisely one pole, which is at $\tau = i\infty$.) Using the invariance of J under $SL(2, \mathbb{Z})$ and (6.8), one can show that $J(\tau)$ takes real values on the boundary ∂D. In particular, it turns out that on the arc $\tau = e^{i\theta}$, $J(\tau)$ decreases monotonically from 984 to -744 as θ runs from $\pi/2$ to $\pi/3$. This is the arc we called C above. On the arcs $x = \pm 1/2$, J decreases monotonically from -744 to $-\infty$ as y runs from $\sqrt{3}/2$ to ∞.

Although the exact form of $J(q)$ is quite complicated, in many cases the "tree-level approximation"

$$J \sim q^{-1} + \dots \tag{6.10}$$

is very useful. To this end, we note that for any given value of $\tau = x + iy$, we have the bound

$$|J(\tau) - q^{-1}| = |\sum_{m \geq 1} c(m)q^m| \leq \sum_{m \geq 1} c(m)|q|^m = J(iy) - e^{2\pi y}. \tag{6.11}$$

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The function $\sum_{m \geq 1} c(m)|q|^m$ depends only on y and is a monotonically decreasing function of y, so in the fundamental domain D, it is bounded above by its value at $y = \sqrt{3}/2$, which is the minimum of y in D. This value is

$$M = J(i\sqrt{3}/2) - e^{\pi\sqrt{3}} \approx 1335. \tag{6.12}$$

So in D, we have

$$|J(\tau) - q^{-1}| \leq M. \tag{6.13}$$

Applying the triangle inequality, it follows that, throughout D, we can bound the value of J by

$$|J(\tau)| \leq e^{2\pi y} + M. \tag{6.14}$$

The approximation (6.10) is a good one when the imaginary part of τ is large, i.e. when τ is close to the cusp at $\tau \to \infty$. By considering modular transformations of the equation (6.13) we arrive at other approximations which are good in other regions of the upper half plane. In particular, for any element $\gamma \in SL(2, \mathbb{Z})$, if $\gamma\tau$ is in the fundamental domain D we have

$$|J(\tau) - e^{-2\pi i \gamma\tau}| \leq M, \tag{6.15}$$

and therefore

$$|J(\tau)| \leq e^{2\pi \text{Im } \gamma\tau} + M. \tag{6.16}$$

This final bound is particularly useful, for the following reason. For a given value of τ, consider the set of possible values of Im $\gamma\tau$, for all $\gamma \in SL(2, \mathbb{Z})$. This set has a finite maximum value which occurs when $\gamma\tau$ is in the fundamental domain or one of its images under the map $\tau \to \tau + n$. So, using the formula (3.9) for Im $\gamma\tau$, we conclude that for any value of $\tau = x + iy$ in the upper half plane we have the bound

$$|J(\tau)| \leq \exp\left\{ \max_{c,d} \frac{2\pi \text{ Im } \tau}{|c\tau + d|^2} \right\} + M. \tag{6.17}$$

Here the maximum is taken over all relatively prime c and d.

6.2. Properties of $T_n J$

We will now consider the action of the Hecke operator T_n on J, defined by

$$T_n J = \sum_{\delta|n} \sum_{\beta=0}^{\delta-1} J((n\tau + \beta\delta)/\delta^2). \tag{6.18}$$

This is a new modular invariant function with an n-th order pole at $q = \infty$. Our normalization (which differs by a factor of n from that in the literature) has been chosen so that the $q-$expansion of $T_n J$ starts with q^{-n}:

$$T_n J = \sum_{m=-n}^{\infty} c_n(m) q^m = q^{-n} + \mathcal{O}(q). \tag{6.19}$$

The Fourier coefficients $c_n(m)$ of $T_n J$ are related to the coefficients $c(m)$ of J by

$$c_n(m) = \sum_{\delta|(m,n)} \frac{n}{\delta} c\left(mn/\delta^2\right) q^m. \tag{6.20}$$

From this it follows that the Fourier coefficients of $T_n J$ are positive integers, just like those of J.

From these facts, it is straightforward to see that $T_n J$ satisfies many of the same properties as J. In particular, $T_n J$ is real along the imaginary axis, as well as on the boundary of the fundamental domain ∂D. Moreover, just as with J, the "tree-level approximation"

$$T_n J \sim q^{-n} + \ldots \tag{6.21}$$

is good enough for many purposes. This approximation is very good when $\tau \to i\infty$, but actually it can be extended to understand the zeroes of $T_n J$, as was shown in [37]. We will summarize their argument here. In section 6.3, we apply the results derived below to more general modular-invariant partition functions.

The basic idea is to apply either (6.15) or (6.17) to each term in the sum (6.18). This allows us to bound the value of $T_n J(\sigma)$ for a point $\sigma = x + iy$ on the arc C defined by $|\sigma| = 1$ and $0 \leq x \leq 1/2$. Let us first consider the $\delta = 1$ term in (6.18), which is $J(n\sigma)$. For any σ in C, we can find an integer m such that $n\sigma + m$ lies in the fundamental domain. This allows us to apply (6.15) with $\tau = n\sigma$ and $\gamma\tau = n\sigma + m$ to get

$$|J(n\sigma) - e^{-2\pi i n\sigma}| \leq M. \tag{6.22}$$

Now, consider the $\delta = n$, $\beta = 0$ term in (6.18), which is $J(\sigma/n)$. Since σ lies in C, we can find an integer m such that $-n/\sigma + m$ lies in the fundamental domain. So we may apply (6.15), with $\tau = \sigma/n$ and $\gamma = -n/\sigma + m$ to get

$$|J(\sigma/n) - e^{-2\pi in/\sigma}| \le M. \tag{6.23}$$

Using these two equations, we may apply the triangle inequality to (6.18) to get

$$|T_nJ(\sigma) - e^{-2\pi in\sigma} - e^{-2\pi in/\sigma}| \le 2M + \sum{}' \left| J\left(\frac{n\sigma + \beta\delta}{\delta^2}\right) \right|. \tag{6.24}$$

Here the sum $\sum'$ is over the same set of β and δ as in (6.18), except we have dropped the two terms with $\delta = 1$ and $\delta = n, \beta = 0$. We will now apply (6.17) to each term on the right hand side of (6.24). Consider first the term with $\delta = n$, $\beta = n - 1$, which is $J\left(\frac{\sigma-1}{n}\right)$. We would like to apply (6.17) with $\tau = \frac{\sigma-1}{n}$, so we must ask what possible values $|c\tau + d|$ can take for this value of τ. Now, since σ lies on C, it follows that $|\sigma - 1| > 1$ and hence $|\tau| = |\frac{\sigma-1}{n}| > 1/n$. This implies that $|c\tau + d| \ge 1/n$ for all possible choices of c and d. Since Im $\tau = y/n$, equation (6.17) gives

$$\left| J\left(\frac{\sigma + n - 1}{n}\right) \right| \le e^{2\pi ny} + M. \tag{6.25}$$

Let us consider the case where $\delta = n$ and $0 < \beta < n - 1$, where we may apply (6.17) with $\tau = \frac{\sigma+\beta}{n}$. For this range of β, $|\sigma + \beta| > \sqrt{2}$, so $|\tau| > \sqrt{2}/n$. Hence $|c\tau + d| > \sqrt{2}/n$ and (6.15) gives

$$\left| J\left(\frac{\sigma + \beta}{n}\right) \right| \le e^{\pi ny} + M. \tag{6.26}$$

Finally, we consider the cases where $1 < \delta < n$. In this case we apply (6.17) with $\tau = (n\sigma + \beta\delta)/\delta^2$. The fact that σ lies on C implies that $|c\tau + d| > \sqrt{3}n/\delta^2$. So we end up with the bound

$$\left| J\left(\frac{\sigma + \beta}{n}\right) \right| \le e^{\frac{2\pi n}{3}y} + M. \tag{6.27}$$

Putting this all together, equation (6.24) becomes

$$|T_nJ(\sigma) - e^{-2\pi in\sigma} - e^{-2\pi in/\sigma}| \le e^{2\pi ny} + ne^{\pi ny} + n^2 e^{\frac{2\pi n}{3}y} + n^2 M. \tag{6.28}$$

The factors of n and n^2 in (6.28) come from the simple fact that there are less than n terms with $\delta = n$, and less than n^2 terms with $1 < \delta < n$. Multiplying both sides of (6.28) by $e^{-2\pi ny}$, and using the fact that $e^{-2\pi in/\sigma} = e^{2\pi in\bar{\sigma}}$ for points on the curve C, this becomes

$$|T_nJe^{-2\pi ny} - 2\cos(2\pi nx)| \le 1 + ne^{-\pi ny} + n^2 e^{-\frac{4\pi n}{3}y} + Mn^2 e^{-2\pi ny}. \tag{6.29}$$

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For the moment, we concentrate on the case $n \geq 2$. Since $y \geq \sqrt{3}/2$ for any point on C, we may evaluate this right hand side of (6.29) to get

$$|T_n J e^{-2\pi n y} - 2\cos(2\pi n x)| \leq 1.12. \tag{6.30}$$

This formula is valid for any point on the arc C and $n \geq 2$.

Equation (6.30) is an important result. It places strong constraints on the location of the zeros of $T_n J$. To see this, note that on each of the n intervals I_m defined by $\frac{m}{n} < x < \frac{m+1}{n}$, $m = 0, \ldots, n-1$, the function $2\cos(2\pi n x)$ varies monotonically from -2 to 2. The right hand side of (6.30), however, is less than 2. This implies that $T_n J$ must have n distinct zeroes on C, with one in each interval I_m. As $n \to \infty$, the lengths of these intervals vanish and the zeroes of $T_n J$ become dense on C. Finally, note that, as $T_n J$ is an nth order polynomial in J, it has only n zeroes on D. In particular, it has no additional zeroes beyond those described above.

We have only proven the bound (6.30) for $n \geq 2$. For our application, we also need the $n = 0$ and $n = 1$ cases, which we will consider separately. For $n = 0$, we define $T_0 J = 1$. So the bound (6.30) is trivial. For $n = 1$, we have $T_1 J = J$. In this case we have the slightly weaker bound

$$|J(\tau)e^{-2\pi y} - 2\cos(2\pi x)| \leq 1.22 \tag{6.31}$$

for points on C. This is straightforward to verify numerically, although it may also be understood analytically. The point is that the function $J(\tau)e^{-2\pi y} - 2\cos(2\pi x)$ is monotonic along the arc C. So the value of this function on C is bounded by its values at the endpoints $\tau = i$ and $\tau = 1/2 + \sqrt{3}/2i$. Using $J(i) = 984$ and $J(1/2 + \sqrt{3}/2i) = -744$ we may compute the values of the function at these endpoints, giving the bound (6.31).

6.3. Zeroes of Modular Invariant Partition Functions

We are now ready for the general case of a modular invariant partition function at level k:

$$Z_k(\tau) = \sum_{\Delta=-k}^{\infty} F_\Delta q^\Delta = \sum_{\Delta=-k}^{0} F_\Delta T_{|\Delta|} J(\tau) \tag{6.32}$$

where the F_Δ are non-negative integers and $F_{-k} = 1$. As with $J(\tau)$, the fact that $Z_k(\tau)$ has real coefficients and is modular-invariant means that it is real on the imaginary τ axis as well as on the boundary of the fundamental region ∂D.

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In the previous section, we proved the estimate (6.30) for $T_n J$ on C with $n \geq 2$. Along with the estimate (6.31) on C, this implies that for all $n \geq 0$

$$T_n J = e^{2\pi n y}(2\cos(2\pi n x) + E_n) \qquad (6.33)$$

where E_n is an error term obeying

$$|E_n| < 1.22. \qquad (6.34)$$

By simply adding up the inequalities (6.33) for $n = 0, \ldots, k$, with coefficients F_{-n}, and using the fact that $F_{-k} = 1$, we get a a similar estimate for Z_k:

$$Z_k e^{-2\pi k y} - 2\cos(2k\pi x) = E_k + \sum_{\Delta=-k+1}^{0} F_\Delta e^{-2\pi(k+\Delta)y}\left(2\cos(2\pi\Delta x) + E_{|\Delta|}\right). \qquad (6.35)$$

To bound the location of the zeroes of Z_k, we must show that the right hand side is less than 2. Then, as in the previous section, the zeroes of Z_k will lie on C and become dense in the large k limit.

For this to be the case, F_Δ must not grow too quickly with Δ. For example, assume that

$$F_\Delta < A e^{2\pi\alpha(\Delta+k)} \qquad \text{for } -k \leq \Delta \leq 0 \qquad (6.36)$$

where α and A are positive constants. In this case

$$\begin{aligned}
\sum_{\Delta=-k+1}^{0} F_\Delta e^{-2\pi(k+\Delta)y} &< A \sum_{\Delta=-k+1}^{0} e^{2\pi(k+\Delta)(\alpha-y)} \\
&< A \sum_{n=1}^{k} e^{-2\pi n(\alpha-\sqrt{3}/2)} < \frac{A}{e^{2\pi(\alpha-\sqrt{3}/2)} - 1}.
\end{aligned} \qquad (6.37)$$

In the second line, we have used the fact that $y > \sqrt{3}/2$ on C. Since

$$|2\cos(2\pi\Delta x) + E_{|\Delta|}| \leq 2 + |E_{|\Delta|}| < 3.22, \qquad (6.38)$$

it follows from (6.34) that

$$|Z_k e^{-2\pi k y} - 2\cos(2k\pi x)| \leq |E_k| + 3.22\frac{A}{e^{2\pi(\sqrt{3}/2-\alpha)} - 1}, \qquad (6.39)$$

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which is less than 2 for certain values of A and α. In particular, using $|E_k| < 1.22$ and setting $A = 1$, we find that the right hand side is less than 2 provided that

$$\alpha \le 0.61 \tag{6.40}$$

We have used several rather conservative estimates, so it is quite possible that the zeroes of Z_k condense on C even when the coefficients F_Δ do not satisfy the precise bound given above.

It is easy to check that this bound is indeed satisfied for the extremal partition functions of [13]. In fact, the coefficients $Z_k(\Delta)$ grow much more slowly than the required behavior, indicating that many more primary fields could be added without spoiling the existence and nature of the phase transition.

7. Supergravity Partition Functions

Our goal here will be to repeat the analysis of Secs. 2 and 3 for supergravity. We will consider only the basic case of $\mathcal{N} = 1$ supergravity, in which the symmetry group $SL(2, \mathbb{R}) \times SL(2, \mathbb{R})$ of AdS$_3$ is replaced by $OSp(1|2) \times OSp(1|2)$, where $OSp(1|2)$ is a supergroup whose bosonic part is $Sp(2, \mathbb{R}) = SL(2, \mathbb{R})$. The boundary CFT has $(1, 1)$ supersymmetry, that is, $\mathcal{N} = 1$ supersymmetry for both left- and right-movers.

7.1. Formalism

In supergravity, there are a few closely related choices of possible partition function. We can compute either $\mathrm{Tr}\, \exp(-\beta H - i\theta J)$ or $\mathrm{Tr}\, (-1)^F \exp(-\beta H - i\theta J)$. And we can compute this trace in either the Neveu-Schwarz (NS) or the Ramond (R) sector. As usual, we attempt to compute these partition functions by summing over three-manifolds M that are locally AdS$_3$ and whose conformal boundary Σ is a Riemann surface of genus 1. The four possible partition functions (NS or R, with or without an insertion of $(-1)^F$) correspond to the four spin structures[8] on Σ. The three that correspond to odd spin structures are permuted by the action of $SL(2, \mathbb{Z})$, as we further discuss below.

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[8] In fact, using a discrete R-symmetry that appears to be present in the boundary CFT, we could choose different spin structures for left- and right-movers on Σ, but we will not consider this generalization.

Once we pick the spin structure on Σ, we then sum over choices of M such that the given spin structure on Σ does extend over M. For example, what spin structure on Σ is compatible with taking $M = M_{0,1}$ to be the three-manifold related to perturbative excitations of AdS$_3$?

In $M_{0,1}$, the "spatial" circle on Σ is contractible. This means that the NS spin structure on the spatial circle extends over $M_{0,1}$ and the R spin structure does not. Hence $M_{0,1}$ contributes to traces in the NS sector, not the R sector.

What is the spectrum of thermal excitations in the NS sector? In ordinary gravity, the thermal excitations of left-movers are obtained by acting on the ground state $|\Omega\rangle$ with Virasoro generators L_{-n}, $n \geq 2$. When the boundary theory has $\mathcal{N} = 1$ supersymmetry, we can also act on $|\Omega\rangle$ with superconformal generators $G_{-n+1/2}$, $n \geq 2$. (We recall that $G_{-1/2}|\Omega\rangle = 0$.) Writing $-k^*/2$ for the ground state energy, the partition function of left-moving excitations is therefore

$$q^{-k^*/2} \prod_{n=2}^{\infty} \frac{1 + q^{n-1/2}}{1 - q^n}. \tag{7.1}$$

Including both left- and right-moving excitations, the contribution of $M_{0,1}$ to $F(q, \bar{q}) = \mathrm{Tr}_{\mathrm{NS}} \exp(-\beta H - i\theta J)$ is

$$F_{0,1} = \left| q^{-k^*/2} \prod_{n=2}^{\infty} \frac{1 + q^{n-1/2}}{1 - q^n} \right|^2. \tag{7.2}$$

This formula can be justified exactly as in Sec. 2.2 for the bosonic case.

The complete function F, or more precisely the sum of known contributions to it, is evaluated by summing over all those modular images of $M_{0,1}$ over which the relevant spin structure extends. We can represent the four spin structures on the two-torus Σ by a column vector

$$\begin{pmatrix} \mu \\ \nu \end{pmatrix}, \tag{7.3}$$

where μ and ν represent respectively the fermion boundary conditions in the "time" and "space" directions on the two-torus Σ. We consider μ and ν to be valued in $\frac{1}{2}\mathbb{Z}/\mathbb{Z}$, taking the values $1/2$ for antiperiodic (NS) boundary conditions and 0 for periodic (R) ones. An element of $SL(2, \mathbb{Z})$ acts on the spin structures by

$$\begin{pmatrix} \mu \\ \nu \end{pmatrix} \rightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \mu \\ \nu \end{pmatrix}. \tag{7.4}$$

The NS partition function $F(\tau) = \text{Tr}_{\text{NS}} \exp(-\beta H - i\theta J)$ corresponds to $\mu = \nu = 1/2$. This condition is invariant under the subgroup of $SL(2, \mathbb{Z})$ characterized by saying that $c + d$ and $a + b$ are both odd. If $c + d$ is odd, we can make $a + b$ odd by adding to (a, b) a multiple of (c, d). F, or at least the sum of known contributions to it, can therefore be computed by summing $F_{0,1}$ over modular images with $c + d$ odd:

$$F(\tau) = \sum_{c,d|c+d\,\text{odd}} F_{c,d}(\tau) \tag{7.5}$$

or equivalently

$$F(\tau) = \sum_{c,d|c+d\,\text{odd}} F_{0,1}((a\tau + b)/(c\tau + d)). \tag{7.6}$$

For any given c, d such that $c + d$ is odd, we pick a and b such that $ad - bc = 1$ and $a + b$ is odd. Because $F_{0,1}(\tau)$ is invariant under $\tau \to \tau + 2$, the summand in (7.6) does not depend on this choice.

It is also of interest to compute partition functions with other spin structures. However, the three even spin structures on Σ (the ones for which μ and ν are not both zero) are permuted by $SL(2, \mathbb{Z})$ and so the associated partition functions are not really independent functions. If we set $\mu = 0$, $\nu = 1/2$, we get $G(\tau) = \text{Tr}_{\text{NS}} (-1)^F \exp(-\beta H - i\theta J)$. The contribution of $M_{0,1}$ to this partition function is obtained by reversing the sign of all fermionic contributions in (7.2):

$$G_{0,1}(\tau) = \left| q^{-k^*/2} \prod_{n=2}^{\infty} \frac{1 - q^{n-1/2}}{1 - q^n} \right|^2. \tag{7.7}$$

The subgroup of $SL(2, \mathbb{Z})$ that preserves the conditions $\mu = 0$, $\nu = 1/2$ is characterized by requiring that b should be even, which implies that a and d are odd. Hence

$$G(\tau) = \sum_{c,d|d\,\text{odd}} G_{c,d} = \sum_{c,d|d\,\text{odd}} G_{0,1}((a\tau + b)/(c\tau + d)), \tag{7.8}$$

where for given c, d, we pick a, b so that $ad - bc = 1$ and b is even. A modular transformation $T : \tau \to \tau + 1$ exchanges $(\mu, \nu) = (0, 1/2)$ with $(\mu, \nu) = (1/2, 1/2)$, so in particular

$$F(\tau) = G(\tau + 1) = F(\tau + 2), \tag{7.9}$$

and the summand in (7.8) does not depend on the choice of a, b.

The Ramond partition function $K = \text{Tr}_{\text{R}} \exp(-\beta H - i\theta J)$ is computed from $\mu = 1/2$, $\nu = 0$, so

$$K(\tau) = G(-1/\tau). \qquad (7.10)$$

This completes our characterization of three of the four partition functions. Finally, in any supersymmetric theory with discrete spectrum, the fourth partition function $I = \text{Tr}_{\text{R}}(-1)^F \exp(-\beta H - i\theta J)$ is an integer, independent of β and θ, since it can be interpreted as the index of a supersymmetry generator. It must be computed using the odd spin structure, the one with $\mu = \nu = 0$. In three-dimensional gravity, assuming that the partition function can be computed by summing over smooth three-geometries, I vanishes, since the odd spin structure does not extend over any three-manifold with boundary Σ.

General Structure

Some simple but important facts follow from (7.9). Since $F(\tau) = \text{Tr}_{\text{NS}} \exp(-\beta H - i\theta J)$ is invariant under $\tau \to \tau + 2$, it follows that all eigenvalues of $J = L_0 - \bar{L}_0$ take values in $\mathbb{Z}/2$. The transformation $\tau \to \tau + 1$ acts as 1 or -1 on states with integer or half-integer J; thus it acts as $(-1)^{2J}$. On the other hand, if we insert a factor of $(-1)^F$ in the trace, we get $G(\tau) = \text{Tr}_{\text{NS}}(-1)^F \exp(-\beta H - i\theta J)$. Since $G(\tau) = F(\tau + 1)$, the operator $(-1)^F$ is equivalent to $(-1)^{2J}$. In other words, states of integer or half-integer J are bosonic or fermionic, respectively. The exact theory (to the extent that it can be reconstructed from the sum over smooth classical geometries) inherits this property from the perturbative spectrum of Brown-Henneaux excitations.

The general form of $G(\tau)$, assuming that it has a Hilbert space interpretation, should therefore be

$$G(\tau) = \sum_{j \in \mathbb{Z}} \sum_n a_{n,j} \exp(-\beta E_{n,j} + i\theta j) - \sum_{j \in \mathbb{Z}+1/2} \sum_n a_{n,j} \exp(-\beta E_{n,j} + i\theta j). \qquad (7.11)$$

Here $E_{n,j}$, $n = 1, 2, 3, \ldots$, are the energy eigenvalues for states of angular momentum j, and $a_{n,j}$ is the number of states of energy $E_{n,j}$. The $a_{n,j}$ are positive integers; the minus sign preceding the second term in (7.11) reflects the relation $(-1)^{2J} = (-1)^F$.

7.2. The Computation

We we will now compute the partition functions of the previous section by summing over smooth geometries. We will consider the partition function $G(\tau) = \mathrm{Tr}_{\mathrm{NS}}\,(-1)^F \exp(-\beta H - i\theta J)$, as it is technically the simplest; the two other non-zero partition functions are then given by (7.9) and (7.10).

From (7.7) and (7.8) we have

$$G(\tau) = \sum_{c,d|\ d\ \mathrm{odd}} \left(|q|^{-k^*} \prod_{n \geq 2} \left| \frac{1 - q^{n-1/2}}{1 - q^n} \right|^2 \right)\Bigg|_\gamma. \tag{7.12}$$

To understand the modular transformation properties of this sum, it is useful to rewrite the infinite product in terms of Dedekind eta functions. Using the identities

$$\prod_{n=2}^{\infty} (1 - q^n)^{-1} = \frac{q^{1/24}(1 - q)}{\eta(\tau)} \tag{7.13}$$

and

$$\prod_{n=2}^{\infty} (1 - q^{n-1/2}) = \frac{q^{1/48}}{(1 - q^{1/2})} \frac{\eta(\tau/2)}{\eta(\tau)} \tag{7.14}$$

this may be written as

$$G(\tau) = \sum \left(|q|^{-k^*+3/24} |1 + q^{1/2}|^2 \frac{|\eta(\tau/2)|^2}{|\eta(\tau)|^4} \right)\Bigg|_\gamma. \tag{7.15}$$

We may now extract these Dedekind eta functions from the sum, using the fact that $\sqrt{\mathrm{Im}\,\tau}\,|\eta(\tau)|^2$ is modular invariant:

$$\begin{aligned}
G(\tau) &= \frac{|\eta(\tau/2)|^2}{y^{1/2}|\eta(\tau)|^4} \sum \left(y^{1/2} |q|^{-k^*+3/24} |1 - q^{1/2}|^2 \right)_\gamma \\
&= \frac{|\eta(\tau/2)|^2}{y^{1/2}|\eta(\tau)|^4} \Big(\widehat{E}(k^* - 3/24, 0) + \widehat{E}(k^* + 1 - 3/24, 0) + \\
&\qquad \widehat{E}(k^* + 1/2 - 3/24, 1/2) + \widehat{E}(k^* + 1/2 - 3/24, -1/2) \Big)
\end{aligned} \tag{7.16}$$

In the second line we have defined

$$\widehat{E}(\kappa, \mu) = \sum_{c,d|\ d\ \mathrm{odd}} \frac{y^{1/2}}{|c\tau + d|} \exp\left\{ 2\pi\kappa\,\mathrm{Im}\,\gamma\tau + 2\pi i\mu\,\mathrm{Re}\,\gamma\tau \right\}. \tag{7.17}$$

This is the supersymmetric version of the Poincaré series studied in Sec. 3.

This sum is divergent, for the same reasons described in Sec. 3.1. In particular, at large c and d the exponential approaches one and we are left with the linearly divergent sum $\sum_{c,d} |c\tau+d|^{-1}$. The sum may be regularized by considering the more general Poincaré series

$$\widehat{E}(s,\kappa,\mu) = \sum_{c,d|\; d \text{ odd}} \frac{y^s}{|c\tau+d|^{2s}} \exp\left\{2\pi\kappa \,\text{Im}\, \gamma\tau + 2\pi i\mu \,\text{Re}\, \gamma\tau\right\} \qquad (7.18)$$

as an analytic function of s and taking $s \to 1/2$. This regularization scheme can be justified on physical grounds, following the same line of argument presented in Sec. 3.1.

We will now proceed to compute the Fourier coefficients of the sum (7.18). Our calculation is quite similar to that done in Sec. 3. The only differences are that in (7.18), κ and μ are allowed to be half-integer, and that we are summing over c and d such that $(c,d) = 1$ and d is odd. These two conditions on c and d can be combined in to the single condition $(2c,d) = 1$.

We start by letting $d = d' + 2nc$, where $d' = d \bmod 2c$. The sum in (7.18) can be written as a sum over c, d', and n:

$$E(s,\kappa,\mu) = y^s e^{2\pi(\kappa y + i\mu x)} + \sum_{c>0} \sum_{d'\in\mathbb{Z}/2c\mathbb{Z}} \sum_{n\in\mathbb{Z}} f(c,d',n) \qquad (7.19)$$

where

$$f(c,d',n) = \frac{y^s}{|c(\tau+2n)+d'|^{2s}} \exp\left\{\frac{2\pi\kappa y}{|c(\tau+2n)+d'|^2} + 2\pi i\mu \left(\frac{a}{c} - \frac{cx+d}{c|c(\tau+2n)+d'|^2}\right)\right\}. \qquad (7.20)$$

We will now apply the Poisson summation formula to the sum over n, as we did for the bosonic partition function in Sec. 3.2. First, we must compute the Fourier transform

$$\widehat{f}(c,d',\widehat{n}) = \int_{-\infty}^{\infty} dn\; e^{2\pi i n \widehat{n}} f(c,d',n)$$

$$= \frac{1}{2} \exp\left(2\pi i \frac{(2\mu)a - \widehat{n}d'}{2c} - \pi i \widehat{n}x\right) \int_{-\infty}^{\infty} dt\; e^{\pi i \widehat{n}t} \left(\frac{y}{c^2(t^2+y^2)}\right)^s \exp\left\{\frac{2\pi(\kappa\, y - i\mu t)}{c^2(t^2+y^2)}\right\}. \qquad (7.21)$$

We have written the integral in terms of a shifted integration variable $t = 2n + x + \frac{d'}{c}$.

Recall that, as described after equation (7.7), we must chose our integers a and b such that b is even. The determinant condition $ad - bc = 1$ therefore implies that $ad' = 1 \bmod 2c$. Since $2\mu \in \mathbb{Z}$, this implies that the exponential prefactor in (7.21) is a function only of $d' = d \bmod 2c$. We may therefore extract the d' dependence of the sum (7.18) into the Kloosterman sum $S(-\widehat{n}, 2\mu; 2c)$, as defined in (3.21). The integral appearing in (7.21) is

precisely that defined in (3.24). The Fourier cofficients of the Poincaré series (7.18) are therefore given by

$$\widehat{E}(s,\kappa,\mu) = y^s e^{2\pi(\kappa y + i\mu x)} + \sum_{\widehat{n}} e^{\pi i \widehat{n} x} \widehat{E}_{\widehat{n}}(s,\kappa,\mu) \qquad (7.22)$$

where

$$\widehat{E}_{\widehat{n}}(s,\kappa,\mu) = \frac{1}{2} \sum_{m=0}^{\infty} I_{m,\widehat{n}/2}(s,\kappa,\mu) \, y^{1-m-s} \left(\sum_{c=1}^{\infty} c^{-2(m+s)} S(-\widehat{n}, 2\mu; 2c) \right) \qquad (7.23)$$

is defined in terms of the integrals (3.24).

We will now restrict our attention to the $\widehat{n} = 0$ case. First consider $\mu = 0$. In this case, the integrals were given in (3.26). To do the sum over c, note first that the Kloosterman sum $S(0,0,2c)$ is equal to Euler's totient function $\phi(2c)$. The sum over c is

$$\sum_{c>0} c^{-2(m+s)} S(0,0,2c) = \sum_{c>0} c^{-2(m+s)} \phi(2c) = \frac{2^{2(m+s)}}{2^{2(m+s)} - 1} \frac{\zeta(2(m+s)-1)}{\zeta(2(m+s))}. \qquad (7.24)$$

To prove this formula, we first evaluate the sum $\sum_{c \text{ odd}} c^{-\sigma} \phi(c)$, where we have defined $\sigma = 2(m+s)$. Multiplying this sum by

$$\sum_{n \text{ odd}} n^{-\sigma} = (1 - 2^{-\sigma})\zeta(\sigma) \qquad (7.25)$$

gives

$$\begin{aligned}
(1 - 2^{-\sigma})\zeta(\sigma) \sum_{c \text{ odd}} c^{-\sigma}\phi(c) &= \sum_{c,n \text{ odd}} (cn)^{-\sigma}\phi(c) \\
&= \sum_{m \text{ odd}} m^{-\sigma} \left(\sum_{c|m} \phi(c) \right) \qquad (7.26) \\
&= \sum_{m \text{ odd}} m^{-\sigma+1} = (1 - 2^{1-\sigma})\zeta(\sigma - 1)
\end{aligned}$$

In the second line we defined the new variable $m = cn$ to be summed over, and in the third line we have used the basic identity for the totient function $\sum_{c|m} \phi(c) = m$. We thus have

$$\sum_{c \text{ odd}} c^{-\sigma}\phi(c) = \frac{2^\sigma - 2}{2^\sigma - 1} \frac{\zeta(\sigma - 1)}{\zeta(\sigma)}. \qquad (7.27)$$

The sum over even c may be done by recalling the identity that was proved in Sec. 3 by similar methods:

$$\sum_{c>1} c^{-\sigma}\phi(\sigma) = \frac{\zeta(\sigma - 1)}{\zeta(\sigma)} \qquad (7.28)$$

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to get

$$\sum_{c \text{ even}} c^{-\sigma} \phi(c) = \frac{1}{2^{\sigma} - 1} \frac{\zeta(\sigma - 1)}{\zeta(\sigma)}. \tag{7.29}$$

But

$$\sum_{c \text{ even}} c^{-\sigma} \phi(c) = 2^{-\sigma} \sum_{c>0} c^{-\sigma} \phi(2c). \tag{7.30}$$

Combining the last two formulas, we arrive at (7.24).

As in the bosonic case, we must be careful when taking $s \to 1/2$. For $m = 0$ (and $\widehat{n} = 0$), the sum (7.24) vanishes at $s = 1/2$, whereas the integral $I_{0,0}$ has a pole at $m = 0$, $s = 1/2$, as we see in (3.26). The product of the two factors is finite; in fact, $\Gamma(s - 1/2)/\zeta(2s) \to 2$ at $s \to 1/2$. The $m > 0$ terms are finite without any such subtleties. Evaluating the first three terms in the expansion of (7.18) gives

$$\widehat{E}_0(1/2, \kappa, 0) = -y^{1/2} + \left(\frac{8\pi^3}{21\zeta(3)} \kappa \right) y^{-1/2} + \left(\frac{64\pi^6}{4185\zeta(5)} \kappa^2 \right) y^{-3/2} + \mathcal{O}(y^{-5/2}). \tag{7.31}$$

Let us now consider the $\mu = \pm 1/2$ terms. In this case the integrals are only slightly different from those done in Sec. 3, which were evaluated at $\mu = \pm 1$. For $m = 0$, we find that

$$I_{0,0}(s, \kappa, \pm 1/2) = \sqrt{\pi} \frac{\Gamma(s - 1/2)}{\Gamma(s)}. \tag{7.32}$$

This is the analog of equation (3.28) used in the computation of the bosonic partition function. For $m > 0$ the integrals are complicated hypergeometric functions of the sort written down in (3.27). At $s = 1/2$ these hypergeometric functions simply to give

$$I_{m,0}(1/2, \kappa, \pm 1/2) = \frac{2^{1-m} \pi^{m+1/2}}{m\Gamma(m + 1/2)} T_m(2\kappa). \tag{7.33}$$

This is the analog of equation (3.42) used in the bosonic case.

To do the sum over c, we use the fact that the Kloosterman sum $S(0, \pm 1, 2c)$ is equal to the Möbius function $\mu(2c)$. The sum over c is given by

$$\sum_{c>0} c^{-2(m+s)} S(0, \pm 1, 2c) = \sum_c c^{-2(m+s)} \mu(2c) = -\frac{2^{2(m+s)}}{(2^{2(m+s)} - 1)\zeta(2(m + s))}. \tag{7.34}$$

We can obtain this formula as follows. We first compute the sum $\sum_{c \text{ odd}} c^{-\sigma} \mu(c)$. Multiplying this sum by (7.25) we get

$$(1 - 2^{-\sigma})\zeta(\sigma) \sum_{c \text{ odd}} c^{-\sigma} \mu(c) = \sum_{c,n \text{ odd}} (cn)^{-\sigma} \mu(c)$$

$$= \sum_{m \text{ odd}} m^{-\sigma} \left(\sum_{c|m} \mu(c) \right). \tag{7.35}$$

$$= 1$$

In the second line we have let $m = cn$ and in the third line we have used the fact that for any m, $\sum_{c|m} \mu(c) = \delta_{m,1}$. The sum over even c may be done by recalling the similar identity from Sec. 3

$$\sum_{c>1} c^{-\sigma} \mu(\sigma) = \frac{1}{\zeta(\sigma)} \tag{7.36}$$

to get

$$\sum_{c \text{ even}} c^{-\sigma} \phi(c) = -\frac{1}{2^\sigma - 1} \frac{1}{\zeta(\sigma)}, \tag{7.37}$$

which is equivalent to (7.34).

At $s = 1/2$, we again must be careful to cancel the zero in (7.34) at $m = 0$ against a pole in (7.32), using the fact that $\Gamma(s - 1/2)/\zeta(2s) \to 2$ as $s \to 1/2$. Including the next two terms in the series, we find

$$E_0(1/2, \kappa, \pm 1/2) = -2y^{1/2} - \left(\frac{16\pi}{7\zeta(3)}\kappa\right) y^{-1/2} - \left(\frac{16\pi^2}{93\zeta(5)}(8\kappa^2 - 1)\right) y^{-3/2} + \mathcal{O}(y^{-5/2}) \tag{7.38}$$

Putting this all together gives the following expansion for the partition function

$$G(\tau) = G_{0,1}(\tau) + \frac{|\eta(\tau/2)|^2}{|\eta(\tau)|^4} \Big(-6 + \frac{(6 + 16k^*)(\pi^3 - 6\pi)}{21\zeta(3)} y^{-1}$$
$$- \frac{4\pi^2(2880k^{*2} - 16k^*(2\pi^4 - 135) + 45 - 12\pi^4)}{4185\zeta(5)} y^{-2} + \mathcal{O}(y^{-3}) \Big). \tag{7.39}$$

As in the bosonic case, a Hilbert space interpretation is precluded both by the fact that the coefficient of the leading correction to $G_{0,1}(\tau)$ is negative and by the fact that there are additional corrections involving powers of $1/y$. A Hilbert space interpretation would require the leading coefficient to be positive in view of (7.11); the leading correction to the Brown-Henneaux spectrum arises for an integer value of the angular momentum, namely $j = \widehat{n} = 0$.

We have so far considered just the $\widehat{n} = 0$ case. The discussion for the $\widehat{n} \neq 0$ case will be qualitatively similar to the bosonic case discussed in Sec. 3.4. The integrals can be evaluated explicitly – they are the same ones we considered in Sec. 3.4 – and contribute terms to the NS sector partition function which fall off exponentially at large y. As in the bosonic case, the sums over c cannot be expressed in terms of elementary number theoretic functions but are related to an appropriately defined Selberg zeta function. In particular, they should be related to the Selberg zeta function associated to the congruence subgroup described in Sec. 7.1, where b is constrained to be even. As in Sec. 3.4, this should provide

a finite regularization of the sum because the associated Laplacian does not have a discrete eigenfunction with eigenvalue 1/4.

We conclude this section by emphasizing a few possible interpretations of the result (7.39) for the supergravity partition function $G(\tau)$. As in the bosonic case, this partition function does not have the structure of a Hilbert space trace $\text{Tr}_{\text{NS}} \, (-1)^F e^{-\beta H - i\theta J}$. One possible interpretation of this result is that three-dimensional pure supergravity does not exist as a quantum theory, and that additional degrees of freedom must be included in order to render the theory sensible. A second possible implication is that the pure supergravity theory exists, but contains long strings, similar to those described in Sec. 4.1. A third possibility is that additional complex saddle points must be included in the sum over geometries, as described for the bosonic theory in Sec. 4.2. This might lead to a sum over two copies of the modular group, resulting in a holomorphic partition function which might coincide with the extremal supergravity partition functions described in [13].

7.3. Phase Transitions

In this section we will comment briefly on the supergravity generalization of the results of Sec. 6. In particular, we ask whether, for holomorphically factorized supergravity partition functions, Hawking-Page phase transitions occur via a condensation of zeroes along curves in the complex temperature plane. Numerical evidence described to us by M. Kaneko suggests that this may be the case.

Let us first ask what the phase diagram of three-dimensional supergravity should be. We will consider the NS partition function $F(\tau) = \text{Tr}_{\text{NS}} \, e^{-\beta H - i\theta J}$. As described in Sec. 7.1, $F(\tau)$ is invariant under the subgroup of $SL(2, \mathbb{Z})$ defined by the condition that $a + b$ and $c + d$ are both odd; this group is sometimes called Γ_θ. The tessellation of the upper half plane $\mathcal{H}$ by fundamental domains of Γ_θ is shown in Fig. 4a).
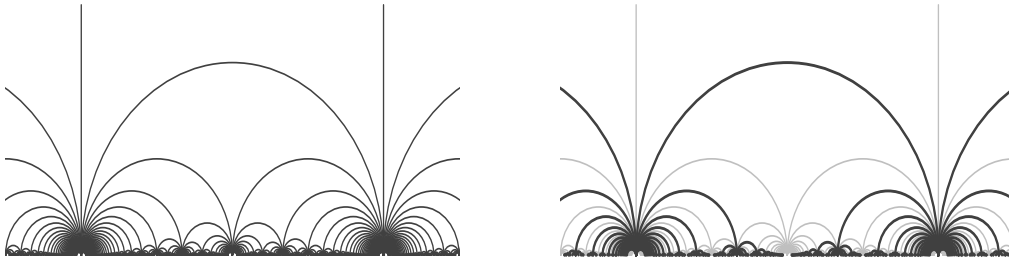

Figure 4: a) The tessellation of the upper half τ-plane by fundamental domains of Γ_θ. b) The subtessellation of the upper half plane corresponding to the coset $\Gamma_\theta/\mathbb{Z}$ (dark lines). This is the phase diagram of three-dimensional supergravity. The tessellation by Γ_θ (light lines) is shown to guide the eye.

Within this modular group Γ_θ, there is a subgroup isomorphic to $\mathbb{Z}$ generated by the translation $\tau \to \tau + 2$. This $\mathbb{Z}$ is the intersection of Γ_θ with the mapping class group of the handlebody $M_{0,1}$, so modular transformations in $\mathbb{Z}$ do not generate new handlebody contributions to the supergravity partition function. There is a unique saddle point contribution to $F(\tau)$ for each element of the coset $\Gamma_\theta/\mathbb{Z}$. The subtessellation of the upper half plane $\mathcal{H}$ associated to this coset $\Gamma_\theta/\mathbb{Z}$ is shown in Fig. 4b). As one moves between tiles in this diagram, we expect a first order Hawking-Page phase transition as different saddles become dominant. So the subtessellation depicted in Fig. 4b) should be interpreted as the phase diagram of three dimensional supergravity.

In [13], holomorphically factorized partition functions for three-dimensional gravity were considered. Numerically, it appears that[9] the zeroes of these partition functions lie on the dark curves in Figure 4b), and become dense on these curves in the large k limit [40]. This indicates that the Hawking-Page transition in supergravity occurs by the mechanism of Lee-Yang and Fischer, as in the bosonic case described in Sec. 6. It may be possible to prove this by extending the analytic arguments of [37] and Sec. 6.

Acknowledgments

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[9] Here we assume the numbers of Ramond and Neveu-Schwarz primary states at the black hole threshold to be small. In [13], there was no natural way to determine these numbers.

References

[1] S. Deser and R. Jackiw, "Three-Dimensional Cosmological Gravity: Dynamics Of Constant Curvature," Annals Phys. **153**, 405 (1984).

[2] J. D. Brown and M. Henneaux, "Central Charges In The Canonical Realization Of Asymptotical Symmetries: An Example From Three-Dimensional Gravity," Commun. Math. Phys. **104** (1986) 207-226.

[3] M. Bañados, C. Teitelboim, and J. Zanelli, "The Black Hole In Three-Dimensional Spacetime," Phys. Rev. Lett. **69** (1992) 1849-1851, hep-th/9204099.

[4] M. Bañados, M. Henneaux, C. Teitelboim and J. Zanelli, "Geometry of the (2+1) Black Hole," Phys. Rev. D **48**, 1506 (1993) [arXiv:gr-qc/9302012].

[5] J. Maldacena, "The Large N Limit Of Superconformal Field Theories And Supergravity," Adv. Theor. Math. Phys. **2** (1998) 231-252, hep-th/9711200.

[6] A. Strominger, "Black Hole Entropy From Near Horizon Microstates," JHEP 9802:009 (1998), hep-th/9712251.

[7] S. Carlip, "Conformal Field Theory, $(2+1)$-Dimensional Gravity, and the BTZ Black Hole," gr-qc/0503022.

[8] G. W. Gibbons, S. W. Hawking and M. J. Perry, "Path Integrals And The Indefiniteness Of The Gravitational Action," Nucl. Phys. B **138**, 141 (1978).

[9] J. M. Maldacena and A. Strominger, "AdS(3) Black Holes and a Stringy Exclusion Principle," JHEP **9812**, 005 (1998) [arXiv:hep-th/9804085].

[10] R. Dijkgraaf, J. Maldacena, G. Moore, and E. Verlinde, "A Black Hole Farey Tail," hep-th/0005003.

[11] D. B. Ray and I. M. Singer, "Analytic Torsion For Complex Manifolds," Annals Math. **98**, 154 (1973).

[12] X. Yin, "Partition Functions of Three-Dimensional Pure Gravity," arXiv:0710.2129 [hep-th].

[13] E. Witten, "Three-Dimensional Gravity Revisited," arXiv:0706.3359 [hep-th].

[14] J. Manschot, "AdS$_3$ Partition Functions Reconstructed," arXiv:0707.1159 [hep-th].

[15] S. W. Hawking and D. N. Page, "Thermodynamics Of Black Holes In Anti-De Sitter Space," Commun. Math. Phys. **87**, 577 (1983).

[16] S. Carlip and C. Teitelboim, "Aspects Of Black Hole Quantum Mechanics And Thermodynamics In (2+1)-Dimensions," Phys. Rev. D **51**, 622 (1995) [arXiv:gr-qc/9405070].

[17] S. Deser, R. Jackiw and G. 't Hooft, "Three-Dimensional Einstein Gravity: Dynamics Of Flat Space," Annals Phys. **152**, 220 (1984).

[18] A. Achúcarro and P. Townsend, "A Chern-Simons Action for Three-Dimensional anti-De Sitter Supergravity Theories," Phys. Lett. **B180** (1986) 89.

[19] E. Witten, "(2+1)-Dimensional Gravity as an Exactly Soluble System," Nucl. Phys. **B311** (1988) 46.

[20] E. Witten, "Coadjoint Orbits Of The Virasoro Group," Commun. Math. Phys. **114** (1988) 1-53.

[21] A. A. Bytsenko, L. Vanzo and S. Zerbini, "Quantum Correction to the Entropy of the (2+1)-Dimensional Black Hole," Phys. Rev. D **57**, 4917 (1998) [arXiv:gr-qc/9710106].

[22] H. Iwaniec, *Spectral Methods of Automorphic Forms*, American Mathematical Society, 2002.

[23] M. Kleban, M. Porrati and R. Rabadan, "Poincare Recurrences and Topological Diversity," JHEP **0410**, 030 (2004) [arXiv:hep-th/0407192].

[24] J. Maldacena, J. Michelson, and A. Strominger, "Anti de Sitter Fragmentation," JHEP 9902:011 (1999), hep-th/9812073.

[25] N. Seiberg and E. Witten, "The D1/D5 System and Singular CFT," JHEP 9904:017 (1999), hep-th/9903224.

[26] K. Krasnov, "On Holomorphic Factorization In Asymptotically AdS 3D Gravity," Class. Quant.Grav. **20** (2003(4015-4042, hep-th/0109198.

[27] G. Höhn, "Selbstduale Vertexoperatorsuperalgebren und das Babymonster," Ph.D. thesis (Bonn 1995), Bonner Mathematische Schriften 286 (1996), 1-85, English translation, arXiv:0706.0236.

[28] D. Gaiotto and X. Yin, "Genus Two Partition Functions of Extremal Conformal Field Theories," JHEP 0708:029 (2007), arXiv:0707.3437.

[29] X. Yin, "Partition Functions of Three-Dimensional Pure Gravity," arXiv:0710.2129.

[30] M. Gaberdiel, " Constraints on Extremal Self-Dual CFTs," arXiv:0707.4073.

[31] D. Gaiotto, to appear.

[32] I. B. Frenkel, J. Lepowsky, and A. Meurman, *Vertex Operator Algebras And The Monster* (Academic Press, Boston, 1988).

[33] S. Carlip, Class. Quant. Grav. **17**, 4175 (2000) [arXiv:gr-qc/0005017].

[34] C. N. Yang and T.-D. Lee, "Statistical Theory Of Equations Of State And Phase Transitions. I. Theory Of Condensation," Phys. Rev. (1952) **87** 404-9.

[35] M. Fisher, in *Lectures In Theoretical Physics* VIIC (University of Colorado Press, Boulder, 1965).

[36] R. Rankin, "The Zeros of Certain Poincaré Series," Compositio Mathmatica, **46** (1982) 255-272.

[37] T. Asai, M. Kaneko, H. Ninomiya, "Zeros of Certain Modular Functions and an Application," Commentarii Math. Univ. Sancti Pauli, vol. 46-1 (1997) 93-101.

[38] J. D. Bekenstein, "A Universal Upper Bound On The Entropy To Energy Ratio For Bounded Systems," Phys. Rev. D **23**, 287 (1981).

[39] T. Apostol, *Modular Functions and Dirichlet Series in Number Theory*, Springer Verlag, 1990.

[40] M. Kaneko, private communication.