Overview Of K-Theory Applied To Strings

Edward Witten

School of Natural Sciences, Institute for Advanced Study Olden Lane, Princeton, NJ 08540, USA

and

Department of Physics, Caltech, Pasadena CA 91125

and

CIT-USC Center For Theoretical Physics, Los Angeles CA

K-theory provides a framework for classifying Ramond-Ramond (RR) charges and fields. K-theory of manifolds has a natural extension to K-theory of noncommutative algebras, such as the algebras considered in noncommutative Yang-Mills theory or in open string field theory. In a number of concrete problems, the K-theory analysis proceeds most naturally if one starts out with an infinite set of D-branes, reduced by tachyon condensation to a finite set. This suggests that string field theory should be reconsidered for $N=\infty$.

1. Introduction And Definition Of K(X)

A D-brane wrapped on a submanifold S of spacetime may carry a nonzero Ramond-Ramond (RR) charge. RR fields are p-forms, and superficially it seems that the conserved charge should be measured by the cohomology class of the RR form (or of the cycle S itself). However, D-branes carry gauge fields; and gauge fields are not natural in (co)homology theory. They are natural in "K-theory." K-theory has been used to answer some questions about RR charges and fields; the aim of the present article is to give an overview of this, along with some speculations (for which I have only very modest evidence) about how one might want to rethink open string field theory in the large N limit. These subjects fill sections 1-3. Some mathematical details have been postponed to section 4.

If X is spacetime and $\mathcal{A}(X)$ is the commutative, associative algebra of continuous complex-valued functions on X, then the K-theory of X can be defined in terms of representations of $\mathcal{A}(X)$. A representation of a ring is usually called a module. Here are some examples of $\mathcal{A}(X)$ -modules.

The most obvious example of an $\mathcal{A}(X)$ -module is $\mathcal{A}(X)$ itself. For $f \in \mathcal{A}(X)$ (regarded as a ring) and $g \in \mathcal{A}(X)$ (regarded as a module), we define

$$f(g) = fg, (1.1)$$

where on the right hand side the multiplication occurs in $\mathcal{A}(X)$. This obviously obeys the defining condition of a module, which is that $(f_1f_2)(g) = f_1(f_2(g))$.

More generally, consider a Dp-brane (or a collection of N Dp-branes for some N > 0) wrapped on a submanifold S of X, with any Chan-Paton gauge bundle W on the D-brane. Let M(S) be the space of sections of W, that is, the space of one-particle states for a charged scalar coupled to the bundle W. Then M(S) is an A(X)-module; for $f \in A(X)$, $g \in M(S)$, we simply set again f(g) = fg. On the right hand side, the multiplication is defined by restricting f (which is a function on X) to S and then multiplying f and g.

So in, say, Type IIB superstring theory, a collection of D9-branes defines a representation or module E of $\mathcal{A}(X)$. A collection of $\overline{D9}$ -branes defines another module F. So any configuration of D9 and $\overline{D9}$ -branes determines a pair (E,F).

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To classify D-brane charge, we want to classify pairs (E, F) modulo physical processes. An important process [1] is brane-antibrane creation and annihilation – the creation or annihilation of a set of D9's and $\overline{D9}$'s each bearing the same gauge bundle G. This amounts to

$$(E,F) \leftrightarrow (E \oplus G, F \oplus G).$$
 (1.2)

The equivalence classes make up a group called K(X) (or K(A(X))) if we want to make the interpretation in terms of A(X)-modules more explicit). The addition law in this group is just

$$(E, F) + (E'F') = (E \oplus E', F \oplus F').$$
 (1.3)

The inverse of (E, F) is (F, E); note that $(E, F) \oplus (F, E) = (E \oplus F, E \oplus F)$, and using the equivalence relation (1.2), this is equivalent to zero. D-branes of Type IIB carry conserved charges that take values in K(X).

In the above definition of K(X), we used only ninebranes, even though, as we explained earlier, an $\mathcal{A}(X)$ module can be constructed using Dp-branes (or antibranes) for any p. In fact, we can classify D-brane charge just using the ninebranes, and then build the Dp-branes of p < 9 via pairs (E, F) with a suitable tachyon condensate. This construction (due to Atiyah, Bott, and Shapiro) was reviewed in [2], section 4, in the D-brane context. A more systematic explanation of the definition of K(X) in terms of modules, clarifying the role of ninebranes, can be found in section 4.

Advantages Of K-Theory Description

What do we gain by knowing that D-brane charge is classified by K-theory?

First of all, it is the right answer. Wherever one looks closely at topological properties of RR charges (or fields), one sees effects that reflect the K-theory structure. For example, there are stable D-brane states (like the nonsupersymmetric D0-branes of Type I) that would not exist if D-brane charge were classified by cohomology instead of K-theory. Conversely, it is possible to have a D-brane state that would be stable if D-brane charge were measured by cohomology, but which is in fact unstable (via a process that involves nucleation of D9- $\overline{D9}$ pairs in an intermediate state). This occurs [3] in Type II superstring theory, in which a D-brane wrapped on a homologically nontrivial cycle in spacetime is in fact in certain cases unstable. Finally, the K-theory interpretation of D-branes is needed [4] to make sense of a certain global worldvolume anomaly.

But I think that there is a deeper reason that it is good to know about the K-theory interpretation of D-branes: it may be naturally adapted for stringly generalizations. In fact

(though some mathematical details have been postponed to section 4), we defined K(X) in terms of representations of the algebra $\mathcal{A}(X)$ of functions on spacetime. We can similarly define $K(\mathcal{A})$ for any noncommutative algebra \mathcal{A} , in terms of pairs (E, F) of \mathcal{A} -modules. By contrast, we would not have an equally useful and convenient notion of "cohomology" if the algebra of functions on spacetime is replaced by a noncommutative ring.

For example, turning on a Neveu-Schwarz B-field, we can make $\mathcal{A}(X)$ noncommutative; the associated $K(\mathcal{A})$ was used by Connes, Douglas, and Schwarz in the original paper on noncommutative Yang-Mills theory applied to string theory [5]. This is an interesting example, even though it involves only the zero modes of the strings. One would much like to have a fully stringy version involving a noncommutative algebra constructed using all of the modes of the string, not just the zero modes.

What is the right noncommutative algebra that uses all of the modes? We do not know, of course. One concrete candidate is the *-algebra of open string field theory, defined in terms of gluing strings together. If I call this algebra \mathcal{A}_{st} , it seems plausible that D-brane charge is naturally labeled by $K(\mathcal{A}_{st})$. (For a manifold of very large volume compared to the string scale, I would conjecture that $K(\mathcal{A}_{st})$ is the same as the ordinary K(X) of topological K-theory.) I will come back to $K(\mathcal{A}_{st})$ in section 3.3. First, I want to finish our survey of known applications of K-theory in string physics.

2. K-Theory And RR Fields

K-theory is relevant to understanding RR fields as well as charges [6-8,3].

Naively speaking, an RR p-form field G_p obeys a Dirac quantization law according to which, for any p-cycle U in spacetime,

$$\int_{U} \frac{G_p}{2\pi} = \text{integer.} \tag{2.1}$$

If that were the right condition, then RR fields would be classified by cohomology.

But that is not the right answer, because the actual quantization condition on RR periods is much more subtle than (2.1). There are a variety of corrections to (2.1) that involve spacetime curvature and the gauge fields on the brane, as well as self-duality and global anomalies [9-13,7,8,3].

The answer, for Type IIB superstrings, turns out to be that RR fields are classified by $K^1(X)$. For our purposes, $K^1(X)$ can be defined as the group of components of the group of continuous maps from X to U(N), for any sufficiently large N.

This statement means that topological classes of RR fields on X are classified by a map $U: X \to U(N)$ for some large N. The relation of G_p to U is roughly $G_p \sim \text{Tr}(U^{-1}dU)^p$; here I have ignored corrections due to spacetime curvature and subtleties associated with self-duality of RR fields.

The physical meaning of U is not clear. For Type IIA, the analog is that RR fields are classified by a U(N) gauge bundle (for some large N) with connection A and curvature F_A , the relation being $G_p \sim \text{Tr } F_A^{p/2}$. The analog for M-theory involves E_8 gauge bundles with connection [13,14,3]. Again, the physical meaning of the U(N) or E_8 gauge fields is not clear.

The value of using K^1 to classify RR fields of Type IIB is that this gives a concise way to summarize the otherwise rather complicated quantization conditions obeyed by the RR fields. In addition, this framework is useful in describing subtle phase factors that enter in the RR partition function. In hindsight, once it is known that RR charges are classified by K-theory, one should have suspected a similar classification for RR fields. After all, RR charges produce RR fields! So the math used to classify RR charges must be similar to the math used to classify RR fields.

Just like K(X), $K^1(X)$ has an analog for any noncommutative algebra \mathcal{A} . Given \mathcal{A} , we let \mathcal{A}_N denote the group of invertible $N \times N$ matrices whose matrix elements are elements of \mathcal{A} . Then $K^1(\mathcal{A})$ is the group of components of \mathcal{A}_N , for large N.

For example, for $\mathcal{A} = \mathcal{A}(X)$ the ring of complex-valued continuous functions on X, \mathcal{A}_N is the group of maps of X to $GL(N, \mathbb{C})$. This is contractible to the group of maps of X to U(N), so for large N the group of components of \mathcal{A}_N is the same as $K^1(X)$, as we defined it initially.

The existence of a generalization of $K^1(X)$ for noncommutative rings means that the description of Type IIB RR fields by $K^1(X)$ in the long distance limit may be a useful starting point for stringy generalizations.

3. $N \to \infty$

In the last section, N was a sufficiently large but finite integer. Our next task will be to describe some things that depend on setting N equal to infinity.

Before doing so, let us recall the role of the $N \to \infty$ limit in physics. It is important in the conjectured link of gauge theory with strings; in the old matrix models that are used to give soluble examples of string theory; in the matrix model of M-theory; and in the correspondence between gravity in an asymptotically AdS spacetime and conformal field theory on the boundary.

My theme here will be to suggest that we should somehow study D-branes with $N = \infty$, with tachyon condensation to annihilate most of the branes and reduce to something more manageable. To motivate this, I will consider two concrete questions that seem to require taking the number of D-branes to be infinite. One question is the relation of Type IIA superstrings to K-theory; the other is the inclusion of a topologically non-trivial NS three-form field H in the K-theory classification of RR charges and fields.

After giving the talk, I became aware of a standard problem that has already been interpreted in terms of an infinite number of D9 and $\overline{D9}$ -branes with tachyon condensation to something manageable (see [15], sections 2.2, 4, and 5). This is a problem involving a D5-brane probe of a D5-D9 system.

3.1. Type IIA

For Type IIB superstrings, we used K(X) to classify RR charges, and $K^1(X)$ to classify RR fields.

The *T*-dual statement is that for Type IIA, $K^1(X)$ should classify RR charges, and K(X) should classify RR fields. (By Bott periodicity, $K^{i+2}(X) = K^i(X)$, so the only K-groups of X are $K^0(X)$, which we have called simply K(X), and $K^1(X)$.)

The most concrete and natural attempt to explain in general why $K^1(X)$ classifies RR charges for Type IIA is that of Horava [16]. The starting point here is to consider a system of N unstable D9-branes of Type IIA. The branes support a U(N) gauge field and a tachyon field T in the adjoint representation of U(N). There is a symmetry $T \to -T$.

The effective potential for the tachyon field is believed to have the general form

$$V(T) = \frac{1}{g_{st}} \operatorname{Tr} F(T), \tag{3.1}$$

where the function F(T) is non-negative and, after scaling T correctly, vanishes precisely if $T = \pm 1$. Hence V(T) is minimized if and only if every eigenvalue of T is ± 1 .

It was argued in [16] that, in flat \mathbf{R}^{10} , one can make supersymmetric Dp-branes (for even p) as solitons of T. For example, to make a D6-brane, we set N=2. Let \vec{x} be the three coordinates normal to the D6-brane, and set

$$T = \frac{\vec{\sigma} \cdot \vec{x}}{|x|} f(|x|), \tag{3.2}$$

where $f(r) \sim r$ for small r, and $f(r) \to \infty$ for $r \to \infty$. So for $|x| \to \infty$, the eigenvalues of T are everywhere ± 1 . Near x = 0, there is a topological knot that we interpret as the D6-brane.

In flat \mathbf{R}^{10} , one can similarly make Dp-branes for other even p. But on a general spacetime, this does not work for arbitrary Dp-branes unless we set $N=\infty$. The problem is most obvious if X, or at least its spatial part, is compact. The tachyon field T, being adjoint-valued, maps X to the Lie algebra of U(N); since the Lie algebra is contractible, T carries no topology. So a map from X to the Lie algebra does not represent an element of $K^1(X)$; indeed, it does not carry topological information at all.

To define an element of $K^1(X)$, we need the group, not the Lie algebra; a map $U: X \to U(N)$ does the job, as I have stated before.

Amazingly, as Atiyah and Singer showed long ago, we get back the right topology from the Lie algebra if we set $N=\infty$! We have to interpret $U(\infty)$ to be the unitary group \mathcal{U} of a Hilbert space \mathcal{H} of countably infinite dimension. (Such a Hilbert space is also called a separable Hilbert space.) We interpret the $N=\infty$ analog of the space of hermitian $N\times N$ matrices to be the space of bounded self-adjoint operators T on \mathcal{H} whose spectrum is as follows: there are infinitely many positive eigenvalues and infinitely many negative ones, and zero is not an accumulation point of the spectrum. (The last condition makes T a "Fredholm operator.") Physically, T should be required to obey these conditions, since they are needed to make the energy and the D8-brane charge finite. In fact, to make the energy finite, almost all the eigenvalues of T are very close to ± 1 . Anyway, with these conditions imposed on T, it turns out that the space of T's has the same topology as that of U(N) for large N.

So we can use tachyon condensation on a system of D9-branes to describe RR charges for Type IIA. But we have to start with infinitely many D9-branes, which then undergo tachyon condensation down to a configuration of finite energy.

Obstruction To Finite N

Let us now explain in more concrete terms the obstruction to making D-branes in this way for finite N, and how it vanishes for $N = \infty$.

Let us go back to the example of a D6-brane constructed with 2 D9-branes. We took the transverse directions to be a copy of \mathbb{R}^3 , and the tachyon field to be

$$T = \frac{\vec{\sigma} \cdot \vec{x}}{|x|} f(|x|). \tag{3.3}$$

If we try to compactify the transverse directions to S^3 , we run into trouble because T is not constant at infinity. The conjugacy class of T is constant at infinity – the eigenvalues of T are everywhere 1 and -1 – but T itself is not constant.

Moreover, T is not homotopic to a constant at infinity. If T were homotopic to a constant near infinity, we would deform it to be constant and then extend it over S^3 . But it is not homotopic to a constant.

The basic obstruction to making T constant at infinity is the "magnetic charge." Let \mathbf{S}^2 be a sphere at infinity in \mathbf{R}^3 . Over \mathbf{S}^2 , we can define a line bundle \mathcal{L}_+ whose fiber is the +1 eigenspace of T, and a line bundle \mathcal{L}_- whose fiber is the -1 eigenspace of T. The line bundles \mathcal{L}_+ and \mathcal{L}_- are topologically nontrivial – their first Chern classes are respectively 1 and -1. As long as we try to deform T preserving the fact that its eigenvalues are 1 and -1, the line bundles \mathcal{L}_+ are well-defined, and their first Chern classes are invariant. So the nontriviality of \mathcal{L}_+ (or \mathcal{L}_-) prevents us from making a homotopy to constant T.

Let us add some additional "spectator" D9-branes, and see if anything changes. Suppose there are M=2k additional branes, so that the total number of branes is N=2+M=2+2k. Let the tachyon field be $T'=T\oplus U$, where T is as above and U is the sum of k copies of the matrix

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{3.4}$$

acting on the 2k additional branes.

Thus T' has near infinity k+1 eigenvalues +1 and k+1 eigenvalues -1. The +1 eigenspace of T' is a vector bundle V_+ of first Chern class 1 (since it is constructed by adding a trivial bundle to \mathcal{L}_+), and the -1 eigenspace of T' is similarly a vector bundle V_- of first Chern class -1. In particular, V_+ and V_- are nontrivial, so we have not gained anything by adding the spectator branes: T' is not homotopic to a constant, and cannot be extended over infinity. The nontriviality of V_+ is controlled by $\pi_1(U(k+1)) = \mathbf{Z}$, which is associated with the existence of a first Chern class.

Instead, what happens if we set $k = \infty$? To be more precise, we take the number of spectator D9-branes to be countably infinite, and assume $T' = T \oplus U$, where U is the direct sum of countably many copies of the matrix in (3.4). We can still define the bundles V_+ and V_- ; their fibers are separable Hilbert spaces (that is, Hilbert spaces of countably infinite dimension). U(k+1) is replaced by \mathcal{U} , the unitary group of a separable Hilbert space. Now we run into the fundamental fact (Kuiper's theorem) that \mathcal{U} is contractible;

its homotopy groups are all zero. Thus, any bundle of separable Hilbert spaces is trivial. In particular, V_+ and V_- are trivial, so T' is homotopic to a constant and can be extended over infinity.

So if the total number of unstable D9-branes is $N = \infty$, we can make a D6-brane localized at a point in \mathbf{S}^3 . More generally, in view of the result of Atiyah and Singer, we can starting at $N = \infty$ build an arbitrary class in $K^1(X)$ via tachyon condensation.

In terms of applying this result to physics, there are a few issues that we should worry about. One question is simply whether it is physically sensible to start with infinitely many branes and rely on tachyon condensation to get us down to something of finite energy. We will have to leave this question to the future (but see [15] for an example).

Quite a different question is whether the answer that we have obtained by setting $N = \infty$ is the right one for physics. In the field of a D6-brane that is localized at a point on S^3 , the equation for the RR two-form field G_2 (of which the D6-brane is a magnetic source) has no solution, since "the flux has nowhere to go."

It seems that the situation is that $N=\infty$ corresponds to the correct answer in classical open string theory, where the effective action comes from worldsheets with the topology of a disc. The RR fields enter as a correction of relative order g_{st} (the closed string coupling constant) coming from worldsheets with cylinder topology, and should be ignored in the classical approximation.

The classification of D-branes by brane creation and annihilation holds at $g_{st} = 0$, and (as we recalled in section 1) leads for Type IIB to a classification of D-brane charge by K(X). To get the analogous answer – namely $K^1(X)$ – for Type IIA via unstable D9-branes and tachyon condensation, we need to start at $N = \infty$.

Intuitively, in the absence of tachyon condensation, $N = \infty$ should correspond to $g_{st} = 0$, since the effective expansion parameter for open strings is $g_{st}N$. If N is infinite, then prior to tachyon condensation, g_{st} must be zero, or the quantum corrections diverge. If we want g_{st} to be nonzero, we need tachyon condensation to reduce to an effective finite value of N.

3.2. Turning On An H-Field

A somewhat analogous problem is to consider D-branes when the Neveu-Schwarz three-form field H is topologically nontrivial. We will carry out this discussion in Type IIB (for Type IIA, we would have to combine what follows with what we said above in the absence of the H-field).

Just as at H = 0, we would like to classify D-brane states by pairs (E, F) (where E is a D9 state and F is a $\overline{D9}$ state) subject to the usual sort of equivalence relation. But there is a problem in having a D9 state in the presence of an H-field.

In fact, when H is topologically non-trivial, one cannot have a single D9-brane. On the D9-brane, there is a U(1) gauge field with field strength F. The relation dF = H shows, at the level of de Rham cohomology, that H must be topologically trivial if a single D9-brane is present. This conclusion actually holds precisely, not just in de Rham cohomology.

There is a special case in which there is a comparatively elementary cure for this difficulty (see [2], section 5.3 and [17]). If H is torsion, that is if there is an integer M > 0 such that MH is topologically trivial, then it is possible to have a set of M D9-branes whose "gauge bundle" actually has structure group $U(M)/\mathbf{Z}_M$, rather than U(M). (The obstruction to lifting the $U(M)/\mathbf{Z}_M$ bundle to a U(M) bundle is determined by H.) We will call such a gauge bundle a twisted bundle. More generally, for any positive integer m, we can have N = mM D9-branes with the structure group of the bundle being $U(mM)/\mathbf{Z}_M$. In such a situation, D-brane charge is classified, as one would guess, by pairs (E, F) of twisted bundles (or D9 and $\overline{D9}$ states) subject to the usual equivalence relation. The equivalence classes make a group $K_H(X)$.

If one wishes to interpret $K_H(X)$ as the K-theory of representations of an algebra, one must pick a particular twisted bundle W and consider a D-brane state with boundary conditions determined by W. The W-W open strings transform in the adjoint representation, so the gauge parameters of the zero mode sector of the open strings are sections of $W \otimes \overline{W}$. Notice that although W is a twisted bundle (with structure group $U(M)/\mathbf{Z}_M$ rather than U(M)), $W \otimes \overline{W}$ is an ordinary bundle, since the center acts trivially in the adjoint representation.

The sections of $W \otimes \overline{W}$ form an algebra, defined as follows: if $s^i{}_j$ and $t^k{}_l$ are sections of $W \otimes \overline{W}$ – where the upper and lower indices are respectively W- and \overline{W} -valued – then their product is $(st)^i{}_l = \sum_k s^i{}_k t^k{}_l$. This is the algebra $\mathcal{A}_W(X)$ of all endomorphisms or linear transformations of the bundle W. The algebra of open string field theory, for W-W open strings, reduces to \mathcal{A}_W if one looks only at the zero modes of the strings. This is a sensible approximation at low energies in a limit in which X is very large compared to the string scale.

If H is zero and W is a trivial rank one complex bundle, then $\mathcal{A}_W(X)$ is our friend $\mathcal{A}(X)$. If H is zero and W is a trivial rank N complex bundle, then including W means

simply that there are $N \times N$ Chan-Paton matrices everywhere. So in this case, $\mathcal{A}_W(X) = \mathcal{A}(X) \otimes M_N$, where M_N is the algebra of $N \times N$ complex-valued matrices. In general, whatever H is, W is always trivial locally, so locally $\mathcal{A}_W(X)$ is isomorphic to $\mathcal{A}(X) \otimes M_N$.

A twisted bundle is equivalent to an \mathcal{A}_W -module, and the group $K_H(X)$ of pairs (E,F) of twisted bundles (modulo the usual equivalence) coincides with $K(\mathcal{A}_W)$, the K-group of \mathcal{A}_W -modules. This will be explained in section 4. This assertion leads to an immediate puzzle; $K_H(X)$ as defined in terms of pairs (E,F) of twisted bundles is manifestly independent of W while $K(\mathcal{A}_W)$ appears to depend on W. Indeed, given any two distinct twisted bundles W and W', the corresponding algebras \mathcal{A}_W and $\mathcal{A}_{W'}$ are distinct, but are "Morita-equivalent." This concept is explained in section 4, where we also show that the Morita equivalence implies that $K(\mathcal{A}_W) = K(\mathcal{A}_{W'})$.

So far, we have only considered the case that H is torsion. A typical example, important in the AdS/CFT correspondence, is the spacetime $X = \text{AdS}_5 \times \text{RP}^5$, where a torsion H-field on RP^5 is used to describe Sp(n) rather than SO(2n) gauge theory in the boundary CFT.

However, in most physical applications, H is not torsion. In that case, we must somehow take a large M limit of what has been said above. The right way to do this has been shown by Bouwknegt and Mathai [18] and Atiyah and Segal [19]. The suitable large M limit of $U(M)/\mathbf{Z}_M$ is $PU(\mathcal{H}) = U(\mathcal{H})/U(1)$. In other words, for $M = \infty$, one replaces U(M) by the unitary group $U(\mathcal{H})$ of a separable Hilbert space \mathcal{H} ; and one replaces \mathbf{Z}_M by U(1). This means, in particular, that when H is not torsion, one cannot have a finite set of D9- or \overline{D} 9-branes, but one can have an infinite set, with a suitable infinite rank twisted gauge bundle E or F. Then D-brane charge is classified by the group K_H of pairs (E,F) modulo the usual equivalence relation. A detailed explanation can be found in [18]. Kuiper's theorem – the contractibility of $\mathcal{U} = U(\mathcal{H})$ – plays an important role, as it did in section 3.1.

This construction, in the $M=\infty$ limit, has the beautiful property, explained in [18], that the noncommutative algebra whose K-group is K_H is unique, independent of any arbitrary choice of twisted bundle W or W'. This really depends on the number of D9 and $\overline{D9}$ branes being infinite.

3.3. Stringy Generalization?

At the risk of going out on a somewhat shaky limb, I will now try to propose a stringy generalization of some of this.

We start with a closed string background and try to form an open string algebra. To do so, we must pick an open string boundary condition – call it α . Then, in open string field theory, the α - α open strings form an algebra \mathcal{A}_{α} .

This algebra is not unique; we could instead pick another boundary condition β and define another algebra \mathcal{A}_{β} .

The example of strings in a background H-field that is torsion, and the associated algebras \mathcal{A}_W , $\mathcal{A}_{W'}$, suggests the nature of the relation between \mathcal{A}_{α} and \mathcal{A}_{β} : they are different algebras, but are Morita-equivalent and hence have the same K-theory.

The assertion that $\mathcal{A}_{\alpha} \neq \mathcal{A}_{\beta}$, if true, is quite troublesome. It is a sharp statement of the lack of manifest background independence of open string field theory. (Background independence in this context means independence of the open string background; of course, classical open string field theory depends on a closed string background in which the open strings propagate.) It means that the formalism depends on which open string background one uses in setting up the theory.

The example of open strings in an H-field suggests a cure: take the number of D9and $\overline{D9}$ -branes to be infinite.

Let $m\alpha$ and $m\beta$ denote m copies of α or β (that is, α or β supplemented with U(m)-valued Chan-Paton factors). Then the conjecture is that $\mathcal{A}_{m\alpha} \neq \mathcal{A}_{m\beta}$ for any finite m, but that they are equal for $m = \infty$. Actually, the statement about what happens for $m = \infty$ can be formulated much more precisely by analogy with statements in [18]. Let \mathcal{K} be the algebra of compact operators in a separable Hilbert space \mathcal{H} . Then the conjecture is that for any α and β ,

$$\mathcal{A}_{\alpha} \otimes \mathcal{K} = \mathcal{A}_{\beta} \otimes \mathcal{K}. \tag{3.5}$$

The main evidence for the conjecture is that, according to [18], the corresponding statement ($\mathcal{A}_W \neq \mathcal{A}_{W'}$ but $\mathcal{A}_W \otimes \mathcal{K} = \mathcal{A}_{W'} \otimes \mathcal{K}$) holds for the zero mode algebra in the presence of a torsion H-field. In addition, if it is necessary to tensor with \mathcal{K} before the algebras become isomorphic, this helps explain why background independence in open string field theory is so hard to understand; in other words it helps explain why the theory constructed with classical solution α and algebra \mathcal{A}_{α} looks different from the theory constructed with classical solution β and algebra \mathcal{A}_{β} . However, the evidence for the conjecture

is quite limited; it may be that whenever α is continuously connected to β , $\mathcal{A}_{\alpha} = \mathcal{A}_{\beta}$, and that the difficulty in understanding background independence in open string field theory is "just" a technical problem.

If the conjecture is right, one would suspect that to get a greater degree of background independence in open string field theory, we should start with infinitely many D9-branes and rely on tachyon condensation to get us down to something reasonable.

For this to be useful, we would need a description of $\mathcal{A}_{\alpha} \otimes \mathcal{K}$ much simpler and more incisive than any description we have today for \mathcal{A}_{α} or \mathcal{A}_{β} . In such a hypothetical new description, the BRST operator Q might be harder to describe. If that happened, we would have to take our lumps!

4. Mathematical Details

This concluding section will be devoted to explaining a few of the mathematical points that we have skimmed over so far.

Let \mathcal{A} be a ring – in fact, let us momentarily assume that \mathcal{A} is commutative and associative. An \mathcal{A} -module M is called "free" if it is a direct sum of copies of \mathcal{A} : $M = \mathcal{A} \oplus \mathcal{A} \oplus \ldots \oplus \mathcal{A}$. An \mathcal{A} -module M is called "projective" if there is another \mathcal{A} -module M' such that $M \oplus M'$ is free.

Let us consider what these definitions mean in case $\mathcal{A} = \mathcal{A}(X)$ is the ring of continuous complex-valued functions on a manifold X. We can think of \mathcal{A} as the space of sections of a trivial complex line bundle \mathcal{O} . Given a trivial rank n complex vector bundle $V = \mathcal{O} \oplus \mathcal{O} \oplus \ldots \oplus \mathcal{O}$, the space of sections of V is a free module M which is the sum of n copies of \mathcal{A} .

Now suppose we are given any complex vector bundle E over X. The space of sections of E is an A-module M(E) (given a function $a \in A(X)$ and a section m of E, we simply define a(m) to be the product am). M(E) is a free module if and only if E is a trivial vector bundle. But M(E) is always projective. Indeed, there is always a "complementary" vector bundle E such that $E \oplus F$ is trivial, and hence $M(E) \oplus M(F) = M(E \oplus F)$ is free. Conversely (by a theorem of Serre and Swan), every projective A(X)-module is M(E) for some complex vector bundle E over X.

² In the informal spirit of our discussion, I have generally used the same name E for a bundle and the corresponding module. In this paragraph only, I distinguish E from M(E) in the notation, to facilitate the statement of the Serre-Swan theorem.

For any ring \mathcal{A} , the projective modules form a semigroup: if E and E' are projective modules, so is $E \oplus E'$. Given any semigroup (with an addition operation that we will write as \oplus), there is a canonical way to form a group. This is done the same way that one builds the group of integers, starting with the semigroup of positive integers. An element of the group is a pair (E, F), with E and F elements of the semigroup (so in our situation, E and E are projective modules) and subject to the equivalence relation $(E, F) \cong (E \oplus G, F \oplus G)$ for any element E in the semigroup. Pairs are added by E0, the equivalence classes form a group (the zero element or additive identity is E1 of or any E2, and the additive inverse of E3, since E4, since E5, since E6, since E6, since E7, and the group formed this way is called E8.

For $\mathcal{A} = \mathcal{A}(X)$, we have in section 1 defined $K(\mathcal{A})$ just in terms of vector bundles or in other words D9- and $\overline{D9}$ -brane configurations. Now we see how to describe this restriction in a way that has broader validity: the ninebranes correspond to projective modules, and the definition of K(X) in terms of ninebranes is a special case of the definition of $K(\mathcal{A})$ for any ring \mathcal{A} in terms of projective modules.

Now let us drop the assumption that \mathcal{A} is commutative (but keep associativity). If \mathcal{A} is a noncommutative ring, one can canonically associate with it a second ring \mathcal{A}^{op} (called the opposite ring) whose elements are in one to one correspondence with elements of \mathcal{A} , but which are multiplied in the opposite order. Thus, for every element $a \in \mathcal{A}$, there is a corresponding $a^{op} \in \mathcal{A}^{op}$, with the multiplication law of \mathcal{A}^{op} being $a^{op}b^{op} = (ba)^{op}$.

For a noncommutative ring, we must distinguish two types of modules, left-modules and right-modules. \mathcal{A} acts on the left on a left-module M, the basic axiom being that for $a, a' \in \mathcal{A}$ and $m \in M$, one has (aa')(m) = a(a'(m)). \mathcal{A} acts on the right on a right-module M, the basic axiom being that $m \cdot (aa') = (m \cdot a) \cdot a'$. A left-module of \mathcal{A} is the same as a right-module of \mathcal{A}^{op} , and vice-versa.

The definition of projective module is the same as in the commutative case: a left or right \mathcal{A} -module M is projective if there is another left or right \mathcal{A} -module N such that $M \oplus N$ is a free left or right \mathcal{A} -module. From the semigroup of projective left \mathcal{A} modules, we form a group called $K(\mathcal{A})$. We can also form a semigroup from the projective right \mathcal{A} -modules; it is the same as the semigroup of projective left \mathcal{A}^{op} -modules, so the associated abelian group is $K(\mathcal{A}^{op})$. For the rings we are interested in, there is generally an operation of complex (or hermitian) conjugation that maps \mathcal{A} to \mathcal{A}^{op} . Consequently, the K-groups formed using left or right A-modules are equivalent to each other under complex conjugation.

Given two rings \mathcal{A} and \mathcal{B} , a bimodule (or $(\mathcal{A}, \mathcal{B})$ -bimodule) is a group M which is simultaneously a left \mathcal{A} -module and a right \mathcal{B} -module, with the left action of \mathcal{A} and the right action of \mathcal{B} commuting. For $a \in \mathcal{A}$, $b \in \mathcal{B}$, and $m \in M$, the left action of \mathcal{A} is denoted a(m) = am, the right action of \mathcal{B} is denoted b(m) = mb, and since we assume that \mathcal{A} and \mathcal{B} commute, we have (am)b = a(mb); we abbreviate these expressions as amb.

We can now define a very important relationship between rings, called Morita equivalence. It was first exploited in connection with D-branes in [5]; for further developments see [20]. Let M be an $(\mathcal{A}, \mathcal{B})$ -bimodule. The choice of M enables us to define a map from left \mathcal{B} -modules to left \mathcal{A} -modules: given a left \mathcal{B} -module W, we map it to the left \mathcal{A} -module $W' = M \otimes_{\mathcal{B}} W$. Similarly, if we are given a $(\mathcal{B}, \mathcal{A})$ -bimodule N, we can map a left \mathcal{A} -module W' to a left \mathcal{B} -module W'' by $W'' = N \otimes_{\mathcal{A}} W'$. If these two operations are inverse to each other, in other words if one has W'' = W for all W, we say that \mathcal{A} and \mathcal{B} are Morita-equivalent. In this case, the semigroup of left \mathcal{A} -modules is isomorphic to the semigroup of left \mathcal{B} -modules, and $K(\mathcal{A}) = K(\mathcal{B})$.

The meaning of this relationship in open string field theory was explained in [21], section 6.4. For any open string boundary condition α , one defines a ring \mathcal{A}_{α} consisting of the α - α open strings. For any other boundary condition γ , the α - γ open strings form a left \mathcal{A}_{α} -module $M_{\alpha\gamma}$, and the γ - α open strings form a right \mathcal{A}_{α} -module $M_{\gamma\alpha}$. Presumably these are in a suitable sense a basis of left and right \mathcal{A}_{α} -modules. Now consider any other open string boundary condition β and the associated algebra \mathcal{A}_{β} . One forms left and right \mathcal{A}_{β} modules $M_{\beta\gamma}$ and $M_{\gamma\beta}$ in the same way, using β - γ and γ - β open strings. Obviously, there is a natural association between left \mathcal{A}_{α} modules and left \mathcal{A}_{β} modules, namely

$$M_{\alpha\gamma} \leftrightarrow M_{\beta\gamma}.$$
 (4.1)

So $K(\mathcal{A}_{\alpha}) = K(\mathcal{A}_{\beta})$ for all α and β . We established this last fact directly without talking about Morita equivalence, but in fact, it seems that the relation between \mathcal{A}_{α} and \mathcal{A}_{β} is that they are Morita-equivalent. Indeed, one has natural $(\mathcal{A}_{\alpha}, \mathcal{A}_{\beta})$ and $(\mathcal{A}_{\beta}, \mathcal{A}_{\alpha})$ bimodules $M_{\alpha\beta}$ and $M_{\beta\alpha}$ constructed from the α - β and β - α open strings; it seems very likely that these bimodules have the right properties to establish a Morita equivalence between \mathcal{A}_{α} and \mathcal{A}_{β} .

Now let us consider a more elementary version of this, with the goal of adding some details to the discussion in section 3.2. Let M_n be the algebra of $n \times n$ complex matrices.

The most obvious left M_n -module is the space W of n-component column vectors. This is a projective module because, as a left module, M_n consists of n copies of W (M_n consists of $n \times n$ matrices; in the left action of M_n on itself, M_n acts separately on each of the n columns making up the $n \times n$ matrix, so these comprise n copies of W). So if W' is the sum of n-1 copies of W, then $W \oplus W'$ is free, and thus W is projective. It can be shown that every projective left M_n module is the sum of copies of W. So the semigroup of projective left M_n modules is isomorphic to the semigroup of non-negative integers, and $K(M_n) = \mathbf{Z}$.

Since $K(M_n)$ is thus independent of n, one might wonder if the M_n 's of different n are Morita-equivalent. This is indeed so. For any positive integers n and k, the most obvious (M_n, M_k) bimodule is the space $C_{n,k}$ of $n \times k$ matrices, with M_n acting on the left and M_k on the right. Likewise, the most obvious (M_k, M_n) bimodule is $C_{k,n}$. It can be shown that these bimodules establish the Morita equivalence between M_n and M_k .

Now, we reconsider the torsion H-field that was discussed in section 3.2. For any twisted bundle W, there is an algebra $\mathcal{A}_W(X)$ generated by the ground states of the W-W open strings. An element of $\mathcal{A}_W(X)$ is a section of $W \otimes \overline{W}$. Given any two different twisted bundles W and W', we can try to make a Morita equivalence between $\mathcal{A}_W(X)$ and $\mathcal{A}_{W'}(X)$ by specializing to this situation the more abstract discussion of general open string algebras \mathcal{A}_{α} and \mathcal{A}_{β} that was given above. This means that we should find $(\mathcal{A}_{W}(X), \mathcal{A}_{W'}(X))$ and $(\mathcal{A}_{W'}(X), \mathcal{A}_{W}(X))$ bimodules by looking at the ground states of the W-W' and W'-W open strings. The ground states of the W-W' open strings are sections of $W \otimes \overline{W'}$; the space of these sections is a $(\mathcal{A}_W(X), \mathcal{A}_{W'}(X))$ bimodule M. Likewise, the space of ground states of the W'-W open strings is a $(\mathcal{A}_{W'}(X), \mathcal{A}_{W}(X))$ bimodule N consisting of the sections of $W' \otimes \overline{W}$. The bimodules M and N do indeed establish a Morita equivalence between $\mathcal{A}_W(X)$ and $\mathcal{A}_{W'}(X)$. Indeed, in proving this one can work locally on X. Locally, W and W' are trivial, and (if n and k are the ranks of W and W') we have locally as noted in section 3.2, $\mathcal{A}_W(X) = \mathcal{A}(X) \otimes M_n$, $\mathcal{A}_{W'}(X) = \mathcal{A}(X) \otimes M_k$. The same computation that gives a Morita equivalence of M_n with M_k shows that $\mathcal{A}_W(X)$ is Morita equivalent to $\mathcal{A}_{W'}(X)$.

A projective right $\mathcal{A}_W(X)$ module is the space of sections of $E \otimes \overline{W}$, for any twisted bundle E. (If $s^{\alpha}{}_{j}$ is such a section – where α is an E index and j a \overline{W} index – and $w^{k}{}_{l}$ is a section of $W \otimes \overline{W}$, then the product of s and w is defined by $(sw)^{\alpha}{}_{j} = \sum_{k} s^{\alpha}{}_{k} w^{k}{}_{j}$.) So projective right $\mathcal{A}_W(X)$ modules are in natural one-to-one correspondence with twisted bundles E, and the K-group of projective right $\mathcal{A}_W(X)$ modules is the group of pairs

(E, F), subject to the usual equivalence relation. The Morita equivalence of \mathcal{A}_W and $\mathcal{A}_{W'}$, for any two twisted bundles W and W' with the same H, is just a fancy way of describing the natural map between \mathcal{A}_W -modules and $\mathcal{A}_{W'}$ -modules that comes from the correspondence

$$E \otimes \overline{W} \leftrightarrow E \otimes \overline{W'}.$$
 (4.2)

This one-to-one map between right \mathcal{A}_W -modules and right $\mathcal{A}_{W'}$ -modules (or the corresponding map for left modules) leads to the conclusion that $K(\mathcal{A}_W) = K(\mathcal{A}_{W'})$.

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