QUANTUM BACKGROUND INDEPENDENCE

IN STRING THEORY

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ABSTRACT

Not only in physical string theories, but also in some highly simplified situations, background independence has been difficult to understand. It is argued that the "holomorphic anomaly" of Bershadsky, Cecotti, Ooguri, and Vafa gives a fundamental explanation of some of the problems. Moreover, their anomaly equation can be interpreted in terms of a rather peculiar quantum version of background independence: in systems afflicted by the anomaly, background independence does not hold order by order in perturbation theory, but the exact partition function as a function of the coupling constants has a background independent interpretation as a state in an auxiliary quantum Hilbert space. The significance of this auxiliary space is otherwise unknown.

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1. Background Independence And The Holomorphic Anomaly

Finding the right framework for an intrinsic, background independent formulation of string theory is one of the main problems in the subject, and so far has remained out of reach. Moreover, some highly simplified special cases or analogs of the problem, which look like they might be studied for practice, have also resisted understanding.

An important example is the problem of understanding the mirror map in the theory of mirror symmetry. In (2,2) compactification on a Calabi-Yau threefold X, one encounters two sets of renormalizable Yukawa couplings, involving modes coming from $H^{1,1}(X)$ and $H^{2,1}(X)$. These are closely related to two twisted topological field theories that can be constructed for a given Calabi-Yau target space X – the A model and the B model.

Mirror symmetry is a relation between two Calabi-Yau manifolds X and Y, in which the Yukawa couplings involving $H^{1,1}(X)$ are identified with those that involve $H^{2,1}(Y)$, and vice-versa. Equivalently, mirror symmetry exchanges the A model of X with the B model of Y, and vice-versa.

The moduli space of sigma models with a Calabi-Yau target space is locally a product of two factors. One factor, the moduli space \mathcal{M}_A of the A model, is (an open set in) $H^{1,1}(X, \mathbb{C}/2\pi i\mathbb{Z})$. The other factor, the moduli space \mathcal{M}_B of the B model, is the moduli space of complex structures on X. Mirror symmetry therefore implies a natural map between $\mathcal{M}_A(X)$ and $\mathcal{M}_B(Y)$. This seems bizarre because – being related to the linear space $H^{1,1}(X,\mathbb{C}) - \mathcal{M}_A$ has a natural "flat" (or really, affine linear) structure, while \mathcal{M}_B , the moduli space of complex structures, has no such natural structure.

Candelas et. al. [1] tried to overcome this problem by using the "special coordinates" on \mathcal{M}_B . Special coordinates are background dependent. They are defined as follows. Pick a complex structure J_0 on X representing a base-point in \mathcal{M}_B . Let Ω_0 be a three-form on X holomorphic with respect to the complex structure

 J_0 . Let J be a variable complex structure on X, and let Ω_J be a holomorphic three-form in the complex structure J, normalized so that if one decomposes Ω_J with respect to the Hodge structure defined by J_0 , then the (3,0) part of Ω_J is cohomologous to Ω_0 . Then [2] the map from \mathcal{M}_B to the (2,1) part of the cohomology class of Ω_J is locally an isomorphism from \mathcal{M}_B to the linear space $H^{2,1}(X_0, \mathbb{C})$ (here X_0 is X with complex structure J_0). This gives the desired "flat" structure on \mathcal{M}_B , the "special coordinates" being the components of the (2,1) part of Ω_J .

The trouble with this is of course the dependence on the base-point J_0 . According to Morrison [3], the mirror of the natural flat structure on $\mathcal{M}_A(X)$ is the flat structure determined on $\mathcal{M}_B(Y)$ by a base-point at infinity. In this context, "infinity" refers to a particular type of degeneration of the complex structure of X; neither existence nor uniqueness of such a degeneration is apparent. (Lack of uniqueness can lead to a multiple mirror phenomenon and topology change [4].) So we come to our first problem:

(1) What is the analog in the A model of choosing a base-point in the B model? Why does the obvious flat structure on the parameter space of the A model correspond to the B model with a base-point at infinity?

Apart from using Calabi-Yau threefolds for compactification of physical string theories, one can use the twisted A or B topological field theories to construct topological string theories which one might think would be a highly simplified laboratory for studying background independence in string theory. Indeed, for open string versions of either the A or B model, there is no problem [5] in identifying the background independent space-time physics, which moreover is local in some important cases. But the natural attempt at extracting an effective space-time field theory for closed topological strings (see §5 of [5]) gives a result that is non-local and is background dependent in the case of the B model. So, even if we put aside the non-locality, we have our second problem of background independence:

[†] Upon contraction with Ω_0 , $H^{2,1}(X_0, \mathbb{C})$ can be identified with $H^1(X_0, T^{1,0}X_0)$, which is the tangent space to \mathcal{M}_B at its chosen base-point X_0 . This is the natural linear space with which one might try to identify \mathcal{M}_B near X_0 .

(2) What is the origin of the background dependence in the space-time field theory of the closed string B model?

A special case of these problems is so simple that it seems worthy of pointing out separately. The usual analysis of the B model appears to show that in genus zero, in expanding around an arbitrary base-point J_0 , the partition function vanishes together with its first two derivatives (while the third derivative gives the Yukawa couplings). So our third question is obviously:

(3) Since background independence appears to require that the choice of basepoint J_0 should play no special role, does not vanishing of the genus zero partition function at J_0 require that function to vanish identically?

Another cluster of questions that may be similar involves soluble string theories in $D \leq 2$. At least in the case of D=2, where one has a two dimensional spacetime, a graviton-dilaton system with a black hole solution, a tachyon scattering matrix, etc., one would hope to find an intrinsic description of the space-time geometry of these theories. Yet this has proved surprisingly elusive. A possible relation of these problems to those discussed above is suggested by the fact that the $D \leq 2$ string theories can be interpreted via twisted N=2 models, as first shown for D < 2 by K. Li [6] and argued more recently for D=2 [7–9].

So our final problem – which we will actually not discuss in this paper – is the following:

(4) Either describe the background independent space-time physics of D=2 (or maybe $D \leq 2$) string theory or describe the obstruction to doing so.

1.1. Role Of The Holomorphic Anomaly

Recently, Bershadsky, Cecotti, Ooguri, and Vafa [10], following earlier work [11,12], described a "holomorphic anomaly" in topological field theories obtained by twisting N=2 models. Their anomaly can be understood as a violation of naive background independence and I will presently argue that it explains most of the puzzles cited above.

Consider an N=2 supersymmetric theory that contains chiral or twisted interactions, in addition to other interactions that we will bury in a Lagrangian L_0 . There is no essential loss in considering the chiral case. If W_a are the chiral primary fields, then the Lagrangian

$$L = L_0 - \sum_a t^a \int d^2x \, d^2\theta \, W_a - \sum_a \overline{t}^a \int d^2x \, d^2\overline{\theta} \, \overline{W}_a, \qquad (1.1)$$

with complex parameters t^a , describes a family of N=2 theories. One can also consider twisted topological field theories with two of the supersymmetries, the ones that generate shifts in $\overline{\theta}$, identified with BRST operators Q_{\pm} . Then the W_a are the physical observables, and $\int \mathrm{d}^2x\,\mathrm{d}^2\overline{\theta}\,\overline{W}_a$ is formally irrelevant being $\{Q_+, [Q_-, \overline{W}_a]\}$. After twisting (1.1), it is natural to add the physical observables as perturbations, considering a more general Lagrangian

$$L = L_0 - \sum_{a} (t^a + u^a) \int d^2x \, d^2\theta \, W_a - \sum_{a} \overline{t}^a \int d^2x \, d^2\overline{\theta} \, \overline{W}_a. \tag{1.2}$$

Formally, the BRST machinery appears to show that the topological observables of the topological field theory (1.2) are independent of \overline{t}^a , and so are only functions of $t^a + u^a$. This is where the holomorphic anomaly comes in; Bershadsky et. al. show that the topological observables really have a dependence on \overline{t}^a determined by their holomorphic anomaly. I want to interpret this as a failure of background independence. The idea is that the choice of \overline{t}^a determines the original "physical" Lagrangian (1.1), which was then twisted and perturbed by topological observables with coefficients u^a . The dependence on \overline{t}^a means that the particular "physical" theory that one started with is not forgotten. It defines a base-point in the space of theories.

Let us change the notation, renaming t+u as t and \overline{t} as t'. Henceforth \overline{t} denotes the complex conjugate of t; t' is an independent complex variable. Now (1.2) takes

a more symmetric-looking form,

$$L = L_0 - \sum_a t^a \int d^2x \, d^2\theta \, W_a - \sum_a t'^a \int d^2x \, d^2\overline{\theta} \, \overline{W}_a. \tag{1.3}$$

The symmetry between t and t' is broken by the choice of twisting. The t'^a determine a base-point in the space of topological couplings, the point at which $\bar{t}^a = t'^a$ and at which the theory can actually be interpreted as a twisting of a physical theory. The BRST argument appears to show that in the twisted model, the topological observables depend only on the t^a and not on the t'^a , but the anomaly obstructs this.

Now we can rather easily dispose of the first three problems identified above:

- (1) The symmetry in structure between A and B models is restored because just as the B model naturally depends on a base-point in the space of complex structures, the A model also depends on a choice of base-point, namely a choice of a point $t'^a \in H^{1,1}(X, \mathbb{C}/2\pi i\mathbb{Z})$. There is therefore a more general family of A-like models than hitherto realized. Bershadsky et. al. show in detail that the traditional A model (in which correlation functions are given by standard instanton sums) corresponds to the case in which $t'^a \to \infty$, that is the case in which the A model is taken to have a base-point at infinity. The mirror of the traditional A model is therefore naturally a B model with base-point at infinity. A more general B model with another base-point would be mirror to an A model with a finite base-point t'^a .
- (2) Because of the holomorphic anomaly, background independence of the field theory of the B model is not expected.
- (3) The paradox involving the genus zero free energy F_0 is also eliminated once one abandons naive background independence. One is merely left with the statement that if the model is constructed with a base-point t'^a , then F_0 vanishes together with its first two derivatives near $t^a = \overline{t}'^a$.

The fourth problem on our above list – the space-time physics of soluble string theories – should perhaps also be reexamined in light of the holomorphic anomaly, and its possible cousins.

1.2. Salvaging Something From Background Independence

Though the interpretation of the holomorphic anomaly as an obstruction to background independence eliminates some thorny puzzles, it is not satisfactory to simply leave matters at this. Is there some more sophisticated sense in which background independence does hold?

In thinking about this question, it is natural to examine the all orders generalization of the holomorphic anomaly equation, which (in the final equation of their paper) Bershadsky et. al. write in the following form. Let F_g be the genus g free energy. Then

$$\overline{\partial}_{i'}F_g = \overline{C}_{i'j'k'}e^{2K}G^{jj'}G^{kk'}\left(D_jD_kF_{g-1} + \frac{1}{2}\sum_r D_jF_r \cdot D_kF_{g-r}\right). \tag{1.4}$$

This equation can be written as a linear equation for $Z = \exp\left(\frac{1}{2}\sum_{g=0}^{\infty}\lambda^{2g-2}F_g\right)$, namely

$$\left(\overline{\partial}_{i'} - \lambda^2 \overline{C}_{i'j'k'} e^{2K} G^{jj'} G^{kk'} D_j D_k\right) Z = 0. \tag{1.5}$$

This linear equation is called a master equation by Bershadsky et. al.; it is similar in structure to the heat equation obeyed by theta functions. (I am here using the notation of Bershadsky et. al., but later, we will make some changes in notation.)

It would be nice to interpret (1.5) as a statement of some sophisticated version of background independence. In thinking about this question, a natural analogy arises with Chern-Simons gauge theory in 2+1 dimensions. In that theory, an initial value surface is a Riemann surface Σ . In the Hamiltonian formulation of the theory, one constructs a Hilbert space \mathcal{H} upon quantization on Σ . \mathcal{H} should be obtained by quantizing a certain classical phase space \mathcal{W} (a moduli space of

flat connections on Σ). Because the underlying Chern-Simons Lagrangian does not depend on a choice of metric, one would like to construct \mathcal{H} in a natural, background independent way. In practice, however, quantization of \mathcal{W} requires a choice of polarization, and there is no natural or background independent choice of polarization.

The best that one can do is to pick a complex structure J on Σ , whereupon \mathcal{W} gets a complex structure. Then a Hilbert space \mathcal{H}_J is constructed as a suitable space of holomorphic functions (really, sections of a line bundle) over \mathcal{W} . We denote such a function as $\psi(a^i;t'^a)$ where a^i are complex coordinates on \mathcal{W} and t'^a are coordinates parametrizing the choice of J.

Now background independence does not hold in a naive sense; ψ cannot be independent of t'^a (given that it is to be holomorphic on \mathcal{W} in a complex structure that depends on t'^a). But there is a more sophisticated sense in which background independence can be formulated [13,14]. The \mathcal{H}_J 's can be identified with each other (projectively) using a (projectively) flat connection over the space of space of J's. This connection ∇ is such that a covariantly constant wave function should have the following property: as J changes, ψ should change by a Bogoliubov transformation, representing the effects of a change in the representation used for the canonical commutation relations.

Using parallel transport by ∇ to identify the various \mathcal{H}_J 's, one can speak of "the" quantum Hilbert space, of which the \mathcal{H}_J 's are realizations determined by a J-dependent choice of representation of the canonical commutators. Background independence of $\psi(a^i;t'^a)$ should be interpreted to mean that the quantum state represented by ψ is independent of the t'^a , or equivalently that ψ is invariant under parallel transport by ∇ . Concretely, this can be written as an equation

$$0 = \left(\frac{\partial}{\partial t'^a} - \frac{1}{4} \left(\frac{\partial J}{\partial t'^a} \omega^{-1}\right)^{ij} \frac{D}{Da^i} \frac{D}{Da^j}\right) \psi \tag{1.6}$$

that is analogous to the heat equation for theta functions. We will discuss this equation further in §2.

(1.5) and (1.6) have an obvious similarity. Our goal in the rest of this paper will be to make this similarity more precise, introducing an auxiliary quantum system with quantum Hilbert space \mathcal{H} and interpreting (1.5) as the statement that the vector in \mathcal{H} determined by the partition function Z is independent of a choice of polarization. For X a Calabi-Yau threefold, the phase space of the auxiliary system will be simply $\mathcal{W} = H^3(X, \mathbb{R})$. \mathcal{W} has a natural symplectic structure given by the intersection pairing

$$\omega(\alpha, \beta) = \int_{X} \alpha \wedge \beta. \tag{1.7}$$

W has no natural complex structure, but every choice of complex structure on X determines a complex structure on W via the Hodge decomposition. Then it turns out (at least up to a c-number factor) that (1.5) can be interpreted to mean that the quantum state determined by the partition function Z is independent of the base-point.

Though this interpretation of the holomorphic anomaly is elegant, its rationale remains obscure. What really is the origin of the phase space W, what is the significance of the Hilbert space \mathcal{H} , and why should it be possible to interpret the partition function as a J-independent vector in \mathcal{H} ? In the case of Chern-Simons theory, these questions are answered by appealing to the underlying three dimensional Chern-Simons action and field theory, but in the present case, it is not clear where an answer would come from.

2. Quantum Background Independence

2.1. Quantization

Before plunging into our specific problem, I will first recall some generalities about quantization of linear spaces. (See the introduction to [14] for more detail.) We consider a linear space $W \cong \mathbb{R}^{2n}$ with a constant symplectic structure, say $\omega = \frac{1}{2}\omega_{ij}\mathrm{d}x^i\mathrm{d}x^j$, with ω_{ij} a constant invertible matrix and the x^i linear coordinates on \mathbb{R}^{2n} . The symbol ω^{-1} will denote the matrix inverse to ω , obeying $\omega_{ij}\omega^{-1jk} = \delta_i{}^k$. To give the problem of quantization its most natural (but perhaps not entirely familiar) formulation [15–18], one begins by introducing a "prequantum line bundle"; this is a unitary line bundle \mathcal{L} with a connection whose curvature is $-i\omega$. Up to isomorphism, there is only one such choice of \mathcal{L} . One can take \mathcal{L} to be the trivial unitary line bundle, with a connection given by the covariant derivatives

$$\frac{D}{Dx^i} = \frac{\partial}{\partial x^i} + \frac{i}{2}\omega_{ij}x^j. \tag{2.1}$$

Then one can introduce the "prequantum Hilbert space" \mathcal{H}_0 which consists of \mathbf{L}^2 sections of \mathcal{L} .

A vector in \mathcal{H}_0 is represented by a function with a rather general dependence on all 2n coordinates x^i . The quantum Hilbert space is instead to be a comparatively "small" subspace of \mathcal{H}_0 consisting of functions that depend freely on only n of the coordinates. There is no natural way to choose which n coordinates are allowed; such a choice is called a choice of polarization.

We will consider a polarization defined by a choice of a complex structure J on W with the following properties:

- (a) J is translation invariant, so it is defined by a constant matrix $J^{i}{}_{j}$ with $J^{2}=-1$.
 - (b) The two-form ω is of type (1,1) in the complex structure J. In components

this means that $J^{i'}{}_{i}J^{j'}{}_{j}\omega_{i'j'}=\omega_{ij}$ or equivalently (using $J^2=-1$)

$$J^{i'}{}_{i}\omega_{i'j} = -J^{j'}{}_{j}\omega_{ij'}. (2.2)$$

Since the curvature of the prequantum line bundle \mathcal{L} is proportional to ω , having ω be of type (1,1) means that the (0,2) part of the curvature vanishes, so that the connection on \mathcal{L} endows it with a complex structure.

(c) J is positive in the sense that the metric g defined by $g(v, w) = \omega(v, Jw)$ is positive.

Given such a J, we define the quantum Hilbert space \mathcal{H}_J to consist of vectors in \mathcal{H}_0 that are represented by functions that are holomorphic in the complex structure J.

 \mathcal{H}_J is a quantization of the symplectic manifold \mathcal{W} ; we want to exhibit a flat connection over the parameter space of the J's that will enable us to identify the \mathcal{H}_J 's. Construction of such a connection enables one to speak of "the" quantum Hilbert space \mathcal{H} which has realizations depending on the choice of J. Actually, the connection will only be projectively flat, so this will only work up to a scalar multiple.

To write down the connection, some notation is useful. First of all, one has projection operators

$$\frac{1}{2}\left(1 \mp iJ\right) \tag{2.3}$$

that project onto the (1,0) and (0,1) parts of a vector. It is convenient (as in [14]) to introduce a special notation for vectors that have been so projected. For any vector v^i , we write

$$v^{\underline{i}} = \frac{1}{2} (1 - iJ)^{i}{}_{j} v^{j}$$

$$v^{\overline{i}} = \frac{1}{2} (1 + iJ)^{i}{}_{j} v^{j}.$$
(2.4)

Similarly for a one-form w_i , we write

$$w_{\underline{j}} = \frac{1}{2} (1 - iJ)^{i}{}_{j} w_{i}$$

$$w_{\overline{j}} = \frac{1}{2} (1 + iJ)^{i}{}_{j} w_{i}.$$
(2.5)

For example, J itself has non-zero components $J^{\underline{i}}_{\underline{j}} = i\delta^{\underline{i}}_{\underline{j}}$, $J^{\overline{i}}_{\overline{j}} = -i\delta^{\overline{i}}_{\overline{j}}$ (which means that projections of J^{i}_{j} and δ^{i}_{j} are proportional).

Let \mathcal{M} be the space of J's obeying conditions (a), (b), and (c) above; it is a copy of the Siegel upper half plane. \mathcal{M} has a natural complex structure, defined as follows. The condition $J^2 = -1$ implies that for δJ a first order variation of J, one must have $J \cdot \delta J + \delta J \cdot J = 0$. This means that the non-zero projections of δJ are $\delta J^{\underline{i}}_{\overline{j}}$ and $\delta J^{\overline{i}}_{\underline{j}}$. We give \mathcal{M} a complex structure by declaring $\delta J^{\underline{i}}_{\overline{j}}$ to be of type (1,0) and $\delta J^{\overline{i}}_{\underline{j}}$ to be of type (0,1).

Over \mathcal{M} we now introduce two Hilbert space bundles. One of them, say \mathcal{H}^0 , is the trivial bundle $\mathcal{M} \times \mathcal{H}_0$ whose fiber is the fixed Hilbert space \mathcal{H}_0 . (Recall that the definition of \mathcal{H}_0 was independent of J.) The second is the bundle, say \mathcal{H}^Q , whose fiber over a point $J \in \mathcal{M}$ is the Hilbert space \mathcal{H}_J . \mathcal{H}^Q is a sub-bundle of \mathcal{H}^0 ; a section of \mathcal{H}^0 is an arbitrary function $\psi(x^i; J)$, while a section of \mathcal{H}^Q is a $\psi(x^i; J)$ which for each given J is, as a function of the x^i , holomorphic in the complex structure defined by J:

$$\frac{D}{Dx^{\overline{i}}}\psi = 0. {2.6}$$

(This equation has a dependence on J coming from the projection operators used in defining $x^{\overline{i}}$.)

A connection on \mathcal{H}^0 restricts to a connection on \mathcal{H}^Q if and only if its commutator with $D_{\overline{i}}$ is a linear combination of the $D_{\overline{j}}$. For instance, since \mathcal{H}^0 is defined

as the product bundle $\mathcal{M} \times \mathcal{H}_0$, there is a trivial connection δ on this bundle:

$$\delta = \sum_{i,j} \mathrm{d}J^i{}_j \frac{\partial}{\partial J^i{}_j}.$$
 (2.7)

Thus, $\psi(x^i; J)$ is annihilated by δ if and only if it is independent of J. We can expand δ in (1,0) and (0,1) pieces, $\delta = \delta^{(1,0)} + \delta^{(0,1)}$, with

$$\delta^{(1,0)} = \sum_{i,j} dJ^{i} \frac{\partial}{\partial J^{i}_{j}}$$

$$\delta^{(0,1)} = \sum_{i,j} dJ^{i} \frac{\partial}{\partial J^{i}_{j}}.$$
(2.8)

The commutation relation

$$\left[\delta, D_{\overline{i}}\right] = \left[\delta, \frac{1}{2} \left(D_i + iJ^k{}_i D_k\right)\right] = \frac{i}{2} dJ^k{}_i D_k \tag{2.9}$$

shows that $[\delta, D_{\overline{i}}]$ is not a linear combination of the $D_{\overline{j}}$ and hence that δ does not descend to a connection on \mathcal{H}^Q .

Rather, we must add an extra term that reflects the effects of the Bogoliubov transformation on the quantum state. The appropriate connection is actually

$$\nabla^{(1,0)} = \delta^{(1,0)} - \frac{1}{4} \left(dJ \omega^{-1} \right)^{\underline{i}\underline{j}} \frac{D}{Dx^{\underline{i}}} \frac{D}{Dx^{\underline{j}}}.$$

$$\nabla^{(0,1)} = \delta^{(0,1)}.$$
(2.10)

Indeed, the commutator $[\nabla^{(0,1)}, D_{\overline{i}}]$ is a linear combination of the $D_{\overline{j}}$, as one can see by using (2.9). The commutator $[\nabla^{(1,0)}, D_{\overline{i}}]$ vanishes. To see this, one uses, in addition to (2.9), the defining relation

$$[D_{\overline{i}}, D_{\underline{j}}] = -i\omega_{\overline{i}\underline{j}} \tag{2.11}$$

of the prequantum line bundle \mathcal{L} , and the relation

$$(\mathrm{d}J\omega^{-1})^{ij} = (\mathrm{d}J\omega^{-1})^{ji},\tag{2.12}$$

which follows from differentiating (2.2).

So ∇ descends to a connection on \mathcal{H}^Q . Now let us compute the curvature of ∇ . The (0,2) part of the curvature vanishes trivially, since $\nabla^{(0,1)} = \delta^{(0,1)}$. The (2,0) part of the curvature can be seen to vanish using $[D_{\underline{i}}, D_{\underline{j}}] = 0$ and also (2.9). Let us work out in detail the (1,1) part of the curvature. This is

$$\left[\nabla^{(0,1)}, \nabla^{(1,0)}\right] = \left[\delta^{(0,1)}, -\frac{1}{4} \left(dJ\omega^{-1}\right)^{\underline{ij}} D_{\underline{i}} D_{\underline{j}}\right]. \tag{2.13}$$

The only J dependence that $\delta^{(0,1)}$ can act on is in the projection operators implicit in the definition of the indices \underline{i}, j . So we make those projection operators explicit:

$$(dJ\omega^{-1})^{\underline{i}\underline{j}}D_{\underline{i}}D_{\underline{j}} = (dJ\omega^{-1})^{ij}\frac{1}{2}(\delta^{i'}{}_{i} - iJ^{i'}{}_{i})\frac{1}{2}(\delta^{j'}{}_{j} - iJ^{j'}{}_{j})D_{i'}D_{j'}.$$
(2.14)

Inserting this in (2.13), we get

$$\left[\nabla^{(0,1)}, \nabla^{(1,0)}\right] = \frac{i}{8} \left(dJ\omega^{-1} \right)^{\underline{i}\underline{j}} \delta^{(0,1)} J^{i'}{}_{i} \left(D_{i'} D_{j} + D_{j} D_{i'} \right). \tag{2.15}$$

To proceed further, we restrict ∇ and its curvature form to \mathcal{H}^Q . According to (2.8), the only non-zero components of $\delta^{(0,1)}J^{i'}{}_j$ are $\mathrm{d}J^{\overline{i'}}{}_{\underline{j}}$. The right hand side of (2.15) can therefore be simplified using the fact that $D_{\overline{i'}}$ annihilates sections of \mathcal{H}^Q and using $[D_{\overline{i'}}, D_j] = -i\omega_{\overline{i'}j}$. One gets finally

$$\left[\nabla^{(0,1)}, \nabla^{(1,0)}\right] = -\frac{1}{8} \mathrm{d}J^{\underline{i}}_{\overline{k}} \mathrm{d}J^{\overline{k}}_{\underline{i}}.$$
 (2.16)

Thus, the curvature is not zero, even when restricted to \mathcal{H}^Q . But it is a c-number, that is, it depends only on J and not on the variables x^i that are being quantized. It is possible to eliminate this central curvature by adding to ∇ a one-form that depends on J only or – to formulate this more invariantly – by tensoring \mathcal{H}^Q by the pullback of a line bundle on \mathcal{M} endowed with a connection whose curvature is minus that of ∇ .

The fact that the curvature of ∇ is a c-number means that parallel transport by ∇ is unique up to a scalar factor (which moreover is of modulus 1 since the curvature is real or more fundamentally since ∇ is unitary). So up to this factor one can identify the various \mathcal{H}_J 's, and regard them as different realizations of "the" quantum Hilbert space \mathcal{H} .

2.2. Application To Calabi-Yau Manifolds

The symplectic manifold that we want to quantize is the linear space $\mathcal{W} = H^3(X, \mathbb{R})$, X being a Calabi-Yau threefold. On \mathcal{W} there is a natural symplectic form,

$$\omega(\alpha, \beta) = \int_{X} \alpha \wedge \beta. \tag{2.17}$$

Every complex structure on X determines a complex structure on W, which can be used to quantize W. So we get a family of quantizations of W, parametrized by the Teichmuller space T of complex structures on X up to isotopy. We will see that the natural connection on the family of quantum Hilbert spaces over T is the anomaly equation of Bershadsky et. al.

First, we recall some facts about variation of Hodge structures on X. (A convenient reference is [19].) The usual complex structure on \mathcal{T} is the one in which the (1,0) part of a variation δJ of the complex structure of X is $\delta J^{\underline{i}}_{\overline{j}}$ or differently put, in which the holomorphic tangent space to \mathcal{T} is the $\overline{\partial}$ cohomology group $H^1(X, T^{1,0}X)$. Let Ω be a holomorphic three-form on X that varies holomorphically in t. A basis of the complexification of \mathcal{W} is given by

$$V_{0} = \Omega$$

$$V_{a} = \frac{\partial \Omega}{\partial t^{a}}$$

$$\overline{V}_{a} = \frac{\partial \overline{\Omega}}{\partial \overline{t}^{a}}$$

$$\overline{V}_{0} = \overline{\Omega}.$$

$$(2.18)$$

In practice, we will be working near some base-point $t \in \mathcal{T}$. One can normalize Ω so

that, at t, the V_a are of type (2,1) (and hence the \overline{V}_b of type (1,2)). If this is done, then ω becomes block diagonal, the non-zero matrix elements being $\omega(V_0, \overline{V}_0)$ and $\omega(V_a, \overline{V}_b)$. The latter are related to the natural metric $g_{\underline{a}\overline{b}}$ on $H^{2,1}(X)$:

$$\omega(V_a, \overline{V}_b) = \int_X \frac{\partial \Omega}{\partial t^a} \wedge \frac{\partial \overline{\Omega}}{\partial \overline{t}^b} = ig_{\underline{a}\overline{b}}.$$
 (2.19)

We will use on \mathcal{T} a similar notation to that which we used on \mathcal{M} – the (1,0) and (0,1) projections of a vector v^a will be written as $v^{\underline{a}}$, $v^{\overline{a}}$.

The metric $g_{\underline{a}\overline{b}}$ in (2.19) is closely related to the Zamolodchikov metric $G_{\underline{a}\overline{b}}$ on the space $H^1(X,T^{1,0}X)$ of physical states. In fact, there is a natural map from $H^1(X,T^{1,0}X)$ to $H^{2,1}(X)$ by contracting with a holomorphic three-form; G and g are related by this map, so $g_{\underline{a}\overline{b}}=G_{\underline{a}\overline{b}}e^{-K}$, where e^{-K} is the natural metric on the space of holomorphic three-forms.

The Yukawa couplings are

$$C_{\underline{a}\underline{b}\underline{c}} = -\int_{Y} \Omega \wedge \frac{\partial^{3}\Omega}{\partial t^{a}\partial t^{b}\partial t^{c}} = \int_{Y} \frac{\partial\Omega}{\partial t^{a}} \wedge \frac{\partial^{2}\Omega}{\partial t^{b}\partial t^{c}}$$
(2.20)

(and other projections of C vanish). The second derivative $\partial^2 \Omega/\partial t^a \partial t^b$ is a linear combination of forms of type (3,0), (2,1), and (1,2). A non-vanishing contribution on the right hand side of (2.20) comes only from the (1,2) part, which is necessarily a linear combination of the (1,2) forms $\partial \overline{\Omega}/\partial \overline{t}^b$. Comparing coefficients, we find

$$\frac{\partial^2 \Omega}{\partial t^a \partial t^b} = -iC_{\underline{a}\underline{b}\underline{c}}g^{\underline{c}\overline{c}}\frac{\partial \overline{\Omega}}{\partial \overline{t}^c} \mod d(\ldots) \oplus H^{(2,1)} \oplus H^{(3,0)}. \tag{2.21}$$

We still need a few more formulas. First of all, since Ω is of type (3,0) and J acts on an index of type (1,0) or (0,1) as multiplication by i or -i, one has

$$J^{i'}{}_{i}J^{j'}{}_{j}J^{k'}{}_{k}\Omega_{i'j'k'} = -i\Omega_{ijk}.$$
 (2.22)

Differentiating this and using the fact that $\partial_a \Omega$ is of type (2,1), we get

$$\frac{\partial J^{i'}_{i}}{\partial t^{a}} \Omega_{i'jk} + \text{cyclic permutations of } ijk = 2i \frac{\partial \Omega_{ijk}}{\partial t^{a}}.$$
 (2.23)

Similarly, using the fact that that $\partial_b \Omega$ is of type (2,1), we have

$$J^{i'}{}_{i}J^{j'}{}_{j}J^{k'}{}_{k}\partial_{b}\Omega_{i'j'k'} = i\partial_{b}\Omega_{ijk}. \tag{2.24}$$

Differentiating this, we get

$$\frac{\partial J^{i'}{}_{i}}{\partial t^{a}} \frac{\partial \Omega_{i'jk}}{\partial t^{b}} + \text{cyclic permutations of } ijk = 2i \frac{\partial^{2} \Omega_{ijk}}{\partial t^{a} \partial t^{b}} \mod H^{2,1} \oplus H^{3,0}. \tag{2.25}$$

Combining this with (2.21), we have

$$\frac{\partial J^{i'}{}_{i}}{\partial t^{a}} \frac{\partial \Omega_{i'jk}}{\partial t^{b}} + \text{cyclic permutations} = 2C_{\underline{a}\underline{b}\underline{c}}g^{\underline{c}\overline{c}} \frac{\partial \overline{\Omega}}{\partial \overline{t}^{c}} \text{mod d}(...) \oplus H^{2,1} \oplus H^{3,0}. \quad (2.26)$$

The complex conjugate of this formula asserts that

$$\frac{\partial J^{i'}{}_{i}}{\partial \overline{t}^{a}} \frac{\partial \overline{\Omega}_{i'jk}}{\partial \overline{t}^{b}} + \text{cyclic permutations} = 2\overline{C}_{\overline{a}\overline{b}\overline{c}} g^{\underline{c}\overline{c}} \frac{\partial \Omega}{\partial t^{c}} \text{mod d}(\ldots) \oplus H^{1,2} \oplus H^{0,3}. \quad (2.27)$$

2.3. Complex Structure Of \mathcal{W}

We now must study the complex structure of \mathcal{W} . An element Θ of $\mathcal{W} = H^3(X, \mathbb{R})$ has an expansion

$$\Theta = \lambda^{-1}\Omega + \sum_{a} u^{a} \frac{\partial \Omega}{\partial t^{a}} + \sum_{a} \overline{u}^{a} \frac{\partial \overline{\Omega}}{\partial \overline{t}^{a}} + \overline{\lambda}^{-1} \overline{\Omega}$$
 (2.28)

in the basis (2.18), with complex coefficients λ , u^a . The object λ appearing here will turn out to be the string coupling constant.

We now have to make explicit the complex structure of W. W has no natural complex structure, but every choice of complex structure J on X enables one to pick a complex structure, which I will call \mathcal{J} , on $W = H^3(X, \mathbb{R})$. We choose \mathcal{J} to be the complex structure on $H^3(X, \mathbb{R})$ in which $H^{3,0}$ and $H^{2,1}$ are considered to be of type (1,0). Hence, \mathcal{J} should be represented by an operator on differential forms that multiplies (3,0) and (2,1) forms by i, and multiplies (0,3) and (1,2) forms by -i. Such an operator is

$$(\mathcal{J}\Theta)_{ijk} = \frac{1}{2}J^{i'}{}_{i}J^{j'}{}_{j}J^{k'}{}_{k}\Theta_{i'j'k'} + \frac{1}{2}\left(J^{i'}{}_{i}\Theta_{i'jk} + \text{cyclic permutations of } ijk\right). \tag{2.29}$$

We now need to compute the \overline{t} dependence of \mathcal{J} . As in §1, we will use the notation t' for \overline{t} . The computation of t' dependence is straightforward:

$$\left(\frac{\partial \mathcal{J}}{\partial t'^{a}}\Theta\right)_{ijk} = \frac{1}{2} \left(\frac{\partial J^{i'}_{i}}{\partial t'^{a}}J^{j'}_{j}J^{k'}_{k}\Theta_{i'j'k'} + \frac{\partial J^{i'}_{i}}{\partial t'^{a}}\Theta_{i'jk}\right) + \text{cyclic permutations of } ijk.$$
(2.30)

In particular, if Θ is of type (1,2), this implies

$$\left(\frac{\partial \mathcal{J}}{\partial t'^{a}}\Theta\right)_{ijk} = \left(\frac{\partial J^{i'}{}_{i}}{\partial t'^{a}}\Theta_{i'jk} + \text{cyclic permutations}\right) \mod H^{1,2} \oplus H^{0,3}.$$
(2.31)

Taking $\Theta = \partial \overline{\Omega} / \partial t'^b$, and using (2.27), we learn that

$$\frac{\partial \mathcal{J}}{\partial t'^a} \left(\frac{\partial \overline{\Omega}}{\partial t'^b} \right) = 2 \overline{C}_{\overline{a}\overline{b}\overline{c}} g^{\underline{c}\overline{c}} \frac{\partial \Omega}{\partial t^c} \mod d(\dots) \oplus H^{1,2} \oplus H^{0,3}. \tag{2.32}$$

2.4. Final Evaluation

To make completely explicit the connection (2.10) for quantization of W, with a family of polarizations parametrized by \mathcal{T} , we need to evaluate $(d\mathcal{J}\omega^{-1})^{\underline{i}\underline{j}}$. The only non-zero matrix element (given that \underline{i} and \underline{j} are to be indices of type (1,0), corresponding to (3,0) or (2,1) forms) comes from (2.32) together with

$$\omega^{-1\overline{b}\underline{a}} = -ig^{\overline{b}\underline{a}},\tag{2.33}$$

which is equivalent to (2.19). Combining these pieces, we get

$$(\mathrm{d}\mathcal{J}\omega^{-1})^{\underline{i}\underline{j}}D_{\underline{i}}D_{\underline{j}} = -2i\sum_{a}\mathrm{d}t'^{a}\overline{C}_{\overline{a}\overline{b}\overline{c}}g^{\underline{b}\overline{b}}g^{\underline{c}\overline{c}}\frac{D}{Du^{b}}\frac{D}{Du^{c}}.$$
 (2.34)

The condition that the quantum state represented by a vector $\Psi(\lambda, u; t')$ is independent of t' can be read off from (2.10) and is

$$\left(\frac{\partial}{\partial t'^{a}} + \frac{i}{2}\overline{C}_{\overline{a}\overline{b}\overline{c}}g^{\underline{b}\overline{b}}g^{\underline{c}\overline{c}}\frac{D}{Du^{b}}\frac{D}{Du^{c}}\right)\Psi = 0$$

$$\frac{\partial}{\partial \overline{t'}^{a}}\Psi = 0.$$
(2.35)

The main point of this paper is that the first equation in (2.35) practically coincides with the holomorphic anomaly equation (1.5) of Bershadsky et. al. A factor of 2i presumably results from a difference in conventions (for instance, as will be clear momentarily, it can be absorbed in the definition of the string coupling constant). Let us analyze the remaining discrepancies.

First of all, Bershadsky et. al. consider the partition function as a function of the string coupling constant and also the complex structure of X. In quantizing

^{*} The second equation in (2.35) is also true in their formalism. They consider t and \overline{t} as independent complex variables and consider functions that are holomorphic in t, \overline{t} . Given that their t, \overline{t} correspond to our u, t', holomorphicity in \overline{t} is the second equation in (2.35) and holomorphicity in t is (2.6).

 \mathcal{W} , our wave functions depend on the variables λ and u introduced in (2.28). λ determines a holomorphic three-form, and in the B model this means that λ should be associated with the string coupling constant. On the other hand, u is a (2,1) cohomology class. To Bershadsky et. al., the natural variables would be the string coupling constant λ and an element t of $H^1(X, T^{1,0}X)$. In a familiar fashion, one can map $H^1(X, T^{1,0}X)$ to $H^{2,1}(X)$ by contracting with a holomorphic three-form, which in this case should be naturally the one determined by λ . This means that the natural relation between u and t is $t = \lambda u$. In terms of t, the first equation in (2.35) would therefore be

$$\left(\frac{\partial}{\partial t'^a} + \frac{i}{2}\lambda^2 \overline{C}_{\overline{a}\overline{b}\overline{c}} g^{\underline{b}\overline{b}} g^{\underline{c}\overline{c}} \frac{D}{Dt^b} \frac{D}{Dt^c}\right) \Psi = 0. \tag{2.36}$$

Now we see the natural appearance of the string coupling constant, as in (1.5).

Another important point is that the connection (2.10) for quantization of an affine space is only projectively flat. Therefore, in attempting to study the t' dependence of a wave function Ψ using (2.35), there would be an undetermined factor of modulus unity – a c-number factor in the sense that it depends only on t' and not on t. Of course, this comes from the fact that Ψ is really a section of the prequantum line bundle rather than a function. Perhaps a trivialization of this line bundle is implicit in [10].

In the definition of the partition function $Z = \exp(\frac{1}{2}\sum_{g=0}^{\infty}\lambda^{2g-2}F_g)$, the genus one term F_1 has the following characteristics: (i) the power of λ multiplying it vanishes; (ii) because of zero mode contributions (analogous to those in Ray-Singer analytic torsion) it is not most naturally interpreted as a "function" but as a section of a certain line bundle. A better understanding of F_1 might clarify the role of the prequantum line bundle.

Despite unresolved questions, the resemblance of the holomorphic anomaly equation to the equation (2.36) of quantum background independence is so close

[†] Recall from §1 that once a base-point is picked, the Teichmuller space of X has a natural local isomorphism with the linear space $H^1(X, T^{1,0}X)$ via special coordinates.

that it is hard to believe that it is a coincidence. This relation seems likely to repay further study.

2.5. A Speculation

For instance, it is amusing to speculate along the following lines. Perhaps the phase space W that has appeared in this discussion should be interpreted as part of the usual classical phase space of a system, and the construction of the auxiliary Hilbert space \mathcal{H} should be regarded as part of the process of quantization. Then what is unusual is that – as opposed to the usual situation in which the choice of a vector in \mathcal{H} is a choice of initial conditions – the physical system here determines a distinguished vector in \mathcal{H} , namely the partition function.

Before identifying this as a cosmologist's dream, in which the initial conditions of the universe are uniquely determined by fundamental theory, we should pause to note that in physical applications of string theory, background independence is realized "normally" and one probably does not want to abandon that since the realization of background independence in general relativity is "normal." "Quantum" background independence as we have encountered it in this paper apparently depends on having a non-trivial cohomology of the b_0, \bar{b}_0 operators (which leads to considerations of $t - \overline{t}$ fusion and the holomorphic anomaly); this is absent in the usual critical string theories. Though one probably would not want "quantum" background independence for transverse gravitons, perhaps there is a modification of the usual string theories in which the BRST cohomology is such that "quantum" background independence holds for the conformal factor in the space-time metric. Then – blindly imitating what we have found above – quantum background independence might dictate that the dependence of the wave function on the conformal factor should be given by the partition function. It has been argued [20] that under such conditions the cosmological constant would vanish, since under some hypotheses the partition function of the universe diverges at zero cosmological constant.

In any event, though "ordinary" background independence appears to suffice (apart from such exotic speculations) for the usual physical applications of string theory, familiarity with "quantum" background independence may be useful in trying to go off-shell. I am reminded of the BV formalism of quantization, which enters on-shell only in exotic string theories in $D \leq 2$ [21,22], but seems to be very valuable in formulating critical string theories off-shell, even at the classical level [23].

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