Anomaly Cancellation On G_2 -Manifolds

Edward Witten

School of Natural Sciences, Institute for Advanced Study Olden Lane, Princeton, NJ 08540, USA

Smooth manifolds of G_2 holonomy, used to compactify M-theory to four dimensions, give only abelian gauge groups without charged matter multiplets. But singular G_2 -manifolds can give abelian or nonabelian gauge groups with chiral fermions. We describe the mechanism of anomaly cancellation in these models, using anomaly inflow from the bulk. We also compare the anomaly predictions to what has been learned by more explicit arguments in some special cases.

1. Introduction

Compactification on a manifold X of G_2 holonomy is a natural framework for reducing eleven-dimensional M-theory to a four-dimensional model with $\mathcal{N}=1$ supersymmetry. This type of model is much harder to study than Calabi-Yau compactification, because Yau's theorem, which gives a useful criterion for existence of Calabi-Yau metrics, has no analog for metrics of G_2 holonomy. Nonetheless, such metrics exist in many cases; for an account of the existence proofs, see [1].

A different kind of problem is that compactification on a large and smooth manifold of G_2 holonomy – the only case in which supergravity is adequate – gives a four-dimensional model with abelian gauge fields only and no massless charged matter multiplets. (The dimensional reduction on a manifold of G_2 holonomy has been worked out most fully in [2].) In fact, since a manifold of G_2 holonomy has no continuous symmetries, gauge fields come only from the dimensional reduction of the three-form field C. Such gauge fields are abelian and, in the supergravity approximation, which is valid for massless fields if X is smooth and large, do not couple to any charged fields at all.

To derive an interesting model of particle physics from a manifold of G_2 holonomy, we therefore must allow singularities. Some special cases of singularities of G_2 manifolds were studied recently from different points of view [3-7], and some of these give nonabelian gauge symmetry and/or chiral fermions.

In this paper, we will explore one route to obtain nonabelian gauge symmetry. In this approach, the generic singularities of X are codimension four A - D - E orbifold singularities, which give gauge symmetry. Chiral fermions arise when the locus of A - D - E singularities passes through isolated points at which X has an isolated conical singularity that is not just an orbifold singularity. This approach to obtaining nonabelian gauge groups and chiral fermions from a singular G_2 -manifold can be motivated by duality with Type IIA orientifolds such as those studied in [6]: in those models, the gauge symmetry is carried by stacks of branes, which lift to A - D - E singularities in M-theory, and the chiral fermions are supported at special points on the branes. The same picture can also be motivated by duality with the heterotic string, as will be explained elsewhere [7]. The heterotic string approach (as will be explained in [7]) involves a G_2 analog of a familiar mechanism to obtain charged hypermultiplets for Type IIA [8].

In section 2, we consider the case that X is smooth except for isolated conical singularities. The gauge group is then abelian, coming from the C-field. Gauge anomalies

can be used to predict that chiral fermions (charged under the abelian gauge group) must be present at conical singularities under a certain topological condition. The anomalies (which cancel by a mechanism involving anomaly inflow from the bulk, rather as in [9]) give a constraint on the chiral fermions. In particular, for examples considered in [5] where the topology has been worked out and spectra of chiral fermions have been proposed, we show that these spectra agree with the anomaly constraints. The fact that isolated singularities of a G_2 -manifold can support charged chiral fermions is somewhat analogous to the fact that charged hypermultiplets can be supported at isolated singularities of a Calabi-Yau threefold in Type IIA [10].

In section 3, we incorporate the A - D - E singularities. Anomaly considerations imply that chiral fermions must arise, under certain conditions, if a singularity of type A passes through an isolated point at which X has a (non-orbifold) conical singularity. Again, the results that come from anomalies can be compared, in special cases, to results obtained in [5]. Anomalies again cancel by a sort of inflow from the bulk.

Some rudimentary model-building observations based on this mechanism for obtaining chiral fermions will appear elsewhere.

Apart from what is discussed in the present paper, another problem of recent interest for which the cone on \mathbb{CP}^3 (discussed in [5] and here) is relevant is the problem of a flop occurring in the strongly coupled limit of the heterotic string on a Calabi-Yau threefold W. When such a flop occurs, the familiar compactification manifold $W \times \mathbb{S}^1/\mathbb{Z}_2$ of the strongly coupled heterotic string is replaced by a more complicated spacetime, studied in [11], with a singularity in the bulk that is a cone on \mathbb{CP}^3 . This might suggest generalizing our discussion to include a G-flux, as in [11], or generalizing the discussion in [11].

In this paper, we consider continuous gauge symmetries and local anomalies only. We do not attempt to analyze global anomalies or anomalies in discrete gauge symmetries.

2. Anomalies And Chiral Fermions From Isolated Conical Singularities

First we consider the case that X is smooth except for isolated conical singularities. Near such a singularity, the metric on X looks like

$$ds^2 = dr^2 + r^2 d\Omega^2, (2.1)$$

where $d\Omega^2$ is a metric on a six-manifold Y. The radial variable r is nonnegative, and the singularity is at r=0.

In general, we want to assume that there are isolated points $P_{\alpha} \in X$, $\alpha = 1, ..., s$ at which X has conical singularities. Near P_{α} , we assume that X looks like a cone on some six-manifold Y_{α} . If we excise from X small open neighborhoods of the P_{α} , we make a smooth manifold-with-boundary X'; its boundary is the union of the Y_{α} .

First let us discuss the gauge group in M-theory on X. If X is smooth, this can be determined by conventional Kaluza-Klein reduction. The gauge group (apart from discrete gauge symmetries coming from symmetries of X) is $H^2(X; U(1))$, coming from the unbroken symmetries of the C-field. The continuous gauge symmetries in this situation can be described particularly simply; if we take a basis w_1, \ldots, w_r of harmonic forms on X (where $r = b_2(X)$, the second Betti number of X), then corresponding massless U(1) gauge fields $A^{(i)}$ arise in four dimensions by the ansatz

$$C = \sum_{i=1}^{r} A^{(i)} \wedge w_i + \dots$$
 (2.2)

To get the right global structure of the gauge group $U(1)^r$, we normalize the w_i to be generators of $H^2(X; \mathbf{Z})$.

In case X has isolated conical singularities, we propose that the gauge group L is $H^2(X'; U(1))$. This is the answer one gets if the harmonic forms w_i are required, near a singular point that is a cone on Y_{α} , to be "pullbacks" from Y_{α} , that is, arbitrary harmonic forms on Y_{α} but independent of the radial variable r. The proposal that $L = H^2(X'; U(1))$ agrees with examples considered in [5] and will lead, as we will see, to an elegant general picture for anomaly cancellation.¹

We will use anomalies to obtain some information about charged chiral fermions that must be supported at the points P_{α} . We consider first purely gauge anomalies (as opposed to mixed gauge-gravitational anomalies). Denoting the curvature of C as G = dC, the anomalies come from the CG^2 interaction of eleven-dimensional supergravity. When properly normalized (so that the periods of the G-field are multiples of 2π), this interaction, which we will call I, is such that²

$$\frac{I}{2\pi} = \frac{1}{6 \cdot (2\pi)^3} \int_M C \wedge G \wedge G. \tag{2.3}$$

¹ It might be that something new would happen if the first Betti number of Y_{α} is nonzero (no examples are known of conical G_2 singularities with this property). Perhaps then we should allow harmonic forms that near a singularity look like dr times a one-form on Y_{α} .

² Together with gravitational corrections, I is defined mod 2π and hence $I/2\pi$ is defined mod 1, as explained in [12]. (This assertion ignores a subtlety about the Rarita-Schwinger determinant that will not be important in the present paper.)

Here M is spacetime, which for our purposes is $\mathbf{R}^4 \times X$. However, this supergravity formula is really only valid away from the singularities of X. So we will really carry the integral only over $M' = \mathbf{R}^4 \times X'$. This will give an anomalous result, and we will cancel the anomaly with a suitable assumption about the nature of the physics at the singularities.

To see the anomaly, we consider a gauge transformation of C, by $C \to C + d\epsilon$, with ϵ a two-form. We want to prove gauge-invariance by integrating by parts and using the fact that dG = 0. In doing this, an anomaly will arise at the singularities. In fact, under $C \to C + d\epsilon$, since the boundary of M' is $\bigcup_{\alpha} \mathbf{R}^4 \times Y_{\alpha}$, I changes by

$$\frac{\delta I}{2\pi} = -\frac{1}{6 \cdot (2\pi)^3} \sum_{\alpha} \int_{\mathbf{R}^4 \times Y_\alpha} \epsilon \wedge G \wedge G. \tag{2.4}$$

Now we consider the case that near a singular point, the ansatz (2.2) is valid; we write $F^{(i)} = dA^{(i)}$ for the field strengths of the four-dimensional fields; and we take

$$\epsilon = \sum_{i} \epsilon^{(i)} w_i, \tag{2.5}$$

where $\epsilon^{(i)}$ are functions on \mathbf{R}^4 . Under these assumptions, the contribution of the α^{th} singular point to the anomaly is

$$\frac{\delta_{\alpha}I}{2\pi} = -\frac{1}{6 \cdot (2\pi)^3} \int_{\mathbf{R}^4} \sum_{i,j,k} \epsilon^{(i)} \wedge F^{(j)} \wedge F^{(k)} \int_{Y_{\alpha}} w_i \wedge w_j \wedge w_k. \tag{2.6}$$

Anomaly cancellation must hold locally on M; hence there must be some additional phenomenon supported at $\mathbf{R}^4 \times P_{\alpha}$ that cancels the anomaly. We will suppose that this additional phenomenon takes the form of charged chiral superfields Φ^{σ} , of charges q_i^{σ} . Here σ takes values in a set T_{α} that depends on α , so the q's depend on α though this is not made explicit in the notation. Since anomalies in four-dimensional gauge theory are derived from the six-form piece of $\exp(F/2\pi)$, which is $(1/6)(F/2\pi)^3$, to cancel the anomaly (2.6) we require that

$$\sum_{\sigma \in T_{\alpha}} q_i^{\sigma} q_j^{\sigma} q_k^{\sigma} = \int_{Y_{\alpha}} w_i \wedge w_j \wedge w_k. \tag{2.7}$$

Thus in particular, for Y_{α} such that the right hand side is not identically zero, there must be massless chiral fermions supported at P_{α} . Now let us verify anomaly cancellation.

Anomaly cancellation means that the anomalies of the massless chiral fermions add up to zero, after summing over α . That is, we want

$$\sum_{\alpha} \sum_{\sigma \in T_{\alpha}} q_i^{\sigma} q_j^{\sigma} q_k^{\sigma} = 0.$$
 (2.8)

In virtue of (2.7), this follows from the fact that $\sum_{\alpha} \int_{Y_{\alpha}} w_i \wedge w_j \wedge w_k = \int_{X'} d(w_i \wedge w_j \wedge w_k) = 0$, as $dw_i = 0$ for all i.

Now let us verify that our result for the local anomaly agrees with the spectra found in [5] in special cases. For $Y = \mathbb{CP}^3$, the second Betti number is 1, and the second cohomology group is generated by a single two-form w with $\int_Y w^3 = 1$. Hence, we expect a chiral spectrum with charges σ such that $\sum_{\sigma} (q^{\sigma})^3 = 1$. This agrees with the claim in [5] that there is a single chiral fermion of charge 1. For $Y = SU(3)/U(1)^2$, the gauge group is $U(1)^2$, conveniently embeddable in $U(1)^3$ in such a way that the charges are (1, -1, 0), (0, 1, -1), and (-1, 0, 1). In this basis, the nonzero elements of the anomaly polynomial $d_{ijk} = \sum_{\sigma} q_i^{\sigma} q_j^{\sigma} q_k^{\sigma}$ are (up to permutations of the indices) $d_{i,i,i-1} = -d_{i,i,i+1} = 1$. This agrees with the intersection form of $SU(3)/U(1)^2$. (A key example in [3-5] was the case $Y = \mathbf{S}^3 \times \mathbf{S}^3$, but that case is not very interesting for the present paper as the second Betti number of Y vanishes.)

Turning things around, we have canceled the anomalies from chiral fermions that "live" at the singularity using a sort of anomaly inflow from the C-field (roughly along the lines of [9]) and without invoking a Green-Schwarz mechanism. The fact that anomaly inflow is the key mechanism is not surprising, since this is the case for intersecting branes in Type II superstrings [13], and such brane intersections are in some cases dual to isolated singularities of a G_2 manifold. The class of models considered here has no Green-Schwarz mechanism in bulk. It might be that the local degrees of freedom at some singularities are more complicated than we have assumed (there might be a nontrivial conformal field theory at a singularity, for example), and perhaps there are some cases in which a description using a local Green-Schwarz mechanism at a singularity is useful.

Mixed Gauge-Gravitational Anomalies

Now let us consider the analogous mechanism for mixed gauge-gravitational anomalies. (There are no purely gravitational anomalies in four dimensions.) The coupling analogous to (2.3) is the gravitational contribution to the Chern-Simons coupling,

$$\frac{I'}{2\pi} = -\frac{1}{48} \int_{M} \frac{C}{2\pi} \wedge \left(p_2 - p_1^2 / 4 \right). \tag{2.9}$$

Here by p_1 and p_2 we mean the differential forms (polynomials in the Riemann tensor) that represent the Pontryagin classes p_i .

Now, under $C \to C + d\epsilon$, we get an additional anomaly:

$$\frac{\delta I'}{2\pi} = \frac{1}{48} \sum_{\alpha} \int_{\mathbf{R}^4 \times Y_\alpha} \frac{\epsilon}{2\pi} \wedge (p_2 - p_1^2/4). \tag{2.10}$$

To extract the four-dimensional gravitational anomaly, we want to evaluate this for fluctuations in the metric of \mathbf{R}^4 that preserve the product form $\mathbf{R}^4 \times X$ (but of course not the flatness of \mathbf{R}^4). For this purpose, if p_1' and p_1'' are the four-forms representing the first Pontryagin classes of \mathbf{R}^4 and X, respectively, we can take $p_1 = p_1' + p_1''$ and $p_2 = p_1' \wedge p_1''$. Also expanding ϵ as in (2.5), the local contribution to the anomaly from the α^{th} singularity is

$$\frac{\delta_{\alpha} I'}{2\pi} = \frac{1}{96} \int_{\mathbf{R}^4} \frac{\epsilon^{(i)}}{2\pi} p_1' \int_{Y_{\alpha}} w_i \wedge p_1''. \tag{2.11}$$

Let us work out the anomaly that is expected due to chiral fermions at Y_{α} of charges q_i^{σ} , $\sigma \in T_{\alpha}$. The anomaly for a chiral fermion of charge 1 in four dimensions is derived from the six-form

$$\frac{1}{6} \left(\frac{F}{2\pi}\right)^3 - \frac{F}{2\pi} \frac{p_1'}{24}.\tag{2.12}$$

So to cancel the local anomaly found in (2.11), the chiral multiplets supported at Y_{α} must have charges such that

$$\sum_{\sigma \in T_{\alpha}} q_i^{\sigma} = \frac{1}{4} \int_{Y_{\alpha}} w_i \wedge p_1''. \tag{2.13}$$

Anomaly cancellation is now established just as for the purely gauge anomalies:

$$\sum_{\alpha} \sum_{\sigma \in T} q_i^{\sigma} = \frac{1}{4} \sum_{\alpha} \int_{Y_{\alpha}} w_i \wedge p_1'' = \frac{1}{4} \int_{X'} d(w_i \wedge p_1'') = 0.$$
 (2.14)

Again, we can compare to the cases considered in [5]. Suppose that one of the Y_{α} 's is a cone on \mathbb{CP}^3 . The relevant gauge group is U(1), associated as above with a two-form w with $\int_{\mathbb{CP}^3} w^3 = 1$. For \mathbb{CP}^3 , we have $c_1 = 4w$, $c_2 = 6w^2$, and $p_1 = c_1^2 - 2c_2 = 4w^2$. So $(1/4) \int_{\mathbb{CP}^3} w \wedge p_1'' = 1$, and (2.13) is compatible with the expectation that there is precisely one chiral multiplet with q = 1.

For the other example, $Y = SU(3)/U(1)^2$, things are more trivial. There is a triality symmetry, exploited in [5], which ensures that the charge generators are traceless (this is clear from the expressions for the charge vectors given above), and likewise ensures that $p_1'' = 0$. (The latter statement holds because $H^4(Y; \mathbf{Z})$ is a rank two lattice that is "rotated" by the triality symmetry, in such a way that there are no nonzero invariant vectors.)

3. Incorporating Nonabelian Gauge Symmetry

M-theory at a simple A-D-E singularity generates gauge symmetry of type A-D-E. (There also are singularities of type D and E with gauge symmetry of reduced rank [14-16], but we will not consider them here.) If X is a manifold of G_2 holonomy with an A-D-E singularity, then, as X has dimension 7 and the A-D-E singularity has codimension four, the singularity is supported on a three-manifold $Q \subset X$. We use the term "manifold" somewhat loosely; like X itself, Q may have singularities. Q is somewhat analogous to a supersymmetric three-cycle in X; in fact, locally, near Q and away from non-orbifold singularities, X is a quotient $X = \widetilde{X}/\Gamma$ where Γ is a finite group, and Q is a supersymmetric three-cycle in \widetilde{X} that is the fixed point set of Γ .

The low energy gauge theory – away from singularities – is supersymmetric A-D-E gauge theory on $\mathbb{R}^4 \times Q$. We want to understand the contributions of singularities, and in particular we want to know what singularities support chiral multiplets in complex representations of the gauge group. We suppose that the singularities are either isolated singularities of Q, or points at which Q is smooth but has a normal bundle with a singularity worse than the generic A-D-E singularity. (In fact, the known examples and additional ones that will be discussed in [7] are of the second type.) In this section, we will use anomalies to give a constraint on chiral fermions from singularities.

To define the A-D-E singularity, we start with \mathbf{R}^4 , acted on by $SO(4) \cong SU(2)_L \times SU(2)_R$. Then we pick a discrete subgroup Γ of $SU(2)_R$, and define the A-D-E singularity as the quotient \mathbf{R}^4/Γ . The A-D-E singularity has as a symmetry group $SU(2)_L \times \Lambda$, where Λ is the subgroup of $SU(2)_R$ that conjugates Γ to itself (thus, $g\Gamma g^{-1} = \Gamma$ for $g \in \Lambda$).

In general, a family of A - D - E singularities, over a base B, can be "twisted" by an arbitrary $SU(2)_L \times \Lambda$ bundle. The case of main interest to us is that $B = \mathbb{R}^4 \times Q$ (where \mathbb{R}^4 is four-dimensional Minkowski space). The twisting by $SU(2)_L$ can be described very directly: the condition that X has G_2 holonomy identifies the $SU(2)_L$ connection with the Riemannian connection of Q. However, G_2 holonomy does not determine how the normal bundle is twisted by Λ , and we must examine this.³

³ To get a rough idea of the group theory here (see [17,18] for a more detailed discussion of analogous problems that depend upon the same group theory), G_2 contains the group $SU(2)_L \times SU(2)_R$, with the **7** of G_2 transforming as $(\mathbf{3},\mathbf{1}) \oplus (\mathbf{2},\mathbf{2})$. The $(\mathbf{3},\mathbf{1})$ is the tangent space to Q and the $(\mathbf{2},\mathbf{2})$ is the normal space to Q (before dividing by Γ to make an orbifold). The last

For singularities of type D or E, the twisting by Λ has been studied in the context of Calabi-Yau compactification and plays an important role [19,20]. In these examples, Γ is a nonabelian group and Λ is a finite group which can be identified with the group of outer automorphisms of the D or E gauge group. Twisting by Λ means, in this case, that the D or E gauge theory can be twisted, as one goes around a non-contractible one-cycle, by an outer automorphism. On either a Calabi-Yau manifold or a manifold of G_2 holonomy, this can give a way to break the gauge group to a non-simply-laced subgroup.

We want to focus here on the case of a singularity of type A, so that the gauge group is SU(N) for some N. In this case, Γ is a \mathbf{Z}_N subgroup of $SU(2)_R$ that we can take to consist of matrices of the form

$$\begin{pmatrix} e^{2\pi ik/N} & 0\\ 0 & e^{-2\pi ik/N} \end{pmatrix}. \tag{3.1}$$

If such a matrix acts on a column vector $\begin{pmatrix} a \\ b \end{pmatrix}$, then the Γ -invariants are $x = a^N$, $y = b^{-N}$, and z = ab, obeying the familiar equation

$$xy = z^N (3.2)$$

of the SU(N) singularity.

The group of $SU(2)_R$ matrices that map Γ to itself is in this case $\Lambda = O(2)$. Here O(2) is generated by a discrete \mathbb{Z}_2 symmetry that exchanges a and b, and a U(1) subgroup of diagonal matrices. The \mathbb{Z}_2 symmetry corresponds to an outer automorphism (complex conjugation) of SU(N), analogous to the discrete symmetries for the D and E groups; the associated physics is similar. We want to focus on the continuous group $\Lambda' \cong U(1)$. It consists of matrices

$$\begin{pmatrix} e^{i\psi/N} & 0\\ 0 & e^{-i\psi/N} \end{pmatrix}, \quad 0 \le \psi \le 2\pi. \tag{3.3}$$

Thus, the action on the invariants x, y, z is

$$(x, y, z) \rightarrow (e^{i\psi}x, e^{-i\psi}y, z).$$
 (3.4)

statement shows that the group $SU(2)_L$ that acts on the normal bundle is the same as the group that acts on the tangent space to Q – which is why the $SU(2)_L$ connection on the normal bundle is the spin connection. It also shows that an arbitrary twisting of the normal bundle by $SU(2)_R$ is compatible with G_2 holonomy.

Now consider M-theory on an eleven-manifold Z with a family of SU(N) singularities on a codimension four submanifold B (in our application, $Z = \mathbb{R}^4 \times X$ and $B = \mathbb{R}^4 \times Q$). The normal space to B might be twisted by Λ' . (There could be a more general twisting by disconnected elements of Λ , but we do not wish to consider that case.) In view of (3.4), this means that the normal space to B can be described by coordinates x, y, z that obey $xy = z^N$; however, they are not functions but sections of certain line bundles over B. In fact, they are sections respectively of \mathcal{L} , \mathcal{L}^{-1} , and \mathcal{O} , where \mathcal{O} is a trivial line bundle and \mathcal{L} is an arbitrary line bundle that incorporates the twisting by Λ' .

In M-theory, \mathcal{L} is not merely an abstract complex line bundle; the metric on Z induces a connection on \mathcal{L} . Let K be the curvature of this connection; then $K/2\pi$ represents the first Chern class of \mathcal{L} and so has integer periods. There is also an SU(N) gauge field A on B, with curvature F; let $\omega_5(A)$ be the Chern-Simons five-form of A (normalized so its periods are gauge-invariant mod 2π). We claim that in the long wavelength limit of M-theory on Z, there is an interaction of the form

$$I = \int_{B} \frac{K}{2\pi} \wedge \omega_{5}(A). \tag{3.5}$$

We will first explore the consequences of this assumption and then show that the interaction must be present.

SU(N) gauge-invariance of I is proved by observing that under an infinitesimal gauge transformation $A \to A - d_A \epsilon$, $\omega_5(A)$ transforms by addition of an exact form, a multiple of $d \operatorname{tr} \epsilon F \wedge F$; upon integrating by parts and using the fact that dK = 0, this suffices to prove invariance of I under infinitesimal gauge transformations. We must also consider the behavior under disconnected gauge transformations; since $\omega_5(A)$ has periods that are gauge-invariant mod 2π , (3.5) has been normalized so that I is invariant mod 2π under disconnected gauge transformations, which is good enough for quantum field theory. Thus, a coefficient multiplying the right hand side of (3.5) must be an integer; the discussion below will show that (with the right choice of orientations) this integer is 1.

Next, let us specialize to $Z = \mathbf{R}^4 \times X$, $B = \mathbf{R}^4 \times Q$ and explore the consequences of having an interaction of the form of (3.5). We suppose that there are finitely many points $P_{\alpha} \in Q$ at which the singularity of the normal space to Q is worse than an SU(N) singularity, so that the line bundle \mathcal{L} is not defined. Away from these points, K is defined and obeys dK = 0, but there might be delta function contributions at the P_{α} :

$$dK = 2\pi \sum_{\alpha} n_{\alpha} \delta_{P_{\alpha}}.$$
 (3.6)

Here $\delta_{P_{\alpha}}$ is a delta function supported at P_{α} . The n_{α} are integers because the period of K, integrated over a small surface $S_{\alpha} \subset Q$ that wraps around P_{α} , is an integer multiple of 2π ; in fact, the restriction of \mathcal{L} to S_{α} has first Chern class n_{α} . By integrating (3.6) over Q, we learn that

$$\sum_{\alpha} n_{\alpha} = 0. \tag{3.7}$$

Is the interaction I gauge-invariant when $n_{\alpha} \neq 0$? Just as in section 2, in the presence of the singularity, we get an anomaly under gauge transformations. In fact, under $A \rightarrow A - d_A \epsilon$, with ω_5 shifted by a multiple of $d \operatorname{tr} \epsilon F \wedge F$, integration by parts shows that the change of I under an infinitesimal gauge transformation is

$$\delta I = -\sum_{\alpha} n_{\alpha} \int_{\mathbf{R}^4 \times P_{\alpha}} \operatorname{tr} \epsilon \, \frac{F \wedge F}{8\pi^2}. \tag{3.8}$$

Hence, gauge-invariance is only maintained if at each P_{α} , there are charged chiral multiplets (or more exotic degrees of freedom) with an $SU(N)^3$ anomaly n_{α} . Rather as in section 2, (3.7) is the condition for anomaly cancellation in the effective four-dimensional theory.

For D and E singularities, Λ is a finite group and we do not get such a mechanism for anomalies in the four-dimensional theory. Such a mechanism is not needed, since in any event the D and E groups admit no anomalies in four dimensions, as their Lie algebras have no cubic symmetric invariant.

An Example

To show now that the interaction (3.5) is really present, it suffices to show independently in one special case that charged degrees of freedom with anomaly n_{α} are really present. For this, we consider the example of a cone on a weighted projective space $Y = \mathbf{WCP}_{N,N,1,1}^3$ as considered in section 3.7 of [5]. We describe the weighted projective space by homogeneous complex coordinates (u_1, u_2, u_3, u_4) , not all zero, modulo

$$(u_1, u_2, u_3, u_4) \to (\lambda^N u_1, \lambda^N u_2, \lambda u_3, \lambda u_4), \ \lambda \neq 0.$$
 (3.9)

Y has \mathbf{Z}_N orbifold singularities on the locus $u_3 = u_4 = 0$, which is a copy U of $\mathbf{CP}^1 = \mathbf{S}^2$. A cone X on the weighted projective space Y is constructed by imposing the equivalence relation (3.9) on the u_i only for $|\lambda| = 1$.

The locus Q of singularities in X is a cone on $U = \mathbf{S}^2$. A cone on \mathbf{S}^2 is \mathbf{R}^3 . So $Q = \mathbf{R}^3$ and in particular is smooth. The generic singularity of X is a \mathbf{Z}_N orbifold singularity, but

at the "origin," the apex P of the cone, the singularity is worse. Though smooth, Q passes through P. The fact that the codimension-four submanifold Q passes through the isolated (non-orbifold) singularity P has a topological explanation: it occurs because the two-sphere U wraps a non-trivial cycle in Y, and hence Q cannot be slipped away from the singularity.

Now let us check that (away from P) the normal bundle to Q is an SU(N) singularity. The normal coordinates to U (or Q) are u_3 and u_4 , but subject to the orbifolding group $(u_3, u_4) \to (\lambda u_3, \lambda u_4)$, where now (to get trivial action on u_1 and u_2) λ is an N^{th} root of unity. Clearly, if we set $a = u_3$, $b = \overline{u}_4$, this coincides with the description of the SU(N) singularity in (3.1).

Now we can determine the line bundle \mathcal{L} . We rewrite (3.9) in the form

$$(u_1, u_2, u_3, u_4) \to (tu_1, tu_2, t^{1/N}u_3, t^{1/N}u_4)$$
 (3.10)

with $t = \lambda^{1/N}$. $U = \mathbf{CP}^1$ is defined by $u_3 = u_4 = 0$ and has u_3 and u_4 as normal coordinates. Actually, u_3 and u_4 are sections of a "line bundle" over U. Because of the exponent 1/N in (3.10), this "line bundle" has first Chern class 1/N (and so the "line bundle" must be defined in an orbifold sense). However, the invariants $x = a^N = u_3^N$ and $y = b^N = \overline{u}_4^N$ transform with exponents ± 1 , and so are functions on $\mathcal{L}^{\pm 1}$, where \mathcal{L} is an ordinary line bundle over U with first Chern class 1. So the integer n associated with this particular singularity is 1.

Hence we expect that charged degrees of freedom with an $SU(N)^3$ anomaly of 1 will be present at this singularity. This agrees with the analysis in section 3.7 of [5], where it was argued that this type of singularity supports a chiral multiplet in the fundamental representation of SU(N).

The interested reader can analyze in a similar fashion the more general weighted projective space $\mathbf{WCP}_{N,N,M,M}^3$ which was considered in [5].

Inclusion Of Abelian Gauge Symmetries

Now let us look at X a little more globally. Away from a finite set of points P_{α} , it is an orbifold. Near P_{α} , X looks like a cone on some six-dimensional orbifold Y_{α} . We can proceed just as in section 2 to analyze unbroken abelian gauge symmetries that come from the M-theory three-form field C. As in section 2, we let X' be a manifold-with-boundary obtained by omitting small neighborhoods of the P_{α} . The boundary of X' is the union of the P_{α} . We propose that the abelian gauge group from the C-field has Lie algebra

 $H^2(X'; \mathbf{R})$ just as in section 2.⁴ As in section 2, we let $w_1, \ldots, w_r, r = b_2(X')$, be a basis of harmonic forms on X'.

The $U(1)^3$ anomalies can be treated just as in section 2; since we worked only at the level of differential forms in section 2, the analysis is unchanged by the fact that X' is an orbifold. We now want to analyze the $U(1) \cdot SU(N)^2$ anomalies. (The following discussion actually applies for all the A - D - E groups, not just SU(N).) For this, the key point is the existence of a (standard) interaction

$$I = \int_{B} C \wedge \frac{\operatorname{tr} F \wedge F}{8\pi^{2}},\tag{3.11}$$

in the long wavelength limit of M-theory, away from singularities. Here $\operatorname{tr} F \wedge F/8\pi^2$ is the (normalized) SU(N) instanton number. This interaction is invariant under a gauge transformation $C \to C + d\epsilon$ if the orbifold locus B is smooth. When B has singularities, we meet anomalies just as in section 2 and above.

In our case, $B = \mathbf{R}^4 \times Q$. In using the low energy interaction (3.11), we must excise small neighborhoods of the singular points P_{α} and integrate only over $\mathbf{R}^4 \times Q'$ where Q' is Q with these neighborhoods removed; we denote the boundaries of these neighborhoods as U_{α} . Transforming $C \to C + d\epsilon$ and integrating by parts, we find that the change in I is a sum of local contributions at the P_{α} :

$$\delta_{\alpha}I = -\int_{\mathbf{R}^4 \times U_{\alpha}} \epsilon \wedge \frac{\operatorname{tr} F \wedge F}{8\pi^2}.$$
 (3.12)

Just as in section 2, to express this in the low energy theory, we consider the case that $\epsilon = \sum_{i=1}^{r} \epsilon^{(i)} w_i$, with $\epsilon^{(i)}$ being functions on \mathbf{R}^4 . The local anomaly is

$$\delta_{\alpha}I = -\sum_{i} \int_{\mathbf{R}^{4}} \epsilon^{(i)} \frac{\operatorname{tr} F \wedge F}{8\pi^{2}} \cdot \int_{U_{\alpha}} w_{i}. \tag{3.13}$$

To cancel this anomaly, the chiral degrees of freedom at $\mathbf{R}^4 \times P_\alpha$ must have a $U(1)_i \cdot SU(N)^2$ anomaly (here $U(1)_i$ is the i^{th} copy of U(1) in the gauge group, generated by the harmonic form w_i) equal to $\int_{U_\alpha} w_i$. Now we can demonstrate anomaly cancellation in the effective

 $^{^4~}H^2(X';\mathbf{R})$ is defined as the ordinary de Rham cohomology of the orbifold. We do not include any "twisted sector" degrees of freedom at the fixed points; they have already been included in the nonabelian gauge symmetry associated with the orbifold. We will not try to analyze the integral structure of $H^2(X')$ or to determine the global form of the gauge group.

four-dimensional theory; rather as in the other examples we have considered, we merely note that

$$\sum_{\alpha} \int_{U_{\alpha}} w_i = \int_{Q'} dw_i = 0.$$
 (3.14)

So the $U(1)_i \cdot SU(N)^2$ anomalies of the chiral fields on the various singularities add up to zero.

We can again illustrate this with the example of the cone on $Y = \mathbf{WCP}_{N,N,1,1}^3$. The second Betti number of Y is 1, so there is a single U(1) to consider, generated by a harmonic two-form w on Y. Such a w has $\int_U w \neq 0$; here U, the locus of orbifold singularities in Y, is a copy of \mathbf{CP}^1 as seen above. So the chiral SU(N) degrees of freedom found above must be charged under the U(1). This agrees with the result in [5], where (using the fact that the cone on Y is dual to a configuration of intersecting branes in \mathbf{R}^6) it was seen that the global form of the gauge group is U(N), not $SU(N) \times U(1)$, and that the chiral degrees of freedom are in the fundamental representation of U(N).

This work was supported in part by NSF Grant PHY-0070928. I would like to thank K. Intriligator and B. Acharya for discussions.

References

- [1] D. Joyce, "Compact Manifolds Of Special Holonomy" (Oxford University Press, 2000).
- [2] P. Townsend and G. Papadopoulos, "Compactification Of D = 11 Supergravity On Spaces Of Exceptional Holonomy," hep-th/9506150. Phys. Lett. **B357** (1995) 472.
- [3] B. Acharya, "On Realising $\mathcal{N} = 1$ Super Yang-Mills In M-Theory," hep-th/0011089.
- [4] M. F. Atiyah, J. Maldacena, and C. Vafa, "An M-Theory Flop As A Large N Duality," hep-th/0011256.
- [5] M. F. Atiyah and E. Witten, "M-Theory Dynamics On A Manifold Of G_2 Holonomy," hep-th/0107177.
- [6] M. Cvetic, G. Shiu, and A. M. Uranga, "Three Family Supersymmetric Standard-Like Models From Intersecting Brane Worlds," hep-th/0107143, "Chiral Four-Dimensional $\mathcal{N}=1$ Supersymmetric Type IIA Orientifolds From Intersecting D6 Branes," hep-th/0107166.
- [7] B. Acharya and E. Witten, to appear.
- [8] S. Katz and C. Vafa, "Matter From Geometry," Nucl. Phys. <u>B</u>497 (1997) 146, hep-th/9606086.
- [9] C. G. Callan, Jr. and J. A. Harvey, "Anomalies And Fermion Zero Modes On Strings And Domain Walls," Nucl. Phys. **B250** (1985) 427.
- [10] A. Strominger, "Massless Black Holes And Conifolds In String Theory," Nucl. Phys. B451 (1995) 96, hep-th/9504090.
- [11] B. R. Greene, K. Schalm, and G. Shiu, "Dynamical Topology Change In M Theory," hep-th/0010207.
- [12] E. Witten, "On Flux Quantization In M-Theory And The Effective Action," hep-th/9609122, J. Geom. Phys. **22** (1997) 1.
- [13] M. B. Green, J. A. Harvey, and G. Moore, "I-Brane Inflow And Anomalous Couplings On D-Branes," Class. Quant. Grav. 14 (1997) 47, hep-th/9605033.
- [14] K. Landsteiner and E. Lopez, "New Curves From Branes," Nucl. Phys. B516 (1998) 273, hep-th/9708118.
- [15] E. Witten, "Toroidal Compactification Without Vector Structure," JHEP **9802:006** (1998) hep-th/9712028.
- [16] J. de Boer, R. Dijkgraaf, K. Hori, A. Keurentjes, J. Morgan, D. R. Morrison, and S. Sethi, "Triples, Fluxes, And Strings," hep-th/0103170.
- [17] J. Maldacena and C. Nunez, "Towards The Large N Limit Of Pure $\mathcal{N}=1$ Super Yang-Mills," Phys. Rev. Lett. **86** (2001) 588, hep-th/0008001.
- [18] B. Acharya, J. P. Gauntlett, and N. Kim, "Fivebranes Wrapped On Associative Three-Cycles," hep-th/0011190.
- [19] P. Aspinwall and M. Gross, "The SO(32) Heterotic String On A K3 Surface," hep-th/9605131.

[20] M. Bershadsky, K. Intriligator, S. Kachru, D. R. Morrison, V. Sadov, and C. Vafa, "Geometric Singularities And Enhanced Gauge Symmetries," Nucl. Phys. B481 (1996) 215.