Multi-Trace Operators, Boundary Conditions, And AdS/CFT Correspondence

Edward Witten

Institute For Advanced Study, Princeton NJ 08540 USA

We argue that multi-trace interactions in quantum field theory on the boundary of AdS space can be incorporated in the AdS/CFT correspondence by using a more general boundary condition for the bulk fields than has been considered hitherto. We illustrate the procedure for a renormalizable four-dimensional field theory with a $(\text{Tr }\Phi^2)^2$ interaction. In this example, we show how the AdS fields with the appropriate boundary condition reproduce the renormalization group effects found in the boundary field theory. We also construct in related examples a line of fixed points with a nonperturbative duality and a flow between two methods of quantization.

1. Introduction

Large N limits for matrix-valued fields have been a subject of considerable interest ever since the large N limit of SU(N) gauge theory was recognized as a likely starting point for understanding the dynamics of four-dimensional quantum gauge theories [1]. Much recent work centers around the AdS/CFT correspondence [2].

In general, given a collection of matrix-valued fields Φ_i , to construct a theory with a large N limit, one considers normalized trace operators

$$\mathcal{O}_{\alpha} = \frac{1}{N} \operatorname{Tr} F_{\alpha}(\Phi_i) \tag{1.1}$$

and an action functional

$$I = N^2 W(\mathcal{O}_{\alpha}). \tag{1.2}$$

Here, the F_{α} are arbitrary functions¹ of the Φ_i and their derivatives, and W is an arbitrary function of the \mathcal{O}_{α} . F_{α} has no explicit dependence on N and is defined without any traces, so \mathcal{O}_{α} is a "single-trace operator," and W likewise has no explicit dependence on N. The powers of N have been chosen to ensure the existence of a large N limit.

If W is a linear function of the \mathcal{O}_{α} 's, then it is a "single-trace action," while non-linear terms in W are "multi-trace interactions." Many important examples, like four-dimensional gauge theory without matter fields, are based on single-trace actions. Theories with multi-trace interactions have, however, also been considered in matrix models of two-dimensional gravity [3-7] and in the AdS/CFT correspondence [8].

The purpose of the present paper is to make a general proposal for how multi-trace interactions can be incorporated in the AdS/CFT correspondence, by generalizing the boundary conditions [9,10] that are used in the case of single-trace actions. This is somewhat subtle for reasons explained in [8]. Multi-trace operators in the CFT correspond to multi-particle states in AdS space, and it is not immediately apparent what is meant by placing a boundary condition at infinity on a multi-particle state. Nevertheless, we will see that the familiar boundary conditions have a natural generalization that serves this purpose.

In section 2, we practice by recalling how multi-trace interactions are treated in the simplest of all matrix models, which is the theory of a single matrix. This model can be solved very directly in the large N limit [11] and the solution can be extended to

¹ In many applications, one considers only polynomial functions.

incorporate multi-trace interactions [3].² In section 3, we present our proposal for the general boundary conditions in the AdS/CFT correspondence and their interpretation in terms of multi-trace interactions. In section 4, we discuss the application of this proposal to a class of four-dimensional examples suggested in [8]. We also describe in related examples a line of fixed points that admits a nonperturbative duality.

The role of boundary conditions in the AdS/CFT correspondence has been reexamined recently in [13]. The paper [14], which appeared on hep-th the same day as the present one, has some results that overlap with those presented here.

2. Review Of The One-Matrix Model

First we practice by recalling the case of a model (in zero space-time dimensions) in which the field variables comprise a single $N \times N$ hermitian matrix Φ . Assuming that we want to consider only polynomial interactions (in this particular model, there is no difficulty in relaxing this assumption), the general single-trace operators are

$$\mathcal{O}_n = \frac{1}{N} \operatorname{Tr} \Phi^n, \quad n = 1, 2, 3, \dots$$
 (2.1)

The action depends on an arbitrary function of the \mathcal{O}_n 's:

$$I = N^2 W(\mathcal{O}_1, \mathcal{O}_2, \ldots). \tag{2.2}$$

One wishes to perform the integral

$$Z = \int d\Phi \, \exp(-I) \tag{2.3}$$

in the large N limit, as a function of W.

The model has U(N) symmetry, acting on Φ by conjugation. Because of this symmetry, the action depends only on the eigenvalues $\lambda_1, \ldots, \lambda_N$ of Φ ; in fact, $\mathcal{O}_n = \frac{1}{N} \sum_i \lambda_i^n$. Up to a constant factor (computable and independent of W) which comes from the volume of the group U(N), the partition function in terms of the eigenvalues becomes

$$Z = \int_{-\infty}^{\infty} d\lambda_1 \dots d\lambda_N \prod_{i < j} (\lambda_i - \lambda_j)^2 \exp(-N^2 W).$$
 (2.4)

² This theory also has scaling limits related to two-dimensional gravity that can be solved in a much more subtle way [12], but we do not need that here.

The integral can be evaluated for large N because the integrand has a sharp maximum at a suitable configuration of the λ 's. This configuration is characterized by a smooth distribution of the λ 's. To find it, we assume that the density of the λ 's is of the form $N\rho(x)$ for some positive function $\rho(x)$ with $\int_{-\infty}^{\infty} dx \, \rho(x) = 1$. The \mathcal{O}_n now become $\mathcal{O}_n = \int_{-\infty}^{\infty} dx \, x^n \rho(x)$, and the integrand in (2.4) is

$$\exp\left(-N^2(W(\mathcal{O}_i) - \int dx \, dx' \rho(x) \rho(x') \ln|x - x'|)\right). \tag{2.5}$$

Maximizing the exponent with the constraint $\int_{-\infty}^{\infty} dx \, \rho(x) = 1$ gives the equation

$$\sum_{n} x^{n} \frac{\partial W}{\partial \mathcal{O}_{n}} - 2 \int dx' \ln|x - x'| \rho(x') + t = 0, \tag{2.6}$$

where t is a Lagrange multiplier for the constraint. How to solve such equations, for suitable W's, is explained in [11].

Let us consider what happens in the single-trace case, for which $W = \sum_{n=1}^{\infty} w_n \mathcal{O}_n$ with constants w_n . Then (2.5) simply reduces to

$$\sum_{n} x^{n} w_{n} - 2 \int dx' \ln|x - x'| \rho(x') + t = 0.$$
 (2.7)

The lesson to be learned is that in the general case, there is a saddle-point equation of the same form as in the single-trace case, except that the coupling constants w_n of the single-trace case are replaced by $\partial W/\partial \mathcal{O}_n$, where \mathcal{O}_n are the observables of the matrix field theory under study and W is the coupling function which can be, in general, an arbitrary function of these observables.

3. The AdS/CFT Case

The remainder of this paper is concerned mainly with adapting that last statement to the AdS/CFT correspondence. However, first we practice with another familiar example of a different type.

Consider a scalar field ϕ on the half-space $x_1 \geq 0$ in \mathbf{R}^n . We take the action for ϕ to be a sum

$$I = \frac{1}{2} \int_{x_1 \ge 0} d^n x \, |d\phi|^2 + \int_{x_1 = 0} d^{n-1} x \, W(\phi, d\phi, \ldots)$$
 (3.1)

of the free kinetic energy for ϕ , integrated over the half-space, plus a boundary interaction W that is an arbitrary local function of ϕ and its derivatives. Now when we vary ϕ , $\phi \to \phi + \delta \phi$, to obtain the Euler-Lagrange equations, we encounter boundary terms

$$\int_{x_1=0} d^{n-1}x \,\delta\phi \left(-\frac{\partial\phi}{\partial x_1} + \frac{\delta W}{\delta\phi} \right). \tag{3.2}$$

Hence, to satisfy the equations of motion, we have to impose a boundary condition

$$\frac{\delta W}{\delta \phi} - \frac{\partial \phi}{\partial x_1} = 0. \tag{3.3}$$

This equation has an analogy with (2.6). The term $\delta W/\delta \phi$ is analogous to the term $\sum_{n} x^{n} \frac{\partial W}{\partial \mathcal{O}_{n}}$ in (2.6), while $-\partial \phi/\partial x_{1}$ corresponds to the terms in (2.6) that do not depend on W.

(3.3) can also be given the following intuitive interpretation. Let us think of x_1 as a "time" direction, even though in a boundary problem like this one with local boundary conditions (and also in the AdS case that we turn to presently) it is usually a spatial direction. Then $p = \partial \phi / \partial x_1$ is naturally regarded as the "momentum." For W = 0, the boundary condition is simply the vanishing of the momentum, p = 0. For general W, the boundary condition $p = \delta W / \delta \phi$ differs by a canonical transformation (generated by W) from the condition p = 0. In the phase space of canonically conjugate variables ϕ and p (defined at $x_1 = 0$), the variables p or equally well $p - \delta W / \delta \phi$ are a maximal set of commuting variables, so their vanishing defines a "Lagrangian submanifold" of the phase space. A Lagrangian submanifold determines a quantum state at least formally (in the present case, the operators $p - \delta W / \delta \phi$ annihiliate the state with wave function e^{-W}). In two-dimensional conformal field theory, the state determined in this way by the boundary conditions is called the boundary state [15].

Analog for Anti de Sitter Space

Anti de Sitter space is somewhat similar, except that the boundary is replaced by a conformal boundary at spatial infinity. We consider AdS space of dimension D = d + 1, with metric

$$ds^2 = \frac{dr^2 + \sum_{i=1}^d dx_i^2}{r^2}$$
 (3.4)

in the region $r \geq 0$. The conformal boundary is at r = 0.

Consider a scalar field ϕ of mass m in AdS space. It behaves near r=0 as

$$\phi = \alpha(x)r^{d-\lambda} + \beta(x)r^{\lambda},\tag{3.5}$$

where we take λ to be the larger root of the equation $\lambda(\lambda + d) = m^2$. (For $\lambda = d/2$, a case we consider in section 4, the roots are equal and the two solutions are $r^{d/2}$ and $r^{d/2} \ln r$.) α and β are canonically conjugate variables, analogous to ϕ and $\partial_1 \phi$ in the example treated above in which the boundary is at finite distance.

In the AdS/CFT correspondence, one interprets $\beta(x)$ as the expectation value of a scalar field \mathcal{O} of dimension λ in the boundary conformal field theory.³ $\alpha(x)$ is related in a way we specify presently to a source for \mathcal{O} . The relationship between the fields and sources is analogous to the relationship between normalizable and unnormalizable operators in Liouville theory [19].

For theories with the familiar sort of large N limit, \mathcal{O} is a single-trace operator. To compute the expectation value of $\exp(-N^2 \int d^n x f(x) \mathcal{O})$ in the boundary conformal field theory, the familiar recipe [9,10] is to compute the AdS partition function with the boundary condition

$$\alpha = f \tag{3.6}$$

on ϕ .

Computing the expectation value of $\exp(-N^2 \int d^n x f(x)\mathcal{O})$ is the same as computing the partition function of the boundary conformal field theory in the presence of a perturbation N^2W added to the Lagrangian, where $W = \int d^n x f(x)\mathcal{O}$. Since β corresponds in the AdS/CFT correspondence to the expectation value of \mathcal{O} , we can symbolically write the boundary coupling as $W = \int d^n x f \beta$. If we do this, then the boundary condition (3.6) can be written

$$\alpha = \frac{\delta W}{\delta \beta}.\tag{3.7}$$

The problem of multi-trace interactions arises if we replace W by a local but nonlinear functional $W(x, \mathcal{O}, d\mathcal{O}, \ldots)$ of \mathcal{O} and its derivatives. Now we can state our proposal for

³ For a certain range of negative values of m^2 , there are two ways to quantize the ϕ field in AdS space [16], and correspondingly \mathcal{O} can have dimension $d - \lambda$ instead of λ [17,18]. This possibility will enter in section 4.

⁴ The factor of N^2 multiplying W is replaced in the string theory by a factor of $1/g_s^2$, where g_s is the string coupling constant. Both the bulk Lagrangian and the boundary coupling have this factor, so it does not show up in the boundary condition.

incorporating multi-trace interactions in the AdS/CFT correspondence: interpret W as a functional of β by replacing \mathcal{O} everywhere with β to get a functional $W(x, \beta, d\beta, \ldots)$ and impose the boundary condition (3.7) whether W is linear or not.

This generalizes immediately to the case of several scalar fields ϕ_i of masses m_i . They behave near r=0 as

$$\phi_i = \alpha_i(x)r^{d-\lambda_i} + \beta_i(x)r^{\lambda_i}. \tag{3.8}$$

The β_i are related to expectation values of operators \mathcal{O}_i of dimension λ_i in the boundary field theory, and the α_i are related to sources for those operators. Given a general multi-trace interaction $W(x, \mathcal{O}_i, d\mathcal{O}_i, \ldots)$ in the boundary theory, to incorporate it in the bulk theory we impose the boundary condition

$$\alpha_i = \frac{\delta W(x, \beta_k, d\beta_k, \ldots)}{\delta \beta_i}.$$
(3.9)

4. $(\operatorname{Tr}\Phi^2)^2$ Interaction In Four Dimensions And Related Examples

To illustrate this proposal, we consider first an example along lines suggested in [8]: a renormalizable field theory in four dimensions with a double-trace interaction.

Let \mathcal{O} be a half-BPS operator of dimension 2 in, for example, $\mathcal{N}=4$ super Yang-Mills theory in four dimensions. (For example, take $\mathcal{O}=\operatorname{Tr}(\Phi_1^2-\Phi_2^2)$ where Φ_1 and Φ_2 are two of the scalar fields.) We want to perturb the theory with the boundary coupling

$$W = \frac{f}{2} \int d^4x \, \mathcal{O}^2. \tag{4.1}$$

f is a dimensionless coupling constant. Classically, turning on $f \neq 0$ preserves conformal invariance while completely breaking supersymmetry. Quantum mechanically, conformal invariance is violated in order f^2 .

In fact, on flat \mathbf{R}^4 at f=0, the two-point function of the operator \mathcal{O} is determined by conformal invariance to be $\langle \mathcal{O}(x)\mathcal{O}(y)\rangle = v/|x-y|^4$, with v>0 by unitarity. This leads to a non-trivial beta function in order f^2 . Indeed, to compute quantum mechanical amplitudes in order f^2 , we would need matrix elements of

$$\frac{f^2}{8} \int d^4x \, d^4y \, \mathcal{O}^2(x) \mathcal{O}^2(y). \tag{4.2}$$

To evaluate the divergent contributions in (4.2), we need to know the operator product expansion of $\mathcal{O}(x)\mathcal{O}(y)$ for $x \to y$. In the large N limit, the structure simplifies drastically:

the two factors of \mathcal{O} in $\mathcal{O}^2(x)$ or $\mathcal{O}^2(y)$ do not "interfere" with each other. A divergent term that renormalizes f and survives in the large N limit comes only from the identity operator appearing in the product $\mathcal{O}(x)\mathcal{O}(y)$ for one pair of \mathcal{O} 's (analogous to a "single contraction" in free field theory). So we have to evaluate the divergent part of

$$\frac{f^2}{2} \int d^4x d^4y \, \mathcal{O}(x) \mathcal{O}(y) \langle \mathcal{O}(x) \mathcal{O}(y) \rangle. \tag{4.3}$$

Setting w = y - x, we encounter the logarithmically divergent integral

$$\int d^4w \mathcal{O}(x) \mathcal{O}(x+w) \langle \mathcal{O}(0)\mathcal{O}(w) \rangle \sim \mathcal{O}(x)^2 \int d^4w \, \frac{v}{|w|^4} \sim 2\pi^2 v \ln \Lambda \cdot \mathcal{O}(x)^2, \tag{4.4}$$

where Λ is a cutoff. This divergence, since it multiplies the operator $\mathcal{O}^2(x)$ that appears in the original interaction (4.1), can be interpreted as a renormalization of f. Because correlation functions factorize in the large N limit, there are no higher order corrections to the beta function; the renormalization just described gives the full answer. Henceforth, we normalize \mathcal{O} so that the beta function coefficient is 1.

Now, let us try to reproduce this behavior in the AdS language. Because \mathcal{O} has dimension 2 = d/2, the corresponding scalar field ϕ in AdS space has an exceptional behavior

$$\phi = \alpha(x)r^2 \ln(\mu r) + \beta(x)r^2 \tag{4.5}$$

near the boundary. Here μ is an arbitrary scale factor that must be included to define the logarithm. Our boundary interaction, interpreted in AdS language, is $W = \frac{f}{2} \int d^4x \, \beta^2$. Hence, the boundary condition (3.7) becomes

$$\alpha = f\beta. \tag{4.6}$$

With this boundary condition, the field behaves near infinity as

$$\phi = \beta r^2 (f \ln(\mu r) + 1). \tag{4.7}$$

Let us try to extract from this the renormalization effects of the boundary field theory. We want to re-express (4.7) in terms of a bare coupling f_0 defined at a cutoff scale $\Lambda >> \mu$, and a bare field β_0 . We expect β to be related to β_0 by multiplicative renormalization, $\beta = F(f_0, \Lambda/\mu)\beta_0$. Since the observable quantity ϕ of the bulk theory must be independent

of the quantity μ that was introduced in defining α and β (or alternatively, it must be independent of the renormalization procedure used on the boundary), we want

$$\beta_0(f_0 \ln(\Lambda r) + 1) = \beta(f \ln(\mu r) + 1).$$
 (4.8)

It follows that $\beta_0 f_0 = \beta f$ and

$$f = \frac{f_0}{1 + f_0 \ln(\Lambda/\mu)}.\tag{4.9}$$

(4.9) is the typical relation between the renormalized coupling and the bare coupling in a theory with only a "one-loop" beta function. Note that f > 0 is required for positivity of the boundary coupling. As in analogous examples of large N bosonic theories discussed in [20], the theory is not asymptotically free for f > 0, though asymptotic freedom would arise formally for f < 0.

A Line Of Fixed Points And A Nonperturbative Duality

Thus, we have shown in a non-trivial example how the more general boundary condition of the bulk theory reproduces the behavior of the boundary field theory. We will now consider another example that is suggested in part by a recent investigation in de Sitter space [21,22]. We consider in D=d+1-dimensional AdS space a theory with two scalar fields ϕ_1, ϕ_2 of equal mass squared. Thus near the boundary

$$\phi_i \sim \alpha_i(x)r^{d-\lambda} + \beta_i(x)r^{\lambda} \tag{4.10}$$

with equal λ . We suppose that λ is in the range $d/2 > \lambda > d/2 - 1$ where [16] two methods of quantization are possible. We adopt one method of quantization for ϕ_1 and the second for ϕ_2 , so that [17,18] ϕ_1 is related in the boundary theory to a conformal primary operator \mathcal{O}_1 of dimension λ but ϕ_2 is related to a conformal primary operator \mathcal{O}'_2 of dimension $d - \lambda$. Because of the reversed method of quantizing ϕ_2 , we write $\beta_2 = \alpha'_2$, $\alpha_2 = \beta'_2$, so (4.10) becomes

$$\phi_1 \sim \alpha_1(x)r^{d-\lambda} + \beta_1(x)r^{\lambda}$$

$$\phi_i \sim \beta_2'(x)r^{d-\lambda} + \alpha_2'(x)r^{\lambda}.$$
(4.11)

Thus, β_1 and β_2' are related to the expectation values of \mathcal{O}_1 and \mathcal{O}_2' , and α_1 and α_2' are related to sources for those operators.

Now we want to perturb the boundary theory by the marginal operator $f\mathcal{O}_1\mathcal{O}_2'$. This means that the boundary functional is to be $W = f \int d^n x \beta_1 \beta_2'$, and hence the boundary condition (3.8) is

$$\alpha_1 = f\beta_2', \quad \alpha_2' = f\beta_1. \tag{4.12}$$

The boundary condition preserves conformal invariance and is compatible with $\phi_1 = \phi_2 = 0$. Because ϕ_1 and ϕ_2 are fields of non-zero mass squared, they vanish in the unperturbed AdS solution at f = 0. That solution still obeys the boundary conditions for $f \neq 0$, and so if stable remains the correct solution for any f. So (modulo stability) we get in this way a line of conformal fixed points, parameterized by f. The dependence of this family on f is non-trivial, since the operator $\mathcal{O}_1\mathcal{O}_2'$ is a nonzero conformal primary, and correlation functions computed using the boundary condition (4.12) will certainly depend on f.

In fact, this line of fixed points admits a nonperturbative duality. The boundary condition (4.12) is invariant under $f \to 1/f$ together with $\alpha_1 \leftrightarrow \beta_1$, $\alpha'_2 \leftrightarrow \beta'_2$. The latter operation is not a symmetry of the full theory, since it is not a symmetry of the $r \to 0$ formulas in (4.11). But suppose that the bulk theory has a symmetry that exchanges ϕ_1 and ϕ_2 . Our method of quantization broke this symmetry, because we quantized ϕ_1 one way and ϕ_2 the other way. But $\phi_1 \leftrightarrow \phi_2$ is a symmetry when combined with $f \leftrightarrow 1/f$. For this operation, which exchanges α_1 with $\alpha_2 = \beta'_2$ and β_1 with $\beta_2 = \alpha'_2$, is a symmetry of both (4.11) and (4.12) as well as being, by hypothesis, a symmetry of the bulk theory.

For $f \to \infty$, the physics is the same as at small f except that ϕ_1 and ϕ_2 are exchanged. So in this limit, ϕ_1 is quantized to give an operator of dimension $d - \lambda$ and ϕ_2 to give an operator of dimension λ , the reverse of the situation for small f. Such a switch in varying f is possible because for f of order one, the two operators are mixed by the $\mathcal{O}_1\mathcal{O}_2'$ perturbation.

In the above, since we assumed $\lambda < d/2$, we could have added a relevant perturbation \mathcal{O}_1^2 , corresponding to a term β_1^2 in W. One can analyze the effects of such a relevant perturbation in a fashion similar to the above. In fact, in doing so, let us for simplicity omit the field ϕ_2 and return to the case of a single scalar field ϕ with expansion

$$\phi \sim \alpha r^{d-\lambda} + \beta r^{\lambda}. \tag{4.13}$$

For $d/2 > \lambda > d/2 - 1$, we can quantize the field with a boundary condition $\alpha = 0$, in which case it corresponds to an operator \mathcal{O} of dimension λ in the boundary theory, or with a boundary condition $\beta = 0$, in which case it corresponds to an operator \mathcal{O}' of dimension $d - \lambda$ in the boundary theory. Let us adopt the first method of quantization but include a relevant perturbation $W = \frac{g}{2}\beta^2$. The boundary condition is $\alpha = g\beta$, and we see that as $g \to \infty$, the boundary condition approaches the condition $\beta = 0$ that is suitable for quantization to get a field of dimension $d - \lambda$. So in fact, the renormalization group flow

leads from one method of quantization to the other. If we quantize with $\beta = 0$ to get an operator of dimension $d - \lambda$, the perturbation that would probe this flow would be $W = \frac{g'}{2}\alpha^2$, which is an irrelevant perturbation by an operator of dimension $2(d - \lambda)$.

The results we have just found are quite similar to results found in the old matrix model [6], where adding a double trace interaction with a suitable coefficient reverses the "gravitational dressing" of the interaction. (For a review, see [7].) Such a reversal is the Liouville analog of switching from λ to $d - \lambda$ the dimension of the operator that is associated with a given bulk field in the boundary theory.

A Remark On Boundary States

In section 3, we noted that the boundary condition (3.8) can be interpreted formally as saying that the boundary condition determines a quantum state, sometimes called the boundary state. In AdS space, this interpretation is somewhat formal because the boundary is at spatial infinity. Especially in the case of a Lorentz signature AdS space, to have a quantum state flowing in from *spatial* infinity does not correspond to standard physical intuition.⁵

The situation is somewhat different in de Sitter space, with positive cosmological constant. Here the boundary is at past and future infinity, and it is perfectly natural to interpret the physical conditions at the boundary as defining an initial or final state $|i\rangle$ or $\langle f|$, and the path integral as a matrix element $\langle f|i\rangle$. From this point of view, any operators that might be inserted in the past and future induce changes in the initial and final states. Hence the boundary correlation functions, studied in [24,25,21,22], arise by expanding the transition matrix element $\langle f|i\rangle$ in "perturbation theory" around the de Sitter invariant state,⁶ a process in which one can extract part of the information contained in that transition matrix element. The correlation functions were approached from this standpoint in [24].

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⁵ Nevertheless, this viewpoint has been adopted in some previous papers, for example [23].

⁶ Existence of more than one de Sitter invariant state, as investigated in [21,22], presumably means that there are different possible Hilbert spaces – inequivalent quantizations of the field – as occurs in flat spacetime in the presence of spontaneous symmetry breaking.

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