

# Algebra of the Infrared: String Field Theoretic Structures in Massive $\mathcal{N} = (2, 2)$ Field Theory In Two Dimensions

Davide Gaiotto,<sup>1</sup> Gregory W. Moore,<sup>2</sup> and Edward Witten<sup>3</sup>

<sup>1</sup>Perimeter Institute for Theoretical Physics

31 Caroline Street North, ON N2L 2Y5, Canada

<sup>2</sup>NHETC and Department of Physics and Astronomy, Rutgers University,  
Piscataway, NJ 08855–0849, USA

<sup>3</sup>School of Natural Sciences, Institute for Advanced Study,  
Princeton, NJ 08540, USA

dgaiotto@gmail.com, gmoore@physics.rutgers.edu, witten@ias.edu

**ABSTRACT:** We introduce a “web-based formalism” for describing the category of half-supersymmetric boundary conditions in  $1 + 1$  dimensional massive field theories with  $\mathcal{N} = (2, 2)$  supersymmetry and unbroken  $U(1)_R$  symmetry. We show that the category can be completely constructed from data available in the far infrared, namely, the vacua, the central charges of soliton sectors, and the spaces of soliton states on  $\mathbb{R}$ , together with certain “interaction and boundary emission amplitudes.” These amplitudes are shown to satisfy a system of algebraic constraints related to the theory of  $A_\infty$  and  $L_\infty$  algebras. The web-based formalism also gives a method of finding the BPS states for the theory on a half-line and on an interval. We investigate half-supersymmetric interfaces between theories and show that they have, in a certain sense, an associative “operator product.” We derive a categorification of wall-crossing formulae. The example of Landau-Ginzburg theories is described in depth drawing on ideas from Morse theory, and its interpretation in terms of supersymmetric quantum mechanics. In this context we show that the web-based category is equivalent to a version of the Fukaya-Seidel  $A_\infty$ -category associated to a holomorphic Lefschetz fibration, and we describe unusual local operators that appear in massive Landau-Ginzburg theories. We indicate potential applications to the theory of surface defects in theories of class S and to the gauge-theoretic approach to knot homology.

---

## Contents

|  |           |
|--|-----------|
| <b>1. Introduction</b>   | <b>5</b>  |
| 1.1 Preliminaries  | 5         |
| 1.2 Branes   | 7         |
| 1.3 Supersymmetric $(\mathfrak{B}_\ell, \mathfrak{B}_r)$ Strings | 8         |
| 1.4 Bulk Vertices  | 11        |
| 1.5 An Algebraic Structure                                       | 12        |
| 1.6 Categorical Wall-Crossing Formula                            | 15        |
| 1.7 A More Detailed Summary                                      | 16        |
| <b>2. Webs</b>   | <b>16</b> |
| 2.1 Plane Webs   | 16        |
| 2.2 Half-Plane Webs  | 24        |
| 2.3 Strip-Webs   | 28        |
| 2.4 Extended Webs  | 31        |
| 2.5 Special Configurations Of Vacuum Weights                     | 32        |
| <b>3. Tensor Algebras Of Webs And Homotopical Algebra</b>        | <b>34</b> |
| 3.1 $L_\infty$ And Plane Webs                                    | 34        |
| 3.1.1 Examples Of Web Algebras                                   | 37        |
| 3.2 Algebraic Structures From Half-Plane Webs                    | 37        |
| 3.2.1 $L_\infty$ -Modules  | 38        |
| 3.2.2 $A_\infty$ -Algebras                                       | 39        |
| 3.2.3 The $LA_\infty$ -Identities                                | 40        |
| 3.2.4 Conceptual Meaning Of The $LA_\infty$ -Identities          | 41        |
| 3.3 Bimodules And Strip Webs                                     | 43        |
| <b>4. Representations Of Webs</b>                                | <b>44</b> |
| 4.1 Web Representations And Plane Webs                           | 45        |
| 4.1.1 Isomorphisms Of Theories                                   | 49        |
| 4.2 Web Representations And Half-Plane Webs                      | 50        |
| 4.3 Web Representations And Strip-Webs                           | 54        |
| 4.4 On Degrees, Fermion Numbers And R-Symmetry                   | 57        |
| 4.5 Representations Of Extended Webs                             | 60        |
| 4.6 A Useful Set Of Examples With Cyclic Vacuum Weights          | 61        |
| 4.6.1 The Theories $\mathcal{T}_\vartheta^N$                     | 61        |
| 4.6.2 The Theories $\mathcal{T}_\vartheta^{SU(N)}$               | 69        |
| 4.6.3 Cyclic Isomorphisms Of The Theories                        | 75        |
| 4.6.4 Relation To Physical Models                                | 76        |

|  |            |
|--|------------|
| <b>5. Categories Of Branes</b>   | <b>79</b>  |
| 5.1 The Vacuum $A_\infty$ -Category  | 79         |
| 5.2 Branes And The Brane Category $\mathfrak{Br}$  | 82         |
| 5.3 Homotopy Equivalence Of Branes   | 83         |
| 5.4 Brane Categories And The Strip   | 85         |
| 5.5 Categorification Of 2d BPS Degeneracies  | 87         |
| 5.6 Continuous Deformations  | 88         |
| 5.7 Vacuum And Brane Categories For The Theories $\mathcal{T}_\vartheta^N$ And $\mathcal{T}_\vartheta^{SU(N)}$ | 89         |
| <b>6. Interfaces</b>   | <b>97</b>  |
| 6.1 Interface Webs   | 97         |
| 6.1.1 Definition And Basic Properties  | 97         |
| 6.1.2 Tensor Algebra Structures  | 99         |
| 6.1.3 Web Representations, Interfaces, And Interface Categories  | 100        |
| 6.1.4 Identity And Isomorphism Interfaces  | 101        |
| 6.1.5 Trivial Theories   | 105        |
| 6.1.6 Tensor Products Of $A_\infty$ -Algebras  | 105        |
| 6.2 Composite Webs And Composition Of Interfaces   | 105        |
| 6.3 Composition Of Three Interfaces  | 116        |
| 6.3.1 Limits   | 118        |
| 6.3.2 Homotopies   | 120        |
| 6.3.3 Homotopies Of Homotopies   | 123        |
| 6.4 Invertible Interfaces And Equivalences Of Theories   | 130        |
| <b>7. Categorical Transport: Simple Examples</b>   | <b>130</b> |
| 7.1 Curved Webs And Vacuum Homotopy  | 132        |
| 7.2 Rotation Interfaces  | 133        |
| 7.3 Wedge Webs   | 134        |
| 7.4 Construction Of Interfaces For Spinning Vacuum Weights   | 136        |
| 7.4.1 Past And Future Stable Binding Points  | 136        |
| 7.4.2 Defining Chan-Paton Spaces And Amplitudes  | 142        |
| 7.4.3 Verification Of Flat Parallel Transport  | 149        |
| 7.4.4 Rigid Rotations And Monodromy  | 150        |
| 7.4.5 The Relation Of Ground States To Local Operators   | 151        |
| 7.5 Locally Trivial Categorical Transport  | 151        |
| 7.6 S-Wall Interfaces  | 153        |
| 7.7 Categorification Of Framed Wall-Crossing   | 161        |
| 7.8 Mutations  | 162        |
| 7.9 Categorical Spectrum Generator And Monodromy   | 166        |
| 7.10 Rotation Interfaces For The Theories $\mathcal{T}_\vartheta^N$ And $\mathcal{T}_\vartheta^{SU(N)}$        | 167        |
| 7.10.1 Powers Of The Rotation Interface  | 182        |

|  |            |
|--|------------|
| <b>8. Categorical Transport And Wall-Crossing</b>  | <b>183</b> |
| 8.1 Preliminary Remarks  | 183        |
| 8.2 Tame Vacuum Homotopies   | 187        |
| 8.3 Wall-Crossing From Exceptional Webs  | 189        |
| 8.3.1 $L_\infty$ -Morphisms And Jumps In The Planar Taut Element                         | 189        |
| 8.3.2 $A_\infty$ -Morphisms And Jumps In The Half-Plane Taut Element                     | 195        |
| 8.3.3 An Interface For Exceptional Walls   | 197        |
| 8.4 Wall-Crossing From Marginal Stability Walls  | 199        |
| <b>9. Local Operators And Webs</b>   | <b>209</b> |
| 9.1 Doubly-Extended Webs And The Complex Of Local Operators On The Plane                 | 209        |
| 9.2 Traces Of Interfaces   | 213        |
| 9.3 Local Operators For The Theories $\mathcal{T}^N$ and $\mathcal{T}^{SU(N)}$           | 214        |
| <b>10. A Review Of Supersymmetric Quantum Mechanics And Its Relation To Morse Theory</b> | <b>217</b> |
| 10.1 The Semiclassical Approximation   | 217        |
| 10.2 The Fermion Number Anomaly  | 221        |
| 10.3 Instantons And The Flow Equation  | 222        |
| 10.4 Lifting The Vacuum Degeneracy   | 224        |
| 10.5 Some Practice   | 228        |
| 10.6 Why $\mathcal{Q}^2 = 0$   | 230        |
| 10.7 Why The Cohomology Does Not Depend On The Superpotential                            | 232        |
| <b>11. Landau-Ginzburg Theory As Supersymmetric Quantum Mechanics</b>                    | <b>238</b> |
| 11.1 Landau-Ginzburg Theory  | 238        |
| 11.2 Boundary Conditions   | 242        |
| 11.2.1 Generalities  | 242        |
| 11.2.2 Hamiltonian Symplectomorphisms  | 246        |
| 11.2.3 Branes With Noncompact Target Spaces  | 250        |
| 11.2.4 $W$ -Dominated Branes   | 251        |
| 11.2.5 Thimbles  | 253        |
| 11.2.6 Another Useful Class Of Branes: Class $T_\kappa$                                  | 254        |
| 11.3 The Fukaya-Seidel Category  | 258        |
| <b>12. MSW Complex On The Real Line: Solitons And Instantons</b>                         | <b>261</b> |
| 12.1 The Fermion Number  | 261        |
| 12.2 Properties Of The $\eta$ -Invariant   | 264        |
| 12.3 Quantum BPS States  | 266        |
| 12.4 Non-Triviality Of The Differential  | 271        |

|   |            |
|---|------------|
| <b>13. MSW Complex On The Half-Line And The Interval</b>                      | <b>274</b> |
| 13.1 The Half-Line  | 274        |
| 13.2 The Strip  | 275        |
| 13.3 The Fermion Number Revisited   | 277        |
| 13.4 Implications Of The Bulk Anomaly   | 283        |
| 13.5 Implications Of The Boundary Anomaly                                     | 286        |
| <b>14. <math>\zeta</math>-Instantons And <math>\zeta</math>-Webs</b>          | <b>288</b> |
| 14.1 Preliminaries  | 288        |
| 14.2 Fan-Like Asymptotics and $\zeta$ -Webs                                   | 290        |
| 14.3 The Index Of The Dirac Operator  | 296        |
| 14.4 More On $\zeta$ -Gluing  | 298        |
| 14.5 The Collective Coordinates   | 299        |
| 14.6 Interior Amplitudes And The Relations They Obey                          | 305        |
| 14.7 $\zeta$ -Instantons On A Half-Space Or A Strip                           | 308        |
| 14.7.1 Preliminaries  | 308        |
| 14.7.2 Boundary Amplitudes And The Relations They Obey                        | 310        |
| 14.7.3 $\zeta$ -Instantons On A Strip   | 312        |
| <b>15. Webs And The Fukaya-Seidel Category</b>                                | <b>314</b> |
| 15.1 Preliminaries  | 314        |
| 15.2 Morphisms  | 316        |
| 15.3 Multiplication   | 322        |
| 15.4 The Higher $A_\infty$ Operations   | 325        |
| <b>16. Local Observables</b>  | <b>328</b> |
| 16.1 The Need For Unfamiliar Local Observables                                | 328        |
| 16.2 Local Open-String Observables  | 330        |
| 16.3 Closed-String Observables  | 333        |
| 16.3.1 Twisted Closed-String States   | 333        |
| 16.3.2 Fans Of Solitons   | 336        |
| 16.3.3 More On The Mirror of $\mathbb{CP}^1$                                  | 340        |
| 16.3.4 Closed-String Amplitudes   | 343        |
| 16.4 Direct Treatment Of Local $A$ -Model Observables                         | 345        |
| <b>17. Interfaces And Forced Flows</b>  | <b>351</b> |
| <b>18. Generalizations, Potential Applications, And Open Problems</b>         | <b>357</b> |
| 18.1 Generalization 1: The Effect Of Twisted Masses                           | 357        |
| 18.2 Generalization 2: Surface Defects, Spectral Networks And Hitchin Systems | 359        |
| 18.3 Generalization 3: Hierarchies Of Scales And Cluster-Induced Webs         | 363        |
| 18.4 Potential Application: Knot Homology                                     | 370        |
| 18.4.1 The Main Point   | 370        |
| 18.4.2 Preliminary Reminder On Gauged Landau-Ginzburg Models                  | 373        |

|           |  |            |
|-----------|--|------------|
| 18.4.3    | Lightning Review: A Gauge-Theoretic Formulation Of Knot Homology                 | 374        |
| 18.4.4    | Reformulation For $M_3 = \mathbb{R} \times C$                                    | 379        |
| 18.4.5    | Boundary Conditions Defining The Fields Of The CSLG Model                        | 382        |
| 18.4.6    | Finite-Dimensional LG Models: The Monopole Model                                 | 385        |
| 18.4.7    | Finite-Dimensional Models: The Yang-Yang Model                                   | 389        |
| 18.4.8    | Knot Homology From Interfaces Between Landau-Ginzburg Models                     | 389        |
| <b>A.</b> | <b>Summary Of Some Homotopical Algebra</b>                                       | <b>392</b> |
| A.1       | Shuffles And Partitions  | 392        |
| A.2       | $A_\infty$ Algebras  | 393        |
| A.3       | $A_\infty$ Morphisms   | 397        |
| A.4       | $A_\infty$ Modules   | 398        |
| A.5       | $L_\infty$ Algebras, Morphisms, And Modules                                      | 400        |
| A.6       | $LA_\infty$ Algebras, Morphisms, And Modules                                     | 402        |
| <b>B.</b> | <b><math>A_\infty</math> Categories And Mutations</b>                            | <b>404</b> |
| B.1       | $A_\infty$ Categories And Exceptional Categories                                 | 404        |
| B.2       | Mutations Of Exceptional Categories  | 407        |
| B.2.1     | Exceptional Pairs And Two Distinguished Branes                                   | 407        |
| B.2.2     | Left And Right Mutations   | 408        |
| <b>C.</b> | <b>Examples Of Categories Of Branes</b>  | <b>412</b> |
| C.1       | One Vacuum   | 412        |
| C.2       | Two Vacua  | 412        |
| <b>D.</b> | <b>Proof Of Equation (7.181)</b>   | <b>415</b> |
| <b>E.</b> | <b>A More Technical Definition Of Fan Boundary Conditions</b>                    | <b>416</b> |
| <b>F.</b> | <b>Signs In The Supersymmetric Quantum Mechanics Formulation Of Morse Theory</b> | <b>417</b> |
| F.1       | Preliminaries  | 418        |
| F.2       | A Mathematical Sign Rule   | 419        |
| F.3       | Approach Via Supersymmetric Quantum Mechanics                                    | 420        |

---

## 1. Introduction

### 1.1 Preliminaries

This paper is devoted to the study of massive two-dimensional theories with  $(2, 2)$  supersymmetry. The supersymmetry operators of positive spacetime chirality are denoted  $Q_+, \bar{Q}_+$  and those of negative chirality by  $Q_-, \bar{Q}_-$ . (The adjoint of an operator  $\mathcal{O}$  is denoted

$\overline{\mathcal{O}}.$ ) It will be important that there is an unbroken  $U(1)$   $R$ -symmetry, whose generator we call  $\mathcal{F}$  or ‘‘fermion number.’’ Supersymmetry generators of  $\mathcal{F} = +1$  are  $Q_-$  and  $\overline{Q}_+$ , and those of  $\mathcal{F} = -1$  are  $Q_+$  and  $\overline{Q}_-$ . In Minkowski space with metric  $d\ell^2 = -dt^2 + dx^2$ , the supersymmetry algebra is

$$\begin{aligned}\{Q_+, \overline{Q}_+\} &= H + P \\ \{Q_-, \overline{Q}_-\} &= H - P \\ \{Q_+, Q_-\} &= \overline{Z} \\ \{\overline{Q}_+, \overline{Q}_-\} &= Z,\end{aligned}\tag{1.1}$$

with other anticommutators vanishing. Here  $H \sim -i\partial_t$  and  $P \sim -i\partial_x$  are the energy and momentum and  $Z$  is a central charge, which commutes with the whole algebra and with all local operators.

Typically, we consider a theory with a finite set  $\mathbb{V}$  of vacua (in some applications, one allows infinitely many vacua) in each of which there is a mass gap. Because  $Z$  is central, in the  $ij$  sector, which is defined as the space of states that interpolate from a vacuum  $i$  at  $x \rightarrow -\infty$  to a vacuum  $j$  at  $x \rightarrow +\infty$ ,  $Z$  is equal to a fixed complex number  $z_{ij}$ . Cluster decomposition implies that for  $i, j, k \in \mathbb{V}$ ,  $z_{ij} + z_{jk} = z_{ik}$ , and therefore there are complex numbers  $W_i$  (unique up to a common additive constant) such that  $W_i - W_j = z_{ij}$ ,  $i, j \in \mathbb{V}$ .  $W_i$  is called the value of the superpotential in vacuum  $i \in \mathbb{V}$ .

A large supply of massive  $\mathcal{N} = 2$  theories with  $U(1)$   $R$ -symmetry can be constructed as Landau-Ginzburg (LG) models with chiral superfields  $\phi_1, \dots, \phi_n$  valued in  $\mathbb{C}^n$ , and a suitable superpotential function  $W(\phi_1, \dots, \phi_n)$ . (Any superpotential at all leads to a theory with a  $U(1)$   $R$ -symmetry. Generically such a theory is massive and we usually call these theories massive  $\mathcal{N} = 2$  theories, leaving the  $R$ -symmetry understood.) Many of our considerations apply to more general LG models with general Kähler target space  $X$  and holomorphic Morse function  $W$ , but our considerations are already quite nontrivial for the case  $X = \mathbb{C}$ , and we restrict attention to  $X = \mathbb{C}^n$  in this introduction. Since a model defined in some other way may have a description as an effective LG theory at low energies, it may be that for some purposes this type of example is universal. In this paper, we describe a framework that we believe applies generally, but on some key points we rely on knowledge of LG models to infer what structure to expect.

Our goal is really to understand what additional information beyond the vacua and their central charges is needed in order to describe the supersymmetric states of a massive  $\mathcal{N} = 2$  theory. The most elementary extra needed information concerns the BPS soliton states in the  $ij$  sector. For useful background and further references see [14, 15, 50]. Let  $\mathcal{Q}_{ij} = Q_- - \zeta_{ij}^{-1} \overline{Q}_+$ , with  $|\zeta| = 1$ . Then for a state of  $P = 0$  in the  $ij$  sector,

$$\{\mathcal{Q}_{ij}, \overline{\mathcal{Q}}_{ij}\} = 2(H - \text{Re}(\zeta_{ij}^{-1} z_{ij})).\tag{1.2}$$

A standard argument shows that BPS states – states annihilated by  $\mathcal{Q}_{ij}$  and  $\overline{\mathcal{Q}}_{ij}$  – can exist only if  $H = |z_{ij}|$  and  $\zeta_{ij} = z_{ij}/|z_{ij}|$ . Such states come in supermultiplets consisting of a pair of states with  $\mathcal{F} = f, f+1$  for some  $f$ . Using cluster decomposition, it can be shown that the values of  $f \bmod \mathbb{Z}$  depend only on the vacua  $i$  and  $j$ . The number of BPS

multiplets for a given value of  $f$  is the most basic observable that goes beyond a knowledge of the set  $\mathbb{V}$  of vacua and the corresponding superpotential values  $W_i$ . In this paper, we write  $\mathcal{R}_{ij}$  for the space of BPS solitons in the  $ij$  sector.

As reviewed in Section §12, in an LG model, a classical approximation to  $\mathcal{R}_{ij}$  is the space  $R_{ij}$  of solutions of a certain supersymmetric soliton equation. In general,  $R_{ij}$  does not necessarily give a basis for the space  $\mathcal{R}_{ij}$  of quantum BPS soliton states. To determine  $\mathcal{R}_{ij}$ , one must in general compute certain instanton corrections to the classical soliton spectrum. The instantons are solutions of a certain nonlinear partial differential equation that we will call the  $\zeta$ -instanton equation.<sup>1</sup> For LG models, the construction of this paper can be developed taking as the starting point either the classical space  $R_{ij}$  of BPS solitons or the corresponding quantum-corrected space  $\mathcal{R}_{ij}$ . However, the construction is probably easier to understand if one starts with the classical space  $R_{ij}$ , so we will use that language in this introduction. For an abstract massive  $\mathcal{N} = 2$  model that is not presented as an LG model (and more generally is not presented with anything one would call a classical limit), there is no space  $R_{ij}$  of classical solitons and one has to make the construction in terms of the space  $\mathcal{R}_{ij}$  of quantum solitons, that is, BPS states in the  $ij$  sector.

## 1.2 Branes

We can get a much richer story by considering also half-BPS branes. We consider our theory on a half-plane  $\mathcal{H}$  defined by  $x \geq 0$  with a boundary condition at  $x = 0$  determined by a brane  $\mathfrak{B}$ . We suppose that, for some complex number  $\zeta_{\mathfrak{B}}$  of modulus 1, the brane  $\mathfrak{B}$  is invariant under the supersymmetry  $\mathcal{Q}_{\mathfrak{B}} = Q_- - \zeta_{\mathfrak{B}}^{-1} \overline{Q}_+$  and its adjoint  $\overline{\mathcal{Q}}_{\mathfrak{B}}$ . We also generally assume that  $\zeta_{\mathfrak{B}}$  does not coincide with any of the  $\zeta_{ij}$ . (This assumption keeps us away from walls at which jumping phenomena occur.)

A basic question about such a brane is as follows. If we formulate a massive  $\mathcal{N} = 2$  theory on the half-plane  $\mathcal{H}$  with the brane  $\mathfrak{B}$  at  $x = 0$  and a vacuum  $i \in \mathbb{V}$  at  $x = \infty$ , then what supersymmetric states are there, and with what values of the fermion number  $\mathcal{F}$ ? We write  $\mathcal{E}_i(\mathfrak{B})$  or just  $\mathcal{E}_i$  for the space of such states. The spaces  $\mathcal{E}_i(\mathfrak{B})$  depend on the brane  $\mathfrak{B}$  and on additional microscopic details of the theory. (In a Landau-Ginzburg theory, as explained in Section §13,  $\mathcal{E}_i(\mathfrak{B})$  can be determined by solving the classical soliton equation with boundary conditions determined by  $\mathfrak{B}$  and computing instanton corrections. The relevant instantons are solutions of the  $\zeta_{\mathfrak{B}}$ -instanton equation obeying certain boundary conditions.)

The assumption that  $\zeta_{\mathfrak{B}}$  does not coincide with any of the  $\zeta_{ij}$  ensures that one cannot make a supersymmetric state consisting of a BPS soliton at rest in the presence of the brane  $\mathfrak{B}$ . However, if we transform to Euclidean signature (by letting  $t = -i\tau$  so that the metric on  $\mathbb{R}^2$  becomes  $ds^2 = d\tau^2 + dx^2$ ), there can potentially exist a BPS configuration consisting of  $\mathfrak{B}$  together with a boosted or more precisely (in Euclidean signature) a rotated soliton. To understand why, recall that  $\zeta_{ij}$  was defined so that an  $ij$  BPS soliton at rest at fixed  $x$  – so that its world line is a straight line in the  $\tau$  direction – is invariant under  $\mathcal{Q}_{ij} = Q_- - \zeta_{ij}^{-1} \overline{Q}_+$ . If we rotate the  $x - \tau$  plane by an angle  $\varphi$ , then  $\mathcal{Q}_{ij}$  transforms

---

<sup>1</sup>For some prior work on this equation, see [88, 21, 41].

to  $\hat{Q}_{ij} = e^{-i\varphi/2} Q_- - \zeta_{ij}^{-1} e^{i\varphi/2} \overline{Q}_+$ . If we pick  $e^{i\varphi} = \zeta_{ij}/\zeta_{\mathfrak{B}}$ , then  $\hat{Q}_{ij}$  is a multiple of  $\mathcal{Q}_{\mathfrak{B}}$ . Hence a soliton whose worldline is a straight line at an angle  $\varphi_{ij} = \text{Arg}(\zeta_{ij}/\zeta_{\mathfrak{B}})$  to the  $\tau$ -axis preserves the same supersymmetry as the brane  $\mathfrak{B}$ . Accordingly, one can ask (Figure 1) whether there is a supersymmetric coupling by which  $\mathfrak{B}$  emits a BPS soliton of type  $ij$  at an angle  $\varphi_{ij}$  to the vertical. The answer to this question is not determined in any elementary way by any data we have mentioned so far. All we can say from the point of view of low energy effective field theory is that in general, the answer depends on the choice of a soliton state in  $R_{ij}$ , and on the assumed initial and final states in  $\mathcal{E}_j(\mathfrak{B})$  and  $\mathcal{E}_i(\mathfrak{B})$ . Thus the answer can be summarized by a linear transformation<sup>2</sup>

$$T_{ij\mathfrak{B}} : \mathcal{E}_j(\mathfrak{B}) \rightarrow \mathcal{E}_i(\mathfrak{B}) \otimes R_{ij}. \quad (1.3)$$

(In a Landau-Ginzburg theory, this linear transformation can be determined, in principle by solving the  $\zeta$ -instanton equation with suitable boundary conditions near the brane and at infinity. This is explained in Section §14.)

### 1.3 Supersymmetric $(\mathfrak{B}_\ell, \mathfrak{B}_r)$ Strings

At first sight, it may not be obvious that the amplitude for a supersymmetric brane to emit a supersymmetric soliton at an angle is related to anything that is usually studied in a supersymmetric theory. To see that it is, replace the half-plane  $x \geq 0$  with a strip  $0 \leq x \leq L$ , where we can take  $L$  to be much greater than the Compton wavelength of any particle in any of the vacua  $i \in \mathbb{V}$ . Let  $\mathfrak{B}_\ell$  and  $\mathfrak{B}_r$  be a pair of mutually BPS branes, meaning that  $\zeta_{\mathfrak{B}_\ell} = \zeta_{\mathfrak{B}_r}$ , so that  $\mathcal{Q}_{\mathfrak{B}_\ell} = \mathcal{Q}_{\mathfrak{B}_r}$ ; we denote them as  $\mathcal{Q}_{\mathfrak{B}}$ . Use  $\mathfrak{B}_\ell$  and  $\mathfrak{B}_r$  to define boundary conditions at  $x = 0$  and  $x = L$ , respectively. By a supersymmetric  $(\mathfrak{B}_\ell, \mathfrak{B}_r)$  state, we mean a state of this system of zero energy or equivalently a state annihilated by  $\mathcal{Q}_{\mathfrak{B}}$  and its adjoint.

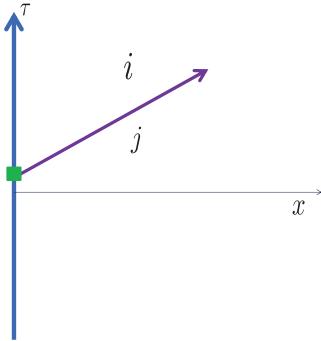
What is the space of these supersymmetric states? There is an obvious approximation for large  $L$ . Far from  $x = 0$  and from  $x = L$ , the system must be exponentially close to one of the vacua  $i \in \mathbb{V}$ . For given  $i$ , near  $x = 0$ , the system is in some state in  $\mathcal{E}_i(\mathfrak{B}_\ell)$  and near  $x = L$ , it is in some state in  $\mathcal{E}_i(\mathfrak{B}_r)$ . The mass gap means that to a good approximation, these two states can be specified independently and thus a large  $L$  approximation to the space of supersymmetric  $(\mathfrak{B}_\ell, \mathfrak{B}_r)$  states is given by

$$\oplus_{i \in \mathbb{V}} \mathcal{E}_i(\mathfrak{B}_\ell) \otimes \mathcal{E}_i(\mathfrak{B}_r). \quad (1.4)$$

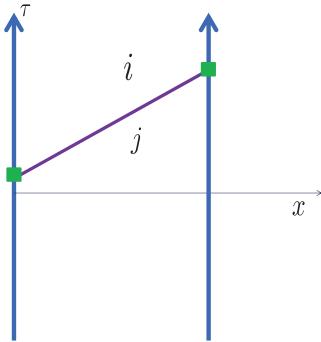
In general, however, eqn. (1.4) only gives an approximation to the space of supersymmetric  $(\mathfrak{B}_\ell, \mathfrak{B}_r)$  states. The states just described have very nearly zero energy, but they may not have precisely zero energy. The reason for this is that, rather as in supersymmetric quantum mechanics [87], instanton corrections can lift some approximate zero energy states away from zero energy, though only by an exponentially small amount. A simple instanton in this context is a process in which brane  $\mathfrak{B}_\ell$  emits a BPS soliton at an angle and that soliton is absorbed by  $\mathfrak{B}_r$  (Figure 2). The angle at which the soliton propagates

---

<sup>2</sup>In the abstract formulation of §4,  $T_{ij\mathfrak{B}}$  is generalized to a multisoliton emission amplitude  $\mathcal{B}$  (an element of the vector space (4.29)) that satisfies a Maurer-Cartan equation (eqn. (4.47)).



**Figure 1:** An  $ij$  soliton emitted from the boundary. This process is supersymmetric if the soliton is emitted at the proper angle.

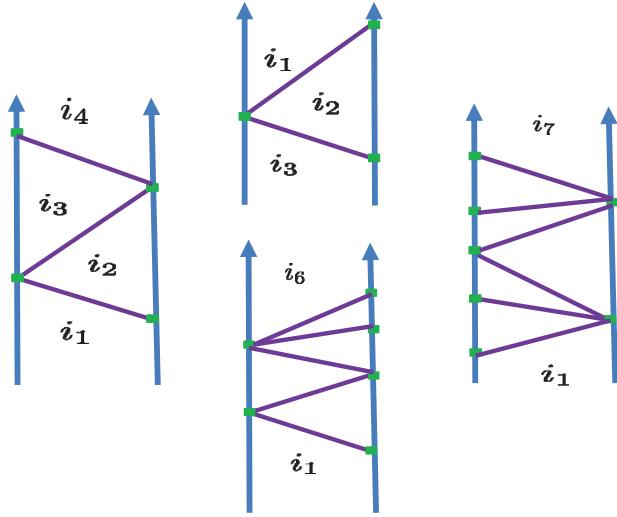


**Figure 2:** A supersymmetric soliton exchanged between the two branes on the left and the right.

in this figure is determined by supersymmetry, and therefore it is reasonable to expect that the instanton sketched in the figure has precisely 1 real modulus, which one can think of as the “time” at which the soliton is exchanged.

In supersymmetric quantum mechanics, an instanton that depends on only 1 real modulus – the time of the instanton event – gives a correction to the matrix element of a supercharge of fermion number  $\mathcal{F} = 1$ . (An anti-instanton that depends on 1 real modulus similarly corrects the matrix element of an  $\mathcal{F} = -1$  supercharge.) More generally, in the field of an instanton that depends on  $k$  real moduli, there are  $k$  fermion zero modes, so to get a non-zero amplitude, one must insert operators that shift  $\mathcal{F}$  by  $k$  units. In massive LG models, the same relationship holds between the dimension of instanton moduli space and the violation of fermion number, as explained in Section §14.

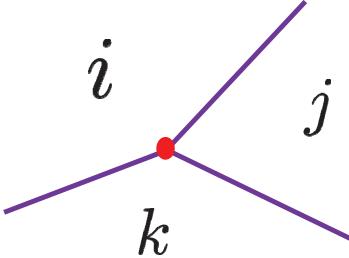
Given the facts stated in the last paragraph, the instanton of Figure 2 describes a



**Figure 3:** This figure shows a variety of rigid strip instantons constructed using boundary vertices only.

process in which  $\mathcal{F}$  changes by 1 and thus this instanton can contribute to the matrix element of  $\mathcal{Q}_{\mathfrak{B}}$  between initial and final states that have zero energy in the approximation of eqn. (1.4). In other words, such instantons can shift some approximately supersymmetric states away from zero energy. They must be taken into account in order to determine the supersymmetric  $(\mathfrak{B}_\ell, \mathfrak{B}_r)$  states.

Once one gets this far, it is not hard to see that additional types of supersymmetric instantons might also be relevant. First of all, there might be amplitudes for the branes  $\mathfrak{B}_\ell$  and/or  $\mathfrak{B}_r$  to emit simultaneously two or more supersymmetric solitons at suitable angles. (In a massive Landau-Ginzburg model, such a multiple emission event is again computed by a solution of the  $\zeta$ -instanton equation with suitable boundary conditions. We consider again the solutions that have only a single real modulus corresponding to overall time translations.) If so, when the theory is formulated on a strip, many additional types of supersymmetric instantons are possible. In particular, the instantons indicated in Figure 3 all depend on only a single real modulus – the overall time of the tunneling event – since the angles are fixed by supersymmetry. Therefore, these instantons must all be taken into account to determine which states of the  $(\mathfrak{B}_\ell, \mathfrak{B}_r)$  system are precisely supersymmetric. We will say that an instanton in the strip is “rigid” if it has no moduli except the one associated to time translations. So the instantons depicted in Figure 3 are all rigid. (Later in this paper, we will make a distinction between “rigid” and “taut” webs of BPS states on  $\mathbb{R}^2$  or on a half-plane, but in the present context of instantons on a strip, this distinction does not arise.)



**Figure 4:** A “bulk” vertex that involves a coupling of three BPS solitons. The vacua involved are  $i, j, k \in \mathbb{V}$ , and the solitons that emanate from the vertex are respectively of types  $ij$ ,  $jk$ , and  $ki$ .

#### 1.4 Bulk Vertices

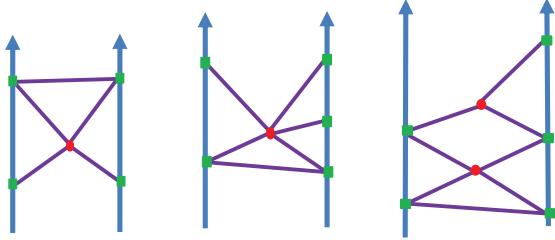
This is still far from the whole story. Certain “closed string” processes must also be considered. Once one realizes that there can be supersymmetric “boundary” vertices in which BPS solitons are emitted from a brane, it is natural to wonder if similarly there can be “bulk” vertices involving the coupling of BPS solitons.<sup>3</sup> The most basic example is a trilinear coupling of three BPS solitons (Figure 4), with each soliton emitted at an angle  $\varphi_{ij} = \text{Arg } \zeta_{ij}/\zeta_{\mathfrak{B}}$ , as above. Low energy effective field theory allows this possibility. (In a Landau-Ginzburg theory, such a coupling arises from a solution of the  $\zeta$ -instanton equation with suitable asymptotic conditions.)

A cubic bulk vertex has at least 2 real moduli, corresponding to the position of the vertex in  $\mathbb{R}^2$ . For our present discussion, the relevant case is that these are the only moduli (otherwise, we will not be able to make a rigid instanton in the strip). We say that a bulk vertex is rigid if it has only the 2 real moduli associated to spacetime translations. Is a rigid cubic bulk vertex, if it exists, relevant to the problem of understanding the supersymmetric  $(\mathfrak{B}_\ell, \mathfrak{B}_r)$  states? One may think the answer is “no” because once a bulk vertex is included, even a rigid one, the number of real moduli will be at least 2.

However, it is not hard to construct strip instantons that depend on only 1 real modulus even though they involve bulk vertices that individually would depend on 2 real moduli

---

<sup>3</sup>Examples of such bulk vertices have been constructed in [13, 39]. No analogous explicit examples of boundary vertices are known.



**Figure 5:** Rigid strip instantons whose construction makes use of bulk vertices. We assume the bulk vertices have no moduli except the ones associated to spacetime translations.

each. Some examples are shown in Figure 5. These web instantons must all be included to determine which  $(\mathcal{B}_\ell, \mathcal{B}_r)$  states have precisely zero energy.

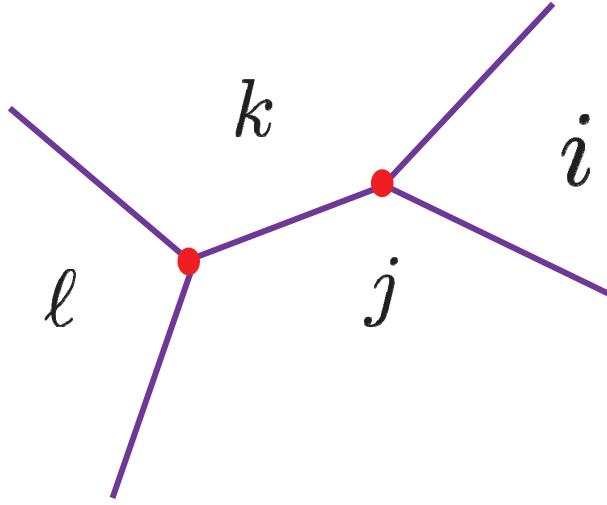
In short, to answer the seemingly simple question of finding the supersymmetric states on an interval in a massive theory, we need a full understanding of all bulk and boundary vertices involving couplings of BPS solitons, and how they can be put together to make what we will call a “web” of BPS solitons.

### 1.5 An Algebraic Structure

At this point, matters may seem bewilderingly complicated. However, there is a hidden simplification: the data that we have described can be combined into a rich algebraic structure that makes things tractable. This structure is the real topic of the present paper.

To illustrate the basic idea, we start with a cubic vertex involving vacua  $i, j, k$  and another cubic vertex involving vacua  $k, j, l$ . We assume that each vertex has only the 2 moduli associated to spacetime translations. If the vertices are far apart, we can make an approximate solution involving all four vacua  $i, j, k, l$  by gluing together the  $jk$  soliton that emerges from one vertex with the  $kj$  soliton that emerges from the other vertex (Figure 6). After slightly adjusting the output of this gluing operation, one gets a family of solutions of the  $\zeta$ -instanton equation with a three-dimensional moduli space that we will call  $\mathcal{M}$ . (One expects that index theory and ellipticity of the LG instanton equation ensure that this adjustment can be made.)

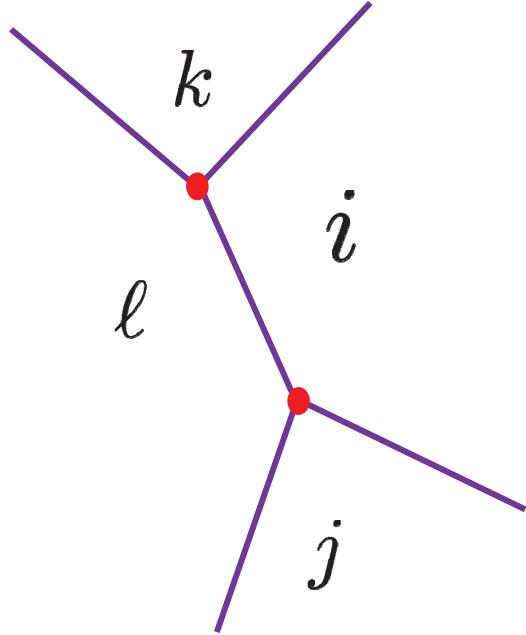
To the extent that we can identify the moduli from the figure, two of them are associated to spacetime translations and the third is the distance  $d$  between the two vertices. So if we rely entirely on this figure, it looks like  $\mathcal{M}$  is a copy of  $\mathbb{R}^2 \times \mathbb{R}_+$ , where  $\mathbb{R}^2$  parametrizes the position of, say, the  $ijk$  vertex, and  $\mathbb{R}_+$  is the half-line  $d \geq 0$ .



**Figure 6:** An “end” of the moduli space of solutions of the  $\zeta$ -instanton equation corresponding to two widely separated vertices of type  $ijk$  and  $kjl$ . Assuming the individual vertices have only the obvious moduli associated to spacetime translations, this component of the moduli space is three-dimensional.

At least in the context of LG models,  $\mathbb{R}^2 \times \mathbb{R}_+$  cannot be the correct answer, since  $\mathbb{R}_+$  has a boundary at  $d = 0$ . Because of the superrenormalizable nature of the LG theory, there will be no such boundary in the moduli space of solutions of the  $\zeta$ -instanton equation. (A technical statement is that the  $\zeta$ -instanton equation is a linear equation plus lower order nonlinear terms. The linear equation with target space  $X = \mathbb{C}^n$  does not admit “bubbling” and the superpotential is a lower order term which does not change that property.) For a generic superpotential, any family of solutions can be continued, with no natural boundaries or singularities, and with ends that arise only when something goes to infinity. In a massive LG theory, the scalar fields cannot go to infinity (since the potential energy grows when they do), so all that can go to infinity are the vertices. This means that the “ends” of the moduli space have a semiclassical picture in terms of a soliton web, as in Figure 6. Moreover, this is also true for the reduced moduli space  $\mathcal{M}' = \mathcal{M}/\mathbb{R}^2$  that is obtained by dividing out by spacetime translations.

In Figure 6, the reduced moduli space, to the extent that we can understand it from the figure, is a copy of  $\mathbb{R}_+$  with one visible end for  $d \rightarrow \infty$ . But the moduli space cannot just end at  $d = 0$ . It has to continue somehow. Because a one-manifold without boundary that



**Figure 7:** The component of moduli space that has one end depicted in Figure 6 must have a second end. The second end might be as depicted here. Note that the “fans” of vacua at infinity are the same in this figure and in Figure 6 so they can appear as parts of the same moduli space.

has at least one noncompact end is a copy of  $\mathbb{R}$ , which has two ends, it must continue to infinity with a second end. A correct view of the figure is that it gives a good approximate picture of a family of solutions of the  $\zeta$ -instanton equation when  $d$  is large. When  $d$  is not large, the semiclassical picture of the solution given in Figure 6 is not valid. But the reduced moduli  $\mathcal{M}'$  must have a second “end” that again has a semiclassical interpretation.

Low energy effective field theory is not powerful enough to predict what this second end will be. In general, there are different possibilities. In the case at hand, a natural possibility (Figure 7) is that in addition to the  $ijk$  and  $kjl$  vertices that we started with, there are also solutions of the  $\zeta$ -instanton equation corresponding to  $ijl$  and  $lki$  vertices. The reason that the soliton webs shown in figs. 6 and 7 can both appear as part of the same moduli space is that they connect to the same “web” of BPS solitons at infinity.

In this situation, if we represent an  $ijk$  vertex by a symbol  $\beta_{ijk}$ , we see that there is some sort of relation between the products  $\beta_{ijk}\beta_{kjl}$  and  $\beta_{ijl}\beta_{klj}$ . What is this relationship precisely?

The answer turns out to be that the bulk vertices are part of an algebraic structure known as an  $L_\infty$  algebra. (More precisely, they define a solution of the Maurer-Cartan equation in an  $L_\infty$  algebra. Such a solution can be used to deform an  $L_\infty$  algebra to a

new  $L_\infty$  algebra.) This structure often appears in closed-string theory (for instance, see [96, 97]), and in related areas of mathematics. The vertices associated to emissions from a brane can similarly be used to define what in open-string theory and related areas is sometimes called an  $A_\infty$  algebra. We expect that these algebraic structures are universal for massive  $\mathcal{N} = 2$  theories, though in motivating them, we have made essential use of LG models. The reason that we expect so is that these algebraic structures are well-adapted to answering our basic question of how to determine the spectrum of supersymmetric states in the presence of branes.

## 1.6 Categorical Wall-Crossing Formula

The spaces  $\mathcal{R}_{ij}$  of quantum BPS soliton states and the spaces of quantum ground states  $\mathcal{E}_i(\mathfrak{B})$  on the half line are objects of independent interest. As the parameters of the underlying theory are varied, these spaces of ground states are expected to jump across certain walls of marginal stability. The standard theory of wall-crossing constrains the variation of the Witten indices of such spaces of states: the BPS and framed BPS degeneracies

$$\mu_{ij} = \text{Tr}_{\mathcal{R}_{ij}}(-1)^F \quad \underline{\Omega}(\mathfrak{B}, i) = \text{Tr}_{\mathcal{E}_i(\mathfrak{B})}(-1)^F \quad (1.5)$$

The framed BPS degeneracies jump across walls where  $\zeta^{\mathfrak{B}}$  aligns to some  $\zeta_{ij}$ . The form of the jump is universal [31]:

$$S_{ij} : \underline{\Omega}(\mathfrak{B}, j) \rightarrow \underline{\Omega}(\mathfrak{B}, j) + \underline{\Omega}(\mathfrak{B}, i)\mu_{ij} \quad (1.6)$$

with all other  $\underline{\Omega}(\mathfrak{B}, k)$  remaining unchanged.

The BPS degeneracies jump across walls where  $\zeta_{ij}$  aligns to some  $\zeta_{jk}$ . The form of the jump is universal [15]

$$\mu_{ik} \rightarrow \mu_{ik} + \mu_{ij}\mu_{jk} \quad (1.7)$$

with all other  $\mu_{kt}$  remaining unchanged. This formula can be derived directly by requiring compatibility with the framed BPS wall-crossing formula: the relation

$$S_{ij}[\mu]S_{ik}[\mu]S_{jk}[\mu] = S_{kj}[\mu']S_{ik}[\mu']S_{ij}[\mu'] \quad (1.8)$$

implies that the  $\mu$  and  $\mu'$  degeneracies are related as in (1.7).

In Sections 7 and 8 we address the problem of providing a categorification of such wall-crossing formulae, i.e. we describe the categorical data which should be added to the vector spaces  $\mathcal{R}_{ij}$  and  $\mathcal{E}_i(\mathfrak{B})$  in order to allow for a universal description of how such vector spaces (and the categorical data itself) jump across walls of marginal stability.

The categorical data which has to be added to the  $\mathcal{E}_i(\mathfrak{B})$  to describe their wall-crossing properties essentially coincides with the amplitudes for the emission of BPS solitons from the boundary condition  $\mathfrak{B}$ , organized into an object of an appropriate category. The categorical wall-crossing of  $\mathcal{E}_i(\mathfrak{B})$  is encoded in a “mutation” of that category.

The categorical wall-crossing of the  $\mathcal{R}_{ij}$  is determined again by requiring compatibility with the categorical framed BPS wall-crossing formula. The existence of a categorical BPS wall-crossing formula is related to the observation that mutations form, in an appropriate sense, a representation of the braid group.

## 1.7 A More Detailed Summary

In this introduction, we have omitted several subjects that are treated in considerable detail in the main text. These include interfaces between theories, as well as bulk and boundary local operators in massive  $\mathcal{N} = 2$  theories.

The curious reader who wants to learn more detail, but is daunted by the length of the present paper, is referred to [35]. These are lecture notes that summarize the entire paper from a broad perspective, and can serve as a detailed introduction.

## 2. Webs

In the previous section we have motivated from qualitative physical considerations the concept of webs associated to a massive two-dimensional supersymmetric QFT. In this section we abstract that idea and discuss in some detail a purely mathematical construction.

### 2.1 Plane Webs

We begin with webs in the plane  $\mathbb{R}^2$ , which we sometimes identify with  $\mathbb{C}$ .

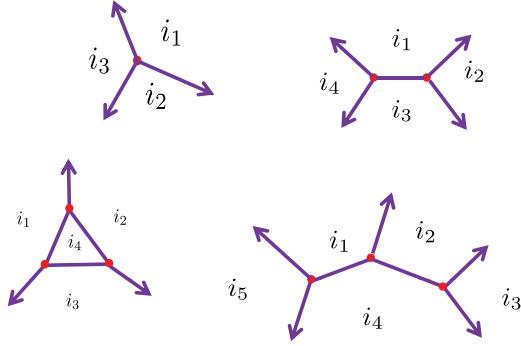
The definition of a web depends on some data. We fix a finite set  $\mathbb{V}$  called the *set of vacua*. Typical elements are denoted  $i, j, \dots \in \mathbb{V}$ . We also fix a set of *weights* associated to these vacua which are complex numbers  $\{z_i\}$ , that is, we fix a map  $z : \mathbb{V} \rightarrow \mathbb{C}$ . We assume that  $z_{ij} \neq 0$  for  $i \neq j$ . The pair  $(\mathbb{V}, z)$  will be called *vacuum data*. The following definition is absolutely fundamental to our formalism:

**Definition:** A *plane web* is a graph in  $\mathbb{R}^2$ , together with a labeling of the *faces* (i.e. connected components of the complement of the graph) by vacua such that the labels across each edge are different and moreover, when oriented with  $i$  on the left and  $j$  on the right the edge is straight and parallel to the complex number  $z_{ij} := z_i - z_j$ . We take plane webs to have all vertices of valence at least three. In Section §2.4 we define a larger class of *extended plane webs* which have two-valent vertices. In Section §9 we will introduce a further generalization to *doubly extended plane webs* by allowing certain zero-valent vertices.

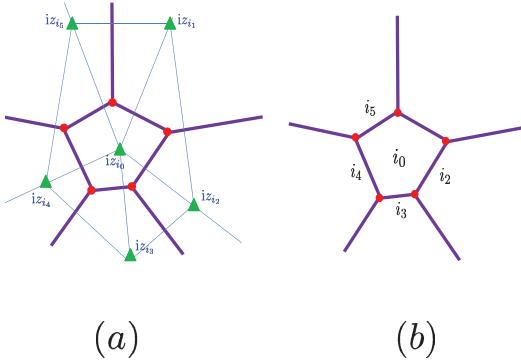
Some examples of webs are shown in Figure 8. We make a number of remarks on some basic properties that immediately follow from this simple definition:

1. Note that the edges do not have an intrinsic orientation. If we reverse the orientation of an edge then  $j$  is on the left and  $i$  is on the right and then the oriented edge is parallel to  $z_{ji} = -z_{ij}$ . Edges which go to infinity are called *external edges* and the remaining edges are *internal edges*. In section §4 we give external edges a canonical outward orientation.
2. At each vertex of a plane web the labels in the angular regions in the *clockwise direction* define a *cyclic fan of vacua*, which is, by definition, an ordered set  $\{i_1, \dots, i_s\}$  so that the phases of  $z_{i_k, i_{k+1}}$ , with  $k$  understood modulo  $s$ , form a clockwise ordered collection of points on the unit circle. Put differently

$$\text{Im} z_{i_{k-1}, i_k} \overline{z_{i_k, i_{k+1}}} > 0 \quad (2.1)$$



**Figure 8:** Some examples of plane webs.



**Figure 9:** (a) A configuration of weights  $z_i$  which is not convex. The green triangles indicate the points  $iz_i$ . The dual graph gives an example of a web shown in (b). Note that the corresponding web has a loop.

for all  $k$ . We generally denote a cyclic fan of vacua by  $I = \{i_1, \dots, i_s\}$  and we say that  $I$  has length  $s$ .

3. A useful intuition is obtained by thinking of the edges as strings under a tension given by  $z_{ij}$ . Then at each vertex we have a no-force condition:

$$z_{i_1, i_2} + z_{i_2, i_3} + \dots + z_{i_n, i_1} = 0 \quad (2.2)$$

It follows that the edges emanating from any vertex cannot lie in any half plane.

4. For plane webs the graph must be connected. Moreover, for a fixed set of weights  $\{z_i\}$

there are only a finite number of plane webs.<sup>4</sup> We can prove these statements with a useful argument which we will call the *line principle*: If we consider any oriented line in the plane which does not go through a vertex of the web then we encounter an ordered set of vacua given by the labels of the regions intersecting the line. The line principle says that no vacuum can appear twice. To prove this note that we can orient all the edges which intersect the line to point into one half-plane cut out by the line. Then a vacuum cannot appear twice since if it did in the sequence  $\{i, j_1, \dots, j_k, i\}$  then, on the one hand, the sum of the tensions  $z_{i,j_1} + \dots + z_{j_k,i} = 0$ , but on the other hand all the terms in the sum point into the same half-plane, which is impossible. Therefore there are only a finite number of possible sequences of vacua. This implies that there are only a finite number of possible vertices. Other corollaries of the line principle are that no vertex can appear twice within any given web and there are at least three external edges.

5. A sequence of weights  $z_{i_k}$  is associated to a cyclic fan of vacua  $I = \{i_1, \dots, i_s\}$  if and only if they are the clockwise ordered vertices of a convex polygon in the complex plane. The topology of a web  $\mathfrak{w}$  is captured by the decomposition of the polygon  $P_\infty$  associated to  $I_\infty(\mathfrak{w})$  into the polygons  $P_v$  associated to the  $I_v(\mathfrak{w})$ . This can be seen by noting that an internal edge of the web connecting vertices  $v_1$  and  $v_2$  corresponds to a shared edge of the polygons  $P_{v_1}$  and  $P_{v_2}$ . On the other hand, each external edge of the web is associated to a single vertex  $v$  and corresponds to an external boundary of  $P_\infty$ . See, for example, Figure 9. Indeed, the decomposition can be identified with a dual graph to the web. This provides an alternative, intuitive explanation of many properties of the webs. The paper [54] of Kapranov, Kontsevich, and Soibelman emphasizes this dual viewpoint and suggests that it is the proper formulation for generalizing the structures we find to higher dimensional field theories.
6. A corollary of the above remark is that for a given set of weights  $\{z_i\}$  there is a web with a closed loop if and only if there is a sequence of weights  $\{z_{i_1}, \dots, z_{i_s}\}$  which are vertices of a convex polygon such that there is a weight  $z_{i_0}$  in the interior of the polygon. The existence of a web with a closed loop implies that there are positive numbers  $\lambda_\alpha$  with

$$\lambda_1 z_{i_1, i_0} + \dots + \lambda_s z_{i_s, i_0} = 0 \quad (2.3)$$

and hence

$$z_{i_0} = t_1 z_{i_1, i_0} + \dots + t_s z_{i_s, i_0} \quad (2.4)$$

with  $t_\alpha = \lambda_\alpha / \sum_\beta \lambda_\beta$ . Conversely, if (2.4) holds with  $\text{Im}(z_{i_{k-1}, i_0} \overline{z_{i_k, i_0}}) > 0$  for all  $k$  then it is easy to show that  $\{i_{k-1}, i_k, i_0\}$  is a cyclic fan of vacua so the web shown in Figure 9(b) exists.

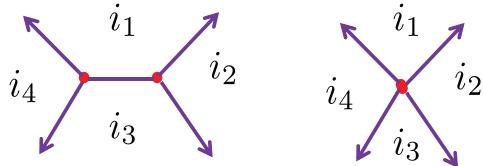
7. *Some notation:* We will denote a web (or rather its “deformation type” - defined below) by a gothic “ $w$ ,” which looks like  $\mathfrak{w}$ . The set of vertices is denoted by  $\mathcal{V}(\mathfrak{w})$ ,

---

<sup>4</sup>This finiteness property is one advantage of the requirement that all vertices have valence bigger than two.

and it has order  $V(\mathfrak{w})$ . Similarly, the set of internal edges is  $\mathcal{E}(\mathfrak{w})$  and has order  $E(\mathfrak{w})$ . At each vertex  $v \in \mathcal{V}(\mathfrak{w})$  there is a cyclic fan of vacua  $I_v(\mathfrak{w})$ . The cyclic fan of vacua at infinity is denoted  $I_\infty(\mathfrak{w})$ .

8. We have the relation  $V(\mathfrak{w}) - E(\mathfrak{w}) + F(\mathfrak{w}) = 1$ , where  $F(\mathfrak{w})$  is the number of bounded faces (and hence the number of internal loops). This follows since if we add a vertex at infinity then the web triangulates  $S^2$ .
9. In the applications to Landau-Ginzburg theories in Sections §§11-17 below the vacua  $\mathbb{V}$  will be the critical points of the superpotential and the vacuum weights  $z_i$  are essentially the critical values of the superpotential. The precise relation, as determined by equation (11.15) and Figure 133 below is  $z_i = \zeta \bar{W}_i$ , where  $W_i$  is the critical value of the superpotential and  $\zeta$  is a phase, introduced in Section §11.
10. Finally, we note that in the application to knot-homology described in §18.4 we will need to relax the constraint that  $\mathbb{V}$  is a finite set. In general, if  $\mathbb{V}$  is infinite one can choose weights  $z_i$  leading to pathologies. (For example, if the weights have an accumulation point in the complex plane, there will be infinite numbers of webs with the same fan at infinity.) In Section §18.1 we describe a class of models where  $\mathbb{V}$  is infinite, but for which our theory still applies. The knot homology examples belong to this class.



**Figure 10:** The two webs shown here are considered to be different deformation types, even though the web on the left can clearly degenerate to the web on the right.

A plane web has a *deformation type*: This is an equivalence class under translation and/or scaling of the lengths of some subset of the internal edges. This scaling must of course be compatible with the constraints that define a web: In terms of the string model of the web mentioned above we are allowed to stretch and translate the strings, but we must not rotate them, and we must maintain the no-force condition. In a deformation type no edge is allowed to be scaled to zero size. See Figure 10. The set of webs with

fixed deformation type  $\mathfrak{w}$  is naturally embedded as a cell  $\mathcal{D}(\mathfrak{w}) \subset (\mathbb{R}^2)^{V(\mathfrak{w})}$  by considering the  $(x, y)$  coordinates of all the vertices of the web. The (internal) edge conditions impose  $E(\mathfrak{w})$  linear relations on these coordinates, together with inequalities requiring that each edge have positive length. When the vacuum weights  $z_i$  are in general position the edge conditions will be independent equations and then  $\mathcal{D}(\mathfrak{w})$  will be a convex cone of dimension

$$d(\mathfrak{w}) := 2V(\mathfrak{w}) - E(\mathfrak{w}). \quad (2.5)$$

We will sometimes refer to  $d(\mathfrak{w})$  as the *degree* of the web. Note that there is a free action of translations on the set of webs of a given deformation type, so  $d(\mathfrak{w}) \geq 2$ . We will refer to the quotient  $\mathcal{D}_r(\mathfrak{w})$  of the moduli space  $\mathcal{D}(\mathfrak{w})$  by the translation group as the *reduced* moduli space. Thus, provided the weights are in general position, the dimension of the reduced moduli space, called the reduced dimension, is  $d_r(\mathfrak{w}) := 2V(\mathfrak{w}) - E(\mathfrak{w}) - 2$ .

For generic configurations of weights  $\{z_i\}$  the boundary of the closure  $\overline{\mathcal{D}}(\mathfrak{w})$  of  $\mathcal{D}(\mathfrak{w})$  in  $\mathbb{R}^{2V(\mathfrak{w})}$  consists of  $d(\mathfrak{w}) - 1$  dimensional cells where some edge inequality is saturated. Thus at each boundary cell two or more vertices of  $\mathfrak{w}$  collapse to a single point  $p$  and  $\mathfrak{w}$  reduces to a simpler web  $\mathfrak{w}_1$  with a marked vertex  $v$  at  $p$ . In a small neighbourhood of such boundary cell  $\mathfrak{w}$  can be recovered from  $\mathfrak{w}_1$  by replacing  $v$  with an infinitesimally small copy of a second web  $\mathfrak{w}_2$  formed by the collapsing vertices and edges. The cyclic fan  $I_\infty(\mathfrak{w}_2)$  coincides with the cyclic fan  $I_p(\mathfrak{w}_1)$ .

In order to formalize the relation between  $\mathfrak{w}$ ,  $\mathfrak{w}_1$  and  $\mathfrak{w}_2$  we introduce the key construction of *convolution of webs*:

**Definition:** Suppose  $\mathfrak{w}$  and  $\mathfrak{w}'$  are two plane webs and there is a vertex  $v \in \mathcal{V}(\mathfrak{w})$  such that

$$I_v(\mathfrak{w}) = I_\infty(\mathfrak{w}'). \quad (2.6)$$

We then define  $\mathfrak{w} *_v \mathfrak{w}'$  to be the deformation type of a web obtained by cutting out a small disk around  $v$  and gluing in a suitably scaled and translated copy of the deformation type of  $\mathfrak{w}'$ . It is important that we only use a deformation type here. In general the external edges of  $\mathfrak{w}'$  do not necessarily meet at a single point when continued inward. However we can deform  $\mathfrak{w}'$  so that the edges literally fit with those of  $\mathfrak{w}$ , provided we take the disk sufficiently small. This and similar statements can be proven trivially from the linear nature of the constraints imposed on the positions of the vertices by the topology of a web.

The procedure is illustrated in Figure 11. When writing convolutions below we always put the ‘‘container web’’ on the left.

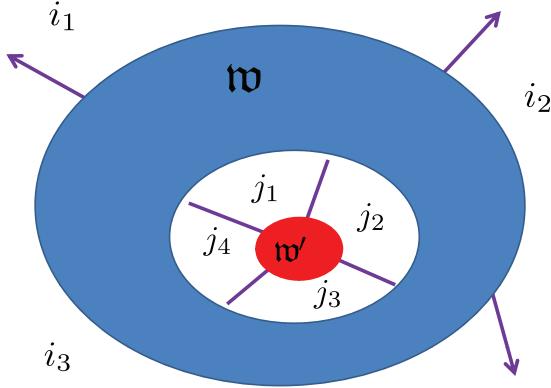
One easily verifies the relations

$$E(\mathfrak{w} *_v \mathfrak{w}') = E(\mathfrak{w}) + E(\mathfrak{w}') \quad (2.7)$$

$$V(\mathfrak{w} *_v \mathfrak{w}') = V(\mathfrak{w}) + V(\mathfrak{w}') - 1 \quad (2.8)$$

and hence we have the important relation

$$d(\mathfrak{w} *_v \mathfrak{w}') = d(\mathfrak{w}) + d(\mathfrak{w}') - 2 \quad (2.9)$$



**Figure 11:** Illustrating the convolution of a web  $\mathfrak{w}$  with internal vertex  $v$  having a fan  $I_v(\mathfrak{w}) = \{j_1, j_2, j_3, j_4\}$  with a web  $\mathfrak{w}'$  having an external fan  $I_\infty(\mathfrak{w}') = \{j_1, j_2, j_3, j_4\}$ .

showing that we can take any sufficiently small representative  $\mathfrak{w}'$  in  $\mathcal{D}_r(\mathfrak{w}')$  and insert it into any given representative  $\mathfrak{w}$  in  $\mathcal{D}_r(\mathfrak{w}')$ .

With these results at hand, it is now clear that, for generic weights  $\{z_i\}$ , the top dimension boundary cells of  $\bar{\mathcal{D}}(\mathfrak{w})$  are in one-to-one correspondence with pairs  $(\mathfrak{w}_1, \mathfrak{w}_2)$  such that  $\mathfrak{w} = \mathfrak{w}_1 *_v \mathfrak{w}_2$  and  $d(\mathfrak{w}_1) = d(\mathfrak{w}) - 1$ . In the neighbourhood of each such boundary cell we have a local isomorphism between  $\mathcal{D}(\mathfrak{w})$  and  $\mathcal{D}(\mathfrak{w}_1) \times \mathcal{D}_r(\mathfrak{w}_2)$ .

We now introduce some special classes of webs which will be of the most use to us:

**Definition:** A *rigid web* is a web with  $d(\mathfrak{w}) = 2$ . A *taut web* is a web with  $d(\mathfrak{w}) = 3$  and a *sliding web* is a web with  $d(\mathfrak{w}) = 4$ .

A rigid web must have  $E(\mathfrak{w}) = 0$  hence  $V(\mathfrak{w}) = 1$  and hence is just a single vertex. Using  $V(\mathfrak{w}) - E(\mathfrak{w}) + F(\mathfrak{w}) = 1$  and eliminating  $E(\mathfrak{w})$  we have

$$V(\mathfrak{w}) = d(\mathfrak{w}) - 1 + F(\mathfrak{w}) \quad (2.10)$$

and hence a taut web has at least two vertices, a sliding web at least three vertices, and so forth.

Let  $\mathcal{W}$  be the free abelian group generated by *oriented* deformation types of webs. By “oriented” we mean that we have chosen an orientation  $o(\mathfrak{w})$  of the cell  $\mathcal{D}(\mathfrak{w})$ . Henceforth the notation  $\mathfrak{w}$  will usually refer to such an oriented deformation type, rather than a specific web. In  $\mathcal{W}$  the object  $-\mathfrak{w}$  is the oriented web with the opposite orientation to  $\mathfrak{w}$ . Henceforth, when working with  $\mathcal{W}$  we will assume the vacuum weights are in generic position. We return to this assumption in Section §2.5 below.

We now define a convolution operation

$$\ast : \mathcal{W} \times \mathcal{W} \rightarrow \mathcal{W} \quad (2.11)$$

by defining  $\mathfrak{w}_1 *_v \mathfrak{w}_2 = 0$  if  $I_\infty(\mathfrak{w}_2) \neq I_v(\mathfrak{w}_1)$  and then setting

$$\mathfrak{w}_1 * \mathfrak{w}_2 := \sum_{v \in \mathcal{V}(\mathfrak{w}_1)} \mathfrak{w}_1 *_v \mathfrak{w}_2. \quad (2.12)$$

Note that, because of the line principle, at most one term on the right hand side of (2.12) can be nonzero. Moreover, in order for this to be well-defined on  $\mathcal{W}$  we must orient  $\mathcal{D}(\mathfrak{w}_1 * \mathfrak{w}_2)$ . If  $o(\mathfrak{w})$  is an orientation of  $\mathcal{D}(\mathfrak{w})$ , thought of as a top-degree form, we can use the freely-acting translation symmetry to define a “reduced orientation” by

$$o_r(\mathfrak{w}) := \iota\left(\frac{\partial}{\partial y}\right)\iota\left(\frac{\partial}{\partial x}\right)o(\mathfrak{w}) \quad (2.13)$$

and then we define

$$o(\mathfrak{w}_1 * \mathfrak{w}_2) := o(\mathfrak{w}_1) \wedge o_r(\mathfrak{w}_2). \quad (2.14)$$

(This uses the product structure near the boundary of the cell where  $\mathfrak{w}_2$  shrinks to a single vertex.)

Since taut webs have a one-dimensional reduced cell we can and will choose a standard orientation for all taut webs to be the orientation with tangent vector in the direction of increasing size. That is, the moduli of the taut web can be taken to be an overall position  $x, y$  together with a scaling modulus  $\ell$ . We take the orientation to be  $dxdy d\ell$ . Now we can define the *taut element*  $\mathfrak{t} \in \mathcal{W}$  to be the sum of all oriented taut webs with standard orientation:

$$\mathfrak{t} := \sum_{d(\mathfrak{w})=3} \mathfrak{w}. \quad (2.15)$$

Including the orientation data, we arrive at our final characterization of the generic codimension one boundaries of  $\overline{\mathcal{D}}(\mathfrak{w})$ : a typical web  $\mathfrak{w}$  looks like a convolution  $\mathfrak{w}_1 *_v \mathfrak{w}_2$  where  $\mathfrak{w}_2$  is a taut web and the orientation of  $\mathfrak{w}$  is written as  $o(\mathfrak{w}) = o(\mathfrak{w}_1) \wedge d\ell_2$ , with  $\ell_2$  oriented towards the interior of the cell  $\overline{\mathcal{D}}(\mathfrak{w})$ . Applying this picture to the case where  $\mathfrak{w}$  is a sliding web we note that  $\mathfrak{w}_1$  is a taut web as well, and the natural orientation  $dxdy d\ell_1 d\ell_2$  might or might not agree with the orientation of  $\mathfrak{w}$ . We should thus write  $\mathfrak{w} = \pm \mathfrak{w}_1 *_v \mathfrak{w}_2$ . Looking carefully at the global structure of the moduli spaces of sliding webs, we deduce our first result:

**Theorem:** We have

$$\mathfrak{t} * \mathfrak{t} = 0. \quad (2.16)$$

*Proof:* Every element  $\mathfrak{w} * \mathfrak{w}'$  in the convolution is a sliding web, since reduced dimension is additive. The reduced moduli space of a sliding web is a two-dimensional cone. Up to a linear transformation it has boundary:

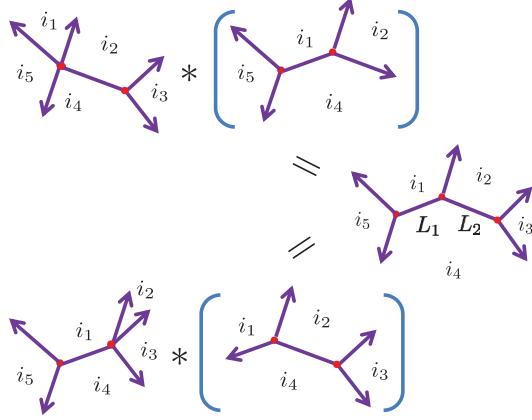
$$\partial \mathbb{R}_+^2 = (\mathbb{R}_{>0} \times \{0\}) \amalg (\{0\} \times \mathbb{R}_{>0}) \amalg \{(0, 0)\}. \quad (2.17)$$

Therefore, the terms can be grouped into pairs, each pair contributing to the same deformation type. If the two boundaries are represented by convolutions of taut webs  $\mathfrak{w}_1 * \mathfrak{w}_2$

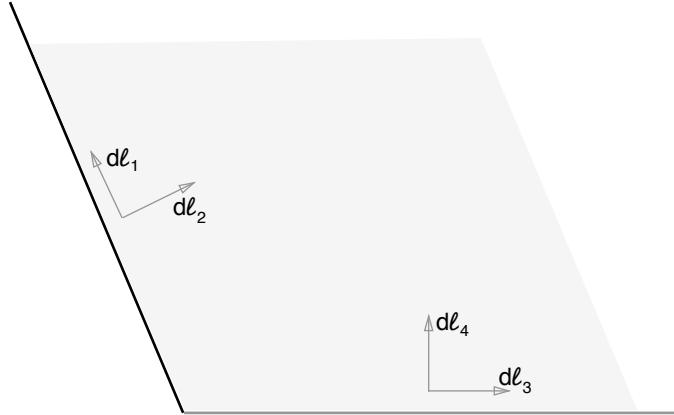
and  $\mathbf{w}_3 * \mathbf{w}_4$  respectively, the corresponding orientations  $d\ell_1 d\ell_2$  and  $d\ell_3 d\ell_4$  of the reduced cell are opposite to each other. Thus

$$\mathbf{w}_1 * \mathbf{w}_2 + \mathbf{w}_3 * \mathbf{w}_4 = 0. \quad (2.18)$$

This concludes the proof. A concrete example illustrating the above argument is shown in Figures 12 and 13.



**Figure 12:** The two boundaries of the deformation type of the sliding web shown on the right correspond to different convolutions shown above and below. If we use the lengths  $L_1, L_2$  of the edges as coordinates then the orientation from the top convolution is  $dL_2 \wedge dL_1$ . On the other hand the orientation from the bottom convolution is  $dL_1 \wedge dL_2$  and hence the sum of these two convolutions is zero. This is the key idea in the demonstration that  $\mathbf{t} * \mathbf{t} = 0$ .



**Figure 13:** A graphical proof that the two boundaries of the reduced moduli space of a sliding web are associated to opposite orientations. It is drawn as a cone since the moduli space of plane webs inherits a metric from the embedding into  $\mathbb{R}^{2V}$ , and with this metric the two boundaries are not orthogonal in general.

## 2.2 Half-Plane Webs

**Definition:**

a.) Let  $\mathcal{H} \subset \mathbb{R}^2$  be a half-plane, whose boundary is not parallel to any of the  $z_{ij}$ . A half-plane web in  $\mathcal{H}$  is a graph in the half-plane, which allows some vertices, but no edges, to be subsets of the boundary. The boundary vertices have valence of at least one. We apply the same rule as for plane webs: Label connected components of the complement of the graph by vacua so that if the edges are oriented with  $i$  on the left and  $j$  on the right then they are parallel to  $z_{ij}$ .

b.) A half-plane fan (often, we will just say, “fan”) is an ordered sequence of vacua  $\{i_1, \dots, i_n\}$  so that the rays from the origin through  $z_{i_k, i_{k+1}}$  are ordered clockwise for increasing  $k$  and  $z_{i_k, i_{k+1}} \in \mathcal{H}$ .<sup>5</sup>

**Remarks:**

1. Unlike plane webs, half-plane webs need not be connected.
2. Let  $\mathfrak{u}$  denote a typical half-plane web. There are now two different kinds of vertices, the boundary vertices  $\mathcal{V}_\partial(\mathfrak{u})$  and the interior vertices  $\mathcal{V}_i(\mathfrak{u})$  with cardinalities  $V_\partial(\mathfrak{u})$  and  $V_i(\mathfrak{u})$ , respectively.
3. We will consider  $\mathcal{V}_\partial(\mathfrak{u})$  to be an *ordered set* and we will use a uniform ordering convention for all half-planes  $\mathcal{H}$  which is invariant under rotation. To this end we choose a direction  $\partial_{\parallel}$  along  $\partial\mathcal{H}$  so that if  $\partial_{\perp}$  is the outward normal to  $\mathcal{H}$  then  $\partial_{\perp} \wedge \partial_{\parallel}$  is the standard orientation of the  $(x, y)$  plane  $\mathbb{R}^2$ , namely  $\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y}$ .<sup>6</sup> Now, our ordering of the boundary vertices

$$\mathcal{V}_\partial(\mathfrak{u}) = \{v_1^\partial, \dots, v_n^\partial\} \quad (2.19)$$

is that reading from left to right proceeds in the direction of  $\partial_{\parallel}$ . In particular, if  $\mathcal{H}_L$  is the *positive* half-plane  $x \geq x_0$ , (with boundary on the left) then  $v_1^\partial, \dots, v_n^\partial$  is a sequence of vertices with decreasing “time”  $y$ , while for the *negative* half-plane  $\mathcal{H}_R$ ,  $x \leq x_0$ , (with boundary on the right) the sequence of vertices is in order of increasing time.

4. We denote general half-plane fans by  $J$ , reserving  $I$  for cyclic fans. We will denote the half-plane fan at infinity by  $J_\infty(\mathfrak{u})$ . Similarly, if  $v \in \mathcal{V}_\partial(\mathfrak{u})$  there is a half-plane fan  $J_v(\mathfrak{u})$ .

We can again speak of a deformation type of a half-plane web  $\mathfrak{u}$ . The set of webs of a given deformation type is denoted  $\mathcal{D}(\mathfrak{u})$ . It has dimension:

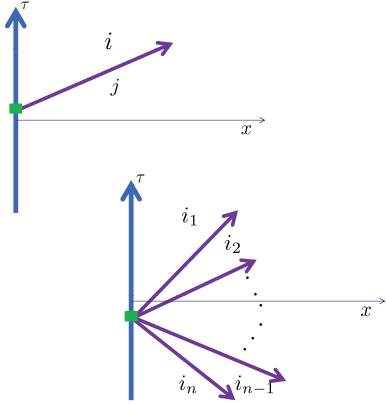
$$d(\mathfrak{u}) := 2V_i(\mathfrak{u}) + V_\partial(\mathfrak{u}) - E(\mathfrak{u}). \quad (2.20)$$

---

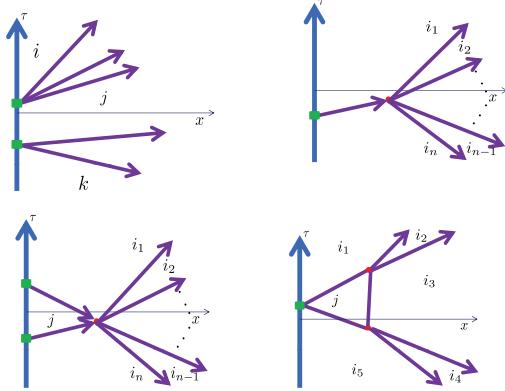
<sup>5</sup>When we write  $z_{ij} \in \mathcal{H}$  for a general half-plane  $\mathcal{H}$  we mean that if we rigidly translate  $\mathcal{H}$  to  $\mathcal{H}'$  so that the origin is on its boundary then  $z_{ij} \in \mathcal{H}'$ . We will use this slightly sloppy notation again later in the paper.

<sup>6</sup>Thus the mnemonic is “Outward-Normal-First” = “One Never Forgets.”

Now translations parallel to the boundary of  $\mathcal{H}$  act freely on  $\mathcal{D}(\mathbf{u})$  and hence  $d(\mathbf{u}) \geq 1$ . Once again we define half-plane webs to be *rigid*, *taut*, and *sliding* if  $d(\mathbf{u}) = 1, 2, 3$ , respectively. Similarly, we can define oriented deformation type in an obvious way and consider the free abelian group  $\mathcal{W}_{\mathcal{H}}$  of oriented deformation types of half-plane webs in the half-plane  $\mathcal{H}$ . Some examples where  $\mathcal{H} = \mathcal{H}_L$  is the positive half-plane are shown in Figures 14, 15, and 16.



**Figure 14:** Two examples of rigid positive-half-plane webs.

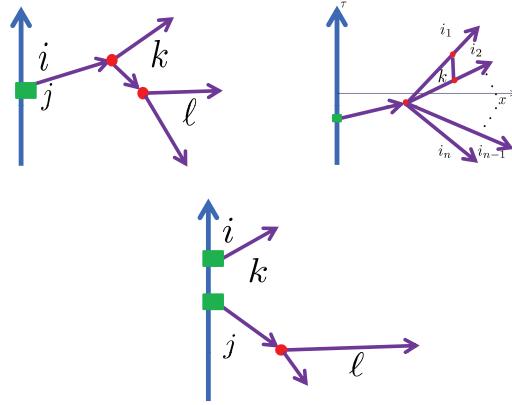


**Figure 15:** Four examples of taut positive-half-plane webs

We can again ask how a half plane web  $\mathbf{u}$  can degenerate near the boundary of the closure  $\overline{\mathcal{D}}(\mathbf{u})$  in  $\mathbb{R}^{2V_i(\mathbf{u})+V_{\partial}(\mathbf{u})}$ . We have now two types of boundary cells: either the collapsing vertices of  $\mathbf{u}$  come together to a point in the interior of  $\mathcal{H}$  or they come together to a point in the boundary  $\mathcal{H}$ . Correspondingly, we can define two kinds of convolution.

If  $\mathbf{u}$  and  $\mathbf{u}'$  are two half-plane fans,  $v^{\partial} \in \mathcal{V}_{\partial}(\mathbf{u})$ , and  $J_{v^{\partial}}(\mathbf{u}) = J_{\infty}(\mathbf{u}')$  then

$$\mathbf{u} *_{v^{\partial}} \mathbf{u}' \tag{2.21}$$



**Figure 16:** Three examples of sliding half-plane webs

is obtained by cutting out a small half-disk around  $v^\partial$  and gluing in a small copy of  $\mathfrak{u}'$ . For this operation  $V_i$  is additive as is  $E$  but  $V_\partial(\mathfrak{u} *_{v^\partial} \mathfrak{u}') = V_\partial(\mathfrak{u}) + V_\partial(\mathfrak{u}') - 1$ , so the dimension behaves like:

$$d(\mathfrak{u} *_{v^\partial} \mathfrak{u}') = d(\mathfrak{u}) + d(\mathfrak{u}') - 1. \quad (2.22)$$

We can extend  $*_{v^\partial}$  to an operation

$$*: \mathcal{W}_\mathcal{H} \times \mathcal{W}_\mathcal{H} \rightarrow \mathcal{W}_\mathcal{H} \quad (2.23)$$

by defining  $\mathfrak{u} *_{v^\partial} \mathfrak{u}' = 0$  if  $J_{v^\partial}(\mathfrak{u}) \neq J_\infty(\mathfrak{u}')$  and then taking

$$\mathfrak{u} * \mathfrak{u}' := \sum_{v^\partial \in \mathcal{V}_\partial(\mathfrak{u})} \mathfrak{u} *_{v^\partial} \mathfrak{u}'. \quad (2.24)$$

Once again, at most one term in this sum can be nonzero. To define the orientation of  $\mathcal{D}(\mathfrak{u} * \mathfrak{u}')$  we again introduce a reduced orientation  $o_r(\mathfrak{u}) := \iota(\partial_\parallel) o(\mathfrak{u})$  by contracting with the vector field  $\partial_\parallel$  described above (2.19) and defining

$$o(\mathfrak{u} * \mathfrak{u}') := o(\mathfrak{u}) \wedge o_r(\mathfrak{u}'). \quad (2.25)$$

Similarly, if  $v \in \mathcal{V}_i(\mathfrak{u})$  is an interior vertex then we can convolve with a plane-web  $\mathfrak{w}$  to produce a deformation type  $\mathfrak{u} *_v \mathfrak{w}$  with orientation  $o(\mathfrak{u}) \wedge o_r(\mathfrak{w})$ . Now  $V_\partial$  and  $E$  are additive but  $V_i(\mathfrak{u} *_v \mathfrak{w}) = V_i(\mathfrak{u}) + V(\mathfrak{w}) - 1$  and hence we now have

$$d(\mathfrak{u} *_v \mathfrak{w}) = d(\mathfrak{u}) + d(\mathfrak{w}) - 2. \quad (2.26)$$

Again we can define

$$*: \mathcal{W}_\mathcal{H} \times \mathcal{W} \rightarrow \mathcal{W}_\mathcal{H} \quad (2.27)$$

by defining  $\mathfrak{u} *_v \mathfrak{w} = 0$  if  $I_v(\mathfrak{u}) \neq I_\infty(\mathfrak{w})$  and then

$$\mathfrak{u} * \mathfrak{w} = \sum_{v \in \mathcal{V}_i(\mathfrak{u})} \mathfrak{u} *_v \mathfrak{w}. \quad (2.28)$$

Once again, by the line principle, at most one term in this sum can be nonzero.

Thus all the boundary cells of  $\bar{\mathcal{D}}(\mathbf{u})$  are associated to some container half plane web  $\mathbf{u}_1$  with either a marked interior vertex  $v$  resolved to a small taut plane web  $\mathbf{w}_2$ , or a marked boundary vertex  $v^\partial$  resolved to a small taut half-plane web  $\mathbf{u}_2$ . We can write either  $\mathbf{u} = \mathbf{u}_1 *_v \mathbf{w}_2$  or  $\mathbf{u} = \mathbf{u}_1 *_{v^\partial} \mathbf{u}_2$  at each boundary cell.

We now define the half-plane taut element

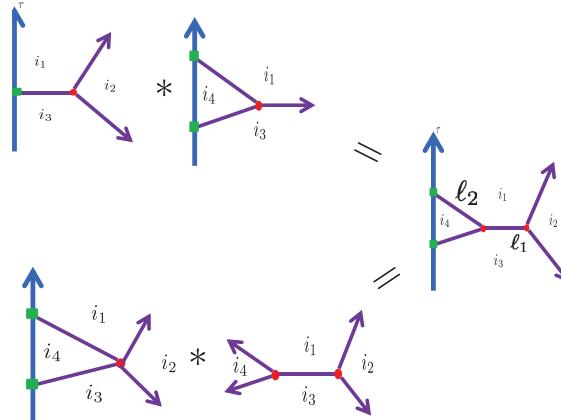
$$\mathbf{t}_{\mathcal{H}} := \sum_{d(\mathbf{u})=2} \mathbf{u} \quad (2.29)$$

There is one scale modulus  $\ell$  so that as  $\ell$  increases the web gets bigger. The canonical orientation of taut elements is then  $dy_\parallel \wedge d\ell$  where  $\partial_\parallel = \frac{\partial}{\partial y_\parallel}$ . Since there are now two kinds of taut elements we henceforth denote the planar taut element (2.15) by  $\mathbf{t}_p$ . We now have:

**Theorem:** Let  $\mathbf{t}_p$  be the taut element for planar webs and  $\mathbf{t}_{\mathcal{H}}$  the taut element for the half-plane  $\mathcal{H}$ . Then, combining the two convolutions (2.23) and (2.27)

$$\mathbf{t}_{\mathcal{H}} * \mathbf{t}_{\mathcal{H}} + \mathbf{t}_{\mathcal{H}} * \mathbf{t}_p = 0. \quad (2.30)$$

*Proof:* The idea of the proof is essentially the same as in the proof of (2.16). The reduced moduli spaces of sliding half-plane webs are still two-dimensional cones, and have paired boundary cells which induce opposite orientations. Thus all terms in 2.30 cancel out in pairs. An example of the argument is shown in Figure 17.



**Figure 17:** An example of the identity on plane and half-plane taut elements. On the right is a sliding half-plane web. Above is a convolution of two taut half-plane webs with orientation  $dy \wedge d\ell_1 \wedge d\ell_2$ . Below is a convolution of a taut half-plane web with a taut plane web. The orientation is  $dy \wedge d\ell_2 \wedge d\ell_1$ . The two convolutions determine the same deformation type but have opposite orientation, and hence cancel.

**Remark:** Since half-plane webs are not connected one might wonder whether we should introduce a new operation of time-convolution in the identity (2.30). Certainly terms

appear which can be interpreted as time-convolutions but these are properly accounted for by the first term of (2.30). On the other hand, will see in §2.3 below that when we replace the half-plane by a strip we do need to introduce a separate time-convolution operation.

### 2.3 Strip-Webs

We now consider webs in the strip  $[x_\ell, x_r] \times \mathbb{R}$ . Again, we assume that the boundary of the strip is not parallel to any of the  $z_{ij}$ . Strip-webs are defined similarly to half-plane webs: We allow some vertices but no edges to lie on the boundary of the strip. Now there are two connected components of the boundary of the strip so the boundary vertices are decomposed as a disjoint union of two sets  $\mathcal{V}_\partial = \mathcal{V}_{\partial,L} \amalg \mathcal{V}_{\partial,R}$ . Every strip web is associated to a certain choice of vacua in the far future ( $y \rightarrow +\infty$ ) and in the far past ( $y \rightarrow -\infty$ ). We can refer to them as the future and past vacua of the strip-web, respectively.

Once again we can speak of deformation type. We denote a generic strip-web, or rather an oriented deformation type, by  $\mathfrak{s}$ . The dimension of the space of strip-webs of fixed deformation type is

$$d(\mathfrak{s}) := 2V_i(\mathfrak{s}) + V_\partial(\mathfrak{s}) - E(\mathfrak{s}). \quad (2.31)$$

Again time translation acts freely on the set  $\mathcal{D}(\mathfrak{s})$  and hence  $d(\mathfrak{s}) \geq 1$ . As overall rescaling is not a symmetry of the problem, the moduli spaces  $\mathcal{D}(\mathfrak{s})$  are not cones anymore.

**Definition:** We define *taut strip-webs* to be those with  $d(\mathfrak{s}) = 1$  and *sliding strip-webs* to be those with  $d(\mathfrak{s}) = 2$ . In other words there is no distinction between rigid and taut strip-webs.

**Remark:** The above definition might be surprising since we did not introduce rigid strip webs. The source of the distinction is the presence or lack of scaling symmetry. When the geometry in which the webs live has a scaling symmetry, such as the plane or half-plane, we distinguish between rigid and taut webs. Otherwise rigid and taut webs are indistinguishable, and have no reduced modulus.

We can define as usual the closure  $\overline{\mathcal{D}}(\mathfrak{s})$  of  $\mathcal{D}(\mathfrak{s})$  in  $\mathbb{R}^{2V_i(\mathfrak{s})+V_\partial(\mathfrak{s})}$ . This introduces three kinds of boundary cells, corresponding to a collection of vertices collapsing to a point in the interior or on either boundary of the strip. Correspondingly, we have now three kinds of convolution. Recall that the “container web” is written on the left. First, we can convolve strip webs with planar webs so that

$$d(\mathfrak{s} *_{v_i} \mathfrak{w}) = d(\mathfrak{s}) + d(\mathfrak{w}) - 2 \quad (2.32)$$

and  $o(\mathfrak{s} *_{v_i} \mathfrak{w}) = o(\mathfrak{s}) \wedge o_r(\mathfrak{w})$ . Next, we can convolve a strip web  $\mathfrak{s}$  with a positive half-plane web with vertices on the left boundary as

$$\mathfrak{s} *_{v^\partial} \mathfrak{u}_L \quad (2.33)$$

where  $v^\partial \in \mathcal{V}_{\partial,L}(\mathfrak{s})$ . Similarly, if  $v^\partial \in \mathcal{V}_{\partial,R}(\mathfrak{s})$  and  $\mathfrak{u}_R$  is a web in the negative-half-plane (so it has boundary vertices on the right) then we write

$$\mathfrak{s} *_{v^\partial} \mathfrak{u}_R. \quad (2.34)$$

We have

$$d(\mathfrak{s} *_{v^\partial} \mathfrak{u}) = d(\mathfrak{s}) + d(\mathfrak{u}) - 1 \quad (2.35)$$

with orientation  $o(\mathfrak{s}) \wedge o_r(\mathfrak{u})$  in both cases. The reduced orientation is defined with the vector field  $\partial_\parallel$  defined in (2.19).

An important aspect in what follows is that we can introduce another operation on strip-webs namely time concatenation: If  $\mathfrak{s}_1$  and  $\mathfrak{s}_2$  are two strip webs such that the future vacuum of  $\mathfrak{s}_2$  coincides with the past vacuum of  $\mathfrak{s}_1$  we define

$$\mathfrak{s}_1 \circ \mathfrak{s}_2 \quad (2.36)$$

to be the deformation type of a strip web where  $\mathfrak{s}_1$  and  $\mathfrak{s}_2$  are disconnected and separated by a line at fixed time, with  $\mathfrak{s}_1$  in the future of  $\mathfrak{s}_2$ . If the future vacuum of  $\mathfrak{s}_2$  and the past vacuum of  $\mathfrak{s}_1$  differ, we define the concatenation  $\mathfrak{s}_1 \circ \mathfrak{s}_2$  to be zero. Note that when  $\mathfrak{s}_1 \circ \mathfrak{s}_2$  is nonzero then

$$d(\mathfrak{s}_1 \circ \mathfrak{s}_2) = d(\mathfrak{s}_1) + d(\mathfrak{s}_2). \quad (2.37)$$

Because of the assumption that none of the  $z_{ij}$  points in the direction along the strip, no connected strip-web may have an arbitrarily large extension along the strip. The only webs which can grow to arbitrarily large size have at least two disconnected components, and can thus be written as the concatenation of simpler webs.

Proceeding as before we define the free abelian group  $\mathcal{W}_S$  generated by oriented deformation types of strip-webs. We extend  $*$  in the usual way to define operations  $* : \mathcal{W}_S \times \mathcal{W} \rightarrow \mathcal{W}_S$  and  $* : \mathcal{W}_S \times \mathcal{W}_{L,R} \rightarrow \mathcal{W}_S$  where  $\mathcal{W}_L$  is the group of positive-half-plane webs (with boundary on the left) and  $\mathcal{W}_R$  is the group of negative-half-plane webs (with boundary on the right). We introduce the taut element for the strip:

$$\mathfrak{t}_s := \sum_{d(\mathfrak{s})=1} \mathfrak{s} \quad (2.38)$$

with  $\mathfrak{s}$  oriented towards the future. That is, choosing any boundary vertex  $v^\partial$  with  $y$ -coordinate  $y^\partial$  the orientation is  $o(\mathfrak{s}) = dy^\partial$ .

It is natural now to study the moduli spaces of sliding webs. There are two possible topologies: the closure of the reduced moduli space  $\overline{\mathcal{D}}_r(\mathfrak{s})$  for a sliding web  $\mathfrak{s}$  can be either a segment or a half-line.<sup>7</sup> At each of the boundaries at finite distance  $\mathfrak{s}$  can be written as the convolution of an appropriate taut strip-web and a taut plane or half-plane web. In all cases, the convolution gives an orientation pointing away from the boundary. On the other hand, the semi-infinite end of a half-line moduli space is associated to a *concatenation* of two taut strip-webs. If we denote the coordinates on the moduli space of two taut strip-webs  $\mathfrak{s}_1$  and  $\mathfrak{s}_2$  as  $y_1$  and  $y_2$ , the orientation of  $\mathfrak{s}_1 \circ \mathfrak{s}_2$  is

$$o(\mathfrak{s}_1 \circ \mathfrak{s}_2) = dy_1 \wedge dy_2 = -dy_1 \wedge d(y_1 - y_2) \quad (2.39)$$

---

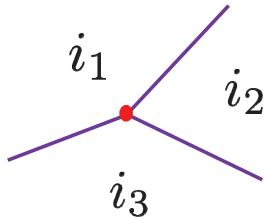
<sup>7</sup>In the case of extended webs introduced below there are some exceptional cases where the moduli space of sliding webs can be  $\mathbb{R}$ . Nevertheless there are two ends with opposite orientation and the convolution identity holds.

In the conventions where  $s_1$  in the future of  $s_2$ ,  $y_1 - y_2$  is the natural coordinate on the reduced moduli space, increasing towards infinity. Thus the concatenation gives an orientation on the half line towards the origin.

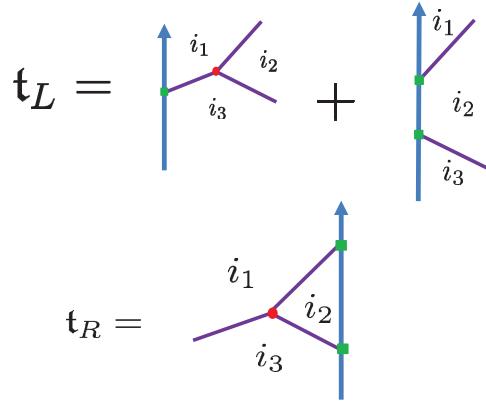
Thus we have the

**Theorem:** Let  $t_p$  be the planar taut element and  $t_L$  and  $t_R$  the taut elements in the positive and negative half-planes, respectively, and  $t_s$  the strip taut element. Then

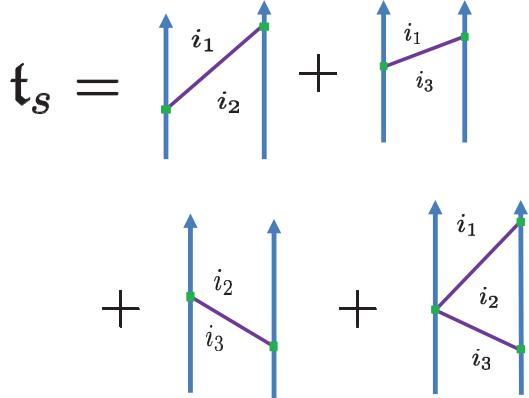
$$t_s * t_L + t_s * t_R + t_s * t_p + t_s \circ t_s = 0. \quad (2.40)$$



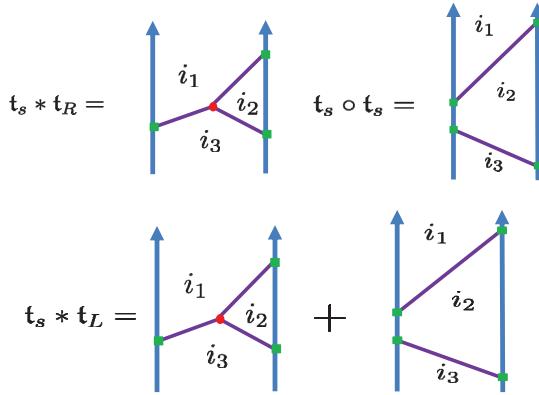
**Figure 18:** The only vertex in a theory with three vacua.



**Figure 19:** The positive and negative half-plane taut elements are illustrated here. Letting  $y$  denote the  $y$ -coordinate of any boundary vertex and  $\ell$  the internal edge length  $t_L$  has orientation  $-dyd\ell$  and  $t_R$  has orientation  $dyd\ell$ .



**Figure 20:** The taut element on the strip with three vacua. Letting  $y$  denote the  $y$  coordinate of any edge vertex the orientation is  $dy$



**Figure 21:** The various terms in the convolution identity on the strip. In this simple example with three vacua  $t_s * t_p = 0$ . The orientations of the three terms are  $-dyd\ell$  on the first line and  $+dyd\ell$  on the second line.

**Example** Suppose there are three vacua in  $\mathbb{V}$ . Then two of the  $\text{Re}z_{ij}$  have the same sign, and without loss of generality we will assume that  $\text{Re}z_{12} > 0$  and  $\text{Re}z_{23} > 0$ . Thus the only planar vertex is of the form shown in Figure 18. The taut element  $t_L$  for the positive-half plane then has two summands while the taut element for the negative half-plane has a single summand as shown in Figure 19. The taut element on the strip is shown in Figure 20. The various convolutions are illustrated in 21 and cancel.

## 2.4 Extended Webs

A small generalization of the webs defined above will turn out to play a role below. In some circumstances it will be useful to relax slightly the restriction that interior vertices

should be at least tri-valent and boundary vertices at least univalent. In particular, we will consider the following two generalizations: First, we can allow two-valent interior vertices and second, we can allow zero-valent boundary vertices. These generalizations weaken somewhat the finiteness properties of webs. Nevertheless, the number of webs of a given degree  $d$  is still finite since the addition of such vertices increases  $d$  by one. This is sufficient to keep our formulae sensible.

We will refer to this larger class of webs as *extended webs* when the distinction is important. Extended webs satisfy most of the same properties as standard webs. The definitions and properties of deformation types, orientation and convolution will all hold true for extended webs. The taut elements for the extended webs satisfy the same convolution identities as the taut elements for standard webs. The whole algebraic structure defined in the next section 3 persists as well if we consider extended webs.

## 2.5 Special Configurations Of Vacuum Weights

At several times in our discussion above we required the set of vacuum weights  $\{z_i\}$  to be in general position. Nevertheless, there are special configurations of weights which will be of some importance in the discussion of certain homotopies in Section §6.3 and in the discussion of wall-crossing in Section §8 below.

We can consider  $\{z_i\}$  to define a point in  $\mathbb{C}^V - \Delta$ , where  $\mathbb{C}^V$  is the space of maps  $V \rightarrow \mathbb{C}$  and  $\Delta$  is the large diagonal where  $z_i = z_j$  for some pair  $i \neq j$ . Within this space are two subspaces of special weights. They are generically of real codimension one but have complicated self-intersections of higher codimension.

The first special codimension one subspace is defined by weights such that some triple of weights  $z_i, z_j, z_k$  for three distinct vacua  $i, j, k$  become colinear:

$$\text{Im}z_{ij}\overline{z_{jk}} = 0. \quad (2.41)$$

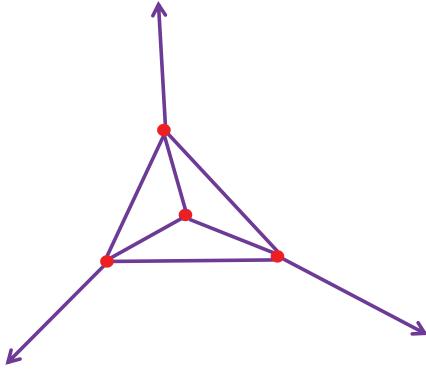
We call these *walls of marginal stability*. Generic one-parameter families of weights will cross such walls. When this happens the set of cyclic vacua and the set of webs changes discontinuously. For example, with four vacua we can pass from a set of webs which are all tree graphs to a set of webs with loops. We will discuss some consequences of such wall-crossings in Section §8, and especially in Section §8.4 below.

A more subtle special configuration of weights is one for which there exist *exceptional webs*. These are, by definition, webs such that

$$D(\mathfrak{w}) := \dim \mathcal{D}(\mathfrak{w}) > d(\mathfrak{w}). \quad (2.42)$$

Such webs can arise because, for some configurations of vacuum weights there can be webs where the edge constraints are not all independent. We say that some edge constraints are ineffective. Let us decompose  $z_{ij}$  into real and imaginary parts  $z_{ij} = u_{ij} + iv_{ij}$ . If an edge  $e$  is of type  $ij$  and has vertices  $(x_e^{(1)}, y_e^{(1)})$  and  $(x_e^{(2)}, y_e^{(2)})$  then the edge constraints are a set of linear equations

$$u_{ij(e)}(y_e^{(2)} - y_e^{(1)}) - v_{ij(e)}(x_e^{(2)} - x_e^{(1)}) = 0 \quad (2.43)$$



**Figure 22:** An example of an exceptional web. There is only one reduced modulus corresponding to overall scaling. Therefore  $\dim \mathcal{D}(\mathfrak{w}) = 3$ . Nevertheless  $V = 4$  and  $E = 6$  so  $d(\mathfrak{w}) = 2V - E = 2$ .

which we can write as  $M(\mathfrak{w})L(\mathfrak{w}) = 0$  for a matrix  $M(\mathfrak{w})$  of edge constraints and a vector  $L(\mathfrak{w})$  of vertex coordinates. The real codimension one *exceptional walls* in  $\mathbb{C}^V - \Delta$  are defined by the loci where the rank of  $M(\mathfrak{w})$  drops from  $E(\mathfrak{w})$  to  $E(\mathfrak{w}) - 1$ . Of course, the closure of the exceptional walls will have many components, intersecting in places where the rank drops further.

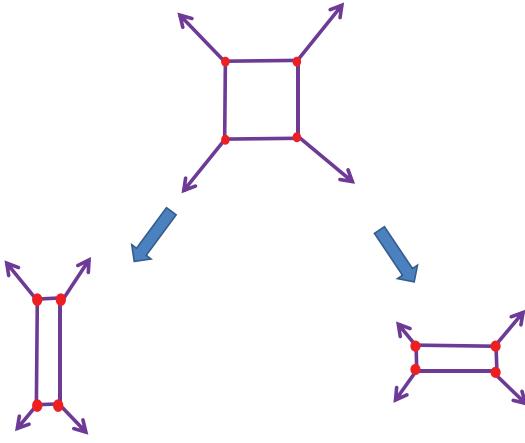
An example of an exceptional web is shown in Figure 22. This is plainly a taut web, so the dimension of its deformation space is three, but  $d(\mathfrak{w}) = 2$ ! When such phenomena arise we will distinguish the true dimension  $D(\mathfrak{w})$  from  $d(\mathfrak{w})$  by calling the latter the *expected dimension*. We use the terminology of index theory because, as we will see in Section §14, this literally does correspond to an issue in index theory.

The example shown in Figure 22 requires at least six vacua. If we hold all but one fixed and vary the last then it is clear that any small perturbation will destroy the web. However, a generic one-parameter family of weights nearby this configuration will have a point admitting such an exceptional web. Further triangulating one of the triangles in Figure 22 reduces the expected dimension by 1 and in this way we can produce examples of exceptional webs with arbitrarily small and negative expected dimension. In general, if a deformation type  $\mathfrak{w}$  is exceptional so that  $\nu := D(\mathfrak{w}) - d(\mathfrak{w}) > 0$  then generic  $\nu$ -parameter families of weights  $\{z_i\}$  will intersect the loci of such webs. We will discuss the consequences of exceptional walls in the wall-crossing story in Section §8.3.

We can now be more precise about the meaning of “generic weights” or “general position” used both above and below. This term implies that the weights are not on walls of marginal stability and do not admit exceptional webs.

### Remarks

- Finally, we remark that there are certain high codimension configurations of weights where webs can degenerate in ways which are not described in terms of convolution at a single vertex. An example is shown in Figure 23.



**Figure 23:** The sliding web shown here has two degenerations, neither of which is of the form  $\mathfrak{w}_1 * \mathfrak{w}_2$ .

2. There are also exceptional vacuum configurations and webs for the half-plane and strip geometries, and these will be used in Section §8.

### 3. Tensor Algebras Of Webs And Homotopical Algebra

In this section we consider algebraic operations defined by webs on various tensor algebras such as <sup>8</sup>

$$T\mathcal{W} := \mathcal{W} \oplus \mathcal{W}^{\otimes 2} \oplus \mathcal{W}^{\otimes 3} \oplus \dots \quad (3.1)$$

and its analogs for  $\mathcal{W}_{\mathcal{H}}$  and  $\mathcal{W}_S$ . Here we take the graded tensor product using the Koszul rule. The grading of a web such as  $\mathfrak{w}, \mathfrak{u}, \mathfrak{s}$  will be given by the dimension  $d(\mathfrak{w}), d(\mathfrak{u}), d(\mathfrak{s})$ , respectively. We will find various algebraic structures familiar from applications of homotopical algebra to string field theory. While these algebraic structures emerge naturally from thinking about webs the reader should be aware that the  $L_{\infty}$  and  $A_{\infty}$  algebras which are used to make contact with the physics only make their appearance when we come to Section §4.

#### 3.1 $L_{\infty}$ And Plane Webs

The convolution of webs is not associative:  $(\mathfrak{w}_1 * \mathfrak{w}_2) * \mathfrak{w}_3 - \mathfrak{w}_1 * (\mathfrak{w}_2 * \mathfrak{w}_3)$  consists of terms where  $\mathfrak{w}_2$  and  $\mathfrak{w}_3$  are glued in at distinct vertices of  $\mathfrak{w}_1$ . One could readily write down a tower of associativity relations for some generalized convolution operations, which insert multiple webs at distinct vertices of a single container web. It turns out that for our applications we only need an operation

$$T(\mathfrak{w}) : \mathcal{W}^{\otimes V(\mathfrak{w})} \rightarrow \mathcal{W} \quad (3.2)$$

---

<sup>8</sup>We take the tensor algebra without a unit, i.e. we do not include the ground ring  $\mathbb{Z}$ .

which replaces *all* the vertices of a plane web with other webs. More precisely, we define  $T(\mathfrak{w})$  as follows:

Given an *ordered* collection of  $\ell = V(\mathfrak{w})$  plane webs  $\{\mathfrak{w}_1, \dots, \mathfrak{w}_\ell\}$ , we seek some permutation  $\sigma$  of an ordered set of vertices  $\{v_1, \dots, v_\ell\}$  of  $\mathfrak{w}$  (with any ordering), such that  $I_{v_{\sigma(a)}}(\mathfrak{w}) = I_\infty(\mathfrak{w}_a)$  for  $a = 1, \dots, \ell$ . If the permutation does not exist, i.e. if the arguments cannot be inserted into  $\mathfrak{w}$  (saturating all the vertices exactly once) then we set  $T(\mathfrak{w})[\mathfrak{w}_1, \dots, \mathfrak{w}_\ell] = 0$ . If the permutation exists, it is unique, since a given cyclic fan of vacua can appear at most once in  $\mathfrak{w}$ . We then define  $T(\mathfrak{w})[\mathfrak{w}_1, \dots, \mathfrak{w}_\ell]$  to be the oriented deformation type obtained by gluing in  $\mathfrak{w}_a$  in small disks cut out around the vertices  $v_{\sigma(a)}$  of  $\mathfrak{w}$ . The orientation is given by  $o(\mathfrak{w}) \wedge o_r(\mathfrak{w}_1) \wedge \dots \wedge o_r(\mathfrak{w}_\ell)$ . This is the only place the ordering of  $\{\mathfrak{w}_1, \dots, \mathfrak{w}_\ell\}$  is used. In particular,  $T(\mathfrak{w})$  is graded symmetric, exactly as we would expect from manipulating graded elements  $\mathfrak{w}_a$  of degree  $d(\mathfrak{w}_a)$  with the Koszul rule. (Since  $d$  and  $d_r$  differ by two the sign is the same.) Now, we regard  $\{\mathfrak{w}_1, \dots, \mathfrak{w}_\ell\}$  as a monomial in  $\mathcal{W}^{\otimes \ell}$  and extend by linearity to define (3.2). Finally, we can extend  $T(\mathfrak{w})$  to a map  $T(\mathfrak{w}) : T\mathcal{W} \rightarrow \mathcal{W}$ , by setting  $T(\mathfrak{w}) : \mathcal{W}^{\otimes n} \rightarrow \mathcal{W}$  to be zero unless  $n = V(\mathfrak{w})$ .

It is useful to recall at this stage the definition of *n-shuffles*. If  $S$  is an ordered set then an *n-shuffle* of  $S$  is an ordered disjoint decomposition into  $n$  ordered subsets

$$S = S_1 \amalg S_2 \amalg \dots \amalg S_n \quad (3.3)$$

where the ordering of each summand  $S_\alpha$  is inherited from the ordering of  $S$  and the  $S_\alpha$  are allowed to be empty. Note that the ordering of the sets  $S_\alpha$  also matters so that  $S_1 \amalg S_2$  and  $S_2 \amalg S_1$  are distinct 2-shuffles of  $S$ . For an ordered set  $S$  we let  $\text{Sh}_n(S)$  denote the set of distinct *n-shuffles* of  $S$ . We can count *n-shuffles* by successively asking each element of  $S$  which set  $S_\alpha$  it belongs to. Hence there are  $n^{|S|}$  such shuffles.

We are now ready to formulate a useful compatibility relation between the  $*$  and  $T$  operations:

$$T(\mathfrak{w} * \mathfrak{w}')[\mathfrak{w}_1, \dots, \mathfrak{w}_n] = \sum_{\text{Sh}_2(S)} \epsilon T(\mathfrak{w})[T(\mathfrak{w}')[\mathfrak{w}_1], \mathfrak{w}_2] \quad (3.4)$$

where we sum over 2-shuffles  $S = S_1 \amalg S_2$  of the ordered set  $S = \{\mathfrak{w}_1, \dots, \mathfrak{w}_n\}$  and we understand that  $T(\mathfrak{w})[\emptyset] = 0$ . The sign  $\epsilon$  in the sum keeps track of the web orientations, and it is determined as follows. We let  $o_r(S_\alpha)$  be the ordered product of reduced orientations of the  $\mathfrak{w}_i$  in  $S_\alpha$  and define

$$\epsilon = \frac{o_r(S_1) \wedge o_r(S_2)}{o_r(\mathfrak{w}_1) \wedge \dots \wedge o_r(\mathfrak{w}_n)} := \epsilon_{S_1, S_2}. \quad (3.5)$$

exactly as we would expect from manipulating graded elements  $\mathfrak{w}$  of degree  $d(\mathfrak{w})$  with the Koszul rule.

Next we extend the map  $\mathfrak{w} \rightarrow T(\mathfrak{w})$  to be a linear map by setting  $T(\mathfrak{w}_1 + \mathfrak{w}_2) := T(\mathfrak{w}_1) + T(\mathfrak{w}_2)$ . It now makes sense to speak of  $T[\mathfrak{t}]$ , which will play a particularly important role for us. The relation (3.4) is bilinear in  $\mathfrak{w}$  and  $\mathfrak{w}'$ . Summing  $\mathfrak{w}$  and  $\mathfrak{w}'$  separately over taut elements and applying  $\mathfrak{t} * \mathfrak{t} = 0$  it follows that, for any ordered set  $S$  (i.e. for any monomial in  $T\mathcal{W}$ ):

$$\sum_{\text{Sh}_2(S)} \epsilon_{S_1, S_2} T(\mathfrak{t})[T(\mathfrak{t})[\mathfrak{w}_1], \mathfrak{w}_2] = 0. \quad (3.6)$$

We can interpret these relations as defining a version of an “ $L_\infty$  algebra.” To make this clear and to lighten the notation let us denote by  $b_n$  (“closed brackets”) the restriction of  $T(\mathbf{t})$  to  $\mathcal{W}^{\otimes n}$ , so

$$b_n : \mathcal{W}^{\otimes n} \rightarrow \mathcal{W} \quad (3.7)$$

has degree  $\deg(b_n) = 3 - 2n$  and satisfies the identities

$$\sum_{\text{Sh}_2(S)} \epsilon_{S_1, S_2} b_{n_2}(b_{n_1}(S_1), S_2) = 0 \quad (3.8)$$

where  $n_i = |S_i|$ ,  $i = 1, 2$ .

### Remarks:

1. The degrees and associativity relations (3.8) coincide with the notion of  $L_\infty$  algebra which appears in other areas of physics, such as closed-string field theory [96, 27]. The degrees and signs used in the mathematical literature are slightly different. Our definitions are known as the  $L_\infty[-1]$  relations. (For a relation of these relations to the more standard  $L_\infty$  relations see [73].)
2. It is worth noting that  $(\mathbf{w}_1 * \mathbf{w}_2) * \mathbf{w}_3 - \mathbf{w}_1 * (\mathbf{w}_2 * \mathbf{w}_3)$  can be interpreted as webs with  $\mathbf{w}_2$  and  $\mathbf{w}_3$  inserted into distinct vertices of  $\mathbf{w}_1$  and is therefore graded symmetric in  $\mathbf{w}_2$  and  $\mathbf{w}_3$ . This is precisely the definition of a (graded) “pre-Lie-algebra,” thus making contact with the papers [16, 63, 78, 8].
3. Note that since every taut web has at least two vertices the differential  $b_1$  is always identically zero. In technical terms these algebras are “minimal” [57].
4. Moreover,  $b_n(\mathbf{w}_1, \dots, \mathbf{w}_n)$  is a web with

$$V = n + \sum_{i=1}^n (V(\mathbf{w}_i) - 1) \quad (3.9)$$

vertices. Since we can grade  $\mathcal{W}$  by the number of vertices it follows that  $b_2$  is a nilpotent multiplication.

5. The  $T$  operation also satisfies a natural associativity relation.

$$T(T(\mathbf{w})[\mathbf{w}_1 \otimes \cdots \otimes \mathbf{w}_n])[\tilde{\mathbf{w}}_1 \otimes \cdots \otimes \tilde{\mathbf{w}}_N] = \sum_{\text{Sh}_n(S)} \epsilon T(\mathbf{w}) [T(\mathbf{w}_1)[S_1], T(\mathbf{w}_2)[S_2], \dots, T(\mathbf{w}_n)[S_n]] \quad (3.10)$$

where the sum over  $n$ -shuffles refers to the ordered set

$$S = \{\tilde{\mathbf{w}}_1, \dots, \tilde{\mathbf{w}}_N\} \quad (3.11)$$

The sign  $\epsilon$  in the sum keeps track as usual of the webs orientations. This relation will not play an important role in the following.

### 3.1.1 Examples Of Web Algebras

We can describe the  $L_\infty[-1]$ -algebra fairly explicitly if there are  $n$  vacua with weights  $z_i$  which are in the set of extremal points of a convex set. We can enumerate the vertices by ordering the vacua so that  $\{1, \dots, n\}$  is a cyclic fan of vacua. Then there is an  $n$ -valent vertex  $\mathfrak{w}_{12\dots n}$  and we can make all other  $(n-j)$ -valent vertices by deleting  $j$  vacua from the cyclic fan  $\{1, \dots, n\}$  to form cyclic fans with smaller numbers of vacua. We must have at least 3 vacua so there are in all:

$$\sum_{j=0}^{n-3} \binom{n}{j} = 2^n - \frac{1}{2}(n^2 + n + 2) \quad (3.12)$$

different vertices. We denote these by  $\mathfrak{w}_I$  where  $I$  is a cyclic fan of vacua, which, in these examples is just a cyclically ordered subset of  $\{1, \dots, n\}$  with at least three elements.

We can make all taut elements by “resolving” the vertices. These will be enumerated by pairs of cyclic fans of vacua  $I_1, I_2$  which are compatible in the sense that they are of the form:

$$I_1 = \{i_1, i_2, \dots, i_k\} \quad I_2 = \{i_k, i_{k+1}, \dots, i_1\} \quad (3.13)$$

and, if we denote

$$I_1 * I_2 = \{i_1, i_2, \dots, i_k, i_{k+1}, \dots, i_1\} \quad (3.14)$$

then  $I_1 * I_2$  is also a cyclic fan. Then we have

$$\mathfrak{t} = \sum \mathfrak{w}_{I_1; I_2} \quad (3.15)$$

where we sum over such compatible pairs of fans. All the taut webs have exactly two vertices and therefore (for such convex configurations of vacuum weights) the higher products  $b_n = 0$  for  $n > 2$ .

The only nonzero products of vertex webs is

$$b_2(\mathfrak{w}_{I_1}, \mathfrak{w}_{I_2}) = \mathfrak{w}_{I_1; I_2} \quad (3.16)$$

However, if a taut web  $\mathfrak{w}_{I_1; I_2}$  has vertices with  $I_1$  or  $I_2$  of length greater than 3 then it can also define products of non-vertex webs which have  $I_\infty(\mathfrak{w}) = I_1$  or  $= I_2$ . Note also that no taut web has any vertex with  $I_v = \{1, 2, \dots, n\}$  and so any web with  $I_\infty(\mathfrak{w}) = \{1, 2, \dots, n\}$  such as the vertex  $\mathfrak{w}_{1,2,\dots,n}$ , and all its resolutions, will be in the annihilator of  $b_2$ . We have an  $(n-2)$ -step nilpotent algebra. That is,  $(n-2)$  applications of  $b_2$  will always vanish.

When the weights  $z_i$  of the vacua are not extremal points of a convex set then the algebras can be more complicated and the higher products  $b_n$  can be nonzero.

### 3.2 Algebraic Structures From Half-Plane Webs

There are three obvious generalizations of  $T$  to half-plane webs: we can either replace all interior vertices of some half-plane webs  $\mathfrak{u}$  with plane webs, replace all boundary vertices with half-plane webs, or both. The latter operation is the composition of the former two: we can first replace the interior vertices, then the boundary vertices (if we try the

opposite, we create new interior vertices at the first step). It is instructive to discuss all three possibilities. The first case, discussed in §3.2.1, shows how half-plane webs provide an  $L_\infty$ -module for the  $L_\infty$ -algebra of plane webs. Then, in §3.2.2 we show that inserting half-plane webs into half-plane webs defines an  $A_\infty$ -algebra structure. When we combine the two operations we end up with a set of identities we call the  $LA_\infty$ -identities in §3.2.3. Finally in §3.2.4 we give a conceptual interpretation of the  $LA_\infty$ -identities in terms of an  $L_\infty$ -morphism between the  $L_\infty$  algebra of plane webs and the  $L_\infty$  algebra of Hochschild cochains on the  $A_\infty$  algebra of half-plane webs.

### 3.2.1 $L_\infty$ -Modules

The first possibility - replacing just interior vertices - is not our essential goal, but it is instructive. We define a multilinear map  $T_i[\mathbf{u}] : T\mathcal{W} \rightarrow \mathcal{W}_H$  which is, as before, zero unless there is some permutation  $\sigma$  which matches the arguments  $\mathbf{w}_a$  of a monomial  $\{\mathbf{w}_1, \dots, \mathbf{w}_\ell\}$  to the interior vertices of  $\mathbf{u}$  in the sense that  $I_{v_{\sigma(a)}^i}(\mathbf{u}) = I_\infty(\mathbf{w}_a)$ , in which case it is the simultaneous convolution with orientation  $o(\mathbf{u}) \wedge o_r(\mathbf{w}_1) \wedge \dots \wedge o_r(\mathbf{w}_\ell)$ . This map has a simple relation to convolutions:

$$\begin{aligned} T_i(\mathbf{u} * \mathbf{w})[\mathbf{w}_1, \dots, \mathbf{w}_n] &= \sum_{S \in \text{Sh}_2(S)} \epsilon T_i(\mathbf{u})[T(\mathbf{w})[S_1], S_2] \\ T_i(\mathbf{u} * \mathbf{u}')[\mathbf{w}_1, \dots, \mathbf{w}_n] &= \sum_{S \in \text{Sh}_2(S)} \epsilon' T_i(\mathbf{u})[S_1] * T_i(\mathbf{u}')[S_2] \end{aligned} \quad (3.17)$$

with the usual definition  $S = \{\mathbf{w}_1, \dots, \mathbf{w}_n\}$ .

The  $\epsilon$  signs keep track of the relative web orientations on the two sides of the equations. It is important to observe that the signs arise from the reorganization of a product of *reduced* web orientations. It would thus be incorrect to assume glibly that  $\epsilon$  coincides with the Koszul rule: we defined the degree of a web as the dimension of the unreduced moduli space. The correct sign rule could be denoted as the “reduced Koszul rule”: treat the symbols as if they had degree given by the reduced dimension of moduli spaces. This subtlety was invisible for bulk webs, for which the reduction of moduli space removes two dimensions, but it is important for half-plane webs. The prime on the  $\epsilon$  in the second equation of (3.17) takes into account that we must bring  $\mathbf{u}'$  across the monomial  $S_1$  in the tensor algebra using the reduced Koszul rule.

If we plug our second theorem  $\mathbf{t}_H * \mathbf{t}_H + \mathbf{t}_H * \mathbf{t}_p = 0$  into (3.17), we get a neat relation

$$\sum_{S \in \text{Sh}_2(S)} \epsilon_{S_1, S_2} T_i(\mathbf{t}_H)[T(\mathbf{t}_p)[S_1], S_2] + \epsilon'_{S_1, S_2} T_i(\mathbf{t}_H)[S_1] * T_i(\mathbf{t}_H)[S_2] = 0 \quad (3.18)$$

This identity can be used to define  $\mathcal{W}_H$  as a “right-module for the  $L_\infty$ -algebra  $\mathcal{W}$ .”

In general, if  $\mathcal{L}$  is an  $L_\infty$ -algebra with products  $b_n^\mathcal{L}$  of degree  $3 - 2n$ , then a left  $L_\infty$ -module  $\mathcal{M}$  is a graded  $\mathbb{Z}$ -module with operations  $b_n^\mathcal{M} : \mathcal{L}^{\otimes n} \otimes \mathcal{M} \rightarrow \mathcal{M}$  defined for  $n \geq 0$  and of degree  $1 - 2n$  so that the analog of the  $L_\infty$ -identities holds, i.e. for all  $S = \{\ell_1, \dots, \ell_s\}$  and  $m \in \mathcal{M}$ :

$$\sum_{S \in \text{Sh}_2(S)} \epsilon b_{s_2+1}^\mathcal{M}(b_{s_1}^\mathcal{L}(S_1), S_2; m) + \sum_{S \in \text{Sh}_2(S)} \epsilon b_{s_1}^\mathcal{M}(S_1; b_{s_2}^\mathcal{M}(S_2; m)) = 0 \quad (3.19)$$

In the present case, the module operations for a set  $S = \{\mathfrak{w}_1, \dots, \mathfrak{w}_n\}$  are

$$\mathfrak{u} \mapsto \mathfrak{u} * T_i(\mathfrak{t}_{\mathcal{H}})[S] \quad (3.20)$$

The module relations are

$$\sum_{\text{Sh}_2(S)} \epsilon_{S_1, S_2} \mathfrak{u} * T_i(\mathfrak{t}_{\mathcal{H}})[T(\mathfrak{t}_p)[S_1], S_2] + \epsilon'_{S_1, S_2} (\mathfrak{u} * T_i(\mathfrak{t}_{\mathcal{H}})[S_1]) * T_i(\mathfrak{t}_{\mathcal{H}})[S_2] = 0 \quad (3.21)$$

and can be proven from (3.18) by convolving with  $\mathfrak{u}$ . The only nontrivial step is the observation that

$$(\mathfrak{u}_1 * \mathfrak{u}_2) * \mathfrak{u}_3 - \mathfrak{u}_1 * (\mathfrak{u}_2 * \mathfrak{u}_3) \quad (3.22)$$

is (reduced Koszul) graded symmetric in  $\mathfrak{u}_2$  and  $\mathfrak{u}_3$ . That is, once again we use the property that convolution defines a pre-Lie algebra structure.

### 3.2.2 $A_\infty$ -Algebras

We can define the second natural multilinear map  $T_{\mathcal{H}}[\mathfrak{u}] : T\mathcal{W}_{\mathcal{H}} \rightarrow \mathcal{W}_{\mathcal{H}}$ , by inserting half-plane webs at all boundary vertices of  $\mathfrak{u}$ . In contrast to the interior vertices, the boundary vertices will always be time ordered, so we define the map to be zero on  $\mathfrak{u}_1 \otimes \dots \otimes \mathfrak{u}_\ell$  unless the arguments match the boundary vertices,  $J_{v_a^\partial}(\mathfrak{u}) = J_\infty(\mathfrak{u}_a)$  for  $a = 1, \dots, \ell$ , *in that order*. (Recall from equation (2.19) that we have chosen an ordering of the boundary vertices.) When this is satisfied we glue in to get a new deformation type in the usual way with the orientation

$$o(\mathfrak{u}) \wedge o_r(\mathfrak{u}_1) \wedge \dots \wedge o_r(\mathfrak{u}_\ell). \quad (3.23)$$

Before stating the compatibility of  $T_{\mathcal{H}}$  with  $*$  it is useful at this point to define a notion of *ordered  $n$ -partitions*. If  $P$  is an ordered set we define an ordered  $n$ -partition of  $P$  to be an ordered disjoint decomposition into  $n$  ordered subsets

$$P = P_1 \amalg P_2 \amalg \dots \amalg P_n \quad (3.24)$$

where the ordering of each summand  $P_\alpha$  is inherited from the ordering of  $P$  and all the elements of  $P_\alpha$  precede all elements of  $P_{\alpha+1}$  inside  $P$ . We allow the  $P_\alpha$  to be the empty set. For an ordered set  $P$  we let  $\text{Pa}_n(P)$  denote the set of distinct  $n$ -partitions of  $P$ . If  $p = |P|$  there are  $\binom{n+p-1}{p}$  such partitions.

Now we can state the compatibility:

$$\begin{aligned} T_{\mathcal{H}}(\mathfrak{u} * \mathfrak{w})[\mathfrak{u}_1, \dots, \mathfrak{u}_n] &= \epsilon (T_{\mathcal{H}}(\mathfrak{u})[\mathfrak{u}_1, \dots, \mathfrak{u}_n]) * \mathfrak{w} - \sum_{m=1}^n \epsilon T_{\mathcal{H}}(\mathfrak{u})[\mathfrak{u}_1, \dots, \mathfrak{u}_m * \mathfrak{w}, \dots, \mathfrak{u}_n] \\ T_{\mathcal{H}}(\mathfrak{u} * \mathfrak{u}')[\mathfrak{u}_1, \dots, \mathfrak{u}_n] &= \sum_{\text{Pa}_3(P)} \epsilon T_{\mathcal{H}}(\mathfrak{u})[P_1, T_{\mathcal{H}}(\mathfrak{u}')][P_2], P_3] \end{aligned} \quad (3.25)$$

In the second identity we have introduced a sum over ordered 3-partitions of an ordered set  $P$  of half-plane webs. As before, we take  $T_{\mathcal{H}}[\mathfrak{u}][P] = 0$  if  $P = \emptyset$ .

Combining this with  $\mathbf{t}_{\mathcal{H}} * \mathbf{t}_{\mathcal{H}} + \mathbf{t}_{\mathcal{H}} * \mathbf{t}_p = 0$  we arrive at the relation

$$\begin{aligned} \epsilon_1(T_{\mathcal{H}}(\mathbf{t}_{\mathcal{H}})[\mathbf{u}_1, \dots, \mathbf{u}_n]) * \mathbf{t}_p - \sum_{m=1}^n \epsilon_2 T_{\mathcal{H}}(\mathbf{t}_{\mathcal{H}})[\mathbf{u}_1, \dots, \mathbf{u}_m * \mathbf{t}_p, \dots, \mathbf{u}_n] \\ + \sum_{\text{Pa}_3(P)} \epsilon_3 T_{\mathcal{H}}(\mathbf{t}_{\mathcal{H}})[P_1, T_{\mathcal{H}}(\mathbf{t}_{\mathcal{H}})[P_2], P_3] = 0. \end{aligned} \quad (3.26)$$

where

$$\epsilon_1 = (-1)^{\sum_s d_r(\mathbf{u}_s)} \quad \epsilon_2 = (-1)^{\sum_{s=1}^m d_r(\mathbf{u}_s)} \quad \epsilon_3 = (-1)^{P_1} := (-1)^{\sum_{\mathbf{u} \in P_1} d_r(\mathbf{u})}. \quad (3.27)$$

We can interpret (3.26) as the standard axioms for an  $A_{\infty}$  algebra structure on  $\mathcal{W}_{\mathcal{H}}$ . To make this clear and to lighten the notation let us denote by  $a_n$  (“open brackets”) the restriction of  $T_{\mathcal{H}}(\mathbf{t}_{\mathcal{H}})$  to  $\mathcal{W}_{\mathcal{H}}^{\otimes n}$  for  $n > 1$  and the operation

$$a_1(\mathbf{u}) = T_{\mathcal{H}}(\mathbf{t}_{\mathcal{H}})[\mathbf{u}] - (-1)^{d_r(\mathbf{u})} \mathbf{u} * \mathbf{t}_p \quad (3.28)$$

for  $n = 1$ . The first of the  $A_{\infty}$ -relations demands that  $a_1(a_1(\mathbf{u})) = 0$ . This works out to be

$$T_{\mathcal{H}}(\mathbf{t}_{\mathcal{H}})(T_{\mathcal{H}}(\mathbf{t}_{\mathcal{H}})[\mathbf{u}]) - (-1)^{d_r(\mathbf{u})} T_{\mathcal{H}}(\mathbf{t}_{\mathcal{H}})[\mathbf{u} * \mathbf{t}_p] + (-1)^{d_r(\mathbf{u})} (T_{\mathcal{H}}(\mathbf{t}_{\mathcal{H}})[\mathbf{u}]) * \mathbf{t}_p - (\mathbf{u} * \mathbf{t}_p) * \mathbf{t}_p = 0 \quad (3.29)$$

Thus to match to (3.26) we also need to check that  $(\mathbf{u} * \mathbf{t}_p) * \mathbf{t}_p = 0$ . Although convolution is not associative, the difference  $(\mathbf{u} * \mathbf{t}_p) * \mathbf{t}_p - \mathbf{u} * (\mathbf{t}_p * \mathbf{t}_p)$  consists of terms where two taut webs  $\mathbf{w}_1$  and  $\mathbf{w}_2$  are inserted separately at two vertices  $v^1$  and  $v^2$  of  $\mathbf{u}$ . Each such term appears twice in the sum, either from  $(\mathbf{u} *_{v^1} \mathbf{w}_1) *_{v^2} \mathbf{w}_2$  or from  $(\mathbf{u} *_{v^2} \mathbf{w}_2) *_{v^1} \mathbf{w}_1$ . The two contributions have opposite orientations and cancel out against each other.

Moving on to the higher identities, the

$$a_n : \mathcal{W}_{\mathcal{H}}^{\otimes n} \rightarrow \mathcal{W}_{\mathcal{H}} \quad n \geq 1 \quad (3.30)$$

have degree  $\deg(a_n) = 2 - n$  and satisfy the identities

$$\sum_{\text{Pa}_3(P)} (-1)^{P_1} a_{p_1+p_3+1}(P_1, a_{p_2}(P_2), P_3) = 0 \quad (3.31)$$

where  $p_i = |P_i|$ ,  $i = 1, 2, 3$ .

### 3.2.3 The $LA_{\infty}$ -Identities

Finally, we can consider the combined operation

$$T(\mathbf{u}) : T\mathcal{W}_{\mathcal{H}} \otimes T\mathcal{W} \rightarrow \mathcal{W}_{\mathcal{H}} \quad (3.32)$$

as

$$T(\mathbf{u})[\mathbf{u}_1, \dots, \mathbf{u}_n; \mathbf{w}_1, \dots, \mathbf{w}_m] := \epsilon T_{\mathcal{H}}(T_i(\mathbf{u})[\mathbf{w}_1, \dots, \mathbf{w}_m]) [\mathbf{u}_1, \dots, \mathbf{u}_n] \quad (3.33)$$

We included a sign, to convert the orientation

$$o(\mathbf{u}) \wedge (o_r(\mathbf{w}_1) \wedge \cdots \wedge o_r(\mathbf{w}_m)) \wedge (o_r(\mathbf{u}_1) \wedge \cdots \wedge o_r(\mathbf{u}_n)) \quad (3.34)$$

in the right hand side to the natural orientation for the order of the arguments on left hand side

$$o(\mathbf{u}) \wedge (o_r(\mathbf{u}_1) \wedge \cdots \wedge o_r(\mathbf{u}_n)) \wedge (o_r(\mathbf{w}_1) \wedge \cdots \wedge o_r(\mathbf{w}_m)). \quad (3.35)$$

Again, we note that convolution and  $T$  interact well together:

$$T(\mathbf{u} * \mathbf{u}')[P; S] = \sum_{\text{Sh}_2(S), \text{Pa}_3(P)} \epsilon T(\mathbf{u})[P_1, T(\mathbf{u}')[P_2; S_1], P_3; S_2] \quad (3.36)$$

where  $S$  is the set of plane web arguments and  $P$  is the set of half-web arguments. The sign is given by the usual reduced Koszul rule.

Similarly

$$T(\mathbf{u} * \mathbf{w})[P; S] = \sum_{\text{Sh}_2(S)} \epsilon_{S_1, S_2} T(\mathbf{u})[P; T(\mathbf{w})[S_1], S_2] \quad (3.37)$$

Combining (3.36) and (3.37) with (2.30) we get some nontrivial algebraic identities

$$\sum_{\text{Sh}_2(S), \text{Pa}_3(P)} \epsilon T(\mathbf{t}_{\mathcal{H}})[P_1, T(\mathbf{t}_{\mathcal{H}})[P_2; S_1], P_3; S_2] + \sum_{\text{Sh}_2(S)} \epsilon T(\mathbf{t}_{\mathcal{H}})[P; T(\mathbf{t}_p)[S_1], S_2] = 0. \quad (3.38)$$

We will refer to this hybrid equation as an “ $LA_\infty$ ” identity. (This is not standard terminology.) It has a somewhat refined algebraic meaning, which we will decode presently. Before that, we would like to remark that both  $T_i$  and  $T_{\mathcal{H}}$  can be recovered from  $T$  by filling either kinds of slots with the sum over all rigid plane webs  $\mathbf{r}$  or all rigid half-plane webs  $\mathbf{r}_{\mathcal{H}}$ :

$$\begin{aligned} T_i[S] &= \sum_n T[\mathbf{r}_{\mathcal{H}}^{\otimes n}; S] \\ T_{\mathcal{H}}[P] &= \sum_n \frac{1}{n!} T[P, \mathbf{r}^{\otimes n}] \end{aligned} \quad (3.39)$$

and we can similarly fill in the slots of (3.38) to get the corresponding equations (3.18) and (3.26).

### 3.2.4 Conceptual Meaning Of The $LA_\infty$ -Identities

We now give a conceptual interpretation of the  $LA_\infty$  identities (3.38). It is useful to organize the equations (3.38) by first considering the special cases where  $S$  has cardinality 0, 1, 2. Then the general structure will become clear. Correspondingly, we can decompose the taut element  $\mathbf{t}_{\mathcal{H}}$  according to the number of interior vertices in the taut webs:

$$\mathbf{t}_{\mathcal{H}} = \mathbf{t}_{\mathcal{H}}^{(0)} + \mathbf{t}_{\mathcal{H}}^{(1)} + \mathbf{t}_{\mathcal{H}}^{(2)} + \cdots \quad (3.40)$$

Note that the only taut half-plane webs with no interior vertices have precisely two boundary vertices and therefore  $\mu := T(\mathbf{t}_{\mathcal{H}}^{(0)})$  is simply a multiplication map

$$\mu : \mathcal{W}_{\mathcal{H}} \times \mathcal{W}_{\mathcal{H}} \rightarrow \mathcal{W}_{\mathcal{H}}. \quad (3.41)$$

When (3.38) is restricted to  $S = \emptyset$  we learn that  $\mu$  is an associative multiplication, up to sign:

$$\mu(\mu(u_1, u_2), u_3) + (-1)^{d(u_1)-1} \mu(u_1, \mu(u_2, u_3)) = 0 \quad (3.42)$$

and therefore  $\tilde{\mu}(u_1, u_2) = (-1)^{d(u_1)-1} \mu(u_1, u_2)$  is strictly associative.

Next, for a fixed planar web  $\mathfrak{w}$  let

$$\mu^{(1)} := T(t^{(1)})[\cdots; \mathfrak{w}] : T\mathcal{W}_{\mathcal{H}} \rightarrow \mathcal{W}_{\mathcal{H}} \quad (3.43)$$

Recall that a Hochschild cochain on an algebra  $\mathcal{A}$  is simply a collection of linear maps  $F_n : \mathcal{A}^{\otimes n} \rightarrow \mathcal{A}$ ,  $n \geq 1$ , or equivalently an element of  $CC^\bullet(\mathcal{A}) := \text{Hom}(T\mathcal{A}, \mathcal{A})$ . Therefore for a fixed  $\mathfrak{w}$  we may view  $\mu^{(1)}$  as a Hochschild cochain on the associative algebra  $\mathcal{W}_{\mathcal{H}}$ . Then, taking  $S = \{\mathfrak{w}\}$ , equation (3.38) becomes

$$\begin{aligned} 0 &= \tilde{\mu}(u_1, \tilde{\mu}^{(1)}(u_2, \dots, u_n)) + \\ &+ \sum_{r=0}^{n-1} (-1)^r \tilde{\mu}^{(1)}(u_1, \dots, u_r, \tilde{\mu}(u_{r+1}, u_{r+2}), u_{r+3}, \dots, u_n) \\ &+ (-1)^n \tilde{\mu}(\tilde{\mu}^{(1)}(u_1, \dots, u_{n-1}), u_n) \\ &=: B^{(1)}(\tilde{\mu}^{(1)})(u_1, \dots, u_n) \end{aligned} \quad (3.44)$$

where  $\tilde{\mu}^{(1)}$  is related to  $\mu^{(1)}$  by signs in a way analogous to the relation of  $\mu$  and  $\tilde{\mu}$ . In the last line of (3.44) we have recognized that the previous lines define the Hochschild differential  $B^{(1)}$  on the Hochschild complex. Thus, our identity (3.38) when  $|S| = 1$  simply says that  $\mu^{(1)}$  is a Hochschild cocycle. In order to discuss the cases  $|S| \geq 2$  we introduce the notation

$$\tilde{\mu}^{(n)}(S) := \epsilon_n T(t^{(n)})[\cdots; S] : T\mathcal{W}_{\mathcal{H}} \rightarrow \mathcal{W}_{\mathcal{H}} \quad (3.45)$$

where  $S = \{\mathfrak{w}_1, \dots, \mathfrak{w}_n\}$ ,  $n \geq 1$ , and  $\epsilon_n$  is again an appropriate sign redefinition. Then the identity for  $|S| = 2$  reads

$$\tilde{\mu}^{(1)}(b_2(\mathfrak{w}_1, \mathfrak{w}_2)) = B^{(1)}\tilde{\mu}^{(2)}(\mathfrak{w}_1, \mathfrak{w}_2) + B^{(2)}(\tilde{\mu}^{(1)}(\mathfrak{w}_1), \tilde{\mu}^{(1)}(\mathfrak{w}_2)) \quad (3.46)$$

where  $b_2$  is the multiplication on  $\mathcal{W}$  defined by  $T(t_p)$ , and we have introduced the Hochschild bracket  $B^{(2)}$  on the Hochschild complex  $CC^\bullet(\mathcal{W}_{\mathcal{H}})$ .

Quite generally the Hochschild bracket on  $CC^\bullet(\mathcal{A})$  may be defined on two cochains  $F, G$  of degree  $n, m$  by

$$\begin{aligned} B^{(2)}(F, G) &:= F \circ G - (-1)^{(|F|-1)(|G|-1)} G \circ F \\ F \circ G &:= \sum_r \epsilon F(u_1, \dots, u_r, G(u_{r+1}, \dots, u_{r+m}), \dots, u_{n+m-1}) \end{aligned} \quad (3.47)$$

where the sign  $\epsilon$  can be found in many papers. See, for examples, [58, 1]. In general, the Hochschild complex is a differential graded Lie algebra with operations  $B^{(1)}$  and  $B^{(2)}$ . A differential graded Lie algebra can be considered to be a special case of an  $L_\infty$ -algebra, so we can speak of the  $L_\infty$  algebra of a Hochschild complex.

To summarize these facts concisely we need the general notion of an  $L_\infty$  morphism between two  $L_\infty$  algebras. See equation (A.52) below.

Now, returning to our example, we consider the map

$$\tilde{\mu} : T\mathcal{W} \rightarrow CC^\bullet(\mathcal{W}_H) \quad (3.48)$$

defined by

$$\mathfrak{w}_1 \otimes \cdots \otimes \mathfrak{w}_n \mapsto T(\mathbf{t}_H[\cdots; \mathfrak{w}_1 \otimes \cdots \otimes \mathfrak{w}_n]). \quad (3.49)$$

Then, separating (3.38) into the cases  $S_1 = \emptyset$ , or  $S_2 = \emptyset$ , or both  $S_1, S_2$  are nonempty, and grouping together the terms  $S_1 \amalg S_2$  with  $S_2 \amalg S_1$  in the latter case we see that the equation (3.38) may be concisely summarized as the statement that  *$\tilde{\mu}$  is an  $L_\infty$ -morphism from the  $L_\infty$ -algebra of planar webs to the  $L_\infty$ -algebra of the Hochschild complex of  $\mathcal{W}_H$ .* This is the conceptual meaning of the  $LA_\infty$  identities.

### 3.3 Bimodules And Strip Webs

Starting from a strip web  $\mathfrak{s}$ , we can define three elementary operations  $T_{i,L,R}(\mathfrak{s})$  which replace either all interior, left boundary or right boundary vertices with plane, positive half-plane or negative half-plane webs, respectively. (Recall that the positive half-plane  $\mathcal{H}_L$  has boundary on the left.) The definitions of these operations are completely parallel to the definitions given in the previous two sub-sections. Due to our choice of ordering of boundary vertices, the arguments of  $T_L(\mathfrak{s})$  should be ordered left to right in decreasing time, and right to left in decreasing time for  $T_R(\mathfrak{s})$ .

We can then define appropriate composite operations

$$\begin{aligned} T_{L,i}(\mathfrak{s})[P; S] &:= \epsilon T_L(T_i(\mathfrak{s})[S])[P] \\ T_{i,R}(\mathfrak{s})[S; P'] &:= T_R(T_i(\mathfrak{s})[S])[P'] \\ T_{L,R}(\mathfrak{s})[P; P'] &:= T_R(T_L(\mathfrak{s})[P])[P'] \\ T(\mathfrak{s})[P; S; P'] &:= \epsilon' T_R(T_L(T_i(\mathfrak{s})[S])[P])[P'] \end{aligned} \quad (3.50)$$

where  $\epsilon, \epsilon'$  are (reduced, as always) Koszul signs for reordering the arguments.

We will now just sketch some of the various algebraic structures which follow from the strip web identity (2.40) which we quote here:

$$\mathbf{t}_s * \mathbf{t}_L + \mathbf{t}_s * \mathbf{t}_R + \mathbf{t}_s * \mathbf{t}_p + \mathbf{t}_s \circ \mathbf{t}_s = 0. \quad (3.51)$$

These structures will involve the notion of an  $A_\infty$ -module.

In general, if  $\mathcal{A}$  is an  $A_\infty$ -algebra with products  $m_n^{\mathcal{A}}$  then a left  $A_\infty$ -module  $\mathcal{M}$  is a graded  $\mathbb{Z}$ -module with operations  $m_n^{\mathcal{M}} : \mathcal{A}^{\otimes n} \otimes \mathcal{M} \rightarrow \mathcal{M}$  defined for  $n \geq 0$  and of degree  $1 - n$  so that the analog of the  $A_\infty$ -identities holds, i.e. for all  $P = \{a_1, \dots, a_p\}$  and  $m \in \mathcal{M}$ :

$$\sum_{\text{Pa}_3(P)} (-1)^{p_1} m_{p_1+p_3+1}^{\mathcal{M}}(P_1, m_{p_2}^{\mathcal{A}}(P_2), P_3; m) + \sum_{\text{Pa}_2(P)} (-1)^{p_1} m_{p_1}^{\mathcal{M}}(P_1; m_{p_2}^{\mathcal{M}}(P_2; m)) = 0 \quad (3.52)$$

Similarly, one can define right  $A_\infty$ -modules as well as bimodules. When we include interior operations  $T_i$  then there will be  $L_\infty$  maps to the  $L_\infty$  algebra of Hochschild cochains with values these modules.

As the simplest example let us consider  $\mathcal{H}_L$ , the positive half-plane and denote  $\mathbf{t}_L := \mathbf{t}_{\mathcal{H}_L}$  and  $\mathcal{W}_L := \mathcal{W}_{\mathcal{H}_L}$ . Proceeding as in the previous section the identity (2.40) implies:

$$\begin{aligned} & \sum_{\text{Pa}_3(P)} \epsilon T_L(\mathbf{t}_s)[P_1, T_{\mathcal{H}_L}(\mathbf{t}_L)[P_2], P_3] + \\ & + \epsilon T_L(\mathbf{t}_s)[P] * \mathbf{t}_R + \epsilon T_L(\mathbf{t}_s)[P] * \mathbf{t}_p - \sum_{m=1}^n \epsilon T_L(\mathbf{t}_s)[\mathbf{u}_1, \dots, \mathbf{u}_m * \mathbf{t}_p, \dots, \mathbf{u}_n] + \\ & + \sum_{\text{Pa}_2(P)} \epsilon T_L(\mathbf{t}_s)[P_1] \circ T_L(\mathbf{t}_s)[P_2] = 0 \end{aligned} \quad (3.53)$$

where, as usual  $P = \{\mathbf{u}_1, \dots, \mathbf{u}_n\}$ .

One can interpret the equations (3.53) as defining a left  $A_\infty$ -module structure on  $\mathcal{M} = \mathcal{W}_S$  for the  $A_\infty$ -algebra  $\mathcal{A} = \mathcal{W}_L$  defined using  $T_{\mathcal{H}_L}(\mathbf{t}_L)$ . To see this define

$$m_0^{\mathcal{M}}(\mathfrak{s}) := (-1)^{d_r(\mathfrak{s})} \mathfrak{s} * (\mathbf{t}_p + \mathbf{t}_R) \quad (3.54)$$

and, for  $n \geq 1$ ,  $m_n^{\mathcal{M}} : \mathcal{W}_L^{\otimes n} \otimes \mathcal{W}_S \rightarrow \mathcal{W}_S$  by

$$m_p^{\mathcal{M}}(P; s) = T_L(\mathbf{t}_s)[P] \circ \mathfrak{s} \quad (3.55)$$

In verifying the module relations we find that  $(m_0^{\mathcal{M}})^2 = 0$  for reasons analogous to those mentioned above. Then we compose the LHS of (3.53) with  $\circ \mathfrak{s}$  and use

$$(-1)^{d_r(\mathfrak{s})} (T_L(\mathbf{t}_s)[P] * (\mathbf{t}_p + \mathbf{t}_R)) \circ \mathfrak{s} = (T_L(\mathbf{t}_s)[P] \circ \mathfrak{s}) * (\mathbf{t}_p + \mathbf{t}_R) - T_L(\mathbf{t}_s)[P] \circ (\mathfrak{s} * (\mathbf{t}_p + \mathbf{t}_R)) \quad (3.56)$$

to recast these equations into the left-module conditions.

In a similar fashion,  $T_R(\mathbf{t}_s)$  gives a right  $A_\infty$  module for the  $A_\infty$  algebra we associated to  $T_{\mathcal{H}_R}(\mathbf{t}_R)$  and  $T_{L,R}(\mathbf{t}_s)$  gives an  $A_\infty$  bi-module, with left and right actions given by the two  $A_\infty$  algebras we associated to  $T_{\mathcal{H}_L}(\mathbf{t}_L)$  and  $T_{\mathcal{H}_R}(\mathbf{t}_R)$ .

As for the interior operation,  $T_i(\mathbf{t}_s)$  will satisfy relations such that the operations  $*T_i(\mathbf{t}_L) + *T_i(\mathbf{t}_R) + *T_i(\mathbf{t}_s)$  define a right  $L_\infty$  module. The three  $T_{L,i}(\mathbf{t}_s)$ ,  $T_{i,R}(\mathbf{t}_s)$ ,  $T(\mathbf{t}_s)$  satisfy lengthy axioms, which essentially define some left, right or bi-module for the  $\mu$  operations (as in Section §3.2.4) associated to either boundary, together with  $L_\infty$  maps from the  $L_\infty$ -algebra of planar webs to the  $L_\infty$ -algebra of the Hochschild complexes of the modules, compatible with the maps defined before.

## 4. Representations Of Webs

**Definition:** Fix a set of vacua  $\mathbb{V}$  and weights  $\{z_i\}_{i \in \mathbb{V}}$ . A *representation of webs* is a pair  $\mathcal{R} = (\{R_{ij}\}, \{K_{ij}\})$  where

- a.)  $R_{ij}$  are  $\mathbb{Z}$ -graded  $\mathbb{Z}$ -modules defined for all ordered pairs  $ij$  of distinct vacua.
- b.)  $K_{ij}$  is a degree  $-1$  symmetric perfect pairing

$$K_{ij} : R_{ij} \otimes R_{ji} \rightarrow \mathbb{Z}. \quad (4.1)$$

By degree  $-1$  we mean that  $K_{ij}(r_{ij}, r'_{ji})$  is only nonzero when  $\deg(r_{ij}) + \deg(r'_{ji}) = +1$  so that the integer  $K_{ij}(r_{ij}, r'_{ji})$  is degree zero. The pairing is symmetric in the sense that

$$K_{ij}(r_{ij}, r'_{ji}) = K_{ji}(r'_{ji}, r_{ij}) \quad (4.2)$$

As the degrees of the arguments differ by one unit, the symmetry of  $K_{ij}$  makes sense. Property (b) of  $K_{ij}$  is motivated by the realization in Landau-Ginzburg models explained near equation (12.18) above. Note that since  $K_{ij}$  is nondegenerate,  $R_{ij}$  and  $R_{ji}$  have the same rank. The property that it is a perfect pairing will be used in several points of the development, for example, in the derivation of equations (7.26), (7.27) below. Often we will drop the subscripts and just write to  $K$  for the pairing when no confusion can arise.

#### 4.1 Web Representations And Plane Webs

Given a representation of webs, for every cyclic fan of vacua  $I = \{i_1, i_2, \dots, i_n\}$  we form

$$R_I := R_{i_1, i_2} \otimes R_{i_2, i_3} \otimes \cdots \otimes R_{i_n, i_1} \quad (4.3)$$

Note that to write this formula we needed to choose a place to start the cyclic sequence. Different choices in the definition of  $R_I$  are related by a canonical isomorphism because the Koszul rule gives a canonical isomorphism

$$R_{i_1, i_2} \otimes R_{i_2, i_3} \otimes \cdots \otimes R_{i_n, i_1} \cong R_{i_2, i_3} \otimes R_{i_3, i_4} \otimes \cdots \otimes R_{i_n, i_1} \otimes R_{i_1, i_2}. \quad (4.4)$$

We will sometimes refer to  $R_I$  as a *representation of a fan*.

Now we collect the representations of all the vertices by forming

$$R^{\text{int}} := \bigoplus_I R_I \quad (4.5)$$

where the sum is over all cyclic fans of vacua. We want to define a map

$$\rho(\mathfrak{w}) : TR^{\text{int}} \rightarrow R^{\text{int}} \quad (4.6)$$

with properties akin to  $T(\mathfrak{w})$ .

As for  $T(\mathfrak{w})$ , we take  $\rho(\mathfrak{w})[r_1, \dots, r_n]$  to be zero unless  $n = V(\mathfrak{w})$  and there exists an order  $\{v_1 \dots v_n\}$  for the vertices of  $\mathfrak{w}$  such that  $r_a \in R_{I_{v_a}}$ . If such an order exists, we will define our map

$$\rho(\mathfrak{w}) : \otimes_{v \in V(\mathfrak{w})} R_{I_v(\mathfrak{w})} \rightarrow R_{I_\infty(\mathfrak{w})} \quad (4.7)$$

as the application of the contraction map  $K$  to all internal edges of the web. Indeed, if an edge joins two vertices  $v_1, v_2 \in V(\mathfrak{w})$  then if  $R_{I_{v_1}(\mathfrak{w})}$  contains a tensor factor  $R_{ij}$  it follows that  $R_{I_{v_2}(\mathfrak{w})}$  contains a tensor factor  $R_{ji}$  and these two factors can be paired by  $K$  as shown in Figure 24.

In order to define  $\rho(\mathfrak{w})$  unambiguously, we need to be very precise about the details of the contraction: Since  $K$  has odd degree, the order of the contractions will affect the sign

of the result! It is useful to denote by  $K_e$  the pairing which we will apply to the tensor factors associated to the edge  $e$ . Then we are attempting to make sense of the overall sign of an expression such as

$$\otimes_{e \in \mathcal{E}(\mathfrak{w})} K_e \circ \otimes_{a=1}^n r_a \quad (4.8)$$

where each of the  $r_a \in R_{I_{v_a}(\mathfrak{w})}$  is a linear combination of tensor products  $r_{i_1, i_2} \otimes \cdots \otimes r_{i_m, i_1}$  if  $I_{v_a}(\mathfrak{w}) = \{i_1, i_2, \dots, i_m\}$ .

Given a specific order of the edges in  $\mathcal{E}(\mathfrak{w})$  and vertices in  $\mathcal{V}(\mathfrak{w})$ , the meaning of 4.8 is clear: we shuffle the symbols around using the Koszul rule until each  $K_e$  is followed by the two tensor factors it is supposed to contract, and then we execute all the contractions. We are left with a sequence of residual tensor factors, which can be reordered again with the Koszul rule until they agree with the order in  $R_{I_\infty(\mathfrak{w})}$ . The final result depends on the initial order we picked for the vertices and edges of  $\mathfrak{w}$  in an obvious way: by the Koszul rule for permuting the  $r_a$  or the degree  $-1$   $K_e$  symbols among themselves. Our aim is to define a graded-symmetric operation. As the  $r_a$  appear in the product in the same order as the arguments of  $\rho(\mathfrak{w})$ , this is automatically true.

The only remaining subtlety is to relate the order for the edges of  $\mathfrak{w}$  and the orientation  $o(\mathfrak{w})$  in such a way that

$$\rho(-\mathfrak{w}) = -\rho(\mathfrak{w}) \quad (4.9)$$

We can do so if we remember that the moduli space of deformations of a web is given by a locus in  $\mathbb{R}^{2V(\mathfrak{w})}$  cut locally by a linear constraint for each edge of the web. We can easily describe a vector field transverse to some edge constraint. For example, we can define  $\partial_e$  by acting with a clockwise rotation on the coordinates of the two endpoints of the edge (the choice of origin for the rotation is immaterial). If we have some order for the edges, we can define an orientation for  $\mathfrak{w}$  from the canonical orientations  $dx_v dy_v$  in each  $\mathbb{R}^2$  factor as

$$\prod_{e \in \mathcal{E}(\mathfrak{w})} \partial_e \circ \prod_{v \in \mathcal{V}(\mathfrak{w})} dx_v dy_v \quad (4.10)$$

where  $\circ$  means we contract the poly-vector field on the left with the differential form on the right.

We are finally ready to give a complete definition: if the arguments are compatible with the vertices of the web

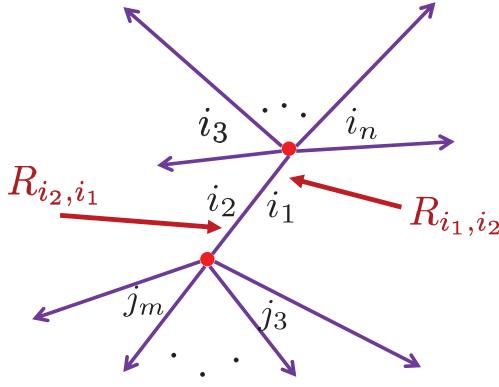
$$\rho(\mathfrak{w})[r_1, \dots, r_n] = \frac{o(\mathfrak{w})}{\prod_{e \in \mathcal{E}(\mathfrak{w})} \partial_e \circ \prod_{v \in \mathcal{V}(\mathfrak{w})} dx_v dy_v} \otimes_{e \in \mathcal{E}(\mathfrak{w})} K_e \circ \otimes_{a=1}^n r_a \quad (4.11)$$

where we use the same ordering of edges for the product over  $\partial_e$  in the denominator and for the product over  $K_e$ . Otherwise,  $\rho(\mathfrak{w})[r_1, \dots, r_n] = 0$ . This map has degree  $-E(\mathfrak{w})$ .

The analogy between  $T$  and  $\rho$  extends to the interplay with the convolution operation.

$$\rho(\mathfrak{w} * \mathfrak{w}')[r_1, \dots, r_n] = \sum_{\text{Sh}_2(S)} \epsilon_{S_1, S_2} \rho(\mathfrak{w})[\rho(\mathfrak{w}')[S_1], S_2] \quad (4.12)$$

where we sum over 2-shuffles of the ordered set  $S = \{r_1, \dots, r_n\}$ . Once again we define  $\rho(\mathfrak{w})[\emptyset] = 0$ . The sign  $\epsilon_{S_1, S_2}$  keeps track as usual of the Koszul signs encountered in the



**Figure 24:** The internal lines of a web naturally pair spaces  $R_{i_1, i_2}$  with  $R_{i_2, i_1}$  in a web representation, as shown here.

shuffling of the arguments  $r_i$ . The only subtlety in checking this relation is the overall sign of each term in the left and right hand sides. It is useful to observe that we can order the edges of  $\mathfrak{w} * \mathfrak{w}'$  by listing all edges of  $\mathfrak{w}$  first, then all edges of  $\mathfrak{w}'$ . Then the order of the  $K$  factors on the two sides of the equation is the same, the order of the  $\partial_e$  vector fields in the denominators is the same, the orientations in the numerators coincide and the reshuffling of the arguments is accounted for by the  $\epsilon$  sign.

On the left hand side, the overall position of the second web is removed (convolution uses the reduced orientation of  $\mathfrak{w}'$ ) from the numerator, the position of the insertion vertex in  $\mathfrak{w}$  is removed from the denominator. As the overall position of the second web is identified naturally with the position of the insertion vertex, this does not introduce any extra sign.

Now we extend  $\rho$  linearly by defining  $\rho(\mathfrak{w} + \mathfrak{w}') := \rho(\mathfrak{w}) + \rho(\mathfrak{w}')$ . In close analogy to the previous section we can plug  $t_p * t_p = 0$  into the relation (4.12) and arrive at the axioms of an  $L_\infty$  algebra  $\rho(t) : TR^{\text{int}} \rightarrow R^{\text{int}}$ :

$$\sum_{S \in \text{Sh}_2(S)} \epsilon_{S_1, S_2} \rho(t_p)[\rho(t_p)[S_1], S_2] = 0 \quad (4.13)$$

The main difference between this algebra and the web algebra  $\mathcal{W}$  we encountered before is that  $R^{\text{int}}$  may have a rich subspace of degree 2, which allows us to discuss solutions to the Maurer-Cartan equation for the  $L_\infty$  algebra:

**Definition:** An *interior amplitude* is an element  $\beta \in R^{\text{int}}$  of degree +2 so that if we define  $e^\beta \in TR^{\text{int}} \otimes \mathbb{Q}$  by

$$e^\beta := \beta + \frac{1}{2!} \beta \otimes \beta + \frac{1}{3!} \beta \otimes \beta \otimes \beta + \dots \quad (4.14)$$

then

$$\rho(t_p)(e^\beta) = 0. \quad (4.15)$$

Note that any taut summand  $\mathfrak{w}$  in  $\mathfrak{t}$  has  $2V(\mathfrak{w}) - E(\mathfrak{w}) = 3$  and then  $\rho(\mathfrak{w})$  has degree  $-E(\mathfrak{w})$ , so evaluated on  $\beta^{\otimes V(\mathfrak{w})}$  we get an element of degree 3. Thus, (4.15) is a nontrivial identity consisting of a sum of elements of degree 3.

**Definition:** A *Theory*  $\mathcal{T}$  consists of a set of vacuum data  $(\mathbb{V}, z)$ , a representation of webs  $\mathcal{R} = (\{R_{ij}\}, \{K_{ij}\})$  and an interior amplitude  $\beta$ . If we want to talk about Theories with different data we can write  $\mathcal{T}(\mathbb{V}, z, \mathcal{R}, \beta)$ . In the remainder of this section we will assume we are working within a specific Theory.

An interesting property of the Maurer-Cartan equation (4.15) for an  $L_\infty$  algebra is that a solution can be used to “shift the origin” of the algebra. If we define

$$\rho_\beta(\mathfrak{w})(r_1, \dots, r_\ell) := \rho(\mathfrak{w})(r_1, \dots, r_\ell, e^\beta) \quad (4.16)$$

then we claim that  $\rho_\beta(\mathfrak{t}_p) : TR^{\text{int}} \rightarrow R^{\text{int}}$  satisfies:

$$\sum_{\text{Sh}_2(S)} \epsilon_{S_1, S_2} \rho_\beta(\mathfrak{t}_p)[\rho_\beta(\mathfrak{t}_p)[S_1], S_2] = 0. \quad (4.17)$$

To prove this note that the 2-shuffles of the ordered set  $\tilde{S} = S \cup \{\beta, \dots, \beta\}$  with  $n$  copies of  $\beta$  appended at the right end of  $S$  include  $\binom{n}{k}$  copies of decompositions of the form

$$\tilde{S} = (S_1 \cup \{\beta, \dots, \beta\}) \amalg (S_2 \cup \{\beta, \dots, \beta\}) \quad (4.18)$$

with  $k$   $\beta$ 's in the first summand and  $n - k$   $\beta$ 's in the second.<sup>9</sup> Now we multiply the  $L_\infty$  axiom for  $\rho(\mathfrak{t}_p)$  applied to  $\tilde{S}$  by  $\frac{1}{n!}$  and sum over  $n$ . Thanks to the above remark the sum can be rearranged to give the left-hand-side of the the  $L_\infty$  axioms for  $\rho_\beta(\mathfrak{t}_p)$  applied to  $S$ . Thus far the argument applies to *any* element  $\beta \in R^{\text{int}}$ . To see what is special about interior amplitudes note that while we defined  $\rho(\mathfrak{w})[S] = 0$  for  $S = \emptyset$ , we have  $\rho_\beta(\mathfrak{w})[\emptyset] \neq 0$  in general! Hence, for general  $\beta$ , the term with  $S_1 = \emptyset$  will contribute an extra “source term” in the identities. However, if  $\beta$  is an interior amplitude then we can drop this term and just sum over shuffles with  $S_1 \neq \emptyset$  to recover the standard  $L_\infty$  relations.

### Remarks:

1. The  $\rho$  and  $T$  operations are compatible:

$$\rho(T(\mathfrak{w})[\mathfrak{w}_1, \dots, \mathfrak{w}_n])(r_1, \dots, r_N) = \sum_{\text{Sh}_n(S)} \epsilon \rho(\mathfrak{w})[\rho(\mathfrak{w}_1)[S_1], \rho(\mathfrak{w}_2)[S_2], \dots, \rho(\mathfrak{w}_n)[S_n]] \quad (4.19)$$

where  $S = \{r_1, \dots, r_N\}$ . This equation is clearly analogous to the  $TT$  associativity relation. In a sense,  $\rho$  behaves as a representation for the algebraic structure defined by  $T$ , hence our terminology.

---

<sup>9</sup>We are being slightly sloppy here about the difference between union and disjoint union. Consider the initially appended  $\beta$ 's as distinct and only identify them after we apply  $\rho(\mathfrak{t}_p)[\tilde{S}_1]$ , etc.

2. The origin of the term “shift the origin” is from the analogy to string theory. Our space  $R^{\text{int}}$  is analogous to the space of closed-string fields, and solutions of the Maurer-Cartan equation are analogous to on-shell backgrounds (at tree level). Now, there is an identity

$$\rho_\beta(S, e^{\beta'}) = \rho(S, e^{\beta+\beta'}) \quad (4.20)$$

which shifts the origin of the space of string fields.

#### 4.1.1 Isomorphisms Of Theories

It is worth giving a careful definition of an isomorphism between two Theories  $\mathcal{T}^{(1)}$  and  $\mathcal{T}^{(2)}$ . First of all, we require that there be a bijection

$$\varphi : \mathbb{V}^{(1)} \rightarrow \mathbb{V}^{(2)} \quad (4.21)$$

so that the weights are mapped into each other. That is, viewing the vacuum weight as a map  $z : \mathbb{V} \rightarrow \mathbb{C}$  we have

$$\varphi^*(z^{(2)}) = z^{(1)} \quad (4.22)$$

It will be convenient to “trivialize”  $\varphi$  so that we identify  $\mathbb{V}^{(1)} = \mathbb{V}^{(2)} = \mathbb{V}$ . Then  $\varphi$  is a bijection of  $\mathbb{V}$  with itself. Because we will discuss successive composition of interfaces from the right it will be convenient to write the action from the *right* so

$$i \mapsto i\varphi \quad (4.23)$$

and the condition on the weights is

$$z_{i\varphi}^{(2)} = z_i^{(1)} \quad \forall i \in \mathbb{V} \quad (4.24)$$

Next, for every distinct pair of vacua  $(i, j)$  we have an isomorphism of graded  $\mathbb{Z}$ -modules:

$$\varphi_{ij} : R_{ij}^{(1)} \rightarrow R_{i\varphi,j\varphi}^{(2)} \quad (4.25)$$

such that

$$(\varphi_{ij} \otimes \varphi_{ji})^*(K_{i\varphi,j\varphi}^{(2)}) = K_{i,j}^{(1)} \quad (4.26)$$

Finally, for any cyclic fan of vacua  $I$  we let  $I\varphi$  be the image cyclic fan of vacua (it is cyclic thanks to (4.24)). Then the  $\varphi_{ij}$  induce an isomorphism  $\varphi_I : R_I^{(1)} \rightarrow R_{I\varphi}^{(2)}$  and we require that

$$\varphi_I(\beta_I^{(1)}) = \beta_{I\varphi}^{(2)} \quad (4.27)$$

These three conditions define an *isomorphism of Theories*.

**Remarks:**

- If  $\varphi^{(12)} : \mathcal{T}^{(1)} \rightarrow \mathcal{T}^{(2)}$  is an isomorphism and  $\varphi^{(23)} : \mathcal{T}^{(2)} \rightarrow \mathcal{T}^{(3)}$  is another isomorphism then  $\varphi^{(12)}\varphi^{(23)}$  is an isomorphism  $\mathcal{T}^{(1)} \rightarrow \mathcal{T}^{(3)}$ .

2. Automorphisms are isomorphisms of a Theory with itself, and these always form a group. Note that a non-identity automorphism must still induce the identity permutation on  $\mathbb{V}$ . For, suppose that  $i\varphi \neq i$  for some  $i$ . Then (4.24) implies  $z_i = z_{i\varphi}$ . But, setting  $j := i\varphi$  we see that this clashes with the condition on vacuum data that  $z_{ij} \neq 0$  for all  $i \neq j$ . The maps  $\varphi_{ij}$  can still be nontrivial so a Theory can still have a nontrivial automorphism group. In the text we make use of some nontrivial isomorphisms which are not automorphisms.

## 4.2 Web Representations And Half-Plane Webs

In Section §5.2 below we will introduce an abstract notion of the Lefschetz thimbles which, in the context of Landau-Ginzburg theory define special branes in the theory associated to each of the vacua. (See Section §11 below.) This motivates the following

**Definition:** Fix a set of vacua  $\mathbb{V}$ . We define *Chan-Paton data* to be an assignment  $i \rightarrow \mathcal{E}_i$  of a graded  $\mathbb{Z}$ -module to each vacuum  $i \in \mathbb{V}$ . The modules  $\mathcal{E}_i$  are often referred to as *Chan-Paton factors*.

Now fix a half-plane  $\mathcal{H}$ . If  $J = \{j_1, \dots, j_n\}$  is a half-plane fan in  $\mathcal{H}$  then we define

$$R_J(\mathcal{E}) := \mathcal{E}_{j_1} \otimes R_{j_1, j_2} \otimes \cdots \otimes R_{j_{n-1}, j_n} \otimes \mathcal{E}_{j_n}^*. \quad (4.28)$$

and the counterpart to (4.5) is

$$R^\partial(\mathcal{E}) := \bigoplus_J R_J(\mathcal{E}) \quad (4.29)$$

where we sum over all half-plane fans in  $\mathcal{H}$ .

We are ready to define the web-representation analogue of  $T(\mathfrak{u})$  defined in (3.32), namely a map

$$\rho(\mathfrak{u}) : TR^\partial(\mathcal{E}) \otimes TR^{\text{int}} \rightarrow TR^\partial(\mathcal{E}) \quad (4.30)$$

graded symmetric on the second tensor factor. As usual, we define the element

$$\rho(\mathfrak{u})[r_1^\partial, \dots, r_m^\partial; r_1, \dots, r_n] \quad (4.31)$$

by contraction. In the equations below we will abbreviate this to  $\rho(\mathfrak{u})[P; S]$  where

$$P = \{r_1^\partial, \dots, r_m^\partial\} \quad S = \{r_1, \dots, r_n\}. \quad (4.32)$$

We define  $\rho(\mathfrak{u})[P; S]$  to be zero unless the following conditions hold:

- The numbers of interior and boundary vertices of  $\mathfrak{u}$  match the number of arguments of either type:  $V_\partial(\mathfrak{u}) = m$  and  $V_i(\mathfrak{u}) = n$ .
- The boundary arguments match in order and type those of the boundary vertices:  $r_a^\partial \in R_{J_{v_a^\partial}(\mathfrak{u})}$  (Recall these are ordered from left to right in the order described in (2.19).).
- We can find an order of the interior vertices  $\mathcal{V}_i(\mathfrak{u}) = \{v_1, \dots, v_n\}$  of  $\mathfrak{u}$  such that they match the order and type of the interior arguments:  $r_a \in R_{I_{v_a}(\mathfrak{u})}$ .

If the above conditions hold, we will simply contract all internal lines with  $K$  and contract the Chan Paton elements of consecutive pairs of  $r_a^\partial$  by the natural pairing  $\mathcal{E}_j \otimes \mathcal{E}_j^* \rightarrow \mathbb{Z}$ . We keep track of signs as before, building an orientation for  $\mathbf{u}$  from the orientation on  $\mathbb{R}^{2V_i(\mathbf{u})+V_\partial(\mathbf{u})}$ , denoting the coordinates of interior vertices as  $x_v, y_v$  and boundary vertices as  $y_\parallel^a$ . The edge vector fields  $\partial_e$  can be built as before, adjusting them so that the boundary vertices remain on the boundary.

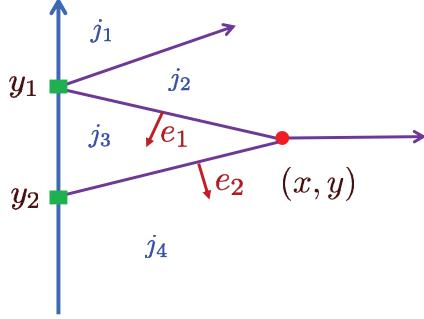
$$\rho(\mathbf{u})[r_1^\partial, \dots, r_m^\partial; r_1, \dots, r_n] = \frac{o(\mathbf{u})}{\left[ \prod_{e \in \mathcal{E}(\mathbf{u})} \partial_e \right] \circ \left[ \prod_{a=1}^m dy_\parallel^a \right] \left[ \prod_{v \in \mathcal{V}_i(\mathbf{u})} dx_v dy_v \right]} \\ \left[ \otimes_{e \in \mathcal{E}(\mathbf{u})} K_e \right] \left[ \prod_{a=1}^m \partial_{\theta_a} \right] \circ \left[ \otimes_{a=1}^m \theta_a r_a^\partial \right] [\otimes_{a=1}^n r_a] \quad (4.33)$$

The ordering of the products over  $dy_\parallel^a$  and  $\partial_{\theta_a}$  follows that specified in (2.19).

In order for the signs to follow as closely as possible the conventions in  $T[\mathbf{u}]$ , we introduced  $m$  auxiliary degree  $-1$  variables  $\theta_a$ , to be contracted with dual  $\partial_{\theta_a}$  to get the final result. The  $\theta_a$  produce useful signs as they are brought across the  $r_a^\partial$  by the Koszul rule. The use of  $\theta_a r_a^\partial$  mimics the use of reduced orientations in the definition of  $T$ . Omitting the  $\theta_a$  auxiliary variables in  $\rho$  would have the same effect as replacing

$$\prod_a o_r(\mathbf{u}_a) \rightarrow \prod_a \partial_{y_\parallel^a} \prod_a o(\mathbf{u}_a) \quad (4.34)$$

in  $T$ , giving rise to somewhat less pleasing sign rules in the various associativity identities.



**Figure 25:** A typical half-plane web. The signs for the contraction are fixed as explained in the example.

**Example:** As an example of how the formalism works consider the half-plane web shown in Figure 25. This half-plane web is taut and hence has a canonical orientation  $o_r(\mathbf{u})$  oriented towards larger webs. Therefore  $o_r(\mathbf{u}) = [dx] = [dy_1] = [-dy_2]$ . (Note that the web gets

larger if we increase  $x$  or  $y_1$  but smaller if we increase  $y_2$ .) Now we can take  $\partial_{\parallel} = -\frac{\partial}{\partial y_1}$  so we get  $o(\mathbf{u}) = [dy_1 dy_2]$ . Similarly,  $\partial_{e_1} \wedge \partial_{e_2} = -\frac{\partial}{\partial y} \wedge \frac{\partial}{\partial x}$ . Thus, the prefactor on the first line of (4.33) works out to

$$\frac{o(\mathbf{u})}{\partial_{e_1} \wedge \partial_{e_2} (-dy_1)(-dy_2)(dxdy)} = \frac{[dy_1 dy_2]}{-[dy_1 dy_2]} = -1 \quad (4.35)$$

Now,  $\rho(\mathbf{u})$  can only be nonzero on sums of vectors of the form  $r_1^{\partial} \otimes r_2^{\partial} \otimes r$  where

$$\begin{aligned} r_1^{\partial} &\in \mathcal{E}_{j_1} \otimes R_{J_1} \otimes \mathcal{E}_{j_3}^* & J_1 &= \{j_1, j_2, j_3\} \\ r_2^{\partial} &\in \mathcal{E}_{j_3} \otimes R_{J_2} \otimes \mathcal{E}_{j_4}^* & J_2 &= \{j_3, j_4\} \\ r &\in R_I & I &= \{j_2, j_4, j_3\} \end{aligned} \quad (4.36)$$

Moreover it suffices to consider monomials of definite degree:

$$\begin{aligned} r_1^{\partial} &= \varepsilon_{j_1} r_{j_1 j_2} r_{j_2 j_3} \varepsilon_{j_3}^* \\ r_2^{\partial} &= \varepsilon'_{j_3} r_{j_3 j_4} \varepsilon_{j_4}^* \\ r &= r_{j_2 j_4} r_{j_4 j_3} r_{j_3 j_2} \end{aligned} \quad (4.37)$$

Therefore

$$\begin{aligned} \rho(\mathbf{u})[r_1^{\partial}, r_2^{\partial}; r] &= -K_{e_1} K_{e_2} \partial_{\theta_1} \partial_{\theta_2} (\theta_1 r_1^{\partial})(\theta_2 r_2^{\partial}) r \\ &= (-1)^{1+|r_1^{\partial}|} K_{j_2 j_3} K_{j_3 j_4} r_1^{\partial} r_2^{\partial} r \end{aligned} \quad (4.38)$$

From here on, we simply apply the Koszul rule. The net result is

$$\rho(\mathbf{u})[r_1^{\partial}, r_2^{\partial}; r] = \kappa \varepsilon_{j_1} \otimes r_{j_1 j_2} \otimes r_{j_2 j_4} \otimes \varepsilon_{j_4}^* \in \mathcal{E}_{j_1} \otimes R_{J_{\infty}} \otimes \mathcal{E}_{j_4}^* \quad (4.39)$$

where  $J_{\infty} = \{j_1, j_2, j_4\}$  and  $\kappa$  is a scalar given by

$$\begin{aligned} \kappa &= (-1)^s \cdot (\varepsilon_{j_3}^* (\varepsilon'_{j_3})) \cdot (K_{j_3 j_4} (r_{j_3 j_4}, r_{j_4 j_3})) \cdot (K_{j_2 j_3} (r_{j_2 j_3}, r_{j_3 j_2})) \\ s &= 1 + |r_1^{\partial}| + |r_{j_2 j_4}|(|r_{j_4 j_3}| + |r_{j_3 j_2}|) + |r_{j_2 j_3}|. \end{aligned} \quad (4.40)$$

With this definition in hand, we can check that  $\rho$  behaves just like  $T$  as far as convolutions are involved. Since the combinatoric structure is the same as for  $T$ , we can focus on the signs. First, we can look at:

$$\rho(\mathbf{u} * \mathbf{w})[P; S] = \sum_{\text{Sh}_2(S)} \epsilon_{S_1, S_2} \rho(\mathbf{u})[P; \rho(\mathbf{w})[S_1], S_2]. \quad (4.41)$$

The orientations in the numerators appear in the same way on both sides of the equation. The  $K$  factors in  $\rho(\mathbf{w})$  on the right hand side are inserted to the right of the  $\theta_a r_a^{\partial}$  factors in  $\rho(\mathbf{u})$  and need to be brought to the left in order to match the left hand side. This reproduces the reduced Koszul rule. We also need to bring the  $\partial_e$  factors in the denominator to the left of the  $dy_{\partial}^a$ , but this cancels against the sign to bring the  $K$  factors to the left of the  $\partial_{\theta_a}$ .

Next, we can look at

$$\rho(\mathbf{u} * \mathbf{u}')[P; S] = \sum_{\text{Sh}_2(S), \text{Pa}_3(P)} \epsilon \rho(\mathbf{u})[P_1, \rho(\mathbf{u}')][P_2; S_1], P_3; S_2]. \quad (4.42)$$

To compare the right hand side to the left hand side, we need to transport the  $K\partial_\theta$  block in  $\rho(\mathbf{u}')$ , together with the  $\theta$  in front of it, to the left of the  $P_1$  arguments  $\theta_a r_a^\partial$  in  $\rho(\mathbf{u})$ . We also need to transport the arguments  $r_a$  of  $S_1$  to the right of the  $P_3$  arguments  $\theta_a r_a^\partial$ . This reproduces the reduced Koszul rule. All denominator manipulations needed to reorganize the  $\partial_e$  and  $dy^a$  give signs which cancel out against the identical manipulations on the  $K$  and  $\partial_{\theta_a}$ .

Plugging in the usual convolution identities for taut elements, we derive the  $LA_\infty$  relation for  $\rho[\mathbf{t}_H]$  analogous to (3.38):

$$\sum_{\text{Sh}_2(S), \text{Pa}_3(P)} \epsilon \rho(\mathbf{t}_H)[P_1, \rho(\mathbf{t}_H)[P_2; S_1], P_3; S_2] + \sum_{\text{Sh}_2(S)} \epsilon \rho(\mathbf{t}_H)[P; \rho(\mathbf{t}_p)[S_1], S_2] = 0. \quad (4.43)$$

The most important consequence of these identities is that if we are given an interior amplitude  $\beta$ , we immediately receive an  $A_\infty$  algebra with operations

$$\rho_\beta(\mathbf{t}_H) : TR^\partial(\mathcal{E}) \rightarrow R^\partial(\mathcal{E}) \quad (4.44)$$

defined by

$$\rho_\beta(\mathbf{t}_H)[r_1^\partial, \dots, r_n^\partial] := \rho(\mathbf{t}_H)[r_1^\partial, \dots, r_n^\partial; e^\beta] \quad (4.45)$$

This is the main object of interest for us. A useful point of view on this derivation is that because  $\rho(\mathbf{t}_p)[e^\beta] = 0$ , any convolution of the form  $\mathbf{u} * \mathbf{t}_p$  will give zero when inserted into  $\rho_\beta$ : applying the convolution identities to  $e^\beta$  we get

$$\rho(\mathbf{u} * \mathbf{t}_p)[P, e^\beta] = \rho(\mathbf{u})[P; \rho(\mathbf{t}_p)[e^\beta], e^\beta] \quad (4.46)$$

We are ready for the the half-plane analog of the interior amplitude:

### Definition

a.) A *boundary amplitude* in a Theory  $\mathcal{T}$  is an element  $\mathcal{B} \in R^\partial(\mathcal{E})$  of degree +1 which solves the Maurer-Cartan equations

$$\sum_{n=1}^{\infty} \rho_\beta(\mathbf{t}_H)[\mathcal{B}^{\otimes n}] = 0. \quad (4.47)$$

b.) A *Brane* in a Theory  $\mathcal{T}$  is a pair  $\mathfrak{B} = (\mathcal{E}, \mathcal{B})$  of Chan-Paton data  $\mathcal{E}$ , together with a compatible boundary amplitude  $\mathcal{B}$ .<sup>10</sup>

---

<sup>10</sup>We will often simply refer to a Brane  $\mathfrak{B}$  by its boundary amplitude  $\mathcal{B}$  when the Chan-Paton data are understood.

We remark that equation (4.47) is a sum of elements of degree 2. Note that we can also define formally<sup>11</sup>

$$\frac{1}{1-\mathcal{B}} = \sum_{n=0}^{\infty} \mathcal{B}^{\otimes n} \quad (4.48)$$

and write the equation as

$$\rho_{\beta}(\mathbf{t}_{\mathcal{H}})\left[\frac{1}{1-\mathcal{B}}\right] = 0. \quad (4.49)$$

### Remarks:

1. In conformal field theory the term “brane” is often used for conformally invariant boundary conditions consistent with a given conformal field theory  $\mathcal{C}$ . These branes form a category. In our context we think of a boundary amplitude as a boundary condition and indeed in the context of Landau-Ginzburg theories, as described in Sections §§11-14.7 below, we will see that boundary conditions indeed provide a boundary amplitude. We will see that, for a fixed Theory  $\mathcal{T}$ , the boundary amplitudes, or equivalently the different Branes, also form a category.
2. The higher operations

$$\rho_{\beta}(\mathbf{t}_{\mathcal{H}})[P; S] := \rho(\mathbf{t}_{\mathcal{H}})[P; S, e^{\beta}] \quad (4.50)$$

still satisfy  $LA_{\infty}$  relations. They will not play a further role for us.

3. As for  $\beta$ , we can use  $\mathcal{B}$  to “shift the origin” in the  $A_{\infty}$  algebra. The operations  $\rho_{\beta}^{\mathcal{B}}(\mathbf{t}_{\mathcal{H}}) : TR^{\partial}(\mathcal{E}) \rightarrow R^{\partial}(\mathcal{E})$  defined by

$$\rho_{\beta}^{\mathcal{B}}(\mathbf{t}_{\mathcal{H}})[r_1^{\partial}, \dots, r_n^{\partial}] = \rho_{\beta}(\mathbf{t}_{\mathcal{H}})\left[\frac{1}{1-\mathcal{B}}, r_1^{\partial}, \frac{1}{1-\mathcal{B}}, \dots, \frac{1}{1-\mathcal{B}}, r_n^{\partial}, \frac{1}{1-\mathcal{B}}\right] \quad (4.51)$$

again satisfy the  $A_{\infty}$ -relations if  $\mathcal{B}$  is a boundary amplitude. The proof is similar to that of (4.17). We will identify this  $A_{\infty}$  algebra in §5.2 with an  $A_{\infty}$  algebra of endomorphisms  $\text{Hom}(\mathcal{B}, \mathcal{B})$  (see (5.17)).

### 4.3 Web Representations And Strip-Webs

Now we will explore what implications a representation of webs has when combined with strip webs. Suppose we are given a web representation  $\mathcal{R} = (\{R_{ij}\}, K)$  and Chan-Paton factors  $\mathcal{E}_{L,i}$  and  $\mathcal{E}_{R,i}$  for the left and right boundaries of the strip. We will denote the fans for the left boundary as  $J$  and the fans for the right boundary as  $\tilde{J}$ , with corresponding spaces  $R_J(\mathcal{E}_L)$  and  $R_{\tilde{J}}(\mathcal{E}_R)$ . The direct sum over positive- and negative- half-plane fans with these Chan-Paton spaces will be denoted as  $R_L^{\partial}(\mathcal{E}_L)$  and  $R_R^{\partial}(\mathcal{E}_R)$ , respectively.

**Definition:** We define the space of *approximate ground states* to be

$$\mathcal{E}_{LR} := \bigoplus_{i \in \mathbb{V}} \mathcal{E}_{L,i} \otimes \mathcal{E}_{R,i}^* \quad (4.52)$$

---

<sup>11</sup>The reader is cautioned about a possible notational confusion. Late on, we will introduce an identity element **Id**. Given a multilinear function  $f$ , the first term in the expansion of some  $f(X, \frac{1}{1-\mathcal{B}}, Y)$  is  $f(X, Y)$ , not  $f(X, \mathbf{Id}, Y)$

and a typical element is denoted by  $g$ .

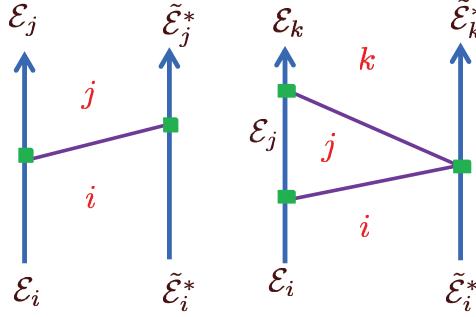
Given a strip web  $\mathfrak{s}$  we plan to define an operation

$$\rho[\mathfrak{s}] : TR_L^\partial(\mathcal{E}_L) \otimes TR^{\text{int}} \otimes \mathcal{E}_{LR} \otimes TR_R^\partial(\mathcal{E}_R) \rightarrow \mathcal{E}_{LR} \quad (4.53)$$

As usual, we take this to be zero unless all the arguments are compatible with the appropriate vertices of the strip web and defined by a familiar formula otherwise:

$$\begin{aligned} \rho(\mathfrak{s})[r_1^\partial, \dots, r_m^\partial; r_1, \dots, r_n; g; \tilde{r}_1^\partial, \dots, \tilde{r}_s^\partial] = \\ o(\mathfrak{u}) \\ \overline{\left[ \prod_{e \in \mathcal{E}(\mathfrak{s})} \partial_e \right] \circ \left[ \prod_{a=1}^m dy_\parallel^a \right] \left[ \prod_{v \in \mathcal{V}_i(\mathfrak{s})} dx_v dy_v \right] \left[ \prod_{a=1}^s d\tilde{y}_\parallel^a \right]} \\ \left[ \otimes_{e \in \mathcal{E}(\mathfrak{s})} K_e \right] \left[ \prod_{a=1}^m \partial_{\theta_a} \right] \left[ \prod_{a=1}^s \partial_{\tilde{\theta}_a} \right] \circ \left[ \otimes_{a=1}^m \theta_a r_a^\partial \right] \left[ \otimes_{a=1}^n r_a \right] \otimes g \left[ \otimes_{a=1}^s \tilde{\theta}_a \tilde{r}_a^\partial \right] \end{aligned} \quad (4.54)$$

Recall that, reading from left to right the  $r_j^\partial$  are inserted on the boundary in order of decreasing  $y$  while the  $\tilde{r}_j$  are inserted in order of increasing time.



**Figure 26:** Strip webs whose contractions are described in the text.

**Example:** The contraction associated with the strip-web on the left in Figure 26 maps

$$R_{ji}(\mathcal{E}) \otimes (\mathcal{E}_i \otimes \tilde{\mathcal{E}}_i^*) \otimes R_{ij}(\tilde{\mathcal{E}}) \rightarrow \mathcal{E}_j \otimes \tilde{\mathcal{E}}_j^* \quad (4.55)$$

It operates on a typical primitive tensor via

$$(v_j \otimes r_{ji} \otimes v_i^*) \otimes (v'_i \otimes \tilde{v}_i^*) \otimes (\tilde{v}_i \otimes \tilde{r}_{ij} \otimes \tilde{v}_j^*) \mapsto \pm (v_i^* \cdot v'_i)(\tilde{v}_i^* \cdot \tilde{v}_i) K(r_{ji}, \tilde{r}_{ji}) v_j \otimes \tilde{v}_j^* \quad (4.56)$$

where the superscript \* indicates a vector is in  $\mathcal{E}^*$  and the sign is determined by the Koszul rule. Similarly, the strip-web on the right in Figure 26 maps

$$\left( R_{kj}(\mathcal{E}) \otimes R_{ji}(\mathcal{E}) \right) \otimes (\mathcal{E}_i \otimes \tilde{\mathcal{E}}_i^*) \otimes R_{ijk}(\tilde{\mathcal{E}}) \rightarrow \mathcal{E}_k \otimes \tilde{\mathcal{E}}_k^*. \quad (4.57)$$

It operates on a typical primitive tensor via

$$(v_k \otimes r_{kj} \otimes v_j^*) \otimes (v'_j \otimes r_{ji} \otimes v_i^{*'}) \otimes (v_i \otimes \tilde{v}_i^*) \otimes (\tilde{v}'_i \otimes \tilde{r}_{ij} \otimes \tilde{r}_{jk} \otimes \tilde{v}_k^*) \xrightarrow{\pm (v_j^* \cdot v'_j)(v_i^{*'} \cdot v_i)(\tilde{v}_i^* \cdot \tilde{v}'_i)K(r_{kj}, \tilde{r}_{jk})K(r_{ji}, \tilde{r}_{ij})v_k \otimes \tilde{v}_k^*} \quad (4.58)$$

where the sign is determined by the Koszul rule.

The full map  $\rho(\mathfrak{s})$  satisfies the same compatibility relations with convolutions as  $T(\mathfrak{s})\circ$ , which combined with the convolution identities for the taut elements tell us that  $\rho(\mathfrak{t}_s)$  satisfies the same lengthy algebraic relations as  $T(\mathfrak{t}_s)\circ$ , described in Section §3.3.

Now let us select a specific choice of interior amplitude  $\beta$ , together with left and right boundary amplitudes  $\mathcal{B}_L$  and  $\mathcal{B}_R$ , respectively, and define an operator  $d_{LR} : \mathcal{E}_{LR} \rightarrow \mathcal{E}_{LR}$  by the equation

$$d_{LR} : g \mapsto \rho(\mathfrak{t}_s)[\frac{1}{1 - \mathcal{B}_L}; e^\beta; g; \frac{1}{1 - \mathcal{B}_R}]. \quad (4.59)$$

The  $\mathfrak{t}_s \circ \mathfrak{t}_s + \dots = 0$  identity (2.40) reduces to the crucial nilpotency

$$d_{LR}^2 = 0 \quad (4.60)$$

essentially because all other terms in the convolution identity give zero when evaluated on  $\frac{1}{1 - \mathcal{B}_L}$ ,  $e^\beta$ , and  $\frac{1}{1 - \mathcal{B}_R}$ . That is,  $d_{LR}$  is a differential on the complex  $\mathcal{E}_{LR}$ .

We believe these considerations amply justify the following

**Definition:** The *complex of ground states* associated to a left and right brane in a given theory is  $(\mathcal{E}_{LR}, d_{LR})$ . The cohomology of this complex gives us the *exact ground states* for this system.

### Remarks:

1. When our formalism is applied to physical theories and physical branes the above definition coincides with the physical notion of groundstates, thus realizing one of the primary objectives of the introductory section §1.
2. We can define operations

$$\rho_{\beta,R}(\mathfrak{t}_s)[P; g] := \rho(\mathfrak{t}_s)[P; e^\beta; g; \frac{1}{1 - \mathcal{B}_R}] \quad (4.61)$$

These endow  $\mathcal{E}_{LR}$  with the structure of an  $A_\infty$  left module for  $\rho_\beta(\mathfrak{t}_L)$ . Similarly,

$$\rho_\beta(\mathfrak{t}_s)[P; g; P'] := \rho(\mathfrak{t}_s)[P; e^\beta; g; P'] \quad (4.62)$$

defines an  $A_\infty$  bimodule structure on  $\mathcal{E}_{LR}$ .

## 4.4 On Degrees, Fermion Numbers And R-Symmetry

Throughout this section, and in later sections, we define the  $R_{ij}$  as graded vector spaces, with a  $\mathbb{Z}$  valued degree which determines the Grassmann parity of objects and allows us to use the Koszul rule in our manipulations.

The integral grading of objects such as the complex of ground states  $\mathcal{E}_{LR}$ , the  $R^{\text{int}}$  and  $R^\partial[\mathcal{E}]$  spaces used in defining the interior and boundary amplitudes should be canonically well-defined, as these objects are expected to be in correspondence to objects in a physical theory which have a well-defined, integral grading given by the conserved R-charge.

On the other hand, the individual  $R_{ij}$  and  $\mathcal{E}_i$  spaces are expected to be in correspondence with objects in a physical theory for which the definition of R-charge is possibly not integral and ambiguous, due to contributions from boundary terms at infinity. Concretely, the R-charge operators  $\hat{q}_{ij}$  and  $\hat{q}_i$  on  $R_{ij}$  and  $\mathcal{E}_i$  are defined up to a constant shift

$$\hat{q}_{ij} \rightarrow \hat{q}_{ij} + f_i - f_j \quad \hat{q}_i \rightarrow \hat{q}_i - f_i \quad (4.63)$$

which leaves the R-charges of  $\mathcal{E}_{LR}$ ,  $R^{\text{int}}$  and  $R^\partial[\mathcal{E}]$  invariant.

When we attempt to associate a web representation to a certain physical theory, we can always select some choice of  $f_i$  such that the R-charges are integral, and can be used to define integral degrees. Such a choice, though, it is not unique, and may break some symmetry of the theory. Different choices are related by shifts with integral  $f_i$ .

As changes in degrees affect the Koszul rules, a shift in degree in the  $R_{ij}$  will lead to sign changes in the definition of  $\rho$ . Furthermore, it may affect the signs in the MC equations for interior and boundary amplitudes. In order for our algebraic structures to behave well under degree shifts, we would like to be able to reabsorb such signs into sign redefinitions in the  $\mathcal{E}_{LR}$ ,  $R^{\text{int}}$  and  $R^\partial[\mathcal{E}]$  spaces and in the  $K$  pairing.

More precisely, we would like to define a new web representation in terms of some  ${}^\vee R_{ij}$  isomorphic, perhaps not canonically, to the degree-shifted  $R_{ij}^{[f_i-f_j]}$ , and CP factors  ${}^\vee \mathcal{E}_i$  isomorphic, perhaps not canonically, to the degree-shifted  $\mathcal{E}_i^{[-f_i]}$  such that we have canonical isomorphisms

$$\mathcal{E}_{LR} \cong {}^\vee \mathcal{E}_{LR} \quad R^{\text{int}} \cong {}^\vee R^{\text{int}} \quad R^\partial[\mathcal{E}] \cong {}^\vee R^\partial[\mathcal{E}] \quad (4.64)$$

which intertwine between  $\rho$  defined by the original representation, and  ${}^\vee \rho$  defined by the new representation and map interior and boundary amplitudes for the original representation to interior and boundary amplitudes for the new representation.

There is a natural, physical way to find such maps, but the story has an unexpected twist: in order to relate naturally  $\rho$  and  ${}^\vee \rho$ , we need to also act with an automorphism of the web algebra, i.e. a linear map  $f_W : \mathfrak{w} \rightarrow \mathfrak{w}$  which commutes with all web convolution operations. Such a map will *not*, in general, preserve the taut element and thus will *not* map interior amplitudes to interior amplitudes, except in some special cases we will describe below, unless we generalized the notion of taut element and interior amplitude slightly.

Lets first describe our degree-shift maps. The maps will act as  $\pm 1$  on each summand  $\mathcal{E}_i \otimes \mathcal{E}_i^*$ ,  $R_I$ ,  $R_J[\mathcal{E}]$ . Consider some one-dimensional graded vector spaces  $V_i$  of degree  $f_i$

and their duals  $V_i^*$ , with a canonical isomorphism  $V_i^* \otimes V_i \cong \mathbb{Z}$ . Define

$${}^\vee R_{ij} = V_i \otimes R_{ij} \otimes V_j^* \quad {}^\vee \mathcal{E}_i = \mathcal{E}_i \otimes V_i^* \quad (4.65)$$

We can focus on  $R^{\text{int}}$  and the plane web representation. The same analysis holds for half-plane and strip web representations. Consider the  ${}^\vee R_I$ . We can apply the canonical isomorphism  $V_i^* \otimes V_i \cong \mathbb{Z}$  to relate canonically

$${}^\vee R_{i_1 \dots i_n} \cong V_{i_1} \otimes R_{i_1 \dots i_n} \otimes V_{i_1}^* \quad (4.66)$$

We can then define a canonical isomorphism  ${}^\vee R_{i_1 \dots i_n} \cong R_{i_1 \dots i_n}$  by Koszul-commuting  $V_{i_1}$  all the way to the right and applying the canonical isomorphism. It is easy to see that such canonical isomorphism makes a neat commutative diagram with the isomorphisms  $R_{i_1 i_2 \dots i_n} \cong R_{i_2 \dots i_n i_1}$  and  ${}^\vee R_{i_1 i_2 \dots i_n} \cong {}^\vee R_{i_2 \dots i_n i_1}$  defined in 4.1.

We should define  ${}^\vee K$  as well. Of course,  ${}^\vee R_{ij} \otimes {}^\vee R_{ji}$  is canonically isomorphic to  $V_i \otimes R_{ij} \otimes R_{ji} \otimes V_i^*$ . As the degrees of the middle factors add up to 1, there is no sign to pay to bring  $V_i$  all the way to the right and apply the canonical isomorphism again to  $R_{ij} \otimes R_{ji}$ . Thus we can take  ${}^\vee K$  to coincide with  $K$  up to this canonical isomorphism.

Lets compose  ${}^\vee \rho$  with the canonical isomorphisms: we take the arguments  $r_a$  in  $R_{I_a}$ , map them canonically to elements in  ${}^\vee R_{I_a}$  and do our contractions with  ${}^\vee K$ , which means we contract the  $R_{ij}$  elements with  $K$  and the  $V_i, V_i^*$  pairwise according to the same pattern.

Effectively, the only difference between  ${}^\vee \rho$  and  $\rho$  is the composition of a bunch of canonical “pair creation” maps  $\mathbb{Z} \rightarrow V_i^* \otimes V_i$  and “annihilation” maps  $V_i^* \otimes V_i \cong \mathbb{Z}$ , along a pattern dictated by the topology of the web. It is easy to see that the chain of contractions produces a loop for every internal face of the web. Thus  ${}^\vee \rho$  and  $\rho$  differ by a factor of  $\prod_{i \in \text{faces}[\mathfrak{w}]} (-1)^{f_i}$ .

We can absorb the difference into a linear map

$$\mathfrak{w} \rightarrow f[\mathfrak{w}] \mathfrak{w} \equiv \left[ \prod_{i \in \text{faces}[\mathfrak{w}]} (-1)^{f_i} \right] \mathfrak{w} \quad (4.67)$$

Thus the web representation transforms canonically under the degree shift combined with the action of this linear map on the space of webs.

The sign  $f[\mathfrak{w}]$  has a striking property:

$$f[\mathfrak{w}_1 *_v \mathfrak{w}_2] = f[\mathfrak{w}_1] f[\mathfrak{w}_2] \quad (4.68)$$

as convolution does not create new internal faces. Thus the map commutes with all web algebraic operations. We can call a collections of numbers with such property a *cocycle* for the web algebra.

It should be clear that the taut element  $\mathfrak{t}$  is not invariant under twisting by a general cocycle  $\sigma[\mathfrak{w}]$ : <sup>12</sup>

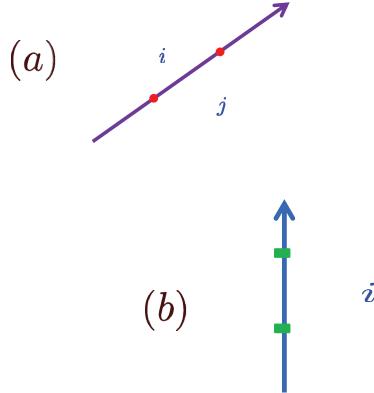
$$\mathfrak{t} \rightarrow \mathfrak{t}_\sigma = \sum_{\text{taut}\mathfrak{w}} \sigma[\mathfrak{w}] \mathfrak{w}. \quad (4.69)$$

---

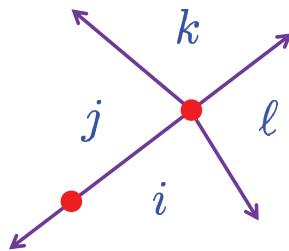
<sup>12</sup>A notable exception is a case of vacuum weights which form a convex polygon: the special cocycles are trivial because there are no internal faces. Half-plane and strip taut webs may have internal faces bounded by one of the boundaries of the space, but the extra sign can be reabsorbed in a re-definition of  $R^\partial[\mathcal{E}]$ . This will be important in later examples

On the other hand, the twisted taut element  $t_\sigma$  is still nilpotent, and we could extend our definition of theory by replacing  $t$  with  $t_\sigma$  in our definition of interior amplitudes, etc. Although we will suppress this possibility in the remainder of the paper, it is likely relevant to concrete applications.

Our final statement is that degree shifts  $f_i$  in the  $R_{ij}$  relate canonically a theory associated to a cocycle  $\sigma$  and a theory associated to a cocycle  $\sigma f$ .



**Figure 27:** In figure (a) we show an extended taut planar web. The contribution of this web to the equation for an interior amplitude shows that such an (extended) interior amplitude can be used to define a differential on  $R_{ij}$ . Similarly, in figure (b) we show a taut extended half-plane web. Its contribution to the Maurer-Cartan equation for the corresponding  $A_\infty$  algebra shows that a component of the (extended) boundary amplitude defines a differential on  $\mathcal{E}_i$ .



**Figure 28:** A bivalent vertex can be added to any leg of any vertex to produce a taut extended web, as shown here.

## 4.5 Representations Of Extended Webs

We can extend the definition of web representations to extended webs. For plane webs, we have new fans available, with two vacua only, and associated vector spaces

$$R_{(ij)} = R_{ij} \otimes R_{ji} \quad (4.70)$$

The interior amplitude includes now a component  $\beta_{ij}$  in  $R_{(ij)}$ . We can use  $K$  to “raise an index” of  $\beta_{ij}$  to define a degree 1 map

$$Q_{ij} : R_{ij} \rightarrow R_{ij} \quad (4.71)$$

by

$$Q_{ij}(r_{ij}) := (1 \otimes K_{23})(\beta_{ij} \otimes r_{ij}). \quad (4.72)$$

where the subscript 23 means that  $K$  is contracting the second and third factors in  $R_{ij} \otimes R_{ji} \otimes R_{ij}$ . The equation satisfied by  $\beta$  implies that  $Q_{ij}$  is a nilpotent differential, making  $R_{ij}$  into a complex. This is illustrated in Figure 27(a).<sup>13</sup> As the two-valent interior vertex only appears in taut plane webs with two vertices, the equations for  $\beta$  differ from the standard case only by terms where some  $Q_{ij}$  acts on an external  $ij$  leg of  $\beta$ . The relevant kinds of taut webs are illustrated in Figure 28.

For half-plane extended webs, we have a single-vacuum fan available, and associated vector spaces

$$R_i(\mathcal{E}) := \mathcal{E}_i \otimes \mathcal{E}_i^* \quad (4.73)$$

associated to a half-plane “web” consisting of one vertex on the boundary, with no ingoing lines. When working with extended webs the definition of  $R^\partial(\mathcal{E})$  in (4.29) now reads

$$R^\partial(\mathcal{E}) = \bigoplus_i \mathcal{E}_i \otimes \mathcal{E}_i^* \oplus \bigoplus_{z_{ij} \in \mathcal{H}} (\mathcal{E}_i \otimes R_{ij} \otimes \mathcal{E}_j^*) \oplus \dots \quad (4.74)$$

When speaking of elements of  $R^\partial(\mathcal{E})$  we refer to the new summand  $\bigoplus_i \mathcal{E}_i \otimes \mathcal{E}_i^*$  in the definition of  $R^\partial(\mathcal{E})$  as the *scalar part*. Thus the boundary amplitude includes now a scalar part  $Q_i$  in  $R_i(\mathcal{E})$ . The Maurer-Cartan equation now includes a taut web with two zero-valent vertices and this contribution requires the scalar part  $Q_i$  to be a differential on  $\mathcal{E}_i$ , making the Chan-Paton factors into a complex. See Figure 27(b). Moreover, the zero-valent boundary vertices only appear in taut half-plane webs with two vertices, so the equations for  $Q_i + \mathcal{B}$  differ from the equations for  $\mathcal{B}$  with unextended webs only by an anti-commutator between some  $Q_i$  and  $\mathcal{B}$ .

It is also interesting to observe that we can define an element  $\mathbf{Id}_i$  as the canonical identity element in  $R_i$ . Then we set

$$\mathbf{Id} := \bigoplus_i \mathbf{Id}_i. \quad (4.75)$$

---

<sup>13</sup>To give a little more detail: The interior amplitude identity says that  $K_{23}(\beta_{ij} \otimes \beta_{ij}) = 0$ , where again the subscript 23 indicates which factors the  $K$  acts upon. Then, using similar notation, to check  $Q_{ij}^2 = 0$  we need to verify  $K_{23}(\beta_{ij} \otimes K_{45}(\beta_{ij} \otimes r_{ij})) = 0$ . Since  $K_{23}$  and  $K_{45}$  act on different spaces and hence (anti)commute we can first contract  $K_{23}(\beta_{ij} \otimes \beta_{ij})$  to get zero.

The **Id** element behaves as a graded identity for the  $A_\infty$  algebra  $\rho_\beta(\mathbf{t}_\mathcal{H})$ , i.e.  $\rho_\beta(\mathbf{t}_\mathcal{H})[\mathbf{Id}] = 0$ , (since there is no taut web with a single boundary vertex) while, using the conventions of (4.33),

$$\rho_\beta(\mathbf{t}_\mathcal{H})[\mathbf{Id}, r] = r \quad \rho_\beta(\mathbf{t}_\mathcal{H})[r, \mathbf{Id}] = (-1)^{|r|} r, \quad (4.76)$$

while  $\rho_\beta(\mathbf{t}_\mathcal{H})[P_1, \mathbf{Id}, P_2] = 0$  if both  $P_1, P_2$  are nonempty (simply because there are no taut webs of the appropriate kind). This feature alone can make extended webs useful.

For extended strip webs, the new vertices only appear in simple taut (=rigid) webs consisting of a single zero-valent vertex on either boundary. Thus we should simply add

$$\oplus_{i \in \mathbb{V}} [Q_{L,i} \otimes 1 + 1 \otimes Q_{R,i}] \quad (4.77)$$

to the differential  $d_{L,R}$ .

#### 4.6 A Useful Set Of Examples With Cyclic Vacuum Weights

We will now describe an infinite family of non-trivial examples of Theories and Branes which will be used again in Sections §5.7 and §7.10 to illustrate our formal constructions. As explained in Section §4.6.4, they are also of physical interest.

Fix a positive integer  $N$  and consider the vacuum data

$$\mathbb{V}_\vartheta^N : z_k = e^{-i\vartheta - \frac{2\pi i}{N}k} \quad k = 0, \dots, N-1 \quad (4.78)$$

For convenience, in this section, we choose a small positive  $\vartheta$ , so that  $z_0$  has the most positive real part among all vacua, and a small negative imaginary part. The vacua are a regular sequence of points on the unit circle ordered in the clockwise direction. In particular they form a regular convex polygon and hence there are no webs with loops. Let us enumerate the rigid vertices. Note that if  $i, j, k$  are three successive vacua in a cyclic fan then, by our conventions, they label regions in the clockwise order and hence  $z_{jk}$  rotates counterclockwise through an angle less than  $\pi$  to point in the direction of  $z_{ij}$ . It follows that the corresponding vertices  $z_i, z_j, z_k$  on the unit circle must be clockwise ordered. From this we can conclude that the rigid vertices are in one-one correspondence with increasing (reading left to right) sequences of numbers between 0 and  $N-1$ . In particular, the trivalent vertices are labeled by triples of vacua with  $0 \leq i < j < k \leq N-1$ .

In what follows we will consider two examples of web representations  $\mathcal{R}$ . We will analyze the resulting  $L_\infty$  MC equation and, for a specific choice of interior amplitude  $\beta$  we will analyze the  $A_\infty$  MC equations and Branes in these Theories. While the development is a purely formal illustration of the mathematical constructions developed above, the two classes of models are meant to correspond to two physical models, as explained in Section §4.6.4 below.

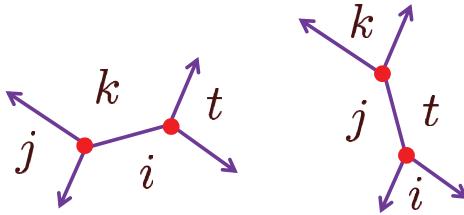
##### 4.6.1 The Theories $\mathcal{T}_\vartheta^N$

Our first class of Theories, denoted  $\mathcal{T}_\vartheta^N$  have web representations such that  $R_{ij}$  is one-dimensional space, with degree (or “fermion number”) 0 or 1:

$$\begin{aligned} R_{ij} &= \mathbb{Z}^{[1]} & i < j \\ R_{ij} &= \mathbb{Z} & i > j \end{aligned} \quad (4.79)$$

Moreover, we take  $K$  to be the natural degree  $-1$  map  $K : \mathbb{Z}^{[1]} \otimes \mathbb{Z} \rightarrow \mathbb{Z}$  given by multiplication.<sup>14</sup>

This restriction on the degree has a neat consequence: the degree of  $R_I$  equals  $|I| - 1$ , where  $|I|$  is the number of vacua in the fan  $I$ . Thus the interior amplitude, which must have degree 2, is concentrated on trivalent vertices only. We can therefore label the independent components of the interior amplitude by  $\beta_{ijk} \in R_{ij} \otimes R_{jk} \otimes R_{ki}$  with  $i < j < k$ . To complete the definition of the theory we must choose a specific interior amplitude. Therefore, let us examine the  $L_\infty$  MC equation.



**Figure 29:** The two terms in the component of the  $L_\infty$  equations for  $i < j < k < t$ .

The only taut webs with only trivalent vertices have four vacua at infinity. For each increasing sequence of four vacua,  $i < j < k < t$ , there are two taut webs corresponding to the two ways to resolve the 4-valent vertex as shown in Figure 29. Therefore the  $L_\infty$  MC equation is a collection of separate equations, one for each such increasing sequence, of the form

$$\rho(\mathbf{t}_p)[\beta_{ijk}, \beta_{ikt}] + \rho(\mathbf{t}_p)[\beta_{ijt}, \beta_{jkt}] = K_{ik} \circ (\beta_{ijk} \otimes \beta_{ikt}) + K_{jt} \circ (\beta_{ijt} \otimes \beta_{jkt}) = 0 \quad (4.80)$$

where  $K_{ik}$  is the contraction of the  $R_{ik} \otimes R_{ki}$  factors and so forth. We used the fact that for a canonically oriented taut web  $\mathbf{w}$  with two vertices,

$$\partial_e \circ (dx_1 dy_1 dx_2 dy_2) = dx_1 dy_2 d(y_1 - y_2) = o(\mathbf{w}) \quad (4.81)$$

to deal with the orientation ratio in  $\rho$ . (See Section §4.1 for the definition of  $\partial_e$ .)

In order to compute the relative signs in the two terms of the MC equation, we remember that

$$\beta_{ijk} \otimes \beta_{ikt} \in R_{ij} \otimes R_{jk} \otimes R_{ki} \otimes R_{ik} \otimes R_{kt} \otimes R_{ti} \quad (4.82)$$

<sup>14</sup>One could introduce an extra multiplicative factor in the definition of  $K$ . Invertibility over the integers constrains it to be  $\pm 1$ . One could extend it to be a nonzero rational number and tensor the representations over  $\mathbb{Q}$ . When counting the  $\zeta$ -webs of Section §14 it is natural to take the multiplicative factor to be 1.

and thus  $K_{ik}$  has to go through two degree 1 spaces  $R_{ij} \otimes R_{jk}$  to contract  $R_{ki} \otimes R_{ik}$ , while

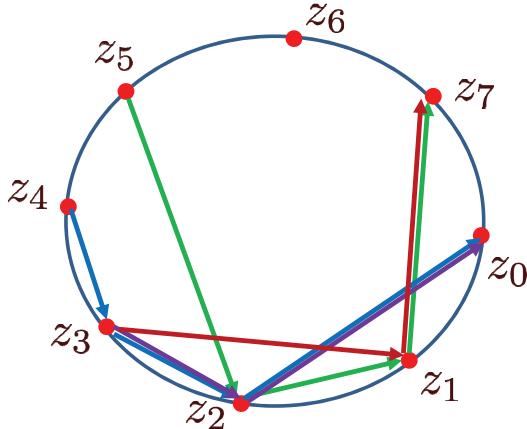
$$\beta_{ijt} \otimes \beta_{jkt} \in R_{ij} \otimes R_{jt} \otimes R_{ti} \otimes R_{jk} \otimes R_{kt} \otimes R_{tj} \quad (4.83)$$

As  $R_{tj}$  has degree 0, we can carry it through the other factors to the right of  $R_{jt}$ , and then  $K_{jt}$  has to go through a single degree 1 factor  $R_{ij}$  in order to contract  $R_{jt} \otimes R_{tj}$ . Therefore, if we identify now  $\beta_{ijk}$  with an integer  $b_{ijk}$  then the  $L_\infty$  MC equations becomes the system of quadratic equations:

$$b_{ijk} b_{ikt} - b_{ijt} b_{jkt} = 0 \quad i < j < k < t \quad (4.84)$$

For example, for  $N = 4$  we have a single equation familiar from the conifold. On the other hand, for large  $N$  there are more equations than variables so it is nontrivial to have solutions at all, let alone integral solutions.<sup>15</sup> Fortunately there is a simple canonical solution given by  $b_{ijk} = 1$  for all  $i < j < k$ . This choice of interior amplitude  $\beta$  defines the Theory we will call  $\mathcal{T}_\vartheta^N$ .

Next, we can look at half-plane webs and Branes. In general the half-plane taut element will depend on the relative choice of  $\vartheta$  and of the slope for the half-plane  $\mathcal{H}$ . We have already chosen  $\vartheta$  to be small and positive and we will now choose  $\mathcal{H}$  to be the positive half-plane.



**Figure 30:** Paths defining positive-half-plane fans for the cyclic weights (4.78). These can be divided into four types according to whether the vacua in  $J = \{i, \dots, j\}$  are down-type vacua  $i, j \in [0, \frac{N}{2})$  or up-type vacua  $i, j \in [\frac{N}{2}, N-1]$ . (We use the fact that  $\vartheta$  is small and positive here.) Reading the paths in the direction of the arrows gives the sequence of vacua encountered reading the fan in the counterclockwise direction, and this corresponds to reading the vacua in  $J = \{i, \dots, j\}$  from right to left. Shown here is the case  $N = 8$ . The vacua  $z_0, \dots, z_3$  are lower vacua. The vacua  $z_4, \dots, z_7$  are upper vacua. The green path is of type  $u \dots u$ . The maroon path is of type  $u \dots d$ . The blue path is of type  $d \dots u$ . Finally the purple path is of type  $d \dots d$ .

Let us now enumerate the possible positive half-plane fans  $J = \{j_1, \dots, j_s\}$ , where we recall that reading from left to right we encounter the vacua in the clockwise direction. The

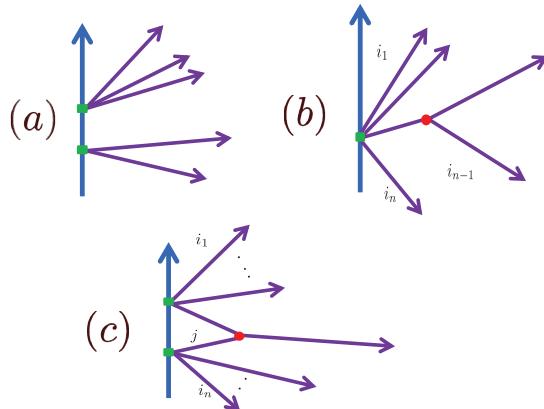
---

<sup>15</sup>From the relation to Landau-Ginzburg theory discussed in Section §14 we expect integral solutions.

presence of the boundary breaks the cyclic symmetry so that when speaking of half-plane webs and Branes it is very useful to distinguish “upper vacua” or “up vacua” from “lower vacua” or “down vacua.” The up vacua have positive imaginary part and the down vacua have negative imaginary part. Thus, the down vacua correspond to  $i$  with  $0 \leq i < \frac{N}{2}$  and the up vacua correspond to  $i$  with  $\frac{N}{2} \leq i < N$ .

In order to enumerate the positive half-plane fans we begin by noting two points: First,  $z_{j_p, j_{p+1}}$  has to have positive real part, and hence  $\text{Re}(z_{j_p}) > \text{Re}(z_{j_{p+1}})$ . Second,  $z_{j_p, j_{p+1}}$  must rotate counterclockwise to point in the direction of  $z_{j_{p-1}, j_p}$ . Now read the list of vacua counterclockwise, i.e. from bottom to top, and from *right to left* in  $J$ . The vacua then define a path of points on the unit circle, and the half-plane fans are enumerated by paths which move to the right so that the successive segments rotate counterclockwise. One way to classify these paths is the following: Denoting generic up- and down-type vacua as  $u$  or  $d$  respectively note that the segments  $dud$  or  $uuu$  rotate clockwise and are excluded. Therefore the half-plane fans  $J$  must be sequences of the type  $\{u_1 \cdots u_2\}$ ,  $\{u_1 \cdots d_2\}$ ,  $\{d_1 \cdots u_2\}$ ,  $\{d_1 \cdots d_2\}$  where in each case the ellipsis  $\cdots$ , if nonempty, is an ordered sequence of down-type vacua. An example is shown in Figure 30.

Since our interior amplitude is supported on trivalent vertices, to write the  $A_\infty$  MC equations we need only list all the taut half-plane webs with trivalent vertices. This can be done, with some effort. Indeed, if we associate to each half plane fan the sequence of edges in the weight plane between the corresponding vacua, then each half-plane web with non-zero representation can be associated to a triangulation of the polygon defined by the sequences of edges for half-fans at boundary vertices together with the half-fan at infinity.



**Figure 31:** The three kinds of taut half-plane webs which can contribute to the  $A_\infty$ -Maurer-Cartan equation in the examples with cyclic weights.

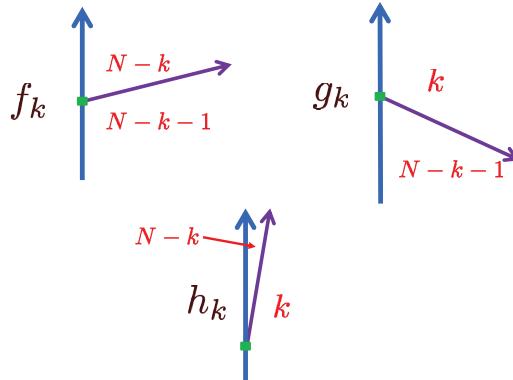
If the half-plane web does not include intermediate upper vacua in the sequence of vacua along the boundary, all we can have is a disconnected taut web (Figure 31(a)) or a web with a single boundary vertex, an edge of which splits at an interior vertex to give a half-fan at infinity with one extra edge than the half fan at the boundary vertex (Figure

31(b)). If the half plane web includes a single intermediate upper vacua, the only relevant taut half-plane web has two half-fans and a single interior vertex at which the last edge of one fan and the first of the next fan join to a new semi-infinite line (Figure 31(c)). Webs with more intermediate upper vacua cannot be taut.

Solving the MC equation is still a rather formidable task, and therefore (with some later applications in mind), we will constrain the problem further, and impose a simple but powerful constraint on the degrees assigned to the (nonzero) Chan-Paton factors  $\mathcal{E}_i$ : we will choose the degrees to be decreasing as we move clockwise around the lower vacua and decreasing as we move counterclockwise around the upper vacua. Moreover, we require a reflection symmetry on the degrees of nonzero Chan-Paton factors. So we take:

$$\begin{aligned} \deg \mathcal{E}_k &= n - k & 0 \leq k < N/2 \\ \deg \mathcal{E}_{N-k-1} &= n - k & 0 \leq k < N/2 \end{aligned} \tag{4.85}$$

for some  $n$ . (The integer  $n$  is not really needed here but cannot be shifted away in the related example (4.117) below.)



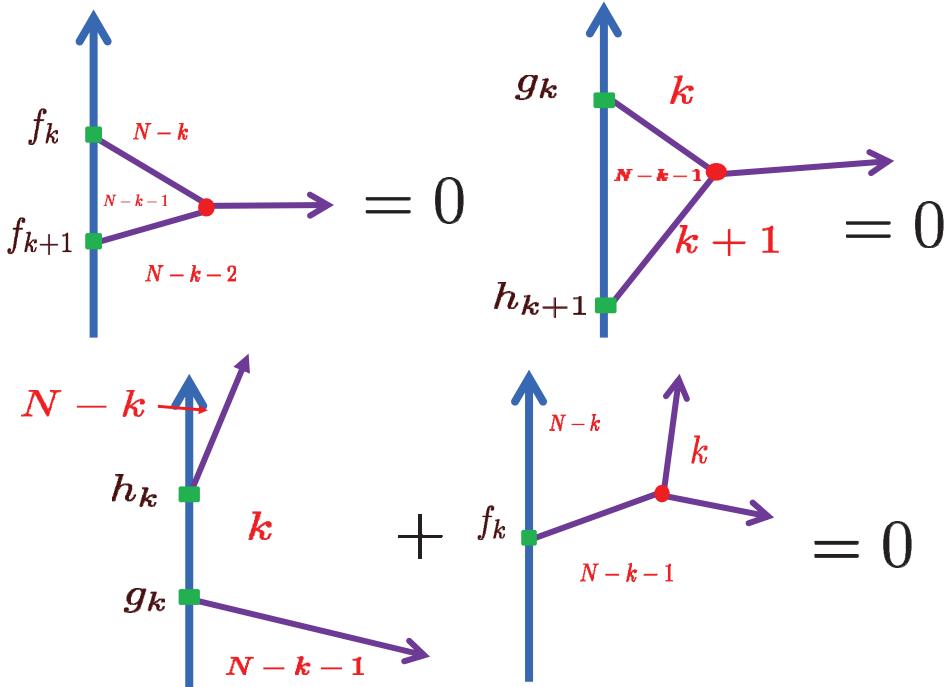
**Figure 32:** Three nontrivial components of a boundary amplitude in the cyclic theories. Note that for small  $\vartheta$  the slope  $z_{N-k,k}$  is nearly vertical,  $z_{k,N-k-1}$  always points downwards, while  $z_{N-k,N-k-1}$  can point upwards or downwards, depending on  $k$ .

The restriction (4.85) strongly constrains which half-plane fans can support a nonzero boundary amplitude  $\mathcal{B}$  (since  $\mathcal{B}$  must have degree 1). It will allow a simple analysis of the MC equation for a boundary amplitude with such Chan-Paton factors. Suppose we consider a component  $\mathcal{B}_{ij} \in R_{\{i\dots j\}}(\mathcal{E})$ . It is useful to look at first at fans which include two vacua only so  $\mathcal{B}_{ij} \in \mathcal{E}_i \otimes R_{ij} \otimes \mathcal{E}_j^*$ . If  $i > j$  then by (4.79)  $R_{ij}$  has degree 0 and hence the degree of  $\mathcal{E}_i \otimes \mathcal{E}_j^*$  must be 1. The only two possibilities are  $i = N - k$  and  $j = k$ , with  $1 \leq k < N/2$  or  $i = N - k$  and  $j = N - k - 1$ , with  $1 \leq k < N/2$ . In both cases  $i$  is  $u$ -type. On the other hand, if  $i < j$  then by (4.79)  $R_{ij}$  has degree 1 and hence the degree of  $\mathcal{E}_i \otimes \mathcal{E}_j^*$  must be zero. Then the only possibility is  $i = k$ ,  $j = N - k - 1$  with  $0 \leq k < N/2$ . In this

case  $i$  is  $d$ -type. It turns out that no half-fans with more than two vacua may contribute to a degree 1 amplitude. For example, if we consider  $\mathcal{E}_{N-k} \otimes R_{N-k,\ell} \otimes R_{\ell,k} \otimes \mathcal{E}_k^*$  with  $k < \ell < N - k$  then  $\{N - k, \ell, k\}$  is never a valid half-plane fan. The reason is that the real part of  $z_{N-k,k}$  is  $\sin(\vartheta) \sin(2\pi k/N)$  and is arbitrarily small, and will be smaller than  $\text{Re}(z_{\ell,k})$  whenever  $\text{Re}(z_{\ell,k}) > 0$ . We conclude that the only potentially nonzero components of a boundary amplitude are of the form:

$$\begin{aligned}\mathcal{B}_{N-k,N-k-1} &\leftrightarrow f_k \in \text{Hom}(\mathcal{E}_{N-k-1}, \mathcal{E}_{N-k}) & 1 \leq k < N/2 \\ \mathcal{B}_{k,N-k-1} &\leftrightarrow g_k \in \text{Hom}(\mathcal{E}_{N-k-1}, \mathcal{E}_k) & 0 \leq k < N/2 \\ \mathcal{B}_{N-k,k} &\leftrightarrow h_k \in \text{Hom}(\mathcal{E}_k, \mathcal{E}_{N-k}) & 1 \leq k < N/2\end{aligned}\quad (4.86)$$

where in the second column we have interpreted the indicated component of  $\mathcal{B}$  in terms of linear transformations  $f_k, g_k, h_k$ . See Figure 32.



**Figure 33:** Three nontrivial equations in the  $A_\infty$  MC equation.

We can now write out the nontrivial components of the  $A_\infty$  Maurer-Cartan equation for  $\mathcal{B}$ . We organize them by the type of the half-plane fan  $J_\infty$ . The  $uu$  component arises from a taut web with a single interior vertex and two boundary vertices as in Figure 31(c). It takes the form

$$\rho(t_H)[\mathcal{B}_{N-k,N-k-1}, \mathcal{B}_{N-k-1,N-k-2}; \beta_{N-k-2,N-k-1,N-k}] = 0 \quad (4.87)$$

and it tells us that  $f_k f_{k+1} = 0$  in the notation of (4.86).

In a similar fashion, the  $dd$  component takes the form

$$\rho(\mathbf{t}_{\mathcal{H}})[\mathcal{B}_{k,N-k-1}, \mathcal{B}_{N-k-1,k+1}; \beta_{k,k+1,N-k-1}] = 0 \quad (4.88)$$

and it tells us that if  $g_k h_{k+1} = 0$  in the notation of (4.86) (where  $k < N/2 - 1$ ).

Finally, the  $udu$  component involves two kinds of webs: a web with two boundary vertices only as in Figure 31(a), and a web with a single boundary vertex and a single interior vertex as in Figure 31(b). Both webs contribute to a term in the MC equation with a half fan of three vacua, which takes the form  $N - k, k, N - k - 1$ . Thus we need to solve

$$\rho(\mathbf{t}_{\mathcal{H}})[\mathcal{B}_{N-k,k}, \mathcal{B}_{k,N-k-1}] + \rho(\mathbf{t}_{\mathcal{H}})[\mathcal{B}_{N-k,N-k-1}; \beta_{k,N-k-1,N-k}] = 0 \quad (4.89)$$

which tells us that  $f_k = h_k g_k$ .

Note that the  $f_k f_{k+1} = 0$  constraint follows from the other constraints. Thus the general solution of the  $A_\infty$  MC equation subject to the constraint (4.85) is given by a set of linear transformations  $\{f_k, g_k, h_k\}$  as in (4.86) subject to the two conditions  $g_k h_{k+1} = 0$  and  $f_k = h_k g_k$  (for values of  $k$  for which this makes sense). These equations are illustrated in Figure 33.

Let us consider two simple types of solutions. The first are the “thimbles”  $\mathfrak{T}_i$  defined by the Chan-Paton factors

$$\mathcal{E}(\mathfrak{T}_i)_j = \delta_{ij} \mathbb{Z} \quad (4.90)$$

with  $f_j = g_j = h_j = 0$ . The second kind are the Branes  $\mathfrak{C}_k$ , with  $1 \leq k < N/2$ . These are defined by taking Chan-Paton spaces

$$\mathcal{E}(\mathfrak{C}_k)_{N-k} = \mathbb{Z}^{[1]} \quad \mathcal{E}(\mathfrak{C}_k)_{N-k-1} = \mathbb{Z} \quad \mathcal{E}(\mathfrak{C}_k)_k = \mathbb{Z} \quad (4.91)$$

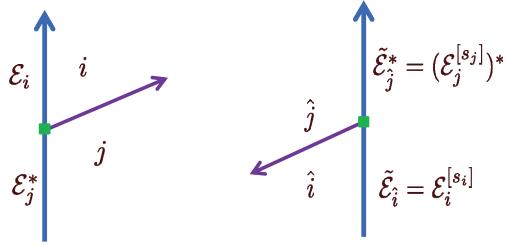
with all other Chan-Paton spaces  $\mathcal{E}(\mathfrak{C}_k)_j$  equal to zero. For the boundary amplitude we take  $f_k, g_k, h_k$  to be multiplication by 1 (and of course  $f_j, g_j, h_j$  vanish for  $j \neq k$ ) and hence  $f_j = h_j g_j$  and  $g_j h_{j+1} = 0$  for all  $j$  is satisfied. The motivation for writing down these branes is that they are generated from rotational interfaces as described at length in Section §7.10 below.

In order to look at the strip we need to define some boundary amplitudes and Branes for the negative half-plane. One convenient way to do this is to use the  $\mathbb{Z}_N$  symmetry of the model and rotate the half-planes by  $\pi$ . We can construct a family of cyclic Theories by simultaneous rotation of all the vacuum weights  $z_i \rightarrow e^{-i\phi} z_i$ . As the edges parallel to  $z_{ij}$  rotate we continue to associate the same spaces  $R_{ij}$  to them, together with the same interior amplitudes, and thus we obtain a family of Theories. A rotation by a multiple of  $2\pi/N$  leaves the set of vacuum weights invariant. Hence there is an isomorphism of the rotated planar theory with the original theory. However, because of the degree assignments in equation (4.79) this isomorphism involves an interesting degree shift on the web representation. If  $z_i \rightarrow \omega^{-d} z_i$  with  $\omega = \exp[2\pi i/N]$  then the isomorphism acts by  $i \rightarrow \hat{i}$  where  $\hat{i} = (i + d)\text{mod}N$  with  $0 \leq \hat{i} < N$  and the rotated web representation  $\Phi_d^*(R_{ij})$  is related to the old one by

$$\Phi_d^*(R_{ij}) = R_{\hat{i}\hat{j}}^{[s_{ij}]} \quad (4.92)$$

with

$$[s_{ij}] = \begin{cases} 0 & i < j \quad \& \quad \hat{i} < \hat{j} \\ 0 & i > j \quad \& \quad \hat{i} > \hat{j} \\ +1 & i < j \quad \& \quad \hat{i} > \hat{j} \\ -1 & i > j \quad \& \quad \hat{i} < \hat{j} \end{cases} \quad (4.93)$$



**Figure 34:** Rotating the theory for  $N$  even by  $\pi$  maps left Branes to right Branes using the above rule.

In general the rotation takes one half-plane theory to another half-plane theory. If  $N$  is even we can use a rotation by  $\omega^{-N/2} = -1$  to take the positive half-plane theory to the negative half-plane theory as in Figure 34. Our rule for mapping Chan-Paton spaces will be that

$$\tilde{\mathcal{E}}_i = \mathcal{E}_i^{[s_i]} \quad (4.94)$$

and the degree shifts are chosen so that there is a degree zero isomorphism

$$\mathcal{E}_i \otimes \Phi_{N/2}^*(R_{ij}) \otimes \mathcal{E}_j^* \cong \tilde{\mathcal{E}}_i \otimes R_{\hat{i}\hat{j}} \otimes \tilde{\mathcal{E}}_j^* \quad (4.95)$$

and hence the degree-shifts on the Chan-Paton spaces are determined, up to an overall shift, by

$$[s_{ij}] = [s_i] - [s_j]. \quad (4.96)$$

The general solution to (4.96) is

$$s_i = \begin{cases} s+1 & 0 \leq i < \frac{N}{2} \\ s & \frac{N}{2} \leq i \leq N-1 \end{cases} \quad (4.97)$$

for some  $s$ . These maps define an isomorphism of theories as in 4.1.1. See also 4.6.3.

In particular, applying this procedure to the Branes  $\mathfrak{C}_k$  produces a collection of Branes for the right boundary,  $\tilde{\mathfrak{C}}_k$ ,  $1 \leq k < N/2$  with

$$\mathcal{E}(\tilde{\mathfrak{C}}_k)_{N-k} = \mathbb{Z}^{[s]} \quad \mathcal{E}(\tilde{\mathfrak{C}}_k)_{k-1} = \mathbb{Z}^{[s-1]} \quad \mathcal{E}(\tilde{\mathfrak{C}}_k)_k = \mathbb{Z}^{[s]} \quad (4.98)$$

and the simplest choice is to take  $s = 1$ . In the above we have renamed the Brane  $\mathfrak{C}_k$  rotated by  $\pi$  to be  $\tilde{\mathfrak{C}}_{\frac{N}{2}-k}$ . The components of the boundary amplitude of  $\tilde{\mathfrak{C}}_k$  are obtained from those of  $\mathfrak{C}_{N/2-k}$  and are  $\pm 1$ .

The strip complex for the Branes  $\mathfrak{C}_k$  and  $\tilde{\mathfrak{C}}_t$  is non-empty only if  $k = t$  or  $k = t - 1$ . In either case, it is a two-dimensional complex, with differential “1”. For example for  $k = t$  we have the complex of approximate ground states

$$\begin{aligned}\mathcal{E}_{LR} &= \bigoplus_i \mathcal{E}(\mathfrak{C}_k)_i \otimes \mathcal{E}(\tilde{\mathfrak{C}}_k)_i^* \\ &\cong \mathbb{Z} \otimes \mathbb{Z}^{[-s]} \oplus \mathbb{Z}^{[1]} \otimes \mathbb{Z}^{[-s]}\end{aligned}\tag{4.99}$$

There is only one taut web which contributes to  $d_{LR}$  given by the amplitude  $h_k$  and hence  $d_{LR}(m, n) = (0, m)$ . The cohomology is therefore zero: there are no exact ground states on the strip between  $\mathfrak{C}_k$  and  $\tilde{\mathfrak{C}}_t$ . In particular, in a physical manifestation of this example, although there are good approximations to supersymmetric groundstates on the interval in fact instanton effects break supersymmetry.

#### 4.6.2 The Theories $\mathcal{T}_\vartheta^{SU(N)}$

There is an interesting, and much richer, variant of the Theories we have described above: we keep the same weights, but define

$$\begin{aligned}R_{ij} &= A_{j-i}^{[1]} \quad i < j \\ R_{ij} &= A_{N+j-i} \quad i > j\end{aligned}\tag{4.100}$$

where  $A_\ell$  is the  $\ell$ -th antisymmetric power of a fundamental representation of  $SU(N)$ .<sup>16</sup> We will show now how to define an  $SU(N)$ -invariant interior amplitude, and thus a family of Theories  $\mathcal{T}_\vartheta^{SU(N)}$  whose algebraic structures will be  $SU(N)$  covariant. We choose an orientation on the fundamental representation  $A_1$ , or equivalently, a nonzero vector in  $A_N$ , denoted by  $\text{vol}$ . It will also be convenient in some formulae to choose an oriented orthonormal basis  $\{e_1, \dots, e_N\}$  and, for multi-indices  $S = \{a_1 < a_2 < \dots < a_\ell\}$  the corresponding vector  $e_S = e_{a_1} \wedge e_{a_2} \wedge \dots \wedge e_{a_\ell}$ . These vectors form an orthonormal basis for  $A_\ell$ .

The fan spaces  $R_I$  are the product of  $SU(N)$  representations whose Young tableaux have a total of  $N$  boxes, and thus contain a *unique*  $SU(N)$  invariant line: Taking the outer product of the antisymmetric tensors from each of the factors we antisymmetrize on all the indices.

The pairing  $K_{ij} : R_{ij} \otimes R_{ji} \rightarrow \mathbb{Z}$  is uniquely determined by  $SU(N)$  invariance to be

$$K_{ij}(v_1 \otimes v_2) = \kappa_{ij} \frac{v_1 \wedge v_2}{\text{vol}}\tag{4.101}$$

where  $\kappa_{ij}$  is a nonvanishing normalization factor. For simplicity we will take the  $\kappa_{ij}$  to be given by a single factor  $\kappa$  for  $i < j$ . Then, by the natural isomorphism  $R_{ij} \otimes R_{ji} \cong R_{ji} \otimes R_{ij}$   $\kappa_{ji} = \pm \kappa$  with a sign determined by  $i, j$ .

---

<sup>16</sup>If we want to work over  $\mathbb{Z}$  or  $\mathbb{Q}$  we should replace  $SU(N)$  with  $SL(N, \mathbb{Z})$  or  $SL(N, \mathbb{Q})$ . We will informally write  $SU(N)$ , since this is what appears in the main physical applications.

Since the degree assignments of (4.100) are the same as those of (4.79) the interior amplitude only has components on trivalent vertices. We will assume our interior amplitudes are valued in the invariant line in  $R_I$  and therefore, for  $i < j < k$  the amplitude must be of the form

$$\beta_{ijk} = b_{ijk} \sum_{\text{Sh}_3^{ijk}} \frac{e_{S_1} e_{S_2} e_{S_3}}{\text{vol}} e_{S_1} \otimes e_{S_2} \otimes e_{S_3} \quad (4.102)$$

where  $b_{ijk}$  is a scalar, exterior multiplication is understood in the first factor in the sum on the RHS, and  $\text{Sh}_3^{ijk}$  is the set of 3-shuffles of  $S = \{1, \dots, N\}$  such that  $|S_1| = j - i$  and  $|S_2| = k - j$ .

Now let us write the  $L_\infty$  MC equation. When we compute  $K_{ik}(\beta_{ijk} \otimes \beta_{ikt})$  we apply the contraction to a sum over pairs of 3-shuffles  $S_1 \amalg S_2 \amalg S_3 \in \text{Sh}_3^{ijk}$  and  $S'_1 \amalg S'_2 \amalg S'_3 \in \text{Sh}_3^{ikt}$ . In fact, the contraction turns out to be valued in the  $SU(N)$  invariant line in  $R_{\{i,j,k,t\}}$  because the contraction of three epsilon tensors is proportional to an epsilon tensor. In our notation we have the identity:

$$\frac{e_{S_1} e_{S_2} e_{S_3}}{\text{vol}} \frac{e_{S'_1} e_{S'_2} e_{S'_3}}{\text{vol}} \frac{e_{S'_1} e_{S_3}}{\text{vol}} = \begin{cases} \frac{e_{S_1} e_{S_2} e_{S'_2} e_{S'_3}}{\text{vol}} & S_1 \amalg S_2 \amalg S'_2 \amalg S'_3 \in \text{Sh}_4^{ijkt} \\ 0 & \text{else} \end{cases} \quad (4.103)$$

where  $\text{Sh}_4^{ijkt}$  is a sum over 4-shuffles with lengths  $j - i, k - j, t - k, N + i - t$ . Note that one must be careful to contract  $R_{ik} \otimes R_{ki} \rightarrow \mathbb{Z}$  in that order. It therefore follows that

$$K_{ik}(\beta_{ijk} \otimes \beta_{ikt}) = \kappa b_{ijk} b_{ikt} \sum_{\text{Sh}_4^{ijkt}} \frac{e_{S_1} e_{S_2} e_{S'_2} e_{S'_3}}{\text{vol}} e_{S_1} \otimes e_{S_2} \otimes e_{S'_2} \otimes e_{S'_3} \quad (4.104)$$

A similar result holds for  $K_{jt}(\beta_{ijt} \otimes \beta_{jkt})$ , which turns out to be (when considered as an element of  $R_{ij} \otimes R_{jk} \otimes R_{kt} \otimes R_{ti}$ )

$$K_{jt}(\beta_{ijt} \otimes \beta_{jkt}) = -\kappa b_{ijt} b_{jkt} \sum_{\text{Sh}_4^{ijkt}} \frac{e_{S_1} e_{S'_1} e_{S'_2} e_{S_3}}{\text{vol}} e_{S_1} \otimes e_{S'_1} \otimes e_{S'_2} \otimes e_{S_3}. \quad (4.105)$$

The overall minus sign has the same origin as in the second term of (4.84). Therefore the  $L_\infty$  MC equations are

$$b_{ijk} b_{ikt} - b_{ijt} b_{jkt} = 0 \quad i < j < k < t. \quad (4.106)$$

These are the same equations as before. For general  $N$  they are overdetermined, and once again we take the canonical solution  $b_{ijk} = 1$  to define the Theories  $\mathcal{T}_\vartheta^{SU(N)}$ .

We can impose the same restrictions on the degree of Chan-Paton factors as in (4.85). The same reasoning as before implies that we can interpret the possible nonzero components of the boundary amplitude as linear transformations. We give them the same names as before, but now equation (4.86) is generalized to give a set of three maps

$$\begin{aligned} \mathcal{B}_{N-k, N-k-1} &\leftrightarrow f_k \in \text{Hom}(\mathcal{E}_{N-k-1}, \mathcal{E}_{N-k} \otimes A_{N-1}) & 1 \leq k < N/2 \\ \mathcal{B}_{k, N-k-1} &\leftrightarrow g_k \in \text{Hom}(\mathcal{E}_{N-k-1}, \mathcal{E}_k \otimes A_{N-2k-1}^{[1]}) & 0 \leq k < N/2 \\ \mathcal{B}_{N-k, k} &\leftrightarrow h_k \in \text{Hom}(\mathcal{E}_k, \mathcal{E}_{N-k} \otimes A_{2k}) & 1 \leq k < N/2 \end{aligned} \quad (4.107)$$

There are two independent MC equations. The first (see the northeast corner of Figure 33) says that

$$\begin{aligned}
K_{26} K_{35} (1 \otimes \beta_{k,k+1,N-k-1}) g_k h_{k+1} : \mathcal{E}_{k+1} &\rightarrow \mathcal{E}_{N-k-1} \otimes A_{2k+2} \\
&\rightarrow \mathcal{E}_k \otimes A_{N-2k-1}^{[1]} \otimes A_{2k+2} \\
&\rightarrow \mathcal{E}_k \otimes A_{N-2k-1}^{[1]} \otimes A_{2k+2} \otimes A_1^{[1]} \otimes A_{N-2k-2}^{[1]} \otimes A_{2k+1} \\
&\rightarrow \mathcal{E}_k \otimes A_1
\end{aligned} \tag{4.108}$$

must vanish. Written out in components this means the following. Choose bases  $\{v_{\alpha_k}\}$  for  $\mathcal{E}_k$  and define matrix elements:

$$\begin{aligned}
h_k(v_{\alpha_k}) &:= \sum_{\beta_{N-k}, |I|=2k} h_{\beta_{N-k}, I | \alpha_k} v_{\beta_{N-k}} \otimes e_I \\
g_k(v_{\beta_{N-k-1}}) &:= \sum_{\gamma_k, |I|=N-2k-1} g_{\gamma_k, I | \beta_{N-k-1}} v_{\gamma_k} \otimes e_I
\end{aligned} \tag{4.109}$$

and the MC equation becomes (assuming  $b_{k,k+1,N-k-1} \neq 0$  as is true for the canonical interior amplitude):

$$0 = \sum_{\beta_{N-k-1}, I_1, I_2} \varepsilon_{i, I_1, I_2} g_{\gamma_k, I_2 | \beta_{N-k-1}} h_{\beta_{N-k-1}, I_1 | \alpha_{k+1}} \tag{4.110}$$

The sum is over all multi-indices  $I_1, I_2$  with  $|I_1| = 2k + 2, |I_2| = N - 2k - 1$ . The equation is meant to hold for all  $\gamma_k, \alpha_{k+1}$ , and  $1 \leq i \leq N$ . The factor  $\varepsilon_{i, I_1, I_2} \in \{0, \pm 1\}$  comes from contracting 3 epsilon tensors. The explicit formula is

$$\varepsilon_{i, I_1, I_2} := \left( \frac{e_i e_{I'_2} e_{I'_1}}{\text{vol}} \right) \varepsilon_{I_2} \varepsilon_{I_1} \tag{4.111}$$

where  $I'$  denotes the complementary multi-index to  $I$  in  $\{1, \dots, N\}$  and  $\varepsilon_I := \frac{e_I \wedge e_{I'}}{\text{vol}}$ .

The second MC equation says that the sum of the two diagrams on the bottom of Figure 33) must vanish. Thus,

$$\begin{aligned}
h_k g_k : \mathcal{E}_{N-k-1} &\rightarrow \mathcal{E}_k \otimes A_{N-2k-1}^{[1]} \\
&\rightarrow \mathcal{E}_{N-k} \otimes A_{2k} \otimes A_{N-2k-1}^{[1]}
\end{aligned} \tag{4.112}$$

plus

$$\begin{aligned}
K_{24} (1 \otimes \beta) f_k : \mathcal{E}_{N-k-1} &\rightarrow \mathcal{E}_{N-k} \otimes A_{N-1} \\
&\rightarrow \mathcal{E}_{N-k} \otimes A_{N-1} \otimes A_{N-2k-1}^{[1]} \otimes A_1^{[1]} \otimes A_{2k} \\
&\rightarrow \mathcal{E}_{N-k} \otimes A_{N-2k-1}^{[1]} \otimes A_{2k}
\end{aligned} \tag{4.113}$$

must vanish. When written out in terms of the matrix elements this means that, for all  $v_{\alpha_{N-k-1}}$  and all  $v_{\beta_{N-k}}$  and all multi-indices  $|I_1| = N - 2k - 1$  and  $|I_2| = 2k$  we must have

$$\xi_{I_1, I_2} f_{\beta_{N-2k}, \widehat{i} | \alpha_{N-k-1}} = \sum_{\gamma_k} h_{\beta_{N-k}, I_2 | \gamma_k} g_{\gamma_k, I_1 | \alpha_{N-k-1}} \tag{4.114}$$

where  $\xi_{I_1, I_2}$  is a constant given by

$$\xi_{I_1, I_2} = (-1)^{n-k+1} \left( \frac{e_{I_1} \wedge e_i \wedge e_{I_2}}{\text{vol}} \right) \left( \frac{e_i \wedge e_{\hat{i}}}{\text{vol}} \right) \kappa b_{k, N-k-1, N-k} \quad (4.115)$$

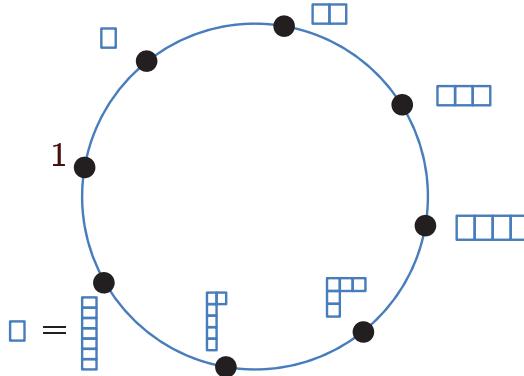
This constant is zero unless  $I_1 \amalg I_2 = \hat{i}$  for some  $i \in \{1, \dots, N\}$ , and  $\hat{i}$  is the multi-index of length  $N - 1$  complementary to  $i$ .

The last MC equation follows from the first two, as in the  $\mathcal{T}_\vartheta^N$  theories.  $f_k$  is a map  $f_k : \mathcal{E}_{N-k-1} \rightarrow \mathcal{E}_{N-k} \otimes A_{N-1}$ . We identify  $A_{N-1} \cong A_1^*$  and then the component of the MC given by the upper left diagram in Figure 33 becomes a map

$$[f_k f_{k+1}] : \mathcal{E}_{N-k-2} \rightarrow \mathcal{E}_{N-k} \otimes A_2^* \quad (4.116)$$

obtained by antisymmetrizing the two  $A_1^*$  indices to an  $A_2^*$  index. The antisymmetrization is due to the contraction with  $\beta_{N-k-2, N-k-1, N-k}$ . The form of  $f_k$  determined above in 4.114 shows that  $[f_k f_{k+1}] = 0$ .

Writing down solutions of these  $A_\infty$  MC equations is considerably less trivial than in the  $\mathcal{T}_\vartheta^N$  theories! It is now natural to impose a requirement of  $SU(N)$  invariance on the boundary amplitude component so that  $\mathcal{B}_{i,j}$  is in the invariant subspace of  $\mathcal{E}_i \otimes R_{ij} \otimes \mathcal{E}_j^*$ . Equivalently, we require that  $f_k, g_k, h_k$  be *intertwiners*.



**Figure 35:** The CP factors for the Brane  $\mathfrak{N}_n$  in the  $\mathcal{T}_\vartheta^{SU(N)}$  theory, for the case  $N = 8$  and  $n = 3$ .

Based on this observation and a certain degree of guesswork using the rotational interfaces discussed in Section 7 below, we have found a neat class of Branes  $\mathfrak{N}_n$  for this model: They are generated from thimbles by using the rotational interfaces. See equation (7.161) et. seq. below. The Chan-Paton factors of  $\mathfrak{N}_n$  are

$$\mathcal{E}(\mathfrak{N}_n)_k = \begin{cases} L_{2k+1, n-k+1}^{[n-k]} & 0 \leq k < N/2 \\ L_{1, n+k+1-N}^{[n+1-N+k]} & \frac{N}{2} - 1 < k \leq N - 1 \end{cases} \quad (4.117)$$

The superscript indicates the degree in which the complex is concentrated, as usual. Here  $L_{\ell,m}$  are the representations of  $SU(N)$  labelled by an (upside down)  $L$ -shaped Young

diagram with a column of height  $\ell$ , a row of length  $m$  and a total of  $\ell+m-1$  boxes. They have dimension

$$\dim L_{\ell,m} = \frac{(N+m-1)!}{(N-\ell)!(\ell-1)!(m-1)!} \frac{1}{\ell+m-1} \quad (4.118)$$

Note that  $L_{1,m} = S_m$ , the  $m^{\text{th}}$  symmetric power of  $A_1$ . Note that we also have  $L_{N,m+1} \cong S_m$  and  $L_{\ell,1} = A_\ell$ . For the lower-vacua, moving clockwise the  $L$  shrinks in width and gets taller. For the upper-vacua, the representation is always a symmetric power, and the Young diagram gets longer as the vacua move clockwise. Note that the two sets of cases in equation (4.117) overlap for  $k = (N-1)/2$ . The upper representation is  $L_{N,1+n-(N-1)/2}^{[n-(N-1)/2]}$  and the lower one is  $L_{1,n-(N-1)/2}^{[n-(N-1)/2]}$  and these are isomorphic. In order for the representations to make sense the definition requires us to take  $n$  sufficiently large. In particular,  $n \geq [\frac{N}{2}] + 2$  will suffice. We will extend it to all integer  $n$  in a later section 7.10.

To define the Brane we must specify the maps

$$\begin{aligned} f_k : S_{n-k} &\rightarrow S_{n-k+1} \otimes A_{N-1} \\ g_k : S_{n-k} &\rightarrow L_{2k+1,n-k+1} \otimes A_{N-2k-1} \\ h_k : L_{2k+1,n-k+1} &\rightarrow S_{n-k+1} \otimes A_{2k}, \quad 0 \leq k < N/2 \end{aligned} \quad (4.119)$$

such that the MC equations (4.110) and (4.114) are satisfied. It will be convenient to use the volume form to perform a partial dualization on  $f_k$  and  $g_k$  and instead use the equivalent maps

$$\begin{aligned} \hat{f}_k : S_{n-k} \otimes A_1 &\rightarrow S_{n-k+1} \\ \hat{g}_k : S_{n-k} \otimes A_{2k+1} &\rightarrow L_{2k+1,n-k+1} \end{aligned} \quad (4.120)$$

Our choice of Chan Paton factors is such that we have a unique, non-zero  $SU(N)$  invariant line where to choose the maps  $f_k$ ,  $g_k$ ,  $h_k$  maps. Indeed we note the isomorphism

$$A_\ell \otimes S_m \cong L_{\ell+1,m} \oplus L_{\ell,m+1} \quad (4.121)$$

and we choose nonzero intertwiners (projection operators)

$$\begin{aligned} \Pi_{\ell,m}^1 : A_\ell \otimes S_m &\rightarrow L_{\ell+1,m} \\ \Pi_{\ell,m}^2 : A_\ell \otimes S_m &\rightarrow L_{\ell,m+1} \end{aligned} \quad (4.122)$$

These projection operators are very easily understood. Given a tensor product of an antisymmetric and symmetric tensor,  $t^a \otimes t^s$ , we could antisymmetrize one index of  $t^s$  with the indices of  $t^a$  to obtain  $\Pi_{\ell,m}^1$  or we could symmetrize one index of  $t^a$  with the indices of  $t^s$  to obtain  $\Pi_{\ell,m}^2$ .

Now we can make  $SU(N)$ -equivariant maps by declaring:

$$\begin{aligned} \hat{f}_k &:= \nu_k \Pi_{1,n-k}^2 \\ \hat{g}_k &:= \gamma_k \Pi_{2k+1,n-k}^2 \\ \Pi^1 \circ h_k &:= \eta_k 1 \quad \Pi^2 \circ h_k = 0 \end{aligned} \quad (4.123)$$

where  $\nu_k, \gamma_k$  and  $\eta_k$  are scalars. The map  $\hat{f}_k$  is particularly easy to write when viewed as a map  $S_{n-k} \rightarrow S_{n-k+1} \otimes A_1^*$  since it is just given by symmetrization:

$$f_k : \text{Sym}(e_{a_1} \otimes \cdots e_{a_{n-k}}) \rightarrow \nu_k \sum_b \text{Sym}(e_{a_1} \otimes \cdots e_{a_{n-k}} \otimes e_b) \otimes e^b \quad (4.124)$$

where  $\{e^b\}$  is a dual basis to  $\{e_a\}$ . From this viewpoint the  $[f_k f_{k+1}] = 0$  follows easily as we are antisymmetrizing two indices which had been symmetrized. For the  $[g_k h_{k+1}] = 0$  equation we note that the composition (given by contracting with the interior amplitude) is a map  $L_{2k+3, n-k} \rightarrow L_{2k+1, n-k+1} \otimes A_1$  but working out the tensor product there is no nonzero intertwiner, by Schur's lemma. Finally,  $f_k$  can be defined in terms of  $h_k g_k$ , and since the relevant space of intertwiners is one-dimensional it is always possible to choose  $\nu_k$  appropriately given  $\gamma_k$  and  $\eta_k$ . To be precise

$$\Pi_{2k+1, n-k}^2(f_k \otimes 1) = \eta_k \hat{g}_k \quad (4.125)$$

When the scalars are related in this way the  $A_\infty$  MC equations are solved.

It is not hard to build a sequence of Branes  $\bar{\mathfrak{N}}_n$  based on the conjugate representations  $\bar{S}_m, \bar{L}_{n,m}$ . They have Chan-Paton factors

$$\mathcal{E}(\bar{\mathfrak{N}}_n)_k = \begin{cases} \bar{L}_{N-2k, n+k+1}^{[-k-n]} & 0 \leq k < \frac{N}{2} \\ \bar{S}_{n+N-k}^{[k+1-n-N]} & \frac{N}{2} \leq k \leq N-1 \end{cases} \quad (4.126)$$

Finally, we can look at strip complex. We take  $N$  even so that we can produce Branes for the negative half-plane using rotation by 180 degrees, as in the  $\mathcal{T}_\vartheta^N$  theories. The rotation exchanges upper and lower vacua and, taking into account the degree shift (4.97) we produce Branes  $\tilde{\mathfrak{N}}_n$ :

$$\begin{aligned} \mathcal{E}(\tilde{\mathfrak{N}}_n)_k &= S_{\tilde{n}+k}^{[\tilde{n}+k+s+1]} & 0 \leq k \leq \frac{N}{2}-1 \\ \mathcal{E}(\tilde{\mathfrak{N}}_n)_{N-k-1} &= L_{N-2k-1, \tilde{n}+k}^{[\tilde{n}+k+s]} & 0 \leq k \leq \frac{N}{2}-1 \end{aligned} \quad (4.127)$$

where  $\tilde{n} = n - \frac{N}{2} + 1$  and  $s$  is an arbitrary degree shift.

The complex for the segment with Branes  $\mathfrak{N}_{n_L}$  and  $\tilde{\mathfrak{N}}_{n_R}$  is a somewhat forbidding direct sum of tensor products of  $SU(N)$  representations:

$$\left( \bigoplus_{k=0}^{\frac{N}{2}-1} L_{2k+1, n+1-k}^{[n_L-k]} \otimes S_{\tilde{n}_R+k}^{[\tilde{n}_R+k+s+1]} \right) \oplus \left( \bigoplus_{k=0}^{\frac{N}{2}-1} S_{n_L-k}^{[n_L-k]} \otimes L_{N-2k-1, \tilde{n}_R+k}^{[\tilde{n}_R+k+s]} \right) \quad (4.128)$$

where the left and right sums come from the lower and upper vacua, respectively. Using this block form the differential is schematically of the form

$$\bigoplus_{k=0}^{\frac{N}{2}-1} \begin{pmatrix} 0 & g_k \otimes \tilde{h}_k \\ h_k \otimes \tilde{g}_k & 0 \end{pmatrix} \quad (4.129)$$

The techniques explained in Section §7.10 below (see especially equations (7.172) et. seq.) can probably be used to evaluate the cohomology of (4.128).

### 4.6.3 Cyclic Isomorphisms Of The Theories

In Section §4.1.1 we gave a formal definition of an isomorphism of Theories. If we relax the constraint that  $\vartheta$  be small and positive we can illustrate that notion with some nontrivial isomorphisms which will be extremely useful to us in Section §7.10 below.

If we consider the Theories  $\mathcal{T}_\vartheta^N$  and  $\mathcal{T}_{\vartheta \pm \frac{2\pi}{N}}^N$  then there are isomorphisms between them. For example

$$\varphi^\pm : \mathcal{T}_\vartheta^N \rightarrow \mathcal{T}_{\vartheta \pm \frac{2\pi}{N}}^N \quad (4.130)$$

satisfies

$$j\varphi^\pm = j \mp 1 \bmod N \quad (4.131)$$

The map  $\varphi_{ij}^+ : R_{i,j} \rightarrow R_{i-1,j-1}$  is a map  $\mathbb{Z} \rightarrow \mathbb{Z}$  or  $\mathbb{Z}^{[1]} \rightarrow \mathbb{Z}^{[1]}$  and consequently has degree zero, *except* when  $i = 0$  or  $j = 0$ . If  $i = 0$  then

$$\varphi_{0,j} : R_{0,j} \rightarrow R_{N-1,j-1} \quad (4.132)$$

is a map  $\mathbb{Z}^{[1]} \rightarrow \mathbb{Z}$  for all  $j = 1, \dots, N-1$ . This necessarily has degree  $-1$ . Similarly,

$$\varphi_{i,0} : R_{i,0} \rightarrow R_{i-1,N-1} \quad (4.133)$$

is a map  $\mathbb{Z} \rightarrow \mathbb{Z}^{[1]}$  for all  $i = 1, \dots, N-1$ . This necessarily has degree  $+1$ . Note that  $\varphi_I$  has degree zero for a cyclic fan of vacua. It is natural to take  $\varphi_{ij}$  to be multiplication by 1, together with an appropriate degree shift. Since this map preserves cyclic ordering, and  $b_{ijk}$  all have the same value, the condition (4.27) is satisfied.

If we consider the Theories  $\mathcal{T}_\vartheta^{SU(N)}$  and  $\mathcal{T}_{\vartheta \pm \frac{2\pi}{N}}^{SU(N)}$  then there are isomorphisms between them. Once again

$$\varphi^\pm : \mathcal{T}_\vartheta^{SU(N)} \rightarrow \mathcal{T}_{\vartheta \pm \frac{2\pi}{N}}^{SU(N)} \quad (4.134)$$

satisfies

$$j\varphi^\pm = j \mp 1 \bmod N \quad (4.135)$$

For  $\varphi_{ij}$ , so long as  $i \neq 0, j \neq 0$  we have

$$\begin{aligned} \varphi_{ij} &: A_{j-i}^{[1]} \rightarrow A_{j-i}^{[1]} & i < j \\ \varphi_{ij} &: A_{N+j-i}^{[1]} \rightarrow A_{N+j-i}^{[1]} & i > j \end{aligned} \quad (4.136)$$

and for  $i = 0$  or  $j = 0$  the representations are cunningly chosen so that

$$\begin{aligned} \varphi_{0,j} &: A_j^{[1]} \rightarrow A_j & j = 1, \dots, N-1 \\ \varphi_{i,0} &: A_{N-i} \rightarrow A_{N-i}^{[1]} & j = 1, \dots, N-1 \end{aligned} \quad (4.137)$$

Again, it makes sense to take all of these to be the identity map, up to the appropriate degree shift.

#### 4.6.4 Relation To Physical Models

We now briefly explain how the above models arise in a physical context. The first class of examples,  $\mathcal{T}_\vartheta^N$ , is meant to correspond to the physics of the simple LG model with target space  $X = \mathbb{C}$  with Euclidean metric and superpotential

$$W = \zeta \frac{N+1}{N} \left( \phi - e^{-iN\vartheta} \frac{\phi^{N+1}}{N+1} \right). \quad (4.138)$$

The prefactor is introduced so that, with the relation  $z_k = \zeta \bar{W}_k$  the vacuum weights coincide with those in (4.78). Indeed the vacua are the critical points of the superpotential  $\phi_k = e^{i\vartheta + \frac{2\pi ik}{N}}$ . (Here  $\zeta$  is a phase introduced in Section §11. In those sections we fix a superpotential and vary  $\zeta$ .) The  $S$ -matrix and soliton spectrum have been worked out in [23]. All the data agree with the theory  $\mathcal{T}_\vartheta^N$ .

We refer to Section §11.2 for a general discussion of boundary conditions in LG models. For now, we can describe the specialization to this simple model. Following Section §11.2.4, we want  $\text{Im}(\zeta^{-1}W) \rightarrow +\infty$  at large  $|\phi|$  for branes on a boundary of the positive half-plane and  $\text{Im}(\zeta^{-1}W) \rightarrow -\infty$  at large  $|\phi|$  for branes on a boundary of the negative half-plane. The sign of  $\text{Im}(\zeta^{-1}W)$  at large  $|\phi|$  is governed by that of  $-\sin((N+1)\arg\phi - N\vartheta)$ . Accordingly, the  $\phi$ -plane at infinity is subdivided into a sequence of  $2N+2$  angular sectors of width  $\frac{\pi}{N+1}$  with boundaries  $\arg\phi = \frac{N}{N+1}\vartheta + \frac{2s}{2N+2}\pi$ ,  $s \in \mathbb{Z}$ . Typical boundary conditions are represented by open curves in the  $\phi$  plane whose endpoints go to infinity in sectors labeled by  $s \in \mathbb{Z}$  such that for a boundary of a positive half-plane we have:

$$\text{Left boundary : } \frac{4s-2}{2N+2}\pi + \frac{N}{N+1}\vartheta < \arg\phi < \frac{4s}{2N+2}\pi + \frac{N}{N+1}\vartheta \quad (4.139)$$

and for the boundary of a negative half-plane we have:

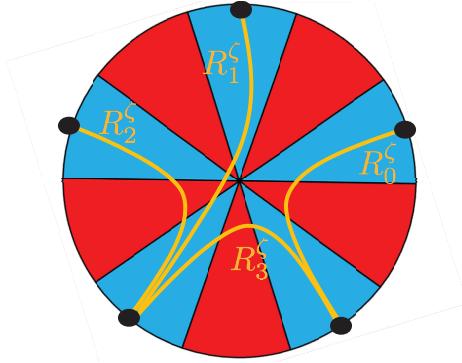
$$\text{Right boundary : } \frac{4s}{2N+2}\pi + \frac{N}{N+1}\vartheta < \arg\phi < \frac{4s+2}{2N+2}\pi + \frac{N}{N+1}\vartheta \quad (4.140)$$

It is instructive to try to draw the  $\mathfrak{C}_k$  branes in the  $\phi$ -plane. In Section §7.10 we construct a family of Branes for the positive half-plane  $\widehat{\mathfrak{B}}_k$  such that  $\widehat{\mathfrak{B}}_0$  is the left Lefshetz thimble<sup>17</sup>  $\mathfrak{T}_0$  and successive brane are obtained by rotation by  $\frac{2\pi}{N}$  in the space-time plane. It is shown in Section §7.10 that for  $2 \leq k \leq [\frac{N}{2}]$  we have  $\widehat{\mathfrak{B}}_k = \mathfrak{C}_{k-1}^{[1]}$ . Now equation (4.91) only defines  $\mathfrak{C}_k$  for  $k < N/2$  so the rotation procedure extends the definition to larger values of  $k$ . It turns out we extend  $\mathfrak{C}_k$  by a sequence of down-vacua thimbles, and  $\widetilde{\mathfrak{C}}_k$  by a sequence of up-type thimbles. When this is done for the  $\mathcal{T}_\vartheta^N$  Theory the rotation operation is periodic and  $\widehat{\mathfrak{B}}_{N+1} = \widehat{\mathfrak{B}}_0$ . In order to draw figures of the Branes we should bear in mind that the Chan-Paton spaces for LG branes are obtained from intersections with the right Lefshetz thimbles. See equation (13.2) below. Using this one can deduce that the  $\mathfrak{C}_k$  branes are Lagrangians stretching between sectors labeled  $s$  and  $s+1$  in (4.139), where  $s$  depends on  $k$ .

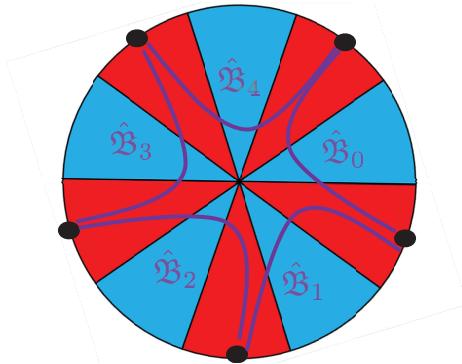
Let us illustrate the procedure for  $N = 4$ . The stability sectors and a basis (for the relevant relative homology group) of right Lefshetz thimbles is shown in Figure 36. For

---

<sup>17</sup>For a definition see Section §11.2.5.



**Figure 36:** This figure illustrates the stability sectors in the  $\phi$ -plane for the  $\mathcal{T}_\vartheta^N$  Theory for  $N = 4$  and small positive  $\vartheta$ . Branes for left boundaries must asymptote to infinity in the red regions. Branes for right boundaries must asymptote to infinity in the blue regions. These regions are divided equally by the rays along which Lefshetz thimbles asymptote. The right thimbles are shown in gold.



**Figure 37:** A system of elementary Branes for the  $\mathcal{T}_\vartheta^N$  Theory for  $N = 4$  and small positive  $\vartheta$ . The Brane  $\mathfrak{C}_1$  is the Brane  $\widehat{\mathfrak{B}}[2]$ .

$N = 4$  the only  $\mathfrak{C}_k$  Brane defined by (4.91) has  $k = 1$ , and corresponds to  $\widehat{\mathfrak{B}}_2^{[-1]}$ . From the intersection numbers we see that it has the shape of  $\widehat{\mathfrak{B}}_2$  in Figure 37. The Brane  $\widehat{\mathfrak{B}}_1$  is neither a left thimble, nor a  $\mathfrak{C}_k$  Brane. The Branes  $\widehat{\mathfrak{B}}_3$  and  $\widehat{\mathfrak{B}}_4$  turn out to be thimbles in this case. Finally, the Branes are related by  $2\pi/5$  rotations in the  $\phi$ -plane. There is a similar picture for Branes  $\widetilde{\mathfrak{C}}_k$  on the right boundary of a negative half-plane obtained by rotating by  $\pi/10$ . In particular,  $\widetilde{\mathfrak{C}}_1$  stretches from  $\arg \phi = \pi/10$  at infinity to  $\arg \phi = 5\pi/10$  at infinity. We conjecture that the obvious generalization of 37 holds for all values of  $N$ . It seems quite likely that the Branes described here are closely related to those studied in (in the conformal theory) in [69] and in section 6.4 of [74].

One can now go on to consider the local operators between these Branes. This is worked out in detail for several pairs of Branes in the  $\mathfrak{C}_\vartheta^N$  Theories in Section §5.7.

The complex of ground states on a segment is built from a vector space generated

by intersections between the left and right branes. The intersection number, i.e. the Witten index of the complex of ground states on the segment, is robust under continuous deformation thanks to the boundary conditions at infinity. Note that  $\mathfrak{C}_1$ , as depicted in Figure 37 is equivalent to a sum of Lefshetz thimbles  $L_1^\zeta + L_2^\zeta + L_3^\zeta$  while  $\tilde{\mathfrak{C}}_1$  is analogously equivalent to  $R_0^\zeta + R_1^\zeta + R_3^\zeta$ . The representation in terms of sums of thimbles gives the complex (4.99), while the brane  $\mathfrak{C}_1$  shown in Figure 37 and its analogue for  $\tilde{\mathfrak{C}}_1$  have zero intersection. These are consistent because (4.99) has no cohomology.

Using Remark 5 near equation (7.53) we can go on to compute the actual spaces of BPS states more generally once we know both the space of local operators between Branes, and how the Branes behave under rotation (in the  $(x, \tau)$  plane) by  $\pi$ .

The second class of examples is meant to correspond to the A-model twist of the affine Toda theory, i.e. the Landau-Ginzburg model whose target space is the subvariety of  $X \subset (\mathbb{C}^*)^N$  defined by

$$Y_1 \cdots Y_N = q \quad (4.141)$$

and

$$W = Y_1 + \cdots + Y_N \quad (4.142)$$

In our case  $q = e^{-i\frac{N}{N-1}\vartheta}$ . The expected  $SU(N)$  global symmetry is not fully manifest in this LG model. Clearly there is a symmetry by the Weyl group of  $SU(N)$ , and the  $N$  critical points are permuted by the center of  $SU(N)$ . The fundamental group of  $X$  can be identified with the root lattice of  $SU(N)$  and the solitons have a “winding number” of  $\log Y_i$  which is independent of  $i$  and equal to  $k/N$  modulo integers for some  $k$ . Thus, the winding numbers can be identified with the weight lattice of  $SU(N)$ .

According to [48, 50] the B-model mirror of this A-model is the supersymmetric  $\mathbb{C}P^{N-1}$ , with B-twist. This model has manifest  $SU(N)$  global symmetry.<sup>18</sup> The study of this model goes back some time [17][86]. and the solitons of type  $(i, i+1)$  are in the fundamental representation of  $SU(N)$ , while those of type  $(i, i+k)$  are boundstates of  $k$  fundamental solitons. This justifies nicely our choice of the  $R_{ij}$  used in (4.100) above. The exact S-matrix for the  $\mathbb{C}P^N$  model was written in [60]. The Witten indices  $\mu_{ij}$  were computed in [15, 48].

A nice geometrical interpretation of the Branes should be available in the  $\mathbb{C}P^{N-1}$  model in terms of homogeneous vector bundles. We propose that the thimbles may be identified as

$$\mathfrak{T}_j = \mathcal{O}(-j) \quad 0 \leq j < \frac{N}{2} \quad (4.143)$$

for the down-vacua, and

$$\mathfrak{T}_{N-j} = \Lambda^{N-2j} TX \otimes \mathcal{O}(j-N) \quad \frac{N}{2} \leq j \leq N-1 \quad (4.144)$$

for the up-vacua. The main justification for this is that the space  $Q$ -closed local operators between these thimbles coincides with the results (5.92)-(5.94) found below using formal techniques. The branes  $\mathfrak{N}_n$  are obtained from the down-vacua thimbles by rotations, which

---

<sup>18</sup>Indeed, the mirror transformation is a T-duality along the  $(\mathbb{C}^*)^N$  isometries.

can be interpreted as B field shifts, and hence the integer  $n$  with the first Chern class on the bundles.

We should note a few differences between the purely formal presentation of this section and the physical models. First, in the physical models the fermion number of the  $ij$  soliton sector is<sup>19</sup>

$$f_{ij} = \left\{ \frac{i-j}{N} \right\} \quad (4.145)$$

where, for a real number  $x$ ,  $\{x\} \in [0, 1]$  is the fractional part of  $x$ . That is,  $x = [x] + \{x\}$ . In general, it is possible to add an exact one-form to a conserved fermion number current  $J^F$  and thereby define a different consistent fermion number. In the LG model this means that  $J^F \rightarrow J^F + df$ , and if the function  $f$  has values  $f(\phi_i) = f_i$  at the critical points of  $W$  then the fermion number  $\int_{\mathbb{R}} *J^F$  is shifted according to  $f_{ij} \rightarrow f_{ij} - f_i + f_j$ . Since we wish to apply the Koszul rule it is useful to make such a redefinition to get integral fermion numbers. One choice which achieves this is  $f_i = i/N$ . In that case we obtain the fermion number assignments used in (4.79) and (4.100) above. Of course, this still leaves the ambiguity further integral shifts of  $f_i$ , but those modifications are dealt with in Section §4.4 above.

Another difference from the physical models is that we work over  $\mathbb{Z}$ . We make a canonical choice of interior amplitude  $\beta$  and it would be very gratifying to know if it corresponds to that which applies to the physical models.

## 5. Categories Of Branes

### 5.1 The Vacuum $A_\infty$ -Category

In this section we would like to discuss the properties of Branes associated to a Theory  $\mathcal{T}$  and a half-plane  $\mathcal{H}$ . Since we will be comparing branes with different Chan-Paton factors we should recall the definition of  $R_J$  from Section §4.2. We denote by  $R_J(\mathcal{E})$  and  $R^\partial(\mathcal{E})$  the  $\mathbb{Z}$ -modules built as in (4.28) from some Chan Paton factors  $\mathcal{E}_i$  and in this section  $R_J$  will denote the “bare space” with trivial CP factors

$$R_J := R_{j_1, j_2} \otimes \cdots \otimes R_{j_{n-1}, j_n}. \quad (5.1)$$

In defining the morphisms of the vacuum category below we will make use of the space  $\widehat{R}_{ij}$  defined as the direct sum of  $R_J$  over all half-plane fans of the form  $J = \{i, \dots, j\}$ , that is:

$$\widehat{R}_{ij} := R_{ij} \oplus (\bigoplus'_k R_{ik} \otimes R_{kj}) \oplus (\bigoplus'_{k_1, k_2} R_{ik_1} \otimes R_{k_1 k_2} \otimes R_{k_2 j}) \oplus \cdots \quad (5.2)$$

where the prime in the direct sum indicates that we only sum over half-plane fans in  $\mathcal{H}$ .  
<sup>20</sup> This allows us to write the  $A_\infty$ -algebra defined by (4.44) as

$$R^\partial(\mathcal{E}) = \bigoplus_{z_{ij} \in \mathcal{H}} \mathcal{E}_i \otimes \widehat{R}_{ij} \otimes \mathcal{E}_j^*. \quad (5.3)$$

---

<sup>19</sup>Incidentally, this gives an example where the adiabatic formula for the fermion number, (12.8) below, which is often used in the literature, is in fact not correct.

<sup>20</sup>Note that, as opposed to  $R_{ij}$ , the expression  $\widehat{R}_{ij}$  depends on a choice of half-plane  $\mathcal{H}$  and is only defined for half of the pairs  $(i, j)$ . It will also be convenient to define  $\widehat{R}_{ii} = \mathbb{Z}$ , concentrated in degree zero. Care should be taken when using this notation to distinguish webs from extended webs.

Until now, we always extended all our maps to give zero whenever some of the arguments cannot be fit together. This is self-consistent, but it hides some structure which is sometimes useful to make manifest. The natural way to discuss operations which only make sense (or are only nontrivial) for certain pairs of arguments is to use a categorical language. In the present case, we need an  $A_\infty$  category:

**Definition:** Suppose we are given the data of a Theory  $\mathcal{T}$  and a half-plane  $\mathcal{H}$ . The associated *Vacuum  $A_\infty$ -category*  $\mathfrak{Vac}$  has as objects the vacua  $i, j, \dots \in \mathbb{V}$  while the space of morphisms is given by

$$\text{Hom}(j, i) = \begin{cases} \widehat{R}_{ij} & z_{ij} \in \mathcal{H} \\ \widehat{R}_{ii} = \mathbb{Z} & i = j \\ 0 & i \neq j \quad \text{and} \quad z_{ij} \notin \mathcal{H} \end{cases} \quad (5.4)$$

The  $A_\infty$ -compositions  $m^{\mathfrak{Vac}}$  are given by  $\rho_\beta(\mathbf{t}_\mathcal{H})$  (defined as before, but without the contractions of Chan Paton factors). See (5.8) below for a more precise statement.

We should make a number of remarks about this definition

1. Recall that  $z_{ij} \in \mathcal{H}$  means that if  $\mathcal{H}$  is translated so that the origin is on its boundary then  $z_{ij}$  is in the translated copy of  $\mathcal{H}$ .
2. The category depends on the data  $\mathcal{T}$  and  $\mathcal{H}$ . We will generally suppress this dependence in the notation but if we wish to stress the dependence or distinguish different choices of data then we will indicate them in the argument and write some or all of the data by writing  $\mathfrak{Vac}(\mathcal{T}, \mathcal{H})$  and so forth.
3. Note that we defined  $\mathfrak{Vac}$  with the “bare”  $R_J$ . We can “add” Chan-Paton factors to define a new  $A_\infty$ -category which we will denote as  $\mathfrak{Vac}(\mathcal{E})$  (when  $\mathcal{T}, \mathcal{H}$  are understood, and as  $\mathfrak{Vac}(\mathcal{T}, \mathcal{H}, \mathcal{E})$  when they are not). The morphism spaces are determined by replacing  $\widehat{R}_{ij} \rightarrow \mathcal{E}_i \otimes \widehat{R}_{ij} \otimes \mathcal{E}_j^*$  in (5.4). To define the  $A_\infty$ -multiplications we tensor  $m^{\mathfrak{Vac}}$  with the obvious contraction operations  $(\mathcal{E}_1 \otimes \mathcal{E}_2^*) \otimes (\mathcal{E}_2 \otimes \mathcal{E}_3^*) \rightarrow (\mathcal{E}_1 \otimes \mathcal{E}_3^*)$ .
4. Note that  $\text{Hom}(i, i) = \mathbb{Z}$  means the complex concentrated in degree zero, consisting of scalar multiples of the graded identity element  $\mathbf{Id}_i$ . The definition fits in well with (5.8) if we use extended webs. Recall that the  $A_\infty$ -multiplications involving  $\mathbf{Id}_i$  are defined near (4.76).
5. There are three interlocking conventions for composition of morphisms, which we will now spell out somewhat pedantically. First, one can compose morphisms successively on the left or on the right. Second, one can read an equation from left to right, or vice versa. Third, the time ordering implicit in successive operations might or might not agree with the geometrical time ordering of increasing  $y$  in the  $(x, y)$  plane. In equation (5.4),  $\text{Hom}(j, i)$  refers to the set of arrows which go *from* object  $j$  *to* object  $i$ . It is useful to define

$$\text{Hop}(i, j) := \text{Hom}(j, i) \quad (5.5)$$

where Hop is an abbreviation for  $\text{Hom}^{\text{opp}}$ . Equations with Hop should generally be read from right to left. Including the Chan-Paton factors we define Hom-spaces by using

$$\text{Hop}^{\mathcal{E}}(i, j) := \mathcal{E}_i \otimes \text{Hop}(i, j) \otimes \mathcal{E}_j^* \quad (5.6)$$

so that the  $A_\infty$ -multiplications

$$\text{Hop}^{\mathcal{E}}(i_0, i_1) \otimes \text{Hop}^{\mathcal{E}}(i_1, i_2) \otimes \cdots \otimes \text{Hop}^{\mathcal{E}}(i_{n-1}, i_n) \rightarrow \text{Hop}^{\mathcal{E}}(i_0, i_n) \quad (5.7)$$

are computed from

$$\rho_\beta(\mathbf{t}_{\mathcal{H}})(r_1, \dots, r_n) \in \text{Hop}^{\mathcal{E}}(i_0, i_n) \quad (5.8)$$

with  $r_s \in \text{Hop}^{\mathcal{E}}(i_{s-1}, i_s)$ . Successive morphisms are composed on the left and the successive composition of arrows should be read from right to left. Now, if  $\mathcal{H}$  is the positive half-plane  $x \geq x_0$  with boundary on the left then this successive composition of morphisms can be visualized as taking place forward in “time”  $y$ . However, by the same token, if  $\mathcal{H}$  is the negative half-plane  $x \leq x_0$  with boundary on the right then successive composition on the left and reading from right to left corresponds to going *backwards* in time  $y$ . This leads to some awkwardness when we discuss web representations and categories associated to strips and interfaces, since these involves both positive and negative half-plane webs. (Of course, for the negative half-plane, reading the composition of morphisms from left to right corresponds to going forward in time, but reversing the time ordering corresponds to composition of morphisms in the *opposite* category.) We will adopt the convention that successive operations on  $\mathcal{E}_{LR}$  of (4.52) act on the left and correspond to transitions forward in time. This makes  $\mathcal{E}_{LR}$  an  $R_L^\partial \otimes (R_R^\partial)^{\text{opp}}$   $A_\infty$ -module. (That is, an  $R_L^\partial - R_R^\partial$  bimodule.) Similarly,  $i \rightarrow \mathcal{E}_{L,i} \otimes \mathcal{E}_{R,i}^*$  defines a  $\mathfrak{Vac}(\mathcal{E}_L) \times \mathfrak{Vac}(\mathcal{E}_R)^{\text{opp}}$  module of  $A_\infty$ -categories, in the sense of Definition 8.14 of [5].

## Examples

1. Consider the top left taut positive half-plane web in Figure 15. Choose arbitrary elements  $r_1 \in \widehat{R}_{ij}$  and  $r_2 \in \widehat{R}_{jk}$ . In this case the composition  $m^{\mathfrak{vac}}(r_1, r_2)$  is simply  $r_1 \otimes r_2 \in \widehat{R}_{ik}$ .
2. Now consider the bottom left taut web in Figure 15. This taut web leads to a contribution to  $m_2^{\mathfrak{vac}}$  which we can illustrate as follows. If  $r_1 \in R_{i_1,j}$  and  $r_2 \in R_{j,i_n}$  then the contribution of this web to

$$m_2^{\mathfrak{vac}}(r_1, r_2) \in \text{Hop}(i_1, i_n) = \widehat{R}_{i_1, i_n} \quad (5.9)$$

is computed as follows. The interior vertex has a cyclic fan of vacua

$$I = \{i_n, j, i_1, i_2, \dots, i_{n-1}\}. \quad (5.10)$$

Let  $\beta_I$  be the component of  $\beta$  in  $R_I$  for this cyclic fan. Then we consider

$$K_{14}K_{23}(r_1 \otimes r_2 \otimes \beta_I) \quad (5.11)$$

where  $K_{23}$  contracts  $R_{j,i_n} \otimes R_{i_n,j} \rightarrow \mathbb{Z}$  and then  $K_{14}$  contracts  $R_{i_1,j} \otimes R_{j,i_1} \rightarrow \mathbb{Z}$  leaving behind an element of

$$R_{i_1,i_2} \otimes \cdots \otimes R_{i_{n-1},i_n} \subset \hat{R}_{i_1,i_n} \quad (5.12)$$

3. Similarly, the two taut webs on the right in Figure 15 contribute to  $m_1$ . The first at order  $\beta$  with one contraction of  $K$  and the second at order  $\beta^2$  with three contractions of  $K$ .

## 5.2 Branes And The Brane Category $\mathfrak{Br}$

We are now ready to define an  $A_\infty$  category of branes  $\mathfrak{Br}$ , associated to a given choice of Theory  $\mathcal{T}$  and half-plane  $\mathcal{H}$ . The objects of  $\mathfrak{Br}$  are Branes, i.e. pairs  $\mathfrak{B} = (\mathcal{E}, \mathcal{B})$  of *some* choice of Chan Paton data  $\mathcal{E}$  together with a compatible boundary amplitude  $\mathcal{B}$ . Recall that a boundary amplitude  $\mathcal{B}$  is a degree one element

$$\mathcal{B} \in \bigoplus_{i,j \in \mathbb{V}} \mathcal{E}_i \otimes \text{Hop}(i,j) \otimes \mathcal{E}_j^* \quad (5.13)$$

which satisfies the Maurer-Cartan equation in  $\mathfrak{Vac}(\mathcal{E})$ :

$$\sum_{n=1}^{\infty} \rho_\beta(\mathbf{t}_{\mathcal{H}})(\underbrace{\mathcal{B}, \dots, \mathcal{B}}_{n \text{ times}}) = 0. \quad (5.14)$$

The space of morphisms from  $\mathfrak{B}_2 = (\mathcal{E}^2, \mathcal{B}_2)$ , a Brane in  $\mathfrak{Vac}(\mathcal{E}^2)$  to  $\mathfrak{B}_1 = (\mathcal{E}^1, \mathcal{B}_1)$ , a Brane in  $\mathfrak{Vac}(\mathcal{E}^1)$  is defined to be

$$\text{Hop}(\mathfrak{B}_1, \mathfrak{B}_2) := \bigoplus_{i,j \in \mathbb{V}} \mathcal{E}_i^1 \otimes \text{Hop}(i,j) \otimes (\mathcal{E}_j^2)^*. \quad (5.15)$$

In order to define the composition of morphisms

$$\delta_1 \in \text{Hop}(\mathfrak{B}_0, \mathfrak{B}_1), \quad \delta_2 \in \text{Hop}(\mathfrak{B}_1, \mathfrak{B}_2), \dots, \delta_n \in \text{Hop}(\mathfrak{B}_{n-1}, \mathfrak{B}_n) \quad (5.16)$$

It is useful to observe that the boundary amplitude  $\mathcal{B}$  of a Brane  $\mathfrak{B}$  is, thanks to the definition (5.15), a morphism in  $\text{Hop}(\mathfrak{B}, \mathfrak{B})$ . With this in mind it makes sense to define the multiplication operations in  $\mathfrak{Br}$  using the formula

$$M_n(\delta_1, \dots, \delta_n) := m \left( \frac{1}{1 - \mathcal{B}_0}, \delta_1, \frac{1}{1 - \mathcal{B}_1}, \delta_2, \dots, \delta_n, \frac{1}{1 - \mathcal{B}_n} \right). \quad (5.17)$$

where  $m$  is the tensor product of  $m^{\mathfrak{Vac}}$  with the natural contraction on CP spaces. Note that  $M_n(\delta_1, \dots, \delta_n) \in \text{Hop}(\mathfrak{B}_0, \mathfrak{B}_n)$ .

After some work (making repeated use of the fact that the  $\mathcal{B}_a$  solve the Maurer-Cartan equation) one can show that the  $M_n$  satisfy the  $A_\infty$ -relations and hence  $\mathfrak{Br}$  is an  $A_\infty$ -category. Although this is well-known we provide a simple proof in Appendix B, below equation (B.4).

**Remarks:**

- As in our notation for  $\mathfrak{Vac}$ , the  $A_\infty$ -category  $\mathfrak{Br}$  implicitly depends on the Theory  $\mathcal{T}(\mathbb{V}, z, \mathcal{R}, \beta)$ , as well as the half-plane  $\mathcal{H}$ . When we wish to stress this dependence we will write these as arguments. Note that  $\mathfrak{Br}(\mathcal{T}, \mathcal{H})$  does not depend on any specific choice of Chan-Paton spaces  $\mathcal{E}$ .
- We should note that the passage from  $\mathfrak{Vac}$  to  $\mathfrak{Br}$  is quite standard in the theory of homotopical algebra where a Brane is known as a “twisted complex.” See, for examples, Section 3l, p.43 of [81] or Definition 8.16, p.614 of [5]. Our category  $\mathfrak{Br}$  would be denoted  $\text{Tw}(\mathfrak{Vac})$  in the mathematics literature.
- It will be useful to extend the Brane category to include the extended webs of Sections 2.44.5. If we use extended webs then the extended web representations (4.74) mean that the morphisms now have a “scalar part”  $Q_i$  with  $Q_i^2 = 0$ . Now we define the morphism spaces to be

$$\text{Hop}(\mathfrak{B}', \mathfrak{B}) = (\bigoplus_i \mathcal{E}'_i \otimes \mathcal{E}_i^*) \oplus \left( \bigoplus_{z_{ij} \in \mathcal{H}} \mathcal{E}'_i \otimes \hat{R}_{ij} \otimes \mathcal{E}_j^* \right) \quad (5.18)$$

and we again refer to the component in  $\bigoplus_i \mathcal{E}'_i \otimes \mathcal{E}_i^*$  as the “scalar part.” The scalar part of  $\delta \in \text{Hop}(\mathfrak{B}', \mathfrak{B})$  is just a collection of linear maps  $f_i : \mathcal{E}(\mathfrak{B})_i \rightarrow \mathcal{E}(\mathfrak{B}')_i$ . Those maps can be composed or applied to (5.15) and hence we can define the multiplications (5.17) by using the extended webs in the taut element  $t_{\mathcal{H}}$  in the definition (5.8) of  $m^{\mathfrak{Vac}}$ . We will generally work with this extended Brane category.

- The category of vacua is naturally a full subcategory of the category of Branes: each vacuum  $i$  maps to “thimble Branes”  $\mathfrak{T}_i$  (also called simply “thimbles”). The reason for the name is explained in §11.2.5. The Chan-Paton spaces of  $\mathfrak{T}_i$  are defined by  $\mathcal{E}(\mathfrak{T}_i)_j = \delta_{ji}\mathbb{Z}$ , in degree zero. It follows that  $\text{Hop}(\mathfrak{T}_i, \mathfrak{T}_j) = \text{Hop}(i, j)$ . Moreover, the boundary amplitude  $\mathcal{B}_i$  of  $\mathfrak{T}_i$  is taken to be zero (even if we used extended webs). Thus the insertion of  $\frac{1}{1-\mathfrak{T}_i}$  in (5.17) has no effect on the contractions and hence  $M_n = m_n^{\mathfrak{Vac}}$  on chains of morphisms between thimbles.
- The thimbles form an “exceptional collection,” from which all other branes, by definition, arise as twisted complexes. Other exceptional collections will exist in  $\mathfrak{Br}$ , especially the ones obtained from mutations of the thimble collection. We will discuss these mutations briefly in Section §7.8 below.
- Finally, we note that  $M_1^2 = 0$  and hence each space of morphisms  $\text{Hop}(\mathfrak{B}_1, \mathfrak{B}_2)$  is a chain complex. In the physical applications the cohomology of the differential  $M_1$  on this complex is interpreted as BRST-invariant local operators which can be inserted at the boundary of the half-plane  $\mathcal{H}$  and change the boundary conditions. See the end of Section §7.4.1, Remark 5 for further discussion.

### 5.3 Homotopy Equivalence Of Branes

Working in the framework of the extended Brane category we can define a notion of homotopy equivalence which will be extremely useful in Sections §6-8.

Because  $\delta \rightarrow M_1(\delta)$  for  $\delta \in \text{Hop}(\mathfrak{B}', \mathfrak{B})$  is a differential we can consider  $M_1$ -exact and  $M_1$ -closed morphisms. The composition  $M_2$  is compatible with  $M_1$ . We can use  $M_1$  and  $M_2$  to define useful notions such as homotopy and homotopy equivalence, treating the branes as an analogue of a chain complex.

We define two  $M_1$ -closed morphisms  $\delta_{1,2}$  to be *homotopic* if they differ by an  $M_1$ -exact morphism, i.e.

$$\delta_1 \sim \delta_2 \quad \longleftrightarrow \quad \delta_1 - \delta_2 = M_1(\delta_3) \quad (5.19)$$

for some  $\delta_3$ . Similarly, we define two branes  $\mathfrak{B}$  and  $\mathfrak{B}'$  to be *homotopy equivalent*, denoted,

$$\mathfrak{B} \sim \mathfrak{B}', \quad (5.20)$$

if there are two  $M_1$ -closed morphisms  $\delta : \mathfrak{B} \rightarrow \mathfrak{B}'$  and  $\delta' : \mathfrak{B}' \rightarrow \mathfrak{B}$  which are inverses up to homotopy. That is:

$$M_2(\delta, \delta') \sim \mathbf{Id} \quad M_2(\delta', \delta) \sim \mathbf{Id}. \quad (5.21)$$

Recall that **Id** is the graded identity defined in (4.75). <sup>21</sup>

If we have two morphisms  $\delta_1$  and  $\delta_2$  in  $\text{Hop}(\mathfrak{B}', \mathfrak{B})$  and  $\text{Hop}(\mathfrak{B}'', \mathfrak{B}')$ , respectively, with scalar parts given by collections of maps  $f_{1,i}$  and  $f_{2,i}$ , the scalar part of the composition  $M_2(\delta_1, \delta_2)$  is simply given by the composition of the scalar parts  $f_{2,i}f_{1,i}$ . Similarly, if the scalar parts of the boundary amplitudes of  $\mathfrak{B}'$  and  $\mathfrak{B}$  are  $Q_i$  and  $Q'_i$ , respectively, and the scalar part of a morphism  $\delta : \mathfrak{B} \rightarrow \mathfrak{B}'$  is  $f_i$ , then the scalar part of  $M_1(\delta)$  is

$$Q'_i f_i \pm f_i Q_i \quad (5.22)$$

Thus the scalar part of an  $M_1$ -closed morphism is a collection of chain maps between the Chan Paton factors, homotopic morphisms have homotopic scalar parts and homotopy equivalent branes have homotopy equivalent Chan Paton factors.

The following will prove to be a very useful criterion for homotopy equivalence between branes  $\mathfrak{B}_1$  and  $\mathfrak{B}_2$  in our discussions in Sections §§6,7,8. (See, for examples equation (6.65) and the discussion at the end of Section §7.4.1.) We consider the special case where the two Branes have the *same* Chan-Paton data:  $\mathcal{E}(\mathfrak{B}_1) = \mathcal{E}(\mathfrak{B}_2)$ , and where the homotopy equivalence can be written as a morphism of the form  $\mathbf{Id} + \epsilon$ , where  $\epsilon$  is a degree zero element in  $\text{Hop}(\mathfrak{B}_2, \mathfrak{B}_1)$  with no scalar part. Note that  $\mathbf{Id} \in \text{Hop}(\mathfrak{B}_2, \mathfrak{B}_1)$  makes sense because the Chan-Paton factors are assumed to be the same. If we require such a morphism between two branes  $\mathfrak{B}_1$  and  $\mathfrak{B}_2$  to be  $M_1$ -closed, we find the relation:

$$\begin{aligned} M_1(\mathbf{Id} + \epsilon) &:= \rho_\beta(t_H)[\frac{1}{1 - \mathcal{B}_2}, \mathbf{Id} + \epsilon, \frac{1}{1 - \mathcal{B}_1}] \\ &= \rho_\beta(t_H)[\mathcal{B}_2, \mathbf{Id}] + \rho_\beta(t_H)[\mathbf{Id}, \mathcal{B}_1] + \rho_\beta(t_H)[\frac{1}{1 - \mathcal{B}_2}, \epsilon, \frac{1}{1 - \mathcal{B}_1}] \\ &= \mathcal{B}_1 - \mathcal{B}_2 + \rho_\beta(t_H)[\frac{1}{1 - \mathcal{B}_2}, \epsilon, \frac{1}{1 - \mathcal{B}_1}] = 0, \end{aligned} \quad (5.23)$$

---

<sup>21</sup>Physically,  $\delta$  and  $\delta'$  correspond to boundary-changing local operators whose OPE is the identity operator up to exact boundary operators. Homotopy equivalent branes essentially represent the same D-brane in a topologically twisted physical model.

where in the last line we used the condition that  $\mathcal{B}_2$  has degree one and the third term refers to unextended webs. Recall that we can regard  $\mathcal{B}_1$  and  $\mathcal{B}_2$  to be themselves elements of  $\text{Hop}(\mathfrak{B}_2, \mathfrak{B}_1)$ , so the equation (5.23) makes sense.

We must also guarantee the invertibility up to homotopy (5.21) of  $\mathbf{Id} + \epsilon$ . This is actually automatic in our setup. Indeed, we can solve  $M_2(\mathbf{Id} + \epsilon, \mathbf{Id} - \epsilon') = \mathbf{Id}$  recursively by

$$\epsilon' = \epsilon - M_2(\epsilon, \epsilon') \quad (5.24)$$

The recursion will stabilize after a finite number of iterations thanks to the finiteness properties of the webs involved in  $M_2$ . As  $M_2$  is associative up to homotopy,  $\mathbf{Id} - \epsilon'$  is also a left inverse up to homotopy. More generally, if we consider a morphism  $f + \epsilon$  with scalar part  $f$  and  $g$  is an inverse of  $f$  up to homotopy then we can find  $\epsilon'$  so that  $g - \epsilon'$  is an inverse of  $f + \epsilon$ , up to homotopy.

For examples of homotopy equivalent pair of branes, see Sections §7.6 and §7.10.

In all the constructions of homotopy equivalences in the rest of the paper, we will actually produce both morphisms  $\mathbf{Id} + \epsilon, \mathbf{Id} - \epsilon'$  explicitly and will not rely on these finiteness properties to argue for the existence of the inverse. Aside from being philosophically more satisfying, this will be useful because we can then extend the results to cases of interest with an infinite number of vacua (such as those relevant to knot homology and the 2d/4d wall-crossing-formula).

#### 5.4 Brane Categories And The Strip

Given a Theory  $\mathcal{T}$  we can associate two  $A_\infty$ -categories to the strip geometry  $[x_\ell, x_r] \times \mathbb{R}$ . We have the category of Branes  $\mathfrak{Br}_L$  associated to the left boundary, controlled by the operation  $\rho_\beta(\mathfrak{t}_L)$  and the category of Branes  $\mathfrak{Br}_R$  associated to the right boundary, controlled by the operation  $\rho_\beta(\mathfrak{t}_R)$ .

Our strip bimodule operation  $\rho_\beta(\mathfrak{t}_s)$  defined in (4.62) above can be given several different useful interpretations. We will use the concept of a “module for an  $A_\infty$ -category” as defined in 8.14 of [5]. In general if  $\mathcal{A}$  is an  $A_\infty$ -category with objects  $x_\alpha$  then a (left) module over  $\mathcal{A}$  consists of a choice of graded  $\mathbb{Z}$ -module  $\mathcal{M}(x_\alpha)$  for each object together with a collection of maps

$$m_n^{\mathcal{M}} : \text{Hop}(x_0, x_1) \otimes \cdots \otimes \text{Hop}(x_{n-1}, x_n) \otimes \mathcal{M}(x_n) \rightarrow \mathcal{M}(x_0) \quad (5.25)$$

which are defined for  $n \geq 1$ , are of degree  $2 - n$ , and satisfy the categorical analog of the identities (3.52). As usual, a bimodule for a pair of  $A_\infty$ -categories  $(\mathcal{A}, \mathcal{B})$  is a module for the  $A_\infty$ -category  $\mathcal{A} \times \mathcal{B}^{\text{opp}}$ .

Our first interpretation is that  $\rho_\beta(\mathfrak{t}_s)$  can be used to define a bimodule for the pair of vacuum categories  $(\mathfrak{Vac}_L, \mathfrak{Vac}_R)$ . The objects of  $\mathfrak{Vac}_L \times \mathfrak{Vac}_R^{\text{opp}}$  are pairs of vacua  $(i, j)$  and we take  $\mathcal{M}(i, j) = \delta_{i,j} \mathbb{Z}$ , with  $\mathbb{Z}$  in degree zero.

As a second interpretation, we can define a bimodule for the pair of *Brane* categories  $(\mathfrak{Br}_L, \mathfrak{Br}_R)$ . The objects of  $\mathfrak{Br}_L \times \mathfrak{Br}_R^{\text{opp}}$  are pairs of Branes and now we take

$$\mathcal{M}(\mathcal{B}_L, \mathcal{B}_R) := \mathcal{E}_{LR} = \bigoplus_{i \in \mathbb{V}} \mathcal{E}_{L,i} \otimes \mathcal{E}_{R,i}^* \quad (5.26)$$

To define the module maps (5.25) in this case we take

$$m^{\mathcal{M}}(\delta_1, \dots, \delta_n; g; \delta'_1, \dots, \delta'_{n'}) := \rho_{\beta}(\mathbf{t}_s) \left( \frac{1}{1 - \mathcal{B}_{L,0}}, \delta_1, \dots, \delta_n, \frac{1}{1 - \mathcal{B}_{L,n}}; g; \frac{1}{1 - \mathcal{B}_{R,0}}, \delta'_1, \dots, \delta'_{n'}, \frac{1}{1 - \mathcal{B}_{R,n'}} \right). \quad (5.27)$$

where

$$\delta_1 \in \text{Hop}(\mathcal{B}_{L,0}, \mathcal{B}_{L,1}), \dots, \delta_n \in \text{Hop}(\mathcal{B}_{L,n-1}, \mathcal{B}_{L,n}) \quad (5.28)$$

$$\delta'_1 \in \text{Hop}(\mathcal{B}_{R,0}, \mathcal{B}_{R,1}), \dots, \delta'_{n'} \in \text{Hop}(\mathcal{B}_{R,n'-1}, \mathcal{B}_{R,n'}). \quad (5.29)$$

The maps  $\delta_1 \rightarrow m^{\mathcal{M}}(\delta_1; \cdot; \cdot)$  and  $\delta'_1 \rightarrow m^{\mathcal{M}}(\cdot; \cdot; \delta'_1)$  are particularly interesting for us. Indeed, we have

$$m^{\mathcal{M}}(M_1(\delta_1); \cdot; \cdot) = \pm d_{LR} m^{\mathcal{M}}(\delta_1; \cdot; \cdot) \pm m^{\mathcal{M}}(\delta_1; \cdot; \cdot) d_{LR} \quad (5.30)$$

and a similar equation for  $d(\cdot; \cdot; M_1(\delta'_1))$ .

Thus  $M_1$ -closed morphisms map to chain maps for the chain complexes of approximate ground states  $(\mathcal{E}_{LR}, d_{LR})$ , homotopic morphisms map to homotopic chain maps and homotopy equivalent Branes give rise to homotopy equivalent chain complexes. In particular, homotopy equivalent Branes have isomorphic spaces of exact ground states on the segment. This is a special case of a general principle: homotopy equivalent Branes are the “same” Brane for most purposes.

We can also use these constructions to interpret the right-Branes as elements of a standard category along the following lines. Given a right Brane  $\mathcal{B}_R$  we can define a module for the  $A_{\infty}$ -category  $\mathfrak{V}\mathfrak{ac}_L$  by the assignment

$$\mathcal{M}(i) = \mathcal{E}_i^* \quad (5.31)$$

Then the module maps (5.25) are just

$$r_1 \otimes \cdots \otimes r_n \otimes g \rightarrow \rho_{\beta}(\mathbf{t}_s)[r_1, \dots, r_n; g; \frac{1}{1 - \mathcal{B}_R}] \quad (5.32)$$

which thus defines a family (parametrized by  $\mathcal{B}_R$ ) of  $A_{\infty}$  modules for  $\mathfrak{V}\mathfrak{ac}_L$  with  $i \rightarrow \mathcal{E}_i^*$ .

Notice that in this formula we have Chan-Paton factors on the right boundary arguments, but not the left boundary arguments. The data (5.32) defines a set of maps

$$\widehat{R}_{i_0, i_1} \otimes \widehat{R}_{i_1, i_2} \cdots \widehat{R}_{i_{n-1}, i_n} \rightarrow \mathcal{E}_{R, i_0}^* \otimes \mathcal{E}_{R, i_n}. \quad (5.33)$$

This observation suggests the following definition of a *mapping category*:<sup>22</sup> The objects will be sets of Chan-Paton data  $\mathcal{E} = \{\mathcal{E}_i\}_{i \in \mathbb{V}}$  and the morphisms  $\text{Hop}(\mathcal{E}, \mathcal{E}')$  will be the set of collections of linear maps

$$\widehat{R}_{i_0, i_1} \otimes \widehat{R}_{i_1, i_2} \cdots \widehat{R}_{i_{n-1}, i_n} \rightarrow (\mathcal{E}'_{i_0})^* \otimes \mathcal{E}_{i_n}. \quad (5.34)$$

---

<sup>22</sup>The term “mapping category” is nonstandard. It appears to be closely related to the notion of a Koszul dual to an  $A_{\infty}$ -category. See, for example, [68].

The difference between the collection of maps (5.25) and the collection of maps of the form (5.34) defining a generic morphism in the mapping category is that the latter need not satisfy the  $A_\infty$ -relations. Let us denote a collection of such linear maps by  $\mathfrak{m}$  and the value on a monomial of the form  $P = \hat{r}_{i_0,i_1} \otimes \hat{r}_{i_1,i_2} \cdots \hat{r}_{i_{n-1},i_n}$  by  $\mathfrak{m}[P]$ . Two morphisms  $\mathfrak{m}_1 \in \text{Hop}(\mathcal{E}, \mathcal{E}')$  and  $\mathfrak{m}_2 \in \text{Hop}(\mathcal{E}', \mathcal{E}'')$  are composed as

$$(\mathfrak{m}_1 \circ \mathfrak{m}_2)[P] = \sum_{\text{Pa}_2(P)} \mathfrak{m}_1[P_1] \mathfrak{m}_2[P_2] \quad (5.35)$$

where on the right-hand-side we contract the spaces  $(\mathcal{E}'_i) \otimes (\mathcal{E}'_j)^*$  in the natural way. This composition is associative, thus making the mapping category an *ordinary* category.

There is also a differential on the morphism spaces of the mapping category given by

$$\mathfrak{d}\mathfrak{m}[P] = \sum_{\text{Pa}_3(P)} \epsilon \mathfrak{m}[P_1, m^{\mathfrak{Vac}_L}[P_2], P_3]. \quad (5.36)$$

It is compatible with the composition. Collections of maps annihilated by the differential make the assignment  $i \rightarrow \mathcal{E}_i$  into an  $A_\infty$ -module for  $\mathfrak{Vac}_L$ .

Now, as noted above, for every Brane  $\mathcal{B}_R \in \mathfrak{Br}_R$  we can define an object in the mapping category, namely the Chan-Paton data of the Brane for the negative half-plane. Moreover, if we regard the mapping category as a very degenerate version of an  $A_\infty$ -category then we can define an  $A_\infty$ -functor from  $\mathfrak{Br}_R^{\text{opp}}$  to the mapping category. The image of morphisms  $\delta_1, \dots, \delta_m$  is the collection of maps

$$\mathfrak{m}[r_1, \dots, r_n] = \rho_\beta(t_s)[r_1, \dots, r_n; \cdot; \frac{1}{1 - \mathcal{B}_{R,0}}, \delta_1, \dots, \delta_m, \frac{1}{1 - \mathcal{B}_{R,m}}] \quad (5.37)$$

The usual convolution identity for strip webs gives the functor property.

This allows us to identify the right Branes as elements of a standard category, if needed.

## 5.5 Categorification Of 2d BPS Degeneracies

There is a very nice way to define the complexes  $\widehat{R}_{ij}$  using matrices of chain complexes. Suppose there are  $N$  vacua so we can identify  $\mathbb{V} = \{1, \dots, N\}$ . Introduce the elementary  $N \times N$  matrices  $e_{ij}$  with a 1 in the  $i^{th}$  row,  $j^{th}$  column and zero elsewhere. Then we can define  $\widehat{R}_{ij}$  from the formal product

$$\mathbb{Z} \cdot \mathbf{1} + \bigoplus_{z_{ij} \in \mathcal{H}} \widehat{R}_{ij} e_{ij} = \bigotimes_{z_{ij} \in \mathcal{H}} (\mathbb{Z} \cdot \mathbf{1} + R_{ij} e_{ij}) \quad (5.38)$$

where  $\mathbf{1}$  is the  $N \times N$  unit matrix and in the tensor product we order the factors left to right by the clockwise order of the argument of  $z_{ij}$ . Phase-ordered products of this kind involving operators, rather than complexes, have appeared in the work of Cecotti and Vafa [15] and in the work of Kontsevich and Soibelman [61, 62] on wall-crossing.<sup>23</sup> Our work here can be considered as a “categorification” of the wall-crossing formulae. This will be discussed further in the sections on wall-crossing §7.7 and §8.

---

<sup>23</sup>Of course, such phase ordered products have also appeared in many previous works on Stokes data.

In the physical context one finds that  $R_{ij}$  are complexes of approximate groundstates and the Witten indices

$$\mu_{ij} := \text{Tr}_{R_{ij}}(-1)^F \quad (5.39)$$

are known as the (two-dimensional) BPS degeneracies. With  $F$  denote the integer fermion number, coinciding with the integer degree we defined on  $R_{ij}$ .

They were extensively studied in [23, 15]. Since fermion number behaves well under tensor product we can take a trace (on the web representations) of (5.38) to obtain

$$1 + \bigoplus_{z_{ij} \in \mathcal{H}} \widehat{\mu}_{ij} e_{ij} = \bigotimes_{z_{ij} \in \mathcal{H}} (1 + \mu_{ij} e_{ij}). \quad (5.40)$$

where

$$\widehat{\mu}_{ij} := \text{Tr}_{\widehat{R}_{ij}}(-1)^F. \quad (5.41)$$

The Cecotti-Vafa-Kontsevich-Soibelman wall-crossing formula states that certain continuous deformations of Theories lead to jumps in the BPS degeneracies  $\mu_{ij}$  while the  $\widehat{\mu}_{ij}$  remain constant. In Section §8 we will discuss the categorified version of that statement.

**Remark:** In LG theories,  $\widehat{\mu}_{ij}$  can be computed by intersecting infinitesimally rotated Lefschetz thimbles [15][48]. See Section §7.9 for a categorized version of that statement.

## 5.6 Continuous Deformations

In this section we elaborate a bit on the meaning of equation. (5.23) (repeated here)

$$\mathcal{B}' - \mathcal{B} + \rho_\beta(\mathbf{t}_\mathcal{I})[\frac{1}{1-\mathcal{B}'}; \epsilon(s); \frac{1}{1-\mathcal{B}}] = 0 \quad (5.42)$$

This material will not be used later and the reader should feel free to skip it.

If we are given a brane  $\mathcal{B}$  and some degree zero morphism  $\epsilon$  with no scalar part we can solve the constraint recursively to find some new  $\mathcal{B}'$ . It is interesting to verify that the result of such recursion is indeed a boundary amplitude: we can compute

$$\begin{aligned} \rho_\beta(\mathbf{t}_\mathcal{I})[\frac{1}{1-\mathcal{B}'}] &= \rho_\beta(\mathbf{t}_\mathcal{I})[\frac{1}{1-\mathcal{B}}] + \rho_\beta(\mathbf{t}_\mathcal{I})[\frac{1}{1-\mathcal{B}'}; \mathcal{B}' - \mathcal{B}; \frac{1}{1-\mathcal{B}}] = \\ &= -\rho_\beta(\mathbf{t}_\mathcal{I})[\frac{1}{1-\mathcal{B}'}; \rho_\beta(\mathbf{t}_\mathcal{I})[\frac{1}{1-\mathcal{B}'}; \epsilon; \frac{1}{1-\mathcal{B}}]; \frac{1}{1-\mathcal{B}}] = \\ &= \rho_\beta(\mathbf{t}_\mathcal{I})[\frac{1}{1-\mathcal{B}'}; \rho_\beta(\mathbf{t}_\mathcal{I})[\frac{1}{1-\mathcal{B}'}; \frac{1}{1-\mathcal{B}'}; \epsilon; \frac{1}{1-\mathcal{B}}] + \\ &\quad + \rho_\beta(\mathbf{t}_\mathcal{I})[\frac{1}{1-\mathcal{B}'}; \epsilon; \frac{1}{1-\mathcal{B}} \rho_\beta(\mathbf{t}_\mathcal{I})[\frac{1}{1-\mathcal{B}}], \frac{1}{1-\mathcal{B}}] \end{aligned} \quad (5.43)$$

This relation proves recursively that  $\rho_\beta(\mathbf{t}_\mathcal{I})[\frac{1}{1-\mathcal{B}'}] = 0$ .

If we have a continuous family  $\epsilon(s)$ , the corresponding family of branes  $\mathcal{B}(s)$  is the exponentiation of an exact deformation. Start from

$$\dot{\mathcal{B}}(s) + \rho_\beta(\mathbf{t}_\mathcal{I})[\frac{1}{1-\mathcal{B}(s)}; \dot{\mathcal{B}}(s); \frac{1}{1-\mathcal{B}(s)}; \epsilon(s); \frac{1}{1-\mathcal{B}}] + \rho_\beta(\mathbf{t}_\mathcal{I})[\frac{1}{1-\mathcal{B}(s)}; \dot{\epsilon}(s); \frac{1}{1-\mathcal{B}}] = 0 \quad (5.44)$$

This equation is solved by  $\dot{\mathcal{B}}(s) = -\rho_\beta(\mathbf{t}_\mathcal{I})[\frac{1}{1-\mathcal{B}(s)}; \dot{\epsilon}(s); \frac{1}{1-\mathcal{B}(s)}]$ . We can exclude other solutions recursively, by writing  $\dot{\mathcal{B}}(s) = r - \rho_\beta(\mathbf{t}_\mathcal{I})[\frac{1}{1-\mathcal{B}(s)}; \dot{\epsilon}(s); \frac{1}{1-\mathcal{B}(s)}]$  and plugging into the second term in the equation. Conversely, the exponentiation of an exact deformation is a family of isomorphisms. Of course, the notion of isomorphism is sensible even when boundary amplitudes were defined, say, on  $\mathbb{Z}$ .

## 5.7 Vacuum And Brane Categories For The Theories $\mathcal{T}_\vartheta^N$ And $\mathcal{T}_\vartheta^{SU(N)}$

As an illustration of the above definitions we comment on  $\mathfrak{Vac}$  and  $\mathfrak{Br}$  for the Theories  $\mathcal{T}_\vartheta^N$  and  $\mathcal{T}_\vartheta^{SU(N)}$  described in Section 4.6. Again, we will work with very small, positive  $\vartheta$ .

Let us consider first the morphism spaces  $\text{Hop}(i, j)$  of  $\mathfrak{Vac}$ , for either theory. In principle these could be worked out from equation (5.38), but it is easier to enumerate the half-plane fans and work out the complexes in special cases. These divide into 4 cases because (for small positive  $\vartheta$ ) there are four distinct kinds of half-plane fans as enumerated below Figure 30. Recall these depend on whether  $i, j$  are upper or lower vacua, determined by the sign of the imaginary parts of  $z_i$  and  $z_j$ . Of course we always have  $\text{Hop}(i, i) = \widehat{R}_{ii} = \mathbb{Z}$  in degree zero. On the other hand, when  $i \neq j$  we have:

1.  $\{d \dots d\}$ . If both  $i$  and  $j$  are lower vacua, then  $\text{Hop}(i, j)$  is non-empty only if  $i \leq j$ . If  $i < j$  the possible half-plane fans  $J$  between  $i$  and  $j$  can be enumerated by all strictly increasing sequences of integers beginning with  $i$  and ending with  $j$ . The phase  $\vartheta_{ij}$  of  $z_{ij}$  is given very nearly (for  $\vartheta \rightarrow 0^+$ ) by  $\tan(\vartheta_{ij}) = \cot(\pi \frac{i+j}{N})$  so the product (5.38) simplifies considerably and we may write

$$\mathbb{Z} \cdot \mathbf{1} + \bigoplus_{0 \leq i < j < \frac{N}{2}} \widehat{R}_{ij} e_{ij} = \bigotimes_{0 \leq i < j < \frac{N}{2}} (\mathbb{Z} \cdot \mathbf{1} + R_{ij} e_{ij}) \quad (5.45)$$

where the product on the RHS is ordered left to right by increasing values of  $i + j$  (since we also have  $i < j$  the product is then well-defined). In terms of  $\widehat{R}_{ij}$  we have

$$\text{Hop}(i, j) = \begin{cases} \widehat{R}_{ij} & 0 \leq i \leq j < \frac{N}{2} \\ 0 & 0 \leq j < i < \frac{N}{2} \end{cases} \quad (5.46)$$

2.  $\{u \dots d\}$ . Similarly, if  $i = N - k$  is an upper vacuum, so  $1 \leq k \leq \frac{N}{2}$  and  $0 \leq j < \frac{N}{2}$  is a lower vacuum then

$$\text{Hop}(N - k, j) = \begin{cases} 0 & 0 \leq j < k \\ \widehat{R}_{N-k,j} & k \leq j < \frac{N}{2} \end{cases} \quad (5.47)$$

where for  $k \leq j$  we have

$$\widehat{R}_{N-k,j} = \bigoplus_{\ell=k}^j R_{N-k,\ell} \otimes \widehat{R}_{\ell,j} \quad (5.48)$$

3.  $\{d \dots u\}$ . Now if  $i$  is a lower vacuum and  $j = N - k$  is upper then

$$\text{Hop}(i, N - k) = \begin{cases} 0 & k \leq i < \frac{N}{2} \\ \widehat{R}_{i,N-k} & 0 \leq i < k \end{cases} \quad (5.49)$$

where for  $i < k$

$$\widehat{R}_{i,N-k} := \bigoplus_{\ell=i}^{k-1} \widehat{R}_{i,\ell} \otimes R_{\ell,N-k} \quad (5.50)$$

4.  $\{u \dots u\}$ . We label a pair of up vacua by  $(N-k, N-t)$  with  $1 \leq k, t \leq \frac{N}{2}$ . Then

$$\text{Hop}(N-k, N-t) = \begin{cases} 0 & t < k \\ \mathbb{Z} & t = k \\ \widehat{R}_{N-k,N-t} & k \leq \ell \leq s \leq t-1 \end{cases} \quad (5.51)$$

where for  $k \leq t-1$ :

$$\widehat{R}_{N-k,N-t} := R_{N-k,N-t} \oplus \bigoplus_{k \leq \ell \leq s \leq t-1} R_{N-k,\ell} \otimes \widehat{R}_{\ell,s} \otimes R_{s,N-t}. \quad (5.52)$$

Now let us describe the multiplication operations on  $\mathfrak{Vac}$ , again for both theories. Since the only taut webs which contribute to  $\rho_\beta$  have one or two boundary vertices the  $A_\infty$ -category  $\mathfrak{Vac}$  is in fact just a differential graded algebra. The differential  $m_1$  arises from Figure 31(b). The multiplication  $m_2$  arises from Figures 31(a) and 31(c). In 31(a)  $J_1, J_2$  share a common d-type vacuum. In Figure 31(c) the intermediate vacuum is an up-type vacuum,  $j_s \in [\frac{N}{2}, N-1]$ .

Now let us specialize the discussion to  $\mathfrak{Vac}(\mathcal{T}_\vartheta^N)$ . In this case the spaces  $R_{ij}$  are very simple and given by (4.79). The key spaces  $\widehat{R}_{ij}$  for a pair of down vacua with  $i < j$  is - as we have said above - a sum over all increasing sequences of integers beginning with  $i$  and ending with  $j$ . Each sequence contributes a summand  $R_J \cong \mathbb{Z}^{[|J|-1]}$  to  $\widehat{R}_{ij}$ . Denoting generators of  $\text{Hop}(i,j)$  by  $e_J$ , where  $J$  is a positive half-plane fan,  $m_1(e_J)$  is a signed sum of generators obtained by inserting a down-type vacuum into  $J$  in all possible ways. For example, if  $J = \{d_1, \dots, d_s\}$  with  $0 \leq d_1 < d_s < \frac{N}{2}$  then

$$m_1(e_{d_1 \dots d_s}) = \sum_{n=1}^{s-1} \sum_{d=d_n+1}^{d_{n+1}-1} (-1)^{n-1} e_{d_1 \dots d_n d d_{n+1} \dots d_s} \quad (5.53)$$

The sign is determined by the usual patient commutation of  $K_{d_n d_{n+1}}$  through the first  $(n-1)$   $R_{ij}$  factors in  $R_J \otimes R_{d_n d} \otimes R_{d d_{n+1}} \otimes R_{d_{n+1} d_n}$ . Note that  $R_{d_{n+1} d_n}$  has degree zero and can be brought from the far right into the relevant place in the product to effect the contraction by  $K$  in the definition of  $\rho_\beta$ . It is easy to check that  $m_1$  is nilpotent. Similar formulae hold for the other three types of half-plane fans  $J$ : The operation  $m_1$  is a signed sum of all fans where we insert one extra down vacuum.

Now consider the multiplication of  $e_{J_1}$  and  $e_{J_2}$  with

$$J_1 = \{j_1, \dots, j_{s-1}, j_s\} \quad J_2 = \{j_s, j_{s+1}, \dots, j_n\} \quad (5.54)$$

If the intermediate vacuum is down-type  $0 \leq j_s < \frac{N}{2}$  then

$$m_2(e_{J_1}, e_{J_2}) = e_{J_1 * J_2} \quad (5.55)$$

with

$$J_1 * J_2 := \{j_1, \dots, j_{s-1}, j_s, j_{s+1}, \dots, j_n\} \quad (5.56)$$

If the intermediate vacuum  $j_s$  is an up-type vacuum,  $j_s \in [\frac{N}{2}, N - 1]$  then

$$\rho(\mathbf{t}_{\mathcal{H}})(e_{J_1}, e_{J_2}) = K_{j_s, j_{s-1}} K_{j_s, j_{s+1}} (e_{J_1} \otimes e_{J_2} \otimes \beta_{j_{s-1}, j_s, j_{s+1}}) = e_{J_1 \wedge J_2} \quad (5.57)$$

with

$$J_1 \wedge J_2 := \{j_1, \dots, j_{s-1}, j_{s+1}, \dots, j_n\}. \quad (5.58)$$

Next let us turn to the Brane category  $\mathfrak{Br}(\mathcal{T}_\vartheta^N)$ . We will limit ourselves to the description of some of the Hom-spaces and their cohomology using the differential  $M_1$ . As noted above, the  $M_1$  cohomology will have the physical interpretation as the space of  $\mathcal{Q}$ -invariant local operators changing boundary conditions of one Brane into another.

The easiest class of Branes to consider are of course the thimbles  $\mathfrak{T}_i$  since

$$\text{Hop}(\mathfrak{T}_i, \mathfrak{T}_j) = \text{Hop}(i, j) \quad (5.59)$$

and in this case  $M_n = m_n^{\mathfrak{vac}}$ . In particular we can study the cohomology of  $m_1$ . It is easy to see that the cohomology is nonzero in general. This happens when the spaces (5.46)-(5.51) consist of a single summand  $R_{ij}$ . Then  $R_{ij}$  has a definite degree so  $m_1$  must be identically zero, and hence the cohomology is the space  $R_{ij}$  itself. Thus, for example,  $\text{Hop}(i, i)$ ,  $\text{Hop}(i, i + 1)$ ,  $\text{Hop}(N - k, k)$ ,  $\text{Hop}(k - 1, N - k)$  and  $\text{Hop}(N - k, N - k - 1)$  have  $m_1 = 0$  and are equal to their own cohomologies. On the other hand, in the other cases we can construct a contracting homotopy so that the cohomology vanishes. Consider, for example, the case where  $i < j$  are two down-type vacua and  $i + 1 < j$ . Then define

$$\kappa(e_{id_2 \dots d_{s-1} j}) := \begin{cases} e_{id_3 \dots d_{s-1} j} & d_2 = i + 1 \\ 0 & d_2 > i + 1 \end{cases} \quad (5.60)$$

One can check that  $\kappa m_1 + m_1 \kappa = \text{Id}$  and hence the cohomology vanishes. Similarly, for  $\text{Hop}(N - k, j)$  with  $j > k$  we can define a contracting homotopy operator

$$\kappa(e_{N-k, d_2 \dots d_{s-1} j}) := \begin{cases} e_{N-k, d_3 \dots d_{s-1} j} & d_2 = k \\ 0 & d_2 > k \end{cases} \quad (5.61)$$

and so forth.

In conclusion: *The cohomology of  $\text{Hop}(\mathfrak{T}_i, \mathfrak{T}_j)$  is nonzero if and only if the only half-plane fan of the form  $J = \{i, \dots, j\}$  is in fact  $J = \{i, j\}$ .* We leave the calculation of the appropriate homotopy contractions to the enthusiastic reader.

As a test, we present the matrix of Poincaré polynomials for the  $\text{Hop}(\mathfrak{T}_i, \mathfrak{T}_j)$  for  $N =$

10:

$$\begin{pmatrix} 1 & y & y(y+1) & y(y+1)^2 & y(y+1)^3 & y(y+1)^4 & y(y+1)^3 & y(y+1)^2 & y(y+1) & y \\ 0 & 1 & y & y(y+1) & y(y+1)^2 & y(y+1)^3 & y(y+1)^2 & y(y+1) & y & 0 \\ 0 & 0 & 1 & y & y(y+1) & y(y+1)^2 & y(y+1) & y & 0 & 0 \\ 0 & 0 & 0 & 1 & y & y(y+1) & y & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & y & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & y+1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & y+1 & (y+1)^2 & y+1 & 1 & 0 & 0 \\ 0 & 0 & 1 & y+1 & (y+1)^2 & (y+1)^3 & (y+1)^2 & y+1 & 1 & 0 \\ 0 & 1 & y+1 & (y+1)^2 & (y+1)^3 & (y+1)^4 & (y+1)^3 & (y+1)^2 & y+1 & 1 \end{pmatrix} \quad (5.62)$$

Setting  $y = -1$  we recover the Witten indices of  $\text{Hop}(\mathfrak{T}_i, \mathfrak{T}_j)$ , i.e. the  $\hat{\mu}_{ij}$ :

$$\begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad (5.63)$$

This supports our statement. Equation (5.63), and similar examples below follow a pattern that leads to a conjectural formula for the general case. We leave it as a challenge to the reader to give a proof of these formulae.

Next we can look at the morphisms between the Branes  $\mathfrak{C}_k$  defined in Section §4.6 and the thimbles. Using the formula (4.91) for the Chan-Paton spaces of  $\mathfrak{C}_k$  we have

$$\text{Hop}(\mathfrak{C}_k, \mathfrak{T}_j) = \text{Hop}(N - k, j)^{[1]} \oplus \text{Hop}(N - k - 1, j) \oplus \text{Hop}(k, j) \quad (5.64)$$

Consider the multiplications  $M_n$  for this pair of Branes. Since the only webs which contribute to  $\mathfrak{t}_{\mathcal{H}}$  are those shown in Figure 31 the only nonzero multiplications  $m_n^{\mathfrak{V}\mathfrak{ac}}$  are  $m_1$  and  $m_2$  and hence in the category of Branes likewise only  $M_1$  and  $M_2$  can be nonzero. Moreover  $M_2$  coincides with  $m_2$ . However, because  $\mathfrak{C}_k$  has nontrivial boundary amplitudes  $\mathcal{B}(\mathfrak{C}_k)$  with components  $\mathcal{B}_{N-k, N-k-1}$ ,  $\mathcal{B}_{k, N-k-1}$ , and  $\mathcal{B}_{N-k, k}$  there will be an important difference between  $m_1$  and  $M_1$ . In particular,

$$\begin{aligned} M_1(\delta) &= m^{\mathfrak{V}\mathfrak{ac}}\left(\frac{1}{1 - \mathcal{B}(\mathfrak{C}_k)}, \delta, \frac{1}{1 - \mathcal{B}(\mathfrak{T}_j)}\right) \\ &= m^{\mathfrak{V}\mathfrak{ac}}\left(\frac{1}{1 - \mathcal{B}(\mathfrak{C}_k)}, \delta\right) \\ &= m_1(\delta) + m_2(\mathcal{B}(\mathfrak{C}_k), \delta) \end{aligned} \quad (5.65)$$

Now, in order to analyze the  $M_1$ -cohomology of (5.64) recall that  $\mathfrak{C}_k$  are only defined when  $k$  is a down-type vacuum. Then, when  $j$  is a down-type vacuum we can use (5.46) et. seq. above to find the morphism spaces:

$$\text{Hop}(\mathfrak{C}_k, \mathfrak{T}_j) = \begin{cases} \widehat{R}_{N-k,j}^{[1]} \oplus \widehat{R}_{N-k-1,j} \oplus \widehat{R}_{k,j} & k+1 \leq j < \frac{N}{2} \\ R_{N-k,k}^{[1]} \oplus \mathbb{Z} & 0 \leq k = j < \frac{N}{2} \\ 0 & 0 \leq j < k < \frac{N}{2} \end{cases} \quad (5.66)$$

If  $k = j$  then  $\text{Hop}(\mathfrak{C}_k, \mathfrak{T}_k) \cong \mathbb{Z}^{[1]} \oplus \mathbb{Z}$  and under this isomorphism  $M_1(x \oplus y) = y \oplus 0$ . The cohomology is therefore zero. If  $k+1 \leq j$  and we write the three components of  $\delta$  as

$$\delta = \delta_{N-k,j} \oplus \delta_{k,j} \oplus \delta_{N-k-1,j} = (\delta_{N-k,j}, \delta_{k,j}, \delta_{N-k-1,j}) \quad (5.67)$$

and then we have

$$\begin{aligned} M_1(\delta_{N-k,j}, 0, 0) &= (m_1(\delta_{N-k,j}), 0, 0) \\ M_1(0, \delta_{k,j}, 0) &= (m_2(\mathcal{B}_{N-k,k}, \delta_{k,j}), m_1(\delta_{k,j}), 0) \\ &= (h_k m_2(e_{N-k,k}, \delta_{k,j}), m_1(\delta_{k,j}), 0) \\ M_1(0, 0, \delta_{N-k-1,j}) &= (m_2(\mathcal{B}_{N-k,N-k-1}, \delta_{N-k-1,j}), m_2(\mathcal{B}_{k,N-k-1}, \delta_{N-k-1,j}), m_1(\delta_{N-k-1,j})) \\ &= (f_k m_2(e_{N-k,N-k-1}, \delta_{N-k-1,j}), g_k m_2(e_{k,N-k-1}, \delta_{N-k-1,j}), m_1(\delta_{N-k-1,j})) \end{aligned} \quad (5.68)$$

The ordering is chosen so that the upper-triangular structure is clear. It is a good exercise to show that  $M_1^2 = 0$  (one must use  $f_k = h_k g_k$ ).

Now we claim that there is a homotopy contraction of  $M_1$  of the form

$$\begin{pmatrix} \kappa & \kappa_{12} & \kappa_{13} \\ 0 & \kappa & \kappa_{23} \\ 0 & 0 & \kappa \end{pmatrix} \quad (5.69)$$

For this to be a homotopy contraction we need (we use the property that  $h_k$  is multiplication by 1 here):

$$\kappa_{12}(m_1(\delta_{k,j})) + m_1(\kappa_{12}(\delta_{k,j})) + \kappa(m_2(e_{N-k,k}, \delta_{k,j})) + m_2(e_{N-k,k}, \kappa(\delta_{k,j})) = 0 \quad (5.70)$$

with similar equations for  $\kappa_{13}$  and  $\kappa_{23}$ . One can solve this using

$$\kappa_{12}(e_{k,d_2,d_3,\dots,j}) = \begin{cases} -e_{N-k,d_3,\dots,j} & d_2 = k+1 \\ 0 & \text{else} \end{cases} \quad (5.71)$$

Thus, there are never boundary-condition-changing operators from  $\mathfrak{C}_k$  to down-type thimbles.

Turning now to the case of thimbles for up-type vacua we write, for  $1 \leq t \leq \frac{N}{2}$ ,

$$\text{Hop}(\mathfrak{C}_k, \mathfrak{T}_{N-t}) = \text{Hop}(N-k, N-t)^{[1]} \oplus \text{Hop}(N-k-1, N-t) \oplus \text{Hop}(k, N-t) \quad (5.72)$$

Using (5.46) et. seq. above we find that this vanishes when  $k > t$ . When  $t = k$  only the first summand, namely,

$$\text{Hop}(N - k, N - k)^{[1]} \cong \mathbb{Z}^{[1]}, \quad (5.73)$$

is nonzero so the complex is concentrated in a single degree, therefore  $M_1 = 0$  and the cohomology is nontrivial. For  $k \leq t - 1$  all three summands are nonzero and we expect that a nontrivial analysis like that above for down-type thimbles shows there are no other cases with nonzero cohomology.

In conclusion we have given considerable evidence for the claim that  $\text{Hop}(\mathfrak{C}_k, \mathfrak{T}_j)$  only has nontrivial cohomology when  $j = N - k$ , in which case the cohomology is one-dimensional.

As a test, we provide the matrix of Poincaré polynomials for  $\text{Hop}(\mathfrak{C}_k, \mathfrak{T}_j)$  for  $N = 10$ :

$$\begin{pmatrix} 0 & y+1 & (y+1)^2 & (y+1)^3 & (y+1)^4 & (y+1)^5 & (y+1)^4 & (y+1)^3 & (y+1)^2 & y \\ 0 & 0 & y+1 & (y+1)^2 & (y+1)^3 & (y+1)^4 & (y+1)^3 & (y+1)^2 & y & 0 \\ 0 & 0 & 0 & y+1 & (y+1)^2 & (y+1)^3 & (y+1)^2 & y & 0 & 0 \\ 0 & 0 & 0 & 0 & y+1 & (y+1)^2 & y & 0 & 0 & 0 \end{pmatrix} \quad (5.74)$$

We expect that a very similar story holds for

$$\text{Hop}(\mathfrak{T}_i, \mathfrak{C}_k) = \text{Hop}(i, N - k)^{[1]} \oplus \text{Hop}(i, N - k - 1) \oplus \text{Hop}(i, k). \quad (5.75)$$

It is straightforward to check that for  $i = k$  there are only two nonzero summands and the complex is isomorphic to  $\mathbb{Z}^{[1]} \oplus \mathbb{Z}$  with  $M_1(x \oplus y) = y \oplus 0$ , and hence the cohomology vanishes. The other easy case is  $i = N - k - 1$ . Then only the middle summand is nonzero so we get nonzero cohomology. We expect that for the other values,  $i \neq k, N - k - 1$ , a detailed analysis like that we did above for the other order would show that the cohomology vanishes, but we have not confirmed this. In any case, we expect that *the only nonzero cohomology appears for  $\text{Hop}(\mathfrak{T}_{N-k-1}, \mathfrak{C}_k)$ , in which case it is one-dimensional*. Again, the cohomology is limited to half-plane fans of length 2.

As a test, we provide the matrix of Poincaré polynomials for  $\text{Hop}(\mathfrak{T}_i, \mathfrak{C}_k)$  for  $N = 10$ :

$$\begin{pmatrix} (y+1)^2 & y+1 & 0 & 0 & 0 & 0 & 0 & 1 & \frac{(y+1)^2}{y} \\ (y+1)^3 & (y+1)^2 & y+1 & 0 & 0 & 0 & 1 & \frac{(y+1)^2}{y} & \frac{(y+1)^3}{y} \\ (y+1)^4 & (y+1)^3 & (y+1)^2 & y+1 & 0 & 0 & 1 & \frac{(y+1)^2}{y} & \frac{(y+1)^3}{y} \\ (y+1)^5 & (y+1)^4 & (y+1)^3 & (y+1)^2 & y+1 & 1 & \frac{(y+1)^2}{y} & \frac{(y+1)^3}{y} & \frac{(y+1)^4}{y} \end{pmatrix} \quad (5.76)$$

Finally, we could look at the morphisms  $\text{Hop}(\mathfrak{C}_k, \mathfrak{C}_t)$ . Each Brane has Chan-Paton spaces with nonzero support at three vacua and hence we now get nine summands:

$$\begin{aligned} \text{Hop}(\mathfrak{C}_k, \mathfrak{C}_t) &= \text{Hop}(N - k, N - t) \oplus \text{Hop}(N - k - 1, N - t)^{[-1]} \oplus \text{Hop}(k, N - t)^{[-1]} \\ &\oplus \text{Hop}(N - k, N - t - 1)^{[1]} \oplus \text{Hop}(N - k - 1, N - t - 1) \oplus \text{Hop}(k, N - t - 1) \\ &\oplus \text{Hop}(N - k, t)^{[1]} \oplus \text{Hop}(N - k - 1, t) \oplus \text{Hop}(k, t) \end{aligned} \quad (5.77)$$

and the differential is

$$M_1(\delta) = m_1(\delta) + m_2(\mathcal{B}(\mathfrak{C}_k), \delta) + m_2(\delta, \mathcal{B}(\mathfrak{C}_t)). \quad (5.78)$$

A systematic analysis of the cohomology would be tedious. So we will limit ourselves to the cases  $t \leq k$ . If  $t < k - 1$  then  $\text{Hop}(\mathfrak{C}_k, \mathfrak{C}_t) = 0$ . If  $t = k - 1$  then the morphism space is concentrated in one degree and is  $\text{Hop}(N - k, N - k)^{[1]}$ . Therefore the cohomology is nonzero in this case. For  $t = k$  we must carry out a nontrivial computation. There are six nonvanishing morphism spaces

$$\begin{aligned} & \text{Hop}(N - k, N - k) \oplus \text{Hop}(N - k - 1, N - k - 1) \oplus \text{Hop}(k, k) \\ & \oplus \text{Hop}(N - k, N - k - 1)^{[1]} \oplus \text{Hop}(k, N - k - 1) \oplus \text{Hop}(N - k, k)^{[1]} \end{aligned} \quad (5.79)$$

This space has rank 7 because

$$\text{Hop}(N - k, N - k - 1)^{[1]} = R_{N-k, N-k-1}^{[1]} \oplus R_{N-k, k, N-k-1}^{[1]} \quad (5.80)$$

has rank 2. A little computation shows that

$$M_1 : \begin{pmatrix} \delta_{N-k, N-k} \\ \delta_{N-k-1, N-k-1} \\ \delta_{k, k} \\ \delta_{N-k, N-k-1} \\ \delta_{N-k, k, N-k-1} \\ \delta_{k, N-k-1} \\ \delta_{N-k, k} \end{pmatrix} \mapsto \begin{pmatrix} 0 \\ 0 \\ 0 \\ \delta_{N-k, N-k} - \delta_{N-k-1, N-k-1} \\ \delta_{k, N-k-1} + \delta_{N-k, k} - \delta_{N-k, N-k-1} \\ \delta_{k, k} - \delta_{N-k-1, N-k-1} \\ \delta_{N-k, N-k} - \delta_{k, k} \end{pmatrix} \quad (5.81)$$

A short computation then shows that the cohomology is rank one and generated by  $(1, 1, 1, 0, 0, 0, 0)$ .

In a similar way we conjecture the absence of cohomology when  $t > k$ . As a test, we provide the matrix of Poincaré polynomials for  $\text{Hop}(\mathfrak{C}_k, \mathfrak{C}_t)$  for  $N = 10$ :

$$\begin{pmatrix} y^2 + 3y + 3 & \frac{(y+1)^4}{y} & \frac{(y+1)^5}{y} & \frac{(y+1)^6}{y} \\ y & y^2 + 3y + 3 & \frac{(y+1)^4}{y} & \frac{(y+1)^5}{y} \\ 0 & y & y^2 + 3y + 3 & \frac{(y+1)^4}{y} \\ 0 & 0 & y & y^2 + 3y + 3 \end{pmatrix} \quad (5.82)$$

Now let us consider briefly the  $\mathcal{T}_\vartheta^{SU(N)}$  theories of Section §4.6. The formulae (5.46)-(5.51) apply to this case as well. In this way we find the following morphism spaces:

For two lower vacua,  $0 \leq i < j < \frac{N}{2}$  we have:

$$\text{Hop}(i, j) = \sum_{n \geq 1} \sum_{\sum_{s=1}^n d_s = j-i} \bigotimes_{s=1}^n A_{d_s}^{[1]} \quad (5.83)$$

If  $i = N - k$  is an upper vacuum and  $j$  a lower vacuum, we need  $j \geq k$  for a nonzero morphism space. In this case we have:

$$\text{Hop}(N - k, j) = A_{j+k} \oplus \sum_{n > 1} \sum_{\sum_{s=1}^n d_s = j-k+1} A_{2k-1+d_1} \otimes \bigotimes_{s=2}^n A_{d_s}^{[1]}. \quad (5.84)$$

If  $i$  is a lower vacuum, and  $j = N - k$  an upper vacuum, similar considerations apply with  $i < k$ :

$$\text{Hop}(i, N - k) = A_{N-k-i}^{[1]} \oplus \sum_{n>1} \sum_{\sum_{s=1}^n d_s = k-i} \bigotimes_{s=1}^{n-1} A_{d_s}^{[1]} \otimes A_{N-2k+d_n}^{[1]} \quad (5.85)$$

Finally if we have two upper vacua an  $t > k$  then

$$\text{Hop}(N - k, N - t) = A_{N+k-t} \oplus \sum_{n>1} \sum_{\sum_{s=1}^n d_s = t-k+1} A_{2k-1+d_1} \otimes \bigotimes_{s=1}^{n-1} A_{d_s}^{[1]} \otimes A_{N-2t+d_n}^{[1]} \quad (5.86)$$

In all four cases (5.83)-(5.86) the sums are over partitions with  $d_s > 0$ .

Let us turn now to the differential  $m_1$  in the vacuum category  $\mathfrak{Vac}(\mathcal{T}_\vartheta^{SU(N)})$ . We use the contraction (4.101) and the interior amplitude (4.102) with  $b_{ijk} = b$  for all  $i < j < k$ . For  $\text{Hop}(i, j)$  in (5.83)  $m_1$  is a signed sum of operations on each of the tensor factors in the product. On a factor of the form  $R_{\ell, \ell+d} \cong A_d^{[1]}$  with  $d > 0$  it acts as an intertwiner:

$$m_1 : A_d \rightarrow \bigoplus_{d_1+d_2=d} A_{d_1} \otimes A_{d_2} \quad (5.87)$$

Note that for each decomposition  $d = d_1 + d_2$  there is a canonical intertwiner  $\Pi_{d_1, d_2} : A_{d_1} \otimes A_{d_2} \rightarrow A_d$  given by the wedge product. The components  $m_1^{(d_1, d_2)}$  of  $m_1$  in (5.87) are such that

$$\Pi_{d_1, d_2} \circ m_1^{(d_1, d_2)} e_S = \kappa b \binom{d}{d_1} \varepsilon_S e_S \quad (5.88)$$

(Recall that  $\varepsilon_S = \frac{e_S \wedge e_{S'}}{\text{vol}}$  where  $S'$  is the complementary multi-index to  $S$ .) In formulae

$$m_1(e_S) = \kappa b \varepsilon_S \sum_{d_1=1}^{d-1} \sum_{\text{Sh}_2(S): |S_1|=d_1} \frac{e_{S_1} e_{S_2} e_{S'}}{\text{vol}} e_{S_1} \otimes e_{S_2} \quad (5.89)$$

To describe the multiplication  $m_2$  in the vacuum category we need to distinguish between the two cases where the intermediate vacuum is down-type, as in Figure 31(a) or up-type, as in Figure 31(c). In the first case we simply take a tensor product. In the second case we must use the contraction. Here we are taking a product

$$m_2 : \text{Hop}(i, N - k) \otimes \text{Hop}(N - k, j) \rightarrow \text{Hop}(i, j) \quad (5.90)$$

so we must combine the final factors in equation (5.85) with the initial factors in equation (5.84). The main step is captured by the map  $R_{i, N-k} \otimes R_{N-k, j} \rightarrow R_{i, j}$  with  $i < j < N - k$ , namely

$$A_{N-k-i}^{[1]} \otimes A_{k+j} \rightarrow A_{j-i}^{[1]}. \quad (5.91)$$

This is simply taking the dual of the wedge product of the duals (up to a factor of  $b\kappa^2$ ).

Turning now to the Brane category  $\mathfrak{Br}(\mathcal{T}_\vartheta^{SU(N)})$  we can use the above complexes to compute the space of boundary-changing operators between thimbles,  $H^*(\text{Hop}(\mathfrak{T}_i, \mathfrak{T}_j), M_1)$ . In the  $\mathcal{T}_\vartheta^{SU(N)}$  theories these will be representations of  $SU(N)$ . We conjecture the following

### Conjecture

a.) If  $i, j$  are lower vacua with  $i < j$  then

$$H^*(\text{Hop}(\mathfrak{T}_i, \mathfrak{T}_j), M_1) \cong S_{j-i}^{[j-i]} \quad (5.92)$$

b.) If  $i = N - k$  is an upper vacuum and  $j$  a lower vacuum with  $j \geq k$

$$H^*(\text{Hop}(\mathfrak{T}_{N-k}, \mathfrak{T}_j), M_1) \cong L_{2k, j-k+1}^{[j-k]} \quad (5.93)$$

c.) If  $i$  is a lower vacuum, and  $j = N - k$  an upper vacuum with  $i < k$ :

$$H^*(\text{Hop}(\mathfrak{T}_i, \mathfrak{T}_{N-k}), M_1) \cong L_{N-2k+1, k-i}^{[k-i]} \quad (5.94)$$

The above conjecture is easily checked for the simple cases in which Hop is concentrated in a single degree, but in general appears to be an extremely challenging computation. We will deduce equation (5.92) using the rotational interfaces of Section §7. See equation (7.172) below.

One could contemplate computing the morphisms spaces involving the Branes  $\mathfrak{N}_n$  defined by the Chan-Paton factors (4.117) and amplitudes (4.119) et. seq. We leave this exercise to the truly energetic reader (with lots of time to spare). Since these Branes are generated from thimbles by rotational Interfaces (see equation (7.161) et. seq.) it is conceivable that arguments along the lines of (7.172) lead to a derivation of these cohomology spaces. It would also be interesting to see if these results can be checked using the  $\sigma$ -model or Fukaya-Seidel viewpoint described in Sections §§11-15.

## 6. Interfaces

### 6.1 Interface Webs

#### 6.1.1 Definition And Basic Properties

We now consider webs in the presence of an ‘‘Interface,’’ a notion we will define precisely just below equation (6.7). Roughly speaking, an Interface is a domain wall separating two Theories. For simplicity we take the wall to be localized on the line  $D$  described by  $x = x_0$  in the  $(x, y)$  plane.<sup>24</sup> We now consider two sets of vacuum data,  $(\mathbb{V}^-, z^-)$ , associated with the negative half-plane  $x \leq x_0$  and  $(\mathbb{V}^+, z^+)$  associated with the positive half-plane  $x \geq x_0$ . The data  $(\mathbb{V}^\pm, z^\pm, x_0)$  will be collectively denoted by  $\mathcal{I}$ . As in the half-plane case, we assume that none of the  $z_{ij}^\pm$  are parallel to  $D$ .

#### Definition:

a.) An *interface web* is a union of half-plane webs  $(\mathfrak{u}^-, \mathfrak{u}^+)$ , with  $\mathfrak{u}^-$  a negative half-plane web and  $\mathfrak{u}^+$  a positive half-plane web, where the half-planes share a common boundary  $D$  at  $x = x_0$ . The webs are determined by the vacuum data  $(\mathbb{V}^\pm, z^\pm)$ , respectively.<sup>25</sup>

---

<sup>24</sup>More generally one could rotate our construction in the plane.

<sup>25</sup>We stress that at this point  $\mathfrak{u}^-$  and  $\mathfrak{u}^+$  are webs, not deformation types of webs.

- b.) An *interface fan* is the union of a fan for the negative half-plane data and a fan for the positive half-plane data. It will be denoted  $J = (J^+, J^-)$  where one of  $J^\pm$  (but not both) is allowed to be the empty set.

**Remarks:**

1. Let  $\mathfrak{d}$  denote a typical interface web.<sup>26</sup> We divide up the set of vertices of  $\mathfrak{d}$  into the set of *wall vertices*  $\mathcal{V}_\partial(\mathfrak{d})$  located on  $D$  and *interior vertices*  $\mathcal{V}_i^\pm(\mathfrak{d})$  in either half-plane with cardinalities  $V_\partial(\mathfrak{d})$  and  $V_i^\pm(\mathfrak{d})$ , respectively. The interior edges are subsets of the negative or positive half-planes, cannot lie in  $D$ , and do not go to infinity. The sets of interior edges are denoted  $\mathcal{E}^\pm(\mathfrak{d})$  and have cardinality  $E^\pm(\mathfrak{d})$ .
2. We will consider  $\mathcal{V}_\partial(\mathfrak{d})$  to be an *ordered set*. Our convention is that reading left to right we order the vertices from future to past, as we would for a left boundary of a positive half-plane.
3. An interface web has an interface fan at infinity  $J_\infty(\mathfrak{d})$ . If  $\mathfrak{d} = (\mathfrak{u}^-, \mathfrak{u}^+)$  then  $J_\infty(\mathfrak{d}) = \{J_\infty(\mathfrak{u}^+); J_\infty(\mathfrak{u}^-)\}$ . Similarly, if  $v \in \mathcal{V}_\partial(\mathfrak{d})$  we can define local interface fans  $J_v(\mathfrak{d})$ .
4. Using a standard reflection trick we could make a precise correspondence between interface webs and half-plane webs for the “disjoint union” of the vacuum data (a term we will not try to make precise). Note that any half-plane web could be seen as an interface web with trivial vacuum data on one side of  $D$ . As we will see, interface webs behave very much like half-plane webs.
5. We can speak of a deformation type of an interface web  $\mathfrak{d}$ . In order to avoid confusion, let us stress that the deformation type  $\mathfrak{d}$  of an interface web is *not* just a pair of deformation types of negative and positive half-plane webs. The reason is that when vertices of the negative and positive half-plane webs coincide deformations must maintain this identification. When they do *not* coincide, deformations must preserve the relative order. See, for example, Figure 38. In particular note that one cannot unambiguously combine deformation types of negative and positive half-plane webs into a deformation type of an interface web. Thus if we identify  $\mathfrak{d}$  with  $(\mathfrak{u}^-, \mathfrak{u}^+)$  we must bear in mind that  $\mathfrak{u}^\pm$  represent webs, not deformation types, so there is further data specifying how the webs are combined, in particular, how their boundary vertices are ordered and/or identified to form the set of wall vertices of  $\mathfrak{d}$ . When the vacuum data are in general position, the moduli space  $\mathcal{D}(\mathfrak{d})$  of interface webs of a fixed deformation type has dimension

$$d(\mathfrak{d}) := (2V_i^-(\mathfrak{d}) - E^-(\mathfrak{d})) + (2V_i^+(\mathfrak{d}) - E^+(\mathfrak{d})) + V_\partial(\mathfrak{d}). \quad (6.1)$$

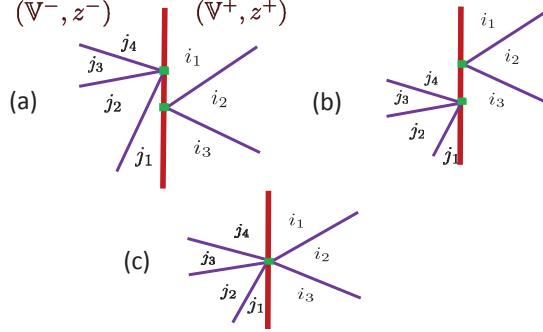
For nongeneric vacuum data there can be exceptional webs with  $\dim \mathcal{D}(\mathfrak{d}) > d(\mathfrak{d})$ .

6. We define interface webs to be *rigid*, *taut*, and *sliding* if  $d(\mathfrak{d}) = 1, 2, 3$ , respectively, just as for half-plane webs. Similarly, we define oriented deformation type and consider the free abelian group  $\mathcal{W}_{\mathcal{I}}$  generated by oriented deformation types of interface

---

<sup>26</sup>The gothic “d,” which looks like  $\mathfrak{d}$ , is for “domain wall,” although in the course of our work that term has been deprecated in favor of “interface.”

webs. The taut webs have a canonical orientation (towards larger webs) and we denote the sum of taut canonically oriented interface webs by  $\mathbf{t}_{\mathcal{I}}^{-,+}$  or, usually, just  $\mathbf{t}^{-,+}$  when the data  $\mathcal{I}$  is understood. We can also denote by  $\mathcal{W}_p^{\pm}$  the group of plane webs associated to the data in the positive and negative half-planes respectively.



**Figure 38:** The three interface webs shown here have different deformation types. The webs (a) and (b) are taut, while (c) is rigid. In all three webs  $J_{\infty}(\mathfrak{d}) = \{i_1, i_2, i_3; j_1, j_2, j_3, j_4\}$ . In Figure (b) the top vertex has  $J_v(\mathfrak{d}) = \{i_1, i_2, i_3; j_4\}$ .

There are natural convolution operations inserting elements of  $\mathcal{W}_p^{\pm}$ ,  $\mathcal{W}_{\mathcal{I}}$  at interior vertices in the appropriate half-planes or at wall vertices respectively and we have the natural

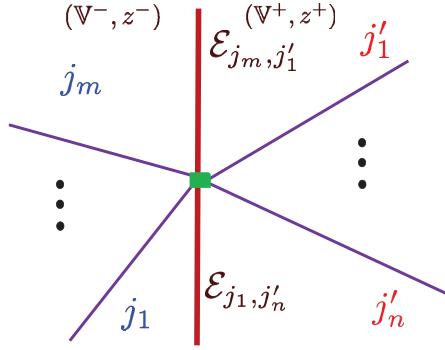
**Theorem:** Let  $\mathbf{t}_{\mathcal{I}}^{-,+}$  be the interface taut element and  $\mathbf{t}_p^{\pm}$  the plane taut elements associated to the data in the two half-planes. It is useful to define the formal sum  $\mathbf{t}_p = \mathbf{t}_p^+ + \mathbf{t}_p^-$ . We have a familiar-looking convolution identity:

$$\mathbf{t}_{\mathcal{I}}^{-,+} * \mathbf{t}_{\mathcal{I}}^{-,+} + \mathbf{t}_{\mathcal{I}}^{-,+} * \mathbf{t}_p = 0. \quad (6.2)$$

The proof is closely modeled on that of the half-plane case (2.30).

### 6.1.2 Tensor Algebra Structures

Turning now to the tensor algebras of webs, we also have natural operations associated to the insertion of appropriate webs at all interior vertices on either half plane and/or at the wall vertices. As usual, the operations involving interior vertices define complicated  $L_{\infty}$ -type structures. The basic operation  $T_{\partial}(\mathfrak{d})$  on  $T\mathcal{W}_{\mathcal{I}}$ , denoted  $T_{\partial}(\mathfrak{d})[\mathfrak{d}_1, \dots, \mathfrak{d}_n]$  is defined as usual by replacing all wall vertices of  $\mathfrak{d}$  on  $D$  with appropriate interface webs with  $J_{v_a}(\mathfrak{d}) = J_{\infty}(\mathfrak{d}_a)$ . To repeat, the ordering of vertices  $v_a$  is toward decreasing  $y$ . This behaves as it did for half-plane webs and in particular applying the reasoning of (3.26) et seq. the operator  $T_{\partial}(\mathbf{t}_{\mathcal{I}}^{-,+}) : T\mathcal{W}_{\mathcal{I}} \rightarrow \mathcal{W}_{\mathcal{I}}$  defines the structure of an  $A_{\infty}$  algebra on  $\mathcal{W}_{\mathcal{I}}$ .



**Figure 39:** Conventions for Chan-Paton factors localized on interfaces. If representation spaces are attached to the rays then this figure would represent a typical summand in  $\text{Hom}(j_m j'_1, j_1 j'_n)$ . We order such vertices from left to right using the conventions of positive half-plane webs.

### 6.1.3 Web Representations, Interfaces, And Interface Categories

We now consider the algebraic structures that arise when we are given a pair of representations of the vacuum data  $(\mathbb{V}^\pm, z^\pm)$ . The discussion closely parallels that for the half-plane theory.

We define a *representation of interface webs* to be a pair of representations

$$\mathcal{R}_{\mathcal{I}} = \left( (\{R_{ij}^-\}, \{K_{ij}^-\}), (\{R_{i'j'}^+\}, \{K_{i'j'}^+\}) \right) \quad (6.3)$$

for the vacuum data  $\mathbb{V}^\pm$ . Similarly, we define *Chan-Paton data* for an interface to be an assignment  $(i, i') \rightarrow \mathcal{E}_{i,i'}$  where the Chan-Paton factors  $\mathcal{E}_{i,i'}$  are graded  $\mathbb{Z}$ -modules. We picture this with a vacuum  $i$  on the negative half plane and  $i'$  on the positive half plane with the Chan-Paton factor located on the boundary  $D$ . Our convention will be that the wall vertices on  $D$  of interface fans  $J = \{j'_1, \dots, j'_n; j_1, \dots, j_m\}$  will be represented by the graded  $\mathbb{Z}$ -module:

$$R_J(\mathcal{E}) := \mathcal{E}_{j_m, j'_1} \otimes R_{j'_1, j'_2}^+ \otimes \cdots \otimes R_{j'_{n-1}, j'_n}^+ \otimes \mathcal{E}_{j_1, j'_n}^* \otimes R_{j_1, j_2}^- \otimes \cdots \otimes R_{j_{m-1}, j_m}^- \quad (6.4)$$

This is illustrated in Figure 39. As usual we define the direct sum over all domain wall fans to be:

$$R^\partial(\mathcal{E}) := \bigoplus_J R_J(\mathcal{E}). \quad (6.5)$$

It is straightforward to define a map  $\rho(\mathfrak{d}) : TR^{\text{int}, -} \otimes TR^\partial(\mathcal{E}) \otimes TR^{\text{int}, +} \rightarrow R^\partial(\mathcal{E})$  with arguments associated respectively to the interior vertices in the negative half plane, wall vertices, and positive half plane interior vertices. Given a Theory  $\mathcal{T}^-$  on the negative half plane and a Theory  $\mathcal{T}^+$  on the positive half-plane we can define  $\rho_\beta(\mathfrak{d}) : TR^\partial(\mathcal{E}) \rightarrow R^\partial(\mathcal{E})$  by

$$\rho_\beta(\mathfrak{d})[r_1^\partial, \dots, r_n^\partial] = \rho_\beta(\mathfrak{d})[e^{\beta^-}; r_1^\partial, \dots, r_n^\partial; e^{\beta^+}] \quad (6.6)$$

where  $\beta^\pm$  are the interior amplitudes of  $\mathcal{T}^\pm$  and  $\beta = (\beta^-; \beta^+)$ . Familiar reasoning shows that this defines an  $A_\infty$ -algebra structure on  $R^\partial(\mathcal{E})$ .

In analogy to the half-plane case we can now define an *interface amplitude* to be an element  $\mathcal{B}_\mathcal{I} \in R^\partial(\mathcal{E})$ , for some  $\mathcal{E}$ , which solves the Maurer-Cartan equations

$$\sum_{n=1}^{\infty} \rho_\beta(\mathbf{t}_\mathcal{I})(\mathcal{B}_\mathcal{I}^{\otimes n}) = \rho_\beta(\mathbf{t}_\mathcal{I})\left[\frac{1}{1 - \mathcal{B}_\mathcal{I}}\right] = 0 \quad (6.7)$$

**Definition:** An *Interface* is a choice of  $D$ , a pair of Theories  $\mathcal{T}^\pm$ , a choice of Chan-Paton data for the interface, and an interface amplitude.

We generally denote an Interface by a capital Gothic letter, such as  $\mathfrak{I}$ . The Chan-Paton data is  $\mathcal{E}(\mathfrak{I})$  and the interface amplitude is  $\mathcal{B}(\mathfrak{I})$ . Occasionally we will simply denote an Interface by its interface amplitude  $\mathcal{B}$ .

As in the half-plane case, given data  $(\mathbb{V}^\pm, z^\pm, x_0)$  and Chan-Paton spaces  $\mathcal{E}_{ii'}$  we can introduce a vacuum category  $\mathfrak{Vac}(\mathcal{T}^-, \mathcal{T}^+, \mathcal{E})$  with morphisms

$$\text{Hom}^\mathcal{E}(jj', ii') = \begin{cases} \mathcal{E}_{ii'} \otimes \widehat{R}_{i'j'}^+ \otimes \mathcal{E}_{jj'}^* \otimes \widehat{R}_{ji}^- & \text{Re}(z_{i'j'}) > 0 \quad \text{and} \quad \text{Re}(z_{ij}) > 0 \\ \mathbb{Z} & i = j \quad \text{and} \quad i' = j' \\ 0 & \text{else} \end{cases} \quad (6.8)$$

See Figure 39 for a typical summand. The superscripts  $\pm$  remind us that the  $\widehat{R}$ 's are defined with respect to positive and negative half-planes, respectively. As before, if we just take  $\mathcal{E}_{ii'} = \mathbb{Z}$  for all  $i, i'$  then we get the “bare” Interface vacuum category  $\mathfrak{Vac}(\mathcal{T}^-, \mathcal{T}^+)$ .

Now, taking all Chan-Paton spaces into account we can define an  $A_\infty$ -category of Interfaces, denoted  $\mathfrak{Br}(\mathcal{T}^-, \mathcal{T}^+)$ , following closely the definitions of  $\mathfrak{Br}(\mathcal{T})$  in Section §5.2. The objects are Interfaces and the space of morphisms from the Interface  $\mathfrak{I}_2$  to the Interface  $\mathfrak{I}_1$  is the natural generalization of (5.15):

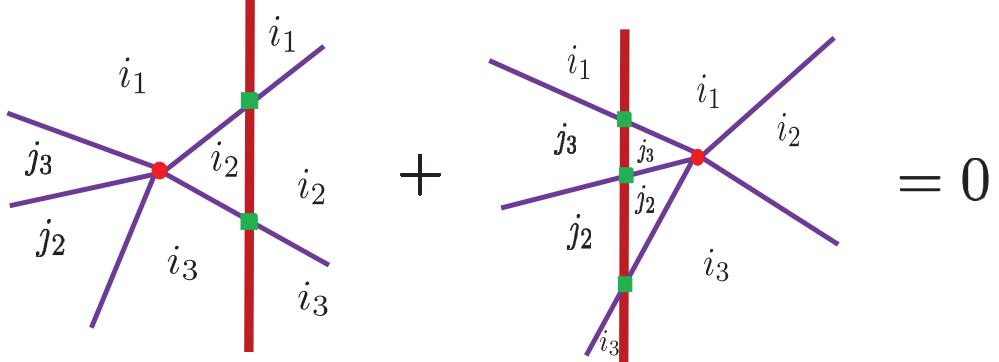
$$\text{Hop}(\mathfrak{I}_1, \mathfrak{I}_2) := \bigoplus_{ii', jj'} \mathcal{E}(\mathfrak{I}_1)_{ii'} \otimes \text{Hop}(ii', jj') \otimes (\mathcal{E}(\mathfrak{I}_2)_{jj'})^*. \quad (6.9)$$

where  $\text{Hop}(ii', jj')$  refers to the morphisms (6.8) of the “bare” category with  $\mathcal{E}_{ii'} = \mathbb{Z}$  for all  $i, i'$ . The  $A_\infty$ -multiplications are given by the natural generalization of equation (5.17). There is no difficulty defining the formalism for extended webs so, as in the discussion of (5.19) et. seq., we can use compositions  $M_1$  and  $M_2$  to define notions of homotopic morphisms and of homotopic Interfaces.

#### 6.1.4 Identity And Isomorphism Interfaces

There is a very simple, universal, and instructive example of an interface between a Theory and itself: the *identity interface*  $\mathfrak{Id}$ . We can pick as Chan-Paton factors  $\mathcal{E}_{ij} = \delta_{ij}\mathbb{Z}$ . With such a choice of CP factors amplitudes are valued in

$$R^\partial(\mathcal{E}) = \bigoplus_{z_{ij} \in \mathcal{H}^+} \widehat{R}_{ij}^+ \otimes \widehat{R}_{ji}^- \quad (6.10)$$



**Figure 40:** Examples of taut interface webs which contribute to the Maurer-Cartan equation for the identity interface  $\mathfrak{Id}$  between a Theory and itself.

Recall that the notation  $\widehat{R}_{ij}$  implies a choice of half-plane. We use the positive half-plane for the left factor and the negative half-plane for the right factor. To define the interface we take  $\mathcal{B}(\mathfrak{Id})$  to have nonzero component only in summands of the form  $R_{ij} \otimes R_{ji}$  corresponding to the fan  $\{i, j; j, i\}$ . The vertex looks like a straight line of a fixed slope running through the domain wall. The specific component of  $\mathcal{B}(\mathfrak{Id})$  in  $R_{ij} \otimes R_{ji}$  will be  $-K_{ij}^{-1}$ , where  $K_{ij}^{-1}$  is defined as follows:

The element  $K_{ij}^{-1} \in R_{ij} \otimes R_{ji}$  uniquely characterized by the property that the map

$$R_{ij} \rightarrow R_{ij} \otimes R_{ji} \otimes R_{ij} \rightarrow R_{ij} \quad (6.11)$$

defined by

$$r \rightarrow K_{ij}^{-1} \otimes r \rightarrow (1 \otimes K_{ji})(K_{ij}^{-1} \otimes r) \quad (6.12)$$

is simply the identity transformation  $r \rightarrow r$ .<sup>27</sup> It is worth expanding  $K_{ij}^{-1}$  in terms of a basis. We introduce bases  $\{v_\alpha\}$  and  $\{v_{\alpha'}\}$  for  $R_{ij}$  and  $R_{ji}$ , respectively, where  $v_\alpha, v_{\alpha'}$  are assumed to have definite degree. Then  $K_{\alpha'\alpha} = K_{ji}(v_{\alpha'}, v_\alpha)$  and  $K_{\alpha\alpha'} = K_{ij}(v_\alpha, v_{\alpha'})$  are related by  $K_{\alpha\alpha'} = K_{\alpha'\alpha}$  since  $K$  is symmetric. The element  $K_{ij}^{-1}$  defined above is

$$K_{ij}^{-1} = (-1)^{\deg(v_\alpha)} K^{\alpha\alpha'} v_\alpha \otimes v_{\alpha'}, \quad (6.13)$$

---

<sup>27</sup>Warning: If we map  $r' \in R_{ji}$  by  $r' \rightarrow r' \otimes K_{ij}^{-1}$  and then contract on the first two factors the result is  $r' \mapsto (-1)^{F+1} r'$ .

where  $K^{\alpha\alpha'}$  is the matrix inverse of  $K_{\alpha'\alpha}$ . That is  $K^{\alpha\alpha'} K_{\alpha'\beta} = \delta_\beta^\alpha$ . Note well that under the natural isomorphism  $R_{ij} \otimes R_{ji} \rightarrow R_{ji} \otimes R_{ij}$  we have  $K_{ij}^{-1} \rightarrow -K_{ji}^{-1}$ . Hence, in this sense,  $K_{ij}^{-1}$  is *antisymmetric*, a fact that will be useful in Sections §§7 and 9. Thus, the component of  $\mathcal{B}(\mathfrak{I}\mathfrak{d})$  in  $R_{ji} \otimes R_{ij}$ , where  $ji$  is the fan in the negative half-plane, is just  $K_{ji}^{-1}$ .

Now that we have defined  $\mathfrak{I}\mathfrak{d}$  let us verify that the interface amplitude indeed satisfies the Maurer-Cartan equation. The only non-zero contributions to  $\rho_\beta(t_{\mathcal{I}})(\frac{1}{1-\mathcal{B}})$  arise from taut webs with a single bulk vertex in either the positive or negative half-plane, but not both, as shown in Figure 40. The interior vertex is saturated by the interior amplitude  $\beta$ . These vertices form pairs obtained by “transporting” the vertex across the wall  $D$ . These pairs of taut webs cancel out together in verifying the Maurer-Cartan equation for  $\mathfrak{I}\mathfrak{d}$ . In slightly more detail, suppose that  $I = \{i_1, \dots, i_n\}$  is a cyclic fan so that the vacuum amplitude  $\beta$  has component  $\beta_I \in R_I$ . Since (by assumption) none of the  $z_{ij}$  for  $i, j \in \mathbb{V}$  is pure imaginary we can choose to start the fan so that  $z_{i_1, i_2}, \dots, z_{i_{m-1}, i_m}$  point into the positive half-plane and  $z_{i_m, i_{m+1}}, \dots, z_{i_n, i_1}$  point into the negative half-plane. The taut web with the vertex in the negative half-plane contributes (up to a sign, determined by equation (4.33))

$$K_{i_1, i_2} \cdots K_{i_{m-1}, i_m} \left( K_{i_1, i_2}^{-1} \otimes \cdots \otimes K_{i_{m-1}, i_m}^{-1} \otimes \beta_I \right) \quad (6.14)$$

to the Maurer-Cartan equation, where the  $K^{-1}$ 's come from  $\mathcal{B}_{\mathcal{I}}$  while the taut web with the same vertex in the positive half-plane contributes (up to a sign, determined by equation (4.33))

$$K_{i_{m+1}, i_m} \cdots K_{i_n, i_1} \left( K_{i_{m+1}, i_m}^{-1} \otimes \cdots \otimes K_{i_1, i_n}^{-1} \otimes \beta_I \right). \quad (6.15)$$

Both of the webs in Figure 40 are taut and hence canonically oriented. If  $x$  is the coordinate of the interaction vertex then the one on the left has  $o_r(u) = -[dx]$  and the one on the right has  $o_r(u) = +[dx]$ . We claim all the other sign factors cancel out and hence the two expressions in fact sum to zero. Essentially, this follows from the fact that the edge vector fields associated to  $K_{i_n, i_{n+1}}$  get contracted with the one-forms for the wall vertex associated to  $K_{i_n, i_{n+1}}^{-1}$ , and the relative order of the vector fields and one forms mimic the relative order of the  $K$  and  $K^{-1}$  symbols.

From this description it is clear that the existence of this vertex strongly uses the fact that the left and right interior amplitudes are assumed to be equal. This property of the MC equation anticipates a theme which will recur later in the paper: the existence of interfaces with given properties can encode relations between two theories. See Section §8 for an implementation of this idea.

There is a very useful generalization of the identity Interface. Suppose there is an isomorphism  $\varphi : \mathcal{T}^{(1)} \rightarrow \mathcal{T}^{(2)}$ , as defined in Section §4.1.1. Then we can construct an almost-canonical invertible Interface

$$\mathfrak{I}\mathfrak{d}^\varphi \in \mathfrak{Br}(\mathcal{T}^{(1)}, \mathcal{T}^{(2)}) \quad (6.16)$$

which we will call an *isomorphism Interface*. The Chan-Paton factors are defined by

$$\mathcal{E}(\mathfrak{I}\mathfrak{d}^\varphi)_{i,j} = \delta_{j,i\varphi} \mathbb{Z}^{[e_i]} \quad (6.17)$$

where  $e_i$  is a degree-shift which will be fixed, up to a common shift  $e_i \rightarrow e_i + s$ , below. (This ambiguity is the reason we say the interface is only “almost” canonical.)

In order to define the amplitudes we define a set of canonical elements

$$K_{ij}^{-1,\varphi} \in R_{ij}^{(1)} \otimes R_{j\varphi,i\varphi}^{(2)} \quad (6.18)$$

labeled by pairs of distinct vacua. This element can be defined by requiring commutativity of the diagram

$$\begin{array}{ccc} R_{ij}^{(1)} & \xrightarrow{1 \otimes K_{ji}^{-1,\varphi}} & R_{ij}^{(1)} \otimes R_{ji}^{(1)} \otimes R_{i\varphi,j\varphi}^{(2)} \\ & \searrow \varphi_{ij} & \downarrow K_{ij}^{(1)} \otimes 1 \\ & & R_{i\varphi,j\varphi}^{(2)} \end{array} \quad (6.19)$$

This is equivalent to the condition

$$\begin{array}{ccc} R_{j\varphi,i\varphi}^{(2)} & \xrightarrow{K_{ji}^{-1,\varphi} \otimes 1} & R_{ji}^{(1)} \otimes R_{i\varphi,j\varphi}^{(2)} \otimes R_{j\varphi,i\varphi}^{(2)} \\ & \swarrow \varphi_{ji} & \downarrow 1 \otimes K_{i\varphi,j\varphi}^{(2)} \\ & & R_{ji}^{(1)} \end{array} \quad (6.20)$$

thanks to (4.26).

In order to give an explicit formula for  $K_{ij}^{-1,\varphi}$  we choose bases  $v_\alpha^{(ij)}$  for the  $R_{ij}^{(1)}$  and similarly for  $R_{ij}^{(2)}$ . We write linear transformations  $v \mapsto v\varphi_{ij}$  so that the matrix elements relative to a basis are defined by  $v_\alpha\varphi = \varphi_{\alpha\beta}w_\beta$ . Then the composition of linear transformations  $\varphi_1\varphi_2$  is represented by the standard matrix product  $(\varphi_1)_{\alpha\gamma}(\varphi_2)_{\gamma\beta}$ . With this understood we have the formula

$$K_{ij}^{-1,\varphi} = (-1)^{\deg(v_\alpha^{(ij)})}(K_{ij}^{-1,\varphi})^{\alpha,\beta}v_\alpha^{(ij)} \otimes v_\beta^{(j\varphi,i\varphi)} \quad (6.21)$$

with

$$(K_{ij}^{-1,\varphi})^{\alpha,\beta} = (K_{ij}^{(1)})^{-1,\alpha,\gamma}(\varphi_{ij})_{\gamma,\beta} \quad (6.22)$$

Now the boundary amplitudes for  $\mathfrak{I}\mathfrak{d}^\varphi$  are valued in

$$\mathcal{E}_{i,i\varphi} \otimes \widehat{R}_{i\varphi,j\varphi}^{(2)} \otimes \mathcal{E}_{j,j\varphi}^* \otimes \widehat{R}_{ji}^{(1)} \quad (6.23)$$

and these are taken to be  $\mathcal{B} = K_{ij}^{-1,\varphi}$  up to degree shifts. The degree shifts are used to ensure that  $\mathcal{B}$  has degree one. Pictorially we have a bivalent vertex on the domain wall with a straight line going through it from one half-plane to the other, just as in Figure 40.

The demonstration that these boundary amplitudes satisfy the MC equation is closely analogous to that of  $\mathfrak{I}\mathfrak{d}$ .

### 6.1.5 Trivial Theories

Once we speak of interfaces it is useful to introduce a formal concept of a *trivial theory*  $\mathcal{T}_{\text{triv}}$ . This is a Theory whose vacuum data is a set  $\mathbb{V}$  with a single element  $v$ . The corresponding vacuum weight  $z$  is irrelevant. There are no planar webs. There is a unique, trivial web representation, as there are no  $R_{ij}$ . Of course,  $\widehat{R}_{v,v} = \mathbb{Z}$ . However, there are extended half-plane webs: They are simply a collection of vertices on the boundary of the half-plane. The Chan-Paton data consists simply of a graded  $\mathbb{Z}$ -module  $\mathcal{E}$ . The boundary amplitude consists entirely of its scalar part  $B \in \mathcal{E} \otimes \mathcal{E}^*$ , which can be viewed as an operator  $Q \in \text{Hom}(\mathcal{E})$  of degree one. The taut web is the case of two boundary vertices so the MC equation simply says that  $Q^2 = 0$ . Thus, giving a Brane for the trivial Theory is equivalent to giving a chain complex over  $\mathbb{Z}$ .

Now, an Interface between the trivial Theory  $\mathcal{T}^- = \mathcal{T}_{\text{triv}}$  and  $\mathcal{T}$  is a Brane for the theory  $\mathcal{T}$  on the positive half-plane. An Interface between the trivial Theory and itself is therefore, once again, a chain complex over  $\mathbb{Z}$ .

### 6.1.6 Tensor Products Of $A_\infty$ -Algebras

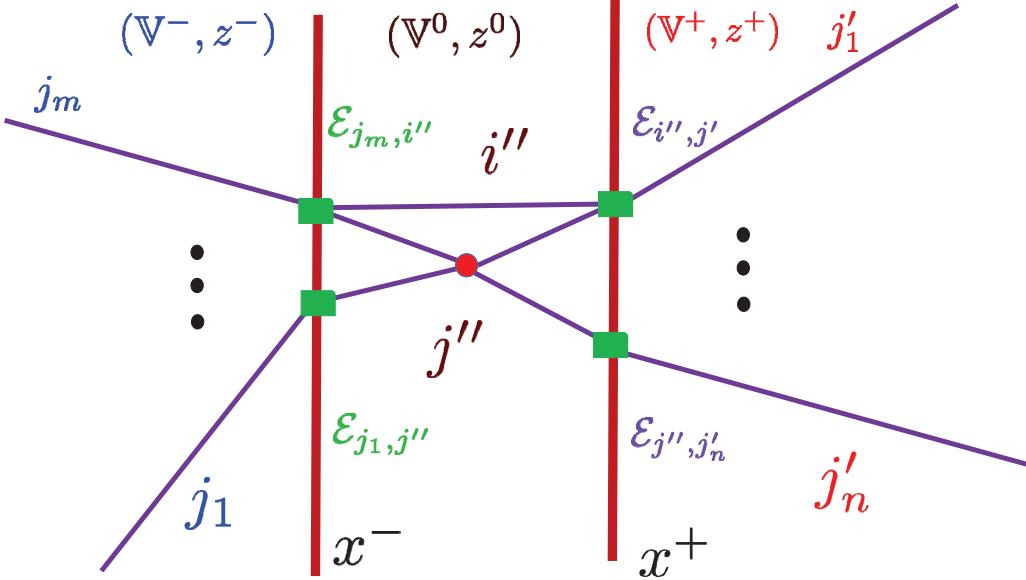
We remark in passing that our representations of interface webs lead to a nice mathematical construction of tensor products of  $A_\infty$ -algebras. In general the problem of defining a tensor product structure on  $A_\infty$ -algebras is nontrivial and has been discussed, for examples, in [4, 70, 67]. In general there is a moduli space of possible tensor products. From our present viewpoint, at least for a pair of  $A_\infty$ -algebras of the form  $\mathfrak{Vac}(\mathcal{T}^-, \mathcal{E}^-)$ ,  $\mathfrak{Vac}(\mathcal{T}^+, \mathcal{E}^+)$  we can choose our interface Chan-Paton spaces to be tensor products  $\mathcal{E}_{i,i'} = \mathcal{E}_i^- \otimes \mathcal{E}_{i'}^+$  and then our construction defines a canonical tensor product structure on  $R^\partial(\mathcal{E}) \cong (R^\partial(\mathcal{E}^-))^{\text{opp}} \otimes R^\partial(\mathcal{E}^+)$ . This is a distinguished  $A_\infty$ -algebra structure on the tensor product, given the vacuum weights.

## 6.2 Composite Webs And Composition Of Interfaces

The crucial property of Interfaces, which goes beyond the properties of Branes, is that they can be composed. We will discuss here the composition of two Interfaces. The composition of an Interface and a Brane is a special case of that. Physically, we are defining a notion of operator product expansion of supersymmetric interfaces.

The composition of Interfaces is based on a generalization of the strip geometry. Choose  $x^- < x^+$  and define a tripartite geometry  $G_2$  to be the union of the negative half-plane  $x \leq x^-$ , the strip  $x^- \leq x \leq x^+$ , and the positive half-plane  $x^+ \leq x$ . To these three regions we associate the three vacuum data  $(\mathbb{V}^-, z^-)$ ,  $(\mathbb{V}^0, z^0)$ , and  $(\mathbb{V}^+, z^+)$ , respectively. See Figure 41.

By definition a *composite* web for this tripartite geometry is a triplet  $\mathbf{c} = (\mathbf{u}^-, \mathbf{s}, \mathbf{u}^+)$  of half-plane and strip webs on  $G_2$  which are based on the appropriate vacuum data in each connected component. (The definition can clearly be generalized to regions with multiple strips with data  $(\mathbb{V}^-, z^-), (\mathbb{V}^0, z^0), \dots, (\mathbb{V}^n, z^n), (\mathbb{V}^+, z^+)$ .) Once again, the moduli space of deformation types is not just the product  $\mathcal{D}(\mathbf{c}) \neq \mathcal{D}(\mathbf{u}^-) \times \mathcal{D}(\mathbf{s}) \times \mathcal{D}(\mathbf{u}^+)$  because the deformations follow the rules for interface webs between  $(\mathbb{V}^-, z^-)$  and  $(\mathbb{V}^0, z^0)$  and between



**Figure 41:** An example of a composite web, together with conventions for Chan-Paton factors. In this web the fan of vacua at infinity has  $J_\infty(\mathfrak{c}) = \{j'_1, \dots, j'_n; j_1, \dots, j_m\}$  and  $\check{J}_\infty(\mathfrak{c}) = \{i''; j'_1, \dots, j'_n; j''; j_1, \dots, j_m\}$ . Reading from left to right the indices are in clockwise order.

$(V^0, z^0)$ , and  $(V^+, z^+)$ .<sup>28</sup> In general we denote the free abelian group generated by oriented deformation types of composite webs as  $\mathcal{W}_C[V^-, V^0, \dots, V^n, V^+]$  (with the vacuum weights understood). As for strip webs, the geometry has no scale invariance, and thus reduced moduli spaces are quotiented by time translations only. Thus the definitions of taut and sliding composite webs are analogous to those for strip webs:

**Definition:** Assuming that at least two of the Theories  $(\mathcal{T}^-, \mathcal{T}^0, \mathcal{T}^+)$  are nontrivial, composite webs with  $d(\mathfrak{c}) = 1$  are called *taut* (or *rigid*) and composite webs with  $d(\mathfrak{c}) = 2$  are called *sliding*.

For composite webs there are two senses in which we can speak of the fans of vacua at infinity. If  $\mathfrak{c} = (\mathfrak{u}^-, \mathfrak{s}, \mathfrak{u}^+)$  then we could define

$$J_\infty(\mathfrak{c}) := \{J_\infty(\mathfrak{u}^+); J_\infty(\mathfrak{u}^-)\}. \quad (6.24)$$

Notice that this has the same structure as boundary vertex fans for an interface web between vacua  $V^-$  and  $V^+$ , a fact which will be useful presently. Sometimes it can be

---

<sup>28</sup>Once again,  $\mathfrak{u}^-, \mathfrak{s}, \mathfrak{u}^+$  are webs, not deformation types. Given three such deformation types there are several ways to combine them into a deformation type of a composite web.

useful to include the past and future vacua  $j^-(\mathfrak{s})$  and  $j^+(\mathfrak{s})$  of  $\mathfrak{s}$ , respectively. Then we define

$$\check{J}_\infty(\mathfrak{c}) := \{j^+(\mathfrak{s}); J_\infty(\mathfrak{u}^+); j^-(\mathfrak{s}); J_\infty(\mathfrak{u}^-)\} \quad (6.25)$$

See for example Figure 41.

We will now describe the convolution identity for  $\mathcal{W}_C$ , the free abelian group generated by the oriented deformation types of composite webs. Here a novel feature arises and the identity itself involves a tensor operation. As usual we consider the possible boundaries of deformation types of sliding composite webs. We encounter again the same phenomenon as for strip webs: some components of the moduli space of taut composite webs are not segments, but half lines. While the boundaries at finite distance are accounted for by convolutions, we need a different operation to account for boundaries at infinity.

For strip webs, the new operation was time convolution: a large strip sliding web takes the form of two taut strip webs separated by a long stretch of time. For composite webs, we can do something similar, but there is an important difference: as a web becomes large in size, the restriction to the central strip will consist of two or more components of finite extent and well separated in time, but the components in the left and right half-planes may simply grow to arbitrarily large size. See Figure 42 for an example.

We can make this statement precise by separating the “bound” vertices whose distance from the boundaries stabilizes as the web keeps growing from the “scaling” vertices whose distance from the boundaries scales linearly with the distance from the boundaries. The bound vertices form clumps consisting of vertices whose distance in time remains bounded as the web grows. We can take a sliding web of some large size  $L$ , and re-scale all coordinates by a factor of  $L$ . The intermediate strip is now of very small width and the composite web is well approximated by an interface web  $\mathfrak{d}$  between  $\mathcal{T}^-$  and  $\mathcal{T}^+$  whose interior vertices correspond to the scaling vertices of the original composite web and whose boundary vertices correspond to each of the clumps of bound vertices of the original composite web.

We can put these heuristic pictures on a firm footing by considering a tensor operation where a composite web is obtained from an interface web  $\mathfrak{d}$  by convolving composite webs into the wall-vertices of the interface web  $\mathfrak{d}$ . To be more precise, suppose  $\mathfrak{d}$  is an interface web between  $(\mathbb{V}^-, z^-)$  and  $(\mathbb{V}^+, z^+)$ . We can define an operation

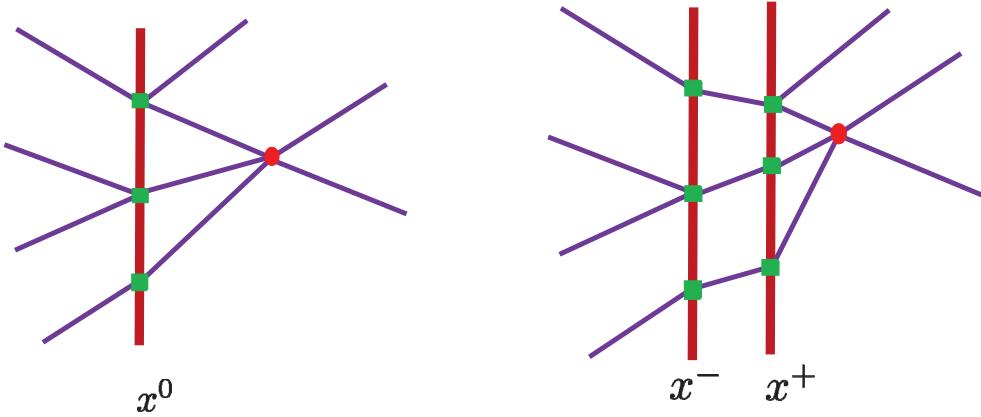
$$T_\partial(\mathfrak{d}) : T\mathcal{W}_C \rightarrow \mathcal{W}_C \quad (6.26)$$

whose nonzero values on monomials  $\mathfrak{c}_1 \otimes \cdots \otimes \mathfrak{c}_n$  are obtained by inserting the  $\mathfrak{c}_a$  (in the correct time order<sup>29</sup>) into the wall vertices  $v_a^\partial$  of  $\mathfrak{d}$  provided  $J_{v_a^\partial}(\mathfrak{d}) = J_\infty(\mathfrak{c}_a)$  and provided that the past strip vacuum of  $\mathfrak{c}_a$  agrees with the future strip vacuum of  $\mathfrak{c}_{a+1}$ . We orient the resulting web in the standard way, wedging the reduced orientations of the arguments in the same order as the arguments themselves. It is easy to check that the dimensions of the deformation spaces of the webs are related by

$$d(T_\partial(\mathfrak{d})[\mathfrak{c}_1, \dots, \mathfrak{c}_n]) = d(\mathfrak{d}) + \sum_{a=1}^n (d(\mathfrak{c}_a) - 1). \quad (6.27)$$

---

<sup>29</sup>By convention we use the order set by the positive half-plane webs. Therefore reading from left to right corresponds to vertices with decreasing  $y$ .



**Figure 42:** The taut interface web on the left can be convolved with three taut (=rigid) composite webs at the three green wall vertices to form the sliding composite web on the right. This represents one type of degeneration in the convolution identity for the taut composite web. The region at infinity is represented by the limit in which the red vertex moves off to infinity in the positive half-plane.

If we take into account the positions of the walls it is most natural to take the position  $x^0$  of the wall for  $\mathfrak{d}$  to be somewhere in the open interval  $x^- < x^0 < x^+$ , and all the composite webs  $\mathfrak{c}_a$  have the same positions  $(x^-, x^+)$ . Since we are defining an operation on deformation types the precise choice of  $x^0$  does not matter.

According to (6.27) if all the composite webs are taut, so  $d(\mathfrak{c}_a) = 1$ , and if the interface web  $\mathfrak{d}$  is taut, so  $d(\mathfrak{d}) = 2$ , then  $T_\partial(\mathfrak{d})[\mathfrak{c}_1, \dots, \mathfrak{c}_n]$  has  $d = 2$  and is hence a *sliding* composite web. In this way the generic *sliding* composite web is associated to a *taut* interface web, with insertions of *an arbitrary number* of taut composite webs. We thus claim that the regions at infinity of the reduced moduli space of sliding composite webs are well described by

$$T_\partial(\mathfrak{t}^{-,+}) \left[ \frac{1}{1 - \mathfrak{t}_c} \right] \quad (6.28)$$

where  $\mathfrak{t}_c$  is the taut element for composite webs and  $\mathfrak{t}^{-,+}$  is the taut element for interface webs between  $(\mathbb{V}^-, z^-)$  and  $(\mathbb{V}^+, z^+)$ . It is worth noting that the time convolution of two composite webs is a special case of this operation. It arises from the taut interface webs with precisely two wall vertices. For another example see Figure 42. Note that this is

qualitatively different from all the previous convolution operations we have encountered because the “more primitive” structure plays the role of the “container web.”

To write down the full convolution identity for the taut composite webs we should also take into account other degenerations at finite distance in the reduced moduli space. To do this let

$$\mathbf{t}_{pl} := \mathbf{t}_p^- + \mathbf{t}_p^0 + \mathbf{t}_p^+ \quad (6.29)$$

be the formal sum of plane web taut elements for the three vacuum data. Similarly, let

$$\mathbf{t}_{\mathcal{I}} := \mathbf{t}^{-,0} + \mathbf{t}^{0,+} \quad (6.30)$$

where  $\mathbf{t}^{-,0}$  is the taut element associated to the interface webs between vacuum data  $(\mathbb{V}^-, z^-)$  and  $(\mathbb{V}^0, z^0)$ , while  $\mathbf{t}^{0,+}$  is the taut element associated to the interface webs between vacuum data  $(\mathbb{V}^0, z^0)$  and  $(\mathbb{V}^+, z^+)$ . The convolution identity for composite webs is

$$\mathbf{t}_c * \mathbf{t}_{pl} + \mathbf{t}_c * \mathbf{t}_{\mathcal{I}} + T_{\partial}(\mathbf{t}^{-,+})[\frac{1}{1 - \mathbf{t}_c}] = 0. \quad (6.31)$$

Following the discussion of Section 3.2.2 we can identify this equation as the Maurer-Cartan equation for an  $A_\infty$  algebra structure on  $\mathcal{W}_C$  with operations  $T_{\partial}(\mathbf{t}^{-,+})$ , plus an extra differential  $*\mathbf{t}_{pl} + *\mathbf{t}_{\mathcal{I}}$ .

We now consider a triplet of Theories  $(\mathcal{T}^-, \mathcal{T}^0, \mathcal{T}^+)$  associated to the three regions of  $G_2$ . Given Interfaces  $\mathfrak{I}^{-,0} \in \mathfrak{Br}(\mathcal{T}^-, \mathcal{T}^0)$  and  $\mathfrak{I}^{0,+} \in \mathfrak{Br}(\mathcal{T}^0, \mathcal{T}^+)$  we want to define a product Interface,  $\mathfrak{I}^{-,0} \boxtimes \mathfrak{I}^{0,+} \in \mathfrak{Br}(\mathcal{T}^-, \mathcal{T}^+)$ .

We first determine the Chan-Paton factors of  $\mathfrak{I}^{-,0} \boxtimes \mathfrak{I}^{0,+}$ . The choice of Theories  $(\mathcal{T}^-, \mathcal{T}^0, \mathcal{T}^+)$  implies a choice of three representations  $(\mathcal{R}^-, \mathcal{R}^0, \mathcal{R}^+)$  of vacuum data. The Interfaces  $\mathfrak{I}^{-,0}$  and  $\mathfrak{I}^{0,+}$  have Chan-Paton spaces  $\mathcal{E}_{i,i''}^{-,0}$  and  $\mathcal{E}_{i'',i'}^{0,+}$ , respectively. Define the Chan-Paton data for the product Interface between vacua  $\mathcal{T}^-$  and  $\mathcal{T}^+$  as

$$\mathcal{E}(\mathfrak{I}^{-,0} \boxtimes \mathfrak{I}^{0,+})_{ii'} := \mathcal{E}_{ii'}^{-,+} := \bigoplus_{i'' \in \mathbb{V}^0} \mathcal{E}_{i,i''}^{-,0} \otimes \mathcal{E}_{i'',i'}^{0,+} \quad (6.32)$$

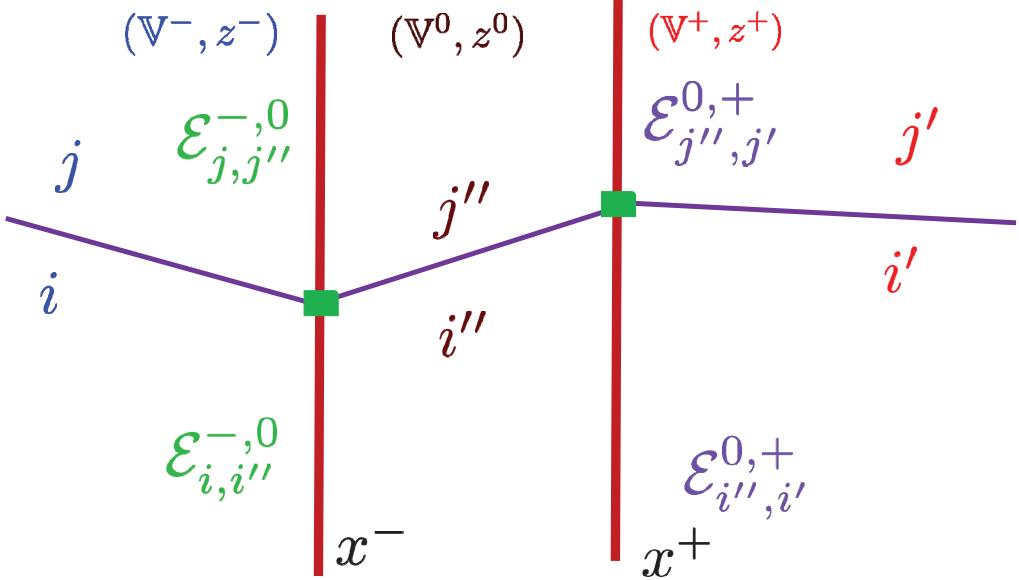
Note that  $\mathcal{E}^{-,+} := \bigoplus_{i \in \mathbb{V}^-, i' \in \mathbb{V}^+} \mathcal{E}_{ii'}^{-,+}$  is a generalization of the approximate ground states on the strip of equation (4.52).

Now, in order to define the interface amplitudes of  $\mathfrak{I}^{-,0} \boxtimes \mathfrak{I}^{0,+}$  we need some more preliminaries. Viewing  $\mathcal{E}_{ii'}^{-,+}$  as Chan-Paton factors for an interface between  $\mathcal{T}^-$  and  $\mathcal{T}^+$  we can formulate the spaces  $R^\partial(\mathcal{E}^{-,+})$  using equation (6.4). For a composite web  $\mathfrak{c}$  we follow the usual procedure and define

$$\rho_\beta(\mathfrak{c}) : TR^\partial(\mathcal{E}^{-,0}) \otimes TR^\partial(\mathcal{E}^{0,+}) \rightarrow R^\partial(\mathcal{E}^{-,+}), \quad (6.33)$$

where  $\mathcal{E}^{-,+}$  is given by (6.32), by inserting  $\beta = (\beta^-, \beta^0, \beta^+)$  into the interior vertices of  $\mathfrak{c}$ , so that

$$\begin{aligned} \rho_\beta(\mathfrak{c})[r_1^{-,0}, \dots, r_n^{-,0}; r_1^{0,+}, \dots, r_m^{0,+}] &:= \rho(\mathfrak{c})[e^{\beta^-}; r_1^{-,0}, \dots, r_n^{-,0}; e^{\beta^0}; r_1^{0,+}, \dots, r_m^{0,+}; e^{\beta^+}] \\ &\in R^\partial(\mathcal{E}^{-,+}). \end{aligned} \quad (6.34)$$



**Figure 43:** A simple taut web is illustrated here. It leads to the contractions described in the example below.

**Example:** As an example consider the taut composite web  $\mathbf{c}$  shown in Figure 43. The action of  $\rho_\beta(\mathbf{c})$  is zero on every component of  $TR^\partial(\mathcal{E}^{-,0}) \otimes TR^\partial(\mathcal{E}^{0,+})$  except on

$$\left( \mathcal{E}_{j,j''}^{-,0} \otimes R_{j'',i''}^0 \otimes (\mathcal{E}_{i,i''}^{-,0})^* \otimes R_{i,j}^- \right) \otimes \left( \mathcal{E}_{j'',j'}^{0,+} \otimes R_{j',i'}^+ \otimes (\mathcal{E}_{i'',i'}^{0,+})^* \otimes R_{i'',j''}^0 \right) \quad (6.35)$$

The superscripts on the  $R$ 's indicates which Theory we are speaking of, and there is no sum on any of the indices. The action of  $\rho_\beta(\mathbf{c})$  on this summand uses the contraction

$$K_{j'',i''}^0 : R_{j'',i''}^0 \otimes R_{i'',j''}^0 \rightarrow \mathbb{Z} \quad (6.36)$$

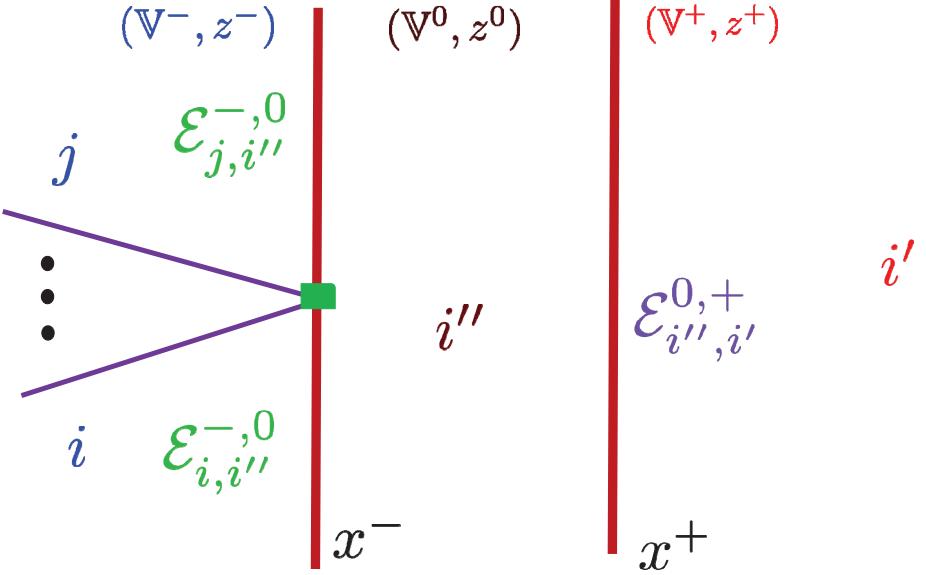
together with the Koszul rule to map an element of (6.35) to

$$\left( \mathcal{E}_{j,j''}^{-,0} \otimes \mathcal{E}_{j'',j'}^{0,+} \right) \otimes R_{j',i'}^+ \otimes \left( \mathcal{E}_{i,i''}^{-,0} \otimes \mathcal{E}_{i'',i'}^{0,+} \right)^* \otimes R_{i,j}^- \quad (6.37)$$

Now note that (6.37) is a summand of  $R^\partial(\mathcal{E}^{-,+})$ .

There is a special case of (6.34) we must deal with separately, namely when  $n = 0$  or  $m = 0$ . The reason is that we can have composite webs with *no* vertices one one of the two interfaces. See for example the taut web in Figure 44. For such webs  $\rho_\beta(\mathbf{c})$  will map

$$\rho_\beta(\mathbf{c}) : TR^\partial(\mathcal{E}^{-,0}) \rightarrow R^\partial(\mathcal{E}^{-,+}), \quad (6.38)$$



**Figure 44:** A taut web with no vertices on the  $\mathcal{T}^0, \mathcal{T}^+$  boundary.

by taking  $\rho_\beta(\mathfrak{c})[r_1, \dots, r_n; \emptyset]$  to have a value only in the component:

$$\left( \mathcal{E}_{j,i''}^{-,0} \otimes \mathcal{E}_{i'',i'}^{0,+} \right) \otimes \widehat{R}_{i',i'}^+ \otimes \left( \mathcal{E}_{i,i''}^{-,0} \otimes \mathcal{E}_{i'',i'}^{0,+} \right)^* \otimes \widehat{R}_{i,j}^- \cong \left( \mathcal{E}_{j,i''}^{-,0} \otimes \widehat{R}_{i'',i''}^0 \otimes (\mathcal{E}_{i,i''}^{-,0})^* \otimes \widehat{R}_{i,j}^- \right) \otimes \left( \mathcal{E}_{i',i'}^{0,+} \otimes (\mathcal{E}_{i'',i'}^{0,+})^* \right) \quad (6.39)$$

with a value given by

$$\rho_\beta(\mathfrak{c})[r_1, \dots, r_n] \otimes \mathbf{Id}_{i'',i'} \quad (6.40)$$

where  $\rho_\beta(\mathfrak{c})[r_1, \dots, r_n]$  is just the contraction for an interface web between  $\mathcal{T}^-$  and  $\mathcal{T}^0$ . We make a similar definition with webs that have no vertices on the  $(\mathcal{T}^-, \mathcal{T}^0)$  boundary.

Now we can define the interface amplitude of  $\mathfrak{I}^{-,0} \boxtimes \mathfrak{I}^{0,+}$ . Suppose that  $\mathfrak{I}^{-,0}$  and  $\mathfrak{I}^{0,+}$  have interface amplitudes  $\mathcal{B}^{-,0}$  and  $\mathcal{B}^{0,+}$ , respectively. We claim that

$$\mathcal{B}(\mathfrak{I}^{-,0} \boxtimes \mathfrak{I}^{0,+}) := \rho_\beta(\mathfrak{t}_c) \left[ \frac{1}{1 - \mathcal{B}^{-,0}}; \frac{1}{1 - \mathcal{B}^{0,+}} \right] \quad (6.41)$$

satisfies the Maurer Cartan equation for an interface amplitude between the theories  $\mathcal{T}^-$  and  $\mathcal{T}^+$  with Chan-Paton spaces (6.32). To prove this claim we first note that

$$\rho_\beta(\mathfrak{t}^{-,+}) \left[ \frac{1}{1 - \mathcal{B}(\mathfrak{I}^{-,0} \boxtimes \mathfrak{I}^{0,+})} \right] = \rho_\beta \left( T_\partial(\mathfrak{t}^{-,+}) \left[ \frac{1}{1 - \mathfrak{t}_c} \right] \right) \left[ \frac{1}{1 - \mathcal{B}^{-,0}}, \frac{1}{1 - \mathcal{B}^{0,+}} \right] \quad (6.42)$$

This forbidding identity has a simple meaning. On the right hand side we are computing the amplitude of composite webs produced by inserting  $\mathfrak{t}_c$  in all possible ways in  $\mathfrak{t}^{-,+}$ . On

the left hand side we compute the amplitude for the individual  $t_c$  sub webs first, and then insert that in  $t^{-,+}$ . Finally, we apply the convolution identity (6.31) and use the fact that  $\beta$  is an interior amplitude and  $\mathcal{B}^{-,0}$  and  $\mathcal{B}^{0,+}$  are interface amplitudes, thus establishing that (6.41) is an interface amplitude.

It follows from the above discussion that Interfaces can be composed. In fact, the product  $\boxtimes$  can be extended to define an  $A_\infty$  bi-functor from the Cartesian product of Interface categories  $\mathfrak{Br}(\mathcal{T}^-, \mathcal{T}^0) \times \mathfrak{Br}(\mathcal{T}^0, \mathcal{T}^+)$  to the Interface category  $\mathfrak{Br}(\mathcal{T}^-, \mathcal{T}^+)$ . That means that if we have

1. A sequence of interface amplitudes  $\mathfrak{I}_0^{-,0}, \dots, \mathfrak{I}_n^{-,0}$  in  $\mathfrak{Br}(\mathcal{T}^-, \mathcal{T}^0)$  together with morphisms  $\delta_1, \dots, \delta_n$  between them, and
2. similarly, we have interface amplitudes  $\mathfrak{I}_0^{0,+}, \dots, \mathfrak{I}_{n'}^{0,+}$  in  $\mathfrak{Br}(\mathcal{T}^0, \mathcal{T}^+)$ , together with morphisms  $\delta'_1, \dots, \delta'_{n'}$  between them,

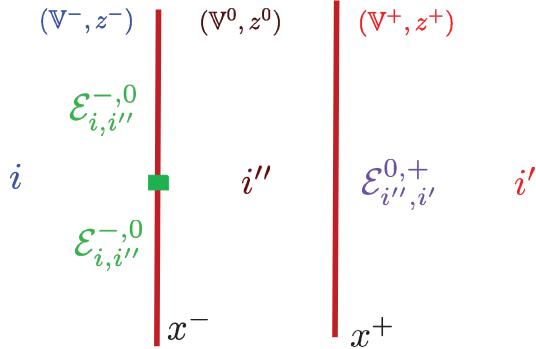
then we can produce an element:

$$\nu(\delta_1, \dots, \delta_n; \delta'_1, \dots, \delta'_{n'}) \in \text{Hop} \left( \mathfrak{I}_0^{-,0} \boxtimes \mathfrak{I}_0^{0,+}, \mathfrak{I}_n^{-,0} \boxtimes \mathfrak{I}_{n'}^{0,+} \right) \quad (6.43)$$

such that the  $A_\infty$ -relations are satisfied separately in the two sets of arguments. The element  $\nu(\delta_1, \dots, \delta_n; \delta'_1, \dots, \delta'_{n'})$  is defined by  $\rho_\beta(t_c)$ :

$$\rho_\beta(t_c) \left[ \frac{1}{1 - \mathcal{B}_0^{-,0}}, \delta_1, \dots, \delta_n \frac{1}{1 - \mathcal{B}_n^{-,0}}; \frac{1}{1 - \mathcal{B}_0^{0,+}}, \delta'_1, \dots, \delta'_{n'} \frac{1}{1 - \mathcal{B}_{n'}^{0,+}} \right] \quad (6.44)$$

This extends the composition of Interfaces to a full  $A_\infty$  bi-functor. (We have not written out the full details of a proof that this is in fact a bi-functor.)



**Figure 45:** The (extended) taut web shown here contributes to  $\nu(\mathbf{Id}; \emptyset)$ .

If we specialize the above discussion to  $n = 1$  and  $n' = 0$  or  $n = 0$  and  $n' = 1$  then we obtain an interesting interplay with notions of homotopy equivalent branes and interfaces. In particular, even though  $\mathfrak{I}^{-,0} \boxtimes \mathfrak{I}^{0,+}$  is not a bilinear operation, we claim that if  $\mathfrak{I}_0^{-,0}$

is homotopy equivalent to  $\mathfrak{I}_1^{-,0}$  then  $\mathfrak{I}_0^{-,0} \boxtimes \mathfrak{I}^{0,+}$  is homotopy equivalent to  $\mathfrak{I}_1^{-,0} \boxtimes \mathfrak{I}^{0,+}$ . There is a completely parallel result for homotopy equivalences of  $\mathfrak{I}^{0,+}$  holding  $\mathfrak{I}^{-,0}$  fixed. To prove this we note that we have identities like the commutativity (up to sign) of the diagram:

$$\begin{array}{ccc} \text{Hop}(\mathfrak{I}_0^{-,0}, \mathfrak{I}_1^{-,0}) & \xrightarrow{\nu} & \text{Hop}(\mathfrak{I}_0^{-,0} \boxtimes \mathfrak{I}^{0,+}, \mathfrak{I}_1^{-,0} \boxtimes \mathfrak{I}^{0,+}) \\ \downarrow M_1^{-,0} & & \downarrow M_1^{-,+} \\ \text{Hop}(\mathfrak{I}_0^{-,0}, \mathfrak{I}_1^{-,0}) & \xrightarrow{\nu} & \text{Hop}(\mathfrak{I}_0^{-,0} \boxtimes \mathfrak{I}^{0,+}, \mathfrak{I}_1^{-,0} \boxtimes \mathfrak{I}^{0,+}) \end{array} \quad (6.45)$$

where  $\nu$  is the map obtained by specializing (6.44) to  $n = 1$  and  $n' = 0$ . Of course there is a similar identity for  $n = 0$  and  $n' = 1$ . This follows by applying the representation of the identity (6.31) to the sequence of arguments

$$e^{\beta^-}; \frac{1}{1 - \mathcal{B}_0^{-,0}}, \delta, \frac{1}{1 - \mathcal{B}_1^{-,0}}; e^{\beta^0}; \frac{1}{1 - \mathcal{B}^{0,+}}; e^{\beta^+} \quad (6.46)$$

where  $\delta \in \text{Hop}(\mathfrak{I}_0^{-,0}, \mathfrak{I}_1^{-,0})$ . It follows from (6.45) that  $\nu$  maps an  $M_1$ -closed or exact morphism between  $\mathfrak{I}_0^{-,0}$  and  $\mathfrak{I}_1^{-,0}$  to an  $M_1$ -closed or exact morphism between  $\mathfrak{I}_0^{-,0} \boxtimes \mathfrak{I}^{0,+}$  and  $\mathfrak{I}_1^{-,0} \boxtimes \mathfrak{I}^{0,+}$ . Now note that if **Id** is the graded identity element of equation (4.75) then  $\nu(\mathbf{Id}; \emptyset) = \mathbf{Id}$ . To prove this note that the only taut webs which can contribute to

$$\rho_\beta(t_c)[\frac{1}{1 - \mathcal{B}^{-,0}}, \mathbf{Id}, \frac{1}{1 - \mathcal{B}^{-,0}}; \frac{1}{1 - \mathcal{B}^{0,+}}] \quad (6.47)$$

are those with a single vertex on the boundary between  $\mathcal{T}^-$  and  $\mathcal{T}^0$ , and a single vacuum  $i''$  in the region  $x^- \leq x \leq x^+$ , as shown in Figure 45. Now, using the definitions (6.38)-(6.40) we can check that the sum over such taut webs gives  $\nu(\mathbf{Id}; \emptyset) = \mathbf{Id}$ . Similarly,  $\nu(\emptyset; \mathbf{Id}) = \mathbf{Id}$  comes from taut webs with a single vertex on the  $\mathcal{T}^0, \mathcal{T}^+$  boundary.

Now, since  $\nu$  is an  $A_\infty$ -functor, if  $\delta, \delta'$  define a homotopy equivalence between  $\mathfrak{I}_0^{-,0}$  and  $\mathfrak{I}_1^{-,0}$  then <sup>30</sup>

$$\begin{aligned} M_2(\nu(\delta), \nu(\delta')) &= \nu(M_2(\delta, \delta')) \pm \nu(M_1(\delta), \delta') \pm \nu(\delta, M_1(\delta')) \pm M_1(\nu(\delta, \delta')) \\ &= \nu(M_2(\delta, \delta')) \pm M_1(\nu(\delta, \delta')) \\ &= \nu(\mathbf{Id} + M_1(\delta'')) \pm M_1(\nu(\delta, \delta')) \\ &= \mathbf{Id} + M_1(\nu(\delta'') \pm \nu(\delta, \delta')) \end{aligned} \quad (6.48)$$

In the first line we used the definition of an  $A_\infty$ -functor. In the second line we used the hypothesis that  $\delta, \delta'$  are closed, in the third line we used the hypothesis that they define a homotopy equivalence. This finally completes the proof that homotopy equivalence is nicely compatible with  $\boxtimes$ .

In the special case where  $\mathcal{T}^-$  is the trivial Theory, we have a useful result: each Interface in  $\mathfrak{Br}(\mathcal{T}^0, \mathcal{T}^+)$  gives us an  $A_\infty$  functor from  $\mathfrak{Br}(\mathcal{T}^0)$  to  $\mathfrak{Br}(\mathcal{T}^+)$ . This will be important for us later in Section §7 so let us spell it out a bit more. If  $\mathfrak{I}^{0,+}$  is a fixed Interface we define an  $A_\infty$ -functor by declaring that on objects

$$\mathcal{F}_{\mathfrak{I}^{0,+}}(\mathfrak{B}) := \mathfrak{B} \boxtimes \mathfrak{I}^{0,+} \quad (6.49)$$

---

<sup>30</sup>In this equation we got lazy about the signs.

and if  $\delta_1, \dots, \delta_n$  is a composable set of morphisms between Branes  $\mathfrak{B}_0, \dots, \mathfrak{B}_n$  in  $\mathfrak{Br}(\mathcal{T}^0)$  then

$$\mathcal{F}_{\mathfrak{I}^{0,+}}(\delta_0, \dots, \delta_n) := \delta \in \text{Hop}(\mathcal{F}_{\mathfrak{I}^{0,+}}(\mathfrak{B}_0), \mathcal{F}_{\mathfrak{I}^{0,+}}(\mathfrak{B}_n)) \quad (6.50)$$

is given by

$$\delta = \rho_\beta(\mathbf{t}_c)[\frac{1}{1 - \mathcal{B}_0}, \delta_1, \dots, \delta_n \frac{1}{1 - \mathcal{B}_n}; \frac{1}{1 - \mathcal{B}^{0,+}}] \quad (6.51)$$

and we claim, moreover, that the  $A_\infty$ -relations defining an  $A_\infty$ -functor are satisfied:

$$\begin{aligned} & \sum_k \sum_{\text{Pa}_k(P)} \rho_{\beta^+}(\mathbf{t}_H^+) (\mathcal{F}_{\mathfrak{I}^{0,+}}(P_1), \dots, \mathcal{F}_{\mathfrak{I}^{0,+}}(P_k)) \\ &= \sum_{\text{Pa}_3(P)} \epsilon_{P_1, P_2, P_3} \mathcal{F}_{\mathfrak{I}^{0,+}}(P_1, \rho_{\beta^0}(\mathbf{t}_H^0)(P_2), P_3) \end{aligned} \quad (6.52)$$

where  $P = \{\delta_1, \dots, \delta_n\}$ ,  $\mathcal{F}_{\mathfrak{I}^{0,+}}(\emptyset) = 0$ ,  $\epsilon_{P_1, P_2, P_3}$  is an appropriate sign, and we note that  $\mathbf{t}^{-,+} = \mathbf{t}_H^+$  is the taut element of the Theory  $\mathcal{T}^+$  in the positive half-plane while  $\mathbf{t}^{-,0} = \mathbf{t}_H^0$  is the taut element of the Theory  $\mathcal{T}^0$  in the positive half-plane.

Moreover, suppose that  $\psi \in \text{Hop}(\mathfrak{I}_1^{0,+}, \mathfrak{I}_2^{0,+})$  is a morphism between Interfaces. Then we claim that there is an  $A_\infty$ -natural transformation  $\tau(\psi)$  between the corresponding functors  $\mathcal{F}_{\mathfrak{I}_1^{0,+}}$  and  $\mathcal{F}_{\mathfrak{I}_2^{0,+}}$ . That is, for every  $\mathfrak{B} \in \mathfrak{Br}(\mathcal{T}^0)$  we can define

$$\tau(\psi)_\mathfrak{B} \in \text{Hop}(\mathcal{F}_{\mathfrak{I}_1^{0,+}}(\mathfrak{B}), \mathcal{F}_{\mathfrak{I}_2^{0,+}}(\mathfrak{B})) \quad (6.53)$$

so that, if  $\delta_1, \dots, \delta_n$  is a composable sequence of morphisms between Branes  $\mathfrak{B}_0, \dots, \mathfrak{B}_n$  in  $\mathcal{T}^0$ , then

$$\begin{array}{ccc} \mathcal{F}_{\mathfrak{I}_2^{0,+}}(\mathfrak{B}_n) & \xrightarrow{\tau(\psi)_{\mathfrak{B}_n}} & \mathcal{F}_{\mathfrak{I}_1^{0,+}}(\mathfrak{B}_n) \\ \downarrow \mathcal{F}_{\mathfrak{I}_2^{0,+}}(\delta_0, \dots, \delta_n) & & \downarrow \mathcal{F}_{\mathfrak{I}_1^{0,+}}(\delta_0, \dots, \delta_n) \\ \mathcal{F}_{\mathfrak{I}_2^{0,+}}(\mathfrak{B}_0) & \xrightarrow{\tau(\psi)_{\mathfrak{B}_0}} & \mathcal{F}_{\mathfrak{I}_1^{0,+}}(\mathfrak{B}_0) \end{array} \quad (6.54)$$

is a commutative diagram. The formula for  $\tau(\psi)_\mathfrak{B}$  is just

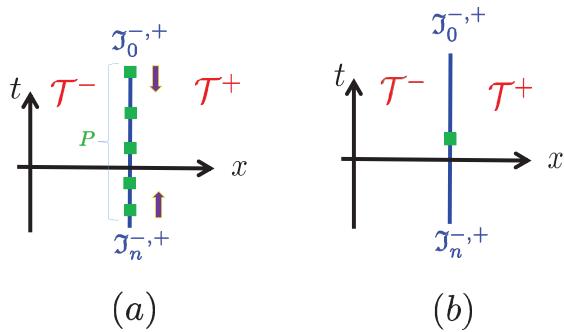
$$\tau(\psi)_\mathfrak{B} := \rho_\beta(\mathbf{t}_c) \left[ \frac{1}{1 - \mathcal{B}}; \frac{1}{1 - \mathcal{B}_1^{0,+}}, \psi, \frac{1}{1 - \mathcal{B}_2^{0,+}} \right]. \quad (6.55)$$

where  $\mathcal{B}$  is the boundary amplitude of  $\mathfrak{B}$  and  $\mathcal{B}_1^{0,+}, \mathcal{B}_2^{0,+}$  are the interface amplitudes of  $\mathfrak{I}_1^{0,+}, \mathfrak{I}_2^{0,+}$ , respectively. Moreover, a natural transformation  $\tau$  between an Interface and itself is homotopic to the identity if, for every  $\mathfrak{B}$ ,  $\tau_\mathfrak{B} = \mathbf{Id} + M_1(\delta)$ . Two  $A_\infty$ -functors can be regarded as homotopy equivalent if they are related by two natural transformations whose composition is homotopic to the identity. Now

$$M_2(\tau(\psi_1)_\mathfrak{B}, \tau(\psi_2)_\mathfrak{B}) = \tau(M_2(\psi_1, \psi_2))_\mathfrak{B} + \dots, \quad (6.56)$$

where the extra terms in  $\dots$  involve  $M_1$ . This simply follows from the observation that  $\tau(\psi)_\mathfrak{B} = \nu(\emptyset; \psi)$  (compare equation (6.44)) and is part of the statement that  $\nu$  is an  $A_\infty$ -functor in its second set of arguments. Hence a homotopy equivalence between Interfaces  $\mathfrak{I}_1^{0,+}$  and  $\mathfrak{I}_2^{0,+}$  leads to a homotopy equivalence of the corresponding  $A_\infty$ -functors.

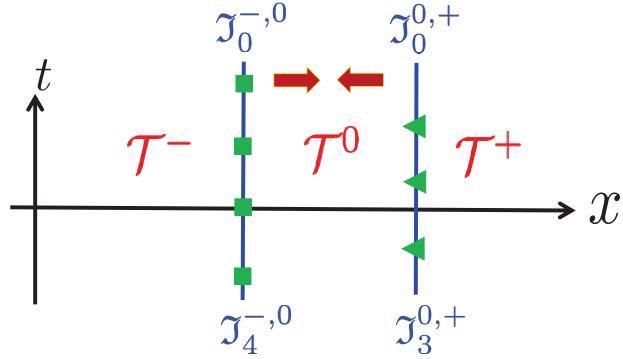
If both  $\mathcal{T}^-$  and  $\mathcal{T}^+$  are trivial, the above results reduce to the previous results for the strip. For example, equation (2.40) is a special case of equation (6.31): The only nontrivial taut element in  $\mathfrak{t}^{-,+}$  consists of two boundary vertices, and the composition operation then gives concatenation of strip-webs. As noted previously, the Chan-Paton space (6.32) becomes the space of approximate ground states (4.52). Moreover equation (6.33) is equivalent to equation (4.53), when we bring  $\mathcal{E}_{LR}$  from the LHS to  $\mathcal{E}_{LR}^*$  on the RHS of the latter equation. The differential  $d_{LR}$  on the complex of approximate ground states can be thought as a solution of a trivial MC equation, where only the composition of the Chan-Paton factors remain interesting.



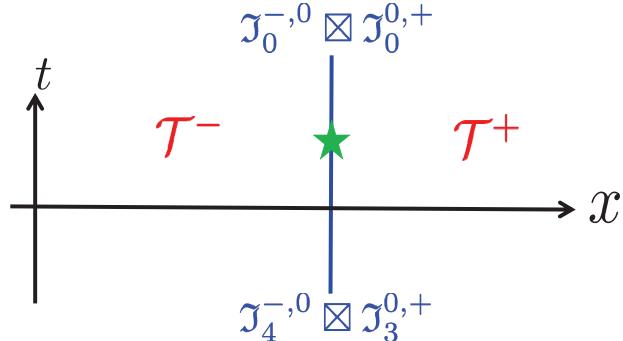
**Figure 46:** Illustrating the  $A_\infty$ -multiplications on the local operators on an Interface. Figure (a) shows a number of local operators between different Interfaces. The vertical purple arrows indicate that they are to be multiplied. The result is a single local operator between the initial and final Interfaces, as illustrated in Figure (b).

### Remarks:

1. In principle we should keep track of the positions  $x^{-,0}$  and  $x^{0,+}$  of the original Interfaces as well as the position  $x^{-,+}$  of the final product Interface. Given the translation invariance and homotopy equivalence we can be a bit sloppy about this, but it is relevant to the sense in which the product is associative. We take that issue up in the next Section §6.3.
2. It might help to restate some of the above formal expressions in more physical terms. First of all, the basic  $A_\infty$ -product on local operators on Interfaces is illustrated in Figure 46. Physically the product  $\mathfrak{I}^{-,0} \boxtimes \mathfrak{I}^{0,+}$  is a kind of operator product of supersymmetric interfaces. In the physical models there is no “Casimir force” between the Interfaces, even with the insertions of (a suitable class of) local operators. Therefore they can be adiabatically brought together as illustrated in Figure 47 to produce a new Interface with a single local operator, as illustrated in Figure 48. An illustration of the statement that  $\nu$  is an  $A_\infty$ -functor (for fixed  $\delta'_1, \dots, \delta'_{n'}$ ) is shown in Figure 49



**Figure 47:** Illustrating the bifunctor  $\nu$ . Here the green squares illustrate 4 local operator insertions  $\delta_1, \delta_2, \delta_3, \delta_4$  between Interfaces from Theory  $\mathcal{T}^-$  to Theory  $\mathcal{T}^0$ . Similarly, the green triangles represent 3 local operator insertions  $\delta'_1, \delta'_2, \delta'_3$  between Interfaces from Theory  $\mathcal{T}^0$  to Theory  $\mathcal{T}^+$ . The maroon arrows indicate that the two interfaces with their local operator insertions are being moved together (adiabatically).

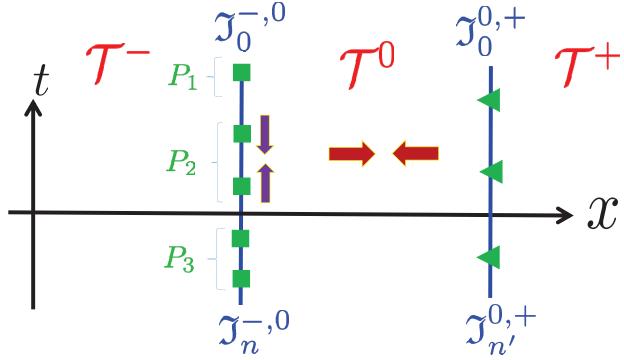


**Figure 48:** The result of the process described in Figure 47 is a single local operator, described by the green star, between the products of the initial and final Interfaces. This is the operator  $\nu(\delta_1, \delta_2, \delta_3, \delta_4; \delta'_1, \delta'_2, \delta'_3)$ .

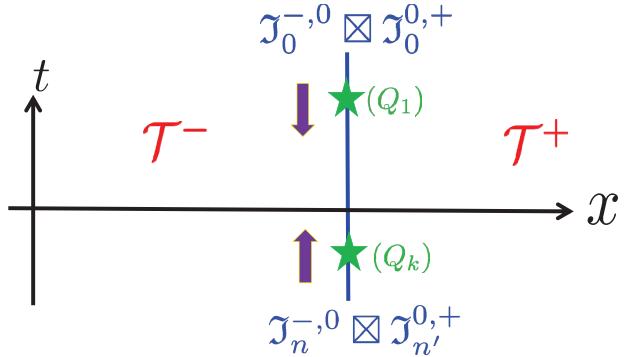
and Figure 50. Of course, an analogous statement can also be made holding  $\delta_1, \dots, \delta_n$  fixed.

### 6.3 Composition Of Three Interfaces

We can now consider a geometry  $G_3^\eta$  with three interfaces, set at  $x = -L$ ,  $x = \eta L$ ,  $x = L$ ,  $-1 < \eta < 1$ , with vacuum data  $(\mathbb{V}^\alpha, z^\alpha)$ ,  $\alpha \in \{-, 0, 1, +\}$ , in the negative half plane, the two strips and the positive half plane respectively. The composite webs in this geometry



**Figure 49:** This figure represents one side of the equation stating that  $\nu$  is a bifunctor. We first take the “operator product” of the ordered set  $P_2$  of local operators on Interfaces between  $\mathcal{T}^-$  and  $\mathcal{T}^0$ , as indicated by the vertical purple arrows, and then apply the Interface product, as indicated by the horizontal maroon arrows. We sum over all decompositions of the local operators on the left Interface into  $P_1 \amalg P_2 \amalg P_3$ .



**Figure 50:** This figure represents the other side of the equation stating that  $\nu$  is a bifunctor. We consider all ordered decompositions  $Q_1 \amalg \dots \amalg Q_k$  of local operators on the left Interface (keeping  $\delta'_1, \dots, \delta'_{n'}$  fixed). We apply the Interface product separately to these collections to produce the Interfaces and local operators indicated by  $\star(Q_1), \dots, \star(Q_k)$ . Then we take the product of these local operators, as indicated by the vertical purple arrows.

have essentially the same properties as in the case with two interfaces, with a convolution identity of the same general form.

Following through the same derivation, we arrive at the statement that given four Theories  $\mathcal{T}^-, \mathcal{T}^0, \mathcal{T}^1, \mathcal{T}^+$  we obtain a triple  $A_\infty$  functor from  $\mathfrak{Br}(\mathcal{T}^-, \mathcal{T}^0) \times \mathfrak{Br}(\mathcal{T}^0, \mathcal{T}^1) \times \mathfrak{Br}(\mathcal{T}^1, \mathcal{T}^+)$  to  $\mathfrak{Br}(\mathcal{T}^-, \mathcal{T}^+)$ , composing three consecutive Interfaces  $\mathfrak{I}^{-,0}, \mathfrak{I}^{0,1}, \mathfrak{I}^{1,+}$  to a

single Interface we can denote as

$$\mathfrak{I}_\eta^{-,+} := (\mathfrak{I}^{-,0} \mathfrak{I}^{0,1} \mathfrak{I}^{1,+})_\eta. \quad (6.57)$$

It has boundary amplitude

$$\mathcal{B}(\mathfrak{I}_\eta^{-,+}) := \rho_\beta(t_c) \left( \frac{1}{1 - \mathcal{B}^{-,0}}, \frac{1}{1 - \mathcal{B}^{0,1}}, \frac{1}{1 - \mathcal{B}^{1,+}} \right) \quad (6.58)$$

where  $t_c$  is the taut composite element in  $G_3^\eta$ .

Thus we get a family of triple compositions, parameterized by  $\eta$ , and we may obviously wonder how would they compare to the repeated compositions  $(\mathfrak{I}^{-,0} \boxtimes \mathfrak{I}^{0,1}) \boxtimes \mathfrak{I}^{1,+}$  and  $\mathfrak{I}^{-,0} \boxtimes (\mathfrak{I}^{0,1} \boxtimes \mathfrak{I}^{1,+})$ . We want to argue that there is an homotopy equivalence between any pair of interfaces  $(\mathfrak{I}^{-,0} \mathfrak{I}^{0,1} \mathfrak{I}^{1,+})_\eta$  and  $(\mathfrak{I}^{-,0} \mathfrak{I}^{0,1} \mathfrak{I}^{1,+})_{\tilde{\eta}}$  and that moreover there are well-defined limits such that:

$$\begin{aligned} (\mathfrak{I}^{-,0} \mathfrak{I}^{0,1} \mathfrak{I}^{1,+})_{\eta \rightarrow -1} &= (\mathfrak{I}^{-,0} \boxtimes \mathfrak{I}^{0,1}) \boxtimes \mathfrak{I}^{1,+} \\ (\mathfrak{I}^{-,0} \mathfrak{I}^{0,1} \mathfrak{I}^{1,+})_{\eta \rightarrow 1} &= \mathfrak{I}^{-,0} \boxtimes (\mathfrak{I}^{0,1} \boxtimes \mathfrak{I}^{1,+}) \end{aligned} \quad (6.59)$$

It then follows that  $(\mathfrak{I}^{-,0} \boxtimes \mathfrak{I}^{0,1}) \boxtimes \mathfrak{I}^{1,+}$  and  $\mathfrak{I}^{-,0} \boxtimes (\mathfrak{I}^{0,1} \boxtimes \mathfrak{I}^{1,+})$  themselves are homotopy equivalent.

### 6.3.1 Limits

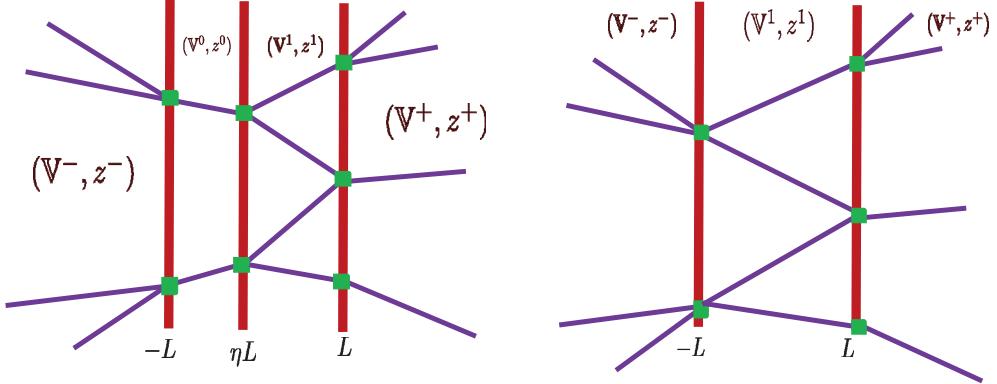
We can start from the analysis of the  $\eta \rightarrow -1$  limit of the triple composition. We need to study the fate of a composite taut web in  $G_3^\eta$  when  $\eta$  is sent to  $-1$ . The taut element itself may jump in the process. As there are only finitely many possible taut web topologies, as we make  $\eta$  sufficiently close to  $-1$ , the taut element will ultimately stabilize.

After the taut element has stabilized, we can analyze the problem in the same way as we did for composite webs. As the left strip shrinks, some vertices will remain at finite distances from the boundaries, while the distance from the left boundary of some other “bound” vertices will scale as  $1 + \eta$ . The bound vertices will form clumps at finite locations along the left boundary. The whole web is well-approximated by a composite taut web in the  $G_2$  geometry with vacua  $\mathbb{V}^-$ ,  $\mathbb{V}^1$ ,  $\mathbb{V}^+$ , with boundary vertices on the left boundary replaced by tiny taut webs in a local  $G_2$  geometry with vacua  $\mathbb{V}^-$ ,  $\mathbb{V}^0$ ,  $\mathbb{V}^1$ .

If  $\mathfrak{c}$  is any composite web in the  $G_2$  geometry with vacua  $\mathbb{V}^-$ ,  $\mathbb{V}^1$ ,  $\mathbb{V}^+$ , so  $\mathfrak{c} \in \mathcal{W}_C[\mathbb{V}^-, \mathbb{V}^1, \mathbb{V}^+]$  then we can define an operation

$$T_\partial(\mathfrak{c}) : T\mathcal{W}_C[\mathbb{V}^-, \mathbb{V}^0, \mathbb{V}^1] \rightarrow \mathcal{W}_C[\mathbb{V}^-, \mathbb{V}^0, \mathbb{V}^1, \mathbb{V}^+] \quad (6.60)$$

whose nonzero values on monomials  $\mathfrak{c}_1 \otimes \cdots \otimes \mathfrak{c}_n$  are obtained by inserting the  $\mathfrak{c}_a$  into the left boundary vertices  $v_a^\partial$  of  $\mathfrak{c}$  provided  $J_{v_a^\partial}(\mathfrak{c}) = J_\infty(\mathfrak{c}_a)$  and provided that the past strip vacuum of  $\mathfrak{c}_a$  agrees with the future strip vacuum of  $\mathfrak{c}_{a+1}$ . We orient the resulting web in the standard way, wedging the reduced orientations of the arguments in the same order as the arguments themselves.



**Figure 51:** In the limit that  $\eta \rightarrow -1$  the taut composite web on the left degenerates to that on the right. We can view this as a contribution of  $T_\partial(\mathbf{t}^{-,1,+})[\mathbf{t}^{-,0,1}, \mathbf{t}^{-,0,1}]$  to  $\mathbf{t}_\eta^{-,0,1,+}$ .

Thus with these definitions, if  $\mathbf{t}_\eta^{-,0,1,+}$  is the taut element in  $G_3^\eta$  and  $\mathbf{t}^{-,1,+}$  and  $\mathbf{t}^{-,0,1}$  the taut elements for the two  $G_2$  geometries respectively,

$$\lim_{\eta \rightarrow -1} \mathbf{t}_\eta^{-,0,1,+} = T_\partial[\mathbf{t}^{-,1,+}]\left(\frac{1}{1 - \mathbf{t}^{-,0,1}}\right) \quad (6.61)$$

See, for example, Figure 51.

Now we can compute

$$(\mathcal{B}^{-,0}\mathcal{B}^{0,1}\mathcal{B}^{1,+})_{\eta \rightarrow -1} = \rho_\beta\left(\lim_{\eta \rightarrow -1} \mathbf{t}_\eta^{-,0,1,+}\right)\left[\frac{1}{1 - \mathcal{B}^{-,0}}; \frac{1}{1 - \mathcal{B}^{0,1}}; \frac{1}{1 - \mathcal{B}^{1,+}}\right] \quad (6.62)$$

from the representations of webs and interface amplitudes. We can write

$$\begin{aligned} \rho_\beta(T_\partial[\mathbf{t}^{-,1,+}]\left(\frac{1}{1 - \mathbf{t}^{-,0,1}}\right))\left[\frac{1}{1 - \mathcal{B}^{-,0}}; \frac{1}{1 - \mathcal{B}^{0,1}}; \frac{1}{1 - \mathcal{B}^{1,+}}\right] = \\ \rho_\beta(\mathbf{t}^{-,1,+})\left[\frac{1}{1 - \rho_\beta(\mathbf{t}^{-,0,1})\left[\frac{1}{1 - \mathcal{B}^{-,0}}; \frac{1}{1 - \mathcal{B}^{0,1}}\right]}; \frac{1}{1 - \mathcal{B}^{1,+}}\right] \end{aligned} \quad (6.63)$$

Thus

$$(\mathfrak{I}^{-,0}\mathfrak{I}^{0,1}\mathfrak{I}^{1,+})_{\eta \rightarrow -1} = (\mathfrak{I}^{-,0} \boxtimes \mathfrak{I}^{0,1}) \boxtimes \mathfrak{I}^{1,+} \quad (6.64)$$

A similar analysis holds for  $\eta \rightarrow 1$ .

At this point, we are left with the task of proving the homotopy equivalence of the triple compositions for different values  $\eta_{p,f}$  of  $\eta$ . The Chan-Paton factors are independent of  $\eta$ . If we define  $\mathfrak{I}_{p,f} = (\mathfrak{I}^{-,0}\mathfrak{I}^{0,1}\mathfrak{I}^{1,+})_{\eta_{p,f}}$  then, according to (5.23), we need to find  $\delta_h \in R^\partial(\mathcal{E}^{-,+})$  such that

$$\mathcal{B}_f - \mathcal{B}_p + \rho_\beta(\mathfrak{t}^{-,+})[\frac{1}{1-\mathcal{B}_f}; \delta_h; \frac{1}{1-\mathcal{B}_p}] = 0 \quad (6.65)$$

where

$$\mathcal{E}_{i_-, i_+}^{-,+} := \oplus_{i_0, i_1} \mathcal{E}_{i_-, i_0}^{-,0} \otimes \mathcal{E}_{i_0, i_1}^{0,1} \otimes \mathcal{E}_{i_1, i_+}^{1,+} \quad (6.66)$$

As the interface amplitudes  $\mathcal{B}_{p,f} \in \text{Hop}(\mathfrak{I}_p, \mathfrak{I}_f)$  are computed directly from the corresponding composite taut elements  $\mathfrak{t}_{p,f}^{-,0,1,+}$  we will first try to find a similar identity for the difference between these two taut elements.

### 6.3.2 Homotopies

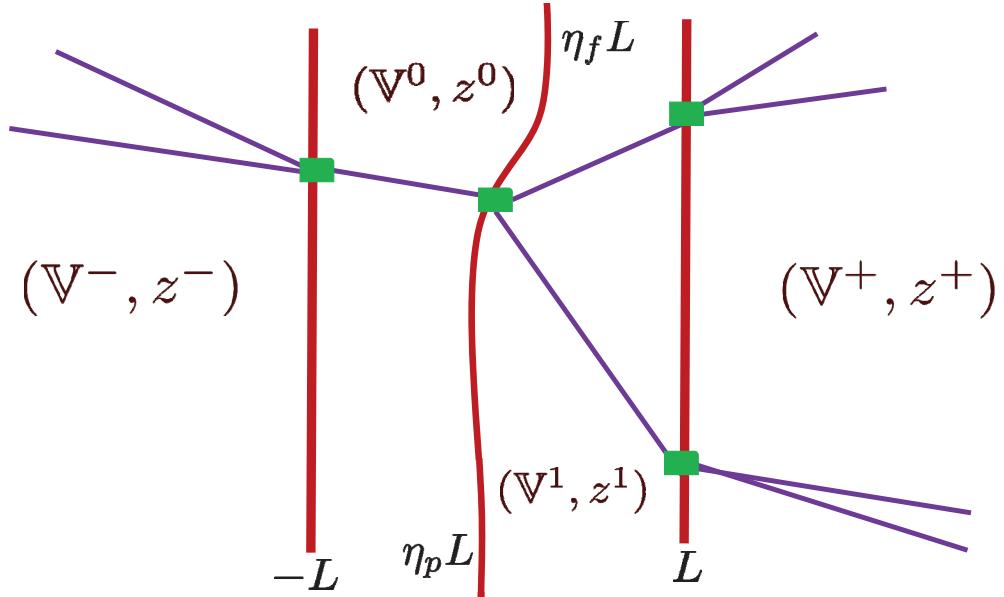
How can we compare the taut elements for different values of  $\eta$ ? One way would be to vary  $\eta$  continuously, and study the special values at which the taut element jumps. This is a somewhat subtle but interesting analysis, and we will come back to it at the very end of the section. Here we will use a different, more effective strategy, which produces precisely the desired result.

We can encode the problem in a simple geometric setup: a time-dependent setup where  $\eta$  depends very slowly on the time direction. We want  $\eta(y)$  to vary slowly enough that the deviation from the vertical of the slope of the boundary between the  $\mathbb{V}^0$  and  $\mathbb{V}^1$  at any  $y$  does not affect the local interface taut element between  $\mathbb{V}^0$  and  $\mathbb{V}^1$ . In particular the slope should remain close enough to vertical so that it never crosses the slopes of the weights  $z_{ij}^0$  and  $z_{i'j'}^1$  for any  $i, j, i', j'$ . We also want the variation of  $\eta$  to be restricted to some compact region  $y \in [y_p, y_f]$  with fixed values  $\eta_p$  and  $\eta_f$  in the past and the future, respectively. We will call such functions  $\eta(y)$  *tame*.

As soon as these conditions are met, we can consider composite webs in the time-dependent geometry.<sup>31</sup> The definitions for composite webs from Section §6.2 have straightforward generalizations, with one important exception noted in the next paragraph. The only slightly new point is that the wall vertices between  $\mathcal{T}^0$  and  $\mathcal{T}^1$  do not sit at a definite value of  $x$  since the boundary between these theories is  $y$ -dependent. Nevertheless, one can define deformation types of time-dependent composite webs. The only new point is that wall vertices between  $\mathcal{T}^0$  and  $\mathcal{T}^1$  must slide along the wall  $x = \eta(y)L$ . If a composite web  $\mathfrak{c}$  has  $n$  wall-vertices on the boundary between  $\mathcal{T}^0$  and  $\mathcal{T}^1$  and  $\mathfrak{d}_1, \dots, \mathfrak{d}_n$  are  $n$  interface webs between  $\mathcal{T}^0$  and  $\mathcal{T}^1$  then the convolution  $T[\mathfrak{c}](\mathfrak{d}_1, \dots, \mathfrak{d}_n)$  makes sense as a composite web in the time-dependent geometry. Moreover, we claim that if  $\eta(y)$  satisfies the above conditions then the deformation type of  $T[\mathfrak{c}](\mathfrak{d}_1, \dots, \mathfrak{d}_n)$  only depends on the deformation types of  $\mathfrak{c}$  and  $\mathfrak{d}_1, \dots, \mathfrak{d}_n$ . To prove this we note that there are only a finite number of possible webs  $\mathfrak{c}, \mathfrak{d}_1, \dots, \mathfrak{d}_n$ . But then note that the homotopy which straightens out the

---

<sup>31</sup>Actually, we could consider the positions of all three interfaces to be time-dependent. The resulting discussion would be similar to what we give here.



**Figure 52:** A smooth adiabatic variation of  $\eta$  as a function of  $y$  can relate the products of interfaces at different values of  $\eta$ .

boundary between  $\mathcal{T}^0, \mathcal{T}^1$  to a vertical line in the neighbourhood of each interface web will change the edge lengths by an amount which can be made arbitrarily small by making the slope of the boundary arbitrarily close to vertical.

The one new point in the definitions for composite webs in time-dependent geometries is that we must change the definitions of rigid, taut and sliding webs from what we used in Section 6.2. Webs in the time-dependent geometry with *no moduli* are *rigid* or *taut*, and webs with a *single modulus* are called *sliding*. An example of a sliding web in a time-dependent geometry is a rigid composite web for  $\eta = \eta_f$  whose support lies in  $y \geq y_f$  or, similarly, a rigid composite web for  $\eta = \eta_p$  whose support lies in  $y \leq y_p$ .

We can now repeat our usual exercise: We define  $t[\eta(y)]$  to be the taut (= rigid) element in  $\mathcal{W}_C$  in the time-dependent geometry determined by  $\eta(y)$  and find its convolution identity by examining the end-points of the moduli spaces of sliding webs in this time-dependent geometry. We encounter standard boundaries at finite distance: some edge inequalities get saturated, some subset of vertices collapse to a point in the interior or at any of the interfaces. These endpoints are enumerated by convolutions of appropriate taut elements.

Boundaries at infinite distance are also rather standard. Much as for the time-independent geometry we can consider directions along which a web grows to large size. These boundaries are accounted for by a standard tensor operation, inserting taut com-

posite webs at the boundary vertices of a taut interface web. In order for the result to have only one modulus in the time-dependent geometry, exactly one of the composite webs must be localized in the compact region where  $\eta$  varies. The others will be sliding along the regions of constant  $\eta$  in the past or future. Thus these endpoints are enumerated by the usual  $T_\partial(\mathbf{t}^{-,+})$  operation with  $\mathbf{t}^{-,+}$  being the taut element for interface webs between  $\mathbb{V}^-$  and  $\mathbb{V}^+$ , acting on a collection of taut webs in the far past or future, together with a single rigid/taut web stuck somewhere in the region  $y_p \leq y \leq y_f$ .

The only new terms are very simple: a single taut web for the  $G_3^{\eta_p}$  geometry inserted far in the past, or a taut web for the  $G_3^{\eta_f}$  geometry inserted far in the future. We are now ready to write the convolution identity for the taut element  $\mathbf{t}[\eta(y)]$  in the time-dependent geometry. Let  $\mathbf{t}_{pl} = \mathbf{t}_p^- + \mathbf{t}_p^0 + \mathbf{t}_p^1 + \mathbf{t}_p^+$  denote the sum of the planar taut elements for the four theories. Similarly, let  $\mathbf{t}_\partial$  denote the sum of the taut interface elements for the three boundaries between pairs of theories:  $\mathbf{t}_\partial = \mathbf{t}^{-,0} + \mathbf{t}^{0,1} + \mathbf{t}^{1,+}$ , and let  $\mathbf{t}_c^{p,f}$  be the taut composite elements for the initial and final  $G_3^{\eta_{p,f}}$  geometries. If  $\mathbf{t}_c^f$  is understood to be made of webs with all edges and vertices in the region  $y \geq y_f$  and likewise  $\mathbf{t}_c^p$  has all lines and vertices in the region  $y \leq y_p$ , then we can consider these as elements of the web group of the time-dependent geometry. With this understanding we have the convolution identity

$$\mathbf{t}[\eta(y)] * \mathbf{t}_{pl} + \mathbf{t}[\eta(y)] * \mathbf{t}_\partial + \mathbf{t}_c^f - \mathbf{t}_c^p + T_\partial(\mathbf{t}^{-,+})\left[\frac{1}{1-\mathbf{t}_c^f}; \mathbf{t}[\eta(y)]; \frac{1}{1-\mathbf{t}_c^p}\right] = 0. \quad (6.67)$$

The  $\mathbf{t}_c^f - \mathbf{t}_c^p$  could be absorbed into the  $T_\partial(\mathbf{t}^{-,+})$  term if we include the empty web (no vertices nor edges) into  $\mathbf{t}[\eta(y)]$ .

Now we combine equation (6.67) together with a representation of the composite webs and apply the result to

$$e^{\beta^-}; \frac{1}{1-\mathcal{B}^{-,0}}; e^{\beta^0}; \frac{1}{1-\mathcal{B}^{0,1}}; e^{\beta^1}; \frac{1}{1-\mathcal{B}^{1,+}}; e^{\beta^+} \quad (6.68)$$

The first two terms of (6.67) give zero. The next two terms give  $\mathcal{B}_f - \mathcal{B}_p$  using the definition (6.58), and, using identities analogous to (6.42), the last term in (6.67) gives

$$\rho_\beta(\mathbf{t}^{-,+}) \left[ \frac{1}{1-\mathcal{B}_f}; \delta[\eta(y)]; \frac{1}{1-\mathcal{B}_p} \right] \quad (6.69)$$

with

$$\delta[\eta(y)] := \rho_\beta(\mathbf{t}[\eta(y)]) \left[ \frac{1}{1-\mathcal{B}^{-,0}}; \frac{1}{1-\mathcal{B}^{0,1}}; \frac{1}{1-\mathcal{B}^{1,+}} \right] \quad (6.70)$$

finally leading to the desired relation (6.65). Recall from (5.23) that (6.65) implies that the morphism  $\mathbf{Id} + \delta[\eta(y)]$  (or just  $\delta[\eta(y)]$  if we include the empty web into  $\mathbf{t}[\eta(y)]$ ) from  $\mathcal{B}_p$  to  $\mathcal{B}_f$  is  $M_1$ -closed.

Next, we need to show that  $\mathbf{Id} + \delta[\eta(y)]$  has a closed inverse up to homotopy. In principle, we could cheat a bit. The morphism  $\delta[\eta(y)]$  can have a non-trivial scalar part, from composite webs with no external edges. As the vacua are naturally ordered, in the sense that for any pair of vacua only one of the two can be to the future of the other, this scalar contribution is upper triangular and thus the scalar part of  $\mathbf{Id} + \delta[\eta(y)]$  is invertible.

We could build the required inverse recursively, as described in Section 5.2. However, there is a much better way to proceed.

In order to prepare some tools which will be useful later, we will instead prove directly that the closed morphism  $\mathbf{Id} + \delta[\eta(y)]$  associated to the  $\eta(y)$  deformation and the closed morphism  $\mathbf{Id} + \delta[\eta(-y)]$  associated to the time-reversed deformation are inverse of each other up to homotopy.

The first observation is that if we have two continuous deformations  $\eta_1(y)$  and  $\eta_2(y)$ , with the same value  $\eta_{1,p} = \eta_{2,f}$  in the past of  $\eta_1$  and in the future of  $\eta_2$ , we can build a “shifted time composition”  $\eta_1 \circ_T \eta_2$  where  $\eta_1$  is placed at some large time  $T$  after  $\eta_2$ . If  $T$  is sufficiently large, it is easy to see that

$$\mathbf{Id} + \delta[\eta_1 \circ_T \eta_2] = M_2(\mathbf{Id} + \delta[\eta_1], \mathbf{Id} + \delta[\eta_2]) \quad (6.71)$$

where  $M_2$  is computed in the Interface category  $\mathfrak{Br}(\mathcal{T}^-, \mathcal{T}^+)$ , and  $\delta[\eta_1 \circ_T \eta_2]$ ,  $\delta[\eta_1]$ , and  $\delta[\eta_2]$  are computed in different time-dependent geometries.

In order to establish (6.71) we observe that the most general rigid web in the composite geometry can be approximated by a large interface web with a boundary vertex resolved into a rigid web in the region of  $\eta_1$  variation, a boundary vertex resolved into a rigid web in the region of  $\eta_2$  variation and any number of boundary vertices resolved to rigid webs in the regions of constant  $\eta$ . Thus we have:

$$\mathbf{t}[\eta_1 \circ_T \eta_2] = \mathbf{t}[\eta_1] + \mathbf{t}[\eta_2] + T_\partial(\mathbf{t}^{-,+})\left[\frac{1}{1 - \mathbf{t}_c^{1,f}}; \mathbf{t}[\eta_1]; \frac{1}{1 - \mathbf{t}_c^{1,p}}; \mathbf{t}[\eta_2]; \frac{1}{1 - \mathbf{t}_c^{2,p}}\right] \quad (6.72)$$

where  $\mathbf{t}_c^{1,f}$  is the taut composite element for the value  $\eta_{1,f}$  and we recall that  $\mathbf{t}_c^{1,p} = \mathbf{t}_c^{2,f}$ . This equation is to be thought of as valued in the web group of the time-dependent geometry described by  $\eta_1 \circ_T \eta_2$ . Now, again using identities similar to (6.42) we learn that (6.72) implies (6.71). Thus, the desired result

$$\mathfrak{I}_{\eta_p}^{-,+} \sim \mathfrak{I}_{\eta_f}^{-,+} \quad (6.73)$$

will follow by proving that

$$\mathbf{Id} + \delta[\eta(y) \circ_T \eta(-y)] \sim \mathbf{Id}. \quad (6.74)$$

We will prove (6.74) in the next Section.

### 6.3.3 Homotopies Of Homotopies

The required identity (6.74) will follow from a more general result: Suppose that  $\eta_1(y)$  and  $\eta_2(y)$  define two tame time-dependent geometries such that  $\eta_1(y) = \eta_2(y) = \eta_f$  for  $y \geq y_f$  and  $\eta_1(y) = \eta_2(y) = \eta_p$  for  $y \leq y_p$ . Suppose moreover that  $\eta(y; s)$  is a homotopy between these functions. Then we claim that

$$\delta[\eta_1(y)] \sim \delta[\eta_2(y)] \quad (6.75)$$

are homotopic morphisms in  $\mathfrak{Br}(\mathcal{T}^-, \mathcal{T}^+)$ . When writing  $\eta(y; s)$  we take  $s \in \mathbb{R}$  and assume that  $\frac{\partial}{\partial s} \eta(y; s)$  has compact support in some interval  $(s_1, s_2)$  with  $\eta(y; s) = \eta_1(y)$  for  $s \leq s_1$

and  $\eta(y; s) = \eta_2(y)$  for  $s \geq s_2$ . We will also assume that the homotopy  $\eta(y; s)$  is tame for all fixed  $s$  and is furthermore generic.

We will establish (6.75) by studying how the taut element  $t[\eta(y; s)]$  jumps as we vary  $s$ . Some of the ideas we introduce here will be very useful in Section §8 on wall-crossing as well as in Sections §10.7 and §15 on Landau-Ginzburg models.

The basic idea is to allow deformations in the geometry  $G_3[s] := G_3^{\eta(\cdot; s)}$  determined by  $\eta(y; s)$  in our notion of “deformation type”. We will call this enlarged notion “deformation type up to homotopy” or just  $h$ -type, for brevity. Thus, we allow the usual translations and dilation of internal edges at fixed  $s$ , but also we allow the parameter  $s$  to be adjusted. For readers who demand more precision we will spell this out more formally. Less fastidious readers can skip to the examples below equation (6.77) below.

Define a *continuous family of webs*  $\mathfrak{w}[s]$ , labeled by  $s_- \leq s \leq s_+$  to be a family of webs such that

1. For each fixed  $s \in [s_-, s_+]$ ,  $\mathfrak{w}[s]$  is a web in the geometry  $G_3[s]$ .
2. As  $s$  varies the vertices of  $\mathfrak{w}[s]$  in the plane vary continuously, no edge shrinks to zero length, and no boundary vertices collide.

For a fixed  $s \in \mathbb{R}$  let  $\text{Web}[G_3[s]]$  denote the set of all webs in the geometry  $G_3[s]$ . According to the definition in Section §2.1, a deformation type of a web in  $G_3[s]$  is a subset  $\mathcal{D}(\mathfrak{w}) \subset \text{Web}[G_3[s]]$ , defined by an equivalence relation under translation and dilation of internal edges. We now consider the union

$$\text{WEB} := \cup_{s \in \mathbb{R}} \text{Web}[G_3[s]] \quad (6.76)$$

and define a *deformation type up to homotopy* (or, just an  $h$ -type, for brevity) to be an equivalence class of webs  $\mathcal{D}^h \subset \text{WEB}$  with the following two equivalence relations:

1. If  $\mathfrak{w}_a, \mathfrak{w}_b \in \text{Web}[G_3[s]]$  for the same value of  $s$  then, if they define the same deformation type within  $\text{Web}[G_3[s]]$ , they are equivalent.
2. If there exists a continuous family of webs  $\mathfrak{w}[s]$ ,  $s_- \leq s \leq s_+$  interpolating between  $\mathfrak{w}_-$  and  $\mathfrak{w}_+$ , then the web  $\mathfrak{w}_- \in \text{Web}[G_3[s_-]]$  is equivalent to the web  $\mathfrak{w}_+ \in \text{Web}[G_3[s_+]]$ .

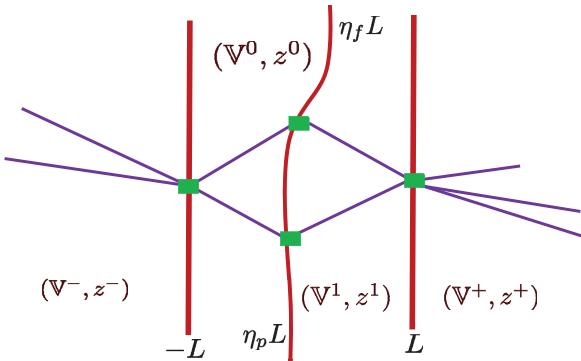
As usual, we will let  $\mathcal{D}^h(\mathfrak{w})$  denote the set of deformation types up to homotopy of any given web  $\mathfrak{w} \in \text{Web}[G_3[s]]$  for some  $s$ .

Of course, there is a projection  $\pi : \text{WEB} \rightarrow \mathbb{R}$  given by the  $s$ -coordinate so an  $h$ -type  $\mathcal{D}^h$  will be fibered over a connected subset of some interval  $[s_-, s_+] \in \mathbb{R}$ . See, Figure 59 below. The fiber of the projection  $\pi : \mathcal{D}^h \rightarrow [s_-, s_+]$  above some  $s \in [s_-, s_+]$  will be a subset of  $\mathbb{R}^{2V_i(\mathfrak{w})+V_\partial(\mathfrak{w})}$  given by the data of the vertices of the web. We can therefore regard  $\mathcal{D}^h(\mathfrak{w}) \subset \mathbb{R}^{2V_i(\mathfrak{w})+V_\partial(\mathfrak{w})+1}$ . This motivates us to define the *expected h-dimension* (or just  $h$ -dimension, for brevity) by

$$d^h(\mathfrak{w}) := 2V_i(\mathfrak{w}) + V_\partial(\mathfrak{w}) + 1 - E(\mathfrak{w}). \quad (6.77)$$

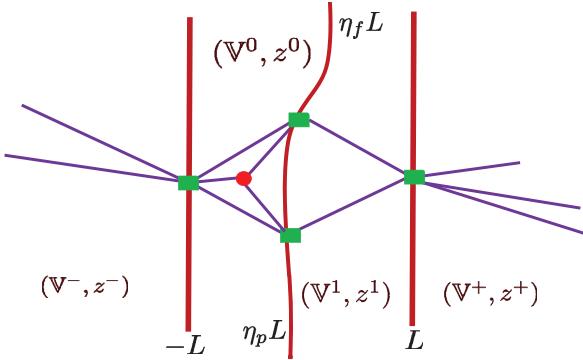
The set  $\mathcal{D}^h(\mathfrak{w})$  is orientable. Oriented  $h$ -types are denoted by  $\mathfrak{w}^h$ . If  $\mathfrak{w} \in \text{Web}[G_3[s]]$  for some  $s$  then we let  $\mathfrak{w}^h$  denote the induced  $h$ -type with orientation  $o(\mathfrak{w}^h) = o(\mathfrak{w}) \wedge ds$ . It can happen that the projection of  $\mathcal{D}^h$  under  $\pi$  is a single point  $s_* \in \mathbb{R}$ . In this case we take the orientation to be  $o(\mathfrak{w}^h) = ds$ . Oriented  $h$ -types again generate a free abelian group  $\mathcal{W}^h$ .

We now consider the convolution identity in  $\mathcal{W}^h$ . As usual we examine the moduli space of sliding  $h$ -types, i.e. those with  $d^h(\mathfrak{w}) = 1$ . What are the possible boundary regions? Provided that  $\eta(y; s)$  is tame and generic these can be listed as follows:

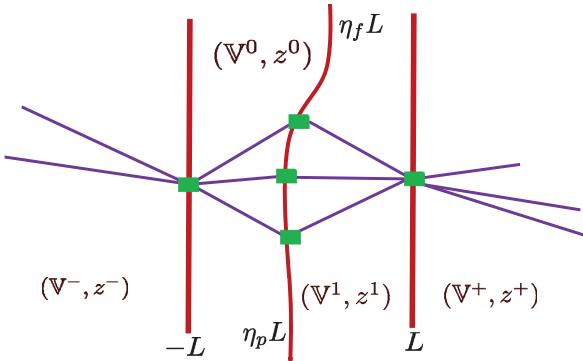


**Figure 53:** At fixed  $s$  the web shown here has expected and true dimension zero: There are four boundary vertices and four internal edges. If the edge constraints can be satisfied then the equations are all independent. At fixed values of  $s$  the web cannot be deformed at all. If it exists it will only exist for a finite set of possible  $y$ -coordinates of the left vertex. However, if it exists, for tame and generic homotopies  $\eta(y, s)$  if we deform  $s$  we can deform the web so that it will generically define a deformation type up to homotopy of expected and true dimension  $d^h = 1$ . This is a typical contribution to both  $\mathbf{t}^h[\eta_1(y)]$  and  $\mathbf{t}^h[\eta_2(y)]$ .

1. First, for  $s \leq s_1$  or  $s \geq s_2$  any of the summands in  $\mathbf{t}^h[\eta(y; s)]$  are sliding. Recall that we assume  $\eta(y; s)$  has nontrivial  $y$ -dependence, so the taut elements are the same as the rigid elements. That is, they have  $d = 0$ . For fixed  $s$  the moduli space of such deformation types is either empty or a finite set. (It could be identified, for example, with the finite set of  $y$  values of the vertex on the left boundary.) On the other hand, the image under  $\pi$  of such an  $h$ -type is a semi-infinite interval containing  $(-\infty, s_1]$  or  $[s_2, +\infty)$ . The boundaries at infinity of these webs contributes  $\mathbf{t}^h[\eta_1(y)] - \mathbf{t}^h[\eta_2(y)]$  to the convolution identity, where  $\mathbf{t}^h[\eta_1(y)], \mathbf{t}^h[\eta_2(y)]$  denote sums of oriented  $h$ -types of webs. For a typical example, see Figure 53. We are interested in the *difference*  $\mathbf{t}^h[\eta_1(y)] - \mathbf{t}^h[\eta_2(y)]$ . It can be nonzero because some webs can appear and disappear by shrinking to zero and then violating edge constraints, or by blowing up to infinity.
2. It can happen that, at some special values of  $s$ , exceptional webs exist. See, for example Figures 54 and 55. Now at those special values of  $s$ ,  $d(\mathfrak{w}) = -1$  and  $d^h(\mathfrak{w}) =$

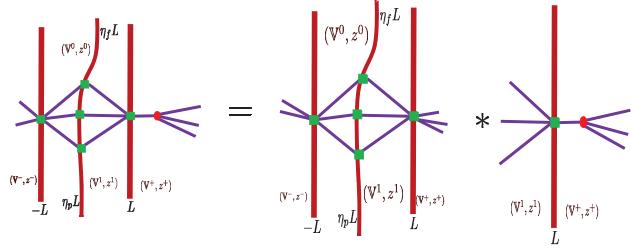


**Figure 54:** This web will only exist at special values of  $s$ . It has dimension  $2V_i + V_\partial - E = 2 + 4 - 7 = -1$  at fixed  $s$  and  $h$ -dimension  $d^h = 0$ .

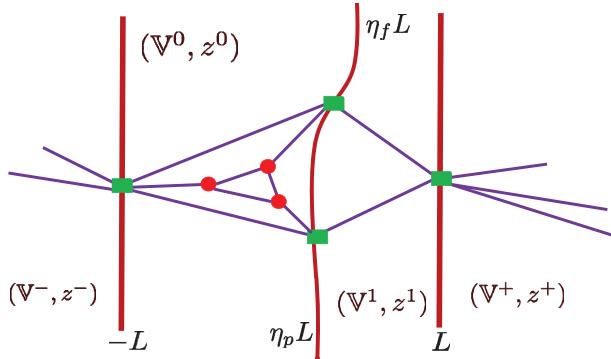


**Figure 55:** This web will only exist at special values of  $s$ . It has dimension  $V_\partial - E = 5 - 6 = -1$  at fixed  $s$  and  $h$ -dimension  $d^h = 0$ .

0. Such webs will exist only at isolated values  $s_*$ . Such exceptional configurations can be convolved with taut elements  $t_p$  or  $t_\partial$  to produce  $h$ -types of  $h$ -dimension one. The case of a convolution with  $t_\partial$  is illustrated in Figure 56. This  $h$ -type projects under  $\pi$  to a single point in the  $s$ -line. If we denote by  $\epsilon^h$  the  $h$ -types of all the exceptional webs of  $h$ -dimension 0 then this represents a contribution of  $\epsilon^h * t_\partial$  to the convolution identity.
3. By contrast in Figure 57 we show an  $h$ -type of  $h$ -dimension 1 which will project to an interval in the  $s$ -line. The boundary of this  $h$ -type contributes to one of the terms  $\epsilon^h * t_{pl}$  in the convolution identity.
4. Finally, there are boundary regions at infinity: a very large exceptional sliding web,



**Figure 56:** Here we show a typical example of a contribution  $\epsilon^h * t_\partial$  to the convolution identity. The sliding  $h$ -type on the left projects to a single value  $s_* \in \mathbb{R}$ .

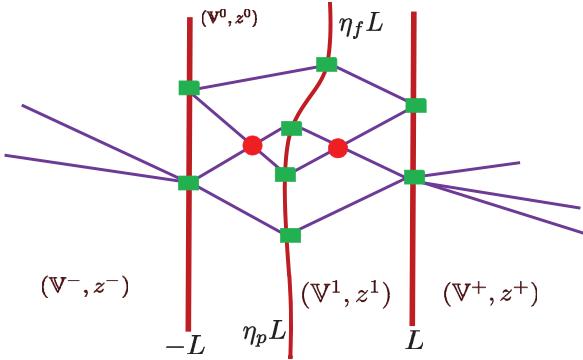


**Figure 57:** The web shown here has expected dimension  $d = 0$  at fixed  $s$  and  $h$ -dimension  $d^h = 1$ . If it exists it will project to an interval of the  $s$ -line. At the boundary of the interval the inner triangle shrinks and the figure degenerates to an exceptional web which only exists at a fixed value  $s_*$ . The nearby sliding  $h$ -types represent contributions to  $\epsilon^h * t_{pl}$  in the convolution identity.

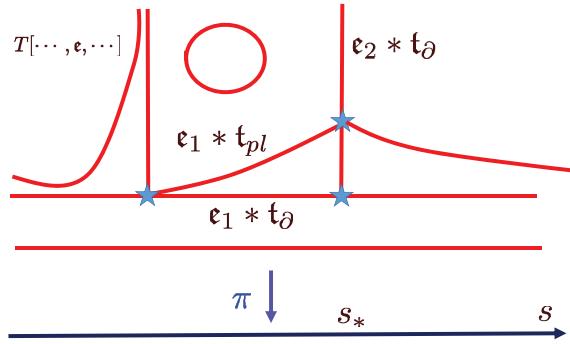
which can be described as usual by the tensor operation  $T_\partial(t^{-,+})[\frac{1}{1-t_c^f}; \epsilon^h; \frac{1}{1-t_c^p}]$ .

5. There can also be contributions to the moduli space of sliding  $h$ -types which are circles with no boundary. This is illustrated in Figure 58. Altogether, the moduli space of sliding  $h$ -types can be schematically pictured as shown in Figure 59.

Putting these various boundaries together we can write the resulting convolution iden-



**Figure 58:** The web shown here has expected dimension  $d = 0$  at fixed  $s$  and  $h$ -dimension  $d^h = 1$ . As  $s$  varies the two “causal diamonds” can merge and exchange places, leading to a component of the moduli space of sliding webs which is a circle.



**Figure 59:** A schematic picture of the set of sliding  $h$ -types. The horizontal axis at the bottom is the  $s$ -line but no meaning is assigned to the vertical direction.

ity as <sup>32</sup>

$$t^h[\eta_1(y)] - t^h[\eta_2(y)] + e^h * t_{pl} + e^h * t_{\partial} + T_{\partial}(t^{-,+})\left[\frac{1}{1-t_c^f}; e^h; \frac{1}{1-t_c^p}\right] = 0 \quad (6.78)$$

where, again,  $e^h$  includes the sum of oriented  $h$ -types of all possible exceptional webs encountered along the deformation.

If we apply a web representation to equation (6.78), and again use identities of the

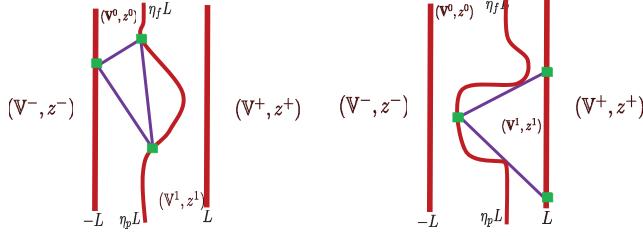
---

<sup>32</sup>We will not try to give a formal definition of a convolution  $\mathcal{W}^h \times \mathcal{W} \rightarrow \mathcal{W}^h$ , and so forth for all possible webs.

form (6.42) to evaluate  $\rho_\beta(T(\mathbf{t}^{-,+}))$  then

$$\delta[\eta_1(y)] - \delta[\eta_2(y)] + M_1 \left( \rho_\beta(\mathbf{e}^h) \left[ \frac{1}{1 - \mathcal{B}^{-,0}}; \frac{1}{1 - \mathcal{B}^{0,1}}; \frac{1}{1 - \mathcal{B}^{1,+}} \right] \right) = 0 \quad (6.79)$$

This finally concludes our proof of equation (6.75).



**Figure 60:** If the homotopy  $\eta(y; s)$  is not tame then configurations can occur which do not fit into our convolution identity. On the left we have sliding webs. On the right we have an exceptional web with  $d^h = 0$ . It could be decorated to make a sliding  $h$ -type.

### Remarks

1. We stress that the assumption that we have a tame homotopy is crucial to our argument above. Otherwise configurations such as those shown in Figure 60 can occur which do not fit into our convolution identity.
2. All these results are easily generalizable to the composition of any number of interfaces.
3. Notice that we could have attempted the approach based on deformation type up to homotopy in order to compare the triple compositions for different values of  $\eta$ , by considering instead a variation  $\eta(s)$  which is not encoded as a time-dependent configuration, but as a family of  $G_3$  geometries. The problem with such an approach is that the corresponding convolution identity would look like

$$\mathbf{e}' * \mathbf{t}_p + \mathbf{e}' * \mathbf{t}_\partial + \mathbf{t}_c^f - \mathbf{t}_c^p + T_\partial(\mathbf{t}^{-,+}) \left[ \frac{1}{1 - \mathbf{t}_c(s)}; \mathbf{e}'; \frac{1}{1 - \mathbf{t}_c(s)} \right] = 0 \quad (6.80)$$

where the tensor operation involving some exceptional web in  $\mathbf{e}'$  picks the taut element  $\mathbf{t}_c(s)$  for the value of  $s$  at which the exceptional web exists. This convolution identity is clearly less useful than 6.67.

4. We have found that Theories and Interfaces produce a very interesting mathematical structure which should, perhaps, be called an  $A_\infty$ -2-category. The objects (“zero-morphisms”) are Theories. The space of (one-)morphisms between two Theories  $\mathcal{T}^-$  and  $\mathcal{T}^+$ , is just the category of Interfaces:  $\text{Hom}(\mathcal{T}^-, \mathcal{T}^+) = \mathfrak{Br}(\mathcal{T}^-, \mathcal{T}^+)$ . The morphisms between two objects of this category  $\mathfrak{I}_1^{-,+}$  and  $\mathfrak{I}_2^{-,+}$  are the two-morphisms. Rather than associativity we have  $A_\infty$ -type axioms for the composition of morphisms.

#### 6.4 Invertible Interfaces And Equivalences Of Theories

The existence of the identity  $\mathfrak{Id}$  interface allows us to make an obvious definition: an interface  $\mathfrak{I} \in \mathfrak{Br}(\mathcal{T}^-, \mathcal{T}^+)$  has a right inverse  $\tilde{\mathfrak{I}} \in \mathfrak{Br}(\mathcal{T}^+, \mathcal{T}^-)$  if  $\mathfrak{I} \boxtimes \tilde{\mathfrak{I}} = \mathfrak{Id}$ . This is an extremely strong condition. For example it implies that the Chan-Paton spaces satisfy

$$\sum_{i' \in \mathbb{V}^+} \mathcal{E}_{ii'}^{-+} \otimes \tilde{\mathcal{E}}_{i'j}^{+-} = \delta_{ij} \mathbb{Z} \quad (6.81)$$

To give a nontrivial example of inverses, suppose that we have isomorphisms  $\varphi^{(12)} : \mathcal{T}^{(1)} \rightarrow \mathcal{T}^{(2)}$  and  $\varphi^{(23)} : \mathcal{T}^{(2)} \rightarrow \mathcal{T}^{(3)}$ . Then define  $\varphi^{(13)} := \varphi^{(12)}\varphi^{(23)}$ . We claim that

$$\mathfrak{I}^{\varphi^{(12)}} \boxtimes \mathfrak{I}^{\varphi^{(23)}} = \mathfrak{I}^{\varphi^{(13)}} \quad (6.82)$$

The key identity needed to establish this claim is that

$$K_{i\varphi^{(12)}, j\varphi^{(12)}}^{(2)} \left( K_{ji}^{-1, \varphi^{(12)}} \otimes K_{j\varphi^{(12)}, i\varphi^{(12)}}^{-1, \varphi^{(23)}} \right) = K_{ji}^{-1, \varphi^{(13)}} \quad (6.83)$$

With some patience this can be proven using (6.22) and (4.26).

We will need a more flexible notion of invertibility of Interfaces. It turns out to be much more useful to define a right-inverse up to homotopy if  $\mathfrak{I} \boxtimes \tilde{\mathfrak{I}}$  is homotopy equivalent to the identity interface  $\mathfrak{Id}$ . Similar definitions hold for left inverses. Because of associativity up to homotopy, a left inverse and a right inverse up to homotopy are equal up to homotopy.

**Definition:** We will refer to an interface which has right and left inverse up to homotopy as an *invertible interface*.

A good example of invertible Interfaces which are only invertible up to homotopy are the rotation Interfaces  $\mathfrak{R}[\vartheta_\ell, \vartheta_r]$  discussed at length in Section 7.

The existence of an invertible Interface between two Theories  $\mathcal{T}^-$  and  $\mathcal{T}^+$  implies a strong relation between the Theories. It defines a functor between the categories of Branes or Interfaces which is invertible up to natural transformations. Concretely, it allows one to identify the spaces of exact ground states between branes of one theory and the other: sandwiching the interface and its inverse between two branes and using associativity we get a quasi-isomorphism between the complex of ground states for the two branes and the complex of ground states for their image in the other theory. In Section §8 we will pursue this idea further to define a notion of equivalence of Theories.

### 7. Categorical Transport: Simple Examples

In the previous Section §6 we have constructed an  $A_\infty$ -2-category of Interfaces. The arguments used in the construction employed (geometric) homotopies of the interface geometries

and their relation to (algebraic) homotopy equivalences of Interfaces. A crucial result was that the composition of consecutive Interfaces along the  $x$ -axis is associative up to homotopy equivalence. We can therefore consider compositions of several Interfaces, closely spaced along the  $x$ -axis. The resulting composition is well-defined up to homotopy equivalence. This suggests consideration of a third kind of homotopy, namely homotopies of the data used to define Theories and Interfaces. Thus, without trying to make the notion too precise at the moment, we can imagine continuous families of vacuum weights  $z_i$ , contractions  $K_{ij}$ , interior amplitudes  $\beta$  and so forth. Moreover, we could generalize the set of vacua  $\mathbb{V}$  to be a discrete (possibly branched) cover over some space of parameters. In the same spirit we could generalize web representations  $R_{ij}$  to bundles of  $\mathbb{Z}$ -modules, etc. Let us denote the relevant parameter spaces for these data generically as  $\mathcal{C}$ . The results of Section §6 thus suggest that given a continuous path  $\varphi$  from some interval  $[s_\ell, s_r] \subset \mathbb{R}$  to  $\mathcal{C}$  there might be a way to “map” the Theory  $\mathcal{T}^\ell$  at  $s_\ell$  to the Theory  $\mathcal{T}^r$  at  $s_r$  so that the Branes for the positive half-plane with data  $\varphi(s_\ell)$  are “coherently mapped” to Branes for the positive half-plane with data  $\varphi(s_r)$ .

To be slightly more precise, for a continuous map  $\varphi$  as above, suppose we have a definite law for constructing  $\mathcal{T}^r$  given  $\mathcal{T}^\ell$ . Then we are aiming to define an  $A_\infty$ -functor

$$\mathcal{F}(\varphi) : \mathfrak{Br}(\mathcal{T}^\ell, \mathcal{H}) \rightarrow \mathfrak{Br}(\mathcal{T}^r, \mathcal{H}) \quad (7.1)$$

where  $\mathcal{H}$  is, say, the positive half-plane. The functor  $\mathcal{F}(\varphi)$  is meant to be a categorical version of parallel transport. Thus, it should be an  $A_\infty$ -equivalence of categories, and moreover, if  $\varphi_1$  and  $\varphi_2$  are two paths which can be composed then there should be an invertible natural transformation between  $\mathcal{F}(\varphi_1 \circ \varphi_2)$  and the composition of the  $A_\infty$ -functors  $\mathcal{F}(\varphi_1)$  and  $\mathcal{F}(\varphi_2)$ . Of course, if  $\varphi$  is the constant path then  $\mathcal{F}(\varphi)$  should be the identity functor. We will refer to such a family of functors as a *categorical parallel transport law*, or just *categorical transport* for brevity. We aim to show, furthermore, that the “connection” defining this transport law is in fact a flat connection. That is, if  $\varphi_1$  is homotopic to  $\varphi_2$  in an appropriate space of parameters  $\mathcal{C}$  then there is an invertible natural transformation between  $\mathcal{F}(\varphi_1)$  and  $\mathcal{F}(\varphi_2)$ .

Given the above motivation our general strategy for constructing the functors  $\mathcal{F}(\varphi)$  will be to regard the variation  $\varphi(s)$  as a *spatially dependent variation of parameters* allowing us to construct families of Interfaces  $\mathfrak{I}[\varphi]$  which satisfy homotopy properties analogous to  $\mathcal{F}(\varphi)$ . In particular, we require two key properties:

1. *Parallel transport*: If  $\varphi_1$  and  $\varphi_2$  are composable paths then there must be a homotopy equivalence of Interfaces:

$$\mathfrak{I}[\varphi_1] \boxtimes \mathfrak{I}[\varphi_2] \sim \mathfrak{I}[\varphi_1 \circ \varphi_2]. \quad (7.2)$$

2. *Flatness*: If  $\varphi_1 \sim \varphi_2$  are homotopy equivalent, in a suitable sense, then correspondingly there must be a homotopy equivalence of Interfaces:

$$\mathfrak{I}[\varphi_1] \sim \mathfrak{I}[\varphi_2]. \quad (7.3)$$

Given a family of Interfaces satisfying (7.2) and (7.3) we can then invoke the construction of equations (6.49), (6.50), and (6.51) to produce the corresponding flat parallel transport  $A_\infty$ -functors.

In this Section and in Section §8 we will make some of these general ideas much more precise for certain variations of the data used to define Theories and Interfaces. For further discussion see the introduction to Section §8.

## 7.1 Curved Webs And Vacuum Homotopy

One of the most important physical examples of the variations of parameters described above are variations of vacuum weights. Thus we consider paths of weights  $\varphi : \mathbb{R} \rightarrow \mathbb{C}^V - \Delta$ , where  $\Delta$  is the large diagonal. If  $\varphi$  is continuous and nonconstant only within some finite interval we will call it a *vacuum homotopy*. Such a collection of maps  $z_i : \mathbb{R} \rightarrow \mathbb{C}$  for  $i \in V$  can be used to define a set of spatially-dependent vacuum weights  $z_i(x)$ . If the maps are continuously differentiable and suitably generic then in the context of Landau-Ginzburg theories such collections of maps can be used to define interesting supersymmetric interfaces. This is described in more detail in Section §17 below.

The notion of webs for spatially-dependent weights  $z_i(x)$  still makes sense: They are again graphs in  $\mathbb{R}^2$  but now the edges are allowed to be smooth non-self-intersecting curves. The connected components of the complement of the graph are labeled by vacua and edges separate regions labeled by pairs of distinct vacua. If  $z_i(x)$  are all continuously differentiable in the neighborhood of a point  $x_0$  then we can orient the  $ij$  edges in a strip centered on  $x_0$  and the tangent vector to an edge at a point  $(x, y)$  in this strip, with  $i$  on the left and  $j$  on the right, is parallel to  $z_{ij}(x) := z_i(x) - z_j(x)$ . Such webs are called *curved webs*.<sup>33</sup> In fact, when discussing homotopies of paths it is useful to generalize still further and consider spacetime dependent weights  $z_i(x, y)$  with the natural generalized definition of curved webs. We will see many examples of such curved webs below. See, for examples, the figures in Section §7.4.1.

In the remainder of Section 7 we will discuss a very simple class of curved webs such that

$$z_i(x) = e^{-i\vartheta(x)} z_i \quad (7.4)$$

where  $\vartheta : \mathbb{R} \rightarrow \mathbb{R}$  is a smooth function with compact support for  $\vartheta'$ . We will call these *spinning weights* because they are all related by a uniform (albeit  $x$ -dependent) rotation. If we wish to distinguish their webs from general curved webs we call them *spinning webs*. We will already find quite a rich set of phenomena in this case.

One advantage of the spinning webs is that we can consider the interior amplitude  $\beta_I$  to be  $x$ -independent. In a general composite web with several interfaces the interior amplitude can vary discontinuously across the interfaces. However, the particular family of “spinning” vacuum weights (7.4) have the property that the set of cyclic fans  $I$  does not change with  $x$ . Therefore there is a natural choice of interior amplitude where we take  $\beta_I$

---

<sup>33</sup>We will assume that  $z_i(x)$  are sufficiently generic that edges have transverse intersections, except at special points called “binding points,” defined below.

to be  $x$ -independent. In the remainder of Section §7 we make that choice. In Section §8 we consider more general situations.

There is another very nice way to motivate the study of spinning vacuum weights and describe that in the next two subsections.

## 7.2 Rotation Interfaces

Given a choice of a Theory  $\mathcal{T}$  and a half-plane  $\mathcal{H}$ , we have defined a category of Branes  $\mathfrak{Br}[\mathcal{T}, \mathcal{H}]$ . It is natural to wonder how this category depends on the choice of  $\mathcal{H}$  for fixed  $\mathcal{T}$ . Certainly it is literally unchanged if we apply a translation to  $\mathcal{H}$ . On the other hand, there is an interesting change if we apply a rotation to  $\mathcal{H}$ . It is useful to denote the phase of the normal to the boundary pointing into  $\mathcal{H}$  as  $\vartheta$  and the corresponding half-plane (defined up to translation) as  $\mathcal{H}_\vartheta$ . The corresponding category of Branes will be denoted  $\mathfrak{Br}_\vartheta$ . The notion of half-plane webs is ill-defined for those special values of  $\vartheta$  such that the boundary of the half-plane aligns with  $z_{ij}$  for some  $i, j \in \mathbb{V}$ . We define an  $S_{ij}$ -ray to be the ray in the complex plane<sup>34</sup> through the angle  $e^{i\vartheta_{ij}}$  such that the canonically oriented boundary has direction  $-z_{ij}$ . In formulae, the  $S_{ij}$ -ray is the ray through  $e^{i\vartheta_{ij}}$  such that

$$\operatorname{Re}\left(e^{-i\vartheta_{ij}} z_{ij}\right) = 0 \quad \& \quad \operatorname{Im}\left(e^{-i\vartheta_{ij}} z_{ij}\right) > 0. \quad (7.5)$$

Thus we have a well-defined notion of  $\mathfrak{Br}_\vartheta$  when  $e^{i\vartheta}$  is in the complement of the union, over all pairs  $(i, j)$  of distinct vacua, of the  $S_{ij}$ -rays.

We would like to define an  $A_\infty$ -equivalence of the categories of Branes  $\mathfrak{Br}_\vartheta$  for different values of  $\vartheta$ . The Branes associated to the  $\mathcal{H}_\vartheta$  half-plane for some Theory  $\mathcal{T}$  with vacuum data  $(\mathbb{V}, z)$  can be re-interpreted as Branes associated to the positive half-plane for a Theory  $\mathcal{T}^\vartheta$ . To define  $\mathcal{T}^\vartheta$  we choose the same set of vacua  $\mathbb{V}$ , but now the weights are rotated:

$$z_j^\vartheta := e^{-i\vartheta} z_j. \quad (7.6)$$

The interior amplitude  $\beta$  and the web representation  $\mathcal{R}$  can be taken to be the same. In other words,  $\mathfrak{Br}_\vartheta[\mathcal{T}] = \mathfrak{Br}_0[\mathcal{T}^\vartheta]$  and we can focus on relations between Branes for rotated Theories in the positive half plane. The  $A_\infty$ -equivalences we seek will be given by the functors associated (via equations (6.49) et. seq.) to a family of invertible Interfaces  $\mathfrak{R}[\vartheta_\ell, \vartheta_r]$  between any pair of rotated Theories  $\mathcal{T}^{\vartheta_\ell}$  and  $\mathcal{T}^{\vartheta_r}$ . Here we must regard  $\vartheta_\ell$  and  $\vartheta_r$  as belonging to the real numbers. Although the Theories  $\mathcal{T}^\vartheta$  only depend on  $\vartheta \bmod 2\pi$  there are interesting monodromy phenomena associated with interpolations that have  $|\vartheta_\ell - \vartheta_r| \geq 2\pi$ . (See Section §7.4.4 below.)

The definition of  $\mathfrak{R}[\vartheta_\ell, \vartheta_r]$  appears in equation (7.50) at the end of Section §7.4.4 and uses the key definition (7.46) below. We now give some motivation for the somewhat elaborate definitions which follow.

As discussed above, we want our Interfaces to behave as a categorical version of parallel transport: the composition  $\mathfrak{R}[\vartheta_1, \vartheta_2] \boxtimes \mathfrak{R}[\vartheta_2, \vartheta_3]$  should be homotopy equivalent to

---

<sup>34</sup>The terminology is motivated by the relation to the theory of spectral networks [31, 33, 34]. The relation to spectral networks is discussed in more detail in Section §18.2 below.

$\mathfrak{R}[\vartheta_1, \vartheta_3]$ , and  $\mathfrak{R}[\vartheta, \vartheta]$  should be the identity interface  $\mathfrak{Id}$ . Once the problem is approached from the point of view of the composition of interfaces, the variation of  $\vartheta$  becomes naturally tied to the space direction. In order to define an interface  $\mathfrak{R}[\vartheta_\ell, \vartheta_r]$  we could then hope that there is a simple definition when  $\vartheta_\ell$  and  $\vartheta_r$  are infinitesimally close and then imagine subdividing the interval  $[\vartheta_\ell, \vartheta_r]$  into a very large number of small sub-intervals  $[\vartheta_k, \vartheta_{k+1}]$ , and use the definition of the infinitesimal interfaces. This line of reasoning naturally leads to the idea that given any continuous interpolation  $\vartheta(x)$  with  $\vartheta(x) = \vartheta_\ell$  for  $x < -L$  and  $\vartheta(x) = \vartheta_r$  for  $x > L$  there is a corresponding Interface  $\mathfrak{I}[\vartheta(x)]$ . In Section §7.4.2 we will indeed define such an interface. (See equation (7.46) below.) We will need to define Chan-Paton factors and amplitudes. The main tool is to use representations of spinning webs. We will also use curved webs with space-time dependent weights to show that homotopy equivalent interpolations will give us homotopy equivalent interfaces.

Before giving the complete set of rules to deal with such curved webs, we gain some intuition by looking at a special case which can be reduced to standard webs with straight edges. Choosing real lifts so that  $\vartheta_\ell > \vartheta_r$  and  $|\vartheta_\ell - \vartheta_r| \leq \pi$  we can consider the smooth interpolation  $\vartheta(x) = -x$  (where the sign is chosen for future convenience) defined on the interval  $-\vartheta_\ell \leq x \leq -\vartheta_r$ . The advantage of this is that if we apply the exponential map  $u + iv := e^{-ix+y}$  taking the strip  $-\vartheta_\ell \leq x \leq -\vartheta_r$  in the  $x + iy$  plane to a wedge, denoted  $\mathcal{H}[\vartheta_\ell, \vartheta_r]$ , in the  $u + iv$  plane, then curves  $x(s) + iy(s)$  with tangent  $\frac{d}{ds}(x(s) + iy(s)) = e^{ix(s)}z_{ij}$  in the  $x + iy$  plane are mapped to curves  $u(s) + iv(s)$  satisfying  $\frac{d}{ds}(u(s) + iv(s)) = -iz_{ij}$ . These will be straight rays (or line segments) parallel to  $-iz_{ij}$  inside the wedge  $\mathcal{H}[\vartheta_\ell, \vartheta_r]$ . Therefore, a curved web on the strip will map to a web with straight edges in the wedge  $\mathcal{H}[\vartheta_\ell, \vartheta_r]$ .

Now, given a left Brane  $\mathfrak{B}$  for  $\mathcal{T}^{\vartheta_\ell}$  and a right Brane  $\tilde{\mathfrak{B}}$  for  $\mathcal{T}^{\vartheta_r}$ , we can define a complex of approximate ground states by sandwiching an  $\mathfrak{R}[\vartheta_\ell, \vartheta_r]$  Interface between the two Branes, i.e. by looking at the composition  $(\mathfrak{B}\mathfrak{R}[\vartheta_\ell, \vartheta_r]\tilde{\mathfrak{B}})_\eta$  computed with a  $G_3$  geometry as in Section §6.3. Recall that an Interface between a trivial Theory and itself is just a chain complex. We could equally well consider the chain complex of approximate groundstates (recall the definition from Section §4.3) for the strip with  $\mathfrak{B} \boxtimes \mathfrak{R}[\vartheta_\ell, \vartheta_r]$  on the left and  $\tilde{\mathfrak{B}}$  on the right or that with  $\mathfrak{B}$  on the left and  $\mathfrak{R}[\vartheta_\ell, \vartheta_r] \boxtimes \tilde{\mathfrak{B}}$  on the right. All of these chain complexes are homotopy equivalent. Our main heuristic is that these chain complexes should also be computed by a natural generalization of the complex of approximate groundstates associated to webs in the wedge geometry. In the next Section §7.3 we will describe that generalization, and then the requirement that there be a homotopy equivalence to the chain complexes  $(\mathfrak{B}\mathfrak{R}[\vartheta_\ell, \vartheta_r]\tilde{\mathfrak{B}})_\eta$  will help us figure out how to compute the Chan-Paton factors for  $\mathfrak{R}[\vartheta_\ell, \vartheta_r]$ .

### 7.3 Wedge Webs

A wedge geometry consists of the conical region  $\mathcal{H}[\vartheta_\ell, \vartheta_r]$  of  $\mathbb{R}^2$  included between the two rays  $\vartheta_\ell$  and  $\vartheta_r$ ,  $\vartheta_\ell > \vartheta_r$ , clockwise from  $\vartheta_\ell$ . A wedge web is a web with vertices which may lie in the interior, on the two boundary rays or at the origin of the wedge. The interior vertices of a wedge web are associated to standard interior fans and the left and right boundary vertices to half-plane fans for  $\mathcal{H}_{\vartheta_\ell}$  and  $\mathcal{H}_{\pi+\vartheta_r}$  respectively. The possible vertex

at the origin is associated to a wedge fan, a sequence of vacua compatible with edges lying in the wedge. The same type of wedge fan labels the external edges of the web, i.e. the fan at infinity. It is convenient to include the trivial wedge fan, with a single vacuum and no edges.

We can define as usual deformation types of wedge webs  $\mathbf{v}$ , moduli spaces of deformations, orientations, etc. We can define the convolution and tensor operations taking a wedge web as a container and inserting appropriate plane or half-plane webs or another wedge web at the vertex at the origin, as long as the fans match.

The wedge geometry has a scaling symmetry and thus a taut wedge web has a single modulus associated to the scale transformations, oriented towards larger webs. The wedge taut element  $\mathbf{t}_w$  satisfies the usual type of convolution identity

$$\mathbf{t}_w * \mathbf{t}_w + \mathbf{t}_w * \mathbf{t}_{pl} + \mathbf{t}_w * \mathbf{t}_{\partial,\ell} + \mathbf{t}_w * \mathbf{t}_{\partial,r} = 0 \quad (7.7)$$

In the first term we are convolving summands from  $\mathbf{t}_w$  into summands from  $\mathbf{t}_w$  at the origin of the wedge.

Given a choice of Theory  $\mathcal{T}$  and of Chan Paton factors  $\mathcal{E}_i, \tilde{\mathcal{E}}_i$  at the two boundary rays, we can define in a standard way a representation of wedge webs. We can associate to a wedge fan  $\{i_1, \dots, i_n\}$  a vector space  $\mathcal{E}_{i_1} \otimes R_{i_1, i_2} \otimes \dots \otimes R_{i_{n-1}, i_n} \otimes \tilde{\mathcal{E}}_{i_n}^*$  and collect all these vector spaces into a single

$$R_w^{\vartheta_\ell, \vartheta_r} [\mathcal{E}, \tilde{\mathcal{E}}] = \bigoplus_{z_{ij} \in \mathcal{H}[\vartheta_\ell, \vartheta_r]} \mathcal{E}_i \otimes \hat{R}_{ij}^{\vartheta_\ell, \vartheta_r} \otimes \tilde{\mathcal{E}}_j^* \quad (7.8)$$

with

$$\mathbb{Z} \cdot \mathbf{1} + \bigoplus_{z_{ij} \in \mathcal{H}[\vartheta_\ell, \vartheta_r]} \hat{R}_{ij}^{\vartheta_\ell, \vartheta_r} e_{ij} := \bigotimes_{z_{ij} \in \mathcal{H}[\vartheta_\ell, \vartheta_r]} (\mathbb{Z} \cdot \mathbf{1} + R_{ij} e_{ij}) \quad (7.9)$$

As usual, given a wedge web  $\mathbf{v}$  we have an associated multilinear map:

$$\rho_\beta(\mathbf{v}) : TR^{\partial, \ell} [\mathcal{E}] \otimes R_w^{\vartheta_\ell, \vartheta_r} [\mathcal{E}, \tilde{\mathcal{E}}] \otimes TR^{\partial, r} [\tilde{\mathcal{E}}] \rightarrow R_w^{\vartheta_\ell, \vartheta_r} [\mathcal{E}, \tilde{\mathcal{E}}] \quad (7.10)$$

The convolution identity tells us that  $\rho_\beta(\mathbf{t}_w)$  defines an  $A_\infty$  bimodule, satisfying the same type of  $A_\infty$  axioms as the strip-web operation  $\rho_\beta(\mathbf{t}_s)$  did, but with a larger vector space  $R_w^{\vartheta_\ell, \vartheta_r} [\mathcal{E}, \tilde{\mathcal{E}}]$  instead of the  $\mathcal{E}_{LR}$  we found for the strip. In particular, for any pair of Branes  $\mathfrak{B}$  and  $\tilde{\mathfrak{B}}$  in the  $\mathfrak{Br}_{\vartheta_\ell}$  and  $\mathfrak{Br}_{\vartheta_r + \pi}$  categories we have a chain complex  $R_w^{\vartheta_\ell, \vartheta_r} [\mathcal{E}, \tilde{\mathcal{E}}]$  with differential

$$g \rightarrow \rho_\beta(\mathbf{t}_w) \left[ \frac{1}{1 - \mathcal{B}}, g, \frac{1}{1 - \tilde{\mathcal{B}}} \right]. \quad (7.11)$$

This complex should be thought of as describing local operators placed at the tip of the wedge.

If we want the chain complex  $R_w^{\vartheta_\ell, \vartheta_r} [\mathcal{E}, \tilde{\mathcal{E}}]$  to be homotopy equivalent to  $(\mathfrak{BR}[\vartheta_\ell, \vartheta_r] \tilde{\mathcal{B}})_\eta$ , as suggested by the exponential map, we are led to the idea that the Chan Paton factors of the interface  $\mathfrak{R}[\vartheta_\ell, \vartheta_r]$  should coincide with the  $\hat{R}_{ij}^{\vartheta_\ell, \vartheta_r}$ . The exponential map relates the straight edges of a wedge web which go to the origin or to infinity in the wedge geometry to external edges of the curvilinear web which go to infinity in the Euclidean time direction

$y$ , either sitting in the far past or far future at values of  $x$  such that  $e^{i\vartheta_{ij}(x)}$  lies on an  $S_{ij}$  ray for the Theory  $\mathcal{T}$ .

This suggests that the Chan Paton factors for the  $\Re[\vartheta_\ell, \vartheta_r]$  interface should be built from individual factors of  $R_{ij}$  associated to such “vertical” external edges. With this hint we are now ready to propose the full set of rules for spinning webs.

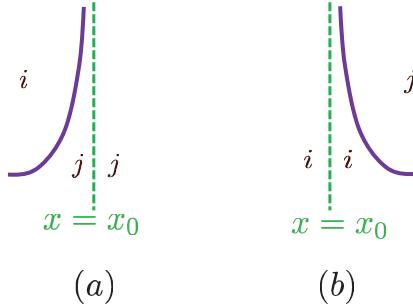
## 7.4 Construction Of Interfaces For Spinning Vacuum Weights

Let us now return to the general case of spinning vacuum weights of the form (7.4), determined by a generic, smooth function  $\vartheta : \mathbb{R} \rightarrow \mathbb{R}$  so that  $\vartheta'$  has compact support.

In what follows we choose some  $L$  so that the support of  $\vartheta'$  is in  $(-L, L)$ . We set  $\vartheta(x) = \vartheta_\ell$  for  $x \leq -L$  and  $\vartheta(x) = \vartheta_r$  for  $x \geq L$ . Moreover, we assume that none of the complex numbers  $e^{-i\vartheta_\ell} z_{ij}$ ,  $e^{-i\vartheta_r} z_{ij}$  is pure imaginary for  $i \neq j$ . Our goal is to define an Interface

$$\Im[\vartheta(x)] \in \mathfrak{Br}(\mathcal{T}^{\vartheta_\ell}, \mathcal{T}^{\vartheta_r}) \quad (7.12)$$

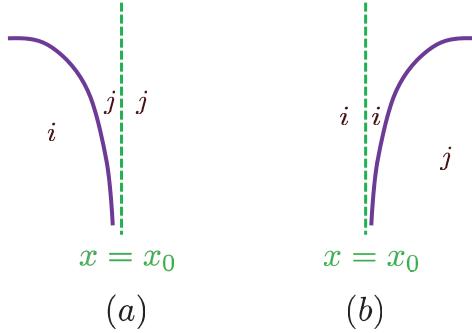
with the flat parallel transport properties spelled out in equations (7.2) and (7.3). To implement (7.3) we define  $\vartheta^1(x) \sim \vartheta^2(x)$  to be homotopic if the functions  $e^{-i\vartheta^a(x)}$ ,  $a = 1, 2$ , define homotopic maps from the real line into the circle.



**Figure 61:** Near a future stable binding point  $x_0$  of type  $ij$  the edges separating vacuum  $i$  from  $j$  asymptote to the dashed green line  $x = x_0$  in the future. Figures (a) and (b) show two possible behaviors of such lines. The phase  $e^{-i\vartheta(x)} z_{ij}$  rotates through the positive imaginary axis in the counterclockwise direction.

### 7.4.1 Past And Future Stable Binding Points

Let us first describe some special properties of the spinning webs. They share some characteristics of plane webs, of interface webs for data  $(\mathbb{V}^-, z^-) = (\mathbb{V}, z^{\vartheta_\ell})$  and  $(\mathbb{V}^+, z^+) = (\mathbb{V}, z^{\vartheta_r})$ , and also of composite and strip webs. In general, curved webs will be denoted by  $\mathfrak{z}$ . All of the vertices  $v$  in  $\mathfrak{z}$  will be equipped with a cyclic fan of vacua  $I_v(\mathfrak{z})$  drawn from  $\mathbb{V}$ , just as for plane webs. Nevertheless, there are two very different kinds of external edges. To see this let us consider what the external edges of the web might look like. The external edges with support in the regions  $x \geq L$  or  $x \leq -L$  extend to  $x \rightarrow +\infty$  or  $x \rightarrow -\infty$  and



**Figure 62:** Near a past stable binding point  $x_0$  of type  $ij$  the edges separating vacuum  $i$  from  $j$  asymptote to the dashed green line  $x = x_0$  in the past. Figures (a) and (b) show two possible behaviors of such lines. The phase  $e^{-i\vartheta(x)}z_{ij}$  rotates through the positive imaginary axis in the clockwise direction.

define positive and negative half-plane fans  $J^+$  and  $J^-$ , respectively. On the other hand, a novel aspect of curved webs is that there can also be vertical external edges supported in the region  $-L < x < L$ . We now describe this important phenomenon in some detail.

At any fixed  $x$  we can apply the line principle to a vertical line through  $x$ , and hence the edges will be graphs of functions over certain intervals. We define an “edge of type  $ij$ ” to be an edge that separates vacua  $i$  and  $j$ . (There is no distinction between an edge of type  $ij$  and type  $ji$ .) Locally, there is a parametrization of the edge with tangent vector oriented so that vacuum  $i$  is on the left and  $j$  is on the right. With this parametrization  $\frac{d}{ds}(x_{ij}(s) + iy_{ij}(s)) = e^{-i\vartheta(x)}z_{ij}$  and hence locally the edge is the graph of a function  $y_{ij}(x)$  satisfying

$$\frac{dy_{ij}}{dx} = \tan(\alpha_{ij} - \vartheta(x)) \quad (7.13)$$

where the phase  $\alpha_{ij}$  is defined by  $z_{ij} := |z_{ij}|e^{i\alpha_{ij}}$ . Note that  $\alpha_{ji} = \alpha_{ij} + \pi$ . This differential equation is singular at those values of  $x$  for which  $\alpha_{ij} - \vartheta(x) \in \frac{\pi}{2} + \pi\mathbb{Z}$ . Suppose that when  $x$  is near  $x_0$  we have

$$\alpha_{ij} - \vartheta(x) = \frac{\pi}{2} + \frac{(x - x_0)}{\kappa} + \mathcal{O}((x - x_0)^2) + 2\pi n \quad (7.14)$$

for some  $n \in \frac{1}{2}\mathbb{Z}$  and some nonzero real number  $\kappa$ .<sup>35</sup> Then near  $x_0$  we must have

$$y_{ij}(x) \cong -\kappa \log|x - x_0| + \text{const} + \dots \quad (7.15)$$

The local behavior of edges separating vacuum  $i$  from  $j$  in the neighborhood of  $x = x_0$  (but not at  $x = x_0$ ) is governed by four cases according to whether  $n$  in (7.14) is integer or half-integer and the sign of  $\kappa$ . It is useful to make the following definition:<sup>36</sup>

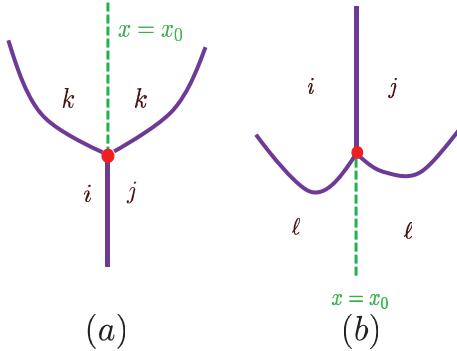
<sup>35</sup>We are using here the assumption that  $\vartheta(x)$  is suitably generic.

<sup>36</sup>Once again the terminology here is motivated by the theory of spectral networks [31, 33, 34]. As discussed in Section §17 and §18.2 below, the “binding points” represent places where supersymmetric domain walls in  $1+1$  dimensional theories (or, more generally, on surface defects) can form boundstates with two-dimensional BPS particles.

**Definition:** Given vacuum weights of the form (7.4) we define a point  $x_0 \in \mathbb{R}$  to be a *binding point of type  $ij$*  if  $\vartheta$  satisfies (7.14) with some integer  $n$ . It is called a *future stable binding point* if  $\kappa > 0$  and a *past stable binding point* if  $\kappa < 0$ . We denote the set of future stable binding points of type  $ij$  by  $\lambda_{ij}$  and the set of past stable binding points of type  $ij$  by  $\gamma_{ij}$ . If  $x_0$  is a (future- or past- stable) binding point of type  $ij$  we call the line  $x = x_0$  in the  $(x, y)$ -plane a *(future- or past- stable) binding wall of type  $ij$* .

**Remarks:**

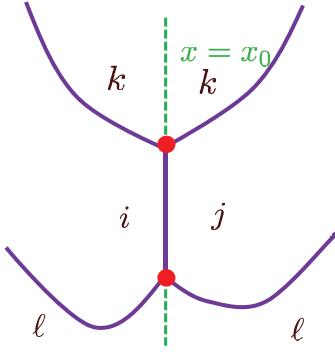
1. Note well that the set  $\gamma_{ij} \cup \lambda_{ij}$  of all binding points of type  $ij$  can be characterized as precisely those positions  $x_0$  on the real line where  $e^{i\vartheta(x_0)}$  defines an  $S_{ij}$ -ray. (Recall the definition of  $S_{ij}$ -rays in (7.5).)
2. The differential equation for  $y_{ij}$  is also singular when  $n$  is a half-integer  $n \in \frac{1}{2} + \mathbb{Z}$  in (7.14). In this case  $x_0$  is a binding point of type  $ji$ .
3. The sign of the derivative  $\vartheta'(x_0)$  determines whether the point is past or future stable. As  $x$  increases past  $x_0$ , the complex number  $z_{ij}(x)$  goes through the positive imaginary axis in the counter-clockwise direction for a future stable binding point, and in the clockwise direction for a past stable binding point.



**Figure 63:** When a vertex for a line separating vacua  $(i, j)$  has an  $x$ -coordinate which is a binding point  $x_0$  of type  $ij$  then it can be “frozen”. In Figure (a) the vertex cannot move off the binding wall  $x = x_0$  if  $x_0$  is a future-stable binding point, since the figure cannot smoothly deform to Figure 61. In Figure (b) the vertex cannot move off the binding wall  $x = x_0$  if  $x_0$  is a past-stable binding point, since the figure cannot smoothly deform to Figure 62. On the other hand, in Figure (a) the vertex is not frozen if  $x_0$  is past stable and in Figure (b) the vertex is not frozen if the  $x_0$  is future stable.

Putting these remarks together we see that edges of type  $ij$  have a behavior in the neighborhood of a binding point  $x_0$  of type  $ij$  of the form shown in Figure 61 for future stable binding points and shown in Figure 62 for past stable binding points.

It is also possible for an  $ij$ -edge to sit within the vertical line  $x = x_0$ . This is illustrated in Figures 63 and 64. As explained in the caption of Figure 63 a vertex can be *frozen* in



**Figure 64:** A vertical internal edge at an  $ij$  binding point. The internal edge can be deformed away from the binding wall  $x = x_0$ .

the sense that it cannot be translated in the  $x$  direction. Such vertices are connected to frozen external edges. We therefore split the vertices of  $\mathfrak{z}$  into *free* and *frozen* subsets  $\mathcal{V}(\mathfrak{z}) = \mathcal{V}_{\text{free}}(\mathfrak{z}) \cup \mathcal{V}_{\text{frozen}}(\mathfrak{z})$ . In an analogous way the external edges can be divided up as  $\mathcal{E}^{\text{ext}}(\mathfrak{z}) = \mathcal{E}_{\text{free}}^{\text{ext}}(\mathfrak{z}) \cup \mathcal{E}_{\text{frozen}}^{\text{ext}}(\mathfrak{z})$ .

Now we can define the fan at infinity  $I_{\infty}(\mathfrak{z})$  for a curved web  $\mathfrak{z}$ . As we have said, edges extending outside  $-L < x < L$  define negative- and positive- half-plane fans  $J^-$ ,  $J^+$ , respectively. Suppose these two fans are  $J^- = \{j_1, \dots, j_m\}$  and  $J^+ = \{j'_1, \dots, j'_n\}$  as in Figure 39. Then the vacua in the regions encountered moving from  $x = -L$  to  $x = +L$  will stabilize, for sufficiently large positive  $y$ , to a set

$$\mathcal{J}^+ = \{j_m, f_1, \dots, f_s, j'_1\} \quad (7.16)$$

while the vacua encountered moving from  $x = +L$  to  $x = -L$  will stabilize, for sufficiently large negative  $y$ , to a set

$$\mathcal{J}^- = \{j'_n, i_1, \dots, i_t, j_1\}. \quad (7.17)$$

Therefore, reading left to right, the vacua encountered in a clockwise traversal at infinity are

$$I_{\infty}(\mathfrak{z}) = \{\mathcal{J}^+, J^+, \mathcal{J}^-, J^-\}. \quad (7.18)$$

We are now ready to define the oriented deformation type of a curved web. Given a curved web  $\mathfrak{z}$  we can deform it by varying the positions of the vertices subject to the edge constraints and subject to the condition that the web does not change topology, i.e., that no edge collapses to zero length. Free vertices contribute two degrees of freedom but frozen vertices only contribute one (it can be taken as the  $y$  coordinate of the vertex). Thus the expected dimension of the deformation space of a generic curved web  $\mathfrak{z}$  is

$$d(\mathfrak{z}) = 2V_{\text{free}}(\mathfrak{z}) + V_{\text{frozen}}(\mathfrak{z}) - E^{\text{int}}(\mathfrak{z}). \quad (7.19)$$

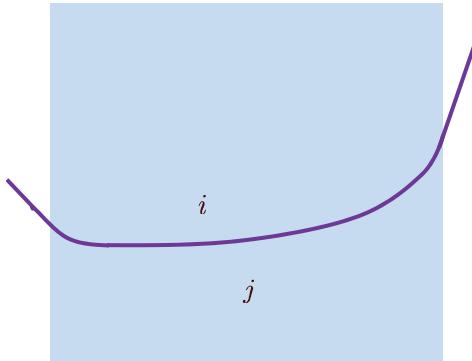
This can also be written as:

$$d(\mathfrak{z}) = 2V(\mathfrak{z}) - E^{\text{int}}(\mathfrak{z}) - E_{\text{frozen}}^{\text{ext}}(\mathfrak{z}). \quad (7.20)$$

since each frozen vertex is uniquely associated with a frozen edge. As usual, we are assuming  $\vartheta(x)$  is sufficiently generic when we make this definition. There will sometimes be exceptional webs  $\epsilon$  where some of the edge constraints are ineffective and the dimension of the deformation space is larger than  $d(\epsilon)$ . In addition to this, the above discussion assumed that the set of vertices  $\mathcal{V}(\mathfrak{z})$  is nonempty. Actually, we can have curved webs with no vertices at all, and in fact these will play an important role below. (See, for examples, Figure 65 and Figures 69- 73 below.) In that case the expected dimension is the true dimension of the deformation space and is simply the number of components of the web.

Unlike the deformation spaces we have considered until now the moduli spaces  $\mathcal{D}(\mathfrak{z})$  do not have piecewise linear boundaries. Nevertheless, they are cells, and for generic  $\mathfrak{z}$  they will have a dimension  $d(\mathfrak{z})$ . We can give them an orientation to define an oriented deformation type  $\mathfrak{z}$  and consider the free abelian group  $\mathcal{W}_{\text{curv}}$  generated by the oriented deformation types of curved webs. When we form the tensor algebra  $T\mathcal{W}_{\text{curv}}$  we give  $\mathfrak{z}$  the degree  $d(\mathfrak{z})$ .

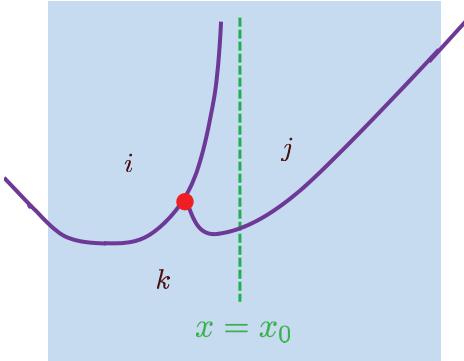
Curved webs have a translation symmetry in the  $y$ -direction but no scaling symmetry. In this sense they are very much like strip webs and composite webs, and indeed can be thought of as a continuous version of composite webs where many closely spaced interfaces have been joined together. We therefore adopt the definitions of Section §6.2 and define  $\mathfrak{z}$  to be *rigid* or *taut* if  $d(\mathfrak{z}) = 1$  and to be *sliding* if  $d(\mathfrak{z}) = 2$ . The canonical orientation for the taut webs is  $dy$  where  $y$  is a measure of the  $y$  position of the web. We denote the taut element in  $\mathcal{W}_{\text{curv}}$  by  $t$



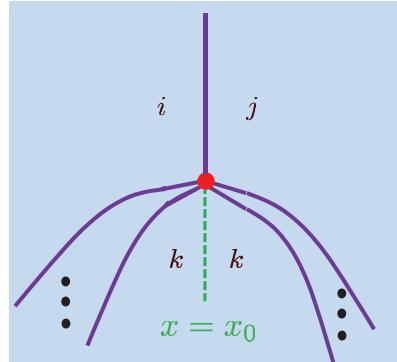
**Figure 65:** An example of a taut curved web consisting of a single free edge of type  $ij$ . This contributes two external free edges. In this and similar figures the light blue shaded region indicates the support of  $\vartheta'(x)$ .

## Examples

1. If there is no binding point of type  $ij$  then an edge of type  $ij$  can stretch from  $x = -\infty$  to  $x = +\infty$  with no vertices. This is a taut curved web  $\mathfrak{z}$ . See Figure 65.
2. If there is a future stable binding point of type  $ij$  then an edge of type  $ij$  can come in from the negative half-plane or from the positive half plane, as shown in Figure 61.



**Figure 66:** An example of a curved sliding web. If  $x = x_0$  is a future stable binding wall of type  $ij$  then the vertex can move across the line.



**Figure 67:** An example of a curved taut web in the case that  $x = x_0$  is a past stable binding wall of type  $ij$ . The edge of type  $ij$  is a frozen external edge; the only degree of freedom corresponds to moving the vertex vertically along the  $ij$  binding wall, and hence the web is taut. Similarly, if an  $ij$  edge extends vertically to  $-\infty$  at a future stable binding point then the external edge is frozen and the web is a taut web.

A curved with with a single component, as shown in either Figure 61(a) or Figure 61(b) is a taut curved web.

3. Similarly if there is a past stable binding point of type  $ij$  then an edge of type  $ij$  can come in from the negative or positive half-plane, as shown in Figure 62. A curved with with a single component, as shown in either Figure 62(a) or Figure 62(b) is a taut curved web.
4. Any vertex of the Theories  $\mathcal{T}^{\vartheta_\ell}$  or  $\mathcal{T}^{\vartheta_r}$  defines a sliding web. See Figure 66.
5. However, if a vertex of the Theories  $\mathcal{T}^{\vartheta_\ell}$  or  $\mathcal{T}^{\vartheta_r}$  has an edge of type  $ij$  then it might also define a taut curved web located on binding walls of type  $ij$ . The case of a past stable binding wall is shown in Figure 67.

6. Finally, it is possible to have a completely rigid curved web with no moduli at all. These correspond to entire lines of type  $ij$ , parallel to the  $y$ -axis and located at  $ij$  binding points. They will play some role in Section 9 below.

Let us now discuss what kinds of convolution are possible with curved webs. Every web  $\mathfrak{w}$  for the vacuum data  $(\mathbb{V}, z^{\vartheta_\ell})$  can be continued to a web for any  $(\mathbb{V}, z^{\vartheta(x)})$  simply by a rotation, until it becomes a web for  $(\mathbb{V}, z^{\vartheta_r})$ . If  $\mathfrak{z}$  is a curved web it makes sense to consider the convolution  $\mathfrak{z} *_v \mathfrak{w}$ , declaring the convolution to be nonzero when the fan coincide and rotating  $\mathfrak{w}$  by the appropriate angle before inserting it at  $v$ .

Now suppose that  $\mathfrak{d}$  is an interface web between  $\mathcal{T}^{\vartheta_\ell}$  and  $\mathcal{T}^{\vartheta_r}$  and  $\mathfrak{z}_1, \dots, \mathfrak{z}_n$  are a collection of curved webs. Then we can define a curved web  $T_\partial(\mathfrak{d})[\mathfrak{z}_1, \dots, \mathfrak{z}_n] \in \mathcal{W}_{\text{curv}}$ . We think of the interface as placed somewhere in the region  $-L < x < L$  and we replaced each of the wall vertices in  $\mathfrak{d}$  with the sequence of curved webs  $\mathfrak{z}_1, \dots, \mathfrak{z}_n$ , defining the convolution to be zero when the curved webs do not fit in properly with the set of wall vertices of  $\mathfrak{d}$ . In particular the operation is zero unless the left and right half-fans of the  $\mathfrak{z}_a$  are compatible with the fan for the corresponding vertex of  $\mathfrak{d}$ . Moreover it is zero unless the (transpose of the) past fan  $\mathcal{J}^-(\mathfrak{z}_a)$  coincides with the future fan  $\mathcal{J}^+(\mathfrak{z}_{a+1})$ , including the specific location for each vertical external edge. The vertical external edges of consecutive arguments can be connected into a single internal edge. It is important to observe that one always connects frozen vertical edges of some curved web with un-frozen vertical edges of another curved web.

One can show that the moduli space of deformations of  $T_\partial(\mathfrak{d})[\mathfrak{z}_1, \dots, \mathfrak{z}_n]$  is locally the product of the deformation space of  $\mathfrak{d}$  times the product of reduced moduli spaces of the  $\mathfrak{z}_a$ . In particular,

$$d(T_\partial(\mathfrak{d})[\mathfrak{z}_1, \dots, \mathfrak{z}_n]) = d(\mathfrak{d}) + \sum_{a=1}^n (d(\mathfrak{z}_a) - 1) \quad (7.21)$$

just like equation (6.27). Thus, the operation is defined on the oriented deformation types of these webs.

Now we can write the convolution identity for the taut element  $\mathbf{t}$  in  $\mathcal{W}_{\text{curv}}$ . Let  $\mathbf{t}_{pl} = \mathbf{t}^{\vartheta_\ell} + \mathbf{t}^{\vartheta_r}$  be the formal sum of taut elements for the Theories  $\mathcal{T}^{\vartheta_\ell}$  and  $\mathcal{T}^{\vartheta_r}$  respectively. Similarly, let  $\mathbf{t}_\mathcal{I}$  be the taut interface element between  $\mathcal{T}^{\vartheta_\ell}$  and  $\mathcal{T}^{\vartheta_r}$ . Then, examining the ends of the moduli spaces of sliding webs, as usual, produces

$$\mathbf{t} * \mathbf{t}_{pl} + T_\partial(\mathbf{t}_\mathcal{I}) \left[ \frac{1}{1 - \mathbf{t}} \right] = 0 \quad (7.22)$$

This is to be compared with equation (6.31). The difference is that all vertices of  $\mathbf{t}$  are now interior.

#### 7.4.2 Defining Chan-Paton Spaces And Amplitudes

Let us now consider a representation of webs  $\mathcal{R}$ . The usual definition of representations of webs only makes use of the set of vacua  $\mathbb{V}$  and not the weights, so it essentially carries over immediately to the case of curved webs, up to an important subtlety concerning frozen external edges. Frozen external edges impose an extra edge constraint on the deformation

space of a curved web  $\mathfrak{z}$  and thus a good definition of  $\rho(\mathfrak{z})$  should include some extra degree  $-1$  map  $\check{K}_{\check{e}}$ , to be defined momentarily, for each frozen external edge  $\check{e}$ :

$$\begin{aligned} \rho(\mathfrak{z})[r_1, \dots, r_n] = & \frac{o(\mathfrak{w})}{\left[ \prod_{\check{e} \in \mathcal{E}_{\text{frozen}}^{\text{ext}}(\mathfrak{z})} \partial_{\check{e}} \prod_{e \in \mathcal{E}(\mathfrak{w})} \partial_e \right] \circ \prod_{v \in \mathcal{V}(\mathfrak{w})} dx_v dy_v} \\ & \cdot \left[ \otimes_{\check{e} \in \mathcal{E}_{\text{frozen}}^{\text{ext}}(\mathfrak{z})} \check{K}_{\check{e}} \otimes_{e \in \mathcal{E}(\mathfrak{w})} K_e \right] \circ \otimes_{a=1}^n r_a \end{aligned} \quad (7.23)$$

We would like to define some interface Chan-Paton data  $\mathcal{E}_{j,j'}$  so that  $\rho(\mathfrak{z})$ , for  $\mathfrak{z}$  a web with  $I_{\infty}(\mathfrak{z})$  of the form (7.18) can be understood as a map

$$\rho(\mathfrak{z}) : \otimes_{v \in \mathcal{V}(\mathfrak{z})} R_{I_v(\mathfrak{z})} \rightarrow \mathcal{E}_{j_m, j'_1} \otimes \hat{R}_{j'_1, j'_m}^+ \otimes (\mathcal{E}_{j_1, j'_n})^* \otimes \hat{R}_{j_1, j_m}^- \quad (7.24)$$

whose output is an interface vertex for some Interface in  $\mathfrak{Br}(\mathcal{T}^{\vartheta_\ell}, \mathcal{T}^{\vartheta_r})$ .

The definitions of  $\mathcal{E}_{j,j'}$  and  $\rho(\mathfrak{z})$  should be compatible with the convolution operation  $T_{\partial}(\mathfrak{d})[\mathfrak{z}_1, \dots, \mathfrak{z}_n]$ :

$$\rho(T_{\partial}(\mathfrak{d})[\mathfrak{z}_1, \dots, \mathfrak{z}_n])(S) = \sum_{\text{Sh}_{n+2}(S)} \epsilon \rho(\mathfrak{d}) [S_1; \rho(\mathfrak{z}_1)[S_2], \dots, \rho(\mathfrak{w}_n)[S_{n+1}]; S_{n+2}] \quad (7.25)$$

That means that the contraction of vertical external edges on the left hand side of this equation should match the contraction of  $\mathcal{E}_{j,j'}^*$  and  $\mathcal{E}_{j,j'}$  on the right hand side. We need an independent summand in  $\mathcal{E}_{j_m, j'_1}$  for any possible  $\mathcal{J}^+(\mathfrak{z})$  in order to encode the representation data attached to vertical edges and, dually, a summand in  $(\mathcal{E}_{j_1, j'_n})^*$  for any possible  $\mathcal{J}^-(\mathfrak{z})$  in order to encode the representation data attached to past vertical edges.

We define the relevant Chan-Paton data using a construction very similar to the product rule construction of  $\hat{R}_{ij}$  in equation (5.38) above:

For each binding point  $x_0$  of type  $ij$  introduce a matrix with chain-complex entries. It depends on whether  $x_0$  is future-stable or past stable:

$$S_{ij}^f(x_0) := \mathbb{Z} \cdot \mathbf{1} + R_{ij} e_{ij} \quad \text{Future stable } ij \text{ binding point} \quad (7.26)$$

$$S_{ij}^p(x_0) := \mathbb{Z} \cdot \mathbf{1} + R_{ji}^* e_{ij} \quad \text{Past stable } ij \text{ binding point.} \quad (7.27)$$

We will refer to  $S_{ij}(x_0)$  as a *categorified  $S_{ij}$ -factor*, or just as an  *$S_{ij}$ -factor*, for short. The future and past stable  $S$ -factors are related by  $S_{ij}^p = (S_{ji}^f)^{tr,*}$ .

We define

$$\bigoplus_{j,j' \in \mathbb{V}} \mathcal{E}_{j,j'} e_{j,j'} := \bigotimes_{i \neq j} \bigotimes_{x_0 \in \gamma_{ij} \cup \lambda_{ij}} S_{ij}(x_0) \quad (7.28)$$

where the tensor product on the RHS of (7.28) is ordered from left to right by increasing values of  $x_0$  and  $S_{ij}(x_0)$  is the future- or past-stable factor as appropriate to the binding point. (Note that the rule (7.28) reduces to the product for wedges (7.9) if  $\vartheta(x) = -x$ .)

Given this definition, we can associate each  $\mathcal{J}^+(\mathfrak{z})$  to a summand in  $\mathcal{E}_{j,j'}$  given as an ordered tensor product with a factor of  $R_{ij}$  for every vertical external unfrozen  $ij$  edge and a factor of  $R_{ji}^*$  for every vertical external frozen  $ij$  edge. These two choices differ by one unit of degree, which matches the extra edge constraint for frozen edges and is crucial in defining a degree one boundary amplitude from the taut curved element  $\mathfrak{t}$  below.

Dually, we can associate each  $\mathcal{J}^-(\mathfrak{z})$  to a summand in  $(\mathcal{E}_{j_1, j_n})^*$  given again as an ordered tensor product with a factor of  $R_{ij}$  for every vertical external unfrozen  $ij$  edge and a factor of  $R_{ji}^*$  for every vertical external frozen  $ij$  edge. To see this take the dual and transpose of the RHS of (7.28). This takes the transpose of the matrix units  $e_{ij}$  and takes  $R_{ij} \rightarrow R_{ij}^*$ . The factors are now ordered with decreasing values of  $x_0$ . Now use the relation  $S_{ij}^p = (S_{ji}^f)^{tr,*}$ .

We have encoded the full fan at infinity  $I_\infty(\mathfrak{z})$  for any curved web  $\mathfrak{z}$  into a summand of the interface factor  $R_J(\mathcal{E})$  of equation (6.4).

Recall that  $K_{ij} : R_{ij} \otimes R_{ji} \rightarrow \mathbb{Z}$  is a *perfect pairing* and since  $K_{ij}$  has degree  $-1$  we can define a degree  $-1$  isomorphism of  $\mathbb{Z}$ -graded modules:

$$\check{K}_{ij} : R_{ij} \rightarrow R_{ji}^* \quad (7.29)$$

by  $\check{K}_{ij}(r_{ij})(\cdot) := K_{ij}(r_{ij}, \cdot)$ . In terms of the notation in equation (6.13) we have

$$\check{K}_{ij}(v_\alpha) = \sum_{\alpha'} K_{\alpha\alpha'} v_{\alpha'}^* \quad (7.30)$$

We identify the map  $\check{K}_{\check{e}}$  in 7.23 associated to a frozen  $ij$  external edge  $\check{e}$  with  $\check{K}_{ij}$ .

In order to complete our definition of  $\rho(\mathfrak{z})$ , we only need to define carefully the vector field  $\partial_{\check{e}}$  associated to the corresponding external edge constraint, in such a way that the compatibility condition 7.25 holds true. We define  $\partial_{\check{e}}$  to be a translation of the frozen vertex in the positive  $x$  direction. When we contract two vertical external edges at a future-stable binding point, this coincides with the  $\partial_e$  vector field for the resulting internal edge. This agrees with the absence of relative sign in  $\check{K}_{ij}(r_{ij}) \cdot r_{ji} := K_{ij}(r_{ij}, r_{ji})$ . On the other hand, when we contract two vertical external edges at a past-stable binding point, this has the opposite orientation to the  $\partial_e$  vector field for the resulting internal edge. This agrees with the relative sign in  $r_{ji} \cdot \check{K}_{ij}(r_{ij}) := -K_{ij}(r_{ij}, r_{ji})$ .

Now that we have defined the Chan-Paton spaces (7.28) for the Interface we turn to the definition of the interface amplitude. This will be defined by using taut curved webs. Recall from Section §7.1 that we are taking the interior amplitude  $\beta$  to be constant. In particular we take the same interior amplitudes  $\beta_I$  for  $\mathcal{T}^{\vartheta_\ell}$  and for  $\mathcal{T}^{\vartheta_r}$ .<sup>37</sup> With this understood, for any curved web  $\mathfrak{z}$  we can define the operator  $\rho_\beta(\mathfrak{z})$  following the usual insertion of  $e^\beta$ . We will be particularly interested in the special element  $\rho_\beta^0(\mathfrak{z})$  given by inserting  $\beta$  into every interior vertex of  $\mathfrak{z}$ .

$$\rho_\beta^0(\mathfrak{z}) = \rho(\mathfrak{z})[e^\beta] \quad (7.31)$$

which is valued in (7.24), and has degree  $2V(\mathfrak{z}) - E^{\text{int}}(\mathfrak{z}) - E_{\text{frozen}}^{\text{ext}}(\mathfrak{z})$ . It follows from (7.20) that if  $\mathfrak{z}$  is a taut web then (7.31) indeed has degree 1, as befits an interface amplitude.

**Example:** An example, which will be useful to us later, is given in Figure 67. Here a vertex of the theory  $\mathcal{T}^{\vartheta_\ell}$  has been rotated so that the  $ij$  edge goes to the future at a past

---

<sup>37</sup>Note that the interior amplitude is *not* defined as a solution of an  $L_\infty$  Maurer-Cartan equation  $\rho(t)(e^\beta) = 0$ , where  $t$  is the curved taut element!

stable binding point of type  $ij$ . Suppose the cyclic fan of the vertex is  $I$ . It can be written as

$$I = \{j, \dots, k, \dots, i\} = J^+ * J^- \quad (7.32)$$

where  $J^+ = \{j, \dots, k\}$  is a positive half-plane fan for  $\mathcal{T}^{\vartheta_r}$  and  $J^- = \{k, \dots, i\}$  is a negative half-plane fan for  $\mathcal{T}^{\vartheta_\ell}$  and the amalgamation  $J^+ * J^-$  is regarded as a cyclic fan. Now, the interior amplitude  $\beta_I$  is a degree two element of  $R_I$  and (after a cyclic transformation) it will be convenient to regard this as

$$\beta_I \in R_{ij} \otimes (R_{J^+} \otimes R_{J^-}). \quad (7.33)$$

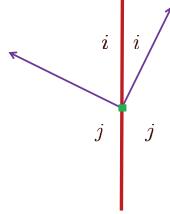
We wish to interpret  $\rho_\beta^0(\mathfrak{z})$  as an interface amplitude valued in  $R_J(\mathcal{E})$ . The Chan-Paton data for  $\rho_\beta^0(\mathfrak{z})$  are defined by (7.28) and the relevant factors for this web are  $\mathcal{E}_{ij} = R_{ji}^*$  (coming from (7.27)) and  $\mathcal{E}_{kk}^* = \mathbb{Z}$ . Therefore the interface amplitude must be valued in

$$\rho_\beta^0(\mathfrak{z}) \in \mathcal{E}_{ij} \otimes R_{J^+} \otimes \mathcal{E}_{kk}^* \otimes R_{J^-} \cong R_{ji}^* \otimes (R_{J^+} \otimes R_{J^-}). \quad (7.34)$$

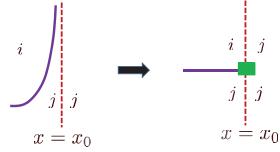
We identify the interface amplitude as

$$\rho_\beta^0(\mathfrak{z}) := (\check{K}_{ij} \otimes 1)(\beta_I) \quad (7.35)$$

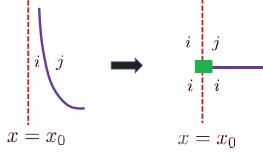
for this particular taut web  $\mathfrak{z}$ . Note that it indeed has degree 1. We do not need any extra sign on the right hand side, as  $i_{\partial_e}(dx \wedge dy) = dy$ .



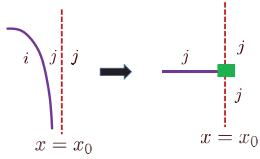
**Figure 68:** The taut curved web of Figure 65 leads to a nonzero interface amplitude for the rigid interface web shown here.



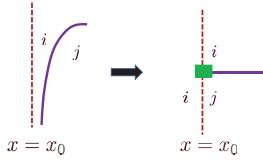
**Figure 69:** The taut curved web of Figure 61 (a) leads to a nonzero interface amplitude for the rigid interface web shown here.



**Figure 70:** The taut curved web of Figure 61(b) leads to a nonzero interface amplitude for the rigid interface web shown here.



**Figure 71:** The taut curved web of Figure 62(a) leads to a nonzero interface amplitude for the rigid interface web shown here.



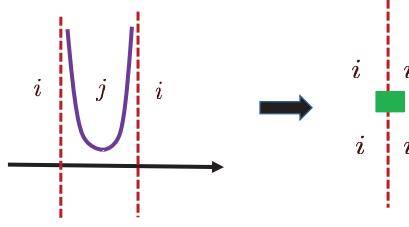
**Figure 72:** The taut curved web of Figure 62(b) leads to a nonzero interface amplitude for the rigid interface web shown here.

The above discussion has assumed that the taut curved web  $\mathfrak{z}$  has vertices. However, as we have already noted, it is possible to have taut curved webs with no vertices. These require special consideration when defining  $\rho_\beta^0(\mathfrak{z})$  (again, considered as an interface amplitude for  $\Im[\vartheta(x)]$ ). We must consider a few cases here.

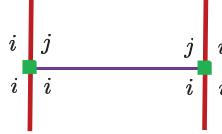
1. The most basic case is for taut webs of the form shown in Figure 65. In the case of Figure 65 there are no binding walls so the Chan-Paton data defined by (7.28) give simply  $\mathcal{E}_{k,\ell} = \delta_{k,\ell} \mathbb{Z}$  for all  $k, \ell \in \mathbb{V}$ . The taut curved web of Figure 65 contributes to  $\rho_\beta^0(\mathfrak{z})$  as an amplitude associated with the rigid interface web shown in Figure 68. The interface amplitude associated with such a web must be valued in  $R_{ji} \otimes R_{ij}$  (where we put the negative half-plane factor first) and we take it to be

$$\rho_\beta^0(\mathfrak{z}) := K_{ji}^{-1}. \quad (7.36)$$

This choice can be shown to be required by demanding the composition property of Interfaces (7.2) or by the Maurer-Cartan equation for the  $S$ -wall Interfaces discussed



**Figure 73:** A taut curved web with no vertices between future stable  $ij$  and  $ji$  binding points.



**Figure 74:** The amplitude associated with the taut curved web of Figure 73 is defined by the interface product shown here.

in Section §7.6 below. Note that the interface amplitudes for  $\mathfrak{I}\mathfrak{d}$  are a special case of this equation.

2. In addition the future- and past-stable taut curved webs shown in Figures 61 and 62 lead to basic amplitudes shown in Figures 69–72. By insisting that the amplitudes define solutions of the Maurer-Cartan equation for the future- and past-stable S-wall interfaces  $\mathfrak{S}_{ij}^{f,p}$  described in Section §7.6 below we derive the following:

- The amplitude of Figure 69,  $\mathcal{B}_{jj}^{ij} \in \mathcal{E}_{ij} \otimes R_{ji} \cong R_{ij} \otimes R_{ji}$  is

$$\mathcal{B}_{jj}^{ij} = -K_{ij}^{-1} \quad (7.37)$$

- The amplitude of Figure 70,  $\mathcal{B}_{ii}^{ij} \in \mathcal{E}_{ij} \otimes R_{ji} \cong R_{ij} \otimes R_{ji}$  is

$$\mathcal{B}_{ii}^{ij} = K_{ij}^{-1} \quad (7.38)$$

- The amplitude of Figure 71,  $\mathcal{B}_{ij}^{jj} \in R_{ij} \otimes \mathcal{E}_{ij}^* \cong R_{ij} \otimes R_{ji}$  is

$$\mathcal{B}_{ij}^{jj} = (-1)^F K_{ij}^{-1} = K^{\alpha\alpha'} v_\alpha \otimes v_{\alpha'} \quad (7.39)$$

- The amplitude of Figure 72,  $\mathcal{B}_{ij}^{ii} \in R_{ij} \otimes \mathcal{E}_{ij}^* \cong R_{ij} \otimes R_{ji}$  is

$$\mathcal{B}_{ij}^{ii} = -(-1)^F K_{ij}^{-1} = -K^{\alpha\alpha'} v_\alpha \otimes v_{\alpha'} \quad (7.40)$$

3. There can also be taut curved webs trapped between two binding points. These should be defined so that the composition property (7.2) holds with an equality. One example, which will be of significance later in Section §9, is shown in Figure 73. The result is an amplitude  $\mathcal{B}_{ii}^{ii} \in \text{End}(\mathcal{E}_{ii})$  where the Chan-Paton space  $\mathcal{E}_{ii}$  is derived from multiplying

$$\begin{pmatrix} \mathbb{Z} & R_{ij} \\ 0 & \mathbb{Z} \end{pmatrix} \begin{pmatrix} \mathbb{Z} & 0 \\ R_{ji} & \mathbb{Z} \end{pmatrix} \quad (7.41)$$

and hence  $\mathcal{E}_{ii} \cong \mathbb{Z} \oplus R_{ij} \otimes R_{ji}$ . We can define  $\mathcal{B}_{ii}^{ii}$  from the interface product  $\boxtimes$  of an Interface  $\mathfrak{I}_1$  of Figure 70 with an Interface  $\mathfrak{I}_2$  of Figure 69 (with  $i$  and  $j$  switched). The relevant diagram is shown in Figure 74. The contraction in this diagram involves:

$$K_{ji} : (\mathcal{E}(\mathfrak{I}_1)_{ij} \otimes R_{ji}) \otimes (R_{ij} \otimes \mathcal{E}(\mathfrak{I}_2)_{ji}) \rightarrow \mathcal{E}_{ij}(\mathfrak{I}_1) \otimes \mathcal{E}(\mathfrak{I}_2)_{ji} \cong R_{ij} \otimes R_{ji} \quad (7.42)$$

and the value is

$$K_{ji}((K_{ij}^{-1}) \otimes (-K_{ji}^{-1})) = K_{ij}^{-1} \quad (7.43)$$

We view the amplitude  $\mathcal{B}_{ii}^{ii}$  as a map

$$\mathbb{Z} \cong \mathcal{E}(\mathfrak{I}_1)_{ii} \otimes \mathcal{E}(\mathfrak{I}_2)_{ii} \rightarrow \mathcal{E}(\mathfrak{I}_1)_{ij} \otimes \mathcal{E}(\mathfrak{I}_2)_{ji} \cong R_{ij} \otimes R_{ji} \quad (7.44)$$

taking 1 to  $K_{ij}^{-1}$  and annihilating  $R_{ij} \otimes R_{ji}$ . It is indeed a degree one differential on  $\mathcal{E}(\mathfrak{I}_1 \boxtimes \mathfrak{I}_2)_{ii}$ , as required by the Maurer-Cartan equation.

4. As we noted above, there can be completely rigid curved webs with no moduli (these are vertical lines at binding points). These can be “added” to generic curved webs at otherwise empty binding points to get new webs with the same number of moduli. They contribute to  $\rho_\beta(\cdot)$  an extra tensor factor acting as the identity on the corresponding  $R_{ij}$  or  $R_{ji}^*$  factors in  $\mathcal{E}_{j,j'}$ . It is important to include these contributions in the total curved taut element  $\mathbf{t}$ .

We have now completely defined  $\rho_\beta^0(\mathbf{t})$ . The operator  $\rho_\beta(\mathbf{z})$  is compatible by construction with convolutions and the tensor operation  $T_\partial$ , and thus the convolution identity gives us

$$\rho_\beta(\mathbf{t}_\mathcal{I}) \left[ \frac{1}{1 - \rho_\beta^0(\mathbf{t})} \right] = 0 \quad (7.45)$$

In other words, we have the key observation that  $\rho_\beta^0(\mathbf{t})$  is an interface amplitude and, together with (7.28), it defines an Interface

$$\mathfrak{I}[\vartheta(x)] \in \mathfrak{Br}(\mathcal{T}^{\vartheta_\ell}, \mathcal{T}^{\vartheta_r}). \quad (7.46)$$

associated to a function  $\vartheta(x)$  defining spinning vacuum weights (7.4).

As a special case, note that if  $\vartheta(x) = \vartheta$  is constant then  $\mathfrak{I}[\vartheta(x)] = \mathfrak{Id}$  is the identity Interface in  $\mathfrak{Br}(\mathcal{T}^\vartheta, \mathcal{T}^\vartheta)$ . Since, by assumption  $e^{-i\vartheta} z_{ij}$  is never imaginary the Chan-Paton factors are indeed given by  $\mathcal{E}_{ij} = \delta_{ij} \mathbb{Z}$ . The amplitude  $\rho_\beta^0(t)$  is defined by (7.36), which is also the definition of the amplitudes for  $\mathfrak{Id}$ .

The above construction can be generalized to discuss curved webs in the presence of interfaces by making  $\vartheta(x)$  piecewise differentiable and placing interfaces at positions  $x$  where  $\vartheta'(x)$  has a discontinuity. For example, suppose we have Interfaces  $\mathfrak{I}^{-,\vartheta_\ell} \in \mathfrak{Br}(\mathcal{T}^-, \mathcal{T}^{\vartheta_\ell})$  at  $x_\ell$  and  $\mathfrak{I}^{\vartheta_r,+} \in \mathfrak{Br}(\mathcal{T}^{\vartheta_r}, \mathcal{T}^+)$  at  $x_r$ . Then we can consider curved webs in the strip  $x_\ell \leq x \leq x_r$  given by  $\vartheta(x)$  interpolating between  $\vartheta_\ell$  and  $\vartheta_r$ . We could repeat our above discussion and define an interface web  $\mathfrak{I}^{\text{strip}}[\vartheta(x)] \in \mathfrak{Br}(\mathcal{T}^-, \mathcal{T}^+)$ . On the other hand, we could instead extend  $\vartheta(x)$  to a function on the real line to define  $\mathfrak{I}[\vartheta(x)]$  as in equation (7.46) above. Then, using a  $G_3$ -geometry as in Section 6.3 we can consider  $(\mathfrak{I}^{-,\vartheta_\ell} \mathfrak{I}[\vartheta(x)] \mathfrak{I}^{\vartheta_r,+})_\eta$  as in (6.58). We claim these Interfaces are all homotopy equivalent. An appropriate space-time dependent geometry can realize an homotopy equivalence with any setup where the interpolation happens throughout the whole region in between the locations of  $\mathfrak{I}^{-,\vartheta_\ell}$  and  $\mathfrak{I}^{\vartheta_r,+}$ . This leads, in particular, to the connection to the wedge geometries sketched above.

#### 7.4.3 Verification Of Flat Parallel Transport

Now that we have defined  $\mathfrak{I}[\vartheta(x)]$  let us verify the two key properties (7.2) and (7.3) for defining flat parallel transport.

First, the composition property (7.2) is straightforward. Suppose  $\vartheta^1(x)$  smoothly interpolates from  $\vartheta^-$  to  $\vartheta^0$  and  $\vartheta^2(x)$  smoothly interpolates from  $\vartheta^0$  to  $\vartheta^+$ . Then, on the one hand, we have defined Interfaces  $\mathfrak{I}[\vartheta^1(x)] \in \mathfrak{Br}(\mathcal{T}^{\vartheta^-}, \mathcal{T}^{\vartheta^0})$  and  $\mathfrak{I}[\vartheta^2(x)] \in \mathfrak{Br}(\mathcal{T}^{\vartheta^0}, \mathcal{T}^{\vartheta^+})$  which can be composed as in equation (6.41). On the other hand, the functions can be concatenated to define a smooth interpolation  $\vartheta^1 \circ \vartheta^2(x)$  from  $\vartheta^-$  to  $\vartheta^+$  and we wish to show:

$$\mathfrak{I}[\vartheta^1(x)] \boxtimes \mathfrak{I}[\vartheta^2(x)] \sim \mathfrak{I}[\vartheta^1 \circ \vartheta^2(x)] \quad (7.47)$$

where  $\sim$  means homotopy equivalence. The definition (7.28) as an ordered product along the real line shows that the Interfaces on the left- and right-hand sides of (7.47) have identical Chan-Paton spaces. As for the amplitude, if we cut a taut curved web (contributing the the interface amplitude of  $\mathfrak{I}[\vartheta^1 \circ \vartheta^2(x)]$ ) into two pieces along a vertical line  $x = x_0$ , then there is a corresponding taut composite web used in the definition of the interface amplitude of  $\mathfrak{I}[\vartheta^1(x)] \boxtimes \mathfrak{I}[\vartheta^2(x)]$  and the two amplitudes match. Next, as we have seen, in the case of taut curved webs with no vertices the amplitudes are defined in terms of elementary ones so that the composition property holds. In fact, often, one can replace the homotopy equivalence in (7.47) by an equality sign. In general, we should write a homotopy equivalence because  $\boxtimes$  is only associative up to homotopy equivalence.

Thanks to the composition property the general Interface  $\mathfrak{I}[\vartheta(x)]$  can be decomposed as a product of elementary Interfaces. We discuss these elementary Interfaces in detail in Sections §§7.5 and 7.6 below.

Next, we need to study the behaviour of  $\mathfrak{I}[\vartheta(x)]$  under homotopy of  $\vartheta(x)$ . Accordingly, let  $\vartheta(x, y)$  be an homotopy interpolating between two functions  $\vartheta_f(x)$  at very large positive

$y$  and  $\vartheta_p(x)$  at very large negative  $y$  with the same endpoints  $\vartheta_{\ell,r}$ . Again, our choice of argument is intentional: we interpret the homotopy as an actual space-time dependent configuration.

We can define curved webs with space-time dependent weights  $e^{-i\vartheta(x,y)}z_i$  in an obvious way. Their properties are somewhat similar to the ones we looked at to prove associativity of composition of interfaces in Section §6.3. Looking at sliding webs, we get the convolution identity closely analogous to equation (6.67)

$$\mathbf{t}_{st} * \mathbf{t}_{pl} + \mathbf{t}^f - \mathbf{t}^p + T_\partial(\mathbf{t}_I)[\frac{1}{1-\mathbf{t}^f}; \mathbf{t}_{st}; \frac{1}{1-\mathbf{t}^p}] = 0 \quad (7.48)$$

where  $\mathbf{t}^{p,f}$  are the taut elements for curved webs with the past and future spinning weights  $e^{-i\vartheta_p(x)}z_i$ ,  $e^{-i\vartheta_f(x)}z_i$ , respectively, and  $\mathbf{t}_{st}$  is now the space-time dependent taut element (not including the empty web). The notations  $\mathbf{t}_{pl}$  and  $\mathbf{t}_I$  are as in (7.22).

Applying a web representation, we find an identity of the form of equation (5.23) and therefore using the discussion of that result we conclude that  $\delta[\vartheta(x,y)] := \rho_\beta^0(\mathbf{t}_{st})$  defines a closed morphism  $\mathbf{Id} + \delta[\vartheta(x,y)]$  between  $\mathfrak{I}[\vartheta_f(x)]$  and  $\mathfrak{I}[\vartheta_p(x)]$ .

Next, we need to show that the closed morphisms  $\mathbf{Id} + \delta[\vartheta(x,y)]$  and  $\mathbf{Id} + \delta[\vartheta(x,-y)]$  are inverse up to homotopy. We can use the same strategy as for the proof of associativity of composition of interfaces up to homotopy: show that given a continuous interpolation  $\vartheta(x,y,s)$  between two homotopies  $\vartheta^1(x,y)$  and  $\vartheta^2(x,y)$  the closed morphism  $\mathbf{Id} + \delta[\vartheta(x,y,s)]$  varies by an exact amount. This follows from a convolution identity for curved web homotopies similar to the ones we have already discussed above.

To summarize, we have proven that given two homotopic maps  $\vartheta^1(x)$  and  $\vartheta^2(x)$  with the same endpoints the corresponding Interfaces  $\mathfrak{I}[\vartheta^1(x)]$  and  $\mathfrak{I}[\vartheta^2(x)]$  are homotopy equivalent.

#### 7.4.4 Rigid Rotations And Monodromy

We can now deliver on a promise made in Section §7.2 and define a canonical Interface  $\mathfrak{R}[\vartheta^\ell, \vartheta^r]$  between Theories  $\mathcal{T}^{\vartheta_\ell}$  and  $\mathcal{T}^{\vartheta_r}$ . Namely, choose lifts of  $\vartheta_\ell, \vartheta_r$  to  $\mathbb{R}$  so that  $\vartheta_\ell > \vartheta_r$  and  $|\vartheta_\ell - \vartheta_r| \leq \pi$ . Then, as in Section §7.2 we can use

$$\vartheta(x) = \begin{cases} \vartheta_\ell & x \leq -\vartheta_\ell \\ -x & -\vartheta_\ell < x < -\vartheta_r \\ \vartheta_r & x > -\vartheta_\ell \end{cases} \quad (7.49)$$

to define

$$\mathfrak{R}[\vartheta_\ell, \vartheta_r] := \mathfrak{I}[\vartheta(x)]. \quad (7.50)$$

Now, it follows from (7.47) that

$$\mathfrak{R}[\vartheta_1, \vartheta_2] \boxtimes \mathfrak{R}[\vartheta_2, \vartheta_3] = \mathfrak{R}[\vartheta_1, \vartheta_2] \quad (7.51)$$

as desired.

If we try to extend the above discussion to intervals larger than  $2\pi$  then we encounter the interesting phenomenon of monodromy. Consider the function

$$\vartheta(x) = \begin{cases} \vartheta_* & x \leq -\vartheta_* \\ -x & -\vartheta_* < x < -\vartheta_* + 2\pi \\ \vartheta_* - 2\pi & x > -\vartheta_* \end{cases} \quad (7.52)$$

The invertible interface  $\mathfrak{I}[\vartheta(x)]$  in this case defines an  $A_\infty$ -functor from  $\mathfrak{Br}(\mathcal{T}^{\vartheta_*})$  to itself. Note that there is precisely one binding wall of type  $ij$  for each pair distinct pair of vacua  $(i, j)$ . This can be viewed as a monodromy transformation on the category of Branes. Indeed, we can see that the Interface for rigid rotation through an angle  $2\pi$  is not equivalent to the identity Interface using the discussion of equations (7.41)-(7.44) above. The cohomology of  $\mathcal{E}_{ii}$  is the quotient of  $R_{ij} \otimes R_{ji}$  by the one dimensional line spanned by  $K_{ij}^{-1}$ , and is in general nontrivial. See Section §7.9 below for further discussion.

#### 7.4.5 The Relation Of Ground States To Local Operators

The rigid rotation Interfaces can be used to describe the precise relation between the complex of ground states on an interval and the complex of local operators on a half-plane.

Let us return to the motivation in Sections §7.2 and 7.3. There is a very useful special case of the exponential map, namely when the wedge has opening angle  $\pi$ . In this case we find that if  $\mathfrak{B}_1, \mathfrak{B}_2 \in \mathfrak{Br}(\mathcal{T}^\vartheta)$  then  $\text{Hop}(\mathfrak{B}_1, \mathfrak{B}_2)$  with differential  $M_1$  is literally the same as the complex of approximate groundstates on the interval with left-Brane  $\mathfrak{B}_1$  and right-Brane  $\mathfrak{B}_2[\pi] := \mathfrak{B}_2 \boxtimes \mathfrak{R}[\vartheta, \vartheta + \pi]$ , with differential (4.59). Indeed, note that on an interval of  $\pi$  for every unordered pair of vacua there will be precisely one binding wall. (It is important to define the rotation Interface so that  $\vartheta$  increases, and hence all the binding walls are past-stable.) The Chan-Paton factors of  $\mathfrak{R}[\vartheta, \vartheta + \pi]$  provide all the relevant half-plane fans and we conclude that

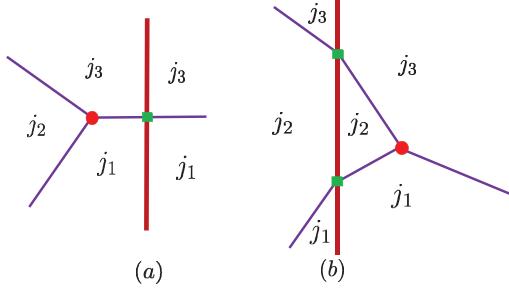
$$H^*(\text{Hop}(\mathfrak{B}_1, \mathfrak{B}_2), M_1) \cong H^*(\mathcal{E}_{LR}(\mathfrak{B}_1, \mathfrak{B}_2[\pi]), d_{LR}). \quad (7.53)$$

This result will be very useful in Section §7.10 below. Physically, it states that the space of BPS states between branes on an interval is isomorphic to the local boundary-changing operators on a half-plane. For more discussion of the relation to local observables see Section §16 below.

#### 7.5 Locally Trivial Categorical Transport

Let us consider a function  $\vartheta(x)$  interpolating from  $\vartheta_\ell$  to  $\vartheta_r$  with  $\vartheta_\ell = \vartheta_r = \vartheta_*$ . We assume, moreover that there is a homotopy  $\vartheta(x, y)$  to the constant function  $\vartheta(x) = \vartheta_*$  such that  $\vartheta(x, y)$  has no binding points (as a function of  $x$  at fixed  $y$ , for all  $y$ ). Applying the discussion of (7.48) et. seq. we conclude that  $\mathfrak{I}[\vartheta(x)]$  is homotopy equivalent to the identify interface  $\mathfrak{Id}$  on  $\mathcal{T}^{\vartheta_*}$ .

More generally when the function  $\vartheta(x)$  has no binding points we say that  $\mathfrak{I}[\vartheta(x)]$  defines a *locally trivial categorical transport*. The Interface and its associated  $A_\infty$ -functor



**Figure 75:** Two contributions to the Maurer-Cartan equation for an interface defined by locally trivial transport, such as that induced by taut webs of the form of Figure 75.

(via equation (6.49)) are particularly simple in this case. Since there are no binding walls equation (7.28) says that the Chan-Paton factors are identical to those of the identity interface  $\mathfrak{Id}$ , namely  $\mathcal{E}_{ij} = \delta_{ij}\mathbb{Z}$ .

We will say that  $\vartheta(x)$  and its associated Interface  $\mathfrak{I}[\vartheta(x)]$  are *simple* if the only curved taut webs look like those in Figure 65 (one for each pair of distinct vacua  $(i, j)$ ). The interface amplitudes for such a simple Interface are then those given by equation (7.36). The demonstration that the Maurer-Cartan equation is satisfied, that is, the demonstration that (7.45) holds in this case, is very similar to that used to show that the identity Interface  $\mathfrak{Id}$  between a theory and itself satisfies the Maurer-Cartan equation. Indeed, we should compare Figure 40 with Figure 75. The main difference is that when we move an interior vertex across the interface it gets rotated in order to be compatible with Figure 68. Algebraically, the demonstration that the Maurer-Cartan equation is satisfied is identical to the case of  $\mathfrak{Id}$ . Nevertheless, we should not identify it with  $\mathfrak{Id}$  because it is in general an Interface between different theories  $\mathcal{T}^{\vartheta_\ell}$  and  $\mathcal{T}^{\vartheta_r}$ .

In general if  $\vartheta(x)$  has no binding points but is not simple then the Chan-Paton factors are still given by  $\mathcal{E}_{ij} = \delta_{ij}\mathbb{Z}$  but in principal there could be exceptional taut webs leading to different interface amplitudes. In this case we can divide up the region of support into a union of small regions  $[x_i, x_{i+1}]$  so that  $\vartheta(x)$  is simple in each region. Then, invoking equation (7.47) we learn that locally trivial transport is always homotopy equivalent to simple locally trivial transport.

Using the discussion of (6.49) et. seq. we see that a locally trivial Interface  $\mathfrak{I}[\vartheta(x)]$  defines an  $A_\infty$ -functor  $\mathcal{F}_{\vartheta(x)} : \mathfrak{Br}(\mathcal{T}^{\vartheta_\ell}) \rightarrow \mathfrak{Br}(\mathcal{T}^{\vartheta_r})$ . However, thanks to the very simple Chan-Paton data, it preserves the vacuum subcategory whose objects are just the thimbles  $\mathfrak{T}_i$ . That is, we can think of locally trivial transport as induced from a simpler functor  $\mathcal{F}_{\vartheta(x)} : \mathfrak{Vac}(\mathcal{T}^{\vartheta_\ell}) \rightarrow \mathfrak{Vac}(\mathcal{T}^{\vartheta_r})$ .

Note well that in the above discussion we have strongly used the fact that there are no binding walls. If, on the other hand, the rotation of some edge from  $e^{-i\vartheta_\ell} z_{ij}$  to  $e^{-i\vartheta_r} z_{ij}$  passes through the positive imaginary axis then one of the vertices of  $\mathcal{T}^{\vartheta_\ell}$  cannot be

transported through the region of support of  $\vartheta'(x)$  to produce the corresponding vertex of  $\mathcal{T}^{\vartheta_r}$ . The Maurer-Cartan equation (7.45) for the simple interface amplitudes (7.36) will fail in this case. The next subsection §7.6 explains in detail how the correct Interface  $\mathfrak{I}[\vartheta(x)]$  corrects the simple amplitudes to produce a solution of the Maurer-Cartan equation.

## 7.6 S-Wall Interfaces

As we mentioned above, thanks to the composition property (7.47) the general Interface  $\mathfrak{I}[\vartheta(x)]$  can be decomposed into elementary factors. These consist of locally trivial parallel transport together with the “S-wall Interfaces” that are described in detail in this Section. The *S-wall Interfaces* are defined (up to homotopy equivalence) by functions  $\vartheta(x)$  which interpolate from  $\vartheta_{ij} \pm \epsilon$  to  $\vartheta_{ij} \mp \epsilon$ , where  $e^{i\vartheta_{ij}}$  defines an  $S_{ij}$ -ray and  $\epsilon$  is a sufficiently small positive number.<sup>38</sup>

To be concrete, we define an Interface  $\mathfrak{S}_{ij}^p$  (up to homotopy equivalence) by choosing  $\vartheta(x)$  to interpolate from  $\vartheta_{ij} - \epsilon$  to  $\vartheta_{ij} + \epsilon$  on some interval  $(x_0 - \delta, x_0 + \delta)$ . Here  $\delta$  is a positive number (and not a morphism!) and, for definiteness, we choose a linear interpolation with  $\epsilon \ll \delta$ . The region of support of  $\vartheta'(x)$  contains a *past stable* binding point  $x_0$  of type  $ij$  and no other binding points. The vacuum weights  $e^{-i\vartheta(x)} z_k$  all rotate clockwise through an angle  $2\epsilon$  and  $e^{-i\vartheta(x)} z_{ij}$  rotates clockwise through the positive imaginary axis.

Similarly, we will define an Interface  $\mathfrak{S}_{ij}^f$  (up to homotopy equivalence) by choosing  $\vartheta(x)$  to interpolate from  $\vartheta_{ij} + \epsilon$  to  $\vartheta_{ij} - \epsilon$  on some interval  $(x_0 - \delta, x_0 + \delta)$ . The region of support of  $\vartheta'(x)$  contains a *future stable* binding point  $x_0$  of type  $ij$  and no other binding points. The vacuum weights  $e^{-i\vartheta(x)} z_k$  all rotate counter-clockwise through an angle  $2\epsilon$  and  $e^{-i\vartheta(x)} z_{ij}$  rotates counter-clockwise through the positive imaginary axis.

The name *S-wall Interfaces* is apt because the path in the complex plane  $e^{-i\vartheta(x)} z_{ij}$  crosses an  $S_{ij}$ -ray (see equation (7.5)).

According to the definition (7.28) the Chan-Paton data for these Interfaces are given by

$$\mathcal{E}(\mathfrak{S}_{ij}^p)_{kl} = \begin{cases} R_{ji}^* & (k, l) = (i, j) \\ \delta_{k,l} \mathbb{Z} & \text{else} \end{cases} \quad (7.54)$$

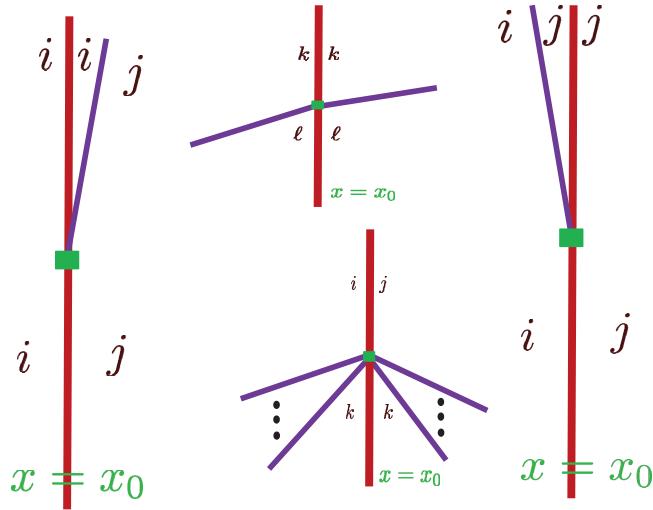
$$\mathcal{E}(\mathfrak{S}_{ij}^f)_{kl} = \begin{cases} R_{ij} & (k, l) = (i, j) \\ \delta_{k,l} \mathbb{Z} & \text{else} \end{cases} \quad (7.55)$$

The interface amplitudes have already been described in Section §7.4.2 above. We summarize the nonzero amplitudes in Figure 76 for  $\mathfrak{S}_{ij}^p$  and in Figure 77 for  $\mathfrak{S}_{ij}^f$ .

It is instructive to check explicitly the claim that the interface amplitude satisfies the Maurer-Cartan equation (7.45). We will do so for  $\mathfrak{S}_{ij}^p$ , and the check for  $\mathfrak{S}_{ij}^f$  is very similar. For webs not involving  $ij$  lines the check is identical to the verification of the Maurer-Cartan equation for the identity Interface. However, there are some new taut webs that arise because  $\mathcal{E}_{ij}$  is nonzero and because there are new vertices involving  $ij$  lines.

---

<sup>38</sup>Note that while the variation in  $\vartheta_{ij}$  is small we have said nothing about how large the region of support of  $\vartheta'(x)$  is. This can be changed, up to homotopy equivalence, and could be taken to be very large or very small.

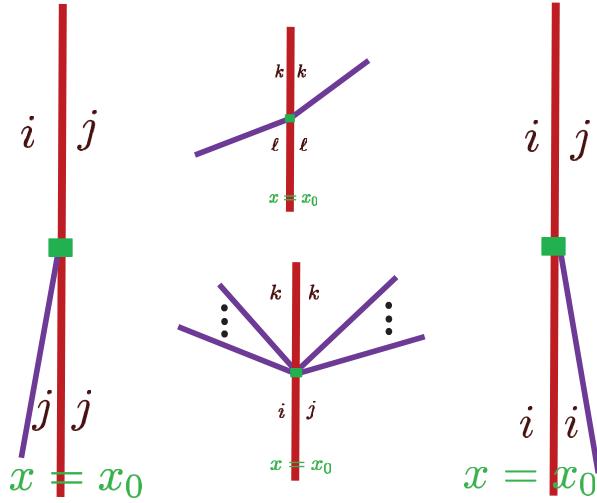


**Figure 76:** This figure shows the nonzero components of the interface amplitude for  $\mathfrak{S}_{ij}^p$ . For every pair of vacua  $(k, \ell)$  there is an amplitude of the form shown in the top middle. The acute angle between the lines in the negative and positive half-planes is  $2\epsilon$ . The left and right figures show new amplitudes relative to the locally trivial case. The acute angle here is  $\epsilon$ . The amplitudes for all the above three cases are given up to sign by  $K^{-1}$ , suitably interpreted. See equations (7.36), (7.39), and (7.40) for the precise formulae. The lower middle figure is a new amplitude associated to any interior vertex with an  $ij$  edge pointing to the future. If  $\beta_I$  is the interior amplitude associated with that vertex then the corresponding interface amplitude is  $(\check{K} \otimes 1)(\beta_I)$ .

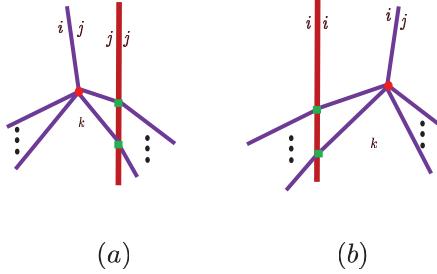
First consider interior vertices with an external  $ij$  line that goes to the future. When this vertex is “moved” through the Interface the lines rotate and we obtain taut interface webs such as those shown in Figure 78. The amplitude for Figure 78(a) is valued in  $\mathcal{E}_{ii} \otimes \hat{R}_{ik}^+ \otimes \mathcal{E}_{kk}^* \otimes \hat{R}_{ki}^-$  while that for Figure 78(b) is valued in  $\mathcal{E}_{jj} \otimes \hat{R}_{jk}^+ \otimes \mathcal{E}_{kk}^* \otimes \hat{R}_{kj}^-$ . The extra rotation, compared to the identity Interface, has led to amplitudes valued in different spaces which therefore cannot cancel. However, thanks to the “new” interface amplitudes for  $\mathfrak{S}_{ij}^p$  there are also two new taut webs shown in Figure 79(a) and Figure 79(b). The amplitude for Figure 78(a) cancels that for Figure 79(a) and similarly the amplitude for Figure 78(b) cancels that for Figure 79(b). In fact, demanding this cancellation gives the derivation of the basic amplitudes (7.39) and (7.40) above.

Next, consider interior vertices of  $\mathcal{T}^{\vartheta_\ell}$  with an external  $ij$  line that goes to the past. Some new taut webs constructed with such a vertex in either half-plane are shown in Figure 80 (for the case of a trivalent vertex). Working patiently and carefully with all the sign conventions we have explained one can check that these two amplitudes do indeed cancel.

Finally, recall that in the identity for the taut planar element  $t_{pl} * t_{pl} = 0$  the terms canceled in pairs. For each such pair involving a vertex with an  $ij$  edge we can construct two taut webs. For example, consider the component of the Maurer-Cartan equation shown in Figure 81. This is an identity for the interior amplitudes  $\beta_I$ . There is a corresponding pair of taut interface webs shown in Figure 82. Since the vertices with  $ij$  lines in Figure 82



**Figure 77:** This figure shows the nonzero components of the interface amplitude for  $\mathfrak{S}_{ij}^f$  analogous to those for  $\mathfrak{S}_{ij}^p$ .



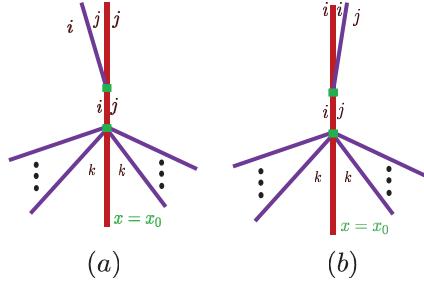
**Figure 78:** These two contributions to the MC equation for  $\mathfrak{S}_{ij}^p$  are analogous to canceling contributions for  $\mathfrak{I}$  but cannot cancel in this case because now the amplitudes are valued in different spaces, as explained in the text.

are defined by  $\check{K}(\beta)$  the  $A_\infty$  Maurer-Cartan equation for the Interface will be satisfied if  $\beta$  satisfies the corresponding  $L_\infty$  Maurer-Cartan equation. This completes the verification of the Maurer-Cartan equation for the Interface  $\mathfrak{S}_{ij}^p$ .

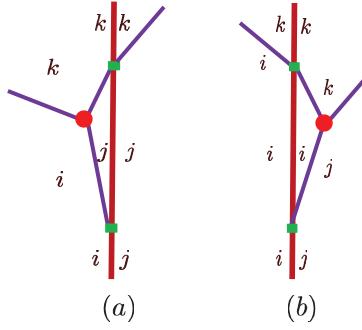
The general arguments we gave in Section §7.4.3 imply that if we compose past and future S-wall Interfaces the result is homotopy equivalent to the identity Interface. Nevertheless, it is instructive to examine this homotopy equivalence in some detail and we turn to this next. Let

$$\mathcal{T}^+ = \mathcal{T}^{\vartheta_{ij} + \epsilon} \quad \mathcal{T}^- = \mathcal{T}^{\vartheta_{ij} - \epsilon} \quad (7.56)$$

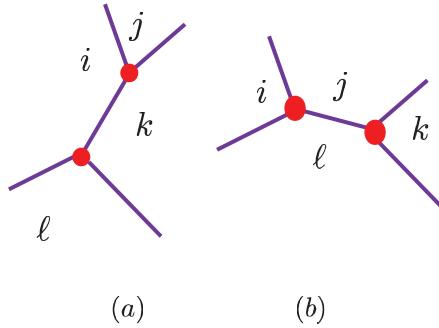
$$\mathfrak{I}^{fp} = \mathfrak{S}_{ij}^f \boxtimes \mathfrak{S}_{ij}^p \in \mathfrak{Br}(\mathcal{T}^+, \mathcal{T}^+) \quad \mathfrak{I}^{pf} = \mathfrak{S}_{ij}^p \boxtimes \mathfrak{S}_{ij}^f \in \mathfrak{Br}(\mathcal{T}^-, \mathcal{T}^-) \quad (7.57)$$



**Figure 79:** Contributions (a) and (b) to the MC equation for  $\mathfrak{S}_{ij}^p$  cancel those of Figure 78(a) and Figure 78(b), respectively.



**Figure 80:** Contributions (a) and (b) to the MC equation for  $\mathfrak{S}_{ij}^p$  cancel.

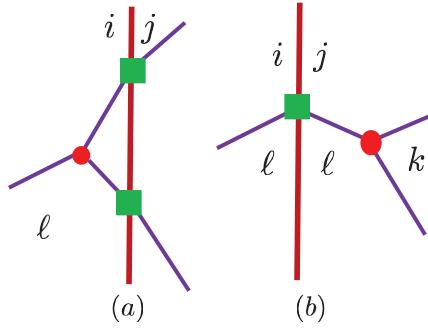


**Figure 81:** A cancelling pair in the Maurer-Cartan equation for the Theory  $\mathcal{T}^{\vartheta_{ij}-\epsilon}$ .

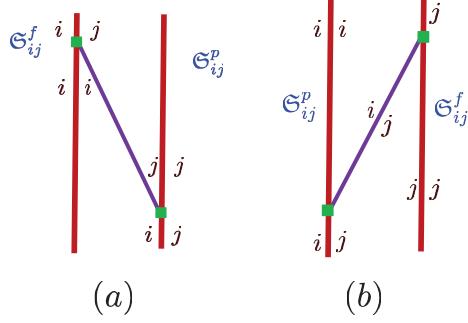
Then we claim that

$$\mathfrak{I}^{fp} \sim \mathfrak{I}\mathfrak{d}_{\mathcal{T}^+} \quad \& \quad \mathfrak{I}^{pf} \sim \mathfrak{I}\mathfrak{d}_{\mathcal{T}^-}. \quad (7.58)$$

The Chan-Paton data of the Interfaces  $\mathfrak{I}^{fp}$  and  $\mathfrak{I}^{pf}$  are the same and are easily com-



**Figure 82:** A component of the  $L_\infty$  MC equation of Figure 81 has a corresponding component for the  $A_\infty$  MC equation for  $\mathfrak{S}_{ij}^p$  shown here.



**Figure 83:** The two composite webs shown here lead to differentials on the  $\mathcal{E}_{ij}$  Chan-Paton space of  $\mathfrak{S}_{ij}^f \boxtimes \mathfrak{S}_{ij}^p$  and  $\mathfrak{S}_{ij}^p \boxtimes \mathfrak{S}_{ij}^f$ , respectively. In each case there is a chain homotopy of the identity morphism to zero so that the cohomology of  $\mathcal{E}_{ij}$  vanishes.

puted from equations (6.32), (7.54), and (7.55), with the result

$$\mathcal{E}(\mathfrak{I}^{fp})_{kl} = \mathcal{E}(\mathfrak{I}^{pf})_{kl} = \begin{cases} \mathcal{E}_{ij}^f \oplus \mathcal{E}_{ij}^p & (k, l) = (i, j) \\ \delta_{k,l} \mathbb{Z} & \text{else.} \end{cases} \quad (7.59)$$

Here we have denoted  $\mathcal{E}_{ij}^f := \mathcal{E}(\mathfrak{S}_{ij}^f)_{ij} \cong R_{ij}$  and  $\mathcal{E}_{ij}^p := \mathcal{E}(\mathfrak{S}_{ij}^p)_{ij} \cong R_{ji}^*$ . The first check of a homotopy equivalence to the identity Interface is that the cohomology of the Chan-Paton space of type  $ij$  should vanish. In fact, the interface product  $\boxtimes$  leads to a nontrivial differential associated with the taut composite webs shown in Figure 83. We explain this in detail for  $\mathfrak{I}^{fp}$ . The taut composite web of Figure 83(a) gives an amplitude valued in

$$\mathcal{E}(\mathfrak{I}^{fp})_{ij} \otimes \mathcal{E}(\mathfrak{I}^{fp})_{ij}^* \cong \text{End}(\mathcal{E}(\mathfrak{I}^{fp})_{ij}). \quad (7.60)$$

Now the endomorphisms of  $\mathcal{E}(\mathfrak{I}^{fp})_{ij}$  can be organized into block matrices using the sum-

mands  $R_{ij}$  and  $R_{ji}^*$  so that elements are valued in the block matrix:

$$\begin{pmatrix} \text{End}(R_{ij}) & \text{Hom}(R_{ij}, R_{ji}^*) \\ \text{Hom}(R_{ji}^*, R_{ij}) & \text{End}(R_{ji}^*) \end{pmatrix} \quad (7.61)$$

With this understood, the amplitude defined by Figure 83(a) is

$$\mathcal{B}(\mathfrak{I}^{fp})_{ij}^{ij} = \begin{pmatrix} 0 & 0 \\ (-1)^F K_{ij}^{-1} & 0 \end{pmatrix} \quad (7.62)$$

Note that  $(-1)^F K_{ij}^{-1} = K^{\alpha\alpha'} v_\alpha \otimes v_{\alpha'} \in \text{Hom}(R_{ji}^*, R_{ij})$  really has degree +1 since the matrix elements are only nonzero when  $\deg(v_\alpha) + \deg(v_{\alpha'}) = +1$ .<sup>39</sup>

Now, to exhibit a homotopy equivalence  $\mathfrak{I}^{fp} \sim \mathfrak{I}\mathfrak{d}$  we need to produce four morphisms:

$$\begin{aligned} \delta_1 &\in \text{Hop}(\mathfrak{I}^{fp}, \mathfrak{I}\mathfrak{d}) \\ \delta_2 &\in \text{Hop}(\mathfrak{I}\mathfrak{d}, \mathfrak{I}^{fp}) \\ \delta_3 &\in \text{Hop}(\mathfrak{I}^{fp}, \mathfrak{I}^{fp}) \\ \delta_4 &\in \text{Hop}(\mathfrak{I}\mathfrak{d}, \mathfrak{I}\mathfrak{d}) \end{aligned} \quad (7.63)$$

such that

$$M_2(\delta_1, \delta_2) = \mathbf{Id}_{\mathfrak{I}^{fp}} + M_1(\delta_3) \quad (7.64)$$

$$M_2(\delta_2, \delta_1) = \mathbf{Id}_{\mathfrak{I}\mathfrak{d}} + M_1(\delta_4) \quad (7.65)$$

First of all, we take  $(\delta_1)_{kk}^{kk} = (\delta_2)_{kk}^{kk} = 1$  for all  $k \in \mathbb{V}$ . Now, because  $\mathfrak{I}^{fp}$  has a CP space

$$\mathcal{E}_{ij} = \mathcal{E}_{ij}^f \oplus \mathcal{E}_{ij}^p \cong R_{ij} \oplus R_{ji}^* \quad (7.66)$$

the identity morphism  $\mathbf{Id}_{\mathfrak{I}^{fp}}$  has a component of type  $(ij, ij)$ , namely, the identity transformation  $\mathbf{Id}_{\mathcal{E}_{ij}} \in \text{Hom}(\mathcal{E}_{ij})$  that is impossible to produce from  $M_2(\delta_1, \delta_2)$ . It is impossible to produce  $\mathbf{Id}_{\mathcal{E}_{ij}}$  simply because if  $\delta_1 \in \text{Hop}(\mathfrak{I}^{fp}, \mathfrak{I}\mathfrak{d}_{T+})$  has an  $ij$  line in the future then it must have a  $jj$  line in the past but if  $\delta_2 \in \text{Hop}(\mathfrak{I}\mathfrak{d}_{T+}, \mathfrak{I}^{fp})$  has an  $ij$  line in the past it must have an  $ii$  line in its future. Therefore, no composition of  $\delta_1$  and  $\delta_2$  can produce an element of  $\text{Hom}(\mathcal{E}_{ij})$ . Therefore the  $(ij, ij)$  component of  $\mathbf{Id}_{\mathfrak{I}^{fp}}$  must come from  $M_1(\delta_3)$ . We choose  $\delta_3$  to have only one nonzero component, valued in  $\text{End}(\mathcal{E}_{ij})$ , and given by

$$(\delta_3)_{ij}^{ij} = \begin{pmatrix} 0 & (-1)^F K_{ji} \\ 0 & 0 \end{pmatrix} \quad (7.67)$$

where we interpret  $(-1)^F K_{ji} = (-1)^{v_{\alpha'}} K_{\alpha'\alpha} v_{\alpha'}^* \otimes v_\alpha^* \in \mathcal{E}_{ij}^p \otimes (\mathcal{E}_{ij}^f)^*$ . Then, when computing  $M_1(\delta_3)$  we meet

$$\rho(\mathfrak{t}_{\mathcal{I}})(\mathcal{B}, \delta_3) + \rho(\mathfrak{t}_{\mathcal{I}})(\delta_3, \mathcal{B}). \quad (7.68)$$

---

<sup>39</sup>It is a nice and rather subtle exercise to check that the product Interface does indeed satisfy the MC equation.

Using  $\mathcal{B}_{ij}^{ij}$  above, and taking proper care of signs<sup>40</sup> one finds that the first summand in (7.68) gives  $\mathbf{Id}_{\mathcal{E}_{ij}^f}$  and the second gives  $\mathbf{Id}_{\mathcal{E}_{ij}^p}$  so the sum gives the desired identity morphism on  $\mathcal{E}(\mathfrak{J}^{fp})_{ij}$ . Thus,  $\delta_3$  provides an explicit chain homotopy equivalence of the identity morphism on  $\mathcal{E}_{ij}$  to the zero morphism.

Since  $\delta_3$  is nonzero there are induced vertices of type  $(ii, ij)$  and  $(ij, jj)$  in  $M_1(\delta_3)$  but these can be cancelled against  $M_2(\delta_1, \delta_2)$  by choosing

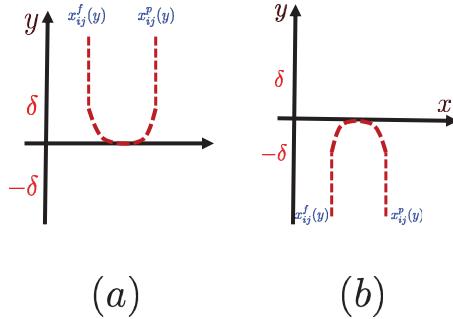
$$(\delta_2)_{ij}^{ii} = +\mathbf{Id}_{\mathcal{E}_{ij}^f} \quad (7.69)$$

$$(\delta_1)_{jj}^{ij} = -\mathbf{Id}_{\mathcal{E}_{ij}^p} \quad (7.70)$$

Now we must check that the other vertices of  $\mathcal{B}(\mathfrak{J}^{fp})$  do not lead to new components of  $M_1(\delta_3)$  or new components of  $M_2(\delta_1, \delta_2)$  or  $M_2(\delta_2, \delta_1)$  which would complicate the homotopy equivalence. There are several potential contractions which would considerably complicate the discussion, but happily they all give zero because, after careful examination, they all involve contractions of  $(\mathcal{E}_{ij}^f)^*$  with  $\mathcal{E}_{ij}^p$  or  $(\mathcal{E}_{ij}^p)^*$  with  $(\mathcal{E}_{ij}^f)$ , and these contractions vanish. Now is easy to check that

$$M_2(\delta_2, \delta_1) = \mathbf{Id}_{\mathfrak{J}^{fp}} \quad (7.71)$$

so we can take  $\delta_4 = 0$ .

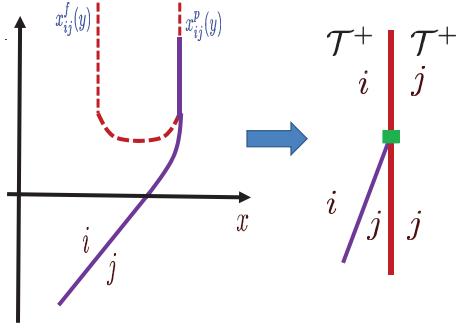


**Figure 84:** The dashed curves show the behavior of binding points under a homotopy between  $\vartheta^{fp}(x)$  and the constant. In (a) the homotopy  $\vartheta(x, y)$  has the property that for  $y \leq -\delta$  it is just the constant  $\vartheta(x, y) = -\epsilon$  and for  $y \geq \delta$  it is  $\vartheta(x, y) = \vartheta^{fp}(x)$ . As  $y$  decreases from  $\delta$  to  $0$ , the binding points approach each other and annihilate. In (b) we show the location of the binding points for the time-reversed homotopy  $\vartheta(x, -y)$ .

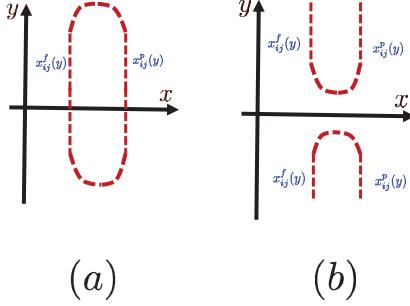
Finally, it is instructive to see how the general argument of Section §7.4.3 produces the above explicit homotopy equivalence of  $\mathfrak{J}^{fp}$  with the identity Interface. Let  $\vartheta^{fp}(x)$  define a vacuum homotopy corresponding to  $\mathfrak{J}^{fp}$ . Thus, on some interval,  $z_{ij}(x)$  rotates from  $i e^{-i\epsilon}$  to  $i e^{+i\epsilon}$  and then back to  $i e^{-i\epsilon}$ , where  $\epsilon > 0$ . Let  $\vartheta(x, y)$  be a homotopy of  $\vartheta^{fp}(x)$  to

---

<sup>40</sup>Define the sign of the dual so that  $v_\alpha^* \cdot v_\beta = \delta_{\alpha, \beta}$ .



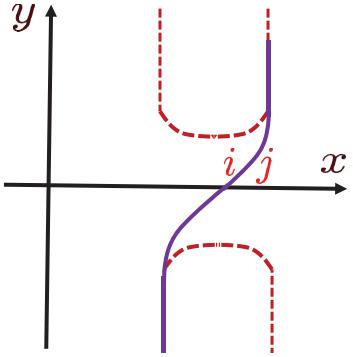
**Figure 85:** The taut (= rigid) web on the left produces a space-time curved web which contributes to the component  $(\delta_1)_{jj}^{ij}$  in the morphism describing the homotopy equivalence of  $\mathcal{I}^{fp}$  and  $\mathcal{D}_{T^+}$ .



**Figure 86:** A vacuum homotopy  $\tilde{\vartheta}(x, y)$  used to compute  $M_2(\delta_2, \delta_1)$  leads to past and future binding points shown in Figure (a). We do not expect any exceptional webs when we consider a homotopy  $\tilde{\vartheta}(x, y; s)$  to the constant function. On the other hand, a vacuum homotopy  $\tilde{\vartheta}(x, y)$  used to compute  $M_2(\delta_1, \delta_2)$  leads to past and future binding points shown in Figure (b). In the corresponding homotopy  $\tilde{\vartheta}(x, y; s)$  to the constant the two dashed curves must start far apart, then merge and turn into two parallel dashed lines. At some point  $s = s_*$  there will be an exceptional web, illustrated in Figure 87 leading to the morphism  $\delta_3$ .

the constant path, so that, for  $y \geq \delta$  we have  $\vartheta(x, y) = \vartheta^{fp}(x)$  and for  $y \leq -\delta$  we have  $\vartheta(x, y) = \epsilon$ . For example, as  $y$  decreases from  $\delta$  to  $-\delta$ , the path  $z_{ij}(x)$  could rotate more and more slowly so that at  $-\delta$  it becomes constant. See Figure 84 for an illustration of how the future and past  $ij$  binding points evolve. Note that at some intermediate time, say  $y = 0$  the vacuum weight  $z_{ij}(x; y)$ , as a function of  $x$  fails to rotate past the positive imaginary axis. Then the past and future binding points annihilate.

As described in Section §7.4.3, the morphism  $\delta_1$  can be constructed from  $\rho_\beta^0(t_{st})$  where  $t_{st}$  is the taut element in the space-time curved webs described by  $\vartheta(x, y)$ . In particular, the nontrivial component of equation (7.70) arises from the taut (= rigid) web shown in Figure 85. One can similarly derive the nontrivial component (7.69) for  $\delta_2$ . Finally, when we



**Figure 87:** When the concatenation time  $\circ_T$  is positive it is impossible to draw an  $ij$  line such as that shown here. In the homotopy  $\check{\vartheta}(x, y; s)$  there will be a critical value of  $s$  where an  $ij$  line of the type shown here will exist. This is an exceptional web  $\epsilon$  that leads to the morphism  $\delta_3$ .

compute  $M_2(\delta_1, \delta_2)$  and  $M_2(\delta_2, \delta_1)$  we should use equation (6.71), valid for concatenations  $\circ_T$  with a large time interval  $T$ . In the case of  $M_2(\delta_2, \delta_1)$  we have a spacetime configuration with binding points evolving as in Figure 86(a). This can be homotoped to the constant function without producing any exceptional webs. On the other hand, when computing  $M_2(\delta_1, \delta_2)$  we use Figure 86(b). In the homotopy  $\check{\vartheta}(x, y; \epsilon)$  to the function  $\vartheta^{fp}(x)$  the dashed line of binding points merges and then turns into two parallel dashed lines. When the concatenation time  $T$  is positive it is impossible to draw an  $ij$  line, and when the two components are too close it is again impossible to draw an  $ij$  line. At a critical value  $s_*$  there will be an exceptional web  $\epsilon$  such as that shown in Figure 87 leading to  $\delta_3 = \rho_\beta^0(\epsilon)$ , and producing an amplitude of the type (7.67).

## 7.7 Categorification Of Framed Wall-Crossing

As we have mentioned, the Interfaces  $\mathfrak{S}_{ij}^{p,f}$  may be regarded as a categorification of the “ $S$ -factors” which play an important role in the theory of spectral networks [31, 33, 34, 75]. The relation to spectral networks is discussed in more detail in Section §18.2 below. In order to recover wall-crossing formulae for *framed* BPS states we replace  $R_{ij}$  by its Witten index  $\mu_{ij}$  to obtain two-dimensional soliton BPS counts. For an Interface  $\mathfrak{I}^{-,+}$  we define the *framed BPS degeneracies* to be the Witten indices of the Chan-Paton factors

$$\underline{\Omega}(\mathfrak{I}^{-,+}, ij') := \text{Tr}_{\mathcal{E}(\mathfrak{I}^{-,+})_{ij'}}(-1)^F \quad (7.72)$$

To compare with [31, 33, 34, 75], note that the role of the line defect is played by  $\mathfrak{I}^{-,+}$  and the IR charge, usually denoted  $\gamma_{ij'}$  is here simply the pair  $ij'$ .

To illustrate the relation to wall-crossing let us consider a vacuum homotopy that crosses and  $S_{ij}$ -wall. For  $x_1 < x_2$  define  $\mathfrak{I}[x_1, x_2]$  to be  $\mathfrak{I}[\vartheta(x; x_1, x_2)]$  where

$$\vartheta(x; x_1, x_2) = \begin{cases} \vartheta(x_1) & x \leq x_1 \\ \vartheta(x) & x_1 \leq x \leq x_2 \\ \vartheta(x_2) & x \geq x_2 \end{cases} \quad (7.73)$$

The family of interfaces satisfies  $\mathfrak{I}[x_1, x_3] \sim \mathfrak{I}[x_1, x_2] \boxtimes \mathfrak{I}[x_2, x_3]$  for  $x_1 < x_2 < x_3$ . In particular, if  $x_{ij}$  is a binding point of type  $ij$  then

$$\mathfrak{I}[x_1, x_{ij} + \delta] \sim \mathfrak{I}[x_1, x_{ij} - \delta] \boxtimes \mathfrak{I}[x_{ij} - \delta, x_{ij} + \delta] = \mathfrak{I}[x_1, x_{ij} - \delta] \boxtimes \mathfrak{S}_{ij}^{p,f} \quad (7.74)$$

where we choose  $\mathfrak{S}_{ij}^{p,f}$  depending on whether  $x_{ij}$  is past or future stable, respectively. This is the categorified  $S$ -wall-crossing formula.

If we consider the Chan-Paton data to be a matrix of complexes then we have the homotopy equivalence of matrices of chain complexes:

$$\mathcal{E}(\mathfrak{I}[x_1, x_{ij} + \delta]) \sim \mathcal{E}(\mathfrak{I}[x_1, x_{ij} - \delta])\mathcal{E}(\mathfrak{S}_{ij}^{p,f}) \quad (7.75)$$

To relate this to the standard wall-crossing formula note that if we take the Witten index of the matrix of Chan-Paton spaces we produce the generating function for framed BPS degeneracies of the Interface:

$$F[\mathfrak{I}[x_1, x_2]] := \text{Tr}_{\mathcal{E}(\mathfrak{I}[x_1, x_2])}(-1)^F = \sum_{k,\ell} \overline{\Omega}(\mathfrak{I}[x_1, x_2], k\ell) e_{k,\ell}. \quad (7.76)$$

This matrix-valued function will be continuous in  $x_1, x_2$  for locally trivial transport, but when  $x$  crosses a binding point of type  $ij$  we have framed wall-crossing formula:

$$F \mapsto \begin{cases} F \cdot (\mathbf{1} + \mu_{ij} e_{ij}) & x_{ij} \in \lambda_{ij} \\ F \cdot (\mathbf{1} - \mu_{ij} e_{ij}) & x_{ij} \in \gamma_{ij} \end{cases} \quad (7.77)$$

In the second line we have used the degree  $-1$  isomorphism of  $R_{ji}^*$  with  $R_{ij}$  to identify the Witten index of  $R_{ji}^*$  with  $-\mu_{ij}$ . Equation (7.77) is precisely the framed wall-crossing formula of [31].

## 7.8 Mutations

Categorical transport by  $\mathfrak{S}_{ij}^{p,f}$  makes contact with the theory of mutations and exceptional collections in category theory. The relation between mutations of exceptional collections and D-branes in Landau-Ginzburg models has been discussed at length in [95, 48, 81]. We briefly make contact with these works.

In general there is no natural order on the set of vacua  $\mathbb{V}$  because the vacuum weights  $z_i$  are points in the complex plane. If we choose a direction in the plane, say parallel to a complex number  $\zeta$ , then that direction defines a height function on the plane and, so long as  $\zeta$  is not parallel to any of the  $z_{ij}$  for  $i, j \in \mathbb{V}$  we can order the vacua by increasing (or decreasing) height. Note this is the same as the condition that  $\zeta$  is not orthogonal to any  $S_{ij}$  ray. Considering  $\zeta$  to the normal direction to a half-plane, the corresponding thimbles  $\mathfrak{T}_i$  can now be ordered:

$$\mathfrak{T}_i < \mathfrak{T}_j \iff \text{Re}(\zeta^{-1} z_{ij}) > 0. \quad (7.78)$$

Note that with such an ordering we can write

$$\text{Hop}(\mathfrak{T}_i, \mathfrak{T}_j) = \begin{cases} \hat{R}_{ij} & \mathfrak{T}_i < \mathfrak{T}_j \\ \hat{R}_{ii} \cong \mathbb{Z} & i = j \\ 0 & \mathfrak{T}_i > \mathfrak{T}_j \end{cases}. \quad (7.79)$$

where  $\hat{R}_{ij}$  is defined with respect to a half-plane with inward-pointing normal vector  $\zeta$ . Since the thimbles generate the category of Branes, they form what is known in category theory as an *exceptional collection*, and the vacuum category  $\mathfrak{Vac}(\mathcal{T}, \mathcal{H})$  is an *exceptional category* as defined in Appendix B below.

Now consider rotating  $\zeta$  or, equivalently, fix  $\zeta = +1$ , take the positive half-plane  $\mathcal{H}^+$ , and consider a family of spinning weights  $e^{-i\vartheta(x)} z_i$ . When  $e^{i\vartheta(x)}$  passes through an  $S_{ij}$  wall an ordered pair of thimbles  $(\mathfrak{T}_i, \mathfrak{T}_j)$  exchanges its ordering and the old exceptional collection is no longer an exceptional collection in the new Theory. A natural question is whether one can construct a new exceptional collection of Branes in the new Theory out of the old exceptional collection of the old Theory. The answer to this question is “yes,” and the procedure is called a *mutation*. We will explain how mutations work in our formalism.

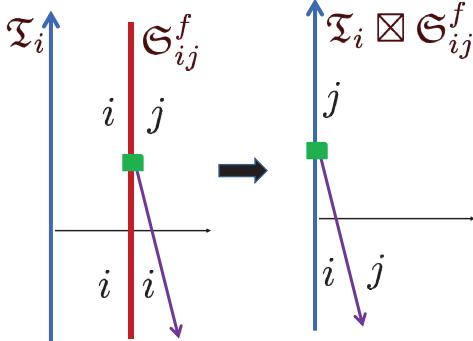
If our path of Theories crosses a future stable  $S_{ij}$  wall then interface product with the  $S$ -wall Interface  $\mathfrak{S}_{ij}^f$  defines, as usual, an  $A_\infty$ -functor  $\mathfrak{Br}(\mathcal{T}^{\vartheta_{ij}+\epsilon}) \rightarrow \mathfrak{Br}(\mathcal{T}^{\vartheta_{ij}-\epsilon})$ . The original Theory,  $\mathcal{T}^{\vartheta_{ij}+\epsilon}$  has  $\text{Re}(z_{ij}) > 0$  so  $\mathfrak{T}_i < \mathfrak{T}_j$ . So, the original exceptional collection is

$$\dots, \mathfrak{T}_i, \mathfrak{T}_j, \dots \quad \text{Future stable} \quad (7.80)$$

and the final Theory,  $\mathcal{T}^{\vartheta_{ij}-\epsilon}$  has  $\text{Re}(z_{ij}) < 0$ . Similarly, if the path of Theories crosses a past stable  $S_{ij}$  wall then we use  $\mathfrak{S}_{ij}^p$  to define an  $A_\infty$ -functor  $\mathfrak{Br}(\mathcal{T}^{\vartheta_{ij}-\epsilon}) \rightarrow \mathfrak{Br}(\mathcal{T}^{\vartheta_{ij}+\epsilon})$ . The original Theory,  $\mathcal{T}^{\vartheta_{ij}-\epsilon}$  has  $\text{Re}(z_{ij}) < 0$  so  $\mathfrak{T}_i > \mathfrak{T}_j$ . So, the original exceptional collection is

$$\dots, \mathfrak{T}_j, \mathfrak{T}_i, \dots \quad \text{Past stable} \quad (7.81)$$

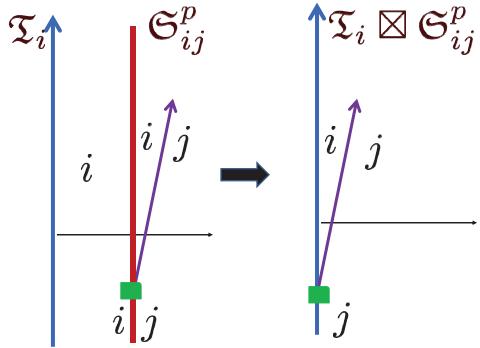
and the final Theory,  $\mathcal{T}^{\vartheta_{ij}+\epsilon}$  has  $\text{Re}(z_{ij}) > 0$ .



**Figure 88:** The Brane  $\mathfrak{T}_i \boxtimes \mathfrak{S}_{ij}^f$  has one nonvanishing boundary amplitude. Under the isomorphism  $\mathcal{E}_i \otimes \hat{R}_{ji} \otimes \mathcal{E}_j^* \cong R_{ij} \otimes R_{ji}$  it is  $K_{ij}^{-1}$ .

Let us examine the action of the  $S$ -wall functors on the exceptional collections of the original Theory in these two cases. It is easy to see that

$$\mathfrak{T}_k \boxtimes \mathfrak{S}_{ij}^{p,f} = \mathfrak{T}_k \quad k \neq i \quad (7.82)$$



**Figure 89:** The Brane  $\mathfrak{T}_i \boxtimes S_{ij}^p$  has one nonvanishing boundary amplitude. Under the isomorphism  $\mathcal{E}_i \otimes \widehat{R}_{ij} \otimes \mathcal{E}_j^* \cong R_{ij} \otimes R_{ji}$  it is  $-(-1)^F K_{ij}^{-1}$ .

since none of the amplitudes in Figures 77 or 76 can contract with  $\mathfrak{T}_k$ . On the other hand,  $\mathfrak{T}_i \boxtimes S_{ij}^{p,f}$  is nontrivial. Indeed we can compute the Chan-Paton factors:

$$\begin{aligned} \mathcal{E}(\mathfrak{T}_k \boxtimes S_{ij}^p)_\ell &= \mathcal{E}(\mathfrak{T}_k)_\ell \oplus \delta_{k,i} R_{ji}^* \otimes \mathcal{E}(\mathfrak{T}_j)_\ell \\ \mathcal{E}(\mathfrak{T}_k \boxtimes S_{ij}^f)_\ell &= \mathcal{E}(\mathfrak{T}_k)_\ell \oplus \delta_{k,i} R_{ij} \otimes \mathcal{E}(\mathfrak{T}_j)_\ell \end{aligned} \quad (7.83)$$

Moreover, each of the Branes  $\mathfrak{T}_i \boxtimes S_{ij}^{p,f}$  has a single nonvanishing boundary amplitude, illustrated in Figures 88 and 89.

Let us consider the case of crossing an  $S_{ij}$  wall in the future-stable direction. We now would like to introduce a new generating set of Branes, replacing the old exceptional collection (7.80) by the new collection of Branes

$$\dots, \mathfrak{T}_j, \mathfrak{T}_i \boxtimes S_{ij}^f, \dots \quad (7.84)$$

In the language of Appendix B this corresponds to a left-mutation at  $j$ . Similarly, when crossing an  $S_{ij}$ -wall in the past-stable direction we would like to introduce a new generating set of Branes, replacing the old exceptional collection (7.81) by the new collection of Branes

$$\dots, \mathfrak{T}_i \boxtimes S_{ij}^p, \mathfrak{T}_j, \dots \quad (7.85)$$

In the language of Appendix B this corresponds to a right-mutation at  $j$ . The idea is that the “missing” Brane  $\mathfrak{T}_i$  can be expressed as a boundstate of the Branes  $\mathfrak{T}_j$  and  $\mathfrak{T}_i \boxtimes S_{ij}^{f,p}$  by condensing local boundary operators. One way of expressing this, often found in the literature, is to relate the three Branes by an exact triangle. We explain this momentarily.

On general grounds, mutations of exceptional collections are expected to provide a representation of the braid group, up to homotopy. Such a braid group representation is intimately connected to the theory of categorical wall-crossing we develop in Section 8.

Before writing our exact triangles we first note that the category of Branes forms a module over the category of  $\mathbb{Z}$ -modules. Given any  $\mathbb{Z}$ -module  $V$  and any Brane  $\mathfrak{B}$  we define

$V \otimes \mathfrak{B}$  to be the Brane which has Chan-Paton spaces

$$\mathcal{E}(V \otimes \mathfrak{B})_i := V \otimes \mathcal{E}(\mathfrak{B})_i \quad (7.86)$$

Then the amplitude of  $V \otimes \mathfrak{B}$  is supposed to live in

$$\oplus_{z_{ij} \in \mathcal{H}} \mathcal{E}(V \otimes \mathfrak{B})_i \otimes \widehat{R}_{ij} \otimes \mathcal{E}(V \otimes \mathfrak{B})_j^* = (V \otimes V^*) \otimes \oplus_{z_{ij} \in \mathcal{H}} \mathcal{E}(\mathfrak{B})_i \otimes \widehat{R}_{ij} \otimes \mathcal{E}(\mathfrak{B})_j^* \quad (7.87)$$

and we take the amplitude (up to an appropriate sign) to be  $\text{Id}_V \otimes \mathcal{B}$  where  $\mathcal{B}$  is the amplitude of  $\mathfrak{B}$ . The hom-spaces of the category of Branes satisfy

$$\text{Hop}(V_1 \otimes \mathfrak{B}_1, V_2 \otimes \mathfrak{B}_2) = V_1 \otimes \text{Hop}(\mathfrak{B}_1, \mathfrak{B}_2) \otimes V_2^*. \quad (7.88)$$

Now, consider a path crossing an  $S_{ij}$ -wall in the future-stable direction and consider the triple of Branes  $R_{ij} \otimes \mathfrak{T}_j, \mathfrak{T}_i, \mathfrak{T}_i \boxtimes \mathfrak{S}_{ij}^f$ . We compute the hom-spaces for the positive half-plane in the new Theory  $\mathcal{T}^{\theta_{ij}-\epsilon}$ :

$$\text{Hop}(R_{ij} \otimes \mathfrak{T}_j, \mathfrak{T}_i) = R_{ij} \otimes \widehat{R}_{ji} \quad (7.89)$$

$$\text{Hop}(\mathfrak{T}_i \boxtimes \mathfrak{S}_{ij}^f, \mathfrak{T}_i) = \text{Hop}(i, i) \oplus R_{ij} \otimes \widehat{R}_{ji} \quad (7.90)$$

$$\text{Hop}(R_{ij} \otimes \mathfrak{T}_j, \mathfrak{T}_i \boxtimes \mathfrak{S}_{ij}^f) = R_{ij} \otimes \widehat{R}_{ji} \oplus R_{ij} \otimes R_{ij}^* \quad (7.91)$$

Each of the summands above contains a canonical element. In (7.89) we have the element  $K_{ij}^{-1}$ , in (7.90) the first summand has the identity and the second has  $K^{-1}$ , and in (7.91) the first summand has  $K^{-1}$  and the second summand has the identity  $\text{Id}_{R_{ij}}$ . Using the definition (5.17) (contraction with the taut element) we can check the exact triangle of morphisms

$$\begin{array}{ccc} R_{ij} \otimes \mathfrak{T}_j & \xleftarrow{K^{-1}} & \mathfrak{T}_i \\ & \swarrow \text{Id}_{R_{ij}} & \searrow K^{-1} \\ & \mathfrak{T}_i \boxtimes \mathfrak{S}_{ij}^f & \end{array} \quad (7.92)$$

is a commutative diagram.

Similarly, for  $\mathfrak{T}_i \boxtimes \mathfrak{S}_{ij}^p$  we compute (again with  $z_{ij}$  in the positive half-plane)

$$\begin{aligned} \text{Hop}(\mathfrak{T}_i, R_{ji}^* \otimes \mathfrak{T}_j) &= \widehat{R}_{ij} \otimes R_{ji} \\ \text{Hop}(\mathfrak{T}_i, \mathfrak{T}_i \boxtimes \mathfrak{S}_{ij}^p) &= \text{Hop}(i, i) \oplus \widehat{R}_{ij} R_{ji} \\ \text{Hop}(\mathfrak{T}_i \boxtimes \mathfrak{S}_{ij}^p, R_{ji}^* \otimes \mathfrak{T}_j) &= \widehat{R}_{ij} \otimes R_{ji} \oplus R_{ji}^* \otimes R_{ji} \end{aligned} \quad (7.93)$$

Once again, using suitable canonical elements from the summands we can construct the exact triangle:

$$\begin{array}{ccc} \mathfrak{T}_i & \xleftarrow{K^{-1}} & R_{ji}^* \otimes \mathfrak{T}_j \\ & \swarrow K^{-1} & \searrow \text{Id}_{R_{ji}^*} \\ & \mathfrak{T}_i \boxtimes \mathfrak{S}_{ij}^p & \end{array} \quad (7.94)$$

These are the kinds of exact triangles that appear in discussions of mutations of exceptional collections found in the literature.

When comparing with the general discussion of Appendix B we should note that some of the factors  $R_{ij}$  above should really be viewed as Hop spaces  $\widehat{R}_{ij}$ . However, near an  $S_{ij}$ -wall these two spaces can be identified.

## 7.9 Categorical Spectrum Generator And Monodromy

Let us now return to (7.52). We define  $\mathfrak{R}[\vartheta, \vartheta - \pi]$  to be the *categorical spectrum generator*. The name is apt because in this case  $\vartheta(x)$  has a unique future stable binding point for each pair of vacua with  $z_{ij}$  in a suitable half-plane. Taking  $\vartheta$  to be small we can rewrite (7.28) as

$$\bigoplus_{j,j' \in \mathbb{V}} \mathcal{E}_{j,j'} e_{j,j'} := \bigotimes_{\text{Re}(z_{ij}) > 0} S_{ij}(x_{ij}) \quad (7.95)$$

where the ordering in the product from left to write is the clockwise ordering of the phases of  $z_{ij}$ . If we consider the Witten index of this product we produce precisely the spectrum generator as defined in [28, 31]. In the case of 2d Landau-Ginzburg models this is precisely the matrix  $S$  defined long ago by Cecotti and Vafa. (See equation (2.11) in [15].)

In our case we have the general result that

$$\mathfrak{R}[\vartheta, \vartheta - 2\pi] \sim \mathfrak{R}[\vartheta, \vartheta - \pi] \boxtimes \mathfrak{R}[\vartheta - \pi, \vartheta - 2\pi] \quad (7.96)$$

Examining the Chan-Paton factors for the Interface  $\mathfrak{R}[\vartheta, \vartheta - 2\pi]$  motivates the interpretation of  $\mathfrak{R}[\vartheta, \vartheta - 2\pi]$  as a categorified version of Cecotti and Vafa's "monodromy"  $SS^{tr,-1}$  (which in turn is motivated by the monodromy of the cohomology of a Milnor fiber in singularity theory). <sup>41</sup> Indeed, if  $\vartheta(x) = -x$  for  $x \in [0, 2\pi]$  then all the  $S$ -walls are future stable and, for every pair  $ij$  with  $i \neq j$  there will be precisely two future stable binding points  $x_{ij}$  and  $x_{ji}$  in the interval of length  $2\pi$ , and moreover  $|x_{ij} - x_{ji}| = \pi$ . Again, with a suitable choice of half-plane  $\mathcal{H}$  (or choosing  $\vartheta$  to be small) we can write the Chan-Paton factors of  $\mathfrak{R}[\vartheta, \vartheta - 2\pi]$  as

$$\cdots \otimes (\mathbb{Z}\mathbf{1} \oplus R_{ij}e_{ij}) \otimes \cdots \otimes (\mathbb{Z}\mathbf{1} \oplus R_{ji}e_{ji}) \otimes \cdots = \mathbb{S} \otimes \mathbb{S}^{\text{opp}} \quad (7.97)$$

where  $\mathbb{S}$  is the clockwise phase-ordered product for  $z_{ij}$  in one half-plane and  $\mathbb{S}^{\text{opp}}$  is the clockwise phase-ordered product in the opposite half-plane. Now, if we take the Witten index to decategorify  $\mathbb{S} \rightarrow S$  then we map the factors in  $\mathbb{S}$  via  $(\mathbb{Z}\mathbf{1} \oplus R_{ij}e_{ij}) \rightarrow (\mathbf{1} + \mu_{ij}e_{ij})$ . Now we need the relation

$$\mu_{ji} = -\mu_{ij}^* \quad (7.98)$$

which follows from the existence of the degree  $-1$  pairing  $K_{ij}$ . (See also the discussion in Landau-Ginzburg theory in Section §12.3 below.) If we define the fermion number to be integral (using the gauge freedom discussed in Sections §4.4 and §4.6.4) then  $\mu_{ij}$  is real and hence if  $\mathbb{S} \rightarrow S$  then  $\mathbb{S}^{\text{opp}} \rightarrow S^{tr,-1}$ .

There is a known relation between properties of the UV theory and the eigenvalues of the matrix  $SS^{tr,-1}$  [15]. If the massive theory flows from a UV SCFT, as is the case for  $\mathcal{T}^N$ , the eigenvalues take the form  $\exp 2\pi i q_a$ , where  $q_a$  are the charges of UV B-model operators under the R-charge broken by the massive deformation of the SCFT. If the UV theory is asymptotically free, as is the case for  $\mathcal{T}^{SU(N)}$ ,  $SS^{tr,-1}$  typically has Jordan blocks. In Section §7.10 below we will construct some rotation Interfaces for  $\mathcal{T}^N$  and  $\mathcal{T}^{SU(N)}$  and

---

<sup>41</sup>We can also regard  $\mathfrak{R}[\vartheta, \vartheta - 2\pi]$  as a categorified version of a Stokes matrix. We will not pursue that very interesting direction in the present paper.

we will see that indeed a sufficiently high power of the rotation Interface is trivial in the former case, but not in the latter case. Moreover, since  $\mathfrak{R}[\vartheta, \vartheta - 2\pi]$  is a categorical lift of  $SS^{tr,-1}$  it is natural to wonder if it somehow reconstructs some other properties of the UV theory. Indeed we will see that it is an essential ingredient in the construction of local operators (on the plane) in Section §9.2.

It would be interesting understand whether there is a categorical generalization of the relation of the eigenvalues of  $SS^{tr,-1}$  to R-charges. Perhaps this can be done by introducing a notion of an “eigen-interface” for  $\mathfrak{R}[\vartheta, \vartheta - 2\pi]$  under interface product  $\boxtimes$ , but we will leave this idea for future work.

### 7.10 Rotation Interfaces For The Theories $\mathcal{T}_\vartheta^N$ And $\mathcal{T}_\vartheta^{SU(N)}$

In this section we construct some interfaces in the Theories  $\mathcal{T}_\vartheta^{N,SU(N)}$  which give a very useful construction of nontrivial Branes from the simple thimbles. (Much of the discussion can be developed in parallel for the two families of Theories  $\mathcal{T}_\vartheta^N$  and  $\mathcal{T}_\vartheta^{SU(N)}$ . So we will delay separating the cases as long as possible.) We will reveal how one could discover the Branes  $\mathfrak{C}_k$  and  $\mathfrak{N}_n$  of the Theories  $\mathcal{T}_\vartheta^N$  and  $\mathcal{T}_\vartheta^{SU(N)}$ , respectively. (See Section §4.6 above.) This construction also leads to a very neat computation of the space of boundary-condition-changing operators  $H^*(\text{Hop}(\mathfrak{B}_1, \mathfrak{B}_2), M_1)$  for certain pairs of Branes, thus justifying several claims made in Section 5.7.

Recall that the Theories  $\mathcal{T}_\vartheta^{N,SU(N)}$  are based on the vacuum weights (making a slight change of notation from §4.6):

$$z_j^\vartheta := e^{-i\vartheta - \frac{2\pi i}{N}j} \quad (7.99)$$

Here  $j$  is an integer modulo  $N$  and we will always choose it to be in the fundamental domain  $0 \leq j \leq N - 1$ . As described in section §4.6.3, there are nontrivial isomorphisms

$$\varphi^\pm : \mathcal{T}_\vartheta^{N,SU(N)} \rightarrow \mathcal{T}_{\vartheta \pm \frac{2\pi}{N}}^{N,SU(N)} \quad (7.100)$$

and the corresponding isomorphism Interfaces will be denoted  $\mathfrak{I}\mathfrak{d}^\pm := \mathfrak{I}\mathfrak{d}^{\varphi^\pm}$ . In particular, we have  $j\varphi^\pm = (j \mp 1)\text{mod}N$  and

$$\mathcal{E}(\mathfrak{I}\mathfrak{d}^+)_j = \begin{cases} \delta_{k,N-1} \mathbb{Z}^{[1]} & j = 0 \\ \delta_{k,j-1} \mathbb{Z} & 1 \leq j \leq N - 1 \end{cases} \quad (7.101)$$

The choice of degree shift here is the simplest one such that the boundary amplitudes

$$K_{ij}^{-1,\varphi^+} \in \mathcal{E}_{i,i-1} \otimes R_{i-1,j-1} \otimes \mathcal{E}_{j,j-1}^* \otimes R_{ji} \quad (7.102)$$

all have degree one. In  $\mathcal{T}_\vartheta^N$  this is just  $\pm 1 \in \mathbb{Z}^{[1]}$  and in  $\mathcal{T}_\vartheta^{SU(N)}$  it is of the form

$$\pm \sum_I \varepsilon_I (e_I \otimes e_{I'})^{[1]} \quad (7.103)$$

where the sum is over multi-indices of length  $|i - j|$  and  $\varepsilon_I$  is a sign defined under (4.110).

We can work out  $\mathfrak{I}^+$  similarly. It is useful to think of the Chan-Paton data as a matrix with chain complexes as entries and in these terms we have

$$\mathcal{E}(\mathfrak{I}^+) = \mathbb{Z}^{[1]} e_{0,N-1} \oplus \bigoplus_{j=1}^{N-1} \mathbb{Z} e_{j,j-1} \quad (7.104)$$

$$\mathcal{E}(\mathfrak{I}^-) = \mathbb{Z}^{[-1]} e_{N-1,0} \oplus \bigoplus_{j=0}^{N-2} \mathbb{Z} e_{j,j+1} \quad (7.105)$$

We now consider rotation Interfaces  $\mathfrak{R}[\vartheta_\ell, \vartheta_r]$  relating the Theories  $\mathcal{T}_\vartheta^{N,SU(N)}$  for different values of  $\vartheta$ . Thus,  $\vartheta(x)$  varies linearly and the binding walls are determined by the equations

$$\text{Re}(z_{jk}^\vartheta) = 2 \sin\left(\vartheta + \frac{\pi}{N}(k+j)\right) \sin\left(\frac{\pi}{N}(k-j)\right) = 0 \quad (7.106)$$

$$\text{Im}(z_{jk}^\vartheta) = 2 \cos\left(\vartheta + \frac{\pi}{N}(k+j)\right) \sin\left(\frac{\pi}{N}(k-j)\right) > 0 \quad (7.107)$$

To be more specific we consider the Theories

$$\mathcal{T}^+ := \mathcal{T}_\epsilon^{N,SU(N)} \quad \mathcal{T}^- := \mathcal{T}_{\frac{2\pi}{N}-\epsilon}^{N,SU(N)} \quad (7.108)$$

where  $\epsilon$  is a small positive phase with  $\epsilon \ll \frac{\pi}{N}$ . (Actually,  $0 < \epsilon < \frac{\pi}{N}$  will already suffice.)

We now consider functors between the categories of Branes  $\mathfrak{Br}(\mathcal{T}^+)$  and  $\mathfrak{Br}(\mathcal{T}^-)$ . They will be induced by Interfaces as discussed in equation (6.49) et. seq. There are four natural ways to relate these Theories. Indeed, for clockwise rotations, we can rotate  $\mathcal{T}_\epsilon^N$  into  $\mathcal{T}_{\frac{2\pi}{N}-\epsilon}^N$ , or rotate  $\mathcal{T}_{\frac{2\pi}{N}-\epsilon}^N$  into  $\mathcal{T}_{\frac{2\pi}{N}+\epsilon}^N$  and then act with a symmetry interface  $\mathfrak{I}^-$  to bring it back to  $\mathcal{T}_\epsilon^N$ . For counterclockwise rotations, we can rotate  $\mathcal{T}_{\frac{2\pi}{N}-\epsilon}^N$  into  $\mathcal{T}_\epsilon^N$ , or rotate  $\mathcal{T}_\epsilon^N$  into  $\mathcal{T}_{-\epsilon}^N$  and then act with a symmetry interface  $\mathfrak{I}^+$  to bring it back to  $\mathcal{T}_{\frac{2\pi}{N}-\epsilon}^N$ .

We can thus define

$$\begin{aligned} \tilde{\mathfrak{I}}^{+-} &:= \mathfrak{R}\left[\epsilon, \frac{2\pi}{N} - \epsilon\right] \in \mathfrak{Br}(\mathcal{T}^+, \mathcal{T}^-) \\ \tilde{\mathfrak{I}}^{-+} &:= \mathfrak{R}\left[\frac{2\pi}{N} - \epsilon, \frac{2\pi}{N} + \epsilon\right] \boxtimes \mathfrak{I}^- \in \mathfrak{Br}(\mathcal{T}^-, \mathcal{T}^+) \\ \mathfrak{I}^{+-} &:= \mathfrak{R}\left[\epsilon, -\epsilon\right] \boxtimes \mathfrak{I}^+ \in \mathfrak{Br}(\mathcal{T}^+, \mathcal{T}^-) \\ \mathfrak{I}^{-+} &:= \mathfrak{R}\left[\frac{2\pi}{N} - \epsilon, \epsilon\right] \in \mathfrak{Br}(\mathcal{T}^-, \mathcal{T}^+). \end{aligned} \quad (7.109)$$

Next, we work out the effect of convolution by these Interfaces on the Chan-Paton factors of Branes. We begin by computing the binding walls.

Consider the case of  $\mathfrak{I}^{-+}$ . Equation (7.106) is equivalent to  $\vartheta + \frac{\pi}{N}(k+j) = n\pi$ , with  $n \in \mathbb{Z}$ . But, given the range of  $\vartheta$ , we must have  $\vartheta = \frac{\pi}{N}$  and hence  $k+j = N-1$  and  $n=1$ . Then the positivity constraint (7.107) implies  $k < j$ , and hence  $0 \leq k < \frac{N-1}{2}$ . Similar results hold for the other three cases with minor variations. We have binding walls of type

$jk$  where, roughly speaking,  $j$  is an upper vacuum and  $k$  is the lower vacuum vertically below it.

There is a small subtlety in this computation because all the binding walls of type  $jk$  with  $j+k = N-1$  are at the same value of  $x$ , thanks to the very symmetric choice of vacuum weights we made for these examples. However, the matrices  $e_{j,k}$  for  $j+k = N-1$  and  $0 \leq k < \frac{N-1}{2}$  all commute with one another and hence the product (7.28) is unambiguous. Indeed, any small deformation of the vacuum weights will split the walls, leading to a multiple convolution of the walls  $\mathfrak{S}_{j,k}^f$  for these values of  $j, k$ . The different orderings of the convolutions will be homotopy equivalent. Similar remarks apply to the other three Interfaces.

Since the product of two matrices of the type  $e_{j,k}$  for  $j+k = N-1$  and  $0 \leq k < \frac{N-1}{2}$  is zero (7.28) simplifies to the lower triangular matrix

$$\mathcal{E}(\mathfrak{I}^{-+}) = \mathbb{Z}\mathbf{1} \oplus \bigoplus_{\frac{N-1}{2} < j \leq N-1} R_{j,N-j-1} e_{j,N-j-1} \quad (7.110)$$

Explicitly, for  $N = 2, 3, 4, 5$  we have

$$\mathcal{E}(\mathfrak{I}^{-+}) = \begin{pmatrix} \mathbb{Z} & 0 \\ R_{1,0} & \mathbb{Z} \end{pmatrix} \quad (7.111)$$

$$\mathcal{E}(\mathfrak{I}^{-+}) = \begin{pmatrix} \mathbb{Z} & 0 & 0 \\ 0 & \mathbb{Z} & 0 \\ R_{2,0} & 0 & \mathbb{Z} \end{pmatrix} \quad (7.112)$$

$$\mathcal{E}(\mathfrak{I}^{-+}) = \begin{pmatrix} \mathbb{Z} & 0 & 0 & 0 \\ 0 & \mathbb{Z} & 0 & 0 \\ 0 & R_{2,1} & \mathbb{Z} & 0 \\ R_{3,0} & 0 & 0 & \mathbb{Z} \end{pmatrix} \quad (7.113)$$

$$\mathcal{E}(\mathfrak{I}^{-+}) = \begin{pmatrix} \mathbb{Z} & 0 & 0 & 0 & 0 \\ 0 & \mathbb{Z} & 0 & 0 & 0 \\ 0 & 0 & \mathbb{Z} & 0 & 0 \\ 0 & R_{3,1} & 0 & \mathbb{Z} & 0 \\ R_{4,0} & 0 & 0 & 0 & \mathbb{Z} \end{pmatrix} \quad (7.114)$$

The case of  $\mathfrak{I}^{+-}$  is slightly more elaborate. The binding walls of  $\mathfrak{R}[\epsilon, -\epsilon]$  are obtained from (7.106) with  $\vartheta = 0$  and hence  $j+k = N$ . Again  $n = 1$  and then (7.107) implies  $1 \leq k < \frac{N}{2}$ . We must then convolve with the isomorphism Interface  $\mathfrak{Id}^+$ . The net result is

$$\mathcal{E}(\mathfrak{I}^{+-}) = \mathbb{Z}^{[1]} e_{0,N-1} \oplus_{j=1}^{N-1} \mathbb{Z} e_{j,j-1} \oplus_{1 \leq k < \frac{N}{2}} R_{N-k,k} e_{N-k,k-1} \quad (7.115)$$

Again, to get a feel for these, we list the cases  $N = 2, 3, 4, 5$ :

$$\mathcal{E}(\mathfrak{I}^{+-}) = \begin{pmatrix} 0 & \mathbb{Z}^{[1]} \\ \mathbb{Z} & 0 \end{pmatrix} \quad (7.116)$$

$$\mathcal{E}(\mathfrak{I}^{+-}) = \begin{pmatrix} 0 & 0 & \mathbb{Z}^{[1]} \\ \mathbb{Z} & 0 & 0 \\ R_{2,1} & \mathbb{Z} & 0 \end{pmatrix} \quad (7.117)$$

$$\mathcal{E}(\mathfrak{I}^{+-}) = \begin{pmatrix} 0 & 0 & 0 & \mathbb{Z}^{[1]} \\ \mathbb{Z} & 0 & 0 & 0 \\ 0 & \mathbb{Z} & 0 & 0 \\ R_{3,1} & 0 & \mathbb{Z} & 0 \end{pmatrix} \quad (7.118)$$

$$\mathcal{E}(\mathfrak{I}^{+-}) = \begin{pmatrix} 0 & 0 & 0 & 0 & \mathbb{Z}^{[1]} \\ \mathbb{Z} & 0 & 0 & 0 & 0 \\ 0 & \mathbb{Z} & 0 & 0 & 0 \\ 0 & R_{3,2} & \mathbb{Z} & 0 & 0 \\ R_{4,1} & 0 & 0 & \mathbb{Z} & 0 \end{pmatrix} \quad (7.119)$$

Using these formulae it easily follows that if  $\mathfrak{B} \in \mathfrak{Br}(\mathcal{T}^-)$  then

$$\mathcal{E}(\mathfrak{B} \boxtimes \mathfrak{I}^{-+})_j = \begin{cases} \mathcal{E}(\mathfrak{B})_j \oplus \mathcal{E}(\mathfrak{B})_{N-j-1} \otimes R_{N-j-1,j} & 0 \leq j < \frac{N-1}{2} \\ \mathcal{E}(\mathfrak{B})_j & \frac{N-1}{2} \leq j \leq N-1 \end{cases} \quad (7.120)$$

and similarly if  $\mathfrak{B} \in \mathfrak{Br}(\mathcal{T}^+)$  then the Chan-Paton factors change by

$$\mathcal{E}(\mathfrak{B} \boxtimes \mathfrak{I}^{+-})_j = \begin{cases} \mathcal{E}(\mathfrak{B})_{j+1} \oplus \mathcal{E}(\mathfrak{B})_{N-j-1} \otimes R_{N-j-1,j+1} & 0 \leq j < \frac{N}{2}-1 \\ \mathcal{E}(\mathfrak{B})_{j+1} & \frac{N}{2}-1 \leq j \leq N-2 \\ \mathcal{E}(\mathfrak{B})_0^{[1]} & j = N-1 \end{cases} \quad (7.121)$$

Entirely analogous remarks apply to the Interfaces  $\tilde{\mathfrak{I}}^{\pm\mp}$ . The main difference is that since  $\vartheta(x)$  increases in the rotation Interfaces the binding walls are past stable. In this way we find that if  $\mathfrak{B} \in \mathfrak{Br}(\mathcal{T}^+)$  then

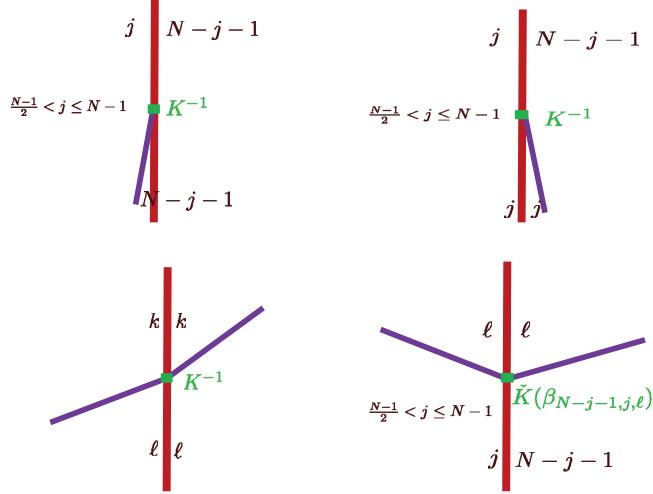
$$\mathcal{E}(\mathfrak{B} \boxtimes \tilde{\mathfrak{I}}^{+-})_j = \begin{cases} \mathcal{E}(\mathfrak{B})_j \oplus \mathcal{E}(\mathfrak{B})_{N-j-1} \otimes R_{j,N-j-1}^* & 0 \leq j < \frac{N-1}{2} \\ \mathcal{E}(\mathfrak{B})_j & \frac{N-1}{2} \leq j \leq N-1 \end{cases} \quad (7.122)$$

and if  $\mathfrak{B} \in \mathfrak{Br}(\mathcal{T}^-)$  then

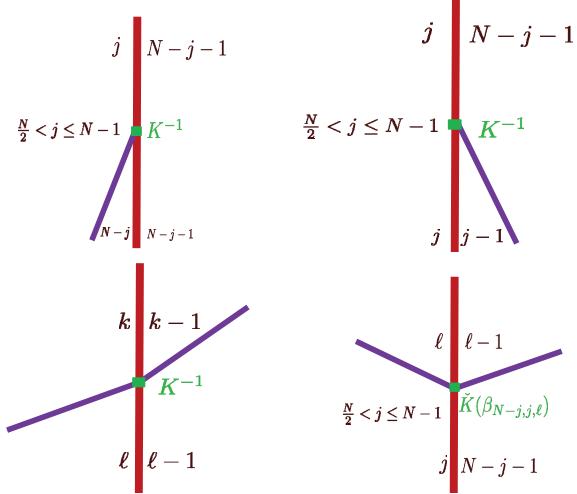
$$\mathcal{E}(\mathfrak{B} \boxtimes \tilde{\mathfrak{I}}^{-+})_j = \begin{cases} \mathcal{E}(\mathfrak{B})_{N-1}^{[-1]} & j = 0 \\ \mathcal{E}(\mathfrak{B})_{j-1} \oplus \mathcal{E}(\mathfrak{B})_{N-j-1} \otimes R_{j-1,N-j-1}^* & 1 \leq j < \frac{N}{2} \\ \mathcal{E}(\mathfrak{B})_{j-1} & \frac{N}{2} \leq j \leq N-1 \end{cases} \quad (7.123)$$

The boundary amplitudes for  $\mathfrak{I}^{-+}$  and  $\mathfrak{I}^{+-}$  are shown in Figures 90 and 91, respectively. Similar results hold for  $\tilde{\mathfrak{I}}^{-+}$  and  $\tilde{\mathfrak{I}}^{+-}$ .

Now, successive application of these Interfaces generates a sequence of Branes in  $\mathcal{T}^-, \mathcal{T}^+$  starting with one Brane in either Theory. To be specific, suppose  $\mathfrak{B} \in \mathfrak{Br}(\mathcal{T}^+)$ .



**Figure 90:** Nonzero boundary amplitudes for  $\mathfrak{I}^{-+}$ .



**Figure 91:** Nonzero boundary amplitudes for  $\mathfrak{I}^{+-}$ .

We then generate a sequence of Branes  $\mathfrak{B}[n] \in \mathfrak{Br}(\mathcal{T}^+)$  and  $\mathfrak{B}[n + \frac{1}{2}] \in \mathfrak{Br}(\mathcal{T}^-)$  with  $n \in \mathbb{Z}$  by setting  $\mathfrak{B}[0] := \mathfrak{B}$  and then defining recursively, for  $n \geq 0$

$$\mathfrak{B}[n + \frac{1}{2}] := \begin{cases} \mathfrak{B}[n] \boxtimes \mathfrak{I}^{+-} & n \in \mathbb{Z} \\ \mathfrak{B}[n] \boxtimes \mathfrak{I}^{-+} & n \in \mathbb{Z} + \frac{1}{2} \end{cases} \quad (7.124)$$

while for  $n \leq 0$  we take

$$\mathfrak{B}[n - \frac{1}{2}] := \begin{cases} \mathfrak{B}[n] \boxtimes \tilde{\mathfrak{I}}^{+-} & n \in \mathbb{Z} \\ \mathfrak{B}[n] \boxtimes \tilde{\mathfrak{I}}^{-+} & n \in \mathbb{Z} + \frac{1}{2} \end{cases} \quad (7.125)$$

Note that for  $n$  positive we are thus taking successive convolutions (well-defined up to homotopy equivalence)

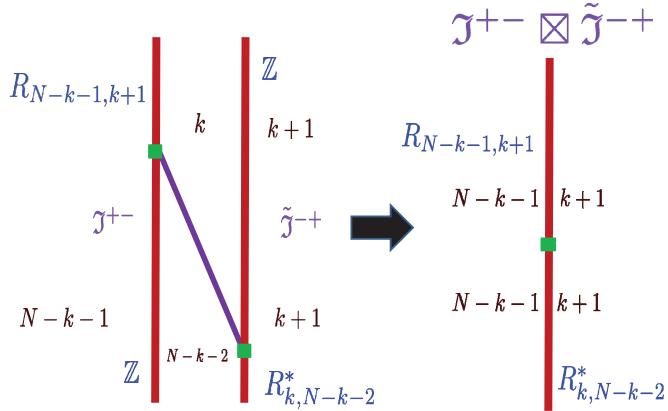
$$\mathfrak{I}^{+-} \boxtimes \mathfrak{I}^{-+} \boxtimes \mathfrak{I}^{+-} \dots \quad (7.126)$$

and similarly for  $n$  negative with  $\tilde{\mathfrak{I}}^{\pm\mp}$ . We do not get anything interesting by alternating  $\mathfrak{I}^{\pm\mp}$  and  $\tilde{\mathfrak{I}}^{\pm\mp}$  because of the homotopy equivalences:

$$\tilde{\mathfrak{I}}^{+-} \boxtimes \mathfrak{I}^{-+} \sim \mathfrak{I}\mathfrak{d}_{\mathcal{T}^+} \quad \mathfrak{I}^{-+} \boxtimes \tilde{\mathfrak{I}}^{+-} \sim \mathfrak{I}\mathfrak{d}_{\mathcal{T}^-} \quad (7.127)$$

and

$$\mathfrak{I}^{+-} \boxtimes \tilde{\mathfrak{I}}^{-+} \sim \mathfrak{I}\mathfrak{d}_{\mathcal{T}^+} \quad \tilde{\mathfrak{I}}^{-+} \boxtimes \mathfrak{I}^{+-} \sim \mathfrak{I}\mathfrak{d}_{\mathcal{T}^-} \quad (7.128)$$



**Figure 92:** A boundary amplitude for  $\mathfrak{I}^{+-} \boxtimes \tilde{\mathfrak{I}}^{-+}$  contributing to the extended web shown on the right arises from the convolution of boundary amplitudes with the taut web shown on the left. This defines a differential which eliminates the Chan-Paton factors not present in the identity Interface, upon taking cohomology.

The homotopy equivalences (7.127) are straightforward given our previous discussion on categorical parallel transport. The equivalences (7.128) require more discussion. A short computation show that the Chan-Paton data for  $\mathfrak{I}^{+-} \boxtimes \tilde{\mathfrak{I}}^{-+}$  is

$$\mathcal{E}(\mathfrak{I}^{+-} \boxtimes \tilde{\mathfrak{I}}^{-+}) = \mathbb{Z}\mathbf{1} \oplus \bigoplus_{0 \leq k < \frac{N}{2}} (R_{k,N-2-k}^* \oplus R_{N-k-1,k+1}) e_{N-k-1,k+1} \quad (7.129)$$

Plainly, the Chan-Paton data differ from those of the identity Interface.

A glance at Figure 92 shows that the problematic chain complexes  $\mathcal{E}(\mathfrak{I}^{+-} \boxtimes \tilde{\mathfrak{I}}^{-+})_{N-k-1,k+1}$  have a nonzero differential. Indeed, using the symmetry isomorphism and  $\check{K}$  we have a degree zero isomorphism:

$$R_{k,N-2-k}^* \cong R_{k+1,N-k-1}^{[-1]} \quad (7.130)$$

and with this understood the differential given by Figure 92 acts on  $R_{k+1,N-k-1}^{[-1]} \oplus R_{N-k-1,k+1}$  as  $(r^{[-1]}, s) \mapsto (0, r)$ . Thus, the cohomology of the problematic terms in the Chan-Paton data vanishes.

Of course, the above quasi-isomorphism would be a simple consequence of the homotopy equivalence (7.128), but, as we have seen with the homotopy equivalence of  $\mathfrak{S}_{ij}^f \boxtimes \mathfrak{S}_{ij}^p$  with the identity Interface, explained at length in Section §7.6, more discussion is needed to establish a homotopy equivalence. The argument in this case is very similar to that for  $\mathfrak{S}_{ij}^f \boxtimes \mathfrak{S}_{ij}^p$ .

Let us now study the sequence of Branes  $\mathfrak{B}[n]$  for some simple choices of  $\mathfrak{B} = \mathfrak{B}[0]$ . It is already quite interesting for thimbles. To begin, suppose that  $\mathfrak{T}_\ell \in \mathfrak{Br}(\mathcal{T}^+)$  is a thimble with  $1 \leq \ell \leq \frac{N}{2}$ , i.e. a down-vacuum thimble. Then one can easily show that

$$\mathcal{E}(\mathfrak{T}_\ell \boxtimes \mathfrak{I}^{+-})_k = \delta_{k,\ell-1} \mathbb{Z} \quad (7.131)$$

and moreover the boundary amplitudes are all zero. To see this note that since the boundary amplitudes of  $\mathfrak{T}_\ell$  are all zero the only possible boundary amplitude in the convolution would use the amplitude in the upper right of Figure 91, but for  $\ell$  in the range  $1 \leq \ell \leq \frac{N}{2}$  there is no such nonzero amplitude. It follows that we have

$$\mathfrak{T}_\ell \boxtimes \mathfrak{I}^{+-} = \mathfrak{T}_{\ell-1} \quad 1 \leq \ell \leq \frac{N}{2} \quad (7.132)$$

$$\mathfrak{T}_\ell \boxtimes \mathfrak{I}^{-+} = \mathfrak{T}_\ell \quad \mathfrak{T}_\ell \in \mathfrak{Br}(\mathcal{T}^-), \quad 0 \leq \ell \leq \frac{N-1}{2}. \quad (7.133)$$

$$\mathfrak{T}_\ell \boxtimes \tilde{\mathfrak{I}}^{+-} = \mathfrak{T}_\ell \quad \mathfrak{T}_\ell \in \mathfrak{Br}(\mathcal{T}^+), \quad 0 \leq \ell \leq \frac{N-1}{2}. \quad (7.134)$$

$$\mathfrak{T}_\ell \boxtimes \tilde{\mathfrak{I}}^{-+} = \mathfrak{T}_{\ell+1} \quad \mathfrak{T}_\ell \in \mathfrak{Br}(\mathcal{T}^-), \quad 0 \leq \ell \leq \frac{N}{2} - 1. \quad (7.135)$$

where the results (7.133)-(7.135) are obtained in an entirely analogous fashion.

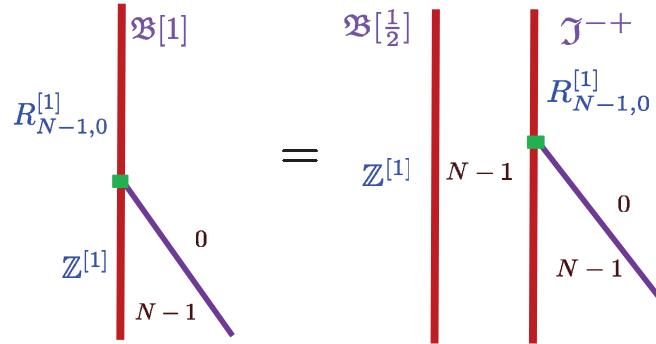
Equations (7.132) and (7.133) cannot be applied to thimbles for upper vacua nor to the case of  $\ell = 0$ . Let us focus on the latter case and consider the more nontrivial sequence of Branes produced when we take  $\mathfrak{B}[0] = \mathfrak{T}_0$ . Then  $\mathfrak{B}[\frac{1}{2}] = \mathfrak{T}_{N-1}^{[1]}$  is a shifted thimble. However, at the next step we find

$$\mathcal{E}(\mathfrak{B}[1])_j = \begin{cases} R_{N-1,0}^{[1]} & j = 0 \\ \mathbb{Z}^{[1]} & j = N-1 \\ 0 & \text{else} \end{cases} \quad (7.136)$$

Moreover,  $\mathfrak{B}[1]$  now acquires a nonzero amplitude for the fan  $J = \{0, N-1\}$  given by  $K_{N-1,0}^{-1} \in R_{N-1,0}^{[1]} \otimes R_{0,N-1} \otimes (\mathbb{Z}^{[1]})^*$ . See Figure 93.

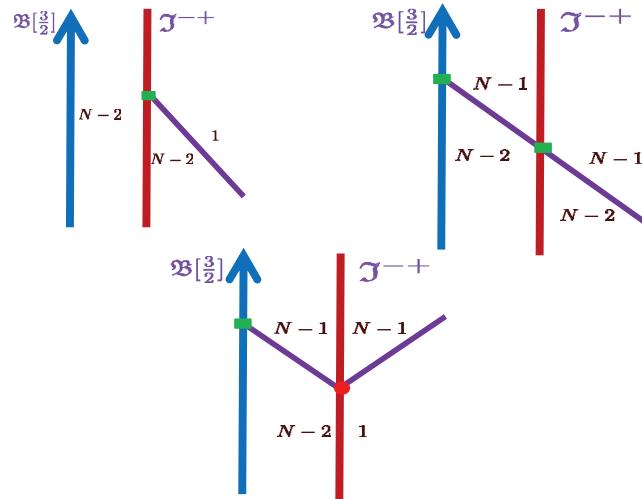
Proceeding to compute  $\mathfrak{B}[\frac{3}{2}] = \mathfrak{B}[1] \boxtimes \mathfrak{I}^{+-}$  we get

$$\mathcal{E}(\mathfrak{B}[\frac{3}{2}])_j = \begin{cases} R_{N-1,1}^{[1]} & j = 0 \\ \mathbb{Z}^{[1]} & j = N-2 \\ R_{N-1,0}^{[2]} & j = N-1 \\ 0 & \text{else} \end{cases} \quad (7.137)$$

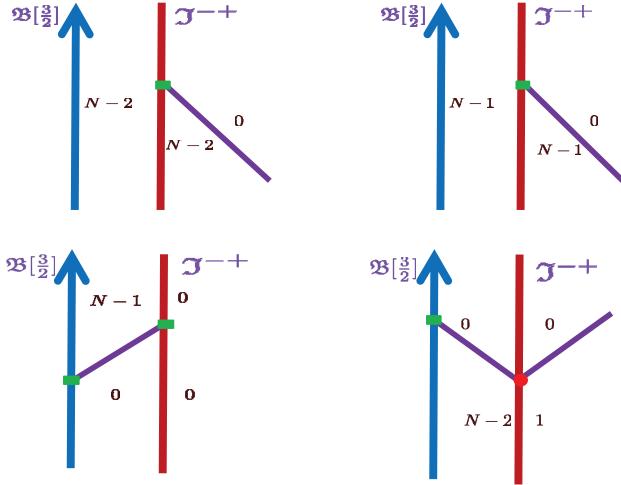


**Figure 93:** This figure shows how a nonzero boundary amplitude can be generated from the thimble.

Moreover, computing the boundary amplitudes we find three fans have nontrivial amplitudes. They can be interpreted as  $K^{-1}$  for fans  $\{0, N-2\}$  and  $\{N-1, N-2\}$  and  $\check{K}(\beta_{0,1,N-1})$  for  $\{N-1, 0\}$ . (These come from using Figure 91, upper right with  $j = N-1$ , lower left with  $k = 0$ ,  $\ell = N-1$ , and lower right, with  $\ell = 0$ ,  $j = N-1$ , respectively.)



**Figure 94:** These amplitudes are common to  $\mathfrak{B}[2]$  and  $\widehat{\mathfrak{B}}[2]$ .



**Figure 95:** These amplitudes are present for  $\mathfrak{B}[2]$  but not for  $\widehat{\mathfrak{B}}[2]$ .

If we move on to  $\mathfrak{B}[2] = \mathfrak{B}[\frac{3}{2}] \boxtimes \mathcal{I}^{-+}$  then

$$\mathcal{E}(\mathfrak{B}[2])_j = \begin{cases} R_{N-1,1}^{[1]} \oplus R_{N-1,0}^{[2]} \otimes R_{N-1,0} & j = 0 \\ R_{N-2,1}^{[1]} & j = 1 \\ \mathbb{Z}^{[1]} & j = N-2 \\ R_{N-1,0}^{[2]} & j = N-1 \\ 0 & \text{else} \end{cases} \quad (7.138)$$

The nontrivial amplitudes are illustrated in Figures 94 and 95. Note that  $\{N-1, 0\}$  is a positive-half-plane fan for the Theory  $\mathcal{T}^-$  but  $\{0, N-1\}$  is a positive-half-plane fan for the Theory  $\mathcal{T}^+$ . Thus the “emission line” in the lower left of Figure 95 does not continue into the positive-half-plane.

We would now like to replace  $\mathfrak{B}[2]$  with a simpler, but homotopy equivalent, Brane, denoted by  $\widehat{\mathfrak{B}}[2]$ . At this stage we must distinguish between the Theories  $\mathcal{T}_\epsilon^N$  and  $\mathcal{T}_\epsilon^{SU(N)}$  since we need to use special properties of the  $R_{ij}$ . We first discuss the case of  $\mathcal{T}_\epsilon^N$ . We then return and pick up the thread for  $\mathcal{T}_\epsilon^{SU(N)}$  at this point.

For  $\mathcal{T}_\epsilon^N$  equation (7.138) simplifies to

$$\mathcal{E}(\mathfrak{B}[2])_j = \begin{cases} \mathbb{Z}^{[1]} \oplus \mathbb{Z}^{[2]} & j = 0 \\ \mathbb{Z}^{[1]} & j = 1 \\ \mathbb{Z}^{[1]} & j = N-2 \\ \mathbb{Z}^{[2]} & j = N-1 \\ 0 & \text{else} \end{cases} \quad (7.139)$$

We want to eliminate the  $j = 0$  Chan-Paton space so we define  $\widehat{\mathfrak{B}}[2]$  to have Chan-Paton data:

$$\mathcal{E}(\widehat{\mathfrak{B}}[2])_j = \begin{cases} \mathbb{Z}^{[1]} & j = 1 \\ \mathbb{Z}^{[1]} & j = N - 2 \\ \mathbb{Z}^{[2]} & j = N - 1 \\ 0 & \text{else} \end{cases} \quad (7.140)$$

with the same boundary amplitudes as in Figure 94.

Now we describe the homotopy equivalence  $\widehat{\mathfrak{B}}[2] \sim \mathfrak{B}[2]$ . Note that with nonempty positive-half-plane fans 0 is always in the future. Moreover,  $\{0, i\}$ ,  $i = 1, N - 2, N - 1$  are all positive-half-plane fans for  $\mathcal{T}^+$ . To construct the homotopy equivalence we need closed morphisms

$$\delta_1 \in \text{Hop}(\widehat{\mathfrak{B}}[2], \mathfrak{B}[2]) = \text{Hop}(\widehat{\mathfrak{B}}[2], \widehat{\mathfrak{B}}[2]) \quad (7.141)$$

$$\delta_2 \in \text{Hop}(\mathfrak{B}[2], \widehat{\mathfrak{B}}[2]) = \text{Hop}(\widehat{\mathfrak{B}}[2], \widehat{\mathfrak{B}}[2]) \oplus \mathcal{D} \quad (7.142)$$

$$\mathcal{D} := \bigoplus_{i=1, N-2, N-1} \mathcal{E}(\mathfrak{B}[2])_0 \otimes \widehat{R}_{0,i} \otimes (\mathcal{E}(\widehat{\mathfrak{B}}[2])_i)^* \quad (7.143)$$

such that the products  $M_2(\delta_1, \delta_2)$  and  $M_2(\delta_2, \delta_1)$  are homotopy equivalent to **Id**:

$$\begin{aligned} M_2(\delta_1, \delta_2) &= \mathbf{Id}_{\widehat{\mathfrak{B}}[2]} + M_1(\delta_3) \\ M_2(\delta_2, \delta_1) &= \mathbf{Id}_{\mathfrak{B}[2]} + M_1(\delta_4). \end{aligned} \quad (7.144)$$

It will be useful below to compare the boundary amplitudes of  $\mathfrak{B}[2]$  and  $\widehat{\mathfrak{B}}[2]$  and write

$$\mathcal{B}(\mathfrak{B}[2]) = \mathcal{B}(\widehat{\mathfrak{B}}[2]) + \Delta \mathcal{B} \quad (7.145)$$

The multiplications  $M_k$  are computed using the taut half-plane webs, and the only ones with at least two boundary vertices in fact have at most two boundary vertices. (See, for example, Figure 31 for the case of unextended webs.) This property considerably simplifies the computation of  $M_2$ . To begin we take  $\delta_1 = \mathbf{Id}_{\widehat{\mathfrak{B}}[2]}$  and  $\delta_2 = \mathbf{Id}_{\widehat{\mathfrak{B}}[2]} \oplus 0$ , where the direct sum refers to the decomposition in (7.142). The equation  $M_2(\delta_1, \delta_2) = \mathbf{Id}_{\widehat{\mathfrak{B}}[2]}$  works nicely with  $\delta_3 = 0$ . On the other hand, since  $\mathfrak{B}[2]$  has a nonzero Chan-Paton space for  $i = 0$  and  $\widehat{\mathfrak{B}}[2]$  does not,  $M_2(\delta_2, \delta_1)$  cannot possibly reproduce  $\mathbf{Id}_{\mathcal{E}(\mathfrak{B}[2])_0}$ . Therefore,  $\delta_4 \in \text{Hop}(\mathfrak{B}[2], \mathfrak{B}[2])$  must be nonzero. We take it to have only nonzero scalar component in  $\text{End}(\mathcal{E}(\mathfrak{B}[2])_0)$ . Then the scalar component of the differential is simply given by matrix multiplication

$$M_1(\delta_4) = \mathcal{B}_{00}\delta_4 + \delta_4\mathcal{B}_{00} \quad (7.146)$$

where  $\mathcal{B}_{00} \in \text{End}(\mathcal{E}(\mathfrak{B}[2])_0)$  is the boundary amplitude induced by the lower left diagram of Figure 95. Writing vectors in  $\mathcal{E}(\mathfrak{B}[2])_0$  in the form  $r_1 \oplus r_2$  where  $r_1 \in \mathbb{Z}^{[1]}$  and  $r_2 \in \mathbb{Z}^{[2]}$  we easily compute that the boundary amplitude is the linear transformation:

$$\mathcal{B}_{00} : r_1 \oplus r_2 \mapsto 0 \oplus r_1^{[1]}. \quad (7.147)$$

Therefore, we can take  $\delta_4$  to be the chain-homotopy inverse

$$\delta_4 : r_1 \oplus r_2 \mapsto r_2^{[-1]} \oplus 0 \quad (7.148)$$

so that the scalar component of  $M_1(\delta_4)$  is the missing component  $\mathbf{Id}_{\mathcal{E}(\mathfrak{B}[2])_0}$ . There will be further contributions from  $M_1(\delta_4)$ , producing amplitudes with positive half-plane fans of type  $\{0, 1\}$ ,  $\{0, N - 2\}$ , and  $\{0, N - 1\}$ . They are given (essentially) by  $\Delta\mathcal{B}$ , and can be cancelled by adding (essentially)  $-\Delta\mathcal{B}$  to  $\delta_2$ . Now we have established the required homotopy equivalence  $\widehat{\mathfrak{B}}[2] \sim \mathfrak{B}[2]$ .

In what follows we will need to employ repeatedly a maneuver very similar to what we just explained: We will find a pair of Branes  $\mathfrak{B}$  and  $\widehat{\mathfrak{B}}$  all of whose Chan-Paton spaces are identical except for one vacuum  $j_*$  (the vacuum  $j_* = 0$  in the example above) and whose boundary amplitudes are identical for all amplitudes not involving this distinguished vacuum. Moreover, we have

$$\mathcal{E}(\mathfrak{B})_{j_*} = \mathcal{E}(\widehat{\mathfrak{B}})_{j_*} \oplus V \oplus V^{[1]} \quad (7.149)$$

and there is a boundary amplitude in  $\text{Hop}(\mathfrak{B}, \mathfrak{B})$  in  $\text{End}(\mathcal{E}(\mathfrak{B})_{j_*})$  taking

$$v_1 \oplus v_2 \oplus v_3 \rightarrow v_1 \oplus 0 \oplus v_2^{[1]} \quad (7.150)$$

In this case we can find a homotopy equivalence  $\widehat{\mathfrak{B}} \sim \mathfrak{B}$ , exactly as in the above example. This will allow us to replace  $\mathfrak{B}$  by the simpler Brane  $\widehat{\mathfrak{B}}$ . We call this the *cancellation lemma* below.

Returning to our sequence of Branes  $\mathfrak{B}[n]$  in the Theory  $\mathcal{T}_\vartheta^N$  we next observe that  $\widehat{\mathfrak{B}}[2]$  is the nontrivial Brane  $\mathfrak{C}_{k=1}^{[1]}$  of equation (4.91) above. We now proceed inductively. Suppose that  $k < \frac{N}{2}$  and that  $\mathfrak{B}[k]$  is homotopy equivalent to  $\widehat{\mathfrak{B}}[k] = \mathfrak{C}_{k-1}^{[1]}$ . Therefore

$$\mathcal{E}(\widehat{\mathfrak{B}}[k])_j = \begin{cases} \mathbb{Z}^{[1]} & j = k - 1 \\ \mathbb{Z}^{[1]} & j = N - k \\ \mathbb{Z}^{[2]} & j = N - k + 1 \\ 0 & \text{else} \end{cases} \quad (7.151)$$

so we compute

$$\mathcal{E}(\widehat{\mathfrak{B}}[k] \boxtimes \mathfrak{I}^{+-})_j = \begin{cases} \mathbb{Z}^{[1]} \oplus \mathbb{Z}^{[2]} & j = k - 2 \\ \mathbb{Z}^{[1]} & j = k - 1 \\ \mathbb{Z}^{[1]} & j = N - k - 1 \\ \mathbb{Z}^{[2]} & j = N - k \\ 0 & \text{else} \end{cases} \quad (7.152)$$

Once again there is a component of the boundary amplitude which acts as a differential on the Chan-Paton space with  $j = k - 2$  and eliminates it upon passing to cohomology. Our cancellation lemma allows us to replace this Brane with a homotopy equivalent Brane

$\widehat{\mathfrak{B}}[k + \frac{1}{2}]$  with

$$\mathcal{E}(\widehat{\mathfrak{B}}[k + \frac{1}{2}])_j = \begin{cases} \mathbb{Z}^{[1]} & j = k - 1 \\ \mathbb{Z}^{[1]} & j = N - k - 1 \\ \mathbb{Z}^{[2]} & j = N - k \\ 0 & \text{else} \end{cases} \quad (7.153)$$

Now we compute again

$$\mathcal{E}(\widehat{\mathfrak{B}}[k + \frac{1}{2}] \boxtimes \mathfrak{I}^{-+})_j = \begin{cases} \mathbb{Z}^{[1]} \oplus \mathbb{Z}^{[2]} & j = k - 1 \\ \mathbb{Z}^{[1]} & j = k \\ \mathbb{Z}^{[1]} & j = N - k - 1 \\ \mathbb{Z}^{[2]} & j = N - k \\ 0 & \text{else} \end{cases} \quad (7.154)$$

and again the cancellation lemma gives us

$$\widehat{\mathfrak{B}}[k + \frac{1}{2}] \boxtimes \mathfrak{I}^{-+} \sim \widehat{\mathfrak{B}}[k + 1] = \mathfrak{C}_k^{[1]}. \quad (7.155)$$

completing the inductive step.

The inductive step works until we produce  $\widehat{\mathfrak{B}}[k] \in \mathfrak{Br}(\mathcal{T}^+)$  for  $k = [\frac{N}{2}]$ . For simplicity assume first that  $N$  is even. Then we compute

$$\mathcal{E}(\widehat{\mathfrak{B}}[\frac{N}{2}] \boxtimes \mathfrak{I}^{+-})_j = \begin{cases} \mathbb{Z}^{[1]} \oplus \mathbb{Z}^{[2]} & j = \frac{N}{2} - 2 \\ \mathbb{Z}^{[1]} & j = \frac{N}{2} - 1 \\ \mathbb{Z}^{[2]} & j = \frac{N}{2} \\ 0 & \text{else} \end{cases} \quad (7.156)$$

The cancellation lemma produces a homotopy equivalent Brane with Chan-Paton factors

$$\mathcal{E}(\widehat{\mathfrak{B}}[\frac{N}{2} + \frac{1}{2}])_j = \begin{cases} \mathbb{Z}^{[1]} & j = \frac{N}{2} - 1 \\ \mathbb{Z}^{[2]} & j = \frac{N}{2} \\ 0 & \text{else} \end{cases} \quad (7.157)$$

Then we compute again

$$\mathcal{E}(\widehat{\mathfrak{B}}[\frac{N}{2} + \frac{1}{2}] \boxtimes \mathfrak{I}^{-+})_j = \begin{cases} \mathbb{Z}^{[1]} \oplus \mathbb{Z}^{[2]} & j = \frac{N}{2} - 1 \\ \mathbb{Z}^{[2]} & j = \frac{N}{2} \\ 0 & \text{else} \end{cases} \quad (7.158)$$

which, by the cancellation lemma, is homotopy equivalent to a shifted thimble! That is, we have  $\widehat{\mathfrak{B}}[\frac{N}{2} + 1] = \mathfrak{T}_{N/2}^{[2]}$ .

Now using equations (7.132) and (7.133) we can continue the procedure to produce a sequence of thimbles of down-type vacua, until we get to  $\mathfrak{T}_0^{[2]}$ .

It follows from the above discussion that

$$\mathfrak{T}_\ell \boxtimes (\mathfrak{I}^{+-} \boxtimes \mathfrak{I}^{-+})^{N+1} \sim \mathfrak{T}_\ell \quad 0 \leq \ell \leq \frac{N}{2} \quad (7.159)$$

a similar story holds if  $N$  is odd. In this case the induction works until we get to  $\widehat{\mathfrak{B}}[\frac{N-1}{2}]$ . Then  $\widehat{\mathfrak{B}}[\frac{N+1}{2}]$  has two Chan-Paton factors  $\mathbb{Z}^{[1]}$  and  $\mathbb{Z}^{[2]}$  at  $j = (N-1)/2$  and  $j = (N+1)/2$ , respectively, and  $\widehat{\mathfrak{B}}[\frac{N+3}{2}] = \mathfrak{T}_{(N-1)/2}^{[2]}$ .

We have not worked out the sequence of Branes generated by thimbles for upper vacua.

We next turn our attention to the Theory  $\mathcal{T}_\epsilon^{SU(N)}$ . We begin with the sequence of Branes  $\mathfrak{B}[k]$ ,  $k \geq 0$  generated by the thimble  $\mathfrak{T}_0$  in the Theory  $\mathcal{T}_\epsilon^{SU(N)}$ . We have already described the general story up to the Brane  $\mathfrak{B}[2]$ , as in equation (7.138). Now, however we have

$$\begin{aligned} \mathcal{E}(\mathfrak{B}[2])_0 &= R_{N-1,1}^{[1]} \oplus R_{N-1,0}^{[2]} \otimes R_{N-1,0} \\ &= A_2^{[1]} \oplus A_1^{[2]} \otimes A_1 \end{aligned} \quad (7.160)$$

Next  $A_1^{[2]} \otimes A_1 \cong A_2^{[2]} \oplus S_2^{[2]}$ . It is natural to expect that the differential computed by the lower left diagram of Figure 95 maps  $A_2^{[1]} \rightarrow A_2^{[2]}$  as a degree shift, since we know the amplitudes are all  $SU(N)$ -covariant. We will assume this to be the case and proceed, although we have not checked the boundary amplitudes in detail.

Passing to cohomology we eliminate the two summands of  $A_2$  and using the cancellation lemma we claim a homotopy equivalence to  $\widehat{\mathfrak{B}}[2]$  with

$$\mathcal{E}(\widehat{\mathfrak{B}}[2])_j = \begin{cases} S_2^{[2]} & j = 0 \\ A_3^{[1]} & j = 1 \\ \mathbb{Z}^{[1]} & j = N-2 \\ A_1^{[2]} & j = N-1 \\ 0 & \text{else} \end{cases} \quad (7.161)$$

Referring to equation (4.117) we identify these as the Chan-Paton factors of  $\mathfrak{N}_1^{[1]}$ . We expect that the boundary amplitudes coincide with those of  $\mathfrak{N}_1^{[1]}$ , although we have not checked in detail. In this and similar equations below we must interpret

$$\begin{aligned} L_{n,m} &= 0 & n > 1, m \leq 0 \\ L_{1,0} &= S_0 \cong \mathbb{Z} & \\ S_m &= 0 & m < 0 \end{aligned} \quad (7.162)$$

Again, we can proceed inductively. Suppose that  $\mathfrak{B}[n] \sim \widehat{\mathfrak{B}}[n]$  where  $\widehat{\mathfrak{B}}[n+1] = \mathfrak{N}_n^{[1]}$ . It is useful to employ the isomorphism  $L_{N,m+1} \cong S_m$  and rewrite (4.117) as

$$\mathcal{E}(\mathfrak{N}_n)_j = \begin{cases} L_{2j+1,n+1-j}^{[n-j]} & 0 \leq j < \frac{N-1}{2} \\ S_{n+j+1-N}^{[n+j+1-N]} & \frac{N-1}{2} \leq j \leq N-1 \end{cases} \quad (7.163)$$

The boundary amplitudes were described in section §4.6. Now it is straightforward to compute

$$\mathcal{E}(\mathfrak{N}_n \boxtimes \mathfrak{I}^{+-})_j = \begin{cases} L_{2j+3,n-j}^{[n-j-1]} \oplus S_{n-j}^{[n-j]} \otimes R_{N-j-1,j+1} & 0 \leq j < \frac{N}{2} - 1 \\ S_{n+j+2-N}^{[n+j+2-N]} & \frac{N}{2} - 1 \leq j \leq N - 1 \end{cases} \quad (7.164)$$

Thus, using (4.100) and (4.121) we learn that for  $0 \leq j < \frac{N}{2} - 1$

$$\mathcal{E}(\mathfrak{N}_n \boxtimes \mathfrak{I}^{+-})_j \cong L_{2j+3,n-j}^{[n-j-1]} \oplus L_{2j+2,n-j+1}^{[n-j]} \oplus L_{2j+3,n-j}^{[n-j]} \quad (7.165)$$

Using the cancellation lemma we claim a homotopy equivalence to  $\widehat{\mathfrak{B}}[n + \frac{3}{2}]$  with

$$\mathcal{E}(\widehat{\mathfrak{B}}[n + \frac{3}{2}])_j = \begin{cases} L_{2j+2,n-j+1}^{[n-j]} & 0 \leq j < \frac{N}{2} - 1 \\ S_{n+j+2-N}^{[n+j+2-N]} & \frac{N}{2} - 1 \leq j \leq N - 1 \end{cases} \quad (7.166)$$

Then a similar argument shows that  $\widehat{\mathfrak{B}}[n + \frac{3}{2}] \boxtimes \mathfrak{I}^{-+} \sim \mathfrak{N}_{n+1}^{[1]}$  thus completing the inductive step. Unlike the case of the Theory  $\mathcal{T}_\epsilon^N$ , the sequence does not simplify and there is no periodicity as we increase  $n$ . Physically,  $n$  is related to a first Chern class of an equivariant bundle on  $\mathbb{CP}^{N-1}$ , and no such periodicity is expected. See Section §7.10.1 below for further discussion of this non-periodicity.

If, instead, we use the interfaces  $\tilde{\mathfrak{I}}^{+-}, \tilde{\mathfrak{I}}^{-+}$  we find that  $n$  is reduced by successive composition. If  $\mathfrak{B}[0] = \mathfrak{N}_n$  with  $n > \frac{N}{2}$  then the recursion relations (7.125) give a sequence of Branes  $\widehat{\mathfrak{B}}[\ell]$  for  $\ell < 0$  and  $\ell \in \mathbb{Z} + \frac{1}{2}$ . With this sequence we find for  $\ell$  negative integral (and not too negative)  $\widehat{\mathfrak{B}}[\ell] \sim \mathfrak{N}_{n+\ell}$ :

$$\mathfrak{N}_n \boxtimes (\tilde{\mathfrak{I}}^{+-} \boxtimes \tilde{\mathfrak{I}}^{-+})^{|\ell|} \sim \mathfrak{N}_{n+\ell} \quad (7.167)$$

At  $n - |\ell| = \frac{N}{2} - 1$  the Chan-Paton factor becomes  $\mathbb{Z}$  for  $j = \frac{N}{2}$ . Proceeding to lower  $N$  the usual cancellation argument in expressions like (7.165) produces zero. Making use of equation (7.162) note that equation (7.163) makes sense for  $n \geq -1$  and we can proceed to reduce  $n$  until we get to  $\mathfrak{N}_{-1} = \mathfrak{T}_0^{[-1]}$ . We can then continue the recursion (7.125) using (7.134) and (7.135) to define  $\mathfrak{N}_n$  for lower values of  $n$  in terms of down-type thimbles. The process then stops when we arrive at  $\mathfrak{N}_{-1-N/2} = \mathfrak{T}_{N/2}^{[-1]}$  (taking  $N$  to be even, for simplicity). At this point we recall the Branes  $\overline{\mathfrak{N}}_n$  of section §4.6 with

$$\mathcal{E}(\overline{\mathfrak{N}}_n)_j = \begin{cases} \overline{L}_{N-2j,n+j+1}^{[-j-n]} & 0 \leq j < \frac{N}{2} \\ \overline{S}_{n+N-j}^{[j+1-n-N]} & \frac{N}{2} \leq j \leq N - 1 \end{cases} \quad (7.168)$$

which makes sense for  $n \geq -N/2$ . Note that  $\overline{\mathfrak{N}}_{-N/2} = \mathfrak{T}_{N/2}^{[1]}$ . The usual arguments then show that

$$\overline{\mathfrak{N}}_n \boxtimes (\tilde{\mathfrak{I}}^{+-} \boxtimes \tilde{\mathfrak{I}}^{-+})^\ell \sim \overline{\mathfrak{N}}_{n+\ell} \quad (7.169)$$

and hence we can continue to define  $\mathfrak{N}_n$  for values below  $-N/2 - 1$  by taking

$$\mathfrak{N}_{-\frac{N}{2}-n} \cong \overline{\mathfrak{N}}_{-1-\frac{N}{2}+n}^{[-2]} \quad n \geq 1 \quad (7.170)$$

(We have not checked the above equations at the level of boundary amplitudes.)

The relation of the Branes  $\mathfrak{N}_n$  to thimbles for a certain range of  $n$  allows us to fill in a gap from Section 5.7, namely the proof of equation (5.92). The key idea is that, thanks to the  $A_\infty$ -bifunctor of Section §6.2 the complex between two Branes  $\text{Hop}(\mathfrak{B}_1, \mathfrak{B}_2)$  is quasi-isomorphic to the complex obtained by composition with an invertible interface. If we apply that to the present case then we can write, for the case of two lower vacua  $i, j < N/2$

$$\begin{aligned}\text{Hop}(\mathfrak{T}_i, \mathfrak{T}_j) &= \text{Hop}(\mathfrak{N}_{-1-i}^{[1]}, \mathfrak{N}_{-1-j}^{[1]}) \\ &=_{q.i.} \text{Hop}(\mathfrak{N}_{-1}^{[1]}, \mathfrak{N}_{-1-j+i}^{[1]}) \\ &= \text{Hop}(\mathfrak{T}_0, \mathfrak{N}_{-1+i-j}^{[1]})\end{aligned}\tag{7.171}$$

where  $=_{q.i.}$  means we have a quasi-isomorphism.

Now assume for simplicity that  $N$  is even. We next use the observation of equation (7.53) to say that,

$$H^*(\text{Hop}(\mathfrak{T}_i, \mathfrak{T}_j), M_1) \cong H^*(\mathcal{E}_{LR}(\mathfrak{T}_0, \mathfrak{N}_{-1+i-j}^{[1]}[\pi]), d_{LR})\tag{7.172}$$

Recall from the discussion of equation (7.53) that we should rotate in the direction of increasing  $\vartheta$ , so we should compose with  $(\tilde{\mathfrak{I}}^{+-} \boxtimes \tilde{\mathfrak{I}}^{-+})^{N/2}$  to rotate the Brane by  $\pi$  and hence

$$\mathfrak{N}_{-1+i-j}^{[1]}[\pi] = \mathfrak{N}_{-1+i-j-N/2}^{[1]}\tag{7.173}$$

The complex of groundstates is very simple when one of the Branes is a thimble. In this case there is no differential and the cohomology is the Chan-Paton factor itself. In our case

$$H^*(\mathcal{E}_{LR}(\mathfrak{T}_0, \mathfrak{N}_{-1+i-j-N/2}^{[1]}), d_{LR}) \cong \left( \mathcal{E}(\mathfrak{N}_{-1+i-j-N/2}^{[1]})_{N/2} \right)^*\tag{7.174}$$

Note that although we have the thimble for the vacuum  $i = 0$  on the left side of the strip, thanks to the rotation by  $\pi$  we should take the Chan-Paton space with vacuum  $N/2$  on the right-Brane. Next, assuming that  $j > i$  we can use equation (7.170) to say

$$\mathcal{E}(\mathfrak{N}_{-1+i-j-N/2}^{[1]})_{N/2} = \mathcal{E}(\overline{\mathfrak{N}}_{j-i-N/2}^{[-1]})_{N/2}\tag{7.175}$$

Finally, using (7.168) we have

$$\mathcal{E}(\overline{\mathfrak{N}}_{j-i-N/2}^{[-1]})_{N/2} = \overline{S}_{j-i}^{[i-j]}\tag{7.176}$$

Putting together equations (7.171)-(7.176) we finally arrive at a proof of equation (5.92).

We expect that similar manipulations allow a computation of the cohomologies of the groundstate complexes such as (4.128) and the spaces of local operators in (5.93) and (5.94).

### 7.10.1 Powers Of The Rotation Interface

It is rather interesting to examine the powers of the Interface  $\mathfrak{I}^{++-} := \mathfrak{I}^{+-} \boxtimes \mathfrak{I}^{-+}$  that corresponds to a rotation by  $2\pi/N$  in the worldvolume of the Theory. We will discuss this at the level of Chan-Paton factors, without investigating the boundary amplitudes.

The Chan-Paton data of  $\mathfrak{I}^{++-}$  is the matrix of complexes:

$$\begin{aligned} \mathcal{E}(\mathfrak{I}^{++-}) &= R_{N-1,0}^{[1]} e_{0,0} \oplus \bigoplus_{j=0}^{N-2} \mathbb{Z} e_{j+1,j} \oplus \mathbb{Z}^{[1]} e_{0,N-1} \\ &\oplus \bigoplus_{1 \leq j < (N-1)/2} R_{N-1-j,j} e_{N-j,j} \oplus \bigoplus_{0 \leq j < (N-2)/2} R_{N-1-j,j+1} e_{N-1-j,j} \end{aligned} \quad (7.177)$$

Explicitly, for  $N = 2$  this is

$$\mathcal{E}(\mathfrak{I}^{++-}) = \begin{pmatrix} R_{1,0}^{[1]} & \mathbb{Z}^{[1]} \\ \mathbb{Z} & 0 \end{pmatrix} \quad (7.178)$$

and for  $N = 3$ ,

$$\mathcal{E}(\mathfrak{I}^{++-}) = \begin{pmatrix} R_{2,0}^{[1]} & 0 & \mathbb{Z}^{[1]} \\ \mathbb{Z} & 0 & 0 \\ R_{2,1} & \mathbb{Z} & 0 \end{pmatrix} \quad (7.179)$$

The equation (7.159) suggests that for  $\mathcal{T}_\vartheta^N$  the  $(N+1)^{th}$  power is homotopy equivalent to a (shifted) Identity interface. Indeed, one easily checks that the third power of (7.178) is just

$$\begin{pmatrix} \mathbb{Z}^{[2]} \oplus \mathbb{Z}^{[2]} \oplus \mathbb{Z}^{[3]} & \mathbb{Z}^{[2]} \oplus \mathbb{Z}^{[3]} \\ \mathbb{Z}^{[2]} \oplus \mathbb{Z}^{[3]} & \mathbb{Z}^{[2]} \end{pmatrix} \quad (7.180)$$

and is quasi-isomorphic to  $\mathbb{Z}^{[2]} \mathbf{1}_2$ . Similarly, a check by hand shows that the fourth power of (7.179) is quasi-isomorphic to  $\mathbb{Z}^{[2]} \mathbf{1}_3$ , and we conjecture that for all  $N$ ,  $(\mathfrak{I}^{++-})^{\boxtimes(N+1)}$  is homotopy equivalent to the isomorphism Interface given by a degree shift of 2. There is a simple intuitive explanation in the LG theory 4.138 for this result. The effect of convolution with  $\mathfrak{I}^{+-}$  or  $\mathfrak{I}^{-+}$  on geometric branes simply rotates the sectors at infinity by one unit and deforms the geometric brane accordingly by a rigid rotation by  $2\pi/(N+1)$  in the  $\phi$  plane.<sup>42</sup> As there are  $2N+2$  sectors, the  $(N+1)$  power of the interface  $\mathfrak{I}^{+-} \mathfrak{I}^{-+}$  acts geometrically on the branes by rotating it by  $2\pi$  back to itself in the  $\phi$  plane.

In fact, the characteristic polynomial of  $\mathcal{E}(\mathfrak{I}^{++-})$  is given by the remarkable formula:

$$\det(x\mathbf{1}_N - \mathcal{E}(\mathfrak{I}^{++-})) = x^N + \sum_{j=1}^{N-1} R_j x^j + 1 \quad (7.181)$$

where we interpret a shift by [1] as a minus sign and we use the property that  $R_{a,b} = R_{a-b}$  only depends on the difference  $a - b$ . For a proof see Appendix §D

For the Theory  $\mathcal{T}_\vartheta^N$  we have  $R_j = \mathbb{Z}$ , and hence the “eigenBranes” of  $\mathfrak{I}^{++-}$  have eigenvalues given by the  $N$  nontrivial  $(N+1)^{th}$  roots of unity. This proves that  $(\mathfrak{I}^{++-})^{N+1}$  is homotopy equivalent to the identity (up to an even degree shift).

---

<sup>42</sup>Do not confuse this with the origin of the Interfaces from rotations by  $2\pi/N$  in the  $(x, \tau)$  plane.

We can also apply equation (7.181) to the Theory  $\mathcal{T}_\vartheta^{SU(N)}$ . Now we have  $R_j = A_{N-j}$  so, at the level of the Witten index we can factorize equation (7.181) to get the character

$$\prod_{i=1}^N (x + t_i) \tag{7.182}$$

where  $t = \text{Diag}\{t_1, \dots, t_N\}$  is a generic element in the diagonal Cartan subgroup of  $SU(N)$ . It is natural to suspect that the “eigen-Branes” can be interpreted in the  $\mathbb{CP}^{N-1}$   $B$ -model as Dirichlet branes located at the  $N$  fixed points of the natural  $SU(N)$  action on the homogeneous coordinates, with eigenvalue  $t_i$ . The result (7.182) will be very useful when we discuss local operators in Section §9.3 below.

## 8. Categorical Transport And Wall-Crossing

### 8.1 Preliminary Remarks

We now return to the general situation discussed at the beginning of Section §7. In Section §7 we considered in detail categorical transport of Brane categories associated to paths of weights  $\varphi$  given by spinning weights (7.4). In this section we consider more general vacuum homotopies. In particular we will consider three kinds of vacuum homotopies:

1. Vacuum homotopies  $\{z_i(s)\}$  which are more general than (7.4) but do not cross the real codimension one walls of special webs described in Section §2.5. In this case the webs behave in a very similar way to those of (7.4). We will call these *tame vacuum homotopies*. They are discussed in Section §8.2.
2. Vacuum homotopies  $\{z_i(s)\}$  which cross the exceptional walls described in §2.5 above. This is discussed in Section §8.3.
3. Vacuum homotopies  $\{z_i(s)\}$  which cross walls of marginal stability described in §2.5. This is discussed in Section §8.4.

In Section §7 we constructed Interfaces  $\mathfrak{I}[\vartheta(x)] \in \mathfrak{Br}(\mathcal{T}^\ell, \mathcal{T}^r)$ . Given  $\mathcal{T}^\ell$  there was a canonical choice for  $\mathcal{T}^r$  given by taking “constant” web representation  $\mathcal{R}$  and interior amplitude  $\beta$ . In this section we will see that the more general paths of weights listed above make a canonical determination of  $\mathcal{T}^r$  given  $\mathcal{T}^\ell$  somewhat more problematical. The reason for this is that there can be *wall-crossing* phenomena associated to the data  $(\mathcal{R}, \beta)$  used to define a Theory. In particular, we will see that if  $\varphi(s)$  crosses an exceptional wall then the  $L_\infty$  algebra of closed webs will in general change because the taut element will in general change. Therefore, in general the interior amplitude will change. If  $\varphi(s)$  crosses a wall of marginal stability then the set of cyclic fans will change and hence  $R^{\text{int}}$  must change. Indeed, in general when crossing a wall of marginal stability both the interior amplitude and the web representation  $\mathcal{R}$  will change. The rules for the discontinuity of  $\mathcal{R}$  lead to a categorification of the Cecotti-Vafa-Kontsevich-Soibelman wall-crossing formula.

We will now make the notion of a “change of Theory” somewhat more precise. Given a vacuum homotopy  $\varphi : \mathbb{R} \rightarrow \mathbb{C}^V - \Delta$  we say that a family of Theories  $\mathcal{T}(s)$  is continuously

defined over  $\varphi$  if for all  $i \neq j$  the  $R_{ij}$  form a continuous vector bundle with connection over  $\varphi$  such that  $K_{ij}$  and  $\beta$  are parallel-transported. If we can and do trivialize the bundle with connection then  $R_{ij}, K_{ij}, \beta$  are all constant. As mentioned above, when the path  $\varphi(s)$  crosses walls with special configurations of weights there will be obstructions to defining a continuous family of theories over that path. Instead, we can only define *piecewise-continuous* paths of theories over  $\varphi$ . Naively, the only discontinuities are located at the walls of special weights described in Section §2.5. We will assume this for the moment, but that assumption will need to be revised for reasons described in Remark 2 below. A formula for the discontinuity of  $\mathcal{T}$  is a *wall-crossing formula*. Let  $\mathcal{T}^-$  denote the theory just before the wall and let  $\mathcal{T}^+$  denote the theory just after the wall. Thus, a wall-crossing formula is, in its simplest incarnation, just a prescription for determining  $(\mathcal{R}^+, \beta^+)$  from  $(\mathcal{R}^-, \beta^-)$ .

Given such a wall-crossing rule, if we have path of vacuum weights  $\varphi$  then, given  $\mathcal{T}^\ell$  we can construct a corresponding piecewise-continuous path  $\mathcal{T}(s)$  of theories. The wall-crossing rule should then be constrained by requiring that the path  $\mathcal{T}(s)$  behave suitably with respect to homotopy and concatenation of paths of vacuum weights  $\varphi$ . Heuristically speaking, we want to define a “flat connection on theories.” However a little thought quickly shows such a parallel transport rule must be defined on a suitable equivalence class of theories. In order to motivate the relevant notion of equivalence let us say a little more about how we propose to approach the wall-crossing formula.

As in Section §7 our theme will be to interpret the variation of parameters  $\mathcal{T}(s)$  as spatial-variation of parameters, so we will have spatially dependent vacuum weights  $\varphi(x)$  and spatially-dependent data of theories  $\mathcal{T}(x)$ . Then it is quite natural to interpret a discontinuity of theories across some point  $x_*$  in terms of a suitable “wall-crossing Interface”  $\mathfrak{I}^{wc} \in \mathfrak{Br}(\mathcal{T}^-, \mathcal{T}^+)$ .

Let us make this slightly more precise. We assume that there exist  $x_\ell$  and  $x_r$  so that  $\varphi(x)$  is constant for  $x \leq x_\ell$  and  $x \geq x_r$ . Choose such points and let  $z^\ell : \mathbb{V} \rightarrow \mathbb{C}$  and  $z^r : \mathbb{V} \rightarrow \mathbb{C}$  be the corresponding weight functions in these regions. Then there should be a corresponding piecewise-continuous family of theories  $\mathcal{T}(x)$  interpolating between  $\mathcal{T}^\ell$  and  $\mathcal{T}^r$  together with an interface

$$\mathfrak{I}[\mathcal{T}(x)] \in \mathfrak{Br}(\mathcal{T}^\ell, \mathcal{T}^r) \quad (8.1)$$

generalizing equation (7.46). As before, such a family of interfaces allows us to define a functor of brane categories  $\mathcal{F} : \mathfrak{Br}(\mathcal{T}^\ell) \rightarrow \mathfrak{Br}(\mathcal{T}^r)$  and hence define a categorical transport law on brane categories.

When trying to construct the relevant interfaces we will keep in mind the following three useful guiding principles:

1. The interfaces for paths such that  $z_{ij}(x)$  is never pure imaginary should already define functors between the vacuum categories  $\mathfrak{Vac}(\mathbb{V}, z^\ell)$  and  $\mathfrak{Vac}(\mathbb{V}, z^r)$ . In particular the Chan-Paton factors of  $\mathfrak{I}[\mathcal{T}(x)]$  will be  $\mathcal{E}_{ii'} = \delta_{ii'} \mathbb{Z}$ .
2. We must have properties (7.2) and (7.3): First, if there are two piecewise-continuous families of theories  $\mathcal{T}^1(x)$  and  $\mathcal{T}^2(x)$  that can be concatenated at a point of continuity

then

$$\mathfrak{I}[\mathcal{T}^1(x)] \boxtimes \mathfrak{I}[\mathcal{T}^2(x)] \sim \mathfrak{I}[\mathcal{T}^1 \circ \mathcal{T}^2(x)]. \quad (8.2)$$

Second, let  $\mathfrak{E}$  be the exceptional set of weights in  $\mathbb{C}^\mathbb{V} - \Delta$  described in Section §2.5. That is, the subset  $\{z_i\}$  where some subset of three or more weights is colinear, or where there are exceptional webs. Then, if  $\wp^p(x)$  and  $\wp^f(x)$  are paths in  $\mathbb{C}^\mathbb{V} - \Delta - \mathfrak{E}$  homotopic in  $\mathbb{C}^\mathbb{V} - \Delta - \mathfrak{E}$  through a homotopy keeping fixed  $(\mathbb{V}, z^\ell)$  and  $(\mathbb{V}, z^r)$  for  $x \leq x^\ell$  and  $x \geq x^r$  and  $\mathcal{T}^p(x)$  and  $\mathcal{T}^f(x)$  are corresponding paths of Theories then

$$\mathfrak{I}[\mathcal{T}^p(x)] \sim \mathfrak{I}[\mathcal{T}^f(x)] \quad (8.3)$$

are homotopy equivalent Interfaces. As before, given such Interfaces we have a notion of flat parallel transport along  $\wp$  from the category of Branes  $\mathfrak{Br}(\mathcal{T}^\ell)$  to the category of Branes  $\mathfrak{Br}(\mathcal{T}^r)$ , generalizing what was constructed in Section §7.

3. Because the underlying physical theory is rotationally invariant the Interfaces should come in a  $\vartheta$ -dependent family, intertwined by the rotation interfaces. To be more precise, for any path  $\vartheta(x)$  of real numbers from 0 to  $\vartheta$  we can concatenate the path  $\wp(x)$  from  $\{z_i^\ell\}$  to  $\{z_i^r\}$  with a path  $e^{-i\vartheta(x)} z_i^r$  from  $z_i^r$  to  $e^{-i\vartheta} z_i^r$ . Alternatively we can concatenate the path  $e^{-i\vartheta(x)} z_i^\ell$  with the path  $e^{-i\vartheta} \wp(x)$ . The homotopy  $z_i(x) e^{-i\vartheta(y)}$  shows that these two vacuum homotopies are homotopic and hence it follows from (8.2) that <sup>43</sup>

$$\mathfrak{R}[0, \vartheta] \boxtimes \mathfrak{I}[e^{-i\vartheta} \mathcal{T}(x)] \sim \mathfrak{I}[\mathcal{T}(x)] \boxtimes \mathfrak{R}[0, \vartheta]. \quad (8.4)$$

More geometrically, we can imagine rotating the plane by angle  $\vartheta$  and defining interfaces for the rotated Theories for vacuum homotopies defined along the rotated  $x$ -axis. Of course, this should not essentially change the parallel transport, and that is what equation (8.4) is meant to express. Note in particular that if we set  $\vartheta = 2\pi$  then  $\mathfrak{I}[e^{-2\pi i} \mathcal{T}(x)] = \mathfrak{I}[\mathcal{T}(x)]$  are literally equal, but  $\mathfrak{R}[0, 2\pi]$  might well be nontrivial.

We can now say what our notion of equivalence of Theories will be. We say that Theories  $\mathcal{T}^1$  and  $\mathcal{T}^2$  are *equivalent Theories* if there exists a *periodic* family of invertible Interfaces  $\mathfrak{I}^\vartheta$  between  $\mathcal{T}^{1,\vartheta}$  and  $\mathcal{T}^{2,\vartheta}$  which intertwine with the rotational Interfaces in the sense of equation (8.4). The auto-equivalences of a Theory with itself form a kind of “gauge symmetry” of the “flat connection on Theories.” We should only hope to define parallel transport up to such “gauge symmetry.” As a simple example, the dependence of equation (8.1) on  $x_\ell$  and  $x_r$  is only up to equivalence of Theories in this sense.

Conditions 1,2,3 above on the Interfaces  $\mathfrak{I}[\mathcal{T}(x)]$  are certainly rather restrictive. We do not know if they are defining properties.

In the remainder of Section §8 we will construct Interfaces and rules for constructing  $\mathcal{T}(x)$  for paths  $\wp$  which cross exceptional walls and walls of marginal stability. We will

---

<sup>43</sup>The notation  $\mathfrak{R}[0, \vartheta]$  is slightly ambiguous since the interface actually depends on the initial Theory, just like in equation (8.1). The two appearances of this Interface in (8.4) are hence slightly different. Also the notation  $e^{-i\vartheta} \mathcal{T}(x)$  means that we take the same continuous family of  $(\mathcal{R}, \beta)$  but the vacuum weights are  $e^{-i\vartheta} z_i(x)$ .

see that the existence of such Interfaces imposes strong constraints on how the interior amplitude and the web representation can vary along the family. The problem of finding the Interfaces is over-determined and thus invertibility of the Interfaces, compatibility with rotations, homotopy invariance, etc. give constraints on the family  $\mathcal{T}(x)$ . This phenomenon is a categorical version of the derivation of the wall-crossing formula for the  $\mu_{ij}$  from the properties of framed BPS degeneracies under deformations of parameters [30, 31, 75].

### Remarks

1. A more ambitious formulation of a wall-crossing formula is to give an  $L_\infty$  morphism  $\gamma^{\text{web}}$  between the planar web algebras determined by  $(\mathbb{V}^\pm, z^\pm)$  compatible with an  $L_\infty$  morphism  $\gamma$  between the  $L_\infty$ -algebras associated with the Theories  $\mathcal{T}^\pm$ . We will, in fact, do this for crossing exceptional walls. One should probably go further and construct an “ $LA_\infty$  morphism” of “ $LA_\infty$  algebras.” which is compatible with a functor  $\mathcal{F} : \mathfrak{Br}(\mathcal{T}^-) \rightarrow \mathfrak{Br}(\mathcal{T}^+)$ . We have not done that. It is not clear to us if such data is uniquely determined by giving families of Interfaces (8.1).
2. There is a natural notion of “inner auto-equivalence” between Theories  $\mathcal{T}^\pm$  which have the same vacuum data and representation of webs, but interior amplitudes which differ by an exact amount:

$$\beta^+ = \beta^- + \rho_{\beta_-}(\mathbf{t})[\epsilon] \quad (8.5)$$

where  $\epsilon$  is a degree 1 element in  $R^{\text{int}}$  supported on a single fan  $I_\epsilon$ . Because of the line principle,  $\rho(e^{\beta^+}) = \rho_{\beta_-}(e^{\rho_{\beta_-}(\epsilon)}) = 0$  if  $\rho(e^{\beta^-}) = 0$ . It is straightforward to map the Brane categories of the two Theories into each other, simply by shifting boundary amplitudes in a similar fashion:

$$\mathcal{B}^+ = \mathcal{B}^- + \rho_{\beta_-}(\mathbf{t}_\partial) \left[ \frac{1}{1 - \mathcal{B}^-}; \epsilon \right] \quad (8.6)$$

This transformation can also be implemented by a family of interfaces  $\mathcal{I}_\epsilon^\vartheta$  which differs from the identity interfaces only by a shift by  $\epsilon$  of the boundary amplitude. It is possible to show that these interfaces are truly invertible (i.e. not just up to homotopy) and commute with rotation interfaces. Indeed, the two sides of the commutation relation 8.4 for these interfaces only differ by an exact term added to the boundary amplitude. It is also possible to recast these relations in the form of an  $LA_\infty$  algebra isomorphism.<sup>44</sup>

3. Inner auto-equivalences play a role in the relation between Theories and concrete physical theories: although we expect to have a direct map from physical theories

---

<sup>44</sup>In order to prove these statements, it is useful to observe that the amplitude for a web defined in the presence of an  $\mathcal{I}_\epsilon^\vartheta$  interface which includes an insertion of  $\epsilon$  at an interface vertex is identical to the amplitude for a web with the same geometry defined in the absence of the  $\mathcal{I}_\epsilon^\vartheta$  interface. By this identification, the MC equation for  $\mathcal{I}_\epsilon^\vartheta$  becomes 8.5, the definition of  $\mathfrak{B} \boxtimes \mathcal{I}_\epsilon^\vartheta$  maps to 8.6 and 8.4 maps to the convolution identity for curved webs.

to Theories, the image of the map may change by inner auto-equivalences as we vary the parameters of the underlying physical theory, even if the corresponding vacuum homotopy does not cross the exceptional set  $\mathfrak{E}$ . These jumps will occur at co-dimension one walls whose position depends on the detail of the underlying theory, possibly including D-term deformations. We will therefore call these *phantom walls*. Because of the possibility of phantom walls, we should really map physical theories to equivalence classes of Theories up to inner auto-equivalences. Correspondingly, in physical applications any “categorical wall-crossing formula” should be understood up to inner auto-equivalences. It is also possible to envision another class of phantom walls, across which the  $R_{ij}$  themselves may change to a homotopy equivalent complex, which would require one to quotient the space of Theories further in order to define a robust map from physical theories. We leave open the problem to identify which type of equivalences between theories should be associated to the most general possible phantom walls.

4. We can elaborate further on the possibility of phantom walls in the context of physical theories such as the Landau-Ginzburg theories we discuss in Sections §§11-17. Suppose we are given a one-parameter family of superpotentials  $W(\phi; s)$ , say with  $s \in \mathbb{R}$ . Following, Remark 9 of Section §2.1 we obtain a vacuum homotopy  $\{z_i(s)\}$ . In general, there will be isolated points  $s_*$  where  $W_{s_*}$  admits exceptional  $\zeta$ -instantons with fan boundary conditions. (See Section §14 below for a discussion of  $\zeta$ -instantons.) Such exceptional instantons will have a moduli space whose formal dimension (given by the index  $\iota(L)$ , discussed in Section §14.3 below) is 1. This means that the amplitude associated with the path integral with fan boundary conditions (as discussed in Section §14.6) will define an element  $\gamma \in R^{\text{int}}$  with fermion number +1. It can be inserted into some taut webs in  $t_{s_*}$  to produce an element  $\rho_\beta[t_{s_*}](\gamma)$  of fermion number +2. This will contribute to a jump in the interior amplitude  $\beta$  as  $s$  passes through  $s_*$ . In terms of  $\zeta$ -instantons, as  $s \rightarrow s_*$  the size of the relevant  $\zeta$ -web that can accomodate the exceptional instanton at a vertex will grow to infinity. This is an infrared phenomenon associated with working on a noncompact spacetime; it has no counterpart to invariants associated with topological field theory integrals defined on compact manifolds.
5. We should note that these are by no means the most general continuous families we could consider. An important variation on the above ideas involves replacing the vacua  $\mathbb{V}$  with the fibers of a branched covering  $\pi : \Sigma \rightarrow C$ , where  $C$  is a space of Theories. This is the setup appearing in the 2d4d wall-crossing formula of [31]. It should be possible to extend the ideas of the present paper to the more general setting of a branched cover, but, beyond some remarks in Section §18.2, that lies beyond the scope of this paper.

## 8.2 Tame Vacuum Homotopies

We define a *tame vacuum homotopy* to be a vacuum homotopy  $\{z_i(x)\}$  such that:

1. For all  $x$  the set of weights  $\{z_i(x)\}$  is in general position, in the sense of Section §2.5.
2. Each taut web in  $\text{WEB}[x]$  (the set of plane webs determined by the vacuum weights  $\{z_i(x)\}$ ) fits into a continuous family of webs, in the sense defined in Section §6.3.3. Moreover, no taut web is created as  $x$  varies.

Given these criteria we can speak of a single web group  $\mathcal{W}$  and there is a continuously varying planar taut element  $t_{pl}(x)$ . We can also define curved webs precisely as in Section §7.1 and hence we can define the curved taut element  $t$  to be the sum of oriented deformation types of curved taut webs, i.e. those with expected dimension  $d = 1$ . We can then write a convolution identity for  $t$ . We make the assumptions contained in the paragraph containing equation (8.1). In particular, the vacuum data  $(\mathbb{V}, z^\ell)$  and  $(\mathbb{V}, z^r)$  define web groups  $\mathcal{W}^\ell$  and  $\mathcal{W}^r$ . We can define  $t_{pl} = t_{pl}^\ell + t_{pl}^r$  to be the formal sum of the planar taut elements in the web groups. Similarly, we let  $t^{\ell,r}$  denote the taut interface element for an interface separating vacuum data  $(\mathbb{V}, z^\ell)$  and  $(\mathbb{V}, z^r)$ . Then we have the analogue of equation (7.22)

$$t * t_{pl} + T_\partial(t^{\ell,r}) \left[ \frac{1}{1-t} \right] = 0. \quad (8.7)$$

We choose the representations  $\mathcal{R}^\ell$  and  $\mathcal{R}^r$  of the webs determined by  $(\mathbb{V}, z^\ell)$  and  $(\mathbb{V}, z^r)$  to be the same, indeed we can think of a constant, i.e.  $x$ -independent representation of the vacuum data  $(\mathbb{V}, z(x))$ . For a tame vacuum homotopy the set of cyclic fans is constant so  $R^{\text{int}}$  is constant. Moreover, since the taut element  $t_{pl}(x)$  varies continuously it makes sense to speak of an  $x$ -independent interior amplitude  $\beta$ . We can therefore define the contraction operation  $\rho_\beta$  on curved webs and then (8.7) implies that

$$\rho_\beta(t^{\ell,r}) \left[ \frac{1}{1 - \rho_\beta^0(t)} \right] = 0 \quad (8.8)$$

where the superscript 0 on  $\rho_\beta^0(t)$  indicates that the interior amplitude  $\beta$  is inserted at all vertices of the *curved* taut element  $t$ . It follows that  $\rho_\beta^0(t)$  can be regarded as an interface amplitude, where the Chan-Paton factors for the interface are given once again by the formula (7.28), repeated here:

$$\bigoplus_{j,j' \in \mathbb{V}} \mathcal{E}_{j,j'} e_{j,j'} := \bigotimes_{i \neq j} \bigotimes_{x_0 \in \gamma_{ij} \cup \lambda_{ij}} S_{ij}(x_0), \quad (8.9)$$

where we just take  $x_0$  in the interval  $(x^\ell, x^r)$ . Strictly speaking we should define binding points and binding walls in the more general context of tame vacuum homotopies, but the definitions of Section 7.4.1 are essentially the same and will not be repeated.

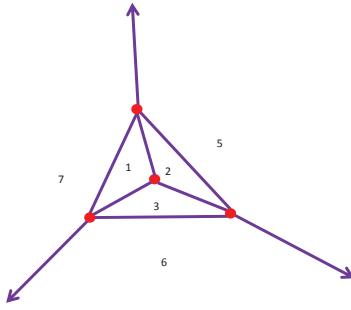
Therefore, to a tame vacuum homotopy and a choice of points  $x_\ell, x_r$ , with corresponding Theories  $\mathcal{T}^\ell$  and  $\mathcal{T}^r$  determined by a constant web representation  $\mathcal{R}$  and vacuum amplitude  $\beta$  we can construct an Interface between the theories.

We can now imitate closely the ideas used in Section §6.3. If we are given a tame homotopy  $\phi(x, y)$  between two tame vacuum homotopies  $\phi^p(x)$  and  $\phi^f(x)$  then there is a set of weights  $z_i(x, y)$  which we can regard as space-time dependent with  $z_i^p(x)$  in the far

past and  $z_i^f(x)$  in the far future. Curved webs again make sense with these space-time dependent weights and we can use them, together with a constant web representation and interior amplitude to construct a closed invertible morphism  $\mathbf{Id} + \delta[\wp(x, y)]$  between  $\mathfrak{I}[\wp^p(x)]$  and  $\mathfrak{I}[\wp^f(x)]$ . Again we can define the time-concatenation  $\circ_T$  of two such homotopies and we claim that

$$\mathbf{Id} + \delta[\wp^1 \circ_T \wp^2] = M_2(\mathbf{Id} + \delta[\wp^1], \mathbf{Id} + \delta[\wp^2]) \quad (8.10)$$

Finally, as in Section 6.3.3 a homotopy of homotopies  $\wp(x, y; s)$  with fixed vacuum weights  $z^\ell$  for  $x \leq x_\ell$  and  $z^r$  for  $x \geq x_r$ , and fixed  $\wp^p(x)$  in the far past and  $\wp^f(x)$  in the far future defines a homotopy equivalence between the morphisms  $\mathbf{Id} + \delta[\wp^p(x)]$  and  $\mathbf{Id} + \delta[\wp^f(x)]$ . It then follows from (8.10) that homotopies between vacuum homotopies lead to homotopy-equivalent Interfaces thus checking (8.3) for tame homotopies between tame vacuum homotopies. In a similar way we can also check equation (8.2) for concatenation of tame vacuum homotopies.



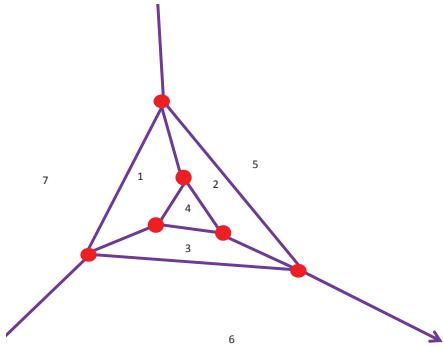
**Figure 96:** An exceptional web which appears only at  $s = s_*$  in a family of webs defined by  $\{z_i(s)\}$ ,  $s \in \mathbb{R}$ .

### 8.3 Wall-Crossing From Exceptional Webs

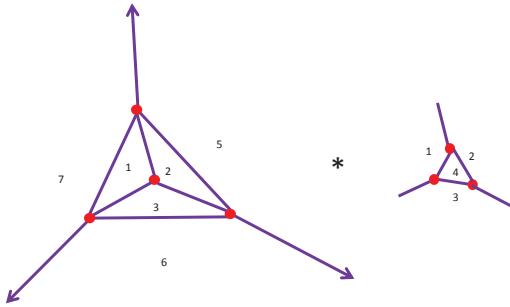
We now come to a different kind of path of vacuum weights where  $\wp(x)$  crosses a wall of exceptional webs. For simplicity assume first that exceptional webs exist only at a single point  $x_*$ . It will be useful to consider first an abstract family of weights  $\{z_i(s)\}$ , with  $s \in \mathbb{R}$  and consider the  $h$ -types of webs for this family rather than curved webs. (Recall the definition of  $h$ -type in Section §6.3.3.) The exceptional webs appear only at  $s = s_*$ . We will describe how such families lead to  $L_\infty$ -morphisms of the  $L_\infty$  algebras  $(\mathcal{W}, T(t))$  of Section §3 and  $(R^{\text{int}}, \rho_\beta(t))$  of Section §4.1, as well as  $A_\infty$ -morphisms of the  $A_\infty$  algebras defined by  $t_\partial$ . Then, when we consider the family as an  $x$ -dependent family of weights we describe the wall-crossing in terms of a suitable Interface.

#### 8.3.1 $L_\infty$ -Morphisms And Jumps In The Planar Taut Element

Let us begin with an example.



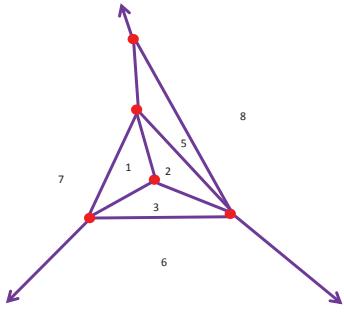
**Figure 97:** A non-exceptional web which can degenerate as  $s \rightarrow s_*$  to the exceptional web shown in Figure 96.



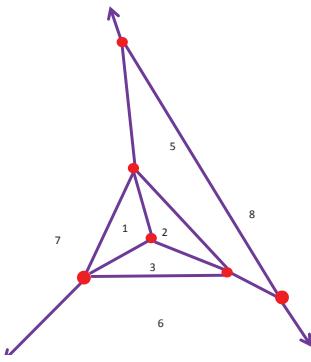
**Figure 98:** The taut web of Figure 97 disappears for  $s > s_*$ , and near  $s = s_*$  its  $h$ -type can be written as a convolution of an exceptinol web with a taut web.

Suppose that the family of weights  $\{z_i(s)\}$  admits an exceptional web such as that shown in Figure 96 at  $s = s_*$ . If we study the behavior of webs in the neighborhood of  $s_*$  several things can happen. The vertex with fan  $I = \{1, 2, 3\}$  will continue to exist for all  $s$ , but it will typically happen that it will not “fit” into any larger triangle with fan  $I_\infty = \{5, 6, 7\}$ . When this happens there are several different subcases:

1. It can happen that the exceptional web of Figure 96 is not a degeneration of any web that exists for  $s \neq s_*$ . Such a web then has no effect on the  $L_\infty$  algebras  $(\mathcal{W}, T(\mathbf{t}))$  and  $(R^{\text{int}}, \rho_\beta(\mathbf{t}))$ .
2. It can also happen that there is a vacuum with weight  $z_4(s)$  so that the web of Figure 96 can be viewed as a degeneration of a nearby non-exceptional web, such as that



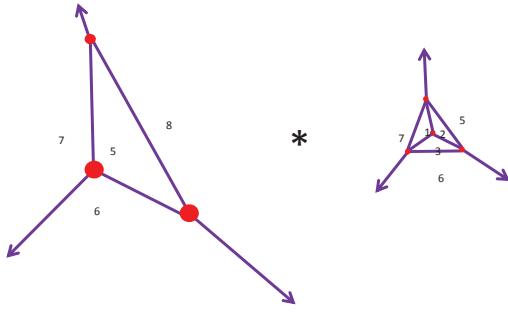
**Figure 99:** Another exceptional web in the exceptional class of the web shown in Figure 96.



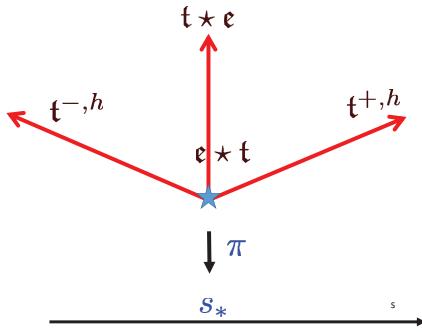
**Figure 100:** An exceptional sliding web that will appear in the convolution  $\epsilon * t$ . To see this convolve the vertex with fan  $\{2, 5, 8, 6, 3\}$  of Figure 99 with a suitable taut web.

shown in Figure 97. Here, some set of edge constraints are effective for  $s \neq s_*$  but become ineffective, or linearly dependent, at  $s = s_*$ . Geometrically, some set of edges and vertices shrinks to a single vertex. Generically, we will have a triangle shrink to a single vertex, reducing the contribution of this set of edges and vertices to the expected dimension from 3 to 2. The result is an exceptional web. In this case there are two further subcases we must consider:

3. It can happen that such nonexceptional degenerating webs exist both for  $s < s_*$  and  $s > s_*$ . Again, when this happens there might or might not be a difference in the  $L_\infty$  algebras  $(\mathcal{W}, T(t))$  and  $(R^{\text{int}}, \rho_\beta(t))$  defined by webs for  $s < s_*$  and  $s > s_*$ . This can be understood once we understand the next and last case.
4. On the other hand, it can also happen that the degenerating web of Figure 97 exists



**Figure 101:** Terms in  $t * e$  such as this cancel the exceptional sliding webs such as those shown in Figure 100.



**Figure 102:** A typical component of the moduli space of sliding  $h$ -types for the family  $\{z_i(s)\}$ . (Three dimensions for translation and dilation have been factored out.) It can happen that the only nonzero component is just  $t^+$  or  $t^-$ , as could happen for the case of Figure 96. Or it can happen that there are several branches meeting at  $s = s_*$ , as for the case of Figure 99. Comparing the boundaries of this dimension one complex leads to the convolution identity for exceptional webs.

for  $s > s_*$ , but not for  $s < s_*$ , or vice versa. Note that the web of Figure 97 is a *taut* web. If it exists for  $s > s_*$  and not for  $s < s_*$  (say), then there must be a change in the taut element and hence a change in the  $L_\infty$  algebras  $(\mathcal{W}, T(t))$  and  $(R^{\text{int}}, \rho_\beta(t))$ . We are most interested in this fourth case.

In order to understand how the algebras  $(\mathcal{W}, T(t))$  and  $(R^{\text{int}}, \rho_\beta(t))$  change in the fourth case above let us first note that the web of Figure 97 can be written as a convolution as in Figure 98. Thus, a convolution of an exceptional web with a non-exceptional web can be non-exceptional. Moreover, given the rule (2.9), in such a case the convolution of a taut

exceptional web with a taut non-exceptional web will be a *taut* non-exceptional web. This example suggests that we can write the change in the taut element  $t^+ - t^-$ , where  $t^\pm$  are the taut elements for  $s > s_*$  and  $s < s_*$  respectively, in terms of  $\epsilon * t$  where  $\epsilon$  is the sum of oriented exceptional taut webs and  $t$  is the sum of taut webs that do not change from  $s > s_*$  to  $s < s_*$ .

There is a problem with expressing  $t^+ - t^-$  in terms of  $\epsilon * t$ . The problem arises from the fact that there can be further exceptional taut webs such as that shown in Figure 99. Indeed, we will refer to the set of exceptional webs obtained by shrinking the same small triangle as an *exceptional class* of webs. Generically  $\epsilon$  will be the sum over one exceptional class. Quite similarly to the case of Figure 96, the web of Figure 99, and indeed every web in the exceptional class, can be viewed as a degeneration of a nonexceptional taut web obtained by taking the convolution of the  $\{1, 2, 3\}$  vertex as in Figure 98. However, the problem is that the convolution  $\epsilon * t$  will also typically contain exceptional sliding webs. An example is shown in Figure 100. Such terms must clearly be cancelled off from  $\epsilon * t$  since  $t^+ - t^-$  contains no exceptional webs. We can do this by noting that Figure 100 can also be degenerated by shrinking the exceptional triangular web, producing a boundary in the form of a convolution as shown in Figure 101. This suggests that we should subtract  $t * \epsilon$  from  $\epsilon * t$ . Note that  $t * \epsilon$  is always exceptional, and hence will always produce exceptional sliding webs.

We now generalize the above example by considering the  $h$ -types of the webs defined by  $\{z_i(s)\}$ . The moduli space of sliding  $h$ -types (that is,  $h$ -types of  $h$ -dimension 4) will have typical components that (for the doubly-reduced moduli space) look like Figure 102, which the reader should compare with Figure 59. Comparing the boundaries of this space leads to the convolution identity for jumps in the taut element due to exceptional webs:

$$t^+ - t^- = \epsilon * t - t * \epsilon \quad (8.11)$$

where, again,  $\epsilon$  is the sum of exceptional taut webs at  $s = s_*$  and  $t$  is the sum of taut webs which do not change across  $s_*$ . In the next paragraph we explain that the taut element  $t$  on the right-hand side can be either  $t^+$  or  $t^-$ .

At this point we need some properties of exceptional webs. Let us call the difference between the dimension of the moduli space of a web and its expected dimension, that is,  $D(\mathfrak{w}) - d(\mathfrak{w})$ , the *excess* dimension. In a generic one-parameter family of webs the excess dimension will only jump by  $\pm 1$ . Thus, for example, in a family where the web of Figure 97 degenerates to Figure 96 the excess dimension jumps by  $+1$ . Let us call the fans at infinity  $I_\infty$  for the exceptional webs *exceptional fans*. In any web, the set of local fans  $I_v(\mathfrak{w})$ , for  $v \in \mathcal{V}(\mathfrak{w})$  will contain at most one exceptional fan. Otherwise the excess dimension would jump by more than  $\pm 1$ . Moreover, no local vertex  $I_v$  of an exceptional web can be an exceptional fan, since resolving such a web would change the excess dimension by more than  $\pm 1$ . It follows that the taut webs which do jump across  $s_*$  cannot have exceptional fans as local vertex fans. Therefore, we can replace  $t$  on the right-hand-side of (8.11) by either  $t^+$  or  $t^-$ .

Given the change in the taut element (8.11) how can we express the change in the  $L_\infty$  algebras  $(\mathcal{W}, T(t))$  and  $(R^{\text{int}}, \rho_\beta(t))$ ? We will focus on  $R^{\text{int}}$ , which is somewhat simpler and

just remark on the former case at the end of this section. It is natural to try to relate the two  $L_\infty$  algebras for  $s > s_*$  and  $s < s_*$  using an  $L_\infty$ -morphism.

Recall that, in general, given two  $L_\infty$  algebras  $(\mathcal{L}^\pm, b^\pm)$  an  $L_\infty$  morphism  $\gamma : (\mathcal{L}^-, b^-) \rightarrow (\mathcal{L}^+, b^+)$  is a map  $\gamma : T\mathcal{L}^- \rightarrow \mathcal{L}^+$  such that, for all monomials  $S \in T\mathcal{L}^-$  we have

$$\sum_k \sum_{\text{Sh}_k(S)} \epsilon b^+(\gamma(S_1), \dots, \gamma(S_k)) = \sum_{\text{Sh}_2(S)} \epsilon \gamma(b^-(S_1), S_2) \quad (8.12)$$

where  $\epsilon$  are signs following from the Koszul rule. See Appendix A below for more precise definitions. It is easy to show that, given an  $L_\infty$  morphism  $\gamma$  and a solution  $\beta^-$  of the  $L_\infty$  MC equation for  $(\mathcal{L}^-, b^-)$  we automatically get a solution

$$\beta^+ = \gamma(e^{\beta^-}) \quad (8.13)$$

of the MC equation for  $(\mathcal{L}^+, b^+)$ .

Now we claim that

$$\gamma = \mathbf{1} + \rho[\mathfrak{e}] \quad (8.14)$$

is an  $L_\infty$ -morphism, where  $\mathbf{1}$  is the identity on  $R^{\text{int}}$  and vanishes on the higher tensors  $(R^{\text{int}})^{\otimes n}$  with  $n > 1$ . To prove this first note that if it is an  $L_\infty$  morphism then we must have

$$\begin{aligned} \beta^+ &= \gamma(e^{\beta^-}) \\ &= \beta^- + \rho[\mathfrak{e}](e^{\beta^-}) \\ &= \beta^- + \rho_{\beta^-}^0[\mathfrak{e}]. \end{aligned} \quad (8.15)$$

Again,  $\beta^+$  and  $\beta^-$  will only differ on summands  $R_I$  where  $I$  is an exceptional fan. As we have just explained, these are never the fans at vertices of an exceptional web so we may write  $\rho_{\beta^-}^0[\mathfrak{e}] = \rho_{\beta^+}^0[\mathfrak{e}]$  and hence it is also true that  $\beta^- = \beta^+ - \rho_{\beta^+}^0[\mathfrak{e}]$ . Note that (8.15) is compatible with equation (8.11) because

$$\begin{aligned} \rho[\mathfrak{t}^+](e^{\beta^+}) &= \rho[\mathfrak{t}^-](e^{\beta^+}) + \rho[\mathfrak{e} * \mathfrak{t}](e^{\beta^+}) - \rho[\mathfrak{t} * \mathfrak{e}](e^{\beta^+}) \\ &= \rho[\mathfrak{t}^-](\rho_{\beta^-}^0[\mathfrak{e}]) + \rho_{\beta^+}[\mathfrak{e}](\rho_{\beta^+}^0[\mathfrak{t}]) - \rho_{\beta^+}[\mathfrak{t}](\rho_{\beta^+}^0[\mathfrak{e}]) \\ &= 0 \end{aligned} \quad (8.16)$$

To get to the last line the first and third terms of the second line cancel and the middle term vanishes, after using the definition of an interior amplitude.

Now to prove that (8.14) is in fact an  $L_\infty$  morphism recall that taut webs can have at most one exceptional fan as a local fan  $I_v$  at its vertices. Therefore, in (8.12) on the left-hand-side  $\rho[\mathfrak{e}]$  can appear at most linearly. (It might appear linearly through the expansion of  $e^{\beta^+}$  using (8.15) or it might act on the arguments of  $S$ .) This, together with the properties of  $\mathbf{1}$  simplifies the sum over  $k$ -shuffles considerably and the required identity follows from applying a representation of webs to the convolution identity (8.11).

## Remarks

1. The final arguments using equations (8.15) and (8.16) made use of the finiteness properties of  $\mathbb{V}$  and the line principle. In general we would like to have more general arguments since some of the main applications, namely knot homology and categorified spectral networks will not enjoy those finiteness principles. We expect that in general there will be an  $L_\infty$  morphism to express the change of the interior amplitude.
2. Let us return briefly to discuss the change in the  $L_\infty$  algebra of planar webs  $(\mathcal{W}, T[\mathbf{t}])$ . It would be preferable to describe the jump in this algebra and then apply web representations to obtain the jump in  $(R^{\text{int}}, \rho_\beta[\mathbf{t}])$ . Roughly speaking, we expect that there will be an equation of the form

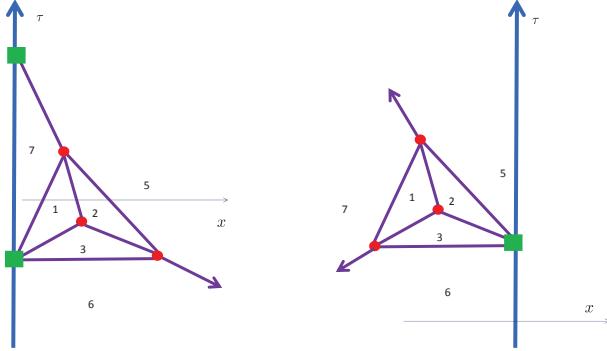
$$T[\mathbf{t}_+](e^\mathbf{g}) = \mathbf{g} * \mathbf{t}_- \quad (8.17)$$

for some object  $\mathbf{g}$  generalizing  $\mathbf{r} + \mathbf{e}$  (where  $\mathbf{r}$  is the rigid element). Some further thought suggests that in order to give a direct geometric meaning to such a formula, and in particular to  $\mathbf{g}$  itself, we need to cook up a setup which is translation invariant, but not scale invariant:  $\mathbf{g}$  has degree number 2! For example, we could let the slope of an edge depend on its length, so that long edges are controlled by the vacuum data for  $s > s_*$  and short edges by the vacuum data for  $s < s_*$ . Then the set of rigid webs in such a setup would give us a degree 2 object  $\mathbf{g}$ . Sliding webs in such a setup would give the desired convolution identity: large web endpoints of moduli spaces will look like a large taut web in  $\mathbf{t}_+$  with all vertices solved to  $\mathbf{g}$  rigid webs, while small web endpoints will look like a rigid web in  $\mathbf{g}$  with a single vertex resolved into a taut web in  $\mathbf{t}_-$ . The advantage of this complicated construction is that it would probably work in situations with weaker finiteness properties. The disadvantage is that the prescription seems somewhat *ad hoc* and unphysical.

3. Returning to the path  $\varphi(x)$  defining  $x$ -dependent weights and curved webs, when  $x$  passes through  $x_*$  we will introduce in Section §8.3.3 below an Interface whose  $(A_\infty)$  Maurer-Cartan equation is equivalent to the condition (8.15) above. Then, if there are several values of  $x$  where  $\varphi(x)$  passes through an exceptional wall we simply take the convolution of the Interfaces.

### 8.3.2 $A_\infty$ -Morphisms And Jumps In The Half-Plane Taut Element

Let us fix a half-plane  $\mathcal{H}$ , for example the positive or negative half-plane, and continue to consider the family of weights  $\{z_i(s)\}$  with exceptional half-plane webs appearing at  $s = s_*$ , and for no other value of  $s$ . It is possible for both plane and half plane exceptional webs to appear at the same value  $s = s_*$ . Indeed, for any class of planar exceptional webs we can make half-plane exceptional webs by taking an extremal vertex of the planar exceptional and interpreting it as a boundary vertex. See, for example, Figure 103. In this way we can construct taut exceptional half-plane webs from taut planar webs. We will denote the sum of oriented taut half-plane exceptional webs at  $s = s_*$  by  $\mathbf{e}_\partial$  and the sum of oriented taut plane exceptional webs, if present at  $s = s_*$ , by  $\mathbf{e}$ .



**Figure 103:** Exceptional half-plane webs for the positive and negative half-planes, whose existence follows from Figure 96.

If we consider the moduli space of sliding  $h$ -types of half-plane webs we derive the convolution identity:

$$\mathbf{t}_\partial^+ - \mathbf{t}_\partial^- = \mathbf{e}_\partial * \mathbf{t}_{pl} + \mathbf{e}_\partial * \mathbf{t}_\partial - \mathbf{t}_\partial * \mathbf{e} - \mathbf{t}_\partial * \mathbf{e}_\partial. \quad (8.18)$$

Adopting the usual arguments based on finiteness and the line principle, it does not matter whether we take  $\mathbf{t}_{pl}^\pm$  or  $\mathbf{t}_\partial^\pm$  on the right hand side.

Let us now suppose that  $\mathcal{B}^-$  is a boundary amplitude for the positive half-plane for a Theory  $\mathcal{T}^-$  with vacuum weights  $z_i(s_-)$  with  $s_- < s_*$ . We must also choose  $\mathcal{R}$  and Chan-Paton data  $\mathcal{E}$ . Now, holding  $\mathcal{R}$  and  $\mathcal{E}$  fixed, consider a Theory  $\mathcal{T}^+$  for  $z_i(s_+)$  with  $s_+ > s_*$ . We can construct a new solution  $\mathcal{B}^+$  to the MC equation of  $\mathcal{T}^+$  if we set

$$\mathcal{B}^+ := \mathcal{B}^- + \rho_\beta(\mathbf{e}_\partial) \left[ \frac{1}{1 - \mathcal{B}^-} \right]. \quad (8.19)$$

Again, we are using heavily the finiteness principle to insure that this expression is well-defined, and independent of the choice of  $\mathcal{B}^\pm$  or  $\beta^\pm$  on the right hand side. To verify it one must take

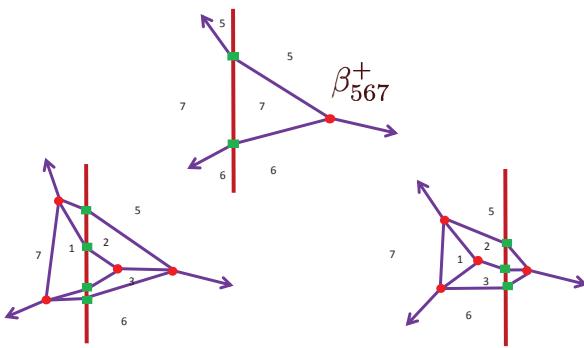
$$\rho(\mathbf{t}_\partial^+) \left[ \frac{1}{1 - \mathcal{B}^+}; e^{\beta^+} \right] \quad (8.20)$$

and expand everything in terms of amplitudes and webs for  $s < s_*$  using (8.15), (8.18) and (8.19). After a few lines of computation, using the finiteness properties and the fact that  $\mathcal{B}^-$  and  $\beta^-$  are boundary and interior amplitudes one finds that (8.20) is indeed zero.

An obvious way to extend this analysis to more general situations would be to think in terms of an  $A_\infty$  morphism  $\gamma_\partial$  from the  $A_\infty$ -algebra  $(R^\partial, \rho_{\beta^-}(\mathbf{t}_\partial^-))$  to the  $A_\infty$ -algebra  $(R^\partial, \rho_{\beta^+}(\mathbf{t}_\partial^+))$ , mapping  $\mathcal{B}^-$  to  $\mathcal{B}^+$ . In the case with a single class of exceptional webs, the morphism is  $\gamma_\partial = \mathbf{1} + \rho_\beta(\mathbf{e}_\partial)$ , in close analogy to the case of planar webs.

### Remarks

1. We could probably extend the above discussion to define an  $LA_\infty$  morphism from  $\rho(\mathfrak{t}_\partial^-)$  to  $\rho(\mathfrak{t}_\partial^+)$ , which coincides in the simple case with  $\mathbf{1} + \rho_\beta(\mathfrak{e}_\partial)$ . (See Appendix §A.6 for the definition of an  $LA_\infty$ -morphism.)
2. As in the planar case we could give a geometric meaning to these general structures by using the same trick to break scale invariance, considering half-plane webs with edges whose slope depends on the length. Rigid webs in such a setup would define an element  $\mathfrak{g}_\partial$  mapped by a web representation to  $\gamma_p$  and satisfying automatically the required axioms.



**Figure 104:** We consider an interface with weights  $\{z_i(s_* + \epsilon)\}$  in the positive half-plane and weights  $\{z_i(s_* - \epsilon)\}$  in the negative half-plane. In the upper center we show a typical interior amplitude which is discontinuous at  $s_*$  because of the exceptional web of Figure 96. There is a unique corresponding taut interface web obtained by placing a vertical slice through an adjusted version of the exceptional web. Two (out of three) possible places for the placement of this vertical slice are shown in the lower left and lower right. Only one of the three possible vertical lines will actually admit a solution to the edge constraints.

### 8.3.3 An Interface For Exceptional Walls

We will now construct an Interface  $\mathfrak{I}^{\text{exc}}$  whose Maurer-Cartan equation is equivalent to the discontinuity (8.15) of the interior amplitude and which induces a functor on brane categories reproducing the discontinuity (8.19) of boundary amplitudes.

We continue to consider the continuous family  $\{z_i(s)\}$  of vacuum weights crossing an exceptional wall at  $s_*$ . For simplicity we restrict ourselves to the generic situation where a single class of exceptional webs appear at the jump locus, and use all the necessary finiteness constraints.

Our Interface  $\mathfrak{I}^{\text{exc}}$  will separate two Theories with vacuum weights  $\{z_i(s_* - \epsilon)\}$  in the negative half-plane and  $\{z_i(s_* + \epsilon)\}$  in the positive half-plane. The Interface will be formally the same as the identity Interface  $\mathfrak{Id}$ . That is the Chan-Paton factors are simply  $\delta_{i,j}\mathbb{Z}$  and the nonzero interface amplitudes are all 2-valent and take the value  $K_{ij}^{-1} \in R_{ij} \otimes R_{ji}$ . The

main difference from the case of the identity Interface  $\mathfrak{I}\mathfrak{d}$ , is that the interface taut element is now more interesting and the Maurer-Cartan equation satisfied by  $\mathfrak{I}^{\text{exc}}$  is more subtle.

Quite generally, given any continuous family of weights  $\{z_i(s)\}$ , by taking a vertical slice through a planar web  $\mathfrak{w}$  with weights  $z_i(s_0)$ , where the slice does not go through any of the vertices of  $\mathfrak{w}$ , one can make a corresponding interface web  $\mathfrak{w}^{\text{ifc}}$  (adding 2-valent vertices where the lines intersect the vertical slice). One could then deform the interface web so the slopes have weights  $z_i(s_0 + \epsilon)$  in the positive half-plane and  $z_i(s_0 - \epsilon)$  in the negative half-plane. This procedure will never change the expected dimension: We always add a boundary vertex and an internal edge so  $d(\mathfrak{w}) = d(\mathfrak{w}^{\text{ifc}})$ . In general, the procedure also does not change the true dimension:  $D(\mathfrak{w}) = D(\mathfrak{w}^{\text{ifc}})$ . Thus, in general, if we apply the procedure to planar taut webs we get interface sliding webs. However, in the special case when we apply this procedure to an *exceptional* web with  $s_0 = s_*$  the resulting interface web is non-exceptional:  $D(\mathfrak{w}^{\text{ifc}}) = d(\mathfrak{w}^{\text{ifc}})$ . In particular, if we apply the procedure to an exceptional taut planar web we then produce a taut interface web. We claim, moreover, that for each deformation class of exceptional taut web at  $s = s_*$  the procedure will yield a *unique* deformation class of taut interface web separating weights  $z_i(s_0 \pm \epsilon)$ . See Figure 104.

It then follows that the Interface  $\mathfrak{I}^{\text{exc}}$  has a more subtle Maurer-Cartan equation than that of  $\mathfrak{I}\mathfrak{d}$ . Since some interior amplitudes  $\beta_I$  are discontinuous we cannot apply the simple argument of Figure 40. But we know from equation (8.15) that  $\beta_I$  will only be discontinuous when  $I$  is an exceptional fan. These extra terms are precisely compensated by the taut interface webs such as those shown in Figure 104! The Maurer-Cartan equation thus becomes

$$\beta^+ - \beta^- = \rho_\beta^0[\mathfrak{e}] \quad (8.21)$$

and thus the MC equation for  $\mathfrak{I}^{\text{exc}}$  is equivalent to the discontinuity condition (8.15), as was to be shown.

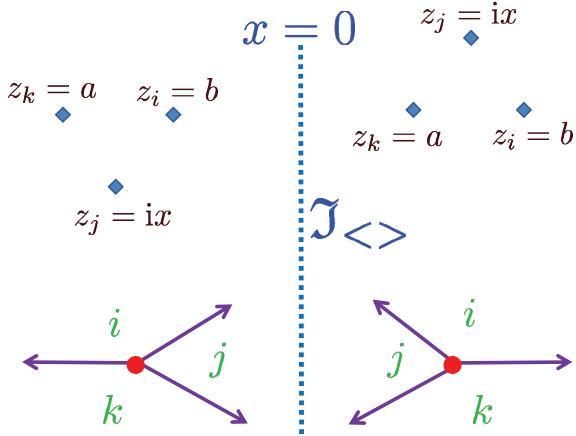
In a similar way, if we try to compose our Interface with a Brane, the only non-trivial taut (=rigid) composite webs will be in one-one correspondence with exceptional taut half-plane webs through a similar procedure of introducing a vertical slice. Thus the  $A_\infty$ -functor  $\mathcal{F}_{\mathfrak{I}^{\text{exc}}}$  defined in (6.49) implements the discontinuity equation

$$\mathcal{B}^+ - \mathcal{B}^- = \rho_\beta(\mathfrak{e}_\partial)[\frac{1}{1 - \mathcal{B}}]. \quad (8.22)$$

The Interface  $\mathfrak{I}^{\text{exc}}$  we have just constructed represents the discontinuity of Theories for crossing an exceptional wall. It is thus similar to the Interfaces  $\mathfrak{S}_{ij}^{p,f}$  for crossing  $S$ -walls. In a way analogous to the general Interface for spinning webs of Section 7, we can use the concatenation property (8.2) to define Interfaces for more general paths  $\varphi(x)$  which can cross several exceptional walls by taking suitable compositions of the Interfaces such as  $\mathfrak{I}^{\text{exc}}$  with interfaces for tame vacuum homotopies. At this point one should engage in an extensive discussion of homotopy equivalence, well-definedness of concatenation of homotopies up to homotopy equivalence etc., but we will not spell out the details here. It is useful to point out, however, that when working with homotopies of homotopies special codimension two loci in the space of weights  $\mathbb{C}^\mathbb{V} - \Delta$  can become important. In

particular, one one should treat with care points where two distinct co-dimension one walls of exceptional weights intersect.

Another loose end which we leave to the reader's imagination is to show that the Interface  $\mathfrak{I}^{\text{exc}}$  satisfies the MC equations also in the more general setup we defined with  $L_\infty$  morphisms and non-scale invariant configurations. This can be done with a setup where the edge slopes depend on their length, but only on the positive half-plane. Then the MC equation for the Interface with  $\beta^+$  realized by rigid webs represents a large sliding web in the setup. The other endpoints can be represented by convolutions with standard planar taut elements, and these terms will vanish. Similar interpolations show that the  $A_\infty$  morphism matches the composition with the trivial interface.

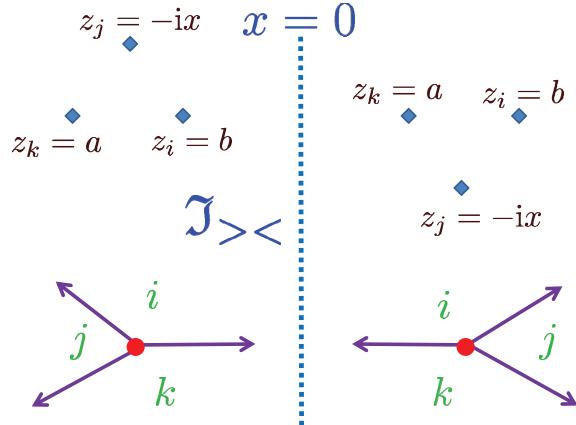


**Figure 105:** An example of a continuous path of vacuum weights crossing a wall of marginal stability. Here  $z_k = a$  and  $z_i = b$  with  $a, b$  real and  $a < 0 < b$ . They do not depend on  $x$ , while  $z_j(x) = ix$ . We show typical vacuum weights for negative and positive  $x$  and the associated trivalent vertex. All other vacuum weights are assumed to be independent of  $x$ . As  $x$  passes through zero the vertex degenerates with  $z_{jk}(x)$  and  $z_{ij}(x)$  becoming real. Note that with this path of weights the  $\{i, j, k\}$  form a *positive* half-plane fan in the negative half-plane, while  $\{k, j, i\}$  form a *negative* half-plane fan in the positive half-plane. If we choose  $x_\ell < 0 < x_r$  there is an associated interface  $\mathfrak{I}_{<>}$ . (We suppress the dependence on  $x_\ell, x_r$  in the notation.) The only vertices are divalent vertices. These are all the standard amplitude  $K^{-1}$  familiar from the identity Interface  $\mathfrak{Id}$ , except for  $\alpha_{<>}^{--} \in R_{ik}^{(2)} \otimes R_{ki}^{(1)}$ .

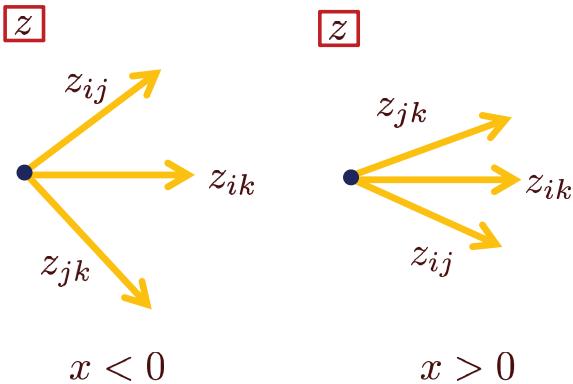
#### 8.4 Wall-Crossing From Marginal Stability Walls

One of the most interesting wall-crossing phenomena occurs when the path of vacuum weights goes through a wall of marginal stability, such as equation (2.41). In this section we examine some important examples of such wall-crossing, but we do not give a completely general wall-crossing prescription.

One way to cross such a wall is illustrated in Figures 105 and 106. While we have chosen a very concrete set of weights our analysis applies to general configurations where the fans behave as described in the captions, and so long as none of the  $z_{ij}, z_{jk}, z_{ki}$  become



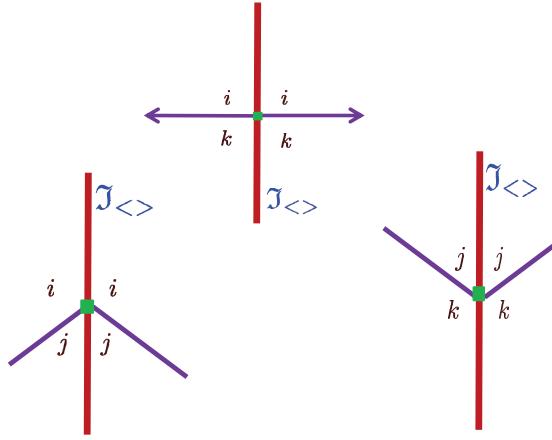
**Figure 106:** In this figure the path of weights shown in Figure 105 is reversed. Again,  $z_k = a$  and  $z_i = b$  with  $a, b$  real and  $a < 0 < b$ , but now  $z_j(x) = -ix$ . We show typical vacuum weights for negative and positive  $x$  and the associated trivalent vertex. All other vacuum weights are assumed to be independent of  $x$ . Note that with this path of weights the  $\{i, j, k\}$  form a *positive* half-plane fan in the positive half-plane, while  $\{k, j, i\}$  form a *negative* half-plane fan in the negative half-plane. In order to define an interface we choose initial and final points for the path  $-x_r < 0 < -x_\ell$  so that, after translation, it can be composed with the path defining  $\mathcal{J}_{<>}$ . The interface  $\mathcal{J}_{<}$  has several nontrivial vertices. See Figure 111.



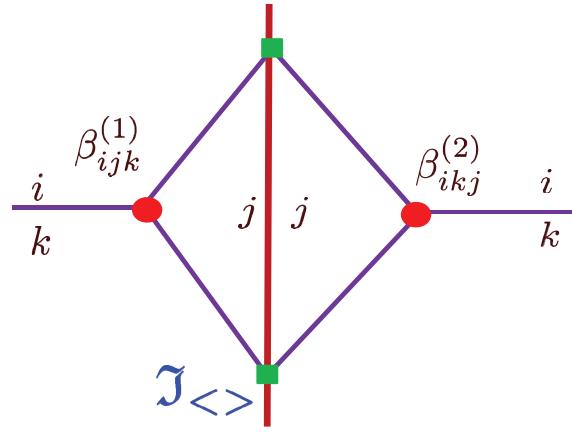
**Figure 107:** For the path of vacuum weights in Figure 105 we have BPS rays crossing as in the standard marginal stability analysis of the two-dimensional wall-crossing formula.

pure imaginary. If our Theory has more than three vacua we assume that all other vacuum weights are constant and just the  $i, j, k$  ‘‘subsector’’ of the Theory is changing.

Let us begin by recalling the well-known standard Cecotti-Vafa-Kontsevich-Soibelman result for this situation. Referring to the path of Figure 105 we have the standard transformation of BPS rays shown in Figure 107. Thus the equality of phase-ordered products



**Figure 108:** Vertices for the interface amplitude  $\mathfrak{J}_{<>}$  described in Figure 105. The bottom left and right amplitudes are the standard  $K^{-1}$ -type of the Interface  $\mathfrak{J}\mathfrak{d}$ , but the middle amplitude is a nontrivial amplitude  $\alpha_{<>}^{--} \in R_{ik}^{(2)} \otimes R_{ki}^{(1)}$  in the wall-crossing identity.



**Figure 109:** There is one nontrivial taut interface web in the MC equation for  $\mathfrak{J}_{<>}$  which only involves vacua  $i, j, k$ .

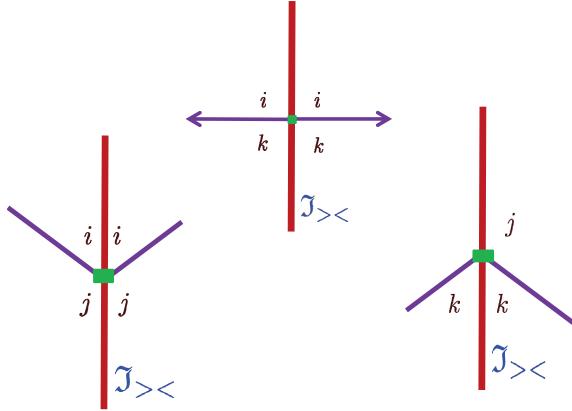
(5.40) where  $\mathcal{H}$  is the positive half-plane gives:

$$(1 + \mu_{ij}^- e_{ij})(1 + \mu_{ik}^- e_{ik})(1 + \mu_{jk}^- e_{jk}) = (1 + \mu_{jk}^+ e_{jk})(1 + \mu_{ik}^+ e_{ik})(1 + \mu_{ij}^+ e_{ij}) \quad (8.23)$$

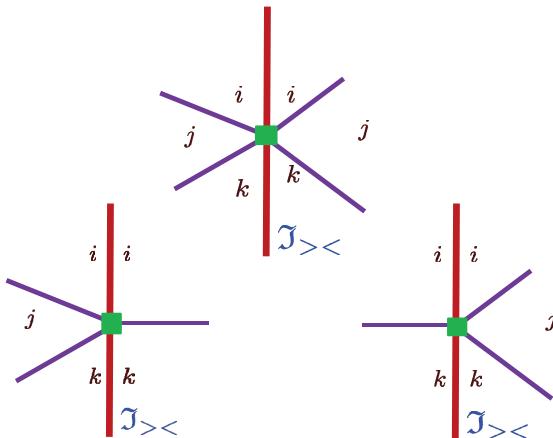
and so we obtain Cecotti and Vafa's wall-crossing result:

$$\begin{aligned} \mu_{ij}^- &= \mu_{ij}^+ \\ \mu_{jk}^- &= \mu_{jk}^+ \\ \mu_{ik}^- + \mu_{ij}^- \mu_{jk}^- &= \mu_{ik}^+. \end{aligned} \quad (8.24)$$

Of course, the inverse transformation is obtained by considering the path shown in Figure 106.



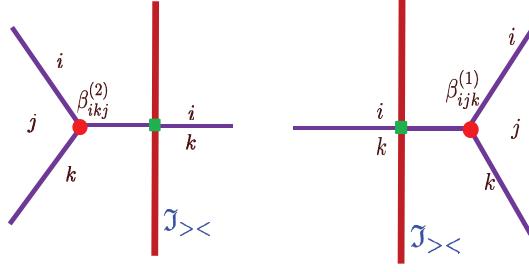
**Figure 110:** Vertices for the interface amplitude  $\mathfrak{J}_{><}$  described in Figure 106. The bottom left and right amplitudes are the standard  $K^{-1}$ -type of the Interface  $\mathfrak{J}\mathfrak{d}$ , but the middle amplitude is a nontrivial amplitude  $\alpha_{><}^{--} \in R_{ik}^{(1)} \otimes R_{ki}^{(2)}$  in the wall-crossing identity.



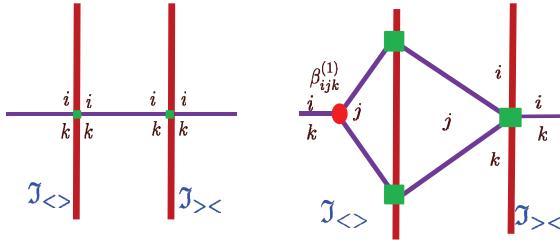
**Figure 111:** There are three other vertices for the interface  $\mathfrak{J}_{><}$  described in Figure 106, shown here. The lower left is an amplitude  $\alpha_{><}^{>-} \in R_{ik}^{(1)} \otimes R_{kj} \otimes R_{ji}$ . The lower right is an amplitude  $\alpha_{><}^{-<} \in R_{ij} \otimes R_{jk} \otimes R_{ki}^{(2)}$ . The middle amplitude is  $\alpha_{><}^{><} \in R_{ij} \otimes R_{jk} \otimes R_{kj} \otimes R_{ji}$ .

At a minimum, a ‘‘categorification’’ of the wall-crossing formula (8.24) should describe the discontinuous change in the web representation  $\mathcal{R}$  of the Theories defined by the weights of Figure 105 at  $x_\ell < 0$  and  $x_r > 0$ . As we have seen with the paths involving exceptional webs we should also allow for a change in the interior amplitude and indeed this is quite necessary in the present case since the set of cyclic fans must change by replacing  $\{i, j, k\}$  with  $\{i, k, j\}$ .

The simplest hypothesis for how  $\mathcal{R}$  changes, which is compatible with the change of Witten indices (8.24), is that  $R_{ik}, R_{ki}, K_{ik}, K_{ki}$  change while all other representation spaces and contractions remain unchanged. Similarly, the component  $\beta_{ijk}$  of the interior



**Figure 112:** There are two nontrivial taut interface webs in the MC equation for  $\mathfrak{I}_{><}$  which only involves vacua  $i, j, k$ .

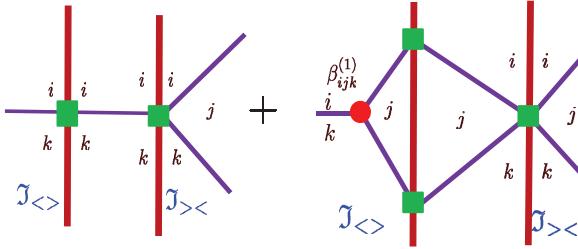


**Figure 113:** The first nontrivial equation in the identity  $\mathfrak{I}_{<>} \boxtimes \mathfrak{I}_{><} \sim \mathfrak{I}_{\partial}$ . The simplest possibility is to take the amplitude to be  $K_{ik}^{(1), -1}$ .

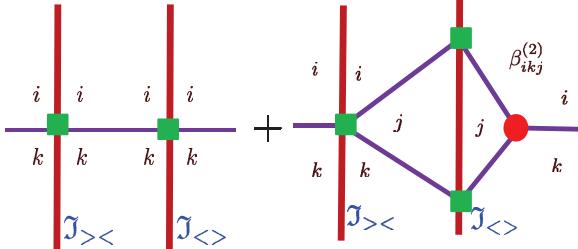
amplitude can only exist on one side of the wall while  $\beta_{ijk}$  can only exist on the other. We assume all other interior amplitudes are unchanged.

We will again seek to characterize an Interface which implements (via the discussion of Section 6.2) the desired  $A_\infty$  functor between Brane categories. The Interface  $\mathfrak{I}_{<>}$  relates the Theory with vacuum weights at  $x_\ell$ , web representation,  $R_{ik}^{(1)}, R_{ki}^{(1)}, K_{ik}^{(1)}, K_{ki}^{(1)}$ , and interior amplitude  $\beta_{ijk}^{(1)}$  on the left and the Theory with vacuum weights at  $x_r$ , web representation  $R_{ik}^{(2)}, R_{ki}^{(2)}, K_{ik}^{(2)}, K_{ki}^{(2)}$  and interior amplitude  $\beta_{ijk}^{(2)}$  on the right. The interface  $\mathfrak{I}_{><}$  is then defined by the choice of path in Figure 106 beginning at  $-x_r$  and ending at  $-x_\ell$ . Thus, we seek to define Interfaces:

$$\mathfrak{I}_{<>} \in \mathfrak{Br}(\mathcal{T}^\ell, \mathcal{T}^r) \quad \& \quad \mathfrak{I}_{><} \in \mathfrak{Br}(\mathcal{T}^r, \mathcal{T}^\ell) \quad (8.25)$$



**Figure 114:** The second nontrivial equation in the identity  $\mathfrak{I}_{<>} \boxtimes \mathfrak{I}_{><} \sim \mathfrak{I}\mathfrak{d}$ . The simplest possibility is to take the amplitude to vanish.



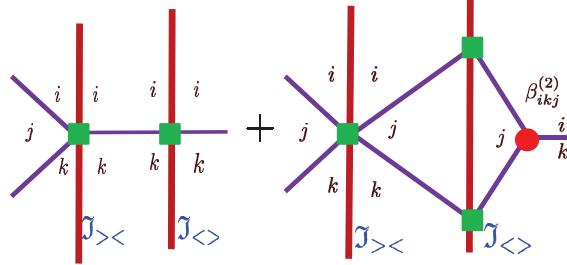
**Figure 115:** The first nontrivial equation in the identity  $\mathfrak{I}_{><} \boxtimes \mathfrak{I}_{<>} \sim \mathfrak{I}\mathfrak{d}$ . The simplest possibility is to take the amplitude to be  $K_{ik}^{(2), -1}$ .

(where the notation is meant to remind us how the half-plane fans are configured in the negative and positive half-planes). Now, the essential statement constraining these Interfaces is that, after a suitable translation of an Interface to the left or right so that they can be composed, the composition of the Interfaces should be homotopy equivalent to the identity Interface:

$$\mathfrak{I}_{<>} \boxtimes \mathfrak{I}_{><} \sim \mathfrak{I}\mathfrak{d}_{\mathcal{T}^\ell} \quad & \quad \mathfrak{I}_{><} \boxtimes \mathfrak{I}_{<>} \sim \mathfrak{I}\mathfrak{d}_{\mathcal{T}^r}. \quad (8.26)$$

The Interfaces only depend on  $x_\ell < 0$  and  $x_r > 0$  through composition with invertible Interfaces.

We now construct such Interfaces  $\mathfrak{I}_{><}$  and  $\mathfrak{I}_{<>}$ . The simplest hypothesis is that the Chan-Paton data of the Interfaces  $\mathfrak{I}_{<>}$  and  $\mathfrak{I}_{><}$  is identical to that of  $\mathfrak{I}\mathfrak{d}$  and we will adopt



**Figure 116:** The second nontrivial equation in the identity  $\mathfrak{J}_{><} \otimes \mathfrak{J}_{><} \sim \mathfrak{J}\partial$ . The simplest possibility is to take the amplitude to vanish.

these. As explained in Figure 108 the Interface  $\mathfrak{J}_{><}$  has amplitudes coinciding with those of the Identity interface  $\mathfrak{J}\partial$ , except for

$$\alpha_{><}^{--} \in R_{ki}^{(1)} \otimes R_{ik}^{(2)} \quad (8.27)$$

where the notation is again meant to be suggestive of the picture. The Mauer-Cartan equation for this interface is illustrated in Figure 109. It constrains the interior amplitudes through the condition:

$$K_{ij} \otimes K_{jk} \left( \beta_{ijk}^{(1)} \otimes \beta_{ikj}^{(2)} \right) = 0 \quad (8.28)$$

where we have used repeatedly the defining properties of  $K^{-1}$ . There is no constraint on  $\alpha_{><}^{--}$  from taut webs involving only vacua  $i, j, k$ . Of course, if there are other vacua then this amplitude might well be involved in other components of the MC equation.

Similarly, as explained in Figure 110 and 111 the Interface  $\mathfrak{J}\partial_{><}$  has a more intricate set of amplitudes. In Figure 110 we have

$$\alpha_{><}^{--} \in R_{ki}^{(2)} \otimes R_{ik}^{(1)} \quad (8.29)$$

and in Figure 111 we have

$$\begin{aligned} \alpha_{><}^{<-} &\in R_{ij} \otimes R_{jk} \otimes R_{ki}^{(2)} \\ \alpha_{><}^{>-} &\in R_{ik}^{(1)} \otimes R_{kj} \otimes R_{ji} \\ \alpha_{><}^{><} &\in R_{ij} \otimes R_{jk} \otimes R_{kj} \otimes R_{ji} \end{aligned} \quad (8.30)$$

Once again, the notation is meant to be a mnemonic for the picture. There are now two nontrivial components to the MC equation, illustrated in Figure 112. These lead to equations

$$K_{ik}^{(2)} \left( \beta_{ikj}^{(2)} \otimes \alpha_{><}^{--} \right) = 0 \quad (8.31)$$

$$K_{ik}^{(1)} (\alpha_{<>}^{--} \otimes \beta_{ijk}^{(1)}) = 0 \quad (8.32)$$

In order to investigate (8.26) we work out the nontrivial amplitudes of the composition of the two Interfaces. For  $\mathfrak{I}_{<>} \boxtimes \mathfrak{I}_{><}$  there are two nontrivial amplitudes. The first, illustrated by Figure 113 is given by <sup>45</sup>

$$K_{ik}^{(2)} (\alpha_{<>}^{--} \otimes \alpha_{><}^{--}) + K_{ij} \otimes K_{jk} (\beta_{ijk}^{(1)} \otimes \alpha_{><}^{>-}) \in R_{ik}^{(1)} \otimes R_{ki}^{(1)}. \quad (8.33)$$

In this formula, and in the similar ones to follow we have used the basic defining property (6.11), (6.12) of  $K^{-1}$  several times. The second nontrivial amplitude, illustrated by Figure 114, is given by

$$K_{ik}^{(2)} (\alpha_{<>}^{--} \otimes \alpha_{><}^{<-}) + K_{ij} \otimes K_{jk} (\beta_{ijk}^{(1)} \otimes \alpha_{><}^{><}) \in R_{ij} \otimes R_{jk} \otimes R_{ki}^{(1)}. \quad (8.34)$$

Similarly, for  $\mathfrak{I}_{><} \boxtimes \mathfrak{I}_{<>}$  there are likewise two nontrivial amplitudes. The first, illustrated by Figure 115 is given by

$$K_{ik}^{(1)} (\alpha_{><}^{--} \otimes \alpha_{<>}^{--}) + K_{ij} \otimes K_{jk} (\alpha_{><}^{<-} \otimes \beta_{ikj}^{(2)}) \in R_{ik}^{(2)} \otimes R_{ki}^{(2)}. \quad (8.35)$$

The second nontrivial amplitude, illustrated by Figure 116, is given by

$$K_{ik}^{(1)} (\alpha_{><}^{--} \otimes \alpha_{<>}^{--}) + K_{ij} \otimes K_{jk} (\alpha_{><}^{<-} \otimes \beta_{ikj}^{(2)}) \in R_{kj} \otimes R_{ji} \otimes R_{ik}^{(2)}. \quad (8.36)$$

In order to illustrate the categorified wall-crossing we will content ourselves with constructing a consistent pair of Interfaces  $\mathfrak{I}_{<>}$  and  $\mathfrak{I}_{><}$  satisfying all the above criteria. We will not try to construct the most general Interface consistent with all the criteria. In this spirit we will therefore try to construct these Interfaces so that equation (8.26) is satisfied with equality, rather than homotopy equivalence. This leads to the four equations:

$$K_{ik}^{(2)} (\alpha_{<>}^{--} \otimes \alpha_{><}^{--}) + K_{ij} \otimes K_{jk} (\beta_{ijk}^{(1)} \otimes \alpha_{><}^{>-}) = K_{ik}^{(1), -1} \quad (8.37)$$

$$K_{ik}^{(2)} (\alpha_{<>}^{--} \otimes \alpha_{><}^{<-}) + K_{ij} \otimes K_{jk} (\beta_{ijk}^{(1)} \otimes \alpha_{><}^{><}) = 0 \quad (8.38)$$

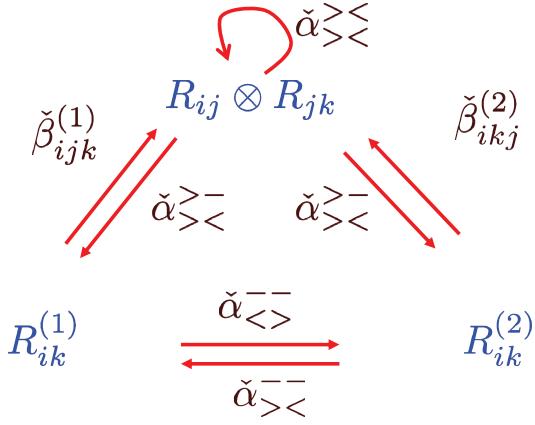
$$K_{ik}^{(1)} (\alpha_{><}^{--} \otimes \alpha_{<>}^{--}) + K_{ij} \otimes K_{jk} (\alpha_{><}^{<-} \otimes \beta_{ikj}^{(2)}) = K_{ik}^{(2), -1} \quad (8.39)$$

$$K_{ik}^{(1)} (\alpha_{><}^{--} \otimes \alpha_{<>}^{--}) + K_{ij} \otimes K_{jk} (\alpha_{><}^{<-} \otimes \beta_{ikj}^{(2)}) = 0 \quad (8.40)$$

The conditions (8.37)-(8.40) are rather opaque. They can be considerably simplified by using the property that  $K$  is a *nondegenerate pairing* to define a degree minus one isomorphism  $R_{ji} \rightarrow R_{ij}^*$ . In this way we can reinterpret the amplitudes (8.27)-(8.30) as 7

---

<sup>45</sup>In these, and similar formulae below we have not attempted to get the relative signs in the equations right.



**Figure 117:** A quiver-like figure illustrating the various linear transformations appearing in the categorified wall-crossing formula.

linear transformations between three different vector spaces:

$$\begin{aligned}
 \check{\alpha}_{><}^{--} &\in \text{Hom}(R_{ik}^{(1)}, R_{ik}^{(2)}) \\
 \check{\alpha}_{><}^{--} &\in \text{Hom}(R_{ik}^{(2)}, R_{ik}^{(1)}) \\
 \check{\alpha}_{><}^{-\leq} &\in \text{Hom}(R_{ik}^{(2)}, R_{ij} \otimes R_{jk}) \\
 \check{\alpha}_{><}^{\geq-} &\in \text{Hom}(R_{ij} \otimes R_{jk}, R_{ik}^{(1)}) \\
 \check{\alpha}_{><}^{\geq\leq} &\in \text{Hom}(R_{ij} \otimes R_{jk}, R_{ij} \otimes R_{jk}) \\
 \check{\beta}_{ijk}^{(1)} &\in \text{Hom}(R_{ik}^{(1)}, R_{ij} \otimes R_{jk}) \\
 \check{\beta}_{ikj}^{(2)} &\in \text{Hom}(R_{ij} \otimes R_{jk}, R_{ik}^{(2)})
 \end{aligned} \tag{8.41}$$

In these terms the Maurer-Cartan equations become 3 simple conditions on the linear transformations:<sup>46</sup>

$$\begin{aligned}
 \check{\beta}_{ijk}^{(1)} \check{\beta}_{ikj}^{(2)} &= 0 \\
 \check{\alpha}_{><}^{--} \check{\beta}_{ijk}^{(1)} &= 0 \\
 \check{\beta}_{ikj}^{(2)} \check{\alpha}_{><}^{--} &= 0,
 \end{aligned} \tag{8.42}$$

while the equivalence of the composition with the identity Interface, equations (8.37)-(8.40) become the four somewhat more tractable equations:

$$\begin{aligned}
 \check{\alpha}_{><}^{--} \check{\alpha}_{><}^{--} + \check{\beta}_{ijk}^{(1)} \check{\alpha}_{><}^{\geq-} &= \mathbf{Id}_{R_{ik}^{(1)}} \\
 \check{\alpha}_{><}^{--} \check{\alpha}_{><}^{--} + \check{\alpha}_{><}^{-\leq} \check{\beta}_{ikj}^{(2)} &= \mathbf{Id}_{R_{ik}^{(2)}} \\
 \check{\alpha}_{><}^{--} \check{\alpha}_{><}^{-\leq} + \check{\beta}_{ijk}^{(1)} \check{\alpha}_{><}^{\geq\leq} &= 0 \\
 \check{\alpha}_{><}^{\geq-} \check{\alpha}_{><}^{--} + \check{\alpha}_{><}^{\geq\leq} \check{\beta}_{ikj}^{(2)} &= 0.
 \end{aligned} \tag{8.43}$$

---

<sup>46</sup>Our convention here is that subsequent composition of linear transformations are written on the *right*.

It helps to draw a quiver-like diagram to represent the linear transformations and their constraints as shown in Figure 117.

There will be a moduli space of solutions to these constraints. Some general facts are readily deduced. For example it is an easy exercise to show from these equations that  $\check{\alpha}_{<>}^--\check{\alpha}_{><}^-$  and  $\check{\alpha}_{><}^-\check{\alpha}_{<>}^-$  are projection operators onto subspaces  $V_1 \subset R_{ik}^{(1)}$  and  $V_2 \subset R_{ik}^{(2)}$  and that  $\check{\alpha}_{<>}^-$  is an isomorphism from  $V_1$  to  $V_2$ . We may therefore take

$$R_{ik}^{(1)} = V \oplus W_1 \quad R_{ik}^{(2)} = V \oplus W_2 \quad (8.44)$$

Therefore  $\check{\beta}_{ijk}^{(1)}\check{\alpha}_{><}^-$  and  $\check{\alpha}_{><}^-\check{\beta}_{ikj}^{(2)}$  are orthogonal projectors onto  $W_1$  and  $W_2$ , respectively. We will not try to give the most general solution to the constraints. The simplest solution of all our constraints is obtained when

$$R_{ij} \otimes R_{jk} = W_1 \oplus W_2 \oplus U \quad (8.45)$$

and then to take  $\check{\alpha}_{><}^{><} = 0$  and <sup>47</sup>

$$\begin{aligned} \check{\alpha}_{><}^- &= \check{\alpha}_{<>}^- = P_V \\ \check{\alpha}_{><}^< &= \check{\beta}_{ikj}^{(2)} = P_{W_2} \\ \check{\beta}_{ijk}^{(1)} &= P_{W_1}^{[1]} \quad \check{\alpha}_{><}^> = P_{W_1}^{[-1]} \end{aligned} \quad (8.46)$$

The superscripts in the last line indicate a degree-shift. Indeed, when passing from the amplitude  $\alpha$  to the linear transformation  $\check{\alpha}$  we must use the degree  $-1$  isomorphism of  $R_{ij} \rightarrow R_{ji}^*$ , and so on. Therefore,  $\check{\alpha}_{><}^-$ ,  $\check{\alpha}_{<>}^-$ ,  $\check{\alpha}_{><}^<$ , and  $\check{\beta}_{ikj}^{(2)}$  all have degree 0, etc.

In terms of physics,  $V$  represents  $ik$  solitons which are unchanged by the wall-crossing, while  $W_1$  and  $W_2$  are sets of solitons which are gained or lost during the wall-crossing. Those subspaces are isomorphic to subspaces of  $R_{ij} \otimes R_{jk}$  (and indeed correspond to boundstates). Thanks to the degree assignments of  $\check{\alpha}$  and  $\check{\beta}$  we see that  $W_1$  and  $W_2$  contribute with opposite signs in computing the index on  $R_{ij} \otimes R_{jk}$  (while  $U$  is a subspace which contributes zero) and in this sense we can say, informally, that the categorified Cecotti-Vafa-Kontsevich-Soibelman wall-crossing formula is

$$\begin{aligned} R_{ik}^{(2)} - R_{ik}^{(1)} &= (R_{ij} \otimes R_{jk})^+ - (R_{ij} \otimes R_{jk})^- \\ &= (R_{ij}^+ - R_{ij}^-) \otimes (R_{jk}^+ - R_{jk}^-) \end{aligned} \quad (8.47)$$

where the superscript  $\pm$  on the right hand side refers to the sign of  $(-1)^F$ .

It is time to stop and assess our results. We have given an explicit description of a pair of “minimal” wall-crossing interfaces  $\mathfrak{I}_{<>}$  and  $\mathfrak{I}_{><}$ , which exist as long as the web representations before and after wall-crossing are related in a natural way, as described by the above decomposition. We have not checked that  $\mathfrak{I}_{<>}$  and  $\mathfrak{I}_{><}$  intertwine with rotation interfaces, nor that one can encode the relation between the two theories enforced by  $\mathfrak{I}_{<>}$

---

<sup>47</sup>There is a slight abuse of notation here.  $P_V$  here denotes the projection to  $V$  composed with the identity map to the subspace  $V$  in the codomain. We have suppressed this in an attempt to keep the equations readable.

and  $\mathfrak{I}_{><}$  into an  $L_\infty$  (or better, an  $LA_\infty$ ) morphism. We leave these problems to future work.

In the context of LG theories, as described in Section §12.3 the  $R_{ik}$  spaces are generated by certain solitons interpolating between the critical points of the superpotential associated to the vacua  $i$  and  $k$ . At a wall-crossing, such solitons only appear and disappear generically when the critical point  $j$  hits the soliton, splitting it into  $ij$  and  $jk$  solitons. The subspaces  $V$ ,  $W_1$  and  $W_2$  should be generated by solitons which respectively are not hit by  $j$ , or are hit when approaching the wall in parameter space from either side. It should also be possible to test our solution for the jump in interior amplitudes for LG theories. We leave that problem, as well, to future work.

We expect our proposal for the wall-crossing of the  $R_{ij}$  spaces to hold universally for massive  $(2, 2)$  theories, in the sense that the “true” wall-crossing interfaces should always factor through our  $\mathfrak{I}_{<>}$  or  $\mathfrak{I}_{><}$ , up to inner auto-equivalences or other equivalences associated to phantom walls (See Remarks 2,3 and 4 at the end of Section §8.1.)

## 9. Local Operators And Webs

This section develops some formalism for discussing local operators on the plane, in the context of the web formalism.

We have already identified the local boundary operators on the half-plane between two Branes  $\mathfrak{B}_1$  and  $\mathfrak{B}_2$  with  $\text{Hop}(\mathfrak{B}_1, \mathfrak{B}_2)$ . More precisely, using the first  $A_\infty$  multiplication  $M_1$ ,  $\text{Hop}(\mathfrak{B}_1, \mathfrak{B}_2)$  is a complex whose cohomology is meant to be the space of  $\mathcal{Q}$ -cohomology classes of local operators preserving suitable supersymmetries. As explained in Section §11.2.1 below, the physical context for these operators is the “A-model with superpotential.” As explained in Section §16.2, this space of local operators includes both order and disorder operators and is slightly unusual in discussions of Landau-Ginzburg models.

Now, in Section §16.3 we show that further new ideas are needed to discuss local operators in the bulk. This proves to be the case in the approach from the web-based formalism as well. We should stress one point: In the Landau-Ginzburg model it is quite natural to look at the Jacobian ideal  $\mathbb{C}[\phi^I]/(dW)$  (or its generalizations with curved target space). This is the chiral ring for the B-twisted model, and is *not* the local operators relevant to the “A-model with superpotential.” The latter has a subspace of local operators given by the DeRham cohomology of the target, although these are only the order operators, and in principle there will be other, disorder operators, in the space of local operators.

### 9.1 Doubly-Extended Webs And The Complex Of Local Operators On The Plane

The guiding principle for generalizing the complex  $\text{Hop}(\mathfrak{B}_1, \mathfrak{B}_2)$  to the case of operators on the plane will be the relation to the complex of groundstates provided by the exponential map

$$u + iv = e^{-ix+y}. \tag{9.1}$$

Recall that, for boundary operators, the spinning webs, (7.4), with uniform rotation  $\vartheta(x) = -x$  on an interval of length  $\pi$  map to half-plane webs with a marked point on the boundary

at  $u = v = 0$ . This point corresponds to the far past  $y \rightarrow -\infty$  on the strip. The relation of the complex of local operators to the complex of groundstates is summarized in equation (7.53).

We now imitate the above discussion for closed strings. Accordingly, we will map the infinite cylinder with coordinates  $(x, y)$  and  $x \sim x + 2\pi$  to the plane using the exponential map (9.1). Equivalently, we can consider periodic webs on the strip in the  $(x, y)$  plane with  $x_\ell \leq x \leq x_\ell + 2\pi$ . We again consider curved spinning webs with uniform rotation  $\vartheta(x) = -x$ . The complex of groundstates on the strip will be given by the trace of the matrix of Chan-Paton factors of the rotation Interface  $\Re[\vartheta_\ell, \vartheta_\ell - 2\pi]$ . (See the discussion in Section §7.3, and further development in Section §9.2 below.) All binding points are future stable and, from equation (7.28) see see that each *cyclic* fan of vacua will fit on the cylinder. We thus might expect the complex of groundstates to be simply the complex  $R^{\text{int}} = \bigoplus_I R_I$  we have met before. This is not quite right since the constant vacuum  $i$ , corresponding to the “fan”  $\{i\}$  is also an approximate groundstate. Therefore, for each vacuum  $i$  we introduce a module  $R_i \cong \mathbb{Z}$ , in degree zero, and we define

$$\begin{aligned} R_c &:= [\bigoplus_{i \in \mathbb{V}} R_i] \oplus R^{\text{int}} \\ &= [\bigoplus_{i \in \mathbb{V}} R_i] \oplus [\bigoplus_I R_I] \end{aligned} \tag{9.2}$$

The complex  $R_c$  defined in equation (9.2) is nicely in accord with the MSW complex of semiclassical twisted ground states discussed in Section §16.3.1 below. The summands  $R_i$  correspond to states in the constant vacuum  $\phi_i$  that sits at a critical point of the superpotential. The summands  $R_I$  where  $I$  has length greater than one correspond, for large radius of the cylinder, to the fans of solitons.

Now, we would like to define a differential  $d_c$  on  $R_c$  to make it into a complex. We follow the lead of the complex of approximate ground states defined in Section §4.3 above. We should contract incoming states at  $y = -\infty$  on the strip with all taut webs, saturating all boundary and interior vertices with boundary and interior amplitudes, as in equation (4.59).

The image under the exponential map (9.1) of a taut spinning periodic web will be one of two types, illustrated in Figures 118 and 119. In the first type, there is a fan of vacua  $I$  at  $y \rightarrow -\infty$  of length larger than one. In the second type, the fan at  $y \rightarrow -\infty$  consists of a single vacuum  $\{i\}$ . In the first case the image of the taut curved web in the  $(u + iv)$ -plane is a taut web with one vertex at the origin, as shown in Figure 118. If  $I$  is the fan of vacua at  $y \rightarrow -\infty$  and if  $r_I \in R_I$ , then we define

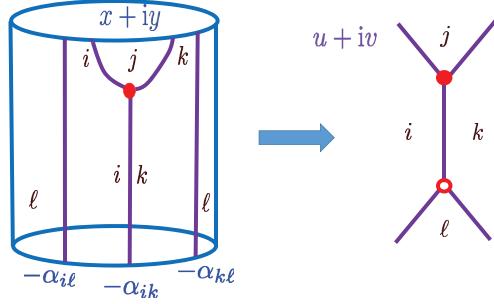
$$d_c(r_I) := \rho_\beta[\mathbf{t}_{pl}](r_I), \tag{9.3}$$

where  $\mathbf{t}_{pl}$  is the taut planar element on the  $(u + iv)$ -plane. Thus, all vertices except for the one at the origin are saturated with the interior amplitude  $\beta$ . The second type of taut spinning periodic web will lead to a map

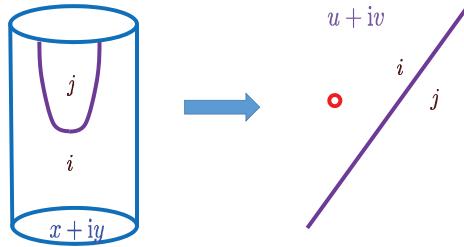
$$R_i \rightarrow \bigoplus_{j \neq i} R_{ij} \otimes R_{ji}. \tag{9.4}$$

To define this map return to Figure 73 and use equations (7.42)-(7.44) to write

$$d_c(\phi_i) := \bigoplus_{j \neq i} K_{ij}^{-1} \tag{9.5}$$



**Figure 118:** This figure represents the relation between a taut curved web on the  $x + iy$  cylinder and a taut extended web on the  $u + iv$  plane. The open red circle is the origin of the plane and corresponds to the  $y \rightarrow -\infty$  limit of the cylinder under the map  $u + iv = e^{-ix+y}$ . The vertical lines extending from  $y = -\infty$  to  $+\infty$  at  $x = -\alpha_{il}$  and  $x = -\alpha_{ie}$  have no moduli, and the vertex can only move vertically at  $x = -\alpha_{ik}$ . Taut webs are used to define a differential on the complex  $R_c$ . Contraction with the taut web shown here takes an element  $r \in R_{\{i,k,\ell\}}$  to  $\rho_\beta[\mathfrak{t}](r)$  which in this case is just  $\rho[\mathfrak{t}](r \otimes \beta_{ijk})$ . The cylindrical picture is meant to motivate this operation as a transition amplitude from an approximate ground state in the far past of the cylinder to a state in the future.



**Figure 119:** The web on the cylinder has one modulus and hence is taut as a curved web. It can be considered as a map from a state in  $R_i$  to a state in  $R_{ij} \otimes R_{ji}$ . It is natural to give this map the amplitude  $K_{ij}^{-1}$  and indeed that completes the operation of Figure 118 to a differential. The image under the exponential map shows that we should broaden our notion of extended webs to doubly-extended webs by including a new kind of vertex in the faces of the webs.

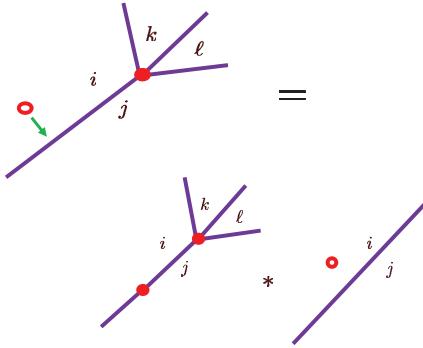
where  $\phi_i$  is a generator of  $R_i \cong \mathbb{Z}$  defined below. The crucial property  $d_c^2 = 0$  will follow from our discussion of “doubly-extended webs” below.

The cohomology  $H^*(R_c, d_c)$  is to be identified with the space of local operators. Recall

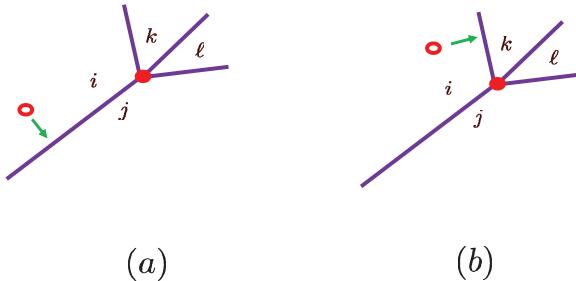
that under the isomorphism  $R_{ij} \otimes R_{ji} \rightarrow R_{ji} \otimes R_{ij}$ , the element  $K_{ij}^{-1}$  is antisymmetric. It therefore follows that

$$\mathbf{1} := \oplus_{i \in \mathbb{V}} \phi_i \quad (9.6)$$

is always closed and defines a canonical element of the cohomology. This element simply corresponds to the unit operator.



**Figure 120:** The pictorial demonstration of a convolution for doubly-extended webs. The web with the closed vertex in vacuum  $i$  has four moduli - two for each vertex and is therefore a sliding web. Near one of the boundaries of its moduli space it can be expressed as a convolution of two taut webs.



**Figure 121:** The two boundaries illustrated in (a) and (b) correspond to two boundaries leading to cancelling contributions in the contribution of  $d(d(\phi_i))$  to the summand of  $R_c$  with fan  $I = \{i, k, \ell, j\}$ .

We now show that  $R_c$  is indeed a complex. In fact, it is an  $L_\infty$  algebra, extending the  $L_\infty$  algebra structure on the set of interior vectors  $R^{\text{int}}$ . To this end we introduce a notion

of *doubly-extended webs*. These are plane webs, defined as before, but now we introduce a new kind of vertex, called a *closed vertex* that can be inserted into the interior of the faces of an ordinary extended plane web. These closed vertices are denoted by open circles in Figures 120 and 121. We can now define oriented deformation type and convolution in straightforward ways. Each closed vertex adds two moduli. The boundaries of the moduli space include ends where a closed vertex approaches a bounding edge of a face. Given a web  $\mathfrak{w}$ , let  $\tilde{\mathfrak{w}}$  be the new doubly-extended web where a closed vertex has been inserted into some face at a point  $(u_o, v_o)$ . Then the orientations of the two webs are related by

$$o(\tilde{\mathfrak{w}}) = o(\mathfrak{w}) \wedge (du_o dv_o) \quad (9.7)$$

All the convolution identities work as for (extended) plane webs. An illustration of an important convolution involving a closed vertex is shown in Figure 120. An example of cancelling ends in a convolution identity is shown in Figure 121.

With a representation of webs we can define generators  $\phi_j$  of  $R_j$ ,  $j \in \mathbb{V}$ , by

$$\rho(\tilde{\mathfrak{w}})[S_1, \phi_j, S_2] := \delta_{i,j} \rho(\mathfrak{w})[S_1, S_2] \quad (9.8)$$

where  $\mathfrak{w}$  and  $\tilde{\mathfrak{w}}$  are related as before and the closed vertex in  $\tilde{\mathfrak{w}}$  is inserted in a face marked with the vacuum  $i$ . Here  $S_1, S_2 \in TR_c$ . The generator  $\phi_i$  of  $R_i$  is the same as that used in equation (9.5), as one can check by carefully comparing orientations in Figure 120. The demonstration of the  $L_\infty$  algebra structure on  $R_c$  completely parallels that used before for  $R^{\text{int}}$ . In particular, the demonstration that  $d(d(\phi_i)) = 0$  follows from the consideration of ends of moduli space such as those shown in Figure 121.

## 9.2 Traces Of Interfaces

There is a useful, alternative perspective on the cylinder geometry with  $\vartheta(x) = -x$ . Up to an homotopy, we can deform the  $\vartheta(x)$  profile so that the variation happens on a small scale compared to the size of the circle. Thus the geometry reduces to a cylinder with essentially constant vacuum weights, and an interface  $\Re[2\pi, 0]$  inserted at  $x = 0$ . The complex of approximate ground states is essentially the same as  $R_c$ , though the differential will only be chain-homotopic to  $d_c$ .

This is a special case of a construction which is available every time we have an interface  $\mathcal{I} \in \mathfrak{Br}(\mathcal{T}, \mathcal{T})$  between a Theory  $\mathcal{T}$  and itself. Let us define the *trace of the Interface*  $\mathcal{I}$ , denoted  $\text{Tr}(\mathcal{I})$ , to be the complex of approximate groundstates on the cylinder with  $\mathcal{I}$  running along the axis of the cylinder. Thus, the underlying  $\mathbb{Z}$ -module of the complex is

$$\text{Tr}(\mathcal{I}) = \bigoplus_{i \in \mathbb{V}} \mathcal{E}(\mathcal{I})_{ii} \quad (9.9)$$

To define the differential we consider “periodic webs.” These are webs in  $\mathbb{R} \times S^1$  where the edges have constant slope. In other words, if we unroll the cylinder by cutting along a vertical line to get a strip, then the edges of the webs are straight lines in the strip. Periodic webs are a close analogue to strip webs or composite webs on a strip geometry.

Taut webs have two moduli (one of which is translation along the axis of the cylinder).<sup>48</sup> The taut element  $\mathbf{t}_c$  satisfies a convolution identity

$$\mathbf{t}_c \circ \mathbf{t}_c + \mathbf{t}_c * \mathbf{t}_{pl} + \mathbf{t}_c * \mathbf{t}_{\mathcal{I}} = 0 \quad (9.10)$$

and hence  $d_c = \rho_{\beta}[\mathbf{t}_c](\frac{1}{1-\mathcal{B}})$ , where  $\mathcal{B}$  is the boundary amplitude of the Interface, is a differential.

In general we claim that  $\text{Tr}(\mathcal{I} \boxtimes \mathfrak{R}[0, \pm 2\pi])$  is homotopy equivalent to the complex of groundstates on the strip  $x \sim x + 2\pi$  with  $\mathcal{I}$  inserted and with spinning vacuum weights, spinning by  $\vartheta(x) = -x$ . In particular, taking  $\mathcal{I}$  to be the identity Interface, and the definition (7.28) we see that the Chan-Paton data of  $\mathfrak{R}[\vartheta, \vartheta - 2\pi]$  are given by

$$\oplus_{i \in \mathbb{V}} (R_i \oplus \widehat{R}'_{ii}) e_{ii} \oplus \oplus_{i < j} \widehat{R}_{ij}^+ e_{ij} \oplus \oplus_{i > j} \widehat{R}_{ij}^- e_{ij} \quad (9.11)$$

where  $\widehat{R}'_{ii}$  is the set of cyclic fans with vacuum  $i$  at  $x = 0$ . (See equation (7.41) above for the case of two vacua.) The trace of this matrix of complexes recovers the complex  $R_c$  of equation (9.2). Thus, given the results of the previous section, the space of local operators in a Theory can be described in terms of the trace of  $\mathfrak{R}[\vartheta, \vartheta - 2\pi]$ .

The above construction can be generalized. If we have a sequence of Interfaces  $\mathcal{I}_i \in \mathfrak{Br}(\mathcal{T}^i, \mathcal{T}^{i+1})$  at locations  $x_i$ ,  $i = 1, \dots, n$  along the periodic direction  $x \rightarrow x + 2\pi$ , interpolating between a periodic sequence of Theories  $\mathcal{T}^i$  then we can consider the trace of the product

$$\text{Tr}(\mathcal{I}_1 \boxtimes \cdots \boxtimes \mathcal{I}_n) = \oplus_{j_1, \dots, j_n} \otimes_{i=1}^n \mathcal{E}(\mathcal{I}_i)_{j_i, j_{i+1}} \quad (9.12)$$

There will be many different homotopy-equivalent differentials. If we consider again the taut element for periodic composite webs  $\mathbf{t}_c$  (generalizing that used above for a single Interface) then the convolution identity generalizes to

$$\mathbf{t}_c \circ \mathbf{t}_c + \mathbf{t}_c * \mathbf{t}_{pl} + \sum_{i=1}^n \mathbf{t}_c * \mathbf{t}_{\mathcal{I}_i} = 0 \quad (9.13)$$

and therefore  $d_c = \rho_{\beta}[\mathbf{t}_c](\otimes_{i=1}^n \frac{1}{1-\mathcal{B}_i})$  defines a differential on (9.12).

### 9.3 Local Operators For The Theories $\mathcal{T}^N$ and $\mathcal{T}^{SU(N)}$

We comment briefly on the computation of the cohomology of  $R_c$  for the two examples of Theories  $\mathcal{T}^N$  and  $\mathcal{T}^{SU(N)}$  discussed throughout this text.

Let us consider first  $\mathcal{T}^N$ . According to Section §4.6.4 this is meant to coincide with the A-model with target space  $X = \mathbb{C}$  and superpotential (4.138). For the strict A-model the target space cohomology  $H_{DR}^*(X)$  has a single operator in degree zero corresponding to the unit operator. Nevertheless, as discussed in Section §16.3 the full space of local operators can in principle include disorder operators, and  $H_{DR}^*(X)$  is only a subspace of the space of local operators. In fact, for the  $\mathcal{T}^N$  Theories, the cohomology of  $R_c$  is one-dimensional, and spanned by the unit operator  $\mathbf{1}$  as the following computation shows.

---

<sup>48</sup>In the absence of the Interface  $\mathcal{I}$ , the line principle shows the only webs would be unions of closed loops wrapping around the cylinder.

The complex  $R_c$  takes the form

$$\oplus_i R_i \rightarrow \oplus_{i < j} R_{ij} \otimes R_{ji} \rightarrow \oplus_{i < j < k} R_{\{i,j,k\}} \rightarrow \cdots \rightarrow \oplus_i R_{\widehat{i}} \rightarrow R_{\{0,1,\dots,N-1\}} \rightarrow 0 \quad (9.14)$$

where in degree  $N - 2$ ,  $\widehat{i}$  denotes the fan that omits  $i$  from  $\{0, \dots, N - 1\}$ . Using (4.79) this becomes

$$\oplus^N \mathbb{Z} \rightarrow \oplus^{\binom{N}{2}} \mathbb{Z}^{[1]} \rightarrow \cdots \rightarrow \oplus^{\binom{N}{j}} \mathbb{Z}^{[j-1]} \rightarrow \cdots \rightarrow \mathbb{Z}^{[N-1]} \rightarrow 0 \quad (9.15)$$

The Witten index is thus automatically 1. We can in fact do better and compute the cohomology as follows. We can identify the complex with a subspace of a Grassmann algebra  $\mathcal{G} = \mathbb{Z}[\theta_0, \dots, \theta_{N-1}]/\mathcal{I}$  where the ideal  $\mathcal{I}$  is generated by  $\theta_i \theta_j + \theta_j \theta_i = 0$ , for all  $i, j$  and the  $\theta_i$  have degree +1. We identify  $R_c$  with the subspace of  $\mathcal{G}$  of elements of degree at least one and then shift the degree by  $-1$ . To see this, identify a generator of  $R_{i_1, i_2} \otimes \cdots R_{i_{k-1}, i_k} \otimes R_{i_k, i_1}$ , where  $i_1 < i_2 < \cdots < i_k$  with  $\theta_{i_1} \theta_{i_2} \cdots \theta_{i_k}[-1]$ . Then, for any fan, the differential acts by diagrams like those of Figure 118. Since the interior amplitude is only nonzero for 3-valent vertices with  $b_{ijk} = 1$  for  $i < j < k$  it is easy to see that the differential  $d_c$  is the same as the action of multiplication by  $\Theta = \theta_0 + \cdots + \theta_{N-1}$ . There is thus a clear chain-homotopy inverse between the zero map and the projection operator onto on elements of  $R_c$  of positive degree. It is given by:

$$\kappa(r) := \begin{cases} \left( \frac{\partial}{\partial \theta_0} + \cdots + \frac{\partial}{\partial \theta_{N-1}} \right) r & \deg(r) > 0 \\ 0 & \deg(r) = 0 \end{cases} \quad (9.16)$$

Thus,

$$(\kappa d_c + d_c \kappa)(r) := \begin{cases} Nr & \deg(r) > 0 \\ Nr - \text{tr}(r)\mathbf{1} & \deg(r) = 0 \end{cases} \quad (9.17)$$

Here  $\mathbf{1} = \sum_{i \in V} \phi_i$  is the unit operator discussed above. In this example  $\mathbf{1} = \sum_{i=0}^{N-1} \theta_i[-1]$ . Moreover, if  $r = \sum x_i \theta_i[-1]$  is of degree zero we define  $\text{tr}(r) = \sum_i x_i$ . Therefore the cohomology is generated by the unit operator.<sup>49</sup>

Turning now to the  $\mathcal{T}^{SU(N)}$  Theory the physical expectation from Section §4.6.4 is that it should correspond to an A-model with superpotential  $W = \sum_{i=1}^N Y_i$  on the space  $\Xi$  defined by  $\Xi = \{(Y_1, \dots, Y_N) | Y_1 \cdots Y_N = q\} \subset (\mathbb{C}^*)^N$  where  $q \neq 0$ . By [47] this should be mirror to the B-model on  $\mathbb{CP}^{N-1}$ . We use this mirror dual pair to check our proposal for the local operators using the complex  $R_c$  constructed from the representation of webs described in equation (4.100) above.

In general, the B-model with target space  $X$  has a space of local operators

$$\oplus_{p,q} H^p(X, \Lambda^q T^{1,0} X). \quad (9.18)$$

---

<sup>49</sup>This argument only suffices to determine the cohomology up to  $N$ -torsion. From examples we find that the cohomology is in fact isomorphic to  $\mathbb{Z}$  in degree zero. In physics the space of local operators is a vector space over  $\mathbb{C}$ , but in the web formalism one can work over  $\mathbb{Z}$ . This raises the interesting question of whether there can be torsion in the cohomology of  $R_c$ , and what its physical meaning would be, if any. We leave that for another time.

In the present case we compute the cohomology, as a representation of  $\text{su}(N)$  to be:

$$H^p(X, \Lambda^q T^{1,0} \mathbb{CP}^{N-1}) = \begin{cases} L_{N-q,q+1} & p = 0 \\ 0 & p > 0 \end{cases} \quad (9.19)$$

where the  $L_{n,m}$  were defined in Section §4.6. The result is in fact very intuitive. Cohomology classes with  $p = 0$  can be represented by global sections

$$C_{i_1 \dots i_q}^{j_1 \dots j_q} X^{i_1} \dots X^{i_q} \frac{\partial}{\partial X^{j_1}} \wedge \dots \wedge \frac{\partial}{\partial X^{j_q}}, \quad (9.20)$$

where  $[X^1 : \dots : X^N]$  are homogeneous coordinates. The  $SU(N)$  tensor  $C_{i_1 \dots i_q}^{j_1 \dots j_q}$  is totally symmetric in  $i_k$  and totally antisymmetric in  $j_k$  and therefore in  $A_{N-q} \otimes S_q$ . Now we must identify by the image of holomorphic vector fields and this requires  $C_{i_1 \dots i_q}^{j_1 \dots j_q}$  to be traceless. Referring to the decomposition (4.121), the traceless part is the second summand. The cohomology is one-dimensional in degree  $q = 0$  and isomorphic to the adjoint in degree  $q = 1$ . Indeed, the one-dimensional cohomology should be a Lie algebra of global symmetries of the theory on *a priori* grounds and that is the  $\text{su}(N)$  symmetry of the present example.

We leave it as an interesting and nontrivial challenge to reproduce (9.19) from the cohomology of  $R_c$ , as defined in equation (9.2).

We can perform one nontrivial check on this identification by examining the character-valued index. Let  $t = \text{Diag}(t_1, \dots, t_N)$  be a generic diagonal element of  $SU(N)$ . It acts naturally on the homogeneous coordinates of  $\mathbb{CP}^{N-1}$  thereby inducing an action on  $H^p(X, \Lambda^q T^{1,0} X)$ . By the Atiyah-Bott fixed point formula we have<sup>50</sup>

$$\begin{aligned} F_N(x) &:= \sum_{q=0}^{N-1} x^q \sum_{p=0}^{N-1} (-1)^p \text{Tr}_{H^p(X, \Lambda^q T^{1,0} \mathbb{CP}^{N-1})} t \\ &= \sum_{i=1}^N \frac{\prod_{j \neq i} (1 + xt_i/t_j)}{\prod_{j \neq i} (1 - t_j/t_i)} \end{aligned} \quad (9.22)$$

In particular, the character-valued index is

$$F_N(-1) = \sum_{i=1}^N \frac{\prod_{j \neq i} (1 - t_i/t_j)}{\prod_{j \neq i} (1 - t_j/t_i)} = (-1)^{N-1} (t_1 \cdots t_N)^{-1} \sum_{i=1}^N t_i^N = (-1)^{N-1} \sum_{i=1}^N t_i^N \quad (9.23)$$

where in the second equality we used the property that  $t \in SU(N)$ .

---

<sup>50</sup>Incidentally, there is an elegant argument to recover the representations of (9.19) from this formula. Consider

$$U_N(z, x) := \frac{1}{z} \frac{\prod_{j=1}^N (1 + xz/t_j)}{\prod_{j=1}^N (1 - t_j/z)} dz \quad (9.21)$$

By equating the residue at  $z = \infty$  with the sum of the finite residues, using the generating functions for characters of symmetric and antisymmetric representations, and using the decomposition (4.121) one can reproduce (9.19).

On the other hand, by direct computation of the character-valued index of  $R_c$ , using the characters of the anti-symmetric representations  $R_{ij}$  we find

$$\mathrm{Tr}_{R_c}(-1)^F = N - N \prod_{i=1}^N t_i + (-1)^{N-1} \sum_{i=1}^N t_i^N = (-1)^{N-1} \sum_{i=1}^N t_i^N \quad (9.24)$$

where in the second equality we used the property that  $t \in SU(N)$ . We checked equation (9.24) for  $N = 2, 3, 4, 5, 6$ , but giving a direct proof of this equation looks difficult. Fortunately the methods of Section §7.10.1 above can be used to give a proof: The Interface  $\mathfrak{I}^{++}$  of that section implements a rotation by  $2\pi/N$ , and hence its  $N^{th}$  power gives the full Interface for rotation by  $2\pi$ . The eigenvalues of the character-valued index of the Chan-Paton data of this Interface follow from (7.182), and are simply  $(-t_i)$ ,  $i = 1, \dots, N$ . Using the relation of the complex  $R_c$  to the trace of the Interface explained in Section §9.2 we arrive at equation (9.24).<sup>51</sup>

Note that the cohomology is much larger than the naive DeRham cohomology of  $\Xi \cong (\mathbb{C}^*)^{N-1}$  that one might associate to the  $A$ -model on  $\Xi$ . Indeed, the dimension of the  $H^*(R_c, d_c)$  for  $\mathcal{T}^{SU(N)}$  is given by  ${}_2F_1(1-N, N+1, 1; -1)$ , and this grows with  $N$  far more rapidly than  $2^{N-1}$ . Thus, there are many disorder operators. Indeed, we can already see the need for disorder operators for the case  $N = 2$  discussed in detail in Section §16.3.3 below.

## 10. A Review Of Supersymmetric Quantum Mechanics And Its Relation To Morse Theory

In Sections §§11-17 below we will sketch our main physical application of the formalism we have developed. That application is based in turn on standard ideas about the interpretation of Morse theory in terms of supersymmetric quantum mechanics [87]. While this material is well-known, and is nicely reviewed, for example, in [50], we would like to emphasize several key points which are of particular importance in our application.

### 10.1 The Semiclassical Approximation

We start by reviewing supersymmetric quantum mechanics and its relation to Morse theory (see [87] and section 10 of [50]), since much of our subject can be developed in close parallel to this. Much of this material may be familiar to many readers, but in section 10.6 we explain a point that may be less familiar and that is crucial background for the present paper.

We begin with a Riemannian manifold  $M$  of dimension  $n$ , with local coordinates  $u^a$ ,  $a = 1, \dots, n$ , a metric tensor  $g_{ab}$ , and a smooth real-valued function  $h$ , called the superpotential. A critical point of  $h$  is a point at which its gradient vanishes,  $\partial h / \partial u^a = 0$ ,  $a = 1, \dots, n$ , and

---

<sup>51</sup>Actually, there is a disagreement by an  $N$ -independent minus sign. We have not sorted out the explanation of this sign.

a critical point is called nondegenerate if at this point the matrix of second derivatives<sup>52</sup>  $\partial^2 h / \partial u^a \partial u^b$  is nondegenerate. (This matrix is sometimes called the Hessian.) We will assume that  $h$  is a Morse function. A Morse function is simply a smooth function such that all critical points  $\phi_i$  are nondegenerate. For the moment, we assume  $M$  to be compact, in which case the number of critical points of  $h$  is finite, but sometimes one wishes to relax the compactness assumption and also allow infinitely many critical points.

From this data, we construct a supersymmetric quantum mechanics model, describing maps from  $\mathbb{R}^{1|2}$  – a supersymmetric worldline with a real coordinate  $t$  and odd coordinates  $\theta$  and  $\bar{\theta}$  – to  $M$ . The supersymmetry algebra<sup>53</sup> is generated by the odd vector fields on  $M$

$$\mathcal{Q} = \frac{\partial}{\partial \bar{\theta}} + i\theta \frac{\partial}{\partial t}, \quad \bar{\mathcal{Q}} = \frac{\partial}{\partial \theta} + i\bar{\theta} \frac{\partial}{\partial t}; \quad (10.1)$$

the only nonzero anticommutator of these operators is

$$\{\mathcal{Q}, \bar{\mathcal{Q}}\} = 2H, \quad H = i\partial_t. \quad (10.2)$$

This supersymmetry algebra admits a group  $U(1)$  of outer automorphisms called the  $R$ -symmetry group, generated by a charge  $\mathcal{F}$  that assigns the values 1 and  $-1$  to  $\theta$  and  $\bar{\theta}$ , respectively. This will be a symmetry of the models we consider. The supersymmetry algebra commutes in the  $\mathbb{Z}_2$ -graded sense with the operators

$$D = \frac{\partial}{\partial \bar{\theta}} - i\theta \frac{\partial}{\partial t}, \quad \bar{D} = \frac{\partial}{\partial \theta} - i\bar{\theta} \frac{\partial}{\partial t}, \quad (10.3)$$

which are used in writing Lagrangians.

To construct a supersymmetric model that describes maps from  $\mathbb{R}^{1|2}$  to  $X$ , we promote the local coordinates  $u^a$  on  $M$  to superfields  $X^a(t, \theta, \bar{\theta}) = u^a(t) + i\bar{\theta}\psi^a(t) + i\theta\bar{\psi}^a(t) + \bar{\theta}\theta F^a(t)$ , where  $\psi^a$  and  $\bar{\psi}^a$  are Fermi fields with  $\mathcal{F} = 1$  and  $\mathcal{F} = -1$ , respectively, and the  $F^a$  are auxiliary fields.<sup>54</sup> It follows from this that

$$\begin{aligned} \{\mathcal{Q}, \bar{\psi}^a\} &= i\dot{u}^a + F^a \\ \{\bar{\mathcal{Q}}, \psi^a\} &= -i\dot{u}^a + F^a. \end{aligned} \quad (10.4)$$

One takes the action to be

$$I = \frac{1}{\lambda} \int dt d^2\theta \left( \frac{1}{2} g_{ab}(X^k) DX^a \bar{D} X^b - h(X^c) \right) \quad (10.5)$$

---

<sup>52</sup>In general, to define the second derivative of a function  $h$  on a Riemannian manifold  $M$ , we need to use the affine connection of  $M$ ; the only natural second derivative is  $D^2 h / Du^a Du^b$ , where  $D/Du^a$  is a covariant derivative. But at a critical point,  $D^2 h / Du^a Du^b$  reduces to the more naive  $\partial^2 h / \partial u^a \partial u^b$ , a fact that we incorporate in some formulas below.

<sup>53</sup>We write  $\mathcal{Q}$  and  $\bar{\mathcal{Q}}$  (rather than  $Q$  and  $\bar{Q}$ ) for the supercharges of the quantum mechanical model, since this will be more convenient in discussing the generalization to two-dimensional LG theories.

<sup>54</sup>When working with complex superalgebras we let  $*$  denote the complex anti-linear involution which acts on odd variables according to the rule  $(\theta_1 \theta_2)^* = \theta_2^* \theta_1^*$ . Our notation is such that  $\bar{\theta} = (\theta)^*$ , and so on. Hence  $X^i$  is a real superfield when  $F^i$  is real. We are using the same letter  $\mathcal{Q}$  for an operator on fields and for an odd vector field on a supermanifold, so the supersymmetry transformation rule is  $[i\mathcal{Q}, X^i] = \mathcal{Q}X^i$ . Unfortunately, our conventions differ from those in [50] by an exchange of  $\psi \leftrightarrow \bar{\psi}$ .

This action describes a supersymmetric  $\sigma$ -model in which the target space is  $M$ . Perturbation theory is a good approximation if  $\lambda$  is small, but  $\lambda$  can be eliminated from the formulas by rescaling  $g_{ab}$  and  $h$  (and to avoid clutter we do so in what follows). After integrating over  $\theta$  and  $\bar{\theta}$ , the action becomes<sup>55</sup>

$$I = \int dt \left( \frac{1}{2} g_{ab} \dot{u}^a \dot{u}^b + i g_{ab} \bar{\psi}^a \frac{D}{Dt} \psi^b + \frac{1}{2} g_{ab} F^a F^b - F^a \partial_a h + \bar{\psi}^a \psi^b \frac{D^2 h}{Du^a Du^b} + \dots \right), \quad (10.6)$$

where the covariant derivative  $D/Dt$  is defined using the pullback of the Levi-Civita connection of  $M$ , and we omit four-fermi terms. One can eliminate the auxiliary field  $F^a$  via its equation of motion

$$F^a = g^{ab} \partial_b h, \quad (10.7)$$

and for the ordinary potential energy, one finds

$$V(u^k) = \frac{1}{2} |\nabla h|^2 = \frac{1}{2} g^{ab} \partial_a h \partial_b h. \quad (10.8)$$

A classical ground state is therefore a critical point of  $h$ . Since we have assumed that  $h$  is a Morse function, there is a finite set  $\mathbb{V}$  of such critical points.

The fermion mass term that arises in expanding around a critical point can be read off from the action:

$$H_{\bar{\psi}\psi} = \frac{1}{2} [\psi^a, \bar{\psi}^b] \frac{\partial^2 h}{\partial u^a \partial u^b}. \quad (10.9)$$

(The bracket appearing here is an ordinary commutator, not a graded commutator, and accounts for an important normal ordering convention to preserve supersymmetry.) Since we assume that  $h$  is a Morse function, the fermion mass matrix  $m_{ab} = \partial^2 h / \partial u^a \partial u^b$  is nondegenerate at each critical point. In Morse theory the number of negative eigenvalues of the Hessian at a critical point  $p$  is called the *Morse index*. We denote it by  $n_p$ . Thus the number of negative eigenvalues of the mass matrix is  $n_p$ . Similarly the bosonic mass squared matrix that arises in expanding around  $p$  is positive-definite. So in expanding around a given critical point, all bosonic and fermionic modes are massive. Hence, from the standpoint of perturbation theory, there is precisely one minimum energy state  $\Phi_p$  for every critical point  $p \in \mathbb{V}$ . In perturbation theory, this state has zero energy and is annihilated by the supercharges  $Q$  and  $\bar{Q}$ . Indeed, the supersymmetry algebra

$$\{Q, \bar{Q}\} = 2H, \quad Q^2 = \bar{Q}^2 = 0, \quad (10.10)$$

implies that eigenstates of  $H$  with nonzero eigenvalue come in pairs, and therefore the fact that in expanding around the critical point  $p$  one finds only a unique ground state  $\Phi_p$  implies that in perturbation theory,  $\Phi_p$  is annihilated by  $Q$ ,  $\bar{Q}$ , and  $H$ , so it is a supersymmetric state of zero energy. (Beyond perturbation theory, as we discuss in section 10.4, nonperturbative effects can modify this statement.)

The states in the  $\sigma$ -model are not functions on  $M$ , but differential forms on  $M$ . To see this, we observe that a state of smallest fermion number must be annihilated by the  $\bar{\psi}^a$

---

<sup>55</sup>Here  $\int d^2 \theta \bar{\theta} \theta = \int d\theta d\bar{\theta} \bar{\theta} \theta = +1$ .

operators, but may have an arbitrary dependence on the bosonic variables  $u^a$ . So letting  $|\Omega\rangle$  denote a state annihilated by all  $\bar{\psi}^a$  and independent of the  $u^a$ , a state of minimum possible fermion number is  $f(u)|\Omega\rangle$ , where  $f(u)$  is an arbitrary function on  $M$ . A state whose fermion number is greater by  $n$  is then  $\sum_{a_1\dots a_n} f_{a_1 a_2 \dots a_n}(u) \psi^{a_1} \psi^{a_2} \dots \psi^{a_n} |\Omega\rangle$ , where in the language of differential geometry,  $\sum_{a_1\dots a_n} f_{a_1 a_2 \dots a_n}(u) du^{a_1} du^{a_2} \dots du^{a_n}$  is called an  $n$ -form on  $M$ . Thus quantum states in the  $\sigma$ -model correspond to differential forms on  $M$  and, up to an additive constant (which is the fermion number we assign to the state  $|\Omega\rangle$ ), the fermion number of a state is the degree of the corresponding differential form. In differential geometry, it is customary to define the degree of a differential form on a  $d$ -manifold to vary from 0 to  $d$ . However, the theory (10.5) has a symmetry  $\psi \leftrightarrow \bar{\psi}$  (“charge conjugation,” which in differential geometry is called the Hodge star operator on differential forms).  $\mathcal{F}$  is odd under this symmetry at the classical level, and to maintain this property quantum mechanically, we subtract an overall constant  $-d/2$  and say that an  $n$ -form corresponds to a state of fermion number  $\mathcal{F} = -d/2 + n$ .

To determine the fermion number of the low energy state  $\Phi_p$  associated to a given critical point  $p$ , we need to determine which modes of  $\psi$  and  $\bar{\psi}$  annihilate  $\Phi_p$ . All we need to know is that for a real number  $m$  and a single pair of fermion modes  $\psi, \bar{\psi}$ , the operator  $H_0 = \frac{m}{2}[\psi, \bar{\psi}]$  has (i) an eigenstate  $|\downarrow\rangle$  annihilated by  $\bar{\psi}$  with  $H_0 = -\frac{m}{2}$  and (ii) an eigenstate  $|\uparrow\rangle$  annihilated by  $\psi$  with  $H_0 = +\frac{m}{2}$ . For  $m > 0$ , the ground state of  $H_0$  is annihilated by  $\bar{\psi}$  but for  $m < 0$ , it is annihilated by  $\psi$ . So the number of modes of  $\psi$  that annihilate the ground state of the fermion mass operator  $H_{\bar{\psi}\psi}$  of eqn. (10.9) is equal to the number of negative eigenvalues of the matrix  $\partial^2 h / \partial u^a \partial u^b$ , or in other words, the Morse index of the critical point  $p$ . So if  $p$  has Morse index  $n_p$ , then  $\Phi_p$  is an  $n_p$ -form.

Thus if  $n_+$  and  $n_-$  are the number of positive and negative eigenvalues of the fermion mass matrix in expanding around a given critical point  $p$  (so  $d = n_+ + n_-$  and the Morse index of  $p$  is  $n_p = n_-$ ), then the fermion number of  $\Phi_p$  is

$$f_p = -\frac{1}{2} (n_+ - n_-). \quad (10.11)$$

A standard argument using the supersymmetry algebra (10.10) shows that the space of supersymmetric states – states annihilated by  $\mathcal{Q}, \bar{\mathcal{Q}}$ , and  $H$  – can be naturally identified with the cohomology of  $\mathcal{Q}$  (the kernel of  $\mathcal{Q}$  divided by its image). Here  $\mathcal{Q}$  is an operator mapping  $n$ -forms to  $n+1$ -forms and obeying  $\mathcal{Q}^2 = 0$ . In differential geometry, there is a standard operator with this property, the exterior derivative  $d$ . In differential geometry, it is usual written  $d = du^a \partial_{u^a}$ , which in our language would be  $\psi^a \partial_{u^a}$ . Hence  $\{d, \bar{\psi}^a\} = g^{ab} \partial_{u^b}$ . Recalling that in canonical quantization,  $\dot{u}^a$  maps to  $-ig^{ab} \partial_{u^b}$ , eqn. (10.4) tells us that  $\{\mathcal{Q}, \bar{\psi}^a\} = g^{ab} (\partial_{u^b} + \partial_b h)$ . So  $\mathcal{Q}$  does not coincide with  $\psi^a \partial_{u^a} = d$ ; rather,

$$\mathcal{Q} = \psi^a (\partial_{u^a} + \partial_a h) = e^{-h} de^h. \quad (10.12)$$

Thus  $\mathcal{Q}$  does not coincide with the exterior derivative  $d$ , but rather is conjugate to it.

This means that the cohomology of  $\mathcal{Q}$  is naturally isomorphic to the cohomology of  $d$ , which is usually called the de Rham cohomology of  $M$ : the cohomology of  $\mathcal{Q}$  is obtained from that of  $d$  by multiplying by the operator  $e^{-h}$ . In particular, the number of states of

precisely zero energy with fermion number  $-d/2 + n$  is the corresponding Betti number  $b_n$  (defined as the rank of the de Rham cohomology for  $n$ -forms) and does not depend on the choice of  $h$ . By contrast, the number of zero energy states found in perturbation theory for given  $n$  is the number of critical points of  $h$  of Morse index  $n$  and definitely does depend on  $h$ . (For example, if  $M$  is the circle  $0 \leq \varphi \leq 2\pi$ , the Morse function  $h = \cos k\varphi$  has  $2k$  critical points, half with index 0 and half with index 1.) So there will have to be nonperturbative effects that in general eliminate some of the vacuum degeneracy.

Since  $\overline{\mathcal{Q}}$  is the adjoint of  $\mathcal{Q}$ , it follows from (10.12) that

$$\overline{\mathcal{Q}} = \overline{\psi}^a (-\partial_{u^a} + \partial_a h) = e^h d^\dagger e^{-h}, \quad (10.13)$$

where  $d^\dagger$  is the adjoint of  $d$  in the standard  $L^2$  metric on differential forms. The Hamiltonian  $H = \{\mathcal{Q}, \overline{\mathcal{Q}}\}/2$  definitely does depend upon  $h$ , though the number of its zero energy states for each value of  $\mathcal{F}$  does not.

## 10.2 The Fermion Number Anomaly

The next topic we must understand is the fermion number anomaly. For this computation, we transform to Euclidean signature via  $t = -i\tau$ . The (linearized) Dirac equation for  $\psi$  and  $\overline{\psi}$  becomes

$$L\psi = 0 = L^\dagger \overline{\psi}, \quad (10.14)$$

with

$$\begin{aligned} (L\psi)^a &= \frac{D\psi^a}{D\tau} - g^{ab} \frac{D^2 h}{Du^b Du^c} \psi^c \\ (L^\dagger \overline{\psi})^a &= -\frac{D\overline{\psi}^a}{D\tau} - g^{ab} \frac{D^2 h}{Du^b Du^c} \overline{\psi}^c. \end{aligned} \quad (10.15)$$

We consider expanding around a path  $\ell \subset M$  that starts at one critical point  $q$  in the far past and ends at another critical point  $p$  in the far future.

Let  $n_q$  and  $n_p$  be the Morse indices of these critical points. We want to compute a vacuum-to-vacuum amplitude, that is, a transition between the initial state  $\Phi_q$ , of  $\mathcal{F} = -d/2 + n_q$ , and the final state  $\Phi_p$ , of fermion number  $\mathcal{F} = -d/2 + n_p$ . The fermion numbers of the initial and final states differ by  $n_p - n_q$ , so the amplitude must vanish unless one inserts operators that carry a net fermion number  $n_p - n_q$ .

As usual, the mechanism for this is that the index<sup>56</sup> of the operator  $L$ , which is defined as the number of zero-modes of  $L$  minus the number of zero-modes of its adjoint  $L^\dagger$ , is equal to  $n_p - n_q$ . Let us verify directly that this is true. Since the index is invariant under smooth deformations (which preserve the mass gap at infinity), it suffices to consider the case that the Levi-Civita connection of  $M$  is trivial along  $\ell$  and that the fermion mass matrix is diagonal along  $\ell$ . Thus it suffices to consider the case of a single pair of fermions

---

<sup>56</sup>There is a slight clash in the standard terminology here; the “index” of an operator should not be confused with the “Morse index” of a critical point of a function.

$\psi, \bar{\psi}$  with

$$\begin{aligned} L\psi(\tau) &= \frac{d\psi(\tau)}{d\tau} - w(\tau)\psi(\tau) \\ L^\dagger\bar{\psi}(\tau) &= -\frac{d\bar{\psi}(\tau)}{d\tau} - w(\tau)\bar{\psi}(\tau) \end{aligned} \quad (10.16)$$

where in the one-component case, we abbreviate  $D^2h/Du^2$  as  $w$ . We assume that the function  $w$  is nonzero for  $\tau \rightarrow \pm\infty$ . The equations  $L\psi = 0$  and  $L^\dagger\bar{\psi} = 0$  imply respectively

$$\begin{aligned} \psi(\tau) &= C \exp\left(\int_0^\tau d\tau' w(\tau')\right) \\ \bar{\psi}(\tau) &= C' \exp\left(-\int_0^\tau d\tau' w(\tau')\right), \end{aligned} \quad (10.17)$$

with constants  $C, C'$ . If  $w$  has the same sign for  $\tau \gg 0$  as for  $\tau \ll 0$ , then neither solution is square integrable. If  $w$  is positive for  $\tau \ll 0$  and negative for  $\tau \gg 0$ , then  $\psi$  has a normalizable zero-mode but not  $\bar{\psi}$ ; if  $w$  is negative for  $\tau \ll 0$  and positive for  $\tau \gg 0$ , then  $\bar{\psi}$  has a normalizable zero-mode but not  $\psi$ . In all cases, the index of the  $1 \times 1$  operator  $L$  for a single pair  $\psi, \bar{\psi}$  is the contribution of this pair to  $n_p - n_q$ . Summing over all pairs, the index of  $L$  equals  $n_p - n_q$ , as expected.

The index  $\iota(L)$  of the operator  $L$  always determines the difference between the number of  $\psi$  and  $\bar{\psi}$  zero-modes, but in fact generically one of these numbers vanishes and the other equals  $|\iota(L)|$ . For example, if  $\iota(L) \geq 0$ , generically there are no  $\bar{\psi}$  zero-modes, and the space of  $\psi$  zero-modes has dimension  $\iota(L)$ ; if  $\iota(L) \leq 0$ , generically there are no  $\psi$  zero-modes and the space of  $\bar{\psi}$  zero-modes has dimension  $-\iota(L)$ . The explicit calculation in the last paragraph shows that these statements are always true in the  $1 \times 1$  case; in fact, they hold generically. Informally,  $\iota(L)$  is a regularized difference between the number of variables and the number of conditions in the equation  $L\psi = 0$ . So for example if  $\iota(L) > 0$ , the equation  $L\psi = 0$  is analogous to a finite-dimensional linear problem with  $\iota(L)$  more variables than equations and generically has a space of solutions precisely of dimension  $\iota(L)$ .

### 10.3 Instantons And The Flow Equation

In general, a Morse function on  $M$  has too many critical points to match the de Rham cohomology of  $M$ , so there must be nonperturbative effects that shift some of the perturbatively supersymmetric states  $\Phi_q$  away from zero energy. For this to happen, the supercharges  $\mathcal{Q}$  and  $\bar{\mathcal{Q}}$ , instead of annihilating the  $\Phi_q$ , must have nonzero matrix elements  $\langle \Phi_p | \mathcal{Q} | \Phi_q \rangle$  or  $\langle \Phi_p | \bar{\mathcal{Q}} | \Phi_q \rangle$ , for distinct critical points  $p$  and  $q$ .

So we have to analyze tunneling events that involve transitions between two critical points  $q$  and  $p$ . As a preliminary, we evaluate the action for a trajectory  $u(\tau)$  that starts at  $q$  for  $\tau \rightarrow -\infty$  and ends at  $p$  for  $\tau \rightarrow +\infty$ . For such a trajectory, in Euclidean signature, and with the auxiliary fields eliminated via (10.7), the bosonic part of the action (10.6) is

$$\begin{aligned} I &= \frac{1}{2} \int_{-\infty}^{\infty} d\tau \left( g_{ab} \frac{du^a}{d\tau} \frac{du^b}{d\tau} + g^{ab} \partial_a h \partial_b h \right) \\ &= \frac{1}{2} \int_{-\infty}^{\infty} d\tau g_{ab} \left( \frac{du^a}{d\tau} \pm g^{ac} \partial_c h \right) \left( \frac{du^b}{d\tau} \pm g^{be} \partial_e h \right) \mp (h(p) - h(q)), \end{aligned} \quad (10.18)$$

after integrating by parts. The action is therefore minimized by a trajectory that obeys

$$\frac{du^a}{d\tau} - g^{ab}\partial_b h = 0, \quad h(p) > h(q), \quad (10.19)$$

or

$$\frac{du^a}{d\tau} + g^{ab}\partial_b h = 0, \quad h(p) < h(q). \quad (10.20)$$

(For a given sign of  $h(p) - h(q)$ , only one of these equations may have a solution, since the left hand side of eqn. (10.18) is non-negative. If  $h(q) = h(p)$ , with  $q \neq p$ , neither equation has a solution.) These equations are called gradient flow equations; we can write them

$$\frac{d\vec{u}}{d\tau} = \pm \vec{\nabla} h, \quad (10.21)$$

where the “flow” vector  $d\vec{u}/d\tau$  has components  $\partial_\tau u^a$ , and the “gradient” vector  $\vec{\nabla} h$  has components  $g^{ab}\partial_b h$ .

The gradient flow equations have another interpretation. The Lorentz signature supersymmetry transformations (10.4) can be transformed to Euclidean signature via  $t = -i\tau$ , so that  $i\dot{u}^a = i du^a/dt$  is replaced by  $-du^a/d\tau$ . After also eliminating the auxiliary fields, the supersymmetry transformations in Euclidean signature read

$$\begin{aligned} \{\mathcal{Q}, \bar{\psi}^a\} &= -\frac{du^a}{d\tau} + g^{ab}\partial_b h \\ \{\bar{\mathcal{Q}}, \psi^a\} &= \frac{du^a}{d\tau} + g^{ab}\partial_b h. \end{aligned} \quad (10.22)$$

The condition for a trajectory to be  $\mathcal{Q}$ -invariant is that  $\{\mathcal{Q}, \bar{\psi}^a\}$  vanishes for that trajectory. So  $\mathcal{Q}$ -invariant trajectories obey

$$\frac{du^a}{d\tau} = g^{ab}\partial_b h, \quad (10.23)$$

and we call these ascending gradient lines since the flow (for increasing  $\tau$ ) is in the direction of steepest ascent for  $h$ . And  $\bar{\mathcal{Q}}$ -invariant trajectories are similarly descending gradient flow lines, obeying

$$\frac{du^a}{d\tau} = -g^{ab}\partial_b h. \quad (10.24)$$

Now let us discuss the moduli space  $\mathcal{M}_{qp}$  of ascending gradient flow lines from  $q$  in the past to  $p$  in the future. The tangent space of  $\mathcal{M}_{qp}$  at the point corresponding to a given solution of the ascending flow equation (10.23) is the space of solutions of the linear equation found by linearizing the ascending flow equation around the given solution. The linearization of the ascending flow equation is simply the equation  $L\psi = 0$ , where  $L$  is the fermion kinetic operator defined in eqn. (10.16). As explained in section 10.2, the index of  $L$  is  $\iota(L) = n_p - n_q$ , and generically, when this number is nonnegative, it equals the dimension of the kernel of  $L$  or in other words of the tangent space of  $\mathcal{M}_{qp}$ . Generically (that is, for a generic metric  $g_{ab}$  on  $M$  and a generic Morse function  $h$ ),  $\mathcal{M}_{qp}$  is a smooth (not necessarily connected) manifold of dimension  $n_p - n_q$ , assuming that this number is positive. For this reason, one calls  $n_p - n_q$  the expected dimension of  $\mathcal{M}_{qp}$ . If the expected dimension is negative, generically  $\mathcal{M}_{qp}$  is empty.

To understand what happens when  $n_q = n_p$ , we first need the following general comment. As long as  $p \neq q$ , an ascending flow from  $q$  to  $p$  cannot be invariant under time translations, so there is always a free action of the group  $\mathbb{R}$  of time translations on  $\mathcal{M}_{qp}$ . So for  $p \neq q$ , we can define a reduced moduli space  $\mathcal{M}_{qp,\text{red}}$  as the quotient  $\mathcal{M}_{qp}/\mathbb{R}$ .  $\mathcal{M}_{qp}$  is a fiber bundle over  $\mathcal{M}_{qp,\text{red}}$  with fibers  $\mathbb{R}$ . The expected dimension of  $\mathcal{M}_{qp,\text{red}}$  is  $n_p - n_q - 1$ . So if  $n_p = n_q$  and  $p \neq q$ , the expected dimension of  $\mathcal{M}_{qp,\text{red}}$  is  $-1$ . This means that generically  $\mathcal{M}_{qp,\text{red}}$  is empty, in which case  $\mathcal{M}_{qp}$  is likewise empty. If instead  $p = q$ , the only ascending or descending flow from  $q$  to  $p$  is the trivial flow in which  $u^a(\tau)$  does not depend on  $\tau$ . (This is an easy consequence of the fact that the right hand side of (10.18) vanishes for such a flow.) So  $\mathcal{M}_{pp}$  is always a point.

One last comment along these lines is that even if  $\mathcal{M}_{qp}$  has a positive expected dimension, it is empty if  $h(p) \leq h(q)$ . This follows easily from (10.18), whose right hand side would be non-positive for an ascending flow from  $q$  to  $p$ .

We conclude with a more elementary way to determine the dimension of  $\mathcal{M}_{qp}$ . Near a nondegenerate critical point  $m \in M$ , we can find Riemann normal coordinates  $u^a$  such that the metric tensor is just  $\sum_a (du^a)^2 + \mathcal{O}(u^2)$ , and

$$h = h_0 + \frac{1}{2} \sum_a f_a u_a^2, \quad (10.25)$$

where the  $f_a$  are all nonzero and the number of negative  $f_a$  equals the Morse index at  $m$ . The flows (10.23) near  $m$  look like

$$u^a(\tau) = \sum_a c_a e^{f_a \tau}, \quad (10.26)$$

so the flows that depart from  $m$  for  $\tau \rightarrow -\infty$  (or approach  $m$  for  $\tau \rightarrow +\infty$ ) are those in which  $c_a = 0$  for  $f_a < 0$  (or  $f_a > 0$ ). The space of all gradient flows has dimension  $d$  (since a flow is determined by its value at a specified time). After imposing  $n_q$  conditions to ensure that a flow starts at  $q$  and  $d - n_p$  conditions to ensure that a flow ends at  $p$ , we find that the expected dimension of the space of flows from  $q$  to  $p$  is  $n_p - n_q$ .

## 10.4 Lifting The Vacuum Degeneracy

Now we want to see how instantons can lift the vacuum degeneracy that is present at tree level. We consider the matrix element of the supercharge  $\mathcal{Q}$  between perturbative vacuum states  $\Phi_q$  and  $\Phi_p$ :

$$\langle \Phi_p | \mathcal{Q} | \Phi_q \rangle. \quad (10.27)$$

For this matrix element to be nonzero, the fermion number of  $|\Phi_p\rangle$  must exceed that of  $|\Phi_q\rangle$  by 1. In other words, the Morse index of  $p$  exceeds that of  $q$  by 1. To compute the matrix element, we perform a path integral over trajectories that begin at  $q$  at  $\tau = -\infty$  and end at  $p$  at  $\tau = +\infty$ . Since  $\mathcal{Q}$  is a conserved quantity (or since  $\Phi_p$  and  $\Phi_q$  both have zero energy in the approximation that is the input to this computation), the time at which  $\mathcal{Q}$  is inserted does not matter.

The action for trajectories from  $q$  to  $p$  is  $\mathcal{Q}$ -exact modulo an additive constant that depends only on the values of the superpotential at the critical points  $p$  and  $q$ :

$$I = \{\mathcal{Q}, V\} + \frac{1}{\lambda}(h(p) - h(q)) \quad (10.28)$$

This statement is a supersymmetric extension of eqn. (10.18). In eqn. (10.28),  $V$  is proportional to  $1/\lambda$ . By rescaling  $V$ , one goes to arbitrarily weak coupling while only changing  $I$  by  $\mathcal{Q}$ -exact terms. So the path integral with insertion of arbitrary  $\mathcal{Q}$ -exact operators, such as  $\mathcal{Q}$  itself, can be computed in the weak coupling limit.

Since  $\mathcal{Q}$  is a symmetry of the action and obeys  $\mathcal{Q}^2 = 0$ , the path integral with insertion of arbitrary  $\mathcal{Q}$ -invariant operators – such as  $\mathcal{Q}$  itself – localizes on  $\mathcal{Q}$ -invariant fields. As we deduced from eqn. (10.22), a  $\mathcal{Q}$ -invariant field is a solution of the ascending gradient flow equation. These are the appropriate instantons in our problem, and the desired matrix element can be computed as a sum of instanton contributions. Since the Morse indices of  $p$  and  $q$  differ by 1, the moduli space  $\mathcal{M}$  of solutions of the gradient flow equation is 1-dimensional and the reduced moduli space  $\mathcal{M}_{\text{red}}$  is a finite set of points. The desired matrix element can be computed by summing over those points.

There is, however, a subtlety in the computation. To see why, it helps to restore the loop counting parameter  $\lambda$  in the original action (10.5). When we do this,  $\mathcal{Q}$ ,  $\overline{\mathcal{Q}}$ , and  $H$  are all proportional to  $1/\lambda$  if expressed in terms of classical variables  $u$ ,  $\dot{u}$ ,  $\psi$ , and  $\bar{\psi}$ . For example

$$\mathcal{Q} = \frac{1}{\lambda} \left( g_{ab} \frac{du^b}{d\tau} - \partial_a h \right) \psi^a. \quad (10.29)$$

The supersymmetry algebra  $\{\mathcal{Q}, \overline{\mathcal{Q}}\} = 2H$  is obeyed (with no factor of  $\lambda$ ) since the canonical commutators are proportional to  $\lambda$ .

In computing a transition from  $\Phi_q$  to  $\Phi_p$  with an insertion of  $\mathcal{Q}$  (or any other  $\mathcal{Q}$ -invariant operator), there will be a factor of  $\exp(-(h(p) - h(q))/\lambda)$  that comes from the value of the classical action for the instanton trajectory. We will use the phrase “reduced matrix element” to refer to a matrix element for a transition from  $\Phi_q$  to  $\Phi_p$  with this elementary factor of  $\exp(-(h(p) - h(q))/\lambda)$  removed.

Though  $\mathcal{Q}$  is of order  $1/\lambda$ , its reduced matrix element is of order 1 and comes from a 1-loop computation around the classical instanton trajectory. There are two ways to see this. First, if we try to do a leading order calculation in the instanton field, we immediately get 0, since the instanton is a solution of the gradient flow equation  $g_{ab} \frac{du^b}{d\tau} - \partial_a h = 0$ , and  $\mathcal{Q}$  is proportional to the left hand side of this equation. So the reduced matrix element must come from a 1-loop calculation. Second, an elegant calculation explained in section 10.5.1 of [50]) shows directly that the reduced matrix is of order  $\lambda^0$ .

This is shown not by doing the 1-loop computation <sup>57</sup> but by an interesting shortcut that avoids the need for such a calculation. Instead of computing the reduced matrix element of  $\mathcal{Q}$ , we pick any function  $f$  that has different values at the critical points  $p$  and  $q$  and compute the reduced matrix element of the commutator  $[\mathcal{Q}, f]$ . (For example, we

---

<sup>57</sup>It turns out that a direct calculation is actually quite tricky.

could take  $f = h$ , and this is the choice actually made in [50].) This contains the same information, for the following reason. When we compute the matrix element of an equal time commutator

$$\langle \Phi_p | (\mathcal{Q}f - f\mathcal{Q}) | \Phi_q \rangle, \quad (10.30)$$

we can as usual assume that the operator that is inserted on the left is inserted at a slightly greater time than the one that is inserted on the right. But as  $\mathcal{Q}$  is a conserved quantity, the time at which it is inserted does not matter and we can take this to be much greater or much less than the time at which  $f$  is inserted. So

$$\langle \Phi_p | [\mathcal{Q}, f] | \Phi_q \rangle = \langle \Phi_p | (\mathcal{Q}(\tau)f(\tau') - f(\tau)\mathcal{Q}(\tau')) | \Phi_q \rangle, \quad (10.31)$$

where we can take  $\tau - \tau'$  to be very large. (Only the difference  $\tau - \tau'$  matters, since the initial and final states have zero energy.) When we evaluate the right hand side of (10.31) by integrating over instanton moduli space, the instanton must occur at a time close to the time at which  $\mathcal{Q}$  is inserted. This means that  $f$  is inserted in the initial or final state  $\Phi_p$  or  $\Phi_q$ . To lowest order in  $\lambda$ , we set  $f$  to  $f(q)$  or  $f(p)$  in the initial or final state. (In a moment, it will be clear that higher order terms are not relevant.) So

$$\langle \Phi_p | \mathcal{Q} | \Phi_q \rangle = \frac{1}{f(q) - f(p)} \langle \Phi_p | [\mathcal{Q}, f] | \Phi_q \rangle. \quad (10.32)$$

What we have gained from this is that  $[\mathcal{Q}, f] = \partial_a f \psi^a$  is independent of  $\lambda$ , so the right hand side of (10.32) can be evaluated classically.

The actual calculation is explained in [50]. We insert  $[\mathcal{Q}, f]$  at, say,  $\tau = 0$ . The instanton trajectory is  $u^a(\tau) = u_{\text{cl}}^a(\tau - v)$ , where  $v$  is a collective coordinate over which we must integrate. The corresponding classical fermion zero-mode is  $\psi_{\text{cl}}^a(\tau) = \partial_v u_{\text{cl}}^a(\tau - v)$ . At the classical level in an instanton field, we simply set  $[\mathcal{Q}, f](0) = \partial_a f \psi^a|_{\tau=0}$  to its value with  $u^a(\tau)$  set equal to the classical trajectory  $u_{\text{cl}}^a(\tau - v)$  and  $\psi^a$  set equal to  $\psi_{\text{cl}}^a$ , both evaluated at  $\tau = 0$ . Thus  $[\mathcal{Q}, f](0) = \partial_v u_{\text{cl}}^a(-v) \partial_a f(u(-v)) = \partial_v f(u(-v))$ . This must be integrated over  $v$  and multiplied by the ratio of fermion and boson determinants. By time translation symmetry, these determinants do not depend on  $v$ , so we can integrate over  $v$  first, giving  $\int_{-\infty}^{\infty} dv \partial_v f(u(-v)) = f(q) - f(p)$ . Inserting this in (10.32) the factor of  $f(q) - f(p)$  cancels, and we find that the contribution of a given instanton to the reduced matrix element is simply the ratio of fermion and boson determinants. The boson and fermion determinant are equal up to sign because of a pairing of nonzero eigenvalues by supersymmetry, so their ratio gives a factor of  $\pm 1$ . In more detail, the ratio of determinants is

$$\frac{\det'(L)}{(\det'(L^\dagger L))^{1/2}}, \quad (10.33)$$

where  $\det'$  is a determinant in the space orthogonal to the zero-modes. The numerator in (10.33) is real, since  $L$  is a real operator, but is not necessarily positive. The denominator is positive since  $L^\dagger L$  is non-negative and is positive-definite once the zero-mode is removed. The cancellation of numerator and denominator up to sign occurs because for a real operator  $L$ ,  $\det'(L) = \det'(L^\dagger)$  so  $\det'(L^\dagger L) = \det'(L^\dagger) \det'(L) = (\det'(L))^2$ .

Actually, as usual, the path integral on  $\mathbb{R}$  is not a number but a transition amplitude between initial and final states. This is clear in eqn. (10.34), where reversing the sign of the initial or final state  $\Phi_q$  or  $\Phi_p$  would certainly reverse the sign of the matrix element  $m_{qp}$ . In the mathematical theory, one says that the regularized fermion path integral is not a number but a section of a real determinant line bundle that can be trivialized by choosing the signs of the initial and final states. We give an introduction to this point of view in Appendix F, but here we give a less technical explanation.

Restoring the factor of  $\exp(-(h(p) - h(q))/\lambda)$ ,  $\mathcal{Q}$  acts on the states of approximately zero energy by

$$\mathcal{Q}\Phi_q = \sum_{p|n_p=n_q+1} \exp(-(h(p) - h(q))/\lambda) m_{qp} \Phi_p, \quad (10.34)$$

where the sum runs over all critical points  $p$  of Morse index  $n_q + 1$ . The matrix element  $m_{qp}$  vanishes if there are no flows from  $q$  to  $p$  and receives a contribution of  $+1$  or  $-1$  for each ascending flow line from  $q$  to  $p$ . We will describe this loosely by saying that  $m_{qp}$  is computed by “counting” the instanton trajectories from  $q$  to  $p$ . We always understand that “counting” means “counting with signs.”

Alternatively, denoting as  $M$  a matrix that multiplies  $\Phi_q$  by  $\exp(-h(q)/\lambda)$ , we find that  $\widehat{\mathcal{Q}} = M^{-1} \mathcal{Q} M$  acts by

$$\widehat{\mathcal{Q}}\Phi_q = \sum_{p|n_p=n_q+1} m_{qp} \Phi_p, \quad (10.35)$$

Thus matrix elements of  $\widehat{\mathcal{Q}}$  are reduced matrix elements of  $\mathcal{Q}$ . Of course, the cohomology of  $\mathcal{Q}$  is naturally isomorphic to that of  $\widehat{\mathcal{Q}}$ , but eqn. (10.35) has the advantage of making it manifest that the cohomology is defined over  $\mathbb{Z}$ . We call the complex with basis  $\Phi_q$  and differential  $\widehat{\mathcal{Q}}$  the MSW (Morse-Smale-Witten) complex.

Now let us see what we can say about the sign of the fermion path integral, based only on very general ideas. It is simplest to assume first that the target space  $M$  is simply-connected. The fermion kinetic operator  $L$  makes sense for expanding around an arbitrary path from  $q$  to  $p$ , not necessarily a classical solution, and likewise it makes sense to discuss the sign of the measure in the fermion path integral for an arbitrary path. For a particular path, there is no natural way to pick the sign of the fermion measure, but it makes sense to ask that the sign of the fermion measure should vary continuously as we vary the path. For simply-connected  $X$ , any two paths from  $q$  to  $p$  are homotopic, and therefore the sign of the fermion measure for any such path is uniquely determined up to an overall choice of sign that depends only on the choice of  $q$  and  $p$  and not on a particular path between them.

From this, it seems that the reduced matrix element  $m_{qp}$  in (10.35) or (10.34) is well-defined up to an overall sign that is the same for all trajectories from  $q$  to  $p$ , say up to multiplication by  $(-1)^{f(p,q)}$  where  $f(p,q)$  equals 0 or 1 for each pair  $p,q$ . However, there is one more ingredient to consider and this is cluster decomposition. Let  $p, q$ , and  $r$  be three critical points. Consider a path from  $q$  to  $r$  that consists of a path that first travels from  $q$  to  $p$  and then, after a long time, travels from  $p$  to  $r$ . (Such a trajectory is called a broken path and plays a further role that we will explain in section 10.6.) Cluster

decomposition – that is, the condition that the fermion measure should factorize naturally in this situation – gives the constraint  $(-1)^{f(p,q)}(-1)^{f(r,p)} = (-1)^{f(r,q)}$ . This implies that  $(-1)^{f(p,q)} = (-1)^{a(p)+a(q)}$  for some function  $a$  on the set of  $\mathbb{V}$  of critical points. But a factor of  $(-1)^{a(p)+a(q)}$  in the reduced matrix element can be eliminated by multiplying the state  $\Phi_s$ , for any critical point  $s$ , by  $(-1)^{a(s)}$ . In short, for simply-connected  $M$ , the fermion measure in all sectors is uniquely determined up to signs that reflect the choices of sign in the initial and final state wavefunctions.

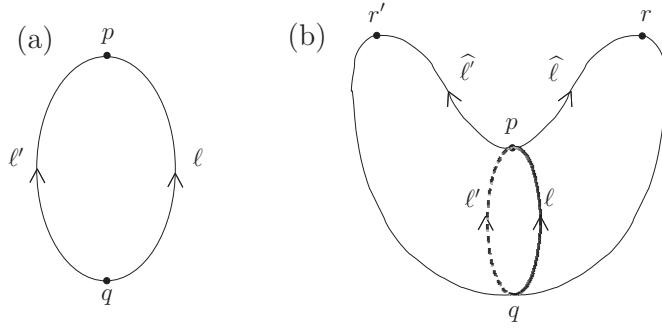
In case  $M$  is not simply-connected, the principles we have used do not necessarily give a unique answer because in general the answer is not unique. The signs of the fermion measure in the different sectors might not be uniquely determined (even after allowing for the freedom to change the external states) because one is free to twist the theory by considering differential forms on  $M$  valued in a flat real line bundle, rather than ordinary differential forms. This will change the signs of the various transition amplitudes. (If one wishes to weight the different sectors of the path integral by arbitrary complex phases, as opposed to arbitrary minus signs as assumed above, one would find that the construction is unique up to the possibility of considering differential forms on  $M$  valued in a flat complex line bundle.)

The attentive reader might notice one subtlety that was ignored in the above discussion. If  $M$  is simply-connected, we can interpolate between any two paths from  $q$  to  $p$ . However, if  $\pi_2(M) \neq 0$ , there are different homotopy types of such interpolations. If different ways to interpolate between one path and another would give different signs for the fermion measure, we would say that the theory has a global anomaly and is inconsistent. This does not happen in the class of supersymmetric quantum mechanical models considered here (basically because the theory of differential forms can be defined on any manifold  $M$ ), but it does happen in other classes of supersymmetric quantum mechanical models.

In the simple remarks just made, we have explained, just from general principles, that the path integral for trajectories from  $q$  to  $p$  is well-defined as a transition amplitude from  $\Phi_q$  to  $\Phi_p$ . But we have not given a recipe to compute the overall sign of this transition amplitude. The reader interested in this should consult Appendix F.

## 10.5 Some Practice

To help orient the reader and as background for section 10.6, we will here give some simple examples involving flow lines and Morse theory. In our first example, we take the target space of the  $\sigma$ -model to be  $N = S^1$  with a Morse function  $h$  that has only two critical points – a maximum  $p$  and a minimum  $q$ . Accordingly, the supersymmetric quantum mechanics with this target space and Morse function has precisely two states whose energy vanishes in perturbation theory – a state  $\Phi_q$  that corresponds to a 0-form and a state  $\Phi_p$  that corresponds to a 1-form. Since the cohomology of  $S^1$  has rank 2, the states  $\Phi_p$  and  $\Phi_q$  must survive in the exact quantum theory as supersymmetric states of precisely zero energy. Since  $h$  has the minimum number of critical points needed to reproduce the cohomology of  $M$ , it is called a perfect Morse function. Because  $\widehat{\mathcal{Q}}$  increases the fermion



**Figure 122:** Some example of flow lines in Morse theory. (a) A circle  $N = S^1$  is embedded in  $\mathbb{R}^2$  in such a way that the “height” function is a perfect Morse function  $h$ , with only the minimum possible set of critical points – a maximum  $p$  and a minimum  $q$ . (b) A two-sphere  $M = S^2$  is embedded in  $\mathbb{R}^3$  in such a way that the height function is *not* a perfect Morse function. It has two local maxima  $r$  and  $r'$ , a saddle point  $p$ , and a minimum  $q$ . The arrows on the flow lines show the directions along which  $h$  increases.

number by 1, its only possibly non-zero matrix element is

$$\hat{\mathcal{Q}}\Phi_q = m_{qp}\Phi_p, \quad (10.36)$$

where  $m_{qp}$  is a sum over ascending flow lines from  $q$  to  $p$ , weighted by the sign of the fermion determinant. However, since  $\Phi_q$  and  $\Phi_p$  must remain at precisely zero energy, we expect  $m_{qp} = 0$ . Concretely, as sketched in Figure 122(a), there are two steepest descent or ascent trajectories from  $p$  to  $q$ , labeled  $\ell$  and  $\ell'$  in the figure.

Each of them, if properly parametrized by the Euclidean time  $\tau$ , gives a solution of the equation for ascending gradient flow from  $q$  to  $p$ . Each of these trajectories contributes  $\pm 1$  to  $m_{qp}$ . The expected result  $\mathcal{Q}\Phi_q = 0$  arises because the two trajectories contribute with opposite signs: whatever orientation we pick at  $p$ , the orientation of time translations along one of the two trajectories will agree with it, along the other trajectory will disagree with it. (Again, see Appendix F for a technical description of the signs.)

For a slightly more elaborate example, we consider  $M = S^2$ , but now with a decidedly non-perfect Morse function that has two local maxima  $r$  and  $r'$ , a saddle point  $p$ , and a minimum  $q$  (Figure 122(b)). The MSW complex now has rank 4, generated by the 2-forms  $\Phi_r$  and  $\Phi_{r'}$ , the 1-form  $\Phi_p$ , and the 0-form  $\Phi_q$ . *A priori*, the possible matrix elements of  $\mathcal{Q}$  are

$$\begin{aligned} \hat{\mathcal{Q}}\Phi_q &= m_{qp}\Phi_p \\ \hat{\mathcal{Q}}\Phi_p &= m_{pr}\Phi_r + m_{pr'}\Phi'_{r'}. \end{aligned} \quad (10.37)$$

As shown in Figure 122(b), there are two flow lines  $\ell$  and  $\ell'$  from  $q$  to  $p$ . However, they cancel just as in the previous example. There is, however, just a single flow line  $\hat{\ell}$  or  $\hat{\ell}'$  from  $p$  to  $r$  or  $r'$ , so no cancellation is possible. We can pick the signs of the states so that

$m_{pr} = m_{pr'} = 1$  and then we have

$$\begin{aligned}\widehat{\mathcal{Q}}\Phi_q &= 0 \\ \widehat{\mathcal{Q}}\Phi_p &= \Phi_r + \Phi'_r.\end{aligned}\tag{10.38}$$

The cohomology of  $S^2$  is therefore of rank 2, generated by a 0-form  $\Phi_q$  and a 2-form that we can take to be  $\Phi_r$  or  $\Phi_{r'}$ .

The result (10.38) makes it clear that in this example  $\mathcal{Q}^2 = 0$ . Considerations of fermion number force  $\mathcal{Q}^2$  to annihilate all the basis vectors except  $\Phi_q$ , but  $\mathcal{Q}^2\Phi_q = 0$  since  $\mathcal{Q}\Phi_q = 0$ .

## 10.6 Why $\mathcal{Q}^2 = 0$

We now want to explain in terms of Morse theory and gradient flow lines why  $\mathcal{Q}^2 = 0$  in general (See [53] for a fairly accessible rigorous explanation.) The explanation gives important motivation for many of the constructions in the present paper.

Consider in general a target space  $M$  with Morse function  $h$ . Let  $q$  be a critical point of Morse index  $n$ . The general form of  $\widehat{\mathcal{Q}}\Phi_q$  from eqn. (10.35) is

$$\widehat{\mathcal{Q}}\Phi_q = \sum_{p_i | n_{p_i} = n+1} m_{qp_i} \Phi_{p_i},\tag{10.39}$$

where the sum runs over critical points  $p_i$  of index  $n+1$  and  $m_{qp_i}$  is the usual sum of contributions  $\pm 1$  from ascending flows from  $q$  to  $p_i$ . And similarly the general form of  $\widehat{\mathcal{Q}}^2\Phi_q$  is

$$\widehat{\mathcal{Q}}^2\Phi_q = \sum_{r_\alpha | n_{r_\alpha} = n+2} \sum_{p_i | n_{p_i} = n+1} m_{qp_i} m_{p_i r_\alpha} \Phi_{r_\alpha},\tag{10.40}$$

where  $r_\alpha$  ranges over the critical points of index  $n+2$ . So the statement that  $\mathcal{Q}^2\Phi_q = 0$  amounts to the statement that for each  $\alpha$ ,

$$\sum_i m_{qp_i} m_{p_i r_\alpha} = 0.\tag{10.41}$$

The only possibly dangerous case is the case that there is, for some  $i$ , an ascending gradient trajectory from  $q$  to  $p_i$  and also an ascending gradient trajectory from  $p_i$  to  $r_\alpha$ . (Otherwise, either  $m_{qp_i}$  or  $m_{p_i r_\alpha}$  vanishes for all  $i$ .) So let us assume this to be the case. Rather as in Figure 122(b), let  $\ell$  be an ascending trajectory from  $q$  to  $p_i$  and let  $\widehat{\ell}$  be an ascending trajectory from  $p_i$  to  $r_\alpha$ . (We assume that as usual these flows depend on no moduli except those associated to time translations.) We can make an approximate ascending gradient flow trajectory from  $q$  to  $r_\alpha$  as follows. Start at  $q$  at  $\tau = -\infty$ . After lingering near  $q$  until time  $\tau_1$ , flow from  $q$  to  $p_i$  along the trajectory  $\ell$ . Linger near  $p_i$  until some much later time  $\tau_2$ , and then flow to  $r_\alpha$  along the trajectory  $\widehat{\ell}$ . For  $\tau_2 - \tau_1 \gg 0$ , this gives a very good approximate solution of the flow equation, depending on the two parameters  $\tau_2$  and  $\tau_1$ .

Index theory and the theory of differential equations can be used to show that these approximate solutions can be corrected to give a family of exact solutions of the flow equations, also depending on 2 parameters.<sup>58</sup>

Since  $n_{r_\alpha} - n_q = 2$  by hypothesis, 2 is the expected dimension of the moduli space  $\mathcal{M}_{qr_\alpha}$  of ascending flows from  $q$  to  $r_\alpha$ . What we have found is a component  $\mathcal{M}_{qr_\alpha}^*$  of  $\mathcal{M}_{qr_\alpha}$  that has this expected dimension. ( $\mathcal{M}_{qr_\alpha}$  is not necessarily connected; in general,  $\mathcal{M}_{qr_\alpha}^*$  is one of its connected components.)

As usual, the group  $\mathbb{R}$  of time translations acts on  $\mathcal{M}_{qr_\alpha}^*$ ; the quotient is a 1-manifold  $\mathcal{M}_{qr_\alpha, \text{red}}^*$ . For generic metric  $g_{ij}$  on  $M$  and Morse function  $h$ , the 2-manifold  $\mathcal{M}_{qr_\alpha}^*$  and the 1-manifold  $\mathcal{M}_{qr_\alpha, \text{red}}^*$  are smooth manifolds without boundary. This follows again from general considerations about index theory and differential equations.

A smooth 1-manifold without boundary is either a circle or a copy of  $\mathbb{R}$ . However,  $\mathcal{M}_{qr_\alpha, \text{red}}^*$  is not compact, since it has an “end” corresponding to  $\tau_2 - \tau_1 \rightarrow \infty$ . Therefore,  $\mathcal{M}_{qr_\alpha, \text{red}}^*$  must be a copy of  $\mathbb{R}$ . And since  $\mathbb{R}$  has two ends,  $\mathcal{M}_{qr_\alpha, \text{red}}^*$  must have a second end, in addition to the one that we know about.

The end of  $\mathcal{M}_{qr_\alpha, \text{red}}^*$  that we know about is sometimes called a broken trajectory or a broken flow line. It corresponds to the limit of a gradient flow line from  $q$  to  $r_\alpha$  that breaks up into a widely separated pair consisting of a flow from  $q$  to another critical point  $p_i$  followed by a flow from  $p_i$  to  $r_\alpha$ , where the Morse index increases by 1 at each step.

It again follows from very general considerations that any end of any component of  $\mathcal{M}_{qr_\alpha, \text{red}}$  corresponds to a broken trajectory from  $q$  to  $r_\alpha$ . So in addition to the broken trajectory from  $q$  to  $r_\alpha$  that appears at the end of  $\mathcal{M}_{qr_\alpha, \text{red}}^*$  that we know about,  $\mathcal{M}_{qr_\alpha, \text{red}}^*$  must have a second end that corresponds to a second broken trajectory from  $q$  to  $r_\alpha$ .

The sum on the left hand side of eqn. (10.41) can be regarded as a sum over contributions from broken trajectories. The contribution of each broken trajectory is  $\pm 1$ . The mechanism by which the sum always vanishes is that broken trajectories appear in pairs, corresponding to the two ends of a 1-dimensional reduced moduli space such as  $\mathcal{M}_{qr_\alpha, \text{red}}^*$ . Careful attention to signs show that they do indeed provide canceling contributions. (See Appendix F.)

To conclude, we should elaborate on the claim that an end of a 1-dimensional reduced moduli space must correspond to a broken trajectory. Roughly speaking, an end of a moduli space of solutions of a differential equation corresponds in general to an ultraviolet effect (something blows up at short distances or times), a large field effect (some fields go to infinity), or an infrared effect (something happens at large distances or times). An example of an ultraviolet effect is the shrinking of an instanton to zero size in four-dimensional gauge theory. This has no analog in our problem, because at short times the gradient flow equation reduces to  $du^a/d\tau = 0$  (the term  $g^{ab}\partial_b h$  in the equation is subleading at small times), and the solutions of this equation do not show any ultraviolet singularity. If  $M$  is

---

<sup>58</sup>Schematically, if we linearize the problem of correcting the approximate solution  $u_0$  to an exact solution we find an inhomogeneous linear problem

$$L \delta u = -Lu_0 \tag{10.42}$$

The obstruction to inverting  $L$  on the space of normalizable fluctuations is controlled by the kernel of  $L^\dagger$ , which is generically empty.

compact, we do not have to worry about  $u^a$  becoming large; if  $M$  is not compact (as will actually be the case in our main application), it is necessary to analyze this possibility. Finally, as we are considering a massive theory, the only interesting effect that is possible at long times is a broken trajectory. In a massive theory in Euclidean signature, long times do not come into play unless the trajectory becomes broken.

A final comment is that actually, this subject is one area in which rigorous mathematical theorems are illuminating for physics. From a physicist's point of view, perturbation theory gives an approximation to the space of supersymmetric ground states of supersymmetric quantum mechanics, and the inclusion of instantons gives a better approximation. Does inclusion of instantons give the exact answer, or could there be nonperturbative corrections that near the classical limit are even smaller than instantons? One answer to this question is that the rigorous theorems, described in [53], show that inclusion of instantons gives the exact answer for the space of supersymmetric states.

## 10.7 Why The Cohomology Does Not Depend On The Superpotential

In equation (10.12), we showed that the supercharge  $\mathcal{Q}$  is conjugate to the de Rham exterior derivative  $d$ . This implies that the cohomology of  $\mathcal{Q}$  is canonically isomorphic to the de Rham cohomology, which is defined without any choice of metric  $g$  or superpotential  $h$ . Hence, the cohomology of  $\mathcal{Q}$  does not depend on the choice of  $g$  or  $h$ .

However, this sort of proof is not available when we get to quantum field theory with spacetime dimension  $> 1$ . We will give another explanation here that does generalize. This explanation relies on counting of gradient flow trajectories. (For a much more precise account, see Section 4 of [53].)

Before beginning the technical explanation, let us explain the physical framework that should make one expect such an explanation to exist. According to eqn. (10.28), the Euclidean action of supersymmetric quantum mechanics is  $\{\mathcal{Q}, V\}$  plus a surface term, where

$$V = \frac{1}{2\lambda} \int_{-\infty}^{\infty} d\tau g_{ab} \bar{\psi}^a \left( \frac{du^b}{d\tau} - g^{bc} \frac{\partial h}{\partial u^c} \right). \quad (10.43)$$

The definition of  $V$  makes sense if  $g$  and  $h$  have an explicit  $\tau$ -dependence, rather than being functions of  $u^a$  only, and in this more general situation, we can generalize (10.28) to the supersymmetric action

$$I = \{\mathcal{Q}, V\} + \frac{1}{\lambda} \int_{-\infty}^{\infty} d\tau \frac{\partial}{\partial \tau} h(u; \tau). \quad (10.44)$$

The fact that it is possible to give  $h$  and  $g$  an explicit time-dependence while maintaining  $\mathcal{Q}$ -invariance can be regarded as an explanation of the technical construction that we will describe in the rest of this section. This technical construction has numerous analogs that are important in the rest of this paper, notably in section 15 where we explain the relation between the Fukaya-Seidel category and the web-based construction of the abstract part of this paper.

Here is the technical explanation. To compare the cohomology of  $\mathcal{Q}$  computed using one metric and superpotential  $g$  and  $h$  to that computed using another pair  $g', h'$ , we

proceed as follows. Let  $\mathcal{C}$  be the set of critical points of  $h$  and  $\mathcal{C}'$  the set of critical points of  $h'$ . In the classical limit, the system based on  $g, h$  has a supersymmetric state  $\Phi_p$  for each  $p \in \mathcal{C}$ , furnishing a basis of the space  $\mathcal{V}$  of approximately supersymmetric states (the MSW complex). Similarly, the system based on  $g', h'$  has an approximately supersymmetric state  $\Phi_{p'}$  for each  $p' \in \mathcal{C}'$ , furnishing a basis of the analogous space  $\mathcal{V}'$ . As in eqn. (10.35), by counting gradient flow trajectories for the Morse function  $h$  with metric  $g$ , we define a normalized differential  $\widehat{\mathcal{Q}}$  acting on  $\mathcal{V}$ , and similarly by counting trajectories for  $g', h'$ , we define a normalized differential  $\widehat{\mathcal{Q}}'$  acting on  $\mathcal{V}'$ . We write  $\mathcal{H}$  and  $\mathcal{H}'$  for the cohomology of  $\widehat{\mathcal{Q}}$  and  $\widehat{\mathcal{Q}}'$ , respectively. We want to define a degree-preserving linear map  $\mathcal{U} : \mathcal{V} \rightarrow \mathcal{V}'$  that will establish an isomorphism between  $\mathcal{H}$  and  $\mathcal{H}'$ .

A degree  $d$  linear transformation  $\mathcal{U} : \mathcal{V} \rightarrow \mathcal{V}'$  will induce a map  $\widehat{\mathcal{U}} : \mathcal{H} \rightarrow \mathcal{H}'$  if it is a “chain map,” meaning that

$$\widehat{\mathcal{Q}}'\mathcal{U} = (-1)^d \mathcal{U}\widehat{\mathcal{Q}}. \quad (10.45)$$

This ensures that if  $\psi \in \mathcal{V}$  represents a cohomology class of  $\widehat{\mathcal{Q}}$ , meaning that  $\widehat{\mathcal{Q}}\psi = 0$ , then  $\mathcal{U}\psi$  represents a cohomology class of  $\widehat{\mathcal{Q}}'$ , since  $\widehat{\mathcal{Q}}'\mathcal{U}\psi = \mathcal{U}\widehat{\mathcal{Q}}\psi = 0$ . Moreover the class of  $\mathcal{U}\psi$  only depends on the class of  $\psi$ , since if we replace  $\psi$  by  $\psi + \widehat{\mathcal{Q}}\chi$ , then  $\mathcal{U}\psi$  is replaced by  $\mathcal{U}\psi + \mathcal{U}\widehat{\mathcal{Q}}\chi = \mathcal{U}\psi + (-1)^d \widehat{\mathcal{Q}}'(\mathcal{U}\chi)$ . We define  $\widehat{\mathcal{U}} : \mathcal{H} \rightarrow \mathcal{H}'$  by saying that if a class in  $\mathcal{H}$  is represented by a state  $\psi$ , then  $\widehat{\mathcal{U}}$  maps this state to the class of  $\mathcal{U}(\psi)$ . To show that such a  $\widehat{\mathcal{U}} : \mathcal{H} \rightarrow \mathcal{H}'$  is an isomorphism, we will want a map  $\mathcal{U}' : \mathcal{V}' \rightarrow \mathcal{V}$  in the opposite direction, obeying the analog of (10.45) and inducing an inverse on cohomology.

To construct  $\mathcal{U}$ , we first pick an interpolation from the pair  $g, h$  to the pair  $g', h'$ . We do this by letting  $g$  and  $h$  depend on a real-valued “time” coordinate  $\tau$ , such that  $(g(\tau), h(\tau))$  approach  $(g, h)$  for  $\tau \rightarrow -\infty$  and approach  $(g', h')$  for  $\tau \rightarrow +\infty$ . We can assume that  $g(\tau)$  and  $h(\tau)$  are nearly (or even exactly) independent of  $\tau$  except near some time  $\tau_0$ . Now we consider the gradient flow equation with a time-dependent metric and superpotential:

$$\frac{du^a}{d\tau} = g^{ab}(u; \tau) \frac{\partial h(u; \tau)}{\partial u^b}. \quad (10.46)$$

The boundary conditions are that  $u(\tau) \rightarrow p \in \mathcal{C}$  for  $\tau \rightarrow -\infty$  and  $u(\tau) \rightarrow p' \in \mathcal{C}'$  for  $\tau \rightarrow +\infty$ . The index determining the expected dimension of the space of solutions of this equation is simply the difference between the Morse indices  $n_p$  and  $n_{p'}$  of  $p$  and  $p'$ . (We know that this is the answer if  $g = g'$  and  $h = h'$ , and the index does not change under continuous evolution of  $g', h'$ , and  $p'$ .) So the index is 0 if  $p$  and  $p'$  have the same Morse index. For a generic interpolation, the moduli spaces have dimensions equal to their expected dimensions; we consider an interpolation that is generic in this sense. So the index zero condition means that the moduli space consists of finitely many points. (Since  $g$  and  $h$  are time-dependent, there is no time translation symmetry to force the existence of a modulus.) These points correspond to solutions in which  $u^a(\tau)$  is almost constant except near some time  $\tau = \tau_0$ . We let  $u_{pp'}$  be the “number” of such solutions (as usual weighting each solution with the sign of the fermion determinant), and define  $\mathcal{U} : \mathcal{V} \rightarrow \mathcal{V}'$  by

$$\mathcal{U}\Phi_p = \sum_{p'|n_{p'}=n_p} u_{pp'} \Phi_{p'}. \quad (10.47)$$

The condition that  $n_{p'} = n_p$  means that  $\mathcal{U}$  preserves the grading of  $\mathcal{V}$  and  $\mathcal{V}'$ . We will need to know that  $\mathcal{U}$  defined by (10.47) is nonzero, and in fact satisfies (10.45). This will be justified at the end of this section.

The left and right hand sides of the equation  $\widehat{\mathcal{Q}}'\mathcal{U} = (-1)^d\mathcal{U}\widehat{\mathcal{Q}}$  both increase the degree (or grading) by 1. So to prove this identity, we have to look at moduli spaces of time-dependent gradient flows with  $n_{p'} = n_p + 1$ . The reasoning we need is very similar to the proof that  $\widehat{\mathcal{Q}}^2 = 0$  in section 10.6. Let  $\mathcal{M}$  be a component of the moduli space of solutions of the time-dependent gradient flow equation, interpolating from  $p$  in the past to  $p'$  in the future.  $\mathcal{M}$  is a one-manifold without boundary, and so is either a copy of  $S^1$ , with no ends at all, or a copy of  $\mathbb{R}$ , with two ends. In the following, only the case that  $\mathcal{M}$  is a copy of  $\mathbb{R}$  is relevant.

Just as in section 10.6, an end of  $\mathcal{M}$  is a broken path, in which a solution breaks up into two pieces localized at widely different times. The building blocks of a broken path in the present context are of the following types:

(A) One ingredient is familiar from section 10.6. In the far past or the far future, where the equation (10.46) has no explicit time-dependence, we may have a solution of the time-independent gradient flow equation that interpolates between two critical points of  $h$  or two critical points of  $h'$  with Morse index differing by 1. Let us say that such a solution is of type  $(A_-)$  if the transition occurs in the past and of type  $(A_+)$  if it occurs in the future.

(B) The new ingredient in the present context is a solution that interpolates from a critical point of  $h$  to a critical point of  $h'$ , near the time  $\tau_0$ .

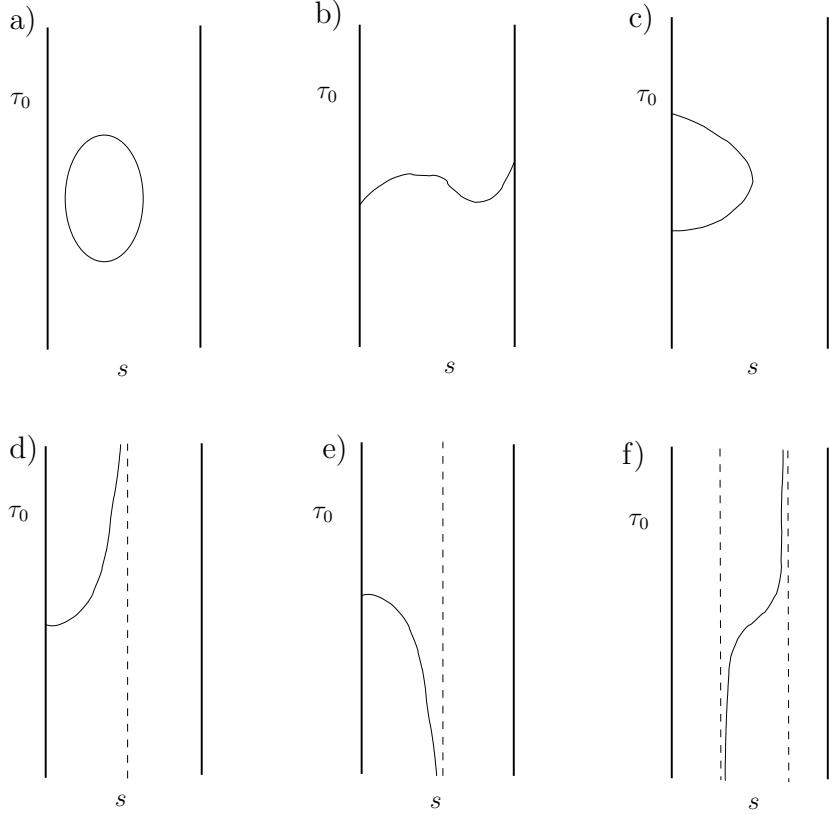
Ends of  $\mathcal{M}$  correspond to broken paths of two possible types:

(i) One type consists of a trajectory of type  $(A_-)$  in the far past, interpolating between two critical points of  $h$ , followed by a trajectory of type  $(B)$  at  $\tau \cong \tau_0$ , interpolating from a critical point of  $h$  to one of  $h'$ .

(ii) The other type consists of a trajectory of type  $(B)$  at  $\tau \cong \tau_0$ , interpolating from a critical point of  $h$  to one of  $h'$ , followed by a trajectory of type  $(A_+)$  interpolating to another critical point of  $h'$ .

Conversely, every broken path of either of these types arises at one end of one component of  $\mathcal{M}$ , and this component has a second end that is either of the same type or of opposite type. This is true for reasons similar to what we explained in showing that  $\widehat{\mathcal{Q}}^2 = 0$ .

A broken path of type (i) contributes  $\pm 1$  (depending on the sign of the fermion determinant) to a matrix element of  $\mathcal{U}\widehat{\mathcal{Q}}$ , and a broken path of type (ii) contributes  $\pm 1$  to the matrix element of  $\widehat{\mathcal{Q}}'\mathcal{U}$  between the same initial and final states. If  $\mathcal{M}$  has two ends that are both of type (i) or both of type (ii), then the corresponding contributions to  $\mathcal{U}\widehat{\mathcal{Q}}$  or to  $\widehat{\mathcal{Q}}'\mathcal{U}$  cancel. On the other hand, if  $\mathcal{M}$  has one end of each type, then these ends make equal contributions to  $\mathcal{U}\widehat{\mathcal{Q}}$  and to  $\widehat{\mathcal{Q}}'\mathcal{U}$ . Both statements follow from the observation that the sign of a contribution is equivalent to a choice of orientation of the corresponding component of  $\mathcal{M}$ , and all contributions are oriented canonically towards the future. After summing these statements over all components of  $\mathcal{M}$ , we arrive at the desired identity (10.45).



**Figure 123:** Some possibilities for a component of the moduli space  $\mathcal{M}$  of solutions to (10.49) that exist for *some*  $s$  in the case where  $n_p = n_{p'}$ . (Some additional possibilities are omitted). The picture is only schematic: while the value of  $s$  at which a solution exists is precisely defined, the corresponding value of  $\tau_0$  which characterizes the solution is not really well-defined. What is well-defined is only whether a sequence of solutions goes to  $\tau_0 = \pm\infty$ . A component of  $\mathcal{M}$  might be compact and without boundary, as in (a). Otherwise, it has two boundaries and/or ends. Each boundary or end contributes to one of the four terms in equation (10.48), and the two boundaries and/or ends of  $\mathcal{M}$  make compensating contributions to this identity. In (d), (e), and (f), the vertical dotted lines represent values of  $s$  at which there is an exceptional gradient flow solution contributing to the matrix  $E$ . These are flows that reduce the Morse index by 1.

One question about this is whether the map  $\mathcal{U} : \mathcal{V} \rightarrow \mathcal{V}'$  (and hence possibly the induced map  $\widehat{\mathcal{U}}$  on cohomology) depends on the specific choice of a generic time-dependent interpolation from  $g, h$  to  $g', h'$  (we call an interpolation generic if the moduli spaces have their expected dimensions). In general, the counting (with signs) of the solutions of an elliptic differential equation is invariant under continuous variations of the parameters in the equation, as long as solutions cannot go to infinity. If we could assume in the present context that solutions cannot go to infinity, then the numbers  $u_{pp'}$  would be independent of the choice of a generic interpolation and  $\mathcal{U}$  would likewise not depend on the interpolation.

This is actually not so in general. If  $\mathcal{U}_0$  and  $\mathcal{U}_1$  are the maps determined by two different generic interpolations, then in general the relationship between them is not  $\mathcal{U}_0 = \mathcal{U}_1$  but

rather

$$\mathcal{U}_1 - \mathcal{U}_0 = \widehat{\mathcal{Q}}' \mathsf{E} - \mathsf{E} \widehat{\mathcal{Q}}, \quad (10.48)$$

where  $\mathsf{E}$  is a linear transformation  $\mathsf{E} : \mathcal{V} \rightarrow \mathcal{V}'$  that reduces the degree by 1. This is enough to ensure that the induced maps on cohomology are equal,  $\widehat{\mathcal{U}}_1 = \widehat{\mathcal{U}}_0$ . The correction terms on the right hand side of eqn. (10.48) should be expected, for the following reason. The action of our system is  $\mathcal{Q}$ -exact up to a surface term (eqn. (10.44)), and when we change the interpolation from  $g, h$  to  $g', h'$  (without changing  $g$  or  $h$  at  $\tau = \pm\infty$ ) we change the action by a  $\mathcal{Q}$ -exact term. After integration by parts, this results in contributions in which  $\mathcal{Q}$  acts on initial and final states, and the transition amplitude changes by  $\widehat{\mathcal{Q}}' \mathsf{E} - \mathsf{E} \widehat{\mathcal{Q}}$  for some  $\mathsf{E}$  (as usual the shift from  $\mathcal{Q}$  to  $\widehat{\mathcal{Q}}$  and  $\widehat{\mathcal{Q}}'$  results from absorbing the surface terms in the action in the normalization of the initial and final states).

Technically,  $\mathsf{E}$  can be found as follows. Given two generic interpolations from  $g, h$  to  $g', h'$ , we first select an interpolation between the two interpolations. This means that we choose a metric  $g(u; \tau, s)$  and superpotential  $h(u; \tau, s)$  that depends not only on the time but on another parameter  $s$ , with  $0 \leq s \leq 1$ , such that the restriction to  $s = 0$  or to  $s = 1$  gives the two interpolations that we want to compare. Now we look for solutions of the gradient flow equation in  $\tau$  at a fixed value of  $s$

$$\frac{du^a}{d\tau} = g^{ab}(u; \tau, s) \frac{\partial h(u; \tau, s)}{\partial u^b} \quad (10.49)$$

flowing from a critical point  $p$  in the past to a critical point  $p'$  in the future in such a way that the Morse index is reduced by 1:  $n_{p'} = n_p - 1$ . For fixed  $s$ , the expected dimension of the moduli space is  $-1$ , meaning that for generic  $s$  there are no solutions (and there are in fact no solutions at  $s = 0$  or  $s = 1$  since we have assumed  $g(u; \tau, s)$  and  $h(u; \tau, s)$  to be generic at  $s = 0, 1$ ). However, by allowing  $s$  to vary – or in other words including  $s$  as an additional variable – we increase the expected dimension by 1. The expected dimension of the moduli space of solutions flowing from  $p$  to  $p'$  at some unspecified value of  $s$  is 0, and we define an integer  $e_{pp'}$  as the “number” of such solutions, weighted by the sign of an appropriate fermion determinant. (We address the sign issues here briefly in Appendix F.)

The matrix  $\mathsf{E}$  is defined by

$$\mathsf{E} \Phi_p = \sum_{p' | n_{p'} = n_p - 1} e_{pp'} \Phi_{p'}. \quad (10.50)$$

Let also  $\mathcal{U}_0 : \mathcal{V} \rightarrow \mathcal{V}'$  and  $\mathcal{U}_1 : \mathcal{V} \rightarrow \mathcal{V}'$  be the maps defined via eqn. (10.49) at  $s = 0$  and  $s = 1$ . We claim that  $\mathcal{U}_0$ ,  $\mathcal{U}_1$ , and  $\mathsf{E}$  satisfy (10.48).

As usual, to justify the claim we analyze the moduli spaces of solutions of the gradient flow equation. We consider a matrix element of eqn. (10.48) from  $\Phi_p$  to  $\Phi_{p'}$ , where  $n_p = n_{p'}$ . The moduli space of gradient flows from  $p$  to  $p'$  at some unspecified value of  $s$  is 1-dimensional. Some illustrative possibilities for what a component  $\mathcal{M}$  of this moduli space might look like are indicated in Figure 123. As usual (Figure 123(a)),  $\mathcal{M}$  might be compact and without boundary, but such a component does not contribute to the discussion. If  $\mathcal{M}$  is not of this type, then it has two boundaries or ends that may be either at  $s = 0, s = 1$ ,

$\tau = -\infty$ , or  $\tau = +\infty$ . Boundaries or ends of  $\mathcal{M}$  of the four possible types contribute to the four terms in eqn. (10.48), and the two ends of  $\mathcal{M}$  make canceling contributions in this identity.

We already know that an endpoint of  $\mathcal{M}$  at  $s = 0$  or  $s = 1$  contributes to  $\mathcal{U}_0$  or  $\mathcal{U}_1$ . What remains is to explain why  $\mathcal{M}$  can have an end at  $\tau_0 = \pm\infty$  and why such ends correspond to matrix elements of  $\widehat{\mathcal{Q}}' E$  or  $E \widehat{\mathcal{Q}}$ . For example, suppose that  $\widehat{\mathcal{Q}}' E$  has a matrix element from  $\Phi_p$  to  $\Phi_{p'}$ , where  $n_{p'} = n_p$ . This means that  $E$  has a matrix element from  $\Phi_p$  to  $\Phi_{q'}$ , where  $q'$  is a critical point of  $h'$  with  $n_{q'} = n_p - 1$ , and  $\widehat{\mathcal{Q}}'$  has a matrix element from  $\Phi_{q'}$  to  $\Phi_{p'}$ . The matrix element of  $E$  comes from a gradient flow from  $p$  to  $q'$  (for the time-dependent superpotential  $h(u; \tau, s)$ ) that exists at some value  $s = s_0$  (this solution is localized near some time  $\tau = \tau_0$ ). The matrix element of  $\widehat{\mathcal{Q}}'$  comes from a flow from  $q'$  to  $p'$  (for the time-independent superpotential  $h'$ ) that exists at generic  $s$  (and any  $\tau$ ). We can try to convert the “broken path”  $p \rightarrow q' \rightarrow p'$  into an exact gradient flow that interpolates from  $p$  to  $q'$  near time  $\tau_0$  and then from  $q'$  to  $p'$  at some much later time  $\tau_1$ . For very large  $\tau_1 - \tau_0$ , we can certainly make a very good approximate solution like this. However, in contrast to examples that were considered before, general considerations of index theory do *not* predict that this approximate solution can be corrected to an exact solution at the same value of  $s$ . The reason is that the initial flow from  $p$  to  $q'$  has virtual dimension  $-1$ , meaning that the linearization of the gradient flow equation near this trajectory is not surjective; the gradient flow equation that this solution satisfies has one more equation than unknown and a generic perturbation of the equation causes the solution not to exist. A generic perturbation can be made by either changing  $s$  or including the second flow near time  $\tau_1 \gg \tau_0$ . However, the fact that the index is  $-1$  means that the space of potential obstructions to deforming a solution is 1-dimensional, so we can compensate for existence of the second flow at very large  $\tau_1$  by perturbing  $s$  slightly away from  $s_0$ . For  $\tau_1 \rightarrow \infty$  (so that the perturbation by the second flow goes to zero), we must take  $s \rightarrow s_0$  (so that the perturbation by  $s$  goes to 0). This is why a matrix element of  $\widehat{\mathcal{Q}}' E$  or  $E \widehat{\mathcal{Q}}$  corresponds to an infinite end of the moduli space, as indicated in Figure 123(d,e,f).

Now that we know that the map induced on cohomology by a generic interpolation from  $g, h$  to  $g', h'$  does not depend on the interpolation, it is straightforward to show that this map is invertible. Just as before, we pick a time-dependent interpolation from  $g', h'$  back to  $g, h$  and use the counting of trajectories to define a map  $\mathcal{U}' : \mathcal{V}' \rightarrow \mathcal{V}$  in the opposite direction. We can compute the product map  $\mathcal{U}'\mathcal{U} : \mathcal{V} \rightarrow \mathcal{V}$  by considering an interpolation from  $g, h$  to itself in which we first interpolate from  $g, h$  to  $g', h'$  near some time  $\tau_0$  and then interpolate back to  $g, h$  near some much later time  $\tau_1$ . Thus the product  $\mathcal{U}'\mathcal{U}$  is computed by counting the trajectories for some interpolation from  $g, h$  to itself. As we have just explained, the map  $\mathcal{U}'\mathcal{U}$  will in general depend on the interpolation, but the induced map  $\widehat{\mathcal{U}}'\widehat{\mathcal{U}}$  on cohomology will not. So we can compute it for the trivial, time-independent interpolation from  $g, h$  to itself. The map on cohomology associated to the trivial interpolation is certainly the identity, so it follows that in general  $\widehat{\mathcal{U}}'\widehat{\mathcal{U}} = 1$ .

## 11. Landau-Ginzburg Theory As Supersymmetric Quantum Mechanics

Now we turn to our real topic – massive theories in two dimensions. Our purpose is to give a concrete realization of the abstract algebraic structures described in Sections §2 - §8 in the context of massive Landau-Ginzburg theories.

Our analysis will proceed roughly in the opposite order as in the abstract context. As discussed in the introduction 1, our starting point is the complex of ground states for a two-dimensional theory compactified on a strip with supersymmetric boundary conditions. In a limit where the segment is made very long, we expect to be able to reconstruct this complex in terms of a web representation and interior and boundary amplitudes which encode properties of the same theory on the plane and on the left and right half-planes. The advantage of working with LG theories is that we can formulate our questions in the language of Supersymmetric Quantum Mechanics and Morse theory. As a result, the complex of ground states on the strip, the web representation data, the interior and boundary amplitudes will all be defined in terms of counting problems for solutions of certain differential equations on the strip, plane and half planes.

In this section we review the basic data required to define a massive LG theory, some supersymmetric boundary conditions, the relation to Supersymmetric Quantum Mechanics and Morse theory and the corresponding differential equations. We also explain the relation between the supersymmetric boundary conditions discussed in this paper and the Fukaya-Seidel category or more precisely the Fukaya category of the superpotential.

### 11.1 Landau-Ginzburg Theory

Let us recall the basic data needed to formulate a  $1 + 1$  dimensional LG theory with  $\mathcal{N} = (2, 2)$  supersymmetry. We require a Kähler manifold  $X$ , together with a holomorphic superpotential  $W : X \rightarrow \mathbb{C}$ .  $X$  has a Kähler form  $\omega$ , making it a symplectic manifold, and a corresponding Kähler metric. A vacuum state corresponds to a critical point of  $W$ , and at such a critical point, the fermion mass matrix is the matrix of second derivatives of  $W$ , also called the Hessian matrix. So an LG theory is massive precisely if the Hessian matrix is nondegenerate at every critical point of  $W$ . In this case, we say that  $W$  is a Morse function in the holomorphic sense. (Using the Cauchy-Riemann equations, this is equivalent to the condition that any nontrivial real linear combination of  $\text{Re } W$  and  $\text{Im } W$  is a Morse function in the ordinary real sense.) Since we assume that the theory is massive in every vacuum, and in addition the soliton states that interpolate between different vacua will also be massive, there is a characteristic length scale  $\ell_W$  beyond which the theory should always be, in some sense, close to a vacuum configuration. As usual, we write  $\mathbb{V}$  for the set of vacua or equivalently the set of critical points of  $W$ .

There are many familiar ways to formulate the standard  $1 + 1$ -dimensional Landau-Ginzburg model associated to the above data. A slightly less familiar approach will be convenient for us. We will formulate the LG model as a special case of the supersymmetric quantum mechanics construction of section 10, but now with an infinite-dimensional target space.<sup>59</sup> This construction will make manifest not all four supersymmetries of the

---

<sup>59</sup>The ability to do this uses  $(2, 2)$  supersymmetry in an essential way. A two-dimensional  $\sigma$ -model

$(2, 2)$  model, but only a subalgebra consisting of two supercharges whose anticommutators generate time translations but not spatial translations. Such a subalgebra is not uniquely determined. It will be important to have an unbroken  $U(1)_R$  symmetry that acts on this superalgebra. This will be an axial  $U(1)$  charge normalized so that the negative-chirality supercharge  $Q_-$  and the positive chirality supercharge  $\bar{Q}_+$  both have axial  $U(1)_R$ -charge  $+1$  while the positive-chirality supercharge  $Q_+$  and the negative-chirality supercharge  $\bar{Q}_-$  have axial  $U(1)_R$ -charge  $-1$ . We will refer to this axial  $U(1)_R$  charge as “fermion number.”

As already explained in section 1, a subalgebra satisfying these conditions depends on the choice of a complex number  $\zeta$  of modulus 1. The subalgebra that we make manifest via the quantum mechanical construction is generated by

$$\mathcal{Q}_\zeta := Q_- - \zeta^{-1} \bar{Q}_+, \quad \bar{\mathcal{Q}}_\zeta := \bar{Q}_- - \zeta Q_+. \quad (11.1)$$

The nonzero anticommutators are

$$\{\mathcal{Q}_\zeta, \bar{\mathcal{Q}}_\zeta\} = 2\mathcal{H} - 2\text{Re}(\zeta^{-1}Z), \quad (11.2)$$

where  $\mathcal{H}$  is the Hamiltonian of the quantum field theory and  $Z$  is the central charge. We will call this a small subalgebra of the supersymmetry algebra.

To present the two-dimensional LG model as an abstract quantum mechanical model, we formulate it on a two-manifold of the form  $\mathbb{R} \times D$ , where  $\mathbb{R}$  is parametrized by the “time,” and  $D$  is a 1-manifold that represents “space.” It could be  $D = \mathbb{R}$ , or the half-lines  $D = [x_\ell, \infty)$  or  $D = (-\infty, x_r]$  or the interval  $D = [x_\ell, x_r]$ .

The target space of the supersymmetric quantum mechanics model is going to be the space of all  $X$ -valued fields on  $D$ , or in other words

$$\mathcal{X} = \text{Maps}(D \rightarrow X). \quad (11.3)$$

$\mathcal{X}$  inherits a natural metric from the Kähler metric of  $X$ :

$$d\ell^2 = \frac{1}{2} \left( g_{I\bar{J}} d\phi^I \otimes d\bar{\phi}^{\bar{J}} + \text{complex conjugate} \right) \quad (11.4)$$

where  $\phi^I$  are local holomorphic coordinates on  $X$ . Thus  $\mathcal{X}$  has metric:

$$|\delta\phi|^2 = \frac{1}{2} \int_D dx \left( g_{I\bar{J}} \delta\phi^I \delta\bar{\phi}^{\bar{J}} + \text{complex conjugate} \right). \quad (11.5)$$

Our motivating example is  $X = \mathbb{C}^n$  with a flat Kähler metric

$$\frac{1}{2} \sum_K \left( d\phi^K \otimes d\bar{\phi}^{\bar{K}} + d\bar{\phi}^{\bar{K}} \otimes d\phi^K \right). \quad (11.6)$$

In this case, the Kähler form  $\omega = \frac{i}{2} \sum_K d\phi^K \wedge d\bar{\phi}^{\bar{K}}$  is exact,  $\omega = d\lambda$  with  $\lambda = \text{Re}(\frac{i}{2} \sum_K \phi^K d\bar{\phi}^{\bar{K}})$ . The following construction applies whenever  $\omega$  is exact. When this is not the case, some

---

with  $(1, 1)$  supersymmetry actually cannot be viewed as an infinite-dimensional version of the construction reviewed in section 10. For example, such a  $\sigma$ -model in general does not have an additively conserved fermion number.

slight modifications are needed since the superpotential  $h$  that we introduce momentarily is not single-valued. In this case one should replace  $\mathcal{X}$  by a suitable cover on which  $h$  is single-valued.

To put the  $\sigma$ -model in the supersymmetric quantum mechanics framework of section 10, all we need is to define the superpotential  $h$ . We take this to be

$$h := -\frac{1}{2} \int_D dx \operatorname{Re} \left( i \sum_I \phi^I \frac{\partial}{\partial x} \bar{\phi}^{\bar{I}} - \zeta^{-1} W \right). \quad (11.7)$$

for the case that  $X = \mathbb{C}^n$  with  $\omega = \frac{i}{2} \sum_I d\phi^I \wedge d\bar{\phi}^{\bar{I}}$ . In general, one replaces  $\operatorname{Re}(\frac{i}{2} \sum \phi^I d\bar{\phi}^{\bar{I}})$  with any 1-form  $\lambda$  such that  $\omega = d\lambda$ . Parametrizing  $X$  by an arbitrary set of real coordinates  $u^a$ , we write  $\lambda = \lambda_a du^a$  and then

$$h := - \int_D dx \left( \lambda_a \frac{du^a}{dx} - \frac{1}{2} \operatorname{Re} (\zeta^{-1} W) \right). \quad (11.8)$$

(If one transforms  $\lambda$  to  $\lambda + d\alpha$  for some function  $\alpha$  on  $X$ ,  $h$  is modified by boundary terms that we will discuss in section 11.2.)

The construction reviewed in section 10, applied to any Riemannian manifold (in this case  $\mathcal{X}$ ), with any superpotential (in this case  $h$ ), gives a quantum mechanical model with two supercharges  $\mathcal{Q}_\zeta$  and  $\bar{\mathcal{Q}}_\zeta$  of fermion number  $\mathcal{F} = 1$  and  $-1$ , respectively. The supersymmetry algebra of the quantum mechanical model includes time translations, but of course it does not include spatial translations, which are not defined in the general quantum mechanical framework. From eqn. (10.6), the kinetic energy of the quantum mechanical model is  $T = \frac{1}{2} g_{ab} \dot{u}^a \dot{u}^b$ , which in the present context becomes

$$T = \int_D dx \frac{1}{2} g_{I\bar{J}} \frac{d\phi^I}{dt} \frac{d\bar{\phi}^{\bar{J}}}{dt}. \quad (11.9)$$

The potential energy of the quantum mechanical system, again from (10.6), is  $V = \frac{1}{2} g^{ab} \partial_a h \partial_b h$ . In the present case, this becomes

$$V = \frac{1}{2} \int_D dx \left| \frac{d\phi^I}{dx} - \frac{i\zeta}{2} g^{I\bar{J}} \frac{\partial \bar{W}}{\partial \bar{\phi}^{\bar{J}}} \right|^2, \quad (11.10)$$

or, after integration by parts,

$$V = \int_D dx \frac{1}{2} \left( g_{I\bar{J}} \frac{d\phi^I}{dx} \frac{d\bar{\phi}^{\bar{J}}}{dx} + \frac{1}{4} g^{I\bar{J}} \frac{\partial W}{\partial \phi^I} \frac{\partial \bar{W}}{\partial \bar{\phi}^{\bar{J}}} \right) + \frac{1}{2} \left[ \operatorname{Im}(\zeta^{-1} W) \right]_{\partial_r D}^{\partial_l D}. \quad (11.11)$$

Here  $\partial_r D$  and  $\partial_l D$  are the left and right boundaries of  $D$ , which for the moment we assume to be at  $x \rightarrow \pm\infty$ . (In case  $D$  has boundaries at finite points  $x_\ell$  and/or  $x_r$ , the same formula holds after imposing some further conditions that we discuss in section 11.2.)

Still assuming that  $D = \mathbb{R}$ , and assuming a reasonable behavior at infinity as discussed in section 11.2), the boundary terms in (11.11) are just constants that depend on the boundary conditions. These constants are responsible for the central charge term in the

small supersymmetry algebra (11.2). Apart from this constant term, the potential energy  $V$  of the  $\sigma$ -model is independent of  $\zeta$ . Moreover, the sum  $T + V$  is simply the bosonic part of the Hamiltonian of the standard LG model with superpotential  $W$ . When one adds in the fermionic terms in the Hamiltonian of the quantum mechanical model, one simply gets the full supersymmetric LG Hamiltonian.

What we have achieved via this construction of the LG model is to make manifest an arbitrary  $\zeta$ -dependent small subalgebra of the supersymmetry algebra. This is useful because we are primarily interested in branes and supersymmetric states that are invariant under such a small subalgebra but not under the full  $\mathcal{N} = 2$  supersymmetry algebra.

As an immediate application, let us discuss the states that are annihilated by  $\mathcal{Q}_\zeta$  and  $\overline{\mathcal{Q}}_\zeta$ . From the general quantum mechanical discussion of section 10, we know that such states<sup>60</sup> correspond in the classical limit to critical points of  $h$ . A simple computation shows that stationary points of  $h$  must satisfy

$$\frac{d}{dx}\phi^I = g^{IJ}\frac{i\zeta}{2}\frac{\partial \bar{W}}{\partial \bar{\phi}^J} \quad (11.12)$$

We call this equation the  $\zeta$ -soliton equation. Not coincidentally, the potential energy is written in eqn. (11.10) as the integral of the square of the left hand side of this equation. For  $D = \mathbb{R}$ , with the fields required to approach specified vacua  $i, j, \in \mathbb{V}$  at the two ends of  $D$ , a solution of this equation gives the classical approximation to an  $ij$  BPS soliton [15].

Another view of the  $\zeta$ -soliton equation is as follows. We can think of  $h$ , as defined in eqn. (11.8), as the action of a classical mechanical system in which the symplectic form is  $\omega = d\lambda$  and the Hamiltonian is  $H = -\frac{1}{2}\text{Re}(\zeta^{-1}W)$ .<sup>61</sup> So the  $\zeta$ -soliton equation is a Hamiltonian flow equation

$$\omega_{ab}\frac{du^b}{dx} + \frac{\partial H}{\partial u^a} = 0. \quad (11.13)$$

Since the Hamiltonian is a conserved quantity in a Hamiltonian flow, an immediate consequence is that  $H = -\frac{1}{2}\text{Re}(\zeta^{-1}W)$  is independent of  $x$  in a solution of the  $\zeta$ -soliton equation. On a Kähler manifold, Hamiltonian flow for a Hamiltonian that is the real part of a holomorphic function is the same as gradient flow with respect to the imaginary part of the same holomorphic function. So it is also possible to write the  $\zeta$ -soliton equation as a gradient flow equation:

$$\frac{du^a}{dx} = g^{ab}\frac{\partial}{\partial u^b}\text{Im}\left(\frac{1}{2}\zeta^{-1}W\right). \quad (11.14)$$

(Concretely, the equivalence of these two forms of the  $\zeta$ -soliton equation is proved using the Cauchy-Riemann equation for the holomorphic function  $\zeta^{-1}W$ .) Hence  $\text{Im}(\zeta^{-1}W)$  is an increasing function of  $x$  for any flow. Combining these statements, it follows, for example,

<sup>60</sup>States annihilated by the full supersymmetry algebra with four supercharges, as opposed to the small subalgebra generated by  $\mathcal{Q}_\zeta$  and  $\overline{\mathcal{Q}}_\zeta$ , are the supersymmetric vacua of the theory and correspond, of course, to critical points of  $W$ .

<sup>61</sup>This  $H$  is distinct from the Hamiltonian  $\mathcal{H}$  in equation (11.2). Note too that one often splits the real coordinates  $u^a$  into canonically conjugate pairs  $p_i$  and  $q^i$ , with  $\omega = \sum_i dq^i \wedge dp_i$  and  $\lambda = -\sum_i p_idq^i$ . This might make  $h = \int(p_idq^i - Hdx)$  look more familiar.

that an  $ij$  soliton (interpolating from  $\phi_i$  at  $\tau = -\infty$  to  $\phi_j$  at  $\tau = +\infty$ ) can only exist for a particular value of  $\zeta$ :

$$i\zeta = i\zeta_{ji} := \frac{W_j - W_i}{|W_j - W_i|} \quad (11.15)$$

We recall that the central charge in this sector is  $Z_{ji} = W_j - W_i$ .

In general, a solution of the  $\zeta$ -soliton equation gives only a classical approximation to a quantum BPS state in the  $ij$  sector. To get the exact spectrum of quantum BPS states, one needs to modify the classical approximation by instanton corrections. We defer the details to section 12, and for now merely remark that the framework to compute instanton corrections is simply the standard framework for instanton corrections in supersymmetric quantum mechanics, as described in section 10. Thus, one needs to count (with signs) the solutions of the instanton equation of the supersymmetric quantum mechanics. The general instanton equation (10.23) of supersymmetric quantum mechanics, specialized to the case that the target space is Kähler, is

$$\frac{d\phi^I}{d\tau} = 2g^{I\bar{J}} \frac{\delta h}{\delta \phi^{\bar{J}}}. \quad (11.16)$$

For the case that the target space is  $\mathcal{X}$  and with our choice of  $h$ , this becomes

$$\left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial \tau} \right) \phi^I = \frac{i\zeta}{2} g^{I\bar{J}} \frac{\partial \bar{W}}{\partial \bar{\phi}^{\bar{J}}}, \quad (11.17)$$

or alternatively

$$\frac{\partial \phi^I}{\partial s} = \frac{i\zeta}{4} g^{I\bar{J}} \frac{\partial \bar{W}}{\partial \bar{\phi}^{\bar{J}}}, \quad (11.18)$$

where  $s = x + i\tau$ . We call this the  $\zeta$ -instanton equation, and we call its solutions  $\zeta$ -instantons.

**Remark:** It is sometimes useful to have a clear idea about how the discrete spacetime symmetries  $P$  and  $PT$  are implemented in Landau-Ginzburg theory.  $PT$  corresponds to a rotation by  $\pi$  in Euclidean space and is therefore a symmetry of the theory. Under  $PT$  the bosonic fields transform as  $\phi(x, \tau) \rightarrow \phi(-x, -\tau)$ , and the fermion fields transform as  $\psi_{\pm} \rightarrow \mp i\psi_{\pm}$ . This is not a symmetry of the  $\zeta$ -instanton equation but rather transforms it by  $\zeta \rightarrow -\zeta$ . Parity, on the other hand, is in general not a symmetry of the Landau-Ginzburg theory. Under parity we must have  $\phi(x, \tau) \rightarrow \bar{\phi}(-x, \tau)$ , while the fermionic fields transform as  $\psi_{\pm} \rightarrow e^{\pm i\alpha} \bar{\psi}_{\mp}$ . In general, this transformation will map one Landau-Ginzburg model to another. A sufficient criterion for parity invariance is  $W(\bar{\phi}) = (W(\phi))^*$ . In particular, if  $W$  is a polynomial it should have real coefficients.

## 11.2 Boundary Conditions

### 11.2.1 Generalities

The nature of the boundary conditions we impose on (11.12) depends on the domain  $D$ .

At an infinite end of  $D$ , to keep the energy of the LG model finite, the fields must approach one of the critical points  $\phi_i$ ,  $i \in \mathbb{V}$ . So if  $D$  extends to  $-\infty$ , then we require

$$\lim_{x \rightarrow -\infty} \phi = \phi_i \quad (11.19)$$

where  $\phi_i$  is some critical point of  $W$ . Similarly, if  $D$  extends to  $+\infty$  then we require

$$\lim_{x \rightarrow +\infty} \phi = \phi_j \quad (11.20)$$

where again  $\phi_j$  is a critical point of  $W$ .<sup>62</sup>

Let us now consider boundaries of  $D$  at finite distance. We want to describe boundary conditions that preserve the small supersymmetry algebra. Up to a certain point, the abstract quantum mechanical construction of the LG model tells us how to do that. Formally, with an arbitrary target space and an arbitrary real superpotential, we can make a quantum mechanical model with 2 supersymmetries. So from that point of view, we can take the target space to be  $\mathcal{X} = \text{Maps}_*(D, X)$  where the notation  $\text{Maps}_*$  means that at the boundary of  $D$ , we place a restriction of our choice on the map from  $D$  to  $X$ . For example, if  $D = [x_\ell, x_r]$  is a compact interval (the analog if  $D \cong \mathbb{R}_+$  has only one boundary point is obvious), we can pick submanifolds  $U_\ell, U_r \subset X$  and require that  $x_\ell$  maps to  $U_\ell$  and  $x_r$  to  $U_r$ . Similarly, from a formal point of view, we can add any boundary terms that we want to the bulk superpotential  $h$  defined in (11.7). In the spirit of LG models, we will take the boundary terms to be functions of the fields  $\phi^I$  only and not their derivatives. So we pick arbitrary real-valued functions  $k_\ell$  on  $U_\ell$  and  $k_r$  on  $U_r$  and add the corresponding boundary terms to  $h$  to get:

$$h := -\frac{1}{2} \int_D dx \left( 2\lambda_a \frac{du^a}{dx} - \text{Re}(\zeta^{-1} W) \right) - k_\ell(u(x_\ell)) + k_r(u(x_r)). \quad (11.21)$$

There is no natural choice of  $\lambda$ ; we are always free to transform  $\lambda \rightarrow \lambda + d\alpha$  for any 0-form  $\alpha$ . But in (11.21), it is clear that such a redefinition of  $\lambda$  can be absorbed in  $k_\ell \rightarrow k_\ell + \alpha_\ell$ ,  $k_r \rightarrow k_r + \alpha_r$ . So, since we allow any  $k_\ell, k_r$ , a shift of  $\lambda$  by an exact form does not matter.

The general quantum mechanical construction gives us an action with target  $\text{Maps}_*(D, X)$  and superpotential  $h$ ; the action is invariant under two supercharges  $\mathcal{Q}_\zeta, \bar{\mathcal{Q}}_\zeta$  for any choices of  $U_\ell, U_r, k_\ell$  and  $k_r$ . However, here we have to be careful because not every action function constructed from infinitely many variables can be quantized in a sensible way. For example, if we simply drop the  $\frac{du^a}{dx}$  term in (11.21), we would lose the corresponding  $|\partial_x \phi|^2$  term in the Hamiltonian (or in the potential energy of eqn. (11.11)). The theory would then be “ultra-local,” with no energetic cost in fluctuations of short wavelength, and we would not expect to be able to quantize it sensibly.

---

<sup>62</sup>The integral (11.7) defining  $h$  is infinite if  $\text{Re}(\zeta^{-1} W(\phi))$  is nonzero at an infinite end of  $D$ . However, in each sector defined by the choices of critical points at infinity,  $h$  can be naturally defined up to an overall constant; the variation of  $h$  (under a local variation of  $\phi(x)$ ) and more generally the differences in the values of  $h$  for different fields in the same sector are finite and well-defined. Actually, for  $D = \mathbb{R}$ , in a sector that contains BPS solitons, we can do better. In such a sector,  $\text{Re}(\zeta^{-1} W(\phi))$  has the same value for  $\phi = \phi_i$  or  $\phi_j$ , and the problem in defining  $h$  can be eliminated by subtracting a constant from  $W$  to ensure that this value is 0. If  $D$  has only one infinite end, one can do the same for each choice of vacuum at infinity.

A more subtle variant of this problem arises in the present context unless  $U_\ell$  and  $U_r$  are middle-dimensional in  $X$ . Quantization will only be possible if the choices of  $U_\ell$  and  $U_r$  lead to elliptic boundary conditions on the Dirac equation of the two-dimensional  $\sigma$ -model. A simple elliptic boundary condition on the Dirac equation on a two-manifold  $\Sigma$  requires that one-half of the fermion components vanish on the boundary of  $\Sigma$ . In the present context, in the abstract quantum mechanical model of section 10, the fermions  $\psi, \bar{\psi}$  take values in (the pullback to the worldline of) the tangent space of the target space  $M$ . In the present context, with  $M = \mathcal{X} = \text{Maps}_*(D, X)$ , this means that the restrictions of  $\psi, \bar{\psi}$  to the boundaries of  $D$  take values in (the pullbacks of) the tangent bundles of  $U_\ell, U_r$ . Thus the condition that the boundary values of the fermions take values in a middle-dimensional subspace means that  $U_\ell, U_r$  must be middle-dimensional. If we do not obey this condition, we can write down a supersymmetric action, but we cannot quantize it in a supersymmetric fashion.

Actually,  $U_\ell$  and  $U_r$  are subject to a much stronger constraint, which we can discover from the condition for a critical point of  $h$ . Asking for  $h$  to be stationary under variations of  $\phi^i$  or  $u^a$  that vanish at the boundary of  $D$  will give the  $\zeta$ -soliton equation that we have already discussed. But there are also boundary terms to consider in the variation of  $h$ . These terms are

$$\frac{\delta h}{\delta u^a(x)} = \dots + \delta(x - x_\ell) \left( \lambda_a(x) - \frac{\partial k_\ell}{\partial u^a} \right) - \delta(x - x_r) \left( \lambda_a(x) - \frac{\partial k_r}{\partial u^a} \right). \quad (11.22)$$

In general, in supersymmetric quantum mechanics,  $h$  is certainly not stationary at a generic point in field space. But in the particular case of the infinite-dimensional target space  $\text{Maps}_*(D, X)$ , to get a sensible model, we do need to work in a function space in which the delta function terms in the variation of  $h$  vanish. Otherwise, when we compute the potential energy  $\frac{1}{2}|dh|^2$ , we will find terms proportional to  $\delta(0)$ .

In the present context, the only way to eliminate the delta function terms in the variation of  $h$  is to constrain suitably  $U_\ell, U_r, k_\ell$ , and  $k_r$ . Writing  $\lambda|_U$  for the restriction of  $\lambda$  to  $U \subset X$ , the conditions we need are

$$\lambda|_{U_\ell} = dk_\ell, \quad \lambda|_{U_r} = dk_r. \quad (11.23)$$

In particular,  $\lambda$  restricted to  $U_\ell$  or  $U_r$  is exact, and therefore  $\omega = d\lambda$  vanishes when restricted to  $U_\ell$  or  $U_r$ . Since  $U_\ell$  and  $U_r$  are middle-dimensional, this means that  $U_\ell$  and  $U_r$  are ‘‘Lagrangian submanifolds’’ of  $X$ . To emphasize that they are Lagrangian, we will henceforth denote them as  $\mathcal{L}_\ell$  and  $\mathcal{L}_r$  rather than  $U_\ell$  and  $U_r$ . Moreover, eqn. (11.23) imply that (up to inessential additive constants)  $k_\ell$  and  $k_r$  are uniquely determined by  $\lambda$ .

The condition on  $U_\ell$  and  $U_r$  that we have found is simply independent of  $W$ , so it must agree with what happens at  $W = 0$ . Indeed, at  $W = 0$ , a brane invariant under the small supersymmetry algebra is usually called an  $A$ -brane, and the usual  $A$ -branes are supported on Lagrangian submanifolds.  $\zeta$  does not enter in standard discussions of  $A$ -branes; the reason for this is that if  $W = 0$ , there is an extra  $U(1)$   $R$ -symmetry that can be used to rotate away  $\zeta$ . The  $A$ -model at  $W = 0$  in general may have ‘‘coisotropic’’ branes [55] as well as the usual Lagrangian branes, but we will not try to generalize them in the presence of a

superpotential. The fact that the usual Lagrangian  $A$ -branes at  $W = 0$  can be generalized so as to preserve the small supersymmetry algebra also for  $W \neq 0$  has been shown before in a somewhat different way (see section 7.1 of [44]).

Just as it is not strictly necessary to assume that  $\omega$  is globally exact and therefore that  $\lambda$  is globally-defined, similarly it is not strictly necessary to assume that  $k_\ell$  and  $k_r$  are globally-defined. Only their derivatives appear in the Lagrangian, and one can consider the case that  $k_\ell$  and  $k_r$  are defined only up to additive constants. In this case, one must develop the theory with a multivalued superpotential  $h$  or else replace  $\text{Maps}_*(D, X)$  by a cover on which  $h$  is single-valued. However, the theory has particularly simple properties if one assumes that  $\lambda$ ,  $k_\ell$ , and  $k_r$  are all single-valued, and this assumption is rather natural in the context of LG models. Therefore, in this paper we will usually make that assumption.

Under this assumption, eqn. (11.23) says that  $\lambda|_{\mathcal{L}_\ell}$  and  $\lambda|_{\mathcal{L}_r}$  are globally exact. On a symplectic manifold  $X$  with exact symplectic form  $\omega = d\lambda$ , one says that a Lagrangian submanifold  $\mathcal{L} \subset X$  is exact if  $\lambda|_{\mathcal{L}}$  is exact. One implication of exactness in the standard  $A$ -model at  $W = 0$  is as follows. In general, a (closed) Lagrangian submanifold  $\mathcal{L} \subset X$  determines a classical boundary condition in the  $A$ -model, but because of disc instanton effects, this classical boundary condition might not really correspond to a supersymmetric  $A$ -brane. In fact, disc instanton effects can cause  $Q_\zeta^2$  to be nonzero in the presence of such a brane, as we explain in section 13.5. A disc instanton is a holomorphic map  $\psi : H \rightarrow X$ , where  $H$  is a disc, such that  $\psi(\partial H) \subset \mathcal{L}$ . Disc instantons do not exist if  $\omega$  and  $\mathcal{L}$  are exact, that is if  $\omega = d\lambda$  and  $\lambda|_{\mathcal{L}} = dk$  with  $\lambda$  and  $k$  globally defined, for then the area of a hypothetical disc instanton would have to vanish:

$$\int_H \psi^*(\omega) = \int_{\partial H} \psi^*(\lambda) = \int_{\partial H} \psi^*(dk) = \int_{\partial H} d\psi^*(k) = 0. \quad (11.24)$$

So in the usual  $A$ -model, an exact Lagrangian submanifold always does correspond to a supersymmetric brane. The same is true in the presence of a superpotential, since as we will explain in section 13.5, the disc instantons that are important here are “small” ones (localized near a particular boundary point) that are not affected by a superpotential.

We conclude this introductory discussion with some general remarks.. The  $(2, 2)$  supersymmetric sigma model with Kähler target  $X$  and no superpotential has two classical  $R$ -symmetries.  $U(1)_{\text{Axial}}$  rotates  $Q_-, \bar{Q}_+$  with a phase and  $Q_+, \bar{Q}_-$  with the opposite phase, while  $U(1)_{\text{Vector}}$  rotates  $Q_-, Q_+$  with a phase and  $\bar{Q}_-, \bar{Q}_+$  with the opposite phase. The topological  $A$ -model is obtained by topological twisting using the current generating  $U(1)_{\text{Vector}}$ .  $A$ -branes are the branes in this topological field theory. When we turn on a generic<sup>63</sup> superpotential  $W$ , the  $U(1)_{\text{Axial}}$  is unbroken classically but the  $U(1)_{\text{Vector}}$  symmetry is broken classically and we cannot twist to make a topological  $A$ -model.

Nevertheless, as we have seen, it is still possible to define branes that correspond rather closely to the usual  $A$ -branes at  $W = 0$ . We will just call them  $A$ -branes. Moreover, as we will learn in section 11.3, it is also possible with  $W \neq 0$  to define tree-level amplitudes

---

<sup>63</sup>In the presence of a quasihomogeneous superpotential, meaning that there is an action on  $X$  of a group  $U(1)_X$  under which  $W$  has “charge 1,” there is still an  $R$ -symmetry that acts on the supersymmetries as  $U(1)_{\text{Vector}}$ . It is a diagonal combination of  $U(1)_{\text{Vector}}$  with  $U(1)_X$ .

with many of the properties of standard  $A$ -model amplitudes (though not the usual cyclic symmetry). We will refer to the model that can be constructed with  $W \neq 0$  as the  *$A$ -model with superpotential*. This theory is not a topological field theory, but shares many features of a topological field theory. In what follows, many statements are equally applicable to either a standard  $A$ -model (with a compact target space, or with branes required to be compact, or some other condition placed on branes for reasons explained in section 11.2.3) or to the  $A$ -model with superpotential  $W \neq 0$ . When we refer loosely to the  $A$ -model, we are making statements that apply equally to the different cases.

At  $W = 0$ , the standard definition of a Lagrangian  $A$ -brane involves specifying not just the support  $\mathcal{L}$  of the brane, but also a flat unitary Chan-Paton vector bundle<sup>64</sup> over  $\mathcal{L}$ . This part of the brane story is not affected by introducing  $W$ , and does not interact in a very interesting way with what we will describe below. In principle we should always denote a brane by  $\mathfrak{B}$  to distinguish it from its support  $\mathcal{L}$ . Nevertheless, we will sometimes trust to the reader's indulgence and simply refer to a brane by  $\mathcal{L}$ .

### 11.2.2 Hamiltonian Symplectomorphisms

In the conventional topological  $A$ -model – with a compact target space, for example – the brane determined by a Lagrangian submanifold  $\mathcal{L}$  is supposed to be invariant under deformations of  $\mathcal{L}$  that are induced by Hamiltonian symplectomorphisms of  $X$ , provided the Hamiltonian symplectomorphisms are isotopic to the identity. The group of Hamiltonian symplectomorphisms is the group generated by Hamiltonian flows with single-valued Hamiltonian functions  $H$ . To a function  $H$ , we associate the Hamiltonian vector field

$$V_H^a = \omega^{ab} \partial_b H. \quad (11.25)$$

The group of Hamiltonian symplectomorphisms is the group generated by these vector fields.

To determine the infinitesimal motion of a Lagrangian submanifold  $\mathcal{L}$  generated by a given Hamiltonian function  $H$ , we only care about the corresponding Hamiltonian vector field  $V_H$  modulo vectors that are tangent to  $\mathcal{L}$ , since a vector field tangent to  $\mathcal{L}$  generates a reparametrization of  $\mathcal{L}$ , rather than a motion of  $\mathcal{L}$  in  $X$ . To determine  $V_H$  modulo vector fields tangent to  $\mathcal{L}$ , we only need to know the first derivatives of  $H$  along  $\mathcal{L}$  (rather than its derivatives in the normal direction). So if we are given a function  $H$  that is defined just on  $\mathcal{L}$  (and not on all of  $X$ ), this suffices to determine a motion of  $\mathcal{L}$  in  $X$  to first order, though of course only to first order.

In the mathematical literature on the topological  $A$ -model, going all the way back to the work of A. Floer in the mid-1980's, invariance of  $A$ -branes under Hamiltonian symplectomorphisms is one of the most central properties. Yet this fact is relatively little-known among physicists. The reason that the statement is not more familiar to string theorists is the following. In a class of Lagrangian submanifolds that are equivalent under Hamiltonian

---

<sup>64</sup>This description is a little over-simplified, as one knows from the K-theory interpretation of  $D$ -branes. The Chan-Paton bundle on a brane is not quite a flat unitary vector bundle but is twisted by a gerbe of order 2 associated to  $w_2(\mathcal{L})$ .

diffeomorphisms that are isotopic to the identity, there is at most one special Lagrangian representative, and this is the representative that is important in most physical applications of the  $A$ -model. If there is no special Lagrangian representative in the given class, then the branes in question are “unstable,” analogous to unstable holomorphic bundles on the  $B$ -model side, and cannot be used in a superconformal construction.

Actually, invariance of the  $A$ -model under Hamiltonian symplectomorphisms of branes is mirror dual to invariance of the  $B$ -model under complex gauge transformations of the Chan-Paton gauge field of a brane. One expects to be able to make Hamiltonian symplectomorphisms independently for each  $A$ -brane, just as in the  $B$ -model, one can make separate complex gauge transformations for each  $B$ -brane.

Here is a more detailed explanation. First we consider the  $B$ -model, and then we will consider the  $A$ -model in parallel. In a  $\sigma$ -model, a brane  $\mathfrak{B}$  is described by a submanifold  $Y \subset X$  that is equipped with a Chan-Paton vector bundle  $E \rightarrow Y$  that is endowed with a unitary connection  $A$ . A unitary gauge transformation of  $A$  just changes the worldsheet action by a total derivative, so it is trivially a symmetry. However, the  $B$ -model is actually invariant under complex gauge transformations of  $A$ , not just unitary ones. The way that this happens is that the change in the action when  $A$  is changed by a gauge transformation with an imaginary generator is  $\mathcal{Q}$ -exact. This is a special case of the fact that a  $D$ -term in the action is always  $\mathcal{Q}$ -exact. For simplicity, in both the  $A$ -model and the  $B$ -model, we will consider only branes of rank 1; for the  $B$ -model, this means that  $A$  is a  $U(1)$  gauge field. The boundary  $D$ -terms of lowest dimension take the form

$$\int_{\partial\Sigma} dt d^2\theta k(u^a), \quad (11.26)$$

where  $X$  is parametrized locally by some functions  $u^a$  and  $k$  is a real-valued function on the support  $\mathcal{L}$  of a brane  $\mathfrak{B}$ . The integral runs over the portion of  $\partial\Sigma$  that is labeled by a particular brane. After performing the  $\theta$  integrals, in the case of the  $B$ -model, one can recognize (11.26) as the change in the action under an infinitesimal gauge transformation of  $A$  with imaginary gauge parameter  $ik$ .

To imitate this in the  $A$ -model, we proceed in exactly the same way, using the same  $D$ -term (11.26). But this is actually not anything new. We already allowed for such a boundary  $D$ -term in eqn. (11.21), where we included in the Morse function  $h$  a boundary contribution involving an *a priori* arbitrary function  $k$  on  $\mathcal{L}$ . (In writing this equation, we allowed for the possibility of separate Lagrangian submanifolds  $\mathcal{L}_\ell$  and  $\mathcal{L}_r$  at the two ends, with separate functions  $k_\ell$  and  $k_r$ .) The contribution of the Morse function to the action is the  $D$ -term  $\int dt d^2\theta h$ , so the dependence of a brane on this interaction is  $\mathcal{Q}_\zeta$ -exact. However, we learned in the subsequent analysis that actually  $k$  cannot be specified independently of the choice of  $\mathcal{L}$ . Having specified once and for all a one-form  $\lambda$  with  $d\lambda = \omega$ , the restriction of  $\lambda$  to  $\mathcal{L}$  must be related to  $k$  by eqn. (11.23):

$$\lambda|_{\mathcal{L}} = dk. \quad (11.27)$$

This means that, up to an additive constant,  $k$  cannot be varied independently of  $\mathcal{L}$ . If we change  $\mathcal{L}$  to first order by a Hamiltonian vector field  $V$ , then the change in  $\lambda|_{\mathcal{L}}$  is  $i_V \omega$ ,

where  $\mathbf{i}_V$  is the contraction operation (in coordinates,  $\mathbf{i}_V\omega = V^a\omega_{ab}du^b$ ). Thus eqn. (11.27) tells us that if we want to change  $k$  by an amount  $\delta k$ , then we also need to move  $\mathcal{L}$  by the Hamiltonian vector field

$$V^a = \omega^{ab}\partial_b(\delta k). \quad (11.28)$$

For a more complete picture, suppose first that we are given a Hamiltonian function  $H$  that is defined throughout  $X$ . We consider making a change of variables in the theory with  $\delta u^a = V_H^a$ . In first order, this moves each Lagrangian submanifold  $\mathcal{L}$  by the restriction to  $\mathcal{L}$  of  $V_H$ . To decide if this is an invariance of the  $A$ -model, we must see what happens to the Morse function  $h$ . This is a sum of three terms

$$\begin{aligned} h_1 &= -\int_D \lambda_a du^a \\ h_2 &= \int_D dx \frac{1}{2} \text{Re}(\zeta^{-1}W) \\ h_3 &= -k_\ell(u_\ell) + k_r(u_r). \end{aligned} \quad (11.29)$$

None of these terms is separately invariant under  $\delta u^a = V_H^a$ . For example,  $h_2$  changes by a bulk integral

$$\delta h_2 = \int_D dx \frac{1}{2} \{H, \text{Re}(\zeta^{-1}W)\}, \quad (11.30)$$

where  $\{f, g\}$  is the Poisson bracket of two functions  $f$  and  $g$ . Because this is purely a bulk integral with no delta function terms at the end, the corresponding contribution  $\int dt d^2\theta \delta h_2$  is a harmless  $D$ -term. By contrast, the change in  $h_1$  is only a boundary term:

$$\delta h_1 = \int_D \delta u^a \omega_{ab} du^b = \int_D \omega^{ac} \partial_c H \omega_{ab} du^b = - \int_D dH = -H(x_r) + H(x_\ell). \quad (11.31)$$

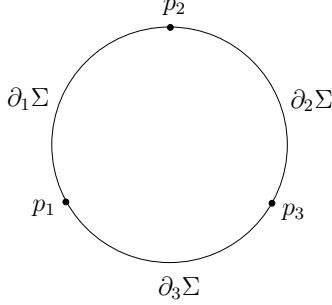
A change in the Morse function by boundary terms will make  $\delta(0)$  contributions to the potential energy  $|dh|^2$  of the  $\sigma$ -model. So these terms must be canceled by the variation of  $h_3$ . To do this, we compensate for the change of variables  $\delta u^a = V_H^a$  by allowing variations in  $k_r$  and  $k_\ell$ :

$$\delta h_3 = -\delta k_\ell(u_\ell) + \delta k_r(u_r). \quad (11.32)$$

To cancel the boundary terms in  $\delta h$ , we simply choose  $\delta k_r$  and  $\delta k_\ell$  to equal  $H(x_r)$  and  $H(x_\ell)$ , respectively, as was already explained in the last paragraph.

The result just described is not general enough, since modulo  $D$ -terms, we are supposed to be able to make *separate* Hamiltonian symplectomorphisms at the two ends of an open string. Suppose that we want to transform  $\mathcal{L}_\ell$  by a Hamiltonian function  $H_\ell$  and  $\mathcal{L}_r$  by another Hamiltonian function  $H_r$ . We pick some more general function  $H(u^a; x)$  that depends explicitly on the point  $x$  along a string, chosen to coincide with  $H_\ell$  in a neighborhood of  $x = x_\ell$  and with  $H_r$  in a neighborhood of  $x = x_r$ . The important boundary terms in the preceding analysis are unaffected. The only change in the analysis is that  $\delta h_1$  becomes more complicated, with an additional bulk contribution that involves the explicit  $x$ -dependence of  $H(u^a; x)$  and contributes a harmless  $D$ -term.

There are still a few loose ends to tie up. First, in addition to the term  $|dh|^2$ , the  $\sigma$ -model action contains another  $D$ -term, the kinetic energy of the  $\sigma$ -model. This is not



**Figure 124:** A disc with its boundary divided in three segments  $\partial_i\Sigma$ ,  $i = 1, 2, 3$ , labeled by different branes.

invariant under a generic Hamiltonian symplectomorphism. However, since it is a  $D$ -term, its variation is also a  $D$ -term, and moreover a harmless one, like  $\delta h_2$ , with no boundary contributions. That is so simply because the kinetic energy has time derivatives only, and no spatial derivatives, so there is no way for it to generate a boundary term.

Second, the  $A$ -model action also has a topological term that is not  $\mathcal{Q}_\zeta$ -exact:

$$I' = \int_{\Sigma} \Phi^*(\omega) = \int_{\Sigma} \omega_{ab} du^a \wedge du^b. \quad (11.33)$$

In discussing this term, we can assume an arbitrary  $\Sigma$ , not necessarily a strip in the plane.  $I'$  is a topological invariant – and hence in particular is  $\mathcal{Q}_\zeta$ -invariant – if  $\Phi(\partial\Sigma)$  is contained in a Lagrangian submanifold  $\mathcal{L}$ . What happens under a Hamiltonian symplectomorphism  $\delta u^a = V_H^a$  that changes the map  $\Phi$  and also changes  $\mathcal{L}$ ? The change in  $I'$  is

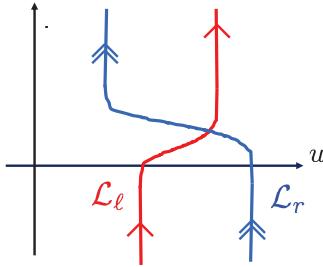
$$\delta I' = \int_{\Sigma} d(\delta u^a \omega_{ab} du^b) = \int_{\partial\Sigma} V_H^a \omega_{ab} du^b = - \int_{\partial\Sigma} dH. \quad (11.34)$$

This vanishes if all of  $\partial\Sigma$  is mapped to the same Lagrangian submanifold with the same  $H$  (since  $H$  is single-valued). More generally,  $\partial\Sigma$  may be a union of segments  $\partial_i\Sigma$  whose left and right endpoints we call  $p_i$  and  $p_{i+1}$  (Figure 124). We assume that the  $\partial_i\Sigma$  are mapped to different Lagrangian submanifolds  $\mathcal{L}_i$ , which we want to deform using different Hamiltonian function  $H_i$ . The generalization of eqn. (11.34) is

$$\delta I' = - \sum_i \int_{\partial_i\Sigma} dH_i = \sum_i (H_i(p_i) - H_{i-1}(p_i)). \quad (11.35)$$

At each intersection point  $p_i$ , a vertex operator is inserted for an external string state, and the contribution  $H_i(p_i) - H_{i-1}(p_i)$  to the action can be absorbed in the normalization of this vertex operator.

Finally, in our discussion of the Morse function and kinetic energy of the  $\sigma$ -model, we considered only a time-independent situation in which  $\Sigma$  is a strip in the plane. This makes it possible to consider  $A$ -branes and strings that preserve two supersymmetries, namely  $\mathcal{Q}_\zeta$  and its adjoint  $\overline{\mathcal{Q}}_\zeta$ . In that context, we have shown that varying  $\mathcal{L}_\ell$  and  $\mathcal{L}_r$  by



**Figure 125:** A pair of Lagrangian submanifolds  $\mathcal{L}_\ell$ ,  $\mathcal{L}_r$  embedded in the  $u - v$  plane.  $\mathcal{L}_\ell$  and  $\mathcal{L}_r$  intersect at the one point indicated.  $u$  is plotted horizontally and we assume that  $\mathcal{L}_\ell$ ,  $\mathcal{L}_r$  are embedded in the half-plane  $u > 0$ .

independent Hamiltonian symplectomorphisms is equivalent to adding to the action a  $D$ -term – a term that is both  $\mathcal{Q}_\zeta$ -exact and  $\overline{\mathcal{Q}}_\zeta$ -exact. In a more general situation in which  $\Sigma$  is not simply a strip (for example, in the study of tree-level amplitudes described in section 11.3), there is no time-translation invariance and one cannot maintain both  $\mathcal{Q}_\zeta$  and  $\overline{\mathcal{Q}}_\zeta$  symmetry. Instead of writing the response to Hamiltonian symplectomorphisms of branes as  $\int_{\Sigma} dt dx d\theta d\bar{\theta} F$  (where  $F$  was described in the above construction), we have to write it simply as  $\int_{\Sigma} d^2x \int d\theta \hat{F}$ , where in the time-independent case  $\hat{F} = \int d\bar{\theta} F$ . In general, we would pick an  $\hat{F}$  that everywhere near the boundary of  $\Sigma$  looks like the functional the  $\hat{F}$  that we used in analyzing the problem on a strip. In this way, we would establish the desired invariance.

### 11.2.3 Branes With Noncompact Target Spaces

In the  $A$ -model with a compact symplectic manifold  $X$  as target, one defines an  $A$ -brane supported on any (closed) Lagrangian submanifold  $\mathcal{L}$ , subject to some mild restrictions that are not pertinent at the moment.<sup>65</sup> However, a compact  $X$  is not relevant for the present paper, since if  $X$  is compact, it is not possible to introduce a nonconstant holomorphic superpotential  $W$ .

Once  $X$  is not compact, one usually wants to impose some sort of condition on the behavior of a Lagrangian submanifold at infinity. The most basic reason is that otherwise the space of supersymmetric states in quantization on a strip with boundary conditions set at the ends by a pair of Lagrangian submanifolds  $\mathcal{L}_\ell$ ,  $\mathcal{L}_r$  will not have the expected behavior. To see what will go wrong in general, consider the case that  $X = \mathbb{R}^2$  with the standard symplectic form  $\omega = du \wedge dv$ , and with  $\mathcal{L}_\ell$  and  $\mathcal{L}_r$  as depicted in Figure 125. We assume that the  $u$ -axis runs horizontally in the figure, and that  $\mathcal{L}_\ell$  and  $\mathcal{L}_r$  are embedded in the half-plane  $u > 0$ . Let us consider this system first in the ordinary  $A$ -model without a superpotential. The classical approximation to a supersymmetric state of the  $(\mathcal{L}_\ell, \mathcal{L}_r)$  system is given by an intersection point of the Lagrangian submanifolds  $\mathcal{L}_\ell$  and  $\mathcal{L}_r$ . In the

---

<sup>65</sup>One restriction is associated to the  $K$ -theory interpretation of  $D$ -branes (the normal bundle to  $\mathcal{L}$  in  $X$  must admit a  $\text{Spin}_c$  structure [24]. Another restriction, described in section 13.5, involves disc instantons.

example shown in the figure, there is precisely one such intersection point. Since the MSW complex is thus of rank 1, its differential necessarily vanishes and the space of quantum supersymmetric states of the  $(\mathcal{L}_\ell, \mathcal{L}_r)$  system is one-dimensional.

Now let us introduce a superpotential and quantize the theory on a strip of width  $w = x_r - x_\ell$ . The classical approximation to a supersymmetric state of the  $(\mathcal{L}_\ell, \mathcal{L}_r)$  system is given, just as in the usual  $A$ -model, by a time-independent supersymmetric state, but now the condition for supersymmetry is the  $\zeta$ -soliton equation. So supersymmetric states, in the classical approximation, correspond to solutions of the  $\zeta$ -soliton equation that start somewhere on  $\mathcal{L}_\ell$  at  $x = x_\ell$  and end somewhere on  $\mathcal{L}_r$  at  $x = x_r$ . (For more on this, see section 13.) In this discussion, we consider only solutions that are independent of the usual time coordinate  $\tau$  and it is convenient to refer to the usual spatial coordinate  $x$  as “time.” Let  $\mathcal{L}_\ell^w$  parametrize the points that can be reached by starting somewhere on  $\mathcal{L}_\ell$  and evolving for time  $w$  via the  $\zeta$ -soliton equation. Then  $\mathcal{L}_\ell^w$  is a Lagrangian submanifold that is very close to  $\mathcal{L}_\ell$  if  $w$  is small. ( $\mathcal{L}_\ell^w$  is Lagrangian because the  $\zeta$ -soliton equation describes Hamiltonian flow with the Hamiltonian  $-\frac{1}{2}\text{Re}(\zeta^{-1}W)$ .)  $\zeta$ -soliton solutions that flow from  $\mathcal{L}_\ell$  to  $\mathcal{L}_r$  in “time”  $w$  are simply intersections of  $\mathcal{L}_\ell^w$  with  $\mathcal{L}_r$ . For small enough  $w$ , the difference between  $\mathcal{L}_\ell$  and  $\mathcal{L}_\ell^w$  is unimportant and the classical supersymmetric states with  $W \neq 0$  correspond naturally to those with  $W = 0$ .

However, with everything being noncompact, intersection points of  $\mathcal{L}_\ell^w$  and  $\mathcal{L}_r$  can flow to infinity at finite  $w$ . This will actually happen in the example of the figure if we take  $\zeta^{-1}W = i\phi^2$  (where  $\phi = u + iv$  with real  $u, v$  and we take the Kahler metric of the  $\phi$ -plane to be  $d\ell^2 = du^2 + dv^2$ ). So  $\text{Im}(\zeta^{-1}W) = (u^2 - v^2)$  and the  $\zeta$ -soliton equation is  $\partial_x u = u, \partial_x v = -v$ . In particular,  $\mathcal{L}_\ell^w$  is obtained from  $\mathcal{L}_\ell$  by  $(u, v) \rightarrow (e^w u, e^{-w} v)$ . In the figure (in which  $u$  is plotted horizontally), we can assume that  $\mathcal{L}_\ell$  and  $\mathcal{L}_r$  are contained in a strip  $u_0 < u < u_1$  with  $u_0 > 0$ . Then for large enough  $w$ ,  $\mathcal{L}_\ell^w$  is entirely to the right of the strip and has no intersections with  $\mathcal{L}_r$ . Thus in this example, for large enough  $w$ , supersymmetry is broken, even though for small  $w$  it is unbroken (with precisely one supersymmetric ground state). For some purposes this might be an interesting example of supersymmetry breaking. However in the present context it is a problem. What has gone wrong is that the intersection of  $\mathcal{L}_\ell^w$  with  $\mathcal{L}_r$  goes to infinity at a finite value of  $w$ .

A related problem arises even in the absence of a superpotential if we consider the fact that the  $A$ -model is supposed to be invariant under Hamiltonian symplectomorphisms applied separately to each brane, as we described in section 11.2.2. This fails if we consider branes and Hamiltonian symplectomorphisms with no restriction on their behavior at infinity. For instance, in our example, we can eliminate the intersection point of  $\mathcal{L}_\ell$  with  $\mathcal{L}_r$  by transforming  $\mathcal{L}_\ell$  via a Hamiltonian symplectomorphism  $(u, v) \rightarrow (u + c, v)$  for a large constant  $c$  (the single-valued Hamiltonian that generates this symplectomorphism is simply  $v$ ). So the space of supersymmetric states of the  $(\mathcal{L}_\ell, \mathcal{L}_r)$  system is not invariant under Hamiltonian symplectomorphisms applied separately to  $\mathcal{L}_\ell$  or  $\mathcal{L}_r$ .

#### 11.2.4 $W$ -Dominated Branes

To avoid both of these problems, we will place some conditions on  $\mathcal{L}_\ell$  and  $\mathcal{L}_r$ , to prevent their intersections from going to infinity. In the present section, we describe a class of branes

for which intersections are bounded and for which the machinery of the present paper applies naturally. In defining this class of branes, we will make use of the superpotential  $W$ . However, for Lagrangian submanifolds  $\mathcal{L}_\ell$  and  $\mathcal{L}_r$  obeying the conditions that we will state momentarily, the space of supersymmetric  $(\mathcal{L}_\ell, \mathcal{L}_r)$  states makes sense in the ordinary  $A$ -model without a superpotential, and is unchanged when one turns on the superpotential  $W$ .

We simply require that  $\text{Im}(\zeta^{-1}W)$  goes to  $+\infty$  at infinity along  $\mathcal{L}_\ell$ , and to  $-\infty$  at infinity along  $\mathcal{L}_r$ . We will refer to left- and right- branes obeying this condition as  $W$ -dominated branes. (The reversal of sign between  $\mathcal{L}_\ell$  and  $\mathcal{L}_r$ , which might look peculiar at first sight, is natural in our formalism because a  $\pi$  rotation of the plane, which exchanges the left and right boundaries, reverses the sign of  $\zeta$  and hence of  $\text{Im}(\zeta^{-1}W)$ .)

One might think that this condition is required for bounding the surface terms in eqn. (11.11). But actually, that is not necessary; the potential  $V$  is positive-definite in any case since it can be written as in eqn. (11.10).

The real virtue of the  $W$ -dominated branes is that the growth condition on  $\text{Im}(\zeta^{-1}W)$  prevents the intersections of left- and right- branes from going to infinity. As one varies  $\mathcal{L}_\ell$  and/or  $\mathcal{L}_r$  to make an intersection point  $p$  go to infinity,  $\text{Im}(\zeta^{-1}W)$  would have to go to  $+\infty$  (since  $p \in \mathcal{L}_\ell$ ) and to  $-\infty$  (since  $p \in \mathcal{L}_r$ ). So intersection points do not go to infinity<sup>66</sup> and the space of  $(\mathcal{L}_\ell, \mathcal{L}_r)$  strings is well-defined in the ordinary  $A$ -model without a superpotential. (In other words, the space of  $(\mathcal{L}_\ell, \mathcal{L}_r)$  strings is well-defined even if we only use  $W$  for guidance in deciding what  $\mathcal{L}_\ell$  and  $\mathcal{L}_r$  to allow and do not actually turn on  $W$  as a contribution to the Lagrangian.)

Also, if  $\text{Im}(\zeta^{-1}W)$  diverges at infinity on  $\mathcal{L}_\ell$ , then the same is true on  $\mathcal{L}_\ell^w$  for any  $w > 0$ , since the  $\zeta$ -soliton equation is ascending gradient flow for  $\text{Im}(\zeta^{-1}W)$ . So intersection points of  $\mathcal{L}_\ell^w$  with  $\mathcal{L}_r$  do not go to infinity with increasing  $w$ . This means that intersection points cannot flow in from or out to infinity when  $w$  is turned on, so that, at the level of cohomology, the space of supersymmetric  $(\mathcal{L}_\ell, \mathcal{L}_r)$  strings is unchanged when  $W$  is turned on and is independent of  $w$ .

A further virtue of  $W$ -dominated branes is that the spaces of supersymmetric states are finite-dimensional spaces. We will phrase our argument here for the standard  $A$ -model without a superpotential; including a superpotential simply replaces  $\mathcal{L}_\ell$  with  $\mathcal{L}_\ell^w$  in what follows. In quantization of the  $(\mathcal{L}_\ell, \mathcal{L}_r)$  system, if  $\mathcal{L}_\ell$  and  $\mathcal{L}_r$  intersect in a discrete set of points  $\mathbb{T}(\mathcal{L}_\ell, \mathcal{L}_r)$ , then in the classical approximation, there is one supersymmetric state  $\Phi_\alpha$  for each  $\alpha \in \mathbb{T}(\mathcal{L}_\ell, \mathcal{L}_r)$ . The theory has a much simpler flavor if the sets  $\mathbb{T}(\mathcal{L}_\ell, \mathcal{L}_r)$  are always finite, and more generally the intersections  $\mathcal{L}_\ell \cap \mathcal{L}_r$  are always compact. Otherwise, one has to deal with infinite-dimensional spaces of supersymmetric states, in the classical approximation and perhaps in the exact theory. A simple example of what one would like to avoid is provided by again taking  $X$  to be the  $u - v$  plane, with symplectic form  $du \wedge dv$ , and taking for  $\mathcal{L}_\ell$  and  $\mathcal{L}_r$  the  $u$ -axis and the curve  $v = \sin u$ . Here the intersection  $\mathcal{L}_\ell \cap \mathcal{L}_r$  consists of infinitely many points. It is easy to construct wilder examples involving spirals

---

<sup>66</sup>We consider a family of branes parametrized by a compact parameter space. In that situation, the upper and lower bounds on  $\text{Im}(\zeta^{-1}W)$  hold uniformly. The same comment is relevant at several points below.

in the plane. For  $W$ -dominated branes, compactness of the intersection  $\mathcal{L}_\ell \cap \mathcal{L}_r$  is insured since  $\text{Im}(\zeta^{-1}W)$  goes to  $+\infty$  at infinity on  $\mathcal{L}_\ell$  and to  $-\infty$  at infinity on  $\mathcal{L}_r$ . In this case, the subset  $\Delta \subset \mathcal{L}_\ell$  on which  $\text{Im}(\zeta^{-1}W)$  is less than its upper bound on  $\mathcal{L}_r$  is compact.<sup>67</sup> The intersection  $\mathcal{L}_\ell \cap \mathcal{L}_r$  is a closed and therefore compact subspace of  $\Delta$  and consequently the space of supersymmetric states of the  $(\mathcal{L}_\ell, \mathcal{L}_r)$  system will always be finite-dimensional.

In our example of §11.2.3 with  $\text{Im}(\zeta^{-1}W) = (u^2 - v^2)$ , the brane  $\mathcal{L}_r$  of Figure 125 has the desired property, since  $(u^2 - v^2)$  goes to  $-\infty$  at infinity along  $\mathcal{L}_r$ , but  $\mathcal{L}_\ell$  does not obey the appropriate condition at infinity. To ensure that  $(u^2 - v^2)$  goes to  $+\infty$  at infinity along  $\mathcal{L}_\ell$ , we could rotate  $\mathcal{L}_\ell$  by  $\pm\pi/2$ . In this case, the pathologies noted in Section of §11.2.3 would disappear. Once we restrict the class of branes so that  $\text{Im}(\zeta^{-1}W)$  goes to  $+\infty$  or  $-\infty$  at infinity on  $\mathcal{L}_\ell$  or  $\mathcal{L}_r$ , we must also restrict the class of gauge transformations to include only those Hamiltonian symplectomorphisms that preserve these conditions. In particular, a  $\pi/2$  rotation, although symplectic, is not an allowed gauge transformation.

In this section, we have explained one natural answer to the question, “For what kind of branes  $\mathcal{L}_\ell, \mathcal{L}_r$  is the space of supersymmetric  $(\mathcal{L}_\ell, \mathcal{L}_r)$  states well-defined?” However, this question has at least one more interesting answer, which we return to in section 11.2.6 after introducing the concept of a thimble.

### 11.2.5 Thimbles

Generically, for every critical point  $\phi_i$ , there is a canonical example of a left-brane  $L_i^\zeta$  and also of a right-brane  $R_i^\zeta$  satisfying the conditions of section 11.2.4. To construct  $L_i^\zeta$ , we consider the  $\zeta$ -soliton equation on the half-line  $(-\infty, 0]$  with the boundary condition that the solution approaches a critical point  $\phi_i$  at  $x = -\infty$ . Regarding the  $\zeta$ -soliton equation as a gradient flow equation, the flows of this type are parametrized by the constants  $c_i$  in eqn. (10.26) with  $f_i > 0$ , so the dimension of  $L_i^\zeta$  is the Morse index  $y$  of the function  $\text{Im}(\zeta^{-1}W)$  at its critical point, and  $L_i^\zeta$  is a copy of  $\mathbb{R}^y$ . Like the real or imaginary part of any holomorphic function that has a nondegenerate critical point, this function has middle-dimensional Morse index, so in particular  $y = \dim_{\mathbb{C}} X$ . By mapping an ascending flow on  $(-\infty, 0]$  that starts at  $\phi_i$  to its value at  $x = 0$  (here we include the trivial flow line that sits at  $\phi_i$  at all times), we can interpret the space of such flows as a middle-dimensional submanifold  $L_i^\zeta \subset X$ . Since the  $\zeta$ -soliton equation is translationally-invariant, the value  $\phi(x_0)$  of an ascending flow from  $\phi_i$  at any point  $x_0$  with  $-\infty < x_0 < \infty$  is on the submanifold  $L_i^\zeta$ . So  $L_i^\zeta$  can be viewed as the union of all ascending flow lines that start at the critical point  $\phi_i$  in the far past. If all flow lines that start at  $\phi_i$  in the past flow to infinity in the future, then  $L_i^\zeta$  is a closed submanifold of  $X$ , and in this case we call it a Lefschetz thimble.

There is an important circumstance in which this can fail. Suppose that  $\zeta = \zeta_{ji}$  for some  $j$ . Then an  $ij$  soliton may exist. Such a soliton is a flow line that starts arbitrarily close to  $\phi_i$  in the past and flows arbitrarily close to  $\phi_j$  in the future, but never reaches it. So in this case, the point  $\phi_j$  is contained in the closure of  $L_i^\zeta$  but not in  $L_i^\zeta$  itself. (One

---

<sup>67</sup>We consider a family of branes parametrized by a compact parameter space. In that situation, the upper and lower bounds on  $\text{Im}(\zeta^{-1}W)$  hold uniformly and all intersections occur in a fixed compact set  $\Delta \subset X$  that is independent of the parameters.

could of course replace  $L_i^\zeta$  with its closure, but this does not work well; for example, for  $\dim_{\mathbb{C}} X > 1$ , the closure is generically not a manifold, and for  $\dim_{\mathbb{C}} X = 1$  it may be a manifold with boundary. The fundamental reason that there is no good definition of a Lefschetz thimble at  $\zeta = \zeta_{ji}$  is really that in crossing such a value, the topology of the Lefschetz thimble can jump.)

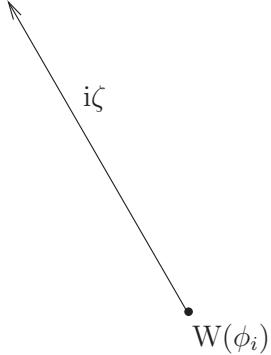
As long as  $\zeta$  does not equal any of the  $\zeta_{ij}$ , we do not meet this problem and  $L_i^\zeta$  is a closed submanifold of  $X$ . We have already seen that  $L_i^\zeta$  is middle-dimensional and topologically  $\mathbb{R}^y$ . To show that  $L_i^\zeta$  is Lagrangian, we use the fact that the  $\zeta$ -soliton equation is Hamiltonian flow (eqn. (11.13)). Using the identification of  $L_i^\zeta$  with the value  $u^a(0)$  of a flow at time  $x = 0$ , and writing the symplectic form of  $X$  as  $\omega_{ab} du^a du^b$ , the restriction of this form to  $L_i^\zeta$  is  $\omega|_{L_i^\zeta} = \omega_{ab} du^a(0) du^b(0)$ . But Hamiltonian flow preserves the symplectic form, so we can equally write  $\omega|_{L_i^\zeta} = \omega_{ab} du^a(x) du^b(x)$  for any  $x$ . Taking  $x \rightarrow -\infty$ , this vanishes for flows that start at  $\phi_i$ , so  $\omega|_{L_i^\zeta} = 0$  and  $L_i^\zeta$  is Lagrangian. Since  $L_i^\zeta$  is topologically  $\mathbb{R}^y$ , any closed form on  $L_i^\zeta$  is exact, and  $L_i^\zeta$  is exact Lagrangian. Finally, because  $L_i^\zeta$  was defined by ascending gradient flow from  $\phi_i$ ,  $\text{Im}(\zeta^{-1}W)$  is bounded below along  $L_i^\zeta$  by its value at  $\phi_i$ . If the Kähler metric of  $X$  is complete,  $\text{Im}(\zeta^{-1}W)$  goes to infinity at infinity along  $L_i^\zeta$ . (Completeness of the metric and the fact that each ascending flow from  $\phi_i$  goes to infinity in  $X$  implies that each ascending flow line has infinite length; this plus the ascending flow equation implies that  $\text{Im}(\zeta^{-1}W)$  goes to infinity along each such line.) The corresponding right-brane  $R_j^\zeta$ , which we call a right Lefschetz thimble, is defined in precisely the same way, with similar properties. It parametrizes ascending gradient flows on the half-line  $[0, \infty)$  that approach  $\phi_j$  for  $x \rightarrow \infty$ , and can be identified with the value of such a flow at  $x = 0$ . Clearly  $\text{Im}(\zeta^{-1}W)$  is bounded above along  $R_j^\zeta$  by its value at  $\phi_j$ .

As a simple example we return to  $\zeta^{-1}W = i\phi^2$  so  $\text{Im}(\zeta^{-1}W) = (u^2 - v^2)$ . There is a single critical point at  $\phi = 0$ . The left-Lefshetz thimble is the  $u$ -axis and the right-Lefshetz thimble is the  $v$ -axis.

#### 11.2.6 Another Useful Class Of Branes: Class $T_\kappa$

In any Hamiltonian flow, the Hamiltonian is a conserved quantity. So in particular, the Hamiltonian  $H = -\frac{1}{2}\text{Re}(\zeta^{-1}W)$  is a conserved quantity for the  $\zeta$ -soliton equation. Moreover,  $\text{Im}(\zeta^{-1}W)$  is an increasing function of  $x$  along any non-constant solution of this equation. Putting these facts together, the values of  $W$  along the left thimble  $L_i^\zeta$  lie on a ray that begins at the point  $W(\phi_i)$  and extends in the direction  $i\zeta$  in the complex  $W$ -plane (Figure 126). (The following discussion could be presented in terms of right thimbles rather than left thimbles, but this would add nothing as it would be equivalent to replacing  $\zeta$  by  $-\zeta$ .)

The images in the  $W$ -plane of all the left thimbles  $L_j^\zeta$ ,  $j \in \mathbb{V}$ , form a collection of parallel rays, starting at the critical values  $W(\phi_j)$ . Assuming that  $\mathbb{V}$  is a finite set, the images of these thimbles are all contained in a semi-infinite strip  $T_\zeta$  of finite width in the  $W$ -plane. (This is sketched in Figure 127, except that, for reasons that will soon be apparent, in the figure  $\zeta$  is replaced by a complex number  $\kappa$  of modulus 1, not necessarily



**Figure 126:** A ray in the complex  $W$ -plane, starting at  $W(\phi_i)$  and running in the  $i\zeta$  direction.

equal to  $\zeta$ .)  $T_\zeta$  is defined by

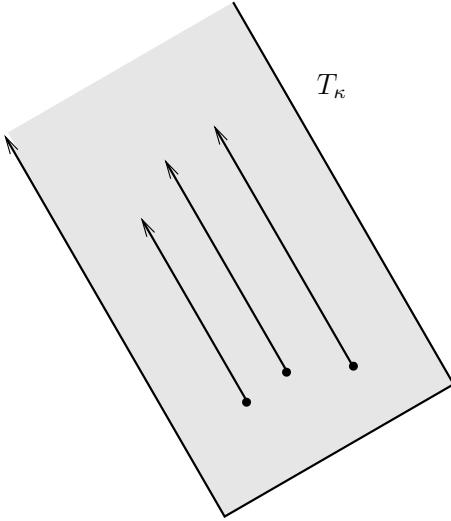
$$\begin{aligned} |\operatorname{Re}(\zeta^{-1}W)| &\leq c \\ \operatorname{Im}(\zeta^{-1}W) &\geq c', \end{aligned} \tag{11.36}$$

for some constants  $c, c'$ . We say that a Lagrangian submanifold  $\mathcal{L}$  – or a brane supported on  $\mathcal{L}$  – is of class  $T_\zeta$  if  $W$  restricted to  $\mathcal{L}$  is valued in  $T_\zeta$ . Branes of class  $T_\zeta$  (or obeying an equivalent condition) are considered in the mathematical theory of the Fukaya-Seidel category [81]; see section 11.3. This is a mathematical theory related to what in physical terms is the  $A$ -model with a superpotential  $W$ . In addition to some reasoning that is described below, the construction of the Fukaya-Seidel category is motivated by mirror symmetry.

Let  $\mathcal{L}$  and  $\mathcal{L}'$  be Lagrangian submanifolds of class  $T_\zeta$ . In the  $A$ -model without a superpotential, the space of supersymmetric  $(\mathcal{L}, \mathcal{L}')$  states is not well-defined, because the strip  $T_\zeta$  is not compact and intersections of  $\mathcal{L}$  and  $\mathcal{L}'$  can go off to infinity (if, for example, a Hamiltonian symplectomorphism is applied to  $\mathcal{L}$  or  $\mathcal{L}'$ ). What happens if we turn on a superpotential  $W$ ? By itself, this does not help. In studying supersymmetric  $(\mathcal{L}, \mathcal{L}')$  states on a strip of width  $w > 0$ , the effect of turning on the superpotential is that instead of looking at intersections  $\mathcal{L} \cap \mathcal{L}'$ , we have to look at intersections  $\mathcal{L}^w \cap \mathcal{L}'$ , where  $\mathcal{L}^w$  is obtained from  $\mathcal{L}$  by ascending gradient flow with respect to  $\operatorname{Im}(\zeta^{-1}W)$ . The image under  $W$  of  $\mathcal{L}^w$  is contained in the strip  $T_\zeta$  itself, so  $\mathcal{L}^w$  is again of class  $T_\zeta$  and the intersections  $\mathcal{L}^w \cap \mathcal{L}'$  are not well-behaved.

However, a simple variant of this idea does work. We pick a complex number  $\kappa$  of modulus 1, but not equal to  $\pm\zeta$ . Then instead of branes of class  $T_\zeta$ , we consider branes of class  $T_\kappa$ . These branes are characterized by the condition (11.36), but with  $\zeta$  replaced by  $\kappa$  (as is actually shown in Figure 127). Omitting the points  $\pm\zeta$  divides the unit circle into two connected components, and it is convenient to make a choice that  $\kappa$  lies to the “left” of  $\zeta$  (meaning that  $\pi > \operatorname{Arg}(\zeta^{-1}\kappa) > 0$ ).

The classical approximation to a supersymmetric  $(\mathcal{L}, \mathcal{L}')$  state on a strip of width  $w$  is now given by a solution of the  $\zeta$ -soliton equation, starting somewhere on  $\mathcal{L}$  at the left end



**Figure 127:** The rays in the complex  $W$ -plane that start at critical points and all run in the  $i\kappa$  direction fit into the semi-infinite strip  $T_\kappa$ , which is shown as a shaded region.

of the strip and ending somewhere on  $\mathcal{L}'$  at the right end. Differently put, the classical approximation to such a supersymmetric state is an intersection point of  $\mathcal{L}^w$  with  $\mathcal{L}'$ , where  $\mathcal{L}^w$  is obtained from  $\mathcal{L}$  by evolving for a “time”  $w$  via the  $\zeta$ -soliton equation.

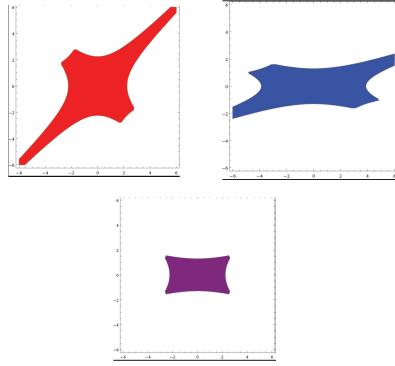
This evolution is in the direction of increasing  $\text{Im}(\zeta^{-1}W)$  (with  $\text{Re}(\zeta^{-1}W)$  fixed). Because of our hypothesis that  $\kappa \neq \pm\zeta$ , this evolution tends to move the image of  $\mathcal{L}^w$  out of the strip  $T_\kappa$ . As we will see momentarily, under reasonable conditions on the growth of  $W$  at infinity, the intersection  $\mathcal{L}^w \cap \mathcal{L}'$  is bounded, so the space of supersymmetric  $(\mathcal{L}, \mathcal{L}')$  states is well-defined.

To show the boundedness, we proceed as follows. Let  $X_\kappa$  be the portion of the target space  $X$  of the  $\sigma$ -model in which  $W$  takes values in  $T_\kappa$ . And let  $X_\kappa^w$  be the subset of  $X$  that  $X_\kappa$  flows to under  $\zeta$ -soliton flow for “time”  $w$ . The key point is now to show that under suitable conditions,  $X_\kappa^w \cap X_\kappa$  is compact for all  $w > 0$ . For if  $\mathcal{L}$  and  $\mathcal{L}'$  are any (closed) Lagrangian submanifolds of class  $T_\kappa$ , then the intersection  $\mathcal{L}^w \cap \mathcal{L}'$  is a closed subset of  $X_\kappa^w \cap X_\kappa$ , and so is compact if  $X_\kappa^w \cap X_\kappa$  is compact. As usual, the compactness of  $\mathcal{L}^w \cap \mathcal{L}'$  will ensure that the space of supersymmetric states of the  $(\mathcal{L}, \mathcal{L}')$  system is well-defined. See Figure 128.

To understand the compactness of  $X_\kappa^w \cap X_\kappa$  without any excessive clutter, let us take  $\zeta = 1$  and  $\kappa = i$ . So the  $\zeta$ -instanton equation is Hamiltonian flow for the Hamiltonian  $H = -\frac{1}{2}\text{Re}(\zeta^{-1}W) = -\frac{1}{2}\text{Re}W$ . The “time”-dependence of  $\text{Re}(\kappa^{-1}W) = \text{Im}W$  along the flow is

$$\frac{d}{dx}\text{Re}(\kappa^{-1}W) = \frac{d}{dx}\text{Im}W = \{H, \text{Im}W\} = -\frac{1}{2}\{\text{Re}W, \text{Im}W\} = -\frac{1}{4}|\text{d}W|^2. \quad (11.37)$$

Here  $\{\ , \}$  is the Poisson bracket computed using the symplectic form of the target space  $X$ , and we have evaluated this Poisson bracket using the Cauchy-Riemann equations obeyed



**Figure 128:** Illustrating the regions  $X_\kappa$ ,  $X_\kappa^w$  and their intersection. Here we choose a single chiral superfield  $\phi$  with  $W = i\phi^2$  and  $\zeta = 1$ . The region  $X_\kappa$  for  $\kappa = i$ ,  $c = 5$ , and  $c' = -5$  is illustrated in the upper left figure. The region is noncompact, with northeast and southwest boundaries asymptoting to the line  $v = u$ . Under the flow  $u \rightarrow e^w u$ ,  $v \rightarrow e^{-w} v$  the region evolves (for  $e^{2w} = 3$ ) to the blue region shown in the upper right figure. Again this region is noncompact. The intersection, shown below in purple is compact for all  $w > 0$  and decompactifies as  $w \rightarrow 0$ .

by the holomorphic function  $W$ . (In general we have  $\frac{d}{dx} \text{Re}(\kappa^{-1} W) = \frac{1}{4} \text{Im}(\frac{\zeta}{\kappa}) |dW|^2$ .)

Usually, we are interested in models in which  $|dW|^2$  goes to infinity at infinity along  $X$ , and hence also along  $X_\kappa$ . (For example, in the most standard Landau-Ginzburg model,  $X = \mathbb{C}^n$  for some  $n$  and  $W$  is a polynomial that is sufficiently generic so that  $|dW|^2$  grows polynomially at infinity.) In this case, eqn. (11.37) implies that the rate at which  $\text{Re}(\kappa^{-1} W)$  increases under  $\zeta$ -soliton flow increases near infinity in  $X_\kappa$ . This means that  $\zeta$ -soliton flow for any positive “time”  $w$  maps a neighborhood of infinity in  $X_\kappa$  strictly outside of  $X_\kappa$  (this neighborhood depends on  $w$ ), and hence  $X_\kappa^w \cap X_\kappa$  is indeed compact for all  $w > 0$ .

In general, we might not want to assume that  $|dW|^2$  goes to infinity at infinity along  $X$ , but it is always reasonable to assume that  $|dW|^2$  is bounded above 0 near infinity. (Otherwise,  $W$  has a critical point at infinity and one should not expect to get a good description based only on the set  $\mathbb{V}$  of finite critical points.) With  $|dW|^2$  bounded above 0, the same reasoning as before shows that if  $w$  is sufficiently large, then  $X_\kappa^w \cap X_\kappa$  is compact.

Putting these statements together, under reasonable conditions, the space of supersymmetric  $(\mathcal{L}, \mathcal{L}')$  states is well-defined for any  $\mathcal{L}, \mathcal{L}'$  of class  $T_\kappa$ . In the above, we took  $\kappa = i\zeta$ , but the same reasoning applies as long as  $\kappa \neq \pm\zeta$ . Saying that the space of  $(\mathcal{L}, \mathcal{L}')$  strings is well-defined means that it invariant under Hamiltonian symplectomorphisms of  $X_\kappa$  (applied separately to  $\mathcal{L}$  and  $\mathcal{L}'$ ), invariant under changes in the Kähler metric of  $X$  (as long as this is not changed too drastically at infinity) and invariant under changes in  $\kappa$  (as long as one keeps away from  $\kappa = \pm\zeta$ ).

### 11.3 The Fukaya-Seidel Category

Having come this far, it is not too hard to understand how to go farther and define open-string tree amplitudes for branes of class  $T_\kappa$ . (By contrast, one cannot do this for  $W$ -dominated branes, as we will soon explain.)

These open-string tree amplitudes – when specialized to the case of just one string in the future, as discussed below – give what would be called mathematically an  $A_\infty$  algebra if one considers just one brane of class  $T_\kappa$ , or an  $A_\infty$  category if one considers all of them. The  $A_\infty$  category that we obtain is presumably the Fukaya-Seidel category [81, 82, 83, 80], or its close cousin, the Fukaya category of the superpotential. (These are expected<sup>68</sup> to have the same derived categories of branes.)

It seems that, mathematically, it is understood that the Fukaya-Seidel category should have a definition along the lines of what we sketch below, but this has not yet appeared in the literature because of analytical details. The existing literature is thus based on alternative approaches that circumvent some analytical difficulties but will be less transparent to a quantum field theorist. Also, some of the details of the setup we use here seem fairly natural from a quantum field theory point of view, but a rigorous approach might use somewhat different definitions because purely from the standpoint of partial differential equations, one can make some more general choices and this freedom might be useful. For example, instead of branes of class  $T_\kappa$ , one could consider branes whose image in the  $W$ -plane is the union of a semi-infinite ray and a compact set. Similarly, instead of using the global  $\zeta$ -instanton equation, one can consider a more general equation that looks like the  $\zeta$ -instanton equation near the infinite ends of the worldsheet. Despite some detailed differences in approach, we expect the open-string amplitudes that we define to have essentially the same content as the Fukaya-Seidel category.

To define open-string amplitudes, it is important to spell out a consequence of the restriction  $\kappa \neq \pm\zeta$ . Concretely, to compute the space of supersymmetric  $(\mathcal{L}, \mathcal{L}')$  strings, we quantize the  $\sigma$ -model on a strip  $S$  in the  $x - \tau$  plane that is defined by  $x_\ell \leq x \leq x_r$  with  $x_r - x_\ell = w$ . This is a strip that runs in the  $\tau$  direction. We construct an MSW complex as usual with a basis given by solutions of the  $\zeta$ -soliton equation and a differential found by counting solutions of the  $\zeta$ -instanton equation. The reason that we specified that the strip  $S$  runs in the  $\tau$  direction is that, unlike the equation for a pseudoholomorphic curve that is usually considered in the  $A$ -model, the  $\zeta$ -instanton equation is not invariant under rotation of the  $x - \tau$  plane. If we rotate  $S$  in the  $x - \tau$  plane, so that  $S$  is at an angle  $\vartheta$  to the  $\tau$ -axis, this would be equivalent to replacing  $\zeta$  by  $\zeta e^{i\vartheta}$ . Since the only restriction on  $\zeta$  is  $\zeta \neq \pm\kappa$ , we may rotate  $S$  by an angle  $\vartheta$  as long as  $\zeta e^{i\vartheta} \neq \pm\kappa$ . For example, if  $\kappa = i\zeta$ , we may take  $S$  to be a strip propagating in any direction in the  $x - \tau$  plane except the horizontal. This restriction on the slope of  $S$  means that  $S$  has a well-defined “past” (an end with  $\tau \rightarrow -\infty$ ) and “future” (an end with  $\tau \rightarrow +\infty$ ).

The procedure for defining the open-string amplitudes in this situation is standard, except for a few key details. Let us recall that in the usual  $A$ -model without a superpotential

---

<sup>68</sup>For example, see the end of section 2 of [80], where the Fukaya category of the superpotential is called  $F(\pi)$  and the Fukaya-Seidel category is called  $A$ . This and other matters described in the next paragraph were explained to us by N. Sheridan.

(or in physical string theory), to compute a tree-level amplitude with  $q$  open strings, one takes the string worldsheet to be a disc  $H$  with  $q$  marked points on its boundary. The regions on the boundary between the marked points are labeled by branes and the marked points are labeled by vertex operators. Modulo conformal transformations,  $H$  depends on  $q - 3$  real moduli. To compute the usual  $A$ -model amplitudes, we count (with signs) all pseudomolomorphic maps from  $H$  to the target space  $X$ , obeying conditions determined by the choices of branes and vertex operators. In the counting, we do not specify *a priori* the conformal structure of  $H$  and we include pseudoholomorphic curves with any values of their moduli.

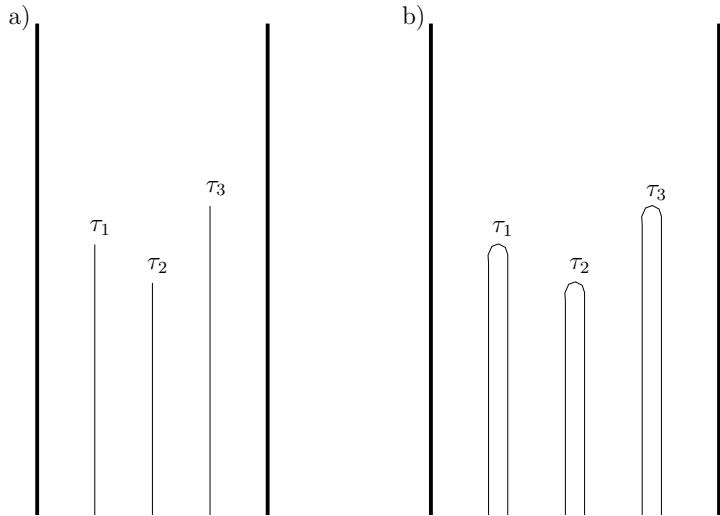
To define open-string amplitudes in the present context, roughly speaking, we do the same thing, with the equation for a pseudoholomorphic map replaced by the  $\zeta$ -instanton equation. There are some changes because the  $\zeta$ -instanton equation is not conformally-invariant or even rotation-invariant. We take  $H$  to be a region in the complex plane, where we know how to define the  $\zeta$ -instanton equation.<sup>69</sup> Without conformal invariance, there is no direct equivalence between states and vertex operators, so we represent the external strings by semi-infinite strips of specified widths. These strips are not allowed to be horizontal, because then in the case of branes of class  $T_\kappa$ , the external string states would not be well-defined, as was just explained. So in contrast to physical string theory (or the usual  $A$ -model), there is a well-defined distinction between string states that come in from the “past” ( $\tau \rightarrow -\infty$ ) and those that go out to the future ( $\tau \rightarrow +\infty$ ). There are well-defined amplitudes with any number  $n$  of strings coming in from the past and any number  $m$  going out to the future. However, in the context of the Fukaya-Seidel category or the Fukaya category of the superpotential, it is usual to consider only the case  $m = 1$ . (This case leads to amplitudes that can be put in a convenient algebraic framework – an  $A_\infty$  algebra – and whose counterparts under mirror symmetry are relatively well-understood.)

For brevity, we will consider only the case  $m = 1$ . The total number of external string states is therefore  $q = n + 1$ , and the number of real moduli is  $q - 3 = n - 2$ . To define the  $n \rightarrow 1$  amplitude, we need a family of regions  $H$  in the  $x - \tau$  plane that depend on the usual  $n - 2$  real moduli of a disc with  $n + 1$  marked points on its boundary. These regions are far from being uniquely determined. One convenient choice is the parametrization of the moduli space of a disc with  $n + 1$  punctures that in ordinary string theory is used in light cone gauge (for the case that all incoming particles have equal  $p_+$ ). The strings coming from the past all have equal width  $w$  and the string going out to the future has width  $nw$ ; as usual in light cone gauge, the moduli are the differences between the values of  $\tau$  at which two strings join. This is depicted in Figure 129(a). If one prefers, one can use the less singular worldsheets of Figure 129(b).

As long as all branes considered are of class  $T_\kappa$ , the counting of  $\zeta$ -instanton solutions to define string amplitudes in this situation is well-defined, basically because the properties of the branes that make the external string states well-defined also ensure that solutions of the  $\zeta$ -instanton equation cannot go to infinity. The resulting tree amplitudes have all

---

<sup>69</sup>It is possible but not necessary for our purposes here to generalize this slightly –  $H$  could be a Riemann surface with boundary with strip-like ends with a not necessarily holomorphic trivialization of its canonical line bundle and some conditions on how the trivialization behaves at infinity and along the boundary.



**Figure 129:** (a) An open-string worldsheet  $H$  in a form familiar in light-cone gauge.  $n$  open strings all of width  $w$  come in from the past ( $\tau = -\infty$ ) and a single one of width  $nw$  goes out to the future ( $\tau = +\infty$ ). There are  $n - 1$  values of  $\tau$  at which two open strings combine to one. The linearly independent differences between these critical values of  $\tau$  are the  $n - 2$  real moduli of this worldsheet. (b) The picture in (a) can be slightly modified in this fashion – if one wishes – so that  $H$  becomes smooth. The moduli are still the differences between the critical values of  $\tau$ .

the usual properties except cyclic symmetry. Lack of cyclic symmetry means that these amplitudes cannot be derived in a natural way from a  $\mathcal{Q}$ -invariant effective action (except possibly by introducing separate fields to represent incoming and outgoing strings) but can be interpreted as constructing a nonlinear  $\mathcal{Q}$  operator (which acts on a Fock space of open strings). Mathematically, lack of cyclic symmetry means that one gets an  $A_\infty$  algebra without a trace.

This construction would not work for  $W$ -dominated branes.  $W$ -dominated branes lead to well-defined spaces of BPS states, but not to tree amplitudes. The reason is that in trying to define a tree amplitude for  $W$ -dominated branes, there is no natural way to decide if we should use a left-brane (with  $\text{Im}(\zeta^{-1}W) \rightarrow +\infty$  at infinity) or a right-brane (with  $\text{Im}(\zeta^{-1}W) \rightarrow -\infty$  at infinity) on the intermediate boundaries in Figure 129(b). Moreover, neither choice leads to a well-controlled counting.

On the other hand, for branes of class  $T_\kappa$ , the definition of  $n \rightarrow 1$  amplitudes by counting of  $\zeta$ -instantons works fine for any  $\kappa \in U(1) - \{\pm\zeta\}$ . If  $\kappa$  and  $\kappa'$  are in the same connected component of  $U(1) - \{\pm\zeta\}$ , the categories associated to  $\kappa$  and  $\kappa'$  are naturally equivalent to each other via rotation. Thus, using branes of class  $T_\kappa$  (with the same  $\kappa$  on all connected components of all boundaries), by counting  $\zeta$ -instantons in Figure 129 we can define two  $A_\infty$  categories  $\mathfrak{Br}_{\pm\kappa}$ . In Section §15 below we will argue that the two categories  $\mathfrak{Br}_{\pm\kappa}$  are  $A_\infty$ -equivalent to the two categories of branes constructed in the abstract part of this paper using left- and right-thimbles of class  $T_\zeta$  and positive or negative half-plane webs.

We should stress that even for branes of class  $T_\kappa$ , while we can define tree-level amplitudes in parallel with standard tree-level  $A$ -model amplitudes, we cannot define analogs of higher genus  $A$ -model amplitudes. This is basically because with  $W \neq 0$ , the standard  $A$ -model twisting is not available. In the future, to avoid repeating ourselves many times, we will use the phrase “ $A$ -model” to refer to either a standard  $A$ -model with  $W = 0$ , or a partial  $A$ -model with  $W \neq 0$  in which one considers only tree amplitudes.

## 12. MSW Complex On The Real Line: Solitons And Instantons

Up to this point, our preliminary discussion of BPS solitons has been purely classical. We now want to study these BPS solitons at the quantum level. Everything about this problem will closely parallel the general quantum mechanical analysis of section 10, except that we have to take into account certain zero-modes associated to broken bosonic and fermionic symmetries; these do not have a close analog in the generic quantum mechanical case.

Let  $\mathcal{S}_{ij}$  denote the set of classical  $ij$  solitons, that is solutions of the  $\zeta$ -soliton equation that interpolate from  $\phi_i$  at  $x = -\infty$  to  $\phi_j$  at  $x = +\infty$ . From eqn. (11.15), we know that such solutions exist only for  $\zeta = \zeta_{ji}$ , so in what follows we choose that value of  $\zeta$ . The group  $\mathbb{R}$  of translations of the  $x$ -axis acts freely on the space of classical  $ij$  solitons, and generically an  $ij$  soliton has no bosonic zero-mode except the one associated to translation invariance. We will assume that we are in this situation. A classical  $ij$  soliton always has a pair of fermionic zero-modes, one of fermion number  $\mathcal{F} = 1$  and one of fermion number  $-1$ , generated by the 2 supercharges that are not in the small subalgebra. When the zero-mode associated to translation symmetry is the only bosonic zero-mode, the 2 zero-modes associated to broken supersymmetries are the only fermionic ones.

In this situation, the quantization of a classical  $ij$  soliton solution, in perturbation theory, is relatively straightforward. The non-zero bosonic and fermionic modes are simply placed in their ground state. The only subtlety here is that one must determine the fermion number  $f_0$  of the ground state of the nonzero modes of the fermions. We consider this question presently. The quantization of the bosonic and fermionic zero-modes is slightly subtle but is well-known. If we write  $a$  for the translational zero-mode of the soliton, then a wavefunction  $e^{ipa}$  for this mode describes a soliton in a state of arbitrary momentum  $p$ . A quantum BPS state invariant under the small supersymmetry algebra generated by  $\mathcal{Q}_\zeta$  and  $\overline{\mathcal{Q}}_\zeta$  arises for  $p = 0$ . (A soliton in a momentum eigenstate with  $p \neq 0$  is invariant under a boosted version of this algebra.) The fermion zero-modes are a pair of operators  $\chi_0, \overline{\chi}_0$  of  $\mathcal{F} = \pm 1$  generating a two-dimensional Clifford algebra; the representation of this algebra gives a pair of states of fermion number  $\mathcal{F} = \pm 1/2$ . So overall, the quantized soliton in perturbation theory can have any momentum and has fermion number  $f_0 \pm 1/2$ .

### 12.1 The Fermion Number

The fermion number  $f_0$  of the Fock vacuum (of non-zero fermion modes) is formally the fermion number of the filled Fermi sea. This of course diverges and needs to be regularized. The physically natural approach is to define first the renormalized, conserved, Lorentz-covariant fermion number current (for example, via a process of point-splitting and normal-

ordering) and then compute the matrix element in the soliton state, in perturbation theory, of the integral of the fermion charge density. The point-splitting and normal ordering deal with ultraviolet divergences in the definition of  $f_0$ . And even if  $D$  has infinite ends (for instance  $D = \mathbb{R}$ ), there is no infrared problem in defining the integrated fermion number because, by virtue of Lorentz invariance, the expectation value of the fermion charge density vanishes in the massive vacua at  $x \rightarrow \pm\infty$ . (It turns out that there is a subtlety at finite distance boundaries of  $D$  rather than at ends at infinite distance; see section 13.3.)

An alternative procedure is more convenient for some purposes. In its standard form that we describe first, this procedure deals with ultraviolet divergences but not infrared ones, so it applies for the case that space is a compact 1-manifold  $D = [x_\ell, x_r]$  (with boundary conditions as in section 11.2). In what follows, by a “fermion state,” we mean an energy level of the single-particle Dirac equation that governs the fermions of  $\mathcal{F} = 1$ . (We need not discuss separately the single-particle  $\mathcal{F} = -1$  modes; they are canonically conjugate to the  $\mathcal{F} = 1$  modes, and the Fock vacuum can be completely characterized by saying which  $\mathcal{F} = 1$  modes annihilate it.) From  $f_0$ , which formally is the number of filled states of negative energy, formally we subtract a constant, namely  $1/2$  the total number of fermion states. Subtracting this constant can be thought of as measuring the fermion number of the soliton relative to the fermion number of the vacuum. Still formally, with this subtraction,  $f_0$  is  $1/2$  of the number of  $\mathcal{F} = 1$  fermion modes of negative energy (the ones that are filled in the Fock vacuum) minus  $1/2$  the number of  $\mathcal{F} = 1$  states of positive energy (the ones that are unfilled). This result, whose quantum mechanical analog is eqn. (10.11), still needs to be regulated. We weight a mode of energy  $E$  by a factor of  $\exp(-\varepsilon|E|)$ , for small positive  $\varepsilon$ , and take  $\varepsilon \rightarrow 0$  at the end of the calculation. Thus if  $\mathbb{T}$  is the set of all *non-zero energy* fermion modes of  $\mathcal{F} = 1$ , we define  $f_0$  as

$$f_0 = -\frac{1}{2} \lim_{\varepsilon \rightarrow 0} \sum_{i \in \mathbb{T}} \exp(-\varepsilon|E_i|) \text{sign}(E_i). \quad (12.1)$$

This formula is the general formula (10.11) of supersymmetric quantum mechanics for the fermion number of a state associated to a critical point, except that in infinite dimensions we require a regulator, such as  $\exp(-\varepsilon|E|)$ , and in the finite-dimensional problem with a nondegenerate Morse function, there is no need to discuss fermion zero-modes. The  $\eta$ -invariant of the single-particle Dirac Hamiltonian  $\mathcal{D}$  (this is the operator whose eigenvalues are the  $E_i$ ; see eqn. (12.5)), is usually defined as the “limit”<sup>70</sup>

$$\eta(\mathcal{D}) = \lim_{\varepsilon \rightarrow 0} \sum_{i \in \mathbb{T}} |E_i|^{-\varepsilon} \text{sign}(E_i), \quad (12.2)$$

but the precise choice of regulator does not matter; one can here replace  $|E|^{-\varepsilon} = \exp(-\varepsilon \log |E|)$  with  $\exp(-\varepsilon|E|)$ , as in (12.1). So

$$f_0 = -\frac{\eta(\mathcal{D})}{2}. \quad (12.3)$$

---

<sup>70</sup>Unlike the sum in (12.1), the one in (12.2) is not absolutely convergent. So we should specify that the meaning of the “limit” in this equation is that the sum over states on the right hand side of eqn. (12.2) converges for sufficiently large  $\text{Re } \varepsilon$  and defines an analytic function of  $\varepsilon$  that has an analytic continuation to  $\varepsilon = 0$  where it is non-singular. Also,  $\eta$  is sometimes defined to include a contribution from the zero-modes, but here it will be more convenient to omit them.

Including the contributions of the zero-modes, the soliton has two states of fermion number  $f_0 \pm 1/2$ . In other words, the two states, which we will call  $\Psi_{ij}^f(p)$  and  $\Psi_{ij}^{f+1}(p)$  (where  $p \in \mathcal{S}_{ij}$  labels a particular classical  $ij$  soliton), have fermion numbers  $f$  and  $f + 1$ , with

$$f = f_0 - \frac{1}{2} = -\frac{\eta(\mathcal{D}) + 1}{2} = -\frac{\eta(\mathcal{D} + \varepsilon)}{2}. \quad (12.4)$$

This is the proper formula for  $f$  on a compact manifold without boundary. In our application,  $D$  always has boundaries and/or infinite ends. In the presence of an infinite end, an infrared regularization of  $\eta$  is required, as in eqn. (12.7) below. In the presence of a boundary, the formula for  $f$  requires a boundary correction that is explained in section 13.3.

A shortcut to find the appropriate Dirac operator  $\mathcal{D}$  whose  $\eta$ -invariant enters this formula is to use the general formalism of supersymmetric quantum mechanics, as reviewed in section 10. In general, the Hamiltonian operator acting on the fermions (which in the quantum mechanical context is the fermion mass matrix  $\partial^2 h / \partial u^i \partial u^j$ ) is the operator that arises in linearizing the equation  $(\partial h / \partial u^i = 0)$  for a critical point. In our present context, the equation for a critical point is the  $\zeta$ -soliton equation and its linearization is the condition

$$\frac{\partial}{\partial x} \delta\phi^I - \frac{i\zeta g^{I\bar{J}}}{2} \frac{\partial^2 \bar{W}}{\partial \bar{\phi}^J \partial \bar{\phi}^K} \delta\bar{\phi}^K = 0, \quad (12.5)$$

together with the complex conjugate of this equation. Writing the left hand side of (12.5) as a linear operator acting on the pair  $\begin{pmatrix} \delta\phi^I \\ \delta\bar{\phi}^J \end{pmatrix}$  and including the complex conjugate equation, we arrive at a formula for the appropriate Dirac operator:

$$\mathcal{D} = \sigma^3 i \frac{d}{dx} + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \frac{\zeta^{-1}}{2} g^{J\bar{I}} \frac{\partial^2 W}{\partial \phi^J \partial \phi^K} + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \frac{\zeta}{2} g^{I\bar{J}} \frac{\partial^2 \bar{W}}{\partial \bar{\phi}^J \partial \bar{\phi}^K}. \quad (12.6)$$

Here the Hessians of  $W$  and  $\bar{W}$  are evaluated on the soliton configuration and  $\sigma^3$  is short for

$$\begin{pmatrix} \delta^I_K & 0 \\ 0 & -\delta^{\bar{I}}_{\bar{K}} \end{pmatrix}.$$

The standard definition of the  $\eta$ -invariant that we have given above assumes that  $\mathcal{D}$  has a discrete spectrum, so it applies for quantization on  $D = [x_\ell, x_r]$  (where, however, it does not quite give the complete answer for the fermion number, as we explain in section 13.3) but not otherwise. In the case of a BPS soliton with  $D = \mathbb{R}$ , because there is a mass gap at infinity,  $\mathcal{D}$  has a discrete spectrum near zero energy, but it has a continuous spectrum above some threshold. One needs to generalize slightly the definition of the  $\eta$ -invariant in this situation (for a rigorous treatment, see [76]). The contribution of the discrete spectrum does not need any change (there are only finitely many states in the discrete spectrum, and their contribution to the  $\eta$ -invariant is actually simply the number of positive energy normalizable eigenstates of  $\mathcal{D}$  minus the number of negative energy normalizable eigenstates of  $\mathcal{D}$ ). The contribution of the continuous spectrum needs to be

defined more precisely. If  $P$  is the orthogonal projector onto the continuous spectrum of  $\mathcal{D}$ , the contribution of the continuous spectrum to the  $\eta$ -invariant is

$$\eta^{\text{cont}}(\mathcal{D}) = \lim_{\varepsilon \rightarrow 0} \int_{-\infty}^{\infty} dx \sum_{s=1}^{2d} \langle x, s | P \text{sign}(\mathcal{D}) \exp(-\varepsilon|\mathcal{D}|) |x, s\rangle. \quad (12.7)$$

As usual,  $|x\rangle$  is a state with delta-function support at a point  $x \in \mathbb{R}$ , and  $s$  parametrizes the additional labels carried by such a state.<sup>71</sup> The definition of the operator  $\text{sign}(\mathcal{D})$  is potentially troublesome because of zero-modes of  $\mathcal{D}$ , but the zero-modes are in the discrete spectrum and so are annihilated by  $P$ , and hence there is no problem in defining the product  $P \text{sign}(\mathcal{D})$ . The fact that is being generalized in the formula (12.7) is that if  $M$  is an operator of finite rank or more generally of “trace class” (represented by an  $x$ -space kernel that we also call  $M$ ), then  $\text{Tr } M = \int_{-\infty}^{\infty} dx \sum_s \langle x, s | M | x, s \rangle$ . The contribution of the continuous spectrum to the  $\eta$ -invariant is supposed to be, naively,  $\lim_{\varepsilon \rightarrow 0} \text{Tr } P \text{sign}(\mathcal{D}) \exp(-\varepsilon|\mathcal{D}|)$ , but this trace is only conditionally-convergent (the operator whose trace we are trying to take is not trace class) and is not well-defined. The formula (12.7) is a well-defined, regularized version of this trace. It is well-defined because the integrand  $\sum_s \langle x, s | P \text{sign}(\mathcal{D}) \exp(-\varepsilon|\mathcal{D}|) | x, s \rangle$  vanishes exponentially for  $x \rightarrow \pm\infty$ . This is so for the same reason that there is no infrared problem in the approach to defining  $f_0$  via point-splitting and normal-ordering: the fermion number density vanishes in the vacua at  $\pm\infty$ .

## 12.2 Properties Of The $\eta$ -Invariant

What can we say about the invariant  $\eta(\mathcal{D})$ ? There is no simple formula for  $\eta(\mathcal{D})$  for a particular  $ij$  soliton solution, but there is a useful general statement comparing the values of  $\eta(\mathcal{D})$  for different  $ij$  solitons. (Since we have not yet analyzed boundary contributions to the fermion number, the following analysis applies strictly for the case  $D = \mathbb{R}$ , but similar statements hold for other cases. See section 13.3.)

Suppose that  $p, p' \in \mathcal{S}_{ij}$  are two different  $ij$  soliton solutions, interpolating between the same vacua at both ends. They have two different Dirac operators  $\mathcal{D}^p$  and  $\mathcal{D}^{p'}$  and so two different  $\eta$ -invariants  $\eta(\mathcal{D}^p)$  and  $\eta(\mathcal{D}^{p'})$  and two different fermion numbers  $f^p$  and  $f^{p'}$ . These can differ, but their difference  $f^{p'} - f^p$  is always an integer. The proof can be expressed in either mathematical or physical language. To express the proof in physical language first, we observe that the bosonic fields of the LG model do not carry fermion number and hence  $\mathcal{F}$  is conserved in the propagation of the fermion fields in an arbitrary time-dependent background constructed from the bosons. So in particular (here we assume that  $X$  is simply-connected), we can construct a time-dependent background that interpolates from the soliton solution  $p$  in the far past to the soliton solution  $p'$  in the far future; moreover, we can do this with fields that are time-independent at spatial infinity. So time evolution gives an  $\mathcal{F}$ -conserving mapping from the fermion Hilbert space  $\mathcal{H}^p$  constructed in

---

<sup>71</sup>The operator  $\mathcal{D}$  acts on a fermi field valued in the tensor product of a two-dimensional Clifford module with the pullback of the complex tangent bundle of the target space  $X$ , which has rank  $d = \dim_{\mathbb{C}} X$ . So  $s$  runs over an orthonormal basis of a  $2d$ -dimensional vector space.

the past by expanding around soliton  $p$  to the corresponding Hilbert space  $\mathcal{H}^p$  constructed in the future by expanding around  $p'$ . All states in  $\mathcal{H}^p$  have  $\mathcal{F} = f^p \bmod \mathbb{Z}$ , since the modes of the fermion field, which act irreducibly in  $\mathcal{H}^p$ , carry  $\mathcal{F} = \pm 1$ . Likewise all states in  $\mathcal{H}^{p'}$  have  $\mathcal{F} = f^{p'} \bmod \mathbb{Z}$ . So the existence of an  $\mathcal{F}$ -conserving map between these two Hilbert spaces implies that  $f^p = f^{p'} \bmod \mathbb{Z}$ .

For a more mathematical version of this argument, observe first that in general, changing only finitely many eigenvalues of an operator  $\mathcal{D}$  does not change  $\eta(\mathcal{D}) \bmod 2$ , since for  $\varepsilon \rightarrow 0$ , each eigenvalue contributes  $\pm 1$  to  $\eta(\mathcal{D})$ . In varying finitely many eigenvalues,  $\eta(\mathcal{D})$  only changes when an eigenvalue changes sign, in which case it jumps by  $\pm 2$ . The same is true if one varies infinitely many eigenvalues provided that the change in the  $n^{\text{th}}$  eigenvalue vanishes rapidly enough for  $n \rightarrow \infty$ . We are in this situation if we change  $\mathcal{D}$  by varying the  $x$ -dependent matrix  $\partial^2 W / \partial \phi^I \partial \phi^J$  in an arbitrary fashion (replacing it with an arbitrary  $m_{IJ}(x)$ , not necessarily derived from an LG field), keeping fixed its behavior for  $x \rightarrow \pm\infty$ . Since we do not change  $\mathcal{D}$  at spatial infinity, we do not change the energies of states at large  $|x|$ ; since we do not change the term  $\sigma^3 \text{id}/dx$  in  $\mathcal{D}$  that dominates at high energies, we do not change the eigenvalues of high energy. So in such a variation, only a finite number of eigenvalues change substantially, and  $\eta(\mathcal{D})$  only changes when an eigenvalue passes through 0. When that happens,  $\eta(\mathcal{D})$  jumps by  $\pm 2$ , so that  $f = -(1/2)\eta(\mathcal{D})$  is constant mod 1. Note that in this argument (as opposed to the physical argument), we do not need to assume that  $X$  is simply-connected; we can interpolate between the Dirac operators  $\mathcal{D}^p$  and  $\mathcal{D}^{p'}$  whether or not we can interpolate between the solitons  $p$  and  $p'$ .

Clearly, there is a topological invariant, namely the value of  $-\frac{1}{2}\eta(\mathcal{D}) \bmod \mathbb{Z}$ , which only depends on the matrices  $m_{IJ} = \partial^2 W / \partial \phi^I \partial \phi^J$  at  $x = \pm\infty$ . How can one compute this topological invariant? One way to compute the value of  $-\frac{1}{2}\eta(\mathcal{D}) \bmod \mathbb{Z}$  is to deform to the case that  $m_{IJ}(x)$  varies adiabatically as a function of  $x$ , between its given limiting values at  $x = \pm\infty$ . The adiabatic condition is that if  $|m|$  is the smallest eigenvalue of  $m_{IJ}$ , then  $|m|^{-2} dm_{IJ}/dx$  is everywhere small. Under this condition,  $\eta(\mathcal{D})$  can be computed as the integral over  $x$  of a universal local expression constructed from  $m$  and its first derivative.

A straightforward computation using perturbation theory then yields the formula

$$f = \frac{1}{2\pi} \left( \arg \det \left. \frac{\partial^2 W}{\partial \phi^I \partial \phi^J} \right|_{\phi_j} - \arg \det \left. \frac{\partial^2 W}{\partial \phi^I \partial \phi^J} \right|_{\phi_i} \right) \bmod \mathbb{Z}. \quad (12.8)$$

This agrees with the formulae stated in [23, 14, 15]. Incidentally, in the case of solitons on the real line, one might wonder if one can write an exact formula for  $f$ , not just a mod  $\mathbb{Z}$  formula, as the integrated “winding number” of the matrix of second derivatives of  $W$ :

$$f \stackrel{?}{=} \frac{1}{2\pi} \int_{-\infty}^{\infty} dx \frac{d}{dx} \arg \det \left. \frac{\partial^2 W}{\partial \phi^I \partial \phi^J} \right|_{\phi_i}. \quad (12.9)$$

This formula actually does not make sense, since in general  $\det \partial^2 W / \partial \phi^I \partial \phi^J$  can have zeroes; moreover, as one varies the Kähler metric of  $X$ , a  $\zeta$ -instanton trajectory can cross such a zero, whereupon the right hand side of (12.9) would jump, contradicting fermion number conservation if eqn. (12.9) were valid.

The continuous variation between the soliton solutions  $p$  and  $p'$  can also be used, in principle, to compute the integer  $f^{p'} - f^p$ . This integer is the “spectral flow,” the net number of eigenvalues of  $\mathcal{D}$  that pass through 0 in the downward direction, in interpolating from  $p$  to  $p'$ . This spectral flow is a regularized version of the difference in the Morse index of the soliton  $p'$  and the soliton  $p$ .

A standard argument in index theory says that the spectral flow of a Dirac operator in  $d$  dimensions gives the index of a Dirac operator in  $d+1$  dimensions. We actually explained this argument for  $d = 0$  in section 10.2, and this particular argument is independent of  $d$ . The argument uses the fact that the  $d+1$ -dimensional Dirac equation can be written  $L\psi = 0$ , where  $L = \partial_\tau + \mathcal{D}$ ; here  $\mathcal{D}$  is a  $d$ -dimensional self-adjoint Dirac operator (for  $d = 0$ , as in section 10.2,  $\mathcal{D}$  is simply a finite rank matrix). The index is the number of normalizable solutions of the equation  $L\psi = 0$  minus the number of normalizable solutions of the adjoint equation  $L^\dagger\bar{\psi} = 0$ , where  $L^\dagger = -\partial_\tau + \mathcal{D}$ . By reducing to the case that the eigenvalues of  $\mathcal{D}$  vary adiabatically with  $\tau$  and performing the analysis of eqn. (10.17) for each eigenvalue, one finds that the index of  $L$  is the spectral flow of  $\mathcal{D}$  (defined as the net number of eigenvalues of  $\mathcal{D}$  that pass from positive to negative between  $\tau = -\infty$  and  $\tau = +\infty$ ). In the context of Morse theory, the  $d+1$ -dimensional Dirac operator  $L$  is the linearization of the gradient flow equation. In two-dimensional LG theory, the gradient flow equation is the  $\zeta$ -instanton equation (eqn. (11.18)) and the analog of  $L$  is the linearization of this equation. As is the case in supersymmetric quantum mechanics in general,  $L$  is the kinetic operator for the fermions of  $\mathcal{F} = 1$  (and its adjoint is the corresponding operator for the fermions of  $\mathcal{F} = -1$ ). In a process involving an instanton transition from the soliton  $p$  in its ground state of lower fermion number to a soliton  $p'$  in the analogous state, the fermion number  $\mathcal{F}$  changes by  $f^{p'} - f^p$ ; everything is consistent because this number is the index of  $L$ , and therefore equals the fermion number of the operator insertions that must be made to get a nonzero amplitude for this transition.

### 12.3 Quantum BPS States

The basic framework to study BPS states in the full quantum theory is the same as in section 10. We construct a complex which additively is given by the semiclassical spectrum of BPS solitons, and on this complex, we use instantons to define a differential  $\hat{Q}_\zeta$  (a normalized version of  $Q_\zeta$  with the values of  $h$  for the classical soliton solutions removed). The cohomology of this differential gives the exact quantum spectrum of BPS solitons.

So additively, the complex describing  $ij$  solitons is

$$\mathbb{M}_{ij} = \bigoplus_{p \in \mathcal{S}_{ij}} \left( \mathbb{Z}\Psi_{ij}^f(p) \oplus \mathbb{Z}\Psi_{ij}^{f+1}(p) \right) \quad (12.10)$$

where  $\mathcal{S}_{ij}$  is the set of intersection points of a left thimble of type  $i$  with a right thimble of type  $j$ . The grading of the complex is given by the fermion number  $\mathcal{F}$ . This grading is not really a  $\mathbb{Z}$ -grading, since the values of  $\mathcal{F}$  differ from integers as explained in (12.8), but it is shifted from a  $\mathbb{Z}$ -grading by an overall constant that depends only on  $i$  and  $j$ .

Just as in section 10, the differential on the complex (12.10) arises from counting instantons interpolating between states whose fermion number differs by +1. In other words

we consider solutions of the  $\zeta$ -instanton equation with  $\zeta = \zeta_{ji}$  and boundary conditions

$$\lim_{x \rightarrow -\infty} \phi(x, \tau) = \phi_i \quad \text{Im } \lim_{x \rightarrow +\infty} \phi(x, \tau) = \phi_j \quad (12.11)$$

together with

$$\lim_{\tau \rightarrow -\infty} \phi(x, \tau) = \phi_{ij}^{p_1}(x) \quad \lim_{\tau \rightarrow +\infty} \phi(x, \tau) = \phi_{ij}^{p_2}(x) \quad (12.12)$$

where  $\phi_{ij}^{p_1}(x)$  and  $\phi_{ij}^{p_2}(x)$  are two  $ij$   $\zeta$ -solitons. In section 12.4, we give an example showing that in general there are  $\zeta$ -instantons obeying these boundary conditions and contributing to the differential  $\hat{\mathcal{Q}}_\zeta$ . So not all BPS solitons give rise to true BPS states.

In the general quantum mechanical analysis of section 10, there were only two supersymmetries, one of which was a symmetry of the instanton. The instanton therefore had just 1 fermion zero-mode, and this mode was responsible for the fact that the instanton amplitude increases  $\mathcal{F}$  by 1. In the present situation, the underlying two-dimensional model altogether has four supersymmetries, only two of which (the ones in the small subalgebra) are symmetries of the initial and final states. As a result, the initial and final states both represent a rank 2 Clifford algebra of broken supersymmetries. This is responsible for the doubling of the spectrum: a classical soliton corresponds to two quantum states of fermion numbers  $f, f+1$ . The instanton still preserves only one supersymmetry, so now there are three fermion zero-modes, of which one is normalizable. The two zero-modes that are generated by supersymmetries that are not in the small subalgebra are localized in space but not in time, so they are not normalizable (and do not contribute to the index of the operator  $L$ ). They go over in the far future or past to the fermion zero-modes of the individual solitons. The fact that the zero-modes of the individual solitons can be extended to time-dependent zero-modes in the instanton background means that the instanton amplitudes commute with the action of the Clifford algebra on the initial and final states. The third fermion zero-mode in the field of the instanton is normalizable and is the analog of the single zero-mode of the quantum mechanical analysis. It is localized in space and time and ensures that  $\zeta$ -instantons contribute to the matrix element of the differential  $\hat{\mathcal{Q}}_\zeta$  only in the case  $f^{p_2} - f^{p_1} = 1$ .

The fact that the instanton amplitude commutes with the Clifford algebra means that we can write the complex a little more economically:

$$\mathbb{M}_{ij} = \mathbb{W} \otimes \mathbb{M}'_{ij}. \quad (12.13)$$

Here  $\mathbb{W} \cong \mathbb{Z} \oplus \mathbb{Z}$  is an irreducible module for the Clifford algebra generated by two basis vectors  $|-\frac{1}{2}\rangle, |+\frac{1}{2}\rangle$  with  $\mathcal{F} = -\frac{1}{2}, +\frac{1}{2}$ , respectively, and  $\mathbb{M}'_{ij}$  is a reduced complex. Indeed, we can write  $\Psi_{ij}^f(p) = |-\frac{1}{2}\rangle \otimes m_{ij}^{f_0}(p)$  and  $\Psi_{ij}^{f+1}(p) = |+\frac{1}{2}\rangle \otimes m_{ij}^{f_0}(p)$  where  $f_0 = f + \frac{1}{2}$ . Thus,

$$\mathbb{M}'_{ij} = \bigoplus_{p \in \mathcal{S}_{ij}} \mathbb{Z} m_{ij}^{f_0}(p). \quad (12.14)$$

In Section §14.5 below, we show that in matching to the web-based formalism we should take what there was called  $R_{ij}$  to be the complex generated by the states of upper fermion number:

$$R_{ij} = \bigoplus_{p \in \mathcal{S}_{ij}} \mathbb{Z} \Psi_{ij}^{f+1}(p) \quad (12.15)$$

The instanton-generated differential  $\hat{\mathcal{Q}}_\zeta$  acts on the reduced complexes  $\mathbb{M}'_{ij}$  and  $R_{ij}$ .

Once we get rid of the doubling of the spectrum in this way, the analogy with the general quantum mechanical analysis of section 10 is much closer. As in that analysis, each instanton solution that interpolates from a soliton  $p_1$  in the past to a soliton  $p_2$  in the future – and has no moduli except the minimum possible number – contributes  $\pm 1$  to the relevant matrix element of the normalized differential  $\hat{\mathcal{Q}}_\zeta$  acting on the reduced complex. The sign of this contribution is given by the sign of the fermion determinant. The problem of defining this sign – and how it should be interpreted<sup>72</sup> – is very similar in the present context to what it was in the general quantum mechanical discussion. If  $\pi_2(X)$  is trivial then any two fields obeying the boundary conditions (12.11), (12.12) are homotopic, so the fermion determinant is uniquely determined up to an overall sign. Moreover, cluster decomposition can be used to determine all the signs except for signs that can be absorbed in the definitions of the initial and final soliton states. More generally, if  $\pi_2(X) \neq 0$ , then the general analysis of the signs of the fermion determinants leads to the possibility of discrete theta-angles, similarly to what happens in section 10 when  $\pi_1(M) \neq 0$ .

Now, let us consider an  $ij$  soliton at rest with vacuum  $i$  on the left and vacuum  $j$  on the right. If we turn the picture upside-down, rotating by an angle  $\pi$  in Euclidean signature, an  $ij$  soliton becomes a  $ji$  soliton. The  $\pi$  rotation, which is the Euclidean version of a CPT transformation, also reverses the sign of the fermion number current. So if a multiplet of  $ij$  solitons have fermion numbers  $\mathcal{F} = f, f+1$ , then the rotation gives a pair of  $ji$  solitons of fermion numbers  $\mathcal{F} = -f, -f-1$ . A static picture of an  $ij$  soliton sitting at rest can be viewed as a pairing between an  $ij$  soliton coming in from  $\tau = -\infty$  and a  $ji$  soliton coming in from  $\tau = +\infty$ . The path integral gives a non-degenerate and  $\mathcal{F}$ -conserving pairing

$$\mathbb{M}_{ij} \otimes \mathbb{M}_{ji} \rightarrow \mathbb{Z}. \quad (12.16)$$

Since the full complexes  $\mathbb{M}_{ij}$  admit this  $\mathcal{F}$ -conserving pairing, the state  $\Psi_{ij}^{f+1}$  must pair with the state  $\Psi_{ji}^{-f-1}$  and the state  $\Psi_{ij}^f$  must pair with the state  $\Psi_{ji}^{-f}$ . Therefore, the pairing on  $\mathbb{W}$  is off-diagonal, pairing  $|-\frac{1}{2}\rangle$  with  $|+\frac{1}{2}\rangle$  to give (without loss of generality)  $+1$ . Since the pairing on the Clifford algebra has fermion number 0 and the total pairing has fermion number 0, the induced pairing on the reduced states  $m_{ij}$  and  $m_{ji}$  has fermion number 0. That is, the induced pairing on the states of lower fermion number in each doublet,

$$K' : \mathbb{M}'_{ij} \otimes \mathbb{M}'_{ji} \rightarrow \mathbb{Z}, \quad (12.17)$$

pairs  $m_{ij}^{f_0}(p)$  with  $m_{ji}^{-f_0}(p)$  and hence  $K'$  has fermion number 0.

When we explain the relation of the Landau-Ginzburg theory to the formalism of Section §4 in Section §14.5 below it will turn out that we should not identify the reduced complex of solitons  $\mathbb{M}'_{ij}$  with what in the web treatment was called  $R_{ij}$ . Rather,  $R_{ij}$  will

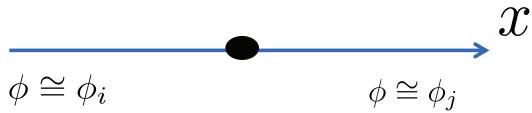
---

<sup>72</sup>The most powerful mathematical framework is provided by the theory of real determinant line bundles, as we discuss in the quantum mechanical case in Appendix F. Here we content ourselves with the following simple remarks showing that as in the general quantum mechanical case, the theory is uniquely determined – up to physically-understood choices – if it works.

actually be identified with the complex of soliton states of “upper” fermion number as in equation (12.15). The pairing  $K'$  induces a similar pairing

$$K : R_{ij} \otimes R_{ji} \rightarrow \mathbb{Z}. \quad (12.18)$$

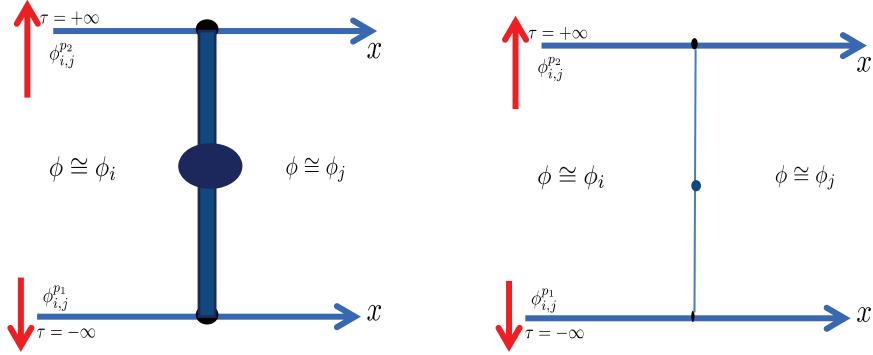
To define  $K$  we simply drop the factor  $| + \frac{1}{2} \rangle$  in both  $\Psi_{ij}^{f+1}(p) = | + \frac{1}{2} \rangle \otimes m_{ij}^{f_0}(p)$  and  $\Psi_{ji}^{-f}(p) = | + \frac{1}{2} \rangle \otimes m_{ji}^{-f_0}(p)$ , and use the pairing  $K'$  on  $m_{ij}^{f_0}(p)$  and  $m_{ji}^{-f_0}(p)$ . Thus,  $K$  is a nondegenerate pairing of fermion number  $-1 + 0 = -1$ , in harmony with the definition of a web representation in Section §4. It is symmetric by CPT invariance.



**Figure 130:** In a massive theory, when viewed from long distance, or in the limit that the mass goes to infinity, the soliton solution is well approximated by a discontinuous solution, discontinuous at some point  $x = x_0$ .

### Remarks

1. A soliton solution  $\phi_{ij}(x)$  is very near  $\phi_i$  or  $\phi_j$  for “most” of the values of  $x$  and only shoots from vacuum  $i$  to  $j$  in a very short interval  $\Delta x$  set by the inverse mass scale  $\ell_W$  of the theory. Thus, “in the infrared limit” where we take the mass scale large the solution can be thought of as a discontinuous function with vacuum  $i$  at  $x < x_0$  and  $j$  at  $x > x_0$ . See Figure 130. Similarly, the instantons can be viewed as stationary soliton worldlines with a small dot inserted as in Figure 131. This is the beginning of the connection to the (extended) webs of previous sections.
2. We will show below that the counting of solitons and instantons leads to web representations. So, starting with the data defining a LG field theory, we can deduce a mathematical structure that is defined over the integers, and this is how it was presented in the first half of the paper. On the other hand, the field theory is defined in terms of complex amplitudes and vector spaces, and hence does not give an entirely natural explanation of why the mathematical structures, such as equation 12.16, are in fact defined over  $\mathbb{Z}$ . (It is natural for topological field theory path integrals to have integral values, but we are not discussing topological field theory here.)



**Figure 131:** Left: An instanton configuration contributing to the differential on the MSW complex. The black regions indicate the locus where the field  $\phi(x, \tau)$  varies significantly from the vacuum configurations  $\phi_i$  or  $\phi_j$ . The length scale here is  $\ell_W$ , set by the superpotential. Right: Viewed from a large distance compared to the length scale  $\ell_W$  the instanton looks like a straight line  $x = x_0$ , where the vacuum changes discontinuously from vacuum  $\phi_i$  to  $\phi_j$ . The nontrivial  $\tau$ -dependence of the instanton configuration, interpolating from a soliton  $p_1$  to another soliton  $p_2$  has been contracted to a single vertex located at  $\tau = \tau_0$ . This illustrates the origin of the 2-valent vertices of extended webs in the context of LG theory.

3. We can now introduce the Witten index, which in this context is known as the BPS index  $\mu_{ij}$ . This is just the Euler character of the complex  $\mathbb{M}_{ij}$  of (12.10) appropriately interpreted to take into account the fact that we are working with a slightly degenerate Morse function. We should compute [14]

$$\mu_{ij} := \text{Tr}_{\mathbb{M}_{ij}} F e^{i\pi F} = - \sum_{p \in L_i^\zeta \cap R_j^\zeta \cap X_{W_0}} e^{i\pi f(p)} \quad (12.19)$$

where  $X_{W_0}$  is the preimage under  $W$  of a regular value  $W_0$  of the superpotential. Here  $\zeta = \zeta_{ji}$  and  $W_0$  lies on the interior of the line segment between the critical values  $W_i$  and  $W_j$ . As we have shown in equation (12.8) the fermion number of a classical soliton  $\phi_{ij}^p$  has the form  $f(p) = f_j - f_i + n_{ij}(p)$  where  $n_{ij}(p)$  is an integer. According to [14, 15, 48] the integer  $n_{ij}(p)$ , reduced mod 2, is the contribution of  $p$

to the oriented intersection number of the Lefshetz thimbles, and hence

$$\mu_{ij} = e^{i\pi(f_j - f_i + 1)} \# L_i^\zeta \cap R_j^\zeta = e^{i\pi(f_j - f_i + 1)} \sum_{p \in L_i^\zeta \cap R_j^\zeta \cap X_{W_0}} (-1)^{\iota(p)} \quad (12.20)$$

where  $\iota(p)$  is the oriented intersection number.

## 12.4 Non-Triviality Of The Differential

Some of the literature on BPS states in two-dimensional LG models studies special cases which might give one the impression that BPS solitons always lead to BPS states. In this section we show that, in general, the differential  $\mathcal{Q}_\zeta$  acting on the space of classical  $\zeta$ -solitons is non-trivial.

The strategy will be to adapt a simple fact in ordinary Morse theory. We consider a family of Morse functions in one variable  $u$  that near  $u = 0$  look like

$$h_\varepsilon(u) = \frac{u^3}{3} - \varepsilon u, \quad (12.21)$$

where  $\varepsilon$  is a real parameter. The equation for a critical point is  $u^2 = \varepsilon$ . It has no real root for  $\varepsilon < 0$  but has a pair of real roots  $u_\pm = \pm\sqrt{\varepsilon}$  for  $\varepsilon > 0$ . For  $\varepsilon < 0$ , there are no classical vacuum states near  $u = 0$ , but for  $\varepsilon > 0$ , there are two of them. However, extra quantum vacua cannot appear as  $\varepsilon$  is varied, so there must be an instanton effect that lifts the two approximately supersymmetric states that appear for  $\varepsilon > 0$ . Indeed, the portion of the  $u$ -axis between  $u_+$  and  $u_-$  is a gradient flow line that connects the two critical points, and the contribution of this gradient flow line to the differential removes from the cohomology the states supported at  $u_+$  and  $u_-$ .

Notice that at  $\varepsilon = 0$ , where the two critical points appear or disappear,  $h_\varepsilon$  is not a Morse function, since its unique critical point at  $u = 0$  is degenerate – the second derivative of  $h_0(u)$  vanishes at  $u = 0$ . Consider in any number of variables a critical point that is degenerate in this way – it is cubic in one variable  $u$  and quadratic in any number of additional variables:

$$h(u, v_1, \dots, v_s, w_1, \dots, w_t) = \frac{u^3}{3} + \sum_{i=1}^s v_i^2 - \sum_{j=1}^t w_j^2. \quad (12.22)$$

Under a generic perturbation, which will include a term linear in  $u$ , such a critical point will either disappear or split into a pair of nondegenerate critical points, depending on the sign of the perturbation. If the sign the perturbation is such as to generate two new critical points, there can be no exact quantum supersymmetric state associated to them, since with the opposite perturbation, these critical points would be absent even classically. So there will always be a gradient flow line removing this pair of approximate ground states from the supersymmetric spectrum.

The  $\zeta$ -soliton equation on the real line is a problem of roughly this nature, with the role of the Morse function played by the functional

$$h := -\frac{1}{2} \int_{-\infty}^{\infty} dx \operatorname{Re} \left( \frac{i}{2} g_{I\bar{J}} \sum_I \phi^I \frac{\partial}{\partial x} \bar{\phi}^{\bar{J}} - \zeta^{-1} W \right). \quad (12.23)$$

in the case that  $g_{I\bar{J}}$  is constant.  $h$  is a function of infinitely many variables, and also we should factor out by translations of  $x$  to think of  $h$  precisely as a Morse function. We will find a situation in which, near a certain critical point (and omitting the mode corresponding to spatial translations),  $h$  will look like (12.22), but with infinitely many  $v$ 's and  $w$ 's. By varying one parameter that will correspond to  $\varepsilon$ , we will be able to make a pair of  $\zeta$ -solitons appear or disappear. When these two classical solitons appear, for the same reasons explained above, there will have to be a  $\zeta$ -instanton interpolating between them and ensuring that the corresponding quantum states are not exactly supersymmetric.

The parameter playing the role of  $\varepsilon$  can actually be either a parameter appearing in the Kähler metric of the target space  $X$  or a parameter in the superpotential. For  $X = \mathbb{C}$ , varying the Kähler metric of  $X$  will never cause a  $\zeta$ -soliton to appear or disappear,<sup>73</sup> but it is not difficult to give an example for  $X = \mathbb{C}^2$ . We parametrize  $\mathbb{C}^2$  with complex variables  $Y, Z$ . To begin with, we take the Kähler metric to be

$$d\ell^2 = |dY|^2 + |dZ|^2. \quad (12.24)$$

Later, we will make a perturbation of this Kähler metric.

We start with a 1-variable superpotential  $W_0(Y)$ , and for brevity we choose  $\zeta^{-1} = i$ , so that the  $\zeta$ -soliton equation is ascending gradient flow for  $\text{Re } W_0$  and a solution will have a fixed value of  $\text{Im } W_0$ . We pick  $W_0$  so that a non-trivial  $ij$   $\zeta$ -soliton  $Y_0(x)$  does exist, for some  $i, j \in \mathbb{V}$ . After adding a constant to  $W_0$ , we can assume that in this  $\zeta$ -soliton,  $W_0(Y_0(x))$  is everywhere real and moreover that  $W_0(Y_0(x))$  is negative for  $x \rightarrow -\infty$  and positive for  $x \rightarrow +\infty$ .

In complex dimension 1, the operator  $L$  obtained by linearizing around a  $\zeta$ -soliton has no zero-mode except the one associated to spatial translations. (This is related to the remark in footnote 73.) To mimic the situation described in eqn. (12.22), we need a  $\zeta$ -soliton that is “degenerate,” meaning that the linearized operator  $L$  has a zero-mode not associated to translation symmetry. To achieve this, we include a second superfield  $Z$  in the discussion, generalizing the superpotential to

$$W_1(Y, Z) = W_0(Y)(1 + Z^2). \quad (12.25)$$

We have picked  $W_1$  so that  $\partial_Z W_1 = 0$  at  $Z = 0$ , ensuring that (with the Kähler metric (12.24)) the  $\zeta$ -soliton equation has the solution  $Y = Y_0(x)$ ,  $Z = 0$ . Now let us expand around this solution. The choice of  $W_1$  ensures that the fluctuations in  $Y$  and  $Z$  obey separate equations.  $Y$  has the one zero-mode associated to spatial translations, and the linearized equation for  $Z$  is

$$\frac{dZ}{dx} = W_0(Y_0(x))\bar{Z}. \quad (12.26)$$

---

<sup>73</sup> For  $X = \mathbb{C}$  and  $i, j \in \mathbb{V}$ ,  $ij$   $\zeta$ -solitons are in 1-1 correspondence with paths in  $X$  from  $i$  to  $j$  with  $\text{Re}(\zeta^{-1}W)$  constant. Such a path, if properly parametrized, becomes a  $\zeta$ -soliton. The Kähler metric of  $X$  enters only in determining the proper parametrization.

Setting  $Z = z_1 + iz_2$  with  $z_1, z_2$  real and recalling that  $W_0(Y_0(x))$  is real, we get the solutions

$$\begin{aligned} z_1(x) &= \exp\left(\int_0^x dx' W_0(Y_0(x'))\right) \\ z_2(x) &= \exp\left(-\int_0^x dx' W_0(Y_0(x'))\right). \end{aligned} \quad (12.27)$$

Here, given our assumptions about  $W_0(Y_0(x))$ ,  $z_1(x)$  blows up exponentially at  $x = \pm\infty$ , but  $z_2(x)$  vanishes exponentially and thus we have found a  $\zeta$ -soliton with a zero-mode that is not associated to translation symmetry.

Now introduce a small parameter  $u$  and consider a family of approximate  $\zeta$ -solitons with

$$(Y, Z) = (Y_0(x), uz_2(x)) + \mathcal{O}(u^2). \quad (12.28)$$

This obeys the  $\zeta$ -soliton equation to order  $u$ , but can we choose the  $\mathcal{O}(u^2)$  terms to obey the equation in order  $u^2$ ? The case of interest is that this is not possible; that will be so if and only if (after integrating out all the non-zero modes), the function  $h$  of eqn. (12.23), when expanded in powers of  $u$ , has a term of order  $u^3$ . (The fact that  $(Y, Z)$  defined in eqn. (12.28) is a  $\zeta$ -soliton up to order  $u^2$  means that  $h$  has no term linear or quadratic in  $u$ .) The model as we have defined it so far has a symmetry  $Z \rightarrow -Z$  which ensures that  $h$  is an even function of  $u$  and hence that there is no  $u^3$  term. However, we can easily modify the model (without affecting the fact that  $h$  has no term linear or quadratic in  $u$ ) so that  $h$  acquires a  $u^3$  term. We simply add to the superpotential a term cubic in  $Z$ . The superpotential thus becomes

$$W_1(Y, Z) = W_0(Y)(1 + Z^2) + A(Y)Z^3, \quad (12.29)$$

for a generic function  $A(Y)$ .

At this point, the solution  $(Y, Z) = (Y_0, 0)$  of the  $\zeta$ -soliton equation is a degenerate critical point of the functional  $h$  that can be modeled as in (12.22) (with infinitely many  $v$ 's and  $w$ 's and an additional zero-mode - due to translation invariance - that does not have a simple finite-dimensional analog). To complete the story, we want to show that a perturbation of the Kähler metric or a further perturbation of the superpotential can add to  $h$  a term linear in  $u$ . By varying the sign of this term, we can then make a pair of classical  $\zeta$ -solitons appear and disappear. For the sign of the perturbation for which the pair of classical solutions exists, there will have to be a  $\zeta$ -instanton interpolating between the two  $\zeta$ -solitons and removing the corresponding quantum states from the supersymmetric spectrum.

A further perturbation of the superpotential that will have the desired effect is simply a term  $\varepsilon B(Y)Z$ , with a generic function  $B(Y)$  and a sufficiently small  $\varepsilon$ . Thus the full superpotential becomes

$$W = W_0(Y)(1 + Z^2) + A(Y)Z^3 + \varepsilon B(Y)Z. \quad (12.30)$$

Alternatively, we can induce the desired linear term in  $h$  by a small perturbation of the Kähler metric (12.24) that breaks the symmetry  $Z \rightarrow -Z$  of this metric. We can choose a

simple perturbation that preserves the flatness of the metric:

$$d\ell^2 \rightarrow d\ell^2 + \varepsilon (dYd\bar{Z} + dZd\bar{Y}). \quad (12.31)$$

For a generic superpotential  $W_1(Y, Z)$  as constructed above, this perturbation adds to  $h$  a term  $\varepsilon u$  and ensures that all of the desired conditions are satisfied.

### 13. MSW Complex On The Half-Line And The Interval

Now we will consider the analogous questions when the LG model is formulated on a two-manifold that is either a half-plane or a strip, in other words it is  $\mathbb{R} \times D$  where  $\mathbb{R}$  parametrizes time and  $D$  is a half-line  $[x_\ell, \infty)$  or  $(-\infty, x_r]$  or a compact interval  $[x_\ell, x_r]$ . An important difference from the discussion of BPS solitons on the real line is that in the discussion of  $ij$  solitons, we set  $\zeta$  to the unique value  $\zeta_{ji}$  at which such solitons may exist. By contrast, when we quantize the LG model on a strip or a half-plane, we pick a generic value of  $\zeta$  at which there are no  $ij$  solitons for any  $i, j$ .<sup>74</sup> Wall-crossing phenomena can occur when  $\zeta$  crosses the special values at which  $ij$  solitons do exist, and it is simplest to work away from these walls.

In quantization on either a half-line or an interval, we need a boundary condition at each finite end of  $D$ . Any such boundary condition will explicitly break the  $\mathcal{N} = (2, 2)$  supersymmetry of the bulk LG model to a subalgebra with at most two supercharges, and we want this to be the small supersymmetry algebra generated by  $\mathcal{Q}_\zeta$  and its adjoint. Since the spatial translations and supersymmetries that are not in the small subalgebra are explicitly broken by the boundary conditions, they do not lead to any zero-modes. This makes the analogy with the generic quantum mechanical analysis of section 10 much more straightforward. Generically, for each choice of boundary condition, the  $\zeta$ -soliton equation has only a finite set of solutions, none of which admit any bosonic or fermionic moduli. Each such solution corresponds to a single approximately supersymmetric state of some fermion number  $f$  (whose definition is analyzed in section 13.3). As usual, we make a complex with a basis corresponding to the solutions of the  $\zeta$ -soliton equation, and on this complex we define a differential  $\hat{\mathcal{Q}}_\zeta$  whose matrix elements are found by counting (with signs that come from the sign of the fermion determinant) the  $\zeta$ -instantons that interpolate between two given  $\zeta$ -solitons.

We now make a few more detailed remarks about the cases of a half-line or a strip.

#### 13.1 The Half-Line

If we do not impose a boundary condition at  $x = 0$ , then the space of  $\zeta$ -solitons on the half-line  $[0, \infty)$  that approach  $\phi_j$  at infinity is the exact Lagrangian submanifold  $R_j^\zeta$  that was described at the end of section 11.2.  $R_j^\zeta$  is naturally embedded in  $X$  by identifying a  $\zeta$ -soliton with the value  $\phi^I(0)$ . To quantize the LG model on  $[0, \infty)$ , we require a boundary condition at  $x = 0$  stating that  $\phi^I(0)$  must lie in a specified Lagrangian submanifold  $\mathcal{L}$ .

---

<sup>74</sup>Using equation (11.15) and Remark 9 of Section §2.1 this statement is equivalent to the criterion, used in the abstract part of the paper, that no difference of vacuum weights  $z_{ij}$  is parallel to the boundary  $\partial\mathcal{H}$ .

$\zeta$ -solitons on  $[0, \infty)$  that obey this condition at  $x = 0$  and also approach  $\phi_j$  at  $x = \infty$  are simply in one-one correspondence with the intersection points  $\mathcal{L} \cap R_j^\zeta$ .

As usual, to find the quantum BPS states, we construct a complex  $\mathbb{M}_{\mathcal{L},j}$  that additively has a basis corresponding to the intersection points just mentioned, with a differential  $\widehat{\mathcal{Q}}_\zeta$  that can be found by counting  $\zeta$ -instantons. The instantons are schematically illustrated in Figure 132.

To study BPS states on the half-line  $(-\infty, 0]$ , again with boundary conditions set by  $\mathcal{L}$ , we proceed in just the same way. The  $\zeta$ -solitons that obey the boundary conditions are the intersections  $L_i^\zeta \cap \mathcal{L}$ , where as in section 11.2,  $L_i^\zeta$  is defined by ascending gradient flows on  $(-\infty, 0]$  that start at  $\phi_i$  at  $x = -\infty$ . This gives a basis for a complex  $\mathbb{M}_{i,\mathcal{L}}$ , whose normalized differential  $\widehat{\mathcal{Q}}_\zeta$  is found by counting  $\zeta$ -instanton solutions.

In general, a Lagrangian submanifold  $\mathcal{L}$  may intersect  $R_j^\zeta$  (or  $L_i^\zeta$ ) in many points. Moreover, if  $\mathcal{L}$  is varied by an exact symplectomorphism of  $X$  (and thus without changing it as an  $A$ -brane), the number of these intersection points may vary. Since the number of exactly supersymmetric states on the half-line does not change if  $\mathcal{L}$  is varied in this way, but the dimension of the complex  $\mathbb{M}_{\mathcal{L},j}$  can jump, the differential  $\widehat{\mathcal{Q}}_\zeta$  is certainly nonzero in general and  $\zeta$ -instantons are important. However, it is difficult to give explicit examples.

We will describe an important example in which it is straightforward to identify the intersections  $\mathcal{L} \cap R_j^\zeta$ . We simply take  $\mathcal{L}$  to be one of the Lagrangian submanifolds  $L_i^\zeta$ . So  $\mathcal{L} \cap R_j^\zeta = L_i^\zeta \cap R_j^\zeta$  consists of points in  $X$  that are boundary values of flows on  $(-\infty, 0]$  that start at  $\phi_i$  and also are boundary values of flows on  $[0, \infty)$  that end at  $\phi_j$ . Gluing these flows together, we get an  $ij$  soliton, but by our hypothesis there are no such solitons for  $i \neq j$ . So  $L_i^\zeta \cap R_j^\zeta$  is empty for  $i \neq j$ . On the other hand, the only solution of the  $\zeta$ -soliton equation interpolating from a vacuum  $i$  to itself is the trivial constant solution (in a non-trivial solution, the function  $\text{Im}(\zeta^{-1}W)$  is strictly increasing), so  $L_i^\zeta \cap R_i^\zeta$  consists of a single point, the critical point  $\phi_i$ . With only a single approximately supersymmetric state, there is no possibility of an instanton transition.

Of course, the analysis of  $L_i^\zeta \cap \mathcal{L}$  is the same if  $\mathcal{L} = R_j^\zeta$ . So

$$\mathbb{M}_{L_i^\zeta, j} = \mathbb{M}_{i, R_j^\zeta} = \begin{cases} 0 & \text{if } j \neq i \\ \mathbb{Z} & \text{if } j = i, \end{cases} \quad (13.1)$$

in each case with trivial differential.

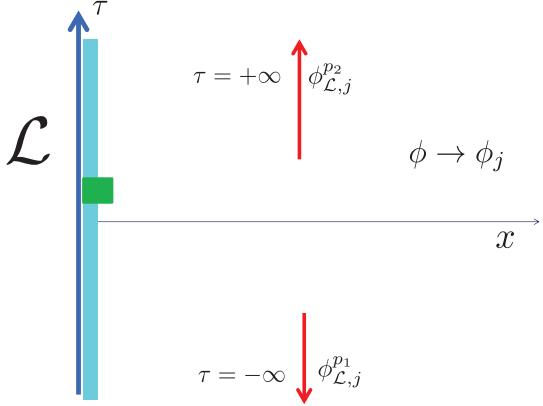
**Remark:** When comparing with the abstract discussion of Section §4.2 we should identify the Chan-Paton factors for a brane defined by a Lagrangian subvariety  $\mathcal{L}$  with

$$\mathcal{E}_j := \mathbb{M}_{\mathcal{L},j} \quad \& \quad \widetilde{\mathcal{E}}_i := \mathbb{M}_{i,\mathcal{L}} \quad (13.2)$$

for left- and right- boundaries, respectively.

### 13.2 The Strip

In quantization on the closed interval  $D = [x_\ell, x_r]$ , with boundary conditions set at the ends by two Lagrangian submanifolds  $\mathcal{L}_\ell, \mathcal{L}_r$ , a classical ground state is a solution of the  $\zeta$ -soliton equation with  $\phi(x_\ell) \in \mathcal{L}_\ell, \phi(x_r) \in \mathcal{L}_r$ . For the case that the strip is very long



**Figure 132:** An instanton in the complex  $\mathbb{M}_{\mathcal{L},j}$ . The solitons corresponding to  $p_1, p_2 \in \mathcal{L} \cap R_j^\zeta$  are exponentially close to the vacuum  $\phi_j$  except for a small region, shown in turquoise, of width  $\ell_W$ . In addition, the instanton transitions from one soliton to another in a time interval of length  $\ell_W$ , indicated by the green square. At large distances the green square becomes the 0-valent vertex used in extended half-plane webs.

compared to the longest Compton wavelength of any massive particle in the theory, we can give a simple counting of such solutions. At long distances from the boundaries, the theory will be close to one of the vacuum states, which correspond to the critical points  $\phi_i$ ,  $i \in \mathbb{V}$ . Given this, near the left boundary, the  $\zeta$ -soliton solution will approximate one of the half-line solutions that contribute basis vectors of  $\mathbb{M}_{\mathcal{L}_\ell, i}$ ; and near the right boundary, it will approximate one of the half-line solutions that contribute basis vectors of  $\mathbb{M}_{i, \mathcal{L}_r}$ . Conversely, if the strip is wide, then generalities of index theory and elliptic operators imply that every pair of left- and right- half-line solutions arises in this way from a solution on the strip.<sup>75</sup> So additively, the complex  $\mathbb{M}_{\mathcal{L}_\ell, \mathcal{L}_r}$  of the two-sided problem has a simple description:

$$\mathbb{M}_{\mathcal{L}_\ell, \mathcal{L}_r} \cong \bigoplus_{i \in \mathbb{V}} \mathbb{M}_{\mathcal{L}_\ell, i} \otimes \mathbb{M}_{i, \mathcal{L}_r} \quad (13.3)$$

At the end of section 13.3, we explain why the grading of  $\mathbb{M}_{\mathcal{L}_\ell, \mathcal{L}_r}$  by fermion number is related to those of  $\mathbb{M}_{\mathcal{L}_\ell, i}$  and  $\mathbb{M}_{i, \mathcal{L}_r}$  in the way suggested by this decomposition.

As usual, the normalized differential  $\widehat{\mathcal{Q}}_\zeta$  on this complex is computed by counting  $\zeta$ -instanton solutions. Some such instantons are localized at one end of the strip or the other. These instantons contribute the differentials of individual factors  $\mathbb{M}_{\mathcal{L}_\ell, i}$  and  $\mathbb{M}_{i, \mathcal{L}_r}$  in the sum over  $i \in \mathbb{V}$  in (13.3). If they were the only instantons on the strip, then the cohomology of the complex  $\mathbb{M}_{\mathcal{L}_\ell, \mathcal{L}_r}$  would have the same sort of decomposition as the complex itself:

$$H^*(\mathbb{M}_{\mathcal{L}_\ell, \mathcal{L}_r}) \stackrel{?}{\cong} \bigoplus_{i \in \mathbb{V}} H^*(\mathbb{M}_{\mathcal{L}_\ell, i}) \otimes H^*(\mathbb{M}_{i, \mathcal{L}_r}). \quad (13.4)$$

<sup>75</sup>If the strip is wide, then a simple gluing starting with a left- and a right- half-line solution comes exponentially close to an exact solution on the strip. Using the assumed nondegeneracy of the solutions and the fact that the relevant index vanishes because  $\mathcal{L}_\ell$  and  $\mathcal{L}_r$  are both middle-dimensional, this approximate solution can be corrected to an exact solution in a unique way.

As discussed qualitatively in the introductory section 1, in general the differential for the problem on the strip is much more complicated and this simple formula is not correct. The differential can receive corrections from instantons that cannot be localized at one end of the strip or the other. To a large extent, the goal of this paper is to understand these unlocalized contributions to the differential, which prevent an elementary description of the space of physical states on the strip. Starting in Section §14, we will seek to understand these contributions in the context of LG models. But first we pause for another interlude on the fermion number (Section §13.3).

Although there is no simple statement about the cohomology on the strip, there is a simple statement about the Euler characteristic of the cohomology. The Euler characteristic of the cohomology of a complex does not depend on the differential (the effect of the differential is eliminate pairs of states of opposite statistics that anyway make no net contribution to the Euler characteristic), so the supersymmetric index of this problem factorizes nicely in the fashion suggested by (13.3). If we define the index

$$\mu_{\mathcal{L}_\ell, i} = \text{Tr}_{\mathbb{M}_{\mathcal{L}, i}} e^{i\pi\mathcal{F}} \quad (13.5)$$

on the half-line, with a similar definition for  $\mu_{i, \mathcal{L}_r}$  and  $\mu_{\mathcal{L}_\ell, \mathcal{L}_r}$ , then, provided the fermion numbers are coherently related, it follows from equation (13.3) that these indices nicely factorize:

$$\mu_{\mathcal{L}_\ell, \mathcal{L}_r} = \sum_{i \in \mathbb{V}} \mu_{\mathcal{L}_\ell, i} \mu_{i, \mathcal{L}_r} \quad (13.6)$$

### 13.3 The Fermion Number Revisited

Much of our analysis of the fermion number and the  $\eta$ -invariant, starting in section 12.1, was equally valid whether the spatial manifold  $D$  on which we quantize is the real line, a half-line, or a compact interval. However, we will now describe some special features that occur when  $D$  has a boundary (as opposed to an infinite end). In doing so, it is convenient to assume that  $D$  has only boundaries and no infinite ends – the general case is a mixture of the two problems.

We begin by considering, in the absence of a superpotential, the standard  $A$ -model of maps  $\phi : \Sigma \rightarrow X$ , where  $\Sigma$  is a compact Riemann surface without boundary of Euler characteristic  $\chi(\Sigma)$  and  $X$  is a compact Kähler manifold. In general, fermion number is not conserved in this model. The net violation of fermion number is given by the index of the two-dimensional Dirac operator coupled to  $\phi^*(T^{1,0}X)$ . After topological twisting<sup>76</sup> the resulting Dirac operator  $L$  has index:

$$\iota(L) = \chi(\Sigma) \dim_{\mathbb{C}} X - 2 \int_{\Sigma} c_1(\phi^*(K_X)), \quad (13.7)$$

where  $K_X$  is the canonical bundle of  $X$  – the bundle of  $(n, 0)$  forms on  $X$ , where  $n = \dim_{\mathbb{C}} X$ . (In the untwisted  $\sigma$ -model, one would drop the first term on the right-hand side of (13.7).) The index (13.7) is the expected real dimension of the moduli space of holomorphic

---

<sup>76</sup>After this twisting, the relevant Dirac operator is naturally regarded as a  $\bar{\partial}$  or Dolbeault operator.

maps  $\phi : \Sigma \rightarrow X$ . Equivalently, it is the net fermion number  $\mathcal{F}$  of operators that must be inserted to get a non-zero  $A$ -model amplitude. For our purposes, the important term in the index formula is the one involving  $c_1(\phi^*(K_X))$ . This term depends on  $\phi$ , so when it is nonzero, different  $A$ -model amplitudes on the same  $\Sigma$  violate the fermion number  $\mathcal{F}$  by different amounts and we say that the  $A$ -model does not have fermion number symmetry. The  $\chi(\Sigma) \dim_{\mathbb{C}}(X)$  term in the index formula, since it is independent of  $\phi$ , is a sort of  $c$ -number anomaly, and actually, in our eventual applications, we will have  $\chi(\Sigma) = 0$ . (The  $A$ -model still makes sense – and is much studied – as a topological quantum field theory even when it does not conserve an integer-valued fermion number. Because of the factor of 2 multiplying the  $\phi$ -dependent term, the  $A$ -model always conserves fermion number modulo  $2k$  for some integer  $k$ , and similarly in the analysis below, even when there is no  $\mathbb{Z}$ -graded MSW complex, there is always a  $\mathbb{Z}_{2k}$ -graded one.)

What happens if we turn on a superpotential? Then the equation  $\partial_{\bar{s}}\phi^I = 0$  for a holomorphic map is deformed to the  $\zeta$ -instanton equation

$$\frac{\partial\phi^I}{\partial\bar{s}} = \frac{i\zeta}{4}g^{I\bar{J}}\frac{\partial\bar{W}}{\partial\phi^{\bar{J}}}.$$
 (13.8)

This equation depends on the choice of local complex parameter  $s$  on  $\Sigma$  and does not make sense on an arbitrary  $\Sigma$ . The left hand side of (13.8) is a  $(0, 1)$ -form and the right hand side is a 0-form; to set these equal implies picking a trivialization of the bundle of  $(0, 1)$ -forms. For  $\Sigma$  of genus 1, this is given by the globally-defined  $(0, 1)$ -form  $d\bar{s}$ , but for  $\chi(\Sigma) \neq 0$ , there is a topological obstruction to the definition. However, whenever the deformation from the usual  $A$ -model equation to the  $\zeta$ -instanton equation is possible, this deformation certainly does not affect the index, since the superpotential  $W$  is a lower order term in the  $\zeta$ -instanton equation, or alternatively since the index is an integer and (for compact  $\Sigma$ ) the spectrum of  $L$  varies smoothly when  $W$  is turned on.

Now let us allow  $\Sigma$  to have a boundary, labeled by a brane. To define a boundary condition on the equation for a holomorphic map, or on the  $\zeta$ -instanton equation, we pick a Lagrangian submanifold  $\mathcal{L} \subset X$  and require that  $\phi(\partial\Sigma) \subset \mathcal{L}$ , as discussed in section 11.2. In addition we choose local boundary conditions on the fermions which then follow from the preservation of  $\mathcal{Q}_\zeta$ -supersymmetry. In this case, the index of the Dirac/Dolbeault operator  $L$  still makes sense and still determines the expected dimension of the moduli space and the net violation of fermion number in amplitudes. Moreover, if  $\Sigma$  is compact, and if it makes sense to turn on a superpotential  $W$ , the index is independent of  $W$ , since  $W$  contributes a lower order term in the  $\zeta$ -instanton equation and does not affect the boundary condition. For similar reasons, when we discuss later the  $\eta$ -invariant on a one-manifold  $D$  with boundary (but no infinite ends),  $W$  will be irrelevant and can be set to 0. Accordingly, the following discussion of the index theorem when  $\Sigma$  has a boundary, and later of the  $\eta$ -invariant when  $D$  has a boundary, can be carried out at  $W = 0$  and is simply part of the analysis of the usual  $A$ -model. By contrast, if  $\Sigma$  or  $D$  has an infinite end, then  $W$  affects the asymptotic behavior at the end and does affect the index theory and the  $\eta$ -invariant, as for instance in eqn. (12.8).

So setting  $W = 0$ , we now consider the formula for the index when  $\Sigma$  has a boundary. The formula (13.7) does not quite make sense without some explanation because to define a first Chern class of the line bundle  $\phi^*(K_X)$  over  $\Sigma$ , one needs a trivialization of  $\phi^*(K_X)$  on  $\partial\Sigma$ . The proper interpretation is as follows. For simplicity, we assume that the Lagrangian submanifold  $\mathcal{L}$  is orientable and pick an orientation.  $\mathcal{L}$  acquires a Riemannian metric from its embedding in the Kähler manifold  $X$ , so it has a volume form  $\mathbf{vol}$ . Setting  $n = \dim_{\mathbb{C}} X$ ,  $\mathbf{vol}$  is a real-valued  $n$ -form on  $\mathcal{L}$ . On the other hand, a section of  $K_X$  is an  $(n, 0)$ -form whose restriction to  $\mathcal{L}$  is a complex-valued  $n$ -form on  $\mathcal{L}$ . A real-valued  $n$ -form is a special case of a complex-valued one, so we can regard  $\mathbf{vol}$  as a trivialization of  $K_X|_{\mathcal{L}}$  (the restriction of  $K_X$  to  $\mathcal{L}$ ). Given a map  $\phi : \Sigma \rightarrow X$  that maps  $\partial\Sigma$  to  $\mathcal{L}$ , the trivialization of  $K_X|_{\mathcal{L}}$  pulls back to a trivialization of  $\phi^*(K_X)|_{\partial\Sigma}$ . We define  $\int_{\Sigma} c_1(\phi^*(K_X))$  using this trivialization of  $\phi^*(K_X)$  on  $\partial\Sigma$ . With this interpretation, the index formula (13.7) remains valid. (For mathematical background on this and related statements, see [40], especially section 2.1.D, and [79], especially section 5.) To emphasize the role of the trivialization, we write the index formula as

$$\iota(L) = \chi(\Sigma) \dim_{\mathbb{C}} X - 2 \int_{\Sigma} c_1(\phi^*(K_X))|_{\mathbf{vol}}, \quad (13.9)$$

where the notation is meant to remind us that the first Chern class is defined using the trivialization via  $\mathbf{vol}$  on the boundary. If one asks “why” the trivialization we have used is the right one in the index formula, one answer is that this trivialization is the only one that can be described in a universal local way, and the heat kernel proof of the index theorem makes clear that there must be a universal local formula.

Now let us ask under what conditions the  $A$ -model has fermion number symmetry, in the sense that the index  $\iota(L)$  does not depend on  $\phi$ . A necessary condition is certainly that  $c_1(K_X) = 0$ . Otherwise, the  $A$ -model has no fermion number symmetry even in the absence of a boundary. However, even if  $c_1(K_X) = 0$ , it does not necessarily follow that  $c_1(\phi^*(K_X))|_{\mathbf{vol}}$  vanishes. This involves a condition on  $\mathcal{L}$ , and the  $A$ -model has fermion number symmetry only when this condition is obeyed.

Let us consider an example. We take  $X$  to be the complex  $\phi$ -plane, and we take  $\mathcal{L}$  to be the unit circle in the  $\phi$ -plane.<sup>77</sup> We take  $\Sigma$  to be the unit disc in the complex  $z$ -plane. A degree 0 map from  $\Sigma$  to  $X$ , mapping  $\partial\Sigma$  to  $\mathcal{L}$ , is a constant map, depending on 1 real parameter (the choice of a point in  $\mathcal{L}$ ). A degree 1 map from  $\Sigma$  to  $X$ , mapping  $\partial\Sigma$  to  $\mathcal{L}$ , is a fractional linear transformation  $\phi = (az + b)/(\bar{b}z + \bar{a})$ , depending on 3 real parameters (the real and imaginary parts of  $a, b$  with the equivalence  $a, b \cong \lambda a, \lambda b$  for  $\lambda > 0$ ; one also requires  $|b| < |a|$ ). More generally, a degree  $r$  map is  $\phi = p(z)/z^r \bar{p}(1/z)$ , where  $p$  is a degree  $r$  polynomial which depends on  $2r + 1$  parameters (after allowing for an equivalence  $p \cong \lambda p$ ,  $\lambda \in \mathbb{R}$ ;  $p(z)$  must have all its zeroes inside the unit disc). So the index  $\iota(L)$  equals  $2r + 1$ . Since  $\iota(L)$  depends on  $\phi$  in this example, this means that with this choice of  $\mathcal{L}$ , the  $A$ -model does not have fermion number symmetry. (In fact,  $\mathcal{F}$  is conserved only mod 2 in this situation, since by changing  $r$  one changes  $\iota(L)$  be an arbitrary multiple of 2.)

---

<sup>77</sup>This particular Lagrangian submanifold does not actually correspond to a quantum  $A$ -brane, because of disc instanton effects. But we can still use it to illustrate the index theorem.

Now let us try to understand the dimension of the moduli space from the index formula (13.9). Since  $\chi(\Sigma) = \dim_{\mathbb{C}} X = 1$ , to get  $\iota(\mathcal{D}) = 2r + 1$ , we need  $\int_{\Sigma} c_1(\phi^*(K_X))|_{\text{vol}} = -r$ . The basic case to understand is  $r = 1$ , as the general case will then follow by taking an  $r$ -fold cover. For  $r = 1$ , we can take the map from  $\Sigma$  to  $X$  to be  $\phi = z$ , so we identify  $\Sigma$  with the unit disc  $Y \subset X$  and perform the calculation there. In terms of polar coordinates  $\phi = re^{i\alpha}$ , the volume form of  $\mathcal{L}$  is  $\text{vol} = d\alpha$ . We want to calculate  $\int_Y c_1(K_X)|_{\text{vol}}$ , where we interpret  $\text{vol}$  as a trivialization of  $K_X|_{\mathcal{L}}$ . One way to do the computation is to extend  $\text{vol} = d\alpha$  to a meromorphic section  $\Upsilon$  of  $K_X$  over  $Y$ . Then the difference between the number of zeroes and poles of  $\Upsilon$  in  $Y$  is  $\int_Y c_1(K_X)|_{\text{vol}}$ . The formula  $(dz/z)|_{|z|=1} = id\alpha$  shows that we can take  $\Upsilon = -idz/z$ , with one pole and no zeroes in the unit disc, so  $\int_Y c_1(K_X)|_{\text{vol}} = -1$ , as expected.

When  $c_1(K_X) = 0$ , the relevant part of the index formula can be written as follows as an integral over  $\partial\Sigma$ . Let  $\Omega$  be a holomorphic  $n$ -form on  $X$ , normalized so that  $|\Omega \wedge \bar{\Omega}|$  agrees with the Riemannian volume form of  $X$ . Then when restricted to  $\mathcal{L}$ , the expression

$$e^{i\vartheta} = \frac{\text{vol}}{\Omega|_{\mathcal{L}}} \quad (13.10)$$

is a  $U(1)$ -valued function on  $\mathcal{L}$ . ( $\mathcal{L}$  is said to be special Lagrangian if this function is constant.) If  $\Sigma$  has several boundary components  $\Sigma_s$  mapping to Lagrangian submanifolds  $\mathcal{L}_s$ , then each has its own volume-form  $\text{vol}_s$  and we set  $e^{i\vartheta_s} = \text{vol}_s/\Omega|_{\mathcal{L}_s}$ . The index formula is then

$$\iota(L) = \chi(\Sigma) \dim_{\mathbb{C}} X - 2 \sum_s \oint_{\partial_s \Sigma} \frac{d\varphi_s}{2\pi}. \quad (13.11)$$

where we define  $\varphi_s := \phi_s^*(\vartheta_s)$  and  $\phi_s$  is the restriction of  $\phi$  to  $\partial_s \Sigma$ . To get from (13.9) to this formula, we observe that if we were to trivialize  $\phi^*(K_X)$  via the everywhere nonzero section  $\phi^*(\Omega)$ , then  $\int_{\Sigma} c_1(\phi^*(K_X))$  would vanish. The actual definition involves a trivialization on each  $\partial_s \Sigma$  via  $\phi^*(\text{vol}_s)$ , and then  $\int_{\Sigma} c_1(\phi^*(K_X))$  is a sum of boundary contributions, where the contribution of  $\partial_s \Sigma$  is made by comparing the two trivializations of  $\phi^*(K_X)|_{\partial_s \Sigma}$ .

Finally this gives a necessary and sufficient condition for the  $A$ -model with branes defined by Lagrangian submanifolds  $\mathcal{L}_s$  to have conserved fermion number. The condition under which the index does not depend on the map  $\phi$  is that on each  $\mathcal{L}_s$ , it is possible to define  $\vartheta_s$  as a real-valued function, not just an angle-valued function.

Now we turn to our real interest, which is to analyze the fermion number of physical states in quantization of the model. For this, we take  $\Sigma = \mathbb{R} \times D$ , where  $D$  is a compact interval, for instance the interval<sup>78</sup>  $[0, \pi]$ . We label the two ends of the interval with Lagrangian submanifolds  $\mathcal{L}_l$  and  $\mathcal{L}_r$ . Assuming  $c_1(X) = 0$ , we want to define the conserved fermion number of the  $A$ -model. From section 12.1, we certainly expect that the  $\eta$ -invariant of the one-dimensional Dirac operator will be part of the answer. But since there is a boundary correction in the index formula (13.11), it is not surprising that a boundary contribution is needed in the definition of a conserved fermion number.

---

<sup>78</sup>For  $W = 0$ , conformal invariance ensures that the width of interval  $D$  does not matter. It is convenient to take an interval of width  $\pi$ .

Rather than developing a general theory, we will continue with the example  $X = \mathbb{C}$ . (The interested reader should be able to generalize the following computation.) We parametrize  $X$  by a single complex field  $\phi$ . To start with, we take  $\mathcal{L}_r$  to be a straight line in  $X$  at an angle  $-\vartheta_r$  to the  $\text{Re } \phi$  axis. The equation defining this straight line is

$$\bar{\phi} = e^{2i\vartheta_r} \phi. \quad (13.12)$$

We similarly take  $\mathcal{L}_\ell$  to be a straight line in  $\mathbb{C}$  at an angle  $\varphi_\ell$  to the real axis, and so described by

$$\bar{\phi} = e^{2i\vartheta_\ell} \phi. \quad (13.13)$$

For compact  $D$ , we can set  $W$  to 0 without modifying the fermion number. When we do this, the Dirac operator (12.6) is simply

$$\mathcal{D} = \sigma^3 i \frac{d}{dx}. \quad (13.14)$$

We recall that this was written in a basis  $\begin{pmatrix} \delta\phi \\ \delta\bar{\phi} \end{pmatrix}$ . We write  $\psi_+$ ,  $\psi_-$  for fermions corresponding respectively to  $\delta\phi$ ,  $\delta\bar{\phi}$ ,<sup>79</sup> so  $\mathcal{D}$  acts on the fermions as

$$\mathcal{D}\psi_\pm = \pm i \frac{d}{dx} \psi_\pm. \quad (13.15)$$

The boundary conditions (13.12) and (13.13) imply that variations of  $\phi$  obey  $\delta\bar{\phi} = e^{2i\vartheta_r} \delta\phi$  at  $x = \pi$  and  $\delta\bar{\phi} = e^{2i\vartheta_\ell} \delta\phi$  at  $x = 0$ . So the boundary conditions on the fermions are

$$\psi_- = \psi_+ \cdot \begin{cases} e^{2i\vartheta_r}, & x = \pi \\ e^{2i\vartheta_\ell}, & x = 0. \end{cases} \quad (13.16)$$

With these boundary conditions, and the specific form of the Dirac operator  $\mathcal{D}$  in (13.15), we can glue  $\psi_+$ ,  $\psi_-$  to a single fermi field

$$\psi(x) = \begin{cases} \psi_+(x) & 0 \leq x \leq \pi \\ e^{-2i\vartheta_r} \psi_-(2\pi - x) & \pi \leq x \leq 2\pi \end{cases}$$

that obeys

$$\mathcal{D}\psi = i \frac{d}{dx} \psi \quad (13.17)$$

and

$$\psi(x + 2\pi) = e^{-2i(\vartheta_r - \vartheta_\ell)} \psi(x). \quad (13.18)$$

The eigenvalues of the Dirac operator acting on states with this periodicity condition are

$$\lambda_n = n + \frac{\vartheta_r - \vartheta_\ell}{\pi}, \quad n \in \mathbb{Z}. \quad (13.19)$$

---

<sup>79</sup>These are actually linear combinations of the fields  $\psi_\pm$  and  $\bar{\psi}_\pm$  used in the standard Lagrangian.

Since this spectrum is a periodic function of  $\Delta = \frac{\vartheta_r - \vartheta_\ell}{\pi}$  with period 1,  $\eta(\mathcal{D})$  is also a periodic function of  $\Delta$  with that period. Using the Hurwitz zeta function  $\zeta(s, \alpha) = \sum_{n=0}^{\infty} \frac{1}{(n+\alpha)^s}$  and its analytic continuation to  $s = 0$ , given by  $\zeta(0, \alpha) = \frac{1}{2} - \alpha$  (and valid for  $\text{Re}(\alpha) > 0$ ) one easily shows that for  $0 < \Delta < 1$ ,  $\eta(\mathcal{D}) = 1 - 2\Delta$ . In general this is correct mod 2,

$$\eta(\mathcal{D}) = 1 - 4 \frac{\vartheta_r - \vartheta_\ell}{2\pi} \pmod{2\mathbb{Z}}, \quad (13.20)$$

with even integer jumps so that  $\eta(\mathcal{D})$  is periodic.

Let us write  $f^\psi$  for the fermion number of the filled fermi sea. By equation (12.3), this is  $-\eta(\mathcal{D})/2$ , which in view of eqn. (13.20) leads to

$$f^\psi = -\frac{1}{2} + 2 \frac{\vartheta_r - \vartheta_\ell}{2\pi} \pmod{\mathbb{Z}}. \quad (13.21)$$

To see why the fermion number needs an additional contribution, let  $\mathcal{L}_\ell$  and  $\mathcal{L}_r$  be more general Lagrangian submanifolds of  $X$ , not necessarily straight lines. Pick a time-independent bosonic field of the LG model; this is just a map  $\phi : [0, \pi] \rightarrow X$  with  $\phi(0) \in \mathcal{L}_\ell$  and  $\phi(\pi) \in \mathcal{L}_r$ . Since the fermions are free for the case that  $X = \mathbb{C}$ , the computation of the  $\eta$ -invariant is the same as above, with  $\varphi_\ell = \vartheta_\ell(\phi(0))$  and  $\varphi_r = \vartheta_r(\phi(\pi))$  where  $\vartheta$  is defined by (13.10). Another way to say this is that for any choice of  $\phi : [0, \pi] \rightarrow X$ , the calculation is the same as above, with  $\mathcal{L}_\ell$  and  $\mathcal{L}_r$  replaced by their tangent lines at the points  $\phi(0) \in \mathcal{L}_\ell$  and  $\phi(\pi) \in \mathcal{L}_r$ . Now give  $\phi$  a slow time dependence. The formula for  $f^\psi$  remains the same, but now it is time-dependent, since  $\varphi_r$  and  $\varphi_\ell$  are time-dependent. To define a conserved fermion number, we have to add to  $f^\psi$  a boundary correction that cancels this time-dependence. As one might guess from the analysis of the index, this is only possible if one can define  $\varphi_\ell$  and  $\varphi_r$  as real-valued functions (rather than angle-valued functions) and in that case one can take the conserved fermion number to be

$$f = f^\psi - 2 \frac{\varphi_r - \varphi_\ell}{2\pi} = -\frac{1}{2} \eta(\mathcal{D}) - 2 \frac{\varphi_r - \varphi_\ell}{2\pi}. \quad (13.22)$$

In view of (13.21),  $f$  takes values in  $-1/2 + \mathbb{Z}$ . The generalization of this to higher complex dimension is  $-\frac{1}{2} \dim_{\mathbb{C}} X + \mathbb{Z}$ . From the point of view of the physical approach described at the beginning of section 12.1, the reason that a boundary correction to the fermion number is needed is that although one can define a conserved current that is bilinear in the fermion fields, its flux through the boundary may be nonzero.

We note, however, that for a Lagrangian submanifold  $\mathcal{L}$ , if the angle-valued function  $\vartheta$  can be lifted to an integer-valued function, then this lift is only uniquely determined mod  $2\pi\mathbb{Z}$  and the choice of lift (which in (13.22) is made independently for  $\mathcal{L}_\ell$  and  $\mathcal{L}_r$ ) affects the definition of the fermion number. Moreover, if we change the orientation of  $\mathcal{L}$ , this changes the sign of  $\text{vol}_s$  and hence shifts  $\vartheta_s$  by  $\pi$ , again changing the fermion number. So an orientation of  $\mathcal{L}$  and a choice of lift of  $\vartheta$  have to be regarded as part of the definition of an  $A$ -brane. A change in orientation or lift of either  $\mathcal{L}_\ell$  or  $\mathcal{L}_r$  changes  $f$  by an integer. (We do not need to worry about the dependence of  $\vartheta$  on the choice of  $\Omega$ , because, assuming we use the same  $\Omega$  to define  $\vartheta_s$  for all  $\mathcal{L}_s$ , a change in  $\Omega$  shifts all  $\vartheta_s$  by the same amount and never contributes to  $\varphi_r - \varphi_\ell$ .)

The physical application of this is as usual. In quantization on  $D = [0, \pi]$ , in a transition between two  $\zeta$ -solitons, the integer-valued conserved fermion number will in general change by an integer. This integer will match the index of the two-dimensional Dirac operator  $L$ , and this index gives the net fermion number of operators that must be inserted to make this transition possible.

Now let us restore  $W$  and take the interval  $D = [x_\ell, x_r]$  to be much longer than the Compton wavelength of any massive particle. Then for each  $\zeta$ -soliton, the fermion number can be written as a sum of contributions from the two ends of the strip, as we assumed in discussing (13.3). The boundary corrections are manifestly a sum of contributions from the two ends, and the  $\eta$ -invariant has the same property, since it can be written rather as in (12.7) as an integral over  $D$ :

$$\eta^{\text{cont}}(\mathcal{D}) = \lim_{\varepsilon \rightarrow 0} \int_D dx \sum_{s=1}^{2d} \langle x, s | \text{sign}(\mathcal{D}) \exp(-\varepsilon |\mathcal{D}|) | x, s \rangle. \quad (13.23)$$

The integral receives a contribution only from the regions near the boundaries of  $D$ , because the integrand vanishes exponentially fast away from the boundaries. The last statement just reflects the fact that the expectation value of the Lorentz-invariant fermion number current is 0 in a Lorentz-invariant vacuum, and moreover, in a massive theory, the approach to the vacuum is exponentially fast.

Thus far, we have assumed  $D$  is compact. If we take  $D$  to be a half-line  $[0, \infty)$  or  $(-\infty, 0]$ , then because of the infinite end of  $D$ , we cannot simply set  $W$  to 0. However,  $W$  does not affect the boundary contributions at the finite end of  $D$  to the index or the fermion number.

In summary, although we have based our discussion on a representative example, in general the fermion number of the MSW complex on the interval is defined by

$$f = -\frac{1}{2}\eta(\mathcal{D}) - 2\frac{\varphi_r - \varphi_\ell}{2\pi} \quad (13.24)$$

where  $\mathcal{D}$  is the Dirac operator obtained from linearizing the  $\zeta$ -soliton equation. We choose local  $\mathcal{Q}_\zeta$ -preserving boundary conditions for the fermions and  $\varphi_\ell = \vartheta_\ell(\phi(x_\ell))$  while  $\varphi_r = \vartheta_r(\phi(x_r))$ . On the interval we can simplify the Dirac operator by setting  $W = 0$ . On the half-line we drop  $\varphi_r$  or  $\varphi_\ell$ , as appropriate, and we cannot set  $W = 0$ .

### 13.4 Implications Of The Bulk Anomaly

What are the implications of the fermion number anomaly? Our next goal is to explain that in general, the framework of the present paper only applies when the fermion number anomaly vanishes. We discuss first the bulk anomaly and then (in section 13.5) the boundary anomaly.

The bulk anomaly is proportional to the first Chern class  $c_1(X)$  of the target space  $X$  of the  $\sigma$ -model, so it is absent for simple Landau-Ginzburg models with  $X = \mathbb{C}^n$ . We pause to give a few illustrative examples of massive  $\sigma$ -models with more complicated target spaces  $X$ , constructed from a  $U(1)$  gauge theory coupled to chiral superfields. Here are two examples:

(A) For our first model, we consider three chiral superfields  $u, v, b$  with  $U(1)$  charges  $1, 1, -2$ . The  $D$ -term constraint coming from the  $U(1)$  gauge-invariance is  $|u|^2 + |v|^2 - 2|b|^2 = r$ . If the constant  $r$  is large and positive, the model is equivalent at low energies to a  $\sigma$ -model in which the target space  $X$  is the Eguchi-Hansen manifold, the total space of the line bundle  $\mathcal{O}(-2) \rightarrow \mathbb{CP}^1$ , where  $\mathbb{CP}^1$  is embedded as the locus  $b = 0$ . We take the superpotential to be  $W = buv + b^2G(u, v)$  where  $G$  is a generic homogeneous polynomial of degree 4. This gives a massive model with 2 vacua at  $b = 0$  (with  $u = 0$  or  $v = 0$ ) and more at  $b \neq 0$ .

(B) For our second model, we consider three chiral superfields  $u, v, b$  with  $U(1)$  charges  $1, 1, -1$ . The  $D$ -term constraint is  $|u|^2 + |v|^2 - |b|^2 = r$ , and if  $r \gg 0$ , the target space  $X$  is the total space of the line bundle  $\mathcal{O}(-1) \rightarrow \mathbb{CP}^1$ , where again  $\mathbb{CP}^1$  is embedded in  $X$  as the locus  $b = 0$ . We take the superpotential to be  $W = bu$ . There is a unique massive vacuum at  $b = u = 0$ .

In general, in a theory of a  $U(1)$  vector multiplet coupled to chiral multiplets of charge  $q_i$ , the condition for a fermion number anomaly is  $\sum_i q_i \neq 0$ . Accordingly, model (A) actually does not have a fermion number anomaly, since the sum of the chosen  $U(1)$  charges vanishes, while model (B) does have an anomaly.

Parenthetically, we remark that while model (A) does not have an anomaly, nevertheless it does illustrate another point of interest in the present paper: the superpotential  $W$  has equal values at both of the  $b = 0$  vacua. This does not reflect global symmetries: for generic  $G(u, v)$ , there are none. Rather, it reflects the fact that the two vacua are contained in a common holomorphically embedded  $\mathbb{CP}^1$ . In general, if  $W$  is restricted to any *compact and holomorphic* subvariety of  $X$  it must be constant there, and hence the critical values of any two critical points on such a subvariety are equal.<sup>80</sup> So in general in a massive  $(2, 2)$  model in two dimensions, it is not true that parameters can be varied so that the superpotential takes generic values in the different vacua. Hence the genericity assumption that we have made throughout this paper needs to be treated with care. However, the case that everything is generic except that two vacua unavoidably have the same value of  $W$  fits into our framework with a minor modification: precisely because the two vacua have the same value of  $W$ , there are no BPS solitons connecting them, and in many statements we can simply conflate the two vacua in question.

Returning to the consequences of a nonvanishing fermion anomaly, as we have noted, model (B) actually does have a fermion number anomaly, since in this model  $q := \sum_i q_i = 1$ . To understand the consequences, recall that in the theory of  $\mathcal{N} = (2, 2)$  vector multiplets

---

<sup>80</sup>In the particular case of a  $(2, 2)$  supersymmetric  $\sigma$ -model with a hyperkähler target  $X$ , there is a possible modulus that is not easily visible in the linear  $\sigma$ -model language and that removes holomorphic subvarieties. The modulus in question is a rotation of the complex structure of  $X$  away from the one that is visible in the linear  $\sigma$ -model. In particular, in model (A), the target has the Eguchi-Hanson metric and is hyperkähler. A rotation of complex structure removes the holomorphically embedded  $\mathbb{CP}^1$ . For such a rotation (with a corresponding change of  $W$  to remain holomorphic) the values of  $W$  in the two vacua will in general become unequal. We can still illustrate the main point since a more elaborate model of this type has no such modulus. An example is a  $U(1)$  gauge theory with chiral superfields  $u, v, w, b, b'$  of charges  $1, 1, 1, -1, -2$  and superpotential  $W = bu + b'vw$ . There are two massive vacua with  $b = b' = u = 0$  and  $v$  or  $w$  vanishing, and  $W = 0$  in each vacuum.

coupled to chiral multiplets in two dimensions, the vector multiplet field strength is a twisted chiral superfield  $\Sigma$ . Classically, the twisted chiral superpotential for this field is  $\widetilde{W} = t\Sigma$ . Here  $t$  is a complex constant, the Kähler modulus, whose imaginary part is the constant  $r$  that enters the classical  $D$ -term constraint; its real part is  $\theta/2\pi$ , with  $\theta$  the quantum mechanical  $\theta$ -angle. But at 1-loop order, when  $q \neq 0$ , the twisted chiral superpotential becomes  $\widetilde{W} = t\Sigma + q\Sigma \log \Sigma/2\pi i$ . In our model  $(B)$ , with  $q = 1$ , this  $\widetilde{W}$  turns out to have a unique critical point, which corresponds to the unique massive vacuum of the model.<sup>81</sup> The value of  $\widetilde{W}$  at the critical point depends on the Kähler modulus  $t$ . If we take model  $(B)$  as it stands, since it has only one vacuum, the value of  $\widetilde{W}$  in this vacuum does not really matter (an additive constant in either  $W$  or  $\widetilde{W}$  is irrelevant), and moreover with only one vacuum, all considerations of the present paper are trivial. To make a more interesting variant of model  $(B)$ , we first observe that model  $(B)$  can be interpreted as a  $\sigma$ -model whose target  $X$  is  $\mathbb{C}^2$  with a point blown up. (As a  $\sigma$ -model, this model is asymptotically free, since  $\sum_i q_i > 0$ , so this description is good even in the ultraviolet.) Here  $\mathbb{C}^2$  is parametrized by  $x = ub$  and  $y = vb$ , and the point  $x = y = 0$  is blown up in  $X$  to the copy of  $\mathbb{CP}^1$  defined by  $b = 0$ . In this description, the superpotential is  $W = x$ . Though the function  $x$  has no critical point on  $\mathbb{C}^2$ , when one blows up a point, it acquires a nondegenerate critical point on the resulting exceptional divisor  $\mathbb{CP}^1$ . Now let us blow up  $k$  points  $p_1, \dots, p_k \in \mathbb{C}^2$ . After the blowup, each  $p_i$  is replaced by a copy of  $\mathbb{CP}^1$ , and on each of these  $\mathbb{CP}^1$  the superpotential  $W$  has a nondegenerate critical point  $\phi_i$ , with the value of  $W(\phi_i)$  being simply the  $x$ -coordinate of the point  $p_i$ ; in particular, these values can be generic. Each  $\mathbb{CP}^1$  also has its own Kähler modulus  $t_i$ . The value of the twisted chiral superpotential  $\widetilde{W}$  at the critical point  $\phi_i$  depends on  $t_i$  just as if the other  $\mathbb{CP}^1$ 's did not exist (it can be computed in  $A$ -model terms by counting holomorphic maps to the given  $\mathbb{CP}^1$ , and this counting does not “know” whether other points have been blown up). In particular, in this model, both the ordinary superpotential  $W$  and the twisted superpotential  $\widetilde{W}$  take generic values in the massive vacua.<sup>82</sup>

The significance of this arises when we consider solitons in the  $ij$  sector. In addition to the familiar central charge  $W(\phi_i) - W(\phi_j)$ , the supersymmetry algebra acquires another central charge term  $\widetilde{W}(\phi_i) - \widetilde{W}(\phi_j)$ . The supercharge  $\mathcal{Q}_\zeta$  whose cohomology we have been analyzing is still conserved, but it is no longer nilpotent in a soliton sector; rather  $\mathcal{Q}_\zeta^2$  is proportional to  $\zeta^{-1} (\widetilde{W}(\phi_i) - \widetilde{W}(\phi_j))$ . The supersymmetric model still exists, but it cannot be studied using the methods of this paper, which assume that  $\mathcal{Q}_\zeta^2 = 0$ . Generically, fermion number conservation is the only obvious mechanism to ensure that the  $\mathcal{F} = 1$  supercharge  $\mathcal{Q}_\zeta$  obeys  $\mathcal{Q}_\zeta^2 = 0$  in a soliton sector. That is why we require fermion number conservation.

As usual, the relation between  $W$  and  $\widetilde{W}$  is actually symmetrical. *A priori*, an  $\mathcal{N} = (2, 2)$  model in two dimensions can have separate  $R$ -symmetry groups acting on positive

---

<sup>81</sup>We are eliding a few points, which are explained for example in section 13.6 of [18]. The semiclassical method described in the text to evaluate  $\widetilde{W}$  in a massive vacuum is valid when  $|r|$  is large and the signs of  $\sum_i q_i$  and of  $r$  are such that the value of  $\Sigma$  is large in this vacuum. Even when this derivation is not valid, the answer it gives for the value of  $\widetilde{W}$  in a massive vacuum can be justified using holomorphy.

<sup>82</sup>The techniques described in [71] could be used to make the above construction quite concrete.

chirality and negative chirality supercharges. Taking  $W \neq 0$ ,  $\widetilde{W} = 0$  leaves one linear combination of these as a possible symmetry, and this is what we have called fermion number. Taking  $W = 0$ ,  $\widetilde{W} \neq 0$  leaves a different symmetry and a different supercharge  $\tilde{\mathcal{Q}}$  that obeys  $\tilde{\mathcal{Q}}^2 = 0$  in soliton sectors. The reasoning of this present paper applies equally well to a model with  $W = 0$ ,  $\widetilde{W} \neq 0$ .

### 13.5 Implications Of The Boundary Anomaly

Even if the bulk fermion number anomaly vanishes, it is natural now to suspect that a similar phenomenon can arise from the boundary contribution to the fermion number anomaly. Indeed, this is the case. As an example, we consider the model that we already used to illustrate the boundary anomaly, with  $X$  being the complex  $\phi$ -plane.<sup>83</sup> We take the superpotential to be  $\zeta^{-1}W = i\phi^2$ , which has just one critical point at  $\phi = 0$ . The right thimble  $R^\zeta$  is the imaginary  $\phi$  axis. We consider the  $\sigma$ -model on the half-plane  $x \geq 0$  in the  $x - \tau$  plane with boundary condition at  $x = 0$  set by a Lagrangian submanifold  $\mathcal{L}$ . As in section 13.1, the classical approximation to a supersymmetric state on the half-plane is given by an intersection  $\mathcal{L} \cap R^\zeta$ . We take  $\mathcal{L}$  to be the unit circle in the  $\phi$ -plane. This is an example where the phase function  $e^{i\vartheta}$  defined in (13.10) does not have a well-defined logarithm, so we should expect trouble. The Lagrangian  $\mathcal{L}$  intersects  $R^\zeta$  in the two points  $\phi = \pm i$ , which we call  $p$  and  $p'$ . So in the classical approximation, there are two supersymmetric half-plane states,  $|p\rangle$  and  $|p'\rangle$ . The state  $|p\rangle$  corresponds to a  $\zeta$ -soliton  $\Phi_p$  on the half-line that interpolates from  $\phi = i$  at  $x = 0$  to  $\phi = 0$  at  $x = \infty$ , and similarly  $|p'\rangle$  corresponds to a  $\zeta$ -soliton  $\Phi_{p'}$  that interpolates from  $\phi = -i$  at  $x = 0$  to  $\phi = 0$  at  $x = \infty$ . The model has a classical symmetry  $\kappa : \phi \rightarrow -\phi$  that exchanges the points  $p$  and  $p'$ , and also the states  $|p\rangle$  and  $|p'\rangle$ . What does  $\kappa$  do to the fermion number of a half-plane state? The  $\eta$ -invariant is invariant under  $\kappa$ , but  $\kappa$  increases the boundary contribution to the fermion number by 1. (This is clear from eqn. (13.22): the boundary contribution at  $x = 0$  is  $\varphi_\ell/\pi$  where  $\varphi_\ell$  differs by  $\pi$  between the points  $p$  and  $p'$ .) The fact that  $\kappa$  increases  $\mathcal{F}$  by 1 is compatible with  $\kappa^2 = 1$ , since  $\mathcal{F}$  is only conserved modulo 2. There is no natural way to define a zero of the fermion number and say which of  $|p\rangle$  and  $|p'\rangle$  is even and which is odd.

Quantum mechanically, there must be no supersymmetric states on the half-plane in this problem. This follows from the fact that the space of supersymmetric states must be invariant under Hamiltonian symplectomorphisms acting on the Lagrangian submanifold  $\mathcal{L}$ . We can use such a symplectomorphism to move  $\mathcal{L}$  so that it does not intersect  $R^\zeta$  at all. (This is possible, in part, because the algebraic intersection number of  $\mathcal{L}$  and  $R^\zeta$  vanishes in this example.) Accordingly, it must be that quantum mechanically  $\mathcal{Q}_\zeta$  does not annihilate the states  $|p\rangle$  and  $|p'\rangle$ . In fact,  $\mathcal{Q}_\zeta$  must exchange these two states, as  $\mathcal{Q}_\zeta$  reverses the  $\mathbb{Z}_2$ -valued fermion number and the two states have opposite fermion number. So  $\mathcal{Q}_\zeta$  must act by

$$\begin{aligned}\mathcal{Q}_\zeta|p\rangle &= \lambda|p'\rangle \\ \mathcal{Q}_\zeta|p'\rangle &= \lambda'|p\rangle,\end{aligned}\tag{13.25}$$

---

<sup>83</sup>This example is closely related to Example 1.11 discussed in [7].

where  $\lambda$  and  $\lambda'$  cannot both be zero. However, the symmetry  $\kappa$  ensures that  $\lambda$  is nonzero if and only if  $\lambda'$  is nonzero.<sup>84</sup> Hence in this space of states  $\mathcal{Q}_\zeta^2 = \lambda\lambda' \neq 0$ .

In the topological  $A$ -model, one would describe the fact that  $\mathcal{Q}_\zeta^2 \neq 0$  by saying that the brane under consideration is not a valid  $A$ -brane. It appears that in the physical supersymmetric model, this brane makes sense but that for strings ending on this brane, there is a central charge such that  $\mathcal{Q}_\zeta^2$  is a nonzero constant. In any event, the fact that  $\mathcal{Q}_\zeta^2 \neq 0$  in the presence of this brane means that supersymmetric states involving this brane cannot be studied using the methods of the present paper.

But what goes wrong with the proof that  $\mathcal{Q}_\zeta^2 = 0$ ? We view the  $\sigma$ -model on the half-line as an infinite-dimensional version of Morse theory, with the Morse function of eqn. (11.21). As explained in section 10.6, to try to prove that  $\mathcal{Q}_\zeta^2 = 0$ , we are supposed to look at two-dimensional moduli spaces of the flow equation, which in the present context is the  $\zeta$ -instanton equation on the half-plane  $x \geq 0$ . To be more precise, to try to prove that  $\mathcal{Q}_\zeta^2$  annihilates the state  $|p\rangle$ , we must study solutions of the  $\zeta$ -instanton equation that approach the  $\zeta$ -soliton  $\Phi_p$  for  $\tau \rightarrow \pm\infty$ , approach  $\phi = 0$  for  $x \rightarrow \infty$ , and obey  $|\phi| = 1$  at  $x = 0$ . To get a two-dimensional moduli space, we need a fermion number anomaly of 2, and this means, since the fermion number anomaly is twice the winding number of the boundary, that along the boundary of the half-plane (which is the line  $x = 0$ ),  $\phi$  wraps once around the unit circle.

If  $\lambda, \lambda' \neq 0$ , then solutions obeying the appropriate conditions do exist. For to get a non-zero matrix element  $\mathcal{Q}_\zeta|p\rangle = \lambda|p'\rangle$ , there is a solution of the  $\zeta$ -instanton equation of the half-plane that describes a flow from  $\Phi_p$  in the past to  $\Phi_{p'}$  in the future; and this solution has only the one modulus that results from time-translation symmetry. Similarly, a non-zero matrix element  $\mathcal{Q}_\zeta|p'\rangle = \lambda'|p\rangle$  means that there are  $\zeta$ -instantons describing flows from  $\Phi_{p'}$  back to  $\Phi_p$ , again with only one modulus associated to time translations. As always in Morse theory, by combining these solutions to a broken path  $\Phi_p \rightarrow \Phi_{p'} \rightarrow \Phi_p$  and correcting the broken path slightly to get a family of exact flows, we get a family of solutions of the  $\zeta$ -instanton equation with a two-dimensional moduli space  $\mathcal{M}$  and hence a one-dimensional reduced moduli space  $\mathcal{M}_{\text{red}}$ .  $\mathcal{M}_{\text{red}}$  has one end corresponding to the broken path  $\Phi_p \rightarrow \Phi_{p'} \rightarrow \Phi_p$ . This broken path by itself makes a nonzero contribution to the matrix element of  $\mathcal{Q}_\zeta^2$  from  $|p\rangle$  to itself.

Usually, in Morse theory, the proof that  $\mathcal{Q}_\zeta^2 = 0$  comes by arguing that any such  $\mathcal{M}_{\text{red}}$  must have a second end corresponding to another broken path starting and ending at  $\Phi_p$ , and that the two broken paths make canceling contributions to the matrix element of  $\mathcal{Q}_\zeta^2$ . However, in the present context, the second end of  $\mathcal{M}_{\text{red}}$  is not another broken path; it is a sort of ultraviolet end that is possible because of the fermion number anomaly. It seems to be difficult to find exactly the relevant two-parameter family of  $\zeta$ -instantons (with a quadratic superpotential, the  $\zeta$ -instanton equation is linear but it is difficult to satisfy the boundary conditions). However, we can easily find the end of  $\mathcal{M}_{\text{red}}$  that is not a broken path and so leads to  $\mathcal{Q}_\zeta^2$  being non-zero. This end is described by a  $\zeta$ -instanton

---

<sup>84</sup>One might be surprised to have both  $\lambda$  and  $\lambda'$  nonzero, since one counts ascending gradient flows from  $p$  to  $p'$  and the other counts ascending flows from  $p'$  to  $p$ . How can there be ascending flows in both directions? The point is that we are in a situation in which the superpotential is not single-valued.

that coincides with the  $\zeta$ -soliton  $\Phi_p$  except very near a boundary point  $x = 0$ ,  $\tau = \tau_0$  (for some  $\tau_0$ ). At short distances, the  $\zeta$ -instanton equation can be approximated by the equation  $\partial\phi/\partial\bar{s} = 0$  for a holomorphic map; as usual  $s = x + i\tau$ . A simple family of such holomorphic maps from the right-half plane to the unit disc, mapping the boundary of the right half plane to the boundary of the disc with winding number 1 is given by a fractional linear transformation

$$\phi_{a,\tau_0}(s) = i \frac{(s - i\tau_0) - a}{(s - i\tau_0) + a}, \quad \tau_0 \in \mathbb{R}, \quad a > 0. \quad (13.26)$$

This is a family of holomorphic maps from the right half-plane to the unit disc, mapping the boundary of the half-plane to the boundary of the disc, and depending on the two real parameters  $\tau_0$  and  $a > 0$ . (A third parameter has been fixed by requiring that  $\phi_{a,\tau_0}(\pm i\infty) = i$ .) For very small  $a$ ,  $\phi(s)$  differs substantially from  $i$  only for  $|s - i\tau_0| \lesssim a$ . In the limit  $a \rightarrow 0$ , the reduced moduli space of holomorphic maps has an “end.” For small  $a$ , we can find a family of  $\zeta$ -instantons that coincide with  $\Phi_p$  except very near  $s = i\tau_0$  and with  $\phi_{a,\tau_0}$  near  $s = i\tau_0$ ; the asymptotic behavior of  $\phi_{a,\tau_0}$  has been chosen to make this possible. (As usual, gluing gives a family of approximate solutions and index theory and general properties of nonlinear partial differential equations predict that this can be slightly corrected to a family of exact solutions.) Thus in this example, the  $\zeta$ -instanton moduli space has an end which is not a broken path and which spoils the usual strategy to show that  $Q_\zeta^2 = 0$ .

## 14. $\zeta$ -Instantons And $\zeta$ -Webs

### 14.1 Preliminaries

One of the most important properties of the  $\zeta$ -instanton equation is that in any massive theory, it has no pointlike solutions. To be more precise, the only solution on  $\mathbb{R}^2$  that approaches a prescribed critical point  $\phi = \phi_i$  at infinity is the constant solution with  $\phi = \phi_i$  everywhere.

This is proved by a standard type of argument. Suppose we are given a solution of the  $\zeta$ -instanton equation

$$\frac{\partial\phi^I}{\partial\bar{s}} - \frac{i\zeta}{4}g^{I\bar{J}}\frac{\partial\bar{W}}{\partial\bar{\phi}^{\bar{J}}} = 0 \quad (14.1)$$

on the whole complex  $s$ -plane. Then

$$0 = \int d^2s \left( \frac{\partial\phi^I}{\partial\bar{s}} - \frac{i\zeta}{4}g^{I\bar{K}}\frac{\partial\bar{W}}{\partial\bar{\phi}^{\bar{K}}} \right) \left( \frac{\partial\bar{\phi}^{\bar{J}}}{\partial s} + \frac{i\zeta^{-1}}{4}g^{L\bar{J}}\frac{\partial W}{\partial\phi^L} \right) g_{I\bar{J}}. \quad (14.2)$$

Expanding this out, we obtain

$$0 = \int d^2s \left( \left| \frac{\partial\phi}{\partial\bar{s}} \right|^2 + \frac{1}{16} \left| \frac{\partial W}{\partial\phi} \right|^2 \right) + \frac{1}{2} \text{Im} \left[ \zeta \int d^2s \frac{\partial}{\partial s} \bar{W} \right]. \quad (14.3)$$

Alternatively, for any  $R > 0$ , after integrating by parts, we have

$$0 = \int_{|s| \leq R} d^2s \left( \left| \frac{\partial \phi}{\partial \bar{s}} \right|^2 + \frac{1}{16} \left| \frac{\partial W}{\partial \phi} \right|^2 \right) + \frac{R}{4} \operatorname{Im} \left[ \zeta \int_0^{2\pi} e^{-i\theta} \bar{W} d\theta \right]_{|s|=R}, \quad (14.4)$$

where  $d^2s = \frac{i}{2} ds \wedge d\bar{s}$ ,  $\theta = \arg(s)$ .

After possibly adding a constant to  $W$  (which does not change the  $\zeta$ -instanton equation), we can assume that  $W(\phi_i) = 0$ . In any massive theory, a solution in which  $\phi \rightarrow \phi_i$  at infinity has the property that  $\phi$  approaches  $\phi_i$  exponentially fast. Hence  $W(\phi)$  approaches its limiting value  $W(\phi_i) = 0$  exponentially fast, and therefore the surface term in (14.4) vanishes for  $R \rightarrow \infty$ . Accordingly, these formulas imply that  $\partial W(\phi)/\partial \phi^I$  identically vanishes in any such solution of the  $\zeta$ -instanton equation, so that  $\phi$  is everywhere equal to a critical point of  $W$ . In a massive theory, the critical points are a discrete set, and therefore  $\phi$  must be identically equal to its limiting value at infinity.

Although the  $\zeta$ -instanton equation does not have point-like solutions, it does have solutions localized on lines, as we explained qualitatively in the introduction. We have chosen  $\zeta$  so that for any  $i, j \in \mathbb{V}$ , there does not exist a time-independent solution of the  $\zeta$ -instanton equation interpolating between  $\phi = \phi_i$  at  $x = -\infty$  and  $\phi = \phi_j$  at  $x = +\infty$ . However, such a solution  $\phi_{ij}(x)$  might exist if we replace  $\zeta$  in the  $\zeta$ -soliton equation by an appropriate value  $\zeta_{ji}$ . If  $\phi_{ij}(x)$  obeys the  $\zeta_{ji}$ -soliton equation

$$\frac{d}{dx} \phi_{ij}^I(x) = \frac{i\zeta_{ji}}{2} g^{I\bar{J}} \partial_{\bar{J}} \bar{W}(\phi_{ij}(x)), \quad (14.5)$$

then clearly its rotated counterpart

$$\phi_{ij}^{\text{euc}}(x, \tau) := \phi_{ij}(\cos \mu x + \sin \mu \tau), \quad (14.6)$$

obeys

$$\left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial \tau} \right) \phi_{ij}^{\text{euc}, I}(x, \tau) = \frac{ie^{i\mu} \zeta_{ji}}{2} g^{I\bar{J}} \partial_{\bar{J}} \bar{W}(\phi_{ij}(x)) \quad (14.7)$$

We call this a *boosted soliton*, although in Euclidean signature it might be more apt to speak of a *rotated soliton*. This construction gives a solution to the  $\zeta$ -instanton equation (11.17) provided we choose  $\mu$  so that

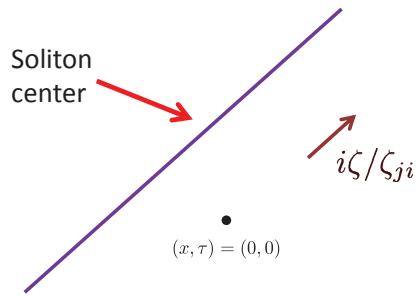
$$e^{i\mu} \zeta_{ji} = \zeta \quad (14.8)$$

The boosted soliton is of course not at rest; rather, it is localized along a line  $L$  in the complex  $s$ -plane. This line is parallel to the complex number  $ie^{i\mu}$  as in Figure 133. Comparing this with the definition of plane webs in Section §2.1 and using equation (11.15) we deduce that the relation between plane web vacuum weights and the critical values of the superpotential is  $z_j = \zeta \bar{W}_j$ .

In saying that the solution is localized on a line, we ignore the width of the BPS solitons. When one looks more closely, these solitons have a width no greater than  $\ell_W = 1/m$ , where  $m$  is the mass of the lightest particle of the theory. Because of this width, though  $L$  has a precisely defined angle, its “impact parameter” is only naturally defined to within a

precision of  $\ell_W$ . A similar remark holds in many statements below about the relation between  $\zeta$ -instantons and webs.

Though there is no natural definition of the impact parameter of a  $\zeta$ -soliton, it is convenient for some purposes to pick a specific definition. First we define for each  $ij$  soliton precisely what we mean by its center of mass. There is no completely natural choice, and we pick some definition. (For instance, for a soliton at rest, we can pick the unique point with equal integrated energy to its left and right.) Having done this, a possibly boosted soliton is centered on a well-defined oriented line  $\ell$ . We pick an arbitrary origin  $\mathbf{v} \in \mathbb{R}^2$  and use the orientation of  $\mathbb{R}^2$  that is built into the  $\zeta$ -instanton equation. Then we define the impact parameter of the given soliton as the signed distance by which  $\ell$  passes to the right of  $\mathbf{v}$  (Figure 133).



**Figure 133:** A “boosted”  $ij$  soliton defines a  $\zeta$ -instanton with “core” along an oriented line  $\ell$  in the  $(x, \tau)$  plane parallel to the phase  $\pm i\zeta/\zeta_{ji}$ . The same line with opposite orientation determines a  $ji$  soliton with opposite impact parameter.

## 14.2 Fan-Like Asymptotics and $\zeta$ -Webs

The  $\zeta$ -instanton equation might also have solutions of a more complicated sort that again was schematically described in the introduction. At large values of  $|s|$  the solutions approach piecewise constant functions on the complement of certain rays in the complex  $s$ -plane. (More precisely the rays should be replaced by certain strips with a width of order  $\ell_W$ .) The constant values of  $\phi$  are a cyclically ordered sequence of vacua  $i_1, \dots, i_p \in \mathbb{V}$ , for some  $p$ . Across a ray separating vacuum  $i_k$  from vacuum  $i_{k+1}$  the field is well approximated by a boosted  $i_k i_{k+1}$  soliton of the type just discussed. The picture is schematically depicted in Figure 178 of Appendix E below (where the boundary conditions are spelled out a bit more precisely). In the picture, in crossing from vacuum  $i_k$  to vacuum  $i_{k+1}$ , the central charge jumps from  $W(\phi_{i_k})$  to  $W(\phi_{i_{k+1}})$ , and the angle  $\mu$  at which the boosted  $i_k i_{k+1}$  soliton emerges is described by  $e^{i\mu} \zeta_{ji} = \zeta$ , as in eqn. (14.8). This angle must be a decreasing function of  $k$  in order for the picture to make sense, and therefore  $i_1, \dots, i_p$  must be a cyclic

fan of vacua in the sense defined in section 2.1.<sup>85</sup> An important detail is that although the angles of the outgoing solitons are directly determined by  $\zeta$  and the critical values  $W_i$ , the impact parameters of the outgoing solitons (defined by some specific procedure like that of section 14.1) are not; they must be found by solving the  $\zeta$ -instanton equation. The outgoing rays are only defined near infinity on the  $s$ -plane, and if continued into the interior of the picture, they do not necessarily meet. Motivated by this picture, we define a *fan of solitons* to be a cyclically-ordered collection of solitons of type<sup>86</sup>  $\{\phi_{i_1, i_2}, \dots, \phi_{i_p, i_1}\}$ .

As was discussed qualitatively in the introduction, low energy field theory (as opposed to a full study of the nonlinear  $\zeta$ -instanton equation) is not powerful enough to determine whether solutions with such fan-like asymptotics at infinity actually exist. The considerations of the present paper are most interesting if they do exist and we proceed assuming this is the case. In fact, the  $\zeta$ -instanton equations with such boundary conditions have been studied previously in the context of domain wall junctions in [13, 39] where, in some special cases, existence proofs are given.

Although we refer to solutions of the  $\zeta$ -instanton equation with fan asymptotics as “ $\zeta$ -instantons,” the fact that such solutions have outgoing solitons means that they are not localized in spacetime and are not instantons in the usual sense. In fact they have infinite action if we use the standard Landau-Ginzburg action discussed (implicitly) in Section §11.1 above. More precisely, using (14.4) one easily shows that the contribution to the standard Landau-Ginzburg action from integrating over a disk of radius  $R$  approaches

$$R \sum_k |W_{i_{k+1}} - W_{i_k}| + \rho + 2 \int_{\mathbb{R}^2} \phi^*(\omega) \quad (14.9)$$

as  $R \rightarrow \infty$ . Here  $\omega$  is the Kähler form on  $X$  and one can show, using (14.4) that  $\rho$  has a finite limit as  $R \rightarrow \infty$ ; it is an interesting function of the fan of solitons. The term linear in  $R$  is just  $R$  times the sum of the masses of the solitons. One could remove this term by a suitable “wavefunction renormalization.” We will, actually, consider a slightly different action for the path integral in our discussion below. Namely, we take the action given by squaring the  $\zeta$ -instanton equation, as in equation (14.2), for the bosonic fields, and then adding the supersymmetric completion. In this theory, the  $\zeta$ -instantons have zero action.

In a model in which solutions with fan-like asymptotics do exist, the most important ones in a certain sense are the “vertices,” the families of irreducible solutions. This concept requires some explanation. Suppose that for a specified fan  $i_1 i_2 \dots i_p$  of vacua, there is a nonempty moduli space  $\mathcal{M} = \mathcal{M}_{i_1 i_2 \dots i_p}$  of solutions of the  $\zeta$ -instanton equation. (In defining  $\mathcal{M}$ , we do not specify the impact parameters of external solitons. See section 14.3 for more on this.)  $\mathcal{M}$  is not necessarily connected and might have components of different dimension. Our considerations are most natural in an  $\mathcal{F}$ -conserving theory, as we have explained in section 13.5. The conservation of the fermion number  $\mathcal{F}$  together with the analysis of Sections §14.3 and §14.4 implies a relation between the expected dimension of

<sup>85</sup>Recall our convention that for a fan of vacua  $i_1, \dots, i_p$ , reading left to right the vacua are located in regions in the clockwise direction.

<sup>86</sup>We will be more precise in section 14.5 about how to treat the fact that each classical soliton corresponds to two quantum states.

$\mathcal{M}$  and the fermion numbers  $(f_a, f_a + 1)$  of the solitons  $\phi_{i_a, i_{a+1}}$  in the fan:

$$\dim \mathcal{M} = \sum_a (f_a + 1) \quad (14.10)$$

In other words, the expected dimension equals the sum of the upper fermion numbers of the outgoing solitons in the fan. See Appendix §E for further details.

The  $\zeta$ -instanton equation has translation symmetry but no additional relevant symmetries. The quotient of  $\mathcal{M}$  by translations is a reduced moduli space  $\mathcal{M}_{\text{red}}$  of dimension  $d - 2$ . A basic question now is whether  $\mathcal{M}_{\text{red}}$  is compact, and if not, what are its “ends”? In other words, after dividing by spatial translations to eliminate a trivial “end” of the moduli space in which a solution moves off to spatial infinity, in what ways can a sequence of solutions with given fan-like asymptotics blow up or diverge?

We make use of the trichotomy that was explained at the end of section 10.6. An end of the moduli space is either an ultraviolet effect (something blows up at short distances), a large field effect (some fields go to infinity), or an infrared effect (something happens at large distances). In a massive theory, we do not anticipate a large field effect, since the potential  $|dW|^2$  blows up if the fields become large. Before discussing ultraviolet effects, we recall some examples. A typical example of an ultraviolet “end” of a moduli space of solutions of a partial differential equation is the small instanton singularity in four-dimensional Yang-Mills theory, or the analogous small instanton singularity in a two-dimensional  $\sigma$ -model with a suitable Kähler target  $X$ . The latter example is more relevant to us, since at short distances the  $\zeta$ -instanton equation can be approximated by the equation  $\partial_{\bar{s}}\phi^I = 0$  for a holomorphic map to  $X$ . In the case of a massive LG model with target  $X = \mathbb{C}^n$ , we do not have to worry about ultraviolet ends of the moduli space, because the equation  $\partial_{\bar{s}}\phi^I = 0$  is linear and does not have an analog of the small-instanton singularity (paying proper attention to domains, a sequence of holomorphic functions does not converge to a meromorphic function with a pole). The considerations of the present paper also apply for more general  $X$ ’s, on which small instanton singularities might in general occur. However, we can assume that the  $A$ -model of  $X$  exists without a superpotential (otherwise turning on a superpotential will hardly help), and this means that the small instanton ends of the moduli space do not contribute to supersymmetric Ward identities. So even if  $X$  is not  $\mathbb{C}^n$ , we do not have to worry about ultraviolet ends of the moduli space.<sup>87</sup>

Just as in the Morse theory problem of section 10.5, we do have to worry about infrared effects. Morse theory involves a massive theory in 1 dimension, and in this case the only interesting infrared effect is a “broken path” involving successive jumps from one critical point to another. The basic example was a two-step jump  $i_1 \rightarrow i_2 \rightarrow i_3$  that was crucial in understanding the MSW complex. To construct a solution of the gradient flow equation representing this two-step jump, we glue together two widely separated solutions

---

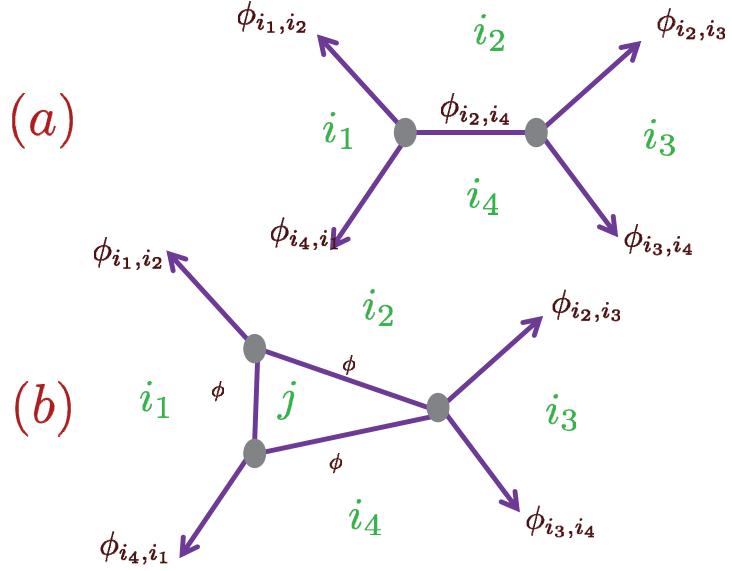
<sup>87</sup>Even for  $X = \mathbb{C}^n$ , if the  $\sigma$ -model is formulated on a two-manifold with boundary, in general the  $\zeta$ -instanton moduli space has ultraviolet ends, as we have explained in section 13.5. But as was also explained there, such effects are not relevant if the boundary conditions are set by a valid  $A$ -brane – or are such that the methods of the present paper are applicable. In that analysis, we made use of the fact that ultraviolet ends of the moduli space are not affected by a superpotential.

representing transitions  $i_1 \rightarrow i_2$  and  $i_2 \rightarrow i_3$ , and deform slightly to make an exact solution corresponding to a broken path. The ends of moduli spaces of gradient flows are given by such broken paths.

In two dimensions, there is a similar operation of combining solutions to make a more complicated solution, but, once we have introduced fan boundary conditions, this operation is much more complicated because there are many more possible gluing operations in two dimensions than in one dimension. The basic gluing operation is made by embedding in  $\mathbb{R}^2$  widely separated  $\zeta$ -instanton sub-solutions each of which has fan-like asymptotics, and which are positioned in such a way that the various soliton lines that are outgoing from the various individual sub-solutions fit into a web. Some examples are sketched in Figure 134. Such a picture represents something that is exponentially close to a solution of the  $\zeta$ -instanton equation, and (modulo technicalities that we will certainly not resolve in the present paper) index theory can be used to predict that this approximate solution can be slightly corrected to make an exact solution. We say a little more on this in section 14.4. We will call the picture representing such a gluing a  $\zeta$ -web.  $\zeta$ -webs are analogous to the abstract webs of section 2.1, but differ in two key ways: First, an edge separating a vacuum  $i$  from vacuum  $j$  is labeled by a choice of classical  $ij$  soliton, as shown in Figure 134. Second, the vertices in a  $\zeta$ -web are not simple points but are rather regions in which the approximation to a boosted soliton breaks down. Within this region the solution should be a solution of the  $\zeta$ -instanton equation with boundary conditions given by the fan of solutions defined by the lines coming out of the vertex. Thus, we may think of the vertex-regions as representing moduli spaces of solutions with fan-like boundary conditions. Since *a priori* we know little about these moduli spaces, a  $\zeta$ -web is a concept that only makes sense in the limit that the vertex regions are small relative to the lengths of the internal lines in the  $\zeta$ -web.

A reduced moduli space of  $\zeta$ -instanton solutions corresponding to a given fan of solitons can have an “end” corresponding to a  $\zeta$ -web. Sketched in Figure 134 are some examples of possible ends of the moduli space  $\mathcal{M}_{\text{red}}$  corresponding to fan boundary conditions  $\{\phi_{i_1,i_2}, \phi_{i_2,i_3}, \phi_{i_3,i_4}, \phi_{i_4,i_1}\}$ . We claim that the only ends of reduced  $\zeta$ -instanton moduli spaces with fan-like asymptotics correspond to such  $\zeta$ -webs. From a physical point of view, we expect this claim to be valid since degeneration to a  $\zeta$ -web with widely separated vertices is the only natural infrared effect in a massive theory. Mathematically, our claim, and even the existence of  $\zeta$ -instanton moduli spaces with fan-like asymptotics, is almost certainly not a direct consequence of any standard theorem and will involve new analysis.

If  $\mathcal{M}_{\text{red}}$  does have an end corresponding to a  $\zeta$ -web such as one of those in Figure 134, then we can repeat the question. Pick one of the blobs corresponding to a sub-solution in this  $\zeta$ -web. This blob is associated to a new fan of vacua  $j_1, \dots, j_q$  and has its own reduced moduli space  $\mathcal{M}'_{\text{red}}$ . We should ask the same question about  $\mathcal{M}'_{\text{red}}$  that we originally asked about  $\mathcal{M}_{\text{red}}$ . Is  $\mathcal{M}'_{\text{red}}$  compact, and if not what are its ends? The same reasoning as before makes us expect that ends of  $\mathcal{M}'_{\text{red}}$  arise just like the ends of  $\mathcal{M}_{\text{red}}$ , by resolving the chosen blob into a new  $\zeta$ -web. This is illustrated in Figure (135). The process of finding an end of a  $\zeta$ -web moduli space by resolving one of the sub-solutions from which this  $\zeta$ -web is constructed into a new  $\zeta$ -web might be called  $\zeta$ -convolution as it is fairly analogous to the



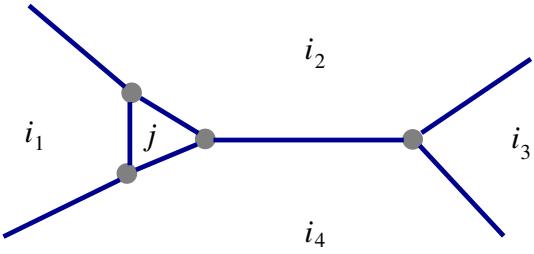
**Figure 134:** Sketched here are some of the possible “ends” of the moduli space  $\mathcal{M}$  of  $\zeta$ -instantons with fan boundary conditions  $\{\phi_{i_1, i_2}, \phi_{i_2, i_3}, \phi_{i_3, i_4}, \phi_{i_4, i_1}\}$ . An end of the moduli space can be described by a web-like picture in which the lines represent BPS solitons that propagate for long distances (compared to  $\ell_W$ ), the regions between the lines are labeled by vacua, and the “blobs” where lines meet represent in their own right moduli spaces of solutions of the  $\zeta$ -instanton equation with fan-like boundary conditions. Thus an end of  $\mathcal{M}$  is associated to a gluing of moduli spaces  $\mathcal{M}_i$  of  $\zeta$ -instantons (the  $\mathcal{M}_i$  have smaller dimension than  $\mathcal{M}$ ) which themselves satisfy fan-like boundary conditions. In subsequent figures of  $\zeta$ -webs, we will generally drop the explicit labeling of the edges by solitons, but they are implicitly there.

ordinary convolution of webs as introduced in section 2.

This process cannot go on indefinitely, because at each step the dimension of the moduli spaces represented by the remaining blobs becomes smaller. (We will be more precise about this in section 14.2.) So eventually we arrive at a web that is made from “ $\zeta$ -vertices.” By a  $\zeta$ -vertex  $\mathcal{V}$  we mean a fan of solitons together with a *compact* and *connected* component of its reduced moduli space. We denote it by  $\mathcal{M}_{\text{red}}(\mathcal{V})$ . Thus a  $\zeta$ -vertex represents a family of solutions of the  $\zeta$ -instanton equation that does not have any ends at which it can be resolved into a non-trivial  $\zeta$ -web (a  $\zeta$ -web with more than one vertex).

There can be several  $\zeta$ -vertices for a given fan of solitons. That is, there can be several compact connected components of the reduced moduli space of  $\zeta$ -instantons with a fixed fan of solitons at infinity. More generally, for a given fan at infinity, one expects the moduli space  $\mathcal{M}$  of  $\zeta$ -instantons to have only finitely many components. This is actually a special case of the statement that  $\mathcal{M}$  is compact except for ends arising from gluings of sub-solutions.

The  $\zeta$ -vertices of reduced dimension 0 play a special role in our construction. We will



**Figure 135:** The ends of  $\mathcal{M}$  depicted in Figure 134. may themselves have ends; these ends would arise by further resolving one of the blobs of that figure as a convolution of  $\zeta$ -instanton moduli spaces of yet smaller dimension. (The particular example shown here can arise by resolving a vertex in either Figure 134(a) or Figure 134(b).) This process only ends when the blobs are  $\zeta$ -vertices, which by definition cannot be further resolved.

call them rigid  $\zeta$ -vertices. The reduced moduli space of a rigid  $\zeta$ -vertex is by definition a point. The algebraic structures constructed in this paper can be understood entirely in terms of rigid  $\zeta$ -vertices. In particular, the rigid  $\zeta$ -vertices suffice for answering one of the basic questions in the present paper, which is to understand in terms of webs the space of supersymmetric states of the  $(\mathfrak{B}_\ell, \mathfrak{B}_r)$  system, where  $\mathfrak{B}_\ell$  and  $\mathfrak{B}_r$  are left and right  $W$ -dominated branes described in section 11.2.4. This statement reflects the following considerations.

A  $\zeta$ -vertex  $\mathcal{V}$ , if we forget the  $\zeta$ -instanton equation, determines in particular a fan of vacua to which we can associate a web  $\mathfrak{w}_\mathcal{V}$  with only one vertex. In the language of section 2.1, since  $\mathfrak{w}_\mathcal{V}$  only has one vertex, its reduced moduli space is a point, of dimension 0. This coincides with  $\mathcal{M}_{\text{red}}(\mathcal{V})$  if  $\mathcal{V}$  is rigid, but if  $\mathcal{V}$  is a non-rigid  $\zeta$ -vertex with  $\mathcal{M}_{\text{red}}(\mathcal{V})$  of dimension  $\varepsilon_\mathcal{V} > 0$ , then  $\mathcal{M}_{\text{red}}(\mathcal{V})$  parametrizes internal degrees of freedom of a family of  $\zeta$ -instanton solutions that cannot be described in terms of webs. We call  $\varepsilon_\mathcal{V}$  the *excess dimension* of  $\mathcal{M}_{\text{red}}(\mathcal{V})$ .

Our definitions and assumptions imply that if  $\mathcal{S}$  is any component of the moduli space of  $\zeta$ -instantons on the  $s$ -plane with fan-like behavior at infinity, then either  $\mathcal{S}$  is the moduli space associated to a  $\zeta$ -vertex  $\mathcal{V}$  or  $\mathcal{S}$  has an end corresponding to a  $\zeta$ -web made from  $\zeta$ -vertices  $\mathcal{V}_i$ . In the latter case, we define  $\varepsilon(\mathcal{S}) = \sum_i \varepsilon(\mathcal{V}_i)$ . (We claim that this sum does not depend on the choice of a particular end of  $\mathcal{S}$ .) As we explain in section 14.4,  $\varepsilon(\mathcal{S})$  is the excess dimension of the moduli space  $\mathcal{M}(\mathcal{S})$  associated to  $\mathcal{S}$ , relative to the dimension of the moduli space of the web  $\mathfrak{w}_\mathcal{S}$ . This excess dimension is always non-negative and vanishes precisely if the ends of  $\mathcal{S}$  are built from rigid  $\zeta$ -vertices.

Finally we can explain the importance of the case that the excess dimension is zero. For the same reasons as in the supersymmetric approach to Morse theory, the dimension of a moduli space of solutions of the  $\zeta$ -instanton equation determines the net violation of fermion number in an amplitude derived from that moduli space. (As in Morse theory, the relevant fact is that the Dirac equation for the fermions of fermion number 1 is the linearization of the  $\zeta$ -instanton equation.) By considering only moduli spaces of zero excess

dimension, we can answer questions that involve the minimum violation of fermion number. Such a question is to determine the space of supersymmetric  $(\mathfrak{B}_\ell, \mathfrak{B}_r)$  states on a strip; the differential of the MSW complex comes from  $\zeta$ -instanton solutions on the strip with no reduced moduli (except the one that is required by time-translation invariance), so that the excess dimension of the moduli space must vanish. Since excess dimensions are non-negative and add under natural operations of combining webs or vertices, to construct the differential of the MSW complex, we only need to study  $\zeta$ -vertices of excess dimension 0, in other words the rigid ones. However, the  $A$ -model can have local observables of positive fermion number, and to compute their matrix elements requires considering moduli spaces with a positive excess dimension; see section 16. For this application, we do need to consider the non-rigid  $\zeta$ -vertices.

We stress again that because a BPS soliton has a nonzero width of order  $\ell_W$ , the solitons emanating from a rigid  $\zeta$ -vertex cannot literally be identified with rays that emanate from a point in the plane. To define rays associated to solitons, one must choose a “center of mass” for each BPS soliton, as in section 14.1. Regardless of those choices, one should not expect the solitons emanating from an arbitrary  $\zeta$ -vertex to correspond to rays that emanate from a point in the plane. This will only work to within a precision of order  $\ell_W$ . In the case of a non-rigid  $\zeta$ -vertex, after making a precise definition of the centers of masses of the solitons, the offsets of the rays representing the solitons will depend on the excess moduli of the vertex, though only by an amount of order  $\ell_W$ . The relationship between  $\zeta$ -solitons and webs is a statement about the infrared limit and always involves ignoring a discrepancy of order  $\ell_W$ .

### 14.3 The Index Of The Dirac Operator

Let  $L$  be the Dirac operator in this theory for fermions of fermion number  $\mathcal{F} = 1$ . The Dirac equation is

$$\left[ \frac{\partial}{\partial r} + i\sigma^3 \frac{1}{r} \frac{\partial}{\partial \theta} + \frac{i}{2} \begin{pmatrix} 0 & -\zeta e^{-i\theta} \bar{W}'' \\ \zeta^{-1} e^{i\theta} W'' & 0 \end{pmatrix} \right] \begin{pmatrix} \delta\phi \\ \delta\bar{\phi} \end{pmatrix} = 0 \quad (14.11)$$

where  $s = x + i\tau = re^{i\theta}$  and for simplicity we take  $X = \mathbb{C}$  with standard Euclidean metric. For the same reasons as in supersymmetric quantum mechanics, the index  $\iota(L)$  is the expected dimension of the moduli space  $\mathcal{M}$  of  $\zeta$ -instantons. This happens because the operator  $L$  is the linearization of the  $\zeta$ -instanton equation.

However, there is a subtlety that does not have an analog in supersymmetric quantum mechanics. When we consider a  $\zeta$ -instanton that is asymptotic at infinity to a fan consisting of  $p \geq 3$  solitons, each soliton has its own “impact parameter.” We have to decide whether in defining a moduli space  $\mathcal{M}$  of such  $\zeta$ -instantons, we want to allow deformations in which these impact parameters change. Such deformations correspond to zero-modes of  $L$  that are asymptotically constant along each outgoing soliton;<sup>88</sup> these represent an asymptotically

---

<sup>88</sup>That is, one could make the ansatz for the Dirac equation  $\delta\phi(x, \tau) = f(r, \theta)\phi'_{ij}(r(\psi_{ij} - \theta))$  in the neighborhood of an outgoing boosted  $ij$  soliton where the soliton ray is parallel to  $e^{i\psi_{ij}} = i\zeta/\zeta_{ji}$ . As  $r \rightarrow \infty$  the Dirac equation rapidly approaches the free Dirac equation for  $f(r, \theta)$ . We require that  $f(r, \theta)$  approaches a constant for  $r \rightarrow \infty$ .

constant displacement of the soliton in question. They are certainly not square-integrable.

For most purposes, it is more natural for us to define a moduli space  $\mathcal{M}$  of  $\zeta$ -instantons in which the impact parameters of outgoing solitons are *not* specified. The dimension of  $\mathcal{M}$  is then the index  $\iota(L)$  of the operator  $L$ , acting on a space of  $\mathcal{F} = 1$  fermion states that are allowed to be asymptotically constant along each outgoing soliton. This motivates our conjectural dimension formula 14.10.

An important detail is that  $\iota(L)$  is the difference in dimension between the kernel of  $L$ , acting on  $\mathcal{F} = 1$  fermions, and the kernel of the adjoint operator  $L^\dagger$ , acting on  $\mathcal{F} = -1$  fermions. The adjoint condition to “constant at infinity” is “vanishing at infinity” so in computing  $\iota(L)$ , one counts  $\mathcal{F} = -1$  zero modes that vanish at infinity.

Although we will not often find this useful, we could alternatively define a moduli space  $\mathcal{M}_{x_1, \dots, x_p}^\diamond$  in which the impact parameters of the solitons are required to take specified values  $x_1, \dots, x_p$ . To make sense of this, as described at the end of section 14.1, we first define for each  $ij$  soliton precisely what we mean by its impact parameter. Then we denote as  $\mathcal{M}_{x_1, \dots, x_p}^\diamond$  the  $\zeta$ -instanton moduli space in which the impact parameters take specified values  $x_1, \dots, x_p$ . The expected dimension of  $\mathcal{M}_{x_1, \dots, x_p}^\diamond$  is  $\iota^\diamond(L)$ , where  $\iota^\diamond(L)$  is the index of  $L$  acting on  $\mathcal{F} = 1$  fermions that vanish at infinity (and dually, on  $\mathcal{F} = -1$  fermions that are allowed to be constant at infinity). The relation between the two notions of the index of  $L$  is

$$\iota^\diamond(L) = \iota(L) - p. \quad (14.12)$$

Indeed, each time we constrain the  $\mathcal{F} = 1$  fermions to vanish at infinity along a particular outgoing soliton, and drop a dual requirement for  $\mathcal{F} = -1$  fermions to vanish at infinity along that soliton, we remove an  $\mathcal{F} = 1$  zero-mode or add an  $\mathcal{F} = -1$  zero-mode, in either case reducing the index by 1.

If  $\iota^\diamond(L) > 0$ , this implies that there are normalizable fermion zero-modes of  $\mathcal{F} = 1$ . All the amplitudes we compute below will then vanish unless we insert local  $A$ -model observables to absorb those zero-modes. In a Landau-Ginzburg model with target  $X = \mathbb{C}^n$ , there are no such observables in bulk, but more general models can have such observables. (Even for target  $\mathbb{C}^n$ , there can be boundary  $A$ -model observables for suitable choices of brane.) Dually, if  $\iota(L) < 0$ , there are normalizable fermion zero-modes of  $\mathcal{F} = -1$ , which will also ensure vanishing contributions to  $A$ -model amplitudes. However,  $\zeta$ -instanton moduli spaces with  $\iota(L) < 0$  have negative expected dimension and generically do not exist. In fact, translation invariance implies that an actual  $\zeta$ -instanton moduli space has dimension at least 2, so  $\zeta$ -instanton moduli spaces with  $\iota(L) < 2$  generically do not exist. Hence we are primarily interested in the case that  $\iota(L) \geq 2$  and (if local  $A$ -model observables are not relevant)  $\iota^\diamond(L) \leq 0$ . The last condition tends to be violated if excess dimensions are too large, so it is part of the reason that  $\zeta$ -instanton moduli spaces with positive excess dimension are not relevant unless we consider local  $A$ -model observables.

When  $\iota(L) = d \geq 2$  for a given component  $\mathcal{M}$  of  $\zeta$ -instanton moduli space, we expect  $\mathcal{M}$  to be generically a smooth manifold of dimension  $d$ . When this is so, if  $\iota^\diamond(L) = -k$  is negative, what this means generically is that  $\mathcal{M}_{x_1, \dots, x_p}^\diamond$  is nonempty only if  $k$  conditions are placed on the impact parameters  $x_1, \dots, x_p$ .

## 14.4 More On $\zeta$ -Gluing

Suppose that we are given a fan of solitons corresponding to vacua  $i_1, \dots, i_p$ ,  $p \geq 3$ . The sum of the fermion numbers of the chosen solitons is an integer. Indeed, the fermion numbers of the individual solitons in the fan are given mod  $\mathbb{Z}$  by certain boundary terms (eqn. (12.8)), and these boundary terms cancel when we add up the fermion numbers of all the solitons in the fan.

Let us suppose that there exist solutions of the  $\zeta$ -instanton equation on the  $s$ -plane that are asymptotic at infinity to the chosen fan of solitons and let  $\mathcal{M}$  be a component of the corresponding moduli space. As already stressed in section 14.3, in defining  $\mathcal{M}$  we do not specify the “impact parameters” of the outgoing solitons. In the notation of that section, when  $\iota(L) = d$ , this means that in trying to solve the  $\zeta$ -instanton equation (without specifying the impact parameters of outgoing solitons), there are in effect  $d$  more unknowns than equations. That is why the moduli space has dimension  $d$ . Now let us consider the gluing operation of section 14.2 from the point of view of index theory. We try to glue various sub-solutions of the  $\zeta$ -instanton equation corresponding to moduli spaces  $\mathcal{M}_i$  of dimensions  $d_i \geq 2$ . We make this gluing via a web  $\mathfrak{w}$ . The corresponding web moduli space  $\mathcal{D}(\mathfrak{w})$  was studied in section 2.1 and (for generic vacuum data) has dimension

$$d(\mathfrak{w}) = 2V - E, \quad (14.13)$$

where  $V$  and  $E$  are respectively the numbers of vertices and internal edges in the web  $\mathfrak{w}$ . Because the three-dimensional group of translations and scalings of  $\mathbb{R}^2$  acts on  $\mathcal{D}(\mathfrak{w})$ , we are usually only interested in gluings leading to webs such that  $d(\mathfrak{w}) \geq 3$ ; other gluings cannot be realized by webs of BPS solitons unless the central charges of the solitons take special values.

The analog of eqn. 14.13 for the dimension of a hypothetical moduli space  $\mathcal{M}^*$  of  $\zeta$ -instantons that arises by gluing of sub-solutions with moduli spaces  $\mathcal{M}_i$  is

$$d(\mathcal{M}^*) = \sum_i d(\mathcal{M}_i) - E. \quad (14.14)$$

For webs, each vertex carries 2 moduli, accounting for the contribution  $2V$  in (14.13). For  $\zeta$ -webs, the contribution of each sub-solution is instead  $d(\mathcal{M}_i)$ , accounting for the formula (14.14). The  $-E$  in (14.14) has the same origin as in (14.13): for every edge in a web or  $\zeta$ -web, there is one constraint so that the two vertices or sub-solutions can be connected by a line or a  $\zeta$ -soliton at the appropriate angle. This result is compatible with our conjectural dimension formula 14.10: each internal edge removes the upper fermion numbers of two CPT-conjugate solitons, which add to 1.

*A priori*, when we glue together widely separated sub-solutions – one for each vertex in the web  $\mathfrak{w}$  – we do not get an exact  $\zeta$ -instanton solution, but we come exponentially close to one. However, index theory and generalities about nonlinear equations make one expect that when the expected dimension  $d(\mathcal{M}^*)$  is positive and the individual moduli spaces that are being glued are smooth (no fermion zero-modes of  $\mathcal{F} = -1$ ), a very good approximate solution can be corrected to a nearby exact solution. The condition on the

expected dimension is satisfied, since  $d(\mathcal{M}^*) \geq d(\mathfrak{w}) \geq 3$ . The formulas (14.13) and (14.14) and the conditions  $d(\mathfrak{w}) \geq 3$ ,  $d(\mathcal{M}_j) \geq 2$  for all  $j$ , imply that  $d(\mathcal{M}^*) > d(\mathcal{M}_i)$  for all  $i$ . This inequality ensures that the process of repeatedly resolving a  $\zeta$ -moduli space by blowing up a sub-solution into a  $\zeta$ -web must terminate, as claimed in section 14.2. By definition, when it terminates, the  $\mathcal{M}_i$  are all  $\zeta$ -vertices. Then  $d(\mathcal{M}_i) = 2 + \varepsilon_i$ , where  $\varepsilon_i$  is the excess dimension defined in section 14.2. So the excess dimension of the  $\zeta$ -web whose moduli space is  $\mathcal{M}^*$  (defined as the difference between  $d(\mathcal{M}^*)$  and the corresponding dimension  $d(\mathfrak{w})$  of the ordinary web moduli space) is

$$d(\mathcal{M}^*) - d(\mathfrak{w}) = \sum_i \varepsilon_i. \quad (14.15)$$

This is the additivity of the excess dimension claimed in section 14.2.

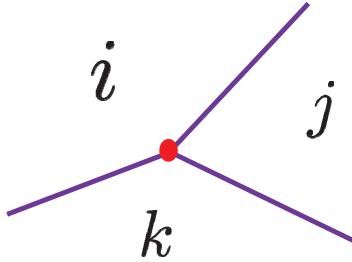
Since  $d(\mathcal{M}_i)$  might be greater than 2,  $d(\mathcal{M}^*)$  might be large even if  $d(\mathfrak{w}) < 3$ . This may make one wonder if gluing of  $\zeta$ -instantons can produce a  $\zeta$ -web that does not have an analog (for generic central charges) in ordinary webs. However, if a given web  $\mathfrak{w}$  cannot be constructed (with a given set of central charges) if the vertices are points, then it also cannot be constructed if the vertices are pointlike within an error of order  $1/\ell_W$ . To construct such a web with vertices that are  $\zeta$ -instanton sub-solutions, one would really have to resolve some of the sub-solutions into webs (so that they become far from pointlike), taking advantage of their moduli spaces  $\mathcal{M}_i$ . What would arise this way is an end of  $\mathcal{M}$  that we would associate not to the web  $\mathfrak{w}$ , but to some other web  $\mathfrak{w}'$  obtained by resolving some of the vertices in  $\mathfrak{w}$ . Thus ends of a moduli space  $\mathcal{M}$  of  $\zeta$ -instantons with fan-like asymptotics can be put in correspondence with ordinary webs with point vertices, the same objects studied in Sections §§2-§9 of this paper.

#### 14.5 The Collective Coordinates

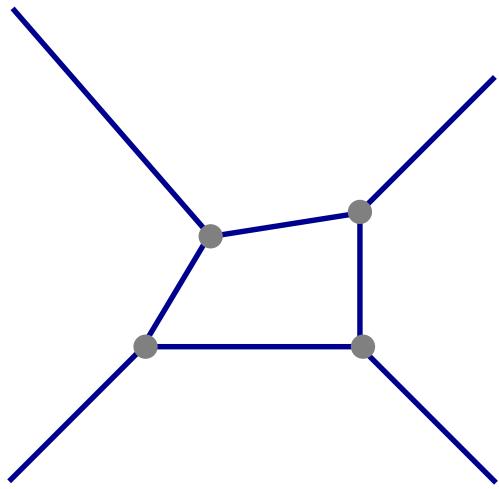
Now let us examine in this context the consequences of the fact that a classical  $\zeta$ -soliton solution actually corresponds to a pair of quantum states of fermion numbers  $f, f+1$  for some  $f$ . The doubling of the spectrum arises from quantizing the center of mass motion of the soliton and its supersymmetric counterpart. It is convenient here, as in section 14.1, to choose a precise definition of the impact parameter  $x$  of each soliton, even though this concept is really only naturally defined up to an additive constant. The fermionic collective coordinate of the soliton is  $\{\mathcal{Q}_\zeta, x\}$ . It is evocative to write this as  $dx$ . Now consider a  $\zeta$ -instanton asymptotic to a fan of, say,  $n$  solitons. Each of the  $n$  outgoing solitons has a bosonic collective coordinate  $x_1, \dots, x_n$ , together with corresponding fermionic collective coordinates  $dx_1, \dots, dx_n$ . The  $\zeta$ -instanton amplitude to create the outgoing solitons is a state  $\Psi(x_1, dx_1; \dots; x_n, dx_n)$ ; we can think of this state as a differential form on a copy of  $\mathbb{R}^n$  parametrized by the  $x_i$ . This differential form is valued in the tensor product

$$\mathbb{M}'_I = \mathbb{M}'_{i_1 i_2} \otimes \mathbb{M}'_{i_2 i_3} \otimes \cdots \otimes \mathbb{M}'_{i_n i_1} \quad (14.16)$$

of the reduced complexes (12.14) that describe BPS solitons without their zero-modes;  $\Psi$  is the product of an ordinary differential form and an element  $m_{i_1 i_2} \otimes m_{i_2 i_3} \otimes \cdots \otimes m_{i_n i_1}$  of



**Figure 136:** This  $\zeta$ -web has only two moduli (assuming the vertex is rigid) so the “impact parameters”  $x_1, x_2, x_3$  of the three outgoing solitons cannot be varied independently. They obey a linear relation  $a_1x_1 + a_2x_2 + a_3x_3 = 0$ , for some constants  $a_i$ .



**Figure 137:** A  $\zeta$ -web with four external solitons and a four-dimensional moduli space. The impact parameters can be varied independently but only in a certain region in  $\mathbb{R}^4$ .

this tensor product <sup>89</sup>

$$\Psi = \psi(x_i, dx_i) \cdot m_{i_1 i_2} \otimes m_{i_2 i_3} \otimes \cdots m_{i_n i_1} \quad (14.17)$$

---

<sup>89</sup>We abbreviate the expression  $m_{ij}^{f_0}(p)$  of equation (12.14) by  $m_{ij}$  and also shift its fermion number by  $-\frac{1}{2}$  so that two Clifford module generators of the module  $\mathbb{W}$  in equation (12.13) have fermion numbers 0 and 1.

$\mathcal{Q}_\zeta$  acts on this  $\Psi$  as the exterior derivative  $d$  on  $\psi$ . As in any instanton calculation, in discussing the state  $\Psi$  generated by a given  $\zeta$ -instanton moduli space, we need not consider higher order  $\zeta$ -instanton effects that generate the differentials for the individual soliton states that appear on the right hand side of the formula for  $\Psi$ . (How they enter will be explained in footnote 90.) Now  $\mathcal{Q}_\zeta$ -invariance of the path integral is the statement that  $d\Psi = 0$ .

To understand  $\Psi$ , we need to know whether, as we vary the  $\zeta$ -instanton moduli, the  $x_i$  can be varied independently. A typical example in which they cannot be varied independently arises from a  $\zeta$ -web with a single trilinear vertex (Figure 136). Assuming the vertex is rigid, it has only two moduli so the  $x_i$  are not independent, but obey a linear relation  $a_1x_1 + a_2x_2 + a_3x_3 = 0$ , with some constants  $a_i$ . (With natural outgoing normalizations for all solitons, we can take the  $a_i$  to be all positive. Even if the vertex is not rigid, the relation is obeyed to within an error of order  $\ell_W$ .) The wavefunction  $\Psi$  is therefore supported on the locus  $a_1x_1 + a_2x_2 + a_3x_3 = 0$ . It takes the form

$$\Psi = \pm \delta(a_1x_1 + a_2x_2 + a_3x_3)(a_1dx_1 + a_2dx_2 + a_3dx_3) \cdot m_{ij} \otimes m_{jk} \otimes m_{ki}, \quad (14.18)$$

where as usual in the  $A$ -model, the signs of boson and fermion determinants cancel, leaving a “constant” wavefunction with a sign  $\pm$  that comes from the sign of the fermion determinant (as usual, this sign depends on the signs chosen for the soliton states  $m_{ij}$ ,  $m_{jk}$ ,  $m_{ki}$ ).

To explain the above formula better, we note that  $A$ -model results are usually expressed as forms on the appropriate moduli space, which here is defined by  $\jmath = 0$  where  $\jmath = a_1x_1 + a_2x_2 + a_3x_3$ . If one does this, the wavefunction would just be  $\pm 1$  (times the tensor product of symbols  $m_{ij}$ , etc., representing the external states). However, in the present context, it is much more natural to express  $\Psi$  as a wavefunction on the space  $\mathbb{R}^3$  that parametrizes the three centers of mass. We do this by multiplying by the form  $\delta(\jmath)d\jmath$ . In general, for any manifold  $X$  with submanifold  $Y$  whose interior points are defined locally by equations  $\jmath_1 = \dots = \jmath_k = 0$ , where an orientation of the normal bundle to  $Y$  is defined by  $d\jmath_1 \wedge \dots \wedge d\jmath_k$ , one defines the  $k$ -form Poincaré dual to  $Y$  by

$$\Theta_Y = \delta(\jmath_1) \dots \delta(\jmath_k) d\jmath_1 \dots d\jmath_k, \quad (14.19)$$

which depends only on  $Y$  and not on the choices of the functions  $\jmath_i$ . In our discussion it is important to consider the case where  $Y$  has a boundary. If  $Y$  has a boundary  $\partial Y$ , and near  $\partial Y$  it is defined locally by equations  $\jmath_1 = \dots = \jmath_k = 0$  together with an inequality  $h \geq 0$ , then (14.19) should be modified to

$$\Theta_Y = \Theta(h)\delta(\jmath_1) \dots \delta(\jmath_k) d\jmath_1 \dots d\jmath_k, \quad (14.20)$$

where  $\Theta(\alpha)$  for  $\alpha$  real is the standard Heaviside function given by 0 for  $\alpha < 0$  and 1 for  $\alpha > 0$ . In this situation,  $\partial Y$  is defined by equations  $\jmath_1 = \dots = \jmath_k = 0 = h$ , so the definitions just given immediately imply that

$$d\Theta_Y = \Theta_{\partial Y}. \quad (14.21)$$

In our example,  $\Psi$  is Poincaré dual to the submanifold  $Y$  defined by  $a_1x_1 + a_2x_2 + a_3x_3 = 0$ ;  $Y$  has no boundary, so  $d\Psi = 0$  in eqn. (14.18). This is a consequence of the underlying  $\mathcal{Q}_\zeta$ -invariance, since as usual in the  $A$ -model,  $\mathcal{Q}_\zeta$  behaves as the exterior derivative  $d$  on moduli spaces of classical solutions.

We note from eqn. (14.18) that although the three-soliton state created by the given  $\zeta$ -instanton has definite fermion number, the individual outgoing solitons are not created in states of definite fermion number: the factors  $dx_1$ ,  $dx_2$ , and  $dx_3$  each have fermion number 1. Another interesting point concerns the total fermion number of the state  $\Psi$ . In an  $\mathcal{F}$ -conserving theory, the outgoing state created by a  $\zeta$ -instanton moduli space must have fermion number 0. So as the explicit factor  $a_1dx_1 + a_2dx_2 + a_3dx_3$  has  $\mathcal{F} = 1$ , the factor  $m_{ij} \otimes m_{jk} \otimes m_{ki}$  must have  $\mathcal{F} = -1$ . Similarly, in eqn. (14.24) below, the factor  $m_{i_1i_2} \otimes m_{i_2i_3} \otimes m_{i_3i_4} \otimes m_{i_4i_1}$  has  $\mathcal{F} = 0$ .

If we want to get a number from this  $\zeta$ -instanton amplitude, we have to pair  $\Psi$  with an external state of the outgoing solitons. Such a state would have to be a two-form, and we want a closed two-form as we want a  $\mathcal{Q}_\zeta$ -invariant state. An example would be the closed two-form  $\Psi' = \delta(x_1 - c_1)dx_1 \delta(x_2 - c_2)dx_2 \cdot m_{ik} \otimes m_{kj} \otimes m_{ji}$ , where  $c_1, c_2$  are constants and, for example,  $m_{ji}$  is the state in the reduced complex  $\mathbb{M}'_{ji}$  that is obtained from  $m_{ij}$  by a  $\pi$  rotation (and so is dual to  $m_{ij}$  under the pairing (12.17)). The pairing

$$Z_{\Psi'} = \int_{\mathbb{R}^3} \mathcal{D}(x_i, dx_i) (\Psi(x_i, dx_i), \Psi'(x_i, dx_i)) \quad (14.22)$$

is clearly nonzero. To evaluate it, we contract out the internal states  $m_{ij}$ ,  $m_{ji}$ , etc., using the pairing (12.17) (this gives a factor  $\pm K'(m_{ij}, m_{ji})K'(m_{jk}, m_{kj})K'(m_{ki}, m_{ik})$ ) and integrate over the other variables using the natural measure  $\mathcal{D}(x_i, dx_i)$  for integration over the pairs of bosonic and fermionic variables  $x_i$  and  $dx_i$ . (So for a function  $f(x, dx)$  defined on  $\mathbb{R}$ , the integral  $\int_{\mathbb{R}} \mathcal{D}(x, dx) f(x, dx)$  is simply the integral over  $\mathbb{R}$  of the differential form  $f(x, dx)$ .)  $Z_{\Psi'}$  is the  $\zeta$ -instanton amplitude to create the outgoing solitons in the state  $\Psi'$ . As usual in an  $A$ -model (with or without a superpotential) this amplitude is the answer to a counting question ( $A$ -model “counting” is always weighted by the sign of a fermion determinant). In the present example, what we have counted (the answer being  $\pm 1$ ) is the number of  $\zeta$ -instantons with the given fan asymptotics that create outgoing solitons with impact parameters obeying the constraints  $x_1 = c_1$  and  $x_2 = c_2$ .

A final comment is that the relation

$$d\Psi = 0, \quad (14.23)$$

which expresses the underlying  $\mathcal{Q}_\zeta$ -invariance, is needed to ensure that the amplitude (14.22) is invariant under  $\Psi' \rightarrow \Psi' + d\chi$  for any  $\chi$ .

Another instructive example is sketched in Figure 137. Here we consider a  $\zeta$ -web with four external soliton lines and four vertices that we will assume rigid; hence this  $\zeta$ -web has a four-dimensional moduli space. The impact parameters  $x_1, \dots, x_4$  of the external solitons can be varied independently, but subject to inequalities that come from the fact that the lengths of the edges in the  $\zeta$ -web must all be positive. Thus the wavefunction  $\Psi$  is nonzero only in a certain (noncompact) polytope  $U \subset \mathbb{R}^4$ . Within  $U$ ,  $\Psi = \pm 1$  (times

a state  $\otimes_{j=1}^4 m_{i_j i_{j+1}}$  in the appropriate reduced complex), expressing the fact that if we specify the desired values of the  $x_i$ , there is a unique point in the given  $\zeta$ -web moduli space that yields these values. (This statement assumes that we keep away from the boundaries of  $U$  by an amount much greater than  $1/m$ , the inevitable error in any statement based on a  $\zeta$ -web.) So the wavefunction of the outgoing solitons is

$$\Psi = \pm \Theta_U \otimes_{j=1}^4 m_{i_j i_{j+1}}, \quad (14.24)$$

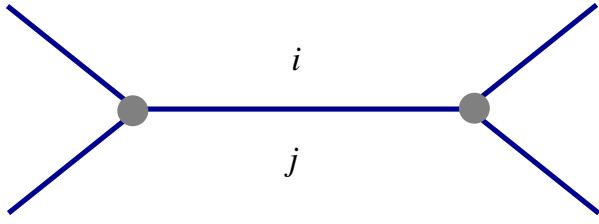
where  $\Theta_U$  is the characteristic function of the region  $U$ , the sign depends on the sign of a fermion determinant, and  $\otimes_j m_{i_j i_{j+1}}$  is the  $\mathcal{Q}$ -invariant state in the reduced complex  $\mathbb{M}'_{i_j i_{j+1}}$  corresponding to the specific outgoing solitons. We immediately see that this wavefunction is not closed:  $d\Psi$  is a delta function supported on the boundaries of  $U$ . In fact, as  $U$  has codimension 0 in  $\mathbb{R}^4$ , the characteristic function  $\Theta_U$  is the Poincaré dual to  $U$  as defined in eqn. (14.19), and a special case of eqn. (14.21) tells us that

$$d\Theta_U = \Theta_{\partial U}, \quad (14.25)$$

where now  $\partial U$  has codimension 1 so  $\Theta_{\partial U}$  is a 1-form. Even though the formula (14.24) is precisely valid only away from the boundaries of  $U$ , modifying  $\Psi$  only near those boundaries will not help in achieving  $d\Psi = 0$ . After all, for a zero-form  $\Psi$  to be annihilated by  $d$ , it must be constant throughout  $\mathbb{R}^4$ .

What is going on here is that the  $\zeta$ -web in question really represents one end of a four-dimensional moduli space  $\mathcal{M}$  of  $\zeta$ -instantons with fan-like asymptotics. If  $\mathcal{M}$  has only one end corresponding to the  $\zeta$ -web in the figure, we will indeed get a contradiction. There must then be other ends of the moduli space that either cancel  $\Psi$  or complete it to a closed 0-form on  $\mathbb{R}^4$ . This paper is really based on the possibility that this cancellation occurs in an interesting way, such that the interior amplitudes defined and studied in section (14.6) are not identically zero. A less interesting but logical possibility is that one of the rigid  $\zeta$ -vertices in Figure 137 can be replaced by a second  $\zeta$ -vertex with the same asymptotics but contributing with the opposite sign of the fermion determinant. (Remember that a choice of a component of the moduli space is part of the definition of a  $\zeta$ -vertex.) This case is less interesting, in the sense that if it occurs, the two  $\zeta$ -vertices in question will always make canceling contributions in all of our considerations. A final comment is that hypothetical ultraviolet ends of  $\mathcal{M}$  – ends supported on a region in which one or more of the internal lines in Figure 137 collapses to a length of order  $1/m$  – could not help in correcting  $\Psi$  to satisfy  $d\Psi = 0$ . But as explained in section 14.2, as long as the corresponding  $A$ -model under discussion exists as a topological field theory, we do not expect contributions from such ultraviolet ends.

Finally, we discuss what happens to the collective coordinates of a given soliton line under gluing. To be specific, consider the  $\zeta$ -web in Figure 138 showing a solution made by convolution of two sub-solutions connected by a single soliton. This soliton is of type  $ij$  if viewed as propagating from left to right in the figure, or of type  $ji$  if we consider it to be propagating from right to left. It has collective coordinates  $x, dx$ . Write generically  $y, dy$  for the collective coordinates of solitons emerging from the subsolution on the left and



**Figure 138:** A  $\zeta$ -web with four external solitons and a four-dimensional moduli space. The impact parameters can be varied independently but only in a certain region in  $\mathbb{R}^4$ .

$z, dz$  for the collective coordinates of solitons emerging on the right. The subsolution on the left creates a fan of solitons in a state that we write generically as  $\Psi_\ell(x, dx; y, dy)$  and the subsolution on the right creates a fan of solitons in a state that we write generically as  $\Psi_r(x, dx; z, dz)$ . The  $\zeta$ -web in the figure has fan-like asymptotics with the outgoing solitons described by the whole collection of variables  $y, dy; z, dz$ . Cluster decomposition in a massive theory tells us that the wavefunction of the solitons emerging from the whole web is the product of the left and right wavefunctions with the collective coordinates  $x, dx$  of the internal soliton integrated out:

$$\Psi(y, dy; z, dz) = \int_{\mathbb{R}} \mathcal{D}(x, dx) \left( \Psi_\ell(x, dx; y, dy), \Psi_r(x, dx; z, dz) \right). \quad (14.26)$$

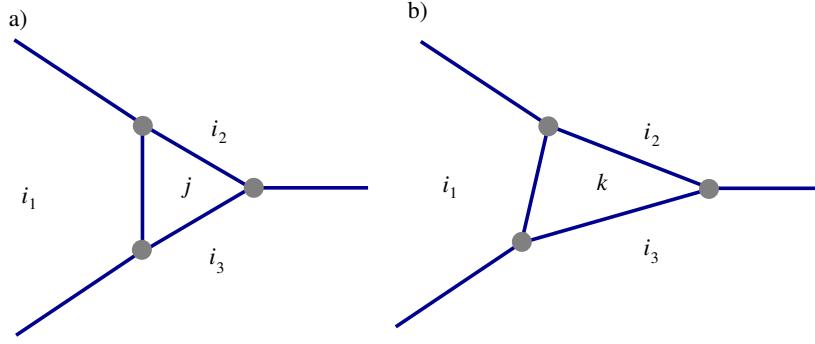
where, again, the pairing on the integrand uses  $K'$ .

Let us formalize this result. The variables  $x$  and  $dx$  are collective coordinates of an  $ij$  soliton emerging from the subsolution on the left of the figure. This  $ij$  soliton is a vector in the reduced complex  $\mathbb{M}'_{ij}$  of classical  $ij$  solitons. Similarly, the  $ji$  soliton emerging from the subsolution on the right of the figure is a vector in the space  $\mathbb{M}'_{ji}$  of classical  $ji$  solitons. The pairing between these states is the nondegenerate pairing of eqn. (12.17). We recall that this pairing is nondegenerate because whatever  $ij$   $\zeta$ -soliton emerges from the left subsolution in the figure can be paired with a unique  $ji$   $\zeta$ -soliton, namely the “same” classical solution rotated by an angle  $\pi$ .

It remains to discuss the integration over the bosonic collective coordinate  $x$ . Let  $\mathcal{M}_\ell$  and  $\mathcal{M}_r$  be the moduli of the sub-solutions on the left and right of the figure and let  $\mathcal{M}$  be the moduli space of the overall  $\zeta$ -web. For a given choice of a point in  $\mathcal{M}_\ell$ , the  $ij$  soliton connecting the two parts of the figure has a definite impact parameter, so it has a wavefunction  $\delta(x - a)$ , where  $a$  is a function on  $\mathcal{M}_\ell$ . Similarly, the wavefunction of the  $ji$  soliton emerging from the right is proportional to  $\delta(x - b)$ , where  $b$  is a function on  $\mathcal{M}_r$ . The integral over  $x$  gives

$$\int_{\mathbb{R}} dx \delta(x - a) \delta(x - b) = \delta(a - b). \quad (14.27)$$

The constraint that  $a = b$  in order for the  $\zeta$ -web in the figure to exist means that  $\dim \mathcal{M} = \dim \mathcal{M}_\ell + \dim \mathcal{M}_r - 1$ . The term  $-E$  in eqn. (14.14) arises because each internal line in a  $\zeta$ -web makes in this way a contribution  $-1$ .



**Figure 139:** These two taut  $\zeta$ -webs might arise as the ends of the same 1-dimensional reduced moduli space  $\mathcal{M}_{\text{red}}$ . (In general, the ends of  $\mathcal{M}_{\text{red}}$  might not be the same topologically, but in this example they are.)

One last comment is that in all of these examples, because we have assumed all  $\zeta$ -vertices to be rigid, there are no normalizable moduli. All deformations of the  $\zeta$ -webs that we have considered change the impact parameters of the external solitons. In the case of a  $\zeta$ -web that has a modulus that can be varied without changing the impact parameters – for example, a non-rigid  $\zeta$ -vertex – there is a normalizable fermion zero-mode. Unless we insert a suitable vertex operator to absorb this zero-mode (and Landau-Ginzburg models with target  $\mathbb{C}^n$  do not have such operators), an amplitude derived from a web with a normalizable zero-mode vanishes. That is why we can restrict our attention here to  $\zeta$ -vertices of zero excess dimension.

#### 14.6 Interior Amplitudes And The Relations They Obey

The goal of this section is to explain the origin of the fundamental relation for representations of plane webs that was presented in eqn. (4.15). Rather like the proof that  $\mathcal{Q}_\zeta^2 = 0$  in the context of the MSW complex (section 10.6), the basic idea is to consider the two ends of 1-dimensional reduced moduli spaces.

Let  $i_1 i_2 \dots i_n$  be a cyclic fan of vacua and let  $\mathbb{M}'_{i_1 \dots i_n} = \mathbb{M}'_{i_1 i_2} \otimes \dots \otimes \mathbb{M}'_{i_n i_1}$  be the tensor product of the reduced complexes of the outgoing solitons. The impact parameters of the outgoing solitons define a point in  $\mathbb{R}^n$ , with one copy of  $\mathbb{R}$  for each soliton. We write  $\Omega^*(\mathbb{R}^n)$  for the space of differential forms on  $\mathbb{R}^n$ .

If  $\mathcal{V}$  is a rigid  $\zeta$ -vertex that is asymptotic to the given fan of vacua, then the path integral associated to this family of  $\zeta$ -instantons determines an element  $B(\mathcal{V}) \in \Omega^*(\mathbb{R}^n) \otimes \mathbb{M}'_{i_1, \dots, i_n}$ . Examples were discussed in section 14.5. A small simplification is that, dividing by overall translations,  $\mathbb{R}^n$  projects to a reduced parameter space  $\mathbb{R}^{n-2}$  and  $B(\mathcal{V})$  is actually always the pullback of an element  $B_{\text{red}}(\mathcal{V}) \in \Omega^*(\mathbb{R}^{n-2}) \otimes \mathbb{M}'_{i_1, \dots, i_n}$ , which we call the reduced state. Some statements are more transparent in terms of  $B_{\text{red}}(\mathcal{V})$ . We note that  $\mathbb{R}^{n-2}$  has a natural origin  $\mathbf{o}$ , corresponding to solitons that all emanate from a common point in  $\mathbb{R}^2$ . Since  $\mathcal{V}$  is rigid, its reduced moduli space is 0-dimensional and  $B_{\text{red}}(\mathcal{V})$  is actually a simple

product

$$B_{\text{red}}(\mathcal{V}) = \Theta_{\mathbf{o}} \cdot B'_{\text{red}}(\mathcal{V}), \quad (14.28)$$

where  $\Theta_{\mathbf{o}}$  is an  $n - 2$ -form with delta function support at  $\mathbf{o} \in \mathbb{R}^{n-2}$  and  $B'_{\text{red}}(\mathcal{V}) \in \mathbb{M}'_{i_1, \dots, i_n}$ . To be more precise,  $B_{\text{red}}(\mathcal{V})$  takes this form within the usual error of order  $1/m$ .  $B_{\text{red}}(\mathcal{V})$  differs from the expression just given by an exact form  $d\chi$ , where  $\chi$  vanishes exponentially fast at distances greater than  $1/m$  from the point  $\mathbf{o}$ .

We will say that a component  $\mathcal{M}$  of the moduli space of  $\zeta$ -instantons is “taut” if (i) it is three-dimensional, so that the corresponding reduced dimension  $\mathcal{M}_{\text{red}} = \mathcal{M}/\mathbb{R}^2$  has dimension 1; and (ii) this reduced space is a copy of  $\mathbb{R}$ . The reason for the definition is that  $\zeta$ -instanton moduli spaces that are taut in this sense will play a role somewhat analogous to the role played by taut webs. Indeed, if  $\mathcal{M}$  is taut, then  $\mathcal{M}_{\text{red}} \cong \mathbb{R}$  has two ends (Figure 139). Each of these ends corresponds to what we will call a taut  $\zeta$ -web, that is a  $\zeta$ -web with a 1-dimensional reduced moduli space. (If  $\mathcal{M}$  obeys condition (i) but not condition (ii), then  $\mathcal{M}_{\text{red}}$  is a copy of  $S^1$ . In this case,  $\mathcal{M}$  is a non-rigid  $\zeta$ -vertex with an excess dimension of 1. Such components of  $\zeta$ -instanton moduli space will play no role in the present section.)

Suppose that  $\mathcal{Z}$  is a taut  $\zeta$ -web asymptotic to the fan of vacua  $i_1 \dots i_n$ . ( $\mathcal{Z}$  is actually asymptotic at infinity to a specific fan of solitons, but it is more convenient here simply to specify the fan of vacua.) Then the path integral for this family of  $\zeta$ -instantons determines a state  $\Psi_{\mathcal{Z}} \in \Omega^*(\mathbb{R}^n) \otimes \mathbb{M}'_{i_1 \dots i_n}$ , which as in the case of a rigid  $\zeta$ -web is a pullback of a reduced state  $\Psi_{\mathcal{Z}, \text{red}} \in \Omega^*(\mathbb{R}^{n-2}) \otimes \mathbb{M}'_{i_1 \dots i_n}$ . Moreover this state is supported on the 1-dimensional reduced moduli space of the web  $\mathcal{Z}$ . By scale-invariance of webs, this reduced moduli space is a ray  $\mathbf{r}$  starting at the origin  $\mathbf{o} \in \mathbb{R}^{n-2}$  (to within the usual error of order  $1/m$ ). As in section 14.5, the state determined by  $\mathcal{Z}$  is the Poincaré dual of  $\mathbf{r}$  times a state  $\Psi_{\mathcal{Z}}^* \in \mathbb{M}'_{i_1 \dots i_n}$ :

$$\Psi_{\mathcal{Z}, \text{red}} = \Theta_{\mathbf{r}} \cdot \Psi_{\mathcal{Z}}^*. \quad (14.29)$$

It immediately follows from this that  $\Psi_{\mathcal{Z}, \text{red}}$  is not closed. We have  $d\Theta_{\mathbf{r}} = \Theta_{\partial \mathbf{r}} = \Theta_{\mathbf{o}}$  (since the boundary of the ray  $\mathbf{r}$  consists of its endpoint  $\mathbf{o}$ ), so

$$d\Psi_{\mathcal{Z}, \text{red}} = \Theta_{\mathbf{o}} \cdot \Psi_{\mathcal{Z}}^*. \quad (14.30)$$

(We have  $d\Psi_{\mathcal{Z}}^* = 0$  since  $\Psi_{\mathcal{Z}}^*$  is just a fixed state in  $\mathbb{M}'_{i_1 \dots i_n}$  corresponding to the relevant fan of solitons.)

The reduced state associated to the whole moduli space  $\mathcal{M}$  will be closed because of the underlying  $\mathcal{Q}_\zeta$ -invariance;  $\Psi_{\mathcal{Z}, \text{red}}$ , which is not closed, is the contribution of just one end of the moduli space.  $\mathcal{M}_{\text{red}}$  has precisely 1 additional end, corresponding to another taut<sup>90</sup> web  $\mathcal{Z}'$ . The state associated to this second taut web is supported on another ray  $\mathbf{r}'$  from the origin, and has the form

$$\Psi_{\mathcal{Z}', \text{red}} = \Theta_{\mathbf{r}'} \cdot \Psi_{\mathcal{Z}'}^*, \quad (14.31)$$

---

<sup>90</sup> Actually, this taut web could be what in the abstract part of this paper was called a taut extended web: that is, it could be constructed from a rigid  $\zeta$ -vertex with a  $\zeta$ -instanton correction to the MSW complex on one of the external lines.

for some  $\Psi_{\mathcal{Z}'}^* \in \mathbb{M}'_{i_1 \dots i_n}$ . Hence

$$d\Psi_{\mathcal{Z}',\text{red}} = \Theta_{\mathbf{o}} \cdot \Psi_{\mathcal{Z}'}^*. \quad (14.32)$$

The condition that  $d(\Psi_{\mathcal{Z},\text{red}} + \Psi_{\mathcal{Z}',\text{red}}^*) = 0$  is thus simply

$$\Psi_{\mathcal{Z}}^* + \Psi_{\mathcal{Z}'}^* = 0. \quad (14.33)$$

This is the identity that leads to the basic algebraic relation for plane webs that was proposed in eqn. (4.15).

Before explaining this claim, we pause to point out that it is oversimplified to expect that the reduced state produced by the moduli space  $\mathcal{M}$  is precisely  $\Psi_{\mathcal{Z},\text{red}} + \Psi_{\mathcal{Z}',\text{red}}$ . The  $\zeta$ -webs  $\mathcal{Z}$  and  $\mathcal{Z}'$  are really only well-defined when they are large, in other words far from the origin  $\mathbf{o} \in \mathbb{R}^{n-2}$ . The state  $\widehat{\Psi}_{\mathcal{M}}$  is not just the pullback of  $\Psi_{\mathcal{Z},\text{red}} + \Psi_{\mathcal{Z}',\text{red}}$ ; it is the pullback of a state  $\widehat{\Psi}_{\mathcal{M},\text{red}}$  that differs from  $\Psi_{\mathcal{Z},\text{red}} + \Psi_{\mathcal{Z}',\text{red}}$  by a state  $\chi$  supported near the point  $\mathbf{o}$ . The reason for the identity (14.33) is that without this identity, the condition  $d(\Psi_{\mathcal{Z},\text{red}} + \Psi_{\mathcal{Z}',\text{red}} + \chi) = 0$  is not satisfied for any compactly supported  $\chi$ . (The obstruction comes from the fact that  $\Theta_{\mathbf{o}}$ , since its integral over  $\mathbb{R}^{n-2}$  is non-zero, represents a non-zero element in the compactly supported cohomology of  $\mathbb{R}^{n-2}$ , so it is not  $d\chi$  for any compactly supported  $\chi$ . Here it does not matter if “compact support” is replaced by “exponential decay at distances large compared to  $1/m$ .”)

To understand the identity (14.33), let us first note that it trivially implies a relation for the sum of contributions of all taut webs asymptotic to the fan  $i_1 i_2 \dots i_n$  of vacua. Let  $\mathcal{S}_{i_1 i_2 \dots i_n}$  be the set of all taut  $\zeta$ -webs asymptotic to the given fan. Then by virtue of (14.33),

$$\sum_{\mathcal{Z} \in \mathcal{S}_{i_1 \dots i_n}} \Psi_{\mathcal{Z}}^* = 0. \quad (14.34)$$

Indeed, each taut  $\zeta$ -web with the given asymptotics is an end of a unique 1-dimensional reduced moduli space of  $\zeta$ -instantons, and, because of (14.33), the two taut  $\zeta$ -webs that are the ends of this reduced moduli space make canceling contributions in (14.34).

The sum in eqn. (14.34) is reminiscent of the sum over ordinary taut webs with given asymptotics in eqn. (4.15). To understand the relationship, we have to analyze the state  $\Psi_{\mathcal{Z}}^* \in \mathbb{M}'_{i_1 \dots i_n}$ . Here the moduli  $\mathcal{Z}$  are not relevant; by definition,  $\Psi_{\mathcal{Z}}^*$  is a state in the product  $\mathbb{M}'_{i_1 i_2} \otimes \dots \otimes \mathbb{M}'_{i_n i_1}$  of the reduced complexes. To determine this state, we can work near infinity in the reduced moduli space of  $\mathcal{Z}$ : thus  $\mathcal{Z}$  is built from widely separated rigid  $\zeta$ -vertices  $\mathcal{V}_a$ ,  $a = 1, \dots, t$ , connected by soliton lines. The path integral for each  $\zeta$ -vertex  $\mathcal{V}_a$  determines a state  $B(\mathcal{V}_a)$  of the solitons emanating from this vertex. When two  $\zeta$ -vertices  $\mathcal{V}_a$  and  $\mathcal{V}_b$  are connected by a soliton line in the  $\zeta$ -web  $\mathcal{Z}$ , this means that the corresponding soliton states emanating from  $\mathcal{V}_a$  and  $\mathcal{V}_b$  have to be contracted via the nondegenerate degree 1 pairing (12.17). The combined operation of computing a state  $B(\mathcal{V}_a)$  and then contracting these states whenever two vertices are joined by lines is precisely the operation  $\rho$  defined in (4.7) et. seq., used in formulating the fundamental algebraic identity (4.15) (this interpretation of  $\rho$  was the motivation for the way it was defined). The terms  $\Psi_{\mathcal{Z}}^*$  that contribute in the identity (14.34) are in natural correspondence with contributions to

the identity (4.15) of the abstract part of the paper: in either case, a contribution is made by arranging rigid vertices as the vertices of a taut web, computing states associated to the vertices and contracting those states whenever two vertices are joined by a line.

In relating our present analysis to the abstract discussion, the object called  $\beta$  in the abstract discussion should be defined as a sum over all rigid  $\zeta$ -vertices  $\mathcal{V}_a$ . If  $\mathcal{V}_a$  is asymptotic to the fan of vacua  $j_1^a \dots j_{s_a}^a$ , and the corresponding solitons have center of mass coordinates  $x_{j_1^a}, \dots, x_{j_s^a}$ , then

$$\beta = \sum_a dx_{j_1^a} \dots dx_{j_s^a} B'_{\text{red}}(\mathcal{V}_a). \quad (14.35)$$

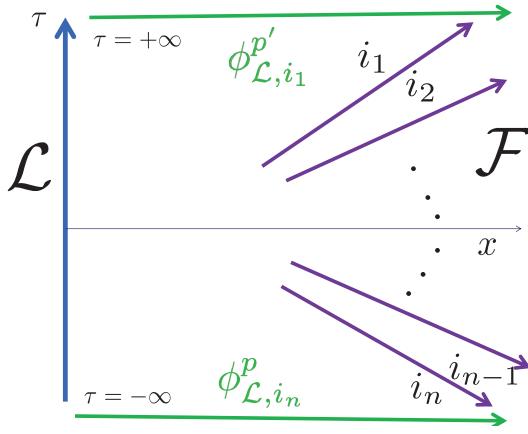
See eqn. (14.28) for the definition of  $B'_{\text{red}}(\mathcal{V}_a)$ . Now,  $B_{\text{red}}$ , being the state produced by the path integral in a fermion-number conserving theory, has  $\mathcal{F} = 0$ ; since  $\Theta_0$  is an  $(n - 2)$ -form,  $B'_{\text{red}}$  has fermion number  $-(n - 2)$ . Including the factors of  $dx_{j_k^a}$  increases the fermion number by  $n$ , so  $\beta$  has fermion number 2, as in the definition (4.15) in Section 4.1. The factors  $dx_{j_k^a}$  have been included in the definition of  $\beta$  for the following reason. In the abstract algebraic treatment, it is convenient to imagine that a classical soliton that has two states of fermion number  $f, f + 1$  is always created in the state of fermion number  $f + 1$ . In the more microscopic path integral treatment, the truth is more complex, as we see in eqn. (14.18), and in the above derivation, it was actually more convenient to use reduced states of fermion number  $f$ . To compensate for this, we include a factor of  $dx$  for each soliton in the formula for  $\beta$ . This also means that the pairing used in the abstract description is not the degree 1 pairing (12.17) but is the degree  $-1$  pairing  $K$  that is obtained by composing (12.17) with the operation that removes the fermion zero-mode  $dx$  from each soliton state.

At this point, using the Landau-Ginzburg theory based on  $(X, W)$ , we have determined vacuum data  $(\mathbb{V}, z)$  (from the superpotential), web representations  $\mathcal{R}$  (from equations (12.15) and (12.18)), and an interior amplitude  $\beta$  (from equation (14.35)), thus defining a Theory in the formal sense of Section §4.1. We have accordingly recovered all the data needed to form the vacuum category  $\mathfrak{Vac}$  using the construction of Section §5.1, once we choose a half-plane  $\mathcal{H}$ . We let  $\mathfrak{Vac}(X, W)$  denote the category for the positive half-plane so that  $\mathfrak{Vac}^{\text{opp}}(X, W)$  is the category for the negative half-plane. It now follows from the purely formal constructions of Section §5.2 above that there is a corresponding  $A_\infty$  category of Branes  $\mathfrak{Br}(X, W)$ , and it is natural to wonder if the corresponding  $A_\infty$  amplitudes are equivalent to those that were constructed in section 11.3 for branes of class  $T_\kappa$ . We will discuss this question in detail in Section §15.

## 14.7 $\zeta$ -Instantons On A Half-Space Or A Strip

### 14.7.1 Preliminaries

In this section, we will first analyze  $\zeta$ -instantons on the half-plane  $x \geq 0$  in the  $x - \tau$  plane, with boundary conditions set by a Lagrangian submanifold  $\mathcal{L}$ . After some preliminaries, we explain in section 14.7.2 how  $\zeta$ -instantons can be used to define a boundary amplitude  $\mathcal{B}$  as introduced in section 4.2 and obeying the key identity 4.47. The considerations here



**Figure 140:** Boundary conditions for general half-plane instantons with fan boundary conditions at  $x \rightarrow +\infty$  and solitons at  $\tau \rightarrow \pm\infty$ .

are very similar to those of sections 14 and 14.6. Then we consider  $\zeta$ -instantons on a strip in section 14.7.3.

We began section 14 by showing that in any massive theory, there are no localized  $\zeta$ -instantons on  $\mathbb{R}^2$ . The analog in a half-plane is more subtle. To formulate the question, we pick a critical point  $\phi_i$  corresponding to a vacuum  $i \in \mathbb{V}$  at  $x \rightarrow \infty$ , and we also choose a half-line  $\zeta$ -soliton  $\phi_{\mathcal{L},i}$  that interpolates from  $\mathcal{L}$  at  $x = 0$  to vacuum  $i$  at  $x = \infty$ . Then we ask if there are  $\zeta$ -instantons on the half-plane that map the  $\tau$ -axis to  $\mathcal{L}$ , approach  $\phi_{\mathcal{L},i}$  for  $\tau \rightarrow \pm\infty$ , and approach  $\phi_i$  for  $x \rightarrow \infty$ . Such a  $\zeta$ -instanton is localized in the sense that it differs substantially from the chosen  $\zeta$ -soliton only in a localized region of the half-plane. In general, depending on  $\mathcal{L}$ , there may be such localized half-plane  $\zeta$ -instantons. We have seen an example in section 13.5. As we learned there, when localized half-plane instantons exist, it often means that the brane in question is not a valid  $A$ -brane and our methods do not apply in its presence.

In this section, we will not consider such localized half-plane  $\zeta$ -instantons. This does not mean that we have to assume they do not exist; they are not relevant to the questions we will consider here. When moduli spaces of localized half-plane  $\zeta$ -instantons exist (but the brane in question is a valid  $A$ -brane), they are somewhat like  $\zeta$ -instanton moduli spaces with positive excess dimension, relevant only if one inserts local  $A$ -model observables. A simple criterion that ensures that there are no localized half-plane  $\zeta$ -instantons is that the superpotential  $h$  of eqn. (11.21) is single-valued. In particular, this is so if the symplectic form  $\omega$  of  $X$  is exact and the Lagrangian submanifold  $\mathcal{L}$  is also exact (for these notions, and their significance, see the discussion of eqn. (11.24); the assumptions of exactness are often made mathematically in the Fukaya-Seidel category). Since the  $\zeta$ -instanton equation is gradient flow for  $h$ ,  $h$  strictly increases with  $\tau$  in any non-trivial  $\zeta$ -instanton. So single-valuedness of  $h$  implies that there are no non-trivial  $\zeta$ -instantons beginning and ending at the same  $\zeta$ -soliton  $\phi_{\mathcal{L},i}$ .

On the other hand, if  $p$  and  $p'$  are two distinct intersection points of  $\mathcal{L}$  with the right thimble of type  $i$ , corresponding to two different half-line solitons, then there can be nontrivial  $\zeta$ -instantons with the boundary condition that  $\phi \rightarrow \phi_{\mathcal{L},i}^p$  for  $\tau \rightarrow -\infty$  and  $\phi \rightarrow \phi_{\mathcal{L},i}^{p'}$  for  $\tau \rightarrow +\infty$ . Indeed, such instantons of fermion number one define the differential on the MSW complex  $\mathbb{M}_{\mathcal{L},i}$ . They are localized both in  $\tau$  and near the boundary and correspond to boundary vertices of valence zero in the extended webs of the web-based formalism. Recall Figure 132.

In analogy with Section §14, our main interest is in  $\zeta$ -instantons on the half-space that are asymptotic to a half-plane fan of solitons, as sketched in Figure 140, with  $i_1 \neq i_n$ . We trust that this notion is clear: as in the figure, we start with a half-plane fan of vacua  $i_1, \dots, i_n$  and then choose suitable half-line  $\zeta$ -solitons at  $\tau \rightarrow \pm\infty$ , and suitable boosted  $\zeta$ -solitons along outgoing lines separating the vacua in the fan. Generally speaking, all notions of Section §14.2 have analogs in this situation. If  $\mathcal{M}$  is a component of the moduli space<sup>91</sup> of half-space  $\zeta$ -instantons with fan-like asymptotics, then the group  $\mathbb{R}$  of time translations acts freely on  $\mathcal{M}$ ; we define the reduced moduli space  $\mathcal{M}_{\text{red}} = \mathcal{M}/\mathbb{R}$ . We call a connected component of  $\mathcal{M}$  a half-space  $\zeta$ -vertex if the corresponding reduced component in  $\mathcal{M}_{\text{red}}$  is compact. If in addition that component is a point, we call it a rigid  $\zeta$ -vertex; if  $\mathcal{M}_{\text{red}}$  has positive dimension, we call its dimension the excess dimension of  $\mathcal{M}$ . We can write again a formula for the expected dimension of  $\mathcal{M}$  in an  $\mathcal{F}$ -conserving system in terms of the upper fermion numbers  $f_a$  of the solitons in the fan and  $f^{p'}, f^p$  of the half-line soliton states  $|p\rangle$  and  $|p'\rangle$ :

$$\dim \mathcal{M} = f^{p'} - f^p + \sum_a (f_a + 1) \quad (14.36)$$

In general, if  $\mathcal{M}_{\text{red}}$  is not compact, it has “ends” that can be constructed by gluing of sub-solutions, as we discussed for  $\zeta$ -instantons with fan-like asymptotics on  $\mathbb{R}^2$  starting in Section §14.2. The sub-solutions can now be  $\zeta$ -instantons with fan-like asymptotics on either  $\mathbb{R}^2$  or the half-space. Thus, in general the “ends” of  $\mathcal{M}$  correspond to half-space  $\zeta$ -webs. After repeatedly resolving the sub-solutions into  $\zeta$ -webs, we eventually learn that any end of  $\mathcal{M}$  can be built by gluing of  $\zeta$ -vertices  $\mathcal{V}_J$ , arranged in a  $\zeta$ -web  $\mathfrak{u}$ . As in eqn. (14.15), the dimension of  $\mathcal{M}$  exceeds the dimension of the moduli space of the half-space web  $\mathfrak{u}$  (for any  $\mathfrak{u}$  associated to an end of  $\mathcal{M}$ ) by the sum of the excess dimensions of the  $\zeta$ -vertices  $\mathcal{V}_J$ .

#### 14.7.2 Boundary Amplitudes And The Relations They Obey

The general analysis of boundary amplitudes and the relations they obey is very similar to what we said in action 14.6 concerning interior amplitudes – so similar that we will be brief.

We start with a boundary condition associated to a Lagrangian submanifold  $\mathcal{L}$ , with a half-plane fan of vacua  $i_1, \dots, i_n$ , as in Figure 140. To this data, we have complexes  $\mathbb{M}_{\mathcal{L},i_1}$  and  $\mathbb{M}_{\mathcal{L},i_n}$  of initial and final classical half-plane states, and reduced complexes

---

<sup>91</sup>As discussed most fully in section 14.3, in defining  $\mathcal{M}$ , we do not specify the impact parameters of outgoing solitons.

$\mathbb{M}'_{i_1 i_2}, \dots, \mathbb{M}'_{i_{n-1} i_n}$  of classical outgoing solitons. The tensor product

$$\mathbb{M}_{i_1 \dots i_n}^{\mathcal{L}} = \mathbb{M}_{\mathcal{L}, i_1} \otimes \mathbb{M}'_{i_1 i_2} \otimes \dots \otimes \mathbb{M}'_{i_{n-1} i_n} \otimes (\mathbb{M}_{\mathcal{L}, i_n})^* \quad (14.37)$$

has a basis corresponding to choices of classical BPS half-line and soliton states. The spaces  $\mathbb{M}_{\mathcal{L}, i}$  of half-line solitons play the role of the Chan-Paton spaces  $\mathcal{E}_i$  of the abstract discussion. The impact parameters of the  $n - 1$  outgoing solitons define a point in  $\mathbb{R}^{n-1}$ , and we write  $\Omega^*(\mathbb{R}^{n-1})$  for the corresponding space of differential forms.

If  $\mathcal{V}$  is a rigid half-plane  $\zeta$ -vertex that is asymptotic to the given fan of vacua, then the corresponding path integral defines an element  $B(\mathcal{V}) \in \Omega^*(\mathbb{R}^{n-1}) \otimes \mathbb{M}_{i_1 \dots i_n}^{\mathcal{L}}$ . Dividing by time translations,  $\mathbb{R}^{n-1}$  projects to  $\mathbb{R}^{n-2}$  and  $B(\mathcal{V})$  is the pullback of an element  $B_{\text{red}}(\mathcal{V}) \in \Omega^*(\mathbb{R}^{n-2}) \otimes \mathbb{M}_{i_1 \dots i_n}^{\mathcal{L}}$ . This copy of  $\mathbb{R}^{n-2}$  has a natural origin  $\mathbf{o}$ , corresponding to a collection of  $n - 1$  solitons that all emanate from a common point on  $\mathbb{R}$ , the boundary of the half-plane. Since  $\mathcal{V}$  is rigid, its reduced moduli space is 0-dimensional and  $B_{\text{red}}(\mathcal{V})$  is a simple product

$$B_{\text{red}}(\mathcal{V}) = \Theta_{\mathbf{o}} \cdot B'_{\text{red}}(\mathcal{V}), \quad (14.38)$$

where  $\Theta_{\mathbf{o}}$  is Poincaré dual to the point  $\mathbf{o} \in \mathbb{R}^{n-2}$  and  $B'_{\text{red}}(\mathcal{V}) \in \mathbb{M}_{i_1 \dots i_n}^{\mathcal{L}}$ . (As usual, such a statement holds modulo a correction  $d\chi$ , where  $\chi$  vanishes rapidly at infinity.)

We say that a component  $\mathcal{M}$  of the moduli space of half-plane  $\zeta$ -instantons is “taut” if (i) it is two-dimensional, so that the reduced space  $\mathcal{M}_{\text{red}} = \mathcal{M}/\mathbb{R}$  has dimension 1; (ii) this reduced space is a copy of  $\mathbb{R}$ . Such components play a role analogous to that played by taut half-plane webs in the abstract discussion of section 4.2. In this situation, each of the two ends of  $\mathcal{M}_{\text{red}} \cong \mathbb{R}$  corresponds to what we will call a taut half-plane  $\zeta$ -web, by which we mean a half-plane  $\zeta$ -web with a 1-dimensional reduced moduli space.

From here, the goal is to show that the objects  $B'_{\text{red}}(\mathcal{V})$  (after dressing as in eqn. (14.35) with 1-forms for outgoing solitons) satisfy the fundamental identity (4.47) or (4.49) of a boundary amplitude, as formulated abstractly in section 4.1. The argument will simply follow what we said for interior amplitudes in section 14.6.

Let  $\mathcal{M}$  be a taut family of half-plane  $\zeta$ -instantons, and let  $\mathcal{Z}, \mathcal{Z}'$  be taut  $\zeta$ -webs representing the two ends of  $\mathcal{M}_{\text{red}}$ . The path integral associated to the family  $\mathcal{Z}$  of  $\zeta$ -instantons determines a state  $\Psi_{\mathcal{Z}} \in \Omega^*(\mathbb{R}^{n-1}) \otimes \mathbb{M}_{i_1 \dots i_n}^{\mathcal{L}}$ , which is the pullback of a reduced state  $\Psi_{\mathcal{Z}, \text{red}} \in \Omega^*(\mathbb{R}^{n-2}) \otimes \mathbb{M}_{i_1 \dots i_n}^{\mathcal{L}}$ . Just as in (14.29), this state is supported on a ray  $\mathbf{r}$  emanating from the origin  $\mathbf{o} \in \mathbb{R}^{n-2}$ :

$$\Psi_{\mathcal{Z}, \text{red}} = \Theta_{\mathbf{r}} \cdot \Psi_{\mathcal{Z}}^*, \quad \Psi_{\mathcal{Z}}^* \in \mathbb{M}_{i_1 \dots i_n}^{\mathcal{L}}. \quad (14.39)$$

Hence

$$d\Psi_{\mathcal{Z}, \text{red}} = \Theta_{\mathbf{o}} \cdot \Psi_{\mathcal{Z}}^*. \quad (14.40)$$

We can make the same construction for  $\mathcal{Z}'$ ;  $\Psi_{\mathcal{Z}', \text{red}}$  is supported on another ray  $\mathbf{r}'$  from  $\mathbf{o}$ , and again

$$d\Psi_{\mathcal{Z}', \text{red}} = \Theta_{\mathbf{o}} \cdot \Psi_{\mathcal{Z}'}^*, \quad (14.41)$$

for some  $\Psi_{\mathcal{Z}'}^* \in \mathbb{M}_{i_1 \dots i_n}^{\mathcal{L}}$ . The state determined by the full moduli space  $\mathcal{M}$  must be closed, and this tells us that

$$\Psi_{\mathcal{Z}}^* + \Psi_{\mathcal{Z}'}^* = 0, \quad (14.42)$$

just as in eqn. (14.33).

From eqn. (14.42), we can deduce a result analogous to eqn. (14.34). Let  $\mathcal{S}_{i_1 \dots i_n}^{\mathcal{L}}$  be the set of all taut  $\zeta$ -webs asymptotic to the given half-plane fan of vacua. Then

$$\sum_{\mathcal{Z} \in \mathcal{S}_{i_1 \dots i_n}^{\mathcal{L}}} \Psi_{\mathcal{Z}}^* = 0. \quad (14.43)$$

The proof mimics the proof of eqn. (14.34). Each taut half-plane  $\zeta$ -web is associated to an end of a 1-dimensional reduced  $\zeta$ -instanton moduli space; each such moduli space has two ends; and by virtue of (14.42), these two ends cancel in pairs in eqn. (14.43).

The relation between (14.43) and the identity (4.47) or (4.49) of the abstract discussion of boundary amplitudes is quite analogous to the corresponding relation between eqn. (14.34) and the identity (4.15) of the abstract discussion of bulk amplitudes.  $\Psi_{\mathcal{Z}}^*$  can be computed by working at infinity in the reduced moduli space of  $\mathcal{Z}$ , where it is built from widely separated bulk and boundary rigid  $\zeta$ -vertices, connected in general by classical solitons that propagate for long distances. Let us denote the bulk and boundary  $\zeta$ -vertices as  $\mathcal{V}_a$  and  $\mathcal{V}_{\alpha}$ , respectively. To compute  $\Psi_{\mathcal{Z}}^*$ , we take the tensor products of all states  $B(\mathcal{V}_a)$  and  $B(\mathcal{V}_{\alpha})$ , and then, whenever two vertices are connected by a soliton line in the  $\zeta$ -web  $\mathcal{Z}$ , we contract out the corresponding soliton states using the degree 1 pairing (12.17). Thus, the left hand side of (14.43) is obtained by summing over all taut half-plane  $\zeta$ -webs  $\mathcal{Z}$ , and for each such web, computing a state  $B(\mathcal{V}_a)$  or  $B(\mathcal{V}_{\alpha})$  for each bulk or boundary vertex and then contracting these states whenever two vertices are joined by lines. All this matches precisely the operation  $\rho_{\beta}$  used in the abstract statements (4.47) or (4.49) and therefore (14.43) essentially matches those identities.

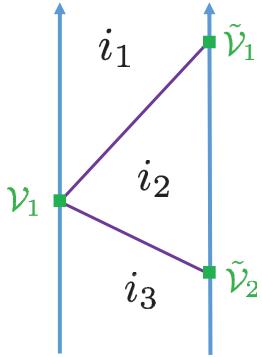
To match the notation used in the abstract discussion, we proceed as in eqn. (14.35). Let  $\mathcal{V}_{\alpha}$  be a rigid half-plane  $\zeta$ -vertex, asymptotic to a half-plane fan with  $s_{\alpha}$  vacua and  $s_{\alpha}-1$  outgoing solitons with impact parameters  $x_1^{\alpha}, \dots, x_{s_{\alpha}-1}^{\alpha}$ . In a fermion-number conserving theory, the state  $B(\mathcal{V}_{\alpha})$  produced by the path integral has fermion number 0. However,  $\Theta_{\mathbf{o}}$ , being Poincaré dual to a point  $\mathbf{o} \in \mathbb{R}^{s_{\alpha}-2}$ , is an  $(s_{\alpha}-2)$ -form, and hence the reduced state  $B'_{\text{red}}(\mathcal{V}_{\alpha})$  has fermion number  $-(s_{\alpha}-2)$ . To match the abstract discussion, we should define

$$\mathcal{B} = \sum_{\alpha} dx_1^{\alpha} \dots dx_{s_{\alpha}-1}^{\alpha} B'_{\text{red}}(\mathcal{V}_{\alpha}), \quad (14.44)$$

where the sum runs over rigid half-plane  $\zeta$  vertices. Clearly,  $\mathcal{B}$  is a sum of terms all of fermion number 1. This is the object that appears in the identity (4.47) or (4.49) of the abstract discussion.

### 14.7.3 $\zeta$ -Instantons On A Strip

Now we return to the problem formulated in section 13.2 of finding physical states when the theory is quantized on an interval  $D = [x_{\ell}, x_r]$ , with boundary conditions at the two ends of the strip set by Lagrangian submanifolds  $\mathcal{L}_{\ell}, \mathcal{L}_r$ . We recall from eqn. (13.3) that, if the width of the strip is much greater than the largest length scale in the theory, there is no problem to describe the classical approximation to the space of physical states on the



**Figure 141:** A rigid strip web  $\mathfrak{s}$ .

strip:

$$\mathbb{M}_{\mathcal{L}_\ell, \mathcal{L}_r} \cong \bigoplus_{i \in \mathbb{V}} \mathbb{M}_{\mathcal{L}_\ell, i} \otimes \mathbb{M}_{i, \mathcal{L}_r} \quad (14.45)$$

The problem raised in section 13.2 was to compute the differential  $\mathcal{Q}_\zeta$  that acts on this complex. Matrix elements of this differential are supposed to be computed by counting  $\zeta$ -instantons on the strip. To be more precise, we are supposed to count families of rigid  $\zeta$ -instantons on the strip, that is families of  $\zeta$ -instantons that have no modulus except the inevitable modulus associated to time translations. To compute a matrix element of  $\mathcal{Q}_\zeta$  between specified initial and final states, we have to count families of rigid  $\zeta$ -instantons that interpolate between specified initial and final  $\zeta$ -solitons – that is elements in the complex  $\mathbb{M}_{\mathcal{L}_\ell, \mathcal{L}_r}$  of (14.45) – in the far past and the far future.

We hope it is now clear how to do the counting. On a very wide strip, the  $\zeta$ -instantons can be represented by webs – or  $\zeta$ -webs – in which the lines are classical solitons and the vertices are bulk and boundary  $\zeta$ -vertices. So if  $\mathcal{M}$  is a component of  $\zeta$ -instanton moduli space on the strip, we can associate to it a strip web  $\mathfrak{s}$  in the sense of section 2.3. The  $\zeta$ -instantons parametrized by  $\mathcal{M}$  are built by gluing bulk and boundary  $\zeta$ -vertices  $\mathcal{V}_a$ ,  $a = 1, \dots, s$  and  $\mathcal{V}_\alpha$ ,  $\alpha = 1, \dots, t$  via the strip web  $\mathfrak{s}$ . The web dimension  $d(\mathfrak{s})$  is always at least 1, because of time translations, and  $\mathfrak{s}$  is called a rigid strip web if  $d(\mathfrak{s}) = 1$ . The dimension of  $\mathcal{M}$  exceeds  $d(\mathfrak{s})$  by the sum of the excess dimensions of the  $\zeta$ -vertices  $\mathcal{V}_a$  and  $\mathcal{V}_\alpha$ . (See eqn. (14.15) for a formula of this type.) So for  $\mathcal{M}$  to be 1-dimensional,  $\mathfrak{s}$  must be rigid and the  $\zeta$ -vertices  $\mathcal{V}_a$  and  $\mathcal{V}_\alpha$  must have 0 excess dimension, that is they must also be rigid. Given a rigid strip web  $\mathfrak{s}$ , such as the one sketched in Figure 141, to count the corresponding  $\zeta$ -instantons, we just sum over all possible labelings of the internal lines in the web by solitons, the boundary segments by half-plane states, and the bulk and boundary vertices  $\mathcal{V}_a$  and  $\mathcal{V}_\alpha$  by the corresponding bulk and boundary amplitudes  $B(\mathcal{V}_a)$  and  $B(\mathcal{V}_\alpha)$ . We have shown that  $B(\mathcal{V}_a)$  and  $B(\mathcal{V}_\alpha)$  obey the algebraic identities that were assumed in section 4.3, and the counting procedure for rigid  $\zeta$ -instantons on the strip coincides with what was used in that section to define a differential. So we conclude that

the procedure of that abstract discussion can indeed be used to determine the differential whose cohomology gives the exact supersymmetric ground states on the strip.

## 15. Webs And The Fukaya-Seidel Category

### 15.1 Preliminaries

In this paper, we have described two algebraic structures associated to open strings in the same massive Landau-Ginzburg model:

(*i*) In the abstract part of this paper, Sections §§2–5, we described an algebraic structure that is built, roughly speaking, by multiplying boundary web vertices. This algebraic structure is an  $A_\infty$  algebra, something that in physical applications is usually associated to open-string amplitudes at tree level.

(*ii*) In section 11.3, we described the Fukaya-Seidel category, or more precisely in our context the Fukaya category of the superpotential, in which an  $A_\infty$  algebra is actually defined from open-string amplitudes.

It would be surprising for two essentially different  $A_\infty$  algebras to arise in the same model, and indeed in the present section we will argue that, once one restricts the web-based construction to a smaller class of branes in a way that we will explain, the two  $A_\infty$  algebras are indeed equivalent. Roughly speaking, (*i*) and (*ii*) correspond, respectively, to methods of describing the same category in the infrared (long distances) and in the ultraviolet (short distances). Naively, the superpotential  $W$  is very important in the infrared and not important in the ultraviolet. Unfortunately, this is a little oversimplified. It is true that in method (*i*), the superpotential plays a very important role, and we will explain below in what sense webs emerge from the  $\zeta$ -instanton equation in the infrared limit. But, unless the branes considered are all compact, there is no ultraviolet limit in which one completely forgets the superpotential; as explained in Section §11.2, it plays a role in controlling the classes of branes that one can consider and in ensuring that counting of solutions is always well-defined.

Approach (*i*) was part of a larger discussion in which the largest class of branes that one can consider are the  $W$ -dominated branes described in section 11.2.4. Recall this means that on left branes  $\text{Im}(\zeta^{-1}W)$  goes to  $+\infty$  at infinity, and on right branes  $\text{Im}(\zeta^{-1}W)$  goes to  $-\infty$  at  $\infty$ . If  $\mathfrak{B}_\ell$  is a left brane and  $\mathfrak{B}_r$  is a right brane, then there is a well-defined space of  $(\mathfrak{B}_\ell, \mathfrak{B}_r)$  strings. This space can be computed by solving the  $\zeta$ -instanton equation on a vertical strip in the  $s = x + i\tau$  plane. To ensure that the space of  $(\mathfrak{B}_\ell, \mathfrak{B}_r)$  strings is well-defined, the strip has to be vertical (as drawn, for example, in Figure 3 from the introduction, and in many other figures in this paper), assuming no restriction except that the branes are  $W$ -dominated. We always think of a left brane as attached to a left vertical boundary of a region in the  $s$ -plane, and a right brane as attached to a right vertical boundary of such a region.

Manipulating boundary vertices by the procedure of Section §5 gives one  $A_\infty$  algebra for the left branes, and another for the right branes. Similarly, there are really two versions of the Fukaya-Seidel category. That follows because in Section §11.3 we considered branes of class  $T_\kappa$  where  $\kappa$  is a complex number of modulus 1 that is required to differ from  $\pm\zeta$ .

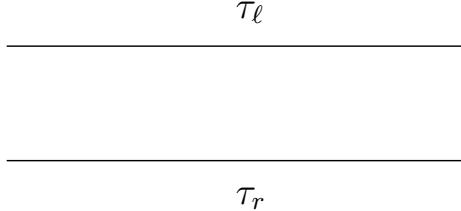
Removing the points  $\pm\zeta$  from the unit circle divides it into two components, and there are really two Fukaya-Seidel categories depending on which component contains  $\kappa$ . We will match the two Fukaya-Seidel categories with the web-based categories for left and right branes.

However, to make contact with the Fukaya-Seidel category, it is not sufficient simply to require that the left and right branes be  $W$ -dominated. Let us therefore discuss what is the smallest class of branes that we could reasonably consider. In the abstract discussion, a brane  $\mathfrak{B}$  has Chan-Paton factors  $\mathcal{E}_i(\mathfrak{B})$  for each vacuum state  $i \in \mathbb{V}$ .  $\mathcal{E}_i(\mathfrak{B})$  is a complex whose cohomology is the space of supersymmetric physical states when the theory is formulated on a half-line with  $\mathfrak{B}$  at the finite end and vacuum  $i$  at infinity. (In the case of a left or right brane, the finite end of the half-line is taken to be on the left or right.) The axioms of the abstract discussion ensure the existence, for each vacuum  $i \in \mathbb{V}$ , of a distinguished left brane  $\mathfrak{T}_i$  whose Chan-Paton factors are  $\delta_{ij}\mathbb{Z}$  (or  $\delta_{ij}\mathbb{C}$  if one considers physical states to be complex vector spaces rather than  $\mathbb{Z}$ -modules); in other words, there are no physical states with brane  $\mathfrak{T}_i$  at the finite end and vacuum  $j$  at infinity, unless  $i = j$ , in which case the space of such states has rank (or dimension) 1. In terms of Landau-Ginzburg models, the  $\mathfrak{T}_i$  are the left thimbles  $L_i^\zeta$  that were introduced in section 11.2.5. The Chan-Paton factors of these thimbles were described in eqn. (13.1) and are as desired. Thus, certainly, in a Landau-Ginzburg construction that is supposed to illustrate the abstract part of this paper, among the left branes we must at least include the left thimbles  $L_i^\zeta$ . (And similarly we must include the right thimbles  $R_i^\zeta$  among the right branes.) The abstract part of this paper also includes a criterion (section 4.2) for what is a brane with more general Chan-Paton factors, and to match this discussion, we need to allow more general branes built from the thimbles. We therefore need a reasonable class of left branes that includes these thimbles, and is closed under the relevant operations (which basically involve building a new brane as an extension of one brane by another) yet is sufficiently small to allow a construction we will come to momentarily.<sup>92</sup> It turns out that a suitable condition is that a left brane must be of class  $T_\zeta$ , as described in section 11.2.6. (We recall that the  $T_\zeta$  condition means that, along the support of the brane,  $W$  must take values in a certain semi-infinite strip that contains the images in the  $W$ -plane of the  $L_i^\zeta$ , as in Figure (127).) The analogous condition on right branes is that they should be of class  $T_{-\zeta}$ .

To compare the web-based approach to the Fukaya-Seidel category, we want to interpret left branes of class  $T_\zeta$  as objects in the Fukaya-Seidel category. At first sight there is some tension here. In section 11.3, we found that to describe the Fukaya-Seidel category using the  $\zeta$ -instanton equation, we had to consider not branes of class  $T_{\pm\zeta}$ , but branes of class  $T_\kappa$  where  $\kappa$  is a complex number of modulus 1 and not equal to  $\pm\zeta$ . But in the web-based procedure using the  $\zeta$ -instanton equation, we definitely want left branes of class  $T_\zeta$ , not of class  $T_\kappa$  with any other  $\kappa$ . (And the right branes should definitely be of class  $T_{-\zeta}$ .) However, there is a simple way to reconcile the different statements. We use the fact that the  $\zeta$ -instanton equation is not invariant under rotations of the  $s$ -plane, and that such a rotation is equivalent to a rotation of the complex number  $\zeta$ . In section 11.3,

---

<sup>92</sup>Mathematically we would certainly want to include the smallest triangulated category containing the thimbles.



**Figure 142:** A horizontal strip  $\tau_\ell > \tau > \tau_r$  in the  $s$ -plane.

we assumed that open-string amplitudes relevant to the Fukaya-Seidel category are to be computed with a quantum field theory defined in a region of the  $s$ -plane whose boundaries are asymptotically vertical, as in Figure 129. In this case, the branes must be of class  $T_\kappa$  with  $\kappa \neq \pm\zeta$ . If we want instead to use branes of class  $T_\zeta$ , we simply have to rotate Figure 129 so that the boundaries of incoming and outgoing open strings are not vertical. For example, it is convenient to rotate the figure by an angle of  $\pm\pi/2$  so that incoming open strings come in from the left of the  $s$ -plane or from the right. (The angle  $\pi/2$  is not essential; any angle other than 0 or  $\pi$  will do.)

As we explained above, there are really two Fukaya-Seidel categories, depending on which component of  $S^1 \setminus \{\pm\zeta\}$  contains  $\kappa$ . After we rotate Figure 129 so that the open strings no longer come in from the bottom (or top) of the figure, the two categories differ by whether the open strings come in from the left half of the  $s$ -plane or from the right half. We will match the Fukaya-Seidel category constructed with open strings that come in from the left to the web-based category of left branes. Similarly, the Fukaya-Seidel category with open strings that come in from the right matches the web-based category of right branes.

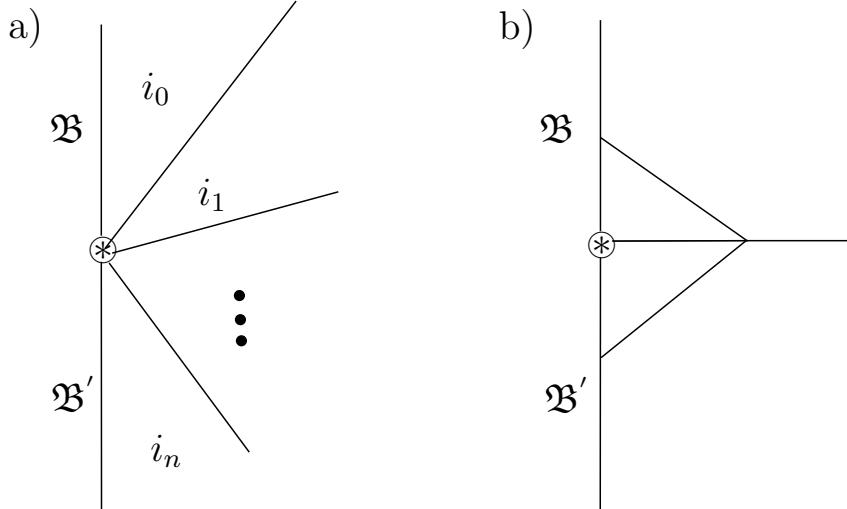
Some of the ideas in the following analysis have been explained in a simpler context in Section §10.7.

## 15.2 Morphisms

If  $\mathfrak{B}$  and  $\mathfrak{B}'$  are two branes of class  $T_\zeta$ , then we can calculate the space of supersymmetric  $(\mathfrak{B}, \mathfrak{B}')$  states on the interval, which we will call  $\mathcal{H}_{\mathfrak{B}, \mathfrak{B}'}$ , in the Fukaya-Seidel category, and we can also compute  $\mathcal{H}_{\mathfrak{B}, \mathfrak{B}'}^{\text{web}} := H^*(\text{Hop}(\mathfrak{B}, \mathfrak{B}'), M_1)$  in the web-based construction of an  $A_\infty$  category. Our goal in this section is to sketch the construction of a natural isomorphism  $\mathcal{H}_{\mathfrak{B}, \mathfrak{B}'} \cong \mathcal{H}_{\mathfrak{B}, \mathfrak{B}'}^{\text{web}}$ . (Of course, like most claims in this paper about the  $\zeta$ -instanton equation, the arguments presented here are not complete mathematically.) When combined with a similar analysis that we will make of the multiplication of string states in section 15.3, and of the higher order operations in section 15.4, this will show the equivalence between the two constructions of an  $A_\infty$  category.<sup>93</sup>

---

<sup>93</sup>In the abstract part of this paper, we introduced (1) a space of  $(\mathfrak{B}, \mathfrak{B}')$  ground states, where  $\mathfrak{B}$  is a left-brane and  $\mathfrak{B}'$  is a right-brane, and also (2) a space  $\mathcal{H}_{\mathfrak{B}, \mathfrak{B}'}^{\text{web}}$  of morphisms between two left-branes (or two right-branes). What is relevant to the present discussion is definitely construction (2). From the  $\sigma$ -model point of view, the definition (1) assumes that  $\mathfrak{B}$  and  $\mathfrak{B}'$  are  $W$ -dominated branes of opposite type, and the definition (2) applies to branes of class  $T_\zeta$ .  $\mathcal{H}_{\mathfrak{B}, \mathfrak{B}'}^{\text{web}}$  can be interpreted as a space of open-string ground



**Figure 143:** (a) The symbol  $\circledast$  represents a half-space fan of solitons, as indicated, or equivalently an element  $\delta \in \text{Hop}(\mathfrak{B}, \mathfrak{B}')$  in the web-based formalism. The symbol  $\circledast$  with the indicated fan of solitons emerging from it represents a possible asymptotic behavior at infinity of a solution of the  $\zeta$ -instanton equation with no knowledge of how the solution behaves in the interior. (b) This picture, which is our first example of a “hybrid”  $\zeta$ -web, would represent a  $\zeta$ -web with a 1-dimensional reduced moduli space, except that one of the vertices is not a  $\zeta$ -vertex but rather is an abstract vertex labeled  $\circledast$ , which represents a fan of solitons rather than a solution of the  $\zeta$ -instanton equation. All other bulk and boundary vertices in the picture are conventional  $\zeta$ -vertices, contracted together by “propagators” in the usual way.

First we review the definition of the two spaces that are supposed to be isomorphic. The definition of  $\mathcal{H}_{\mathfrak{B}, \mathfrak{B}'}$  involves two steps:

(a) We consider the LG model of interest on a strip  $\mathbb{R} \times I$ , where  $I$  is a closed interval, with boundary conditions at the two ends set by  $\mathfrak{B}$  and  $\mathfrak{B}'$ , where  $\mathfrak{B}, \mathfrak{B}'$  are both of class  $T_\zeta$ . For reasons explained in section 15.1, we take this strip to run horizontally, rather than vertically, in the  $s = x + i\tau$  plane (Figure 142). Thus the strip is defined by  $\tau_\ell \geq \tau \geq \tau_r$  (for some  $\tau_\ell, \tau_r$ ), and  $x$  plays the role of “time.” We define a vector space  $\mathbb{M}_{\mathfrak{B}, \mathfrak{B}'}$  (actually a  $\mathbb{Z}$ -module) that has a basis vector for every  $\zeta$ -soliton – that is, for every solution of the  $\zeta$ -instanton equation (11.17) on the strip that depends only on  $\tau$ . This complex is graded by fermion number in the usual way.

(b) On  $\mathbb{M}_{\mathfrak{B}, \mathfrak{B}'}$ , we define a differential  $\widehat{\mathcal{Q}}_\zeta$  by counting suitable 1-parameter families of  $\zeta$ -instantons on the strip. The solutions are required to be independent of  $x$  for  $x \ll 0$  and also for  $x \gg 0$ . This means that the “initial data” (for  $x \ll 0$ ) are associated to a basis vector  $|\phi_\ell\rangle$  of  $\mathbb{M}_{\mathfrak{B}, \mathfrak{B}'}$ , corresponding to some  $\zeta$ -soliton that satisfies the boundary conditions, and the “final state” (for  $x \gg 0$ ) is similarly associated to a possibly different  $\zeta$ -soliton  $|\phi_r\rangle$ . The one-parameter family corresponds to translation in the  $x$ -direction.

---

states in quantization on a horizontal strip as in Figure 142, but since the class of branes is different and both branes are of the same type, these are not the open-string states studied in the abstract part of the paper.

Counting 1-parameter families of  $\zeta$ -instantons with these boundary conditions gives the matrix element of  $\widehat{Q}_\zeta$  from  $|\phi_\ell\rangle$  to  $|\phi_r\rangle$ . (As always, in this ‘‘counting,’’ a  $\zeta$ -instanton is weighted with a factor  $\pm 1$  coming from the sign of the fermion determinant.) The space  $\mathcal{H}_{\mathfrak{B}, \mathfrak{B}'}$  is the cohomology of the differential  $\widehat{Q}_\zeta$ .

The definition of  $\mathcal{H}_{\mathfrak{B}, \mathfrak{B}'}^{\text{web}}$  in the web-based procedure of section 5 involves two analogous steps:

(a') We first choose a half-plane,  $\mathcal{H}$ , and here we choose it to be the positive half  $s$ -plane as in Figure 143(a)). Next we define a complex  $\mathbb{M}_{\mathfrak{B}, \mathfrak{B}'}^{\text{web}}$  with a basis vector associated to the following data: As in Figure 143(a)) we choose a half-plane fan of vacua  $J = \{i_0, i_1, \dots, i_n\}$ . The regions between the lines in the figure are labeled by vacua  $i_0, \dots, i_n \in \mathbb{V}$ , representing constant solutions of the  $\zeta$ -instanton equation. The line separating any two consecutive vacua  $i_k$  and  $i_{k+1}$  is labeled by an  $i_k i_{k+1}$   $\zeta$ -soliton. When restricted to large  $|s|$ ,  $\mathcal{H}$  has both an upper and a lower boundary. The upper boundary is labeled by a half-line  $\zeta$ -soliton interpolating from brane  $\mathfrak{B}$  to vacuum  $i_0$ ; the lower boundary is labeled by a half-line  $\zeta$ -soliton set by the brane  $\mathfrak{B}'$  and the vacuum  $i_n$ . If we compare with the definitions in Section §5.2 then we should identify

$$\mathbb{M}_{\mathfrak{B}, \mathfrak{B}'}^{\text{web}} = \text{Hop}(\mathfrak{B}, \mathfrak{B}') = \text{Hom}(\mathfrak{B}', \mathfrak{B}) = \bigoplus_{i,j \in \mathbb{V}} \mathcal{E}(\mathfrak{B})_i \otimes \widehat{R}_{ij} \otimes (\mathcal{E}(\mathfrak{B}')_j)^*. \quad (15.1)$$

In short,  $\mathbb{M}_{\mathfrak{B}, \mathfrak{B}'}^{\text{web}}$  has a basis in which the basis vectors are half-plane fans of solitons, interpolating between branes  $\mathfrak{B}$  and  $\mathfrak{B}'$ . Such a fan is indicated in Figure 143(a). It represents the asymptotic behavior at infinity of a possible solution of the  $\zeta$ -instanton equation on the half-plane, with no knowledge concerning the behavior in the interior.

In the abstract part of this paper, we studied webs in which the vertices represented elements of  $\mathbb{M}_{\mathfrak{B}, \mathfrak{B}'}^{\text{web}}$ , and its bulk analog. Starting in section 14.2, we have studied  $\zeta$ -webs, in which the vertices are  $\zeta$ -vertices, which represent solutions of the  $\zeta$ -instanton equation with fan-like asymptotics. It will now be useful to consider webs with ingredients of both kinds. We will call these *hybrid  $\zeta$ -webs*. In a hybrid  $\zeta$ -web – our first example is in Figure 143(b) – a vertex labeled by  $\circledast$  represents a fan of solitons, and a vertex not so labeled is a conventional  $\zeta$ -vertex. Thus, the symbol  $\circledast$ , which we call an abstract vertex (in homage to the abstract nature of the web-based construction) represents an element of  $\mathbb{M}_{\mathfrak{B}, \mathfrak{B}'}^{\text{web}}$  associated to a given fan of solitons, while the other vertices are the ones that we have used until now in the  $\sigma$ -model approach. Consideration of these hybrid  $\zeta$ -webs will be helpful in bridging the gap between the two approaches to open-string amplitudes.

The next step is to define a differential on the space  $\mathbb{M}_{\mathfrak{B}, \mathfrak{B}'}^{\text{web}}$ , making it a complex whose cohomology is  $\mathcal{H}_{\mathfrak{B}, \mathfrak{B}'}^{\text{web}}$ . The differential is called  $M_1$  in eqn. (5.17) and may be described as follows:

(b') In the web-based treatment, the differential acting on  $\delta \in \mathbb{M}_{\mathfrak{B}, \mathfrak{B}'}^{\text{web}}$  is defined by inserting  $\delta$  (or the associated half-space fan) at one boundary vertex of a taut half-plane web, and weighting all other interior or boundary vertices by the appropriate interior or boundary amplitudes. We follow the same procedure now, with one difference: we interpret the interior or boundary amplitudes as the ones that are computed in the  $\sigma$ -model, by

counting solutions of the  $\zeta$ -instanton equation. This means that the bulk and boundary vertices – other than the one associated to  $\delta$  – are now going to be  $\zeta$ -vertices.<sup>94</sup>

Accordingly, we consider a hybrid  $\zeta$ -web (Figure 143(b)), of the type just described. We use such hybrid  $\zeta$ -webs, for the time being, merely to give a pictorial representation of some of the abstract web-based constructions, with the one change that bulk and interior amplitudes are now derived by counting  $\zeta$ -instantons. The hybrid  $\zeta$ -webs in Figure 143(b) are required to be taut; that is, they have precisely one reduced modulus, derived from an overall scaling of the half-plane, keeping the abstract vertex fixed.

To a hybrid  $\zeta$ -web with one abstract vertex  $\circledast$  and with fan-like asymptotics at infinity, we can associate a pair of elements of the web-based complex  $\mathbb{M}_{\mathfrak{B}, \mathfrak{B}'}^{\text{web}}$ . By restricting the hybrid  $\zeta$ -web to a neighborhood of the abstract vertex  $v$  we find a labeled fan of vacua  $J_v$  and a corresponding basis vector  $|\phi_F^{\text{web}}(\circledast)\rangle$  of  $\mathbb{M}_{\mathfrak{B}, \mathfrak{B}'}^{\text{web}}$ . (The subscript  $F$  is meant to remind us that  $\phi_F$  depends on a fan of solitons, rather than a single soliton.) Similarly, by restricting the web to a neighborhood of  $|s| = \infty$ , we get a second labeled fan of vacua  $J_\infty$  and correspondingly a second basis vector  $|\phi_F^{\text{web}}(\infty)\rangle$ . It is convenient to refer to a neighborhood of the abstract vertex and a neighborhood of infinity as the two ends of the hybrid  $\zeta$ -web in Figure 143(b). The matrix element of the web-based differential  $M_1 = \hat{Q}_\zeta^{\text{web}}$  from  $|\phi_F^{\text{web}}(\circledast)\rangle$  to  $|\phi_F^{\text{web}}(\infty)\rangle$  is computed by counting the number of hybrid  $\zeta$ -webs, as in Figure 143(b), with precisely one reduced modulus associated to scaling, and with  $|\phi_F^{\text{web}}(\circledast)\rangle$  and  $|\phi_F^{\text{web}}(\infty)\rangle$  as the restrictions to the two ends. The webs are weighted with the signs carefully spelled out in equation (4.33) above. Physically these signs come from the fermion determinants (and they agree because fermion determinants satisfy the gluing laws that were built into the other approach). The algorithm we have just described is precisely the definition of  $M_1(\delta)$  as defined in equation (5.17), where the morphism  $\delta$  is inserted at the abstract vertex  $\circledast$ .

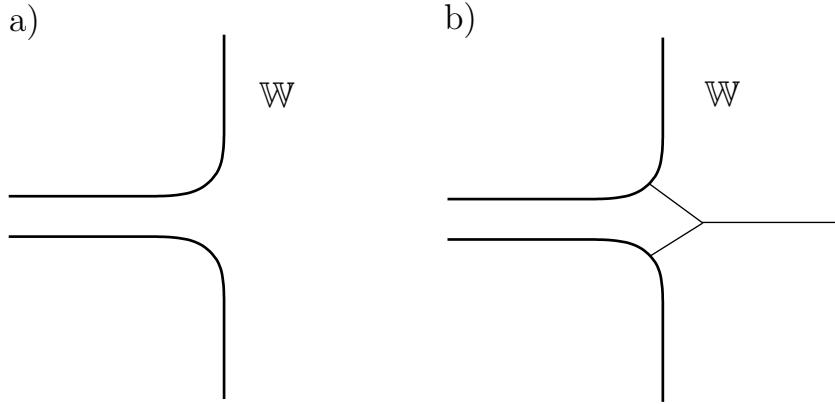
To establish a natural isomorphism between  $\mathcal{H}_{\mathfrak{B}, \mathfrak{B}'}$  and  $\mathcal{H}_{\mathfrak{B}, \mathfrak{B}'}^{\text{web}}$ , we will imitate the procedure explained in section 10.7 for showing that the cohomology of the MSW complex does not depend on the metric or superpotential on  $X$ . First we define a linear map

$$\mathcal{U} : \mathbb{M}_{\mathfrak{B}, \mathfrak{B}'} \rightarrow \mathbb{M}_{\mathfrak{B}, \mathfrak{B}'}^{\text{web}}, \quad (15.2)$$

as follows. We consider a region  $\mathbb{W}$  of the complex  $s$ -plane in which a semi-infinite strip comes in from  $x = -\infty$  and fans out to the positive half plane (Figure 144(a)). We consider solutions of the  $\zeta$ -instanton equation on  $\mathbb{W}$  with boundary conditions determined by  $\mathfrak{B}$  and  $\mathfrak{B}'$  on the upper and lower boundaries, respectively, and which moreover are independent of  $x$  for  $x \rightarrow -\infty$  and finally have fan-like asymptotics for  $|s| \rightarrow \infty$  for large  $\text{Re}(s)$  in the positive half-plane. The behavior of such a solution for  $x \rightarrow -\infty$  gives a basis vector  $|\phi_\ell\rangle$  of  $\mathbb{M}_{\mathfrak{B}, \mathfrak{B}'}$ , and its behavior for  $|s| \rightarrow \infty$ ,  $x \gg 0$  gives a basis vector  $|\phi_F^{\text{web}}(\infty)\rangle$  of  $\mathbb{M}_{\mathfrak{B}, \mathfrak{B}'}^{\text{web}}$ . We consider only the components of the moduli space of such solutions with an expected dimension of 0. The actual dimension is then also 0 if the Kähler metric of the target space  $X$  is generic. Counting (with signs, as always) the solutions that are in

---

<sup>94</sup>Likewise in discussing below other constructions of the web-based approach, all bulk or boundary vertices not labeled by  $\circledast$  will be  $\zeta$ -vertices.



**Figure 144:** (a) A region  $\mathbb{W}$  in the  $s$ -plane; a strip coming in from the left opens out to the half-plane  $\mathcal{H}$ . (b) A  $\zeta$ -web in the region  $\mathbb{W}$ . The web sketched here is a simple one in which two BPS solitons emitted from the boundary join into one. This web corresponds to a moduli space of zero dimension if and only if the indicated boundary vertices are possible only at specific points along the boundary, because the  $\zeta$ -solitons in question can be emitted only when the boundary slopes at a favored angle.

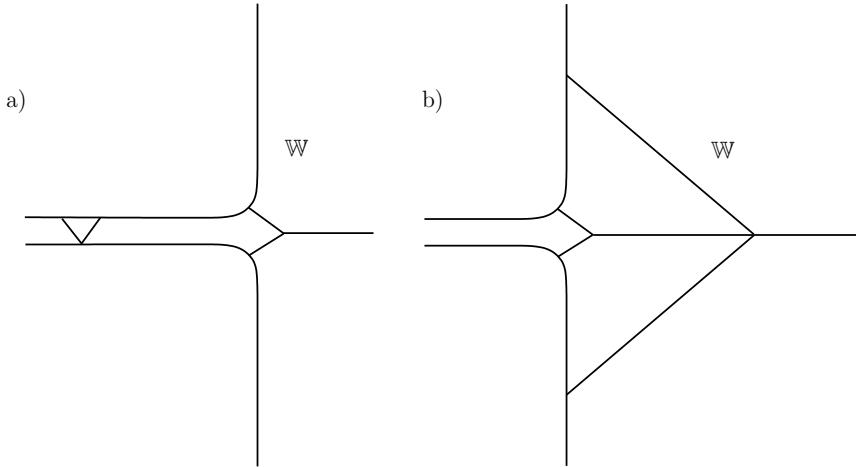
zero-dimensional moduli spaces and with the asymptotics that we have specified gives the matrix element of the desired operator  $\mathcal{U}$  from  $|\phi_\ell\rangle$  to  $|\phi_F^{\text{web}}(\infty)\rangle$ . The map  $\mathcal{U}$  that is defined this way preserves the fermion number since it comes from components of moduli space with expected dimension 0.

In this construction, the width of the strip in the left half plane is arbitrary. In particular, we are free to choose this strip to be much wider than the natural length scale of the massive LG theory under study. If we do this, then a  $\zeta$ -instanton contributing to  $\mathcal{U}$  can everywhere be represented by a  $\zeta$ -web. A possible example is sketched in Figure 144(b). (The  $\zeta$ -web associated to a  $\zeta$ -instanton without moduli extends only a bounded distance to the left down the strip, but given the desired fan-like asymptotics for  $x \gg 0$ , it is unbounded to the right.) Computing the dimension of the moduli space associated to such a web involves some new ingredients, which we will not explore, since some boundary vertices might exist only when the boundary is oriented at a favored angle. If we work with wide strips, then other solutions that we encounter presently are similarly web-like.

As in eqn. (10.45),  $\mathcal{U}$  induces a map on cohomology, in other words a linear transformation  $\widehat{\mathcal{U}} : \mathcal{H}_{\mathfrak{B}, \mathfrak{B}'} \rightarrow \mathcal{H}_{\mathfrak{B}, \mathfrak{B}'}^{\text{web}}$ , because it obeys (up to sign)

$$\widehat{\mathcal{Q}}_\zeta^{\text{web}} \mathcal{U} = \mathcal{U} \widehat{\mathcal{Q}}_\zeta. \quad (15.3)$$

The proof of this formula proceeds like the proof of eqn. (10.45). We consider the 1-dimensional moduli space  $\mathcal{M}$  of solutions of the  $\zeta$ -instanton equation on the region  $\mathbb{W}$ , which we require to have asymptotic behavior corresponding to basis vectors  $|\phi_\ell\rangle$  and  $|\phi_F^{\text{web}}(\infty)\rangle$  (whose fermion numbers now differ by 1, since  $\mathcal{M}$  is 1-dimensional). The moduli space  $\mathcal{M}$  in general has several connected components. As usual the compact components do not affect the following discussion. Alternatively, a given component  $\mathcal{M}_\alpha$  of  $\mathcal{M}$  could



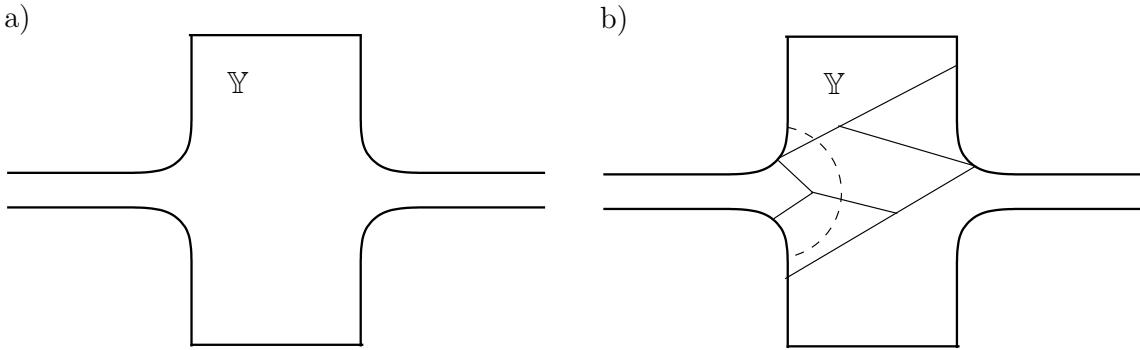
**Figure 145:** The two types of end of a 1-dimensional moduli space of  $\zeta$ -instantons on the region  $\mathbb{W}$  are depicted here. (These webs correspond to 1-dimensional moduli spaces if the web in Figure 144(b) has a zero-dimensional moduli space. As observed in the caption to that figure, this depends on some assumptions about the boundary vertices.) The end in (a) corresponds to the  $\zeta$ -instanton of Figure 144(b), which contributes to  $\mathcal{U}$ , concatenated with a  $\zeta$ -instanton on the strip, which contributes to  $\widehat{\mathcal{Q}}_\zeta$ . An end of this type contributes to the product  $\mathcal{U}\widehat{\mathcal{Q}}_\zeta$ . The end in (b) is made by replacing the abstract vertex in Figure 143(b), which contributes to  $\widehat{\mathcal{Q}}_\zeta^{\text{web}}$ , with the  $\zeta$ -instanton of Figure 144(b), which contributes to  $\mathcal{U}$ . An end of this type contributes to the product  $\widehat{\mathcal{Q}}_\zeta^{\text{web}}\mathcal{U}$ .

instead be a copy of  $\mathbb{R}$ , with two ends. Such ends are of two possible types as illustrated in Figure 145. One type of end contributes to the left hand side of eqn. (15.3), and one contributes to the right hand side. If a given component  $\mathcal{M}_\alpha$  has two ends of the same type, these ends both contribute to the same side of eqn. (15.3) but with opposite signs. If the ends are of opposite types, one end contributes  $\pm 1$  to the left hand side of the identity and one makes an equal contribution to the right hand side. After summing the contributions of all 1-dimensional components  $\mathcal{M}_\alpha$  of  $\mathcal{M}$ , we arrive at the identity (15.3).

The interpolation we have made to define the map  $\mathcal{U}$  involved a number of arbitrary choices: the region  $\mathbb{W}$  was not uniquely determined (we only specified its asymptotic form), the Kähler metric on the target space  $X$  is supposed to be irrelevant, and for that matter the Kähler metric on  $\mathbb{W}$  should likewise be unimportant, as long as its asymptotic behavior is kept fixed. To show that the induced map  $\widehat{\mathcal{U}}$  on cohomology does not depend on the choices, one needs an identity of the same form as eqn. (10.48). If  $\mathcal{U}_0, \mathcal{U}_1 : \mathbb{M}_{\mathfrak{B}, \mathfrak{B}'} \rightarrow \mathbb{M}_{\mathfrak{B}, \mathfrak{B}'}^{\text{web}}$  are computed with two different choices related by homotopy, we need a map  $\mathsf{E} : \mathbb{M}_{\mathfrak{B}, \mathfrak{B}'} \rightarrow \mathbb{M}_{\mathfrak{B}, \mathfrak{B}'}^{\text{web}}$  of fermion number  $-1$  obeying

$$\mathcal{U}_1 - \mathcal{U}_0 = \widehat{\mathcal{Q}}_\zeta^{\text{web}} \mathsf{E} - \mathsf{E} \widehat{\mathcal{Q}}_\zeta. \quad (15.4)$$

This is established by imitating the proof of eqn. (10.48): one interpolates between the choices made to define  $\mathcal{U}_0$  and  $\mathcal{U}_1$ , and then looks at certain 1-dimensional moduli spaces. These moduli spaces have ends of four types that correspond to the four terms in eqn.



**Figure 146:** (a) Since the region  $\mathbb{Y}$  is homotopic to an infinite strip, counting of  $\zeta$ -instantons on  $\mathbb{Y}$  gives a map  $\mathcal{V}$  that induces an isomorphism on cohomology. (b) Restriction of a  $\zeta$ -instanton on  $\mathbb{Y}$  to the dotted semi-circle reveals a fan of solitons that determines a basis vector of  $\mathbb{M}_{\mathfrak{B}, \mathfrak{B}'}^{\text{web}}$ . This implies that the map  $\mathcal{V}$  determined by  $\zeta$ -instantons on  $\mathbb{Y}$  can be factored through  $\mathcal{U}$ , implying that  $\mathcal{U}$  also induces an isomorphism on cohomology.

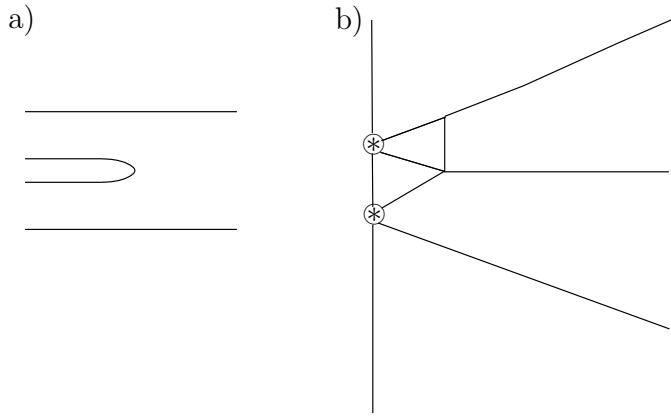
(15.4). The map  $\mathsf{E}$  is defined by counting exceptional solutions of the  $\zeta$ -instanton equation that exist only at specific points during the interpolation from  $\mathcal{U}_0$  to  $\mathcal{U}_1$ .

Finally, we would like to establish that the induced map  $\hat{\mathcal{U}}$  on cohomology is an isomorphism. As in section 10.7, this is most naturally done by finding an inverse map. For this, we consider  $\zeta$ -instantons on the region  $\mathbb{Y}$  sketched in Figure 146(a). A strip coming in from  $x = -\infty$  opens out to a very large portion of the right half-plane (compared to the width of the strip) which is part of  $\mathbb{Y}$ , but eventually this is cut off and the region  $\mathbb{Y}$  reduces back to a strip for  $x \rightarrow +\infty$ . By counting  $\zeta$ -instantons on  $\mathbb{Y}$  that are independent of  $x$  for  $x \rightarrow \pm\infty$ , we get a linear map  $\mathcal{V} : \mathbb{M}_{\mathfrak{B}, \mathfrak{B}'} \rightarrow \mathbb{M}_{\mathfrak{B}, \mathfrak{B}'}$ . The induced map on cohomology is the identity. (If  $\mathbb{Y}$  were replaced by a simple product  $\mathbb{R} \times I$ , rather than a region asymptotic to such a product, then  $\mathcal{V}$  would be the identity even before passing to cohomology; a special case of what is explained in the last paragraph is that the induced map on cohomology is unaffected if  $\mathbb{R} \times I$  is replaced by  $\mathbb{Y}$ .) On the other hand, the  $\zeta$ -instantons contributing to  $\mathcal{V}$  have a web-like description, as illustrated in Figure 146(b). If one of these solutions is restricted to the semi-circle indicated in that figure, it determines a labeled fan of vacua or in other words a basis vector  $|\phi_F^{\text{web}}(\infty)\rangle$  of  $\mathbb{M}_{\mathfrak{B}, \mathfrak{B}'}^{\text{web}}$ . Accordingly, the map  $\mathcal{V}$  can be factored as  $\mathcal{V}' \circ \mathcal{U}$ , where  $\mathcal{U} : \mathbb{M}_{\mathfrak{B}, \mathfrak{B}'} \rightarrow \mathbb{M}_{\mathfrak{B}, \mathfrak{B}'}^{\text{web}}$  was defined above, and  $\mathcal{V}'$  is a map in the other direction. It follows that the map  $\hat{\mathcal{U}}$  on cohomology is an isomorphism.

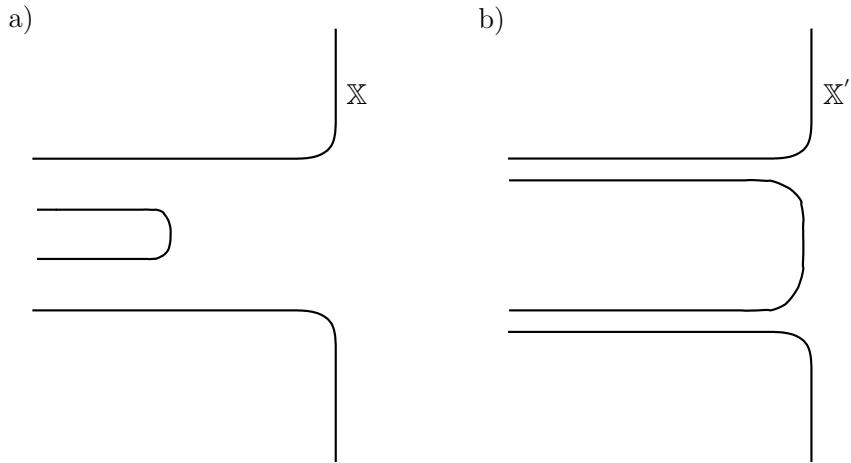
### 15.3 Multiplication

Let  $\mathfrak{B}$ ,  $\mathfrak{B}'$ , and  $\mathfrak{B}''$  be three branes. In the  $\sigma$ -model approach, a string state  $x \in \mathbb{M}_{\mathfrak{B}, \mathfrak{B}'}$  and a string state  $y \in \mathbb{M}_{\mathfrak{B}', \mathfrak{B}''}$  are multiplied by joining the two strings (Figure 147(a)). This operation gives a bilinear map  $m_2 : \mathbb{M}_{\mathfrak{B}, \mathfrak{B}'} \otimes \mathbb{M}_{\mathfrak{B}', \mathfrak{B}''} \rightarrow \mathbb{M}_{\mathfrak{B}, \mathfrak{B}''}$ .

On the other hand, there is also a bilinear product  $M_2 : \mathbb{M}_{\mathfrak{B}, \mathfrak{B}'}^{\text{web}} \otimes \mathbb{M}_{\mathfrak{B}', \mathfrak{B}''}^{\text{web}} \rightarrow \mathbb{M}_{\mathfrak{B}, \mathfrak{B}''}^{\text{web}}$ , defined in eqn. (5.17). The definition is as follows. We recall that an element of  $\mathbb{M}_{\mathfrak{B}, \mathfrak{B}'}^{\text{web}}$  or  $\mathbb{M}_{\mathfrak{B}', \mathfrak{B}''}^{\text{web}}$  represents a half-plane fan of solitons emanating from an abstract vertex. To



**Figure 147:** (a) In the  $\sigma$ -model approach, the multiplication  $M_{\mathcal{B}, \mathcal{B}'} \otimes M_{\mathcal{B}', \mathcal{B}''} \rightarrow M_{\mathcal{B}, \mathcal{B}''}$  is defined by joining open strings. (b) In the web-based approach, the multiplication  $\text{Hom}(\mathcal{B}, \mathcal{B}') \otimes \text{Hom}(\mathcal{B}', \mathcal{B}'') \rightarrow \text{Hom}(\mathcal{B}, \mathcal{B}'')$  is defined by configurations similar to rigid half-plane  $\zeta$ -webs but with abstract vertices (depicted as  $*$  in the figure) at two specified points on the boundary.



**Figure 148:** In (a), we multiply two string states within the  $\sigma$ -model approach, and then apply the map  $\mathcal{U}$  to the web-based category. The string states come in from the left, are multiplied when the two strings join, and are mapped to the web picture when the strip widens out into a half-plane. In (b), we first map each state separately to the web-based category, and then multiply them using the multiplication law of that category. The two results are equivalent modulo a sum of terms in which the differential  $\widehat{\mathcal{Q}}_\zeta$  or  $\widehat{\mathcal{Q}}_\zeta^{\text{web}}$  acts on the initial or final states, because the regions  $X$  and  $X'$  of the  $s$ -plane can be deformed into each other without altering anything at infinity. Under this deformation, the  $A$ -model action changes by an exact term, which after integration by parts is equivalent to a sum of contributions in which the differential acts on external states. In particular, these contributions vanish if we consider the multiplication law on cohomology.

multiply two such fans, we place the two abstract vertices at chosen points on the boundary of the half-plane  $\mathcal{H}$  and draw a picture that is analogous to a rigid  $\zeta$ -web except that two

of the vertices are abstract vertices rather than  $\zeta$ -vertices. (Choosing the points at which the two abstract vertices are inserted on the boundary of  $\mathcal{H}$  can be understood as dividing by the translation and scaling symmetries of  $\mathcal{H}$ .) This procedure is illustrated in Figure 147(b); every line or vertex in this figure, except the two abstract vertices that are denoted as  $\circledast$ , represents a boosted  $\zeta$ -soliton or  $\zeta$ -instanton solution with appropriate asymptotics. Then for  $\delta_1^{\text{web}} \in \mathbb{M}_{\mathfrak{B}, \mathfrak{B}'}^{\text{web}}$ ,  $\delta_2^{\text{web}} \in \mathbb{M}_{\mathfrak{B}', \mathfrak{B}''}^{\text{web}}$ , we want to define  $M_2(\delta_1^{\text{web}} \otimes \delta_2^{\text{web}}) \in \mathbb{M}_{\mathfrak{B}, \mathfrak{B}''}^{\text{web}}$ . The definition is as follows: the coefficient with which a given basis vector of  $\mathbb{M}_{\mathfrak{B}, \mathfrak{B}''}^{\text{web}}$  – corresponding to an outgoing fan of solitons at infinity – appears in  $M_2(\delta_1^{\text{web}} \otimes \delta_2^{\text{web}})$  is given by counting (with signs, as usual) the possible rigid pictures of the type sketched in Figure 147(b).

We now should recall that we also have, for any branes  $\mathfrak{B}_1, \mathfrak{B}_2$ , the natural map  $\mathcal{U} : \mathbb{M}_{\mathfrak{B}_1, \mathfrak{B}_2} \rightarrow \mathbb{M}_{\mathfrak{B}_1, \mathfrak{B}'}^{\text{web}}$ . So given  $\delta_1 \in \mathbb{M}_{\mathfrak{B}, \mathfrak{B}'}$ ,  $\delta_2 \in \mathbb{M}_{\mathfrak{B}', \mathfrak{B}''}$ , we have two ways to produce an element of  $\mathbb{M}_{\mathfrak{B}, \mathfrak{B}''}^{\text{web}}$ :

(i) We first multiply  $\delta_1$  and  $\delta_2$  in the  $\sigma$ -model approach, and use  $\mathcal{U}$  to map the result to the web-based category, to get  $\mathcal{U}(m_2(\delta_1 \otimes \delta_2)) \subset \mathbb{M}_{\mathfrak{B}, \mathfrak{B}''}^{\text{web}}$ .

(ii) Alternatively, we first map  $\delta_1$  and  $\delta_2$  to  $\mathcal{U}\delta_1$  and  $\mathcal{U}\delta_2$  in the web-based category, and then use the multiplication  $M_2$  in that category, to get  $M_2(\mathcal{U}\delta_1 \otimes \mathcal{U}\delta_2)$ .

The relation between these two procedures is that there is a map  $\mathsf{E} : \mathbb{M}_{\mathfrak{B}, \mathfrak{B}'} \otimes \mathbb{M}_{\mathfrak{B}', \mathfrak{B}''} \rightarrow \mathbb{M}_{\mathfrak{B}, \mathfrak{B}''}^{\text{web}}$ , reducing the fermion number by 1, such that

$$M_2 \circ (\mathcal{U} \otimes \mathcal{U}) - \mathcal{U} \circ m_2 = \widehat{\mathcal{Q}}_{\zeta}^{\text{web}} \mathsf{E} - \mathsf{E} \widehat{\mathcal{Q}}_{\zeta}. \quad (15.5)$$

This formula can be understood by considering Figure 148. In part (a) of this figure, two open strings join to a single open string – corresponding to the operation  $m_2$  in the  $\sigma$ -model approach – and then the worldsheet of that string opens out to a half-plane – giving the map  $\mathcal{U}$  to the web-based category. Solving the  $\zeta$ -instanton equation in the region  $\mathbb{X}$  of Figure 148(a), with asymptotic conditions of the standard type at the ends, gives a matrix element of  $\mathcal{U} \circ m_2$ . In part (b), each open-string worldsheet fans out into a region of a half-plane, with opening angle  $\pi$ , to describe the separate action of  $\mathcal{U}$  on  $\delta_1$  and  $\delta_2$  (in other words to describe  $\mathcal{U} \otimes \mathcal{U}$ ), and then, on a larger length scale, the two regions fit into a single half-plane, to describe  $M_2(\mathcal{U} \otimes \mathcal{U})$ . Solving the  $\zeta$ -instanton equation on the resulting region  $\mathbb{X}'$ , with the standard type of asymptotic condition, gives a matrix element of  $M_2 \circ (\mathcal{U} \otimes \mathcal{U})$ . To understand the relation between corresponding matrix elements of  $M_2 \circ (\mathcal{U} \otimes \mathcal{U})$  and of  $\mathcal{U} \circ m_2$ , we need to compare the counting of  $\zeta$ -instantons in the regions  $\mathbb{X}$  and  $\mathbb{X}'$  of figures 148(a) and (b).

Those regions are topologically the same, and differ only by a change of metric; moreover, this change of metric is trivial near infinity. As in the A-model the stress tensor is  $\mathcal{Q}$ -exact, so the change in the action resulting from a change in the metric is  $\mathcal{Q}$ -exact. After integration by parts, a  $\mathcal{Q}$ -exact term in the action leads to contributions in which the differential acts on initial or final states; these contributions make up the right hand side of eqn. (15.5). Thus the origin of the right hand side of this equation is precisely analogous to the origin of the right hand side of eqn. (10.48) of section 10.7.

A detailed explanation of the right hand side of eqn. (15.5) – by describing moduli spaces of solutions of the  $\zeta$ -instanton equation, rather than by a quantum field theory

argument using  $\mathcal{Q}$ -exactness – would proceed much like the explanation of eqn. (15.4). One would pick an interpolation between the two pictures of Figure 148(a,b), involving a new parameter  $u$ . Including  $u$  in the description, one would look at one-parameter families of solutions of the  $\zeta$ -instanton equations. Such families would have ends of four possible types, which would contribute to the four terms in eqn. (15.5). In particular,  $E$  would be computed by counting exceptional solutions of the  $\zeta$ -instanton equation that exist at special values of  $u$ .

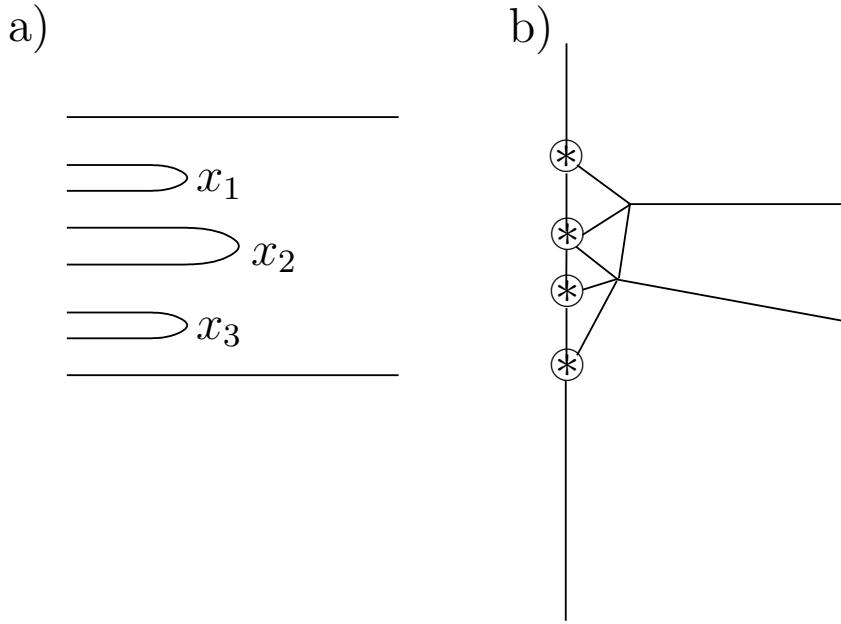
The map  $m_2$  induces a map on cohomology groups,  $\widehat{m}_2 : \mathcal{H}_{\mathfrak{B}, \mathfrak{B}'} \otimes \mathcal{H}_{\mathfrak{B}', \mathfrak{B}''} \rightarrow \mathcal{H}_{\mathfrak{B}, \mathfrak{B}''}$ , and likewise  $M_2$  induces a map  $\widehat{M}_2 : \mathcal{H}_{\mathfrak{B}, \mathfrak{B}'}^{\text{web}} \otimes \mathcal{H}_{\mathfrak{B}', \mathfrak{B}''}^{\text{web}} \rightarrow \mathcal{H}_{\mathfrak{B}, \mathfrak{B}''}^{\text{web}}$ . When we pass to cohomology, the right hand side of eqn. (15.5) drops out and we get  $\widehat{\mathcal{U}} \circ \widehat{m}_2 = \widehat{M}_2 \circ (\widehat{\mathcal{U}} \otimes \widehat{\mathcal{U}})$ . In other words, the isomorphism between the cohomology groups of the two categories coming from  $\widehat{\mathcal{U}}$  extends to an isomorphism between the two multiplication laws.

#### 15.4 The Higher $A_\infty$ Operations

The  $\sigma$ -model and web-based categories also have higher multiplication laws, which we have described in sections 11.3 and 5.2. In either case, one is given branes  $\mathfrak{B}_0, \dots, \mathfrak{B}_n$  of class  $T_\zeta$  and elements  $\delta_i \in \mathbb{M}_{\mathfrak{B}_{i-1}, \mathfrak{B}_i}$  or  $\delta_i^{\text{web}} \in \mathbb{M}_{\mathfrak{B}_{i-1}, \mathfrak{B}_i}^{\text{web}}$ ,  $i = 1, \dots, n$  which can be multiplied to get  $m_n(\delta_1 \otimes \delta_2 \otimes \dots \otimes \delta_n) \in \mathbb{M}_{\mathfrak{B}_0, \mathfrak{B}_n}$  or  $M_n(\delta_1^{\text{web}} \otimes \delta_2^{\text{web}} \otimes \dots \otimes \delta_n^{\text{web}}) \in \mathbb{M}_{\mathfrak{B}_0, \mathfrak{B}_n}^{\text{web}}$ . We want to compare these operations. First let us summarize the two definitions.

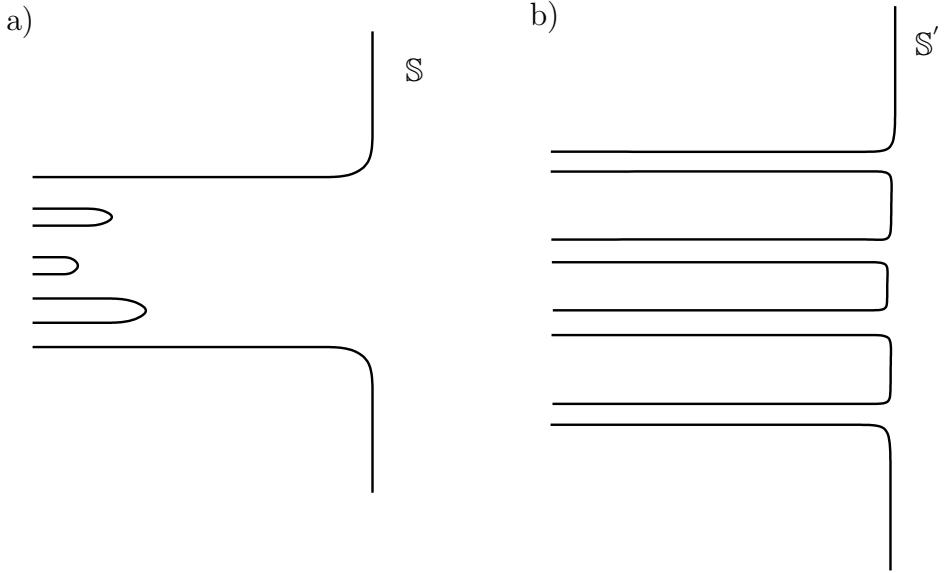
In the  $\sigma$ -model approach,  $m_n$  is defined by joining together  $n$  open strings (via a worldsheet with disc topology) to make a single string; one has to integrate over the  $n - 2$  real moduli that are involved in this gluing. Since we depict strings in the  $\sigma$ -model approach as propagating from left to right, we depict this gluing as in Figure 149(a). The moduli in the gluing are the differences between the horizontal positions  $x_i$  of the joining events. We should point out, however, that in an  $A$ -model, “integrating” over the moduli has a special meaning. Amplitudes are defined in the  $A$ -model by counting (with signs) solutions of the  $\zeta$ -instanton equation that have appropriate asymptotics. If the fermion numbers of external states are such that the matrix element of  $m_n$  that we are trying to compute is not trivially zero, then the expected dimension of the moduli space  $\mathcal{M}$  of  $\zeta$ -instantons is 0. Generically,  $\mathcal{M}$  then actually consists of finitely many (nondegenerate) points, which correspond to solutions that exist only for particular values of the  $n - 2$  moduli of Figure 149(a). Given this, the “integration” that is involved in evaluating a matrix element of  $m_n$  reduces to the sum of finitely many delta-function contributions, corresponding to these solutions.

In the web-based description, the higher multiplication operation  $M_n$  of eqn. (5.17) can be described as follows. The elements  $\delta_i^{\text{web}}$  that we wish to multiply correspond to morphisms attached to the  $\circledast$  symbols in Figure 149(b). To compute a matrix element of  $M_n$  from  $\delta_1^{\text{web}} \otimes \dots \otimes \delta_n^{\text{web}}$  to a given fan at infinity, one counts certain pictures of the form shown in the figure. (Recall the morphisms dictate a particular fan of solitons at each abstract vertex  $\circledast$ .) In the figure, as usual, all vertices are  $\zeta$ -vertices, except that the  $\circledast$  symbols represent abstract vertices. We only care about pictures like that of Figure 149(b) modulo the translation and scaling symmetries of the right half plane. We can use those symmetries, for example, to fix the top and bottom abstract vertices in the figure



**Figure 149:** The purpose of this picture is to sketch the higher  $A_\infty$  operations  $m_n$  and  $M_n$  in the  $\sigma$ -model approach and in the web-based approach. (We take  $n = 4$  for illustration.) (a) In the  $\sigma$ -model approach,  $m_n$  is defined by joining  $n$  open strings and “integrating” over the differences between the horizontal positions  $x_i$  at which this joining takes place. (This is simply Figure 129(b) of §11.3, but rotated by an angle  $\pi/2$ .) (b) In the web-based approach, the multiplication is defined by counting rigid pictures such as that shown here. In this picture, there are  $n$  abstract vertices labeled by the symbol  $\circledast$ . One can use the scaling and translation symmetries of the right half plane to place the first and last of them at specified values of  $\tau$ , say 1 and 0; otherwise, the positions of the abstract vertices are unspecified. As usual, in the web-based approach, the abstract vertices represent morphisms  $\delta_i \in \text{Hop}(\mathfrak{B}_{i-1}, \mathfrak{B}_i)$ . All lines and all other vertices in the figure represent  $\zeta$ -solitons or  $\zeta$ -instantons with appropriate asymptotics.

to specified positions, say  $\tau = 1$  and  $\tau = 0$ . However, the locations of the other abstract vertices are then “moduli,” over which we have to integrate. An integration over this moduli space is hidden in eqn. (5.17), in the following sense. If the fermion numbers of the initial and final fans are chosen so that the matrix element of  $M_n$  is not trivially zero, then (for generic superpotential) pictures of the form sketched in Figure 149(b) are rigid up to the translation and scaling symmetries of the right half plane, and each such picture determines the moduli – the positions of the abstract vertices – uniquely. In eqn. (5.17), no restriction is placed on the moduli and a given matrix element of  $M_n$  is determined by counting (in our present language) all rigid  $\zeta$ -webs with specified fans at the abstract vertices and at infinity. Since the moduli are unspecified, one can think of this procedure as an instruction to integrate over moduli space, where the integrand has a delta function contribution (with coefficient  $\pm 1$ ) at any point in moduli space at which an appropriate picture exists. This “integration” is precisely in parallel with the integration in the  $\sigma$ -model approach to the Fukaya-Seidel category.



**Figure 150:** In (a), we sketch the worldsheet that can be used to multiply  $n$  open strings in the  $\sigma$ -model approach (sketched here for  $n = 4$ ) and then map them to the web-based category. To instead map the open strings to the web-based category before multiplying them, we should use the worldsheet (b). The topological equivalence between the two pictures leads to the  $A_\infty$  equivalence between the two categories.

Now we would like to compare the result of first multiplying  $n$  open strings in the  $\sigma$ -model approach and then mapping to the web description, on the one hand, with the result of first mapping to the web description and then multiplying, on the other hand. In other words we want to compare  $\mathcal{U} \circ m_n$  to  $M_n \circ (\mathcal{U} \otimes \dots \otimes \mathcal{U})$ . The basis for the comparison is shown in Figure 150, in which part (a) shows the worldsheets that must be used to compute  $\mathcal{U} \circ m_n$ , and part (b) shows the worldsheets that must be used to compute  $M_n \circ (\mathcal{U} \otimes \dots \otimes \mathcal{U})$ . As in section 15.3, these worldsheets differ only by having different Kähler metrics (the worldsheets can actually be identified in such a way that the metrics are the same at infinity). Since the Kähler metric is irrelevant in the  $A$ -model, in some sense there is an equivalence between  $\mathcal{U} \circ m_n$  and  $M_n \circ (\mathcal{U} \otimes \dots \otimes \mathcal{U})$ . Technically the equivalence is a morphism of  $A_\infty$  algebras, a fact that can be understood as follows.

A change in Kähler metric changes the  $A$ -model action by a  $\widehat{\mathcal{Q}}_\zeta$ -exact term, which can be analyzed by integration by parts on the worldsheet. In the case of the multiplication law, this integration by parts leads to the terms  $\widehat{\mathcal{Q}}_\zeta^{\text{web}} E - E \widehat{\mathcal{Q}}_\zeta$  on the right hand side of (15.5), and certainly a term of this kind also appears in the difference  $M_n \circ (\mathcal{U} \otimes \dots \otimes \mathcal{U}) - \mathcal{U} \circ m_n$ . What is new for  $n > 2$  is that there are also moduli, and  $\mathcal{Q}$ -exact terms can contribute total derivatives on moduli space, leading to surface terms on moduli space. Thus,  $\mathcal{U}$  is extended to include maps  $\mathcal{U} : \mathbb{M}_{\mathfrak{B}_0, \mathfrak{B}_1} \otimes \dots \otimes \mathbb{M}_{\mathfrak{B}_{n-1}, \mathfrak{B}_n} \rightarrow \mathbb{M}_{\mathfrak{B}_0, \mathfrak{B}_n}^{\text{web}}$  (the map  $E$  in equation (15.5) is already the case  $n = 2$ ), and the result of these surface terms is that there will be extra terms in the identity:

1. The region where  $x_{k+1}, \dots, x_{k+k'} \ll x_1, \dots, x_k, x_{k+k'+1}, \dots, x_n$  in Figure 150(a) leads to terms of the form

$$\mathcal{U}(\delta_1, \dots, \delta_k, m_{k'}(\delta_{k+1}, \dots, \delta_{k+k'}), \delta_{k+k'+1}, \dots, \delta_n) \quad (15.6)$$

with  $k' < n$ .

2. The collision of abstract vertices in Figure 150(b) leads to operations  $\mathcal{U}(\delta_i, \dots, \delta_j)$  and terms of the form

$$M_s(\mathcal{U}(\delta_1, \dots, \delta_{k_1}), \mathcal{U}(\delta_{k_1+1}, \dots, \delta_{k_1+k_2}), \dots, \mathcal{U}(\delta_{k_1+k_2+\dots+k_{s-1}+1}, \dots, \delta_n)) \quad (15.7)$$

Taking all boundaries into account and recalling that  $\widehat{\mathcal{Q}}_\zeta = m_1$  and  $\widehat{\mathcal{Q}}^{\text{web}} = M_1$  we find that  $\mathcal{U}$  is simply an  $A_\infty$ -functor:

$$\sum_s \sum_{\text{Pa}_s(P)} M_s(\mathcal{U}(P_1), \dots, \mathcal{U}(P_s)) = \sum_{\text{Pa}_3(P)} \epsilon_{P_1, P_2, P_3} \mathcal{U}(P_1, m_{p_2}(P_2), P_3) \quad (15.8)$$

where  $P = \{\delta_1, \dots, \delta_n\}$  and  $p_k = |P_k|$ .

From a physical point of view, one would simply summarize the above discussion as follows (in the context of the  $A$ -model): An  $A_\infty$  algebra or category is a way of describing the tree level approximation to an open-string theory. Two open-string theories whose construction differs by  $\widehat{\mathcal{Q}}_\zeta$ -exact terms should be equivalent. A map from one open-string theory to another that maps the operations of one to the operations of the other, modulo  $\widehat{\mathcal{Q}}_\zeta$ -exact terms, should be an equivalence.

## 16. Local Observables

### 16.1 The Need For Unfamiliar Local Observables

The familiar local observables of the  $A$ -model with target  $X$  are associated to the cohomology of  $X$ . One way to describe the local operator corresponding to a given cohomology class makes use of a dual homology cycle. Indeed, if  $H \subset X$  is any oriented submanifold, one defines in the  $A$ -model a local operator  $\mathcal{O}_H$  as follows. In general,  $A$ -model observables on a two-manifold  $\Sigma$  are defined by counting holomorphic maps  $\Phi : \Sigma \rightarrow X$  (or solutions of a corresponding  $\zeta$ -instanton equation) that obey suitable conditions. An insertion of  $\mathcal{O}_H$  at a point  $p \in \Sigma$  imposes the condition that  $\Phi(p)$  must lie in  $H$ . The fermion number of  $\mathcal{O}_H$  is the codimension of  $H$ ; it is also the degree of the cohomology class dual to  $H$ . It is also possible, in a standard way, to define one-form and two-form descendants of  $\mathcal{O}_H$ .

However, even in the absence of a superpotential, these familiar  $A$ -model observables are not the whole story. To see this, let us consider the  $B$ -model of  $\mathbb{C}^* = \mathbb{R} \times \widetilde{S}^1$ , where  $\widetilde{S}^1$  is a circle and we regard  $\mathbb{C}^*$  as the quotient of the complex  $\tilde{\phi}$ -plane by  $\tilde{\phi} \sim \tilde{\phi} + 2\pi i$ . We recall that in general the most familiar local observables of the  $B$ -model with target  $X$  correspond to elements of  $H^p(X, \wedge^q TX)$ , where  $TX$  is the tangent bundle of  $X$ . To be more precise, an element of  $H^p(X, \wedge^q TX)$  corresponds to a local  $B$ -model observable of fermion number  $p+q$ . In particular, setting  $p=q=0$ , we have the holomorphic functions

$\mathcal{O}_n = e^{n\tilde{\phi}}$ ,  $n \in \mathbb{Z}$ , and these correspond to observables of fermion number 0.  $H^0(\mathbb{C}^*, T\mathbb{C}^*)$  is also infinite-dimensional, with sections  $\widehat{\mathcal{O}}_n = e^{n\tilde{\phi}}\partial_{\tilde{\phi}}$ . These correspond to observables of fermion number 1. (This completes the story, since  $H^1(\mathbb{C}^*, \wedge^q T\mathbb{C}^*) = 0$  for all  $q$ .)

By  $T$ -duality on the second factor of  $\mathbb{C}^* = \mathbb{R} \times \widetilde{S}^1$ , we can map the  $B$ -model of  $\mathbb{C}^*$  to the  $A$ -model of a dual  $\mathbb{C}^*$ , namely  $\mathbb{R} \times S^1$ , where  $S^1$  is the circle dual to the original  $\widetilde{S}^1$ . So the  $A$ -model of  $\mathbb{C}^*$  must also have infinitely many local operators of fermion number 0 or 1. The standard construction gives only one local operator of degree 0 and one of degree 1, since the cohomology of  $\mathbb{C}^*$  is of rank 1 in degree 0 or 1 and vanishes otherwise. What are all the other local observables of the  $A$ -model of  $\mathbb{C}^*$ ?

Going back to the  $B$ -model, the observables corresponding to  $\mathcal{O}_0$  and  $\widehat{\mathcal{O}}_0$  have momentum 0 around the circle in  $\mathbb{C}^* = \mathbb{R} \times \widetilde{S}^1$ , and they correspond to the standard  $A$ -model observables, i.e. the cohomology classes mentioned in the first paragraph of this section. But the operators  $\mathcal{O}_n$ ,  $\widehat{\mathcal{O}}_n$  with  $n \neq 0$  carry momentum around the circle. Their duals will have to correspond to  $A$ -model observables with winding number.

What is an  $A$ -model observable with winding number? It is a disorder operator, which creates a certain type of singularity in the holomorphic map (or solution of the  $\zeta$ -instanton equation) that is counted in the  $A$ -model. Let us take the worldsheet  $\Sigma$  to be the complex  $s$ -plane and let us parametrize the dual  $\mathbb{C}^*$  of the  $A$ -model by a complex variable  $\phi$  with  $\phi \sim \phi + 2\pi i$ . An  $A$ -model observable of winding number  $n$ , inserted at a point  $s = s_0$ , imposes a constraint that the holomorphic map  $\phi(s)$  should have a singularity at  $s \rightarrow s_0$

$$\phi(s) \sim n \log(s - s_0), \quad (16.1)$$

where  $n$  is the winding number. Indeed, this singularity of  $\phi(s)$  is characterized by the fact that as  $s$  loops around  $s_0$ ,  $\phi(s)$  loops  $n$  times around the second factor in  $\mathbb{C}^* = \mathbb{R} \times S^1$ .

Actually, the concept of a disorder operator in the  $A$ -model, which creates some sort of singularity in a holomorphic map, is much more general than the concept of an  $A$ -model observable that carries winding number. It is not difficult to give a more sophisticated example of a  $B$ -model with infinitely many observables such that the dual  $A$ -model has a target space with finite rank cohomology that moreover has a trivial or finite fundamental group.<sup>95</sup> In such a situation, most of the  $B$ -model observables will correspond to disorder operators of the  $A$ -model, but these operators cannot be characterized by their winding numbers. To describe them requires more insight about what sort of condition should be placed on the singularity of a holomorphic map.

We will not pursue this issue here. Instead, we will consider a more sophisticated example of an  $A$ -model that must have local observables of an unfamiliar sort. This example will provide a good illustration of ideas we develop later.

Our example is the  $B$ -model of  $\mathbb{CP}^1$ . In this example,  $H^0(\mathbb{CP}^1, \mathcal{O}) \cong \mathbb{C}$ ;  $H^0(\mathbb{CP}^1, T\mathbb{CP}^1)$  is of rank 3 (and corresponds to the Lie algebra of  $SL(2, \mathbb{C})$ ); and  $H^1(\mathbb{CP}^1, \wedge^q T\mathbb{CP}^1) = 0$ ,  $q = 0, 1$ . So the space of local observables has rank 1 in fermion number 0 and rank 3 in fermion number 1.

---

<sup>95</sup>A good example is the  $B$ -model on Hitchin moduli space in a complex structure other than  $\pm I$ . Consider the mirror duals of the characters of holonomies of the flat gauge field parametrized by the moduli space.

The  $B$ -model of  $\mathbb{CP}^1$  has a mirror that can be obtained [47] by  $T$ -duality on the orbits of a  $U(1)$  subgroup of the  $SL(2, \mathbb{C})$  symmetry group of  $\mathbb{CP}^1$ . This dual has target space  $\mathbb{C}^*$ , which we again parametrize by a complex variable  $\phi$  with  $\phi \sim \phi + 2\pi i$ . In the  $\sigma$ -model,  $\phi$  is promoted to a chiral superfield. The difference between the mirror of  $\mathbb{C}^*$  and the mirror of  $\mathbb{CP}^1$  is that in the latter case, there is a superpotential

$$W = e^\phi + e^{-\phi}. \quad (16.2)$$

The  $U(1)$  symmetry of the  $B$ -model that is used in the  $T$ -duality is converted to the winding number symmetry of the  $A$ -model of  $\mathbb{C}^*$ . (Except for the Weyl symmetry  $\phi \leftrightarrow -\phi$ , which will play an important role, the rest of the  $SL(2, \mathbb{C})$  symmetry of the  $B$ -model is not manifest in the  $A$ -model mirror.) So let us enumerate the  $U(1)$  charges of the  $B$ -model observables. This will tell us something about what to expect in the  $A$ -model. The observable of the  $B$ -model of  $\mathbb{CP}^1$  of fermion number 0 corresponds to the constant function 1; it is  $U(1)$ -invariant, so its dual will be an  $A$ -model observable of winding number 0. The three observables of the  $B$ -model of  $\mathbb{CP}^1$  with fermion number 1 correspond to the Lie algebra of  $SL(2, \mathbb{C})$ , so (if we normalize the  $U(1)$  generator to assign charges  $\pm 1/2$  in the two-dimensional representation of  $SL(2, \mathbb{C})$ ) they have  $U(1)$  charges 1, 0,  $-1$ .

This then gives our expectations for the  $A$ -model of  $\mathbb{C}^*$  with the superpotential of eqn. (16.2): there should be one observable of fermion number 0 and winding number 0, and three observables of fermion number 1 and winding number  $-1, 0, 1$ . We return to this example after developing the necessary machinery.

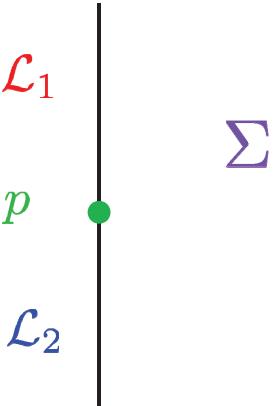
We have explained here one reason, based on  $T$ -duality, that in general closed-string disorder operators must be considered in the  $A$ -model with a noncompact target space. This argument could be adapted for open-string observables, but instead in section 16.2, we give a different explanation of why in the  $A$ -model with a superpotential, the natural class of local open-string observables includes disorder operators.

## 16.2 Local Open-String Observables

Local open-string observables in the presence of a superpotential can be studied rather directly using the construction already described in section 15.2. (To adapt this for closed strings will require some further ideas that are explained in section 16.3.)

An open-string observable is inserted at a point  $p$  in the boundary of a string worldsheet  $\Sigma$ . The point  $p$  locally divides the boundary into two pieces which in general are labeled by branes  $\mathfrak{B}_1$  and  $\mathfrak{B}_2$  that are supported on Lagrangian submanifolds  $\mathcal{L}_1$  and  $\mathcal{L}_2$  (Figure 151). In the ordinary  $A$ -model with compact target space  $X$ , the space of local operators that can be inserted at  $p$  is the same as the space of  $(\mathcal{L}_1, \mathcal{L}_2)$  strings. To argue that this is also true in the presence of a superpotential, we return to Figure 144(a) of section 15.2, where a strip comes in from the left of the  $s$ -plane and attaches to the right half-plane. In the limit that the strip is very narrow, the incoming string can be replaced by a local operator of some sort inserted at a point on the boundary, as in Figure 151. However, there is some subtlety in describing this local operator.

$A$ -model observables are computed by summing contributions of solutions of the  $\zeta$ -instanton equation (or simply the equation for a holomorphic map if the superpotential



**Figure 151:** The vertical line is the boundary of a string worldsheet  $\Sigma$ . A point  $p \in \Sigma$  divides the boundary in portions that are mapped to two different Lagrangian submanifolds  $\mathcal{L}_1$  and  $\mathcal{L}_2$ , which are the supports of branes  $\mathfrak{B}_1$  and  $\mathfrak{B}_2$ .

vanishes). Thus, whatever we compute in the picture of Figure 144(a) involves summing over solutions of the  $\zeta$ -instanton equation in the geometry  $\mathbb{W}$  of that picture. In the limit that the strip that comes in from the left in Figure 144(a) is very narrow,  $\mathbb{W}$  can be replaced by the right half-plane  $\Sigma$  of Figure 151 and a solution of the  $\zeta$ -instanton equation on  $\mathbb{W}$  converges to a solution on  $\Sigma$ . However, the limiting solution on  $\Sigma$  might have a singularity at  $p$ .

If the limiting solution has no singularity at  $p$ , we call the local operator  $\mathcal{O}$  that is inserted at  $p$  an order operator. In this case, the role of  $\mathcal{O}$  in the  $A$ -model is to put a constraint on the solution of the  $\zeta$ -instanton equation. We will explain this more precisely in a moment. Alternatively, the limiting solution might have a singularity at  $p$ . In that case, we call  $\mathcal{O}$  a disorder operator. Such a disorder operator is characterized by the precise type of singularity that occurs at  $p$ ; this is determined by the choice of the open-string state that comes in from the left in the original description on  $\mathbb{W}$ .

In the conventional  $A$ -model with compact target space (and therefore no superpotential), all local open-string (and closed-string) observables are order operators. We will recall why this is true and explain why it is not true in the presence of a superpotential. More fundamentally, we will see that in the presence of a superpotential, the distinction between order and disorder operators is not really natural for open-string observables of the  $A$ -model, in the sense that it is not invariant under independent Hamiltonian symplectomorphisms of  $\mathcal{L}_1$  and  $\mathcal{L}_2$ . This fact will provide useful background for our study of the closed-string case in section 16.3.

Let us look at Figure 151 and ask what an order operator might be. In doing this, we will assume for simplicity that  $\mathcal{L}_1$  and  $\mathcal{L}_2$  intersect transversely at finitely many points

$r_1, \dots, r_k$ . We can expect to reduce to this case<sup>96</sup> by applying suitable Hamiltonian symplectomorphisms to  $\mathcal{L}_1$  and  $\mathcal{L}_2$ . What can a solution of the  $\zeta$ -instanton equation on the half-plane  $\Sigma$  look like in this situation, assuming that it has no singularity at  $p$ ? The upper half of the boundary is mapped to  $\mathcal{L}_1$  and the lower half to  $\mathcal{L}_2$ . So the point  $p$  must be mapped to one of the intersection points  $r_1, \dots, r_k \in \mathcal{L}_1 \cap \mathcal{L}_2$ . For each such point, we can define a local operator  $\mathcal{O}_{r_k}$  that imposes a constraint that the point  $p$  must be mapped to  $r_k$ . It is not true necessarily true that the  $\mathcal{O}_{r_k}$  are  $A$ -model observables, because there might be a nontrivial differential acting on the space spanned by the  $\mathcal{O}_{r_k}$ . (This will happen, in particular, if  $\mathcal{L}_1$  and  $\mathcal{L}_2$  were chosen to have unnecessary intersections that could be deformed away.) Rather, one can define a complex with basis given by the  $\mathcal{O}_{r_k}$  such that the cohomology of this complex is the space of local  $A$ -model open-string observables.

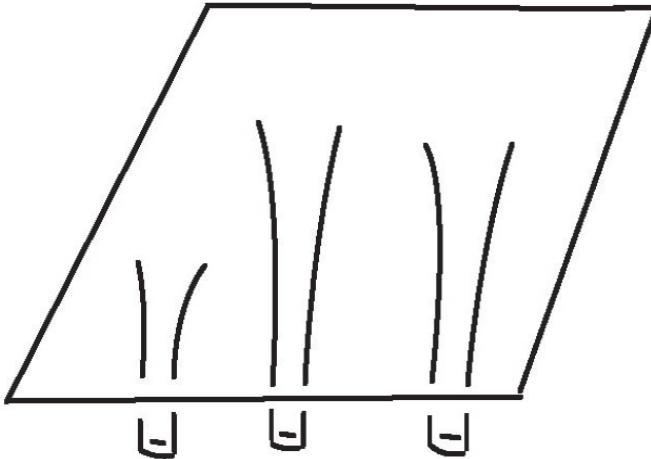
To see that all  $A$ -model open-string observables in the absence of a superpotential are order operators, let us just ask what is the space of  $(\mathfrak{B}_1, \mathfrak{B}_2)$  strings in the ordinary  $A$ -model without a superpotential. A classical zero energy state of an open-string, with the left endpoint mapped to  $\mathcal{L}_1$  and the right endpoint mapped to  $\mathcal{L}_2$ , is given by a constant map of the string to one of the intersection points  $r_k \in \mathcal{L}_1 \cap \mathcal{L}_2$ . So the MSW complex of the open strings has a basis corresponding to the  $r_k$ . The differential acting on this complex is the same as the differential acting on the corresponding space of operators  $\mathcal{O}_{r_k}$ . This follows by a standard argument involving a conformal mapping from a strip  $1 \geq \text{Im } s \geq 0$  in the complex  $s$ -plane, on which one defines the MSW complex, to the half-plane  $\Sigma$  of Figure 151. In this mapping, the left end of the strip (where an initial string state comes in from  $\text{Re } s = -\infty$ , as in Figure 144(a)) is mapped to a boundary point  $p$  at which a local operator is inserted.

The reason that this argument does not apply in the presence of a superpotential is that when there is a nontrivial superpotential the MSW complex does not have a basis corresponding to the intersections of  $\mathcal{L}_1$  and  $\mathcal{L}_2$ . Rather, as we explained in section 11.2.3, it has a basis corresponding to intersection points of  $\mathcal{L}_1^w$  with  $\mathcal{L}_2$ , where  $w$  is the width of the strip on which we quantize and  $\mathcal{L}_1^w$  is obtained from  $\mathcal{L}_1$  by Hamiltonian flow for “time”  $w$  with Hamiltonian  $\frac{1}{2}\text{Re}((i\zeta)^{-1}W)$ . The reason for the “extra” factor of  $-i$  is that far down the funnel of Figure 144(a) the relevant  $\zeta$ -instantons are (to exponentially good accuracy)  $x$ -independent, and therefore solve the soliton equation in the  $\tau$ -direction with phase  $\zeta \rightarrow -i\zeta$ . There is no simple correspondence in general between  $\mathcal{L}_1^w \cap \mathcal{L}_2$  and  $\mathcal{L}_1 \cap \mathcal{L}_2$ .

In order to define a local operator we need to take the limit  $w \rightarrow 0$  of the solution to the  $\zeta$ -instanton equation in the geometry of Figure 144(a). We consider  $(\mathfrak{B}_1, \mathfrak{B}_2)$  strings such that the corresponding supports  $\mathcal{L}_1, \mathcal{L}_2$  are both of class  $T_\zeta$ , as defined in Section §11.2.6 above. As long as  $w > 0$  there will be a finite number of points in  $\mathcal{L}_1^w \cap \mathcal{L}_2$ . However, it can happen that as  $w \rightarrow 0$  some of those points move off to infinity. If the intersection point moves off to infinity as  $w \rightarrow 0$  then the limiting solution  $\phi_{\lim}$  on the positive half-plane must be such that  $\phi_{\lim}(q) \rightarrow \infty$  as  $q \rightarrow p$ . Such solutions correspond to disorder operators. The remaining points in the intersection  $\mathcal{L}_1^w \cap \mathcal{L}_2$  will have smooth limits to intersection

---

<sup>96</sup>We can expect to do this even if  $\mathfrak{B}_1 = \mathfrak{B}_2$ . This case arises in mathematical work on the Arnold-Givental conjecture, and its special case, the Arnold conjecture. The work of A. Floer on the Arnold conjecture in the 1980’s was one of the original mathematical applications of the  $A$ -model.



**Figure 152:** Three tubes have been attached to the complex  $s$ -plane. Closed-string states will propagate in from these tubes. In the limit that the circumference of the tubes goes to zero, these closed-string insertions will converge to local closed-string operators of the  $A$ -model.

points in  $\mathcal{L}_1 \cap \mathcal{L}_2$ . Such solutions correspond to order operators, as in the standard  $A$ -model without superpotential. It can well happen that  $\mathcal{L}_1 \cap \mathcal{L}_2 = \emptyset$  and yet  $\mathcal{L}_1^w \cap \mathcal{L}_2$  is nonempty for all positive  $w$ . (For an example, consider the two Lagrangians  $v = u \pm \epsilon$  in the upper left region of Figure 128.) In this case all the local operators in  $\text{Hop}(\mathfrak{B}_1, \mathfrak{B}_2)$  are disorder operators.

We have just explained that the local operators in  $\text{Hop}(\mathfrak{B}_1, \mathfrak{B}_2)$  will, in general, consist of both order and disorder operators. Actually, this distinction is not invariant under Hamiltonian symplectomorphisms applied separately to  $\mathcal{L}_1$  and  $\mathcal{L}_2$ . The relevant symplectomorphisms  $\varphi$  in this case are those that preserve the region  $X_\zeta$  (and which are isotopic to the identity). Recall from Section §11.2.6 that  $X_\zeta$  is the preimage under  $W$  of the rectangle  $T_\zeta$ . As we saw in the example of Figure 125, such symplectomorphisms can map a pair  $\mathcal{L}_1, \mathcal{L}_2$  with nontrivial intersections to a pair  $\varphi(\mathcal{L}_1), \mathcal{L}_2$  with no intersections. Hence, while there is a well-defined space  $\text{Hop}(\mathfrak{B}_1, \mathfrak{B}_2)$  of boundary-condition-changing local operators, there is no invariant distinction between which operators are order operators and which operators are disorder operators. This fact will be useful background for our study of the closed strings. In that study, we will consider all local operators together, and it will not be very apparent which are order operators and which are disorder operators.

### 16.3 Closed-String Observables

#### 16.3.1 Twisted Closed-String States

For closed-string observables, there is an essentially new ingredient. We associated open-string observables to ordinary open-string states, but closed-string observables will be associated to what one might call twisted closed-string states.

By analogy with Figure 144(a), which was the starting point to define the map from open-string states to the corresponding local observables, we can start by considering a worldsheet  $\Sigma$  – we will take it to be the complex  $s$ -plane – to which semi-infinite tubes are attached (Figure 152). Closed-string states of some kind will be inserted at the ends of these tubes. In the limit that the tubes shrink to zero circumference, the tubes and the incoming closed-string states can be replaced by local operator insertions. The question is what kind of closed-string states we will have to use in this process.

The complex  $s$ -plane  $\Sigma$  with the tubes attached is conformally equivalent to the  $s$ -plane with finitely many points  $s_1, \dots, s_k$  removed. However, as the  $A$ -model with a superpotential is not conformally-invariant, we have to specify a Kahler metric on the punctured  $s$ -plane. We want this metric to be tubelike, so we choose something like

$$d\ell^2 = |ds|^2 \left( 1 + \sum_{i=1}^k \frac{f_i^2}{|s - s_i|^2} \right) \quad (16.3)$$

with small positive constants  $f_i$ .

To formulate the  $A$ -model with superpotential on the punctured  $s$ -plane, we need to study the  $\zeta$ -instanton equation. It will have the form

$$\bar{\partial}\phi^I - \bar{\xi} \frac{i\zeta}{4} g^{I\bar{J}} \frac{\partial \bar{W}}{\partial \bar{\phi}^{\bar{J}}} = 0, \quad (16.4)$$

where  $\bar{\xi}$  will be a  $(0, 1)$ -form that should be everywhere non-zero. In fact  $\bar{\xi}$  should be such that  $\text{vol} := \frac{i}{2}\xi \wedge \bar{\xi}$  is the positive  $(1, 1)$  form associated with the metric  $d\ell^2$  (where  $\xi$  is the complex conjugate of  $\bar{\xi}$ ). Indeed, let  $\bar{\mathcal{E}}^I$  denote the left-hand-side of (16.4). Then one can show that on any Riemann surface  $\Sigma$  with boundary:

$$\begin{aligned} -2i \int_{\Sigma} g_{I\bar{J}} \bar{\mathcal{E}}^I \mathcal{E}^{\bar{J}} &= \int_{\Sigma} g_{I\bar{J}} d\phi^I * d\bar{\phi}^{\bar{J}} + \frac{1}{4} \text{vol } g^{I\bar{J}} \partial_I W \partial_{\bar{J}} \bar{W} - 2 \int_{\Sigma} \phi^*(\omega) \\ &\quad - \int_{\Sigma} \text{Re}(\zeta \bar{W} \partial \bar{\xi}) + 2 \oint_{\partial\Sigma} \text{Re}(\zeta \bar{W} \bar{\xi}) \end{aligned} \quad (16.5)$$

where the Hodge  $*$  in line one is computed with the metric whose  $(1, 1)$ -form is  $\text{vol} = \frac{i}{2}\xi \bar{\xi}$ .

Returning to the metric (16.3), a simple choice for  $\bar{\xi}$  is:

$$\bar{\xi} = d\bar{s} \left( 1 + \sum_{i=1}^k \frac{f_i^2}{|s - s_i|^2} \right)^{1/2}. \quad (16.6)$$

This expression has norm one near the punctures and is free of zeroes. But it leads to “twisted closed-string states,” as we now explain.

To study the closed-string state that is inserted at  $s = s_j$ , we set  $z = i \log(s - s_j)$ , and write

$$z = x + i\tau, \quad 0 \leq x \leq 2\pi, \quad (16.7)$$

so

$$s - s_j = \exp(-iz) = e^\tau e^{-ix}. \quad (16.8)$$

Conventions have been chosen to match with Section §11, with  $x$  and  $\tau$  understood as “space” and “time” coordinates for the closed string and with Section §7 for wedge webs. In the limit that  $s \rightarrow s_i$ , that is, such that  $\tau \rightarrow -\infty$ , the  $\zeta$ -instanton equation (16.4) becomes

$$\frac{\partial \phi^I}{\partial \bar{z}} = -\frac{\zeta e^{ix}}{4} f_i g^{I\bar{J}} \frac{\partial \bar{W}}{\partial \bar{\phi}^J}. \quad (16.9)$$

Apart from a factor of  $f_i$  this differs from the standard form of the  $\zeta$ -instanton equation on the flat  $z$ -plane (i.e., with the substitution  $s \rightarrow z$  in eqn. (11.18)) in one important way:  $\zeta$  is replaced by  $\zeta_{\text{eff}} = i\zeta e^{ix}$ . Thus, effectively all values of  $\zeta$  are realized at a unique point along the circle parametrized by  $x = \text{Re } z$ .

This twisted version of the  $\zeta$ -instanton equation describes gradient flow in the  $\tau = \text{Im } z$  direction, with a superpotential that is the obvious generalization of eqn. (11.8):

$$h := - \int_0^{2\pi} dx \left( \lambda_a \frac{du^a}{dx} - \frac{1}{2} \text{Re}(-i\zeta^{-1} e^{-ix} f_i W) \right). \quad (16.10)$$

We call this a “twisted” version of the usual superpotential, where what is “twisted” is  $\zeta$  as a function of  $x$ .

The standard machinery of supersymmetric quantum mechanics and the MSW complex can be used with this superpotential to determine what we will call the twisted closed-string states. Twisted closed-string states are the ones that correspond to local closed-string observables of the  $A$ -model with a superpotential. In the absence of a superpotential, the twisted  $\zeta$ -instanton equation, just like the ordinary one, would reduce to the equation for a holomorphic map, and the twisted closed-string states would reduce to the standard closed-string states of the  $A$ -model.

For the twisted theory to be satisfactory, and to give a space of twisted closed-string states that has the expected invariances of the  $A$ -model we need to know that the critical points of  $h$  and the flows between them cannot go to or from infinity as one varies the Kahler metric of the target space  $X$  or of the tube parametrized by  $z$ . In later applications, we need an analogous compactness result for solutions of the  $\zeta$ -instanton equation on a more general worldsheet  $\Sigma$ , such as that of Figure 152, with tubes attached, and possibly with boundaries. A sufficient condition for such compactness arguments is that

$$|dW|^2 \gg |W| \quad (16.11)$$

at infinity on  $X$ . This condition is satisfied for most interesting choices of  $X$  (and its Kahler metric) and  $W$ . To show the relevance of the condition (16.11), we refer to equation (16.5) above. On the second line there is an extra term proportional to  $W$ . The condition (16.11) means that near infinity on  $X$ , this extra term is subdominant compared to the  $|dW|^2$  term that appears in the first line of equation (16.5). One expects that in a rigorous mathematical theory, this will give the compactness of moduli spaces of twisted  $\zeta$ -solitons and  $\zeta$ -instantons that is needed to justify the reasoning that follows.

### 16.3.2 Fans Of Solitons

In the above construction, the Kahler metric on the tube is  $R^2|dz|^2$ , where for the  $j^{th}$  closed-string insertion,  $R = f_j$ . Though the  $\sigma$ -model is not conformally-invariant, as always (given the compactness that was just explained) the cohomology of the MSW complex does not depend on the Kahler metric. The connection to local operators arises if  $R$  is small. If  $R$  is large, we can get a useful alternative description of the MSW complex. Almost everywhere along the circle, the  $\sigma$ -model fields will sit at a critical point  $i \in \mathbb{V}$ , corresponding to one of the vacua of the theory. An  $ij$  soliton involving a transition from vacuum  $i$  to vacuum  $j$  can occur at a unique angular position  $x$  along the circle, namely the position at which the effective value  $\zeta_{\text{eff}} = i\zeta e^{ix}$  of  $\zeta$  coincides with the value  $\zeta_{ji} = -i(W_j - W_i)/|W_j - W_i|$  (eqn. (11.15)) at which an  $ij$  soliton can actually occur. The condition for this is  $x = x_{ji}$  with

$$\frac{W_j - W_i}{|W_j - W_i|} = -\zeta e^{ix_{ji}}. \quad (16.12)$$

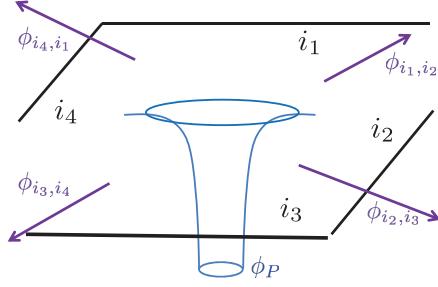
Using the relation between the vacuum weights of the web formalism and the critical values of  $W$ , which in this case becomes  $z_i(x) = \zeta_{\text{eff}} \bar{W}_i$ , we see that the  $x_{ji}$  of equation (16.12) correspond precisely to the binding points of type  $ij$  of Section §7. (See equation (7.5).)

Thus, when the radius of the tube is large compared to  $\ell_W$ , a basis of the MSW complex is given by a cyclic fan of solitons, arranged at preferred locations around the circle. This might not be a complete surprise, since in the abstract discussion of local operators in section 9, such a cyclic fan of solitons gave a basis for the complex that was used to describe local operators. (Vacua in such a fan were enumerated in clockwise order, which in view of eqn. (16.8) means in the order of increasing  $x$ .)

The fact that the solitons have a preferred location along the circle has an important implication for the fermion number of a twisted closed-string state. Usually, a soliton along the real line has an arbitrary position. That is so because the appropriate superpotential (11.7) does not depend on the position of the soliton. The position of the soliton is a bosonic collective coordinate that has two fermionic partners. Quantization of the fermion zero-modes gives a pair of states differing in fermion number by 1, and accordingly the quantum soliton really represents such a pair states.

Things are different for twisted closed-string states. The appropriate superpotential  $h$  of eqn. (16.10) depends on the angular position  $x_*$  of a soliton. For an  $ij$  soliton, it is stationary at  $x_* = x_{ji}$ . Moreover, as a function of  $x$ ,  $h$  has a local maximum at  $x_* = x_{ji}$  if  $R$  is large enough. This splits the degeneracy between the two states of the soliton. In general, in Morse theory, an unstable direction contributes +1 to the fermion number of the quantum state associated to a critical point. So a twisted closed-string state in the classical approximation corresponds to a cyclic fan of solitons, with the upper value of the fermion number chosen for each soliton. This coincides with the recipe of section 9, once we take into account (12.15).

To verify that  $h$  has a local maximum at  $x_* = x_{ji}$ , we will literally evaluate  $h$  as a function of the assumed position  $x_*$  of the soliton along the circle. Since the first term in  $h$



**Figure 153:** A tube is attached to the complex  $s$ -plane. Sketched is a solution of the  $\zeta$ -instanton equation on the  $s$ -plane with fanlike asymptotics at infinity on the  $s$ -plane and with a solution asymptotic to a twisted soliton at the end of the tube. The solution has no simple description in the tube (unless the radius of the tube is large), but asymptotically it is fanlike.

(namely  $-\oint \lambda_a du^a$ ) is rotation-invariant, we only have to examine the  $x_*$ -dependence of

$$\Delta h = \frac{1}{2} \int_0^{2\pi} dx \operatorname{Re} (-i\zeta^{-1} e^{-ix} W). \quad (16.13)$$

In this computation, assuming that the circle is very large compared to the internal structure of the soliton, we can ignore that internal structure and think of the soliton as a discontinuity in the fields, which we assume jump from one vacuum  $i \in \mathbb{V}$  to another vacuum  $j \in \mathbb{V}$  at  $x = x_*$  along the circle. We work in an interval  $I : x_0 \leq x \leq x_1$  that we assume contains only one soliton. More specifically, we assume the fields are in vacuum  $i$  for  $x_0 \leq x < x_*$  and in vacuum  $j$  for  $x_* < x \leq x_1$  for some point  $x_* \in I$ . As usual, we also write  $W_i$  and  $W_j$  for the values of  $W$  in vacua  $i$  and  $j$ . The contribution to  $\Delta h$  from the interval  $I$  is

$$\Delta h_I = \int_{x_0}^{x_*} dx \operatorname{Re} (-i\zeta^{-1} e^{-ix} W_i) + \int_{x_*}^{x_1} dx \operatorname{Re} (-i\zeta^{-1} e^{-ix} W_j). \quad (16.14)$$

The second derivative of this with respect to  $x_*$  is

$$\frac{\partial^2 \Delta h_I}{\partial x_*^2} = \operatorname{Re} (\zeta^{-1} e^{-ix_*} (W_j - W_i)). \quad (16.15)$$

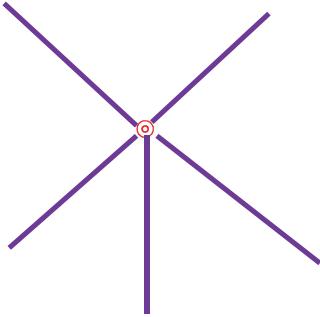
We want to evaluate this at  $x_* = x_{ji}$ , where  $x_{ji}$  satisfies eqn. (16.12). We get simply

$$\left. \frac{\partial^2 \Delta h_I}{\partial x_*^2} \right|_{x_* = x_{ji}} = -|W_j - W_i| < 0. \quad (16.16)$$

So  $h$  has a local maximum as a function of the assumed position of the soliton, and hence the quantum soliton must be placed in the state of upper soliton number.

To map the MSW complex of twisted closed strings to a complex of cyclic fans of solitons, we imitate the open-string construction of Figure 144 and equation (15.2). Thus, we seek to define a chain map

$$\tilde{\mathcal{U}} : \mathbb{M}_{S^1} \rightarrow \mathbb{M}_{S^1}^{\text{web}}. \quad (16.17)$$



**Figure 154:** A cyclic fan of solitons. The solitons emanate from a point that has been labeled  $\circledcirc$ . This is an abstract symbol that does not necessarily have an interpretation in terms of a solution of the  $\zeta$ -instanton equation on the  $s$ -plane, even a singular one.

Here  $\mathbb{M}_{S^1}$  is the MSW complex of twisted closed strings, that is, periodic solutions of the soliton equation (16.9), and

$$\mathbb{M}_{S^1}^{\text{web}} = R_c \quad (16.18)$$

where  $R_c$  is the complex defined in equation (9.2): It is the sum over all cyclic fans of solitons (where the singleton fan  $\{i\}$  corresponds to the vacuum  $\phi(x) = \phi_i$ ). In analogy to equation (15.2) we define  $\tilde{\mathcal{U}}$  by defining its matrix elements using the counting of  $\zeta$ -instantons. If  $\phi_P(x)$  is a twisted closed-string state, and  $\phi_{\mathcal{F}}$  is a cyclic fan of solitons then the matrix element between  $\phi_P(x)$  and  $\phi_{\mathcal{F}}$  is given by counting, with signs, the zero-dimensional components of the moduli space of the solutions to the following  $\zeta$ -instanton equation: We attach a tube to the  $s$ -plane as in Figure 153. We consider the  $\zeta$ -instanton equation (16.4) with the choice of  $\bar{\xi}$  in (16.6) (with  $k = 1$ ). The boundary conditions for  $s \rightarrow s_i$  (i.e.  $\tau \rightarrow -\infty$  in the coordinates (16.8)) are specified by the choice of twisted closed string state  $\phi_P(x)$ , while the boundary conditions for  $s \rightarrow \infty$  are specified by fan boundary conditions with data  $\phi_{\mathcal{F}}$ . We argue below equation (16.19) that  $\tilde{\mathcal{U}}$  is a chain map.

If the radius of the tube is large compared to  $\ell_W$ , a twisted closed-string state in the tube is itself labeled by a fan of solitons, as explained above. The MSW differential  $\widehat{\mathcal{Q}}_{\zeta}^{\text{MSW}}$  itself can be represented by counting webs on the large cylinder. The solution in Figure 153 has a simple description everywhere in terms of solitons: the solitons are located at fixed positions in the tube, and when the tube opens up to the  $s$ -plane, they remain at the corresponding angular positions in the  $s$ -plane.

On the other hand, to understand local closed-string operators, we want to consider the tube to have a small circumference. A solution of the  $\zeta$ -instanton equation in this situation can then be represented by a fan only at large distances on the  $s$ -plane. We

consider all such fans and label the vertex from which a fan emanates as  $\odot$  as shown in (Figure 154). We think of this as an abstract vertex analogous to the abstract vertices that were labeled  $\circledast$  in the open-string discussion.

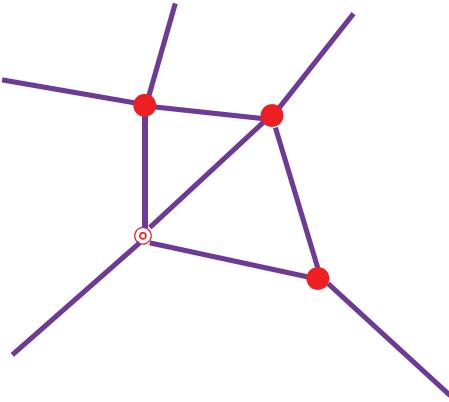
The abstract vertex does not necessarily represent a  $\zeta$ -instanton solution on the  $s$ -plane, not even a singular one. The reason for this statement is that the map  $\tilde{\mathcal{U}}$  from the twisted MSW complex to the set of cyclic fans is not necessarily injective or surjective if the tube has a small circumference. The only general statement, which we will justify in a moment, is that  $\tilde{\mathcal{U}}$  induces an isomorphism on cohomology; there is no simple general statement about how it acts on the complex. In the limit that the tube in Figure 153 has a small radius the  $\zeta$ -instanton sketched in that figure plausibly converges to a singular  $\zeta$ -instanton on the  $s$ -plane. But this does not give a natural way to interpret the  $\odot$  solution in terms of singular  $\zeta$ -instantons; when the radius is small, some cyclic fans of solitons might arise from more than one solution in Figure 153 and some might not arise at all. Just as  $\circledast$  is can be viewed as a receptacle for open string local operators, so too  $\odot$  can be viewed as a receptacle for local closed-string operators.

To prove that the map  $\tilde{\mathcal{U}}$  induces an isomorphism on cohomology, we simply repeat the reasoning from the open string case. First we show that

$$\mathcal{Q}_\zeta^{\text{web}} \tilde{\mathcal{U}} = \tilde{\mathcal{U}} \hat{\mathcal{Q}}_\zeta^{\text{MSW}}, \quad (16.19)$$

where  $\hat{\mathcal{Q}}_\zeta^{\text{MSW}}$  is the differential of the MSW complex, and we will recall in a moment the definition of the corresponding web differential  $\mathcal{Q}_\zeta^{\text{web}}$ . This means that the map  $\tilde{\mathcal{U}}$  between the two complexes induces a map on cohomology. To show eqn. (16.19), we consider 1-parameter families of  $\zeta$ -instantons in the geometry of Figure 153.

We now make an argument of a type that should be familiar; accordingly, we will be brief. The compact components of  $\mathcal{M}$  make no contribution. The noncompact components of  $\mathcal{M}$  will have two ends. Each of these ends be one of two types. One type of end arises from convolving a  $\zeta$ -instanton solution on the punctured plane with a tunneling event between two closed-string solitons far down the tube. Such a tunneling event represents a contribution to  $\hat{\mathcal{Q}}_\zeta^{\text{MSW}}$ . When it occurs far down the tube in the geometry of Figure 153, this gives a contribution to  $\tilde{\mathcal{U}} \hat{\mathcal{Q}}_\zeta^{\text{MSW}}$ , the right hand side of eqn. (16.19). The other possible type of end of  $\mathcal{M}$  arises from convolving a  $\zeta$ -instanton solution on the punctured plane with a taut  $\zeta$ -web, as pictured in Figure 155. In this picture, all vertices except the one labeled  $\odot$  are  $\zeta$ -vertices (i.e. interior amplitudes) and represent solutions of the  $\zeta$ -instanton equation with the indicated asymptotics. (The  $\zeta$ -web in the figure is taut if its only modulus, keeping fixed the position of the special vertex  $\odot$ , is the one derived from scaling.) The vertex labeled  $\odot$  simply represents a cyclic fan of solitons emanating from a  $\zeta$ -instanton solution on the punctured plane. The web differential  $\mathcal{Q}_\zeta^{\text{web}}$  in the abstract treatment of local operators (described in Section §9) was defined as follows. Its matrix element from a given “initial” fan of solitons to a given “final” one is obtained by counting taut webs that interpolate from the initial fan at the origin to the final fan at infinity. In this counting, one considers only taut webs and one weights each vertex by the



**Figure 155:** A cyclic fan of solitons. The solitons emanate from a point that has been labeled  $\circledcirc$ . This is an abstract symbol that does not necessarily have an interpretation in terms of a solution of the  $\zeta$ -instanton equation on the  $s$ -plane, even a singular one.

corresponding interior amplitude. With this definition of  $\mathcal{Q}_\zeta^{\text{web}}$ , an end of  $\mathcal{M}$  of the second type represents a contribution to  $\mathcal{Q}_\zeta^{\text{web}} \tilde{\mathcal{U}}$ .

A given component  $\mathcal{M}$  might have two ends of the same type. If so, they make canceling contributions to the left or to the right of eqn. (16.19). A component with ends of opposite types makes equal contributions to the left and to the right. As usual, any matrix element of either the left or the right can be interpreted as a sum of ends of one-parameter families of  $\zeta$ -instantons.<sup>97</sup> The identity of eqn. (16.19) comes from summing the contributions of ends of 1-parameter families of  $\zeta$ -instantons.

So  $\tilde{\mathcal{U}}$  induces a map,  $\hat{\mathcal{U}}$ , on cohomology. The fact that  $\hat{\mathcal{U}}$  is an isomorphism is proved by imitating the argument of Figure 146: one lets a tube fan out into a plane that then closes back into a tube, to give an inverse to  $\hat{\mathcal{U}}$ . The fact that  $\hat{\mathcal{U}}$  does not depend on arbitrary choices made in the construction is shown by an argument similar to the one discussed in relation to eqn. (15.4).

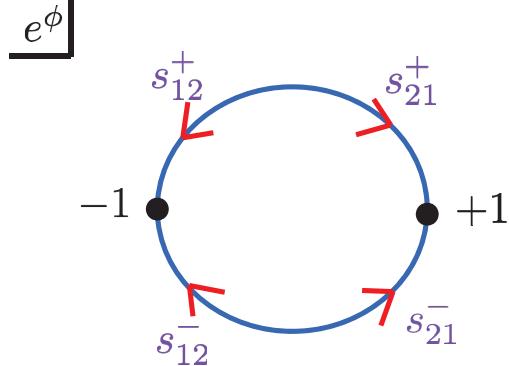
### 16.3.3 More On The Mirror of $\mathbb{CP}^1$

We now return to an example described in section 16.1: the  $A$ -model mirror of the  $B$ -model of  $\mathbb{CP}^1$ . This is a model with a single chiral superfield  $\phi$ , with an equivalence  $\phi \cong \phi + 2\pi i$  so that it parametrizes  $X = \mathbb{C}^*$ , and with superpotential  $W(\phi) = e^\phi + e^{-\phi}$ .

The Weyl group of the original  $SL(2, \mathbb{C})$  symmetry of the  $\mathbb{CP}^1$  acts by  $\phi \rightarrow -\phi$ . This symmetry plays a significant role, so we want to pick a Kahler metric on  $X$  that respects this symmetry. For example, we can use the flat Kahler metric  $d\ell^2 = |d\phi|^2$ .

---

<sup>97</sup>This is proved by the usual type of gluing argument. The convolution of a  $\zeta$ -instanton on the punctured plane with a tunneling event far down the tube or a very large  $\zeta$ -web is exponentially close to a  $\zeta$ -instanton solution and can be slightly perturbed to make an exact solution.



**Figure 156:** Illustrating the path of  $e^{\phi(x)}$  for the four nontrivial solitons.

The superpotential has two critical points at  $\phi = \phi_1 = 0$  and  $\phi = \phi_2 = \pi i$ . The values of the superpotential at the two critical points are  $W_1 = 2$  and  $W_2 = -2$ . In particular,  $W_1 - W_2$  is real, and this means that 12 and 21 solitons only occur when  $\zeta = \pm i$ . In turn this means that  $\text{Im } W$  will be a conserved quantity in the  $\zeta$ -soliton equation. Since  $\text{Im } W = 0$  at each critical point, the soliton trajectory will be one along which  $\text{Im } W$  is identically 0. The locus  $\text{Im } W = 0$  is the union of the real  $\phi$  axis, the line  $\text{Im } \phi = \pi$ , and the circle  $\text{Re } \phi = 0$ . (This is a circle because  $\phi \cong \phi + 2\pi i$ .) A trajectory on the real  $\phi$  axis or on the line  $\text{Im } \phi = \pi$  cannot connect the two critical points at  $\phi = 0$  and  $\phi = \pi i$ , but we can connect them by a trajectory on the circle  $\text{Re } \phi = 0$ . In fact, we can do this in two different ways: the soliton trajectory can be the “upper” segment  $0 \leq \text{Im } \phi \leq \pi$  or “lower” segment  $-\pi \leq \text{Im } \phi \leq 0$  (in each case with  $\text{Re } \phi = 0$ ). The  $\zeta$ -soliton equation on the real  $x$ -line determines a unique parametrization of these segments (as a function of  $x$ , and up to an additive constant) to make a 12 or 21 soliton. So there are two 12 solitons, say  $s_{12}^+$  and  $s_{12}^-$ , where the superscript refers to the upper or lower soliton trajectory, and likewise two 21 solitons,  $s_{21}^\pm$ . See Figure 156.

Now we can enumerate a basis for the MSW complex. There are two states consisting of trivial fans that simply sit at critical point 1 or 2 on the whole circle. And there are four states consisting of a fan of two solitons  $s_{12}^\pm$  and  $s_{21}^\pm$  that sit at antipodal points  $x_{12}, x_{21}$  on the circle, chosen to give the right values of  $\zeta_{\text{eff}}$ .

We would like to know the winding numbers and fermion numbers of these six states. For the winding numbers, this is immediate. The trivial fans have winding number 0. For the four fans  $s_{12}^\pm s_{21}^\pm$ , we observe that two of these four go forwards and then back along the upper or lower half of the circle, producing a net winding of 0. The other two states go all the way around the circle in one direction or the other, traversing first the upper half of the circle and then the lower half, or vice-versa. These two states have a net winding number of 1 and  $-1$ , respectively.

It is almost equally easy to determine the fermion numbers. The trivial fans correspond to states with fermion number 0. This is a universal statement about a massive  $A$ -model with a superpotential. A trivial fan corresponds to a closed-string state that lives in a particular vacuum  $i \in \mathbb{V}$ . Since the fermion number current is a Lorentz vector, its expectation value in any vacuum is 0 in infinite volume. In a massive theory, this statement is not significantly affected by compactification on a very large circle to get a closed-string state, so the state associated to a trivial fan always has fermion number 0.

What about the non-trivial fans of solitons? The fermion number of a fan is the sum of the fermion numbers of the individual solitons. The fermion number of an individual soliton is an  $\eta$ -invariant computed from the non-zero eigenvalues of the Dirac operator, plus a contribution  $\pm 1/2$  from the zero-modes. For the particular model under study here, with  $W = e^\phi + e^{-\phi}$ ,  $\phi$  imaginary in a soliton trajectory, and  $\zeta_{\text{eff}}$  imaginary, the appropriate Dirac operator  $\mathcal{D}$  of eqn. (12.6) is odd under complex conjugation (or alternatively under conjugation with the Pauli matrix  $\sigma^1$ ). This implies the spectrum is invariant under  $E \rightarrow -E$ , and accordingly the  $\eta$ -invariant computed from the non-zero modes vanishes.<sup>98</sup> Hence each soliton has two states of fermion number  $\pm 1/2$  from quantization of the zero-modes. As explained in section 16.3.2, to make a fan we take for each soliton the state of upper fermion number. In the present context, this means that the solitons all have fermion number  $1/2$  and hence that each of the four fans  $s_{12}^\pm s_{21}^\pm$  has a fermion number of 1.

In summary, the MSW complex for this problem has a basis consisting of six states, as follows: (1) there are two states of fermion number 0 and winding 0; (2) there are four states of fermion number 1 and winding number  $0, 0, 1, -1$ .

Since the differential of the MSW complex commutes with winding number, the two states of fermion number 1 and winding number  $\pm 1$  will certainly survive in the cohomology. However, there could be a nonzero differential acting on the states of fermion number 0 and winding number 0, mapping them to fermion number 1 and winding number 0. From the point of view of the MSW complex on the cylinder, it is not obvious what this differential would be. However, this differential was determined in the web-based formalism below equation (9.4), and this answer carries over to the MSW complex, since we identified the two complexes in section 16.3.2. Equation (9.4) shows that when there are only two vacua, the two states corresponding to trivial fans, which in that language are  $R_1$  and  $R_2$ , have up to sign the same non-zero image  $R_{12} \otimes R_{21}$  in fermion number 1. So precisely one linear combination of states of fermion number 0 survives in the cohomology. This corresponds to the identity operator of the field theory. Dually, the cohomology in fermion number 1 and winding 0 is one-dimensional, generated by any state of winding number zero that is not in the image of the one-dimensional space of winding number zero states obtained from the differential acting on  $R_1 \oplus R_2$ .

Accordingly, the cohomology of the MSW complex is as follows: (1) there is one state of fermion number 0 and winding 0; (2) there are three states of fermion number 1 and

---

<sup>98</sup>Since the real line on which the soliton is defined is not compact, the  $\eta$ -invariant is defined not just in terms of the eigenvalues of  $\mathcal{D}$  but also requires a regularization, such as the one in eqn. (12.7). With such a regularization, the fact that  $\mathcal{D}$  is odd under complex conjugation does imply that the nonzero eigenvalues do not contribute to  $\eta(\mathcal{D})$ .

winding  $1, 0, -1$ . This agrees with the prediction from mirror symmetry.

#### 16.3.4 Closed-String Amplitudes

Having come this far, it is not hard to see how to define closed-string amplitudes, making up what mathematically is called an  $L_\infty$  algebra. We simply imitate what we did in section 11.3 for open strings.

Imitating the open-string worldsheets of Figure 129, we need a suitable family of closed-string worldsheets that describes a transition from  $k$  twisted closed strings to a single twisted closed string.<sup>99</sup> For the necessary worldsheet  $\Sigma$ , we simply use the  $s$ -plane with punctures at  $s_1, \dots, s_k$ , but we omit the “1” from the metric and from the  $(0, 1)$ -form  $\bar{\xi}$ . Thus the metric becomes

$$d\ell^2 = |ds|^2 \sum_{i=1}^k \frac{f_i^2}{|s - s_i|^2} \quad (16.20)$$

and  $\bar{\xi}$  becomes

$$\bar{\xi} = d\bar{s} \left( \sum_{i=1}^k \frac{f_i^2}{|s - s_i|^2} \right)^{1/2}. \quad (16.21)$$

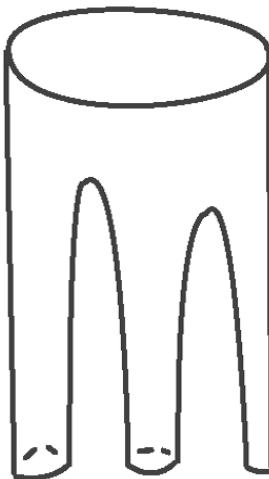
The  $s$ -plane with the metric (16.20) is sketched in Figure 157. The regions near  $s = s_i$ ,  $i = 1, \dots, k$  make up  $k$  tubes that we consider “incoming”; they join into a single “outgoing” tube, represented by the region near  $s = \infty$ .

We define a  $k$ -fold product that maps  $k$  states in the twisted closed-string MSW complex to a single one as follows. We choose  $k$  incoming states in the twisted MSW complex of the closed strings, and one such outgoing state. These  $k+1$  states determine the desired asymptotic behavior of a solution on  $\Sigma$  of the  $\zeta$ -instanton equation. We count the number of solutions (with signs, as always) of that equation, with the desired asymptotics. In the counting, we use the translation and scaling symmetries of the  $s$ -plane to set  $s_1 = 0$ ,  $s_2 = 1$ , but we leave  $s_3, \dots, s_k$  unspecified. The number of such solutions, with the stated asymptotics, and any values of  $s_3, \dots, s_k$ , gives the chosen matrix element of the  $k^{th}$   $L_\infty$  operation. The procedure is sketched in Figure 157.

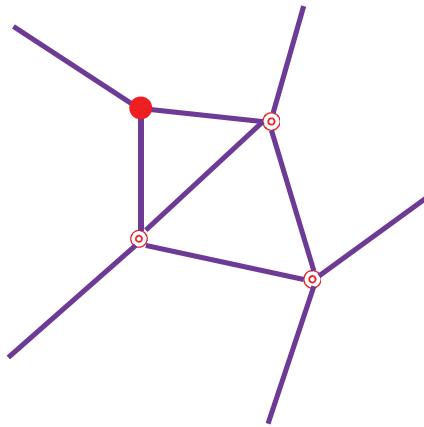
What we have stated is the standard procedure for defining closed-string amplitudes in topological string theory, adapted to a situation in which we need to specify more carefully what worldsheets should be used, since some of the symmetry is missing. This procedure

---

<sup>99</sup>In contrast to the open-string case, it appears difficult to define  $k \rightarrow m$  amplitudes with  $m > 1$ . The singularity of  $\bar{\xi}$  (eqn. (16.21)) near an incoming string at, say,  $s_j = 0$  is  $\bar{\xi} \sim d\bar{s}/|s|$ . We call this an incoming singularity. The singularity near  $s = \infty$ , in terms of  $t = 1/s$ , is  $\bar{\xi} \sim |t|d\bar{t}/t^2$ . We call this an outgoing singularity. For topological reasons, on a surface of genus 0, if  $\bar{\xi}$  has no zeroes and has only incoming and outgoing singularities, the number of outgoing singularities is precisely 1, though there may be any number of incoming singularities. To see this note that a nonzero  $\bar{\xi}$  defines a trivialization of  $T^{(0,1),*}\Sigma$ , where  $\Sigma$  is the surface without the singular points. At incoming singularities  $\bar{\xi}$  has zero winding number, and the trivialization may be extended over these points. On the other hand, at outgoing singularities  $\bar{\xi}$  has a winding number of  $-2$ . The sum of the winding numbers must equal the first Chern class of  $T^{(0,1),*}\bar{\Sigma}$  where  $\bar{\Sigma}$  is the surface with the singular points filled in. This is given by minus the Euler character of  $\bar{\Sigma}$ . Therefore there is precisely one outgoing puncture for  $g = 0$ , there are no outgoing punctures for  $g = 1$ , and for  $g > 1$  a  $\bar{\xi}$  with the assumed properties does not exist.



**Figure 157:**  $k$  tubes joining to a single one, sketched here for  $k = 3$ .



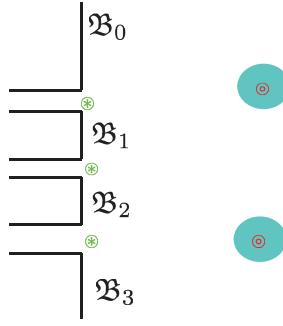
**Figure 158:** The  $s$ -plane with a web containing  $k$  abstract vertices denoted  $\circledcirc$  (here  $k = 3$ ). Additional vertices are  $\zeta$ -vertices. In this example, only one such  $\zeta$ -vertex is shown.

nevertheless defines an  $L_\infty$  algebra structure on the MSW complex  $\mathbb{M}_{S^1}$  of twisted closed string states.

To relate what we have just described to the web-based definition in section 9, we imitate what we did in the open-string case in section 15. First, if we deform the metric on  $\Sigma$  so that the tube in Figure 157 joins on a flat  $s$ -plane then we obtain a sequence of maps

$$\tilde{\mathcal{U}}_k : \mathbb{M}_{S^1}^{\otimes k} \rightarrow \mathbb{M}_{S^1}^{\text{web}} \quad k \geq 1 \quad (16.22)$$

Next we claim that these maps define an  $L_\infty$  morphism. (See Section §A.5.) To see this



**Figure 159:** The web-based  $LA_\infty$  algebra is reproduced by counting  $\zeta$ -instantons with abstract vertices (representing insertions of local operators) Each turquoise disk is removed and the boundary is smoothly attached to a semi-infinite cylinder.

we now deform the metric to a different picture in which  $k$  small tubes join separately to the asymptotically flat  $s$ -plane (this picture was sketched in Figure 152). In a limit in which the tubes are all widely separated compared to their sizes and to the natural length scale of the theory, a solution of the  $\zeta$ -instanton equation becomes weblike, but now with  $k$  abstract vertices as well as possible  $\zeta$ -vertices, as shown in Figure 158. In this description, to compute the  $k$ -fold  $L_\infty$  product, we have to count such webs, where two of the abstract vertices are placed at  $s_1 = 0$  and  $s_2 = 1$ , and the others are at unspecified points  $s_3, \dots, s_k$ . In the counting, each  $\zeta$ -vertex is weighted by the corresponding interior amplitude. But this is precisely the definition of  $\rho(t_p)$  used in defining the planar  $L_\infty$  algebra in Section §4.1 and the  $L_\infty$  algebra  $R_c$  in Section §9. The argument of Section §15.4 can now be imitated to establish that the  $\tilde{\mathcal{U}}_k$  define an  $L_\infty$ -morphism.

We can extend further the maps  $\tilde{\mathcal{U}}_k$  defined above to give an  $LA_\infty$  morphism of open-closed  $LA_\infty$  algebras. (See Section §A.6.) We now attach tubes to the geometries of the form shown in Figure 129 to define  $LA_\infty$  products on the Fukaya-Seidel morphisms  $\mathbb{M}_{\mathfrak{B}, \mathfrak{B}'}$  with  $\mathbb{M}_{S^1}$ . Then by attaching a geometry such as Figure 157 to, say, Figure 149(a) and counting  $\zeta$ -instantons we obtain a sequence of linear maps

$$\tilde{\mathcal{U}}_{n,m} : \mathbb{M}_{\mathfrak{B}_0, \mathfrak{B}_1} \otimes \cdots \otimes \mathbb{M}_{\mathfrak{B}_{n-1}, \mathfrak{B}_n} \otimes \mathbb{M}_{S^1}^{\otimes m} \rightarrow \mathbb{M}_{\mathfrak{B}_0, \mathfrak{B}_n}^{\text{web}} \quad n \geq 1, m \geq 0. \quad (16.23)$$

Then, by deforming these geometries to those such as the ones shown in Figure 159, (corresponding to the case  $n = 3$  and  $m = 2$ ) we can prove that the  $\tilde{\mathcal{U}}_{n,m}$  satisfy the axioms of an  $LA_\infty$ -morphism.

#### 16.4 Direct Treatment Of Local $A$ -Model Observables

This section is somewhat outside our main line of development. We will ask what we can learn by studying standard local  $A$ -model operators in a web framework. This is a little unnatural, for we have learned in sections 16.1 and 16.2 that in the presence of a superpotential, the standard local operators of the  $A$ -model are not the whole story. Still it seemed interesting to ask what we could say by directly studying the standard operators.

We will do this just in the special case of boundary local operators, associated to a particular brane  $\mathfrak{B}$  that is supported on a Lagrangian submanifold  $\mathcal{L}$ . These operators are associated to the cohomology of  $\mathcal{L}$ . Let  $H \subset \mathcal{L}$  be a homology class of real codimension  $r$ . The  $A$ -model has a corresponding boundary observable  $\mathcal{O}_H(\mathbf{w})$  which should be inserted at a point  $\mathbf{w}$  in a boundary  $\partial_{\mathcal{L}}\Sigma$  (of the Riemann surface  $\Sigma$  on which the  $A$ -model is defined) that maps to  $\mathcal{L}$ : this operator imposes a constraint that the point  $\mathbf{w}$  should map to  $H \subset \mathcal{L}$ . The descent procedure gives a corresponding 1-form operator  $\mathcal{O}_H^{(1)}$ , characterized by the condition

$$\{\mathcal{Q}_\zeta, \mathcal{O}_H^{(1)}\} = d\mathcal{O}_H(\mathbf{w}). \quad (16.24)$$

The insertion  $\int_{\partial_{\mathcal{L}}\Sigma} \mathcal{O}_H^{(1)}$  imposes a constraint that some point on  $\partial_{\mathcal{L}}\Sigma$  should map to  $H$ . Modulo  $\{\mathcal{Q}_\zeta, \cdot\}$ , the operators  $\mathcal{O}_H$  and  $\int_{\partial_{\mathcal{L}}\Sigma} \mathcal{O}_H^{(1)}$  depend only on the homology class of  $H$ . The fermion number of the operator  $\mathcal{O}_H$  is the codimension  $r$  of  $H$ , and that of  $\mathcal{O}_H^{(1)}$  is  $r - 1$ .

For a simple example, we will take  $\Sigma$  to be the strip  $[x_\ell, x_r] \times \mathbb{R}$ , and as usual define boundary conditions at the two ends of the strip by Lagrangian submanifolds  $\mathcal{L}_\ell, \mathcal{L}_r$ . We write  $\Sigma_\ell$  for the left boundary at  $x = x_\ell$ . For a given  $H \subset \mathcal{L}_\ell$ , and a choice of the point  $\mathbf{w} \in \Sigma_\ell$ , we attempt to calculate a matrix element

$$\langle f | \mathcal{O}_H(\mathbf{w}) | i \rangle. \quad (16.25)$$

where  $|i\rangle$  and  $\langle f$  are some initial and final states, whose details will not be important in what follows. First, we assume that  $H$  is a hypersurface in  $\mathcal{L}_\ell$ , of codimension 1, so that  $\mathcal{O}_H$  has fermion number 1. In this case, a nonzero contribution to the matrix element in (16.25) must come from a component  $\mathcal{M}$  of  $\zeta$ -instanton moduli space of dimension 1. Thus  $\mathcal{M}$  parametrizes a family of rigid  $\zeta$ -instantons, with time translation as the only modulus. To get a non-zero contribution to the given matrix element, we must adjust this modulus so that the  $\zeta$ -instanton maps the point  $\mathbf{w}$  to  $H$ .

A family of rigid  $\zeta$ -instantons is associated to a  $\zeta$ -web with rigid vertices as in figure 161. Let us discuss how  $\Sigma_\ell$  is mapped to the target space  $X$  by such a  $\zeta$ -instanton. There is a finite set  $P$  of points in  $\mathcal{L}_\ell$  that are boundary values of some half-line  $\zeta$ -soliton that interpolates between  $\mathcal{L}_\ell$  and one of the critical points  $i \in \mathbb{V}$ . Suppose that along  $\Sigma_\ell$ , there are a total of  $s$  rigid  $\zeta$ -vertices  $\mathcal{V}_\alpha$ ,  $\alpha = 1 \dots s$ . The intervals before and after these vertices (including the semi-infinite intervals with  $\tau \rightarrow \pm\infty$ ) are mapped to points in  $P$  (within an exponentially small error). We call these points  $p_0, \dots, p_s$  where  $p_{\alpha-1}$  is just before  $\mathcal{V}_\alpha$  (in time) and  $p_\alpha$  is just after.

Let us choose  $H$ , within its homology class, so that  $H$  does not intersect  $P$ . Then it is only within the vertices  $\mathcal{V}_\alpha$  that we may find a point on  $\Sigma_\ell$  that maps to  $H$ . Each  $\mathcal{V}_\alpha$  defines a half-plane  $\zeta$ -instanton that, when restricted to  $\Sigma_\ell$ , determines a path  $\rho_\alpha$  from  $p_{\alpha-1}$  to  $p_\alpha$ . Let  $n_\alpha^H$  be the intersection number of  $H$  with the path  $\rho_\alpha$ . (This intersection number is defined as usual by counting intersections with signs that depend on relative orientations.) This number depends not only on the homology class of  $H$  in  $\mathcal{L}$  but on its homology class in  $\mathcal{L} \setminus P$ , in other words it depends on how we chose  $H$  so as not to intersect  $P$ . To evaluate the contribution of a family  $\mathcal{M}$  of rigid  $\zeta$ -instantons to the matrix element

(16.25), we have to count (with signs) the points in  $\mathcal{M}$  that parametrize  $\zeta$ -instantons that map the point  $\mathbf{w} \in \Sigma_\ell$  to  $H$ . This counting is the same as the counting of the intersections  $H \cap \rho_\alpha$  (for all possible  $\alpha$ ), since any point  $y \in H \cap \rho_\alpha$  is the image of the point  $\mathbf{w} \in \Sigma_\ell$  for a unique  $\zeta$ -instanton in the family  $\mathcal{M}$ . (One simply slides the  $\zeta$ -instanton forwards and backwards in time until the point in  $\partial_\ell \Sigma$  that maps to  $y$  is  $\mathbf{w}$ .) So the contribution of  $\mathcal{M}$  to the matrix element (16.25) is given by

$$\pm \sum_{\alpha=1}^s n_\alpha^H. \quad (16.26)$$

As usual, the overall sign here is determined by the sign of a fermion determinant (and depends on the signs of the initial and final states  $|i\rangle$  and  $|f\rangle$ ). The full matrix element is obtained by summing this expression over all families  $\mathcal{M}$  of rigid  $\zeta$ -instantons that satisfy the boundary conditions.

An attentive reader might notice a puzzle here. If  $H$  is moved across one of the points  $p_\alpha$ ,  $\alpha = 1, \dots, s-1$ , which label an interval of  $\partial_\ell \Sigma$  between adjacent vertices  $\mathcal{V}_\alpha$  and  $\mathcal{V}_{\alpha+1}$ , then the integers  $n_\alpha^H$  and  $n_{\alpha+1}^H$  make equal and opposite jumps and the sum (16.26) does not change. However, this sum is not invariant if  $H$  crosses one of the points  $p_0$  or  $p_s$  at the ends of the chain, for then there is jumping only of  $n_1^H$  or  $n_s^H$ . The interpretation is as follows. In general, when  $H$  is replaced by another cycle in its homology class,  $\mathcal{O}_H(\mathbf{w})$  changes by  $\{\mathcal{Q}_\zeta, \mathcal{X}(\mathbf{w})\}$  for some  $\mathcal{X}$ . The matrix element  $\langle f | \mathcal{O}_H(\mathbf{w}) | i \rangle$  shifts by  $\langle f | \{\mathcal{Q}_\zeta, \mathcal{X}(\mathbf{w})\} | i \rangle$ , and for this to vanish, the states  $|i\rangle$  and  $|f\rangle$  must be  $\mathcal{Q}_\zeta$ -invariant. However, the family  $\mathcal{M}$  of  $\zeta$ -instantons whose contribution we have been evaluating would by itself contribute  $\pm 1$  to the matrix element  $\langle f | \mathcal{Q}_\zeta | i \rangle$ , since it parametrizes a one-parameter family of  $\zeta$ -instantons that interpolates between the state  $|i\rangle$  in the past and the state  $|f\rangle$  in the future. If it is true that  $\mathcal{Q}_\zeta |i\rangle = 0$ , then there must be a second one-parameter family  $\mathcal{M}'$  of  $\zeta$ -instantons also interpolating from  $|i\rangle$  in the past to  $|f\rangle$  in the future, and contributing to the matrix element of interest by a formula similar to (16.26) (in general with  $s$  replaced by some  $s'$  and the  $\mathcal{V}_1, \dots, \mathcal{V}_s$  replaced by another set of rigid  $\zeta$ -vertices  $\mathcal{V}'_1, \dots, \mathcal{V}'_{s'}$ ) but with a relative minus sign because of an opposite sign of the fermion determinant. In such a situation, the jumping when  $H$  moves across an endpoint of the chain cancels between  $\mathcal{M}$  and  $\mathcal{M}'$ , and the matrix element of  $\mathcal{O}_H(\mathbf{w})$  depends only on the homology class of  $H$ .

The moral of the story is that in evaluating the matrix element  $\langle f | \mathcal{O}_H(\mathbf{w}) | i \rangle$ , the operator  $\mathcal{O}_H(\mathbf{w})$  can be replaced by an instruction to count only  $\zeta$ -instantons in which one of the vertices  $\mathcal{V}_\alpha$  is located at the point  $\mathbf{w}$ , and to weight every such  $\zeta$ -instanton by a factor  $n_\alpha^H$ . Here the coefficients  $n_\alpha$  are not completely natural but depend on how  $H$  is chosen to not intersect the finite set  $P$ .

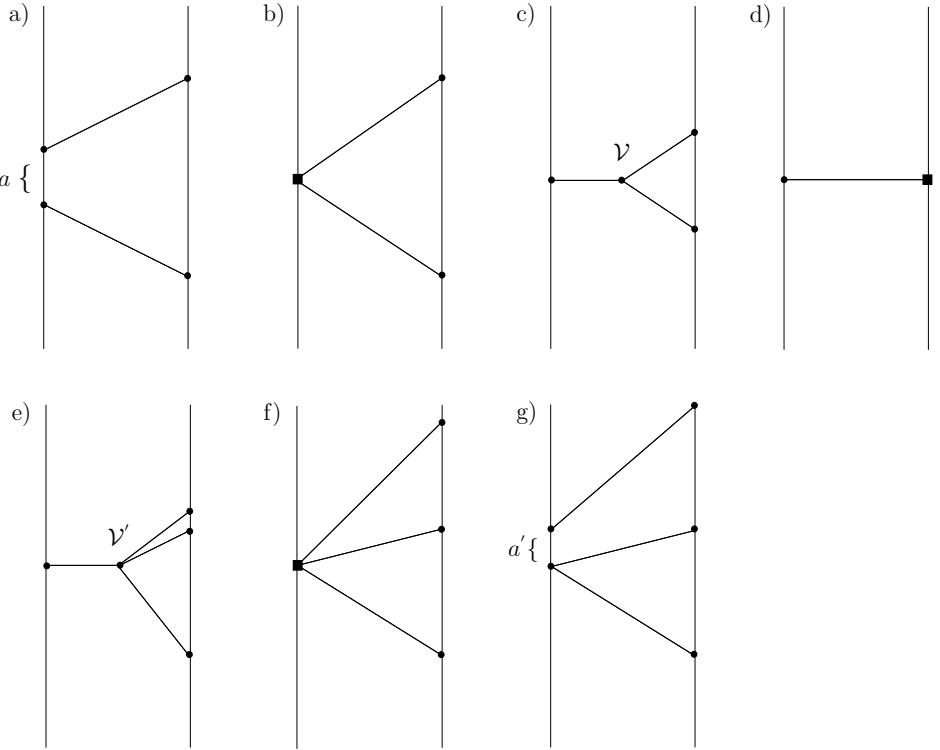
Now let us go on and consider the case that  $H \subset \mathcal{L}_\ell$  is of codimension 2. In this case, the matrix element (16.25) will be a sum of contributions of two-parameter moduli spaces  $\mathcal{M}$  of  $\zeta$ -instantons. There are two types of two-parameter moduli spaces on a strip. (1) In one case, we construct  $\zeta$ -instantons on the strip by gluing  $\zeta$ -vertices, one of which has an excess dimension 1, and the rest of which are rigid, using a rigid web  $\mathfrak{s}$ . (2) In the second case, we construct  $\zeta$ -instantons on the strip by gluing rigid  $\zeta$ -vertices via a strip web  $\mathfrak{s}$  that has a two-dimensional moduli space. (This gives a description of one region in

a two-parameter moduli space  $\mathcal{M}$ , and a full explanation will involve considering some of the other regions, as discussed below.) Each of the two cases can be relevant, in general, to evaluating the matrix element (16.25).

We consider first the case of a moduli space  $\mathcal{M}$  of type (1). This case is particularly interesting as it gives an example (the only example that will be studied in the present paper) in which a non-rigid  $\zeta$ -vertex is relevant. The relevant picture is again that of Figure 161, but now one of the  $\zeta$ -vertices on the left boundary, say  $\mathcal{V}_*$ , has an excess dimension of 1. Since  $H$  has codimension 2 in  $\mathcal{L}$ , we can choose it not to intersect any of the paths  $\rho_\alpha$  associated to the rigid  $\zeta$ -vertices  $\mathcal{V}_\alpha$  on  $\Sigma_\ell$ . With such a choice, the contribution of  $\mathcal{M}$  to the desired matrix element comes entirely from the non-rigid vertex  $\mathcal{V}_*$ . The part of  $\Sigma_\ell$  just before or after  $\mathcal{V}_*$  is mapped to points  $p, p' \in P$ , and  $\mathcal{V}_*$  parametrizes a 1-parameter family of paths from  $p$  to  $p'$ , sweeping out a two-manifold  $D \subset \mathcal{L}$ . Let  $n_*$  be the intersection number  $H \cap D$ . The contribution of  $\mathcal{M}$  to the matrix element (16.25) is then  $\pm n_*$ , where as usual the sign comes from a fermion determinant. This sort of contribution to the matrix element can be described by saying that the insertion of  $\mathcal{O}_H(\mathbf{w})$  receives a contribution in which this operator is replaced by an insertion of the non-rigid  $\zeta$ -vertex  $\mathcal{V}_*$  at  $\mathbf{w}$ , with amplitude  $n_*$ .

Now let us consider the contribution of a two-parameter moduli space  $\mathcal{M}$  of type (2). The reduced space  $\mathcal{M}_{\text{red}}$  of  $\mathcal{M}$  is 1-dimensional. When the strip  $[x_\ell, x_r]$  is very wide,  $\mathcal{M}_{\text{red}}$  is divided into regions each of which can be constructed by gluing of  $\zeta$ -vertices using a strip web  $\mathfrak{s}$  that has a 1-dimensional reduced moduli space. The reduced moduli space of such a  $\mathfrak{s}$  has two ends. At an end, the description of  $\mathcal{M}$  by the strip web  $\mathfrak{s}$  breaks down, but  $\mathcal{M}$  continues past this breakdown, with a new region that is described by a generically different web  $\mathfrak{s}'$ . So overall,  $\mathcal{M}$  has regions related to various webs  $\mathfrak{s}_\sigma$ . See Figure 160 for an example. In this example,  $\mathcal{M}$  has four web-like regions, with three transition regions between them. The transition regions are represented by pictures that look web-like, but they do not really represent  $\zeta$ -webs, since they each contain one “vertex” (labeled in the figure by a small square) that is not what we usually call a  $\zeta$ -vertex (it represents a family of  $\zeta$ -instantons with a one-dimensional reduced moduli space with ends that correspond to  $\zeta$ -webs). The reduced moduli space of  $\mathcal{M}$  might be compact or, as shown in the figure, it might have ends that correspond to time-convolution of webs, an operation considered in section 2.3.

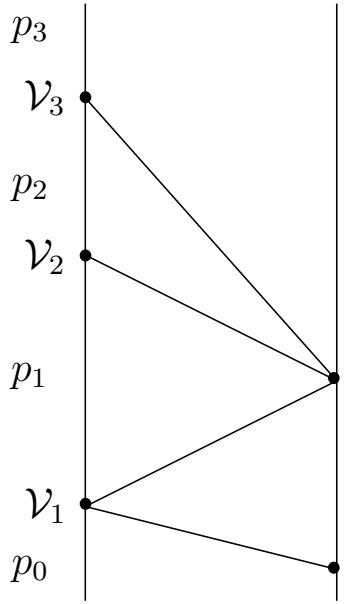
Assuming that  $H$  does not intersect any of the paths  $\rho_\alpha$  associated to  $\zeta$ -vertices, the web-like regions in a type (2) moduli space  $\mathcal{M}$  do not contribute to the matrix element of eqn (16.25). However, in general, the transition regions may contribute. The web-like regions represent  $\zeta$ -instantons that map  $\partial_\ell \Sigma$  to  $\mathcal{L}$  via a sequence of paths  $\rho_s \star \rho_{s-1} \star \dots \star \rho_1$  (where  $\star$  represents amalgamation of paths). A transition region of the type shown in Figure 160(b,f) (where the transition occurs along  $\Sigma_\ell$ ) involves a transition from one sequence  $\rho_k \star \rho_{k-1}$  (for some  $k$ ) between points  $p, p' \in P$  to another sequence  $\rho'_k \star \rho'_{k-1}$ . As the transition is made, the relevant family of  $\zeta$ -instantons sweeps out a two-dimensional surface  $D'$  that interpolates between  $\rho_k \star \rho_{k-1}$  and  $\rho'_k \star \rho'_{k-1}$ . The contribution of the transition region in question to the desired matrix element (16.25) is  $\pm H \cap D'$ , where as usual the sign is the sign of the fermion determinant. The contribution of  $\mathcal{M}$  to the matrix element



**Figure 160:** A two-parameter  $\zeta$ -instanton moduli space  $\mathcal{M}$  of type (2) is sketched here. (a) is a weblike region of this moduli space, with a single reduced modulus  $a$ , as indicated in the picture. For  $a \rightarrow 0$ , the web description breaks down and we get a region of the moduli space indicated in (b); the reduced modulus is hidden in the small square, which represents a family of half-plane  $\zeta$ -instantons with a one-dimensional reduced moduli space (a copy of  $\mathbb{R}$ ) that interpolates between (a) and (c). Here (c) is another web-like region, the reduced modulus being the horizontal position of the vertex labeled  $\nu$ . This web-like description breaks down when  $\nu$  reaches the right boundary; after another transition region (d), the reduced moduli space continues with still another web-like region (e), in which the reduced modulus is the horizontal position of the vertex  $\nu'$ . This description breaks down when  $\nu'$  reaches the left boundary. After one more transition region (f), the moduli space ends in one last web-like region (g), the reduced moduli space here being the distance  $a'$ . The reduced moduli space  $\mathcal{M}$  has two ends, corresponding to  $a \rightarrow \infty$  in (a) and  $a' \rightarrow \infty$  in (g). In the transition regions (b), (d), and (f), the small square on the left or right boundary does not represent a  $\zeta$ -vertex (whose reduced moduli space by definition has no ends) but rather a family of half-plane  $\zeta$ -instantons whose reduced moduli space is a copy of  $\mathbb{R}$ , with two ends, related to two different webs.

is the sum of these contributions, for all the relevant transition regions.

Combining what we have said about moduli spaces of types (1) and (2), an insertion of  $\mathcal{O}_H(\mathbf{w})$  can be represented in a weblike picture as a sum of insertions of effective  $\zeta$ -vertices. The effective  $\zeta$ -vertices in question can be either (1) non-rigid  $\zeta$ -vertices whose excess dimension is 1 and whose internal modulus is fixed by requiring that the point  $\mathbf{w}$  is mapped to  $H$ , or (2) effective  $\zeta$ -vertices that arise by using this constraint that  $\mathbf{w}$  maps to



**Figure 161:** A rigid strip web with left-boundary Lagrangian  $\mathcal{L}_\ell$ . The  $\zeta$ -vertices on the left boundary have been labeled as  $\mathcal{V}_1, \dots, \mathcal{V}_s$  (where  $s = 3$  in this example). A particular  $\zeta$ -instanton will be exponentially close to points  $p_0, \dots, p_s \in \mathcal{L}_\ell$  on the boundary segments between vertices.

$H$  to fix a modulus in a transition region between two  $\zeta$ -webs.

Though we will not try to develop a systematic theory in the present paper, hopefully we have said enough to convince the reader that it is possible to give a recipe to compute matrix elements of standard local observables of the  $A$ -model in terms of  $\zeta$ -webs. The intrepid reader can consider in a similar spirit the case that the codimension of  $H$  is greater than 2, the case that  $\mathcal{O}_H(\mathbf{w})$  is replaced by insertion of an integrated descendant  $\int \mathcal{O}_H^{(1)}$ , or the case that the boundary operator  $\mathcal{O}_H(\mathbf{w})$  is replaced by a bulk local observable of the  $A$ -model.

In this analysis, we found a role for  $\zeta$ -vertices with positive excess dimension. One may be puzzled, as they did not enter the seemingly more general analysis of section 16.2. This has happened because here we considered order operators only and labeled the entire left boundary of the worldsheet by a single Lagrangian submanifold  $\mathcal{L}$ . An order operator then places a constraint on the solution of the  $\zeta$ -instanton equation and can remove the excess moduli associated to a non-rigid  $\zeta$ -vertex. By contrast, in section 16.2, we always assumed that if an open-string observable is inserted at a boundary point  $p$  that separates regions labeled by Lagrangians  $\mathcal{L}_1$  and  $\mathcal{L}_2$  (Figure 151), then  $\mathcal{L}_1$  and  $\mathcal{L}_2$  intersect transversely, even if they are equivalent under Hamiltonian symplectomorphisms. It is then still true that an order operator places a constraint on the solution of the  $\zeta$ -instanton equation, but the interpretation in terms of non-rigid  $\zeta$ -vertices is hidden.

## 17. Interfaces And Forced Flows

Now we study a *family* of massive superpotentials (i.e. holomorphic Morse functions), all defined on a fixed target space  $X$ . Given such a family we can define a set of interesting supersymmetric interfaces between Landau-Ginzburg theories, thus illustrating the abstract ideas of Sections §6 - §8 in the concrete example of Landau-Ginzburg models.

We denote a typical superpotential by  $W(\phi; z)$  where in this section  $z$  is in a parameter space  $C$ . Some notable examples of  $C$  include the case where  $C$  is a Riemann surface, as in [29, 31, 33], or  $C = \text{Sym}^n(C_{uv})$ , with  $C_{uv}$  a Riemann surface, as in [32] and in Section §18.4 below. The parameter  $z$  is not to be confused with a vacuum weight. In fact, for a fixed  $z \in C$  the superpotential  $W(\phi; z)$  has critical points  $\phi_{i,z}$ , labeled by  $i \in \mathbb{V}$  and, introducing a phase  $\zeta$ , the critical values

$$z_i := \zeta \bar{W}(\phi_{i,z}; z) \quad (17.1)$$

define the vacuum weights of the corresponding Theory. The set of vacuum weights is the fiber of an  $N : 1$  covering space of  $C$  which we denote by  $\pi : \Sigma \rightarrow C$ . In general this covering space will have nontrivial monodromy. In the examples discussed in [29, 31, 33] the covering extends to a branched covering  $\pi : \bar{\Sigma} \rightarrow \bar{C}$ , where  $\bar{C}$  is a punctured Riemann surface. The superpotential is not massive at the branch points of this covering, so in this Section we avoid those points and just work on  $C$ . Some preliminary remarks on extending our considerations to the full branched covering are deferred to Section §18.3 below.<sup>100</sup>

Now let  $D \subset \mathbb{R}$  be a spatial domain and consider a continuously differentiable map  $z : D \rightarrow C$  with compact support for  $\frac{d}{dx}z(x)$  contained within the interval  $[x_-, x_+]$ . Just as in Section §11 we can define a  $1 + 1$  dimensional QFT by considering the supersymmetric quantum mechanics with real superpotential

$$h = -\frac{1}{2} \int_D [2\phi^*(\lambda) - \text{Re}(\zeta^{-1}W(\phi; z(x)))dx] \quad (17.2)$$

where  $\lambda = pdq$  is a Liouville one-form for the symplectic form on  $X$ . The resulting  $1 + 1$  dimensional QFT manifestly has the supersymmetries  $\mathcal{Q}_\zeta, \bar{\mathcal{Q}}_\zeta$  of equation (11.1). For  $x \leq x_-$  the integrand of  $h$  coincides with that associated to the LG theory determined by  $W(\phi; z_-)$  and for  $x \geq x_+$  it coincides with that associated to  $W(\phi; z_+)$ . Therefore we can consider this QFT to be the theory of a supersymmetric interface between the theories at  $z = z_-$  and  $z = z_+$ .

To make contact with the abstract part of the paper we denote the path traced out by  $z(x)$  in  $C$  by  $\wp$ . Along  $\wp$  there is a parallel transport of the vacua  $\phi_{i,z}$ , allowing us to define “local vacua”  $\phi_{i,x}$  for the theory at  $z = z(x)$ . The corresponding critical values are denoted  $W_{i,x}$ . Now, defining

$$z_i(x) = \zeta \bar{W}_{i,x} \quad (17.3)$$

we obtain a “vacuum homotopy,” in the language of Sections 7 and 8. It now follows from the results of Sections §§11-14 that we have a family of Theories. The representation of

---

<sup>100</sup>In the papers just cited, what is here called  $C$  would be called  $C'$ , while what is here called  $\bar{C}$  would simply be called  $C$ .

webs for the Theory at  $x$  is provided by the MSW complex for the superpotential at  $z(x)$  and the local interior amplitude  $\beta(x)$  for the Theory  $\mathcal{T}(x)$  is provided by the amplitude (14.35) for the LG theory  $W(\phi; z(x))$ . From the abstract discussion it follows that the resulting family of Theories defined by (17.3) have a corresponding Interface  $\mathfrak{I}[\wp]$ . We claim that this Interface is precisely the theory defined by (17.2). Our goal is now to describe this Interface in conventional Landau-Ginzburg terms.

To this end we follow once again the standard SQM interpretation of Morse theory. When  $D = \mathbb{R}$  we choose boundary conditions

$$\lim_{x \rightarrow +\infty} \phi(x) = \phi_{j', z_+} \quad (17.4)$$

$$\lim_{x \rightarrow -\infty} \phi(x) = \phi_{i, z_-} \quad (17.5)$$

where  $z_{\pm} := \lim_{x \rightarrow \pm\infty} z(x)$  and the vacua  $j'$  for  $x \geq x_+$  are to be compared to the vacua  $j$  for  $x \leq x_-$  by parallel transport on the covering space  $\Sigma$ . The prime on  $j'$  is meant to remind us that  $\phi_{j', z_+}$  are vacua in a Theory  $\mathcal{T}^+$  different from the vacua  $\phi_{i, z_-}$  of the Theory  $\mathcal{T}^-$ .

The stationary points of (17.2) are given by solutions to the differential equation

$$\frac{d}{dx} \phi^I = \frac{i\zeta}{2} g^{IJ} \frac{\partial \bar{W}}{\partial \bar{\phi}^J} (\bar{\phi}; z(x)) \quad (17.6)$$

but now there are some important differences from the  $\zeta$ -soliton equation. Compared to the old equation there is extra  $x$ -dependence on the right hand side of (17.6) due to the explicit  $x$ -dependence of  $z(x)$ . Moreover, the phase  $\zeta$  is a choice fixed from the beginning and we do not take it to be related to the phase  $\zeta_{j'i}$ . We call (17.6) the  $\zeta$ -forced flow equation.<sup>101</sup>

Following the usual Morse-theoretic interpretation of SQM we define the MSW complexes:

$$\mathbb{M}_{ij'}^\bullet(\wp) = \oplus_p \Psi_{ij'}^f(p) \mathbb{Z} \quad (17.7)$$

Once again,  $p$  enumerates the solutions  $\phi_{ij'}^p(x)$  of the forced flow equation with boundary conditions (17.4), (17.5). If  $z(x)$  has nontrivial  $x$ -dependence the equation (17.6) is no longer translation invariant and hence there will in general be a unique BPS ground-state  $\Psi_{ij'}^f(p)$ . The grading/fermion number will be given once again by the eta invariant:  $f = -\frac{1}{2}\eta(\mathcal{D})$  with  $\mathcal{D}$  given by (12.6). The differential on the complex is again given by computing solutions to the  $\zeta$ -instanton equation

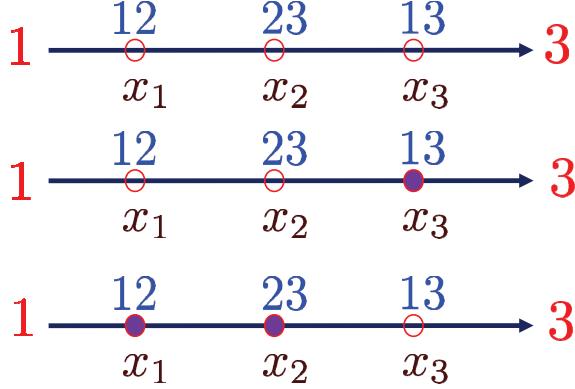
$$\left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial \tau} \right) \phi = \frac{i\zeta}{2} \frac{\partial \bar{W}}{\partial \bar{\phi}} (\bar{\phi}(x, \tau); z(x)) \quad (17.8)$$

interpolating between solutions whose fermion number differs by 1. The Chan-Paton complex of the Interface  $\mathfrak{I}[\wp]$  is now provided by the MSW complex:

$$\mathcal{E}(\mathfrak{I}[\wp])_{ij'} = \mathbb{M}_{ij'}^\bullet(\wp). \quad (17.9)$$

---

<sup>101</sup>The local vacua  $\phi_{i,x}$  are *not* to be confused with solutions to the forced flow equation (17.6).



**Figure 162:** An example of the bound-soliton basis for the complex  $\mathbb{M}_{13'}^\bullet(\phi)$ . In the first line the empty circles indicate potential positions of solitons at binding points  $x_1, x_2, x_3$ . We could potentially glue in a soliton of type 12 at  $x_1$ , 23' at  $x_2$  and 13' at  $x_3$ . In the second line, the 13' soliton is filled at  $x_3$  and the solution is approximately in the vacuum  $\phi_{1,x}$  for  $x < x_3$ . This represents one basis vector for the complex. In the third line the soliton at  $x_1$  of type 12 and that at  $x_2$  of type 23 are filled and the soliton is approximately in the vacuum  $\phi_{3',x}$  for  $x > x_2$ . This represents a second basis vector for the complex. We claim that in this example  $\mathbb{M}_{13'}^\bullet(\phi)$  will have precisely two generators of the space of states that undergo framed wall-crossing. They are represented by the above approximate solutions because there are no other ways the vacua of bound solitons can be compatible with the boundary conditions.

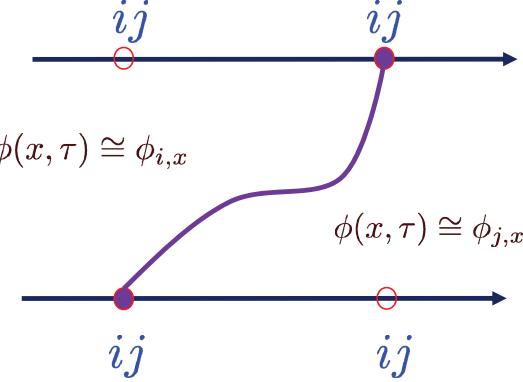
We now describe the origin of curved webs in the present context. We begin with a very useful picture of some elements of the Morse complex  $\mathbb{M}_{ij'}^\bullet(\phi)$ . Starting with the vacua of  $W(\phi; z_-)$  at  $x = -\infty$  we parallel transport the vacua along  $\phi$  to produce  $\phi_{i,x}$  with corresponding critical values  $W_{i,x}$ . Now, near values of  $x = x_0$  such that

$$\frac{W_{j_2,x_0} - W_{j_1,x_0}}{i\zeta} \in \mathbb{R}_+ \quad (17.10)$$

we can produce an approximate solution of the  $\zeta$ -forced flow equation by using a soliton solution of type  $\phi_{j_1,j_2}(x)$  for the superpotential  $W(\phi; z(x_0))$ . Moreover, we can choose the solution to be “centered” near  $x_0$ . (Such a center is only well-defined within a range of order  $1/m$ , where  $m$  is the mass scale of the interface.) Such solutions can be glued together to produce good approximations to true solutions satisfying the boundary conditions (17.4), (17.5) provided the intermediate vacua of subsequent solitons agree. Physically, such solutions correspond to boundstates of the solitons to the interface, binding near the special point  $x = x_0$ . This is the origin of the terminology *binding points* used in Section §7.4.1 and indeed the condition (17.10) is equivalent to the definition used in Section §7.4.1, given the vacuum homotopy (17.3).

A basis for the space of states in the MSW complex  $\mathbb{M}_{ij'}^\bullet(\phi)$  that can undergo framed wall-crossing can be obtained by attaching solitons to the binding points in all ways consistent with boundary conditions as described above. (It is clear that we can produce

solutions to (17.6) by gluing together such solitons, but it is not self-evident that these are the only solutions.) There is a simple pictorial formalism for these generatoris, illustrated in Figure 162.



**Figure 163:** An analog of the boosted soliton for the case of a supersymmetric interface.

Now we are in a position to describe the physical origin of curved webs. There will be time-independent solutions to the forced  $\zeta$ -instanton equation which, at long distances, have vertical worldlines of solitons with a single dot inserted at some  $\tau$ , analogous to Figure 131. In addition, there will be solutions analogous to the boosted solitons of Section §14.1. To describe these suppose we have (in some region of  $\varphi$  in  $C$ ) a family of solitons  $\phi_{ij}(x; z)$  satisfying the (ordinary) soliton equation for  $W(\phi; z)$  with phase  $\zeta_{ji}(z)$  given by the phase of  $W_j(z) - W_i(z)$ . Due to translation invariance of the soliton equation we can, moreover, assume that  $\frac{d}{dx}\phi_{ij}(x; z)$  has its support near  $x \cong 0$ . We now make an ansatz for the forced  $\zeta$ -instanton equation of the form:

$$\phi_{ij}(x - x(\tau); z(x(\tau))) \quad (17.11)$$

for some trajectory  $s(\tau) = x(\tau) + i\tau$  in the complex  $s$ -plane. If

$$\zeta^{-1} \frac{ds(\tau)}{d\tau} \frac{W_{j,x(\tau)} - W_{i,x(\tau)}}{|W_{j,x(\tau)} - W_{i,x(\tau)}|} = -1 \quad (17.12)$$

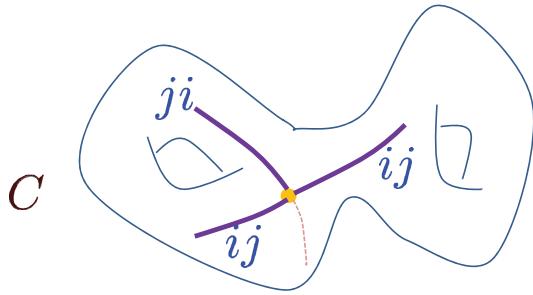
and at the same time

$$|\dot{x}(\tau) (\partial_z \phi_{ij} z' + \partial_{\bar{z}} \phi_{ij} \bar{z}')| \quad (17.13)$$

is small compared to other terms in the  $\zeta$ -instanton equation (as will be guaranteed if  $z(x)$  evolves adiabatically in  $x$ ) then the ansatz (17.11) will be a good approximation to the forced  $\zeta$ -instanton equation. The slope of the curve  $s(\lambda)$  defined by the center of the soliton will be parallel to  $z_{i,x(\lambda)} - z_{j,x(\lambda)}$  where  $z_{i,x}$  are the “local” values of the LG vacuum weights. These instantons may be depicted as in Figure 163 and are the main motivation for the definition of curved webs in Section §7.1. Of course, when there are three or more

vacua we will have local versions of the vertices used to construct  $\zeta$ -webs. These define  $\beta(x)$ , as noted above.

We can now follow the general discussion of Sections §7 and §8. The cohomology of the complex  $H^*(\mathbb{M}_{ij'}(\wp))$  is the space of “framed BPS states,” and its Witten index is the framed BPS index  $\underline{\Omega}(\mathcal{I}[\wp], ij')$ , in the language of [30, 31]. Thus the complex  $\mathbb{M}_{ij'}(\wp)$  “categorifies” the framed BPS indices.



**Figure 164:** When  $\bar{C}$  is the base of a branched cover of vacua there will be a real codimension two branch locus, indicated here by the orange dot. Three  $S$ -walls of type  $ij$  can terminate on a simple branch point of type  $ij$  as shown. There will be a monodromy exchanging vacua  $i$  and  $j$  around the branch locus and the pink dashed line indicates a cut for the trivialization of the cover we have chosen.

#### Remarks:

1. There is a nice interpretation of the meaning of the special binding points defined by (17.10). On the space  $C$  we can define “S-walls of type  $ij$ ” and phase  $\vartheta$  by the equation:

$$\{z : \frac{W_i(z) - W_j(z)}{i\zeta} \in \mathbb{R}_+\} \quad (17.14)$$

where  $\zeta = e^{i\vartheta}$  and the meaning of  $i, j$  depends on choosing a local trivialization of the cover  $\pi : \Sigma \rightarrow C$ . Then the binding points correspond to the values of  $x$  where the path  $\wp$  crosses the various S-walls. In fact, as we have noted, in the physical examples motivating this construction it is useful to consider  $C$  embedded into a family  $\bar{C}$  such that the cover  $\pi : \bar{\Sigma} \rightarrow \bar{C}$  is a branched cover. In this case the  $S$ -walls can end on a branch locus as in Figure 164. What we are describing here is a piece of a spectral network. See Section §18.2 below for further discussion.

2. If we now consider a family of paths  $\wp_s$  with variable endpoint  $z_f(s)$  then when that endpoint passes across an  $S$ -wall there will be wall-crossing of the framed BPS states: Physically, as the parameters of the supersymmetric interface are changed, it will emit or absorb some of the solitons associated to the binding point. This physical

picture was described in [31], but now we can describe it at the level of complexes - so again we have, in some sense, categorified the wall-crossing story. We did this in the abstract half of the paper when we defined the Interfaces  $\mathfrak{S}_{ij}^{p,f}$  in Section §7.6 to describe a categorified notion of “ $S$ -wall crossing.”

3. As a simple example, consider the family of theories with superpotential

$$W = \frac{1}{3}\phi^3 - z\phi \quad (17.15)$$

corresponding to a family of theories of the type  $\mathcal{T}^N$ , with  $N = 2$ , discussed in Section §4.6. The family is parametrized by  $z \in C$  with  $C = \mathbb{C}^*$ . There are two massive vacua at  $\phi_{\pm} = \pm z^{1/2}$  where we choose the principal branch of the logarithm and  $z$  is not a negative real number. We choose a path  $\varphi$  defined by  $z(x)$  in  $\mathbb{C}^*$  where  $x \in [\epsilon, 1 - \epsilon]$  for  $\epsilon$  infinitesimally small and positive with  $z(x) = e^{i(1-2x)\pi}$ . The spinning vacuum weights satisfy  $z_{-+}(x) = \frac{4}{3}\zeta e^{i(2x-1)3\pi/2}$ . If we also take  $\zeta$  to have a small and positive phase then, applying the criterion of (7.14) we find that there are two binding points of type  $+-$  at  $x = 1/3 - 0^+, 1 - 0^+$  and one binding point of type  $-+$  at  $x = 2/3 - 0^+$ . They are all future stable. The two  $S_{+-}$  and one  $S_{-+}$  walls emanate from  $z = 0$  with angle  $2\pi/3$  between them. There is a corresponding family of Interfaces  $\mathfrak{I}_x = \mathfrak{I}[\varphi_x]$  given by the path  $\varphi_x$  that evolves along  $\varphi$  from  $\epsilon$  to  $x$ . The wall-crossing formula for the framed BPS indices amounts to a simple matrix identity:

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad (17.16)$$

where the three factors on the LHS reflect the wall-crossing across the three  $S_{ij}$ -rays, and the matrix on the right accounts for the monodromy of the vacua. (See Section 8.1.1 of [31] for an extended discussion.) We now use equations (7.26) and (7.28) together with the web representation provided by (4.79):

$$\begin{aligned} R_{-+} &= \mathbb{Z}[f_1] \\ R_{+-} &= \mathbb{Z}[f_2] \end{aligned} \quad (17.17)$$

where the fermion number shifts  $f_1, f_2$  must satisfy  $f_1 + f_2 = 1$ , since  $K$  must have degree minus one. The categorification of the wall-crossing identity (17.16), at least at the level of Chan-Paton complexes, is obtained by generalizing the left-hand-side of (17.16) to:

$$\begin{pmatrix} \mathbb{Z} & 0 \\ \mathbb{Z}[f_2] & \mathbb{Z} \end{pmatrix} \begin{pmatrix} \mathbb{Z} & \mathbb{Z}[f_1] \\ 0 & \mathbb{Z} \end{pmatrix} \begin{pmatrix} \mathbb{Z} & 0 \\ \mathbb{Z}[f_2] & \mathbb{Z} \end{pmatrix} = \begin{pmatrix} \mathcal{E}_{--} & \mathcal{E}_{-+} \\ \mathcal{E}_{+-} & \mathcal{E}_{++} \end{pmatrix} \quad (17.18)$$

Here  $\mathcal{E}_{-+} = \mathbb{Z}[f_1]$ , while

$$\mathcal{E}_{--} = \mathcal{E}_{++} = \mathbb{Z} \oplus \mathbb{Z}[f_1 + f_2] \quad (17.19)$$

is a complex with a degree one differential (note that  $f_1 + f_2 = 1$ ) and

$$\mathcal{E}_{+-} = \mathbb{Z}[f_2] \oplus \mathbb{Z}[f_2] \oplus \mathbb{Z}[f_2 + 1] \quad (17.20)$$

is another complex with a degree one differential. The differential comes from combining two boundary amplitudes of the form in Figure 77 in a way similar to what happens in Figure 92. The matrix of complexes (17.18) is quasi-isomorphic to the categorified version of the monodromy:

$$\begin{pmatrix} 0 & \mathbb{Z}[1-f_2] \\ \mathbb{Z}[f_2] & 0 \end{pmatrix}. \quad (17.21)$$

The identity (17.16) is important in the nonabelianization map of [33] when extending the construction across branch points of a spectral covering. We expect the above identity to be important in the extension of this construction to the categorified context.

## 18. Generalizations, Potential Applications, And Open Problems

### 18.1 Generalization 1: The Effect Of Twisted Masses

In the main text of the paper we have encountered examples of Theories enriched by conserved global symmetries, namely the  $\mathcal{T}_\vartheta^{SU(N)}$  Theories. These global symmetries could be identified in their physical counterparts (see Section §4.6.4) either as winding symmetries of an LG theory with a non simply connected target space  $X \subset (\mathbb{C}^*)^N$ , or as isometries of the target space in a mirror description as a  $\mathbb{C}P^{N-1}$  sigma model.

In either case, the underlying physical theories admit a special class of relevant deformations which deform the supersymmetry algebra and are mirror to each other. The LG superpotential can be deformed to a multivalued function with single-valued first derivatives:

$$W = \sum_a Y_a + m_a \log Y_a \quad (18.1)$$

The  $\mathbb{C}P^{N-1}$  sigma model can be deformed by twisted masses  $m_a$  sitting in the Cartan subalgebra of the  $SU(N)$  global symmetry. In either case, the mass parameters  $m_a$  can be interpreted as the expectation value of scalar components of background gauge supermultiplets coupled to the corresponding global symmetries: twisted vector multiplets [38] in the A-type description and vectormultiplets in the mirror B-type description.

Abstractly, the  $\mathcal{N} = 2$  algebra in two dimensions allows a central charge  $Z = \{\bar{Q}_+, \bar{Q}_-\}$  and a twisted central charge  $\tilde{Z} = \{Q_+, \bar{Q}_-\}$ . Either one of these breaks one of the two  $U(1)$   $R$ -symmetries of the theory. As the framework of the present paper requires such a  $U(1)$   $R$ -symmetry, we can include one or the other of these but not both; because of the mirror symmetry of the  $\mathcal{N} = 2$  algebra, it does not matter which we include. By convention, we have assumed in this paper that  $\tilde{Z} = 0$ ,  $Z \neq 0$ .

In general, every  $(2, 2)$  theory equipped with global symmetries which can be coupled to background vectormultiplets will admit twisted mass deformations, which modify the twisted central charge  $\tilde{Z} = \{Q_+, \bar{Q}_-\}$  to include contributions proportional to the corresponding global charges. Dually, every  $(2, 2)$  theory equipped with global symmetries which can be coupled to background twisted vectormultiplets will admit mass deformations which modify the central charge  $Z = \{\bar{Q}_+, \bar{Q}_-\}$  to include contributions proportional

to the corresponding global charges. We are not aware of a specific pre-existing naming convention for the mirror notion to twisted masses. In the context of this paper, it seems reasonable to dub them A-twisted masses and denote the usual twisted masses as B-twisted masses.

In order to keep  $\tilde{Z} = 0$ , we will only allow A-twisted mass deformations. The general expression for the central charge in a massive  $(2, 2)$  theory with A-twisted masses  $M$ , for states which interpolate between vacua  $i$  and  $j$  and carry charge global  $\gamma$  is

$$Z = W_i - W_j + M \cdot \gamma \quad (18.2)$$

Here the charge  $\gamma$  is valued in a lattice  $\Gamma$  of global charges and  $M$  is valued in the Cartan subalgebra of the global symmetry group. This expression is a bit ambiguous: the superpotential vevs  $W_i$  and  $W_j$  are defined up to integral shifts of  $M$ , which are related to the possibility to re-define the global symmetry charge by some  $\gamma \rightarrow \gamma + \gamma^{(i)} - \gamma^{(j)}$ . We can fix the ambiguity by selecting specific values for  $W_i$  and  $W_j$ .<sup>102</sup>

A crucial new physical phenomenon in the presence of A-twisted masses is the existence of charged BPS particles which live in a specific vacuum, rather than interpolating between two vacua. In particular, these charged BPS particles modify the standard Cecotti-Vafa wall-crossing formalism. The correct wall-crossing formula is a specialization of the 2d-4d wall-crossing formula of [31] where the charge lattice is restricted to contain flavor charges only. Intuitively, the  $S_{ij}$  factors for standard BPS solitons are refined to keep track of global charges and new  $K_\gamma$  factors are introduced to account for the contribution to bound states of whole Fock spaces of BPS particles of charge  $\gamma$ .

We are thus presented with the natural problem of extending our formalism to theories with A-twisted masses. In the context of LG theories, there is a simple way to recast the problem which allows one to employ directly much of our standard formalism: one can replace the target space  $X$  with some minimal cover  $\hat{X}$  on which  $W$  is single-valued.

Generically, the fiber of such cover is naturally a torsor for the lattice  $\Gamma^f$  of global charges and  $\Gamma^f$  acts by deck transformations. The theory has a finite set  $\mathbb{V}$  of vacua corresponding to the critical points of  $W$  on  $X$  (these critical points are well-defined even though  $W$  is only single-valued up to an additive constant). The set  $\mathbb{V}$  of vacua is covered by the set  $\hat{\mathbb{V}}$  of critical points of  $W$  regarded as a function on  $\hat{X}$ . A point in  $\hat{\mathbb{V}}$  is a point in  $\mathbb{V}$  – labeling a critical point  $p \in X$  – together with a choice of lift of  $p$  to  $\hat{X}$ . The weights attached to elements of  $\hat{\mathbb{V}}$  which cover a given vacuum  $p$  will take the schematic form  $z_p + m \cdot \gamma$ , where we use the charge  $\gamma$  to label the possible lifts of  $p$ .<sup>103</sup>

The first obstruction one encounters in applying our formalism to build a Theory associated to the infinite set of vacua  $\hat{\mathbb{V}}$  is the infinite proliferation of possible webs. This obstruction, though, must be purely formal. It is clear that any physical calculation, say of a strip differential in the underlying LG theory, must only involve a finite set of terms in each matrix element. After all, the complexes may even be defined over the integer

---

<sup>102</sup>Even better, we can embrace the ambiguity by taking the charges of solitons to live in a torsor for the lattice of global charges, see i.e. [31]

<sup>103</sup>If we pick a reference sheet,  $\gamma$  will be an element of  $\Gamma^f$ . If not,  $\gamma$  will be an element of some torsor  $\Gamma_p^f$ .

numbers. Some finiteness principle must limit the number of webs which may occur as  $\zeta$ -webs in a given physical model.

We expect that such physical restrictions will be encoded in the abstract web formalism by adding some extra selection rules to the vacuum data which select some subset of all possible webs which still satisfies all the required convolution identities but has better finiteness properties. A simple example could be a list of allowed pairs of elements in  $\widehat{\mathbb{V}}$  which restricts which edges are allowed in the webs. Such a restriction is compatible with convolution identities and preserves our algebraic structures.

A second, more significant obstruction is the need to accommodate charged BPS particles within the web representation data. In principle, an abstract edge of slope  $m \cdot \gamma$  may represent the trajectories of multiple particles of charges proportional to  $\gamma$ . Thus the representation data will have to include both  $R_{p,p',\gamma}$  spaces encoding standard BPS solitons and extra Fock spaces encoding charged BPS particles. The notions of pairing, representations of fans, etc. will have to be adjusted accordingly.

A categorical wall-crossing formula adapted to this deformed context will have to include a categorification of  $K_\gamma$  factors. It would be interesting to find out the corresponding generalization of the notions of exceptional collections and mutations. We can sketch here some basic idea for such a generalization, which we can dub “flavoured exceptional collection”.

- We still expect to have a collection of basic objects  $\mathfrak{T}_p$  attached to vacua in  $\mathbb{V}$ . The spaces of morphisms between these objects will be graded by the lattice  $\Gamma^f$ .
- The Hop spaces will have a graded triangular structure, i.e. the  $\gamma$ -graded subspace  $\text{Hop}_\gamma(\mathfrak{T}_p, \mathfrak{T}_{p'})$  vanishes if  $z_{pp'} + m \cdot \gamma \notin \mathcal{H}$ .
- The categorification of  $S$  walls will be encoded by “partial mutations”, involving only the  $\gamma$ -graded subspace  $\text{Hop}_\gamma(\mathfrak{T}_p, \mathfrak{T}_{p'})$  associated to the weight which is entering/exiting  $\mathcal{H}$ .
- The categorification of  $K_\gamma$ , walls will be encoded by “categorical reflections” which completely reorganize the triangular structure of the collection, as  $m \cdot \gamma$  enters/exits  $\mathcal{H}$ .

## 18.2 Generalization 2: Surface Defects, Spectral Networks And Hitchin Systems

One of the main motivations for the present work was the desire to categorify the 2d/4d wall-crossing formula for BPS states associated surface defects in four-dimensional  $\mathcal{N} = 2$  theories. For background see [31, 75]. One way to produce such defects is to consider an embedding of two-dimensional Minkowski space  $\mathbb{M}^{1,1}$  into  $\mathbb{M}^{1,3}$  and to couple a 1 + 1 dimensional field theory with  $(2, 2)$  supersymmetry, supported on the embedded  $\mathbb{M}^{1,1}$ , to the ambient four-dimensional theory.

A  $(2, 2)$  defect with a  $U(1)$   $R$ -symmetry, leftover from the bulk  $SU(2)_R$  symmetry, has much in common with a massive  $(2, 2)$  theory deformed by A-twisted masses. The low

energy bulk gauge symmetries play a similar role to the global symmetries and the central charge includes a contribution from the bulk gauge charges. Schematically,

$$Z = W_i - W_j + Z_\gamma \quad (18.3)$$

where  $Z_\gamma$  is the bulk central charge for a particle of gauge and flavor charge  $\gamma$ .

A simple  $\Omega$ -deformation in the plane orthogonal to the defect (even in the absence of an actual  $(2, 2)$  defect) is expected to reduce the system to an effective  $(2, 2)$  theory [77], breaking down the bulk BPS particles to infinite towers of angular momentum modes, each behaving as a 2d BPS particle.

Based on such an analogy, we expect it should be possible to develop a consistent web formalism to study the space of ground states of the system in the presence of boundary conditions or interfaces for the 2d defect, or even for the bulk theory. Compared to the 2d setup with A-twisted masses of Section §18.1, the new ingredient will be the presence of bulk Abelian gauge fields. Even at the level of the 2d/4d wall-crossing formula the effect of the bulk Abelian gauge fields is rather dramatic: the  $K_\gamma$  factors commute in the 2d setup, but not in the full 2d/4d setup. We expect the effect to be equally dramatic in the full categorical setup. Although we do not know how to construct such a generalized 2d/4d web formalism, we can describe some possible applications, especially those which only involve the categorification of  $S_{ij}$  transformations.

In theories of class S, characterized by a triplet of data  $(\mathfrak{g}, \overline{C}, D)$ , where  $\mathfrak{g}$  is a Lie algebra of ADE type,  $\overline{C}$  is a punctured Riemann surface, and  $D$  is a collection of codimension two defects located at the punctures of  $\overline{C}$ , there is a canonical surface defect  $\mathbb{S}_z$  associated to a point  $z$  on the ultraviolet curve  $\overline{C}$ . Its origin in  $M$  theory is a semi-infinite  $M2$  brane whose boundary is  $\mathbb{M}^{1,1} \times \{z\}$ . In some regions of parameters this surface defect can be viewed as an LG model coupled to the ambient four-dimensional theory.

Let us recall the basic mathematical setup for the theory of canonical surface defects in theories of class  $S$  [42, 3, 29, 31]. We begin with the data of an  $N : 1$  branched cover  $\pi : \overline{\Sigma} \rightarrow \overline{C}$ . As before we let  $C$  be  $\overline{C}$  minus the branch points. Physically,  $\pi : \overline{\Sigma} \rightarrow \overline{C}$  is the covering of the Seiberg-Witten curve over the UV curve  $\overline{C}$  and mathematically  $\overline{\Sigma}$  is the spectral cover associated with a Hitchin system. Families of 1+1-dimensional LG models  $\mathbb{S}_z$  parametrized by  $z \in C$  also fit into this framework. In such cases, the ambient four-dimensional theory is trivial. Whether or not the ambient theory is trivial, vacua of the 1 + 1 dimensional defect theory  $\mathbb{S}_z$  are labeled by the sheets  $z^{(i)}$  of the covering,  $i = 1, \dots, N$ , and hence we identify

$$\mathbb{V}(\mathbb{S}_z) = \pi^{-1}(z). \quad (18.4)$$

In the present paper - with the notable exception of the Section §18.1 - an ordered pair of vacua  $(i, j)$  with  $i, j \in \mathbb{V}$  uniquely determines a soliton sector for the theory on a spatial domain  $\mathbb{R}$ . By contrast, in the theory of the surface defect  $\mathbb{S}_z$ , the soliton sectors are labeled, roughly speaking, by homology classes of paths on  $\overline{\Sigma}$  connecting the vacua  $z^{(i)}$  and  $z^{(j)}$ . To be slightly more precise, they are labeled by equivalence classes of open chains  $\mathbf{c} \subset \overline{\Sigma}$  with the constraint that  $\partial \mathbf{c} = z^{(j)} - z^{(i)}$ . The set of these “charges” is a sublattice

of a relative homology lattice

$$\Gamma(z^{(i)}, z^{(j)}) \subset H_1(\bar{\Sigma}; \mathbb{Z}), \quad (18.5)$$

and is a torsor for a sublattice  $\Gamma$  in  $H_1(\bar{\Sigma}; \mathbb{Z})$ . For example, when  $\mathfrak{g} = A_1$ , and the Seiberg-Witten curve is a two-fold cover,  $\Gamma$  is the anti-invariant sublattice of  $H_1(\bar{\Sigma}; \mathbb{Z})$  under the deck transformation.<sup>104</sup>

The central charge associated to a soliton sector  $\gamma_{ij} \in \Gamma(z^{(i)}, z^{(j)})$  is just

$$Z_{\gamma_{i,j}} = \frac{1}{\pi} \int_{\gamma_{i,j}} \lambda \quad (18.6)$$

where  $\lambda$  is the Seiberg-Witten differential (the canonical Liouville form for the natural holomorphic symplectic structure on  $T^*\bar{C}$ ).

The lattice  $\Gamma$  is, physically, the character lattice of the group which is the product of the unbroken gauge symmetry and the continuous global flavor symmetry of the four-dimensional theory, i.e. the lattice of gauge and global charges of the bulk theory. The central charge  $Z_\gamma$  of the bulk theory is simply given by the contour integral of  $\lambda$  along  $\Gamma$ .

The expression for  $Z_{\gamma_{i,j}}$  is a slightly more canonical version of equation 18.3. Although it does not look like a difference of two weights, all the essential constructions in the paper only involve differences of vacuum weights, and not the vacuum weights themselves, it is not, strictly speaking, necessary to identify particular vacuum weights. The edges of webs can simply be labeled by  $\gamma_{i,j}$  (so that the cyclic sum of charges around a vertex is zero). The phases of  $Z_{\gamma_{i,j}}$  suffice to define the slopes of the edges. Similarly, the generalization of the complexes  $R_{ij}$  used in a representation of webs is a set of complexes  $R_{\gamma_{ij}}$ . The contraction  $K : R_{\gamma_{ij}} \otimes R_{\gamma'_{ji}} \rightarrow \mathbb{Z}$  is a symmetric degree  $-1$  perfect pairing when  $\gamma_{ij} + \gamma'_{ji} = 0$ , and so on.

The theory of Interfaces has a very natural formulation in theories of class S and this was, in fact, one of the main motivations for the discussion in Section §17 above. To each path  $\wp$  in  $C$  connecting  $z_1$  to  $z_2$  and a choice of phase  $\zeta$  one can define a supersymmetric interface  $\mathcal{I}[\wp, \zeta]$  between the theories  $\mathbb{S}_{z_1}$  and  $\mathbb{S}_{z_2}$ . The framed BPS states associated with vacua  $z_1^{(i)}$  and  $z_2^{(j')}$  have a “charge” in the relative homology lattice  $\Gamma(z_1^{(i)}, z_2^{(j')})$ . Just as for the soliton sectors  $\Gamma(z_1^{(i)}, z_2^{(j)})$  the lattice  $\Gamma(z_1^{(i)}, z_2^{(j')})$  is a  $\Gamma$ -torsor of chains with  $\partial \mathbf{c} = z_2^{(j')} - z_1^{(i)}$ , up to homology. The interfaces support framed BPS states and the central charge of these framed BPS states is given by

$$Z_{\gamma_{i,j'}} = \frac{1}{\pi} \int_{\gamma_{i,j'}} \lambda \quad (18.7)$$

If we label vacua by  $z_2^{(j')}$  and  $z_1^{(i)}$  then the Witten index of framed BPS states  $\underline{\Omega}(\mathcal{I}[\wp, \zeta], ij')$  will in general be infinite. However, we can grade the cohomology by characters of a global symmetry group with character lattice  $\Gamma$ . Then we can consider the indices  $\underline{\Omega}(\mathcal{I}[\wp, \zeta], \gamma_{ij'})$  of the subspace transforming in a given representation. From physical reasoning the indices with fixed charge  $\gamma_{ij'}$  are expected to be finite.

---

<sup>104</sup>More generally,  $\Gamma$  might be a subquotient of  $H_1(\bar{\Sigma}; \mathbb{Z})$ .

One of the interesting aspects of the theory of surface defects is that one can construct a “nonabelianization map” which is a converse to the standard abelianization map of the theory of Hitchin systems. (See [33] Section 10 for the definition of the nonabelianization map and [34] for an extended example. See also [75] for more expository remarks.) We now describe how that is related to our theory of Interfaces.

Given a phase  $\zeta = e^{i\vartheta}$  one can construct (WKB)  $S_{ij}$ -walls on  $\overline{C}$  which are essentially the same as the  $S_{ij}$  walls used in this paper. A suitable collection of such walls forms a graph on  $\overline{C}$  known as a “spectral network”  $\mathcal{W}$ , so-called because the combinatorics of the network allow one to construct the spectrum of BPS degeneracies of the 2d4d system [33, 34, 36, 37]. In the theory of Hitchin systems the spectral curve  $\overline{\Sigma}$  is equipped with a holomorphic line bundle  $L$  with a flat connection  $\nabla^{\text{ab}}$  [46, 26]. The map from the nonabelian Hitchin system on  $\overline{C}$  to the flat abelian connection on  $\overline{\Sigma}$  is known as the “abelianization map.” Conversely, given a line bundle  $L$  with flat connection  $\nabla^{\text{ab}}$  on  $\overline{\Sigma}$  we can define the parallel transport  $F(\wp)$  of a flat nonabelian connection on a certain rank  $N$  bundle  $E \rightarrow \overline{C}$  along a path  $\wp \subset \overline{C}$  using

$$F(\wp) = \sum_{\gamma_{ij'}} \overline{\Omega}(\mathfrak{I}[\wp, \zeta], \gamma_{ij'}) \mathcal{Y}_{\gamma_{ij'}}. \quad (18.8)$$

We have  $E \cong \pi_*(L)$  away from the network  $\mathcal{W}$  and  $\mathcal{Y}_{\gamma_{ij'}}$  are the parallel transports using the connection  $\nabla^{\text{ab}}$  on  $L \rightarrow \overline{\Sigma}$ .

If we assume the existence of a categorification of  $S_{ij}$  walls in the full 2d/4d setup, either defined by a direct web construction or through the abstract categorification sketched at the end of Section §18.1, then one will obtain directly a categorification of equation (18.8).  $F(\wp)$  is generalized from the parallel transport operator associated with a flat nonabelian connection to an  $A_\infty$ -functor between Brane categories for the surface defect theories  $\mathbb{S}_{z_1}$  and  $\mathbb{S}_{z_2}$ , implemented via an Interface  $\mathfrak{I}[\wp]$  as in Section §7. The Chan-Paton factors of this Interface provide a “lift” of the framed BPS degeneracies  $\overline{\Omega}(\mathfrak{I}[\wp, \zeta], \gamma_{ij'})$  to complexes  $\mathcal{E}_{\gamma_{ij'}}$ . Our rules for the composition of rotation Interfaces  $\mathfrak{R}[\vartheta(x)]$  which do not cross  $S$ -walls, as well as those for the wall-crossing Interfaces  $\mathfrak{S}_{ij}^{p,f}$ , can be recognized as a categorification of the “detour rules” of [33, 34, 75].

The parallel transport  $F(\wp)$  is that of a flat connection on  $\overline{C}$ , not just  $C$ . That is, it smoothly extends over the branch points of the covering. This is its claim to fame! Now, we discussed the sense in which the corresponding functor  $F[\wp]$  on Brane categories is homotopy invariant in Sections §7 and §8, but we did not discuss the crucial notion of homotopy invariance for deforming  $\wp$  across a branch point of the cover  $\pi : \overline{\Sigma} \rightarrow \overline{C}$ . This will involve extending our formalism to theories which are not completely massive, because at a branch point two vacua have coinciding values of  $W_i$  and hence some solitons become massless. Some preliminary remarks about this issue can be found in Section §18.3. We leave the matter here for the present paper. Clearly, it will be an interesting project to generalize the considerations of this paper to the case where  $\pi : \overline{\Sigma} \rightarrow \overline{C}$  is a branched cover with nontrivial monodromy and where  $\lambda$  has periods densely filling the complex plane.

One generalization of the surface defects  $\mathbb{S}_z$  studied in the literature is of great significance for the applications to knot homology. In the M-theory context we can imagine

several parallel semi-infinite  $M2$  branes ending on the  $M5$ -brane. So the boundary is now  $\mathbb{M}^{1,1} \times \{z_1, z_2, \dots, z_n\}$  where the  $z_i$  are distinct points of  $C$ . Naively, these  $M2$  branes are mutually BPS and would seem to have no effect on each other. However there are in fact interesting “topological interactions” and, as we will see below, these are responsible for nontrivial knot homologies. Indeed, the knot homologies are closely related to the spaces of framed BPS states for the generalized surface defect theories  $\mathbb{S}_{z_1, \dots, z_n}$ . We turn to a more detailed discussion of potential applications to knot homology in Section §18.4.

It is likely that a study of categorical wall-crossing for surface defects would have other interesting applications. It might provide new insights in a variety of interesting subjects, such as quantum Teichmüller theory and the Stokes theory of asymptotics of holomorphic functions on Hitchin moduli space.

### 18.3 Generalization 3: Hierarchies Of Scales And Cluster-Induced Webs

In this Section we sketch an interesting construction that becomes available when there is a hierarchy of scales among sets of vacuum weights. There are several potential applications of the construction described at the end of this Section.

By a hierarchy of scales we mean that we consider Theories in which we can divide up the vacua into a disjoint union:

$$\mathbb{V} = \coprod_{\mu} \mathbb{V}^{(\mu)}, \quad (18.9)$$

where the vacuum weights form well-separated clusters. Here the labels  $\mu$  run over some finite set. (As we will soon see, it is a set of Theories.) The vacua will be denoted as  $(\mu, i)$ ,  $i = 1, \dots, |\mathbb{V}^{(\mu)}|$ . The vacuum weight functions are denoted  $z^{(\mu)} : \mathbb{V}^{(\mu)} \rightarrow \mathbb{C}$  and specific vacuum weights are denoted by  $z_i^{(\mu)}$ ,  $i = 1, \dots, |\mathbb{V}^{(\mu)}|$ . We let  $Z_{\mu}$  be the center of mass of the  $z_i^{(\mu)}$  for fixed  $\mu$ . Thus we can write

$$z_i^{(\mu)} = Z_{\mu} + \delta z_i^{(\mu)} \quad (18.10)$$

To be precise, by a *hierarchy of scales* we mean that, for all  $\mu, i, \nu, \lambda$

$$|\delta z_i^{(\mu)}| \ll |Z_{\nu} - Z_{\lambda}| \quad (18.11)$$

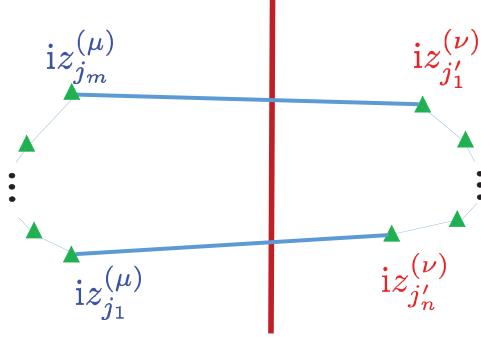
where  $Z_{\nu\lambda} := Z_{\nu} - Z_{\lambda}$ . Note, in particular, that the convex hulls of the images of the  $z^{(\mu)}$  for different  $\mu$  do not intersect.

Let us now assume that we are given a Theory  $\mathcal{T}$ , in the sense of Section §4.1. If the convex hulls of  $z_i^{(\mu)}$  do not intersect (for fixed  $\mu$ ), then by the dual interpretation of webs through convex polygons (Section §2.1, Remark 5) it follows that any web  $\mathfrak{w}$  with a fan  $I_{\infty}$  which involves only vacuum weights of type  $\mu$  will also only have pairs of weights of type  $\mu$  on internal edges as well. Therefore, if we consider the restriction of the web representation  $\mathcal{R}$  and the interior amplitude  $\beta$  to vacua purely of type  $\mu$  then the data  $(\mathbb{V}^{(\mu)}, z^{(\mu)}, \mathcal{R}^{(\mu)}, \beta^{(\mu)})$  by themselves define a Theory, which we will denote  $\mathcal{T}^{(\mu)}$ . In particular,  $\beta^{(\mu)}$  satisfies the  $L_{\infty}$  Maurer-Cartan equation. Note that if  $|\mathbb{V}^{(\mu)}| = 1$  then the Theory  $\mathcal{T}^{(\mu)}$  is trivial.

Let us now consider two sub-Theories  $\mathcal{T}^{(\mu)}$  and  $\mathcal{T}^{(\nu)}$  with  $\mu \neq \nu$ . We claim that the data of the parent Theory  $\mathcal{T}$  allows us to construct an Interface  $\mathfrak{I}^{(\mu, \nu)} \in \mathfrak{Br}(\mathcal{T}^{(\mu)}, \mathcal{T}^{(\nu)})$

between the sub-Theories, where the domain wall  $D_{\mu\nu}$  is parallel to  $Z_{\mu\nu}$ . The Chan-Paton factors of the Interface are given by

$$\mathcal{E}(\mathfrak{I}^{(\mu,\nu)})_{(\mu,i),(\nu,j')} := R_{(\mu,i),(\nu,j')} \quad (18.12)$$



**Figure 165:** The dual of a fan in the Theory  $\mathcal{T}$  which involves two half-plane fans in the sub-Theories  $\mathcal{T}^{(\mu)}$  and  $\mathcal{T}^{(\nu)}$ . This can be interpreted as defining an interface fan between the sub-Theories. Heavy blue lines connect vacua of type  $\mu, \nu$ . Light blue lines connect vacua of the same type. The vertical maroon line is parallel to  $Z_{\mu\nu}$ .

The absorption/emission amplitudes  $\mathcal{B}$  of  $\mathfrak{I}^{(\mu,\nu)}$  are derived from  $K$  and  $\beta$  of the parent theory as follows. Consider a fan of vacua  $I_\infty$  in  $\mathcal{T}$  which involves only the  $\mu$  and  $\nu$ -type vacua. Because the clusters are well-separated, and the fan must involve a convex set of weights it will be of the form  $I_\infty = \{J^+, J^-\}$  where

$$\begin{aligned} J^+ &= \{(\nu, j'_1), (\nu, j'_2), \dots, (\nu, j'_n)\} \\ J^- &= \{(\mu, j_1), (\mu, j_2), \dots, (\mu, j_m)\}. \end{aligned} \quad (18.13)$$

See Figure 165, which should be compared to Figure 39. Now, the component of the interior amplitude  $\beta$  of the parent Theory  $\mathcal{T}$  with this fan at infinity is a degree two element

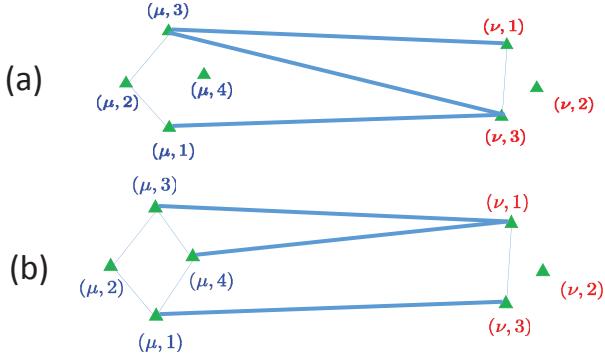
$$\beta_{I_\infty} \in R_{(\mu,j_m),(\nu,j'_1)} \otimes R_{J^+}^+ \otimes R_{(\nu,j'_n),(\mu,j_1)} \otimes R_{J^-}^- \quad (18.14)$$

where

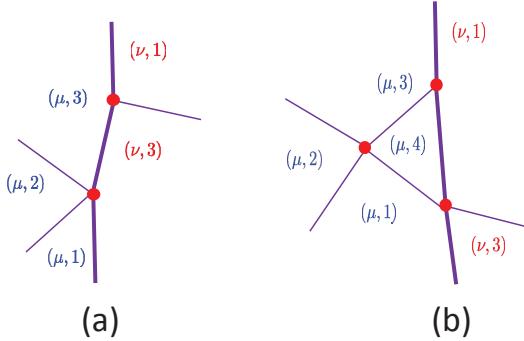
$$\begin{aligned} R_{J^+}^+ &= R_{(\nu,j'_1),(\nu,j'_2)} \otimes \cdots \otimes R_{(\nu,j'_{n-1}),(\nu,j'_n)} \\ R_{J^-}^- &= R_{(\mu,j_1),(\mu,j_2)} \otimes \cdots \otimes R_{(\mu,j_{m-1}),(\mu,j_m)}. \end{aligned} \quad (18.15)$$

If we apply  $\check{K}_{(\nu,j'_n),(\mu,j_1)} : R_{(\nu,j'_n),(\mu,j_1)} \rightarrow R_{(\mu,j_1),(\nu,j'_n)}^*$  then we get a degree one element  $\mathcal{B}_I$  for an interface amplitude appropriate to the Chan-Paton data (18.12). (Compare equation (6.4) above.) We claim that in fact

$$\mathcal{B}_{J^+, J^-} := \check{K}_{(\nu,j'_n),(\mu,j_1)}(\beta_{I_\infty}). \quad (18.16)$$



**Figure 166:** Two representative decompositions of a dual polygon giving a web with  $I_\infty = \{(\nu, 1), (\nu, 3), (\mu, 1), (\mu, 2), (\mu, 3)\}$ .



**Figure 167:** Two webs with  $I_\infty = \{(\nu, 1), (\nu, 3), (\mu, 1), (\mu, 2), (\mu, 3)\}$ . Heavy purple lines separate vacua of type  $\mu, \nu$ . They are approximately vertical and correspond to the domain wall of a corresponding interface. Light purple lines separate vacua of the same type. They correspond to edges of webs in the corresponding sub-Theories.

is an interface amplitude. (We are not being careful about signs here.)

To prove this, let us consider the plane webs  $\omega$  contributing to the  $L_\infty$  identity satisfied by  $\beta$  with fan of vacua  $I_\infty = \{J^+, J^-\}$ . These are in one-one correspondence with decompositions of the convex polygon with vertices

$$iz_{j_m}^{(\mu)}, iz_{j'_1}^{(\nu)}, \dots, iz_{j'_n}^{(\nu)}, iz_{j_1}^{(\mu)}, \dots, iz_{j_m}^{(\mu)} \quad (18.17)$$

into convex polygons with vacuum weights as vertices. We can separate the edges into two types. Edges connecting vacua of different types are called “heavy,” because in the

Landau-Ginzburg incarnation the corresponding classical solitons will be heavy.<sup>105</sup> In the limit that the clusters become infinitely separated all the heavy lines will be parallel to  $Z_{\mu\nu}$ . Edges connecting vacua of the same type are called “light.” The heavy lines are all nearly parallel and their common parallel serves as a locus for the domain wall  $D_{\mu\nu}$ . Note that fixing the transverse position of the domain wall eliminates one degree of freedom so in this mapping of plane webs for  $\mathcal{T}$  to interface webs between  $\mathcal{T}^{(\mu)}$  and  $\mathcal{T}^{(\nu)}$ , taut webs are mapped to taut webs. Conversely, every taut interface web between  $\mathcal{T}^{(\mu)}$  and  $\mathcal{T}^{(\nu)}$  can be “lifted” to a taut plane web for  $\mathcal{T}$ .

The extra factor  $K$  used in defining  $\mathcal{B}_{J^+, J^-}$  in equation (18.16) is precisely what is needed in order to convert the contraction of interior amplitudes for a plane web  $\rho(\mathfrak{w})$  in  $\mathcal{T}$  to the contraction of interface webs  $\rho(\mathfrak{d})$  between Theories  $\mathcal{T}^{(\mu)}$  and  $\mathcal{T}^{(\nu)}$ . Thus the  $L_\infty$  equations satisfied by  $\beta$  in the parent theory become the  $A_\infty$  equations satisfied by the interface amplitude of equation (18.16). The ordering of vertices along the heavy edges nearly parallel to  $Z_{\mu\nu}$  collapses the  $L_\infty$  combinatorics to  $A_\infty$  combinatorics. In conclusion, a pair of far separated clusters of vacua canonically defines an Interface  $\mathfrak{I}^{(\mu, \nu)}$  as claimed above.

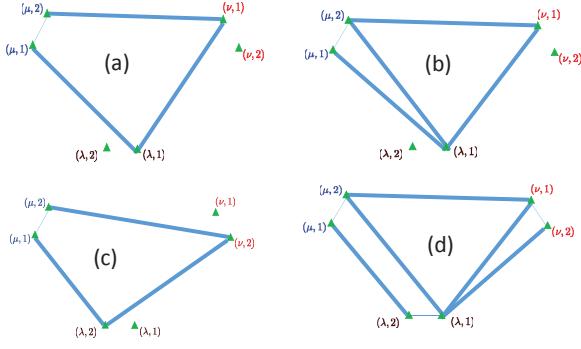
We can now envision a nontrivial generalization of our entire formalism, where vacua are replaced by Theories, and edges of webs support Interfaces. These Interfaces will themselves interact at junctions, thus we again have a system of webs, which we will call *cluster-induced webs*, whose edges are parallel to  $Z_{\mu\nu}$ . Moreover, on a half-plane the Interfaces can have junctions at the boundary of the half-plane, leading to cluster-induced half-plane webs where again edges are associated to Interfaces and segments on the boundaries are associated with Branes within the various Theories. We will next sketch how this idea can be made more precise, but we will leave detailed verification of the full picture for future work.

We first use a key idea from the construction of the Interfaces  $\mathfrak{I}^{(\mu, \nu)}$ . We consider webs  $\mathfrak{W}$  whose set of vacuum labels are the Theories  $\mathcal{T}^{(\mu)}$  and whose vacuum weights are the center of mass coordinates  $Z_\mu$ . These are the *cluster-induced webs* mentioned above. In the limit that the clusters of vacua  $z_i^{(\mu)}$  are well-separated for different  $\mu$ , to every web  $\mathfrak{w}$  of the parent Theory we can associate a cluster-induced web  $\mathfrak{W}[\mathfrak{w}]$ , called the *skeleton* of  $\mathfrak{w}$ . It is defined by keeping the heavy lines in  $\mathfrak{w}$  and collapsing the light lines. Conversely, given a web of type  $\mathfrak{W}$  there will be several webs  $\mathfrak{w}_1, \mathfrak{w}_2, \dots$  in the parent Theory which have the same skeleton  $\mathfrak{W}$ . Instead of web representations, to the edges of  $\mathfrak{W}$  we associate the Interfaces  $\mathfrak{I}^{(\mu, \nu)}$ . See, for example Figures 168, 169, and 170.

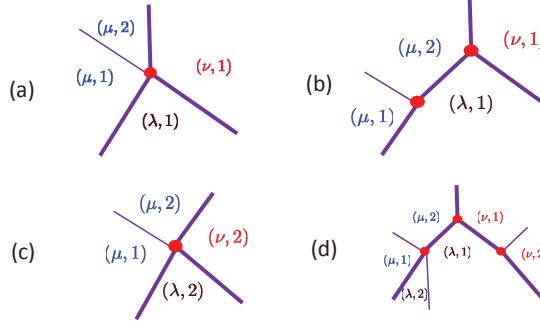
The vertices of  $\mathfrak{W}$  correspond now to junctions of Interfaces. See again Figure 170. Instead of a representation of a fan, the natural object to associate to a cyclic junction of Interfaces  $\mathfrak{I}^{(\mu_1, \mu_2)}, \mathfrak{I}^{(\mu_2, \mu_3)}, \dots, \mathfrak{I}^{(\mu_n, \mu_1)}$  is the chain complex defined by the trace of the composite Interface. Recall that each Interface  $\mathfrak{I}^{(\mu, \nu)}$  is associated with a domain wall parallel to  $Z_{\mu\nu}$ , so let  $\mathfrak{R}_{\mu, \nu}$  be the rotation Interface of Section §7 (defined in the parent Theory  $\mathcal{T}$ ) that rotates  $Z_{\mu\nu}$  through an angle less than  $\pi$  to be vertical (so the Interface

---

<sup>105</sup>There is a potential for confusion here. A given heavy classical soliton which connects vacua of different types can function as a domain wall between the Theories. But our domain walls are labeled by pairs of Theories, and not by specific solitons between the vacua of different types.



**Figure 168:** We illustrate three clusters of vacua of type  $\mu, \nu, \lambda$ . Associated to this “fan of Theories” are several fans of vacua  $I_\infty$  in the parent theory as well as several webs in the parent Theory with fixed  $I_\infty$ . A few examples are shown here.



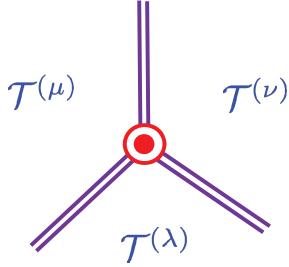
**Figure 169:** This figure shows some of the webs associated with the fan of Theories  $\{\mathcal{T}^{(\mu)}, \mathcal{T}^{(\nu)}, \mathcal{T}^{(\lambda)}\}$ .

domain wall is vertical). Then, instead of associating a fan representation  $R_I$  to a vertex, as we do in the parent theory  $\mathcal{T}$ , to a junction of Interfaces we now associate the chain complex:

$$R_{\mu_1, \dots, \mu_n} := \text{Tr} \left[ \left( \mathfrak{I}^{(\mu_1, \mu_2)} \boxtimes \mathfrak{R}_{\mu_1, \mu_2} \right) \boxtimes \left( \mathfrak{I}^{(\mu_2, \mu_3)} \boxtimes \mathfrak{R}_{\mu_2, \mu_3} \right) \boxtimes \dots \boxtimes \left( \mathfrak{I}^{(\mu_n, \mu_1)} \boxtimes \mathfrak{R}_{\mu_n, \mu_1} \right) \right] \quad (18.18)$$

where the trace of an Interface was defined in Section §9 above.

Now, we conjecture that the data of the interior amplitude  $\beta$  of the parent Theory allows us to construct distinguished elements in the complexes associated to the vertices of  $\mathfrak{W}$ , say  $\underline{\beta}_{\mu_1, \dots, \mu_n} \in R_{\mu_1, \dots, \mu_n}$ . Then, the direct sum over all fans of Theories defines an



**Figure 170:** A junction of Interfaces  $\mathcal{T}^{(\mu,\nu)}, \mathcal{T}^{(\nu,\lambda)}, \mathcal{T}^{(\lambda,\mu)}$ . This serves as a vertex for the cluster-induced webs.

analog  $\underline{R}^{\text{intfc}}$  of  $R^{\text{int}}$ . We further conjecture that  $\underline{R}^{\text{intfc}}$  carries the structure of an  $L_\infty$  algebra and  $\beta_{\mu_1, \dots, \mu_n}$  define a solution to the Maurer-Cartan equation of that  $L_\infty$  algebra.

We can be more specific. The notion of amplitude for a “web of interfaces”  $\mathfrak{W}$  makes perfect sense in our general algebraic setup even outside the context of cluster-induced webs. Given a general collection of Theories  $\mathcal{T}^{(\mu)}$  attached to the faces of the web, of Interfaces  $\mathfrak{J}^{(\mu,\nu)}$  attached to the edges of the web, each belonging to the category of interfaces with the slope of the corresponding edge, and a collection of elements  $\underline{r}_a$  in the chain complexes associated to the vertices of the web, we can *define* the amplitude  $\underline{\rho}(\mathfrak{W})[\underline{r}_a]$  of such a web in a straightforward manner. We can identify  $\mathfrak{W}$  as a sum over composite webs (including edges going into the vertices of  $\mathfrak{W}$ , as for wedge webs) of the underlying  $\mathcal{T}^{(\mu)}$  Theories defined within the corresponding faces, and define  $\underline{\rho}(\mathfrak{W})[\underline{r}_a]$  by inserting the appropriate interior and boundary amplitudes in  $\rho(\mathfrak{W})[\cdots; \cdots; \underline{r}_a]$ .

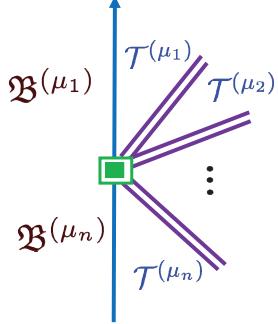
The above conjecture can be stated as the claim that for the Theories and Interfaces defined in this section, there exist a sum of cluster-induced webs  $t_{\text{cluster}}$  such that  $\underline{\rho}(t_{\text{cluster}})[\underline{r}_a]$  defines an  $L_\infty$  algebra and  $\beta_{\mu_1, \dots, \mu_n}$  is a solution to the Maurer-Cartan equation of that  $L_\infty$  algebra. We expect  $t_{\text{cluster}}$  to arise from the decomposition into cluster-induced webs of the taut element  $t$  in the underlying theory.

The plane webs  $\mathfrak{W}$  described above should have half-plane analogs. Let us fix the positive half-plane. Then, in close analogy to the definitions of the Theories  $\mathcal{T}^{(\mu)}$ , a Brane  $\mathfrak{B}$  with Chan-Paton spaces  $\mathcal{E}_{(\mu,i)}$  and boundary amplitudes  $\mathcal{B}_J$  in the parent Theory  $\mathcal{T}$  defines a collection of Branes  $\mathfrak{B}^{(\mu)}$ , one for each Theory  $\mathcal{T}^{(\mu)}$ . The key observation is again that if  $J_\infty$  involves only vacua of type  $\mu$  and all the emission amplitudes only involve vacua of type  $\mu$  then all the interior vertices must also correspond to fans of vacua solely of type  $\mu$ .<sup>106</sup> Consequently, for fixed  $\mu$ , we can define  $\mathfrak{B}^{(\mu)}$  to have Chan-Paton spaces  $\mathcal{E}_{(\mu,i)}$ ,  $i = 1, \dots, |\mathbb{V}^{(\mu)}|$ , with amplitudes  $\mathcal{B}_J$  where  $J$  is a half-plane fan of vacua all of type

---

<sup>106</sup>Intuitively, heavy lines cannot end away from the boundary and cannot turn back.

$\mu$ . If we contract interior vertices with  $\beta^{(\mu)}$  then these will satisfy the  $A_\infty$  identities by themselves, and hence define Branes in  $\mathcal{T}^{(\mu)}$ .



**Figure 171:** Supersymmetric Interfaces can end on boundaries. Shown here is a cluster-induced half-plane web  $\mathfrak{U}$ . The “emission amplitude” that interpolates between Branes of two Theories and joins several interfaces is an amplitude  $\underline{\mathcal{B}}$  constructed from the data of the boundary amplitudes of the underlying parent Theory  $\mathcal{T}$ .

Now, if we consider half-plane fans in the parent theory of type

$$J_\infty = \{(\mu, j_1), \dots, (\mu, j_n), (\nu, j'_1), \dots, (\nu, j'_m)\} \quad (18.19)$$

the corresponding half-plane webs will involve heavy edges, approximately parallel to  $Z_{\mu\nu}$  (which we assume points into the positive half-plane) and terminating at the boundary of  $\mathcal{H}$ . Thus, supersymmetric Interfaces  $\mathfrak{I}^{(\mu,\nu)}$  should also have boundary junctions. Instead of a representation  $R_J$  associated with an emission vertex in the parent Theory  $\mathcal{T}$ , now, to a junction such as that shown in Figure 171 we associate again a complex

$$\underline{R} = \mathfrak{B}^{(\mu_1)} \boxtimes \left( \mathfrak{I}^{(\mu_1, \mu_2)} \boxtimes \mathfrak{R}_{\mu_1, \mu_2} \right) \boxtimes \left( \mathfrak{I}^{(\mu_2, \mu_3)} \boxtimes \mathfrak{R}_{\mu_2, \mu_3} \right) \boxtimes \dots \boxtimes \left( \mathfrak{I}^{(\mu_n, \mu_1)} \boxtimes \mathfrak{R}_{\mu_n, \mu_1} \right) \boxtimes \mathfrak{B}^{(\mu_n)}[\pi] \quad (18.20)$$

The analog of the Chan-Paton factors are now the Branes  $\mathfrak{B}^{(\mu)}$  in the Theories  $\mathcal{T}^{(\mu)}$ . A conjecture analogous to that we made above states that the data of the boundary amplitudes  $\mathcal{B}_J$  of the parent theory allow us to define elements  $\underline{\mathcal{B}}$  of these complexes so that the  $A_\infty$  relations of the half-plane webs  $\mathfrak{U}$  associated to data  $(\mathcal{T}^{(\mu)}, Z_\mu, \underline{R}, \underline{\beta})$  will be satisfied. Again, the operations of the  $A_\infty$  category will be defined from the amplitudes of appropriate webs of Interfaces drawn in the half-plane, including both the  $\mathfrak{B}^{(\mu)}$  Branes and the  $\mathfrak{I}^{(\mu,\nu)}$  Interfaces.

In conclusion, we have sketched how, in the limit that vacuum weights form well-separated clusters we can build a new set of webs, called cluster-induced webs and a “representations of webs” (the chain complexes  $\underline{R}$ ) with elements  $\underline{\beta}$  and  $\underline{\mathcal{B}}$  satisfying the  $LA_\infty$  relations. If, then, there are hierarchies of clusters within clusters there will be

a corresponding hierarchy of these cluster-induced structures. This might be a way to generalize our formalism to infinite collections of vacuum weights with accumulation points. But we leave that for the future.

Finally, let us sketch some of the physical motivations and potential applications of the above construction:

1. In this paper we have heavily used the assumption that the IR vacua of the field theory under study are all massive. It is natural to ask whether the formalism can be extended to include interacting massless vacua, and in particular nontrivial interacting CFT's. We believe the the above construction can be used to define a web-formalism for such interacting CFT's. In the context of LG theories, massless vacua appear when we consider families of superpotentials such that one or more Morse critical points approach a common (non-Morse) critical point, and the cluster-webs should allow a description of the physics of these models. An interesting open problem is whether the formalism then applies to non-Morsifiable singularities.
2. A closely related application is the extension of the application to categorified spectral networks to include the branch points of the spectral cover  $\pi : \bar{\Sigma} \rightarrow \bar{C}$ , as mentioned in Section §18.2 above.
3. A third application is to the construction of creation and fusion Interfaces in the knot homology application of our formalism sketched in Section §18.4 below. See, in particular Section §18.4.8. We turn to these matters next.

## 18.4 Potential Application: Knot Homology

### 18.4.1 The Main Point

Knot homology is an important topic in low-dimensional topology. It has interesting relations to string theory and gauge theory and there have been several interpretations of knot homology in the physics literature. The gauge-theoretic definition of knot homology proposed in [92] can be deformed to a setup which has many properties in common with a massive 2d theory with  $\mathcal{N} = (2, 2)$  supersymmetry compactified on a segment [32].

The objective of this Section is to describe how one might possibly employ the machinery developed in the present paper in order to express the gauge-theoretic definition of knot homology in the language of webs, web representations, Theories and Interfaces.

The immediate payoff of such a translation would be to bridge the conceptual gap between the gauge-theoretic definition of knot homology, which has a direct relationship to the three-dimensional geometry of the knot, and the standard combinatorial definitions of Khovanov cohomology, which lack such a relationship. Ultimately, it should be possible to establish a direct equivalence between some Vacuum  $A_\infty$  categories associated to the gauge-theoretic construction and the categories employed in the combinatorial definitions of Khovanov cohomology.

The gauge-theoretic definition of knot homology employs an MSW complex built from solutions of certain four and five-dimensional  $\mathcal{Q}$ -fixed point equations for a five-dimensional

supersymmetric gauge theory on a five-manifold with boundary:

$$M_5 = \mathbb{R} \times M_3 \times \mathbb{R}_+, \quad (18.21)$$

where  $M_3$  is a three-manifold. The knot resides in  $M_3$  on the boundary and is used to formulate the crucial boundary conditions for the instanton equations of the gauge theory. Referring to the first factor as “time” the solutions of time-independent, four-dimensional, “soliton” equations provide the complex of approximate ground states and the solutions of the five-dimensional “instanton” equations with time-independent boundary conditions provide the differential on the complex. We will review the setup in full detail in Section §18.4.3. The relation to this paper begins to emerge when we realize that the five-dimensional instanton equations are equivalent to the  $\zeta$ -instanton equations for a gauged Landau-Ginzburg model whose “worldsheet” is  $\mathbb{R} \times \mathbb{R}_+$  and whose target space is a space of complexified gauge connections on  $M_3$ , as described in Section §18.4.3. The superpotential is the Chern-Simons functional. (This gauged LG model is referred to as CSLG1 below.) In the case when  $M_3 = \mathbb{R} \times C$ , with  $C$  a Riemann surface, the equations are also the  $\zeta$ -instanton equations for another gauged Landau-Ginzburg model (referred to as CSLG2 below). Referring to the first two coordinates of  $M_5 = \mathbb{R} \times \mathbb{R} \times C \times \mathbb{R}_+$  as  $(x^0, x^1)$  the “worldsheet” of CSLG2 is the  $(x^0, x^1)$  plane, while the target is a space of complexified gauge fields on  $\widetilde{M}_3 = C \times \mathbb{R}_+$ . Again, the superpotential is a Chern-Simons functional. Since the Chern-Simons functional is multivalued,  $dW$  can have periods, and the considerations of Section §18.1 become important. In either case, the knot complex is the MSW complex for the Landau-Ginzburg theory.

In order to illustrate the relation to the web formalism, we focus on the case when  $M_3 = \mathbb{R} \times C$  and use the formulation CSLG2. We should stretch the link along one spatial direction  $x^1$ , and introduce the deformation of the boundary conditions proposed in [32]. In the approximation that the link’s strands are parallel to the  $x^1$  direction, the boundary conditions have 2d translation symmetry and the 5d equations have much in common with the  $\zeta$ -instanton equations of an ungauged massive LG model. In particular, they admit isolated solutions akin to 2d vacua, which are independent of time and the  $x^1$  space direction and only depend on the three remaining space directions. Conjecturally, these vacua are *massive*, with a mass scale controlled by the deformation parameters and the transverse separation between the strands. Such a conjecture implies a familiar structure for solutions of the five-dimensional instanton equations: as long as the strands of the link are approximately parallel away from co-dimension one loci (“interfaces”), which are in turn well-separated from each other, the solutions will be almost everywhere exponentially close to the 2d vacuum solutions, except in the neighbourhood of a BPS web.

Assuming that this conjecture holds true, it should be possible to employ the 5d instanton equations to define the same variety of counting problems as we did for the  $\zeta$ -instanton equation in standard LG theories in Sections §§11-17, as long as we deal with the non-single-valued vacuum weights as sketched in Section §18.1. Thus, we expect that for any collection  $S$  of parallel strands:

- Solutions of the 5d instanton equation which do not depend on  $(x^0, x^1)$  will give the vacuum data  $\mathbb{V}_S$ .

- Solutions of the 5d instanton equation which depend only on the combination  $x^1 \cos \mu + x^0 \sin \mu$  will provide the spaces of solitons which can interpolate between any two given vacua and thus web representations for the vacuum data  $\mathbb{V}_S$ .
- Solutions of the 5d instanton equation with fan-like asymptotics in the  $(x^0, x^1)$  plane will provide interior amplitudes  $\beta_S$  and thus Theories  $\mathcal{T}_S$ .
- If  $S$  is an empty collection the theory  $\mathcal{T}_S$  will be trivial. That is, it will have a unique vacuum.

Similarly, for any “supersymmetric interface”  $\mathcal{I}$ , i.e. a time-independent boundary condition for the 5d equations which involves a set of parallel strands  $S^-$  for  $x^1 \ll -L$  and a set of parallel strands  $S^+$  for  $x^1 \gg L$

- Solutions of the 5d instanton equation which do not depend on time will give Chan-Paton data  $\mathcal{E}_{j,j'}^{\mathcal{I}}$ .
- Solutions of the 5d instanton equation with fan-like asymptotics in the  $(x^0, x^1)$  plane will provide boundary amplitudes  $\mathcal{B}^{\mathcal{I}}$  and thus an Interface  $\mathfrak{I}[\mathcal{I}]$  between Theories  $\mathcal{T}_{S^-}$  and  $\mathcal{T}_{S^+}$ .

We can assume that the stretched link is approximated by a sequence of collections of strands  $S_a$ , starting and ending with the empty collection  $S_0 = S_n = 0$ , separated by interfaces  $\mathcal{I}_{a,a+1}$ . The approximate ground states and instantons of the knot homology complex will literally coincide with the chain complex of the Interface  $\mathfrak{I}(\text{Link})$  between the trivial Theory and itself, defined as the composition of the Interfaces  $\mathfrak{I}[\mathcal{I}_{a,a+1}]$

$$\mathfrak{I}(\text{Link}) := \mathfrak{I}[\mathcal{I}_{0,1}] \boxtimes \cdots \boxtimes \mathfrak{I}[\mathcal{I}_{n-1,n}]. \quad (18.22)$$

This complex is bigraded. One grading is the fermion number used throughout this paper. The second grading is related to the instanton number current in the five-dimensional theory and hence to the multi-valuedness of the Chern-Simons functional and the considerations of Section §18.1.

Furthermore, if we allow the transverse position of the strands to evolve adiabatically in between discrete events such as recombination of strands, according to some profile  $S_a(x^1)$ , we expect the knot homology complex to coincide with the chain complex of an Interface  $\mathfrak{I}(\text{Link})$  which now includes the insertion of the corresponding categorical parallel transport interfaces:

$$\mathfrak{I}(\text{Link}) := \mathfrak{I}[\mathcal{I}_{0,1}] \boxtimes \mathfrak{I}[\mathcal{T}_{S_1(x^1)}] \boxtimes \cdots \boxtimes \mathfrak{I}[\mathcal{T}_{S_{n-1}(x^1)}] \boxtimes \mathfrak{I}[\mathcal{I}_{n-1,n}]. \quad (18.23)$$

In the remainder of this Section, we will review in a little more detail the gauge theory definition of knot homology and the relation between the five-dimensional instanton equations and the  $\zeta$ -instanton equations. We will also review the definitions of a collection of finite-dimensional auxiliary ungauged Landau-Ginzburg models introduced in [32]. (These are the “monopole” and “Yang-Yang” models described below.) These models are expected to provide a low-energy effective description for the full gauge theory model in

the case when the collection  $S$  consists of parallel strands. It might be possible to prove the equivalence of the Theories  $\mathcal{T}_S$  and Interfaces  $\mathfrak{I}[\mathcal{I}]$  with the Theories and Interfaces computed from these finite-dimensional ungauged LG models.

#### 18.4.2 Preliminary Reminder On Gauged Landau-Ginzburg Models

In Sections §§11-17 above we discussed at length  $\mathcal{N} = (2, 2)$  Landau-Ginzburg sigma models with Kähler target  $X$  and holomorphic superpotential  $W$ . In Sections §§18.4.3-18.4.4 we will reformulate the gauge theoretic approach to knot homology in terms of Landau-Ginzburg models. The relevant superpotential, which will be a Chern-Simons functional, will actually be a degenerate superpotential due to gauge invariance and hence we need to generalize the discussion of Section §11 slightly to include the case of *gauged* Landau-Ginzburg models. This is easily done. We briefly summarize the generalization here. (For background see Section 5.1.1 of [91]. We reduce the  $d = 4$ ,  $\mathcal{N} = 1$  gauged nonlinear model of [85], ch.24 following the general procedure of [89].)

Suppose the Kähler manifold  $X$  has a continuous group  $S$  of isometries and suppose moreover that  $S$  is a symmetry of the superpotential  $W$ .<sup>107</sup> The LG model then has a global symmetry and we can gauge it. We do so in a supersymmetric way, coupling to a  $(2, 2)$  vectormultiplet with bosonic fields  $(\sigma, \bar{\sigma}, B, D)$ . All fields are locally valued in the Lie algebra of  $S$ .  $B$  is a gauge field for an  $S$ -bundle on the “worldsheet.” The remaining fields are scalars.  $D$  is an auxiliary field. In order to couple the gauge fields supersymmetrically we assume furthermore that  $S$  acts symplectically on  $X$  so that there is a moment map  $\mu : X \rightarrow \text{Lie}(S)^*$ .

We can define supersymmetries  $\mathcal{Q}_\zeta$  as before. The fixed point equations of the topologically twisted theory are an interesting combination of vortex and  $\zeta$ -instanton equations:

$$\bar{\partial}_B \phi^I = \frac{i\zeta}{4} g^{I\bar{J}} \frac{\partial \bar{W}}{\partial \bar{\phi}^J} d\bar{w} \quad (18.24a)$$

$$*_2 F_B + \mu = 0 \quad (18.24b)$$

where  $d\bar{w} = dx^1 - idx^0$  is a  $(0, 1)$  form on the worldsheet and the covariant derivatives on the scalars can be written:

$$d_B \phi^I = d\phi^I + \langle B, V^I \rangle. \quad (18.25)$$

Here the action of  $S$  on  $X$  defines vector fields  $V$  on  $X$  valued in  $\text{Lie}(S)^*$ , that is, an element  $\mathfrak{x}$  of the Lie algebra of  $S$  generates a vector field

$$V(\mathfrak{x}) = V^I(\mathfrak{x}) \frac{\partial}{\partial \phi^I} + V^{\bar{I}}(\mathfrak{x}) \frac{\partial}{\partial \bar{\phi}^{\bar{I}}}. \quad (18.26)$$

In (18.25) the angle brackets denote contraction with the  $\text{Lie}(S)$ -valued gauge field  $B$ . The second set of equations, (18.24b), minimize the kinetic energy terms of the gauge fields in

---

<sup>107</sup>Or rather, a symmetry of the current generated by the pullback of  $dW$ . The superpotential is allowed to shift by a constant under symmetry transformations.

the action. In the twisted theory one adds a boundary term  $\sim \oint \langle B, \mu \rangle$  to the standard physical action allowing one to complete a square and write the kinetic energy term for the gauge fields as  $\sim \int (*F_B - \mu)^2$ . The final parts of the  $\mathcal{Q}_\zeta$ -fixed point equations require that the Lie algebra valued field  $\sigma$  is covariantly constant and that all the other fields should be invariant under gauge transformation by  $\sigma$ .

An important simplification occurs when  $S$  acts freely and the critical points of  $W$  correspond to isolated, nondegenerate  $S^c$  orbits. In this case the low energy behavior of the gauged model is described by an ordinary LG model on the symplectic quotient. Since the  $S$ -action is fixed point free,  $\sigma = 0$ . Moreover, since  $\mu = 0$  in the symplectic quotient we can gauge away  $B$  and the equation reduces to the ordinary  $\zeta$ -instanton equation. By geometric invariant theory we know that, with a suitable stability condition, the symplectic quotient can be identified, as a complex manifold, with  $X/S^c$  and  $W$  descends to a nondegenerate Morse function on this space. These conditions will hold in our application below since the Nahm pole boundary conditions guarantee that the  $\mathcal{Q}_\zeta$ -fixed points have no symmetries.

Finally, it is interesting to generalize the formulation of a LG model as supersymmetric quantum mechanics, as explained in Section §11 above, to the case of a gauged model. Equation (11.8) is now generalized to

$$h = - \int_D \left[ \phi^*(\lambda) - \langle B, \mu \rangle - \frac{1}{2} \text{Re}(\zeta^{-1} W) dx \right] \quad (18.27)$$

The equation for upwards gradient flow for  $h$  becomes the pair of equations (18.24b) and (18.24a).

#### 18.4.3 Lightning Review: A Gauge-Theoretic Formulation Of Knot Homology

The gauge theoretic formulation of knot homology given in [92] has its origins in the theory of supersymmetric branes in string theory, or alternatively, in the six-dimensional (2,0) superconformal field theory. These motivations are explained at length in [92] and will not be repeated here. See [93, 94] for brief reviews of [92] and related papers. Here we summarize the final mathematical statement arrived at in [92] but approaching the subject via Landau-Ginzburg theory and Morse theory, the topics of such importance in this paper.

Let  $L \subset M_3$  be a knot (or link) in an oriented and framed three-manifold  $M_3$ . We wish to formulate a doubly-graded homology theory  $\mathcal{K}(L)$ . This will be the homology of a complex  $\widehat{\mathcal{K}(L)}$ , which in turn will be a certain MSW complex. In order to formulate the MSW complex and its differential we introduce a metric  $g_{ij} dx^i dx^j$  on  $M_3$ . We also introduce a compact simple Lie group  $G$  with real Lie algebra  $\mathfrak{g}$ , and a principal  $G$  bundle  $E \rightarrow M_3$ . We further let  $\mathcal{U}$  be the space of connections on  $E$  and  $\mathcal{U}^c$  its natural complexification. A generic element  $\mathcal{A}$  of  $\mathcal{U}^c$  can be decomposed into “real and imaginary parts” as  $\mathcal{A} = A + i\phi$  where  $A$  is connection on  $E$  and  $\phi$  is a one-form valued in the adjoint bundle. Locally, they are one-forms valued in the compact real Lie algebra  $\mathfrak{g}$ .<sup>108</sup> The space  $\mathcal{U}^c$  is an infinite-dimensional Kähler manifold with metric

$$d\ell^2 = \int_{M_3} \text{Tr}(\delta\mathcal{A} * \delta\overline{\mathcal{A}}) \quad (18.28)$$

---

<sup>108</sup>In our conventions  $\mathfrak{g}$  is a real subalgebra of a Lie algebra of anti-hermitian matrices.

where  $\text{Tr}$  is a positive-definite Killing form on  $\mathfrak{g}$ . The normalization of the Killing form does not affect the flow equations. The symplectic structure is

$$\omega = \int_{M_3} \text{Tr}(\delta A * \delta\phi). \quad (18.29)$$

and the complex structure maps  $\delta A$  to  $\delta\phi$ .

We will consider a Landau-Ginzburg theory on the half-line  $\mathbb{R}_+$ , parametrized by  $y$ , with target space  $\mathcal{U}^c$ , with certain boundary conditions at  $y = 0, \infty$  that will be sketched below. The data of the knot enters in the boundary conditions at  $y = 0$ . <sup>109</sup>

The superpotential of the model will be the Chern-Simons term

$$W^{cs}(\mathcal{A}) = \int_{M_3} \text{Tr} \left( \mathcal{A} d\mathcal{A} + \frac{2}{3} \mathcal{A}^3 \right) \quad (18.30)$$

Of course this is not single-valued, but  $dW^{cs}$  is single-valued, and this is all that is needed for the construction. Note that we have not chosen a normalization of the Killing form, so the periods of  $W^{cs}(\mathcal{A})$  have not yet been specified. The superpotential  $W^{cs}(\mathcal{A})$  is a degenerate holomorphic Morse function on  $\mathcal{U}^c(\mathcal{BC})$  due to the gauge invariance of  $dW^{cs}$ . Introduce the group  $\mathcal{G}$  of unitary automorphisms of  $E$ . When  $E$  is trivializable, as we will assume,  $\mathcal{G}$  is just the group  $\mathcal{G} = \text{Map}(M_3, G)$ . The group  $\mathcal{G}$  acts as a group of isometries preserving the symplectic structure as well as  $dW^{cs}$ . We are therefore in a position to consider - at least formally - the gauged Landau-Ginzburg model, as described in Section §18.4.2, with symmetry group  $S = \mathcal{G}$ . In particular the group of gauge transformations of this gauged LG model consists of maps from  $\mathbb{R} \times \mathbb{R}_+$ , parametrized by  $(x^0, y)$ , into  $\mathcal{G}$ .

Viewed as a problem in equivariant Morse theory on  $X = \text{Map}(\mathbb{R}_+, \mathcal{U}^c)$  (or rather, on a cover on which  $W^{cs}$  is single-valued) the Morse function is, according to (18.27)

$$h = - \int_{\mathbb{R}_+} dy \int_{M_3} \text{vol}(g) g^{ij} (\phi_i \partial_y A_j - \phi_i D_j B_y) - \frac{1}{2} \text{Re} \left[ ie^{-i\tilde{\vartheta}} CS(\mathcal{A}) \right] \quad (18.31)$$

where  $D_j$  is the covariant derivative with respect to  $A_j$ .

The flow equations, or, equivalently, the  $\mathcal{Q}_\zeta$ -fixed point equations (with  $\zeta = -ie^{i\tilde{\vartheta}}$ ) are now easily written. When covariantized in the time direction they become

$$[(D_y - iD_0), \mathcal{D}] = e^{i\tilde{\vartheta}} *_M \mathcal{F}^* \quad (18.32a)$$

$$[D_0, D_y] + D_A * \phi = 0 \quad (18.32b)$$

The conventions here are the following:  $\mathcal{D}$  is the covariant derivative with respect to  $\mathcal{A}$  on  $M_3$ . In local coordinates  $\mathcal{D} = \sum_{i=1}^3 dx^i (\partial_i + A_i + i\phi_i)$ , and the fieldstrength is  $\mathcal{F}_{ij} = [\mathcal{D}_i, \mathcal{D}_j]$ . The complex conjugation  $*$  is an anti-linear involution acting as  $-1$  on  $\mathfrak{g}$ . The covariant derivatives  $D_0, D_y$  on the “worldsheet”  $\mathbb{R} \times \mathbb{R}_+$  have gauge field  $B = dx^0 B_0 + dy B_y$  valued in the Lie algebra  $\text{Map}(M_3, \mathfrak{g})$ .

---

<sup>109</sup>This LG theory provides a convenient way to derive the gauge theory BPS equations. It should not be confused with the LG theory in the  $(x^0, x^1)$  plane which will be used in Section §18.4.4 to establish the relation to the web formalism

These equations can be rewritten as equations on the five-dimensional space of the form

$$M_5 = \mathbb{R} \times M_3 \times \mathbb{R}_+ \quad (18.33)$$

(with local coordinates  $(x^0, x^i, y)$ ) for a five-dimensional gauge field, locally a  $\mathfrak{g}$ -valued one-form:

$$A^{5d} = B_0 dx^0 + A_i dx^i + B_y dy, \quad (18.34)$$

together with a  $\mathfrak{g}$ -valued field  $\phi$  on  $M_5$  that is cotangent to  $M_3$ . Locally  $\phi = \sum_{i=1}^3 \phi_i dx^i$ . Put more formally,  $\phi \in \Gamma(M_5, \pi^*(T^*M_3) \otimes \text{Ad}(E))$  where  $\pi : M_5 \rightarrow M_3$  is the projection. Note that the first term in the expression for  $h$  in (18.31) can be written as  $\int_{\mathbb{R}_+ \times M_3} \text{Tr}(\phi * F^{5d})$ , showing that the interpretation of  $A^{5d}$  as a five-dimensional gauge field is natural. (Compare with equation (5.42) of [92].)

In local orthonormal coordinates, the  $\zeta$ -instanton equations of the CSLG model are seven “real” equations for 8 real fields on  $M_5$ :

$$F_{yi} + D_0 \phi_i + \frac{1}{2} \epsilon_{ijk} (F - \phi^2)_{jk} + \mathbf{t} \left( D_y \phi_i + F_{i0} + \frac{1}{2} \epsilon_{ijk} (D_A \phi)_{jk} \right) = 0 \quad \forall i = 1, 2, 3 \quad (18.35a)$$

$$F_{yi} + D_0 \phi_i - \frac{1}{2} \epsilon_{ijk} (F - \phi^2)_{jk} - \mathbf{t}^{-1} \left( D_y \phi_i + F_{i0} - \frac{1}{2} \epsilon_{ijk} (D_A \phi)_{jk} \right) = 0 \quad \forall i = 1, 2, 3 \quad (18.35b)$$

$$F_{0y} + D_i \phi_i = 0 \quad (18.35c)$$

where it is useful to introduce the parameter:

$$\mathbf{t} = \frac{\sin \tilde{\vartheta}}{1 + \cos \tilde{\vartheta}} = \tan\left(\frac{1}{2}\tilde{\vartheta}\right). \quad (18.36)$$

We should regard  $\mathbf{t}$  as the stereographic projection of  $e^{i\tilde{\vartheta}}$  on the unit circle to the real line. These equations have some remarkable properties:

1. When  $\mathbf{t} = 1$  they become covariant equations on the four-manifold  $\widehat{M}_4 = \mathbb{R} \times M_3$ , with local coordinates  $(x^0, x^i)$ , where  $\phi_i$  are reinterpreted as the three components of a self-dual 2-form on  $\widehat{M}_4$ . These equations were written in equation (5.36) of [93]. Similar equations were written in [43], and hence we will refer to the equations (18.35) as the  $HW(\mathbf{t})$  equations.
2. For general  $\mathbf{t}$ , if the fields are taken to be time-independent, i.e. if they are pulled back from the four-manifold  $M_4 = M_3 \times \mathbb{R}_+$ , with local coordinates  $(x^i, y)$ , then, if we rename  $B_0 \rightarrow -\phi_y$ , and introduce a new adjoint-valued 1-form:  $\phi^{\text{kw}} = \phi_i dx^i + \phi_y dy$  the equations become the Kapustin-Witten equations with parameter  $\mathbf{t}$ :

$$F - (\phi^{\text{kw}})^2 + \mathbf{t} (d_A \phi^{\text{kw}})^+ - \mathbf{t}^{-1} (d_A \phi^{\text{kw}})^- = 0 \quad (18.37a)$$

$$d_A * \phi^{\text{kw}} = 0 \quad (18.37b)$$

Here the superscript  $\pm$  refer to the self-dual projections with the product metric  $g_{ij}dx^i dx^j + dy^2$  on  $M_4$  and the orientation  $dy dx^1 dx^2 dx^3$ . We will refer to the equations (18.37) as the  $KW(\mathbf{t})$  equations.

The desired knot homology complex  $\widehat{\mathcal{K}(L)}$  will be an MSW complex for the above gauged LG model on the half-plane with coordinates  $(x^0, y)$ . The vacua are solutions of the  $KW(\mathbf{t})$  equations and the instantons between them are the solutions of the  $HW(\mathbf{t})$  equations. That is, the differential on the complex  $\widehat{\mathcal{K}(L)}$  is obtained by counting solutions to the five-dimensional equations. However, to specify the complex precisely we need to specify boundary conditions for  $y \rightarrow 0$  and  $y \rightarrow \infty$ . We now turn to these boundary conditions. They should be viewed as part of the specification of the gauged LG model, including a brane at  $y = 0$ .

In our main application below we will take  $M_3 = \mathbb{R} \times C \cong \mathbb{R}^3$  where  $C$  is the complex plane. In this case, a suitable boundary condition for  $y \rightarrow \infty$  is simply to require that  $A \rightarrow 0$ , and  $\phi \rightarrow \vec{c} \cdot d\vec{x}$ , where  $\vec{c}$  is a chosen triple of commuting elements of  $\mathfrak{g}$ . Thus we can conjugate  $\vec{c}$  to a Cartan subalgebra  $\mathfrak{t} \subset \mathfrak{g}$ . For a more general  $M_3$ , a suitable condition for  $y \rightarrow \infty$  is to require the fields to approach a specified,  $y$ -independent, solution of the  $KW(\mathbf{t})$  equations.

The boundary conditions at  $y = 0$  are more subtle ones and involve specifying a singularity that the fields are supposed to have. One chooses the following data:

1. A homomorphism of Lie algebras  $\rho : \mathfrak{su}(2) \rightarrow \mathfrak{g}$ . In the usual application to knot homology, this is taken to be a principal embedding.
2. A representation  $R^\vee$  of the Langlands or GNO dual group  $G^\vee$  associated to each connected component of  $L$ . In general different components are associated to different representations.

We describe the boundary condition first for the case that  $M_3 = \mathbb{R}^3$  with Euclidean metric  $\sum_i dx_i^2$ , and without knots, and that  $G = SU(2)$ . Also, we take  $\rho$  to be the principal embedding, which for  $\mathfrak{g} = \mathfrak{su}(2)$  is just the identity map  $\mathfrak{su}(2) \rightarrow \mathfrak{su}(2)$ . In this situation, we impose the following<sup>110</sup> “Nahm pole boundary condition.” We require that for  $y \rightarrow 0$ ,  $A$  vanishes and

$$\phi \sim \frac{1}{y} \sum_{i=1}^3 \rho(t_i) dx^i + \mathcal{O}(y). \quad (18.38)$$

This boundary condition has a natural generalization on a general Riemannian three-manifold  $M_3$ . We state this first in the absence of knots. We fix a spin bundle<sup>111</sup>  $\mathcal{S} \rightarrow M_3$ , with structure group  $SU(2)$ , and write  $P_{\mathcal{S}}$  for the corresponding principal bundle. A  $G$ -bundle  $E = M_3$  is then defined by  $E = P_{\mathcal{S}} \times_{SU(2)} G$ , where  $SU(2)$  is embedded in  $G$  via  $\rho$  and acts on  $G$  on the left. Eqn. (18.38) then makes sense for a section  $\phi$  of  $T^*M_3 \otimes \text{ad}(E)$ .

---

<sup>110</sup>The name reflects the fact that the singular behavior that we are about to specify for  $y \rightarrow 0$  was originally introduced by W. Nahm in his study of Nahm’s equations associated to magnetic monopoles.

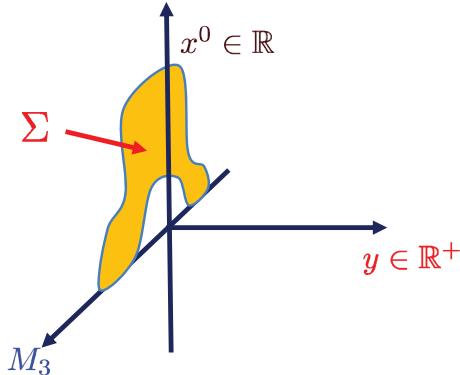
<sup>111</sup>In fact, for the principal embedding the choice of spin structure does not matter since the Lie algebra  $\mathfrak{g}$  pulls back to an integer spin representation.

The gauge field  $A$  is required to approach the spin connection on  $TM_3$  as  $y \rightarrow 0$ , where the spin connection is embedded in  $G$  via  $\rho : \mathfrak{su}(2) \rightarrow \mathfrak{g}$ . The point is that with this choice, the singular part of the  $KW$  equations is obeyed near  $y = 0$ , assuming that  $\mathbf{t} = 1$  (a minor modification is needed for more general  $\mathbf{t}$ ). To generalize the Nahm pole boundary condition in the presence of a knot, one “embeds a Dirac monopole singularity in the Nahm pole singularity.” What this means is that the solution is required to asymptote along  $L$  to a certain model solution of the  $KW(\mathbf{t})$  equations that informally is a combination of a Dirac monopole with a Nahm pole; the model solution depends on the choice of a representation of the dual group  $G^\vee$ . The basic model solution for  $G = SU(2)$ ,  $\rho = \text{Id}$ ,  $R^\vee$  the fundamental representation, and  $\mathbf{t} = 1$  is given in Section 3.6 of [92]. Near  $y = \infty$  the gauge field  $A$  vanishes and  $\phi_i$  approaches a constant determined, up to conjugacy, by the  $c_i$ . In particular for  $M_3 = \mathbb{R}^3$ ,

$$\phi \rightarrow \text{Ad}(g) \left( \sum_{i=1}^3 c_i dx^i \right) + \mathcal{O}(y^{-\delta}) \quad (18.39)$$

for some  $g \in G$  and some positive  $\delta$ . In the presence of a Nahm pole, a gauge transformation is required to be trivial at  $y = 0$ , and this ensures that the gauge group acts freely.

A demonstration that the Nahm pole boundary conditions for  $KW(\mathbf{t})$  and  $HW(\mathbf{t})$  at  $\mathbf{t} = 1$  are elliptic is given in [92, 72] and the generalization to  $\mathbf{t} \neq \pm 1$  and to include singular monopoles is fully expected to hold.



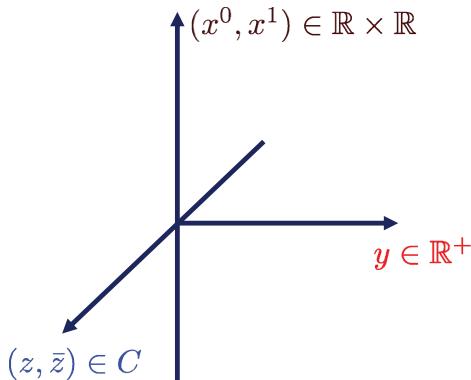
**Figure 172:** The basic setup for the gauge-theoretic approach to knot homology. One considers 5 dimensional SYM on a space  $\mathbb{R} \times M_3 \times \mathbb{R}_+$  where  $M_3$  is an oriented Riemannian 3-fold. For fixed  $x^0$ ,  $M_3$  contains a link  $L$  located at  $y = 0$ . The link evolves as a function of  $x^0$  to produce a knot bordism  $\Sigma$ .

### Remarks:

1. In knot homology an important role is played by morphisms  $\Phi(\Sigma) : \mathcal{K}(L_1) \rightarrow \mathcal{K}(L_2)$  associated with knot bordisms in  $\mathbb{R} \times M_3$  such that  $\partial\Sigma = L_2 - L_1$ . In the gauge

theoretic approach these, too, are located at  $y = 0$  as illustrated in Figure 172. There are corresponding boundary conditions on the  $\zeta$ -instantons, that is, on the  $HW(\mathbf{t})$  equations in the presence of such bordisms.

2. The advantage of the gauge-theoretic approach to knot homology is that the definition of the homology groups  $\mathcal{K}(L)$  does not employ a special choice of direction in  $M_3$  unlike, say, combinatorial definitions based on knot projections (such as Khovanov's original definition [59, 9]). The origin of the equations (18.37) and their 5d counterparts in topological field theory lead us to expect that the knot homology will be independent of the metric  $g_{ij}$ , the parameter  $\mathbf{t}$ , and the symmetry-breaking boundary conditions  $\vec{c}$ .
3. The knot homologies do, of course, depend on the data  $G, R^\vee, \rho$ . In [92] many of the expected properties of knot homologies such as a  $\mathbb{Z} \times \mathbb{Z}$  grading and its behavior with respect to change of framing and knot bordisms were established. (The second factor in the  $\mathbb{Z} \times \mathbb{Z}$  framing depends also on a choice of framing of the manifold  $M_3$  and of the link  $L$ .)



**Figure 173:** In the second formulation of  $\widehat{\mathcal{K}}(L)$  in terms of a gauged Landau-Ginzburg model, we specialize to  $M_3 = \mathbb{R} \times C$ , with  $C$  a Riemann surface. Thus, we consider five-dimensional SYM on a space  $\mathbb{R} \times \mathbb{R} \times C \times \mathbb{R}_+$ . We will eventually take  $C$  to be the complex plane, but the basic picture should hold for a general Riemann surface. (The surface  $C$  must have at least one puncture so that  $M_3$  admits a framing.) The link bordism at  $y = 0$  is not shown here. We use local complex coordinates  $z = x^2 + ix^3$  on  $C$ .

#### 18.4.4 Reformulation For $M_3 = \mathbb{R} \times C$

In order to make contact with the web formalism, we need an alternative formulation of the gauge theory equations which is available in the special case when the three-manifold factorizes  $M_3 = \mathbb{R} \times C$ , where  $C$  is a Riemann surface. Remarkably, the full set of equations  $HW(\mathbf{t})$  in this case can also be described as the  $\zeta$ -instanton equations in a second gauged

Landau-Ginzburg model. Since there are now two gauged Chern-Simons-Landau-Ginzburg models in play we will refer to them as CSLG1 and CSLG2.

The model CSLG2 is again a gauged LG model with a target space of complexified gauge fields, but now they are gauge fields on the three-manifold

$$\widetilde{M}_3 = C \times \mathbb{R}_+, \quad (18.40)$$

where  $\mathbb{R}_+$  is the  $y$ -direction. We must also assume the link  $L$  is translation invariant in the  $x^1$ -direction. Configurations involving a slow  $x^1$  dependence of the link  $L$  or even the Riemann surface  $C$  can be included as in Section §17

See Figure 173. We consider a sigma model of maps from  $\mathbb{R}^2$ , with coordinates  $(x^0, x^1)$ , to a space of complexified  $G^c$  gauge fields on  $\widetilde{M}_3 = C \times \mathbb{R}_+$ . We denote this space of gauge fields by  $\widetilde{\mathcal{U}}^c(\mathcal{BC})$ , where, now, the boundary conditions  $\mathcal{BC}$  serve to define the target space of the LG model, rather than the boundary conditions of the LG model. The conditions  $\mathcal{BC}$  will be specified in Section §18.4.5 below for the case of  $G = SU(2)$  or  $G = SO(3)$ .

In some more detail, we denote the complexified gauge field by

$$\tilde{\mathcal{A}} = \tilde{\mathcal{A}}_2 dx^2 + \tilde{\mathcal{A}}_3 dx^3 + \tilde{\mathcal{A}}_y dy. \quad (18.41)$$

We use the same formulae (18.28) for the metric, (18.29) for the symplectic form and (18.30) for the superpotential, but now with the replacement  $M_3 \rightarrow \widetilde{M}_3$  and  $\mathcal{A} \rightarrow \tilde{\mathcal{A}}$ . Viewed as equivariant Morse theory on  $\text{Map}(\mathbb{R}, \widetilde{\mathcal{U}}^c(\mathcal{BC}))$ , the Morse function is now

$$h = - \int_{\mathbb{R}} dx^1 \int_{\widetilde{M}_3} \text{vol}(g) g^{ij} \left( \tilde{\phi}_i \partial_{x^1} \tilde{A}_j - \tilde{\phi}_i \tilde{D}_j \tilde{B}_1 \right) - \frac{1}{2} \text{Re} \left[ ie^{-i\tilde{\vartheta}} CS(\tilde{\mathcal{A}}) \right] \quad (18.42)$$

and the  $\mathcal{Q}_\zeta$ -fixed point equations (again with  $\zeta = -ie^{i\tilde{\vartheta}}$ ) are again of the form (18.32), but with all covariant derivatives replaced by their corresponding version with a tilde:

$$[(\tilde{D}_1 - i\tilde{D}_0), \tilde{\mathcal{D}}] = e^{i\tilde{\vartheta}} (*_{\widetilde{M}_3} \tilde{\mathcal{F}}^*) \quad (18.43a)$$

$$[\tilde{D}_0, \tilde{D}_1] + \tilde{D} * \tilde{\phi} = 0 \quad (18.43b)$$

We stress that the equivalence of the equations (18.32) and (18.43) is not entirely trivial. The fields of the two CSLG models are different. In CSLG1, which applies for general three-manifolds  $M_3$ , we use one-form-valued fields with components  $\phi_1, \phi_2, \phi_3$ . In CSLG2, which applies for  $M_3 = \mathbb{R} \times C$ , we use one-form-valued fields with components  $\phi_2, \phi_3, \phi_y$ . Nevertheless, the new flow equations (18.43) are in fact equivalent to the original flow equations (18.32). To see this, let  $D_{i'} = \text{Ad}(\phi_i)$ ,  $i = 1, 2, 3$  and  $\tilde{D}_{a'} = \text{Ad}(\tilde{\phi}_a)$ ,  $a = 2, 3, y$ . If we make the replacement:

$$\begin{aligned} D_{1'} &\rightarrow \tilde{D}_0 \\ D_0 &\rightarrow -\tilde{D}_{y'} \end{aligned} \quad (18.44)$$

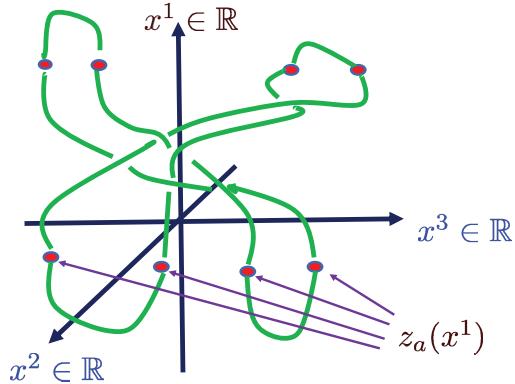
as well as  $D_{2',3'} \rightarrow \tilde{D}_{2',3'}$  and  $D_{1,2,3,y} \rightarrow \tilde{D}_{1,2,3,y}$  in (18.32), then, after some nontrivial rearrangement we obtain precisely the set of equations (18.43).

The great advantage of the second formulation is that the equations for the vacua of the model are much simpler than the  $KW(t)$  equations. Indeed, they simply say that the gauge field is flat and satisfies a moment map equation:<sup>112</sup>

$$[\tilde{\mathcal{D}}_a, \tilde{\mathcal{D}}_b] = 0 \quad (18.45a)$$

$$\sum_{a=2,3,y} [\tilde{\mathcal{D}}_a, \tilde{\mathcal{D}}_a^\dagger] = 0 \quad (18.45b)$$

Indeed, in CSLG1, the vacua cannot be “spatially independent” (i.e.  $y$ -independent) because of the boundary conditions on the LG model. In CSLG2, the vacua can be translationally invariant in the  $x^1$ -direction.



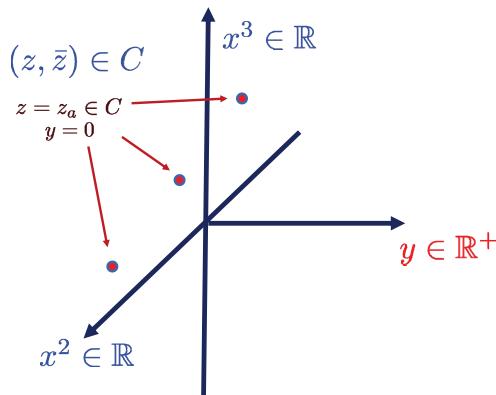
**Figure 174:** This figure depicts the link  $L$  in the boundary at  $y = 0$  at a fixed value of  $x_0$ . It is presented as a tangle evolving in the  $x^1$  direction and therefore can be characterized as a trajectory of points  $z_a(x^1)$  in the complex  $z = x^2 + ix^3$  plane. The points  $z_a$  are decorated with irreducible representations  $R_a$  of  $G^\vee$ . The tangle is closed by “creation” and “annihilation” of the points  $z_a$  in pairs (decorated with dual representations).

As mentioned above, the model CSLG2 is defined when the link  $L$  is translation-invariant in the  $x^1$  direction. Of course most links  $L \subset \mathbb{R} \times C$  do not have this property. An important part of the argument of [32] is to employ topological invariance of the underlying topologically twisted 5d SYM theory to present the link  $L$  as the closure of a tangle, that is, as an adiabatically evolving collection of points in  $C$ . They will be denoted  $z_a(x^1)$ ,  $a = 1, \dots, n$ . The evolution of  $z_a(x^1)$  defines a braid. We may take the evolution to be adiabatic, thus justifying various low-energy approximations used in the physical arguments. However, since we ultimately use a topological field theory this is not strictly necessary. At some critical values of  $x^1$  the number of points  $n$  jumps by  $n \rightarrow n \pm 2$  due to “creation” and “annihilation” processes. Indeed, we assume that for sufficiently large  $|x^1|$  there are no such points at all. That is, all the strands have been closed off in the far future and past. See Figure 174.

---

<sup>112</sup>In our conventions  $A_j, \phi_j \in \mathfrak{g}$  are regarded as anti-hermitian in real local coordinates, so, for example,  $(\partial_j + A_j + i\phi_j)^\dagger = -(\partial_j + A_j - i\phi_j)$ .

Therefore, our strategy will be first, to understand the model CSLG2 (and its finite-dimensional reductions) for the case where the link  $L$  is just a disjoint union of lines  $L = \coprod_{a=1}^n \mathbb{R} \times \{z_a\}$ , where  $z_a$  are a collection of points on the Riemann surface  $C$ . Then we will understand the  $x^1$ -evolution of the points  $z_a$  as defining a path of such theories. Accordingly, we can apply our general theory of Interfaces developed in Sections §§6-7. Ultimately we will have an Interface between the trivial theory and itself. As we saw in Section 6.1.5 above, such an Interface is a complex. This will be our proposal for a knot homology complex  $\widehat{\mathcal{K}}(L)$ . It is meant to be homotopy equivalent to the original MSW complex of CSLG1.



**Figure 175:** At a fixed value of  $x^0, x^1$  we have a 3-fold  $\widetilde{M}_3 = C \times \mathbb{R}_+$  shown here. The equivalent gauged Landau-Ginzburg model is formulated in terms of target space of complexified gauge fields  $\mathcal{A}$  defined on this space and satisfying suitable boundary conditions  $\mathcal{BC}$  at  $y \rightarrow 0, \infty$ , together with extra conditions at  $z = z_a$ .

#### 18.4.5 Boundary Conditions Defining The Fields Of The CSLG Model

Now we must discuss the boundary conditions  $\mathcal{BC}$  used to define the target space  $\widetilde{\mathcal{U}}^c(\mathcal{BC})$  of the model CSLG2. The link  $L$  is encoded in a collection of points  $z_a \in C$  at  $y = 0$ , as shown in Figure 175.

The rotation (18.44) is incompatible with Nahm pole boundary conditions on  $\phi$ . Therefore, we must find a new set of boundary conditions which correctly implements the presence of the knot at  $y \rightarrow 0$ . It is also useful to include boundary conditions at  $y \rightarrow \infty$  that reduce the structure group  $G$  of  $E$  to a maximal torus. Physically, this corresponds to moving onto the Coulomb branch of a gauge theory and resolves some confusing singularities in the moduli space of vacua.

While the papers [92, 93, 94] emphasize the formulation reviewed in Section §18.4.3, in fact, there is some freedom in the choice of which supersymmetry to use in deriving the  $Q$ -fixed point equations, as well as the precise boundary conditions which are imposed on the fields in the formulation of  $\widehat{\mathcal{K}(L)}$ . It should be possible to deform the boundary conditions

and still define an equivalent knot homology. Some of this freedom was employed in [32]. See Appendices A and B of [32] for a detailed discussion.

For technical simplicity, the discussion of [32] is restricted to  $G = SU(2)$  or  $G = SO(3)$ . Moreover, the Riemann surface  $C$  is taken to be the complex plane with Euclidean metric and trivial framing.<sup>113</sup> The irreducible representations  $R_a^\vee$  of  $G^\vee$  at  $z_a$  are described by their dimensions  $k_a + 1$ , where  $k_a$  is a positive integer. If  $G = SU(2)$  then  $k_a$  must be even, and if  $G = SO(3)$  then  $k_a$  can have either parity.<sup>114</sup> It is convenient to package these data into a monic polynomial

$$K(z) := \prod_{a=1}^n (z - z_a)^{k_a}. \quad (18.46)$$

To define the space  $\tilde{\mathcal{U}}^c(\mathcal{BC})$  we consider all  $G^c$  gauge fields on  $C \times \mathbb{R}_+$  that satisfy the following boundary conditions: For  $z \neq z_a$  and  $y \rightarrow 0$  we require that there is a gauge in which

$$\tilde{\mathcal{A}} \rightarrow \frac{1}{2y} \begin{pmatrix} dy & 2dz \\ 0 & -dy \end{pmatrix} + \dots \quad z \neq z_a, y \rightarrow 0 \quad (18.47)$$

This is the complex gauge field analog of the Nahm pole boundary condition.

For  $y \rightarrow \infty$  at fixed  $z$  we require that in some gauge

$$\tilde{\mathcal{A}} \rightarrow \frac{dz}{\xi} \begin{pmatrix} \mathbf{c} & 0 \\ 0 & -\mathbf{c} \end{pmatrix} + \xi d\bar{z} \begin{pmatrix} \bar{\mathbf{c}} & 0 \\ 0 & -\bar{\mathbf{c}} \end{pmatrix} + dy \begin{pmatrix} c_1 & 0 \\ 0 & -c_1 \end{pmatrix} \quad (18.48)$$

where the symmetry-breaking data  $\vec{c}$  is encoded here as three real numbers  $c_i$  and we define  $\mathbf{c} := \frac{1}{2}(c_2 - ic_3)$ . The parameter  $\xi$  that enters here is a complex number  $\xi \neq 0, \infty$  and is used to deform the original Nahm pole boundary conditions of CSLG1.<sup>115</sup> Away from knots, the Nahm pole boundary condition implies the existence of an everywhere nonzero “flat section”  $s$  of the rank 2 associated bundle to  $E^c$  which solves

$$\begin{aligned} \tilde{\mathcal{D}}_{\bar{z}} s &= 0 \\ \tilde{\mathcal{D}}_y s &= 0 \end{aligned} \quad (18.49)$$

Moreover, one can choose  $s$  to grow only polynomially for  $z \rightarrow \infty$  (without this condition, one would be free to multiply  $s$  by an entire function such as  $e^z$ ). If  $\mathcal{F} = 0$  and  $s$  satisfies (18.49), then the quantity  $s \wedge \mathcal{D}_z s$  is  $y$ -independent. Although we refer to  $s$  as a “flat section” in fact  $\mathcal{D}_z s$  is not zero. Knots are most succinctly incorporated by saying that  $s$  has a zero of order  $k_a$  along a knot, leading to

$$s \wedge \mathcal{D}_z s = K(z) \text{vol}(E) \quad (18.50)$$

where  $\text{vol}(E)$  is a fixed constant volume form on the rank two associated bundle. The boundary condition (18.50) encodes the presence of the knots at  $z = z_a$  and  $y = 0$ .

---

<sup>113</sup>It would be interesting to generalize the following considerations to an arbitrary Riemann surface.

<sup>114</sup>In the example relevant to a knot or link, presented as in Figure 174, the  $k_a$  are not independent but always appear in pairs.

<sup>115</sup>In [32] the parameter  $\xi$  was called  $\zeta$ , but we have renamed it here since it should not be confused with the  $\zeta$  used for the Landau-Ginzburg model.

This completes the formulation of the boundary conditions, and hence the specification of the target space  $\tilde{\mathcal{U}}^c(\mathcal{BC})$  of the model CSLG2.

### Remarks

1. We will henceforth write the space of gauge fields  $\tilde{\mathcal{A}}$  determined by these boundary conditions as  $\tilde{\mathcal{U}}^c(z_a, k_a; \xi, \vec{c})$ .
2. In the gauge (18.47) the section  $s$  is the solution of  $\mathcal{D}_y s = 0$  which vanishes for  $y \rightarrow 0$ . Therefore it is called the “small flat section.”
3. Note that even though  $\tilde{\mathcal{A}}$  is a flat connection on the simply connected domain  $\mathbb{R}^2 \times \mathbb{R}_+$  we cannot write a solution to both (18.49) and  $\mathcal{D}_z s = 0$  which satisfies the above boundary conditions. There can therefore be an interesting moduli space of such connections.

Reference [32] argues that one can parametrize the moduli space of solutions to the vacuum equations by making a complex gauge transformation, growing at most polynomially for  $z \rightarrow \infty$ , such that, away from  $y = 0$ ,  $\mathcal{A}$  is given by (18.48) exactly, not just asymptotically, on  $\widetilde{M}_3$ . The moduli space is then parametrized by the data of the small flat section, which, in this gauge, must have the form

$$s = \exp[-(\xi \bar{c} z + c_1 y) \sigma^3] \begin{pmatrix} P(z) \\ Q(z) \end{pmatrix} \quad (18.51)$$

where  $P(z), Q(z)$  are polynomials, unique up to rescaling  $(P, Q) \rightarrow (\lambda P, \lambda^{-1} Q)$ . Then equation (18.50) implies that

$$PQ' - QP' - c_0 PQ = K \quad (18.52)$$

where  $c_0 := -\frac{2c}{\xi}$ . Following [32] we rewrite this as  $e^{-c_0 z} \frac{K}{Q^2} = -\partial_z(e^{-c_0 z} \frac{P}{Q})$  and since  $P, Q$  are polynomials it follows that  $e^{-c_0 z} \frac{K}{Q^2}$  must have zero residue at all the zeroes of  $Q$ . Using the scaling freedom we can assume that  $Q(z)$  is monic, so it must have the form

$$Q(z) = \prod_{i=1}^q (z - w_i) \quad (18.53)$$

and moreover the roots  $w_i$  of  $Q$  are constrained to satisfy

$$\sum_{a=1}^n \frac{k_a}{w_i - z_a} = c_0 + \sum_{j \neq i} \frac{2}{w_i - w_j} \quad i = 1, \dots, q \quad (18.54)$$

These equations are of course the critical points of the function

$$W = \sum_{i,a} k_a \log(w_i - z_a) - \sum_{i \neq j} \log(w_i - w_j) - c_0 \sum_i w_i \quad (18.55)$$

As discussed at length in [32], the equations (18.54) define an oper with monodromy-free singularities. Consequently, there are close connections to integrable systems such

as the Gaudin model (with an irregular singular point at infinity when  $c_0 \neq 0$ ) [25]. In particular, the equations (18.54) are just the Bethe ansatz equations for the Gaudin model. Moreover, after adding a term depending only on  $k_a$  and  $z_a$ :

$$\Delta W = \frac{c_0}{2} \sum_a k_a z_a - \frac{1}{4} \sum_{a \neq b} \log(z_a - z_b) \quad (18.56)$$

the function  $W + \Delta W$  is the Yang-Yang function of the model. It is shown in [22] that there are generically  $\prod_a (k_a + 1)$  solutions to equation (18.54), provided we consider all  $0 \leq q \leq k$ . (This is also the dimension of  $\otimes_a R_a^\vee$ , a statement which is important in the theory of the Bethe ansatz.) In particular the space of vacua generically consists of a finite collection of distinct flat connections.

Moreover, the integral of  $\exp[(W + \Delta W)/b^2]$  over the Lefshetz thimbles associated to (18.55) are the renowned free-field representations of conformal blocks of degenerate representations of the Viraroso algebra. This in turn leads to a demonstration that, in the case of  $M_3 = \mathbb{R} \times \mathbb{C}$  and  $G = SU(2)$ , the Euler character of the MSW complex built on the  $KW(\mathbf{t})$  equations is indeed the Jones polynomial [32], thus providing substantial evidence that  $\mathcal{K}(L)$  is equivalent to Khovanov homology.

All of this strongly suggests that there is a low energy effective description of the model CSLG2 – equivalent at the level of  $LA_\infty$  algebras and the  $A_\infty$  category of Interfaces – in terms of a simpler ungauged LG model with a finite-dimensional target space and with superpotential (18.55). We next proceed with an argument that this is indeed the case.

#### 18.4.6 Finite-Dimensional LG Models: The Monopole Model

The first step in simplifying CSLG2, again described in [32], is to reduce it to an ungauged LG model whose target space is a moduli space of magnetic monopoles. We will refer to that as the *monopole model*. We now briefly sketch how this is done. A full derivation, taking careful account of  $D$ -terms, remains to be done.

A quick route to the monopole model is provided by applying the remark at the end of Section §18.4.2. Since  $S$  acts on  $X$  without fixed points, we can consider an equivalent ungauged Landau-Ginzburg model with target space the symplectic quotient. As a complex manifold this is  $X/S^c = \tilde{\mathcal{U}}^c(z_a, k_a; \xi; \vec{c})/\mathcal{G}^c$  and  $W^{cs}$  (modulo periods) descends to a nondegenerate Morse function on this space. The critical points are, of course, just the gauge equivalence classes of flat connections  $\tilde{\mathcal{A}}$  on  $\tilde{M}_3$  satisfying the boundary conditions (18.47) - (18.50). We claim that, at least when all three components  $c_i \neq 0$  and  $\xi \neq 0, \infty$ , there is a finite set of critical points which can be identified as a finite set of points in a moduli space of smooth magnetic monopoles for  $SU(2)$  of charge

$$m := \frac{1}{2} \sum_a k_a. \quad (18.57)$$

(For  $G = SU(2)$ , the integers  $k_a$  are all even. For  $G = SO(3)$ , the integers  $k_a$  can have any parity, but the sum is constrained to be even.)

To explain this claim in some more detail, we begin with an important preliminary remark. Due to the Nahm pole,  $\tilde{\varphi} := \frac{1}{2}(\tilde{\phi}_2 - i\tilde{\phi}_3)$  is nonzero and hence  $\mathcal{A}_z$  is not unitary.

For the moment take the complex symmetry breaking parameter  $c_0 = 0$  but  $c_1 \neq 0$ . Since  $c_1$  is nonzero the gauge symmetry is spontaneously broken to  $U(1)$  and hence charged fields, such as  $\tilde{\varphi}$ , will decay exponentially fast for  $y \rightarrow \infty$  on a scale set by  $c_1$ . On the other hand, when  $\tilde{\varphi} = 0$  the equations (18.49) are equivalent to the ‘‘holomorphic part’’ of the standard Bogomolnyi equations for ordinary magnetic monopoles in a Yang-Mills-Higgs theory.

We are therefore led to consider the space  $\mathcal{M}(c_1, m)$  of smooth  $SU(2)$  magnetic monopoles with asymptotic Higgs field

$$\tilde{\phi}_y \rightarrow c_1 \mathfrak{h} - \frac{m}{2r} \mathfrak{h} + \dots \quad \tilde{F} \rightarrow \frac{1}{2} m \mathfrak{h} \sin \theta d\theta d\phi + \dots \quad (18.58)$$

where  $\mathfrak{h} = i\sigma^3$  is a simple coroot in a standard Cartan subalgebra of  $su(2)$  and we chose a gauge with  $\tilde{\phi}_y$  constant at infinity. Moreover, we choose the gauge so that  $c_1$  is positive. (This still leaves an  $SO(2)$  subgroup of global gauge transformations unfixed. We will fix it below.) Although  $\mathcal{M}(c_1, m)$  is hyperkähler, the choice of a distinguished direction  $y$  selects a distinguished complex structure on  $\mathcal{M}(c_1, m)$  and we will simply regard it as a Kähler manifold in this complex structure. In order to incorporate the other flatness equations from  $W^{cs}$ , and the fact that  $\tilde{\varphi}$  is not exactly zero we will introduce an effective superpotential on the space  $\mathcal{M}(c_1, m)$ . It will be holomorphic in the complex structure selected by  $y$ .

In order to write the effective superpotential, we will make use of a well-known presentation of the monopole moduli space  $\mathcal{M}(c_1, m)$  [45, 19, 52, 6]. We consider the scattering problem along the  $y$  axis at fixed  $(z, \bar{z})$  for the operator  $\mathcal{D}_y$ . The evolution operator along the  $y$ -axis is just

$$P \exp \left[ - \int_{y_-}^{y_+} (\tilde{A}_y + i\tilde{\phi}_y) dy \right] \quad (18.59)$$

Using (18.58), it follows that there exists a basis of covariantly constant sections with  $y \rightarrow +\infty$  asymptotics

$$\begin{aligned} s^{(+\infty, +)} &\sim e^{c_1 y} y^{-m/2} \begin{pmatrix} 1 \\ 0 \end{pmatrix} (1 + \mathcal{O}(1/y)) \\ s^{(+\infty, -)} &\sim e^{-c_1 y} y^{m/2} \begin{pmatrix} 0 \\ 1 \end{pmatrix} (1 + \mathcal{O}(1/y)) \end{aligned} \quad (18.60)$$

Similarly, there is a basis of such sections with  $y \rightarrow -\infty$  asymptotics

$$\begin{aligned} s^{(-\infty, -)} &\sim e^{c_1 y} |y|^{m/2} \begin{pmatrix} 1 \\ 0 \end{pmatrix} (1 + \mathcal{O}(1/y)) \\ s^{(-\infty, +)} &\sim e^{-c_1 y} |y|^{-m/2} \begin{pmatrix} 0 \\ 1 \end{pmatrix} (1 + \mathcal{O}(1/y)) \end{aligned} \quad (18.61)$$

Since the space of flat sections is two-dimensional we have

$$\begin{aligned} s^{(-\infty, -)} &= Q s^{(+\infty, +)} - \tilde{P} s^{(+\infty, -)} \\ s^{(-\infty, +)} &= P s^{(+\infty, +)} + R s^{(+\infty, -)} \end{aligned} \quad (18.62)$$

where the scattering matrix

$$\begin{pmatrix} Q & P \\ -\tilde{P} & R \end{pmatrix} \quad (18.63)$$

satisfies  $QR + P\tilde{P} = 1$  since the field  $\tilde{\mathcal{A}}_y$  is traceless. The peculiar sign choice in front of  $\tilde{P}$  will be convenient later.

Now, the holomorphic part of the Bogomolnyi equations can be written as

$$[\mathcal{D}_{\bar{z}}, \mathcal{D}_y] = 0. \quad (18.64)$$

We may therefore choose the sections  $s^{(\pm\infty, \pm)}$  to be annihilated by  $\mathcal{D}_{\bar{z}}$  as well as  $\mathcal{D}_y$  and therefore the “S-matrix” (18.63) is holomorphic in  $z$ . The asymptotics (18.60) and (18.61) only determine the bases up to a shift of the growing solution by a multiple of the decaying solution. This multiple can be a holomorphic function of  $z$  and therefore the S-matrix is only determined up to multiplication

$$\begin{pmatrix} Q & P \\ -\tilde{P} & R \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 \\ U_+(z) & 1 \end{pmatrix} \begin{pmatrix} Q & P \\ -\tilde{P} & R \end{pmatrix} \begin{pmatrix} 1 & U_-(z) \\ 0 & 1 \end{pmatrix} \quad (18.65)$$

Note that if the monopoles are all uniformly translated by  $\Delta y$  in the  $y$ -direction then, from the asymptotics (18.60) and (18.61) we see that the scattering matrix is transformed to

$$\begin{pmatrix} Q & P \\ -\tilde{P} & R \end{pmatrix} \rightarrow \begin{pmatrix} e^{c_1 \Delta y} & 0 \\ e^{-c_1 \Delta y} & e^{c_1 \Delta y} \end{pmatrix} \begin{pmatrix} Q & P \\ -\tilde{P} & R \end{pmatrix} \begin{pmatrix} e^{-c_1 \Delta y} & 0 \\ e^{c_1 \Delta y} & e^{-c_1 \Delta y} \end{pmatrix} \quad (18.66)$$

and in particular  $P \rightarrow e^{2c_1 \Delta y} P$ . From equation (18.62) it is clear that the zeroes of  $Q$  correspond to the points  $z = w_i$  where there is a boundstate for the scattering problem, with exponential decay at both ends  $y \rightarrow \pm\infty$ . In the asymptotic region of moduli space with well-separated monopoles these zeroes represent the positions of the monopoles in the complex plane, and the  $S$ -matrix approaches a product of factors  $S_1 S_2 \cdots S_m$  where  $S_i$  are the scattering matrices computed from the one monopole problem

$$S_i = \begin{pmatrix} (z - w_i) & e^{\mathcal{Y}_i} \\ -e^{-\mathcal{Y}_i} & 0 \end{pmatrix} \quad (18.67)$$

and

$$\text{Re}(\mathcal{Y}_1) \ll \text{Re}(\mathcal{Y}_2) \ll \cdots \ll \text{Re}(\mathcal{Y}_m). \quad (18.68)$$

Again,  $\text{Re}(\mathcal{Y})$  shifts by  $2c_1 \Delta y$  under translation of the monopole by  $\Delta y$  in the  $y$ -direction, and hence represents the  $y$ -position of the monopole.

Given the asymptotic factorization of the scattering matrix into factors of the form (18.67) it follows that the components of the matrix (18.63) are polynomial functions of  $z$ . In particular,  $Q$  has degree  $m$ . Using the residual  $SO(2)$  gauge freedom left unfixed from (18.58) we can take  $Q(z)$  to be monic:

$$Q(z) = \prod_{i=1}^m (z - w_i). \quad (18.69)$$

Now we can fix the ambiguity (18.65) by requiring  $P, \tilde{P}$  to have degree  $m - 1$  and  $R$  to have degree  $m - 2$ . Thus, we can uniquely associate to a point in monopole moduli space a rational map  $\tilde{P}/Q$  and, by [19] this is in fact a diffeomorphism with the space of rational maps.

Given the above rational map presentation of  $\mathcal{M}(c_1, m)$  we are now ready to introduce the effective superpotential on the space  $\mathcal{M}(c_1, m)$ . By combining physical arguments with qualitative features of the moduli space of opers, reference [32] proposed the effective superpotential on monopole moduli space to be:

$$W = \sum_{i=1}^m \left( \text{Res}_{z=w_i} \frac{K(z)\tilde{P}(z)dz}{Q(z)} - \log \tilde{P}(w_i) - c_0 w_i \right) + \Delta W(z_a) \quad (18.70)$$

where  $K(z)$  was defined in (18.46) above and the last term  $\Delta W(z_a)$  is independent of the  $w_i$ , and cannot be determined by the above arguments.<sup>116</sup> Note that we have taken into account the complex symmetry breaking by  $c_0$  through the superpotential.

The justification for (18.70) is given by matching the critical points to the moduli space of opers with monodromy-free singularities. Assuming that the points  $w_i$  are all distinct we can change coordinates to  $\tilde{P}(w_i) := e^{-\mathcal{Y}_i}$ . As we have seen, in the asymptotic regions of moduli space the parameters  $\text{Re}(\mathcal{Y}_i)$  measure the positions of  $m$  basic 't Hooft-Polyakov monopoles in the  $y$  direction so long as these values are large and positive. In this limit the fields are heavy and it is justified to integrate them out leaving an effective superpotential for a collection of chiral superfields  $W_i$ , whose leading term is  $w_i$ . The critical points are determined by

$$e^{\mathcal{Y}_i} = \frac{K(w_i)}{Q'(w_i)} \quad (18.71)$$

But, since  $QR + P\tilde{P} = 1$ , we have  $P(w_i) = e^{\mathcal{Y}_i}$ . This matches beautifully with (18.52). Moreover, the effective superpotential for the remaining fields  $w_i$  is precisely (18.55).

Of course,  $W$  is not single-valued due to the terms  $\log \tilde{P}(w_i)$ . In fact  $\pi_1(\mathcal{M}) = \mathbb{Z}$  and the universal cover of the moduli space is  $\widehat{\mathcal{M}} = \mathbb{R}^4 \times \mathcal{M}_0$ , where  $\mathcal{M}_0$  is the simply connected moduli space of centered monopoles. The superpotential will be single-valued on this cover. Note that deck transformations shift  $W$  by elements of  $2\pi i\mathbb{Z}$ . Until now we have not said how the trace  $\text{Tr}$  on  $\mathfrak{g}$  used in (18.28), (18.29), and (18.30) is normalized, but at this point it should be normalized so that the periods of the Chern-Simons form for the unitary gauge field  $A$  are in  $2\pi i\mathbb{Z}$ .

In summary we have the *monopole model*:

1. The Kähler target space is the space of *smooth* monopoles  $\mathcal{M}(c_1, m)$  on  $\mathbb{R}^3$  with magnetic charge  $m\mathfrak{h}$  and Higgs vev at infinity  $c_1\mathfrak{h}$ . Using the standard hyperkähler metric we consider it to be Kähler in the complex structure determined by  $y$ .
2. The superpotential is given by (18.70).

---

<sup>116</sup>In principle,  $\Delta W(z_a)$  could be determined by evaluation of the Chern-Simons functional on the critical flat connections. In [32], it was determined by a relation to conformal field theory. In any case, in the application to webs below, it is a constant shift in the superpotential and will not affect the webs.

### 18.4.7 Finite-Dimensional Models: The Yang-Yang Model

The procedure of integrating out the fields  $\mathcal{Y}_i$  in the monopole model leads to a model we call the *Yang-Yang model*. Integrating out heavy fields is expected to produce a LG model whose associated  $LA_\infty$  algebra and categories of Branes and Interfaces is equivalent to the original one.

To define the *Yang-Yang model*, we choose a collection of  $n$  distinct points  $z_a \in \mathbb{C}$  and label them with positive integers  $k_a$  such that  $k = \sum_a k_a$  is even. Fix an integer  $1 \leq q \leq k$ . (For  $q$  outside this range, the Bethe equations that we arrive at will have no solutions.) The target space of the model is a covering space of the configuration space  $\mathcal{C}(q, S)$  of  $q$  distinct, but indistinguishable points  $w_i$ ,  $i = 1, \dots, q$  on  $S := \mathbb{C} - \{z_1, \dots, z_n\}$ . To define the covering space we introduce the superpotential:

$$W = \sum_{i,a} k_a \log(w_i - z_a) - \sum_{i \neq j} \log(w_i - w_j) - c_0 \sum_i w_i \quad (18.72)$$

The target space  $X$  should be the smallest cover of  $\mathcal{C}(q, S)$  on which  $W$  is single-valued as a function of the  $w_i$ . Thus, explicitly,  $X = \widehat{\mathcal{C}}(q, s)/H$  where  $\widehat{\mathcal{C}}(q, s)$  is the universal cover and  $H$  is the subgroup of  $\pi_1$  given by the kernel of the homomorphism  $\oint dW : \pi_1 \rightarrow 2\pi i\mathbb{Z}$ .  $X$  is thus a Galois cover with covering group  $\pi_1/H \cong \mathbb{Z}$ . We are primarily interested in the case  $q = \frac{k}{2}$ , since this is the case that arises from the monopole model. The derivation of the model from the monopole model suggests that the Kähler metric should be taken to be the metric induced from the hyperkähler metric on  $\mathcal{M}(c_1, m)$ . Using the discussion of Section 10.7 above, the algebraic structures of concern to us will be unaffected if we replace that metric by the much simpler Euclidean metric  $\sum_i |dw_i|^2$  (pulled back to  $X$ ) and we make this choice. In particular, with this choice we can define the model for any value of  $q$ . We denote the Yang-Yang model by  $\mathcal{T}(q, \{k_a, z_a\})$ .

### 18.4.8 Knot Homology From Interfaces Between Landau-Ginzburg Models

We can now use the Yang-Yang model to formulate a knot homology complex. The vacua  $\mathbb{V}$  are in 1-1 correspondence with the (lifts of) the Bethe roots of (18.54). They can be labeled (noncanonically) as  $\vec{w}^{(r,n)}$ , with  $r = 1, \dots, \prod_a (k_a + 1)$  and  $n \in \mathbb{Z}$ . For each Bethe root  $\vec{w}^{(r,n)}$  the vacuum weight is determined by the value of the Yang-Yang function  $W^{r,n} = W(\vec{w}^{(r,0)}) + 2\pi i n$ . In addition the web representation  $\mathcal{R}$ , the interior amplitude  $\beta$ , and, once we choose a phase  $\zeta$  and a half-plane, the category of branes are all determined, in principle, by the physical model.

Now we consider again the link presented as a tangle determined by the functions  $z_a(x^1)$ . The vacuum data evolves with  $x^1$ , as does the web representation and interior amplitude. When the number of strands is conserved as a function of  $x^1$  the theory of Sections §6-8 determines an Interface  $\mathfrak{I}[z_a(x)]$  between the initial and final Theories  $\mathcal{T}(q, \{k_a, z_a^{\text{in}}\})$  and  $\mathcal{T}(q, \{k_a, z_a^{\text{out}}\})$ . As we have explained, this can be used to construct an  $A_\infty$  functor between the initial and final  $A_\infty$  categories of Branes.

In addition, if two points, say  $z_{a_1}(x)$  and  $z_{a_2}(x)$ , carry the same integer,  $k_{a_1} = k_{a_2} = k_1$ , then they can be “annihilated,” changing  $n + 2 \rightarrow n$ . The “time-reversed” process creates

two points and changes  $n \rightarrow n + 2$ . Taking, for simplicity  $q = \frac{1}{2}k$ , the annihilation must lead to an Interface  $\mathfrak{I}^{n+2 \rightarrow n}(a_1, a_2) \in \mathfrak{Br}(\mathcal{T}(q, \{k_a, z_a\}), \mathcal{T}(q', \{k_a, z_a\}'))$ , where  $q' = q - k_1$  and  $\{k_a, z_a\}' = \{k_a, z_a\} - \{k_{a_1}, k_{a_2}, z_{a_1}, z_{a_2}\}$ . Similarly, the creation process must lead to an interface  $\mathfrak{I}^{n \rightarrow n+2}(a_1, a_2) \in \mathfrak{Br}(\mathcal{T}(q', \{k_a, z_a\}'), \mathcal{T}(q, \{k_a, z_a\}))$ . Moreover, these interfaces should be related by parity-reversal.

Up to homotopy equivalence of Interfaces the Interface associated with a tangle can be decomposed into products of Interfaces associated with elementary positive or negative braidings of single pairs of points and associated to creation and annihilation of pairs of points. We are thus led to consider four types of basic interfaces:

1. If the path  $\varphi_{a_1, a_2}^\pm$  braids two points  $z_{a_1}(x)$  and  $z_{a_2}(x)$  while all other points  $z_a(x)$ , for  $a \neq a_1, a_2$  are fixed (on some small interval in  $x$ ) then there will be *braiding Interfaces*  $\mathfrak{I}^\pm(\varphi_{a_1, a_2}^\pm)$  between the theory  $\mathcal{T}(\{z_a\}, \{k_a\})$  and the theory with  $k_{a_1} \leftrightarrow k_{a_2}$ . The superscript indicates whether the braiding is clockwise or counterclockwise. These will be very similar to the S-wall interfaces discussed above.
2. If two points  $z_{a_1}(x)$  and  $z_{a_2}(x)$  annihilate then there will be an *annihilation Interface*  $\mathfrak{I}^{n \rightarrow n+2}(a_1, a_2)$  as described above. Similarly, there will be creation Interfaces  $\mathfrak{I}^{n+2 \rightarrow n}(a_1, a_2)$ .

Now, a tangle such as shown in Figure 174 is an  $x^1$ -ordered instruction of creation of pairs of points, braidings of points, and annihilations of pairs of points. Let us denote the corresponding ordered set of Interfaces for the tangle as  $\mathfrak{I}_1, \dots, \mathfrak{I}_N$ , for some  $N$ , where each  $\mathfrak{I}_s$  is one of the four types of basic interfaces described above. Then we can use the interface product  $\boxtimes$  described above to construct

$$\mathfrak{I}(\text{Tangle}) := \mathfrak{I}_1 \boxtimes \cdots \boxtimes \mathfrak{I}_N. \quad (18.73)$$

The Interface (18.73) is an Interface between the trivial Theory and itself. As we saw in Section §6.1.5 an Interface between the trivial Theory and itself is a chain complex. We propose that this chain complex defines a knot homology complex  $\widehat{\mathcal{K}(L)}$ . Moreover, in the case of  $G = SO(3)$  and all  $k_a = 1$  this should give a theory equivalent to Khovanov homology.

The required double-grading on  $\widehat{\mathcal{K}(L)}$  comes about as follows: The  $R_{ij}$  and Chan-Paton data have the usual grading by Fermion number  $\mathcal{F}$ . The second grading comes from the fact that  $dW$  has periods. As we have seen, the vacua  $\bar{w}^{r,n}$  are labeled (noncanonically) by a sheet index  $n \in \mathbb{Z}$ . It is natural to assign  $q$ -grading  $(n_2 - n_1)$  to the Morse complexes  $R_{(r_1, n_1), (r_2, n_2)}$  and  $q$ -grading  $n$  to the Chan-Paton spaces  $\mathcal{E}_{(r, n)}$  of the Branes and Interfaces. One important statement which is expected, but should be proven, is that the Interface (18.73) does not depend on the tangle presentation of  $L$ , up to homotopy equivalence of Interfaces.

It remains to construct the elementary interfaces. This is beyond the scope of the present work and will be addressed elsewhere. We simply mention that one can get a good analytic understanding of the Bethe roots and the critical values of  $W$  in the limit that  $c_0$

is large but  $\delta = c_0(z_1 - z_2) \rightarrow 0$  as a double expansion in  $1/c_0$  and  $\delta$ . Moreover, the critical values form two clusters of vacua<sup>117</sup> and hence the results on cluster webs from Section §18.3 will be relevant to the construction.

**Example** In order to illustrate some of the issues which must be overcome to make this proposal computationally effective let us consider the construction of the complex  $\widehat{\mathcal{K}(L)}$  for the unknot with  $k_1 = k_2 = 1$ . In this case two points  $z_a$ ,  $a = 1, 2$  are simply created and then annihilated. The superpotential, after creation, is just

$$W = \log(w - z_1) + \log(w - z_2) - c_0 w \quad (18.74)$$

The target space  $X$  is therefore a cyclic cover of the plane with two punctures. Writing  $z_1 = \delta/c_0$  and  $z_2 = -\delta/c_0$ , the two Bethe roots are

$$w^\epsilon = \frac{1}{c_0} + \epsilon \frac{\sqrt{1 + \delta^2}}{c_0} \quad (18.75)$$

where  $\epsilon \in \{\pm 1\}$ . Hence the vacua on  $X$  can be denoted by  $w^{\epsilon, n}$ ,  $n \in \mathbb{Z}$ . The critical values of  $W$  at these vacua are

$$W^{\epsilon, n} = 2 \log \frac{1}{c_0} + \log[2(1 + \epsilon \sqrt{1 + \delta^2})] - 2 + 2\pi i n \quad (18.76)$$

The soliton spaces  $R_{(\epsilon_1, n_1), (\epsilon_2, n_2)}$  clearly only depend on  $n = n_1 - n_2$  so denote we denote them simply by  $R_{\epsilon_1, \epsilon_2, n}$ . This space should have  $q$  grading  $q^n$ . For  $\epsilon_1 \epsilon_2 = +1$  the spaces are nonzero only for  $n = \pm 1$ . When  $\epsilon_1 \epsilon_2 = -1$  there will be an infinite number of nonzero spaces  $R_{\epsilon_1, \epsilon_2, n}$ .<sup>118</sup> Hence there is an infinite number of soliton slopes, with an accumulation slope along the vertical axis. We therefore choose boundary conditions to preserve  $\zeta$ -supersymmetry with  $\zeta \neq \pm 1$ . There are in principle an infinite number of cyclic fans and interior amplitudes, but the  $L_\infty$  Maurer-Cartan equations will be well-defined, as discussed above in Section §18.1. Specifying the creation Interface  $\mathfrak{I}^>$  requires specifying the Chan-Paton spaces  $\mathcal{E}(\mathfrak{I}^>)_\epsilon, n$  together with the boundary amplitudes. Once this is determined the annihilation Interface  $\mathfrak{I}^<$  is just the parity reverse. Thus, altogether, the complex for the unknot is

$$\oplus_{\epsilon, n} \mathcal{E}(\mathfrak{I}^>)_\epsilon, n \otimes \mathcal{E}(\mathfrak{I}^<)_\epsilon, n^* \quad (18.77)$$

with a differential obtained by our formalism using the taut strip-webs and the boundary and interior amplitudes. We hope to return to a more complete analysis on another occasion.

### Remarks

1. The generalization where  $C$  is a Riemann surface makes contact with the theory of surface defects in class S theories. The surface defect theories for  $\mathbb{S}_{z_1, \dots, z_n}$  mentioned

<sup>117</sup>These clusters can be nicely understood in terms of bases of conformal blocks for degenerate representations of the Viraroso algebra.

<sup>118</sup>Whether or not the cohomology of  $R_{\epsilon_1, \epsilon_2, n}$  is nonzero is more subtle. Experience with the closely related  $\mathbb{CP}^1$  model with a twisted mass parameter suggests that this depends on  $c_0$  and  $\delta$ .

above should be closely related to the Landau-Ginzburg models we have just discussed based on the data  $z_a$ . An important lesson we learn is that there are topological interactions between the distinct M2 branes ending on the UV curve, and they cannot be treated independently, even when they are far separated.

2. Knot bordisms can be incorporated into our formalism by using the  $(x^0, x^1)$ -dependent data of Theories discussed in Section §6.3.3 and §8.
3. The categorified version of the skein relations should translate into some interesting relations between the basic Interfaces described above. In order to prove, for example, that the Interface (18.73) is independent of the tangle presentation up to homotopy equivalence it would then suffice to prove these relations on the Interfaces.

## Acknowledgements

We would especially like to thank Nick Sheridan for many useful discussions, especially concerning the Fukaya-Seidel category. We would also like to thank M. Abouzaid, K. Costello, T. Dimofte, D. Galakhov, E. Getzler, M. Kapranov, L. Katzarkov, M. Kontsevich, Kimyeong Lee, S. Lukyanov, Y. Soibelman, and A. Zamolodchikov for useful discussions and correspondence. The research of DG was supported by the Perimeter Institute for Theoretical Physics. Research at Perimeter Institute is supported by the Government of Canada through Industry Canada and by the Province of Ontario through the Ministry of Economic Development and Innovation. The work of GM is supported by the DOE under grant DOE-SC0010008 to Rutgers and NSF Focused Research Group award DMS-1160591. GM also gratefully acknowledges the hospitality of the Institute for Advanced Study, the Perimeter Institute for Theoretical Physics, the Aspen Center for Physics (under NSF Grant No. PHY-1066293), the KITP at UCSB (NSF Grant No. NSF PHY11-25915) and the Simons Center for Geometry and Physics. The work of EW is supported in part by NSF Grant PHY-1314311.

## A. Summary Of Some Homotopical Algebra

This subject is well-reviewed. See, for examples, [5, 57, 81]. We briefly summarize some material to establish our notation and conventions and, in some places, to emphasize a slightly nonstandard viewpoint on this standard material.

Throughout this appendix and the next, the term “module” refers to either a  $\mathbb{Z}$ -module, i.e., an abelian group, or a vector space over a field. All modules are assumed to be  $\mathbb{Z}$ -graded. Moreover, all infinite sums are assumed to be convergent.

### A.1 Shuffles And Partitions

If  $P$  is any ordered set we define an ordered  $n$ -partition of  $P$  to be an ordered disjoint decomposition into  $n$  ordered subsets

$$P = P_1 \amalg P_2 \amalg \cdots \amalg P_n \tag{A.1}$$

where the ordering of each summand  $P_\alpha$  is inherited from the ordering of  $P$  and all the elements of  $P_\alpha$  precede all elements of  $P_{\alpha+1}$  inside  $P$ . We allow the  $P_\alpha$  to be the empty set. For an ordered set  $P$  we let  $\text{Pa}_n(P)$  denote the set of distinct  $n$ -partitions of  $P$ . If  $p = |P|$  there are  $\binom{n+p-1}{p}$  such partitions. For example, if  $n = 2$  there are  $p+1$  different 2-partitions. Each 2-partition is completely determined by specifying the number of elements in  $P_1$ . This number can be any integer from 0 to  $p$ .

If  $S$  is an ordered set then an  $n$ -shuffle of  $S$  is an ordered disjoint decomposition into  $n$  ordered subsets

$$S = S_1 \amalg S_2 \amalg \cdots \amalg S_n \quad (\text{A.2})$$

where the ordering of each summand  $S_\alpha$  is inherited from the ordering of  $S$  and the  $S_\alpha$  are allowed to be empty. Note that the ordering of the sets  $S_\alpha$  also matters so that  $S_1 \amalg S_2$  and  $S_2 \amalg S_1$  are distinct 2-shuffles of  $S$ . For an ordered set  $S$  we let  $\text{Sh}_n(S)$  denote the set of distinct  $n$ -shuffles of  $S$ . We can count  $n$ -shuffles by successively asking each element of  $S$  which set  $S_\alpha$  it belongs to. Hence there are  $n^{|S|}$  such shuffles.

## A.2 $A_\infty$ Algebras

An  $A_\infty$  structure  $\mu$  on a module  $\mathcal{A}$  is defined by a collection of multilinear maps

$$\mu_n : \mathcal{A}^{\otimes n} \rightarrow \mathcal{A} \quad (\text{A.3})$$

of degree  $2 - n$ , which satisfies the following  $A_\infty$  associativity relation:

$$\sum_{P_{1,2,3} \in \text{Pa}_3(P)} \epsilon_P(P_{1,2,3}) \mu(P_1, \mu(P_2), P_3) = 0 \quad (\text{A.4})$$

for all ordered sets  $P$  of elements in  $\mathcal{A}$ . (We identify ordered sets of elements in  $\mathcal{A}$  with monomials in  $\mathcal{A}^{\otimes n}$ .) We define for convenience  $\mu(P_\alpha) := \mu|_{P_\alpha}(P_\alpha)$  and we take  $\mu(\emptyset) := 0$ .<sup>119</sup> We will define the sign  $\epsilon_P(P_{1,2,3})$  momentarily. The pair  $(\mathcal{A}, \mu)$  is called an  $A_\infty$  algebra.

In this paper, we choose the sign convention where  $\epsilon_P(P_{1,2,3}) = (-1)^{\deg_r(P_1)}$ , where the reduced degree  $\deg_r$  for a set  $\{a_1, \dots, a_n\}$  of elements in  $\mathcal{A}$  is

$$\deg_r(\{a_1, \dots, a_n\}) = \sum_{k=1}^n (\deg(a_k) - 1) \quad (\text{A.5})$$

This is an example of the *reduced Koszul rule* we use throughout this paper: in order to determine the sign  $\epsilon_P(P_{1,2,3})$  we assign *reduced degree*  $\deg_r(a) = \deg(a) - 1$  to arguments in  $P$  and  $\deg_r(\mu) = 1$  to  $\mu$  and use the Koszul rule with the reduced degree in order to bring the symbols in (A.4) from a canonical order  $\mu\mu P$  to the order they appear with in

---

<sup>119</sup>It is straightforward, if needed, to relax the axioms for  $A_\infty$  structures by allowing a choice of nonvanishing “source”  $\mu(\emptyset) \in \mathcal{A}$ . Such an algebra is called a “curved  $A_\infty$ -algebra” in the math literature.

the equation. For example, the first few associativity relations are

$$\begin{aligned}
& \mu_1(\mu_1(a)) = 0 \\
& \mu_1(\mu_2(a_1, a_2)) + \mu_2(\mu_1(a_1), a_2) + (-1)^{\deg(a_1)-1} \mu_2(a_1, \mu_1(a_2)) = 0 \\
& \mu_1(\mu_3(a_1, a_2, a_3)) + \mu_2(\mu_2(a_1, a_2), a_3) + (-1)^{\deg(a_1)-1} \mu_2(a_1, \mu_2(a_2, a_3)) + \mu_3(\mu_1(a_1), a_2, a_3) \\
& + (-1)^{\deg(a_1)-1} \mu_3(a_1, \mu_1(a_2), a_3) + (-1)^{\deg(a_1)+\deg(a_2)} \mu_3(a_1, a_2, \mu_1(a_3)) = 0 \\
& \dots
\end{aligned} \tag{A.6}$$

The choice of signs based on the reduced Koszul rule may appear surprising at first sight. In order to understand why it is the “correct” choice, as opposed to naive alternatives, such as a sign based on a standard Koszul rule, it is useful to point out a simple consistency condition on the associativity constraint (A.4). Choose an ordered set of elements  $P$  in  $\mathcal{A}$  and, for any  $P_{1,2,3} \in \text{Pa}_3(P)$  let  $\widehat{P_{1,2,3}}$  denote the ordered set of objects  $P_1 \amalg \{\mu(P_2)\} \amalg P_3$ . We consider the associativity constraint (A.4) for the ordered set  $\widehat{P_{1,2,3}}$ , and then sum that over  $\text{Pa}_3(P)$ , weighted by  $\epsilon_P(P_{1,2,3})$ :

$$\sum_{P_{1,2,3} \in \text{Pa}_3(P)} \epsilon_P(P_{1,2,3}) \sum_{Q_{1,2,3} \in \text{Pa}_3(\widehat{P_{1,2,3}})} \epsilon_{\widehat{P_{1,2,3}}}(Q_{1,2,3}) \mu(Q_1, \mu(Q_2), Q_3) = 0 \tag{A.7}$$

Now, to each nested partition  $Q_{1,2,3}$  we can associate a 5-partition of  $P$ . There are three cases, according to whether  $\mu_2(P_2)$  is in  $Q_1$ ,  $Q_2$ , or  $Q_3$ . In each case, if  $\mu(P_2) \in Q_a$  then we write  $Q_a$  as a disjoint union  $Q_a = Q'_a \amalg \{\mu(P_2)\} \amalg Q''_a$  and to the partition  $Q_{1,2,3}$  we associate the 5-partition of  $P$  given by

$$Q'_1 \amalg P_2 \amalg Q''_1 \amalg Q_2 \amalg Q_3 \quad \mu_2(P) \in Q_1 \tag{A.8}$$

$$Q_1 \amalg Q'_2 \amalg P_2 \amalg Q''_2 \amalg Q_3 \quad \mu_2(P) \in Q_2 \tag{A.9}$$

$$Q_1 \amalg Q_2 \amalg Q'_3 \amalg P_2 \amalg Q''_3 \quad \mu_2(P) \in Q_3 \tag{A.10}$$

We now decompose the sum (A.7) into three terms corresponding to these three cases:

$$\begin{aligned}
0 &= \sum_{\mu_2(P_2) \in Q_1} \epsilon_P(P_{1,2,3}) \epsilon_{\widehat{P_{1,2,3}}}(Q_{1,2,3}) \mu(Q'_1, \mu(P_2), Q''_1, \mu(Q_2), Q_3) \\
&+ \sum_{\mu_2(P_2) \in Q_2} \epsilon_P(P_{1,2,3}) \epsilon_{\widehat{P_{1,2,3}}}(Q_{1,2,3}) \mu(Q_1, \mu(Q'_2, \mu(P_2), Q''_2), Q_3) \\
&+ \sum_{\mu_2(P_2) \in Q_3} \epsilon_P(P_{1,2,3}) \epsilon_{\widehat{P_{1,2,3}}}(Q_{1,2,3}) \mu(Q_1, \mu(Q_2), Q'_3, \mu(P_2), Q''_3)
\end{aligned} \tag{A.11}$$

The middle sum can be rearranged with an inner sum over partitions of the fixed set  $R = Q'_2 \amalg P_2 \amalg Q''_2$ . Since  $\epsilon_P(P_{1,2,3}) \epsilon_{\widehat{P_{1,2,3}}}(Q_{1,2,3}) = (-1)^{\deg_r(Q'_2)}$  the signs are such that the middle sum is zero by (A.4). The first and third lines of (A.11) will cancel each other by exchanging the roles of  $\mu(P_2)$  and  $\mu(Q_2)$ , but the cancellation requires that we use the reduced Koszul rule in formulating the equations. If we had used a standard Koszul rule, rather than the reduced Koszul rule, the terms would not have cancelled out, and the associativity constraints would have been over-constraining.

It is possible to seek for sign redefinitions of the  $\mu_n$  maps which give the axioms a more familiar form. For example, we could define maps  $\tilde{\mu}_n$  by

$$\left[ \prod_{m=1}^n \theta_m \right] \mu_n(a_1, \dots, a_n) = \tilde{\mu}_n(\theta_1 a_1, \dots, \theta_n a_n), \quad (\text{A.12})$$

where  $\theta_m$  are formal variables of degree 1 which are manipulated on the right hand side using a standard Koszul relation. Concretely,

$$\begin{aligned} \mu_1(a_1) &= -\tilde{\mu}_1(a_1) \\ \mu_2(a_1, a_2) &= (-1)^{\deg(a_1)} \tilde{\mu}_2(a_1, a_2) \\ \mu_3(a_1, a_2, a_3) &= -(-1)^{2\deg(a_1)+\deg(a_2)} \tilde{\mu}_3(a_1, a_2, a_3) \\ &\dots \end{aligned} \quad (\text{A.13})$$

Indeed, plugging this into the associativity relations A.6, we get conventional-looking graded associativity relations

$$\begin{aligned} \tilde{\mu}_1(\tilde{\mu}_1(a)) &= 0 \\ \tilde{\mu}_1(\tilde{\mu}_2(a_1, a_2)) - \tilde{\mu}_2(\tilde{\mu}_1(a_1), a_2) - (-1)^{\deg(a_1)} \tilde{\mu}_2(a_1, \tilde{\mu}_1(a_2)) &= 0 \\ \tilde{\mu}_1(\tilde{\mu}_3(a_1, a_2, a_3)) + \tilde{\mu}_2(\tilde{\mu}_2(a_1, a_2), a_3) - \tilde{\mu}_2(a_1, \tilde{\mu}_2(a_2, a_3)) + \tilde{\mu}_3(\tilde{\mu}_1(a_1), a_2, a_3) \\ + (-1)^{\deg(a_1)} \tilde{\mu}_3(a_1, \tilde{\mu}_1(a_2), a_3) + (-1)^{\deg(a_1)+\deg(a_2)} \tilde{\mu}_3(a_1, a_2, \tilde{\mu}_1(a_3)) &= 0 \\ &\dots \end{aligned} \quad (\text{A.14})$$

Although homotopical algebra has its origins in the homotopy theory of H-spaces, it has a highly abstract and algebraic character. It is useful to give the equations a geometrical interpretation. One such interpretation involves odd vector fields on non-commutative manifolds. Here we will emphasize the physical interpretation according to which  $A_\infty$  algebras encode general non-linear gauge symmetries. This interpretation arises naturally in the applications to string field theory (see, for examples, [27, 97, 96]) and constitutes a particularly interesting class of such odd vector fields. In order to make this relation explicit, we should identify the degree 1 elements  $a \in \mathcal{A}$  as “connections,” and define a “covariantized” differential  $\mathcal{A} \rightarrow \mathcal{A}$

$$\mu_a(\tilde{a}) = \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} \mu(a^{\otimes k}, \tilde{a}, a^{\otimes n}). \quad (\text{A.15})$$

The “field strength” for such an  $\mathcal{A}$ -connection is defined naturally as

$$\mathcal{F}_a = \sum_{n=1}^{\infty} \mu(a^{\otimes n}) \quad (\text{A.16})$$

and has degree two.

If we plug  $a^{\otimes n}$  into the  $A_\infty$  associativity relations and sum over  $n$  we find that the field strength is covariantly closed

$$\mu_a(\mathcal{F}_a) = 0. \quad (\text{A.17})$$

If we plug  $a^{\otimes k} \otimes \tilde{a} \otimes a^{\otimes n}$  and sum over  $k, n$ , we find that the square of the covariantized differential is proportional to the field strength:

$$\mu_a(\mu_a(\tilde{a})) + \mu_a(\mathcal{F}_a, \tilde{a}) + (-1)^{\deg_r(\tilde{a})} \mu_a(\tilde{a}, \mathcal{F}_a) = 0 \quad (\text{A.18})$$

where we defined the second deformed operation

$$\mu_a(\tilde{a}_1, \tilde{a}_2) = \sum_{n_0=0}^{\infty} \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \mu(a^{\otimes n_0}, \tilde{a}_1, a^{\otimes n_1}, \tilde{a}_2, a^{\otimes n_2}). \quad (\text{A.19})$$

The first term in A.18 collects all terms in the associativity relation A.4 such that  $\tilde{a} \in P_2$ , the second and third collect all terms where  $\tilde{a} \in P_3$  and  $\tilde{a} \in P_1$  respectively. In particular, the field strength transforms covariantly under the infinitesimal “gauge transformations”  $a \rightarrow a + \mu_a(\tilde{a})$  (where  $\tilde{a}$  has degree zero):

$$\mathcal{F}_a \rightarrow \mathcal{F}_a + \mu_a(\mu_a(\tilde{a})) = \mathcal{F}_a + \mu_a(\tilde{a}, \mathcal{F}_a) - \mu_a(\mathcal{F}_a, \tilde{a}). \quad (\text{A.20})$$

Thus “flat connections” coincide with solutions of the Maurer-Cartan equation for the  $A_\infty$  algebra:

$$\sum_{n=1}^{\infty} \mu(a^{\otimes n}) = 0. \quad (\text{A.21})$$

Given such a flat  $\mathcal{A}$ -connection  $a$ , we can actually define a full finite deformation of all the original  $A_\infty$  operations on  $\mathcal{A}$ :

$$\mu_a(\tilde{a}_1, \dots, \tilde{a}_k) = \sum_{n_0=0}^{\infty} \dots \sum_{n_k=0}^{\infty} \mu(a^{\otimes n_0}, \tilde{a}_1, a^{\otimes n_1}, \dots, a^{\otimes n_{k-1}}, \tilde{a}_k, a^{\otimes n_k}) \quad (\text{A.22})$$

The  $A_\infty$  associativity relation for  $\mu_a$  can be easily rewritten as a linear combination of associativity relations for  $\mu$ .<sup>120</sup> If  $\tilde{a}$  satisfies the MC equations for  $\mu_a$ , then  $a + \tilde{a}$  satisfies the MC equations for  $\mu$ .

It is straightforward to extend this analogy to “matrix-valued connections”. Given any differential graded module  $\mathcal{E}$ , we can extend the  $\mu$  operations to multilinear maps  $\mu^\mathcal{E}$  on  $\mathcal{A}^\mathcal{E} := \mathcal{E} \otimes \mathcal{A} \otimes \mathcal{E}^*$  by matrix multiplication, i.e. simply by contracting the  $\mathcal{E}^*$  and  $\mathcal{E}$  factors of consecutive arguments and acting on the  $\mathcal{A}$  factors with the vanilla  $\mu$  maps. The sign conventions needed to satisfy (A.6) are a little tricky, so we spell them out. The differential  $\mu_1$  on  $\mathcal{A}^\mathcal{E}$  is the natural one induced by that on the three factors. It is crucial to use the convention that the differentials on  $\mathcal{E}$  and  $\mathcal{E}^*$  are related by the sign rule  $\mu_{1,\mathcal{E}^*}(e_1^*) \cdot e_2 = (-1)^{\deg(e_1^*)} e_1^* \cdot \mu_{1,\mathcal{E}}(e_2)$  when checking (A.6). Thus, for example, if  $a \in \mathcal{A}$ ,  $e \in \mathcal{E}$  and  $e^* \in \mathcal{E}^*$  are three independent homogeneous elements then

$$\mu_1(eae^*) = \mu_{1,\mathcal{E}}(e)ae^* + (-1)^{\deg(e)} e\mu_{1,\mathcal{A}}(a)e^* + (-1)^{\deg(e)+\deg(a)} ea\mu_{1,\mathcal{E}^*}(e^*) \quad (\text{A.23})$$

(Notice we use the standard Koszul rule for computations with  $\mu_1$ .) For  $n > 1$  the multiplication is defined by

$$\mu_n((e_1 a_1 e_1^*) \cdots (e_n a_n e_n^*)) = (-1)^{\deg(e_1)} \sigma e_1 \mu_n(a_1, \dots, a_n) e_n^* \quad (\text{A.24})$$

---

<sup>120</sup>The proof of this is a special case of the discussion below (B.4) below. In  $a$  is not flat, we find instead the weaker version of the  $A_\infty$  relations, with non-zero source  $\mu_a(\emptyset) = \mathcal{F}_a$

where  $\sigma$  is the scalar:

$$\sigma = (e_1^* \cdot e_2) \cdots (e_{n-1}^* \cdot e_n) \quad (\text{A.25})$$

Again, the sign in (A.23) is crucial to checking (A.6).

A (flat) matrix-valued  $\mathcal{A}$  connection is defined simply as a (flat)  $\mathcal{A}^\mathcal{E}$ -connection.

### A.3 $A_\infty$ Morphisms

Given two  $A_\infty$  algebras  $\mathcal{A}$  and  $\mathcal{B}$ , with operations  $\mu_{\mathcal{A}}$  and  $\mu_{\mathcal{B}}$ , we can define an  $A_\infty$  morphism from  $\mathcal{A}$  to  $\mathcal{B}$  as a collection of multi-linear maps

$$\phi_n : \mathcal{A}^{\otimes n} \rightarrow \mathcal{B} \quad n \geq 1. \quad (\text{A.26})$$

The degree of  $\phi$  is  $1 - n$  and hence it has reduced degree  $\deg_r(\phi) = 0$ . It must satisfy the following relations:

$$\sum_{\text{Pa}_3(P)} \epsilon_P(P_{1,2,3}) \phi(P_1, \mu_{\mathcal{A}}(P_2), P_3) = \sum_{n \geq 1} \sum_{\text{Pa}_n(P)} \mu_{\mathcal{B}}(\phi(P_1), \dots, \phi(P_n)), \quad (\text{A.27})$$

for all ordered sets  $P$  of elements in  $\mathcal{A}$ , where we defined for convenience  $\phi(P_\alpha) := \phi|_{P_\alpha}(P_\alpha)$  and  $\phi(\emptyset) := 0$ . The identity morphism,  $\mathcal{B} = \mathcal{A}$ , with  $\phi_n = 0$  for  $n > 1$  and  $\phi_1(a) = a$ , is an  $A_\infty$ -morphism.

Given three  $A_\infty$  algebras  $\mathcal{A}$  and  $\mathcal{B}$  and  $\mathcal{C}$ , and  $A_\infty$  morphisms  $\phi$  from  $\mathcal{A}$  to  $\mathcal{B}$  and  $\phi'$  from  $\mathcal{B}$  to  $\mathcal{C}$ , we can define the composition of the two  $A_\infty$  morphisms by:

$$[\phi' \circ \phi](P) := \sum_{n \geq 1} \sum_{\text{Pa}_n(P)} \phi'(\phi(P_1), \dots, \phi(P_n)). \quad (\text{A.28})$$

Composition with the identity morphism is a left- and right- identity element for this composition. One can verify that this composition is associative as follows: The composition  $\phi'' \circ (\phi' \circ \phi)(P)$  involves a sum over nested partitions

$$P = [P_1^{(1)} \amalg \dots \amalg P_1^{(k_1)}] \amalg [P_2^{(1)} \amalg \dots \amalg P_2^{(k_2)}] \amalg \dots \amalg [P_n^{(1)} \amalg \dots \amalg P_n^{(k_n)}] \quad (\text{A.29})$$

where  $n, k_1, k_2, \dots \geq 1$ . On the other hand, the composition  $(\phi'' \circ \phi') \circ \phi(P)$  is a sum over all double partitions, where we first partition  $P$  into  $P_{1,\dots,n}$  and then consider partitions:

$$\{\phi(P_1), \dots, \phi(P_n)\} = Q_1 \amalg \dots \amalg Q_N. \quad (\text{A.30})$$

But there is a 1-1 correspondence between these two kinds of partitions. Similar manipulations confirm that in fact  $\phi' \circ \phi$  does define an  $A_\infty$ -morphism from  $\mathcal{A}$  to  $\mathcal{C}$ . It is simplest to start with the right-hand-side of the identity, identify it as a sum over nested partitions and rearrange that sum as a sum over partitions of the form (A.30):

$$\sum_{n \geq 1} \sum_{\text{Pa}_n(P)} \sum_{N \geq 1} \sum_{\text{Pa}_N(\{\phi(P_1), \dots, \phi(P_n)\})} \mu_{\mathcal{C}}(\phi'(Q_1), \dots, \phi'(Q_N)) \quad (\text{A.31})$$

Next use the fact that  $\phi'$  is an  $A_\infty$ -morphism. The resulting sum can be rearranged to give the left-hand-side of the desired identity. Thus, the set of  $A_\infty$ -morphisms forms a standard category.

A simple observation is that an  $A_\infty$  morphism  $\phi$  from  $\mathcal{A}$  to  $\mathcal{B}$  can be used to map flat  $\mathcal{A}$ -connections to flat  $\mathcal{B}$ -connections: if  $a$  satisfies the MC equations for  $\mathcal{A}$ , then the degree one element

$$b := \phi_a(\emptyset) := \sum_{n=1}^{\infty} \phi(a^{\otimes n}) \quad (\text{A.32})$$

satisfies the MC equations for  $\mathcal{B}$ . This can be shown simply by plugging  $a^{\otimes n}$  in A.27, recalling that  $\deg_r(\phi) = 0$ , and summing over  $n$ . Similarly, the morphism maps gauge transformations to gauge transformations: If we define

$$\tilde{b} := \phi_a(\tilde{a}) = \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} \phi(a^{\otimes k}, \tilde{a}, a^{\otimes n}) \quad (\text{A.33})$$

then the gauge transformation  $a \rightarrow a + \mu_a(\tilde{a})$  induces the transformation

$$\phi_a(\emptyset) \rightarrow \phi_a(\emptyset) + \phi_a(\mu_a(\tilde{a})) = \phi_a(\emptyset) + \mu_{\phi_a(\emptyset)}(\phi_a(\tilde{a})) = b + \mu_{\mathcal{B}, b}(\tilde{b}). \quad (\text{A.34})$$

Similarly multilinear maps  $\phi_a : \mathcal{A}^{\otimes n} \rightarrow \mathcal{B}$  give an  $A_\infty$  morphism from the finite deformation of  $\mathcal{A}$  by  $a$  to the finite deformation of  $\mathcal{B}$  by  $b = \phi_a(\emptyset)$ .

A more surprising observation is that the equations A.27 which define  $A_\infty$  morphisms from  $\mathcal{A}$  to  $\mathcal{B}$  can be reinterpreted as the MC equations for an  $A_\infty$  algebra  $\text{Hop}(\mathcal{A}, \mathcal{B})$ . A degree  $k$  element of the algebra  $\text{Hop}(\mathcal{A}, \mathcal{B})$  is a collection  $\alpha$  of multi-linear maps  $\alpha_n : \mathcal{A}^{\otimes n} \rightarrow \mathcal{B}$  of degree  $k - n$  (i.e. reduced degree  $k - 1$ ). The  $A_\infty$  operations are

$$\begin{aligned} [\mu_{\text{Hop}(\mathcal{A}, \mathcal{B})}(\alpha)](P) &= \mu_{\mathcal{B}}(\alpha(P)) + (-1)^{\deg(\alpha)} \sum_{\text{Pa}_3(P)} \epsilon_P \alpha(P_1, \mu_{\mathcal{A}}(P_2), P_3) \\ [\mu_{\text{Hop}(\mathcal{A}, \mathcal{B})}(\alpha_1, \dots, \alpha_n)](P) &= \sum_{\text{Pa}_n(P)} \mu_{\mathcal{B}}(\alpha_1(P_1), \dots, \alpha_n(P_n)) \end{aligned} \quad (\text{A.35})$$

The  $A_\infty$  associativity relations for  $\mu_{\text{Hop}(\mathcal{A}, \mathcal{B})}$  can be reduced to the relations for  $\mathcal{A}$  and  $\mathcal{B}$  in a straightforward, if tedious, way. The three groups of terms involving two  $\mu_{\mathcal{B}}$ , a  $\mu_{\mathcal{B}}$  and a  $\mu_{\mathcal{A}}$  or two  $\mu_{\mathcal{A}}$  respectively cancel out separately. Thus  $A_\infty$  morphisms from  $\mathcal{A}$  to  $\mathcal{B}$  coincide with flat  $\text{Hop}(\mathcal{A}, \mathcal{B})$ -connections. In particular they inherit the  $A_\infty$  category structures discussed in appendix B.

#### A.4 $A_\infty$ Modules

A (left)  $A_\infty$ -module for an  $A_\infty$ -algebra  $\mathcal{A}$  is a module  $\mathcal{M}$  equipped with a collection of multi-linear maps

$$\nu_n : \mathcal{A}^{\otimes n} \otimes \mathcal{M} \rightarrow \mathcal{M} \quad n \geq 0 \quad (\text{A.36})$$

of degree  $1 - n$ . These maps must satisfy the following relations:

$$\sum_{P_{1,2,3} \in \text{Pa}_3(P)} \epsilon_P(P_{1,2,3}) \nu(P_1, \mu_{\mathcal{A}}(P_2), P_3; m) + \sum_{P_{1,2} \in \text{Pa}_2(P)} \epsilon_P(P_{1,2}) \nu(P_1; \nu(P_2; m)) = 0, \quad (\text{A.37})$$

for all ordered sets  $P$  of elements in  $\mathcal{A}$  and any element  $m$  in  $\mathcal{M}$ . Here we defined for convenience  $\nu(P_\alpha) := \nu|_{P_\alpha}(P_\alpha)$ . We denote the map for  $n = 0$  in (A.36) by  $\nu(m)$  and define

$\nu(\emptyset; m) := \nu(m)$ . The map  $\nu : \mathcal{M} \rightarrow \mathcal{M}$  provides a differential on  $\mathcal{M}$ . The sign  $\epsilon_P(P_{1,2})$  is obtained by starting with the order  $\nu\nu P$  and rearranging using the reduced Koszul rule, where  $\deg_r(\nu_n) = 1$ . As a check on the sign in the second term note that an  $A_\infty$ -algebra should be a left- $A_\infty$ -module over itself with  $\nu_n = \mu_{n+1}$ . With our sign convention this is indeed the case: Apply the relations (A.4) to partitions whose last element is  $m$ . The second term in (A.37) corresponds to the partitions with  $P_3 = \emptyset$ .

Following the analogy between  $A_\infty$  algebras and gauge connections, we can pick a “connection”  $a \in \mathcal{A}$  and define the “covariantized” differential  $\nu_a$  on  $\mathcal{M}$

$$\nu_a(m) := \sum_{n=0}^{\infty} \nu(a^{\otimes n}; m) \quad (\text{A.38})$$

and “gauge transformations” of parameter  $\tilde{a}$ :

$$m \rightarrow m + \nu_a(\tilde{a}; m) \quad (\text{A.39})$$

with

$$\nu_a(\tilde{a}; m) := \sum_{n=0}^{\infty} \sum_{n'=0}^{\infty} \nu(a^{\otimes n}, \tilde{a}, a^{\otimes n'}; m). \quad (\text{A.40})$$

The defining relations for the  $A_\infty$  morphism insure that the differential  $\nu_a$  transforms covariantly under gauge transformations of  $a$  and  $m$  with parameter  $\tilde{a}$ .

$$\nu_a(m) \rightarrow \nu_a(m) + \nu_a(\nu_a(\tilde{a}; m)) + \nu_a(\mu_a(\tilde{a}); m) = \nu_a(m) + \nu_a(\tilde{a}; \nu_a(m)) \quad (\text{A.41})$$

To check the signs here recall that the gauge field  $a$  has reduced degree zero and the gauge parameter  $\tilde{a}$  has reduced degree  $-1$ . The defining equations (A.37) also insure that a flat  $\mathcal{A}$ -connection gives a nilpotent differential  $\nu_a$ . Similarly defined multilinear maps  $\nu_a : \mathcal{A}^{\otimes n} \otimes \mathcal{M} \rightarrow \mathcal{M}$  give  $\mathcal{M}$  the structure of a left  $A_\infty$  module for the finite deformation of  $\mathcal{A}$  by  $a$ .

There are natural relations between  $A_\infty$  morphisms and  $A_\infty$  modules. As noted above,  $\mathcal{A}$  is an  $A_\infty$ -module over itself. Moreover, modules pull back: That is, if  $\nu_{\mathcal{B},n} : \mathcal{B}^{\otimes n} \otimes \mathcal{M} \rightarrow \mathcal{M}$  defines the structure of an  $A_\infty$ -module on  $\mathcal{M}$  for an  $A_\infty$ -algebra  $\mathcal{B}$ , and if  $\phi$  is an  $A_\infty$ -morphism from  $\mathcal{A}$  to  $\mathcal{B}$  then  $\nu_{\mathcal{A},n} : \mathcal{A}^{\otimes n} \otimes \mathcal{M} \rightarrow \mathcal{M}$  defined by

$$\nu_{\mathcal{A},p}(P; m) := \sum_{n \geq 1} \sum_{\text{Pa}_n(P)} \nu_{\mathcal{B}}(\phi(P_1), \dots, \phi(P_n); m) \quad (\text{A.42})$$

where  $p = |P|$ , defines the structure of an  $A_\infty$ - $\mathcal{A}$ -module on  $\mathcal{M}$ . To prove this assertion it is easiest to begin with the  $\nu_{\mathcal{A}}\nu_{\mathcal{A}}Pm$  term, and use the definition to write it in the form  $\nu_{\mathcal{B}}\phi; \nu_{\mathcal{B}}\phi m$ . The sum can be rearranged as a sum over 2-partitions of sets of the form  $\phi(P_1), \dots, \phi(P_n)$ . One then applies the module axiom for  $\nu_{\mathcal{B}}$  and rearranges the sum to be of the form of the first term in the  $\nu_{\mathcal{A}}$ -module axiom. As a corollary, if  $\phi$  is an  $A_\infty$ -morphism from  $\mathcal{A}$  to  $\mathcal{B}$ , then  $\mathcal{B}$  is canonically an  $\mathcal{A}$ -module.

A second relation between  $A_\infty$  morphisms and modules is the following. An  $A_\infty$  module  $\mathcal{M}$  for  $\mathcal{A}$  can be reinterpreted as an  $A_\infty$ -morphism into a very simple  $A_\infty$ -algebra  $\mathcal{B}$ . As a module  $\mathcal{B} = \mathcal{M} \otimes \mathcal{M}^* \cong \text{End}(\mathcal{M})$ . There are only two nontrivial operations on  $\mathcal{B}$ , first,

$$\mu_{\mathcal{B},1}(b) = -(\nu b + (-1)^{\deg_r(b)} b \nu) = -[\nu, b] \quad (\text{A.43})$$

where  $\nu$  is the differential on  $\mathcal{M}$  and second,

$$\mu_{\mathcal{B},2}(b_1, b_2) = (-1)^{\deg_r(b_1)} b_1 \circ b_2 \quad (\text{A.44})$$

where on the right hand side we have ordinary composition of endomorphisms. Again we see that, given an  $A_\infty$  morphism from  $\mathcal{A}$  to  $\mathcal{B}$ , and a  $\mathcal{B}$ -module  $\mathcal{M}$  we get a  $\mathcal{A}$ -module. This is simply a composition of  $A_\infty$  morphisms.

A right  $A_\infty$  module  $\mathcal{M}$  is a vector space equipped with a collection of multilinear maps  $\nu_n : \mathcal{M} \otimes \mathcal{A}^{\otimes n} \rightarrow \mathcal{M}$  of degree  $1 - n$ , which satisfies the following relations:

$$\sum_{P_{1,2,3} \in \text{Pa}_3(P)} \tilde{\epsilon}_P(P_{1,2,3}) \nu(m; P_1, \mu_{\mathcal{A}}(P_2), P_3) + \sum_{\text{Pa}_2(P)} \nu(\nu(m; P_1); P_2) = 0. \quad (\text{A.45})$$

The symbols and reduced degree are defined as before. The sign  $\tilde{\epsilon}_P(P_{1,2,3})$  is computed by starting with the order  $\nu m P \mu$  and then distributing the factors with the reduced Koszul sign rule.

Finally an  $A_\infty$  bimodule  $\mathcal{M}$  is a vector space equipped with a collection of multi-linear maps  $\nu_{m,n} : \mathcal{A}^{\otimes m} \otimes \mathcal{M} \otimes \mathcal{B}^{\otimes n} \rightarrow \mathcal{M}$  of degree  $1 - n - m$ , which satisfies the following relations:

$$\begin{aligned} & \sum_{\text{Pa}_3(P)} \epsilon \nu(P_1, \mu_{\mathcal{A}}(P_2), P_3; m; P') + \sum_{\text{Pa}_3(P')} \epsilon' \nu(P; m; P'_1, \mu_{\mathcal{B}}(P'_2), P'_3) \\ & + \sum_{\text{Pa}_2(P)} \sum_{\text{Pa}_2(P')} \epsilon'' \nu(P_1; \nu(P_2; m; P'_2); P_2) = 0. \end{aligned} \quad (\text{A.46})$$

The symbols and reduced degree are defined as before. The signs are given by combining those of the previous two cases.

Notice that an  $A_\infty$  bimodule maps an  $\mathcal{A}$ -flat connection  $a$  to a right  $\mathcal{B}$  module

$$\nu_a(m; P) := \sum_{n=0}^{\infty} \nu(a^{\otimes n}; m; P). \quad (\text{A.47})$$

### A.5 $L_\infty$ Algebras, Morphisms, And Modules

Roughly speaking, an  $L_\infty$  algebra is a graded-commutative version of an  $A_\infty$  algebra. In the context of the present paper, and in other physical contexts in which  $A_\infty$  and  $L_\infty$  algebras occur, the former are associated to correlation functions of operators located on the boundary of a two-dimensional region, the latter to correlation functions of operators located in the interior. As a consequence, the natural degree assignment for the operations of  $A_\infty$  and  $L_\infty$  algebras in a physical context differ: the former are maps of degree  $2 - n$  acting on  $n$  arguments, while the latter are maps of degree  $3 - 2n$ . Thus in standard mathematical notation our  $L_\infty$  algebras would be denoted as  $L_\infty[-1]$  algebras. Correspondingly, the reduced degree of elements in  $L_\infty$  algebras will coincide with the degree minus 2.

We are ready for our definition. Consider a module  $\mathcal{L}$ . An  $L_\infty$  structure  $\lambda$  on  $\mathcal{L}$  is defined by a collection of multi-linear, graded-commutative maps  $\lambda_n : \mathcal{L}^{\otimes n} \rightarrow \mathcal{L}$  of degree  $3 - 2n$ , which satisfies the following  $L_\infty$  associativity relation:

$$\sum_{S_{1,2} \in \text{Sh}_2(S)} \epsilon_S(S_{1,2}) \lambda(\lambda(S_1), S_2) = 0 \quad (\text{A.48})$$

for all sets  $S$  of elements in  $\mathcal{L}$ , where we defined for convenience  $\lambda(S_\alpha) := \lambda_{|S_\alpha|}(S_\alpha)$  and  $\lambda(\emptyset) := 0$ .<sup>121</sup> The sign  $\epsilon_S(S_{1,2})$  is given again by the reduced Koszul rule, with the symbol  $\lambda$  having degree 1. Of course, in this case this is the same as the standard Koszul rule. The pair  $(\mathcal{L}, \lambda)$  is called an  $L_\infty$  algebra.

The counterpart to the equations (A.6) are

$$\begin{aligned} \lambda_1(\lambda_1(s)) &= 0 \\ \lambda_1(\lambda_2(s_1, s_2)) + \lambda_2(\lambda_1(s_1), s_2) + (-1)^{s_1 s_2} \lambda_2(\lambda_1(s_2), s_1) &= 0 \\ \lambda_3(\lambda_1(s_1), s_2, s_3) + (-1)^{s_1 s_2} \lambda_3(\lambda_1(s_2), s_1, s_3) + (-1)^{s_3(s_1+s_2)} \lambda_3(\lambda_1(s_3), s_1, s_2) + \\ \lambda_2(\lambda_2(s_1, s_2), s_3) + (-1)^{s_2 s_3} \lambda_2(\lambda_2(s_1, s_3), s_2) + (-1)^{s_1(s_2+s_3)} \lambda_2(\lambda_2(s_2, s_3), s_1) + \\ &\quad + \lambda_1(\lambda_3(s_1, s_2, s_3)) = 0 \\ &\quad \dots \end{aligned} \tag{A.49}$$

The third equation is a version of the Jacobi identity, up to homotopy.

The natural MC equation associated to an  $L_\infty$  algebra can be written compactly as

$$\lambda(e^\beta) := \sum_{n=0}^{\infty} \frac{1}{n!} \lambda_n(\beta^{\otimes n}) = 0 \tag{A.50}$$

where  $\beta$  has  $\deg(\beta) = 2$ . In analogy to the main text of the paper, we can denote a solution of such MC equation as an ‘‘interior amplitude’’. We could pursue an analogy between interior amplitudes and flat 2-form connections, but we will not do so. Again, an interior amplitude gives a finite deformation of an  $L_\infty$  algebra  $\mathcal{L}$ , with operations

$$\lambda_\beta(S) = \lambda(e^\beta, S). \tag{A.51}$$

See equation (4.17) for a detailed proof.

We can define  $L_\infty$  morphisms from an  $L_\infty$  algebra  $\mathcal{L}$  to an  $L_\infty$  algebra  $\tilde{\mathcal{L}}$  in an obvious way, as collection of maps  $\varphi_n : \mathcal{L}^{\otimes n} \rightarrow \tilde{\mathcal{L}}$  of degree  $2 - 2n$  (hence  $\deg_r(\varphi) = 0$ ) which satisfy

$$\sum_{S_{1,2} \in \text{Sh}_2(S)} \epsilon_S(S_{1,2}) \varphi(\lambda_{\mathcal{L}}(S_1), S_2) = \sum_{n>0} \frac{1}{n!} \sum_{S_{1,\dots,n} \in \text{Sh}_n(S)} \epsilon_S(S_{1,\dots,n}) \lambda_{\tilde{\mathcal{L}}}(\varphi(S_1), \dots, \varphi(S_n)) \tag{A.52}$$

where the signs are given by the Koszul sign required for rearranging the elements of  $S$ . Again  $\varphi(\emptyset) = 0$ . Such morphisms map any interior amplitude  $\beta$  to an interior amplitude  $\varphi(e^\beta)$ . Two  $L_\infty$  morphisms can also be composed

$$[\varphi \circ \tilde{\varphi}](S) = \sum_{n>0} \frac{1}{n!} \sum_{\text{Sh}_n(S)} \epsilon_S(S_{1,\dots,n}) \varphi(\tilde{\varphi}(S_1), \dots, \tilde{\varphi}(S_n)) \tag{A.53}$$

and composition is associative. Once again the identity morphism is a left- and right-identity for this composition.

---

<sup>121</sup>It is straightforward, if needed, to relax the axioms for  $L_\infty$  structures by allowing a choice of ‘‘source’’  $\lambda(\emptyset) \neq 0$ .

We can also define an  $L_\infty$  module  $\mathcal{M}$ . It is a module  $\mathcal{M}$  together with a collection of maps  $\nu_n : \mathcal{L}^{\otimes n} \otimes \mathcal{M} \rightarrow \mathcal{M}$  of degree  $1 - 2n$  with

$$\sum_{S_{1,2} \in \text{Sh}_2(S)} \epsilon_S(S_{1,2}) \nu(\lambda(S_1), S_2; m) + \sum_{S_{1,2} \in \text{Sh}_2(S)} \epsilon'_S(S_{1,2}) \nu(S_1; \nu(S_2; m)) = 0 \quad (\text{A.54})$$

The sign rule  $\epsilon'_S(S_{1,2})$  can be deduced by requiring that an  $L_\infty$  algebra  $\mathcal{L}$  be a left-module over itself. Then equation (A.54) is the  $L_\infty$ -relation for a set  $S$  whose last element is  $m$ . The first term in (A.54) corresponds to the term in (A.48) where  $m$  is not an element of  $S_1$  and the second term in (A.54) corresponds to the term in (A.48) where  $m$  is an element of  $S_1$ .

## A.6 $LA_\infty$ Algebras, Morphisms, And Modules

In this paper we encounter a neat structure which allows an  $L_\infty$  algebra to control deformations of an  $A_\infty$  algebra. This structure, which we dub an  $LA_\infty$  algebra, is defined by an  $L_\infty$  algebra  $\mathcal{L}$ , together with a module  $\mathcal{A}$  equipped with multilinear operations  $\mu_{k,n} : \mathcal{L}^{\otimes k} \otimes \mathcal{A}^{\otimes n} \rightarrow \mathcal{A}$ ,  $k \geq 0$ ,  $n > 0$ , graded symmetric in the first set of arguments, and of degree  $2 - n - 2k$ . We abbreviate the operation on monomials as  $\mu(S; P)$ . The expression  $\mu(S; P)$  is zero if  $P = \emptyset$ . The operations satisfy the relations

$$\sum_{\text{Sh}_2(S), \text{Pa}_3(P)} \epsilon \mu(S_1; P_1, \mu(S_2; P_2), P_3) + \sum_{\text{Sh}_2(S)} \epsilon_S(S_{1,2}) \mu(\lambda(S_1), S_2; P) = 0. \quad (\text{A.55})$$

Here  $\epsilon$  is the reduced Koszul sign computed from the order  $\mu\mu SP$  and the sign in the second term comes from  $\mu\lambda SP$ . Therefore

$$\epsilon = \epsilon_S(S_{1,2})(-1)^{d_r(S_1) + d_r(P_1) + d_r(P_1)d_r(S_2)}. \quad (\text{A.56})$$

The maps  $\mu_{0,n}$  endow  $\mathcal{A}$  with an  $A_\infty$  algebra structure. The maps  $\mu_{k,1}$  alone give multilinear maps  $\mathcal{L}^{\otimes k} \otimes \mathcal{A} \rightarrow \mathcal{A}$  which endow  $\mathcal{A}$  with the structure of an  $L_\infty$  module. Furthermore, given any interior amplitude  $\beta$  for  $\mathcal{L}$ ,

$$\mu_\beta(P) = \mu(e^\beta; P) \quad (\text{A.57})$$

endow  $\mathcal{A}$  with an  $A_\infty$  algebra structure controlled by  $\beta$ .

The relations involving a non-empty set  $S$  can be expressed as the statement of an  $L_\infty$  morphism from  $\mathcal{L}$  to the *Hochschild complex* of  $\mathcal{A}$ , denoted  $CC^*(\mathcal{A})$ . This complex can be thought of as an  $L_\infty$  algebra whose interior amplitudes are deformations of the  $\mathcal{A}$  operations.

We define a degree  $k$  element in  $CC^k(\mathcal{A})$  as a collection of multilinear maps  $\delta_n : \mathcal{A}^{\otimes n} \rightarrow \mathcal{A}$  of degree  $k - n$ . The Hochschild complex is equipped with linear and quadratic operations, and all higher operations can be taken to be 0 when giving it the structure of an  $L_\infty$  algebra. The first operation on the complex is

$$\lambda_{\tilde{\mathcal{L}}}(\delta)(P) = \sum_{\text{Pa}_3(P)} \epsilon_1 \delta(P_1, \mu(P_2), P_3) - \sum_{\text{Pa}_3(P)} \epsilon_2 \mu(P_1, \delta(P_2), P_3) \quad (\text{A.58})$$

where both  $\epsilon_1$  and  $\epsilon_2$  are determined by using the reduced Koszul rule starting from the order  $\mu\delta P$ , so

$$\epsilon_1 = (-1)^{d_r(\delta)+d_r(P_1)} \quad \epsilon_2 = (-1)^{d_r(\delta)d_r(P_1)} \quad (\text{A.59})$$

The second operation on the Hochschild complex is the graded-symmetric operation:

$$\lambda_{\mathcal{L},2}(\delta_1, \delta_2)(P) = \sum_{\text{Pa}_3(P)} \epsilon_3 \delta_1(P_1, \delta_2(P_2), P_3) + \epsilon_4 \delta_2(P_1, \delta_1(P_2), P_3) \quad (\text{A.60})$$

where

$$\epsilon_3 = (-1)^{d_r(\delta_1)} (-1)^{d_r(P_1)d_r(\delta_2)} \quad (\text{A.61})$$

and

$$\epsilon_4 = (-1)^{d_r(\delta_2)} (-1)^{d_r(\delta_1)d_r(\delta_2)+d_r(P_1)d_r(\delta_1)} \quad (\text{A.62})$$

These are almost but not quite what we would get from the reduced Koszul rule starting from the ordering  $\delta_1\delta_2P$ . (The factor  $(-1)^{d_r(\delta_1)}$  in  $\epsilon_3$  violates the rule, but is needed for the  $L_\infty$  relations. We can restore the ordering rule by shifting  $d_r(P) \rightarrow d_r(P) + 1$ .)

Now, the  $L_\infty$  morphism  $\varphi : \mathcal{L} \rightarrow CC^*(\mathcal{A})$  takes a monomial  $S$  to  $\varphi(S)$  where  $\varphi(S)$  is the Hochschild cochain taking  $P \in \mathcal{A}^{\otimes n}$  to

$$\varphi(S)(P) := \mu(S; P) \quad (\text{A.63})$$

Note that

$$\deg \mu(S; P) = 2 - p - 2s + \deg(P) + \deg(S) = (2 + \deg_r(S)) + \deg_r(P) \quad (\text{A.64})$$

so the Hochschild cochain  $\delta = \varphi(S)$  has degree  $k = 2 + \deg_r(S)$  as we expect if  $\varphi$  is to be an  $L_\infty$  morphism.

Now we interpret the equation (A.55) as the condition for  $\varphi$  to be an  $L_\infty$  morphism. The second term of (A.55) is identified with the left hand side of equation A.52. The first term of equation (A.55) can be identified with (minus) the right hand side of equation A.52. It can be decomposed into terms with  $S_1$  and  $S_2$  both not empty, which give terms quadratic in  $\varphi$ , and terms with  $S_1$  or  $S_2$  empty, which give terms linear in  $\varphi$ . A special case of the relation of  $LA_\infty$  algebras and the Hochschild complex is worked out in detail in Section §3.2.4 above.

Given an  $LA_\infty$  structure  $\mu$  on  $\mathcal{L}$  and  $\mathcal{A}$ , and an  $L_\infty$  morphism  $\varphi$  from  $\tilde{\mathcal{L}}$  to  $\mathcal{L}$ , one gets an  $LA_\infty$  structure  $\mu \circ \varphi$  by composing  $\mu$  and  $\varphi$ , interpreted as an  $L_\infty$  morphism. Concretely,

$$[\mu \circ \varphi](S; P) = \sum_{n>0} \frac{1}{n!} \sum_{\text{Sh}_n(S)} \epsilon_S(S_1, \dots, S_n) \mu(\varphi(S_1), \dots, \varphi(S_n); P) \quad (\text{A.65})$$

An  $LA_\infty$  morphism between  $LA_\infty$  algebras  $(\mathcal{L}_1, \mathcal{A}_1)$  and  $(\mathcal{L}_2, \mathcal{A}_2)$  can be defined as a collection of maps

$$\begin{aligned} \phi_{k,n} : \mathcal{L}_1^{\otimes k} \otimes \mathcal{A}_1^{\otimes n} &\rightarrow \mathcal{A}_2 & n > 0, k \geq 0 \\ \phi_{k,0} : \mathcal{L}_1^{\otimes k} &\rightarrow \mathcal{L}_2 & k > 0 \end{aligned} \quad (\text{A.66})$$

The degree of  $\phi_{k,n}$  is  $1 - n - 2k$  for  $n > 0, k \geq 0$  and  $2 - 2k$  for  $n = 0$ . Moreover the  $\phi_{k,n}$  must satisfy:

$$\begin{aligned} & \sum_{\text{Pa}_3(P), \text{Sh}_2(S)} \epsilon \phi(S_1; P_1, \mu_{\mathcal{L}_1, \mathcal{A}_1}(S_2; P_2), P_3) + \sum_{\text{Sh}_2(S)} \epsilon' \phi(S_1; \mu_{\mathcal{L}_1}(S_2; P)) \\ &= \sum_{n,m \geq 0} \sum_{\text{Pa}_n(P)} \sum_{\text{Sh}_{n+m}(S)} \epsilon'' \mu_{\mathcal{L}_2, \mathcal{A}_2}(\phi(S_1), \dots, \phi(S_m); \phi(S_{m+1}; P_1), \dots, \phi(S_{m+n}; P_n)). \end{aligned} \quad (\text{A.67})$$

Where (we didn't check) the signs should be determined from the reduced Koszul rule starting with canonical orderings. This equation can be understood as a morphism between two  $L_\infty$  morphisms  $\varphi_i : \mathcal{L}_i \rightarrow CC^*(\mathcal{A}_i)$ .

A left  $LA_\infty$  module can be defined as a collection of multi-linear maps

$$\nu_{k,n} : \mathcal{L}^{\otimes k} \otimes \mathcal{A}^{\otimes n} \otimes \mathcal{M} \rightarrow \mathcal{M} \quad n, k \geq 0 \quad (\text{A.68})$$

We write  $\nu(S; P; m)$  on monomials. The structure of an  $LA_\infty$  module allows one to associate to any choice of interior amplitude  $\beta$  an  $A_\infty$  module for the corresponding  $A_\infty$  structure on  $\mathcal{A}$ . The defining relations can be written as:

$$\begin{aligned} 0 = & \sum_{\text{Sh}_2(S), \text{Pa}_3(P)} \epsilon \nu(S_1; P_1, \mu(S_2; P_2), P_3; m) \\ & + \sum_{\text{Sh}_2(S)} \epsilon' \nu(\lambda(S_1), S_2; P; m) + \sum_{\text{Sh}_2(S), \text{Pa}_2(P)} \epsilon'' \nu(S_1; P_1; \nu(S_2; P_2); m). \end{aligned} \quad (\text{A.69})$$

The signs  $\epsilon, \epsilon', \epsilon''$  are determined by using the reduced Koszul rule starting with the orders  $\nu\mu S P m$ ,  $\nu\lambda S P m$  and  $\nu\nu S P m$ , respectively.

Clearly, the  $\nu_{0,m}$  maps give  $\mathcal{M}$  the structure of an  $A_\infty$  module. It would be interesting to interpret the remaining equations as some morphism from  $\mathcal{L}$  to some kind of Hochschild complex for the  $A_\infty$  module  $\mathcal{M}$ , which encodes the possible deformations of the  $A_\infty$  module structure which possibly accompany deformations of the  $A_\infty$  algebra.

Right  $LA_\infty$  modules and  $LA_\infty$  bimodules can be defined in a similar manner.

## B. $A_\infty$ Categories And Mutations

### B.1 $A_\infty$ Categories And Exceptional Categories

An  $A_\infty$  category  $\mathfrak{A}$  consists of the following data:

- A set of objects, which we denote as  $\mathbf{Ob}(\mathfrak{A})$
- For each pair of objects  $A, B \in \mathbf{Ob}(\mathfrak{A})$  a module <sup>122</sup>  $\text{Hom}(A, B) := \text{Hop}(B, A)$

---

<sup>122</sup>See the preface to Appendix A for our use of the word “module.”

- Multilinear composition maps

$$\mu_n : \text{Hop}(A_0, A_1) \otimes \cdots \otimes \text{Hop}(A_{n-1}, A_n) \rightarrow \text{Hop}(A_0, A_n) \quad (\text{B.1})$$

of degree  $2 - n$  which satisfy the  $A_\infty$  associativity axiom.

Note that an  $A_\infty$  algebra  $\mathcal{A}$  is a special case of an  $A_\infty$  category, with a single object  $A$  and  $\mathcal{A} = \text{Hop}(A, A)$ . It is straightforward to refine the definitions we gave for  $A_\infty$  morphisms, modules, etc. to corresponding categorical notions such as  $A_\infty$  functors, etc. The defining relations take an identical form, but the arguments live in compatible sequences of Hop spaces.

As a first example, an  $A_\infty$  functor from  $\mathfrak{A}$  to  $\mathfrak{B}$  is given by a map  $\mathcal{F}$  from objects of  $\mathfrak{A}$  to objects of  $\mathfrak{B}$  together with a collection of multilinear maps

$$\mathcal{F}_n : \text{Hop}(A_0, A_1) \otimes \cdots \otimes \text{Hop}(A_{n-1}, A_n) \rightarrow \text{Hop}(\mathcal{F}(A_0), \mathcal{F}(A_n)) \quad (\text{B.2})$$

of degree  $1 - n$  satisfying relations formally identical to those for morphisms, equation (A.27). As a second example, one can define a categorical analogue to an  $A_\infty$  module by a map  $\mathfrak{M}$  from objects  $A_i$  of  $\mathfrak{A}$  to modules  $\mathfrak{M}(A_i)$  and collections of multilinear maps

$$\nu_n : \text{Hop}(A_0, A_1) \otimes \cdots \otimes \text{Hop}(A_{n-1}, A_n) \otimes \mathfrak{M}(A_n) \rightarrow \mathfrak{M}(A_0) \quad (\text{B.3})$$

of degree  $1 - n$  and satisfying the analogue of equation (A.37). And so forth.

A somewhat surprising observation is that given an  $A_\infty$  algebra  $\mathcal{A}$ , the set of  $\mathcal{A}$ -flat connections forms an  $A_\infty$  category. The spaces  $\text{Hop}(a, a')$  from a Maurer-Cartan element  $a'$  to a Maurer-Cartan element  $a$  all coincide with  $\mathcal{A}$  as a module, and the multilinear operations take the form

$$M(\tilde{a}_1, \dots, \tilde{a}_n) = \sum_{k_i \geq 0} \mu(a_0^{\otimes k_0}, \tilde{a}_1, a_1^{\otimes k_1}, \dots, \tilde{a}_n, a_n^{\otimes k_n}). \quad (\text{B.4})$$

This fact was used in defining the category of Branes using the multiplications in equation (5.17) above. Here we give a simple proof that (B.4) satisfy the  $A_\infty$ -relations. Given  $P = \{\tilde{a}_1, \dots, \tilde{a}_n\}$  we wish to show that

$$\sum_{P_{1,2,3} \in \text{Pa}_3(P)} \epsilon_P(P_{1,2,3}) M(P_1, M(P_2), P_3) = 0. \quad (\text{B.5})$$

For each vector  $\vec{k} = (k_0, \dots, k_n) \in \mathbb{Z}_+^{n+1}$  define the ordered set:

$$P(\vec{k}) = \{a_0^{k_0}, \tilde{a}_1, a_1^{k_1}, \tilde{a}_2, \dots, \tilde{a}_n, a_n^{k_n}\}. \quad (\text{B.6})$$

We then apply the  $A_\infty$ -relations for  $\mu_{\mathcal{A}}$  to the set  $P(\vec{k})$ . Then we sum these over all  $\vec{k}$ . Now we reorganize the terms in the disjoint union

$$\prod_{\vec{k} \in \mathbb{Z}_+^{n+1}} \text{Pa}_3(P(\vec{k})) \quad (\text{B.7})$$

according to

$$\Pi_{P_{1,2,3} \in \text{Pa}_3(P)} \left\{ \Pi_{\vec{k}_1 \in \mathbb{Z}_+^{p_1+1}} \Pi_{\vec{k}_2 \in \mathbb{Z}_+^{p_2+1}} \Pi_{\vec{k}_3 \in \mathbb{Z}_+^{p_3+1}} \{P_1(\vec{k}_1)\} \amalg \{P_2(\vec{k}_2)\} \amalg \{P_3(\vec{k}_3)\} \right\} \quad (\text{B.8})$$

This gives precisely the expression

$$\sum_{\text{Pa}_3(P)} \epsilon_P(P_{1,2,3}) M_{p_1+p_3+1}(P_1, M_{p_2}(P_2), P_3) = 0. \quad (\text{B.9})$$

where we include  $P_2 = \emptyset$ . The one place we use the property that the  $a_i$  solve the Maurer-Cartan equation is that the  $P_2 = \emptyset$  terms can be dropped.

This statement can be extended to the set of matrix-valued  $\mathcal{A}$ -flat connections, given by pairs  $\mathcal{B} := (\mathcal{E}, a \in \mathcal{E} \otimes \mathcal{A} \otimes \mathcal{E}^*)$ . Then  $\text{Hop}(\mathcal{B}_1, \mathcal{B}_2) := \mathcal{E}_1 \otimes \mathcal{A} \otimes \mathcal{E}_2^*$  as a module, with compositions  $\mu$  including the contractions of  $\mathcal{E}_i^*$  and  $\mathcal{E}_i$  factors in consecutive arguments. We use the sign conventions of equation (A.23) et. seq. Taking together all modules  $\mathcal{E}$  (with suitable finiteness properties) the resulting  $A_\infty$  category is denoted as  $\mathfrak{Br}[\mathcal{A}]$ .

Given any  $A_\infty$  category  $\mathfrak{A}$ , we can define a flat  $\mathfrak{A}$  connection  $\mathcal{B}$  starting from a collection of pairs of objects and vector spaces  $(A_i, \mathcal{E}_i)$  and solving the MC equation for an element  $a \in \bigoplus_{i,j} \mathcal{E}_i \otimes \text{Hop}(A_i, A_j) \otimes \mathcal{E}_i^*$ . Flat  $\mathfrak{A}$  connections also form a larger  $A_\infty$  category  $\mathfrak{Br}[\mathfrak{A}]$  with  $\text{Hop}(\mathcal{B}, \mathcal{B}') := \mathcal{E}_i \otimes \text{Hop}(A_i, A_j) \otimes (\mathcal{E}_i^!)^*$ .

In general, spaces of flat  $\mathcal{A}$  or  $\mathfrak{A}$  connections can be very intricate, as they are defined by intricate, possibly non-polynomial, MC equations. The equations greatly simplify if there is an ordering on the set of objects  $\mathbf{Ob}(\mathfrak{A}) = \{T_i\}$  such that there is a triangular structure on the hom-sets. This motivates the

**Definition:** An *exceptional category*  $\mathfrak{E}$  is an  $A_\infty$ -category such that there is an ordering on the (countable) set of objects  $\{T_i\}$  such that

$$\text{Hop}(T_i, T_j) = 0 \quad i > j \quad (\text{B.10})$$

Moreover  $\text{Hop}(T_i, T_i) \cong \mathbb{Z}$  is concentrated in degree zero and the generator  $\mathbf{Id}_i \cong 1$  is a graded identity for  $\mu_2$ :

$$\mu_2(\mathbf{Id}, a) = a \quad \mu_2(a, \mathbf{Id}) = (-1)^{\deg(a)} a \quad (\text{B.11})$$

where  $\mathbf{Id} = \bigoplus_i \mathbf{Id}_i$  and  $a$  is homogeneous. Moreover,  $\mu_n(P) = 0$  whenever  $n \neq 2$  and  $P$  contains a multiple of  $\mathbf{Id}$ .

The MC equations in an exceptional category have a triangular structure and are rather tractable.<sup>123</sup> This is precisely the case which occurs in our paper. Physically, the objects in the exceptional collection coincide with the thimble branes. General branes in a given theory appear as  $\mathfrak{A}$ -flat connections for the exceptional category.

---

<sup>123</sup>In this situation, the definition of flat  $\mathfrak{A}$ -connections coincides with the mathematical notion of twisted complex.

## B.2 Mutations Of Exceptional Categories

### B.2.1 Exceptional Pairs And Two Distinguished Branes

An *exceptional pair* is a pair of objects  $T_1, T_2$  in a category such that

$$\begin{aligned} \text{Hop}(T_1, T_1) &\cong \mathbb{Z} & \text{Hop}(T_1, T_2) &:= \mathcal{H} \\ \text{Hop}(T_2, T_1) &= 0 & \text{Hop}(T_2, T_2) &\cong \mathbb{Z} \end{aligned} \quad (\text{B.12})$$

where  $\mathcal{H}$  is in general nonzero, and is simply some differential graded module.

Given an exceptional pair  $(T_1, T_2)$  and any given differential graded modules  $\mathcal{E}_1 = \mathcal{E}_{T_1}$  and  $\mathcal{E}_2 = \mathcal{E}_{T_2}$  we can put a differential graded associative algebra structure on the set of matrices with elements in

$$\begin{pmatrix} \mathcal{E}_1 \mathcal{E}_1^* & \mathcal{E}_1 \text{Hop}(T_1, T_2) \mathcal{E}_2^* \\ 0 & \mathcal{E}_2 \mathcal{E}_2^* \end{pmatrix} \quad (\text{B.13})$$

To define  $\mu_1$  we simply apply it to each of the matrix elements using the natural induced differential. The definition of  $\mu_2$  requires some care, given the conventions above. If we consider homogeneous matrices of monomials like

$$X = \begin{pmatrix} ee^* & \tilde{e}af^* \\ 0 & ff^* \end{pmatrix} \quad (\text{B.14})$$

then  $\mu_2(X_1, X_2)$  is given by

$$\begin{pmatrix} (-1)^{e_1}(e_1^* \cdot e_2)e_1e_2^* & (-1)^{e_1}(e_1^* \cdot \tilde{e}_2)e_1a_2\tilde{f}_2^* + (-1)^{\tilde{e}_1+a_1}(\tilde{f}_1^* \cdot f_2)\tilde{e}_1a_1f_2^* \\ 0 & (-1)^{f_1}(f_1^* \cdot f_2)f_1f_2^* \end{pmatrix} \quad (\text{B.15})$$

where  $(-1)^v$  is short for  $(-1)^{\deg(v)}$  for a homogeneous vector  $v$ . The extra sign  $(-1)^{a_1}$  in the 12 element comes about because the identity in  $\text{Hop}(T_1, T_1)$  and  $\text{Hop}(T_2, T_2)$  is a graded identity, so  $\mu_2(1, a) = a$  but  $\mu_2(a, 1) = (-1)^a a$ . Taking  $\mu_n = 0$  for  $n > 2$  one can check that the equations (A.6) are indeed satisfied.

Now, using this construction we can construct two distinguished Maurer-Cartan elements in the above differential graded algebra. They will therefore define objects in  $\mathfrak{Br}(\mathfrak{A})$  in any category  $\mathfrak{A}$  which contains the exceptional pair  $(T_1, T_2)$ . We will therefore refer to them as “Branes.”

The first Brane, denoted  $L(T_1, T_2)$  may be denoted

$$L(T_1, T_2) = \mathcal{E}_1 T_1 \oplus \mathcal{E}_2 T_2 = T_1 \oplus \mathcal{H}^{[-1]} T_2 \quad (\text{B.16})$$

where we recall that  $\mathcal{H} = \text{Hop}(T_1, T_2)$ .<sup>124</sup> The Maurer-Cartan element (“boundary amplitude”) is

$$\mathcal{B} = \begin{pmatrix} 0 & \mathcal{B}_{12} \\ 0 & 0 \end{pmatrix} \quad (\text{B.17})$$

with

$$\mathcal{B}_{12} := \sum_s f_s \otimes (f_s[-1])^* \quad (\text{B.18})$$

---

<sup>124</sup>We adopt the convention that for a degree shift by  $s$ ,  $\deg(v[s]) = \deg(v) + s$ .

where we sum over a basis  $\{f_s\}$  for  $\mathcal{H}$ . It is not difficult to check that  $\mu_1(\mathcal{B}_{12}) = 0$  and hence the MC equation is satisfied.

The second Brane, denoted  $R(T_1, T_2)$ , has Chan-Paton factors:

$$R(T_1, T_2) = \mathcal{E}_1 T_1 \oplus \mathcal{E}_2 T_2 = (\mathcal{H}^{[-1]})^* T_1 \oplus T_2. \quad (\text{B.19})$$

The Maurer-Cartan element (“boundary amplitude”) is again of the form

$$\mathcal{B} = \begin{pmatrix} 0 & \mathcal{B}_{12} \\ 0 & 0 \end{pmatrix} \quad (\text{B.20})$$

now with

$$\mathcal{B}_{12} := \sum_s (f_s[-1])^* \otimes f_s \quad (\text{B.21})$$

where we sum over a basis  $\{f_s\}$  for  $\mathcal{H}$ . It is not difficult to check that  $\mu_1(\mathcal{B}_{12}) = 0$  and hence the MC equation is satisfied.

### B.2.2 Left And Right Mutations

Now suppose that  $\mathfrak{E}$  is an exceptional category. We begin with a definition:

**Definition:** A *left mutation at  $j$  of an exceptional category  $\mathfrak{E}$*  is another exceptional category  $\mathfrak{F}$ , with an ordering on its objects  $\{S_i\}$  such that there is an  $A_\infty$ -functor

$$\mathcal{F} : \mathfrak{F} \rightarrow \mathfrak{Br}(\mathfrak{E}) \quad (\text{B.22})$$

with

$$\mathcal{F}(S_i) = \begin{cases} T_i & i \neq j, j+1 \\ T_{j+1} & i = j \\ L(T_j, T_{j+1}) & i = j+1 \end{cases} \quad (\text{B.23})$$

and

$$\mathcal{F}_1 : \text{Hop}(S_i, S_k) \rightarrow \text{Hop}(\mathcal{F}(S_i), \mathcal{F}(S_k)) \quad (\text{B.24})$$

is a quasi-isomorphism for all  $i, k$ .

We now show that left mutations in fact do exist. To do this, we first note that it is not immediately obvious that quasi-isomorphisms of the desired type in fact do exist. Note that if we consider the full subcategory  $L_j(\mathfrak{E})$  of  $\mathfrak{Br}(\mathfrak{E})$  whose set of objects is just  $\{\tilde{T}_i\}$  with

$$\tilde{T}_i = \begin{cases} T_i & i \neq j, j+1 \\ T_{j+1} & i = j \\ L(T_j, T_{j+1}) & i = j+1 \end{cases} \quad (\text{B.25})$$

The resulting subcategory  $L_j(\mathfrak{E})$  is *not* an exceptional category. It does not satisfy the definition because  $\text{Hop}(\tilde{T}_{j+1}, \tilde{T}_{j+1})$  is not isomorphic to  $\mathbb{Z}$  and  $\text{Hop}(\tilde{T}_{j+1}, \tilde{T}_j)$  is not isomorphic to zero. We therefore should check that they are at least quasi-isomorphic to  $\mathbb{Z}$  and zero, respectively.

We first show there is a quasi-isomorphism of  $\text{Hop}(\tilde{T}_{j+1}, \tilde{T}_{j+1})$  with  $\mathbb{Z}$ . Now,

$$\text{Hop}(L(T_j, T_{j+1}), L(T_j, T_{j+1})) \quad (\text{B.26})$$

is the associative matrix algebra associated with the exceptional pair  $(T_j, T_{j+1})$ . So, our definition only makes sense if this matrix algebra is quasi-isomorphic to  $\mathbb{Z}$  with the differential  $M_1$  in the brane category. Let us check that this is indeed the case:

Morphisms  $\delta \in \text{Hop}(L(T_j, T_{j+1}), L(T_j, T_{j+1}))$  can be thought of as being in

$$\begin{pmatrix} \mathbb{Z} & \mathcal{H}_j \otimes (\mathcal{H}_j^{[-1]})^* \\ 0 & (\mathcal{H}_j^{[-1]}) \otimes (\mathcal{H}_j^{[-1]})^* \end{pmatrix} \quad (\text{B.27})$$

where  $\mathcal{H}_j = \text{Hop}(T_j, T_{j+1})$ . We write these as

$$\delta = \begin{pmatrix} \delta_{11} & \delta_{12} \\ 0 & \delta_{22} \end{pmatrix} \quad (\text{B.28})$$

and, using  $\mathcal{B}^2 = 0$  we compute

$$M_1(\delta) = \mu_1(\delta) + \mu_2(\mathcal{B}, \delta) + \mu_2(\delta, \mathcal{B}) \quad (\text{B.29})$$

So

$$M_1(\delta) = \begin{pmatrix} 0 & \mu_1(\delta_{12}) + \mu_2(\mathcal{B}_{12}, \delta_{22}) + \delta_{11}\mu_2(1, \mathcal{B}_{12}) \\ 0 & \mu_1(\delta_{22}) \end{pmatrix} \quad (\text{B.30})$$

It is not difficult to show that the cohomology of  $M_1$  on the subspace of morphisms of the form

$$\delta = \begin{pmatrix} 0 & \delta_{12} \\ 0 & \delta_{22} \end{pmatrix} \quad (\text{B.31})$$

is zero, precisely because the operation  $x \rightarrow \mu_2(\mathcal{B}_{12}, x)$  acts as a “twisted degree shift.” The projection of the kernel of  $M_1$  to  $\delta_{11} \in \mathbb{Z}$  is then the desired quasi-isomorphism.

Now consider  $(\alpha, \beta) = (j+1, j)$  then, on the one hand,  $\text{Hop}(S_{j+1}, S_j) = 0$ , and therefore

$$\begin{aligned} \text{Hop}(L(T_j, T_{j+1}), T_{j+1}) &= \text{Hop}(T_j, T_{j+1}) \oplus \mathcal{E}_{j+1} \otimes \text{Hop}(T_{j+1}, T_{j+1}) \\ &\cong \mathcal{H}_j \oplus \mathcal{H}_j^{[-1]} \end{aligned} \quad (\text{B.32})$$

must be quasi-isomorphic to zero with the differential  $M_1$  in the Brane category. Indeed this is the case. Once again

$$M_1(\delta) = \mu_1(\delta) + \mu_2(\mathcal{B}, \delta) + \mu_2(\delta, \mathcal{B}). \quad (\text{B.33})$$

Writing an element of (B.32) as  $\delta = \delta_1 \oplus \delta_2$  we compute that

$$M_1(\delta) = \left( \mu_1(\delta_1) + \mu_2(\mathcal{B}_{12}, \delta_2) \right) \oplus \mu_1(\delta_2) \quad (\text{B.34})$$

and again  $\mu_2(\mathcal{B}_{12}, \delta_2)$  acts as a twisted degree shift. Using this property it is not difficult to show that the cohomology is zero.

If we consider the fourteen remaining cases of  $\mathcal{F}_1 : \text{Hop}(S_\alpha, S_\beta) \rightarrow \text{Hop}(\tilde{T}_\alpha, \tilde{T}_\beta)$  then many cases are trivially quasi-isomorphisms, and the remaining ones simply constrain the relation of the hom-sets in interesting ways. We will comment on those below.

We can now show the existence of left-mutations, in the sense of our definition. We use the result of Kadeishvili, and of Kontsevich and Soibelman (see [5], pp. 587-593 for a detailed exposition) that there is always a quasi-isomorphism of any  $A_\infty$ -category with another  $A_\infty$ -category with the same set of objects, but whose morphism spaces are the cohomologies of the original category. If we apply the construction of Kadeishvili-Kontsevich-Soibelman to  $L_j(\mathfrak{E})$  we obtain the required exceptional category  $\mathfrak{F}$ .

In an entirely analogous fashion we can give the:

**Definition** A *right mutation at  $j$  of an exceptional category  $\mathfrak{E}$*  is another exceptional category  $\mathfrak{F}$ , with an ordering on its objects  $\{S_i\}$  such that there is an  $A_\infty$ -functor

$$\mathcal{F} : \mathfrak{F} \rightarrow \mathfrak{Br}(\mathfrak{E}) \quad (\text{B.35})$$

such that

$$\mathcal{F}(S_i) = \begin{cases} T_i & i \neq j, j+1 \\ R(T_j, T_{j+1}) & i = j \\ T_j & i = j+1 \end{cases} \quad (\text{B.36})$$

and

$$\mathcal{F}_1 : \text{Hop}(S_i, S_k) \rightarrow \text{Hop}(\mathcal{F}(S_i), \mathcal{F}(S_k)) \quad (\text{B.37})$$

is a quasi-isomorphism for all  $i, k$ .

We would like to conclude with a number of remarks:

1. As a general rule, in a mutation, if we “add” a new brane of the type  $C = A + \mathcal{E}B$  where  $\mathcal{E} \sim \text{Hop}(A, B)$  (up to degree shift and duals) then the new set of objects contains  $C$  and  $B$  but *not*  $A$ .
2. The categorical mutations described here arise from framed-wall-crossing on  $S_{ij}$ -walls, as described in Section §7.8 above.
3. As we remarked, the existence of a left- or right-mutation implies some interesting relations between the spaces  $\text{Hop}(S_\alpha, S_\beta)$  and  $\text{Hop}(T_\alpha, T_\beta)$ . For example, considering the case  $(\alpha, \beta) = (j, j+1)$  for a left-mutation at  $j$  we find that there must be a quasi-isomorphism

$$\mathcal{F}_1 : \text{Hop}(S_j, S_{j+1}) \rightarrow (\text{Hop}(T_j, T_{j+1})^{[-1]})^* \quad (\text{B.38})$$

This is in harmony with the expectations of  $S$ -wall-crossing. Similarly, the case  $(\alpha, j+1)$  for  $\alpha < j$  implies there are quasi-isomorphisms

$$\mathcal{F}_1 : \text{Hop}(S_\alpha, S_{j+1}) \rightarrow \text{Hop}(T_\alpha, L(T_j, T_{j+1})) \cong \text{Hop}(T_\alpha, T_j) \oplus \text{Hop}(T_\alpha, T_{j+1}) \otimes (\mathcal{H}_j^{[-1]})^* \quad (\text{B.39})$$

while  $(j+1, b)$  for  $\beta > j+1$  implies there are quasi-isomorphisms

$$\mathcal{F}_1 : \text{Hop}(S_{j+1}, S_\beta) \rightarrow \text{Hop}(L(T_j, T_{j+1}), T_\beta) = \text{Hop}(T_j, T_\beta) \oplus \mathcal{H}_j^{[-1]} \otimes \text{Hop}(T_{j+1}, T_\beta) \quad (\text{B.40})$$

Again, in harmony with  $S$ -wall-crossing.

4. There is a sense in which left and right mutations are inverse to each other: *If  $\mathfrak{F}$  is a left mutation at  $j$  of  $\mathfrak{E}$ , then  $\mathfrak{E}$  is a right mutation at  $j$  of  $\mathfrak{F}$ .* As a check on this assertion note that  $\mathcal{F}_j^{(R)} \circ \mathcal{F}_j^{(L)}$  takes  $S_j \rightarrow S_j$  but

$$\mathcal{F}_j^{(R)} \circ \mathcal{F}_j^{(L)} : S_{j+1} \rightarrow \mathcal{E}S_j \oplus S_{j+1} \quad (\text{B.41})$$

where

$$\mathcal{E} = (\text{Hop}(S_j, S_{j+1})^{[-1]})^* \oplus \text{Hop}(T_j, T_{j+1})^{[-1]} \quad (\text{B.42})$$

and by equation (B.38) above this space can indeed admit a differential making it quasi-isomorphic to zero. Similarly,  $\mathcal{F}_j^{(L)} \circ \mathcal{F}_j^{(R)}$  takes  $S_{j+1} \rightarrow S_{j+1}$  but

$$\mathcal{F}_j^{(L)} \circ \mathcal{F}_j^{(R)} : S_j \rightarrow S_j \oplus \mathcal{E}S_{j+1} \quad (\text{B.43})$$

where

$$\mathcal{E} = \text{Hop}(S_j, S_{j+1})^{[-1]} \oplus (\text{Hop}(T_j, T_{j+1})^{[-1]})^{[-1]} \quad (\text{B.44})$$

and again (B.38) shows this space can indeed admit a differential making it quasi-isomorphic to zero.

5. Moreover, we expect the left and right mutations to define a braid group action in the following sense. Suppose we have left-mutations:

$$\begin{aligned} \mathcal{F}_j^{(L)} &: \mathfrak{E}_1 \rightarrow \mathfrak{Br}(\mathfrak{E}_2) \\ \mathcal{F}_{j+1}^{(L)} &: \mathfrak{E}_2 \rightarrow \mathfrak{Br}(\mathfrak{E}_3) \\ \mathcal{F}_j^{(L)} &: \mathfrak{E}_3 \rightarrow \mathfrak{Br}(\mathfrak{E}_4) \end{aligned} \quad (\text{B.45})$$

and similarly,

$$\begin{aligned} \mathcal{F}_{j+1}^{(L)} &: \mathfrak{E}_1 \rightarrow \mathfrak{Br}(\mathfrak{E}'_2) \\ \mathcal{F}_j^{(L)} &: \mathfrak{E}'_2 \rightarrow \mathfrak{Br}(\mathfrak{E}'_3) \\ \mathcal{F}_{j+1}^{(L)} &: \mathfrak{E}'_3 \rightarrow \mathfrak{Br}(\mathfrak{E}'_4) \end{aligned} \quad (\text{B.46})$$

then we expect that there is an equivalence of  $A_\infty$ -categories  $\mathcal{G} : \mathfrak{Br}(\mathfrak{E}_4) \rightarrow \mathfrak{Br}(\mathfrak{E}'_4)$  such that there is an isomorphism of  $A_\infty$ -functors:

$$\mathcal{G} \circ \mathcal{F}_j^{(L)} \circ \mathcal{F}_{j+1}^{(L)} \circ \mathcal{F}_j^{(L)} \cong \mathcal{F}_{j+1}^{(L)} \circ \mathcal{F}_j^{(L)} \circ \mathcal{F}_{j+1}^{(L)} \quad (\text{B.47})$$

That is, there is an invertible  $A_\infty$ -natural transformation between these two functors. We have not checked the details of these last assertions, so we'll leave it here for now.

## C. Examples Of Categories Of Branes

For small numbers of vacua it is possible to write out in some generality the full structure of the web-formalism of Sections §2-9. Of course the complexity increases rapidly as the number of vacua is increased.

### C.1 One Vacuum

In this case there are no planar webs. There are no unextended half-plane webs but there are extended half-plane and interface webs. As noted in Section §6.1.5, the category of Branes, and also the category of Interfaces between the trivial Theory and itself is precisely the category of chain complexes.

### C.2 Two Vacua

Suppose that  $\mathbb{V}$  has cardinality 2. Then the entire web formalism can still be written out quite explicitly. We will describe the category of Branes for the positive and negative half-plane and the differential on the strip complex. Therefore we must assume that  $\text{Re}(z_{12}) \neq 0$ . By reordering vacua may assume without loss of generality that  $\text{Re}(z_{12}) > 0$ .

There are no unextended plane webs. In particular there are no vertices of valence three or higher. The only plane webs consist of a single line with arbitrarily many 2-valent vertices on it. For a web representation we are free to choose any pair of  $\mathbb{Z}$ -graded modules  $R_{12}, R_{21}$  together with a perfect degree  $-1$  pairing  $K : R_{12} \otimes R_{21} \rightarrow \mathbb{Z}$ . There is one taut web, illustrated in Figure 27(a). As explained in the text, the  $L_\infty$  Maurer-Cartan equation implies that the interior amplitude defines a differential  $\mathcal{Q}$  on  $R_{12}$  and  $R_{21}$  so that the pairing is  $\mathcal{Q}$ -invariant. Thus, the Theory  $\mathcal{T}$  is entirely characterized by a choice of a complex  $R_{12}$  and a dual complex  $R_{21}$  with a perfect pairing.

The complex  $(R_c, d_c)$  of local operators is

$$0 \rightarrow R_1 \oplus R_2 \rightarrow R_{12} \otimes R_{21} \rightarrow 0 \quad (\text{C.1})$$

where  $R_1, R_2 \cong \mathbb{Z}$  with generators  $\phi_1, \phi_2$  and

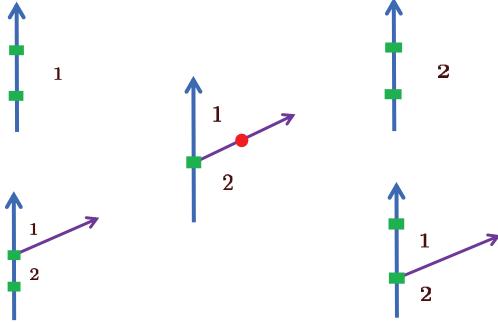
$$\begin{aligned} d_c(\phi_1) &= K_{12}^{-1} \\ d_c(\phi_2) &= K_{21}^{-1} \end{aligned} \quad (\text{C.2})$$

so that the  $\mathcal{Q}$ -invariant local operators on the plane consist of the identity  $\mathbf{1} = \phi_1 + \phi_2$  in degree zero and a space of operators isomorphic to  $(R_{12} \otimes R_{21}) / K_{12}^{-1} \mathbb{Z}$ . The  $\mathbb{CP}^1$  model discussed in Section §16.3.3 above is a special case of this situation.

Now let us consider the category  $\mathfrak{Br}(\mathcal{T}, \mathcal{H}^+)$  in the positive half-plane for the above Theory. The half-plane webs consist of an arbitrary number of zero-valent boundary vertices with one or zero emission lines separating vacua 1 and 2. If there is such a line it can have an arbitrary number of 2-valent vertices on it. In particular, there are five taut half-plane webs shown in Figure 176.

Now we construct positive half-plane Branes. We choose two  $\mathbb{Z}$ -graded Chan-Paton modules  $\mathcal{E}_1, \mathcal{E}_2$ . The boundary amplitude is an element

$$\mathcal{B} \in \bigoplus_{i,j} \mathcal{E}_i \otimes \text{Hop}(i, j) \otimes \mathcal{E}_j^* \quad (\text{C.3})$$



**Figure 176:** In a Theory with two vacua, with  $\text{Re}(z_{12}) > 0$ , there are five taut positive half-plane webs, illustrated here.

and may be thought of as a  $2 \times 2$  matrix:

$$\mathcal{B} = \begin{pmatrix} \mathcal{B}_{11} & \mathcal{B}_{12} \\ 0 & \mathcal{B}_{22} \end{pmatrix} \quad (\text{C.4})$$

As in the discussion of Section §4.5 the Maurer-Cartan equation following from the two taut webs on the top of Figure 176 imply that  $\mathcal{B}_{ii} \in \mathcal{E}_i \otimes \mathcal{E}_i^* \cong \text{End}(\mathcal{E}_i)$  are differentials, while the remaining three taut webs imply that

$$\mathcal{B}_{12} \in \text{Hom}(\mathcal{E}_2, \mathcal{E}_1 \otimes R_{12}) \quad (\text{C.5})$$

is annihilated by the differential induced from that on  $\mathcal{E}_1, \mathcal{E}_2$  and  $R_{12}$ . We conclude that the objects in the category  $\mathfrak{Br}(\mathcal{T}, \mathcal{H}^+)$  consist of a pair of  $\mathbb{Z}$ -graded complexes  $(\mathcal{E}_1, \mathcal{E}_2)$  together with an arbitrary degree one,  $\mathcal{Q}$ -invariant morphism in (C.5).

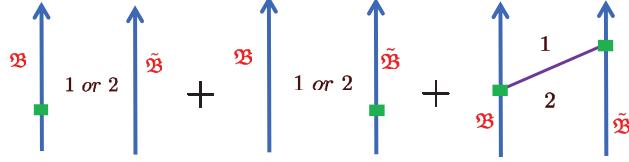
The space  $\text{Hop}(\mathfrak{B}, \mathfrak{B}')$  between two Branes  $\mathfrak{B}, \mathfrak{B}'$  in  $\mathfrak{Br}(\mathcal{T}, \mathcal{H}^+)$  can be thought of as  $2 \times 2$  matrices valued in

$$\begin{pmatrix} \text{Hom}(\mathcal{E}'_1, \mathcal{E}_1) & \text{Hom}(\mathcal{E}'_2, \mathcal{E}_1 \otimes R_{12}) \\ 0 & \text{Hom}(\mathcal{E}'_2, \mathcal{E}_2) \end{pmatrix} \quad (\text{C.6})$$

The differential  $M_1$  on  $\text{Hop}(\mathfrak{B}, \mathfrak{B}')$  computed using equation (5.17) with the taut element shown in Figure 176 is the natural differential acting on (C.6), so the local operators between  $\mathfrak{B}$  and  $\mathfrak{B}'$  can be identified with the  $\mathcal{Q}$ -cohomology of (C.6). The multiplication  $M_2$  on the category is given by the naive multiplication of elements of the form (C.6) since all taut half-plane webs have at most two boundary vertices. Moreover, for this reason, the higher multiplications  $M_n$ ,  $n \geq 3$  all vanish.

Similarly, the objects in the category  $\mathfrak{Br}(\mathcal{T}, \mathcal{H}^-)$  associated with the negative half-plane consist of a pair of  $\mathbb{Z}$ -graded complexes  $(\tilde{\mathcal{E}}_1, \tilde{\mathcal{E}}_2)$  together with an arbitrary degree one,  $\mathcal{Q}$ -invariant morphism

$$\tilde{\mathcal{B}}_{12} \in \text{Hom}(\tilde{\mathcal{E}}_2, R_{21} \otimes \tilde{\mathcal{E}}_1) \quad (\text{C.7})$$



**Figure 177:** In a Theory with two vacua, with  $\text{Re}(z_{12}) > 0$ , there are three taut webs on the strip.

Now, consider the strip with a Brane  $\mathfrak{B} \in \mathfrak{Br}(\mathcal{T}, \mathcal{H}^+)$  on the left-boundary and  $\tilde{\mathfrak{B}} \in \mathfrak{Br}(\mathcal{T}, \mathcal{H}^-)$  on the right boundary. The complex of approximate groundstates is

$$\mathcal{E}_1 \otimes \tilde{\mathcal{E}}_1 \oplus \mathcal{E}_2 \otimes \tilde{\mathcal{E}}_2 \quad (\text{C.8})$$

with a differential induced from the taut strip-webs shown in Figure 177. The first two types of webs give the naive differential on (C.8). Using the block form corresponding to the direct sum decomposition in (C.8) the differential has the form

$$d_{LR} = \begin{pmatrix} \mathcal{Q}_1 \otimes 1 + 1 \otimes \tilde{\mathcal{Q}}_1 & \mathcal{Q}_{12} \\ 0 & \mathcal{Q}_2 \otimes 1 + 1 \otimes \tilde{\mathcal{Q}}_2 \end{pmatrix} \quad (\text{C.9})$$

where, up to sign

$$\mathcal{Q}_{12} = K_{12}(\mathcal{B}_{12} \otimes \tilde{\mathcal{B}}_{12}) \in \text{Hom}(\mathcal{E}_2 \otimes \tilde{\mathcal{E}}_2, \mathcal{E}_1 \otimes \tilde{\mathcal{E}}_1) \quad (\text{C.10})$$

We can compute the cohomology of  $d_{LR}$  by first passing to the naive cohomology of (C.8) using the diagonal elements of (C.9). The operator  $\mathcal{Q}_{12}$  passes to an operator  $\hat{\mathcal{Q}}_{12}$  on the this cohomology. The ‘‘space of exact ground states’’ in the sense of Section §4.3 is therefore

$$\text{Ker}(\hat{\mathcal{Q}}_{12}) \oplus \text{Cok}(\hat{\mathcal{Q}}_{12}). \quad (\text{C.11})$$

Note that if we work over the integers the cokernel can be a finite abelian group, and therefore the space of exact ground states can have torsion. It would be interesting to know if this has a physical interpretation.

One can similarly work out the general category of Interfaces, but to do this one must drop the assumption that  $\text{Re}(z_{12}) > 0$  since there are now two Theories associated (up to locally trivial parallel transport) with  $\text{Re}(z_{12}) > 0$  and  $\text{Re}(z_{12}) < 0$ . It is then possible to write out in full detail the Interfaces  $\mathfrak{I}[\phi]$  for a vacuum homotopy  $z_{12}(x)$  and the  $S$ -wall Interfaces. This is a good exercise that we will leave to the reader.

Similarly, one could move on and write out in full generality the web formalism when  $\mathbb{V}$  has three vacua. Again, we leave this as an extensive exercise to the reader.

## D. Proof Of Equation (7.181)

In this appendix we prove that the characteristic polynomial of the  $N \times N$  matrix

$$\begin{aligned} \mathcal{E} = & -R_{N-1}e_{0,0} + \sum_{j=0}^{N-2} e_{j+1,j} - e_{0,N-1} \\ & + \sum_{1 \leq j < (N-1)/2} R_{N-1-2j}e_{N-j,j} + \sum_{0 \leq j < (N-2)/2} R_{N-2-2j}e_{N-1-j,j} \end{aligned} \quad (\text{D.1})$$

(where we treat  $R_n$  as scalars) is simply

$$\det(x\mathbf{1}_N - \mathcal{E}) = x^N + \sum_{j=1}^{N-1} R_j x^j + 1 \quad (\text{D.2})$$

We prove equation (D.2) by expansion by minors. We use the last column of  $x\mathbf{1}_N - \mathcal{E}$  because it has only two nonzero entries. The minor of the  $(N-1, N-1)$  matrix element is lower triangular and immediately gives

$$x^{N-1}(x + R_{N-1}) \quad (\text{D.3})$$

The minor of the  $(0, N-1)$  matrix element is  $(-1)^{N+1}$  times the determinant of the  $(N-1) \times (N-1)$  dimensional matrix:

$$\begin{aligned} \widetilde{M} = & x \sum_{j=0}^{N-3} e_{j,j+1} - \sum_{j=0}^{N-2} e_{j,j} \\ & - \sum_{1 \leq j < (N-1)/2} R_{N-1-2j}e_{N-1-j,j} - \sum_{0 \leq j < (N-2)/2} R_{N-2-2j}e_{N-2-j,j} \end{aligned} \quad (\text{D.4})$$

Now we use row and column operations to eliminate the  $x$ 's above the diagonal. For simplicity assume that  $N$  is even. Then, adding  $x$  times row  $(N-2)$  to row  $(N-3)$ , then  $x$  times row  $(N-3)$  to row  $(N-4)$  and so forth up to (but not including) row  $N/2$  gives a matrix of the form

$$\begin{pmatrix} A & 0 \\ * & -\mathbf{1}_{\frac{(N-2)}{2}} \end{pmatrix} \quad (\text{D.5})$$

where

$$\begin{aligned} A = & x \sum_{j=0}^{N/2-1} e_{j,j+1} - \sum_{j=0}^{N/2-1} e_{j,j} \\ & - x^{(N-2)/2} R_{N-2} e_{N/2-1,0} - \left( x^{(N-2)/2} R_{N-3} + x^{(N-4)/2} R_{N-4} \right) \\ & - \cdots - (x^2 R_3 + x R_2) e_{N/2-1, N/2-2} - x R_1 e_{N/2-1, N/2-1} \end{aligned} \quad (\text{D.6})$$

Now to determine  $\det A$  use column operations to eliminate the  $x$ 's above the diagonal: First add  $x$  times column 0 to column 1, then  $x$  times column 1 to column 2 and so forth

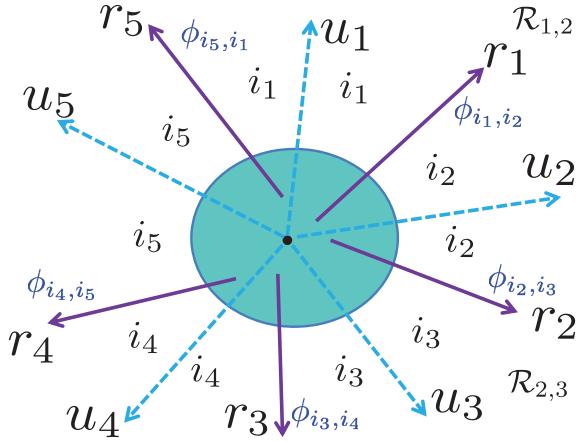
to produce a lower triangular matrix with diagonal with  $-1$  in every element except the  $(\frac{N}{2} - 1, \frac{N}{2} - 1)$  matrix element which is

$$-1 - xR_1 - x^2R_2 - \cdots - x^{N-2}R_{N-2} \quad (\text{D.7})$$

Thus the contribution of the  $(0, N - 1)$  minor in  $x\mathbf{1}_N - \mathcal{E}$  is (recall  $N$  is even):

$$\begin{aligned} -\det M &= (-1)^{N/2} \det A \\ &= -(-1)^{N/2} (-1)^{N/2-1} (1 + xR_1 + x^2R_2 + \cdots + x^{N-2}R_{N-2}) \\ &= 1 + xR_1 + x^2R_2 + \cdots + x^{N-2}R_{N-2} \end{aligned} \quad (\text{D.8})$$

completing the proof of (D.2).



**Figure 178:** In the regions  $\mathcal{R}_{k,k+1}$  which contain the boosted soliton core rays  $r_k$  and are at  $|s| > R$  (the region outside the blue circle) we can write approximate solutions to the instanton equations to exponentially good accuracy.

## E. A More Technical Definition Of Fan Boundary Conditions

In this appendix we describe how to define a set of boundary conditions for the  $\zeta$ -instanton equation (11.18) associated with a fan of solitons

$$\mathcal{F} = \{\phi_{i_1, i_2}^{p_1}, \dots, \phi_{i_n, i_1}^{p_n}\}. \quad (\text{E.1})$$

To begin, we need to define some notation. We choose a set of points  $s_1, \dots, s_n$  in the complex  $s = x + i\tau$  plane. To these points we attach a set of rays  $\mathfrak{r}_k = s_k + z_{i_k, i_{k+1}} \mathbb{R}_+$ . The index  $k$  is now considered modulo  $n$ . Now we also choose an  $R \gg |s_k|$ . Now consider the connected components of the complement of the rays  $\mathfrak{r}_k$  in the region  $|s| > R$ . The region between  $\mathfrak{r}_{k-1}$  and  $\mathfrak{r}_k$  is labeled with a vacuum  $i_k$ . Now, in each such component write the ray  $\mathfrak{v}_k$  with slope bisecting  $\mathfrak{r}_{k-1}$  and  $\mathfrak{r}_k$ . Finally, define a region

$$\mathcal{R}_{k,k+1}(R) := \{s \mid \arg \mathfrak{v}_k > \arg s > \arg \mathfrak{v}_{k+1} \quad |s| \geq R\} \quad (\text{E.2})$$

In the region  $\mathcal{R}_{k-1,k}(R)$  we use a boosted soliton solution  $\phi_{i_{k-1},i_k}^{p_{k-1}}$  with core centered on the ray  $\mathbf{r}_{k-1}$ . These are true solutions of the  $\zeta$ -instanton equation, but only in this region. On the boundary  $\mathbf{v}_k$  between  $\mathcal{R}_{k-1,k}(R)$  and  $\mathcal{R}_{k,k+1}(R)$  there is a discontinuity. However, for large  $R$  and for large mass scale  $m$  of the LG theory both solutions  $\phi_{i_{k-1},i_k}^{p_{k-1}}$  and  $\phi_{i_k,i_{k+1}}^{p_k}$  are within order  $\mathcal{O}(e^{-mR})$  of the vacuum  $\phi_{i_k}$ . Therefore, as  $R \rightarrow \infty$  our solution, which is only defined on the open region  $\cup_k \mathcal{R}_{k-1,k}$  has exponentially small discontinuities at  $\mathbf{v}_k$ . Let us denote these discontinuous solutions by  $\phi_{\mathcal{F},\vec{s},R}$ , where  $\mathcal{F}$  stands for the fan data (E.1),  $\vec{s}$ , is the vector of origins of the rays  $\mathbf{r}_k$ , and  $R$  is large.

Now we state the boundary conditions on the instanton equation (11.18). First, our solutions  $\phi(x, \tau)$  should be continuously differentiable. Next, we require that there is some  $\vec{s}$ , and some  $R_0$ , sufficiently large, and some constant  $C$  so that admissible solutions satisfy

$$|\phi(x, \tau) - \phi_{\mathcal{F},\vec{s},R}| \leq Ce^{-mR} \quad \forall |s| \geq R \quad (\text{E.3})$$

when  $R > R_0$ .

Given a fan of solitons  $\mathcal{F}$  we let  $\mathcal{M}(\mathcal{F})$  denote the moduli space of smooth solutions to the  $\zeta$ -instanton equation (11.18) with fan boundary conditions (E.3). As explained in Sections §§14.2-14.4 when the excess dimension dimension of all zeta vertices vanishes there is a physical expectation that the component of  $\mathcal{M}(\mathcal{F})$  of maximal dimension has dimension given by:

$$\dim \mathcal{M}(\mathcal{F}) = -\frac{1}{2} \sum_k \eta(D_k - \epsilon) \quad (\text{E.4})$$

where  $D_k$  is the Dirac operator (12.6) on  $\mathbb{R}$  with  $W''$  evaluated for the soliton  $\phi_{i_k,i_{k+1}}^{p_k}$ . In particular, it is physically reasonable to expect that the excess dimension vanishes for  $X = \mathbb{C}^n$ . Giving a rigorous proof of (E.4), even in this case, would seem to be a challenging task. One needs to compute the index of the Dirac operator corresponding to the first order variation of the  $\zeta$ -instanton equation

$$\widehat{D} = \frac{\partial}{\partial \tau} - i\sigma^3 \frac{\partial}{\partial x} - \frac{1}{2} \begin{pmatrix} 0 & \zeta \overline{W}'' \\ \bar{\zeta} W'' & 0 \end{pmatrix} \quad (\text{E.5})$$

Superficially this would appear to be a problem of the kind studied in [12, 10]. However, the presence of the center of mass collective coordinates discussed at length in §14.5 implies that the Dirac operator with our boundary conditions is *not* Fredholm, so standard theorems will not apply. Nevertheless, there should be a good theory of the Dirac operator  $\widehat{D}$  with our boundary conditions. In particular, we need to orient the moduli spaces  $\mathcal{M}(\mathcal{F})$ . The relative orientation for different fan boundary conditions should be fixed from a trivialization of the determinant line of  $\widehat{D}$  on the space of all LG fields.

## F. Signs In The Supersymmetric Quantum Mechanics Formulation Of Morse Theory

In this Appendix we briefly remark on the signs by which instantons are weighted in the approach to Morse theory reviewed in Section §10 above. In particular we comment on how

to choose the signs for instanton amplitudes in (10.35) and (10.50). This is closely related to the proper interpretation of the fermion determinant (10.33). The sign conventions for  $m_{qp}$  were already discussed in footnote 3 of the original article [87] (see the end of Section 10.5.3 of [50] for an elaboration of that discussion). We will take our cue from the description of signs on p. 106 of [11] and Section 2 of [53].

## F.1 Preliminaries

We first need a little notation. It will be useful to speak of determinant line bundles. If  $V$  is a finite-dimensional real vector space then  $\text{Det}(V)$  is the real line given by the highest exterior power. The set of nonzero vectors in  $\text{Det}(V)$  has two connected components. A choice of a component is, by definition, an orientation of  $V$ . We can thus identify an orientation of  $V$  with a choice of a nonzero vector in  $\text{Det}(V)$  up to rescaling by positive real numbers. If  $V$  has a metric then a volume form on  $V$  is a nonzero product of orthonormal vectors, and corresponds to a vector of norm one in  $\text{Det}(V)$ . When  $V$  is a vector bundle over a manifold a choice of metric defines a reduction of structure group of  $\text{Det}(V)$  from  $GL(1, \mathbb{R}) \cong \mathbb{R}^*$  to  $O(1) \cong \mathbb{Z}_2$ .

The set of descending flows from a critical point  $p \in M$  is the set of trajectories  $u(\tau)$  satisfying (10.20) with  $\lim_{\tau \rightarrow -\infty} u(\tau) = p$ . The union of the descending flows is a cell  $\mathcal{D}_p \subset M$ . Similarly, the set of ascending flows from  $p$  is the set of trajectories satisfying (10.19) with  $\lim_{\tau \rightarrow \infty} u(\tau) = p$ . Their union is also a cell in  $M$  denoted  $\mathcal{U}_p$ .<sup>125</sup> At a nondegenerate critical point the Hessian, or fermion mass matrix:

$$m_{ij} = \frac{D^2 h}{Du^i Du^j} \quad (\text{F.1})$$

is a symmetric, real, nondegenerate form on  $T_p M$ . Let  $P_+(p)$  be the projector onto the subspace of  $T_p M$  with positive fermion masses and  $P_-(p)$  the projector onto the space with negative fermion masses. We can identity

$$P_+(p)T_p M = T_p \mathcal{U}_p \quad \& \quad P_-(p)T_p M = T_p \mathcal{D}_p \quad (\text{F.2})$$

Thus, assuming  $M$  is finite-dimensional, we can say that  $\mathcal{D}_p$  is a cell of dimension  $n_p$  and  $\mathcal{U}_p$  is a cell of dimension  $d - n_p$ , where  $d = \dim M$ .

If  $(g, h)$  are sufficiently generic then the ascending and descending cells from two critical points  $p, q$  will intersect *transversally* in the following sense: We suppose  $h(p) > h(q)$  so there is a moduli space  $\mathcal{M}_{qp}$  of ascending flows from  $q$  to  $p$ . It will be a manifold, perhaps with many connected components, of dimension  $n_p - n_q$ . We let  $\Gamma_{qp} \subset M$  be the image of these flows in  $M$ . Of course,  $\Gamma_{qp} \subset \mathcal{D}_p \cap \mathcal{U}_q$ . Our primary example is when  $n_p = n_q + 1$

---

<sup>125</sup>Recall that a cell of dimension  $n$  is a topological space homeomorphic to the open  $n$ -dimensional ball in  $\mathbb{R}^n$ . Using the Morse lemma it is easy to see that the union of flows defines a cell in the neighborhood of a critical point. For example, for ascending flows the coefficients  $c_i$  in (10.26) with  $f_i > 0$  provide a system of coordinates. Since we have a first order differential equation the flows will not intersect even when we follow them beyond the neighborhood where the Morse lemma applies. See [2] for the rigorous proof that  $\mathcal{D}_p$  and  $\mathcal{U}_p$  are cells. One of the very useful aspects of Morse theory in topology is that it provides explicit cell decompositions of topological spaces.

so that  $\mathcal{M}_{qp}$  (generically) consists of an isolated set of instanton trajectories. The image  $\Gamma_{qp}$  is a union of disjoint one-manifolds in  $M$  and the closure is a graph with two vertices at  $q$  and  $p$ . We will also need the case  $n_p = n_q + 2$  below. We say that  $(g, h)$  determine transversal flows if there is an exact sequence

$$0 \rightarrow T_{\mathcal{P}}\Gamma_{qp} \rightarrow T_{\mathcal{P}}\mathcal{U}_q \oplus T_{\mathcal{P}}\mathcal{D}_p \rightarrow T_{\mathcal{P}}M \rightarrow 0 \quad (\text{F.3})$$

for any point  $\mathcal{P} \in \Gamma_{qp}$ . Linear algebra now guarantees the existence of a *canonical* isomorphism

$$\text{Det}(T_{\mathcal{P}}\Gamma_{qp}) \otimes \text{Det}(T_{\mathcal{P}}M) \cong \text{Det}(T_{\mathcal{P}}\mathcal{U}_q) \otimes \text{Det}(T_{\mathcal{P}}\mathcal{D}_p). \quad (\text{F.4})$$

This will be the crucial identity in our determination of sign rules.

## F.2 A Mathematical Sign Rule

In many mathematical treatments of the MSW complex (see, for examples [11], [53] Section 2, or [5] Section 8.3) one chooses generators  $[\mathcal{U}_p]$  of the complex to be orientations of  $T_p\mathcal{U}_p$ , that is, a nonzero vector in  $\text{Det}(T_p\mathcal{U}_p)$ , up to positive scaling. The choice of orientation at each critical point  $p$  is made arbitrarily. Since  $\mathcal{U}_p$  is a cell, such a choice determines an orientation of  $T_{\mathcal{P}}\mathcal{U}_p$  for all  $\mathcal{P} \in \mathcal{U}_p$ , say, by parallel transport. Similarly, we have a canonical isomorphism  $\text{Det}(T_q\mathcal{U}_q) \otimes \text{Det}(T_q\mathcal{D}_q) \cong \text{Det}(T_qM)$  and hence an orientation of  $T_q\mathcal{U}_q$  also determines a nonzero vector (up to positive scaling) of  $\text{Det}(T_qM) \otimes (\text{Det}(T_q\mathcal{D}_q))^{-1}$ . Again, the orientation of this line can be extended continuously to an orientation of  $\text{Det}(T_{\mathcal{P}}M) \otimes (\text{Det}(T_{\mathcal{P}}\mathcal{D}_q))^{-1}$  for all  $\mathcal{P} \in \mathcal{U}_q$ . Now, rewrite the canonical isomorphism (F.4) as

$$\text{Det}(T_{\mathcal{P}}\mathcal{U}_q) \cong \text{Det}(T_{\mathcal{P}}\Gamma_{qp}) \otimes (\text{Det}(T_{\mathcal{P}}M) \otimes (\text{Det}(T_{\mathcal{P}}\mathcal{D}_p))^{-1}) \quad (\text{F.5})$$

Now suppose that  $n_p = n_q + 1$ . Then  $\mathcal{M}_{qp}$  is a disjoint union over instantons  $\ell$ . The matrix element  $m_{qp}$  in (10.35) is a sum of contributions  $m_{qp}(\ell) \in \{\pm 1\}$  where, in this approach, one identifies  $\Phi_p = [\mathcal{U}_p]$  in (10.35). For each instanton  $\ell$  choose a point  $\mathcal{P} \neq p, q$  on  $\ell$  and choose the upwards-flowing orientation of  $T_{\mathcal{P}}\Gamma_{qp}$ . Then the arbitrary choices of orientations of  $[\mathcal{U}_p]$  and  $[\mathcal{U}_q]$  will either agree or disagree with the canonical isomorphism (F.5) for a trajectory  $\ell$  containing a point  $\mathcal{P}$ . If the orientations agree then  $m_{qp}(\ell)$  is  $+1$  and if they disagree it is  $-1$ . Note, incidentally, that it was not necessary to orient  $M$ . This is the sign rule given in [11] and [53] and it is equivalent to that given in [87].

As an example let us check that the two upward flows in Figure 122(a) indeed cancel. The ascending cell  $\mathcal{U}_q$  is the circle minus  $p$ , while  $\mathcal{D}_q$  is the 0-cell  $q$  itself. The descending cell  $\mathcal{D}_p$  is the circle minus  $q$  while  $\mathcal{U}_p$  is the 0-cell  $p$  itself. We choose the orientation of  $\mathcal{U}_q$  to be defined by a nonzero horizontal tangent vector pointing left at  $q$  and the orientation of  $\mathcal{D}_p$  to be a nonzero horizontal tangent vector pointing right at  $p$ . We choose the clockwise orientation for  $TM$ . The orientations of the two ascending flows will be given by a nonzero tangent vector in the ascending direction. With these choices we compute from the above sign rule that the contribution to  $m_{qp}$  from the ascending flow on the left is  $+1$  and that of the ascending flow on the right is  $-1$ .

### F.3 Approach Via Supersymmetric Quantum Mechanics

Our goal here is to give a slight reformulation of the above sign rule which makes more direct contact with the physical approach from supersymmetric quantum mechanics. This approach provides a framework for generalization to the case when  $M$  is an infinite-dimensional function space, such as is necessary in applications to quantum field theory (such as Landau-Ginzburg theory).

We must first return to the description of the Fermion state associated with a critical point  $p \in M$  (See the discussion from equations (10.9) to (10.11) above.)

At any point  $\mathcal{P} \in M$ , the span of the Fermions  $\psi^i$  is  $\Pi T_{\mathcal{P}}^* M$  and their canonical conjugates  $\bar{\psi}_i = g_{ij} \bar{\psi}^j$  span  $\Pi T_{\mathcal{P}} M$ . (Here  $\Pi$  indicates parity reversal - these are odd real vector spaces.) The canonical quantization relations define the natural Clifford algebra of  $T_{\mathcal{P}}^* M \oplus T_{\mathcal{P}} M$  associated with the natural signature  $(d, d)$  form given by contraction. In order to write a definite Fermionic Hilbert space we must choose a particular Clifford module and this is done by choosing a maximal isotropic subspace of  $T_{\mathcal{P}}^* M \oplus T_{\mathcal{P}} M$ . If we quantize by choosing the vacuum line to be determined by the maximal isotropic subspace  $T_{\mathcal{P}} M$  then the Hilbert space of Fermionic states at  $\mathcal{P} \in M$  is the Clifford module

$$\mathcal{H}_{\mathcal{P}}^{\text{Fermi}} = \Lambda^*(T_{\mathcal{P}}^* M) \otimes \mathfrak{M}_{\mathcal{P}}. \quad (\text{F.6})$$

where, for any point  $\mathcal{P} \in M$  we have introduced the notation  $\mathfrak{M}_{\mathcal{P}} := \text{Det}(T_{\mathcal{P}} M)$ . Heuristically, this is the real span of the product of the  $\bar{\psi}_i$ . This quantization applies to general manifolds  $M$ , orientable or not. When  $M$  is orientable, there exist smooth trivializations of  $\text{Det}(TM)$ . Since  $M$  is Riemannian there are two natural trivializations of unit norm, corresponding to the two possible orientations of  $M$ . Relative to such a trivialization a vector in (F.6) can be identified with a differential form at  $\mathcal{P}$ , and in this way the entire Hilbert space of the theory is identified with  $\Omega^*(M)$ . When  $M$  is unorientable there is no continuous trivialization of  $\mathfrak{M} = \text{Det}(TM)$ . Rather wavefunctions are sections of  $\Omega^*(M; \mathfrak{M})$ . These are known as densities, and can be integrated on any manifold, oriented or not.

#### Remarks

1. It is useful to note that the supersymmetric quantum mechanics can be coupled to a flat line bundle  $\mathfrak{L}$ . When we do this equation (F.6) is generalized to

$$\mathcal{H}_{\mathcal{P}}^{\text{Fermi}} = \Lambda^*(T_{\mathcal{P}}^* M) \otimes \mathfrak{M}_{\mathcal{P}} \otimes \mathfrak{L}_{\mathcal{P}}. \quad (\text{F.7})$$

The instanton amplitudes  $m_{qp}$  discussed below now have an extra factor corresponding to the parallel transport from  $\mathfrak{L}_q$  to  $\mathfrak{L}_p$ . This leads to the MSW complex twisted by a flat line bundle. If  $\mathfrak{L}$  is a real line bundle then  $\mathcal{H}_{\mathcal{P}}^{\text{Fermi}}$  is a real Hilbert space.

2. There is a dual quantization where the Clifford vacuum is based on the maximal isotropic subspace  $T_{\mathcal{P}}^* M$ . This corresponds to an exchange of  $\psi^i$  for  $\bar{\psi}_i$ . Equation (F.6) treats  $\psi^i$  and  $\bar{\psi}_i$  asymmetrically and moreover differs in our conventions for the relation of Fermion number to the degree of a form. In (F.6) a differential form

of degree  $n$  corresponds naturally to a Fermion state of Fermion number  $n - d$ . If  $M$  is orientable then one can maintain the (Hodge) symmetry between  $\psi^i$  and  $\bar{\psi}_i$  by choosing  $\mathfrak{L}^{-1}$  to be a real line bundle which squares to  $\text{Det}(TM)$ . We can use the metric to reduce the structure group of  $\mathfrak{M}$  to  $O(1) \cong \mathbb{Z}_2$ . If we do so, then a choice of  $\mathfrak{L}$  is a choice of a flat order two real line bundle. If we twist by such a line  $\mathfrak{L}$  then a differential form of degree  $n$  corresponds to a Fermionic state in (F.7) of Fermion number  $n - d/2$ .

Now we wish to define a (perturbative) ground state associated with a critical point  $p$  in the supersymmetric quantum mechanics.

In general, in quantum mechanics, a (pure) state is a one-dimensional complex line in a complex Hilbert space. (In the physics literature such a line is often called a “ray.”) In our problem the Hilbert space is naturally real. The perturbative ground state is a product of real lines for the bosonic and Fermionic degrees of freedom. The bosonic groundstate, which has Gaussian support near  $p$ , poses no sign difficulties and will henceforth be ignored. The Fermionic ground state is therefore considered to be a real line in the real Hilbert space (F.6). Physicists often use the word “state” to refer to either a line or a norm one vector in that line. Since the distinction will be important in what follows, and in order to avoid confusion, we will use the term *Fermion vector* to refer to an actual vector in the line. In Morse theory we are interested in transition matrix elements and not their squares so we must choose a Fermion vector  $\Psi(p) \in \mathcal{H}_p^{\text{Fermi}}$ . Since the Fermion line is a real line there are two normalized vectors  $\pm\Psi(p)$ . The vectors  $\Phi_p$  used in equations (10.27) et. seq. of Section §10.4 are then obtained by taking a product with the (unproblematic) bosonic vector.

Now, for any critical point  $p$ , let us determine the Fermion line in which  $\Psi(p)$  must live. Recall from the analysis above (10.11) that if we choose a coordinate frame diagonalizing  $m_{ij}(p)$  then the ground state is made by acting on the vacuum line of (F.6) with the wedge product of  $\psi^i$  for  $i$  running over the downward directions. Thus, the Fermion ground vector at  $p$  is an element of the real line

$$\mathcal{S}_p := \text{Det}(T_p^* \mathcal{D}_p) \otimes \mathfrak{M}_p. \quad (\text{F.8})$$

Using the canonical isomorphisms

$$\text{Det}(T_p^* \mathcal{D}_p) \cong (\text{Det}(T_p \mathcal{D}_p))^{-1} \quad (\text{F.9})$$

and

$$\mathfrak{M}_p \cong \text{Det}(T_p \mathcal{D}_p) \otimes \text{Det}(T_p \mathcal{U}_p) \quad (\text{F.10})$$

we can equally well write

$$\mathcal{S}_p \cong \text{Det}(T_p \mathcal{U}_p) \quad (\text{F.11})$$

This will be the form of most use to us, and we will find it useful to introduce the notation

$$\mathfrak{U}_p := \text{Det}(T_p \mathcal{U}_p) \quad \& \quad \mathfrak{D}_p := \text{Det}(T_p \mathcal{D}_p). \quad (\text{F.12})$$

Now we can start to make contact with the mathematical formulation of Section §F.2 above. A choice at each critical point  $p$  of  $\Psi(p)$  determines a set of nonzero vectors in  $\mathcal{S}_p$

up to *positive* scaling. That is equivalent to a choice of orientation of  $\mathcal{U}_p$ . The difference is that in physics  $\Psi(p)$  is vector in a Hilbert space and in the mathematical approach  $[\mathcal{U}_p]$  is just an abstract generator of a  $\mathbb{Z}$ -module.

Let us next turn to the matrix element  $m_{qp}$  of equation (10.35). Here  $p, q$  are critical points with  $n_p = n_q + 1$ . The operator  $\widehat{\mathcal{Q}}$  is not a section of a line bundle so we regard  $m_{qp}$  as an element of a line

$$m_{qp} \in \frac{\mathfrak{U}_q}{\mathfrak{U}_p} \quad (\text{F.13})$$

The matrix element  $m_{qp}$  is in the dual line to  $\text{Hom}(\mathcal{S}_q, \mathcal{S}_p)$ . The localization of the path integral states that  $m_{qp} = \sum_{\ell \in \mathcal{M}_{qp}} m_{qp}(\ell)$ , and each  $m_{qp}(\ell)$  is given by the one-loop path integral expanded around the instanton  $\ell$ . This is the Fermionic path integral normalized by the (unproblematic) one loop bosonic determinant. The Fermionic path integral is a section,  $\det(L)$  of the determinant line bundle  $\text{Det}(L)$ . In general, if  $T : \mathcal{H}_1 \rightarrow \mathcal{H}_2$  is a Fredholm operator between two Hilbert spaces we can define  $\text{Det}(T) \cong \text{Det}(\ker L)^{-1} \otimes \text{Det}(\ker L^\dagger)$ . As explained in Section §10.2, since  $\iota(L) = n_p - n_q = 1$  is positive, we generically have  $\ker L^\dagger = \{0\}$  and for simplicity we will assume that this is the case. Thus,  $\det(L) \in \text{Det}(\ker L)^{-1}$ . Indeed, this is where the measure of the Fermion zeromodes should live. On the other hand, for any point  $\mathcal{P} \in \ell$ , we can also identify  $\ker L \cong T_{\mathcal{P}}\ell$ , since the zeremode is just  $u^i(\tau)$ , defining a nonvanishing vector field along  $\ell$ . Thus, we can choose any point  $\mathcal{P} \in \ell$  and use this isomorphism to identify the path integral as an nonzero vector in  $\text{Det}(T_{\mathcal{P}}\ell)$ . Now we have isomorphisms

$$\frac{\mathfrak{U}_q}{\mathfrak{U}_p} \cong \frac{\mathfrak{U}_q \mathfrak{D}_p}{\mathfrak{M}_p} \cong \frac{\text{Det}(T_{\mathcal{P}}\mathcal{U}_q) \text{Det}(T_{\mathcal{P}}\mathcal{D}_p)}{\mathfrak{M}_{\mathcal{P}}} \cong \text{Det}(T_{\mathcal{P}}\Gamma_{qp}) = \text{Det}(T_{\mathcal{P}}\ell). \quad (\text{F.14})$$

where the first isomorphism is canonical, the second uses parallel transport along  $\ell$  to a point  $\mathcal{P} \in \ell$ , and the third is the canonical isomorphism (F.4). In this way, for each  $\ell$  we can map the amplitude to an element  $m_{qp}(\ell) \in \frac{\mathfrak{U}_q}{\mathfrak{U}_p}$  in a common line, where the amplitudes can be sensibly added.

As explained in Section §10.4,  $\det(L)$  is a product of the zeremode measure (valued in  $\text{Det}(T_{\mathcal{P}}\ell)$ , for any choice of  $\mathcal{P} \in \ell$ ) and the ratio of determinants in equation (10.33):

$$\frac{\det' L}{(\det'(L^\dagger L))^{1/2}} \quad (\text{F.15})$$

The ratio of determinants is a number (in general complex) not a section of a line bundle and can be defined, say, by  $\zeta$ -function regularization. In our case, since  $L$  is real this number is  $\pm 1$ . Once we have chosen a trivialization  $\Psi(p), \Psi(q)$  of  $\mathfrak{U}_q/\mathfrak{U}_p$  the contributions of the amplitudes  $m_{qp}(\ell)$  for different instantons differ by  $\pm 1$  where this sign is given by both the ratio of determinants and the ratio of orientations of Fermion measures. This makes contact with the mathematical sign rule of Section §F.2 above.

As an example, consider again Figure 122(a). Let  $\phi \sim \phi + 2\pi$  parametrize the circle and let  $h = \cos(\phi)$ . The two instanton flows are given by  $\tan(\phi/2) = \pm e^{-(\tau-\tau_0)}$  and  $L = -\frac{d}{d\tau} + \cos \phi$ . In this case (F.15) is clearly the same for the two flows, but the Fermion measure is oriented in opposite directions for the two flows, and hence the two instanton amplitudes cancel.

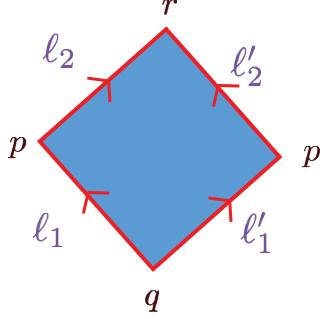
More generally, it is instructive to go back to the localization formula for the vev of some generic operator of the form

$$\widehat{O} = O_{i_1 \dots i_n} \psi^{i_1} \dots \psi^{i_n} \quad (\text{F.16})$$

which is  $\mathcal{Q}$ -closed if the degree  $n$  form  $O = O_{i_1 \dots i_n} du^{i_1} \dots du^{i_n}$  on  $M$  is closed. The same analysis as for  $[\mathcal{Q}, f]$  given in Section §10.4 shows that

$$\langle \Phi_p | \widehat{O}(\tau) | \Phi_q \rangle = \int_{\Gamma_{qp}} O \quad (\text{F.17})$$

Here  $n = n_p - n_q$ , otherwise both sides are zero. The right hand side is a sum over the components  $\Gamma_{qp}^\alpha$  of  $\Gamma_{qp}$ , but how should these components be oriented? Having made a choice of  $\Psi(p)$  and  $\Psi(q)$  to define the left hand side, we can again use (F.14) to determine the orientations of the components  $\Gamma_{qp}^\alpha$ . We define the sign of the integral  $\int_{\Gamma_{qp}^\alpha} O$  using this orientation.



**Figure 179:** A component of  $\mathcal{M}_{qr}$  when  $n_r - n_q = 2$  corresponds to a two-cell  $\Gamma_{qr} \subset M$  depicted schematically here. There are two one-dimensional boundaries of this cell, corresponding two broken paths  $\ell_1 * \ell_2$  and  $\ell'_1 * \ell'_2$  interpolating  $q \rightarrow p \rightarrow r$ , where  $n_p = n_q + 1$ .

Let us now address the cancellations required for equation (10.41) to hold. Suppose that  $n_r = n_p + 1 = n_q + 2$  and focus on a particular component of the moduli space  $\mathcal{M}_{qr}$ . This component will map to a two-dimensional cell  $\Gamma_{qp} \subset M$  with boundaries corresponding to broken paths  $\ell_1 * \ell_2$  and  $\ell'_1 * \ell'_2$  as in Figure 179. The product of instanton amplitudes  $m_{qp} m_{pr}$  for both broken paths is, according to (F.13), valued in

$$m_{qp} m_{pr} \in \frac{\mathfrak{U}_q}{\mathfrak{U}_p} \otimes \frac{\mathfrak{U}_p}{\mathfrak{U}_r} \cong \frac{\mathfrak{U}_q}{\mathfrak{U}_r} \quad (\text{F.18})$$

Now, once again, if we choose a particular two-cell in  $\Gamma_{qr}$  as in Figure 179 we can use the isomorphism (F.14) to map to an element

$$m_{qp} m_{pr} \in \text{Det}(T_{\mathcal{P}} \mathcal{M}_{qr}) \quad (\text{F.19})$$

where  $\mathcal{P} \in \Gamma_{qr}$  is any point in the cell. Thus the contributions of the two bounding broken paths  $\ell_1 * \ell_2$  and  $\ell'_1 * \ell'_2$  lie in the same line and can meaningfully be compared. On the other hand, from (F.14) we see that  $m_{qp}(\ell_1)$  and  $m_{pr}(\ell_2)$  determine orientations of  $\ell_1$  and  $\ell_2$ . Since the broken path  $\ell_1 * \ell_2$  is a limit of smooth paths, all of which can be oriented, there is a natural correlation between the orientation of  $\ell_1$  and  $\ell_2$  so that there is a well-defined orientation of the boundary  $\ell_1 * \ell_2$ . Thus,  $m_{qp}(\ell_1)m_{pr}(\ell_2)$  determines an orientation of the boundary one-cell  $\ell_1 * \ell_2$ . There are only two natural maps into  $\text{Det}(T_{\mathcal{P}}\mathcal{M}_{qr})$ , namely taking a wedge product with the outward or the inward normal to the two-cell  $\Gamma_{qr}$ . Making the same choice for both  $\ell_1 * \ell_2$  and  $\ell'_1 * \ell'_2$  the contributions from the two boundaries cancel. The reason we make the same choice is that we then have a rule analogous to (F.17) for matrix elements of operators of Fermion number 2.

Finally, we discuss the sign choices for the exceptional instantons counted by  $\mathsf{E}$  in equation (10.50). Let  $s_0$  be an isolated value of  $s$  such that equation (10.49) has a solution. The relevant linear operator acts on a Fermion field with one extra component

$$\psi(\tau) = (\psi^i(\tau), \psi^s(\tau)) \quad (\text{F.20})$$

where  $\psi^s(\tau)$  takes into account the first order variation in the  $s$  direction. The relevant Dirac operator is given by

$$(\tilde{L}\psi)^i = \frac{D\psi^i}{D\tau} - g^{ij} \frac{D^2 h}{Du^j Du^k} \psi^k - \frac{\partial}{\partial s} \Big|_{s_0} \left[ g^{ij} \frac{\partial h}{\partial u^j} \right] \psi^s \quad (\text{F.21})$$

where  $g, h$  and their derivatives are evaluated on the exceptional instanton  $u^i(\tau; s_0)$ . The contribution of the exceptional instanton at  $s_0$  to  $e_{pp'}$  in equation (10.50) is, as usual, an element

$$e_{pp'}(s_0) \in \frac{\mathfrak{U}_p}{\mathfrak{U}_{p'}} \otimes T_{s_0}^* I \quad (\text{F.22})$$

where the last factor  $T_{s_0}^* I$  on the right hand side comes from the extra component of the Fermi field,  $\psi^s$ , in the domain of the Dirac operator  $\tilde{L}$ , and  $I = [0, 1]$  is the space parametrized by  $s$  and describing the homotopy between  $(g(\tau), h(\tau))$  and  $(g'(\tau), h'(\tau))$ . Accordingly, once we choose an orientation for  $I$  we can add the contributions of the different exceptional instantons. Reversing the orientation changes the sign of the operator  $\mathsf{E}$ . From equation (10.48) it is clear this must be the case.

## References

- [1] H. Abbaspour, “On the algebraic structures of the Hochschild complex,” e-print arXiv:1302.6534
- [2] R. Abraham and J. Robbin, *Transversal Mappings and Flows*, W.A. Benjamin, 1967
- [3] L. F. Alday, D. Gaiotto, S. Gukov, Y. Tachikawa and H. Verlinde, “Loop and surface operators in N=2 gauge theory and Liouville modular geometry,” JHEP **1001**, 113 (2010) [arXiv:0909.0945 [hep-th]].
- [4] L.J.C. Amorim, “A Künneth theorem in Lagrangian Floer theory,” PhD thesis, Univ. of Wisconsin

- [5] P. Aspinwall, T. Bridgeland, A. Craw, M.R. Douglas, M. Gross, et. al. *Dirichlet Branes and Mirror Symmetry*, Clay Mathematical Monographs, vol. 4, 2009.
- [6] M. Atiyah and N. Hitchin, The geometry and dynamics of magnetic monopoles. M. B. Porter Lectures. Princeton University Press, Princeton, NJ, 1988.
- [7] D. Auroux, “A beginner’s introduction to Fukaya categories,” e-print: arXiv:1301.7056 [math.SG].
- [8] R. Bandiera, “Formality of Kapranov’s Brackets on Pre-Lie Algebras,” arXiv:1307.8066[math.QA]
- [9] D. Bar-Natan, “On Khovanov’s Categorification Of The Jones Polynomial,” arXiv:math/0201043, Alg. Geom. Topology **2** (2002) 337-370.
- [10] R. Bott and R. Seeley, “Some Remarks on the Paper of Callias,” CMP vol. 62, pp. 235-245 (1978).
- [11] R. Bott, “Morse theory indomitable,” Inst. Hautes Etudes Sci. Publ. Math. (1988), no. 68, 99114 (1989).
- [12] C. Callias, “Index Theorems on Open Spaces,” Commun. Math. Phys. **62**, 213 (1978).
- [13] S. M. Carroll, S. Hellerman and M. Trodden, “Domain wall junctions are 1/4 - BPS states,” Phys. Rev. D **61**, 065001 (2000) [hep-th/9905217].
- [14] S. Cecotti, P. Fendley, K. A. Intriligator and C. Vafa, “A New supersymmetric index,” Nucl. Phys. B **386**, 405 (1992) [hep-th/9204102].
- [15] S. Cecotti and C. Vafa, “On classification of  $\mathcal{N} = 2$  supersymmetric theories,” Commun. Math. Phys. **158**, 569 (1993) [hep-th/9211097].
- [16] F. Chapoton and M. Livernet, “Pre-Lie Algebras and the Rooted Trees Operad,” Intl. Math. Res. Notes, 2001, No.8
- [17] A. D’Adda, P. Di Vecchia and M. Luscher, “Confinement and Chiral Symmetry Breaking in CP\*\*n-1 Models with Quarks,” Nucl. Phys. B **152**, 125 (1979).
- [18] P. Deligne, P. Etingof, D. S. Freed, L. C. Jeffrey, D. Kazhdan, J. W. Morgan, D. R. Morrison and E. Witten, “Quantum fields and strings: A course for mathematicians. Vol. 1, 2,” Providence, USA: AMS (1999) 1-1501
- [19] S.K. Donaldson, “Nahm’s equations and the classification of monopoles,” Commun. Math. Phys. **96**, 387-408 (1984)
- [20] N. Dorey, “The BPS spectra of two-dimensional supersymmetric gauge theories with twisted mass terms,” JHEP **9811**, 005 (1998) [hep-th/9806056].
- [21] H. Fan, T. J. Jarvis and Y. Ruan, “The Witten equation, mirror symmetry and quantum singularity theory,” arXiv:0712.4021 [math.AG].
- [22] B. Feigin, E. Frenkel, and V. Toledano-Laredo, Gaudin Models With Irregular Singularities, arXiv:math/0612798., and to appear.
- [23] P. Fendley and K. A. Intriligator, “Scattering and thermodynamics in integrable  $N=2$  theories,” Nucl. Phys. B **380**, 265 (1992) [hep-th/9202011].
- [24] D. Freed and E. Witten, “Anomalies In String Theory With  $D$ -Branes,” Asian J. Math. **3** (1999) 819, arXiv:hep-th/9907189.

- [25] E. Frenkel, “Gaudin model and opers,” math/0407524 [math-qa].
- [26] E. Frenkel and E. Witten, “Geometric endoscopy and mirror symmetry,” Commun. Num. Theor. Phys. **2** (2008) 113 [arXiv:0710.5939 [math.AG]].
- [27] M. R. Gaberdiel and B. Zwiebach, “Tensor constructions of open string theories. 1: Foundations,” Nucl. Phys. B **505**, 569 (1997) [hep-th/9705038].
- [28] D. Gaiotto, G. W. Moore and A. Neitzke, “Wall-crossing, Hitchin Systems, and the WKB Approximation,” arXiv:0907.3987 [hep-th].
- [29] D. Gaiotto, “Surface Operators in  $N = 2$  4d Gauge Theories,” JHEP **1211**, 090 (2012) [arXiv:0911.1316 [hep-th]].
- [30] D. Gaiotto, G. W. Moore and A. Neitzke, “Framed BPS States,” arXiv:1006.0146 [hep-th].
- [31] D. Gaiotto, G. W. Moore and A. Neitzke, “Wall-Crossing in Coupled 2d-4d Systems,” arXiv:1103.2598 [hep-th].
- [32] D. Gaiotto and E. Witten, “Knot Invariants from Four-Dimensional Gauge Theory,” arXiv:1106.4789 [hep-th].
- [33] D. Gaiotto, G. W. Moore and A. Neitzke, “Spectral networks,” arXiv:1204.4824 [hep-th].
- [34] D. Gaiotto, G. W. Moore and A. Neitzke, “Spectral Networks and Snakes,” arXiv:1209.0866 [hep-th].
- [35] D. Gaiotto, G.W. Moore, and E. Witten, “An Introduction To The Web-Based Formalism,” to appear.
- [36] D. Galakhov, P. Longhi, T. Mainiero, G. W. Moore and A. Neitzke, “Wild Wall Crossing and BPS Giants,” JHEP **1311**, 046 (2013) [arXiv:1305.5454 [hep-th]].
- [37] D. Galakhov, P. Longhi and G. W. Moore, “Spectral Networks with Spin,” arXiv:1408.0207 [hep-th].
- [38] S. J. Gates, Jr., C. M. Hull and M. Rocek, “Twisted Multiplets and New Supersymmetric Nonlinear Sigma Models,” Nucl. Phys. B **248**, 157 (1984).
- [39] G. W. Gibbons and P. K. Townsend, “A Bogomolny equation for intersecting domain walls,” Phys. Rev. Lett. **83**, 1727 (1999) [hep-th/9905196].
- [40] M. Gromov, “Pseudo Holomorphic Curves in Symplectic Manifolds,” Invent. Math. **82** (1985) 307-47.
- [41] J. Guffin and E. Sharpe, A-Twisted Landau-Ginzburg Models, J. Geom. Phys. **59** (2009) 1547-80, arXiv:0801.3836.
- [42] S. Gukov and E. Witten, “Gauge Theory, Ramification, And The Geometric Langlands Program,” hep-th/0612073.
- [43] A. Haydys, “Seidel-Fukaya Category And Gauge Theory,” arXiv:1010.2353.
- [44] M. Herbst, K. Hori, and D. Page, “Phases Of  $\mathcal{N} = 2$  Theories In 1+1 Dimensions With Boundary,” arXiv:0803.2045.
- [45] N. Hitchin, “Monopoles and Geodesics,” Commun. Math. Phys. **83**, 579-602 (1982).
- [46] N. J. Hitchin, “The self-duality equations on a Riemann surface,” Proc. London Math. Soc. (3) **55** (1987), no. 1, 59(126).

- [47] K. Hori and C. Vafa, “Mirror symmetry,” hep-th/0002222.
- [48] K. Hori, A. Iqbal and C. Vafa, “D-branes and mirror symmetry,” hep-th/0005247.
- [49] K. Hori, “Linear models of supersymmetric D-branes,” hep-th/0012179.
- [50] K. Hori, S. Katz, A. Klemm, R. Pandharipande, R. Thomas, C. Vafa, R. Vakil and E. Zaslow, “Mirror symmetry,” (Clay mathematics monographs. 1)
- [51] K. Hori, C. Y. Park and Y. Tachikawa, “2d SCFTs from M2-branes,” JHEP **1311**, 147 (2013) [arXiv:1309.3036 [hep-th]].
- [52] J. Hurtubise, “Monopoles and Rational Maps: A Note on a Theorem of Donaldson,” Commun. Math. Phys. **100**, 191-196 (1985)
- [53] M. Hutchings, “Lecture notes On Morse homology (with an eye towards Floer homology and pseudoholomorphic curves),” available at [math.berkeley.edu/hutching](http://math.berkeley.edu/hutching).
- [54] M. Kapranov, M. Kontsevich and Y. Soibelman, “Algebra of the infrared and secondary polytopes,” arXiv:1408.2673 [math.SG].
- [55] A. Kapustin and D. Orlov, “Remarks On A-Branes, Mirror Symmetry, and The Fukaya Category,” J. Geom. Phys. **48** (2003) 84, hep-th/0109098.
- [56] A. Kapustin and E. Witten, “Electric-Magnetic Duality And The Geometric Langlands Program,” Commun. Num. Theor. Phys. **1**, 1 (2007) [hep-th/0604151].
- [57] B. Keller, “Introduction to A-infinity algebras and modules,” arXiv:math/9910179
- [58] B. Keller, Notes for an Introduction to Kontsevich’s Quantization Theorem,”
- [59] M. Khovanov, “A Categorification Of The Jones Polynomial,” Duke. Math. J. **101** (2000) 359-426.
- [60] R. Koberle and V. Kurak, “Solitons in the Supersymmetric CP<sup>\*\*</sup>(n-1) Model,” Phys. Rev. D **36**, 627 (1987).
- [61] M. Kontsevich and Y. Soibelman, “Stability structures, motivic Donaldson-Thomas invariants and cluster transformations,” arXiv:0811.2435 [math.AG].
- [62] M. Kontsevich and Y. Soibelman, “Motivic Donaldson-Thomas invariants: Summary of results,” arXiv:0910.4315 [math.AG].
- [63] T. Lada and M. Markl, “Symmetric Brace Algebras,” Applied Categorical Structures (2005) 13:351-370
- [64] W. Lerche, C. Vafa and N. P. Warner, “Chiral Rings in N=2 Superconformal Theories,” Nucl. Phys. B **324**, 427 (1989).
- [65] W. Lerche and N. P. Warner, “Polytopes and solitons in integrable, N=2 supersymmetric Landau-Ginzburg theories,” Nucl. Phys. B **358**, 571 (1991).
- [66] W. Lerche and N. P. Warner, “Solitons in integrable, N=2 supersymmetric Landau-Ginzburg models,” CALT-68-1747, USC-91-030, C91-05-20.1.
- [67] J.-L. Loday, “The diagonal of the Stasheff polytope,” arXiv:0710.0572
- [68] D.-M. Lu, J. H. Palmieri, Q.-S. Wu, J. J. Zhang, “Koszul Equivalences in  $A_\infty$ -Algebras,” arXiv:0710.5492 [math.RA]

- [69] J. M. Maldacena, G. W. Moore and N. Seiberg, “Geometrical interpretation of D-branes in gauged WZW models,” JHEP **0107**, 046 (2001) [hep-th/0105038].
- [70] M. Markl and S. Shnider, “Associahedra, cellular W construction, and products of  $A_\infty$ -algebras,” arXiv:math/0312277.
- [71] E. J. Martinec and G. W. Moore, “On decay of K theory,” hep-th/0212059.
- [72] R. Mazzeo and E. Witten, “The Nahm Pole Boundary Condition,” arXiv:1311.3167 [math.DG].
- [73] R. Mehta and M. Zambon, “ $L_\infty$ -Algebra Actions,” arXiv:1202.2607.
- [74] G. W. Moore and A. Parnachev, “Localized tachyons and the quantum McKay correspondence,” JHEP **0411**, 086 (2004) [hep-th/0403016].
- [75] G.W. Moore, Felix Klein Lectures: “Applications of the Six-dimensional (2,0) Theory to Physical Mathematics,” October 1 - 11, 2012 at the Hausdorff Institute for Mathematics, Bonn. <http://www.physics.rutgers.edu/~gmoore>
- [76] W. Muller, “Eta Invariants And Manifolds With Boundary,” J. Diff. Geom. **40** (1994) 311-377.
- [77] N. A. Nekrasov and S. L. Shatashvili, “Quantization of Integrable Systems and Four Dimensional Gauge Theories,” arXiv:0908.4052 [hep-th].
- [78] J.-M. Oudom and D. Guin, “On the Lie enveloping algebra of a pre-Lie algebra,” J. K-Theory **2** (2008), 147-167.
- [79] J. Robbin and D. Salamon, “The Spectral Flow And The Maslov Index,” Bull. London Math. Soc. **27** (1995) 1-33.
- [80] P. Seidel, “Symplectic Homology As Hochschild Homology,” arXiv:math/0609037.
- [81] P. Seidel, *Fukaya Categories and Picard-Lefschetz Theory*, European Mathematical Society, Zürich, 2008.
- [82] P. Seidel, “Fukaya  $A_\infty$  Structures Associated to Lefschetz Fibrations, I” arXiv:0912.3932.
- [83] P. Seidel, “Fukaya  $A_\infty$  Structures Associated to Lefschetz Fibrations, II” arXiv:1404.1352.
- [84] D. Tong, “Quantum Vortex Strings: A Review,” Annals Phys. **324**, 30 (2009) [arXiv:0809.5060 [hep-th]].
- [85] J. Wess and J. Bagger, “Supersymmetry and supergravity,” Princeton, USA: Univ. Pr. (1992) 259 p
- [86] E. Witten, “Instantons, the Quark Model, and the 1/n Expansion,” Nucl. Phys. B **149**, 285 (1979).
- [87] E. Witten, “Supersymmetry and Morse theory,” J. Diff. Geom. **17**, 661 (1982).
- [88] E. Witten, Algebraic Geometry Associated With Matrix Models Of Two-Dimensional Gravity, in *Topological Methods In Modern Mathematics* (Publish or Perish, Houston TX, 1993).
- [89] E. Witten, “Phases of N=2 theories in two-dimensions,” Nucl. Phys. B **403**, 159 (1993) [hep-th/9301042].
- [90] E. Witten, “Analytic Continuation Of Chern-Simons Theory,” arXiv:1001.2933 [hep-th].

- [91] E. Witten, “A New Look At The Path Integral Of Quantum Mechanics,” arXiv:1009.6032.
- [92] E. Witten, “Fivebranes and Knots,” arXiv:1101.3216 [hep-th].
- [93] E. Witten, “Khovanov Homology And Gauge Theory,” arXiv:1108.3103 [math.GT].
- [94] E. Witten, “Two Lectures On The Jones Polynomial And Khovanov Homology,” arXiv:1401.6996 [math.GT].
- [95] E. Zaslow, “Solitons and helices: The Search for a math physics bridge,” Commun. Math. Phys. **175**, 337 (1996) [hep-th/9408133].
- [96] B. Zwiebach, “Closed string field theory: Quantum action and the B-V master equation,” Nucl. Phys. B **390**, 33 (1993) [hep-th/9206084].
- [97] B. Zwiebach, “Oriented open - closed string theory revisited,” Annals Phys. **267**, 193 (1998) [hep-th/9705241].