Geometric Langlands From Six Dimensions

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ABSTRACT. Geometric Langlands duality is usually formulated as a statement about Riemann surfaces, but it can be naturally understood as a consequence of electric-magnetic duality of four-dimensional gauge theory. This duality in turn is naturally understood as a consequence of the existence of a certain exotic supersymmetric conformal field theory in six dimensions. The same six-dimensional theory also gives a useful framework for understanding some recent mathematical results involving a counterpart of geometric Langlands duality for complex surfaces. (This article is based on a lecture at the Raoul Bott celebration, Montreal, June 2008.)

1. Introduction

A d-dimensional quantum field theory (QFT) associates a number, known as the partition function $Z(X_d)$, to a closed d-manifold X_d endowed with appropriate structure. Depending on the type of QFT considered, the requisite structure may be a smooth structure, a conformal structure, or a Riemannian metric, possibly together with an orientation or a spin structure, etc. In physical language, the partition function can usually be calculated via a path integral over fields on X. However, this lecture will be partly based on an exception to that statement.

To a closed d-1-dimensional manifold X_{d-1} (again with some suitable structure), a d-dimensional QFT associates a vector space $\mathcal{H}(X_{d-1})$, usually called the space of physical states. In the case of a unitary QFT (such as the one associated with the Standard Model of particle physics), \mathcal{H} is actually a Hilbert space, not just a vector space. The quantum field theories considered in this lecture are not necessarily unitary. The partition function associated to the empty d-manifold is $Z(\varnothing) = 1$, and the vector space associated to the empty d-1-manifold is $\mathcal{H}(\varnothing) = \mathbb{C}$.

There is a natural link between these structures. To a d-manifold X_d with boundary X_{d-1} , a d-dimensional QFT associates a vector $\psi_{X_d} \in \mathcal{H}(X_{d-1})$. (In physical terminology, ψ_{X_d} can usually be computed by performing a path integral for fields on X_d that have prescribed behavior along its boundary.) This generalizes the partition function, since if $X_{d-1} = \emptyset$, then $\psi_{X_d} \in \mathcal{H}(\emptyset) = \mathbb{C}$ is simply a complex number, which is the partition function $Z(X_d)$.

What I have said so far is essentially rather familiar to physicists. (The reason for the word "essentially" in the last sentence is that for most physical applications,

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¹We will slightly relax the usual axioms in section 4.1.

a less abstract formulation is adequate.) Less familiar is that it is possible to continue the above discussion to lower dimensions. The next step in the hierarchy is that to a closed d-2-manifold X_{d-2} (with appropriate structure) one associates a category $C(X_{d-2})$. Then, for example, to a d-1-manifold X_{d-1} with boundary X_{d-2} , one associates an object $P(X_{d-1})$ in the category $C(X_{d-2})$. (For relatively informal accounts of these matters from different points of view, see [1, 2]; for some recent developments, see [3] as well as [4].)

1.1. Categories And Physics. In practice, physicists do not usually specify what should be associated to X_{d-2} . This is not necessary for most purposes – certainly not in standard applications of QFT to particle physics or condensed matter physics. However, before getting to the main subject of this talk, I will briefly explain a few cases in which that language is or might be useful for physicists.

So far, the most striking physical application of the "third tier," that is the extension of QFT to codimension two, is in string theory, where one uses two-dimensional QFT to describe the propagation of a string. In this case, since d=2, a d-2-manifold is just a point. So the extra layer of structure is just that the theory is endowed with a category \mathcal{C} , which is the category of what physicists call boundary conditions in the quantum field theory, or D-branes.

For d=2, a connected d-1-manifold with boundary is simply a closed interval I, whose boundary consists of two points. To define a space $\mathcal{H}(I)$ of physical states of the open string, one needs boundary conditions \mathcal{B} and \mathcal{B}' at the two ends of I. To emphasize the dependence on the boundary conditions, the space of physical states is better denoted as $\mathcal{H}(I;\mathcal{B},\mathcal{B}')$. In category language, this space of physical states is called the space of morphisms in the category, $\mathrm{Hom}_{\mathcal{C}}(\mathcal{B},\mathcal{B}')$. (This construction has two variants that differ by whether the manifolds considered are oriented; they are both relevant to string theory.)

Another case in which the third tier can be usefully invoked, in practice, is three-dimensional Chern-Simons gauge theory. This is a quantum field theory for d=3 with a compact gauge group G and a Lagrangian that is, roughly speaking,² an integer k times the Chern-Simons functional. A closed d-2-manifold is now a circle, and again, the extra layer of structure is that a category $\mathcal C$ is associated to the theory; it is the category of positive energy representations of the loop group of G at level k.

Finally, the state of the Universe in the presence of a black hole or a cosmological horizon is sometimes described in terms of a density matrix rather than an ordinary quantum state, to account for one's ignorance of what lies beyond the horizon. This point of view (which notably has been advocated by Stephen Hawking) can possibly be usefully reformulated or refined in terms of categories. The idea here would be that, in d-dimensional spacetime, the horizon of a black hole (or a cosmological horizon) is a closed d-2-manifold. Indeed, suppose that X_d is a d-dimensional Lorentz signature spacetime with an "initial time" hypersurface X_{d-1} . Suppose further that a black hole is present; its horizon intersects X_{d-1} on a codimension two submanifold X_{d-2} . It is plausible that to X_{d-2} , we should associate a category \mathcal{C} , and then to X_{d-1} we would associate not – as we would in the absence of the black hole – a physical Hilbert space $\mathcal{H}(X_{d-1})$ – but rather an object \mathcal{P} in that category.

²This formulation suffices if G is simple, connected, and simply-connected. In general, k is an element of $H^4(BG,\mathbb{Z})$.

To make this more concrete, suppose for example that \mathcal{C} is the category of representations of an algebra \mathcal{S} . Then \mathcal{P} is an \mathcal{S} -module, which in this context would mean a Hilbert space $\mathcal{H}(X_{d-1}; X_{d-2})$ with an action of \mathcal{S} . Physical operators would be operators on this Hilbert space that commute with \mathcal{S} . Intuitively, \mathcal{S} is generated by operators that act behind the horizon of the black hole. (That cannot be a precise description in quantum gravity, where the position of the horizon can fluctuate.) This point of view is most interesting if the algebra \mathcal{S} is not of Type I, so that it does not have irreducible modules and the category of \mathcal{S} -modules is not equivalent to the category of vector spaces. At any rate, even if the categorical language is relevant to quantum black holes, it may be oversimplified to suppose that \mathcal{C} is the category of representations of some algebra.

1.2. Geometric Langlands. Our aim here, however, is to understand not black holes but the geometric Langlands correspondence. In this subject, one studies a Riemann surface C, but the basic statements that one makes are about categories associated to C. Indeed, the basic statement is that two categories associated to C are equivalent to each other.

For G a simple complex Lie group, let $Y_G(C) = \text{Hom}(\pi_1(C), G)$ be the moduli stack of flat G-bundles over C. And let $Z_G(C)$ be the moduli stack of holomorphic G-bundles over C.

To the group G, we associate its Langlands [5] or GNO [6] dual group G^{\vee} . (The root lattice of G is the coroot lattice of G^{\vee} , and vice-versa.) Then the basic assertion of the geometric Langlands correspondence [7] is that the category of coherent sheaves on $Y_{G^{\vee}}(C)$ is naturally equivalent to the category of \mathcal{D} -modules on $Z_G(C)$.

If we are going to interpret this statement in the context of quantum field theory, we should start with a theory in dimension d=4, so that it will associate a category to a manifold of dimension d-2=2, in this case the two-manifold C. We need then an equivalence between a quantum field theory defined using G and a quantum field theory defined using G^{\vee} , both in four dimensions. In fact, there is a completely canonical theory with the right properties. It is the maximally supersymmetric Yang-Mills theory in four dimensions.

This theory, which has $\mathcal{N}=4$ supersymmetry, depends on the choice of a compact³ gauge group G. It also depends on the choice of a complex-valued quadratic form on the Lie algebra \mathfrak{g} of G; the imaginary part of this quadratic form is required to be positive definite. If G is simple, then Lie theory lets us define a natural invariant quadratic form on \mathfrak{g} (short coroots have length squared 2), and any such form is a complex multiple of this one. We write the multiple as

(1.1)
$$\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{e^2},$$

where e and θ (known as the gauge coupling constant and theta-angle) are real. We call τ the coupling parameter.

The classic statement (which evolved from early ideas of Montonen and Olive [8]) is that $\mathcal{N}=4$ super Yang-Mills theory with gauge group G and coupling

³In the formulation via gauge theory, we begin with a compact gauge group, whose complexification then naturally appears by the time one makes contact with the usual statements about geometric Langlands. Geometric Langlands is usually described in terms of this complexification.

parameter τ is equivalent to the same theory with dual gauge group G^{\vee} and coupling parameter

(Here $n_{\mathfrak{g}}$ is the ratio of length squared of long and short roots of G or G^{\vee} .) The equivalence between the two theories exchanges electric and magnetic fields, in a suitable sense, and is known as electric-magnetic duality. There are also equivalences under

that can be seen semiclassically (as a reflection of the fact that the instanton number of a classical gauge field is integer-valued). The non-classical equivalence (1.2) combines with the semiclassical equivalences (1.3) to an infinite discrete structure. For instance, if G is simply-laced, then $n_{\mathfrak{g}} = 1$, G and G^{\vee} have the same Lie algebra, for many purposes one can ignore the distinction between τ and τ^{\vee} , and the symmetries (1.2) and (1.3) generate an action of the infinite discrete group $SL(2,\mathbb{Z})$ on τ .

There is a "twisting procedure" to construct topological quantum field theories (TQFT's) from physical ones. Applied to $\mathcal{N}=2$ super Yang-Mills theory, this procedure leads to Donaldson theory of smooth four-manifolds. Applied to $\mathcal{N}=4$ super Yang-Mills theory, the twisting procedure leads to three possible constructions. Two of these are quite similar to Donaldson theory in their content, while the third is related to geometric Langlands [9].

The equivalence between this third twisting for the two groups G and G^{\vee} (and with an inversion of the coupling parameter) leads precisely at the level of "categories," that is for two-manifolds, to the geometric Langlands correspondence. (The underlying electric-magnetic duality treats G and G^{\vee} symmetrically. But the twisting depends on a complex parameter; the choice of this parameter breaks the symmetry between G and G^{\vee} . That is why the usual statement of the geometric Langlands correspondence treats G and G^{\vee} asymmetrically.)

So this is the basic reason that geometric Langlands duality, most commonly understood as a statement about Riemann surfaces, arises from a quantum field theory in *four* dimensions.

REMARK 1.1. For another explanation of why four dimensions is a natural starting point for geometric Langlands, see [10]. This explanation uses the fact that the mathematical theory as usually developed is based on moduli stacks rather than moduli spaces; but a two-dimensional sigma model whose target is the moduli stack of bundles is best understood as a four-dimensional gauge theory. This relies on the gauge theory interpretation of the moduli stack, introduced in a well-known paper by Atiyah and Bott [11].

2. Defects Of Various Dimension

In the title of this talk, I promised to get up to six dimensions, not just four. Eventually we will, but first we will survey the role of structures of different dimension in a four-manifold.

Suppose that a quantum field theory on a manifold M is defined by some sort of path integral, schematically

(2.1)
$$\int DA \dots \exp\left(-\int_{M} L\right),$$

where L is a Lagrangian density that depends on some fields A (and perhaps on additional fields that are not written). "Inserting a local operator $\mathcal{O}(p)$ at a point $p \in M$ " means modifying the path integral at that point. This may be done by including a factor in the path integral that depends on the fields and their derivatives only at p. It may also be done in some more exotic way, such as by prescribing a singularity that the fields should have near p.

In addition to local operators, we can also consider modifications of the theory that are supported on a p-dimensional submanifold $N \subset M$. We give some examples shortly. A local operator is the case p = 0. The general case we call a p-manifold operator.

In much of physics, the important operators are local operators. This is also the case in Donaldson theory. The local operators that are important in Donaldson theory are related to characteristic classes of the universal bundle.

I should point out that geometrically, a local operator may be a tensor field of some sort on M; it may be, for example, a q-form for some q. If \mathcal{O}_q is a local operator valued in q-forms, we can integrate it over a q-cycle $W_q \subset M$ to get $\int_{W_q} \mathcal{O}_q$. The most important operators in Donaldson theory are of this kind, with q=2. For our purposes, we need not distinguish a local operator from such an integral of one. (What we call a p-manifold operator cannot be expressed as an integral of q-manifold operators with q < p.)

Local operators also play a role in geometric Langlands. Indeed, a construction analogous to that of Donaldson is relevant. Imitating the construction of Donaldson theory and then applying electric-magnetic duality, one arrives at results, many of which are known in the mathematical literature, comparing group theory of G to cohomology of certain orbits in the affine Grassmannian of G^{\vee} .

But local operators are not the whole story. In gauge theory, for example, given an oriented circle $S \subset M$, and a representation R of G, we can form the trace of the holonomy of the connection A around S in the given representation. Physicists denote this as

(2.2)
$$W_R(S) = \operatorname{Tr}_R P \exp\left(-\oint_S A\right).$$

When included as a factor in a quantum path integral, $W_R(S)$ is known as a Wilson operator. Wilson operators were introduced over thirty years ago in formulating a criterion for quark confinement in the theory of the strong interactions.

 $W_R(S)$ cannot be expressed as the integral over S of a local operator. We call it a one-manifold operator.

Electric-magnetic duality inevitably converts $W_R(S)$ to another one-manifold operator, which was described by 't Hooft in the late 1970's. The 't Hooft operator is defined by prescribing a singularity that the fields should have along S. (See [9] for a review. Operators defined in this way are often called disorder operators, while operators like the Wilson operator that are defined by interpreting a classical expression in quantum mechanics are called order operators.) The possible singularities in G gauge theory are in natural correspondence with representations

 R^{\vee} of the dual group G^{\vee} . Electric-magnetic duality maps a Wilson operator in G^{\vee} gauge theory associated with a representation R^{\vee} to an 't Hooft operator in G gauge theory that is also associated with R^{\vee} .

If one specializes to the situation usually studied in the geometric Langlands correspondence, the 't Hooft operators correspond to the usual geometric Hecke operators of that subject. The electric-magnetic duality between Wilson and 't Hooft operators leads to the usual statement that a coherent sheaf on $Y_{G^{\vee}}(C)$ that is supported at a point is dual to a Hecke eigensheaf on $Z_G(C)$. (Saying that a \mathcal{D} -module on $Z_G(C)$ is a Hecke eigensheaf is the geometric analog of saying that a classical modular form is a Hecke eigenform.)

Moving up the chain, the next step is a two-manifold operator. In general, in d-dimensional gauge theory, one can define a d-2-manifold operator as follows. One omits from M a codimension two submanifold L. Then, fixing a conjugacy class in G, one considers gauge fields on $M \setminus L$ with holonomy around L in the prescribed conjugacy class.

For d=4, we have d-2=2, so L is a two-manifold. Classical gauge theory in the presence of a singularity of this kind has been studied in the context of Donaldson theory by Kronheimer and Mrowka. In geometric Langlands, to get a class of two-manifold operators that is invariant under electric-magnetic duality, one must incorporate certain quantum parameters in addition to the holonomy [12]. Once one does this, one gets a natural quantum field theory framework for understanding "ramification," i.e. the geometric Langlands analog of ramification in number theory.

The next case are operators supported on a three-manifold $W \subset M$. With M being of dimension four, W is of codimension one and locally divides M into two pieces. The theory of such three-manifold operators is extremely rich and [13, 14] there are many interesting constructions, even if one requires that they should preserve the maximum possible amount of supersymmetry (half of the supersymmetry).

For example, the gauge group can jump in crossing W. We may have G gauge theory one side and H gauge theory on the other. If H is a subgroup of G, a construction is possible that is related to what Langlands calls functoriality. Other universal constructions of geometric Langlands – including the universal kernel that implements the duality – are similarly related to supersymmetric three-manifold operators.

As long as we are in four dimensions, this is the end of the road for modifying a theory on a submanifold. A modification in four dimensions would just mean studying a different theory. So to continue the lecture, we will, as promised in the title, try to relate geometric Langlands to a phenomenon above four dimensions.

3. Selfdual Gerbe Theory In Six Dimensions

Until relatively recently, it was believed that four was the maximum dimension for nontrivial (nonlinear or non-Gaussian) quantum field theory. One of the surprising developments coming from string theory is that nontrivial quantum field theories exist up to (at least) six dimensions.

To set the stage, I will begin by sketching a linear, but subtle, quantum field theory in six dimensions. The nonlinear case is discussed in section 4.

In six dimensions, with Lorentz signature -+++++, a real three-form H can be selfdual, obeying $H = \star H$, where \star is the Hodge star operator.⁴ Let us consider such an H and endow it with a hyperbolic equation of motion

$$dH = 0.$$

That equation is analogous to the Bianchi identity $\mathrm{d}F=0$ for the curvature twoform F of a line bundle. It means that (in a mathematical language that physicists generally do not use) H can be interpreted as the curvature of a U(1) gerbe with connection.

In contrast to gauge theory, there is no way to derive this system from an action. The natural candidate for an action, on a six-manifold M_6 , would seem to be $\int_{M_6} H \wedge \star H$, but if H is self-dual this is the same as $\int_{M_6} H \wedge H = 0$.

Nevertheless, there is a quantum field theory of the closed, selfdual H field. To explain how one part of the structure of quantum field theory emerges, suppose that the Lorentz signature six-manifold M_6 admits a global Cauchy hypersurface M_5 . M_5 is thus a five-dimensional Riemannian manifold. Fixing the topological type of a U(1) gerbe in a neighborhood of M_5 , the space of gerbe connections with selfdual curvature, modulo gauge transformations, is an (infinite-dimensional) symplectic manifold in a natural way. (Roughly speaking, if B is the gerbe connection, then the symplectic form is defined by the formula $\omega = \int_{M_5} \delta B \wedge \mathrm{d}\delta B$.) Quantizing this space, we get a Hilbert space associated to M_5 . This association of a Hilbert space to a five-manifold is part of the usual data of a six-dimensional quantum field theory. The rest of the structure can also be found, with some effort. (For a little more detail, see [15, 16, 17].)

An important fact is that the quantum field theory of the H field is conformally invariant. Classically, the equations $H = \star H$, $\mathrm{d}H = 0$, are conformally invariant. The passage to quantum mechanics preserves this property, because the theory is linear.

Now let us consider the special case that our six-manifold⁵ takes the form $M_6 = M_4 \times T^2$, where M_4 is a four-manifold and T^2 is a two-torus. We assume a product conformal structure on $M_4 \times T^2$. After making a conformal rescaling to put the metric on T^2 in a standard form (say a flat metric of unit area), we are left with a Riemannian metric on M^4 . The conformal structure of T^2 is determined by the choice of a point τ in the upper half of the complex plane – modulo the action of $SL(2,\mathbb{Z})$.

Next in $M_4 \times T^2$, let us keep fixed the second factor, with a definite metric, and let the first factor vary. We let M_4 be an arbitrary four-manifold with boundaries, corners, etc. Starting with a conformal field theory on M_6 , this process gives us a four-dimensional quantum field theory (not conformally invariant) that depends on τ as a parameter. Clearly, the induced four-dimensional theory depends on the conformal structure of T^2 only up to isomorphism. So if we parametrize the

 $^{^4}$ The quantum theory of a real selfdual threeform in six dimensions can be analytically continued to Euclidean signature, whereupon H is still selfdual but is no longer real. Such a continuation will be made later. In general, analytic continuation from Lorentz to Euclidean signature and back is an important tool in quantum field theory; the basic reason that it is possible is that in Lorentz signature the energy is non-negative.

⁵Henceforth, and until section 5.3, we generally work in Euclidean signature, using the analytic continuation mentioned in footnote 4.

induced four-dimensional theory by τ , we will have a symmetry under the action of $SL(2,\mathbb{Z})$ on τ .

The induced four-dimensional quantum field theory is actually closely related to U(1) gauge theory, which is its "infrared limit." Let us think of T^2 as \mathbb{C}/Λ , where \mathbb{C} is the complex plane parametrized by z=x+iy and Λ is the lattice generated by complex numbers 1 and τ . Further, make an ansatz

$$(3.2) H = F \wedge dx + \star F \wedge dy,$$

where F is a two-form on M_4 (pulled back to $M_6 = M_4 \times T^2$), and \star is the four-dimensional Hodge star operator. Then the equations dH = 0 become Maxwell's equations

$$dF = d \star F = 0.$$

This gives an embedding of four-dimensional U(1) gauge theory in the sixdimensional theory. To be more precise, we should think of H as the curvature of a U(1) gerbe connection; then F is the curvature of a U(1) connection. Of course, we have described the embedding classically, but it also works quantum mechanically.

This construction is more than an embedding of four-dimensional U(1) gauge theory in a six-dimensional theory. The four-dimensional U(1) gauge theory is the infrared limit of the six-dimensional theory in the following sense. We have endowed M_6 with a product metric g_6 that we can write schematically as $g_6 = g_4 \oplus g_2$, where g_4 and g_2 are metrics on M_4 and T^2 , respectively. Now we modify g_6 to $g_6(t) = t^2g_4 \oplus g_2$, where t is a real parameter. The claim is that for $t \to \infty$, the theory on M_6 converges to U(1) gauge theory on M_4 . (This theory is conformally invariant, so the t^2 factor in the metric of M_4 can be dropped.) This is usually described more briefly by saying that U(1) gauge theory on M_4 is the long distance or infrared limit of the underlying theory on M_6 .

Even though U(1) gauge theory on M_4 gives an effective and useful description of the large t limit of the six-dimensional theory on M_6 , something is obscured in this description. The process of compactifying on T^2 and taking the large t limit is canonical in that it depends only on the geometry of T^2 and not on a choice of coordinates. But to go to a description by U(1) gauge theory, we used the ansatz (3.2), which depended on a choice of coordinates x and y. As a result, some of the underlying symmetry is hidden in the description by U(1) gauge theory.

Concretely, though the six-dimensional theory does not have a Lagrangian, the four-dimensional U(1) gauge theory does have one:

(3.4)
$$I = \frac{1}{4e^2} \int_{M_A} F \wedge \star F + \frac{\theta}{8\pi^2} \int F \wedge F.$$

The coupling parameter

(3.5)
$$\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{e^2}.$$

of the abelian gauge theory is simply the τ -parameter of the T^2 in the underlying six-dimensional description.

The six-dimensional theory depends on τ only modulo the usual $SL(2,\mathbb{Z})$ equivalence $\tau \to (a\tau + b)/(c\tau + d)$, with integers a, b, c, d obeying ad - bc = 1, since values of τ that differ by the action of $SL(2,\mathbb{Z})$ correspond to equivalent tori. Therefore, the limiting four-dimensional U(1) gauge theory must also have $SL(2,\mathbb{Z})$ symmetry. However, there is no such classical symmetry. Manifest $SL(2,\mathbb{Z})$ symmetry was lost

in the reduction from six to four dimensions, because the ansatz (3.2), which was the key step in reducing to four dimensions, is not $SL(2,\mathbb{Z})$ -invariant. Hence this ansatz leads to a four-dimensional theory with a "hidden" $SL(2,\mathbb{Z})$ symmetry, one which relates the description by a U(1) gauge field with curvature F to a different description by a different U(1) gauge field with another curvature form (which, roughly speaking, is related to F by the action of $SL(2,\mathbb{Z})$).

What we get this way is an $SL(2,\mathbb{Z})$ symmetry of quantum U(1) gauge theory that does not arise from a symmetry of the classical theory. To physicists, this symmetry is known as electric-magnetic duality. The name is motivated by the fact that an exchange $(x,y) \to (y,-x)$ in (3.2), which is a special case of $SL(2,\mathbb{Z})$, would exchange F and $\star F$, and thus in nonrelativistic terminology would exchange electric and magnetic fields.

So we have seen that electric-magnetic duality in U(1) gauge theory in four dimensions follows from the existence of a suitable conformal field theory in six dimensions [18]. The starting point in this particularly nice explanation is the existence in six dimensions of a quantum theory of a gerbe with selfdual curvature. (It is also possible to demonstrate the four-dimensional duality by a direct calculation, involving a sort of Fourier transform in field space; see [19].)

4. The Nonabelian Case

Since there is not a good notion classically of a gerbe whose structure group is a simple nonabelian Lie group, one might think that it is too optimistic to look for an analogous explanation of electric-magnetic duality for nonabelian groups. However, it turns out that such an explanation does exist – in the maximally supersymmetric case.

The picture is simplest to describe if G is simply-laced, in which case G and G^{\vee} have the same Lie algebra (and to begin with, we will ignore the difference between them, though this is precisely correct only if $G = E_8$; a more complete picture can be found in section 4.1). For G to be simply-laced is equivalent to the condition that $n_{\mathfrak{g}} = 1$ in eqn. (1.2). For many purposes, we can ignore the difference between τ and τ^{\vee} , and then the quantum duality (1.2) and the semiclassical equivalence (1.3) combine to an action of $SL(2,\mathbb{Z})$ on τ .

For every simply-laced Lie group G, there is a six-dimensional conformal field theory that in some sense is associated with gerbes of type G. The theory is highly supersymmetric, so supersymmetry is essential in what follows. The existence of this theory was discovered in string theory in the mid-1990's. (The first hint [20] came by considering Type IIB superstring theory at an ADE singularity.) Its existence is probably our best explanation of electric-magnetic duality – and therefore, in particular, of geometric Langlands duality. It is, in the jargon of quantum field theory, an isolated, non-Gaussian conformal field theory. This means among other things that it cannot be properly described in terms of classical notions such as partial differential equations.

However, it has two basic properties which in a sense justify thinking of it as a quantum theory of nonabelian gerbes. Each property involves a perturbation of some kind that causes a simplification to a theory that *can* be given a classical description. The two perturbations are as follows:

(1) After a perturbation in the vacuum expectation values of certain fields (which are analogous to the conjectured Higgs field of particle physics), the theory

reduces at low energies to a theory of gerbes, with selfdual curvature, and structure group the maximal torus T of G. This notion does make sense classically, since T is abelian. In fact, the selfdual gerbe theory of T is much like the U(1) theory described in Section 3, with U(1) replaced by T. (Supersymmetry plays a fairly minor role in the abelian case.)

(2) Let $M_6 = M_5 \times S^1$ be the product of a five-manifold M_5 with a circle; we endow it with a product metric $g_6 = g_5 \oplus g_1$. The six-dimensional theory on M_6 has a description (valid at long wavelengths) in terms of G gauge fields (and other fields related to them by supersymmetry) on M_5 , but this description involves a highly nonclassical trick. If the circle factor of $M_6 = M_5 \times S^1$ has radius R, then the effective action for the gauge fields in five dimensions is *inversely* proportional to R:

$$I_5 = \frac{1}{8\pi R} \int_{M_5} \operatorname{Tr} F \wedge \star F.$$

The factor of R^{-1} multiplying the action is a simple consequence of conformal invariance in six dimensions. (Under multiplication of the metric of M_6 by a positive constant t^2 , the Hodge operator \star mapping two-forms to three-forms in five dimensions is multiplied by t, while R is also multiplied by t, so the action in (4.1) is invariant.) Though easily understood, this result is highly nonclassical. Eqn. (4.1) is a classical Lagrangian for gauge fields in five dimensions. Can it arise from a classical Lagrangian for gauge fields on $M_6 = M_5 \times S^1$? Given a six-dimensional Lagrangian for gauge fields, we would reduce to a five-dimensional Lagrangian (for fields that are pulled back from M_5) by integrating over the fibers of the projection $M_5 \times S^1 \to M_5$. This would give a factor of R multiplying the five-dimensional action, not R^{-1} . So a theory that leads to the effective action (4.1) cannot arise in this way. The theory in six dimensions should be, in some sense, not a gauge theory but a gerbe theory instead, but this does not exist classically in the nonabelian case.

What I have said so far is that the same six-dimensional quantum field theory can be simplified to either (i) a six-dimensional theory of abelian gerbes, or (ii) a five-dimensional theory with a simple non-abelian gauge group. The two statements together show that one cannot do justice to this theory in terms of either gauge fields (as opposed to gerbes) or abelian groups (as opposed to non-abelian ones).

Now let us look more closely at the implications of the peculiar factor of 1/R in (4.1). We will study what happens for $M_6 = M_4 \times T^2$, the same decomposition that we used in studying the abelian gerbe theory in section 3. However, for simplicity we will take $T^2 = S^1 \times \widetilde{S}^1$ to be the orthogonal product of a circle S^1 of radius S^1 and a second circle S^1 of radius S^1 . The tau parameter of such a torus (which is made by identifying the sides of a rectangle of height and width $2\pi R$ and $2\pi S$) is

(4.2)
$$\tau = i\frac{S}{R} \text{ or } \tau = i\frac{R}{S},$$

depending on how one identifies the rectangle with a standard one. The two values of τ differ by

$$\tau \to -\frac{1}{\tau}.$$

We first view the six-manifold M_6 as $M_6 = M_5 \times S^1$, where $M_5 = M_4 \times \tilde{S}^1$. The six-dimensional theory on M_6 reduces at long distances to a supersymmetric gauge theory on M_5 . According to (4.1), the action for the gauge fields is

(4.4)
$$I_5 = \frac{1}{8\pi R} \int_{M_4 \times \widetilde{S}^1} \operatorname{Tr} F \wedge \star F.$$

Now if M_4 is much larger than \widetilde{S}^1 , then at long distances we can assume that the fields are invariant under rotation of \widetilde{S}^1 and we can deduce an effective action in four dimensions by integration over the fiber of the projection $M_4 \times \widetilde{S}^1 \to M_4$. This second step is purely classical, so it gives a factor of S. The effective action in four dimensions is thus

$$(4.5) I_4 = \frac{S}{8\pi R} \int_{M_4} \operatorname{Tr} F \wedge \star F.$$

The important point is that this formula is not symmetric in S and R, even though they enter symmetrically in the starting point $M_6 = M_4 \times S^1 \times \widetilde{S}^1$. Had we exchanged the two circles before beginning this procedure, we would have arrived at the same formula for the four-dimensional effective action, but with S/R replaced by R/S.

Looking back to (4.2) and (4.3), we see that the two formulas differ by $\tau \to -1/\tau$. Thus, we have deduced⁶ that for simply-laced G, the four-dimensional gauge theory that corresponds to the maximally supersymmetric completion of (4.5) has a quantum symmetry that acts on the coupling parameter by $\tau \to -1/\tau$. This is the electric-magnetic duality that has many applications in physics and also underlies geometric Langlands duality. What we have gained is a better understanding of why it is true in the nonabelian case.

REMARK 4.1. Unfortunately, despite its importance, there is no illuminating and widely used name for the six-dimensional QFT whose existence underlies duality in this way. According to Nahm's theorem [21], the superconformal symmetry group of a superconformal field theory in six dimensions, when formulated in Minkowski spacetime, is OSp(2,6|2r) for some r. The known examples have r=1 or 2, and the theory with the properties that I have just described is the "maximally symmetric" one with r=2. This theory is rather inelegantly called the six-dimensional (0,2) model of type G, where 2 is the value of r (and the redundant-looking number 0 involves a comparison to six-dimensional models that are supersymmetric but not conformal).

REMARK 4.2. The bosonic subgroup of OSp(2,6|2r) is $SO(2,6) \times Sp(2r)$, where SO(2,6) is the conformal group in six dimensions, and Sp(2r) is an "internal symmetry group" (it acts trivially on spacetime) and is known as the R-symmetry group. Thus, the R-symmetry group of the (0,2) model is Sp(4). This is the group (sometimes called Sp(2)) of 2×2 unitary matrices of quaternions; its fundamental representation is of quaternionic dimension 2, complex dimension 4, or real dimension 8. Sp(4) is also the group that acts on the cohomology of a hyper-Kahler manifold; this is no coincidence, as we will see later.

4.1. The Space Of Conformal Blocks. Among the simple Lie groups, only E_8 is simply-connected and has a trivial center. Equivalently, its root lattice Γ endowed with the usual quadratic form is unimodular, that is, equal to its dual

⁶To keep the derivation simple, we considered only a rectangular torus. It is possible by using eqn. (5.2) to similarly analyze the case of a general torus.

 Γ^{\vee} . In general, if G is simple, simply-laced, and simply-connected, its center is $\mathcal{Z} = \Gamma^{\vee}/\Gamma$, and the quadratic form on Γ leads to a perfect pairing

$$(4.6) \mathcal{Z} \times \mathcal{Z} \to \mathbb{R}/\mathbb{Z} = U(1).$$

(The pairing actually takes values in the subgroup \mathbb{Z}_n of U(1), where n is the smallest integer that annihilates \mathcal{Z} .)

What has been said so far is sufficient for E_8 , but more generally, a refinement is necessary. (Most of this article does not depend on the following details.) For orientation, consider two-dimensional current algebra (that is, the holomorphic part of the WZW model⁷) of the simply-connected and simply-laced group G at level 1. This theory, formulated on a closed Riemann surface W, does not have a unique partition function (which is required in the usual axioms of quantum field theory, as indicated in the introduction to this article). Rather, it has a vector space of possible partition functions, known as the space of conformal blocks. This vector space (in the particular case of a simply-laced group G at level 1) can be constructed as follows. The pairing (4.6) together with the intersection pairing on the cohomology of W leads to a perfect pairing

$$(4.7) H1(W, Z) \times H1(W, Z) \to U(1).$$

This pairing enables us to define a Heisenberg group extension

$$(4.8) 1 \to U(1) \to F \to H^1(W, \mathcal{Z}) \to 0.$$

Up to isomorphism, the group F has a single faithful irreducible representation \mathcal{R} in which U(1) acts in the natural way; it is obtained by "quantizing" the finite group $H^1(W, \mathcal{Z})$. One picks a decomposition of $H^1(W, \mathcal{Z})$ as $A \times B$, where A and B (which can be constructed using a system of A-cycles and B-cycles on W) are maximal subgroups on which the extension (4.8) is trivial. One then lets B act by multiplication – in the sense that \mathcal{R} is the direct sum of all one-dimensional characters of B. Since (4.7) restricts to a perfect pairing $A \times B \to U(1)$, characters of B correspond to elements of A. Thus, \mathcal{R} has a unitary basis consisting of elements ψ_a , $a \in A$; the action of A is $a(\psi_{a'}) = \psi_{aa'}$, while B acts by $b\psi_a = \exp(2\pi i(b,a))\psi_a$ (where $\exp(2\pi i(b,a))$) denotes the pairing between A and B). The dimension of \mathcal{R} is thus $(\#\mathcal{Z})^g$, where g is the genus of W and $\#\mathcal{Z}$ is the order of \mathcal{Z} .

The space of conformal blocks of the level 1 holomorphic WZW model on a Riemann surface W with a simple and simply-laced symmetry group G is isomorphic to \mathcal{R} . Thus, for $G \neq E_8$, the space of conformal blocks has dimension bigger than 1. That means that this theory does not have a distinguished partition function and so does not quite obey the full axioms of quantum field theory. One may either relax the axioms slightly, study the ordinary (non-holomorphic) WZW model, or in some other way include holomorphic or non-holomorphic degrees of freedom so as to be able to define a distinguished partition function.

The situation in the six-dimensional (0,2) theory is similar, with the finite group $H^3(M_6, \mathbb{Z})$ playing the role of $H^1(W, \mathbb{Z})$ in two dimensions. From (4.6) and

⁷ There is no satisfactory terminology in general use. The WZW model is really [22] a two-dimensional quantum field theory that is modular-invariant but neither holomorphic nor antiholomorphic. Its holomorphic part corresponds to what physicists know as two-dimensional current algebra (which is a much older construction than the WZW model). But the phrase "two-dimensional current algebra" is not well-known to mathematicians, and may even be unclear nowadays to physicists.

Poincaré duality, we again have a perfect pairing $H^3(M_6, \mathbb{Z}) \times H^3(M_6, \mathbb{Z}) \to U(1)$, leading to a Heisenberg group extension

(4.9)
$$1 \to U(1) \to F \to H^3(M_6, \mathbb{Z}) \to 0.$$

Again, up to isomorphism, F has a unique faithful irreducible module \mathcal{T} with natural action of U(1). The theory on a general six-manifold has a space of conformal blocks that is isomorphic to \mathcal{T} . For G a simple and simply-laced Lie group that is not of type E_8 , this again represents a slight departure from the usual axioms of quantum field theory. Our options are analogous to what they were in the two-dimensional case: live with it (which will be our choice in the present paper) or consider various more elaborate constructions in which one can avoid the problem.

Now let us consider an illuminating example. We take $M_6 = M_5 \times S^1$. We have a decomposition $H^3(M_6, \mathbb{Z}) = H^2(M_5, \mathbb{Z}) \oplus H^3(M_5, \mathbb{Z})$. Calling the summands A and B, we can as above construct the space T of conformal blocks as the direct sum of characters of B. Hence, as in the two-dimensional case, T has a basis consisting of elements ψ_a , $a \in A = H^2(M_5, \mathbb{Z})$.

On the other hand, the (0,2) model on $M_6 = M_5 \times S^1$ is supposed to be related to gauge theory on M_5 . So in gauge theory on M_5 , we should find a way to define a partition function for every $a \in H^2(M_5, \mathbb{Z})$. This is easily done once one appreciates that one should use the adjoint form of the group, which we will call $G_{\rm ad}$. A $G_{\rm ad}$ bundle over any space X has a characteristic class $a \in H^2(X, \mathbb{Z})$ (where \mathbb{Z} is the center of the simply-connected group G or equivalently the fundamental group of $G_{\rm ad}$). In $G_{\rm ad}$ gauge theory on M_5 , we define for every $a \in H^2(M_5, \mathbb{Z})$ a corresponding partition function Z_a by summing the path integral of the theory over all bundles whose characteristic class equals a.

In defining the Z_a , we are relaxing the usual axioms of quantum field theory a little bit. If the gauge group is supposed to be G, the characteristic class must vanish and the partition function is essentially Z_0 . (I will omit some elementary factors involving the order of \mathcal{Z} .) If the gauge group is supposed to be $G_{\rm ad}$, all values of the characteristic class are allowed and the partition function is $\sum_a Z_a$. For groups intermediate between G and $G_{\rm ad}$, certain formulas intermediate between those two will arise. But for no choice of gauge group is the partition function precisely Z_a , for some fixed and nonzero a. Clearly, on the other hand, it is natural to permit ourselves to study these functions. So this is a situation in which we probably want to be willing to slightly generalize the usual axioms of quantum field theory.

Now as before let us consider the case $M_5 = M_4 \times \widetilde{S}^1$, where \widetilde{S}^1 is another circle, so that $M_6 = M_4 \times S^1 \times \widetilde{S}^1$ can be viewed in more than one way as the product of a circle and a five-manifold. For simplicity, let us assume that $H^1(M_4, \mathbb{Z}) = H^3(M_4, \mathbb{Z}) = 0$. Then $H^3(M_6, \mathbb{Z}) = A \oplus B$, where

(4.10)
$$A = H^2(M_4, \mathbb{Z}) \otimes H^1(S^1, \mathbb{Z}), B = H^2(M_4, \mathbb{Z}) \otimes H^1(\widetilde{S}^1, \mathbb{Z}).$$

The extension is trivial on both A and B. Reasoning as above, the space \mathcal{T} of conformal blocks has a basis ψ_a , $a \in A$. On the other hand, exchanging the roles of A and B, it has a second basis $\widetilde{\psi}_b$, $b \in B$. As is usual in quantization, the relation between these two bases (which are analogous to "position space" and "momentum space") is given by a Fourier transform. In the present case, both A and B can be

identified with $H^2(M_4, \mathbb{Z})$ and the Fourier transform is a finite sum:

(4.11)
$$\widetilde{\psi}_b = C \sum_{a \in H^2(M_4, \mathbb{Z})} \exp(2\pi i (a, b)) \psi_a.$$

Here C is a constant and we write $\exp(2\pi i(a,b))$ for the perfect pairing $H^2(M_4, \mathbb{Z}) \times H^2(M_4, \mathbb{Z}) \to U(1)$.

Let us interpret this formula in four-dimensional gauge theory. In G_{ad} gauge theory on M_4 , we can as before define a partition function Z_a by summing over bundles with a fixed characteristic class $a \in H^2(M_4, \mathbb{Z})$. Identifying these with the ψ_a , we find that under electric-magnetic duality the Z_a must transform by

(4.12)
$$Z_b(-1/\tau) = C \sum_a \exp(2\pi i(b, a)) Z_a(\tau).$$

We have incorporated the fact that (because it exchanges the last two factors in $M_6 = M_4 \times S^1 \times \widetilde{S}^1$) electric-magnetic duality inverts τ , in addition to its action on the label a. This formula was first obtained in purely four-dimensional terms in [23], where more detail can be found. Here we have given a six-dimensional context for this result.

If G is a simply-laced and simply-connected Lie group, then its GNO or Langlands dual group G^{\vee} is precisely the adjoint group $G_{\rm ad}$. Apart from elementary constant factors that are considered in [23], the partition function of the theory with gauge group G is Z_0 (since a must vanish if the gauge group is the simply-connected form G), and the partition function of the theory with gauge group $G_{\rm ad}$ is $\sum_a Z_a$ (since all choices of a are equally allowed if the gauge group is the adjoint form). As noted in [23], a special case of (4.12) is that Z_0 transforms under $\tau \to -1/\tau$ into a constant multiple of $\sum_a Z_a$. This assertion means that in this particular case the G and G^{\vee} theories are dual. Other specializations of (4.12) correspond to duality for forms intermediate between G and $G_{\rm ad}$, but in general (4.12) contains more information than can be extracted from such special cases.

REMARK 4.3. The close analogy between the conformal blocks of the six-dimensional (0,2) model and those of the the level 1 WZW model in two dimensions make one wonder if there might be an analog in six dimensions of the WZW models at higher level. All one can say here is that the usual (0,2) model has appeared in string theory in many ways and as of yet there is no sign of a hypothetical higher level analog.

4.2. What Is Next? In view of what we have said, if we specialize to six-manifolds of the form $M_6 = M_4 \times T^2$, where we keep the two-torus T^2 fixed and let only M_4 vary, the six-dimensional (0,2) theory gives a good framework for understanding geometric Langlands.

We can do other things with this theory, since we are free to consider more general six-manifolds. This will be our topic in Section 5. But perhaps we should first address the following question. Is this the end? Or will physicists come back next year and say that geometric Langlands should be derived from a theory above six dimensions?

There is a precise sense in which six dimensions is the end. It is the maximum dimension for superconformal field theory, according to an old result of Nahm [21]. To get farther, one needs a different kind of theory.

If one wishes to go beyond six dimensions, the next stop is presumably string theory (dimension ten). Indeed, the existence and most of the essential properties of the six-dimensional QFT that underlies four-dimensional electric-magnetic duality are known primarily from the multiple relations this theory has with string theory.

5. Geometric Langlands Duality For Surfaces

5.1. Circle Fibrations. As we have discussed, one of the most basic properties of the six-dimensional (0,2) theory is that when formulated on $M_6 = M_5 \times S^1$, it gives rise at long distances to five-dimensional gauge theory on M_5 .

The simplest generalization⁸ of this is to consider not a product $M_5 \times S^1$, but a fibration over M_5 with S^1 fibers:

$$\begin{array}{ccc} S^1 & \to & M_6 \\ \downarrow & & \downarrow \\ & & M_5 \end{array}$$

(For simplicity, we assume that the fibers are oriented.) In this situation, the long distance limit is still gauge theory on M_5 , with gauge group G. But there is an important modification.

We pick on M_6 a Riemannian metric that is invariant under rotation of the fibers of the U(1) bundle $M_6 \to M_5$. Such a metric determines a connection on this U(1) bundle, and therefore a curvature two-form $f \in \Omega^2(M_5)$. Let A be the gauge field on M_5 (so A is a connection on a G-bundle over M_5), and let $\mathrm{CS}(A)$ be the associated Chern-Simons three-form. (As is customary among physicists, we will normalize this form so that its periods take values in $\mathbb{R}/2\pi\mathbb{Z}$.) Then the twisting of the fibration $M_6 \to M_5$ results in the presence in the long distance effective action of an additional term ΔI that roughly speaking is

(5.2)
$$\Delta I = \frac{i}{2\pi} \int_{M_5} f \wedge \mathrm{CS}(A).$$

To be more precise, one should define $-i\Delta I$ as the integral of a certain Chern-Simons five-form for the group $U(1)\times G$. This Chern-Simons five-form is associated to an invariant cubic form on the Lie algebra of $U(1)\times G$ that is linear on the first factor of this Lie algebra and quadratic on the second. Since ΔI is i times the integral of a Chern-Simons form, ΔI is well-defined and gauge-invariant mod $2\pi i\mathbb{Z}$ assuming that M_5 is a compact manifold with boundary. This ensures that $\exp(-\Delta I)$ is well-defined as a complex number, so that it is possible to include a factor of $\exp(-\Delta I)$ in the integrand of the path integral of five-dimensional supersymmetric gauge theory on M_5 . (Saying that ΔI appears as a term in the effective action means precisely that the integrand of the path integral has such a factor.)

5.2. Allowing Singularities. However, it is natural to relax the conditions that we have imposed so far. Describing M_6 as a U(1) bundle over some base M_5 amounts to exhibiting a free action of the group U(1) on M_6 ; if such an action is given, one simply defines $M_5 = M_6/U(1)$ and then M_6 is a U(1) bundle over M_5 . Clearly, a more general situation is to consider a six-manifold M_6 together with a non-trivial action of the group U(1). After possibly replacing U(1) by a

 $^{^8{\}rm The}$ material in this section was presented in more detail in lectures at the IAS in the spring of 2008. Notes by D. Ben-Zvi can be found at http://www.math.utexas.edu/users/benzvi/GRASP/lectures/IASterm.html.

finite quotient of itself (to eliminate a possible finite subgroup that acts trivially), we can assume that U(1) acts freely on a dense open set in M_6 . The quotient $M_5 = M_6/U(1)$ is a five-manifold possibly with singularities where the U(1) action is non-free.

The above description, with the term (5.2) in the effective action, is applicable away from the non-free locus in M_5 (which consists of the points in M_5 that correspond to non-free orbits in M_6). Along the non-free locus, one should expect the gauge theory description to require some kind of modification. What sort of modification is needed depends on how the U(1) action fails to be free. U(1) may have non-free orbits in codimension 2, 4, or 6, and these non-free orbits may be either fixed points of the whole group, or semi-free orbits whose stabilizer is a finite subgroup of U(1). (To characterize the local behavior, one also needs to specify the action of U(1) in the normal space to the non-free locus. Further, though this will not be important for our purposes, in general one wishes to allow the possibility of a U(1) symmetry that acts via a homomorphism to the R-symmetry group Sp(4), in addition to acting geometrically on M_6 .)

Thus, for a full analysis of this problem, there are many interesting cases to consider, most of which have not been analyzed yet. A simple example is that U(1) may act on M_6 with a fixed point set of codimension 2, in which case M_5 is a manifold with boundary. Thus a natural boundary condition in five-dimensional supersymmetric gauge theory will have to appear.

For our purposes, we will consider just one situation, in which one knows the appropriate modification of the effective field theory that occurs near the exceptional set in M_5 . This is the case of a codimension 4 fixed point locus W such that the action of U(1) on the normal space to W can be modeled by the natural action of U(1) on $\mathbb{C}^2 \cong \mathbb{R}^4$.

Thus, focusing on the normal space to W, we take U(1) to act on \mathbb{C}^2 by $(z_1, z_2) \to (e^{i\theta} z_1, e^{i\theta} z_2)$, for $e^{i\theta} \in U(1)$. Clearly, this gives an action of U(1) on \mathbb{C}^2 that is free except for an isolated fixed point at the origin. Somewhat less obvious – but elementary to prove – is that the quotient $\mathbb{C}^2/U(1)$ is actually a smooth manifold. In fact, it is a copy of \mathbb{R}^3 :

$$(5.3) \mathbb{C}^2/U(1) \cong \mathbb{R}^3.$$

We can get this statement by taking a cone over the Hopf fibration. The Hopf fibration is the U(1) bundle $S^3 \to S^2$. A cone over S^3 is $\mathbb{R}^4 \cong \mathbb{C}^2$, while a cone over S^2 is \mathbb{R}^3 . So, writing 0 for the origin in \mathbb{R}^4 or \mathbb{R}^3 , $\mathbb{R}^4 \setminus \{0\}$ is a U(1) bundle over $\mathbb{R}^3 \setminus \{0\}$. Gluing back in the origin on both sides, we arrive at the assertion (5.3).

It follows from (5.3) that if U(1) acts on M_6 freely except for a codimension 4 fixed point set W as just described, then $M_5 = M_6/U(1)$ is actually a smooth manifold. A few simple facts about the geometry of M_5 deserve attention. One obvious fact is that W is naturally embedded as a codimension 3 submanifold of M_5 . Moreover, it is only away from W that the natural projection $M_6 \to M_5 = M_6/U(1)$ is a U(1) fibration. This projection thus gives a U(1) bundle over $M_5 \setminus W$, which topologically cannot be extended over M_5 . The obstruction to extending the U(1) bundle can be measured as follows. Let S be a small two-sphere in $M_5 \setminus W$ that has linking number 1 with W. (One can construct a suitable S by choosing a normal three-plane S to S at some chosen point S and letting S consist of points in S a distance S from S, for some small S.) Then the S0 bundle over S1 bundle over S2 bundle over S3 bundle over S4.

This fact can be expressed as an equation for the curvature two-form f of the U(1) bundle over $M_5 \setminus W$. As a form on $M_5 \setminus W$, f is closed, obeying df = 0. But f has a singularity along W which can be characterized by the statement

$$(5.4) df = 2\pi \delta_W,$$

where δ_W is the Poincaré dual to W.

5.2.1. Role Of W In The Quantum Theory. In light of this information, let us consider now the (0,2) theory of type G formulated on M_6 , and its reduction to an effective description on M_5 . Away from W, as we have already discussed, the effective theory on M_5 is simply supersymmetric gauge theory with gauge group G, and with the additional interaction (5.2) that reflects the twisting of the fibration $M_6 \to M_5$. The gauge field is a connection on a G-bundle $E \to M_5$.

However, there is an important and very interesting modification along W. This modification results from the fact that the interaction ΔI is not well-defined in the usual sense. We can define a Chern-Simons five-form on $M_5\backslash W$ for the group $U(1)\times G$, but as $M_5\backslash W$ is not compact, the integral of this form is not gauge-invariant, even modulo 2π .

Consequently, $\exp(-\Delta I)$, the corresponding factor in the path integral, is not well-defined as a complex number, but as a section of a certain complex line bundle \mathcal{L} . \mathcal{L} is a line bundle over the space of all G-valued gauge fields, modulo gauge transformations, on W. More exactly, \mathcal{L} is a line bundle over the space of all connections on $E|_W$ modulo gauge transformations ($E|_W$ is simply the restriction to W of the G bundle $E \to M_5$). We write \mathcal{A} for the space of connections on $E|_W$ and \mathcal{G} for the group of gauge transformations; then \mathcal{L} is a line bundle over the quotient \mathcal{A}/\mathcal{G} (or equivalently, a \mathcal{G} -invariant line bundle over \mathcal{A}). In fact, \mathcal{L} is the fundamental line bundle over \mathcal{A}/\mathcal{G} , often loosely called the determinant line bundle. (The motivation for this terminology is that if G = SU(n) or U(n) for some n, then \mathcal{L} can be defined as the determinant line bundle of a $\bar{\partial}$ operator. It can also be defined as the Pfaffian line bundle of a Dirac operator if G = SO(n) or Sp(2n).)

The characterization of \mathcal{L} can be justified as follows. The interaction ΔI as defined in (5.2) does not depend on a choice of gauge for the U(1) bundle $M_6 \setminus W \to M_5 \setminus W$, as it is written explicitly in terms of the U(1) curvature f. On the other hand, under an infinitesimal G gauge transformation $A \to A - \mathrm{d}_A \epsilon$, the Chern-Simons three-form $\mathrm{CS}(A)$ transforms by $\mathrm{CS}(A) \to \mathrm{CS}(A) + \mathrm{d}X_2$, where X_2 is known to physicists as the anomaly two-form (explicitly, $X_2 = (1/4\pi) \operatorname{Tr} \epsilon \mathrm{d}A$). Substituting this gauge transformation law in (5.2), integrating by parts, and using (5.4), we see that under such a gauge transformation, ΔI transforms by

$$\Delta I \to \Delta I - i \int_W X_2,$$

which is equivalent to saying that $\exp(-\Delta I)$ should be understood as a section of the line bundle \mathcal{L} .

Physicists would describe this situation by saying that the factor $\exp(-\Delta I)$ in the path integral has an anomaly under gauge transformations that are non-trivial along W. The anomaly must be canceled by incorporating in the theory another ingredient with an equal and opposite anomaly. This additional ingredient must be supported on W (since away from W we already know what is the right effective field theory). The theory that does the job is the two-dimensional (holomorphic)

WZW model (or in other words, current algebra, as explained in footnote 7) on W, at level 1.

This then is the secret of W: it supports this particular two-dimensional quantum field theory. This is the main fact that we will use in interpreting recent mathematical results [24, 25, 26] about instantons and geometric Langlands for surfaces.

5.2.2. More Concrete Argument. This somewhat abstract argument can be replaced by a much more concrete one if G is a group of classical type, rather than an exceptional group. (A similar analysis has been made independently for somewhat related reasons in [27]. See also [28]. The following discussion requires more detailed input from string theory than the rest of the present article, and the reader may wish to jump to section 5.3.) The simplest case is that G is SU(n) or even better U(n). We use the fact [29] that the (0,2) model of U(n) describes the low energy behavior of a system of n parallel M5-branes. We consider M-theory on $\mathbb{R}^7 \times TN$, where TN is the Taub-NUT space, a certain hyper-Kahler four-manifold that topologically is \mathbb{R}^4 . (It is described in detail in section 5.4.) TN has a U(1)symmetry with $TN/U(1) = \mathbb{R}^3$, as suggested by eqn. (5.3); we denote as 0 the point in \mathbb{R}^3 that corresponds to the U(1) fixed point in TN. Inside $\mathbb{R}^7 \times \text{TN}$, we consider n M5-branes supported on $\mathbb{R}^2 \times TN$ (for some choice of embedding $\mathbb{R}^2 \subset \mathbb{R}^7$); this gives a realization of the (0,2) theory of type U(n) on $\mathbb{R}^2 \times TN$. We want to divide by the U(1) symmetry of TN to reduce the six-dimensional (0,2) model supported on the M5-branes to a five-dimensional description. This may be done straightforwardly. For any seven-manifold Q_7 , M-theory on $Q_7 \times TN$ is equivalent [30] to Type IIA superstring theory on $Q_7 \times \mathbb{R}^3$ with a D6-brane supported on $Q_7 \times \{0\}$. So M-theory on $\mathbb{R}^7 \times TN$ is equivalent to Type IIA on $\mathbb{R}^7 \times \mathbb{R}^3$ with a D6-brane supported at $\mathbb{R}^7 \times \{0\}$. In this reduction, the *n* M5-branes on $\mathbb{R}^2 \times TN$ turn into n D4-branes supported on $\mathbb{R}^2 \times \mathbb{R}^3$. The low energy theory on the D4-branes is $\mathcal{N}=4$ super Yang-Mills theory with gauge group U(n). The D4-branes intersect the D6-brane on the Riemann surface $W = \mathbb{R}^2 \times \{0\}$, and a standard calculation (which uses the fact that the D4-branes and the D6-brane intersect transversely on W) shows the appearance on W of U(n) current algebra at level 1. The behavior of the (0,2) model of type D_n can be analyzed similarly by replacing \mathbb{R}^7 in the starting point with $\mathbb{R}^5/\mathbb{Z}_2 \times \mathbb{R}^2$.

5.3. Compactification On A Hyper-Kahler Manifold. We are going to consider the (0,2) theory in a very special situation. We take $M_6 = \mathbb{R} \times S^1 \times X$, where X will be a hyper-Kahler four-manifold. We think of \mathbb{R} as parametrizing the "time" direction. On M_6 , we take the obvious sort of product metric, giving circumference 2π to S^1 . We could take the metric on M_6 to be of Euclidean signature (which would agree well with some of our earlier formulas), but it is actually more elegant in what follows to use a Lorentz signature metric, that is a metric of signature -++++++, with the negative eigenvalue corresponding to the \mathbb{R} direction.

⁹One of the important general facts about quantum field theory, as remarked in footnote 4, is that in the world of unitary, physically sensible quantum field theories with positive energy – such as the six-dimensional (0, 2) model considered here – it is possible in a natural way to formulate the "same" quantum field theory on a space of Euclidean or Lorentzian signature. In the following analysis, the main thing that we gain by using Lorentz signature is that the supersymmetry generators are hermitian and the energy is bounded below.

The most obvious ordinary or bosonic conserved quantities in this situation are the ones that act geometrically: the Hamiltonian H, which generates translations in the \mathbb{R} direction, the momentum P, which generates rotations of S^1 , and possible additional conserved quantities associated with symmetries of X. The (0,2) model also has a less obvious bosonic symmetry group; this is the R-symmetry group Sp(4), mentioned in Remark 4.2. Because $\mathbb{R} \times S^1$ is flat and X is hyper-Kahler, so that $M_6 = \mathbb{R} \times S^1 \times X$ admits covariantly constant spinor fields, there are also unbroken supersymmetries. In fact, there are eight unbroken supersymmetries Q_{α} , $\alpha = 1, \ldots, 8$; they are hermitian operators that transform in the fundamental representation of the R-symmetry Sp(4) (which has real dimension 8). They commute with H and P, and obey a Clifford-like algebra. With a suitable choice of normalizations and orientations, this algebra reads

$$\{Q_{\alpha}, Q_{\beta}\} = 2\delta_{\alpha\beta} (H - P).$$

Accordingly, the operator H-P is positive semi-definite; it can be written in many different ways as the square of a Hermitian operator. States that are annihilated by H-P are known as BPS states and play a special role in the quantum theory [31]. We write \mathcal{V} for the space of BPS states. \mathcal{V} admits an action of $U(1) \times Sp(4)$ (or possibly a central extension thereof), where U(1) is the group of rotations of S^1 and Sp(4) is the R-symmetry group. The center of Sp(4) is generated by an element of order 2 that we denote as $(-1)^F$; it acts as +1 or -1 on bosonic or fermionic states, respectively. So in particular, \mathcal{V} is \mathbb{Z}_2 -graded by the eigenvalue of $(-1)^F$. We refer to \mathcal{V} , with its action of $U(1) \times Sp(4)$, as the spectrum of BPS states. One important general fact is that P is bounded below as an operator on \mathcal{V} ; indeed, on general grounds, H is bounded below in the full Hilbert space of the (0,2) theory, while H=P when restricted to \mathcal{V} .

Certain features of the spectrum of BPS states are "topological invariants," that is, invariant under continuous deformations of parameters. (In the present problem, the relevant parameters are the moduli of the hyper-Kahler metric of X.) The most obvious such invariant is the "elliptic genus," $F(q) = \text{Tr}_{\mathcal{V}} (-1)^F q^P$, where q is a complex number with |q| < 1. (It has modular properties, since it can be represented by the partition function of the (0,2) model on $\Sigma \times X$, where Σ is an elliptic curve whose modular parameter is $\tau = \ln q/2\pi i$.) F(q) is invariant under smooth deformation of the spectrum by virtue of the same arguments that are usually used to show that the index of a Fredholm operator is invariant under deformation.

In the present problem, the whole spectrum of BPS states, and not only the index, is invariant under deformation of the hyper-Kahler metric of X. One approach to proving this uses the fact that \mathcal{V} can be characterized as the cohomology of \mathcal{Q} , where \mathcal{Q} is any complex linear combination of the hermitian operators Q_{α} that squares to zero. Picking any one complex structure on X (from among the complex structures that make up the hyper-Kahler structure of X), one makes a judicious choice of \mathcal{Q} to show that the spectrum of BPS states is invariant under deformations of the Kahler metric of X (keeping the chosen complex structure fixed). Repeated moves of this kind (specializing at each stage to a different complex structure and therefore a different choice of \mathcal{Q}) can bring about arbitrary changes of the hyper-Kahler metric of X, so the spectrum of BPS states is independent of the moduli of X.

In our discussion in section 5.4, we compare two computations of \mathcal{V} in two different regions of the moduli space of hyper-Kahler metrics on X. The results must be equivalent in view of what has just been described.

Remark 5.1. There is some sleight of hand here, as the arguments above have assumed X to be compact, and we will use the results for noncompact X. So some refinement of the arguments is actually needed.

5.4. Taub-NUT Spaces. Now the question arises of what sort of hyper-Kahler four-manifold X we will select in the above construction.

We will choose X to admit a triholomorphic U(1) symmetry, that is, a U(1) symmetry that preserves the hyper-Kahler structure of X. (Among other things, this ensures that this U(1) also commutes with the unbroken supersymmetries Q_{α} of eqn. (5.6).) Hyper-Kahler four-manifolds with triholomorphic U(1) symmetry are highly constrained [32]. The general form of the metric is

(5.7)
$$ds^{2} = U d\vec{x} \cdot d\vec{x} + \frac{1}{U} (d\theta + \vec{\omega} \cdot d\vec{x})^{2},$$

where \vec{x} parametrizes \mathbb{R}^3 , U is a harmonic function on \mathbb{R}^3 , and (away from singularities of U) θ is an angular variable that parametrizes the U(1) orbits.

This form of the metric shows that the quotient space X/U(1) (assuming X is complete) is equal to \mathbb{R}^3 . Indeed, the natural projection $X \to X/U(1)$, which was considered in section 5.2, has a special interpretation in this situation. It is the hyper-Kahler moment map $\vec{\mu}$ and it is a surjective map to \mathbb{R}^3 :

$$(5.8) \vec{\mu}: X \to \mathbb{R}^3.$$

The most obvious hyper-Kahler four-manifold with a triholomorphic U(1) symmetry is \mathbb{R}^4 . This corresponds to the choice $U=1/2|\vec{x}|$. The U(1) action on \mathbb{R}^4 has a fixed point at the origin (where U has a pole and the radius of the U(1) orbits vanishes, according to (5.7)). This fixed point is precisely of the sort considered in section 5.2. To verify this, begin with the fact that the rotation group of \mathbb{R}^4 has $SU(2)_L \times SU(2)_R$ for a double cover; $SU(2)_L$ and $SU(2)_R$ are two copies of SU(2). We can pick a hyper-Kahler structure on X compatible with its flat metric such that $SU(2)_L$ rotates the three complex structures and $SU(2)_R$ preserves them. We simply take U(1) to be a subgroup of $SU(2)_R$. Then, upon picking a complex structure on \mathbb{R}^4 that is invariant under $SU(2)_L \times U(1)$ (this complex structure is not part of its U(1)-invariant hyper-Kahler structure), we can identify \mathbb{R}^4 with \mathbb{C}^2 and U(1) acts in the natural way $(z_1, z_2) \to (e^{i\theta} z_1, e^{i\theta} z_2)$. This then is the situation that was considered in section 5.2, and the statement (5.8) gives a hyper-Kahler perspective on the fact that the quotient $\mathbb{R}^4/U(1)$ is \mathbb{R}^3 , as was asserted in (5.3).

Although \mathbb{R}^4 has the properties we need from a topological point of view, there is a different hyper-Kahler metric on \mathbb{R}^4 that will be more useful for our application in section 5.5. This is the Taub-NUT space, which we will call TN. To describe TN explicitly, we simply choose U to be

(5.9)
$$U = \frac{1}{R^2} + \frac{1}{2|\vec{x}|},$$

where R is a constant. Looking at (5.7), the interpretation of R is easy to understand: the U(1) orbits have circumference $2\pi/\sqrt{U}$, which at infinity approaches $2\pi R$. The flat metric on \mathbb{R}^4 is recovered in the limit $R \to \infty$; in \mathbb{R}^4 , of course, the circumference of an orbit diverges at infinity.

Accordingly, the hyper-Kahler metric on TN is quite different at infinity from the usual flat hyper-Kahler metric on \mathbb{R}^4 . However, in one sense the difference is subtle. If we pick any one of the complex structures that make up the hyper-Kahler structure, then it can be shown that, as a complex symplectic manifold in this complex structure, TN is equivalent to $\mathbb{R}^4 \cong \mathbb{C}^2$.

A more general choice of X is also important. First of all, naively we could pick an integer k>1 and take $X=\mathbb{R}^4/\mathbb{Z}_k$, where \mathbb{Z}_k is the subgroup of U(1) consisting of points of order k. Certainly $\mathbb{R}^4/\mathbb{Z}_k$ has a (singular) hyper-Kahler metric with a triholomorphic U(1) symmetry. The singularity at the origin of $\mathbb{R}^4/\mathbb{Z}_k$ is known as an A_{k-1} singularity. It is possible to make a hyper-Kahler resolution of this singularity, still with a triholomorphic U(1) symmetry. This is accomplished by picking k points $\vec{x}_1, \dots \vec{x}_k$ in \mathbb{R}^3 and setting $U = \frac{1}{2} \sum_{j=1}^k 1/|\vec{x} - \vec{x}_j|$. This gives a complete hyper-Kahler manifold which is smooth if the \vec{x}_j are distinct. As a complex symplectic manifold in one complex structure, it can be described by an equation

$$(5.10) uv = f(w),$$

where f(w) is a k^{th} order monic polynomial. This is the usual complex resolution of the A_{k-1} singularity. In this description, the holomorphic symplectic form is $du \wedge dv/f'(w)$, and the triholomorphic U(1) symmetry is $u \to \lambda u$, $v \to \lambda^{-1}v$.

However, again, a generalization is more convenient for our application in section 5.5. We simply add a constant to U and take

(5.11)
$$U = \frac{1}{R^2} + \frac{1}{2} \sum_{j=1}^k \frac{1}{|\vec{x} - \vec{x}_j|}.$$

This gives a complete hyper-Kahler manifold, originally constructed in [33], that we call the multi-Taub-NUT space and denote as TN_k .

As a complex symplectic manifold in any one complex structure, TN_k is independent of the parameter R and coincides with the usual resolution (5.10) of the A_{k-1} singularity. However, the addition of a constant to U markedly changes the behavior of the hyper-Kahler metric at infinity. Just as in the k=1 case that was considered earlier, the asymptotic value at infinity of the circumference of the fibers of the fibration $TN_k \to \mathbb{R}^3$ is $2\pi R$.

The space TN_k is smooth as long as the \vec{x}_j are distinct. When r of them coincide, an A_{r-1} singularity develops, that is, an orbifold singularity of type $\mathbb{R}^4/\mathbb{Z}_r$.

In general, for $\vec{x} \to \vec{x}_j$, we have $U \to \infty$. So at those points, and only there, the radius of the U(1) fibers vanishes. The k points $\vec{x} = \vec{x}_j$ are, accordingly, the fixed points of the triholomorphic U(1) action.

5.4.1. A Note On The Second Cohomology. We conclude this subsection with some technical remarks that will be useful in section 5.5 (but which the reader may choose to omit). Topologically, TN_k is, as we have noted, the same as the resolution of the A_{k-1} singularity. A classic result therefore identifies $H^2(TN_k, \mathbb{Z})$ with the root lattice of the group $A_{k-1} = SU(k)$.

However, TN_k is not compact and one should take care with what sort of cohomology one wants to use. It turns out that another natural definition is useful. We define an abelian group Γ_k as follows: an element of Γ_k is a unitary line bundle $\mathcal{L} \to TN_k$ with anti-selfdual and square-integrable curvature and whose connection has trivial holonomy when restricted to a fiber at infinity of $\vec{\mu}: TN_k \to \mathbb{R}^3$. Γ is

a discrete abelian group with a natural and integer-valued quadratic form, defined as follows; if \mathcal{L} is a line bundle with anti-selfdual curvature F, we define $(\mathcal{L}, \mathcal{L}) = -\int_{\text{TN}_k} F \wedge F/4\pi^2$.

It turns out that $\Gamma_k \cong \mathbb{Z}^k$, with the quadratic form corresponding to the quadratic function of k variables $y_1^2 + y_2^2 + \dots + y_k^2$. (A basis of Γ_k is described in section 5.4.2.) Thus, Γ corresponds to the weight lattice of the group U(k). In many string theory problems involving Taub-NUT spaces, one must use Γ_k as a substitute for $H^2(TN_k, \mathbb{Z})$, which does not properly take into account the behavior at infinity.

This is notably true if one considers the (0,2) model on $M_6 = W \times TN_k$, for W a Riemann surface. In section 4.1, we explained that the (0,2) model on a compact six-manifold M_6 has a space of conformal blocks that is obtained by quantizing, in a certain sense, the finite abelian group $H^3(M_6, \mathbb{Z})$. For $M_6 = W \times TN_k$, the appropriate substitute for this group is

(5.12)
$$\widetilde{H}^{3}(W \times TN_{k}, \mathcal{Z}) = H^{1}(W, \mathcal{Z}) \otimes \Gamma_{k}.$$

5.4.2. Basis Of Γ_k . It is furthermore true that Γ_k has a natural basis corresponding to the U(1) fixed points \vec{x}_j , $j=1,\ldots,k$. To show this, we first describe a dual basis of noncompact two-cycles. For $j=1,\ldots,k$, we let ℓ_j be a path in \mathbb{R}^3 from \vec{x}_j to ∞ , not passing through any \vec{x}_r for $r \neq j$. Then we set $C_j = \vec{\mu}^{-1}(\ell_j)$. C_j is a noncompact two-cycle that is topologically \mathbb{R}^2 . A line bundle \mathcal{L} that represents a point in Γ_k is trivialized at infinity on C_j because its connection is trivial on the fibers of $\vec{\mu}$ at infinity. So we can define an integer $\int_{C_j} c_1(\mathcal{L})$. One can pick a basis of Γ_k consisting of line bundles \mathcal{L}_r such that $\int_{C_j} c_1(\mathcal{L}_r) = \delta_{jr}$. (The \mathcal{L}_r are described explicitly in [36].)

Now let us reconsider the definition of $\widetilde{H}^3(W \times TN_k, \mathcal{Z})$ in (5.12). From what we have just said, $H^1(W, \mathcal{Z}) \otimes \Gamma_k$ has a natural decomposition as the direct sum of copies $H^1_{(i)}(W, \mathcal{Z})$ of $H^1(W, \mathcal{Z})$ associated with the fixed points:

(5.13)
$$\widetilde{H}^{3}(W \times TN_{k}) = \bigoplus_{j=1}^{k} H^{1}_{(j)}(W, \mathcal{Z}).$$

Upon quantization, this means that the space of conformal blocks of the (0, 2) model on $W \times TN_k$ is the tensor product of k factors, each of them isomorphic to the space of conformal blocks in the level 1 WZW model (associated with the group G) on W. The factors are naturally associated to the U(1) fixed points.

REMARK 5.2. Similarly, we can enrich the definition of the two-dimensional characteristic class a of a $G_{\rm ad}$ bundle over TN_k . Normally, a takes values in $H^2(\mathrm{TN}_k, \mathcal{Z})$. However, suppose $E \to \mathrm{TN}_k$ is a $G_{\rm ad}$ bundle that is trivialized over each fiber at infinity of $\vec{\mu}: \mathrm{TN}_k \to \mathbb{R}^3$. Then E is trivialized at infinity on each C_j , so one can define a pairing $a_j = \langle a, C_j \rangle$ for each j; the a_j take values in \mathcal{Z} . Equivalently, we can consider a as an element of $\widetilde{H}^2(\mathrm{TN}_k, \mathcal{Z}) = \Gamma_k \otimes_{\mathbb{Z}} \mathcal{Z}$. This also has an analog if we are given a conjugacy class $\mathcal{C} \subset G_{\rm ad}$ and the monodromy of E on each fiber at infinity lies in C. Then one can define a C-dependent torsor for the group $\widetilde{H}^2(\mathrm{TN}_k, \mathcal{Z})$, and one can regard a as taking values in this torsor. Concretely, this means that, once we pick a path in $G_{\rm ad}$ from C to the identity (an operation that trivializes the torsor), we can define the elements $a_j \in \mathcal{Z}$ as before. Two different paths from C to the identity would differ by a closed loop in $G_{\rm ad}$, corresponding to an element $b \in \mathcal{Z}$; if we change the trivialization of the torsor by changing the path by b, then the a_j are shifted to $a_j + b$. (b is the same for all j,

since the regions at infinity in the two-cycles C_j can be identified, by taking the paths ℓ_j to coincide at infinity.)

5.5. Two Ways To Compute The Space Of BPS States. Now we are going to study in two different ways the space of BPS states of the (0,2) model formulated on

$$(5.14) M_6 = \mathbb{R} \times S^1 \times TN_k.$$

The results will automatically be equivalent, as explained at the end of section 5.3. M_6 admits an action of $U(1) \times U(1)'$ (the product of two factors of U(1)), where U(1) acts by rotation of S^1 , and U(1)' is the triholomorphic symmetry of TN_k . We choose a product metric on M_6 , such that S^1 has circumference $2\pi S$, and TN_k has a hyper-Kahler metric in which the U(1)' orbit has asymptotic circumference $2\pi R$. In section 5.3, we took S=1; in any event, because the (0,2) model is conformally invariant, only the ratio R/S is relevant.

U(1) and U(1)' play very different roles in the formalism because of the structure of the unbroken supersymmetry algebra, which we repeat for convenience:

$$\{Q_{\alpha}, Q_{\beta}\} = 2\delta_{\alpha\beta} (H - P).$$

Here P is the generator of the U(1) symmetry. It appears in the definition of the elliptic genus $F(q) = \text{Tr}_{\mathcal{V}} q^P(-1)^F$, where \mathcal{V} is the space of BPS states. The function F(q) has modular properties, so if it is nonzero (as will turn out to be the case), there are BPS states with arbitrarily large eigenvalues of P. By contrast, it turns out that U(1)' acts trivially on \mathcal{V} .

One of our two descriptions of $\mathcal V$ will be good for $S\to 0$ or equivalently $R\to \infty$; the other description will be good for $R\to 0$ or equivalently $S\to \infty$. Comparing them will give a new perspective on the results of [24, 25, 26].

5.5.1. Description I. For $S \to 0$, the low energy description is by gauge theory on $M_6/U(1) = \mathbb{R} \times TN_k$. As U(1) acts freely, we need not be concerned here with the behavior at fixed points. As the metric of M_6 is a simple product $S^1 \times M_5$ (with $M_5 = \mathbb{R} \times TN_k = M_6/U(1)$), we also need not worry about the interaction described in eqn. (5.2). So we simply get maximally supersymmetric Yang-Mills theory on $\mathbb{R} \times TN_k$, with gauge group G.

In formulating gauge theory on $\mathbb{R} \times \mathrm{TN}_k$, we specify up to conjugacy the holonomy U of the gauge field over a fiber at infinity of $\vec{\mu}: \mathrm{TN}_k \to \mathbb{R}^3$. This choice (which has a six-dimensional interpretation) leads to an important bigrading of the physical Hilbert space \mathcal{H} of the theory and in particular of the space \mathcal{V} of BPS states. First of all, let H be the subgroup of G that commutes with U. Classically, one can make a gauge transformation that approaches at infinity a constant element of H; quantum mechanically, to avoid infrared problems, the constant should lie in the center of H. So the center of H acts on \mathcal{H} and \mathcal{V} . We call this the electric grading. (The center of H is, of course, abelian, and the eigenvalues of its generators are called electric charges.)

A second "magnetic" grading arises for topological reasons. When $U \neq 1$, the topological classification of finite energy gauge fields on TN_k becomes more elaborate. Near infinity on TN_k , the monodromy around S^1 reduces the structure group from G to H, and the bundle can be pulled back from an H-bundle over the region near infinity on \mathbb{R}^3 . Infinity on \mathbb{R}^3 is homotopic to S^2 , so we get an H-bundle over S^2 . The Hilbert space of the theory is then graded by the topological type of

the H-bundle. We call this the magnetic grading. (Its components corresponding to U(1) subgroups of H are called magnetic charges.)

According to [6], electric-magnetic duality exchanges the electric and magnetic gradings. In our context, this will mean that the electric grading in Description I matches the magnetic grading in Description II, and viceversa. In the simplest situation, if U is generic, then H is a maximal torus T of G; the electric and magnetic gradings correspond to an action of T and T^{\vee} , respectively.

Actually, to extract the maximum amount of information from the theory, we want to allow an arbitrarily specified value of the two-dimensional characteristic class a. As described in Remark 5.2, a takes values in a certain torsor for $\widetilde{H}^2(\operatorname{TN}_k, \mathbb{Z})$, which means, modulo a trivialization of the torsor, that a assigns an element of \mathbb{Z} to each fixed point. (The origin of a in six dimensions was discussed in section 4.1.) Roughly speaking, allowing arbitrary a means that we do G_{ad} gauge theory, but there is a small twist: to extract the most information, we divide by only those gauge transformations that can be lifted to the simply-connected form G. This means that the monodromy U can be regarded as an element of G (up to conjugacy), and similarly that in Description II, we meet representations of the Kac-Moody group of G (not G_{ad}).

In gauge theory on $\mathbb{R} \times \mathrm{TN}_k$, U(1)' acts geometrically, generating the triholomorphic symmetry of TN_k . But how does U(1) act? The answer to this question is that in this description, the generator P of U(1) is equal to the instanton number I. (This fact is deduced using string theory.) The instanton number is defined via a familiar curvature integral, normalized so that on a compact four-manifold and with a simply-connected gauge group, it takes integer values. In the present context, the values of the instanton number are not necessarily integers, because TN_k is not compact. The analog of integrality in this situation is the following. First, one should add to the instanton number I a certain linear combination of the magnetic charges (with coefficients given by the logarithms of the monodromies). Let us call the sum \widehat{I} . Then there is a fixed real number r, depending only on the monodromy at infinity and the characteristic class a, such that \widehat{I} takes values in $r + \mathbb{Z}$. So in this description, eigenvalues of P are not necessary integers, but (for bundles with a fixed a and U) a certain linear combination of the eigenvalues of P and the magnetic charges are congruent to each other modulo integers.

Since P' generates the U(1)' symmetry of TN_k , one might expect its eigenvalues to be integers, but here we run into the electric charges. There is an operator that generates the triholomorphic symmetry and whose eigenvalues are integers; it is not simply P' but the sum of P' and a central generator of H (this generator is the logarithm of the monodromy at infinity), or in other words the sum of P' and a linear combination of electric charges.

What are BPS states in this description? Classically, the minimum energy fields of given instanton number are the instantons – that is the gauge fields that are independent of time and are anti-selfdual connections on TN_k . Instantons on TN_k have recently been studied by D-brane methods [34, 35, 36]. In particular [34], certain components of the moduli space \mathcal{M} of instantons on TN_k , when regarded as complex symplectic manifolds in one complex structure, coincide with components of the moduli space of instantons on the corresponding ALE space (the resolution of $\mathbb{R}^4/\mathbb{Z}_k$). All components of instanton moduli space on the ALE space arise in this way, but there are also components of instanton moduli space on TN_k that

have no analogs for the ALE space. (According to [34] and as explained to me by the author of that paper, these are the components of nonzero magnetic charge, corresponding to nonzero electric charge in Description II.)

An instanton is a classical BPS configuration, but to construct quantum BPS states, we must, roughly speaking, take the cohomology of the instanton moduli space \mathcal{M} . Actually, \mathcal{M} is not compact and by "cohomology," we mean in this context the space of L² harmonic forms on \mathcal{M} . (These are relevant for essentially the same reasons that they entered in one of the early tests of electric-magnetic duality [37].) So \mathcal{V} is the space of L² harmonic forms on \mathcal{M} . Of course, to construct \mathcal{V} we have to include contributions from all components \mathcal{M}_n of \mathcal{M} :

(5.16)
$$\mathcal{V} = \bigoplus_n H_{\mathsf{L}^2 \text{ harm}}^*(\mathcal{M}_n),$$

where we write $H_{\mathrm{L}^2 \text{ harm}}^*$ for the space of L^2 harmonic forms. The action of P on \mathcal{V} is multiplication by the instanton number, and similarly the magnetic grading is determined by the topological invariants of the bundles parametrized by a given \mathcal{M}_n . P' and the electric charges act trivially on \mathcal{V} because they correspond to continuous symmetries of \mathcal{M}_n that act trivially on its cohomology.

Each \mathcal{M}_n is a hyper-Kahler manifold, and accordingly the group Sp(4) – which in the present context is the R-symmetry group (as explained in Remark 4.2) – acts on the space of L^2 harmonic forms on \mathcal{M}_n and hence on \mathcal{V} . However, as in similar problems [38], it seems likely that Sp(4) acts trivially on these spaces. (This is equivalent to saying that L^2 harmonic forms exist only in the middle dimension and are of type (p,p) for every complex structure.) This would agree with what one sees on the other side of the duality, which we consider next.

REMARK 5.3. If we simply replace TN_k by \mathbb{R}^4 (with its usual metric) in this analysis, we learn in the same way that BPS states of the (0,2) model on $\mathbb{R} \times S^1 \times \mathbb{R}^4$ correspond to L^2 harmonic forms on instanton moduli space on \mathbb{R}^4 , with its usual metric. The same holds with an ALE space instead of \mathbb{R}^4 . The advantage of TN_k over \mathbb{R}^4 or an ALE space is that there is an alternative second description.

5.5.2. Description II. The other option is to take $R \to 0$. In this case, the fibers of $\vec{\mu} : \mathrm{TN}_k \to \mathbb{R}^3$ collapse, so to go over to a gauge theory description, we replace TN_k by \mathbb{R}^3 , with special behavior at the U(1) fixed points $\vec{x}_j, j = 1, \ldots, k$, where holomorphic WZW models will appear. We get a second description, then, in terms of maximally supersymmetric gauge theory on $M_5 = \mathbb{R} \times S^1 \times \mathbb{R}^3$, with level 1 holomorphic WZW models of type G supported on the k two-manifolds $W_j = \mathbb{R} \times S^1 \times \vec{x}_j, j = 1, \ldots, k$.

Once again, we must specify the holonomy U at infinity of the gauge field around S^1 . This is simply the same as the corresponding holonomy at infinity in Description I. Suppose for a moment that U is trivial. Then we also must pick, for each \vec{x}_j , $j=1,\ldots,k$, an integrable representation of the affine Kac-Moody algebra of G at level 1. For a simply-laced and simply-connected group, the integrable representations are classified by characters of the center \mathcal{Z} of G, or, as there is a perfect pairing $\mathcal{Z} \times \mathcal{Z} \to U(1)$, simply by \mathcal{Z} . So for each j, we must give an element $a_j \in \mathcal{Z}$. This is precisely the data that we obtained in Description I from the characteristic class $a \in \widetilde{H}^2(TN_k, \mathcal{Z})$. Since the second homology group of $\mathbb{R} \times S^1 \times \mathbb{R}^3$ vanishes, there is no two-dimensional characteristic class to be chosen

in Description II (matching the fact that there was no Kac-Moody representation in Description I).

More generally, for any U, we can canonically pick up to isomorphism a G-bundle on $\mathbb{R} \times S^1 \times \mathbb{R}^3$ with that monodromy at infinity, namely a flat bundle with holonomy U around S^1 . In the presence of this flat bundle, the Kac-Moody algebra on each $S^1 \times \vec{x}_j$ is twisted; if θ is an angular parameter on S^1 , then instead of the currents obeying $J(\theta + 2\pi) = J(\theta)$, they obey $J(\theta + 2\pi) = UJ(\theta)U^{-1}$. The representations of this twisted Kac-Moody algebra at level 1 are a torsor for Z – the same torsor that we met in Remark 5.2. The torsor is the same for each j since each Kac-Moody algebra is twisted by the same U. (The torsor property means concretely that the representations of the Kac-Moody algebra are permuted if U undergoes monodromy around a noncontractible loop in G_{ad} .)

In Description II, P generates the rotations of S^1 . For reasons that will become apparent, what is important is how P acts on the representations of the Kac-Moody algebra. In the Kac-Moody algebra, P corresponds to the operator – usually called L_0 – that generates a rotation of the circle. First set U=1. Then L_0 has integer eigenvalues in the vacuum representation of the Kac-Moody algebra (that is, the representation whose highest weight is G-invariant). In a more general representation (but still at U=1), L_0 has eigenvalues that are congruent mod \mathbb{Z} to a fixed constant r that depends only on the highest weight. This matches the fact that, in Description I (at U=1) the instanton number takes values in $r + \mathbb{Z}$ where r depends only on the characteristic class a. In the Kac-Moody theory, when the twisting parameter U is varied away from 1, the eigenvalues of L_0 shift. However (recalling that H is the commutant of U in G), one can add to L_0 a linear combination of the generators of H to make an operator L_0 with the property that in a given representation of the twisted Kac-Moody algebra, its eigenvalues are congruent mod \mathbb{Z} . Thus, electric charges play precisely the role in Description II that magnetic charges play in Description I.

On the other hand, in Description II, P' is the instanton number of a G-bundle on the initial value surface $S^1 \times \mathbb{R}^3$. If the monodromy U at infinity is trivial, then P' is integer-valued, just as in Description I. In general, for any U, a certain linear combination of P' and the magnetic charges (with coefficients given as usual by the logarithms of the monodromies) takes integer values. This mirrors the fact that in Description I, a linear combination of P' and the electric charges takes integer values.

What is the space of BPS states in Description II? Supported on $\mathbb{R} \times S^1 \times \vec{x}_j$ for each $j=1,\ldots,k$, there is a level 1 Kac-Moody module \mathcal{W}_j . This module has H=P for all states (mathematically, the representation theory of affine Kac-Moody algebras is usually developed with a single L_0 operator, not two of them), and consists entirely of BPS states. The space of BPS states is simply $\mathcal{V} = \otimes_{j=1}^k \mathcal{W}_j$. In particular, as the R-symmetry group Sp(4) acts trivially on the \mathcal{W}_j , it acts trivially on \mathcal{V} . The analogous statement in Description I was explained at the end of section 5.5.1.

Comparing the results of the two descriptions, we learn that

(5.17)
$$\otimes_{j=1}^{k} \mathcal{W}_{j} = \bigoplus_{n} H_{L^{2} \text{ harm}}^{*}(\mathcal{M}_{n}).$$

The right hand side is graded by instanton number and magnetic charge, and the left hand side by L_0 and electric charge. This equivalence closely parallels a central claim in [24, 25, 26].

A couple of differences may be worthy of note. Our description uses L^2 harmonic forms; different versions of cohomology are used in recent mathematical papers. Also, our instantons live on TN_k with its hyper-Kahler metric, not on the resolution of the A_{k-1} singularity. This does not affect \mathcal{M}_n as a complex symplectic manifold (as long as one considers on the TN_k side judiciously chosen components of the moduli space [34]), but it certainly affects the hyper-Kahler metric of \mathcal{M}_n and therefore the condition for an L^2 harmonic form. The components \mathcal{M}_n of instanton moduli space on TN_k that do not have analogs on the resolution of the A_{k-1} singularity are also presumably important.

One may wonder why we do not get additional BPS states from quantizing the moduli space of instantons, as we did in Description I. This can be understood as follows

Generically, curvature breaks all supersymmetry. In Description I, because the curvature of TN_k is anti-selfdual, it leaves unbroken half of the supersymmetry. The half that survives is precisely the supersymmetry that is preserved by an instanton (since an instanton bundle also has anti-selfdual curvature). Hence instantons are supersymmetric and must be considered in constructing the space of BPS states. By contrast, in Description II, there is no curvature to break supersymmetry. Instead, there is a coupling (5.2), which (when extended to include fields and terms that we have omitted) leaves unbroken half the supersymmetry, but a different half from what is left unbroken by anti-selfdual curvature. The result is that in Description II, instantons are not supersymmetric.

So in Description II, the instanton number and similarly the magnetic charges annihilate any BPS state. This implies that P' and the magnetic charges annihilate \mathcal{V} in Description II, just as P' and the electric charges do in Description I.

5.5.3. A Note On The Dual Group. The reader may be puzzled by the fact that in this analysis of two ways to describe the space of BPS states, we have not mentioned the dual group G^{\vee} . The reason for this is that for simplicity, we have limited ourselves to the case that G is simply-laced. When this is so, G and G^{\vee} have the same Lie algebra. Instead of merely comparing G and G^{\vee} theories, we can learn more, as explained in section 4.1, by considering G_{ad} bundles with an arbitrary two-dimensional characteristic class a. This is what we have done.

For groups that are not simply-laced, a slight variant of the construction that we have used is available [39]. The basic idea is that outer automorphisms of a simply-laced group G can appear as symmetries of the (0,2) model of type G in six dimensions. To a simple but not simply-laced Lie group H, one associates a pair (G,λ) , where G is simply-laced and λ is an outer automorphism of G. The (0,2) model of type G, when formulated on $M_6 = M_5 \times S^1$, can be "twisted" by λ in going around S^1 , in which case the low energy limit on M_5 is maximally supersymmetric gauge theory of type H^{\vee} . Now consider $M_6 = M_4 \times S^1 \times \widetilde{S}^1$, with circles of radius S and R and a twist by λ around S^1 . Repeating the analysis of section 4, we get at long distances on $M_4 \times \widetilde{S}^1$ a description by H^{\vee} gauge theory; this further reduces on M_4 to a description by H^{\vee} gauge theory with coupling parameter $\tau^{\vee} = iS/R$. Alternatively, we get a description by G gauge theory on $M_4 \times S^1$ with a twist around S^1 that reduces G to H; this further reduces on M_4

to H gauge theory with coupling parameter $\tau = iR/S$. The comparison of these two descriptions is essentially the perspective offered in [39] on electric-magnetic duality in four dimensions for non-simply-laced groups.

We can learn more if we do not reduce all the way to four dimensions. We take $M_6 = \mathbb{R} \times S^1 \times \mathrm{TN}_k$, where S^1 has circumference $2\pi S$ and the fiber at infinity of $\vec{\mu}: \mathrm{TN}_k \to \mathbb{R}^3$ has circumference $2\pi R$. For small S, we get Description I, which involves H^\vee instantons on TN_k . For small R, we get Description II, which now involves level 1 modules of the affine Kac-Moody algebra of G, twisted by λ . This is equivalent to the claim of [24] (although the roles of H and H^\vee are reversed in our presentation here). To make contact with the formulation given in [24], one must know that the Langlands or GNO dual of the Kac-Moody group of H^\vee is not the Kac-Moody group of H but the λ -twisted Kac-Moody group of G.

REMARK 5.4. The construction in the last paragraph does not appear to have an analog with a twist around the circle at infinity in TN_k . Precisely because this circle is contractible in the interior of TN_k , it is not possible to twist by a discrete symmetry of the (0,2) model in going around this circle.

5.5.4. Points With Multiplicity. So far we have considered the points \vec{x}_j to be distinct, so that TN_k is smooth. It is important, however, to consider the behavior as some of the \vec{x}_j become coincident. In general, suppose that the \vec{x}_j become coincident for $j=i_1,\ldots,i_r$. Without any essential change in the following remarks, we could allow several subsets of \vec{x}_j to simultaneously become coincident.

In Description I, when this happens, TN_k develops a A_{r-1} singularity, which is an orbifold singularity, locally modeled by $\mathbb{R}^4/\mathbb{Z}_r$. Gauge theory on $\mathbb{R}^4/\mathbb{Z}_r$ is defined as \mathbb{Z}_r -invariant gauge theory on \mathbb{R}^4 , but the notion of \mathbb{Z}_r invariance depends on the choice of how \mathbb{Z}_r acts on the fiber of a G-bundle at the fixed point (the origin in \mathbb{R}^4). Such a choice is a homomorphism $\phi: \mathbb{Z}_r \to G$. When a \mathbb{Z}_r orbifold singularity develops, the space \mathcal{V} of BPS states becomes graded by the choice of ϕ . This is in addition to the grading by the part of the characteristic class a that can be defined on the complement of the singularity.

The dual in Description II is that r of the points \vec{x}_j that support level 1 holomorphic WZW models become coincident, say at a point $\vec{y} \in \mathbb{R}^3$. Then the submanifold $\mathbb{R} \times S^1 \times \vec{y}$ of M_6 supports a level r holomorphic WZW model of type G. The affine Kac-Moody group of G supports several inequivalent integrable highest weight modules of level r, and when the points $\vec{x}_{i_1}, \ldots, \vec{x}_{i_r}$ become coincident, the Hilbert space of the theory decomposes as a direct sum of subspaces transforming in different such representations. This also gives a decomposition of the space $\mathcal V$ of BPS states.

So duality must establish a correspondence between two types of data (ϕ and the relevant part of a on one side; a choice of level r integrable representation on the other side). Such a correspondence is used in [24] and can be described as follows in physical terms. For simplicity, we set $G = E_8$, so that we can dispense with a. Consider three-dimensional Chern-Simons theory with gauge group G at level r. A Wilson loop can be considered in any irreducible representation R that is the highest weight of a level r integrable module of the affine Kac-Moody algebra. On the other hand, around the Wilson loop, the gauge field acquires a monodromy that is an element of G of order r. This gives a correspondence between level r integrable modules and conjugacy classes of order r. It should be possible to use

string theory arguments to show that this is the correspondence that enters in comparing Descriptions I and II, but this will not be attempted here.

The general story is to consider points $\vec{y}_1, \ldots, \vec{y}_s \in \mathbb{R}^3$ with multiplicities r_1, \ldots, r_s . In our presentation, we obtain this case starting with $k = \sum_{i=1}^s r_i$ points, all of multiplicity 1, and letting them coalesce in clumps of the appropriate sizes. Then in Description I, we consider a TN_k space that is constrained to have singularities of type $A_{r_{i-1}}$ for $i = 1, \ldots, s$. Gauge theory at the i^{th} singularity is defined by a choice of homomorphisms $\phi_i : \mathbb{Z}_{r_i} \to G$. Description II is based on supersymmetric gauge theory on $\mathbb{R} \times S^1 \times \mathbb{R}^3$, with each $\mathbb{R} \times S^1 \times \vec{y}_i$ supporting a holomorphic level r_i WZW model, associated with the level r_i integrable representation that corresponds to ϕ_i .

This correspondence, moreover, is compatible with further coalescences of points, in close parallel to the picture of [24]. Any further coalescence of points leads to a further decomposition of \mathcal{V} on both sides.

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References

- D. S. Freed, "Higher Algebraic Structures and Quantization," Commun. Math. Phys. 159 (1994) 343 [arXiv:hep-th/9212115].
- J. Baez and J. Dolan, "Higher Dimensional Algebra And Topological Quantum Field Theory,"
 J. Math. Phys. 36 (1995), 6073-6105 [arXiv:q-alg/9503002].
- [3] J. Lurie, "Expository Article On Topological Field Theories," available at http://math.mit.edu/~lurie/.
- [4] G. Segal, to appear.
- [5] R. Langlands, "Problems In The Theory Of Automorophic Forms," in Lect. Notes in Math. 170 (Springer-Verlag, 1970) 17-61.
- [6] P. Goddard, J. Nuyts and D. I. Olive, "Gauge Theories And Magnetic Charge," Nucl. Phys. B125 (1977) 1.
- [7] A. Beilinson and V. Drinfeld. "Quantization Of Hitchin's Integrable System and Hecke Eigensheaves," preprint (ca. 1995). available http://www.math.uchicago.edu/~mitya/langlands.html.
- [8] C. Montonen and D. I. Olive, "Magnetic Monopoles As Gauge Particles?" Phys. Lett. B72 (1977) 117.
- [9] A. Kapustin and E. Witten, "Electric-Magnetic Duality and the Geometric Langlands Program," arXiv:hep-th/0604151.
- [10] E. Witten, "Mirror Symmetry, Hitchin's Equations, And Langlands Duality," arXiv:0802:0999.
- [11] M. F. Atiyah and R. Bott, "The Yang-Mills Equations Over Riemann Surfaces," Phil. Trans. R. Soc. London A308 (1982) 523-615.
- [12] S. Gukov and E. Witten, "Gauge Theory, Ramification, and the Geometric Langlands Program," arXiv:hep-th/0612073.
- [13] D. Gaiotto and E. Witten, "Supersymmetric Boundary Conditions in $\mathcal{N}=4$ Super Yang-Mills Theory," arXiv:0804.2902 [hep-th].
- [14] D. Gaiotto and E. Witten, "S-Duality of Boundary Conditions In $\mathcal{N}=4$ Super Yang-Mills Theory," arXiv:0807.3720 [hep-th].
- [15] E. Witten, "Conformal Field Theory In Four And Six Dimensions," in U. Tillman, ed., Topology, Geometry and Quantum Field Theory, (Cambridge University Press, 2004) 405-419, arXiv:0712.0157.
- [16] M. Henningson, B. E. W. Nilsson and P. Salomonson, JHEP 9909 (1999) 008 [arXiv:hep-th/9908107].
- [17] M. Henningson, JHEP 0203 (2002) 021 [arXiv:hep-th/0111150].

- [18] E. P. Verlinde, "Global Aspects of Electric-Magnetic Duality," Nucl. Phys. B455 (1995) 211 [arXiv:hep-th/9506011].
- [19] E. Witten, "On S-Duality in Abelian Gauge Theory," Selecta Math. 1 (1995) 383 [arXiv:hep-th/9505186].
- [20] E. Witten, "Some Comments on String Dynamics," in I. Bars et. al., eds., Future Perspectives In String Theory (World Scientific, 1996), arXiv:hep-th/9507121.
- [21] W. Nahm, "Supersymmetries And Their Representations," Nucl. Phys. B135 (1978) 149.
- [22] E. Witten, "Non-Abelian Bosonization In Two Dimensions," Commun. Math. Phys. 92 (1984) 455-472.
- [23] C. Vafa and E. Witten, Nucl. Phys. B 431 (1994) 3 [arXiv:hep-th/9408074].
- [24] A. Braverman and M. Finkelberg, "Pursuing The Double Affine Grassmannian I: Transversal Slices Via Instantons On A_k Singularities," arXiv:0711.2083.
- [25] A. Licata, "Framed Rank r Torsion-free Sheaves on \mathbb{CP}^2 and Representations of the Affine Lie Algebra $\widehat{gl(r)}$," arXiv:math/0607690.
- [26] H. Nakajima, "Quiver Varieties And Branching," SIGMA 5 (2009) 3, arXiv:0809.2605.
- [27] R. Dijkgraaf, L. Hollands, P. Sulkowski and C. Vafa, "Supersymmetric Gauge Theories, Intersecting Branes and Free Fermions," JHEP 0802 (2008) 106 [arXiv:0709.4446 [hep-th]].
- [28] M. C. Tan, "Five-Branes in M-Theory and a Two-Dimensional Geometric Langlands Duality," arXiv:0807.1107 [hep-th].
- [29] A. Strominger, "Open p-Branes," Phys. Lett. B 383 (1996) 44 [arXiv:hep-th/9512059].
- [30] P. K. Townsend, "The Eleven-Dimensional Supermembrane Revisited," Phys. Lett. B 350 (1995) 184 [arXiv:hep-th/9501068].
- [31] D. I. Olive and E. Witten, "Supersymmetry Algebras That Include Topological Charges," Phys. Lett. B78 (1978) 97.
- [32] U. Lindstrom and M. Rocek, "Scalar Tensor Duality And $\mathcal{N}=1,\ \mathcal{N}=2$ Nonlinear Sigma Models," Nucl. Phys. **B222** (1983) 285.
- [33] S. Hawking, "Gravitational Instantons," Phys. Lett. A60 (1977) 81.
- [34] S. A. Cherkis, "Moduli Spaces of Instantons on the Taub-NUT Space," arXiv:0805.1245 [hep-th].
- [35] S. A. Cherkis, "Instantons on the Taub-NUT Space," arXiv:0902.4724 [hep-th].
- [36] E. Witten, "Branes, Instantons, And Taub-NUT Spaces," arXiv:0902.0948 [hep-th].
- [37] A. Sen, "Dyon-Monopole Bound States, Selfdual Harmonic Forms on the Multi-Monopole Moduli Space, and $SL(2,\mathbb{Z})$ Invariance in String Theory," Phys. Lett. **B329** (1994) 217 [arXiv:hep-th/9402032].
- [38] N. Hitchin, "L²-Cohomology of Hyperkahler Quotients," Commun. Math. Phys. 211 (2000), 153-165.
- [39] C. Vafa, "Geometric Origin of Montonen-Olive Duality," Adv. Theor. Math. Phys. 1 (1998) 158 [arXiv:hep-th/9707131].

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