VECTOR BUNDLES AND F THEORY

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To understand in detail duality between heterotic string and F theory compactifications, it is important to understand the construction of holomorphic G bundles over elliptic Calabi-Yau manifolds, for various groups G. In this paper, we develop techniques to describe these bundles, and make several detailed comparisons between the heterotic string and F theory.

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1. Introduction

One of the important recent insights about string duality is that the compactification of the heterotic string on \mathbf{T}^2 is equivalent to the compactification of F theory on an elliptically fibered K3 with a section [1,2]. Extending this idea, one then expects that the heterotic string compactified on an n-fold Z which is elliptically fibered over a base B should be equivalent to F theory compactified on an n + 1-fold X which is fibered with K3 fibers over the same base. This should follow upon fiberwise application of the basic heterotic string/F theory duality on the fibers.

The first non-trivial case of this fiberwise duality is n = 2 – which means in practice that $B = \mathbf{P^1}$, $Z = \mathrm{K3}$, and X is a Calabi-Yau three-fold. In this case, this duality has been successfully used [3] to illuminate many aspects of heterotic string dynamics on K3, including [4] aspects of the strong coupling singularity. A successful extension to n = 3 would be very interesting physically and would raise many new issues such as the possibility of a spacetime superpotential. Several aspects have been discussed so far [5-19].

To understand in detail F theory/heterotic duality, for any value of n, involves understanding and comparing the moduli spaces on the two sides. On the F theory side, the moduli spaces involved have been comparatively well understood [20,21], but on the heterotic string side there is a major gap. In compactification of the heterotic string on a two-torus or on an elliptically fibered manifold of n > 1, a major ingredient is the choice of a suitable $E_8 \times E_8$ (or $Spin(32)/\mathbb{Z}_2$) stable holomorphic bundle. Only limited information about the relevant bundles has been brought to bear so far.

There is, however, an effective framework for understanding stable bundles on elliptically fibered manifolds [22,23]. In this approach, which has been developed in detail for SU(2) bundles on elliptically fibered surfaces (for the purpose of applications to Donaldson theory), one describes bundles on an elliptically fibered manifold by first describing the bundles on a particular elliptic curve, and then working fiberwise. This approach is not limited to Calabi-Yau manifolds. Most of the present paper is devoted to describing this approach mathematically. In the last part of the paper, we specialize to Calabi-Yau manifolds and make some applications to F theory.

Some Generalities About Bundles

Before focusing on our specific problem, we make some general remarks about bundles (in somewhat more detail than really needed to follow the rest of the paper). The bundles of interest, whether over a single elliptic curve or an elliptically fibered manifold, can be viewed in either of two ways: (1) as holomorphic stable bundles (or semistable ones as explained below) with structure group the complexification $G_{\mathbf{C}}$ of a compact Lie group G; (2) as solutions of the hermitian-Yang-Mills equations for a G-valued connection.⁴ The second point of view arises most directly in physics; the first point of view is convenient for analyzing the bundles. The equivalence of the two viewpoints is a theorem of Narasimhan and Seshadri [24] for vector bundles on a Riemann surface, generalized for arbitrary semi-simple gauge groups in [25-27], and of Donaldson [26], and Uhlenbeck and Yau [28], in higher dimensions.

Over a Riemann surface, the hermitian-Yang-Mills equations for a connection simply say that the connection is flat, so the Narasimhan-Seshadri theorem identifies the moduli space of semistable holomorphic $G_{\mathbf{C}}$ bundles on a Riemann surface with the moduli space of flat G-valued connections. The moduli space of such flat connections has an elementary, explicit description: a flat connection on the two-torus is given by a pair of commuting elements in the gauge group G. Two such connections are equivalent if and only if they are isomorphic, which is the same thing as the commuting pairs being conjugate in G.

The description of the same moduli space via semi-stable holomorphic $G_{\mathbf{C}}$ bundles is more subtle in several ways. First of all, the equivalence relation between semistable bundles that is used to build the moduli space, called S-equivalence, is in general weaker than isomorphism. (For example, $\mathcal{O} \oplus \mathcal{O}$ and the non-trivial extension of \mathcal{O} by \mathcal{O} are S-equivalent. But for the generic semi-stable $G_{\mathbf{C}}$ bundle on a torus, S-equivalence is the same as isomorphism.) The Narasimhan-Seshadri theorem tells us that every S-equivalence class contains (up to isomorphism) a unique representative that admits a flat connection. This preferred representative is not always the one that arises on the fibers of an elliptic fibration. In fact, every S-equivalence class has another distinguished representative, a "regular" bundle whose automorphism group has dimension equal to the rank of G. It is the regular representatives that fit together most naturally in families, as was shown for rank two bundles over surfaces in [22,23].

When we refer somewhat loosely to a "G bundle," the context should hopefully make clear whether a given argument is best understood in terms of solutions of the hermitian-Yang-Mills equations with a compact gauge group G, or holomorphic stable (or semistable) $G_{\mathbf{C}}$ bundles. Note that in the important case G = SU(n) the complexification $SU(n)_{\mathbf{C}}$

⁴ These equations say that the (2,0) and (0,2) part of the curvature vanish, and the (1,1) part is traceless.

is customarily called $SL(n, \mathbf{C})$; the complexifications have no special names in the other cases. Hopefully, it will anyway cause no confusion if we refer loosely to G bundles even for G = SU(n).

Finally, let us explain the meaning of the term "semistable" as opposed to "stable." A stable bundle corresponds to a solution of the hermitian-Yang-Mills equations which is irreducible (the holonomy commutes only with the center of the gauge group), while a semistable bundle is associated with a reducible solution of those equations. In many situations, the generic semistable bundle is actually stable, but the case of an elliptic curve E is special; as its fundamental group is abelian, the flat connections over E have holonomy that [29] can be conjugated into a maximal torus (if the gauge group is simply connected and semi-simple) and so are reducible, and correspond to semistable rather than stable bundles. The bundles we will construct on an elliptically fibered manifold E of dimension E 1 are, however, generically stable, if the Kähler class of E is chosen suitably. (A sufficient requirement is, as in [23], that the fiber is sufficiently small compared to the base, justifying an adiabatic argument by which stability is proved.)

Bundles On An Elliptic Curve

Now we turn to our specific problem. In studying semistable bundles on an elliptic curve with general structure group, an important role is played by a theorem of Looijenga [30] (another proof was given by Bernshtein and Shvartsman [31]) which determines the moduli space \mathcal{M} of G bundles on an elliptic curve E for any simple, connected, and simply-connected group G of rank r.⁵ \mathcal{M} is always a weighted projective space $\mathbf{WP}_{s_0,s_1,\ldots,s_r}^r$, where the weights s_0,\ldots,s_r are 1 and the coefficients of the highest coroot of G. (In other words, the weights are the coefficients of the null vector of the *dual* of the untwisted Kac-Moody algebra of G. We will sometimes suppress the weights from the notation and write just \mathbf{WP}^r .) The requisite weights, for the various simple groups, are summarized in figure one.

In this paper, we will develop four approaches to understanding Looijenga's theorem, for different classes of G.

(1) For G = SU(n) or G = Sp(n), the moduli space can be determined by a completely direct computation that we present in section 2. SU(n) and Sp(n) (or A_{n-1} and C_n) are the unique cases in which the weights of the weighted projective space are all 1, so that the

 $^{^{5}}$ There is also a generalization for non-simply-connected G which can be obtained via the method of section 5 and will be presented elsewhere.

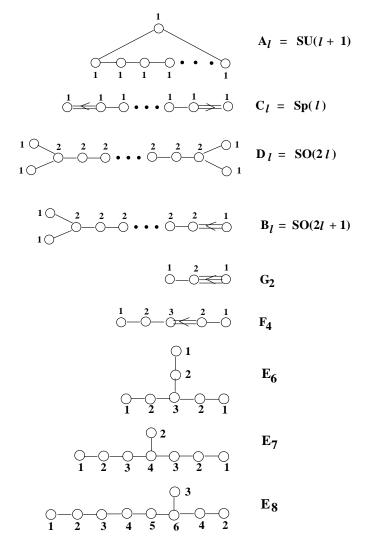


Figure 1. The simple Lie groups together with the *duals* of their untwisted Kac-Moody algebras. The integers labeling the nodes are the weights of the corresponding weighted projective space.

moduli space is actually an ordinary projective space. In these cases, a direct treatment along the general lines of [23] is possible.

(2) Every not necessarily simply-laced group G has a canonical simply-laced subgroup G', generated by the long roots of G. Looijenga's theorem for G is a consequence of Looijenga's theorem for G', as we will show in section 3. We also explain another reduction

to the simply-laced case by embedding G in a suitable simply-laced group.

- (3) For E_6 , E_7 , E_8 , and certain subgroups, Looijenga's theorem can be proved by relating G bundles to del Pezzo surfaces. This approach, which we will explore in section 4, is perhaps closest to Looijenga's original approach. For additional background see [35]. This method gives an attractive way to see the relation between groups and singularities (in this case, between subgroups of G and singularities of the del Pezzo surface) that has been important in the last few years in studies of string duality. The chain of groups related to del Pezzo surfaces is important in applications of F theory [32-34].
- (4) Finally, we explain in section 5 our most general and powerful approach. For any G, Looijenga's theorem can be proved by constructing a distinguished unstable G bundle on E, which has the beautiful property that it can be deformed in a canonical way to any semistable G bundle. (This construction always produces the regular representative of every S-equivalence class [36].)

Each of these approaches is most efficient for understanding some aspects of F theory. For instance, the first approach, as well as being the most elementary, gives (at the present level of our understanding) the most complete information for SU(n) bundles, which enter in most attempts at using the heterotic string to make models of particle physics. The last approach is (at the present level of understanding) the method that enables us to concretely construct the E_8 bundles that are relevant to the easiest applications of F theory.

For our applications, we want to understand G bundles not just on a single elliptic curve E, but on a complex manifold Z that is elliptically fibered over a base B. The basic idea here is to understand Looijenga's fiberwise fiberwise. The fiber of Z over a point $b \in B$ is an elliptic curve E_b (perhaps singular). The moduli space of G bundles on E_b is a weighted projective space \mathbf{WP}_b . The \mathbf{WP}_b fit together, as b varies, to a bundle \mathcal{W} of weighted projective spaces. Any G bundle over E that is sufficiently generic on each fiber determines a section of \mathcal{W} , and in many situations the bundles associated with a given section can be effectively described.

One of our main goals will therefore be to obtain a description of W. We will focus on the case that the elliptic manifold $Z \to B$ has a section, whose normal bundle we call \mathcal{L}^{-1} . (This is the case that arises in the simplest applications of F theory.) We will see that for every case except $G = E_8$, W can be described very simply as the projectivization of a rank r + 1 vector bundle Ω over B which is simply a sum of line bundles. In fact,

$$\Omega = \mathcal{O} \oplus \left(\bigoplus_{j=1}^{r} \mathcal{L}^{-d_j} \right), \tag{1.1}$$

where the d_j are the degrees of the independent Casimir invariants of G. (This assertion is closely related to a result of Wirthmuller [37] who in particular discovered the exceptional status of E_8 .) In dividing the fibers of Ω by \mathbb{C}^* to make the weighted projective space bundle \mathcal{W} , \mathbb{C}^* acts diagonally on the \mathcal{L}^{-d_j} with weights s_j introduced above. The matching of d_j and s_j is described in the table. This determination of \mathcal{W} will serve as the basis in section 6 for an extensive comparison of the moduli space of G bundles on G to appropriate G theory moduli spaces, in the course of which it will be clear from the G theory point of view why G0 should be exceptional.

In the decomposition (1.1), the summand \mathcal{O} plays a distinguished role. The section of \mathcal{W} coming from the constant section 1 of \mathcal{O} corresponds to a bundle on Z whose restriction to each fiber is S-equivalent to the trivial G bundle. The most elementary way to see why Casimir weights appear is actually to look at the behavior near this section.

Of our four approaches, methods (1) and (4) actually enable us to construct G bundles over an elliptically fibered manifold Z and not merely to determine the moduli spaces. When the bundles can be constructed, one has a starting point for addressing more detailed question like the computation of Yukawa couplings. Most such questions will not be considered in this paper. However, in section 7, we make one important application of the construction of bundles, which is to compute the basic characteristic class of these bundles (this is a four-dimensional class which for G = SU(n) is the conventional second Chern class). This computation leads to an important comparison between the heterotic string and F theory; for the case of compactification of the heterotic string on a Calabi-Yau threefold, we will understand from the heterotic string point of view the origin of the threebranes that appear mysteriously on the F theory side [8].

	1	2	3	4
A_n	$0, 2, 3, \ldots, n+1$			
B_n	0, 2, 4	$6, 8, \ldots, 2n$		
C_n	$0,2,4,\ldots,2n$			
D_n	0,2,4,n	$6, 8, 10, \dots, 2n-2$		
G_2	0,2	6		
F_4	0, 2	6,8	12	
E_6	0, 2, 5	6, 8, 9	12	
E_7	0, 2	6, 8, 10	12, 14	18

This table displays the relation between weights s_j and exponents d_j for the simple Lie groups (all those other than E_8) for which \mathcal{W} is the projectivization of some $\Omega = \bigoplus_j \mathcal{L}^{-d_j}$. Weights are plotted horizontally and the entries in the table are the exponents d_j for a given weight. For instance, for the group G_2 , the exponents are 0 and 2 in weight 1 and 6 in weight 2; no other weights appear for this group.

In section 8, we compare the explicit construction of bundles to what could be predicted a priori from index theory.

In this paper, we concentrate on explaining aspects of the problem that seem likely to be most immediately relevant and useful for physicists. A more systematic exposition with full proofs will appear elsewhere [36].

2. Direct Approach For SU(n) and Sp(n)

2.1. Bundles On An Elliptic Curve

For the starting point, we consider bundles on a single elliptic curve E – that is, a two-torus with a complex structure and a distinguished point p called the "origin." p is the identity element in the group law on E.

A stable or semistable holomorphic G bundle on a Riemann surface Σ in general is associated with a representation of the fundamental group of Σ in (the compact form of) G. For the case that the Riemann surface is a two-torus E, the fundamental group is abelian and generated by two elements, so if G is simply-connected, a representation of the fundamental group in G can be conjugated to a representation in the maximal torus of G [29]. As promised in the introduction, the present section is devoted to a direct construction of G bundles on E in certain simple cases. First we take G = SU(n).

In this case, a G bundle determines a rank n complex vector bundle V, of trivial determinant. The fact that V can be derived from a representation of the fundamental group in a maximal torus means that $V = \bigoplus_{i=1}^{n} \mathcal{N}_i$, where the \mathcal{N}_i are holomorphic line bundles. The fact that V is an SU(n) (rather than U(n)) bundle means that $\bigotimes_{i=1}^{n} \mathcal{N}_i = \mathcal{O}$. (\mathcal{O} is a trivial line bundle over E.) For V to be semistable means that the \mathcal{N}_i are all of degree zero. The Weyl group of SU(n) acts by permuting the \mathcal{N}_i , and the \mathcal{N}_i are uniquely determined up to this action.

If \mathcal{N}_i is a degree zero line bundle on E, there is a unique point Q_i in E with the following property: \mathcal{N}_i has a holomorphic section that vanishes only at Q_i and has a pole only at p. So the decomposition $V = \bigoplus_{i=1}^n \mathcal{N}_i$ means that V determines the n-tuple of points Q_1, \ldots, Q_n on E. The fact that $\bigotimes_{i=1}^n \mathcal{N}_i = \mathcal{O}$ means that (using addition with respect to the group law on E) $\sum_i Q_i = 0$. Conversely, every $Q_i \in E$ determines a degree zero line bundle $\mathcal{N}_i = \mathcal{O}(Q_i) \otimes \mathcal{O}(p)^{-1}$ (whose sections are functions on E that are allowed to have a pole at Q_i and required to have a zero at p), and every p-tuple p-tup

The moduli space of \mathcal{M} of semistable SU(n) bundles on E is therefore simply the moduli space of unordered n-tuples of points in E that add to zero. The space of such n-tuples can be conveniently described as follows. If Q_1, \ldots, Q_n is such an n-tuple, then there exists a meromorphic function w which vanishes (to first order) at the Q_i and has poles only at p. (Existence of such a w is equivalent to the vanishing of the sum of the Q_i in the group law on E.) Such a w is unique up to multiplication by a non-zero complex scalar. Conversely, let $W = H^0(E, \mathcal{O}(np))$ be the space of meromorphic functions on E that have a pole of at most n^{th} order at p and no poles elsewhere. Such a function w has p zeroes p which add up to zero (some of these points may be coincident; also, if the pole

at p is of order less than n, we interpret this to mean that some of the Q_i coincide with p).

This correspondence between n-tuples and functions means that \mathcal{M} is a copy of complex projective space \mathbf{P}^{n-1} , obtained by projectivizing W:

$$\mathcal{M} = \mathbf{P}H^0(E, \mathcal{O}(np)). \tag{2.1}$$

Actually, the functions $w \in H^0(E, \mathcal{O}(np))$ can be described very explicitly. If E is described by a Weierstrass equation

$$y^2 = 4x^3 - g_2x - g_3 (2.2)$$

in x - y space, and p is the point $x = y = \infty$, then a meromorphic function w with a pole only at p is simply a polynomial in x and y. As x has a double pole at p and y has a triple pole, w can be written

$$w = a_0 + a_2 x + a_3 y + a_4 x^2 + a_5 x^2 y + \dots, (2.3)$$

where the last term is $a_n x^{n/2}$ for n even, or $a_n x^{(n-3)/2} y$ for n odd. In other words, w is a general polynomial in x and y with at most an n^{th} order pole at infinity, and (modulo the Weierstrass equation) at most a linear dependence on y. To allow for a completely general set of Q_i , one restricts the a_k only by requiring that they are not all identically zero. (For example, a_n vanishes if and only if one of the Q_i is the point p at infinity.) Since the a_k are never identically zero, it makes sense to interpret them as homogeneous coordinates of a complex projective space, and this is the idea behind (2.1).

Sp(n) Bundles

The other case for which G bundles on an elliptic curve can be described explicitly with similar methods is the case G = Sp(n). Using the 2n-dimensional representation of Sp(n), we can think of an Sp(n) bundle as a rank 2n holomorphic vector bundle V equipped with a non-degenerate holomorphic section ω of $\wedge^2 V^*$, reducing the structure group to Sp(n). On an elliptic curve, a stable Sp(n) bundle is simply a direct sum $V = \bigoplus_{i=1}^n (\mathcal{N}_i \oplus \mathcal{N}_i^{-1})$; in this basis, the non-zero matrix elements of ω map $\mathcal{N}_i \otimes \mathcal{N}_i^{-1} \to \mathcal{O}$. We associate each pair $(\mathcal{N}_i, \mathcal{N}_i^{-1})$ with a pair $(Q_i, -Q_i)$ of equal and opposite points in E. The Weyl group acts by permutation of these pairs and by the interchanges $Q_i \leftrightarrow -Q_i$. The moduli space \mathcal{M} of Sp(n) bundles on E is simply the space of n-tuples of unordered pairs $(Q_i, -Q_i)$, up to permutation.

A point Q on E corresponds to a set of values (x, y) obeying the Weierstrass equation (2.2). y is determined by x up to sign. Since the transformation $Q \to -Q$ is the \mathbb{Z}_2 symmetry $y \to -y$ of E, being given not a point Q but a pair (Q, -Q) is tantamount to being given only the value of x. So an n-tuple of pairs $(Q_i, -Q_i)$ is equivalent to an n-tuple of values of x, say x_1, x_2, \ldots, x_n . Because of the Weyl action, the x_i are determined by the bundle only up to permutation.

As in the discussion of SU(n) bundles, the unordered n-tuple x_1, \ldots, x_n is conveniently summarized by giving a polynomial in x whose zeroes are the x_i :

$$t = c_0 + c_1 x + c_2 x^2 + \ldots + c_n x^n. (2.4)$$

Once again, to allow for the possibility that some Q_i are equal to p, the c_i are restricted only to not all be zero. Since a rescaling $t \to \lambda t$ with non-zero complex λ does not change the zeroes, the moduli space \mathcal{M} of Sp(n) bundles on E is again a projective space, in this case the projective space \mathbf{P}^n whose homogeneous coordinates are the c's.

It should be stressed that what the above constructions determine is the moduli space of G bundles on E for the *simply-connected* groups SU(n) and Sp(n). The discussion must be considerably adapted to describe the $SU(n)/\mathbb{Z}_n$ or $Sp(n)/\mathbb{Z}_2$ moduli spaces by similar methods. For example, these moduli spaces have different components (of different dimension) indexed by the topological type of the bundle.

We conclude by briefly comparing Sp(n) bundles to SU(2n) bundles. Given the natural embedding of Sp(n) in SU(2n), the moduli space $\mathcal{M}_{Sp(n)}$ of flat Sp(n) bundles on E can be embedded as a subspace of the moduli space $\mathcal{M}_{SU(n)}$ of flat SU(2n) bundles on E. In fact, according to (2.3), flat SU(2n) bundles are related to polynomials $w = a_0 + a_2x + a_3y + \ldots + a_nx^n$. If we simply set to zero the a_i of odd i (the ones odd under $y \to -y$) such a polynomial takes the form of the Sp(n) polynomial in (2.4). By more carefully examining the above constructions, it can be shown that this identification of polynomials does give the embedding of $\mathcal{M}_{Sp(n)}$ in $\mathcal{M}_{SU(2n)}$. An analogous relation holds for Sp(n) and SU(2n) bundles on elliptic manifolds.

2.2. Bundle Of Projective Spaces

For our applications, we must understand not vector bundles on a single elliptic curve E, but vector bundles on a family of elliptic curves, that is on a complex manifold Z which maps to some base B with the generic fiber being an elliptic curve. We will assume for

simplicity that the map $Z \to B$ has a section (the case most commonly considered in relation to F theory). In that case, Z can be described by a Weierstrass equation. The Weierstrass equation can be written in a \mathbf{P}^2 bundle W over B; W is the projectivization of $\mathcal{L}^2 \oplus \mathcal{L}^3 \oplus \mathcal{O}$, with \mathcal{L} being some line bundle over B. If we describe W by homogeneous coordinates x, y, z (which are sections respectively of $\mathcal{L}^2, \mathcal{L}^3$, and \mathcal{O}), then the Weierstrass equation reads

$$zy^2 = 4x^3 - g_2xz^2 - g_3z^3 (2.5)$$

where g_2 and g_3 are sections of \mathcal{L}^4 and \mathcal{L}^6 , respectively. Often, we will use affine coordinates with z=1. For Z to be a Calabi-Yau manifold – our main interest for the applications in this paper – one needs $\mathcal{L}=K_B^{-1}$, with K_B the canonical bundle of B. However, the description of vector bundles over Z does not require this.

First we consider in some detail SU(n) bundles. On a *single* elliptic curve, we described an SU(n) bundle by giving an n-tuple of points, determined by another equation

$$a_0 + a_2 x + a_3 y + \ldots + a_n x^{n/2} = 0.$$
 (2.6)

(If n is odd, the last term is $x^{(n-3)/2}y$.) The a_i , up to scaling, define a point in a projective space \mathbf{P}^{n-1} that parametrizes SU(n) bundles on E. Given that x and y are sections of \mathcal{L}^2 and \mathcal{L}^3 , one can think of a_i as a section of \mathcal{L}^{-i} .

Now if one has a family of elliptic curves, making up an elliptic manifold $Z \to B$, then over each $b \in B$, there is an elliptic curve E_b and a moduli space \mathbf{P}_b^{n-1} of SU(n) bundles on E_b . The \mathbf{P}_b^{n-1} 's fit together into a \mathbf{P}^{n-1} bundle over B which we will call W. By noting that the a_i can be interpreted as homogeneous coordinates for this bundle, we see that it can be constructed by projectivizing the vector bundle over B

$$\Omega = \mathcal{O} \oplus \mathcal{L}^{-2} \oplus \mathcal{L}^{-3} \oplus \ldots \oplus \mathcal{L}^{-n}. \tag{2.7}$$

Note that the exponents here are 0 and $-s_j$, where $s_j = 2, 3, 4, \ldots, n$ are the degrees of the independent Casimir operators of SU(n) (that is, if ϕ is a vector in the adjoint representation of SU(n), regarded as an $n \times n$ hermitian matrix, then the invariants are $\text{Tr } \phi^k$, for $k = 2, 3, \ldots, n$; and these have degrees $2, 3, 4, \ldots, n$). This is the form for Ω promised in the introduction.

The constant section of \mathcal{O} , when embedded as a section of $\Omega = \mathcal{O} \oplus \ldots$, gives a section of \mathcal{W} that can be characterized by the fact that the homogeneous coordinates a_i all vanish for i > 0. This means that on each fiber all of the Q_i are at infinity; in the

description of bundles by flat unitary connections, such a bundle corresponds to the trivial flat connection. This interpretation of the summand \mathcal{O} was promised in the introduction.

Analogous results for Sp(n) are easily obtained. We found in section 2.1 that an Sp(n) bundle over a single elliptic curve E in Weierstrass form is determined by an equation

$$c_0 + c_1 x + c_2 x^2 + \ldots + c_n x^n = 0. (2.8)$$

The c_i were homogeneous coordinates of a projective space \mathbf{P}^n that parametrizes Sp(n) bundles. If instead one has a family of elliptic curves, making up an elliptic manifold $Z \to B$, then we should think of the c_i as homogeneous coordinates on a \mathbf{P}^n bundle \mathcal{W} over B (whose fiber over $b \in B$ is the moduli space of Sp(n) bundles on the elliptic curve E_b that lies over b). One can think of \mathcal{W} as the projectivization of a vector bundle $\mathcal{O} \oplus \mathcal{L}^{-2} \oplus \mathcal{L}^{-4} \oplus \ldots \oplus \mathcal{L}^{-2n}$ (the exponents are clear if one recalls that x in equation (2.8) is a section of \mathcal{L}^2). Note that the exponents are 0 and -k where $k = 2, 4, 6, \ldots, 2n$ are the degrees of the Casimir invariants of Sp(n). Thus, we obtain again the form for \mathcal{W} promised in the introduction. The section of \mathcal{W} coming from the summand \mathcal{O} again corresponds on each E_b to a bundle that is related to the trivial flat connection.

2.3. Construction Of Bundles Over Elliptic Manifolds

Let us begin with a rank n complex vector bundle over Z with a hermitian-Yang-Mills SU(n) connection. This determines a holomorphic vector bundle V over Z which can be restricted to give a holomorphic bundle on each fiber. If the restriction to each fiber is semistable, it determines a section of the projective space bundle $W \to B$. The section s is not the whole story; there is additional data that we will describe shortly. But first let us explain in some detail how to construct a general section s.

The mapping from $\Omega = \mathcal{O} \oplus \mathcal{L}^{-2} \oplus \mathcal{L}^{-3} \oplus \ldots \oplus \mathcal{L}^{-n}$ to \mathcal{W} (by throwing away the zero section and dividing on each fiber by \mathbb{C}^*) gives a holomorphic line bundle over \mathcal{W} that we will call $\mathcal{O}(-1)$ (it restricts on each fiber \mathbb{P}^{n-1}_b to the \mathbb{C}^* bundle usually known by that name). The homogeneous coordinates a_k ($k = 0, 2, 3, \ldots, n$) are sections of $\mathcal{O}(1) \otimes \mathcal{L}^{-k}$. If $s : B \to \mathcal{W}$ is a section of $\mathcal{W} \to B$, then $s^*(\mathcal{O}(1))$ is a line bundle on B that we will call \mathcal{N} . Different \mathcal{N} 's can arise; the homotopy class of the section of \mathcal{W} is determined by the first Chern class of \mathcal{N} . (We will learn in section 7 how the first Chern class of \mathcal{N} is related to the second Chern class of \mathcal{V} .)

The a_k pull back under s to sections of $\mathcal{N} \otimes \mathcal{L}^{-k}$. This process can also be read in reverse: if one picks an arbitrary line bundle \mathcal{N} on B which is sufficiently ample, and picks

sections a_k of $\mathcal{N} \otimes \mathcal{L}^{-k}$ that are sufficiently generic as to have no common zeroes, then $b \to (a_0(b), a_2(b), \dots, a_n(b))$ gives a section of \mathcal{W} . Two sections coincide if and only if the corresponding a_j are proportional, so the space of sections (of given homotopy class) is itself a projective space \mathbf{P}^m for some m.

So we get an effective way to construct sections of W: pick \mathcal{N} and the a_k . Now, a suitable SU(n) bundle V over Z determines, as we have explained, such a section s. In particular, it determines an \mathcal{N} . However, the section may not uniquely determine the bundle, as we will now explain.

A section of W concretely determines an equation (2.6) (with the a_k now understood as sections over B), and this, together with (2.5), determines a hypersurface C in Z. C is an n-fold ramified cover of B, since for fixed $b \in B$, the equations (2.5) and (2.6) have n solutions. By analogy with similar structures in the theory of integrable systems, we call any such hypersurface in Z that projects to an n-fold cover of B a "spectral cover."

Although a "good" hermitian-Yang-Mills connection on an SU(n) bundle over Z determines in this way a unique spectral cover C, many different bundles may give the same spectral cover. To proceed further, we need to make a digression about the "Poincaré line bundle."

The Poincaré Line Bundle

We have already exploited the following basic fact. If E is an elliptic curve, with a distinguished point p, then the degree zero line bundles on E are parametrized by E itself; a point $Q \in E$ corresponds to the line bundle $\mathcal{L}_Q = \mathcal{O}(Q) \otimes \mathcal{O}(p)^{-1}$. Now consider the product $F = E \times E$, and think of the first factor as parametrizing degree zero line bundles on the second. Then one can aim to construct a line bundle \mathcal{P} on E, whose restriction to E0 E1, will be isomorphic to E2. In fact, one can take E2 to be the line bundle E3, where E4 is the divisor E5, the idea here is that E6 intersects E7 in the divisor E8. However, it is more symmetric to take E8 and E9 in the divisor E9. The idea is now that E9 is isomorphic to E9 if restricted to either E9 in the property just stated will be called a Poincaré line bundle.

We actually want a Poincaré line bundle for a family of elliptic curves. Suppose that one is given an elliptic manifold $\pi: Z \to B$, with a section σ . One forms the "fiber

product" $Z \times_B Z$ which consists of pairs $(z_1, z_2) \in Z \times Z$ such that $\pi(z_1) = \pi(z_2)$.⁶ The equation $z_1 = z_2$ defines a divisor in $Z \times_B Z$ which we will call Δ . $Z \times_B Z$ can be mapped to Z in two ways, by forgetting z_2 or z_1 ; the two maps are called π_1 and π_2 . One can also simply project $Z \times_B Z$ to B by $(z_1, z_2) \to \pi(z_1)$ (which equals $\pi(z_2)$); we will call this map $\widetilde{\pi}$. For any $b \in B$, $\widetilde{\pi}^{-1}(b)$ is a copy of $E_b \times E_b$, where $E_b = \pi^{-1}(b)$.

By a Poincaré line bundle \mathcal{P}_B over $Z \times_B Z$ we mean a line bundle which on each $E_b \times E_b$ is a Poincaré line bundle in the previous sense, and which is trivial when restricted to $\sigma \times Z$ or $Z \times \sigma$. One might think that one should take \mathcal{P}_B to be $\mathcal{O}(D)$, where $D = \Delta - \sigma \times Z - Z \times \sigma$. This line bundle certainly restricts appropriately to each $E_b \times E_b$. Its restriction to $\sigma \times Z$ or $Z \times \sigma$ is however non-trivial – in fact, it is isomorphic to the pullback $\widetilde{\pi}^*(\mathcal{L})$ of $\mathcal{L} \to B$, as we will show in section 7. For the desired Poincaré line bundle, we take $\mathcal{P}_B = \mathcal{O}(D) \otimes \widetilde{\pi}^*(\mathcal{L}^{-1})$.

Bundles From Sections

Now we want to return to our problem of understanding how a vector bundle over Z is to be constructed from a section $s: B \to \mathcal{W}$, or equivalently from the spectral cover C. We start with $Y = C \times_B Z$, which is defined as the subspace of $Z \times_B Z$ with $z_1 \in C$. The map π_2 (forgetting z_1) maps $Y \to Z$. Y is an n-fold cover of Z, since $C \to B$ was an n-fold cover.

Suppose we are given any line bundle \mathcal{R} over Y. Away from branch points of the map $\pi_2: Y \to Z$, one can define a rank n vector bundle V over Z as follows. Lying above any given $z \in Z$, there are n points $y_1, \ldots, y_n \in Y$; take the fiber V_z of V at z to be $\bigoplus_{i=1}^n \mathcal{R}_{y_i}$ (where \mathcal{R}_y is the fiber of \mathcal{R} over $y \in Y$). The bundle V so defined can actually be extended over all of Z by using a more powerful definition based on the "push-forward" operation in algebraic geometry; one defines a section of V over a small open set $U \subset Z$ to be a section of \mathcal{R} over $\pi_2^{-1}(U)$. The resulting vector bundle over Z is denoted $V = \pi_{2*}(\mathcal{R})$.

Here let us point out a technical fact about this construction. The bundles produced in this way have the property that their restrictions to most, but not all, fibers carry flat SU(n) connections. If $b \in B$ is such that its pre-image in the spectral cover C consists of n distinct points, then it is clear from the construction that the restriction of the resulting vector bundle to the fiber E_b is a sum of n line bundles of degree zero (given by the n points in E_b) and hence carries a flat SU(n) connection. At the branch points of V something entirely different happens [23]. For example, if the pre-image of b in the spectral cover

⁶ In this paper, it will be possible to ignore singularities of this fiber product.

C consists of n-2 points of multiplicity one and a point of multiplicity two then the restriction of V to E_b is a direct sum of n-2 line bundles and a rank two bundle that is a non-trivial extension of a line bundle by a second (isomorphic) line bundle. This bundle admits no flat SU(n) connection. So, although the section of \mathcal{W} can be viewed as defining a varying family of holomorphic bundles with flat connections on the fibers of Z over B, to fit these fundles together to make a holomorphic bundle on Z we must replace some of the flat bundles by non-isomorphic, S-equivalent bundles. After fitting these bundles together, we often produce a stable bundle which then carries a hermitian-Yang-Mills connection. But this connection is not obtained by gluing together the original flat connections. In many situations, this construction yields the generic stable bundle over Z.

Reconstruction Of A Bundle From The Spectral Cover

Suppose we start with a vector bundle V over Z, and use it as above to construct a spectral cover C of B. To recover V from C, the basic idea is to start with a suitable line bundle \mathcal{R} over $C \times_B Z$, and obtain V as $\pi_{2*}(\mathcal{R})$.

The instructive first case to consider is that in which $\mathcal{R} = \mathcal{P}_B$, the Poincaré line bundle over $Z \times_B Z$ restricted to $C \times_B Z$. Recall that to construct the spectral cover C from the vector bundle V, the idea was that the restriction of V to E_b was isomorphic⁷ to $\mathcal{L}_{Q_1(b)} \oplus \ldots \oplus \mathcal{L}_{Q_n(b)}$ for some points $Q_1(b), \ldots, Q_n(b) \in E_b$; we then defined C to be an n-sheeted cover of B such that the points over b are $Q_1(b), \ldots, Q_n(b)$. If we define $V' = \pi_{2*}(\mathcal{P}_B)$, then from the definitions of π_{2*} and \mathcal{P}_B , the restriction of V' to E_b is indeed equivalent to $\mathcal{L}_{Q_1} \oplus \ldots \oplus \mathcal{L}_{Q_n}$.

So V and V' are equivalent on each E_b . But this does not necessarily imply that V = V'. In fact, the above construction can be generalized as follows. Let S be any line bundle over C, and let $V' = \pi_{2*}(\mathcal{P}_B \otimes S)$. Then the isomorphism class of the restriction of V' to any E_b is independent of S, since S (being trivial locally along C) is trivial when restricted to a neighborhood of the inverse image of any given $b \in B$.

This is the only ambiguity in the reconstruction of a vector bundle from its spectral cover in the following sense. The main theorem of chapter 7 of [23] asserts that if the base B is one-dimensional, then any generic V can be reconstructed from its spectral cover C as $V = \pi_{2*}(\mathcal{P}_B \times \mathcal{S})$ for some \mathcal{S} . For B of dimension bigger than one, it is too much to expect

 $^{^{7}}$ Or in general S-equivalent.

 $^{^8}$ The argument in that reference is formulated for rank two bundles, but that restriction was needed primarily in giving a precise description of the possible exceptional behavior; in describing a generic V in the above-stated form, there is no restriction to rank two.

this to be true for all bundles, but it is true for the bundles that can be most naturally constructed via spectral covers; and these suffice to construct (open dense subsets of) some components of the moduli space of all bundles. To understand those components – all we will aim for in this paper – we "only" need to understand spectral covers and line bundles over them.

To summarize, we have here described in some detail the construction of bundles from spectral covers for G = SU(n). A similar construction should be possible for G = Sp(n).

2.4. A Note On Jacobians

We will here make a few remarks that are not needed for understanding most of the paper, but are background for the comparison between the moduli space of spectral covers and the moduli space of F theory complex structures that we will make in section six. These remarks concern the role in the construction of stable bundles of certain Jacobians and abelian varieties.

In our applications, B, as the base of a Calabi-Yau fibration, is simply-connected.

If B is of dimension one, and therefore in practice $B = \mathbf{P}^1$, then the n-sheeted cover C of B is a Riemann surface of higher genus. S is then not completely fixed by its first Chern class; any given S could be modified by twisting by a flat line bundle on C. Such flat line bundles are classified by the Jacobian J(C) of C. The moduli space of stable bundles over Z is fibered over the space of C's, with the fiber being this Jacobian.

When B is of dimension one, the Calabi-Yau manifold Z is actually a K3 surface, and the moduli spaces of bundles are hyper-Kähler. The space of sections of W is a Kähler manifold but not hyper-Kähler; in fact, it is a projective space \mathbf{P}^m for some m, as was seen above. The Jacobian J(C) has the same dimension m; in fact the whole moduli space looks locally, near the zero section of the bundle of Jacobians, like the cotangent bundle $T^*\mathbf{P}^m$. (Indeed, C is a curve in Z; if N_C is the normal bundle to C in Z, then the tangent space to the space of spectral covers at C is $H^0(C, N_C)$, which because Z has trivial canonical bundle is dual to $H^1(C, \mathcal{O}_C)$, which is the tangent space to the Jacobian of C.) In heterotic string compactification on Z, the m chiral superfields parametrizing the choice of C combine with m chiral superfields parametrizing the Jacobian of C to make m hypermultiplets.

Though we have so far considered only SU(n) and Sp(n) bundles, an analogous picture holds for any G. The moduli space \mathcal{M} of bundles is fibered over the space \mathcal{Y} of sections of a certain weighted projective space bundle that we will construct; these sections generalize the notion of the spectral cover. \mathcal{Y} is itself a weighted projective space, as we will see. The fiber of the map from \mathcal{M} to \mathcal{Y} is an abelian variety of dimension equal to that of \mathcal{Y} ; the total space \mathcal{M} is hyper-Kähler and looks locally (near a certain "zero section" of $\mathcal{M} \to \mathcal{Y}$) like $T^*\mathcal{Y}$.

Duality with F theory relates the heterotic string on the K3 surface Z to F theory on a Calabi-Yau threefold X that is fibered over B with K3 fibers. The part of the F theory moduli space that is related to the moduli space of bundles on Z must, if the duality is correct, have a fibration analogous to $\mathcal{M} \to \mathcal{Y}$, with the fiber being an abelian variety of dimension equal to the base. In F theory, abelian varieties (and more general complex tori) appear in the moduli space of vacua because an F theory vacuum is parametrized among other things by the choice of a point in the complex torus $H^3(X, \mathbf{R})/H^3(X, \mathbf{Z})$, which is known as the intermediate Jacobian J(X) of X. (The appearance of J(X) is readily seen if one compactifies on another circle to convert to M-theory; in that formulation, J(X) parametrizes the periods of the vacuum expectation value of the three-form potential of eleven-dimensional supergravity.) The F theory moduli space is fibered over the space of complex and Kähler structures on X with fiber J(X).

In duality between the heterotic string and F theory, heterotic string vacua in which the structure group of the $E_8 \times E_8$ gauge bundle reduces to $G \times E_8$ for some $G \subset E_8$ correspond to points in F theory moduli space in which the K3 fibration $X \to B$ has a section θ of singularities; the precise nature of the singularities for arbitrary G was described in [20]. One factor in the heterotic string moduli space is the moduli space \mathcal{M} of G bundles over Z, with its fibration $\mathcal{M} \to \mathcal{Y}$. In the duality, \mathcal{Y} should be mapped to the space of certain data parametrizing the geometry of X near θ ; the details of which parameters should be relevant for given G were worked out in [3,20]. The abelian variety that is the fiber of $\mathcal{M} \to \mathcal{Y}$ should correspond to a certain factor that splits off from J(X)when X develops the prescribed type of singularity.

In section six, we will compare the heterotic string to F theory by comparing the space \mathcal{Y} (as determined by our analysis of G bundles) to the appropriate data describing the behavior of X near θ (as determined in [3,20]). We will not compare the abelian varieties that appear on the two sides, for two roughly parallel reasons. (1) On the heterotic string side, while we will determine the analog of \mathcal{Y} for general G, we do not have equally good control over the abelian varieties except for G = SU(n). (2) On the F theory side, while the complex structure parameters that should be related to G bundles have been determined for general G [3,20], the appropriate factor in J(X) (which presumably involves cycles with

a particular behavior near θ) has not yet been described. Identifying the abelian varieties on the two sides and comparing them is an interesting question.

At the end of section 4, we will state a conjectural description of the abelian variety for the case $G = E_8$.

Fibrations Over Surfaces

Now let us move on to the case that the base B of the elliptic fibration is of dimension bigger than one. In practice, the main case for physics is that B is of dimension two, so that the elliptic manifold $Z \to B$ is a threefold and the K3-fibered manifold $X \to B$ is a fourfold. Much of the discussion above still applies. The moduli spaces \mathcal{M} of bundles on Z have fibrations $\mathcal{M} \to \mathcal{Y}$ where \mathcal{Y} is a space of spectral covers (in a generalized sense) and the fiber is an abelian variety. Likewise, on the F theory side, there is a space of complex moduli of X with fibered over it the abelian variety $H^3(X, \mathbf{R})/H^3(X, \mathbf{Z})$. Our test of the duality in section six involves comparing \mathcal{Y} to the appropriate part of the moduli space of complex structures on X, without trying to compare the abelian varieties.

For the present purposes, the main change in going from B of dimension one to B of dimension two is that the base and fiber of the fibration $\mathcal{M} \to \mathcal{Y}$ need not have equal dimensions, and in particular the fiber vanishes in many of the simplest examples. This is so on the heterotic string side because in many simple examples, the spectral cover C of the base B of an elliptic Calabi-Yau threefold is simply-connected; when that is so, a line bundle $S \to C$ is completely determined by its first Chern class, and the choice of S does not introduce any abelian variety. On the F theory side, Calabi-Yau four-folds can very commonly have $H^3 = 0$, so that the supergravity three-form has no periods, and there is no Jacobian to consider. Thus, in many instances, our check of heterotic string/F theory duality in section 7 is more complete for B a surface than for B a curve, in the sense that the abelian varieties over which one does not have good control are actually trivial.

It would be very interesting, of course, to show that the relevant part of $H^3(X)$ is non-trivial precisely when $H^1(C)$ is non-trivial, and to compare the resulting abelian varieties.

3. Reduction To The Simply-Laced Case

3.1. Simply-Laced Subgroup

We now for the moment think of semi-stable holomorphic $G_{\mathbf{C}}$ bundles on an elliptic curve E in terms of representations of the fundamental group of E in the compact form of

G. Such a representation is determined by two commuting elements of G. For G simply-connected, these two elements can be conjugated into a maximal torus T [29] in a way that is unique up to the action of the Weyl group W. The moduli space of semi-stable G bundles on E is thus isomorphic to $(T \times T)/W$, where W acts in the natural fashion on both factors of T.

We propose to use this in the following situation. Let G be a simple, connected, and simply-connected group that is not simply-laced. G then has a canonical simply-laced subgroup G' that is generated by the long roots of G. The embedding of G' in G gives an isomorphism of the maximal torus of G' with that of G. The Weyl group G' is however a subgroup of the Weyl group G' of G' in fact, it is a normal subgroup; there is a group homomorphism

$$1 \to W' \to W \to \Gamma \to 1 \tag{3.1}$$

for some finite group Γ . Γ is a group of outer automorphisms of G'. If $\mathcal{M} = (T \times T)/W$ and $\mathcal{M}' = (T \times T)/W'$ are the moduli spaces of G and G' bundles over E, then

$$\mathcal{M} = \mathcal{M}'/\Gamma. \tag{3.2}$$

We will use this to describe the moduli space of G bundles given the moduli space of G' bundles, and thus to reduce the description of the moduli space to the simply-laced case.

In practice, there are four examples of this construction:

- (1) For G = Sp(n), $G' = SU(2)^n$. Γ is the group S_n of permutations of the n copies of SU(2) in G'.
- (2) For $G = G_2$, G' = SU(3). Γ is the group \mathbb{Z}_2 of "complex conjugation" that exchanges the three-dimensional representation of SU(3) with its dual.
- (3) For G = Spin(2n+1), G' = Spin(2n). Γ is the group \mathbb{Z}_2 generated by the outer automorphism of Spin(2n) that exchanges the two spin representations of Spin(2n).
- (4) For $G = F_4$, G' = Spin(8). Γ is the triality group S_3 that permutes the three eight-dimensional representations of Spin(8).

We consider the four examples in turn.

Sp(n) Revisited

We consider the elliptic curve E to be given by a Weierstrass equation in the x-y plane. The moduli space of SU(2) bundles on E is parametrized, as we learned in the last section, by the choice of a point x which can be regarded as the root of a spectral equation

$$a_0 + a_2 x = 0. (3.3)$$

A G' bundle for $G' = SU(2)^n$ is therefore given by an ordered n-tuple x_1, x_2, \ldots, x_n . The group Γ acts by permutation of the x_i , so the relation $\mathcal{M} = \mathcal{M}'/\Gamma$ says in this case that the moduli space of Sp(n) bundles over E is the space of unordered n-tuples x_1, x_2, \ldots, x_n . This is the description that we obtained "by hand" in section 2. As we explained there, the space of unordered n-tuples can be identified with the space of spectral equations of the form

$$c_0 + c_1 x + c_2 x^2 + \ldots + c_n x^n = 0. (3.4)$$

Furthermore, in the case of an elliptically fibered manifold $Z \to B$, the c_i are homogeneous coordinates for a projective space bundle $W \to B$, as we explained in section 2.

G_2 Bundles

Now we consider the case that $G = G_2$, G' = SU(3). From what we learned in section 2, an SU(3) bundle over E is determined by a spectral equation

$$a_0 + a_2 x + a_3 y = 0 (3.5)$$

whose roots are three points $Q_1, Q_2, Q_3 \in E$ with $Q_1 + Q_2 + Q_3 = 0$. The moduli space \mathcal{M}' of SU(3) bundles is thus a copy of \mathbf{P}^2 with homogeneous coordinates a_0, a_2 , and a_3 .

The exchange of an SU(3) bundle with its dual amounts to $Q_i \to -Q_i$, or equivalently $y \to -y$. The moduli space of G_2 bundles is therefore $\mathcal{M} = \mathcal{M}'/\mathbb{Z}_2$, where \mathbb{Z}_2 acts on \mathcal{M}' by $a_3 \to -a_3$. Thus \mathcal{M} is a weighted projective space $\mathbf{WP}^2_{1,1,2}$ with homogeneous coordinates a_0 , a_2 , and a_3^2 . This is Looijenga's theorem for G_2 .

In the case of an elliptically fibered manifold $Z \to B$, for each $b \in B$ one has a weighted projective space \mathbf{WP}_b^2 parametrizing G_2 bundles on the fiber E_b of Z over b. The \mathbf{WP}_b^2 fit together as fibers of a \mathbf{WP}^2 bundle \mathcal{W} over B. The objects a_0, a_1 , and a_2 must now be interpreted as sections of \mathcal{O} , \mathcal{L}^{-2} , and \mathcal{L}^{-3} over B. So the homogeneous coordinates a_0, a_2 , and a_3^2 of \mathcal{W} are sections of \mathcal{O} , \mathcal{L}^{-2} , and \mathcal{L}^{-6} . Since the fundamental Casimir invariants of G_2 are of degrees 2 and 6, this confirms for the case $G = G_2$ the claim made in the introduction about the structure of \mathcal{W} .

The section of W coming from the constant section of the summand \mathcal{O} corresponds to the trivial G_2 bundle on each fiber, since this was true for SU(3). (A similar statement holds in the other cases below and will not be repeated.)

Spin(2n+1) Bundles

In the last two cases, G' is a spin group Spin(2n) or Spin(8). We have not yet discussed Looijenga's theorem for the spin groups (we will do so in section 5), but we will here show that by analogy with the cases considered above, Looijenga's theorem for Spin(2n + 1) and for F_4 follows from the corresponding statement for the simply-laced groups Spin(2n) and Spin(8).

The fundamental Casimir invariants of Spin(2n) are of degrees $2, 4, 6, \ldots, 2n-2$ and n. If ϕ is an element of the adjoint representation, regarded as an antisymmetric $2n \times 2n$ matrix, then the invariants are $w_k = \operatorname{Tr} \phi^{2k}$ for $1 \leq k \leq n-1$, of degree 2k, and the Pfaffian $w' = \operatorname{Pf}(\phi)$, which is of degree n. The outer automorphism of Spin(2n), which generates $\Gamma = \mathbf{Z}_2$, acts trivially on all Casimir invariants except w', which changes sign.

Looijenga's theorem says that the moduli space \mathcal{M}' of Spin(2n) bundles is a weighted projective space $\mathbf{WP}_{1,1,1,1,2,...,2}^n$ (four weights one and the rest two). \mathcal{M}' has homogeneous coordinates s_k , $k=0,\ldots,n-1$, in natural correspondence with the invariants w_k , and s', in correspondence with w'. The coordinates s_0, s_1, s_2 , and s' are of weight 1 and the others of weight 2. These statements can be proved by methods of section 5. The Spin(2n+1) moduli space is hence $\mathcal{M}=\mathcal{M}'/\mathbf{Z}_2$, where (in view of its action on the Casimir invariants), the generator of \mathbf{Z}_2 leaves the s_k invariant and maps $s' \to -s'$. Thus \mathcal{M} is a weighted projective space $\mathbf{WP}_{1,1,1,2,...,2}^n$ (three 1's and the rest 2's), with homogeneous coordinates s_k and $(s')^2$ (the weight one coordinates are s_0, s_1, s_2). This is Looijenga's theorem for Spin(2n+1).

In the case of an elliptically fibered manifold $Z \to B$, the s_k and s' become sections of \mathcal{L}^{-2k} and \mathcal{L}^{-n} , respectively. The usual bundle \mathcal{W}' of weighted projective spaces (whose fiber above $b \in B$ is the moduli space of Spin(2n) bundles on the fiber above b) has s_k and s' as homogeneous coordinates. These assertions (which are the Spin(2n) case of the description of \mathcal{W}' claimed in the introduction) can again be proved using the methods of section 5. The analogous weighted projective space bundle \mathcal{W} for Spin(2n+1) therefore has homogeneous coordinates $s_0, s_1, s_2, \ldots, s_{n-1}$, and $(s')^2$, of weights $(1, 1, 1, 2, \ldots, 2)$ and these homogeneous coordinates are sections of \mathcal{L}^{-2k} and \mathcal{L}^{-2n} , respectively. This is the description promised in the introduction of the weighted projective space bundle for

Spin(2n+1). Note that the fundamental Casimir invariants of Spin(2n+1) are of degree $2, 4, 6, \ldots, 2n$.

F_4 Bundles

For $G = F_4$ the story is similar, but slightly more complicated as Γ is the nonabelian triality group S_3 .

In this case G' = Spin(8). The Casimirs are w_k , $1 \le k \le 3$, of degree 2k, and w', of degree 4. Γ acts trivially on w_0, w_1 , and w_3 , but w_2 and w' transform in an irreducible two-dimensional representation.

The moduli space \mathcal{M}' of Spin(8) bundles on an elliptic curve E is, according to Looijenga's theorem, a weighted projective space $\mathbf{WP}_{1,1,1,1,2}^4$ where in notation above the weight one homogeneous coordinates are s_0, s_1, s_2 , and s', while s_3 has weight 2. Because of the behavior of the Casimirs, Γ acts trivially on s_0, s_1 , and s_3 while s_2 and s' transform in an irreducible two-dimensional representation ρ . The ring of invariants in the representation ρ is a polynomial ring generated by a quadratic polynomial $A(s_2, s')$ and a cubic polynomial $B(s_2, s')$.

The F_4 moduli space $\mathcal{M} = \mathcal{M}'/\Gamma$ is hence a weighted projective space $\mathbf{WP}_{1,1,2,2,3}^4$ with homogeneous coordinates $s_0, s_1, s_3, A(s_2, s')$, and $B(s_2, s')$ of weights 1, 1, 2, 2, 3. This is Looijenga's theorem for F_4 .

In the case of an elliptic manifold $Z \to B$, the usual weighted projective space bundle \mathcal{W}' for Spin(8) has homogeneous coordinates s_0, s_1, s_2, s', s_3 (of weights 1, 1, 1, 1, 2) which are sections respectively of $\mathcal{O}, \mathcal{L}^{-2}, \mathcal{L}^{-4}, \mathcal{L}^{-4}$, and \mathcal{L}^{-6} . (These assertions can again be proved using the methods of section 5.) Restricting to the Γ -invariants, the weighted projective space bundle \mathcal{W} for F_4 therefore has homogeneous coordinates $s_0, s_1, s_3, A(s_2, s')$, and $B(s_2, s')$, of weights 1, 1, 2, 2, 3, which are sections respectively of $\mathcal{O}, \mathcal{L}^{-2}, \mathcal{L}^{-6}, \mathcal{L}^{-8}$, and \mathcal{L}^{-12} . This is the promised description of the weighted projective space bundle for F_4 . Note that the fundamental Casimir invariants of F_4 are of degrees 2, 6, 8, and 12.

3.2. Embedding In A Simply-Laced Group

We will now more briefly explain another way to reduce Looijenga's theorem to the simply-laced case.

So far, to understand bundles for a non-simply laced group G, we have compared G bundles to G' bundles, where G' is a canonical simply-laced subgroup of G. An alternative way to reduce Looijenga's theorem to the simply-laced case uses the fact that every

simple and simply-connnected but non-simply-laced group G can be, in a unique fashion, embedded in a simply-laced group G'' in such a way that G is the subgroup of G'' left fixed by an outer automorphism ρ . (This construction has been used in understanding the appearance of non-simply-laced gauge groups in F theory [20,21].) The automorphism ρ will act on the moduli space \mathcal{M}'' of G'' bundles on E, and the desired moduli space \mathcal{M} of G bundles is a component of the subspace of \mathcal{M}'' left fixed by ρ . In fact, \mathcal{M} is the component of the fixed point set that contains the point in \mathcal{M}'' that corresponds to the trivial flat connection.

According to Looijenga's theorem for G'', \mathcal{M}'' is a weighted projective space whose homogeneous coordinates are in correspondence with the identity and the Casimir invariants of G''. The desired component of the fixed point set of ρ has homogeneous coordinates in correspondence with the identity and the ρ -invariant Casimirs of G''. Looijenga's theorem for G is thus a consequence of Looijenga's theorem for G'' together with an appropriate statement about the action of ρ on the Casimirs of G''. Here is how things work out in the four cases:

- (1) For G = Sp(n), G'' = SU(2n) and ρ is the outer automorphism of G'' that acts by "complex conjugation." The Casimirs of G'' are $\operatorname{Tr} \phi^k$ for $k = 2, 3, 4, \ldots, 2n$. ρ acts by multiplication by $(-1)^k$, so the ρ -invariant Casimirs are $\operatorname{Tr} \phi^{2m}$ for $m = 1, 2, \ldots, n$. These are also the Casimirs of Sp(n), and they appear with weight one for both SU(2n) and Sp(n). Indeed, this relation between Sp(n) bundles and SU(2n) bundles was already described at the end of section 2.1.
- (2) For $G = G_2$, G'' = Spin(8), and ρ is the triality automorphism. Of the Casimirs of G'', $\operatorname{Tr} \phi^2$ and $\operatorname{Tr} \phi^6$ are ρ -invariant, and the quartic Casimirs transform non-trivially. So the ρ -invariant homogeneous coordinates for G'' are associated with the identity, $\operatorname{Tr} \phi^2$, and $\operatorname{Tr} \phi^6$, the degrees being 0, 2, 6 and the weights 1, 1, 2. These are the right degrees and weights for G_2 .
- (3) For G = Spin(2n+1), G'' = Spin(2n+2), and ρ is a "reflection of one coordinate" that reverses the sign of the Pfaffian and leaves fixed the other Casimirs. The ρ -invariant homogeneous coordinates for G'' are hence associated with the identity and $\operatorname{Tr} \phi^k$, $k = 2, 4, 6, \ldots, 2n$, and have weights $1, 1, 1, 2, 2, \ldots, 2$. These are the correct degrees and weights for Spin(2n+1).
- (4) The final example is $G = F_4$. For this case, $G'' = E_6$, and ρ is the involution that reverses the sign of the Casimirs of degree 5 and 9 and leaves fixed the others. The

surviving homogeneous coordinates – of weights 1, 1, 2, 2, 3 and associated with Casimirs of degree 0, 2, 6, 8, 12 – have the appropriate degrees and weights for F_4 .

Note that in this construction based on a simply-laced group G'' containing G, we want the ρ -invariant Casimirs, which are homogeneous coordinates on a subspace of \mathcal{M}'' , while in the previous construction based on a simply-laced subgroup G', we wanted the Γ -invariant functions of the Casimirs (not only the linear functions), which are functions on \mathcal{M}'/Γ .

4. Construction Via Del Pezzo Surfaces

We here explain how to construct the moduli space of G bundles on an elliptic curve, for certain G, via del Pezzo surfaces. We first give a somewhat abstract account and then proceed to explicit formulas.

A del Pezzo surface S is a two-dimensional complex surface whose anticanonical line bundle is ample. The second Betti number $b_2(S)$ of such a surface ranges from 1 to 9; we set $k = b_2(S) - 1$. In practice, a smooth del Pezzo surface (we incorporate singularities later) is isomorphic either to $\mathbf{P}^1 \times \mathbf{P}^1$ or to \mathbf{P}^2 with k general points blown up for $0 \le k \le 8$. We will restrict ourselves to the latter case. ($\mathbf{P}^1 \times \mathbf{P}^1$ would be an exception for many of the statements and is not very useful for the applications.)

The intersection form on $L = H^2(S, \mathbf{Z})$ is isomorphic over \mathbf{Z} to the form

$$u_0^2 - u_1^2 - \dots - u_k^2. (4.1)$$

where we can pick coordinates so that u_0 generates the second cohomology of an underlying \mathbf{P}^2 and the u_i , i > 0, are exceptional divisors created by blowing up k points. Note in particular that this gives a basis for L consisting of the classes of algebraic cycles, so that $H^{2,0}(S) = 0$ and every $y \in L$ is the first Chern class of a holomorphic line bundle \mathcal{L}_y .

Let T_S be the tangent bundle to S and $x = c_1(T_S)$. In the coordinates just described

$$x = 3u_0 - u_1 - \dots - u_k. (4.2)$$

(The anticanonical class of \mathbf{P}^2 is $3u_0$, and all exceptional divisors created by blowups enter with coefficient -1.) Evidently $x^2 = 9 - k$ and (as $x^2 > 0$ follows from ampleness of the anticanonical divisor) one sees the restriction to $k \leq 8$. Let L' be the sublattice of L consisting of points y with $x \cdot y = 0$. Then the intersection form on L' is negative

definite and moreover (since all coefficients in (4.2) are odd) is even. Moreover, as L has a unimodular intersection form, the discriminant of L' is equal to $x^2 = 9 - k$.

For k = 8, the intersection form on L' is thus even and unimodular and of rank eight and so (after reversing the sign of the quadratic form to make it positive definite) is the conventional intersection form of the E_8 lattice. More generally, for any $k \leq 8$, L' can be similarly identified with the root (or coroot) lattice of a simply-laced simple Lie group G of rank k which we will call E_k . For k = 6, 7, E_6 and E_7 are the groups usually called by those names, while $E_5 = D_5$, $E_4 = A_4$, etc. In what follows, we mainly consider E_6 , E_7 , and E_8 .

One can also see in a similar way the weight lattice of E_n (which is defined as the dual of the root lattice). It is $L'' = L/x\mathbf{Z}$ (where $x\mathbf{Z}$ is the one-dimensional sublattice of L generated by x). Notice that the pairing on L induces a perfect pairing $L' \otimes L'' \to \mathbf{Z}$ identifying L'' with the dual of L'.

The center of E_n is isomorphic to L''/L'. Because $x^2 = 9 - k$, this is isomorphic to $\mathbf{Z}/(9-k)\mathbf{Z}$.

A Note On Flat Connections

Before explaining how to use del Pezzo surfaces to make bundles on elliptic curves, we first describe a slightly alternative way to think about semistable G bundles on an elliptic curve E, for simply-connected G.

Such a bundle is equivalent to a flat connection A with values in the maximal torus T. Now every point \mathbf{w} in the weight lattice L'' of G determines a representation $\rho_{\mathbf{w}}$ of T and, by taking the flat connection A in the representation $\rho_{\mathbf{w}}$, we get a line bundle $\mathcal{L}_{\mathbf{w}}$ over E. This line bundle determines a point on the Jacobian of E (which of course is isomorphic to E itself).

This correspondence $\mathbf{w} \to \mathcal{L}_{\mathbf{w}}$ determines a homomorphism from L'' to the Jacobian of E. Conversely, from such a homomorphism one can recover a T-valued flat connection A and therefore a G bundle. (Of course, $\operatorname{Hom}(L'', E) \cong (L'')^* \otimes E = L' \otimes E$.)

As $L'' = L/x\mathbf{Z}$, a homomorphism from L'' to E is the same as a homomorphism from L to E that maps x to zero.

A homomorphism to E from the root lattice $L' \subset L''$ would determine the $\mathcal{L}_{\mathbf{w}}$'s for \mathbf{w} a weight of the adjoint representation, but not for all weights. So this would determine a flat bundle on E with structure group $\operatorname{ad}(E_k)$ (which is the quotient of E_k by its center).

The identifications of L' and L'' with the root and weight lattices of $G = E_k$ are natural only up to the action of the Weyl group of E_k . But two T bundles over E that differ by the action of the Weyl group on T determine isomorphic E_k bundles. So homomorphisms from L' or L'' to E do determine well-defined $\operatorname{ad}(E_k)$ and E_k bundles, respectively, over E.

4.1. Bundles From Del Pezzos

Now we are in a position to explain how to build E_k bundles over an elliptic curve given the appropriate del Pezzo.

The anticanonical bundle of a del Pezzo surface S has a non-zero holomorphic section. The existence of such a section can be proved via Riemann-Roch (or seen explicitly, as we do below). In general, on an n-dimensional complex manifold, a section of the anticanonical bundle vanishes on an n-1-dimensional Calabi-Yau submanifold; in the present case, n-1=1, so this Calabi-Yau submanifold is in fact an elliptic curve E.

We have already observed that every point $y \in L = H^2(S, \mathbf{Z})$ is the first Chern class of a holomorphic line bundle \mathcal{L}_y . Of course

$$\mathcal{L}_{y+y'} = \mathcal{L}_y \otimes \mathcal{L}_{y'}. \tag{4.3}$$

Now fix a particular anticanonical divisor E, of genus one. For $y \in L'$, we have $y \cdot x = 0$, and this translates to the fact that the restriction of \mathcal{L}_y to E (which we will simply denote as \mathcal{L}_y) is of degree zero. So \mathcal{L}_y defines a point in the Jacobian of E. Because of (4.3), the map $y \to \mathcal{L}_y$ is a homomorphism from L' to the Jacobian of E. According to the Torelli theorem, the moduli space of such pairs S, E is isomorphic to the set of homomorphisms $L' \to E$ modulo the action on L' of the Weyl group of E_k .

For k = 8, L'' = L', and this homomorphism determines an E_8 bundle over E.

For k < 8, a homomorphism from L' would determine only an $\mathrm{ad}(E_k)$ bundle. But suppose we are given a distinguished $(9-k)^{th}$ root \mathcal{M} of the restriction to E of the anticanonical bundle \mathcal{L}_x of S. Then we can map L to the Jacobian of E by $y \to \mathcal{L}_y \mathcal{M}^{-y \cdot x}$. This homomorphism maps x to zero (since $\mathcal{M}^{-x \cdot x} \otimes \mathcal{L}_x$ is trivial), so it induces a homomorphism from L'' to the Jacobian, which will determine an E_k bundle.

The basic strategy can now be stated. We will fix an anticanonical divisor E in a del Pezzo surface S, and let the complex structure of S vary, keeping fixed E and the $(9-k)^{th}$ root mentioned above. Every complex structure on S will determine an E_k bundle on E,

and by considering a suitable family of complex structures, we will get the moduli space of E_k bundles on S. We will consider this construction in some detail for k = 8, 7, 6.

Up to this point, we have tried to be conceptual, but in what follows we will put more emphasis on being explicit.

4.2. Construction Of Bundles For E_6, E_7, E_8

E_8 Bundles

The del Pezzo surface with k = 8 can be constructed as a hypersurface S of degree six in a weighted projective space $\mathbf{WP}_{1,1,2,3}^3$, with homogeneous coordinates u, v, x, y. S may be defined by an equation such as

$$y^{2} = \alpha x^{3} + \beta x v^{4} + \gamma u^{6} + \delta u^{4} x + \ldots + \epsilon v^{6}.$$
(4.4)

S is a del Pezzo surface simply because the sum of the weights, namely w = 1+1+2+3=7, is bigger than the degree of the hypersurface, which is d = 6. That it has k = 8 can be shown, for instance, by computing the Euler characteristic of S by standard methods.

The anticanonical divisor of S is of degree equal to the difference w - d = 1. So for instance the degree one hypersurface u = 0 is an anticanonical divisor. This divisor is given by an equation of weighted degree six in v, x, and y:

$$y^2 = \alpha x^3 + \beta x v^4 + \epsilon v^6 + \dots \tag{4.5}$$

(Only some representative terms are indicated explicitly.) This equation defines an elliptic curve E in $\mathbf{WP}_{1,2,3}^2$. By an automorphism of $\mathbf{WP}_{1,2,3}^2$, this equation can be put in a standard Weierstrass form

$$y^2 = 4x^3 - q_2xv^4 - q_3v^6. (4.6)$$

Note that this curve is really an elliptic curve; there is a distinguished point on it, namely v = 0.9

⁹ If one blows up the point u=v=0, one gets a surface \widehat{S} which is elliptic (the map that forgets x and y is a map $\widehat{S} \to \mathbf{P}^1$ with elliptic fibers) and has a distinguished section σ consisting of the exceptional divisor produced in the blow-up. Conversely, given such a rational elliptic surface \widehat{S} with section σ , a degree one del Pezzo surface S can be produced by blowing down σ . This gives a natural isomorphism from degree one del Pezzo surfaces to rational elliptic surfaces with section.

As explained above, we want to consider the general deformation of the complex structure of S keeping E fixed. To construct this general deformation, we add to (4.4) all possible terms of degree six that vanish at u = 0, and divide by automorphisms of $\mathbf{WP}^3_{1,1,2,3}$ that vanish at u = 0. The automorphisms in question are (i) u dependent translations of x, y, z such as $y \to y + \epsilon ux + \epsilon' u^3 + \ldots$; and (ii) rescaling of $u, u \to w^{-1}u$ with $u \in \mathbb{C}^*$. Dividing by (i) can be accomplished by suppressing all u-dependent terms divisible by the v, x, and y derivatives of the polynomial

$$P = 4x^3 - g_2xv^4 - g_3v^6 - y^2 (4.7)$$

whose vanishing defines E.

Assuming that $g_3 \neq 0$, dividing by symmetries of type (i) can be accomplished by suppressing all u-dependent terms divisible by y, x^2 , or v^5 . (For $g_3 = 0$, the division by the type (i) symmetries must be accomplished in a somewhat different way; this will have important consequences later.) The general deformation of interest, modulo the automorphisms of type (i), can thus be described by an equation

$$y^{2} = 4x^{3} - g_{2}xv^{4} - g_{3}v^{6} + (\alpha_{6}u^{6} + \alpha_{5}u^{5}v + \alpha_{4}u^{4}v^{2} + \alpha_{3}u^{3}v^{3} + \alpha_{2}u^{2}v^{4}) + (\beta_{4}u^{4} + \beta_{3}u^{3}v + \beta_{2}u^{2}v^{2} + \beta_{1}uv^{3})x.$$

$$(4.8)$$

Nine complex parameters, namely $\alpha_2, \ldots, \alpha_6$ and β_1, \ldots, β_4 , multiply terms that vanish at u = 0. But to construct the desired space of S's, we must divide by the symmetries of type (ii), that is by $u \to w^{-1}u$. The result of this last step is that the α 's and β 's become homogeneous coordinates of a weighted projective space $\mathbf{WP}_{1,2,2,3,3,4,4,5,6}^8$, where the weights come from the fact that α_j and β_j are each of weight j.

Every point in the weighted projective space determines a del Pezzo surface S (possibly with some singularities of A - D - E type). The construction in section 4.1 gives for every point in $\mathbf{WP}_{1,2,2,3,3,4,4,5,6}^8$ an E_8 bundle over E. We thus get a family of such E_8 bundles, parametrized by the weighted projective space. Note that the weights that have appeared are the ones promised by Looijenga's theorem for E_8 , which is indeed equivalent to the statement that the family of bundles just constructed is the universal family of E_8 bundles over E.

The foregoing has the following illuminating interpretation. If we simply set to zero all the α 's and β 's, we get a hypersurface C(E)

$$y^2 = 4x^3 - g_2xv^4 - g_3v^6 (4.9)$$

which is a weighted cone over E. This hypersurface has at v = x = y = 0 a singularity that is known as an elliptic singularity of type \widetilde{E}_8 . From this point of view, the quantity g_3^2/g_2^3 (which is invariant under rescalings of v and determines the j-invariant of E) is a modulus of the singularity. What is considered in (4.8) is the general unfolding of the singularity in which the behavior at infinity is kept fixed. (Or more informally, the modulus is kept fixed.) The parameter space of this unfolding has a \mathbb{C}^* action induced by the \mathbb{C}^* action on C(E) given by $(v, x, y) \to (wv, w^2x, w^3y)$. \mathbb{C}^* acts on this parameter space with all weights of the same sign (the sign is generally taken to be negative in the literature on singularity theory), and the quotient of the parameter space by this \mathbb{C}^* is a weighted projective space.

The hypersurface (4.9) is too singular to define a point on the moduli space of del Pezzo surfaces. But if one wishes to understand the fact that the moduli space of k = 8 del Pezzo surfaces containing a fixed E is a weighted projective space with the weights found above, it is very helpful to start with the singular object and consider its deformations. We will see analogous phenomena in section five in the context of stable bundles.

Reduction Of Structure Group And Singularities

In this construction, one can see at a classical level the relation between unbroken gauge symmetry and singularities that has played an important role in studies of string duality in the last few years. Namely, the bundle induced on an elliptic curve E by its embedding in a k=8 del Pezzo surface S has structure group that commutes with a semi-simple subgroup H of E_8 (which will always be simply-laced) if and only if S contains a singularity of type H.

To make this argument, it is convenient to work not on S but on a smooth almost del Pezzo surface X made by resolving singularities of S (replacing possible A - D - E singularities in S by configurations of rational curves). One reason that this is convenient is that while the cohomology of S drops when S acquires a singularity, that of X remains fixed and thus has the structure we described above for a smooth del Pezzo surface. In considering a possibly singular del Pezzo surface S, we define $L = H^2(X, \mathbf{Z})$, L' as the sublattice orthogonal to the anticanonical divisor x of X, and $L'' = L/x\mathbf{Z}$.

We first prove that if S has an A - D - E singularity, then the induced bundle on E commutes with the corresponding A - D - E subgroup of E_8 . Let E_1 be the sublattice of E generated by rational curves in E of self-intersection E. Let E be such a curve. Since E is an anticanonical divisor, the cohomology class of E is E is E and E on the fact that E implies tha

line bundle $\mathcal{O}(C)$ determined by C is trivial if restricted to E. Thus in the map from L' to the Jacobian of E, L_1 is mapped to zero. This means that the induced bundle on E has a stabilizer of the appropriate A - D - E type.

To justify the last assertion, recall first that the automorphism group H' of the E_8 bundle $V \to E$ has for its Lie algebra $\mathcal{H} = H^0(E, V)$. With V being induced by a homomorphism from L' to E, V is a sum of line bundles of degree zero, and $H^0(E, V)$ is a sum of one-dimensional contributions from trivial subbundles in V. From what was seen in the last paragraph, every length squared -2 point in L_1 corresponds to a trivial line subbundle of V, and hence to a generator of \mathcal{H} . So if S has a singularity of type H, then all roots of H appear in H' and so $H \subset H'$.

The proof of the converse is longer. For \mathcal{N} a line bundle, let $h^i(\mathcal{N}) = \dim H^i(X, \mathcal{N})$. As will become clear, the main step in the argument is to show that if \mathcal{L} is a holomorphic line bundle over X with $c_1(\mathcal{L})^2 = -2$, then $h^0(\mathcal{L}) = h^0(\mathcal{L}^{-1}) = 0$ implies that the restriction of \mathcal{L} to E is non-trivial.

For such an \mathcal{L} , the index of the $\overline{\partial}$ operator with values in $\mathcal{L}^{\pm 1}$ is zero, so in particular

$$0 = h^{0}(\mathcal{L}^{-1}) - h^{1}(\mathcal{L}^{-1}) + h^{2}(\mathcal{L}^{-1}). \tag{4.10}$$

By Serre duality, $h^2(\mathcal{L}^{-1}) = h^0(K \otimes \mathcal{L})$. But vanishing of $h^0(\mathcal{L})$ and existence of a holomorphic section s of K^{-1} (which vanishes on E) imply vanishing of $h^0(K \otimes \mathcal{L})$. (For instance, multiplication by s would map a non-zero holomorphic section of $K \otimes \mathcal{L}$ to a non-zero holomorphic section of \mathcal{L} .) So $h^0(\mathcal{L}) = h^0(\mathcal{L}^{-1}) = 0$ implies $h^1(\mathcal{L}^{-1}) = 0$ and hence by Serre duality $h^1(K \otimes \mathcal{L}) = 0$.

Next look at the exact sequence of sheaves

$$0 \to K \otimes \mathcal{L} \to \mathcal{L} \to \mathcal{L}|_E \to 0, \tag{4.11}$$

where the first map is multiplication by s and the second is restriction to E. The associated long exact sequence of cohomology groups reads in part

$$\dots \to H^0(X, \mathcal{L}) \to H^0(E, \mathcal{L}) \to H^1(X, K \otimes \mathcal{L}) \to \dots$$
 (4.12)

Thus, if $h^0(\mathcal{L}) = h^0(\mathcal{L}^{-1}) = 0$, then from the above $h^1(K \otimes \mathcal{L}) = 0$, so the exact sequence implies that $H^0(E, \mathcal{L}) = 0$. But this implies that the restriction of \mathcal{L} to E is non-trivial.

Now, let L_0 be the sublattice of L' corresponding to line bundles whose restriction to E is trivial. The intersection form on L_0 is even, and the sublattice L_1 of L_0 generated by

the points of length squared -2 is the root lattice of some product of A - D - E groups. From what we have just proved, if $y \in L_1$ has $y^2 = -2$, then \mathcal{L}_y or \mathcal{L}_y^{-1} has a holomorphic section. Such a section vanishes on a holomorphic curve C_y with self-intersection number -2. C_y does not meet E (since $y \cdot x = 0$) so the anticanonical bundle of X is trivial when restricted to C_y . If we go back to S, therefore, the C_y are all blown down, producing the promised singularity of type L_1 .

E_7 Bundles

Now we consider in a precisely similar way the case k = 7. A k = 7 del Pezzo surface S can be constructed as a hypersurface of degree four in a weighted projective space $\mathbf{WP}_{1,1,1,2}^3$, with homogeneous coordinates u, v, x, y. Such a hypersurface is described by an equation of the general form

$$y^2 = ax^4 + bv^4 + cu^4 + \dots (4.13)$$

The difference between the sum of the weights and the degree of the hypersurface is 1+1+1+2-4=1, so the degree 1 hypersurface u=0 is an anticanonical divisor. This divisor is in fact a genus one curve E in a weighted projective space $\mathbf{WP}_{1,1,2}^2$ with homogeneous coordinates v, x, y. By an automorphism of the weighted projective space, E can be put in the form

$$y^2 = 4vx^3 - g_2xv^3 - g_3v^4. (4.14)$$

When put in this form, E is naturally an elliptic curve, with distinguished point p given by (v, x, y) = (0, 0, 1), and the line bundle $\mathcal{O}(p)$ is a square root of the restriction to E of the anticanonical bundle of S.

If (4.14) is regarded as defining a hypersurface in $\mathbf{WP}_{1,1,1,2}^3$, then that hypersurface is a cone over E and has a singularity of type \widetilde{E}_7 at x=y=v=0. The j-invariant of E is a modulus of this singularity. The universal unfolding of the \widetilde{E}_7 singularity preserving this modulus (or more precisely the behavior at infinity) is made by adding to (4.14) terms that vanish at u=0 modulo u-dependent translations of v, x, y, These translations can be taken into account by excluding deformations of the equation divisible by y, x^3 , or vx^2 . (This "gauge fixing condition" can be made uniformly, independent of g_2 and g_3 , an important difference from the E_8 case.) The universal deformation thus looks like

$$y^{2} = 4vx^{3} - g_{2}xv^{3} - g_{3}v^{4} + u(\alpha_{1}v^{3} + \alpha_{2}xv^{2}) + u^{2}(\beta_{1}v^{2} + \beta_{2}xv + \beta_{3}x^{2}) + u^{3}(\gamma_{1}v + \gamma_{2}x) + u^{4}\delta.$$
(4.15)

The moduli space of S's containing the given E is obtained by dividing by the additional symmetry $u \to w^{-1}u$. Under this transformation, the α 's have weight one, the β 's have weight two, the γ 's have weight three, and δ has weight four. The requisite moduli space of S's is thus a weighted projective space $\mathbf{WP}_{1,1,2,2,2,3,3,4}^7$. The construction of section 4.1 gives a family of E_7 bundles over E parametrized by this weighted projective space. The content of Looijenga's theorem for E_7 is that this family is the universal E_7 bundle over E, so that the moduli space of such E_7 bundles is the weighted projective space that we just encountered.

E_6 Bundles

 E_6 is treated similarly. A k=6 del Pezzo surface S can be constructed as a hypersurface of degree four in an ordinary projective space \mathbf{P}^3 , with homogeneous coordinates u, v, x, y. Such a hypersurface is described by a homogeneous cubic equation in u, v, x, and y. The difference between the sum of the weights and the degree of the hypersurface is 1+1+1+1-3=1, so the degree 1 hypersurface u=0 is an anticanonical divisor. This divisor is in fact a genus one curve E in an ordinary projective space \mathbf{P}^2 , with homogeneous coordinates v, x, y. By an automorphism of the projective space, E can be put in the form

$$vy^2 = 4x^3 - g_2xv^2 - g_3v^3. (4.16)$$

This way of writing E exhibits it as an elliptic curve with distinguished point p given by (v, x, y) = (0, 0, 1), and the line bundle $\mathcal{O}(p)$ is a cube root of the restriction to E of the anticanonical bundle of S.

If (4.16) is regarded as defining a hypersurface in \mathbf{P}^3 , then that hypersurface is a cone over E and has a singularity of type \widetilde{E}_6 at x=y=v=0. The j-invariant of E is a modulus of this singularity. The universal unfolding of the \widetilde{E}_6 singularity preserving this modulus (in the sense described earlier) is made by adding to (4.14) terms that vanish at u=0 modulo u-dependent translations of v,x,y, These translations can be taken into account by excluding deformations of the equation divisible by y^2, x^2 and vy. (It is again significant that this "gauge fixing" can be made universally, independent of g_2 and g_3 .) The universal deformation thus looks like

$$vy^{2} = 4x^{3} - g_{2}xv^{2} - g_{3}v^{3} + u(\alpha_{1}v^{2} + \alpha_{2}xv + \alpha_{3}xy) + u^{2}(\beta_{1}v + \beta_{2}x + \beta_{3}y) + u^{3}\gamma.$$
(4.17)

The moduli space of of S's containing the given E is obtained by dividing by the additional symmetry $u \to w^{-1}u$. Under this transformation, the α 's have weight one, the

 β 's have weight two, and γ has weight three. The requisite moduli space of S's is thus a weighted projective space $\mathbf{WP}_{1,1,1,2,2,2,3}^6$. The construction of the last subsection gives a family of E_6 bundles over E parametrized by this weighted projective space. The content of Looijenga's theorem for E_6 is that this family is the universal E_6 bundle over E, so that the moduli space of such E_6 bundles is the weighted projective space that we just encountered.

4.3. Bundles Over Elliptic Manifolds

Now we wish to consider bundles over an elliptically fibered manifold $Z \to B$ with a section σ (whose normal bundle we call \mathcal{L}^{-1}). For $b \in B$, let E_b be the elliptic curve over b. For each gauge group G, there is a weighted projective space bundle $\mathcal{W} \to B$ whose fiber over $b \in B$ is the moduli space of G bundles over E_b . We want to obtain a simple description of \mathcal{W} for $G = E_6$ or E_7 , and to see how the existence of such a simple description is obstructed for E_8 .

The basic idea is to make the above construction with parameters. The only subtlety is that one must give a description of the fibration $Z \to B$ which is adapted to the choice of G. For instance, for $G = E_6$, we regard Z as usual as a hypersurface in a \mathbf{P}^2 bundle over B, which is obtained by projectivizing $\mathcal{O} \oplus \mathcal{L}^2 \oplus \mathcal{L}^3$, with respective homogeneous coordinates v, x, y. The Weierstrass equation defining Z reads

$$vy^2 = 4x^3 - g_2xv^2 - g_3v^3, (4.18)$$

with g_2 and g_3 being now sections of \mathcal{L}^4 and \mathcal{L}^6 . To obtain the desired \mathcal{W} , we simply make fiberwise the construction given above. We embed the \mathbf{P}^2 bundle over B in a \mathbf{P}^3 bundle, obtained by projectivizing $\mathcal{L}^6 \oplus \mathcal{O} \oplus \mathcal{L}^2 \oplus \mathcal{L}^3$, with respective homogeneous coordinates u, v, x, y (the choice of exponent 6 for u is convenient but not essential); we interpret (4.18) as defining a singular hypersurface in this bundle (a sort of cone over Z). We consider deformations of this hypersurface of same form as before:

$$vy^{2} = 4x^{3} - g_{2}xv^{2} - g_{3}v^{3} + u(\alpha_{1}v^{2} + \alpha_{2}xv + \alpha_{3}xy) + u^{2}(\beta_{1}v + \beta_{2}x + \beta_{3}y) + u^{3}\gamma.$$
(4.19)

The α_i , β_j , and γ are now interpreted as homogeneous coordinates for the desired weighted projective space \mathcal{W} ; they are sections of line bundles which are determined by requiring that (4.19) makes sense as an equation with values in \mathcal{L}^6 . \mathcal{W} is therefore a $\mathbf{WP}_{1,1,1,2,2,2,3}^6$ bundle whose successive homogeneous coordinates are sections of $\mathcal{O}, \mathcal{L}^{-2}, \mathcal{L}^{-5}, \mathcal{L}^{-6}, \mathcal{L}^{-8}, \mathcal{L}^{-9}$, and

 \mathcal{L}^{-12} . This is the expected form of \mathcal{W} for E_6 . Note that the Casimir invariants of E_6 are of degree 2, 5, 6, 8, 9, and 12.

 E_7 is treated similarly. The only difference is that here we regard Z as a hypersurface in a $\mathbf{WP}^2_{1,1,2}$ bundle over B, obtained by projectivizing $\mathcal{O} \oplus \mathcal{L}^2 \oplus \mathcal{L}^3$, with homogeneous coordinates v, x, y of weights 1, 1, 2, and Weierstrass equation

$$vy^2 = 4vx^3 - q_2xv^3 - q_3v^4. (4.20)$$

We embed the $\mathbf{WP}_{1,1,2}^2$ bundle in a $\mathbf{WP}_{1,1,1,2}^3$ bundle over B, obtained by projectivizing $\mathcal{L}^6 \oplus \mathcal{O} \oplus \mathcal{L}^2 \oplus \mathcal{L}^3$, with homogeneous coordinates u, v, x, and y. (4.20) describes a singular hypersurface in this larger bundle. We consider deformations of this hypersurface of the form

$$y^{2} = 4vx^{3} - g_{2}xv^{3} - g_{3}v^{4} + u(\alpha_{1}v^{3} + \alpha_{2}xv^{2}) + u^{2}(\beta_{1}v^{2} + \beta_{2}xv + \beta_{3}x^{2}) + u^{3}(\gamma_{1}v + \gamma_{2}x) + u^{4}\delta.$$
(4.21)

The α_i , β_i , γ_i , and δ are now interpreted as homogeneous coordinates for the desired weighted projective space bundle \mathcal{W} , and are again sections of line bundles that are determined by requiring that (4.21) makes sense as an equation with values in \mathcal{L}^6 . \mathcal{W} is thus a $\mathbf{WP}^7_{1,1,2,2,2,3,3,4}$ bundle whose successive homogeneous coordinates are sections of $\mathcal{O} \oplus \mathcal{L}^{-2} \oplus \mathcal{L}^{-6} \oplus \mathcal{L}^{-8} \oplus \mathcal{L}^{-10} \oplus \mathcal{L}^{-12} \oplus \mathcal{L}^{-14} \oplus \mathcal{L}^{-18}$. Note that the Casimir invariants of E_7 are of degree 2, 6, 8, 10, 12, 14, and 18.

One cannot by these methods obtain a description of the weighted projective space bundle for E_8 as the projectivization of a sum of line bundles (and we believe that there is no such description). The reason is that there is no universal way, independent of g_2 and g_3 , to parametrize the unfolding of the \tilde{E}_8 singularity. For $g_3 \neq 0$, a parametrization is given in (4.8), and one could similarly pick a parametrization for $g_2 \neq 0$, but there is no uniform choice. (The natural parametrization of the weighted projective space bundle for $g_3 \neq 0$ is related to an analogous natural parametrization for $g_2 \neq 0$ by a nonlinear automorphism of the weighted projective spaces.) When one considers elliptic manifolds over a base B of dimension at least two, one meets cusp fibers with $g_2 = g_3 = 0$; near such a fiber the description of bundles is really different.

4.4. Relation To Duality In Eight Dimensions

We conclude this section with a discussion of the role of the \widetilde{E}_8 singularity in string duality.

The basic duality between the heterotic string and F theory maps the heterotic string on a two-torus E to F theory on an elliptically-fibered K3. A heterotic string vacuum on a two-torus E is described by a family of conformal field theories depending on 18 complex parameters (plus more one real parameter, the string coupling constant, which determines the Kähler class in F theory). If one asks for unbroken $E_8 \times E_8$ gauge symmetry, 16 complex parameters, which parameterize the flat $E_8 \times E_8$ bundles, are fixed. The remaining two complex parameters are the complex structure and complexified Kähler class of E.

According to Morrison and Vafa (see section 2 of the second paper in [3]), this two-parameter heterotic string locus corresponds in terms of F theory to the family of elliptically-fibered K3's described (in affine coordinates) by the following explicit equation:

$$y^{2} = 4x^{3} - q_{2}t^{4}x + t^{5} - q_{3}t^{6} + t^{7}. (4.22)$$

Here g_2 and g_3 are the two parameters and t is an affine coordinate on the base \mathbf{P}^1 of the elliptic fibration. (The fiber over $t = \infty$ should thus be included.) For given t, g_2, g_3 , (4.22) is a Weierstrass equation defining the elliptic fibration.

Morrison and Vafa further consider the case in which, on the heterotic string side, the area ρ of E becomes large, keeping the complex structure fixed. They show that this corresponds to g_2 and g_3 becoming large with fixed g_2^3/g_3^2 . We can enter this region taking $g_2 \to c^2 g_2$, $g_3 \to c^3 g_3$, where c is to become large. It is convenient to also rescale t by $t \to c^{-1}t$. In this way, we can actually take the limit as $c \to \infty$. This corresponds to decompactification of the heterotic string, with fixed complex structure on E but area going to infinity. Such decompactification of the heterotic string thus corresponds in E theory to the singular K3 fibration described in affine coordinates by the $c = \infty$ limit of (4.22), or

$$y^2 = 4x^3 - q_2 t^4 x - q_3 t^6. (4.23)$$

We see that this has two \widetilde{E}_8 singularities, one at x=y=t=0 and one at x=y=0, $t=\infty$. (To see the latter singularity, set t=1/t', $x=x'/(t')^4$, $y=y'/(t')^6$.)

A surface with these two \widetilde{E}_8 singularities does not correspond to a point on the moduli space of vacua. This is clear on the heterotic string side because such a point can only be reached by decompactification. In the natural metric on the moduli space,

decompactification is at infinite distance; one gets a complete metric on the moduli space without including it. However, the singular surface with the two \widetilde{E}_8 singularities is a convenient starting point in understanding the part of the moduli space where classical geometry is a good approximation, that is, the part where the two-torus has large area. This is essentially what we have done in using the unfolding of the elliptic singularity to describe E_8 bundles.

Stable Version And Behavior In Families

Actually, the degeneration of a K3 surface to produce two \widetilde{E}_8 singularities, as just described, does not correspond to a stable point on the moduli space of K3 surfaces. The stable version is as follows.

Eqn. (4.22) describes a K3 surface X that is elliptically fibered over a base B' which is a copy of \mathbf{P}^1 (parametrized by t) with a section σ . There are 24 points P_i on the t plane over which the fiber degenerates.

To produce the two \widetilde{E}_8 singularities, 12 of the P_i move to t=0 and the other 12 to $t=\infty$. Near this limit, as explained by Morrison and Vafa, the hyper-Kähler metric on B' (which we identify with the section of the elliptic K3) looks like a long cigar with 12 of the P_i at each end; the limit $c\to\infty$ is the limit in which the cigar becomes infinitely long. From the point of view of complex geometry, the stable version of such a degeneration is that in which the B' splits into two components H_1 and H_2 (each isomorphic to \mathbf{P}^1 and sharing a point Q in common) with 12 P_i in each component. In this picture, X degenerates to a union of two rational elliptic surfaces U_1 and U_2 glued together on an elliptic curve E. We write this as $X = U_1 \cup_E U_2$. In terms of the projection $\pi: X \to B'$, one has $U_1 = \pi^{-1}(H_1)$, $U_2 = \pi^{-1}(H_2)$, and $E = \pi^{-1}(Q)$. The section σ of X splits up into sections σ_1 and σ_2 of U_1 and U_2 ; by blowing down the σ_i we can map the U_i to del Pezzo surfaces W_i , glued together along E.

In view of what has been said by Morrison and Vafa and above, the correspondence with the heterotic string is simply that E is the elliptic curve on the heterotic string side, and the two E_8 bundles V_1 and V_2 over E are coded in the complex structures of W_1 and W_2 .

Now, it is easy to extend this formulation to families. On the heterotic string side we replace E by an elliptic n-fold $\pi: Z \to B$. Assuming that the Kahler metric on B is large so that we may make fiberwise duality with F theory, this corresponds in F theory to an n+1-fold that maps to B with K3 fibers, and maps with elliptic fibers to a certain \mathbf{P}^1

bundle B' over B. Now if we also take the area of the fibers of $\pi: Z \to B$ to be large, to reduce to classical geometry, and also blow down the sections, then as was just seen each K3 fiber of the map $X \to B$ will degenerate to a union of two del Pezzo surfaces, glued together along an elliptic curve. Globally, X will degenerate to a union of two n+1-folds glued over an n-fold (which is also fibered over B); in fact, it degenerates to $X = W_1 \cup_Z W_2$, where Z is the Calabi-Yau n-fold seen on the heterotic string side, and W_1 and W_2 are bundles of del Pezzo surfaces over B.¹⁰

Fiberwise application of the correspondence between E_8 bundles on an elliptic curve and del Pezzo surfaces shows that the complex structures of W_1 and W_2 code the isomorphism classes of the restriction to each fiber of the E_8 bundles V_1 and V_2 over Z.

As we saw in section 2 in the case of SU(n) bundles, given a bundle $V \to Z$, V is not uniquely determined, in general, by a knowledge of the isomorphism class of its restriction to each fiber. One can make certain twists by a line bundle on the spectral cover. Similarly for gauge groups other than SU(n), we expect an abelian variety, generalizing the Jacobian of the spectral cover, to enter in the parametrization of bundles. As explained in section 2.4, the additional data should in F theory show up (along with other things) in the intermediate Jacobian of X, $J_X = H^3(X, \mathbf{R})/H^3(X, \mathbf{Z})$.

When X degenerates to $W_1 \cup_Z W_2$, the intermediate Jacobian of X splits off factors isomorphic to the intermediate Jacobians $J_{W_i} = H^3(W_i, \mathbf{R})/H^3(W_i, \mathbf{Z})$. Note that, as $H^{3,0}(W_i) = 0$, the J_{W_i} are abelian varieties (as is J_X when dim B > 1, but generally not when dim B = 1). As the E_8 bundles $V_i \to Z$ are closely related to the structure of the W_i , it is natural to believe that the intermediate Jacobians J_{W_i} , for i = 1, 2, contain the additional information necessary to determine the V_i .

As evidence for this interpretation of the facts, we will show that for the case that $B = \mathbf{P}^1$, the J_{W_i} have the appropriate dimension and in fact the appropriate tangent space.

Let $W=W_i$ for i=1 or 2. The tangent space to the space of deformations of the complex structure of W that preserve the existence of the divisor Z (other deformations involve a change in the complex structure of the heterotic string manifold Z and are not related to bundles on Z) is $\mathcal{T}=H^1(W,T_W\otimes\mathcal{O}(-Z))$. Here T_W is the tangent space to W and $\mathcal{O}(-Z)$) is the line bundle whose holomorphic sections are holomorphic functions

More exactly, as in the case of a single K3, one gets bundles of rational elliptic surfaces that can be blown down to make bundles of del Pezzo surfaces.

that vanish on Z. Because $\mathcal{O}(-Z)$ is the canonical bundle of W, Serre duality says that $\mathcal{T}^* = H^2(W, T_W^*) = H^{1,2}(W)$. But $T_J = H^{1,2}(W)$ is the tangent space to the intermediate Jacobian J_W . This equality of dimensions between \mathcal{T} and T_J , and in fact the duality between them, is expected in view of the complex symplectic structure (and hyper-Kahler structure) of the moduli space of bundles on Z, for the case that $B = \mathbf{P}^1$ and Z is a K3 surface.

Mathematically, it is possible to "twist" the intermediate Jacobian of X by an arbitrary integral class $\alpha \in H^{2,2}(X)$. (The twisted intermediate Jacobians are components of the Deligne cohomology of X.) In physical terminology, using the language of M theory, making such a twist means taking the four-form field strength G of eleven-dimensional supergravity to represent the cohomology class α . It has been shown by K. and M. Becker [9] that in the case that X is a four-fold, introducing α in M theory (and hence also, with some restriction, in F theory) is compatible with space-time supersymmetry provided that α is a primitive element of $H^{2,2}(X)$. (α is defined to be primitive if its contraction with the Kahler class vanishes in cohomology. For X a three-fold, this behavior implies that the image of α vanishes in real cohomology so that α is a torsion class.) This gives physical models, with space-time supersymmetry, in which twisted versions of the intermediate Jacobian of X enter.

It is natural to conjecture that twists by those elements $\alpha \in H^{2,2}(X)$ that are derived (when X reduces to $W_1 \cup_Z W_2$) from an element $\beta \in H^{2,2}(W)$ have the following interpretation. We saw in section two that the moduli space of SU(n) bundles on $Z \to B$ that have a given restriction to each fiber is not necessarily connected, but (depending on the Picard group of the spectral cover $C \to B$) may have different components. We conjecture that a similar result holds for E_8 bundles and that the different components of bundles with a fixed restriction to each fiber correspond in F theory to the twists of J_W by different primitive elements of $H^{2,2}(W)$ or (more physically) to the different values of the cohomology class of the G field.

5. Uniform Approach To Construction Of Bundles

Having reached this far, the reader may yearn for a more uniform approach to the problem. In sections 2,3, and 4, we presented different approaches to understanding the moduli space of G bundles on an elliptic curve E; each approach was effective for a particular class of G's. Is there not a more uniform approach?

In this section, we will explain an approach which does in fact work uniformly for all simple, connected, and simply-connected G.¹¹ The inspiration for this construction comes in part from the construction via del Pezzo surfaces in section 4. We saw there that to understand the moduli space of "good" del Pezzo surfaces, which have at worst singularities of A-D-E type, it helps to start with a "bad" surface with an \widetilde{E}_k singularity. The good surfaces are conveniently constructed as deformations of a bad one.

We will take a similar approach to bundles. Though we are interested mainly in semistable bundles, we will find a distinguished unstable bundle which has the property that the semistable bundles are conveniently constructed as its deformations.

We explain first the idea for G = SU(n). A rank n vector bundle $V \to E$ of degree zero is unstable if and only if it contains a sub-bundle U of positive degree. Such a U fits into an exact sequence

$$0 \to U \to V \to U' \to 0 \tag{5.1}$$

with some U'. To make V just barely unstable, we pick U to be of degree 1 and U' to be of degree -1. We also assume that U' and U are themselves both stable.

The determinant of U is a rank one bundle of the form $\mathcal{O}(p)$ for some point $p \in E$ which we will keep fixed. The determinant of U' is then $\mathcal{O}(p)^{-1}$. Now we specialize to the case that E is of genus one. In this case, a Riemann-Roch argument (using stability of $U^* \otimes U'$) shows that the sequence (5.1) splits, so that in fact $V = U \oplus U'$.

Also, for E of genus one, U is uniquely determined up to translation on E. In fact, up to isomorphism there is for each $k \geq 1$ a unique stable bundle W_k of determinant $\mathcal{O}(p)$ for any given point $p \in E$. For k = 1, $W_1 = \mathcal{O}(p)$, and if W_k is known then W_{k+1} can be constructed inductively as the unique non-split extension

$$0 \to \mathcal{O} \to W_{k+1} \to W_k \to 0. \tag{5.2}$$

The W_k will appear extensively in what follows. The dual of W_k , which we write as W_k^* , is the unique rank k stable bundle over E of degree -1 and determinant $\mathcal{O}(p)^{-1}$.

So for our starting point, we take the unstable bundle

$$V = W_k \oplus W_{n-k}^*, \tag{5.3}$$

¹¹ It is even possible to extend the discussion to non-simply-connected G, by using different parabolic subgroups of G, but we will not make this generalization in the present paper.

with some k in the range $1 \le k \le n-1$. This is, up to translation on E, the unique minimally unstable bundle with summands of the chosen dimension.

The decomposition (5.3) of V enables one to define a group $H \cong \mathbb{C}^*$ of automorphisms of V that acts by scalar multiplication on W_k while acting trivially on W_{n-k}^* . The structure group of V reduces to the subgroup of SU(n) consisting of block diagonal matrices of the form

$$\begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}, \tag{5.4}$$

where the upper left block is $k \times k$ and the lower right block is $(m-k) \times (m-k)$. H acts as a group of automorphisms of the Lie algebra of SU(n). The diagonal blocks transform under H with weight 0, the upper right block has weight 1, and the lower left block has weight -1.

To first order, a deformation of V is determined by an element of $T = H^1(E, ad(V))$, where ad(V) is the adjoint bundle derived from V. Using the facts noted in the last paragraph, T can be decomposed into pieces of weight 1, 0, and -1 under H, as follows:

- (1) The weight 1 piece is $H^1(E, \text{Hom}(W_{n-k}^*, W_k)) = H^1(E, W_{n-k} \otimes W_k)$. The bundle $W_{n-k} \otimes W_k$ is a stable bundle of positive degree. On a curve of genus one, any stable bundle of positive degree has vanishing H^1 , so the weight 1 piece vanishes.
- (2) The weight 0 piece is the tangent space to the space of deformations of W_k and W_{n-k}^* that preserve the decomposition $V = W_k \oplus W_{n-k}^*$ (and the fact that V has trivial determinant). For reasons already noted above, the only such deformation is a motion of the point p such that the determinant of W_k is $\mathcal{O}(p)$. So the weight 0 piece is one-dimensional and can be viewed as the tangent space to E at p.
- (3) Finally, the weight -1 piece is $H^1(E, \text{Hom}(W_k, W_{n-k}^*)) = H^1(E, W_k^* \otimes W_{n-k}^*)$. This will play the starring role in what follows.

Let us compute the dimension of the weight -1 deformation space. By Riemann-Roch, and the fact that the bundle is semi-stable, this is minus the degree of the bundle $W_k^* \otimes W_{n-k}^*$. In what follows, we will many times have to compute the degrees of such tensor products. If the degree -1 bundle W_k^* were the sum of a degree -1 line bundle and k-1 line bundles of degree 0, then in $W_k^* \otimes W_{n-k}^*$, we would see one summand of degree -2, n-2 of degree -1, and the rest of degree 0. The total degree would thus be -n. Actually, although the W^* 's are not such direct sums, the computation just performed can be justified using exact sequences such as (5.2) and its dual, so the degree is really -n.

So the -1 part of the weight space is of dimension n. Now, a deformation of V by an element $\alpha \in H^1(E, W_k^* \otimes W_{n-k}^*)$ produces a bundle V' with an exact sequence

$$0 \to W_{n-k}^* \to V' \to W_k \to 0. \tag{5.5}$$

The existence of such a sequence does not contradict stability of V' since W_{n-k}^* is of negative degree. In fact, a straightforward argument shows that bundles on an elliptic curve constructed by non-trivial extensions of the form (5.5) are all semistable; see [36].

We want to consider only deformations of negative weight, suppressing the weight 0 part of the deformation space. This is analogous to considering only deformations of fixed j in the construction via del Pezzo surfaces.

Structure Of The Deformations

In an explicit description of bundles by an open covering and transition functions, the transition functions for the extension V' look like

$$\begin{pmatrix} * & 0 \\ \alpha & * \end{pmatrix}, \tag{5.6}$$

where the upper left and lower right blocks are the transition functions for W_k and W_{n-k}^* . Because α only appears in the lower left block, the α -dependent terms in the cocycle condition for such transition functions are linear in α . That is why, even though in many problems in geometry H^1 controls only the linearized deformations, the choice of $\alpha \in H^1(E, \text{Hom}(W_k, W_{n-k}^*))$ produces an actual extension V' as described in (5.5), and not just a first order approximation to one.

Closely related is the fact that it does not matter if α is "big" or "small," in the sense that if α is replaced by $t\alpha$ with $t \in \mathbb{C}^*$, then the bundle V' is unchanged, up to isomorphism. This is so because t can be scaled out by using the scaling by H. The point is that the "unperturbed" bundle V has an automorphism group $H \cong \mathbb{C}^*$ that is "broken" by the perturbations. To construct the moduli space of bundles that can be built by perturbations that do not preserve the \mathbb{C}^* , one must divide the space of first order deformations by \mathbb{C}^* .

If $T_- = H^1(E, W_k^* \otimes W_{n-k}^*)$ is the space of negative weight first order deformation, then the family of bundles that can be constructed via such deformations is naturally parametrized by $M = (T_- - \{0\})/\mathbb{C}^*$ (which we abbreviate below as T_-/\mathbb{C}^*). Since T_- is a copy of \mathbb{C}^n and the \mathbb{C}^* acts by scalar multiplication, M is a copy of \mathbb{P}^{n-1} .

We have already seen in section 2 that the moduli space \mathcal{M} of semistable SU(n) bundles on E is a copy of \mathbf{P}^{n-1} . This raises the question of whether one can naturally identify M with \mathcal{M} . It will be proved elsewhere that this is so [36] (for any k in the range from 1 to n-1). In other words, M can be identified with the projective space predicted by Looijenga's theorem in the case of SU(n).

Framework For Generalizations

In the rest of this section, we will show how to make an analogous construction for any simple, connected and simply-connected Lie group G. In each case, we find a distinguished, slightly unstable G bundle V over E with the property that the semistable G bundles over E all arise naturally as deformations of V. Before considering specific examples, we pause for some useful generalities.

A subgroup of $SL(n, \mathbf{C})$ is called parabolic if (perhaps after conjugation) it contains the diagonal and upper triangular matrices:

$$\begin{pmatrix} * & * & * & \dots & * \\ 0 & * & * & \dots & * \\ & \dots & & & \\ 0 & 0 & & * & * \\ 0 & 0 & \dots & 0 & * \end{pmatrix}. \tag{5.7}$$

Any such group P has the property that $SL(n, \mathbb{C})/P$ is compact. The existence of the exact sequence (5.1) is equivalent to a reduction of the structure group to a group P_k of block upper triangular matrices of the form

$$\begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \tag{5.8}$$

(the upper left hand block being $k \times k$). Such a group is certainly parabolic, and in fact it is a maximal parabolic subgroup of $SL(n, \mathbf{C})$. The maximal reductive subgroup¹² of P_k is a group R_k of matrices of the form

$$\begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}. \tag{5.9}$$

The Lie algebra of R_k is that of $SL(k) \times SL(n-k) \times \mathbb{C}^*$. The \mathbb{C}^* plays an important role. If one decomposes the $SL(n, \mathbb{C})$ Lie algebra into eigenspaces of \mathbb{C}^* , then the Lie algebra

¹² For our purposes, a reductive group is a group that can be obtained by complexifying a compact group; equivalently, it is locally a product of simple factors and U(1)'s.

of P_k is the sum of the spaces of non-negative eigenvalue, while the Lie algebra of R_k is the subalgebra that commutes with \mathbb{C}^* .

For any simple Lie group G, a subgroup P of $G_{\mathbf{C}}$ is called parabolic if $G_{\mathbf{C}}/P$ is compact. The maximal parabolic subgroups of $G_{\mathbf{C}}$, up to conjugation, are in one to one correspondence with the nodes on the Dynkin diagram of G. Each node determines a \mathbf{C}^* subgroup U of the complexified maximal torus of G. (The choice of a node in the Dynkin diagram generalizes the fact that for SU(n) one chooses an integer k with $1 \le k \le n - 1$.) One decomposes the Lie algebra of G under U; the sum of the non-negative eigenspaces is the Lie algebra of a maximal parabolic P, and the subalgebra that commutes with U is the Lie algebra of the maximal reductive subgroup R of P.

Let V be a G bundle over E. The structure group of V can be reduced to a maximal parabolic subgroup P in many possible ways. Because of the \mathbb{C}^* factor $U \subset R \subset P$, any such reduction enables one to define a first Chern class. The bundle V is unstable if and only if for some reduction to a maximal parabolic subgroup, the first Chern class is positive.

We will call an unstable bundle "minimally unstable" with respect to a reduction to P if the first Chern class determined by the reduction takes the smallest possible positive value. One might think that, for given P, there would be many minimally unstable bundles. But we will find that for every Lie group G, there exists a choice of P such that a bundle V that is minimally unstable in a reduction to P has the same degree of uniqueness that we found for SU(n): it is unique up to translation on E, that is up to the choice of a distinguished point $P \in E$.

For E of genus one, a Riemann-Roch argument shows that if the structure group of V can be reduced to P in such a way that the first Chern class is positive, then it can be further reduced to R. The importance of this is that R has the center \mathbf{C}^* (the subgroup U that we started with). We can therefore decompose $H^1(E, \mathrm{ad}(V))$ in subspaces of definite weight under the \mathbf{C}^* action. As in the case considered above, the subspace of positive weight vanishes, the subspace of weight zero is one-dimensional, and we want to consider deformations of negative weight. If T_- is the negative weight deformation space, then the family of G bundles built by negative weight deformations of V is a weighted projective space T_-/\mathbf{C}^* . (It is a weighted projective space in general, not an ordinary one, because various weights appear in the action of \mathbf{C}^* on T_- .) The weights turn out to be just the ones predicted for G by Looijenga's theorem. That the family we make this way is indeed the moduli space of semistable G bundles over E will be proved elsewhere [36].

For general G and P, the negative weight part of the Lie algebra is nilpotent (repeated commutators vanish after finitely many steps) but not abelian. It therefore takes some additional argument, which we give elsewhere [36], to identify the linearized deformation space $H^1(E, ad(V))$ with a space of actual deformations of V.

A perhaps surprising difference between SU(n) and other groups is that while for SU(n) we were able to use any maximal parabolic subgroup as the starting point, for other G there is just a unique choice with the right properties. The vertices that work are the ones indicated in figure two.

In the remainder of this section, we carry out this program for the various simple Lie groups. Then we conclude with a few remarks about bundles on elliptic manifolds.`

5.1. Fresh Look At Sp(n)

Sp(n) contains a subgroup U(n) whose complexification is the maximal reductive subgroup R of a maximal parabolic subgroup P of $Sp(n)_{\mathbb{C}}$. If K_n denotes the standard n-dimensional representation of U(n), then the Lie algebra of Sp(n) decomposes under U(n) as

$$u(n) \oplus \operatorname{Sym}^{2}(K_{n}) \oplus \operatorname{Sym}^{2}(K_{n}^{*}),$$
 (5.10)

where u(n) is the adjoint representation of U(n), $\operatorname{Sym}^2(K_n)$ is the symmetric part of $K_n \otimes K_n$, and K_n^* is the dual of K_n . We normalize the U(1) factor in the Lie algebra of U(n) so that the three pieces in (5.10) transform with weights 0, 1, and -1. The pieces of weight 0 and 1 generate the Lie algebra of a maximal parabolic subgroup P of Sp(n).

An Sp(n) bundle can be represented by a rank 2n holomorphic vector bundle with a symplectic pairing. A minimally unstable Sp(n) bundle is

$$V = W_n \oplus W_n^* \tag{5.11}$$

with W_n as before the unique stable bundle of determinant $\mathcal{O}(p)$. The symplectic pairing of V comes from the pairing of W_n with W_n^* . V is unstable because the first Chern class of the summand W_n is positive, and it is minimally unstable because this first Chern class has the smallest positive value. V is unique up to translations on E because W_n has that property.

Now we consider deformations of V. The first order deformations are classified by $H^1(E, \text{ad}(V))$. This can be decomposed using (5.10) in terms of weights 1, 0, and -1. The weight one term would be $H^1(E, \text{Sym}^2 W_n)$, and vanishes because the bundle in question is

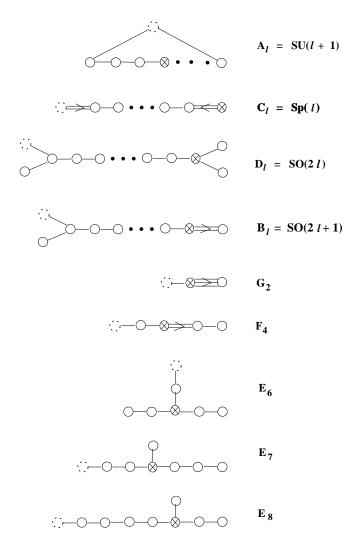


Figure 2. Shown here are the Dynkin diagrams of the simple Lie groups. In each case, one of the nodes has been marked with an \times (the marked node is arbitrary for SU(n) but in the other cases a distinguished node is marked). Also indicated, with dotted lines, is an additional vertex that can be added to produce the extended Dynkin diagram of the same group G. Deleting the distinguished vertex from the ordinary Dynkin diagram produces the Dynkin diagram of the maximal reductive subgroup R of a maximal parabolic subgroup of G. Deleting the distinguished vertex from the extended Dynkin diagram produces the Dynkin diagram of a maximal subgroup of G that contains R.

semistable and of positive degree. A similar vanishing for the deformation space of positive weight holds in all other cases considered below and will not be mentioned subsequently. The weight zero deformations are just the deformations of the bundle W_n in (5.11) and correspond (in view of the uniqueness statement about V) to translations of E. Again, given the uniqueness statement about the unstable bundle, this will have an immediate

analog in all the other cases, and will not be repeated.

Thus in subsequent examples we will focus at once on the negative weight part of the deformation space, which in this Sp(n) example is $H^1(E, \operatorname{Sym}^2(W_n^*))$. This equals minus the degree of the semi-stable bundle $\operatorname{Sym}^2(W_n^*)$. We compute that degree to give one more illustration of the methods for such computations; in subsequent examples we will give only the result. If W_n^* were the sum of a line bundle of degree -1 and n-1 line bundles of degree 0, then $\operatorname{Sym}^2(W_m^*)$ would have one summand of degree -2, n-1 of degree -1, and the rest of degree 0. The degree would thus be -(n+1). Though W_n^* is not actually such a direct sum, this type of computation can be justified by considering the exact sequences involving W_n and W_n^* .

So the negative weight space T_- is of dimension n+1. The unstable bundle V has a U(1) symmetry (coming from the center of U(n)) that is broken by the deformations. So the family of bundles that one builds by perturbing V by a negative weight deformation is parametrized by $M = T_-/\mathbb{C}^*$ which is a copy of \mathbb{P}^n . It will be proved elsewhere that this projective space is actually the projective space predicted by Looijenga's theorem for Sp(n).

5.2. Spin Groups

We will next consider the spin groups. We work first of all at the Lie algebra level, and thus initially we do not distinguish SO from Spin or describe the precise global forms of the various relevant subgroups of the Spin group.

We begin with Spin(2n). We consider a maximal parabolic subgroup of Spin(2n) associated with the "trivalent" node of the Dynkin diagram, as in figure two. The reductive part of the maximal parabolic subgroup associated with the given vertex is $U(2n-2) \times SO(4)$. This group is embedded in Spin(2n) by the chain $U(2n-2) \times SO(4) \subset SO(2n-4) \times SO(4) \subset Spin(2n)$.

An SO(2k) bundle can be regarded as a rank 2k bundle with a nondegenerate holomorphically varying quadratic form. A minimally unstable SO(2n-4) bundle, with respect to a parabolic subgroup of reductive part U(2n-2), would be $W_{n-2} \oplus W_{n-2}^*$; a minimally unstable SO(2n) bundle would be

$$V = W_{n-2} \oplus W_{n-2}^* \oplus Q_4 \tag{5.12}$$

where Q_4 is a stable (or semistable) SO(4) bundle.

Now we have to pay some attention to the global forms of the groups, and a crucial subtlety arises. The SO(2n-4) bundle $W_{n-2} \oplus W_{n-2}^*$ actually has a non-zero second Stiefel-Whitney class w_2 , because the first Chern class of W_{n-2} is odd. Since we want V to lift to a Spin(2n) bundle, we must cancel the obstruction by taking for Q_4 a stable SO(4) bundle which likewise has a non-zero w_2 .

There is a unique such Q_4 , up to isomorphism. As a flat SO(4) bundle, it can be described by saying that the monodromies around two independent one-cycles in the two-torus E are in a suitable basis

$$\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & -1
\end{pmatrix}
\text{ and }
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & -1
\end{pmatrix}.$$
(5.13)

If lifted to Spin(4), these matrices anticommute instead of commuting, so they define a bundle with non-zero w_2 . As a holomorphic bundle, Q_4 can be described as

$$Q_4 = \bigoplus_{\alpha} \mathcal{L}_{\alpha} \tag{5.14}$$

where the sum runs over the four isomorphism classes of line bundles of order two on E. The quadratic form on Q_4 is diagonal with respect to this decomposition and comes from the isomorphisms $\mathcal{L}_{\alpha} \otimes \mathcal{L}_{\alpha} \cong \mathcal{O}$. Q_4 is unique as any deformation of the \mathcal{L}_{α} spoils the existence of this quadratic form. It is "fortunate" that w_2 appeared in this way, because such uniqueness would certainly *not* hold for SO(4) bundles with vanishing w_2 .

Uniqueness of Q_4 and uniqueness up to translation of W_{n-2} mean that the minimally unstable bundle V of equation (5.12) is unique up to translation. Now, let us consider its deformations. V has a \mathbb{C}^* symmetry (coming from the center of U(n-2)) under which W_{n-2}, Q_4 , and W_{n-2}^* have weights 1,0, and -1. The deformations we want to consider are the negative weight deformations of V.

A novelty relative to the previous cases we considered is that in the decomposition of the Lie algebra of Spin(2n) under $U(n-2) \times SO(4)$, two different negative weights appear, not just one. In fact, the Lie algebra has a piece of weight -1, corresponding to $W_{n-2}^* \otimes Q_4$, and a piece of weight -2 corresponding to $\wedge^2(W_{n-2}^*)$ (for a bundle $F, \wedge^2(F)$ will be the antisymmetric part of $F \otimes F$). The negative weight deformation space of V is thus the sum of two terms:

- (1) The weight -1 piece is $T_{-1} = H^1(E, W_{n-2}^* \otimes Q_4)$. As Q_4 has rank four and degree zero, while W_{n-2}^* has degree -1, the semi-stable bundle $W_{n-2} \otimes Q_4$ has degree -4, so $\dim T_{-1} = 4$.
- (2) The weight -2 piece is $T_{-2} = H^1(E, \wedge^2(W_{n-2}^*))$. By methods explained before, one computes that $\wedge^2(W_{n-2}^*)$ has degree -(n-3), so that dim $T_{-2} = n-3$.

The negative weight deformation space of V is $T = T_{-1} \oplus T_{-2}$. We want to consider the family of Spin(2n) bundles parametrized by $M = T_{-}/\mathbb{C}^*$, where \mathbb{C}^* is the symmetry of V broken by the deformations. Because two different weights appear in T_{-} , this is a weighted projective space $M = \mathbf{WP}_{1,1,1,1,2,2,...,2}^n$ (four 1's and the rest 2's) as predicted for Spin(2n) by Looijenga's theorem.

The Odd Case

Spin(2n-1) can be considered with almost no change. The reductive part of a maximal parabolic (obtained by deleting from the Dynkin diagram the vertex indicated in figure two) is $U(n-2) \otimes SO(3)$. A minimally unstable bundle is now

$$V = W_{n-2} \oplus W_{n-2}^* \oplus Q_3 \tag{5.15}$$

where now Q_3 should be a stable SO(3) bundle with non-zero w_2 . There is a unique Q_3 , of the form $Q_3 = \bigoplus_{\alpha \neq 0} \mathcal{L}_{\alpha}$ (the sum runs now over the three *non-trivial* line bundles of order two), so V is unique up to the translations of E, acting on W_{n-2} .

V has a \mathbb{C}^* symmetry for which the three pieces written in (5.15) have weights 1, -1, and 0, respectively. The negative weight deformation space of V is the sum of two terms:

- (1) The weight -1 piece is $T_{-1} = H^1(E, W_{n-2}^* \otimes Q_3)$, and has dimension three.
- (2) The weight -2 piece is $T_{-2} = H^1(E, \wedge^2(W_{n-2}^*))$, and has dimension n-3.

The negative weight deformation space of V is $T_{-} = T_{-1} \oplus T_{-2}$. The family of Spin(2n) bundles parametrized by $M = T_{-}/\mathbb{C}^{*}$, where \mathbb{C}^{*} is the symmetry of V broken by the deformations, is a weighted projective space $M = \mathbf{WP}_{1,1,1,2,2,...,2}^{n}$ (three 1's and the rest 2's) as predicted for Spin(2n-1) by Looijenga's theorem.

5.3. E_8 Bundles

What remain are the exceptional groups. For these we switch notation slightly. For the classical groups, we first described a minimally unstable G bundle V using a distinguished representation of G, and then we considered the adjoint bundle $\operatorname{ad}(V)$. For the exceptional

groups, we will simply start with the adjoint representation from the beginning. So the bundle V will be an adjoint bundle, and the deformation space will be $H^1(E, V)$.

First we consider the simply-laced groups E_8 , E_7 , and E_6 . In each case, we consider the parabolic subgroup associated with the "trivalent" vertex, as in figure two. We first consider E_8 .

Deleting the indicated vertex from the extended Dynkin diagram of E_8 would give the Dynkin diagram of a maximal subgroup of E_8 , namely $H = (SU(6) \times SU(2) \times SU(3))/\mathbf{Z}_6$. (If one thinks of \mathbf{Z}_6 as the group of sixth roots of unity, then the \mathbf{Z}_6 subgroup of $SU(6) \times SU(2) \times SU(3)$ consists in an obvious notation of group elements of the form $\omega \times \omega^3 \times \omega^2$.) In what follows, let C_n be the fundamental n-dimensional representation of SU(n). The adjoint representation of E_8 has the following decomposition under E_8 ; it consists of the adjoint representation of E_8 has the following pieces:

$$\wedge^{3} C_{6} \otimes C_{2}, \quad \wedge^{2} C_{6}^{*} \otimes C_{3}, \quad C_{6} \otimes C_{2} \otimes C_{3},$$

$$\wedge^{2} C_{6} \otimes C_{3}^{*}, \text{ and } C_{6}^{*} \otimes C_{2} \otimes C_{3}^{*}.$$

$$(5.16)$$

This expansion is easily computed using the chain $SU(6) \times SU(2) \times SU(3) \subset E_6 \times SU(3) \subset E_8$.

If the trivalent vertex is deleted from the unextended Dynkin diagram of E_8 , one is left with the Dynkin diagram of $SU(5) \times SU(2) \times SU(3) \times U(1)$ (where we include a U(1) for the deleted node). This is the local form of the reductive part R of the maximal parabolic subgroup of E_8 associated with the given node. To describe the global form of R and the embedding of R in H, note that SU(6) contains a subgroup $U(5) = (SU(5) \times U(1))/\mathbf{Z}_5$, so H has the subgroup $R = (U(5) \times SU(2) \times SU(3))/\mathbf{Z}_6 = (SU(5) \times SU(2) \times SU(3) \times U(1))/\mathbf{Z}_{30}$.

A minimally unstable bundle should have a first Chern class (for the U(1) factor) which is positive and as small as possible. Here and in subsequent examples, it is convenient to work somewhat formally and introduce the "SU(n) bundle"

$$B_n = \mathcal{O}(p)^{-1/n} \otimes W_n. \tag{5.17}$$

The fractional exponents will cancel out of all final formulas; if one wishes one can give a precise meaning to a fractional root of a line bundle in a suitable formal context. That the bundle we construct is minimally unstable will be clear from the fact that its decomposition contains summands of degree 1.

We consider an H bundle in which the SU(2) factor is B_2 , the SU(3) factor is B_3 , and the SU(6) factor is

$$X_6 = (W_5 \oplus \mathcal{O}) \otimes \mathcal{O}(p)^{-1/6}. \tag{5.18}$$

Clearly, such a bundle does not make sense as a $SU(6) \times SU(2) \times SU(3)$ bundle, but it makes sense as an H bundle because the fractional exponents cancel out for all representations of $SU(6) \times SU(2) \times SU(3)$ in which the \mathbb{Z}_6 acts trivially. For instance, the E_8 bundle that we really want can be described as follows. The part coming from the adjoint representation of SU(6) gives

$$ad(W_5) \oplus W_5 \oplus W_5^* \oplus \mathcal{O}, \tag{5.19}$$

while the adjoint representation of $SU(2) \times SU(3)$ gives just

$$ad(W_2) \oplus ad(W_3). \tag{5.20}$$

(By ad (W_n)) we mean the traceless part of $W_n \otimes W_n^*$.) The part of the E_8 bundle coming from (5.16) can be expanded

$$\wedge^{3}C_{6} \otimes C_{2} = \wedge^{3}W_{5} \otimes W_{2} \otimes \mathcal{O}(p)^{-1} \oplus \wedge^{2}W_{5} \otimes W_{2} \otimes \mathcal{O}(p)^{-1}$$

$$\wedge^{2}C_{6}^{*} \otimes C_{3} = \wedge^{2}W_{5}^{*} \otimes W_{3} \oplus W_{5}^{*} \otimes W_{3}$$

$$C_{6} \otimes C_{2} \otimes C_{3} = W_{5} \otimes W_{2} \otimes W_{3} \otimes \mathcal{O}(p)^{-1} \oplus W_{2} \otimes W_{3} \otimes \mathcal{O}(p)^{-1}$$

$$\wedge^{2}C_{6} \otimes C_{3}^{*} = \wedge^{2}W_{5} \otimes W_{3}^{*} \oplus W_{5} \otimes W_{3}^{*}$$

$$C_{6}^{*} \otimes C_{2} \otimes C_{3}^{*} = W_{5}^{*} \otimes W_{2}^{*} \otimes W_{3}^{*} \otimes \mathcal{O}(p) \oplus W_{2}^{*} \otimes W_{3}^{*} \otimes \mathcal{O}(p).$$
(5.21)

This E_8 bundle V is unique up to translations on the elliptic curve because of the corresponding statement for the W_n 's.

Now we want to consider the negative weight deformations of V. The \mathbb{C}^* in question is easy to identify because it originated as a subgroup of the SU(6) factor in H. So it acts trivially on B_2 and B_3 , while in the decomposition (5.18) of X_6 , the \mathbb{C}^* acts on $W_5 \otimes \mathcal{O}(p)^{-1/6}$ with weight 1 and on the other summand $\mathcal{O}(-p)^{-1/6}$ with weight -5. The negative weight deformation space of V can now be analyzed as follows:

- (1) The weight -1 summand of V is $V_{-1} = W_5^* \otimes W_2^* \otimes W_3^* \otimes \mathcal{O}(p)$, of degree -1. So the weight -1 subspace T of $H^1(E, V)$ is $T_{-1} = H^1(E, V_{-1})$, and has dimension 1.
- (2) The weight -2 summand of V is $V_{-2} = \wedge^2 W_5^* \otimes W_3$, of degree -2. So the weight -2 deformation space $T_{-2} = H^1(E, V_{-2})$ has dimension 2.

- (3) The weight -3 summand of V is $V_{-3} = \wedge^2 W_5 \otimes W_2 \otimes \mathcal{O}(p)^{-1}$, of degree -2. So the weight -3 deformation space $T_{-3} = H^1(E, V_{-3})$ has dimension 2.
- (4) The weight -4 summand of V is $V_{-4} = W_5 \otimes W_3^*$, of degree -2. So the weight -4 deformation space $T_{-4} = H^1(E, V_{-4})$ has dimension 2.
- (5) The weight -5 summand of V is $V_{-5} = W_2 \otimes W_3 \otimes \mathcal{O}(p)^{-1}$, of degree -1. So the weight -5 deformation space $T_{-5} = H^1(E, V_{-5})$ has dimension 1.
- (6) The weight -6 summand of V is $V_{-6} = W_5^*$, of degree -1. So the weight -6 deformation space $T_6 = H^1(E, V_{-6})$ has dimension 1.

Putting the pieces together, we can identify the parameter space $M = T_{-}/\mathbf{C}^{*}$ of bundles built by a negative weight deformation of V. It is a weighted projective space $\mathbf{WP}_{1,2,2,3,3,4,4,5,6}^{8}$, as predicted by the E_{8} case of Looijenga's theorem.

5.4. E_7 Bundles

We next consider E_7 in a similar spirit. The "trivalent" node on the extended E_7 Dynkin diagram is associated with the maximal subgroup $H = (SU(4) \times SU(4) \times SU(2))/\mathbb{Z}_4$ of E_7 . The \mathbb{Z}_4 consists of elements of $SU(4) \times SU(4) \times SU(2)$ of the form $\omega \times \omega \times \omega^2$, where $\omega^4 = 1$. The Lie algebra of E_7 decomposes under H as the adjoint representation of H plus

$$C_4 \otimes C_4' \otimes C_2 \oplus C_4^* \otimes C_4'^* \otimes C_2^* \oplus \wedge^2 C_4 \otimes \wedge^2 C_4'. \tag{5.22}$$

(Here C_4 , C'_4 , and C_2 are the basic representations of the three factors in H.)

The reductive part of the maximal parabolic associated with this node is obtained by restricting to a subgroup of the first SU(4) in H that is isomorphic to $U(3) = (SU(3) \times U(1))/\mathbb{Z}_3$. This maximal reductive subgroup is thus $R = (SU(3) \times SU(4) \times SU(2) \times U(1))/\mathbb{Z}_{12}$. (The Dynkin diagram of R is obtained by omitting the trivalent vertex from the unextended E_7 Dynkin diagram, with the missing node understood to represent a U(1) factor.)

We will describe the minimally unstable bundle first of all in terms of $SU(4) \times SU(4) \times SU(2)$. In the first SU(4) we take the bundle to be

$$X_4 = (W_3 \oplus \mathcal{O}) \otimes \mathcal{O}(p)^{-1/4}. \tag{5.23}$$

In the second SU(4), we take $B_4 = W_4 \otimes \mathcal{O}(p)^{-1/4}$, and for the SU(2) factor we take $B_2 = W_2 \otimes \mathcal{O}(p)^{-1/2}$. Just as in the E_8 case, the fractional exponents disappear when one

constructs the associated bundle in any representation of $SU(4) \times SU(4) \times SU(2)$ in which the \mathbb{Z}_4 subgroup acts trivially, that is, any representation of H. So we get an H bundle, and therefore an E_7 bundle. The center of R is a \mathbb{C}^* that acts with weight one on the first summand in (5.23), with weight -3 on the second, and trivially on B_4 and B_2 .

It is now straightforward, using (5.22), to describe in detail the minimally unstable E_7 bundle V. Rather than repeating this in as much detail as we did for E_8 , we will just write down the pieces of negative weight.

- (1) The weight -1 subbundle of V is $V_{-1} = W_3^* \otimes W_4^* \otimes W_2^* \otimes \mathcal{O}(p)$. This has degree -2, so $T_1 = H^1(E, V_{-1})$ has dimension 2.
- (2) The weight -2 subbundle of V is $V_{-2} = W_3 \otimes \wedge^2 W_4 \otimes \mathcal{O}(p)^{-1}$. This has degree -3, so $T_2 = H^1(E, V_{-2})$ has dimension 3.
- (3) The weight -3 subbundle of V is $V_{-3} = W_4 \otimes W_2 \otimes \mathcal{O}(p)^{-1}$. This has degree -2, so $T_3 = H^1(E, V_{-3})$ has dimension 2.
- (4) Finally, the weight -4 subbundle of V is $V_{-4} = W_3^*$. This has degree -1. so $T_4 = H^1(E, V_{-4})$ has dimension 1.

Putting the pieces together, we see that the parameter space $M = T_{-}/\mathbb{C}^{*}$ of negative weight deformations of V is a weighted projective space $\mathbf{WP}_{1,1,2,2,2,3,3,4}^{7}$, as predicted by Looijenga's theorem for E_{7} .

5.5. E_6 Bundles

Now we consider the last simply-laced group E_6 . Removing the trivalent vertex from the extended Dynkin diagram leaves the Dynkin diagram of the maximal subgroup $H = (SU(3) \times SU(3) \times SU(3))/\mathbf{Z}_3$ of E_6 ; the \mathbf{Z}_3 is the diagonal subgroup of the product of the centers of the three SU(3)'s. The Lie algebra of E_6 consists of the adjoint representation of H plus

$$C_3 \otimes C_3' \otimes C_3'' \oplus C_3^* \otimes {C'}_3^* \otimes {C''}_3^*.$$
 (5.24)

Here C_3 , C'_3 , and C''_3 are the three-dimensional representations of the three SU(3)'s.

The maximal reductive subgroup of the corresponding maximal parabolic is obtained by replacing the first SU(3) in H by $U(2) = (SU(2) \times U(1))/\mathbb{Z}_2$. The reductive group is thus $R = (SU(2) \times SU(3) \times SU(3) \times U(1))/\mathbb{Z}_6$.

We describe a minimally unstable bundle first of all in terms of H. In the first SU(3) we take

$$X_3 = (W_2 \oplus \mathcal{O}) \otimes \mathcal{O}(p)^{-1/3}, \tag{5.25}$$

and in the second and third SU(3)'s we take $B_3 = W_3 \otimes \mathcal{O}(p)^{-1/3}$. Once again, this gives something which makes sense as an H bundle, and therefore also as an E_6 bundle. The center of R is a \mathbb{C}^* which acts with respective weights 1 and -2 on the two summands in (5.25) and trivially on factors coming from the other SU(3)'s.

It is straightforward to give a detailed description of the minimally unstable E_6 bundle V. We content ourselves with looking at the pieces of negative weight:

- (1) In weight -1, we have $V_{-1} = W_2^* \otimes W_3^* \otimes W_3^* \otimes \mathcal{O}(p)$, of degree -3. So $T_{-1} = H^1(E, V_{-1})$ has dimension 3.
- (2) In weight -2, we have $V_{-2}=W_3\otimes W_3\otimes \mathcal{O}(p)^{-1}$, of degree -3. So $T_{-2}=H^1(E,V_{-2})$ has dimension 3.
- (3) In weight -3, we have $V_{-3} = W_2*$, of degree -1. So $T_{-3} = H^1(E, V_{-3})$ has dimension 1.

Putting the pieces together, we see that the space $M = T_{-}/\mathbb{C}^{*}$ of negative weight deformations of V is a weighted projective space $\mathbf{WP}_{1,1,1,2,2,2,3}^{6}$, as predicted by Looijenga's theorem for E_{6} .

5.6. G_2 Bundles

We come now to the two exceptional groups that are not simply laced.

 G_2 has a maximal subgroup $H = (SU(2) \times SU(2))/\mathbb{Z}_2$, where the \mathbb{Z}_2 is the diagonal subgroup of the product of the centers of the two SU(2)'s. (The Dynkin diagram of H is obtained from the extended Dynkin diagram of G_2 by omitting the vertex indicated in figure two.) The Lie algebra of G_2 decomposes under H as the sum of the adjoint representation plus

$$C_2 \otimes \operatorname{Sym}^3 C_2'$$
. (5.26)

(Here Sym^3C_2' denotes the symmetric part of $C_2'\otimes C_2'\otimes C_2'$.)

By restricting to a subgroup U(1) of the first SU(2), we get a group $R = (U(1) \times SU(2))/\mathbb{Z}_2$, which is the maximal reductive subgroup of a maximal parabolic subgroup of G_2 .

A minimally unstable G_2 bundle V can be described at the level of $SU(2) \times SU(2)$ by taking the first SU(2) factor to be

$$X_2 = (\mathcal{O}(p) \oplus \mathcal{O}) \otimes \mathcal{O}(p)^{-1/2} \tag{5.27}$$

and the second to be $B_2 = W_2 \otimes \mathcal{O}(p)^{-1/2}$. \mathbb{C}^* acts with weight 1 and -1 on the two summands in (5.27) and trivially on the second SU(2).

 G_2 is so small that we can fairly painlessly write down a detailed description of the minimally unstable bundle V. It is

$$\mathcal{O}(p) \oplus \mathcal{O} \oplus \mathcal{O}(p)^{-1} \oplus \operatorname{Sym}^{2} W_{2} \otimes \mathcal{O}(p)^{-1} \oplus \operatorname{Sym}^{3} W_{2} \otimes \mathcal{O}(p)^{-1} \oplus \operatorname{Sym}^{3} W_{2} \otimes \mathcal{O}(p)^{-2}.$$
 (5.28)

The first three summands come from the Lie algebra of the first SU(2), the fourth from the Lie algebra of the second SU(2), and the last two from decomposing $C_2 \otimes \text{Sym}^3 C_2'$ under R.

In particular, the subbundles of V of negative weight are as follows.

- (1) $V_{-1} = \text{Sym}^3 W_2 \otimes \mathcal{O}(p)^{-2}$, of degree -2. So $T_{-1} = H^1(E, V_{-1})$ is of dimension 2.
- (2) $V_{-2} = \mathcal{O}(p)^{-1}$, of degree -1. So $T_{-2} = H^1(E, V_{-2})$ is of dimension 1.

So the parameter space $M = T_{-}/\mathbb{C}^{*}$ of negative weight deformations of V is isomorphic to a weighted projective space $\mathbf{WP}_{1,1,2}^{2}$, as predicted by Looijenga's theorem for G_{2} .

5.7. F_4 Bundles

We conclude by examining F_4 .

 F_4 has a maximal subgroup (related to the node of the extended Dynkin diagram indicated in figure two) isomorphic to $H = (SU(3) \times SU(3))/\mathbb{Z}_3$ where \mathbb{Z}_3 is the diagonal subgroup of the product of the centers of the two SU(3)'s. The Lie algebra of F_4 decomposes under F_4 as the adjoint representation plus

$$C_3 \otimes \operatorname{Sym}^2 C_3' \oplus C_3^* \otimes \operatorname{Sym}^2 {C_3'}^*, \tag{5.29}$$

where C_3 and C_3' are the basic three-dimensional representations of the two SU(3)'s.

To obtain the reductive subgroup of a maximal parabolic, one restricts to a $U(2) = (SU(2) \times U(1))/\mathbf{Z}_2$ subgroup of the first SU(3). So the reductive group in question is $R = (SU(2) \times SU(2) \times U(1))/\mathbf{Z}_6$.

A minimally unstable bundle V can be obtained at the level of the $SU(3) \times SU(3)$ by choosing in the first SU(3)

$$X_3 = (W_2 \oplus \mathcal{O}) \otimes \mathcal{O}(p)^{-1/3} \tag{5.30}$$

and $B_3 = W_3 \otimes \mathcal{O}(p)^{-1/3}$ in the second. The center \mathbb{C}^* of R acts with weights 1 and -2 on the two summands in (5.30), and trivially on the second SU(3).

The negative weight subbundle of V is explicitly described as follows.

- (1) $V_{-1} = W_2^* \otimes \operatorname{Sym}^2 W_3^* \otimes \mathcal{O}(p)$, of degree -2. So $T_{-1} = H^1(E, V_{-1})$ is of dimension 2.
 - (2) $V_{-2} = \operatorname{Sym}^2 W_3 \otimes \mathcal{O}(p)^{-1}$, of degree -2. So $T_{-2} = H^1(E, V_{-2})$ is of dimension 2.
 - (3) $V_{-3} = W_2^*$, of degree -1. So $T_{-3} = H^1(E, V_{-3})$ is of dimension 1.

So the parameter space $M = T_{-}/\mathbb{C}^{*}$ of negative weight deformations of V is isomorphic to a weighted projective space $\mathbf{WP}_{1,1,2,2,3}^{2}$, as predicted by Looijenga's theorem for F_{4} .

5.8. Bundles Over Elliptic Manifolds

We actually wish to construct G bundles not just over a single elliptic curve but over an elliptically fibered manifold $\pi: Z \to B$ with a section σ . Z is described by a Weierstrass equation

$$y^2 = 4x^3 - g_2x - g_3, (5.31)$$

where x and y are sections of \mathcal{L}^2 and \mathcal{L}^3 , with \mathcal{L} being some line bundle over B. The section σ is given by $x = y = \infty$.

To imitate the above construction in this situation, we would like to construct suitable unstable G bundles over Z, which reduce on every fiber of π to the minimally unstable bundle constructed above, and can be deformed to stable G bundles over V.

The minimally unstable bundles were all built from tensor products and sums of the basic building blocks $\mathcal{O}(p)$ and W_n . ¹³ So all we need is to generalize those to an elliptic manifold.

The global version of $\mathcal{O}(p)$ is just $\mathcal{O}(\sigma)$, since σ intersects each fiber E of π in a distinguished point p. To construct a global version of the W_n , we must go back to the inductive procedure defining them. On a single elliptic curve, we had $W_1 = \mathcal{O}(p)$, so globally we take $W_1 = \mathcal{O}(\sigma)$. W_2 was defined over a single elliptic curve by the existence of an exact sequence

$$0 \to \mathcal{O} \to W_2 \to \mathcal{O}(p) \to 0. \tag{5.32}$$

So globally we ask that W_2 should have an exact sequence

$$0 \to \mathcal{M} \to W_2 \to \mathcal{O}(\sigma) \to 0, \tag{5.33}$$

¹³ In the *Spin* case, we used distinguished bundles Q_3 and Q_4 . It can be shown that $Q_4 = W_2 \otimes W_2^*$, and $Q_3 = \operatorname{ad}(W_2)$.

where \mathcal{M} is the pullback to Z of some line bundle on B. (Thus, \mathcal{M} is trivial on each fiber.) Moreover, we want the extension in (5.32) to be non-trivial when restricted to the fiber E_b over any $b \in B$. This is a strong condition which (up to isomorphism) uniquely determines \mathcal{M} and the extension in (5.33). The condition is equivalent to the statement that the line bundle over B whose fiber at b is $H^1(E_b, \mathcal{O}(\sigma)^{-1} \otimes \mathcal{M})$ should be trivial, so that it has an everywhere non-zero section.

 \mathcal{M} can be determined as follows. We have $H^1(E_b, \mathcal{O}(\sigma)^{-1} \otimes \mathcal{M}) = \mathcal{M}_b \otimes H^1(E_b, \mathcal{O}(\sigma)^{-1})$. (\mathcal{M}_b is the fiber at b.) By Serre duality, $H^1(E_b, \mathcal{O}(\sigma)^{-1})$ is dual to $H^0(E_b, K \otimes \mathcal{O}(\sigma))$ (where K is the canonical bundle); this is generated by dx/y, so a natural generator of $H^1(E_b, \mathcal{O}(\sigma)^{-1})$ can be identified with $y(dx)^{-1}$. For the line bundle whose fiber over b is $\mathcal{M}_b \otimes H^1(E_b, \mathcal{O}(\sigma)^{-1})$ to be trivial over B, $y(dx)^{-1}$ should make sense globally as a section of \mathcal{M} , so we need $\mathcal{M} = \mathcal{L}$.

This type of reasoning can be generalized to get global versions of all the W_n 's. W_n is defined inductively by an exact sequence

$$0 \to \mathcal{L}^{n-1} \to W_n \to W_{n-1} \to 0. \tag{5.34}$$

The line bundle \mathcal{L}^{n-1} in (5.34) is chosen to ensure the existence of an extension that is non-trivial on each fiber.

Global Version Of Unstable G Bundles

Having identified the global versions of $\mathcal{O}(p)$ and the W_n 's, we can construct appropriate global versions of the minimally unstable bundles. We simply replace in all above formulas $\mathcal{O}(p)$ and W_n by their global versions. The only subtlety is that one can twist by additional data coming from B.

Instead of trying to be abstract, let us first write down a concrete example for G = Sp(n). The minimally unstable bundle over a single elliptic curve was $W_n \oplus W_n^*$, and (having explained what we mean by W_n) we could take the same starting point over a general elliptic manifold Z. However, we can generalize slightly, pick an arbitrary line bundle \mathcal{M} over B, and consider the Sp(n) bundle

$$V = W_n \otimes \mathcal{M} \oplus W_n^* \otimes \mathcal{M}^{-1}$$
 (5.35)

which is isomorphic to the minimally unstable bundle $W_n \oplus W_n^*$ on each fiber.

To express this in a language that is more general, a bundle over Z that is isomorphic on each fiber to $W_n \oplus W_n^*$ is not uniquely determined because the bundle $W_n \oplus W_n^*$ has

automorphisms. Let A_b be the automorphism group of the Sp(n) bundle $W_n \oplus W_n^*$ over E_b , and let A be the sheaf of groups over B whose fiber at $b \in B$ is A_b . Then the bundle $W_n \oplus W_n^*$ over Z can be "twisted" by any element of $H^1(B, A)$.

The maximal reductive subgroup of A_b is the center, $U \cong \mathbb{C}^*$, of the reductive group R that was used in building the minimally unstable bundles over the fibers. What we have done in (5.35) is to twist by an element of $H^1(B, U)$.

This discussion can be slightly generalized as follows. If \mathcal{M} is not well-defined as a line bundle, but is the square root of a line bundle, then V is not well-defined as a vector bundle, but associated objects such as $V \otimes V$ and $\mathrm{ad}(V)$ are well-defined as vector bundles with structure group $Sp(n)/\mathbb{Z}_2$. With this starting point, one can use the parabolic construction to construct $Sp(n)/\mathbb{Z}_2$ bundles over an elliptic manifold Z that can be lifted to an Sp(n) bundle on each fiber, but not globally. A similar construction can be made for non-simply-connected forms of groups other than Sp(n).

Weighted Projective Space Bundle Over B

For every G, there is a bundle $\mathcal{W} \to B$ of weighted projective spaces whose fiber over $b \in B$ is the moduli space of semistable G bundles over E_b . We claimed in the introduction that for arbitrary simple, connected, and simply-connected G except E_8 , \mathcal{W} is a bundle of weighted projective spaces that can be obtained by projectivizing a certain sum of line bundles. In sections 2,3, and 4, we exhibited such structures for certain classes of G. Here we will briefly point out a general framework for exhibiting this structure.

Let Ω be the bundle over B whose fiber at $b \in B$ is the negative weight part of $H^1(E_b, \operatorname{ad}(V))$. For each b, the moduli space of G bundles on E_b is simply Ω_b/\mathbb{C}^* . So \mathcal{W} is obtained by projectivizing the vector bundle Ω .

Our claim is that for G other than E_8 , Ω is a certain sum of line bundles over B, in fact a sum of powers of \mathcal{L} , with exponents and \mathbb{C}^* weights that were summarized in the table in the introduction. But we now have a general framework for computing Ω and verifying that this is so. For instance, for G = Sp(n), Ω is the bundle whose fiber at b is $H^1(E_b, \operatorname{Sym}^2(W_n^*))$. By analyzing this cohomology group and its analogs for other G, the decomposition of Ω as a sum of line bundles will be exhibited elsewhere [36].

Note that if it is true that W is obtained by projectivizing a vector bundle Ω with \mathbb{C}^* action, then Ω is not uniquely determined; one could pick an arbitrary line bundle $\mathcal{N} \to B$ and twist the weight k subbundle of Ω by \mathcal{N}^k , without changing the projectivization B.

This freedom is reflected in the fact that W can be determined starting with (5.35) (or its analogs for other G) for arbitrary \mathcal{M} ; Ω depends on \mathcal{M} but \mathcal{W} does not.

Deformation To A Stable Bundle

Stable bundles over Z can (often) be made by deforming the unstable bundle V. First order deformations are classified by $H^1(Z, \operatorname{ad}(V))$. If (following standard notation in algebraic geometry) we denote the bundles on B made by taking the i^{th} cohomology of $\operatorname{ad}(V)$ along the fibers of $\pi: Z \to B$ as $R^i\pi_*(V)$, then in the situation considered here the Leray spectral sequence for $\pi: Z \to B$ degenerates to an exact sequence:

$$0 \to H^1(B, R^0\pi_*(\mathrm{ad}(V))) \to H^1(Z, \mathrm{ad}(V)) \to H^0(B, R^1\pi_*(\mathrm{ad}(V))) \to 0.$$
 (5.36)

(The Leray spectral sequence of a mapping always reduces to such an exact sequence when the cohomology of the fibers is nonzero in only two dimensions.) So the space of deformations of V maps to $H^0(B, R^1\pi_*(\operatorname{ad}(V)))$, which is the space whose projectivization is the space of sections of W. The fiber of the induced map from the space of bundles to the space of sections has for its tangent space $H^1(B, R^0\pi_*(\operatorname{ad}(V)))$. This is the tangent space to an abelian variety which generalizes the Jacobian found for G = SU(n) in section 2. (5.36) thus generalizes part of the structure found in section 2: the moduli space of bundles maps to the space of sections of W, the fiber being an abelian variety.

6. Comparison To F Theory Moduli Spaces

The remainder of this paper mainly aims at using our results to make two tests of duality between the heterotic string and F theory. The first test, in this section, involves comparison of moduli spaces.

We consider on the heterotic string side an elliptically fibered manifold $Z \to B$, with a section σ whose normal bundle we call \mathcal{L}^{-1} . For each G, there is a weighted projective space bundle $\mathcal{W}_G \to B$, which parametrizes G bundles over the fibers of $Z \to B$.

In heterotic string theory, Z is endowed with an $E_8 \times E_8$ bundle. All of our interest will focus on what happens in one of the two E_8 's, say the first one. We consider the locus of heterotic string vacua on Z in which the structure group of this E_8 bundle reduces to a subgroup G, of rank r. To specify a point in this locus requires picking among other things a G bundle over Z. It will become clear that the G bundles relevant to comparison

with the simplest F theory compactifications are semistable when restricted to the generic fiber. Such a G bundle determines a section s of the bundle \mathcal{W}_G .¹⁴

Much of the work of this paper can be summarized by saying that W_G , for $G \neq E_8$, has homogeneous coordinates a_j , j = 0, 1, ..., r, which are sections, respectively, of the line bundles $\mathcal{O}(1)^{s_j} \otimes \mathcal{L}^{-d_j}$. Here $\mathcal{O}(1)$ is a line bundle over W_G which restricts on each fiber to the basic line bundle on the weighted projective space; the numbers s_j are the weights appearing in Looijenga's theorem and the d_j are the degrees of the fundamental Casimir invariants of G. Under a section $s: B \to \mathcal{W}_G$, $\mathcal{O}(1)$ pulls back to a line bundle \mathcal{N} over B, and the a_j pull back to sections of $\mathcal{N}^{s_j} \otimes \mathcal{L}^{-d_j}$. Conversely, sections

$$\widetilde{a}_j \in H^0(B, \mathcal{N}^{s_j} \otimes \mathcal{L}^{-d_j})$$
 (6.1)

which are sufficiently generic (no common zeroes) determine a section s of W_G . The \tilde{a}_j are uniquely determined by s up to

$$\widetilde{a}_j \to \lambda^{s_j} \widetilde{a}_j, \text{ with } \lambda \in \mathbf{C}^*.$$
 (6.2)

The heterotic string compactified on the elliptic manifold $Z \to B$ is believed to be dual to F theory compactified on a K3-fibered manifold $X \to B$. The topology of X depends on the topology of the $E_8 \times E_8$ bundle over Z, in a way first analyzed in [3].

When the structure group of the E_8 bundle reduces to G, the heterotic string acquires an unbroken gauge symmetry H, where H is the commutant of G in E_8 . If B is a point, then H is necessarily simply-laced and unbroken H symmetry of the heterotic string corresponds to the appearance of a singularity of type H in F theory (in a fashion that we analyzed in the del Pezzo context in section 4). For B of positive dimension, unbroken gauge symmetry corresponds in F theory to appearance of a section $\theta: B \to X$ of singularities. In general, the singularity along θ is not of type H (H may not even be simply-laced); it is of type H', where $H' \supset H$ is a simply-laced group, and the H' symmetry is broken to H [20,21] by a monodromy corresponding to an automorphism of the Dynkin diagram of H' whose quotient is the Dynkin diagram of H. (We used this automorphism in section 3.2 to compare H and H' bundles.)

 $^{^{14}}$ s is defined at least over the dense open subset in B that parametrizes fibers over which the bundle actually is semistable. In general, s can be defined everywhere only after some blow-ups of B. When there is enough ampleness, and the rank of G exceeds the dimension of B, such blowups are generically not necessary.

In [20], the precise parameters controlling the complex structure near θ that should be related to bundle data on the heterotic string side were identified, for each H. The correspondence between the two theories was checked by counting parameters on the two sides. Here we will be more precise and actually exhibit a natural map from complex structure parameters in F theory to bundle parameters on the heterotic string side. In fact, we will show that the choice of a section $s: B \to \mathcal{W}_G$ is in natural correspondence with the data identified in [20] in F theory. As we explained in section 2.4, a more complete comparison of the two theories would involve also comparing certain abelian varieties.

In comparing (6.1) and (6.2) to the results of [20], we will actually generalize the statements of [20] in a fairly obvious way. In [20], the case $B = \mathbf{P}^1$ was considered, and an important role was played by two line bundles over B, namely $K_{\mathbf{P}^1}^{-1} = \mathcal{O}(2)$ and an additional line bundle $\mathcal{O}(12+n)$ that enters in constructing the K3 fibration over B. We will generalize to the case that B may have dimension greater than one, will write \mathcal{L} wherever $K_{\mathbf{P}^1}^{-1}$ appears in [20], and will replace the line bundle $\mathcal{O}(12+n) \to \mathbf{P}^1$ used in [20] by a general line bundle $\mathcal{N} \to B$. There is no difficulty in adapting the reasoning and conclusions of [20] to this more general case. We will not attempt here an explanation of the arguments of [20], but will just cite their answers and compare to (6.1) and (6.2).

We have seen in this paper that the description of E_8 bundles on an elliptically fibered manifolds is rather different from the description of G bundles for any G other than E_8 . In terms of heterotic string/F theory duality, this is related to the following. The ability to compare bundle data on the heterotic string side to F theory in the way we will do below depends on considering heterotic string bundles whose structure group is a proper subgroup G of E_8 , so that a singularity appears on the F theory side; the structure of the G bundle is then coded in the behavior near the singularity. The case $G = E_8$ is quite exceptional as then there is no singularity and no way to "localize" the bundle information on the F theory side. The comparison of the heterotic string and F theory moduli spaces then involve many additional issues such as heterotic string T dualities that can mix geometrical and bundle moduli. One way to turn off the T dualities while looking at E_8 bundles is to take the area of the fibers to be big; this option was explored at the end of section 4.

6.1. Comparison Of Moduli Spaces

SU(2) Bundles

First we consider the case of SU(2) bundles. For G = SU(2) and all other groups G considered subsequently, we assume a "minimal" embedding of SU(2) in E_8 , for which the generator of $H^3(E_8, \mathbf{Z})$ pulls back to the smallest possible value. Since the commutant of a minimally embedded SU(2) in E_8 is E_7 , reduction of the structure group of an E_8 bundle to such an SU(2) corresponds in F theory to considering a K3 fibered manifold $X \to B$ with a section θ of E_7 singularities. (E_7 has no outer automorphisms, so there is no monodromy breaking E_7 .)

Only the behavior of X near θ is relevant, and one can write a rather explicit local formula describing X as a hypersurface in a bundle $\mathcal{M} \oplus (\mathcal{L}^2 \otimes \mathcal{M}^2) \oplus (\mathcal{L}^3 \otimes \mathcal{M}^3)$, with coordinates u, x, y; \mathcal{M} is a line bundle over B. (In [20], this is formulated for $B = \mathbf{P}^1$ and $\mathcal{M} = \mathcal{O}(n)$.) Taking θ to be u = x = y = 0, the behavior near θ is given by an equation

$$y^2 = 4x^3 - fxu^3 - gu^5, (6.3)$$

where f and g are sections of the line bundles $\mathcal{L}^4 \otimes \mathcal{M}$ and $\mathcal{L}^6 \otimes \mathcal{M}$ over B. An obvious rescaling of u, x, y (with weights 1, 2, and 3) brings the equivalence $f \to \lambda f$, $g \to \lambda g$. Setting $\mathcal{N} = \mathcal{L}^6 \otimes \mathcal{M}$, and recalling that for SU(2) one has weights $s_j = 1$ and exponents $d_j = 0, 2$, for j = 0, 1, we see that f and g correspond in a natural way to the sections \tilde{a}_j of equations (6.1) and (6.2).

Two remarks should be made about this:

(1) In [20], it is asserted that, in a heterotic string description dual to this F theory model, the instanton number (of the E_8 bundle whose structure group is reducing to SU(2)) is 12+n. To express this in a way that does not assume that the base B is one-dimensional, the assertion is that if V is the SU(2) bundle and $\pi: Z \to B$ the elliptic fibration, then Chern classes of V and \mathcal{N} are related by

$$c_1(\mathcal{N}) = \pi_*(c_2(V)).$$
 (6.4)

This assertion was based in [20] on qualitative properties of the heterotic/F theory duality but without sufficient information about the bundles to actually compute $c_2(V)$ and verify (6.4). Having constructed the bundles, we are in a position to do so. In fact, (6.4) is

equivalent to a result of [23]. We give a proof, together with generalizations to other groups, in section 7.

(2) Generically along B, (6.3) describes a singularity of type E_7 at u = x = y = 0, but the singularity is worse at zeroes of f. It is proposed in [20] that matter fields in the two-dimensional representation of SU(2) come from zeroes of f. Translated into bundle language, the assertion amounts to the mathematical statement that $H^1(Z, V)$ can be computed locally from the behavior at zeroes of f, and that in case B is a curve (so that there are only finitely many zeroes), $H^1(Z, V)$ receives a one-dimensional contribution from each zero. This proposal was originally made on the basis of counting parameters and was further supported by a study of the F theory singularity near zeroes of f [38]. Having constructed the bundles, we are in a position to verify the relation between matter fields and zeroes of f by computing $H^1(Z, V)$ directly. We do so at the end of the present section.

We now compare results of [20] to (6.1) and (6.2) for groups G other than SU(2), taking the groups in the same order as in [20]. The remarks just made have analogs and will not be repeated.

SU(3) Bundles

SU(3) bundles correspond to unbroken E_6 . In the notation of [20], the structure near θ to get unbroken E_6 is

$$y^2 = 4x^3 - fxu^3 - gu^5 - g^2u^4. (6.5)$$

There is a singularity of type E_6 at u=x=y=0, away from zeroes of q. The fact that the coefficient of u^4 is a square prevents a monodromy that would break E_6 to F_4 . g, f and q are sections of $\mathcal{L}^6 \otimes \mathcal{M}$, $\mathcal{L}^4 \otimes \mathcal{M}$, and $\mathcal{L}^3 \otimes \mathcal{M}$, that is of \mathcal{N} , $\mathcal{N} \otimes \mathcal{L}^{-2}$, and $\mathcal{N} \otimes \mathcal{L}^{-3}$. g, f, and g transform with weight 1 when g, g, g are scaled with weights 1, 2, 3.

The weights and exponents just obtained agree with (6.1) and (6.2) for the case of SU(3).

G_2 Bundles

A reduction of the E_8 structure group to G_2 leaves unbroken F_4 . This corresponds in F theory to a section of E_6 singularities with monodromy allowed. The analog of (6.5) is

$$y^2 = 4x^3 - fxu^3 - gu^5 - bu^4. (6.6)$$

The only difference from (6.5) is that the coefficient of u^4 is not required to be a perfect square. g, f, and b have weights 1, 1, 2 under scalings of u, x, y, and are sections of \mathcal{N} , $\mathcal{N} \otimes \mathcal{L}^{-2}$, and $\mathcal{N}^2 \otimes \mathcal{L}^{-6}$, in agreement with expectations for G_2 . In fact, the relation $b = q^2$ between the descriptions for G_2 and for SU(3) was already seen in section 3. The other examples discussed in section three also have analogs in F theory.

In subsequent examples, the precise formulas for behavior along θ analogous to (6.6) become more complicated and will not be presented. Interested readers are referred to [20].

Spin(5) Bundles

The commutant of Spin(5) in E_8 is Spin(11). This corresponds in F theory to having a D_6 singularity along θ , with monodromy allowed.

In comparing to [20] for Spin(5) and the other examples, we use the following conventions. In [20], various objects are written as f_{2n+12} , q_{n+6} , etc. In general, if the subscript is a(n+12)-2b, then in our notation the corresponding object has weight a and is a section of $\mathcal{N}^a \otimes \mathcal{L}^{-b}$.

For instance, according to [20], the F theory locus with Spin(11) gauge symmetry is described by objects g_{12+n} , f_{8+n} , and s_{4+n} . In our notation, these objects all have weight one and are sections of $\mathcal{N} \otimes \mathcal{L}^{-d_j}$ for $d_j = 0, 2, 4$. These are the expected weights and exponents for Spin(5).

Spin(6) Bundles

Spin(6) bundles correspond in E_8 to unbroken Spin(10). In F theory, this corresponds to a section of D_5 singularities without monodromy. According to [20], such a section is described by objects h_{n+4} , q_{n+6} , g_{n+12} , and f_{n+8} , that is to say objects of weight 1 and exponents $d_j = 0, 2, 3, 4$. These are the expected values for Spin(6) = SU(4).

Spin(7) Bundles

We conclude with one more example. (Many more cases are worked out in [20]; the interested reader can verify that in each case, the weights and exponents are as expected from our analysis of bundles.) Spin(7) bundles correspond to unbroken Spin(9) and to a section of D_5 singularities with \mathbb{Z}_2 monodromy. In [20], the moduli are described by objects g_{n+12} , f_{n+8} , h_{n+4} , and g_{2n+12} , or in other words objects of weights 1, 1, 1, 2 and exponents $d_j = 0, 2, 4, 6$. This is as expected for Spin(7).

6.2. Localization Of Cohomology

One of the insights in [20], further explored in [38], was that if V is a G bundle over an elliptic manifold $Z \to B$, then, depending on G, certain cohomology groups of V, which in physics determine the spectrum of light quarks and leptons, appear to be localized on certain subvarieties of B. (The case considered in detail was the case that B is a curve and a subvariety is therefore a finite set of points.) As promised above, we will here explain directly from the bundle point of view why this is so. We will also explain why other cohomology groups are *not* localized in this way.

We illustrate the idea with the case (particularly important in applications) in which G = SU(n). Let V be a rank n complex vector bundle over Z, constructed by a spectral cover as in section 2. Suppose that one wants to compute $H^1(Z, V)$.

If we think of the fibers of $\pi: Z \to B$ as being small, the first step would clearly be to solve the $\overline{\partial}$ equation along the fibers, and then solve for the adiabatic motion along the base. In fact, in complex geometry, there is a systematic procedure (the Leray spectral sequence) to compute $H^i(Z, V)$ starting with a computation of $H^j(E_b, V)$, where E_b , for $b \in B$, is the fiber of Z over b. The result is in particular that $H^i(Z, V)$ is localized along those fibers that have the property that $H^j(E_b, V)$ is non-zero for some $j \leq i$.

In our problem, along a generic fiber, V splits as a sum of line bundles none of which are trivial. It is therefore the case that for generic b, $H^{j}(E_{b}, V) = 0$ for all j. The computation of $H^{i}(Z, V)$ will be localized along the locus in B on which one of the factors of V is trivial.

We assume as usual that Z is presented in Weierstrass form and that the spectral cover is defined by an equation of the form familiar from section 2:

$$a_0 + a_2 x + a_3 y + \ldots + a_n x^{n/2} = 0 (6.7)$$

(if n is odd the last term is slightly different). We want to find the condition on $b \in B$ so that when restricted to E_b , V does have a trivial factor. The condition is simply

$$a_n = 0. (6.8)$$

For $a_n = 0$ is the condition under which one of the roots of (6.7) is at $x = \infty$, which is the point on E_b that corresponds to a trivial line bundle.

So the computation of $H^1(Z, V)$ will be localized on the subvariety of B defined by vanishing of a_n . If Z is a K3 surface and B is a curve, then (6.8) defines a finite set of

points. A universal local computation shows that each simple zero of a_n will contribute a one-dimensional subspace to $H^1(Z, V)$. In a higher-dimensional case, (6.8) defines a hypersurface D in B, and $H^1(Z, V)$ must be computed by solving an appropriate $\overline{\partial}$ equation along D.

Apart from computing the cohomology of V, one also wishes to compute the cohomology of other bundles derived from V, such as the second exterior power $\wedge^2 V$. The basic idea is similar: on a generic E_b , $\wedge^2 V$ splits as a sum of line bundles, and $H^i(Z, \wedge^2 V)$ will be localized along those E_b on which one of the line bundles is trivial.

If along E_b , $V = \bigoplus_{i=1}^n \mathcal{L}_i$, then $\wedge^2 V = \bigoplus_{1 \leq i < j \leq n} \mathcal{L}_i \otimes \mathcal{L}_j$, so $\wedge^2 V$ contains a trivial line bundle if and only if for some i < j, $\mathcal{L}_i = \mathcal{L}_j^{-1}$. Inverse line bundles correspond to points on E_b that differ by $y \to -y$, so the localization will be on b's such that the spectral equation (6.7) has two solutions that differ by $y \to -y$. If in other words we write the spectral equation as

$$P(x) + yQ(x) = 0, (6.9)$$

where P and Q are polynomials in x only, then the condition is that P and Q have a common zero; in other words, the resultant R(P,Q) should vanish. The computation of $H^i(Z, \wedge^2 V)$ will be localized on the hypersurface in B defined by R(P,Q) = 0.

Let us make this more explicit for small values of n (which are of particular interest for applications). The first non-trivial case is n = 4. In this case, vanishing of Q reduces to $a_3 = 0$, and $H^i(Z, \wedge^2 V)$ will be localized on the hypersurface defined by that equation.

The first case in which one really sees the resultant is n = 5, for which

$$P = a_0 + a_2 x + a_4 x^2$$

$$Q = a_3 + a_5 x.$$
(6.10)

Solving the second equation for x and substituting in the first, we see that the condition for a common zero of P and Q is

$$a_0 a_5^2 - a_2 a_3 a_5 + a_4 a_3^2 = 0, (6.11)$$

and this equation defines the hypersurface in B along which $H^2(Z, \wedge^2 V)$ will be localized. The detailed formula for the resultant becomes increasingly complicated for larger n.

It is not true that the cohomology with values in *any* representation has such a localization. For instance, there is no such localization for $H^i(Z, ad(V))$. The reason is that ad(V) contains trivial sub-bundles – associated with the Cartan subalgebra – on a generic

 E_b . In general, the cohomology is localized precisely for those representations that contain no vector invariant under a maximal torus, so that on a generic E_b there is no trivial sub-bundle. This statement holds for arbitrary G, not just the case G = SU(n) where explicit formulas can be worked out using the spectral covers.

All of these assertions are in full agreement with what has been guessed or calculated on the F-theory side in [20,38].

7. Computations Of Characteristic Classes

Our goal in this section is to understand better the G bundles that we have constructed on elliptic manifolds $Z \to B$ by computing their characteristic classes, and to use this information for another test of duality.

To be more precise, every simple Lie group G has $H^3(G, \mathbf{Z}) = \mathbf{Z}$, so that a G bundle always has a four-dimensional characteristic class λ . We focus on the case that G is connected and simply-connected; then the homotopy groups $\pi_i(G)$ vanish for i < 3, and λ is the first non-trivial characteristic class of a G bundle.

For G = SU(n), λ is the usual second Chern class (of the associated rank n complex vector bundle). For G = Spin(n), λ is one half of the usual first Pontryagin class (of the associated rank n vector bundle).

We have given in this paper two constructions of G bundles that can be used to compute their characteristic classes – the constructions via parabolic subgroups and via spectral covers. The parabolic construction gives a simple method to compute characteristic classes for any G; however, it is not completely general at present because we do not understand the analog of "twisting" by a line bundle on the spectral cover. The spectral cover construction in the explicit form discussed in section 2 is limited mainly to SU(n) and Sp(n), and leads to much more complicated computations, but has the virtue that one can incorporate such twists.

In the next subsection, we compute λ via parabolics for two cases: G = SU(n) and $G = E_8$. SU(n) was chosen for illustration and to permit comparison with spectral covers, and E_8 was chosen because the computation of λ for E_8 bundles will make it possible to resolve a longstanding mystery about F theory, which is the appearance of certain "three-branes" in the vacuum. After settling that issue in section 7.2, we go on in section 7.3 to compute λ via spectral covers for SU(n) bundles.

7.1. Computation Via Parabolics

The basic idea of constructing stable G bundles via parabolics is that one first defines a very simple unstable G bundle and then one deforms it to become stable. For the sake of computing characteristic classes, the second step is unnecessary; the topology of the bundle is in any case invariant under deformations. So we can compute directly for the unstable bundle, and this makes things simple.

For instance, for G = SU(n) the starting point is the unstable bundle

$$V = W_k \otimes \mathcal{M} \oplus W_{n-k}^* \otimes \mathcal{M}' \tag{7.1}$$

with some k in the range $1 \leq k \leq n$, and $\mathcal{M}, \mathcal{M}'$ two line bundles over B. For the purposes of computing Chern classes, W_k can be replaced by a direct sum $\mathcal{O}(\sigma) \oplus \mathcal{L} \oplus \mathcal{L}^2 \oplus \ldots \oplus \mathcal{L}^{k-1}$, and likewise W_{n-k}^* can be replaced by $\mathcal{O}(\sigma)^{-1} \oplus \mathcal{L}^{-1} \oplus \mathcal{L}^{-2} \oplus \ldots \oplus \mathcal{L}^{-(n-k-1)}$. (This can be proved using the exact sequences by which W and W^* are defined.) \mathcal{M} and \mathcal{M}' should be constrained so that V has trivial determinant; this means that $\mathcal{M}^k \otimes (\mathcal{M}')^{n-k} \otimes \mathcal{L}^{-\frac{1}{2}(n-1)(n-2k)} \cong \mathcal{O}$.

It is straightforward to compute the second Chern class of V, using the fact that if $V = \bigoplus_{i=1}^{n} \mathcal{L}_{i}$, then

$$c_2(V) = \sum_{i < j} c_1(\mathcal{L}_i) c_1(\mathcal{L}_j). \tag{7.2}$$

Even without computation, it is evident that the answer is a polynomial in $c_1(\mathcal{O}(\sigma))$, $c_1(\mathcal{O}(\mathcal{L}))$, and $c_1(\mathcal{M})$. We exhibit the formula only in a comparatively simple case that we will use later in comparing to results obtained from spectral covers. This is the case that n is even, k = n/2, and $\mathcal{M}' = \mathcal{M}^{-1}$. In this case, if we set $\sigma = c_1(\mathcal{O}(\sigma))$, and

$$c_1(\mathcal{M}) = -\frac{1}{2} \left(\eta - c_1(\mathcal{L}) \right), \tag{7.3}$$

then we get 15

$$c_2(V) = \eta \sigma - \frac{1}{24} c_1(\mathcal{L})^2 (n^3 - n) - \frac{n}{8} \eta (\eta - nc_1(\mathcal{L})).$$
 (7.4)

This formula shows that the interpretation of η is that

$$\eta = \pi_*(c_2(V)). \tag{7.5}$$

¹⁵ Here and for the E_8 calculation given below, one needs to know that $\sigma^2 = -\sigma \cdot c_1(\mathcal{L})$, a relation proved in section 7.2.

Here π_* is the operation of "integrating over the fibers" of the elliptic fibration $\pi: Z \to B$. Clearly, not all values of η are possible; one has

$$\eta \equiv c_1(\mathcal{L}) \text{ modulo } 2.$$
(7.6)

From (7.4), we see that $c_2(V)$ is of the form

$$c_2(V) = \eta \sigma + \pi^*(\omega), \tag{7.7}$$

with $\omega \in H^4(B, \mathbf{Z})$. In this way of writing things, η and ω are uniquely determined. If we fix the elliptic manifold $Z \to B$, so that σ and $c_1(\mathcal{L})$ are fixed, then according to (7.3), η is arbitrary (apart from the mod two condition) and a choice of η fixes \mathcal{M} . There is no additional freedom in the construction; ω is uniquely determined in terms of η and Z by the formula given in (7.4).

In the spectral cover construction, as we see later, this relation can be modified by twisting by a line bundle on the spectral cover. As we will show elsewhere [36], in special cases this freedom can be seen in the construction via parabolics by taking $k \neq n/2$. For other groups, we do not know the analog of twisting by a line bundle on the spectral cover.

Characteristic Class Of E_8 Bundles

Now we consider the case of E_8 .

The starting point in building an E_8 bundle V via parabolics is to consider a bundle whose structure group reduces to a group that is locally $SU(6) \times SU(2) \times SU(3)$ (and even to a subgroup thereof). We thus need to describe SU(6), SU(2), and SU(3) bundles over B that we will call X_6 , X_2 , and X_3 .

The fundamental characteristic class $\lambda(V)$ of an E_8 bundle whose structure group reduces to $SU(6) \times SU(2) \times SU(3)$ can be described very simply: it is 16

$$\lambda(V) = c_2(X_6) + c_2(X_2) + c_2(X_3). \tag{7.8}$$

In section 5, in working on a single elliptic curve, we took $X_2 = W_2 \otimes \mathcal{O}(\sigma)^{-1/2}$. Globally, we must modify this slightly. In view of the exact sequence

$$0 \to \mathcal{L} \to W_2 \to \mathcal{O}(\sigma) \to 0, \tag{7.9}$$

The characteristic class $\lambda(V)$ is simply $c_2(V)/60$. The following formula is computed directly using this fact and the form of the embedding of $SU(6) \times SU(2) \times SU(3)$ in E_8 .

the determinant of W_2 is $\mathcal{O}(\sigma) \otimes \mathcal{L}$, so we take

$$X_2 = W_2 \otimes \mathcal{O}(\sigma)^{-1/2} \otimes \mathcal{L}^{-1/2}, \tag{7.10}$$

which has trivial determinant. Likewise, the definition $X_3 = W_3 \otimes \mathcal{O}(\sigma)^{-1/3}$ used in section 5 must be modified to

$$X_3 = W_3 \otimes \mathcal{O}(\sigma)^{-1/3} \otimes \mathcal{L}^{-1}. \tag{7.11}$$

Finally, in working on a single elliptic curve, the SU(6) bundle was $(W_5 \oplus \mathcal{O}) \otimes \mathcal{O}(\sigma)^{-1/6}$. Here we want to consider a bundle that is isomorphic to this on each fiber and has trivial determinant. The most general possibility is

$$X_6 = (W_5 \otimes \mathcal{S}^{-1} \oplus \mathcal{S}^5 \otimes \mathcal{L}^{-1}) \otimes \mathcal{O}(\sigma)^{-1/6} \otimes \mathcal{L}^{-3/2}, \tag{7.12}$$

with S an arbitrary line bundle on B. Because of the fractional exponents, X_2 , X_3 , and X_6 do not really make sense as SU(n) bundles, but the fractions disappear when one puts together an E_8 bundle (or an $(SU(6) \times SU(2) \times SU(3))/\mathbb{Z}_6$ bundle). The fractions cause no harm in computing Chern classes; one simply uses (7.2) formally, setting $c_1(\mathcal{L}^{\gamma}) = \gamma c_1(\mathcal{L})$ for $\gamma \in \mathbb{Q}$.

If we set

$$\eta = c_1(\mathcal{S}) + 4c_1(\mathcal{L}),\tag{7.13}$$

then after a calculation that is only somewhat tedious, the fundamental characteristic class of the E_8 bundle comes out to be

$$\lambda(V) = \eta \sigma - 15\eta^2 + 135\eta c_1(\mathcal{L}) - 310c_1(\mathcal{L})^2. \tag{7.14}$$

In particular,

$$\eta = \pi_*(\lambda(V)). \tag{7.15}$$

We again see that

$$\lambda = \eta \sigma + \pi^*(\omega), \tag{7.16}$$

for some $\omega \in H^4(B, \mathbf{Z})$. Moreover, as in the SU(n) case, while η can be adjusted independently, ω is determined uniquely in terms of η and Z. If one wishes to vary η and ω independently, or at least more independently, one must learn the analog of twisting by a line bundle on a spectral cover.

In the next subsection, we will give strong evidence that the E_8 bundles that appear in the simplest applications of F theory are actually the ones whose characteristic class we have just computed. This will be done by showing that the formula (7.14), with the strange numbers -15, 135, and -310, agrees with expectations from F theory. Of course, (7.14) is mainly interesting if B is of dimension bigger than one; otherwise, for dimensional reasons, $\omega = 0$ and the discussion collapses. So for this purpose we are interested in the case that B is a surface.

\mathbf{Z}_2 Symmetry

It might at first seem unexpectedly lucky that our simplest construction agrees with F theory. We have definitely not analyzed the most general stable E_8 bundle over Z, perhaps not even the most general one that is semistable on the generic fiber. It may be possible to construct more general bundles by an analog of twisting by a line bundle on the spectral curve.

It seems that one reason for our good fortune has to do with an important bit of physics that we have not yet exploited in this paper. An elliptic manifold Z with a section σ has a \mathbb{Z}_2 symmetry, generated by an "involution" τ that leaves σ invariant and acts as -1 on each fiber. In terms of a Weierstrass model $y^2 = 4x^3 - g_2x - g_3$, τ is just the operation $y \to -y$ with fixed x.

What does this correspond to on the F theory side? The elliptic manifold $Z \to B$ corresponds in F theory to a manifold X that is fibered over B with K3 fibers. The K3's are themselves elliptic, so there is an elliptic fibration with section $\pi': X \to B'$, where B' is a \mathbf{P}^1 bundle over B. On the F theory side, there is therefore a potential \mathbf{Z}_2 symmetry τ' . If one tracks through the duality between the heterotic string and F theory, one can see that τ is mapped to τ' .

Now, τ' is automatically a symmetry in F theory on X unless one turns on modes of the three-form field C of eleven-dimensional supergravity that are odd under τ . Such modes involve either the intermediate Jacobian introduced in section 2.4, or the discrete data discussed by K. and M. Becker [9](for which we proposed an interpretation at the end of section 4). If we suppress the discrete data and if $H^3(X)$ vanishes so that there are no periods, then τ' is automatically a symmetry in F theory; so in the corresponding heterotic string story we want bundles that are invariant under τ .

The E_8 bundles whose characteristic class was computed above are τ invariant, while one would expect that a generic twist or perhaps any twist of the sort that we do not

presently know how to make would break the symmetry. (For instance, when we compute for SU(n), where we do understand the possible twists, we will see that the additional twists give bundles that are not τ -invariant.) So it is natural that the bundles that we know how to construct are the ones that should be compared to the simplest cases of F theory.

For an SU(n) bundle understood as a rank n complex vector bundle, the relevant statement of τ -invariance becomes $\tau^*(V) = V^*$. (Any semistable rank n bundle on a single elliptic curve obeys this relation – since any line bundle does – so it is natural to look for a component of the moduli space of bundles in which every bundle obeys $\tau^*(V) = V^*$.) In the construction via parabolics, starting with $W_k \oplus W_{n-k}^*$, duality exchanges k with n-k, and so the condition $\tau^*(V) = V^*$ is most easily implemented by taking k = n/2, as we did in arriving at (7.4). When we compute via spectral covers where many twists are possible, we will compare (7.4) to the Chern classes of a bundle constructed via spectral curves and obeying $\tau^*(V) = V^*$.

7.2. Origin Of F Theory Threebranes

One of the very surprising features about F theory compactification on a Calabi-Yau four-fold X is that a consistent compactification requires the presence of threebranes in the vacuum [8].¹⁷ The number of threebranes is $I = \chi(X)/24$, where χ is the topological Euler characteristic. If X has an elliptic fibration $\pi': X \to B'$ with a smooth Weierstrass model, then one can prove as in [8] that

$$I = 12 + 15 \int_{B'} c_1 (TB')^3, \tag{7.17}$$

where $c_1(TB')$ is the first Chern class of the tangent bundle of B'.

The theory is defined as Type IIB superstring theory with a coupling constant that varies in space-time. It reduces at long distances to ten-dimensional Type IIB supergravity with (among other things) additional "threebranes." A threebrane is a sort of impurity near which the supergravity description breaks down; its worldvolume is a four-dimensional submanifold of spacetime. In the present discussion, spacetime is $\mathbf{R}^4 \times B'$ (where B' is the base of the elliptic fibration with total space X), and to maintain four-dimensional Poincaré invariance, the four-manifolds in question are of the form $\mathbf{R}^4 \times p_i$, where the p_i are points in B'. The number of threebranes, that is the number of points p_i , was determined in [8] by observing that the supergravity equations have a solution only if the correct number of impurities is included.

Under duality with the heterotic string, the threebranes turn into fivebranes that are wrapped over fibers of $Z \to B$, and the question is why such fivebranes should be present. The explanation depends upon heterotic string anomaly cancellation. Perturbative anomaly cancellation without fivebranes requires an $E_8 \times E_8$ bundle $V_1 \times V_2$ with

$$\lambda(V_1) + \lambda(V_2) = c_2(TZ). \tag{7.18}$$

(TZ) is the tangent bundle of Z.) The general anomaly cancellation condition with fivebranes is

$$\lambda(V_1) + \lambda(V_2) + [W] = c_2(TZ). \tag{7.19}$$

where [W] is the cohomology class of the fivebranes.

It has been suspected in the past that the reason that fivebranes appear is that $\lambda(V_1)$ and $\lambda(V_2)$ cannot be varied freely and that, after adjusting $\pi_*(\lambda(V_1))$ and $\pi_*(\lambda(V_2))$ to specified values whose sum equals $\pi_*(TZ)$, (7.18) would be in error by the pullback of a cohomology class from B. Any such class is of the form h[p], where $h \in \mathbf{Z}$ and [p] is the class of a point on B. Suppose that (7.19) is obeyed with

$$[W] = h[F], \tag{7.20}$$

where $[F] = \pi^*([p])$ is the class of a fiber of the elliptic fibration. In that case, h will be the number of fivebranes, on the heterotic string side, and should coincide with the number of threebranes seen in F theory.

By now, we have seen that the $\lambda(V_i)$ obey restrictions of the appropriate kind, and we have the information in hand to compute h and verify that h = I, finally giving a heterotic string explanation of the number of threebranes. To do this, we will have to make some computations of Chern classes.

Reduction Of F Theory Formula

First, as a preliminary, we need to make a further reduction of the F theory formula (7.17), for the case relevant to (the simplest versions of) heterotic string/F theory duality. This is the case that B' is a \mathbf{P}^1 bundle over B, the \mathbf{P}^1 bundle being the projectivization of a vector bundle $Y = \mathcal{O} \oplus \mathcal{T}$, with \mathcal{T} a line bundle over B. We endow the \mathbf{P}^1 bundle with homogenous coordinates a, b which are sections of $\mathcal{O}(1)$ and $\mathcal{O}(1) \otimes \mathcal{T}$, respectively; here $\mathcal{O}(1)$ is a bundle that restricts on each \mathbf{P}^1 fiber to the line bundle that usually goes by that name. If we set $r = c_1(\mathcal{O}(1))$, $t = c_1(\mathcal{T})$, then the fact that the sections a and b of the

line bundles $\mathcal{O}(1)$ and $\mathcal{O}(1) \otimes \mathcal{T}$ over B' have no common zeroes means that r(r+t) = 0 in the cohomology ring of B'.

Let $c_1(B)$ and $c_2(B)$ denote the Chern classes of the *tangent* bundle of B. When confusion is unlikely we will call these simply c_1 and c_2 , and likewise we write simply c_i , rather than $\pi^*(c_i)$, etc., for the pullbacks of the c_i under the various fibrations such as π and π' .

A standard adjunction formula says that the total Chern class of the tangent bundle of B' is

$$c(B') = (1 + c_1 + c_2)(1 + r)(1 + r + t). (7.21)$$

Hence, the first Chern class of B' is $c'_1 = c_1 + 2r + t$. One can now evaluate (7.17) in terms of the geometry of B. Let $\pi'': B' \to B$ be the projection. Using r(r+t) = 0 to reduce $(c'_1)^3$ to a linear function of r and then using $\pi''_*(r) = 1$, $\pi''_*(1) = 0$, we can compute $\pi''_*((c'_1)^3)$ and thereby get the following formula expressing the number of threebranes in terms of data defined on B:

$$I = 12 + 90 \int_{B} c_1^2 + 30 \int_{B} t^2.$$
 (7.22)

The base B of a Calabi-Yau elliptic fibration is rational and hence obeys

$$12 = \int_{B} \left(c_1^2 + c_2 \right). \tag{7.23}$$

We can combine the last two expressions and write

$$I = \int_{B} \left(c_2 + 91c_1^2 + 30t^2 \right). \tag{7.24}$$

Computations On Heterotic String Side

Now we can compute on the heterotic string side. First of all, the conjectured duality between F theory and the heterotic string says that F theory on the fourfold $X \to B' \to B$, with B' as above, should be compared to the heterotic string on $\pi: Z \to B$ with $E_8 \times E_8$ bundles that are chosen so that

$$\eta_1 = \pi_*(\lambda(V_1)) = 6c_1 + t, \quad \eta_2 = \pi_*(\lambda(V_2)) = 6c_1 - t.$$
(7.25)

This is the generalization of the more familiar statement [3] that F theory on the Hirzebruch surface \mathbf{F}_n corresponds to a heterotic string on K3 with 12 + n instantons in one

 E_8 and 12 - n in the other. In other words, when the base \mathbf{P}^1 of the Hirzebruch fibration $\mathbf{F}_n \to \mathbf{P}^1$ is replaced by a surface B (which is the base of $B' \to B$), the generalization of the number 12 is the cohomology class $6c_1(B)$, and the generalization of the number n is the cohomology class t.

The V_i are uniquely determined (at least within the class of bundles we are considering) by specifying t and hence the η_i , and then via (7.14) the $\lambda(V_i)$ are determined, given $c_1(\mathcal{L})$. Also, it is appropriate now to impose the Calabi-Yau condition $c_1(\mathcal{L}) = c_1(B) = c_1$. We get

$$\lambda(V_1) + \lambda(V_2) = -80c_1^2 + 12\sigma c_1 - 30t^2. \tag{7.26}$$

To proceed further we need $c_2(TZ)$, which is the remaining unknown in (7.19). This can be computed by the same methods we used to arrive at (7.22). The Weierstrass equation $zy^2 = 4x^3 - g_2xz^2 - g_3z^3$ embeds Z in a \mathbf{P}^2 bundle $W \to B$. W is the projectivization of a sum of line bundles \mathcal{L}^2 , \mathcal{L}^3 , and \mathcal{O} . W has homogeneous coordinates x, y, and z which we interpret as sections of $\mathcal{O}(1) \otimes \mathcal{L}^2$, $\mathcal{O}(1) \otimes \mathcal{L}^3$, and $\mathcal{O}(1)$ over W; we set $r = c_1(\mathcal{O}(1))$. The total Chern class $c(Z) = 1 + c_1(Z) + c_2(Z) + \ldots$ of Z is given by adjunction as

$$c(Z) = c(B) \frac{(1+r)(1+r+2c_1(\mathcal{L}))(1+r+3c_1(\mathcal{L}))}{1+3r+6c_1(\mathcal{L})}.$$
 (7.27)

The denominator expresses the fact that the Weierstrass equation is a section of $\mathcal{O}(1)^3 \otimes \mathcal{L}^6$. The fact that x, y, and z have no common zeroes means that $r(r+2c_1(\mathcal{L}))(r+3c_1(\mathcal{L}))=0$ in the cohomology ring of W. Since multiplication by $3(r+2c_1(\mathcal{L}))$ can be understood as restriction from W to Z (which is defined as we just said by vanishing of a section of $\mathcal{O}(1)^3 \otimes \mathcal{L}^6$), the relation for r simplifies in the cohomology ring of Z to $r(r+3c_1(\mathcal{L}))=0$. As the section z of $\mathcal{O}(1)$ vanishes on σ with multiplicity 3 (σ can be described in homogeneous coordinates by (x,y,z)=(0,1,0), and we see that near σ , z has a third order zero, being given by $z \sim x^3$), one has $r=3\sigma$ in the cohomology ring of Z. With a little patience one can use these facts and expand (7.27) to learn that

$$c_2(TZ) = c_2 + 11c_1^2 + 12\sigma c_1. (7.28)$$

Everything is now in place to evaluate (7.19). Using (7.28) and (7.26) we see that (7.19) is obeyed if and only if W = h[F] with

$$h = c_2 + 91c_1^2 + 30t^2. (7.29)$$

Using (7.24), one can see that this amounts to the statement that h, the number of fivebranes required on the heterotic string side, equals I, the number of threebranes required in the F theory description. So, as promised, we have obtained from the heterotic string point of view some understanding of the appearance of threebranes in F theory compactification.

For further use, note that because $r(r + 3c_1(\mathcal{L})) = 0$ and $r = 3\sigma$, we have obtained the relation

$$\sigma^2 = -\sigma c_1(\mathcal{L}) \tag{7.30}$$

which entered at several points in this paper and is further used below. The relation $r = 3\sigma$ that we just exploited means that the line bundle $\mathcal{O}(1)$ over $W \to B$, when restricted to Z, obeys $\mathcal{O}(1) \cong \mathcal{O}(\sigma)^3$. (The assertions of this paragraph do not require the Calabi-Yau condition and hold for any \mathcal{L} .)

7.3. Computation Via Spectral Covers

For the rest of this section, our goal will be to compute Chern classes using the description of bundles over Z by spectral covers. We do not assume that Z is Calabi-Yau, so $c_1(\mathcal{L})$ and $c_1(B)$ are unrelated. We otherwise use the same notation as above; Z is embedded in a \mathbf{P}^2 bundle $W \to B$ via a Weierstrass equation, and has a section σ . We will incorporate the twisting by a line bundle that was explained in section two.

The spectral cover C introduced in section 2 is given by an equation s=0 which defines a hypersurface in Z. s is a section of $\mathcal{O}(\sigma)^n \otimes \mathcal{M}$, where \mathcal{M} is an arbitrary line bundle over B; we set $\eta' = c_1(\mathcal{M})$. (In eqn. (7.70), we will see that η' coincides with η as introduced earlier.) Concretely in affine coordinates with z=1,

$$s = a_0 + a_2 x + a_3 y + \ldots + a_n x^{n/2}$$
(7.31)

(the last term is $x^{(n-3)/2}y$ if n is odd). Here a_0 is a section of \mathcal{M} , and (x and y being sections of \mathcal{L}^2 and \mathcal{L}^3 in the Weierstrass model) a_r is a section of $\mathcal{M} \otimes \mathcal{L}^{-r}$. s has a pole of order n at $x = y = \infty$, which is why it is a section of $\mathcal{M} \otimes \mathcal{O}(\sigma)^n$.

We recall that SU(n) bundles on Z are constructed as follows. Let \mathcal{P}_B be the Poincaré line bundle on $Z \times_B Z$, which we restrict to $C \times_B Z$, and let \mathcal{N} be an arbitrary line bundle over C. Let π_2 be the projection of $C \times_B Z$ to the second factor. The vector bundle over Z that we want to study is then

$$V = \pi_{2*}(\mathcal{N} \otimes \mathcal{P}_B). \tag{7.32}$$

Now let us explain the basic strategy for computing Chern classes. If G is any vector bundle on Z, then the index of the $\overline{\partial}$ operator on $C \times_B Z$, with values in $\mathcal{N} \otimes \mathcal{P}_B \otimes \pi_2^* G$, would be

$$\int_{C \times_B Z} e^{c_1(\mathcal{N} \otimes \mathcal{P}_B)} \operatorname{ch}(\pi_2^* G) \operatorname{Td}(C \times_B Z), \tag{7.33}$$

where ch is the Chern character, Td is the Todd class, and π_2^*G is the "pullback" of G to $C \times_B Z$. The index of the $\overline{\partial}$ operator on Z, with values in $V \otimes G$, is

$$\int_{Z} \operatorname{ch}(V)\operatorname{ch}(G)\operatorname{Td}(Z). \tag{7.34}$$

The Hirzebruch-Riemann-Roch or Atiyah-Singer index theorem says that (with V defined in (7.32)), these are equal. But G is arbitrary, and therefore ch(G) is essentially arbitrary. For (7.33) and (7.34) to be equal for any G implies a relationship between the integrands that is known as the Grothendieck-Riemann-Roch theorem (GRR):

$$\pi_{2*}\left(e^{c_1(\mathcal{N}\otimes\mathcal{P}_B)}\mathrm{Td}(C\times_B Z)\right) = \mathrm{ch}(V)\mathrm{Td}(Z).$$
(7.35)

Here π_{2*} is, at the level of differential forms, the operation of "integrating over the fibers" of the map $\pi_2: C \times_B Z \to Z$. (As this map is an *n*-fold cover, integration over the fibers is in this case somewhat akin to taking a finite sum.) Since everything else in (7.35) can be computed independently, (7.35) will serve to determine the Chern classes of V.

Our main goal is to compute $c_1(V)$ and $c_2(V)$. With this in mind, we expand various factors in (7.35) up to the relevant order. We have

$$ch(V) = n + c_1(V) + \frac{1}{2}c_1(V)^2 - c_2(V) + \dots$$
(7.36)

where n enters because V is of rank n. For any complex manifold X,

$$\operatorname{Td}(X) = 1 + \frac{c_1(X)}{2} + \frac{c_2(X) + c_1(X)^2}{12} + \dots$$
 (7.37)

So given a choice of \mathcal{N} and hence of $c_1(\mathcal{N})$, if we can compute the Chern classes of Z and of $C \times_B Z$, then (7.35) will determine the Chern classes of V.

7.4. The First Chern Class

Our first task is to compute the first Chern class $c_1(V)$. For this purpose, a few simplifications occur. The construction of V in section 2 ensured that V is an SU(n) bundle when restricted to any fiber of $\pi: Z \to B$. So $c_1(V)$ vanishes when restricted to a fiber, and is therefore determined by its restriction to B, that is, to the section σ of π .

So instead of working on $C \times_B Z$, we can restrict to $C \times_B \sigma = C$. Therefore, instead of using GRR for $\pi_2 : C \times_B Z \to Z$, we can use GRR for the projection $\pi : C \to B$ (which is just the restriction to C of $\pi : Z \to B$). In writing GRR for $\pi : C \to B$, we can moreover set $c_1(\mathcal{P}_B)$ to zero, since \mathcal{P}_B was defined to be trivial when restricted to $Z \times_B \sigma$. So we get

$$\pi_* \left(e^{c_1(\mathcal{N})} \mathrm{Td}(C) \right) = \mathrm{ch}(V) \mathrm{Td}(B).$$
 (7.38)

With $\operatorname{Td}(B) = 1 + c_1(B)/2 + \ldots$, along with $e^{c_1(\mathcal{N})} = 1 + c_1(\mathcal{N}) + \ldots$, and $\operatorname{ch}(V) = n + c_1(V) + \ldots$, the formula for the first Chern class becomes

$$\pi_* \left(c_1(\mathcal{N}) + \frac{1}{2} c_1(C) \right) = \frac{n}{2} c_1(B) + c_1(V). \tag{7.39}$$

We want to determine the \mathcal{N} 's for which $c_1(V) = 0$. The condition for this is

$$\pi_*(c_1(\mathcal{N})) = -\frac{1}{2}\pi_*\left(c_1(C) - \pi^*c_1(B)\right). \tag{7.40}$$

 $(\pi^*$ is the operation of "pullback," and because the map $\pi:C\to B$ is an n-fold cover, $\pi_*\pi^*c_1(B)=nc_1(B)$.) This says that

$$c_1(\mathcal{N}) = -\frac{1}{2} \left(c_1(C) - \pi^* c_1(B) \right) + \gamma, \tag{7.41}$$

where γ is a class such that

$$\pi_* \gamma = 0. \tag{7.42}$$

Now, let us discuss the significance of this for the two basic cases: B a curve, and B of dimension greater then one.

B A Curve

If B is a curve, then C is also a curve, and the condition $\pi_* \gamma = 0$ implies that $\gamma = 0$. Hence $c_1(\mathcal{N})$ is determined uniquely by (7.41). If K_B and K_C are the canonical bundles of B and C, then (7.41) means that

$$\mathcal{N} = K_C^{1/2} \otimes K_B^{-1/2} \otimes \mathcal{F} \tag{7.43}$$

where \mathcal{F} is a flat line bundle over C. This possibility of tensoring with a flat line bundle means that the Jacobian of C enters the story, as we saw in section 2. Note that a curve is a spin manifold, so that square roots $K_C^{1/2}$ and $K_B^{1/2}$ do exist.

B Of Higher Dimension

Now, consider the case that B is of dimension greater than one. In this case, in many interesting examples $H^{1,0}(C) = 0$, and the classification of line bundles on C is discrete; \mathcal{N} is then uniquely determined up to isomorphism by its first Chern class.

Now, let us ask what γ can be. $c_1(\mathcal{N})$ will necessarily be an integral class of type (1,1). Such classes on C are relatively scarce for the following reason. Although $H^{1,0}(C) = 0$, $H^{2,0}(C)$ is generically non-zero for the C's of interest (with possible exceptions for small η' and n); this tends to prevent the existence of many (1,1) integral classes.

The only obvious such classes on C are the cohomology class of the section σ , and the pullbacks $\pi^*\beta$ of integral (1,1) classes β on B. We will compute presently (eqn. (7.56)) that

$$\pi_* \sigma = \eta' - nc_1(\mathcal{L}). \tag{7.44}$$

Also,

$$\pi_* \pi^* \beta = n\beta \tag{7.45}$$

because $\pi:C\to B$ is an *n*-sheeted cover. So

$$\pi_*(n\sigma - \pi^*\eta' + n\pi^*c_1(\mathcal{L})) = 0.$$
 (7.46)

And this is the only general construction of a class annihilated by π_* .

So we can take

$$\mathcal{N} = K_C^{1/2} \otimes K_B^{-1/2} \otimes \left(\mathcal{O}(\sigma)^n \otimes \mathcal{M}^{-1} \otimes \mathcal{L}^n \right)^{\lambda}$$
 (7.47)

for suitable λ .

Actually, there is a subtlety here. The square root $(K_C \otimes K_B^{-1})^{1/2}$ may not exist, and if it does not one may not take $\lambda = 0$; in fact, λ must be half-integral, and there is a restriction on \mathcal{M} so that \mathcal{N} actually exists as a line bundle. However, for a reason that

we will now explain, $\lambda = 0$ is the appropriate case for the simplest tests of the duality between F-theory and the heterotic string.

Involution And Duality

We recall that the elliptic fibration $Z \to B$ has a \mathbb{Z}_2 symmetry τ which acts on the Weierstrass model by $y \to -y$, while leaving x and z unchanged. (τ multiplies each fiber of the elliptic fibration by -1.) As we already explained at the end of section 7.1, the most easily seen F theory moduli are all invariant under τ , so in comparison with F theory there is particular interest to construct components of the moduli space of bundles that are entirely τ -invariant. In the case of a rank n complex vector bundle, we want τ -invariance in the sense that $\tau^*(V) = V^*$.

In fact, (7.47) with $\lambda = 0$, that is the existence of an isomorphism

$$\mathcal{N}^2 = K_C \otimes K_B^{-1},\tag{7.48}$$

is the condition for $\tau^*V = V^*$. To see this, note first that the condition $\tau^*V = V^*$ says that there is a non-degenerate map $s: V \otimes \tau^*V \to \mathcal{O}$. This is equivalent to the following. If ω is a meromorphic one-form on Z with poles on a divisor D, there should be a non-degenerate residue map $\phi_{\omega}: V \otimes \tau^*V \to \mathcal{O}_D$ obeying certain standard axioms. One simply defines $\phi_{\omega}(v,w) = \text{Res}(s(v,w)\omega)$ where Res is the residue operation. (7.48) lets us construct ϕ_{ω} as follows.

We let τ act on $C \times_B Z$ through its action on the second factor. Then $\tau^*(\mathcal{N}) = \mathcal{N}$ (since \mathcal{N} is pulled back from C, on which τ acts trivially), and $\tau^*(\mathcal{P}_B) = \mathcal{P}_B^{-1}$ (a basic property of the Poincaré line bundle). Hence $(\mathcal{N} \otimes \mathcal{P}_B) \otimes \tau^*(\mathcal{N} \otimes \mathcal{P}_B) = (\mathcal{N} \otimes \mathcal{P}_B) \otimes (\mathcal{N} \otimes \mathcal{P}_B^{-1}) = \mathcal{N}^2$, and an isomorphism as in (7.48) gives a map of this to $K_C \otimes K_B^{-1}$. We let θ be the composite map $(\mathcal{N} \otimes \mathcal{P}_B) \otimes \tau^*(\mathcal{N} \otimes \mathcal{P}_B) \to K_C \otimes K_B^{-1}$. Given sections v' and v' of $\mathcal{N} \otimes \mathcal{P}_B$ and $\tau^*(\mathcal{N} \otimes \mathcal{P}_B)$, $\theta(v', w') \otimes \omega$ is a meromorphic section of K_C , and we define $\phi'_{\omega}(v', w') = \text{Res}(\theta(v', w') \otimes \omega)$. ϕ'_{ω} takes values in $\mathcal{O}_{D'}$, where D' is the divisor in $C \times_B Z$ that lies over D. We then define the desired object ϕ_{ω} such that if $v = \pi_{2*}(v')$, and $v = \pi_{2*}(v')$, then $\phi_{\omega}(v, w) = \pi_{2*}\phi'_{\omega}(v', w')$. This has the necessary properties to establish the desired duality between V and τ^*V . The main point is to verify non-degeneracy of ϕ_{ω} , which can be checked by a standard local computation, the interesting detail being that this works near branch points of $C \to B$.

Thus, to achieve $\tau^*V = V^*$, we must define \mathcal{N} by (7.47), with $\gamma = 0$. This, however, is possible only if the line bundle $K_C \otimes K_B^{-1}$ has a square root. A computation we will perform

presently shows that $c_1(C) = -\eta' - n\sigma + c_1(B) - c_1(\mathcal{L})$, so $c_1(C) - c_1(B) = -\eta' - c_1(\mathcal{L}) - n\sigma$. The only obvious circumstance in which this is divisible by two is that

$$n \equiv 0 \mod 2$$

 $\eta' \equiv c_1(\mathcal{L}) \mod 2.$ (7.49)

These conditions have already been encountered before. In the construction of bundles via parabolics, to achieve τ invariance, we needed n even (so that we could take $W_n \oplus W_n^*$ as the starting point), and we found in (7.6) that we needed $\eta \equiv c_1(\mathcal{L})$ mod two. So, except that we have not yet proved that $\eta = \eta'$, we have found the same mod two conditions in the two approaches.

Computation Of $c_1(C)$

To justify eqn. (7.49), and for later use, let us calculate $c_1(C)$. We already used in section 7.2 the fact that the cohomology of the \mathbf{P}^2 bundle $W \to B$ is generated by $r = c_1(\mathcal{O}(1))$, with the relation

$$r(r + 2c_1(\mathcal{L}))(r + 3c_1(\mathcal{L})) = 0 (7.50)$$

which simplifies in the cohomology ring of Z to

$$r(r+3c_1(\mathcal{L})) = 0. (7.51)$$

C is defined inside W by the vanishing of the Weierstrass equation, which is a section of $\mathcal{L}^6 \otimes \mathcal{O}(3)$, and of the section s introduced in (7.31), which is a section of $\mathcal{M} \otimes \mathcal{O}(n/3)$. (There is a small sleight of hand here: $\mathcal{O}(1)$ has the cube root $\mathcal{O}(1/3) = \mathcal{O}(\sigma)$ only when restricted to Z, but for the computation that we are about to perform, this is of no moment.) It follows from adjunction that the total Chern classes c(C) and c(B) of C and B are related by

$$c(C) = c(B) \frac{(1+r)(1+r+2c_1(\mathcal{L}))(1+r+3c_1(\mathcal{L}))}{(1+3r+6c_1(\mathcal{L}))(1+\eta'+\frac{n}{3}r)}.$$
 (7.52)

To eliminate the fraction from the denominator, simply recall that when restricted to Z (and therefore also when restricted to C), r is divisible by 3 and $r/3 = \sigma$. From (7.52), we get

$$c_1(C) = -\eta' - n\sigma + c_1(B) - c_1(\mathcal{L}). \tag{7.53}$$

(To be more precise, η' here could be written as $\pi^*\eta'$, but we will henceforth not be so fastidious on this and similar points.) (7.53) is a relation upstairs on C. So the equation (7.41) for $c_1(\mathcal{N})$ becomes

$$c_1(\mathcal{N}) = \frac{1}{2} \left(n\sigma + \eta' + c_1(\mathcal{L}) \right) + \gamma. \tag{7.54}$$

For future use, we note also that by expanding (7.52) to higher order one gets

$$c_2(C) = c_2(B) + 12\sigma c_1(\mathcal{L}) + 11c_1(\mathcal{L})^2 + (\eta' + n\sigma)^2 - (c_1(B) - c_1(\mathcal{L}))(\eta' + n\sigma + c_1(\mathcal{L})).$$
(7.55)

Finally, let us by similar methods compute $\pi_*\sigma$, where π is the map $\pi: C \to B$. σ extends over Z, and C represents in Z the cohomology class $\eta' + n\sigma$. So pushing down σ from C to B is the same as pushing down $\sigma(\eta' + n\sigma)$ from Z to B. We saw earlier that $\sigma^2 = -\sigma c_1(\mathcal{L})$, so $\sigma(\eta' + n\sigma) = \sigma(\eta' - nc_1(\mathcal{L}))$. The pushdown from Z to B is now easy: $\eta' - nc_1(\mathcal{L})$ is a pullback from B, and integration over the fiber of $Z \to B$ maps σ to 1. So finally

$$\pi_*(\sigma) = \eta' - nc_1(\mathcal{L}),\tag{7.56}$$

as promised in (7.44).

7.5. The Poincaré Line Bundle

Before trying to compute $c_2(V)$, we need a digression concerning the Poincaré line bundle \mathcal{P}_B over $Z \times_B Z$. Let $\sigma_1 = \sigma \times_B Z$, $\sigma_2 = Z \times_B \sigma$. We recall that \mathcal{P}_B is defined by saying that it is trivial when restricted to σ_1 or σ_2 , plus the following condition. Given $b \in B$, the inverse image of b in Z is an elliptic curve E_b , and the inverse image in $Z \times_B Z$ is a copy of $E_b \times E_b$. The second defining property of \mathcal{P}_B is that its restriction to each $E_b \times E_b$ is the Poincaré line bundle in the standard sense (which was explained in section 2.3).

 \mathcal{P}_B can be described very explicitly as follows. Its first Chern class is

$$c_1(\mathcal{P}_B) = \Delta - \sigma_1 - \sigma_2 - c_1(\mathcal{L}), \tag{7.57}$$

where Δ is the diagonal in $Z \times_B Z$. To be more precise, one can take

$$\mathcal{P}_B = \mathcal{O}(\Delta - \sigma_1 - \sigma_2) \otimes \mathcal{L}^{-1}. \tag{7.58}$$

This \mathcal{P}_B has the correct restriction to each $E_b \times E_b$, since (being a pullback from B) \mathcal{L}^{-1} is trivial on each $E_b \times E_b$, and $\mathcal{O}(\Delta - \sigma_1 - \sigma_2)$ is the standard Poincaré line bundle on

 $E_b \times E_b$. To show that \mathcal{P}_B is trivial when restricted to σ_1 (or equivalently to σ_2), it suffices to show that $\sigma_1 \cdot c_1(\mathcal{P}_B) = 0$. But we showed above that $\sigma_1 \cdot (\sigma_1 + c_1(\mathcal{L})) = 0$ in the cohomology ring of Z, and $\sigma_1 \cdot (\Delta - \sigma_2) = 0$ for a simple geometrical reason (if one is at the standard section in the first factor of $Z \times_B Z$, then being on the diagonal is equivalent to being at the standard section of the second).

For future use, let us note the fact just exploited:

$$\sigma_1 \cdot \sigma_2 = -\sigma_1 \cdot \Delta = -\sigma_2 \cdot \Delta. \tag{7.59}$$

Likewise, one has

$$\Delta \cdot \Delta = -\Delta \cdot c_1(\mathcal{L}). \tag{7.60}$$

This can be proved by constructing explicitly a section of the normal bundle to Δ with divisor $-c_1(\mathcal{L})$. The idea is that if u is a meromorphic section of \mathcal{L}^{-1} , and x and y are Weierstrass coordinates, then $\psi = uy \ d/dx$ is a meromorphic vector field on Z tangent to the elliptic fibers, whose divisor (being that of u) has first Chern class $-c_1(\mathcal{L})$. On the other hand, this vector field, taken in, say, the second factor of $Z \times_B Z$ gives a section of the normal bundle to Δ , and so $\Delta \cdot \Delta$ is Δ times the divisor of ψ .

7.6. The Second Chern Class

Now we will compute $c_2(V)$. One can see from the GRR formula without any detailed computation that $c_2(V)$ has the general form

$$c_2(V) = \sigma \cdot \pi^*(\eta) + \pi^*(\omega), \tag{7.61}$$

where η and ω are some classes on B.

We will first compute the first term in (7.61), which for dimensional reasons is the only term present if B is a curve. In any event, the first term can be detected by restricting to an arbitrary curve $B' \subset B$. We restrict the elliptic fibration $\pi: Z \to B$ to $Z' = \pi^{-1}(B')$, and we let $C' = C \cap Z'$. We will compute the restriction of $c_2(V)$ to Z' – which sees the $\sigma \cdot \pi^* \eta$ term – by using the GRR theorem for the projection $\pi_2: C' \times_{B'} Z' \to Z'$:

$$\pi_{2*}\left(e^{c_1(\mathcal{N})+c_1(\mathcal{P}_B)}\operatorname{Td}(C'\times_{B'}Z')\right) = \operatorname{ch}(V)\operatorname{Td}(Z'). \tag{7.62}$$

Note that, even if Z is Calabi-Yau, $c_1(Z')$ is generally non-zero. However, because the restriction of $c_1(Z')$ to the elliptic fibers vanishes, $c_1(Z')$ is a pullback from B'.

A drastic simplification occurs in evaluating the left hand side of (7.62) because C' and B' are curves. If α, β are any two two-dimensional classes pulled back from C' or B' – such as $\sigma, \eta', c_1(\mathcal{N})$, or $c_1(Z')$ – then $\alpha\beta = 0$. Also, for such an α ,

$$\alpha \cdot (\Delta - \sigma_2) = 0 \tag{7.63}$$

in cohomology, because the left hand side is annihilated by the projection from $C' \times_{B'} Z'$ to C' (both Δ and σ_2 pick out one point on the fiber of this projection; these contributions cancel). With these simplifications, the left hand side of (7.62) collapses to

$$\pi_{2*} \left(e^{c_1(\mathcal{P}_B)} \left(1 + \frac{c_2(Z')}{12} \right) \right),$$
 (7.64)

and the four-dimensional class obtained by expanding this is

$$\pi_{2*}\left(\frac{c_1(\mathcal{P}_B)^2}{2}\right) + \frac{nc_2(Z')}{12}.$$
 (7.65)

The right hand side of (7.62) gives on the other hand (since we assume $c_1(V) = 0$) the four-dimensional class

$$\frac{nc_2(Z')}{12} - c_2(V). (7.66)$$

So (7.62) reduces to

$$c_2(V) = -\frac{1}{2}\pi_{2*} \left(c_1(\mathcal{P}_B)^2 \right),$$
 (7.67)

and as we are working on the four-manifold Z', no information will be lost if we integrate and write

$$\int_{Z'} c_2(V) = -\frac{1}{2} \int_{C' \times_{B'} Z'} c_1(\mathcal{P}_B)^2.$$
 (7.68)

If we write $c_1(\mathcal{P}_B) = u + v$, with $u = -\sigma_1 - c_1(\mathcal{L})$ and $v = \Delta - \sigma_2$, then $u^2 = 0$ because u is pulled back from B', and uv = 0 because of (7.63). So we reduce to computing v^2 . We found above $\Delta^2 = -\Delta c_1(\mathcal{L})$, so $\int_{C' \times_{B'} Z'} \Delta^2 = -\int_{C' \times_{B'} Z'} \Delta c_1(\mathcal{L})$. Integration over the second factor maps this to $-\int_{C'} c_1(\mathcal{L})$, which, because the map $C' \to B'$ is a n-fold cover, equals $-n\int_{B'} c_1(\mathcal{L})$. Likewise, we had $\sigma_2^2 = -\sigma_2 c_1(\mathcal{L})$, so by similar steps $\int_{C' \times_{B'}} \sigma_2^2 = -n\int_{B'} c_1(\mathcal{L})$. Finally, as $\Delta \sigma_2 = \sigma_1 \sigma_2$, we need $\int_{C' \times_{B'} Z'} \sigma_1 \sigma_2$; integration over the second factor maps this to $\int_{C'} \sigma_1 = \int_{B'} \pi_* \sigma_1 = \int_{B'} (\eta' - nc_1(\mathcal{L}))$. So upon putting the pieces together, (7.68) is equivalent to

$$\int_{Z'} c_2(V) = \int_{B'} \eta'. \tag{7.69}$$

Since B' was arbitrary, this is equivalent to

$$\pi_*(c_2(V)) = \eta', \tag{7.70}$$

or equivalently

$$c_2(V) = \sigma \eta' + \omega, \tag{7.71}$$

where the class ω is annihilated by π_* and is thus a pullback from B. Equation (7.70) (which for n=2 was proved in [23]) was asserted in section 6 as part of the comparison of F theory and the heterotic string. Also, we have now confirmed that η' as we have defined it here should be identified with η as introduced in section 7.1 in the computations via parabolics.

Evaluation Of ω

It remains, then, to evaluate ω . Since ω is a pullback from B, it is determined by its restriction to the canonical section σ of $\pi: Z \to B$, and we will make this restriction. Since the Poincaré line bundle \mathcal{P}_B is trivial when restricted to σ , it can be dropped. As $C \times_B \sigma = C$, the projection $\pi_2: C \times_B Z \to Z$, when restricted to $C \times_B \sigma$, reduces to $\pi: C \to B$. Thus, we will use GRR for $\pi: C \to B$ to compute the restriction of $c_2(V)$ to $\sigma = B$.

We have, therefore,

$$\pi_* \left(e^{c_1(\mathcal{N})} \mathrm{Td}(C) \right) = \mathrm{ch}(V) \mathrm{Td}(B).$$
 (7.72)

Everything needed to make this explicit has already been given: the Chern classes of C are in (7.53) and (7.55), the expansion of the Todd class is in (7.37), and $c_1(\mathcal{N})$ is in (7.54). Also we have $ch(V) = n - c_2(V)$, and to calculate π_* one only needs to know that $\pi_*(1) = n$, $\pi_*(\sigma) = \eta' - nc_1(\mathcal{L})$, $\pi_*\gamma = 0$, and finally that $\sigma^2 = -\sigma c_1(\mathcal{L})$. The explicit evaluation of (7.72) then gives, after some algebra,

$$c_2(V)|_{\sigma} = -\frac{c_1(\mathcal{L})^2(n^3 - n)}{24} - \eta' c_1(\mathcal{L}) - \frac{n\eta'(\eta' - nc_1(\mathcal{L}))}{8} - \frac{\pi_*(\gamma^2)}{2}.$$
 (7.73)

Note that by the Hodge index theorem, γ^2 is always negative, so that $c_2(V)|_{\sigma}$ is minimized for $\gamma = 0$, which we recall is the unique case in which the involution τ acts by $\tau^*V = V^*$. For $\gamma = \lambda(n\sigma - \eta' + nc_1(\mathcal{L}))$ (the only general solution of $\pi_*\gamma = 0$, as we explained above), one computes from formulas summarized above

$$\pi_*(\gamma^2) = -\lambda^2 n \eta' (\eta' - nc_1(\mathcal{L})). \tag{7.74}$$

In combining (7.71) and (7.73) into a general formula for $c_2(V)$, one must recall that $\sigma^2 = -\sigma c_1(\mathcal{L})$, so that $\sigma|_{\sigma} = -c_1(\mathcal{L})|_{\sigma}$. Hence a term $-\eta' c_1(\mathcal{L})$ in (7.73) is the result, in a sense, of restricting the $\eta'\sigma$ term in $c_2(V)$ to σ . The final formula for $c_2(V)$ is thus

$$c_2(V) = \eta' \sigma - \frac{c_1(\mathcal{L})^2 (n^3 - n)}{24} - \frac{n\eta' (\eta' - nc_1(\mathcal{L}))}{8} - \frac{\pi_*(\gamma^2)}{2}.$$
 (7.75)

If we set $\gamma = 0$ to ensure $\tau^*V = V^*$, then this is in happy agreement with (7.4).

Change In n

Before leaving the subject of spectral covers, we will point out an important consequence of the formulas above that determine $c_2(V)|_{\sigma}$ in terms of $\eta = \pi_*c_2(V)$. The relation between them depends on n, the rank of the bundle. This means that if V is a rank n bundle of the sort we have been constructing, then V cannot degenerate by varying parameters to $\mathcal{O} \oplus V'$, where V' is a rank n-1 bundle built from the same type of construction. (Such a degeneration would of course correspond in physical terms to restoration of some gauge symmetry.) In fact, V and $\mathcal{O} \oplus V'$ have different Chern classes.

Let us see concretely where the obstruction is. Rank n bundles were constructed using a spectral cover C defined by

$$a_0 + a_2 x + a_3 y + \ldots + a_n x^{n/2} = 0 (7.76)$$

(or a slightly different formula if n is odd). For a rank n-1 bundle, the equation of the spectral cover is similar except that the last term is absent. So one might hope to reduce V to $\mathcal{O} \oplus V'$ by setting $a_n = 0$. When this is done, C becomes a reducible variety with two branches, one a copy of σ and one an n-1-sheeted spectral cover C'. (σ appears because as $a_n \to 0$, one root of (7.76) goes to $x = \infty$, that is, to σ .) By itself σ , regarded as the trivial spectral cover, would correspond to \mathcal{O} , and C' would likewise correspond to a rank n-1 bundle V'. However, σ and C' intersect, and because of this intersection one gets not a direct sum $\mathcal{O} \oplus V'$ but a more complicated extension of bundles – with, of course, the same Chern classes as V! (V can in fact be obtained as a so-called elementary modification of $\mathcal{O} \oplus V'$.)

The interpretation of gauge theory singularities via string theory five-branes gives a hint of what one might be able to do. Though $\widetilde{\mathcal{O}} \oplus V'$ cannot be deformed to an irreducible rank n bundle V if $\widetilde{\mathcal{O}}$ is a trivial bundle and V' is a rank n-1 bundle, such a deformation of $\widetilde{\mathcal{O}} \oplus V'$ may be possible if $\widetilde{\mathcal{O}}$ is a rank one torsion-free sheaf of $c_1 = 0$ and appropriate non-zero c_2 . In other words, $\widetilde{\mathcal{O}}$ would be the ideal sheaf I_{Γ} for some codimension two subvariety Γ . The codimension two submanifold $\pi^{-1}(\Gamma)$ would be interpreted physically as the world-volume of a set of heterotic string fivebranes, generalizing the adventures that we had with fivebranes for somewhat analogous reasons in section 7.2 above.

8. \mathbb{Z}_2 Index Theorem

We conclude this paper by working out some general information about bundles on elliptic Calabi-Yau threefolds that can be deduced from index theory. (The index computation can be performed for other elliptic threefolds but gives a particularly neat result in the Calabi-Yau case.) It is perhaps slightly anticlimactic to conclude with such generalities after having made detailed constructions of bundles. However, it is still interesting to compare the detailed constructions to general theory.

Given any gauge group G and complex manifold Z, one would hope to at least be able to determine the dimensions of the moduli spaces of G bundles on Z. This can be done via index theory if Z is a surface, but if Z is a Calabi-Yau threefold one runs into difficulty; in general there is no index that determines the dimension of the moduli space of bundles. One can contemplate an index theorem for the alternating sum

$$\sum_{i=0}^{3} (-1)^{i} \dim H^{i}(Z, \operatorname{ad}(V)), \tag{8.1}$$

but this sum is zero because of Serre duality, which asserts that in this problem (because the bundle ad(V) is real and the canonical bundle of Z is trivial) H^i is dual to H^{3-i} , and so has the same dimension.

Physically, the absence of an index theorem reflects the fact that it can be hard to predict whether an approximately massless chiral superfield (coming from an apparent modulus of V) is really exactly massless. One can learn something about the superpotential of the string theory by using the involution τ of Z that was discussed in section 7. Because τ acts as "multiplication by -1" on the fibers of $Z \to B$, while acting trivially on B, it multiplies the canonical bundle of Z by -1. According to the standard arguments about Calabi-Yau compactification of string theory, τ is therefore observed in heterotic string compactification on Z as a \mathbb{Z}_2 R symmetry; so the superpotential is odd under τ .

Suppose then that we work at a τ -invariant point in the moduli space, by which we mean that the action of τ on Z lifts to an action on the adjoint bundle $\operatorname{ad}(V)$; we fix such a lifting. If A and B are chiral superfields that are respectively even and odd under τ , then A^2 and B^2 terms in the superpotential are forbidden by the symmetry. A mass term would necessarily be of the form AB, and gives mass to one even and one odd superfield. If then n_e and n_o are the numbers of even and odd massless chiral superfields, the "index" $I = n_e - n_o$ is invariant under \mathbb{Z}_2 -invariant perturbations of the superpotential and may be much easier to calculate than n_e and n_o separately.

In fact, this difference is governed by an index theorem. We project onto the τ -invariant part of the index problem and consider

$$I = -\sum_{i=0}^{3} (-1)^{i} \operatorname{Tr}_{H^{i}(Z, \operatorname{ad}(V))} \frac{1+\tau}{2}.$$
(8.2)

If we write H_e^i and H_o^i for the subspaces of H^i that are even or odd under τ , then

$$\operatorname{Tr}_{H^i} \frac{1+\tau}{2} = \dim H_e^i \tag{8.3}$$

SO

$$I = -\sum_{i=0}^{3} (-1)^{i} \operatorname{dim} H_{e}^{i}. \tag{8.4}$$

The dimension of H_e^1 is what we called n_e , the number of τ -invariant chiral superfields. On the other hand, Serre duality says in this situation that H_e^i is dual to H_o^{3-i} . (The duality exchanges the even and odd subspaces because it involves multiplying by a holomorphic three-form, which is odd.) So the dimension of H_e^2 is n_o , the number of odd chiral superfields. Moreover, H_e^0 and H_e^3 (the latter is dual to H_o^0), are the number of unbroken gauge generators that are even or odd under τ . If for example the gauge symmetry is completely broken, these numbers vanish and we have simply

$$I = n_e - n_o. (8.5)$$

Even when the gauge symmetry is not completely broken, the correction to (8.5) would be known (if one knows the unbroken gauge group) so that $n_e - n_o$ can always be effectively related to I.

As we noted above, in many examples (those which on the F-theory side are described by four-folds X with $H^3(X) = 0$) one expects $n_o = 0$, so (8.5) will reduce to $I = n_e$. But in any case, it is the difference $n_e - n_o$ that is calculable from index theory.

Since the ordinary index vanishes, the definition of I is equivalent to

$$I = -\frac{1}{2} \sum_{i=0}^{3} (-1)^{i} \operatorname{Tr}_{H^{i}(Z, \operatorname{ad}(V))} \tau.$$
(8.6)

Such a "character-valued index" can be effectively computed by a fixed point theorem (originally obtained in [39]; see [40] for a derivation based on path integrals). In a case such as this one, in which (as we will see) the components of the fixed point set are all

orientable and of codimension two, the fixed point theorem can be stated as follows. Let U_i be the components of the fixed point set, and let N_i be the normal bundle to U_i in Z, regarded as a complex line bundle. Let F_i be the restriction of ad(V) to U_i , and let $F_{i,e}$ and $F_{i,o}$ be the subbundles of ad(V) on which τ acts by 1 or -1; let $s_{i,e}$ and $s_{i,o}$ be the rank of $F_{i,e}$ and $F_{i,o}$. And let ch denote the Chern character and Td the Todd class. Then

$$\sum_{i=0}^{3} (-1)^{i} \operatorname{Tr}_{H^{i}(Z,\operatorname{ad}(V))} \tau = \sum_{i} \int_{U_{i}} \frac{\operatorname{ch}(F_{i,e}) - \operatorname{ch}(F_{i,o})}{1 + e^{c_{1}(N_{i})}} \operatorname{Td}(U_{i}).$$
(8.7)

This can be evaluated more explicitly as follows. In evaluating the Chern characters, we can stop at four-forms because the U_i have dimension four. The bundles $F_{i,e}$ and $F_{i,o}$, being real, have vanishing first Chern class. So we get

$$ch(F_{i,e}) = s_{i,e} - c_2(F_{i,e})$$

$$ch(F_{i,o}) = s_{i,o} - c_2(F_{i,o}).$$
(8.8)

In our actual application, the numbers $s_{i,e}$ and $s_{i,o}$ will be independent of i - let us call them s_e and s_o . That being so, the part of the contribution to (8.7) that is proportional to s_e or s_o can be equated, using the fixed point theorem, with the value of $s_e - s_0$ times $\sum_{i=0}^{3} (-1)^i \operatorname{Tr}_{H^i(Z,\mathcal{O})} \tau$. (In other words, that contribution would be unchanged if $F_{i,e}$ and $F_{i,o}$ were replaced by trivial bundles of the same rank.) That last expression is two (as $H^0(Z,\mathcal{O})$ is one-dimensional and even under τ , $H^3(Z,\mathcal{O})$ is one-dimensional and odd, and $H^1(Z,\mathcal{O}) = H^2(Z,\mathcal{O}) = 0$). So this part of (8.7) is simply $2(s_e - s_o)$. In evaluating the rest of (8.7), we can replace $c_1(N_i)$ in the denominator by zero, since the numerator already contains four-forms, and we can likewise replace the Todd class by 1. So (8.7) becomes

$$\sum_{i=0}^{3} (-1)^{i} \operatorname{Tr}_{H^{i}(Z,\operatorname{ad}(V))} \tau = 2(s_{e} - s_{o}) - \frac{1}{2} \sum_{i} \int_{U_{i}} (c_{2}(F_{i,e}) - c_{2}(F_{i,o})).$$
 (8.9)

Action At Fixed Points

The index formula depends, of course, on how τ acts on the fibers of $\operatorname{ad}(V)$ over the fixed point set. The case of most direct interest for comparing to the constructions of bundles that we have given in this paper is the case in which τ acts as the involution of the Lie algebra of G that is induced from the involution -1 of the root lattice of G. We will call this involution of the Lie algebra ρ . In fact, on a single elliptic curve E, every flat G bundle V has the property that the involution τ of E lifts to $\operatorname{ad}(V)$ in such a way as

to act on fibers over fixed points as ρ . (This is therefore also true for every semistable G bundle up to S-equivalence.) So, given a G bundle V over Z that is semistable on each fiber, it is natural to look for a lifting of τ to $\operatorname{ad}(V)$ so as to act by ρ on fibers over fixed points.

More physically, a lifting with this property is natural because in duality between F theory and the heterotic string, what in F theory is the involution τ corresponds on the heterotic string side to multiplication of the full Narain lattice by -1, and (in a limit where classical geometry applies) this certainly induces the involution ρ of the root lattice.

With this choice of lifting of τ , (8.9) can be made more explicit. The difference $s_e - s_0$ is just the trace of ρ in the adjoint representation of G. In taking this trace, the non-zero weights (which are exchanged by ρ in pairs) do not contribute, so the complete contribution comes from the maximal torus, and is equal to -r (r being the rank of G).

One can also make a reduction of the second Chern classes that appear in (8.9). We want to express the difference $c_2(F_{i,e}) - c_2(F_{i,o})$ in terms of the fundamental characteristic class $\lambda(V)$ of V. We claim that in fact

$$c_2(F_{i,e}) - c_2(F_{i,o}) = -4\lambda(V)|_{U_i}.$$
(8.10)

For SU(n), this means that

$$c_2(F_{i,e}) - c_2(F_{i,o}) = -4c_2(V)|_{U_i}. (8.11)$$

This is proved using the explicit form of ρ to compute traces.

So we can rewrite (8.6) and (8.9) to say that the index $I = n_e - n_o$ of bundle moduli is

$$I = r - \sum_{i} \int_{U_i} \lambda(V). \tag{8.12}$$

8.1. Comparison With Construction Of Bundles

We want to compare this index formula to the actual number of moduli found in our construction of bundles (for the τ -invariant components of the moduli space). For this, we must describe explicitly the U_i and determine the quantities $\lambda(V)|_{U_i}$. Recall that the manifold Z is described by a Weierstrass equation

$$zy^2 = 4x^3 - g_2xz^2 - g_3z^3. (8.13)$$

Moreover, τ is the transformation $y \to -y$, leaving other coordinates fixed. A fixed point is a point at which the homogeneous coordinates x, y, and z are left fixed up to overall scaling. There are thus two components of the fixed point set. One component, U_1 , is given by x = z = 0, $y \neq 0$, and is the section σ of $Z \to B$ that figured extensively above. The other component, U_2 , is given by y = 0. U_2 is thus a triple cover of B, given by the equation

$$0 = 4x^3 - g_2 x z^2 - g_3 z^3 (8.14)$$

in a certain \mathbf{P}^1 bundle W' over B.

We will compare explicitly the index formula (8.12) to our actual construction of bundles only for G = SU(n). The index formula predicts that

$$I = n - 1 - \int_{U_1} c_2(V) - \int_{U_2} c_2(V). \tag{8.15}$$

The cohomology class of U_1 is simply the class of the section σ , while (as y is a section of $\mathcal{O}(1) \otimes \mathcal{L}^3 = \mathcal{O}(\sigma)^3 \otimes \mathcal{L}^3$) the cohomology class of U_2 is $3\sigma + 3c_1(\mathcal{L})$. So (8.15) can be written

$$I = n - 1 - 4 \int_{\sigma} c_2(V)|_{\sigma} - 3 \int_{Z} c_1(\mathcal{L})c_2(V).$$
(8.16)

So, using (7.75) (and of course setting $\gamma = 0$ to ensure τ -invariance), we get

$$I = n - 1 + \int_{B} \left(\frac{(n^3 - n)c_1(\mathcal{L})^2}{6} + \frac{n\eta(\eta - nc_1(\mathcal{L}))}{2} + \eta c_1(\mathcal{L}) \right). \tag{8.17}$$

Now let us count the parameters in our construction of the bundle. For this, we simply count the parameters in the equation (7.31) that defines the spectral cover, and subtract 1 for overall scaling of that equation. As a_r is a section of $\mathcal{M} \otimes \mathcal{L}^{-r}$, the number of parameters, assuming a suitable amount of ampleness so that an index theorem can be used to compute the dimension of $H^0(B, \mathcal{M} \otimes \mathcal{L}^{-r})$, is

$$\widetilde{I} = -1 + \int_{B} e^{\eta} \left(1 + e^{-2c_1(\mathcal{L})} + e^{-3c_1(\mathcal{L})} + \dots + e^{-nc_1(\mathcal{L})} \right) \operatorname{Td}(B).$$
 (8.18)

With a small amount of algebra (and using the Calabi-Yau condition $c_1(B) = c_1(\mathcal{L})$ since we have assumed this in doing the index theory), one finds $\widetilde{I} = I$, as expected.

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