### Gravity in Complex Hermitian Space-Time

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#### Abstract

A generalized theory unifying gravity with electromagnetism was proposed by Einstein in 1945. He considered a Hermitian metric on a real space-time. In this work we review Einstein's idea and generalize it further to consider gravity in a complex Hermitian space-time.

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In the year 1945, Albert Einstein [1], [2] attempted to establish a unified field theory by generalizing the relativistic theory of gravitation. At that time it was thought that the only fundamental forces in nature were gravitation and electromagnetism. Einstein proposed to use a Hermitian metric whose real part is symmetric and describes the gravitational field while the imaginary part is antisymmetric and corresponds to the Maxwell field strengths. The Hermitian symmetry of the metric  $g_{\mu\nu}$  is given by

$$g_{\mu\nu}\left(x\right) = \overline{g_{\nu\mu}\left(x\right)},$$

where

$$g_{\mu\nu}(x) = G_{\mu\nu}(x) + iB_{\mu\nu}(x),$$

so that  $G_{\mu\nu}(x) = G_{\nu\mu}(x)$  and  $B_{\mu\nu}(x) = -B_{\nu\mu}(x)$ . However, the space-time manifold remains real. The connection  $\Gamma^{\rho}_{\mu\nu}$  on the manifold is not symmetric, and is also not unique. A natural choice, adopted by Einstein, is to impose the hermiticity condition on the connection so that  $\Gamma^{\rho}_{\nu\mu} = \overline{\Gamma^{\rho}_{\mu\nu}}$ , which implies that its antisymmetric part is imaginary. The connection  $\Gamma$  is determined as a function of  $g_{\mu\nu}$  by defining the covariant derivative of the metric to be zero

$$0 = g_{\mu\nu,\rho} - g_{\mu\sigma} \Gamma^{\sigma}_{\rho\nu} - \Gamma^{\sigma}_{\mu\rho} g_{\sigma\nu}.$$

This gives a set of 64 equations that matches the number of independent components of  $\Gamma^{\sigma}_{\mu\nu}$  which can then be solved uniquely, provided that the metric  $g_{\mu\nu}$  is not singular. It cannot, however, be expressed in closed form, but only perturbatively in powers of the antisymmetric field  $B_{\mu\nu}$ . There are also two possible contractions of the curvature tensor, and therefore, unlike the real case, the action is not unique. Both fields  $G_{\mu\nu}$  and  $B_{\mu\nu}$  appear explicitly in the action, but the only symmetry present is that of diffeomorphism invariance. Einstein did notice that this unification does not satisfy the criteria that the field  $g_{\mu\nu}$  should appear as a covariant entity with an underlying symmetry principle. It turned out that although the field  $B_{\mu\nu}$ satisfies one equation which is of the Maxwell type, the other equation contains second order derivatives and does not imply that its antisymmetrized field strength  $\partial_{\mu}B_{\nu\rho} + \partial_{\nu}B_{\rho\mu} + \partial_{\rho}B_{\mu\nu}$  vanishes. In other words, the theory with Hermitian metric on a real space-time manifold gives the interactions of the gravitational field  $G_{\mu\nu}$  and a massless field  $B_{\mu\nu}$ . Much later, it was shown that the interactions of the field  $B_{\mu\nu}$  are inconsistent at the non-linear level, because one of the degrees of freedom becomes ghost like [3]. There is an

exception to this in the special case when a cosmological constant is added, in which case the theory is rendered consistent as a mass term for the  $B_{\mu\nu}$  field is acquired [4],[5].

More recently, it was realized that this generalized gravity theory could be formulated elegantly and unambiguously as a gauge theory of the U(1,3) group [6]. A formulation of gravity based on the gauge principle is desirable because such an approach might give a handle on the unification of gravity with the other interactions, all of which are based on gauge theories. This can be achieved by taking the gauge field  $\omega_{\mu a}^{\ \ b}$  to be anti-Hermitian:

$$\overline{\omega_{\mu a}{}^{b}} = -\eta_{c}^{a}\omega_{\mu d}{}^{c}\eta_{b}^{d},$$

where

$$\eta_b^a = \text{diag}(-1, 1, 1, 1),$$

is the Minkowski metric. A complex vielbein  $e^a_{\mu}$  is then introduced which transforms in the fundamental representation of the group U(1,3). The complex conjugate of  $e^a_{\mu}$  is defined by  $e_{\mu a} = \overline{e^a_{\mu}}$ . The curvature associated with the gauge field  $\omega_{\mu a}^{\phantom{a}b}$  is given by

$$R_{\mu\nu a}{}^{b} = \partial_{\mu}\omega_{\nu a}{}^{b} - \partial_{\nu}\omega_{\mu a}{}^{b} + \omega_{\mu a}{}^{c}\omega_{\nu c}{}^{b} - \omega_{\nu a}{}^{c}\omega_{\mu c}{}^{b}.$$

The gauge invariant Hermitian action is uniquely given by

$$I = \int d^4x \, |e| \, e^{\mu a} R_{\mu\nu a}{}^b e_b^{\nu},$$

where

$$|e|^2 = (\det e^a_\mu) (\det e_{\nu a})$$

and the inverse vierbein is defined by

$$e^{\mu}_{a}e^{a}_{\nu}=\delta^{\mu}_{\nu}, \qquad e^{\mu a}=\overline{e^{\mu}_{a}}.$$

This action coincides, in the linearized approximation, with the action proposed by Einstein, but is not identical. The reason is that in going from first order formalism where the field  $\omega_{\mu a}^{\ \ b}$  is taken as an independent field determined by its equations of motion, one gets a non-linear equation which can only be solved perturbatively. A similar situation is met in the Einstein theory where the solution of the metricity condition determines the connection  $\Gamma^{\rho}_{\mu\nu}$  as function of the Hermitian metric  $g_{\mu\nu}$  in a perturbative expansion.

The gauge field  $\omega_{\mu a}^{a}$  associated with the U(1) subgroup of U(1,3) couples only linearly, so that its equation of motion simplifies to

$$\frac{1}{\sqrt{G}}\partial_{\nu}\left(\sqrt{G}\left(e_{a}^{\nu}e^{\mu a}-e_{a}^{\mu}e^{\nu a}\right)\right)=0,$$

where  $G = \det G_{\mu\nu}$ . In the linearized approximation, this equation takes the form  $\partial_{\nu}B^{\mu\nu} = 0$  which was the original motivation for Einstein to identify  $B_{\mu\nu}$  with the Maxwell field [1],[2]. In the gauge formulation the metric arises as a product of the vierbeins  $g_{\mu\nu} = e^a_{\mu}e_{\nu a}$  which satisfies the hermiticity condition  $\overline{g_{\mu\nu}} = g_{\nu\mu}$  as can be easily verified. Decomposing the vierbein into its real and imaginary parts

$$e_{\mu}^{a} = e_{\mu}^{a0} + ie_{\mu}^{a1},$$

and similarly for the anti-Hermitian infinitesimal gauge parameters

$$\Lambda_a^b = \Lambda_a^{b0} + i\Lambda_a^{b1},$$

where  $\Lambda_{ab}^{\ \ 0}=-\Lambda_{ba}^{\ \ 0}$  and  $\Lambda_{ab}^{\ \ 1}=\Lambda_{ba}^{\ \ 1}.$  From the gauge transformations

$$\delta e^a_\mu = \Lambda^a_b e^b_\mu,$$

we see that there exists a gauge where the antisymmetric part of  $e_{\mu}^{a0}$  and the symmetric part of  $e_{\mu}^{a1}$  can be set to zero. This shows that the gauge theory with complex vierbeins is equivalent to the theory with a symmetric metric  $G_{\mu\nu}$  and antisymmetric field  $B_{\mu\nu}$ .

It turns out that the field  $B_{\mu\nu}$  does not have the correct properties to represent the electromagnetic field. Moreover, as noted by Einstein, the fields  $G_{\mu\nu}$  and  $B_{\mu\nu}$  are not unified with respect to a higher symmetry because they appear as independent tensors with respect to general coordinate transformations. In the massless spectrum of string theory the three fields  $G_{\mu\nu}$ ,  $B_{\mu\nu}$  and the dilaton  $\phi$  are always present. The effective action of closed string theory contains, besides the Einstein term for the metric  $G_{\mu\nu}$ , a kinetic term for the field  $B_{\mu\nu}$  such that the later appears only through its field strength

$$H_{\mu\nu\rho} = \partial_{\mu}B_{\nu\rho} + \partial_{\nu}B_{\rho\mu} + \partial_{\rho}B_{\mu\nu}.$$

This implies that there is a hidden symmetry

$$\delta B_{\mu\nu} = \partial_{\mu} \Lambda_{\nu} - \partial_{\nu} \Lambda_{\mu}$$

preventing the explicit appearance of the field  $B_{\mu\nu}$ . As both  $G_{\mu\nu}$  and  $B_{\mu\nu}$  fields are unified in the Hermitian field  $g_{\mu\nu}$ , it will be necessary to combine the diffeomorphism parameter  $\zeta^{\mu}(x)$  and the abelian parameters  $\Lambda_{\mu}(x)$  into one complex parameter. This leads us to consider the idea that the manifold of space-time is complex, but in such a way that at low energies the imaginary parts of the coordinates should be very small compared with the real ones, and become relevant only at energies near the Planck scale. Indeed this idea was first put forward by Witten [7] in his study of topological orbifolds. He was motivated by the observation that string scattering amplitudes at Planckian energies depend on the imaginary parts of the string coordinates [8].

We shall not require the sigma model to be topological. Instead we shall start with the sigma model [9], [10]

$$I = \int d\sigma^{+} d\sigma^{-} g_{\mu \overline{\nu}} \left( Z(\sigma, \overline{\sigma}), \overline{Z}(\sigma, \overline{\sigma}) \right) \partial_{+} Z^{\mu} \partial_{-} Z^{\overline{\nu}},$$

where we have denoted the complex coordinates by  $Z^{\mu}$ ,  $\mu=1,\cdots,d$ , and their complex conjugates by  $\overline{Z^{\mu}}\equiv Z^{\overline{\mu}}$ , and where the world-sheet coordinates are denoted by  $\sigma^{\pm}=\sigma^0\pm\sigma^1$ . We also require that the background metric for the complex d-dimensional manifold M to be Hermitian so that

$$\overline{g_{\mu\overline{\nu}}} = g_{\nu\overline{\mu}}, \qquad g_{\mu\nu} = g_{\overline{\mu}\,\overline{\nu}} = 0.$$

Decomposing the metric into real and imaginary components

$$g_{\mu\overline{\nu}} = G_{\mu\nu} + iB_{\mu\nu},$$

the hermiticity condition implies that  $G_{\mu\nu}$  is symmetric and  $B_{\mu\nu}$  is antisymmetric. This sigma model can be made topological by including additional fields, but this will not be considered here. It can be embedded into a 2d dimensional real sigma model with coordinates of the target manifold denoted by  $Z^i = \{Z^{\mu}, Z^{\overline{\mu}}\}, \ \mu = 1, \dots, d$ , with a background metric  $g_{ij}(Z)$  and antisymmetric tensor  $b_{ij}(Z)$ , with the action

$$I = \int d\sigma^{+} d\sigma^{-} \left( g_{ij} \left( Z \right) + b_{ij} \left( Z \right) \right) \partial_{+} Z^{i} \partial_{-} Z^{j}.$$

The connection is taken to be

$$\Gamma_{ij}^{k} = \mathring{\Gamma}_{ij}^{k} + \frac{1}{2}g^{kl}T_{ijl}, 
\mathring{\Gamma}_{ij}^{k} = \frac{1}{2}g^{kl}\left(\partial_{i}g_{lj} + \partial_{j}g_{il} - \partial_{l}g_{ij}\right), 
T_{ijk} = \left(\partial_{i}b_{jk} + \partial_{j}b_{ki} + \partial_{k}b_{ij}\right),$$

so that the torsion on the target manifold is totally antisymmetric. The embedding is defined by taking

$$\begin{split} g_{\mu\nu} &= 0 = g_{\overline{\mu}\overline{\nu}}, \\ b_{\mu\nu} &= 0 = b_{\overline{\mu}\overline{\nu}}, \\ b_{\overline{\nu}\mu} &= g_{\mu\overline{\nu}} = -b_{\mu\overline{\nu}} = g_{\overline{\nu}\mu} \end{split}$$

so that, as can be easily verified, the only non-zero components of the connections are

$$\Gamma^{\rho}_{\mu\lambda} = g^{\overline{\nu}\rho} \partial_{\lambda} g_{\mu\overline{\nu}}$$

and their complex conjugates.

Having made the identification of how the complex d-dimensional target manifold is embedded into the sigma model with a 2d real target manifold, we can proceed to summarize the geometrical properties of Hermitian non-Kähler manifolds.

The Hermitian manifold M of complex dimensions d is defined as a Riemannian manifold with real dimensions 2d with Riemannian metric  $g_{ij}$  and complex coordinates  $z^i = \{z^{\mu}, z^{\overline{\mu}}\}$  where Latin indices  $i, j, k, \dots$ , run over the range  $1, 2, \dots, d, \overline{1}, \overline{2}, \dots, \overline{d}$ . The invariant line element is then [11]

$$ds^2 = g_{ij}dz^i dz^j,$$

where the metric  $g_{ij}$  is hybrid

$$g_{ij} = \left(\begin{array}{cc} 0 & g_{\mu\overline{\nu}} \\ g_{\nu\overline{\mu}} & 0 \end{array}\right).$$

It has also an integrable complex structure  $J_i^j$  satisfying

$$J_i^k J_k^j = -\delta_i^j,$$

and with a vanishing Nijenhuis tensor

$$N_{ii}^{h} = J_i^t \left( \partial_t J_i^h - \partial_i J_t^h \right) - J_i^t \left( \partial_t J_i^h - \partial_j J_t^h \right).$$

Locally, the complex structure has components

$$J_i^j = \begin{pmatrix} i\delta_\mu^\nu & 0\\ 0 & -i\delta_{\overline{\mu}}^{\overline{\nu}} \end{pmatrix}.$$

The affine connection with torsion  $\Gamma_{ij}^h$  is introduced so that the following two conditions are satisfied

$$\nabla_k g_{ij} = \partial_k g_{ij} - \Gamma^h_{ik} g_{hj} - \Gamma^h_{jk} g_{ih} = 0,$$
  
$$\nabla_k F^j_i = \partial_k F^j_i - \Gamma^h_{ik} F^j_h + \Gamma^j_{hk} F^h_i = 0.$$

These conditions do not determine the affine connection uniquely and there exists several possibilities used in the literature. We shall adopt the Chern connection, which is the one most commonly used, . It is defined by prescribing that the  $(2d)^2$  linear differential forms

$$\omega^i_{\ j} = \Gamma^i_{jk} dz^k,$$

be such that  $\omega^{\mu}_{\ \nu}$  and  $\omega^{\overline{\mu}}_{\ \overline{\nu}}$  are given by [12]

$$\begin{split} & \omega^{\mu}_{\ \nu} = \Gamma^{\mu}_{\nu\rho} dz^{\rho}, \\ & \overline{\omega^{\mu}_{\ \nu}} = \omega^{\overline{\mu}}_{\overline{\nu}} = \Gamma^{\overline{\mu}}_{\overline{\nu}\overline{\rho}} dz^{\overline{\rho}}, \end{split}$$

with the remaining  $(2d)^2$  forms set equal to zero. For  $\omega^{\mu}_{\nu}$  to have a metrical connection the differential of the metric tensor g must be given by

$$dg_{\mu\overline{\nu}} = \omega^{\rho}_{\ \mu} g_{\rho\overline{\nu}} + \omega^{\overline{\rho}}_{\ \overline{\nu}} g_{\mu\overline{\rho}},$$

from which we obtain

$$\partial_{\lambda}g_{\mu\overline{\nu}}dz^{\lambda} + \partial_{\overline{\lambda}}g_{\mu\overline{\nu}}dz^{\overline{\lambda}} = \Gamma^{\rho}_{\mu\lambda}g_{\rho\overline{\nu}}dz^{\lambda} + \Gamma^{\overline{\rho}}_{\overline{\nu}\overline{\lambda}}g_{\mu\overline{\rho}}dz^{\overline{\lambda}},$$

so that

$$\Gamma^{\rho}_{\mu\lambda} = g^{\overline{\nu}\rho} \partial_{\lambda} g_{\mu\overline{\nu}},$$
  
$$\Gamma^{\overline{\rho}}_{\overline{\nu}\overline{\lambda}} = g^{\overline{\rho}\mu} \partial_{\overline{\lambda}} g_{\mu\overline{\nu}},$$

where the inverse metric  $g^{\overline{\nu}\mu}$  is defined by

$$g^{\overline{\nu}\mu}g_{\mu\overline{\kappa}} = \delta^{\overline{\nu}}_{\overline{\kappa}}.$$

Notice that the Chern connection agrees with the connection obtained above by embedding of the non-linear sigma model with Hermitian target manifold into a real one with double the number of dimensions. The condition  $\nabla_k J_i^j = 0$  is automatically satisfied and the connection is metric. The torsion forms are defined by

$$\Theta^{\mu} \equiv -\frac{1}{2} T_{\nu\rho}{}^{\mu} dz^{\nu} \wedge dz^{\rho}$$
$$= \omega^{\mu}{}_{\nu} dz^{\nu} = -\Gamma^{\mu}{}_{\nu\rho} dz^{\nu} \wedge dz^{\rho},$$

which implies that

$$\begin{split} T_{\nu\rho}^{\ \ \mu} &= \Gamma^{\mu}_{\nu\rho} - \Gamma^{\mu}_{\rho\nu} \\ &= g^{\overline{\sigma}\mu} \left( \partial_{\rho} g_{\nu\overline{\sigma}} - \partial_{\nu} g_{\rho\overline{\sigma}} \right). \end{split}$$

The torsion form is related to the differential of the Hermitian form

$$J = \frac{1}{2} J_{ij} dz^i \wedge dz^j,$$

where

$$J_{ij} = J_i^k g_{kj} = -J_{ji},$$

is antisymmetric and satisfy

$$J_{\mu\nu} = 0 = J_{\overline{\mu}\overline{\nu}},$$
  
$$J_{u\overline{\nu}} = ig_{u\overline{\nu}} = -J_{\overline{\nu}\mu},$$

so that

$$J = i g_{\mu \overline{\nu}} dz^{\mu} \wedge dz^{\overline{\nu}}.$$

The differential of the two-form J is then

$$dJ = \frac{1}{6}J_{ijk}dz^i \wedge dz^j \wedge dz^k,$$

so that

$$J_{ijk} = \partial_i J_{jk} + \partial_j J_{ki} + \partial_k J_{ij}.$$

The only non-vanishing components of this tensor are

$$J_{\mu\nu\overline{\rho}} = i \left( \partial_{\mu} g_{\nu\overline{\rho}} - \partial_{\nu} g_{\mu\overline{\rho}} \right) = -i T_{\mu\nu}^{\phantom{\mu\nu}\sigma} g_{\sigma\overline{\rho}} = -i T_{\mu\nu\overline{\rho}},$$

$$J_{\overline{\mu}\,\overline{\nu}\rho} = -i \left( \partial_{\overline{\mu}} g_{\rho\overline{\nu}} - \partial_{\overline{\nu}} g_{\rho\overline{\mu}} \right) = i T_{\overline{\mu}\,\overline{\nu}}^{\phantom{\overline{\nu}}\overline{\sigma}} g_{\rho\overline{\sigma}} = i T_{\overline{\mu}\,\overline{\nu}\rho}.$$

The curvature tensor of the metric connection is constructed in the usual manner

$$\Omega^{i}_{j} = d\omega^{i}_{j} - \omega^{i}_{k} \wedge \omega^{k}_{j},$$

with the only non-vanishing components being  $\Omega^{\nu}_{\mu}$  and  $\Omega^{\overline{\nu}}_{\overline{\mu}}$ . These are given by

$$\Omega^{\nu}_{\ \mu} = -R^{\nu}_{\ \mu\kappa\lambda}dz^{\kappa}\wedge dz^{\lambda} - R^{\nu}_{\ \mu\kappa\overline{\lambda}}dz^{\kappa}\wedge dz^{\overline{\lambda}},$$

where one can show that

$$\begin{split} R^{\nu}_{\ \mu\kappa\lambda} &= 0, \\ R^{\nu}_{\ \mu\kappa\overline{\lambda}} &= g^{\overline{\rho}\nu} \partial_{\kappa} \partial_{\overline{\lambda}} g_{\mu\overline{\rho}} + \partial_{\overline{\lambda}} g^{\overline{\rho}\nu} \partial_{\kappa} g_{\mu\overline{\rho}}. \end{split}$$

Transvecting the last relation with  $g_{\nu\overline{\sigma}}$  we obtain

$$-R_{\nu\overline{\sigma}\kappa\overline{\lambda}} = \partial_{\kappa}\partial_{\overline{\lambda}}g_{\mu\overline{\sigma}} + g_{\nu\overline{\sigma}}\partial_{\overline{\lambda}}g^{\overline{\rho}\nu}\partial_{\kappa}g_{\mu\overline{\rho}}.$$

Therefore the only non-vanishing covariant components of the curvature tensor are

$$R_{\mu\overline{\nu}\kappa\overline{\lambda}}, \quad R_{\mu\overline{\nu}\overline{\kappa}\lambda}, \quad R_{\overline{\mu}\nu\kappa\overline{\lambda}}, \quad R_{\overline{\mu}\nu\overline{\kappa}\lambda},$$

which are related by

$$R_{\mu\overline{\nu}\kappa\overline{\lambda}} = -R_{\overline{\nu}\mu\kappa\overline{\lambda}} = -R_{\mu\overline{\nu}\overline{\lambda}\kappa},$$

and satisfy the first Bianchi identity [12]

$$R^{\nu}_{\mu\kappa\overline{\lambda}} - R^{\nu}_{\kappa\mu\overline{\lambda}} = \nabla_{\overline{\lambda}} T_{\mu\kappa}^{\phantom{\mu\kappa}\nu}.$$

The second Bianchi identity is given by

$$\nabla_{\rho} R_{\mu \overline{\nu} \kappa \overline{\lambda}} - \nabla_{\kappa} R_{\mu \overline{\nu} \rho \overline{\lambda}} = R_{\mu \overline{\nu} \sigma \overline{\lambda}} T_{\rho \kappa}^{\ \ \sigma},$$

together with the conjugate relations. There are three possible contractions for the curvature tensor which are called the Ricci tensors

$$R_{\mu\overline{\nu}} = -g^{\overline{\lambda}\kappa} R_{\mu\overline{\lambda}\kappa\overline{\nu}}, \quad S_{\mu\overline{\nu}} = -g^{\overline{\lambda}\kappa} R_{\mu\overline{\nu}\kappa\overline{\lambda}}, \quad T_{\mu\overline{\nu}} = -g^{\overline{\lambda}\kappa} R_{\kappa\overline{\lambda}\mu\overline{\nu}}.$$

Upon further contraction these result in two possible curvature scalars

$$R = g^{\overline{\nu}\mu} R_{\mu\overline{\nu}}, \quad S = g^{\overline{\nu}\mu} S_{\mu\overline{\nu}} = g^{\overline{\nu}\mu} T_{\mu\overline{\nu}}.$$

Note that when the torsion tensors vanishes, the manifold M becomes Kähler. We shall not impose the Kähler condition as we are interested in Hermitian non-Kählerian geometry. We note that it is also possible to consider the Levi-Civita connection  $\Gamma_{ij}^k$  and the associated Riemann curvature  $K_{kij}^h$  where

$$\mathring{\Gamma}_{ij}^{k} = \frac{1}{2} g^{kl} \left( \partial_{i} g_{lj} + \partial_{j} g_{il} - \partial_{l} g_{ij} \right), 
K_{kij}^{h} = \partial_{k} \mathring{\Gamma}_{ij}^{h} - \partial_{i} \mathring{\Gamma}_{kj}^{h} + \mathring{\Gamma}_{kt}^{h} \mathring{\Gamma}_{ij}^{t} - \mathring{\Gamma}_{it}^{h} \mathring{\Gamma}_{kj}^{t}.$$

The relation between the Chern connection and the Levi-Civita connection is given by

$$\Gamma_{ij}^{k} = \mathring{\Gamma}_{ij}^{k} + \frac{1}{2} \left( T_{ij}^{k} - T_{ij}^{k} - T_{ji}^{k} \right).$$

It can be immediately verified that the Levi-Civita connection of the Hermitian manifold is identical to the one obtained from the non-linear sigma model, but only after the identification of  $b_{\mu\bar{\nu}}$  with  $-g_{\mu\bar{\nu}}$ . The Ricci tensor and curvature scalar are  $K_{ij} = K_{tij}{}^t$  and  $K = g^{ij}K_{ij}$ . Moreover, it is also possible to define  $H_{kj} = K_{kji}{}^t J_t^i$  and  $H = g^{kj}H_{kj}$ . The two scalar curvatures K and H are not independent but related by [13]

$$K - H = \mathring{\nabla}^h J^{ij} \mathring{\nabla}_j J_{ih} - \mathring{\nabla}^k J_{ki} \mathring{\nabla}_h J^{hi} - 2J^{ji} \mathring{\nabla}_j \mathring{\nabla}^k J_{ki}.$$

There are also relations between curvatures of the Chern connection and those of the Levi-Civita connection, mainly [13]

$$\frac{1}{2}K = S - \nabla^{\mu}T_{\mu} - \nabla^{\overline{\mu}}T_{\overline{\mu}} - T_{\mu}T_{\overline{\nu}}g^{\overline{\nu}\mu},$$

where  $T_{\mu} = T_{\mu\nu}^{\nu}$ . There are two natural conditions that can be imposed on the torsion. The first is  $T_{\mu} = 0$  which results in a semi-Kähler manifold. The other is when the torsion is complex analytic so that  $\nabla_{\overline{\lambda}} T_{\mu\kappa}^{\nu} = 0$  implying that the curvature tensor has the same symmetry properties as in the Kähler case. In this work we shall not impose any conditions on the torsion tensor.

We note that the line element

$$ds^2 = 2g_{\mu\overline{\nu}}dz^{\mu}d\overline{z}^{\overline{\nu}},$$

preserves its form under infinitesimal holomorphic transformations

$$z^{\mu} \to z^{\mu} - \zeta^{\mu}(z) ,$$
  
$$z^{\overline{\mu}} \to z^{\overline{\mu}} - \zeta^{\overline{\mu}}(\overline{z}) ,$$

as can be seen from the transformations

$$\delta g_{\mu\overline{\nu}} = \partial_{\mu} \zeta^{\lambda} g_{\lambda\overline{\nu}} + \partial_{\overline{\nu}} \zeta^{\overline{\lambda}} g_{\mu\overline{\lambda}} + \zeta^{\lambda} \partial_{\lambda} g_{\mu\overline{\nu}} + \zeta^{\overline{\lambda}} \partial_{\overline{\lambda}} g_{\mu\overline{\nu}}.$$

It is instructive to express these transformations in terms of the fields  $G_{\mu\nu}(x,y)$  and  $B_{\mu\nu}(x,y)$  by writing

$$\zeta^{\mu}(z) = \alpha^{\mu}(x, y) + i\beta^{\mu}(x, y),$$
  
$$\zeta^{\overline{\mu}}(\overline{z}) = \alpha^{\mu}(x, y) - i\beta^{\mu}(x, y).$$

The holomorphicity conditions on  $\zeta^{\mu}$  and  $\zeta^{\overline{\mu}}$  imply the relations

$$\begin{split} \partial_{\mu}^{y}\beta^{\nu} &= \partial_{\mu}^{x}\alpha^{\nu}, \\ \partial_{\mu}^{y}\alpha^{\nu} &= -\partial_{\mu}^{x}\beta^{\nu}, \end{split}$$

where we have denoted

$$\partial_{\mu}^{y} = \frac{\partial}{\partial y^{\mu}}, \qquad \partial_{\mu}^{x} = \frac{\partial}{\partial x^{\mu}}.$$

The transformations of  $G_{\mu\nu}(x,y)$  and  $B_{\mu\nu}(x,y)$  are then given by

$$\delta G_{\mu\nu}(x,y) = \partial_{\mu}^{x} \alpha^{\lambda} G_{\lambda\nu} + \partial_{\nu}^{x} \alpha^{\lambda} G_{\mu\lambda} + \alpha^{\lambda} \partial_{\lambda}^{x} G_{\mu\nu} - \partial_{\mu}^{x} \beta^{\lambda} B_{\lambda\nu} + \partial_{\nu}^{x} \beta^{\lambda} B_{\mu\lambda} + \beta^{\lambda} \partial_{\lambda}^{y} G_{\mu\nu}, \delta B_{\mu\nu}(x,y) = \partial_{\mu}^{x} \beta^{\lambda} G_{\lambda\nu} - \partial_{\nu}^{x} \beta^{\lambda} G_{\mu\lambda} + \alpha^{\lambda} \partial_{\lambda}^{x} B_{\mu\nu} + \partial_{\mu}^{x} \alpha^{\lambda} B_{\lambda\nu} + \partial_{\nu}^{x} \alpha^{\lambda} B_{\mu\lambda} + \beta^{\lambda} \partial_{\lambda}^{y} B_{\mu\nu}.$$

One readily recognizes that in the vicinity of small  $y^{\mu}$  the fields  $G_{\mu\nu}(x,0)$  and  $B_{\mu\nu}(x,0)$  transform as symmetric and antisymmetric tensors with gauge parameters  $\alpha^{\mu}(x)$  and  $\beta^{\mu}(x)$  where

$$\alpha^{\mu}(x,y) = \alpha^{\mu}(x) - \partial_{\nu}^{x} \beta^{\mu}(x) y^{\nu} + O(y^{2}),$$
  
$$\beta^{\mu}(x,y) = \beta^{\mu}(x) + \partial_{\nu}^{x} \alpha^{\mu}(x) y^{\nu} + O(y^{2}),$$

as implied by the holomorphicity conditions. Therefore, it should be possible to find an action where diffeomorphism invariance in the complex dimensions imply diffeomorphism invariance in the real submanifold and abelian invariance for the field  $B_{\mu\nu}(x)$  to insure that the later only appears through its field strength.

For simplicity, we shall now specialize to four complex dimensions. We start with the most general action limited to derivatives of order two

$$I = \int_{M^4} d^4z d^4\overline{z}g \left(aR + bS + c T_{\mu\nu\overline{\kappa}} T_{\overline{\rho}\,\overline{\sigma}\lambda} g^{\overline{\rho}\mu} g^{\overline{\sigma}\nu} g^{\overline{\kappa}\lambda} + d T_{\mu\nu\overline{\kappa}} T_{\overline{\rho}\,\overline{\sigma}\lambda} g^{\overline{\rho}\mu} g^{\overline{\sigma}\lambda} g^{\overline{\kappa}\nu}\right).$$

One can show that by requiring the linearized action, in the limit  $y \to 0$ , to give the correct kinetic terms for  $G_{\mu\nu}(x)$  and  $B_{\mu\nu}(x)$  relates the coefficients a, b, c, d to each other [14]

$$b = -a,$$
  $d = -1 - a,$   $c = \frac{1}{2}.$ 

In this case the action simplifies to the very elegant form

$$I = -\frac{1}{2} \int_{M} d^{4}z d^{4}\overline{z} \epsilon^{\overline{\kappa}\overline{\lambda}\overline{\sigma}\overline{\eta}} \epsilon^{\mu\nu\rho\tau} g_{\tau\overline{\eta}} \partial_{\mu} g_{\nu\overline{\sigma}} \partial_{\overline{\kappa}} g_{\rho\overline{\lambda}}.$$

which can be expressed in terms of the two-form J,

$$I = \frac{i}{2} \int_{M} J \wedge \partial J \wedge \overline{\partial} J.$$

We stress that this action is only invariant under holomorphic transformations. The equations of motion are given by

$$\epsilon^{\overline{\kappa}\overline{\lambda}\overline{\sigma}\,\overline{\eta}}\epsilon^{\mu\nu\rho\tau}\left(g_{\nu\overline{\sigma}}\partial_{\mu}\partial_{\overline{\kappa}}g_{\rho\overline{\lambda}} + \frac{1}{2}\partial_{\mu}g_{\nu\overline{\sigma}}\partial_{\overline{\kappa}}g_{\rho\overline{\lambda}}\right) = 0,$$

which are trivially satisfied when the metric  $g_{\mu\overline{\nu}}$  is Kähler

$$\partial_{\mu}g_{\nu\overline{\rho}} = \partial_{\nu}g_{\mu\overline{\rho}}, \qquad \partial_{\overline{\sigma}}g_{\nu\overline{\rho}} = \partial_{\overline{\rho}}g_{\nu\overline{\sigma}}.$$

We proceed to evaluate the four-dimensional limit of the action when the imaginary parts of the coordinates are small at low-energy. The action is a

function of the fields  $G_{\mu\nu}(x,y)$  and  $B_{\mu\nu}(x,y)$  which depend continuously on the coordinates  $y^{\mu}$ , implying a continuous spectrum with an infinite number of fields depending on  $x^{\mu}$  only. To obtain a discrete spectrum a certain physical assumption should be made that forces the imaginary coordinates to be small. One idea, suggested by Witten [7], is to suppress the imaginary parts by constructing an orbifold space M' = M/G where G is the group of imaginary shifts

$$z^{\mu} \rightarrow z^{\mu} + i(2\pi k^{\mu}),$$

where  $k^{\mu}$  are real. To maintain invariance under general coordinate transformations we must require  $k^{\mu}(x,y)$  to be coordinate dependent. It is not easy, however, to deal with such an orbifold in field theoretic considerations.

Instead, we shall proceed by examining the dynamical properties of the action which depends on terms not higher than second derivatives of the fields. It is then enough to expand the fields to second order in  $y^{\mu}$  and take the limit  $y \to 0$ . We therefore write

$$G_{\mu\nu}(x,y) = G_{\mu\nu}(x) + G_{\mu\nu\rho}(x)y^{\rho} + \frac{1}{2}G_{\mu\nu\rho\sigma}(x)y^{\rho}y^{\sigma} + O(y^{3}),$$
  

$$B_{\mu\nu}(x,y) = B_{\mu\nu}(x) + B_{\mu\nu\rho}(x)y^{\rho} + \frac{1}{2}B_{\mu\nu\rho\sigma}(x)y^{\rho}y^{\sigma} + O(y^{3}).$$

In the absence of a symmetry principle that determines the fields  $G_{\mu\nu\rho}(x)$ ,  $B_{\mu\nu\rho}(x)$ ,  $G_{\mu\nu\rho\sigma}(x)$  and  $B_{\mu\nu\rho\sigma}(x)$  and all higher terms as functions of  $G_{\mu\nu}(x)$ ,  $B_{\mu\nu}(x)$  we impose boundary conditions, in the limit  $y \to 0$ , on the first and second derivatives of the Hermitian metric. In order to have this action identified with the string effective action, the equations of motion in the  $y \to 0$  limit should reproduce the low-energy limit of the string equations

$$0 = G^{\eta\tau} \left( R \left( G \right) + \frac{1}{6} H_{\mu\nu\rho} H^{\mu\nu\rho} \right) - 2 \left( R^{\eta\tau} \left( G \right) + \frac{1}{4} H^{\eta}_{\nu\rho} H^{\tau\nu\rho} \right),$$
  
$$0 = \nabla^{\mu(G)} H_{\mu\eta\tau}.$$

These equations could be derived from the equations of motion of the Hermitian theory, provided we impose the following boundary conditions on torsion and curvature of the Hermitian manifold:

$$T_{\mu\nu\overline{\rho}}|_{y\to 0} = 2iB_{\mu\nu,\rho}\left(x\right),$$

$$\left[R_{\mu\overline{\sigma}\kappa\overline{\lambda}} - R_{\kappa\overline{\sigma}\mu\overline{\lambda}}\right]_{y\to 0} = -2\left(R_{\mu\kappa\sigma\lambda}\left(G\right) + i\left(\nabla_{\lambda}^{G}H_{\mu\kappa\sigma} - \nabla_{\sigma}^{G}H_{\mu\kappa\lambda}\right)\right).$$

The solution of the torsion constraint gives, to lowest orders,

$$G_{\mu\nu\rho}(x) = \partial_{\nu}B_{\mu\rho}(x) + \partial_{\mu}B_{\nu\rho}(x),$$
  

$$B_{\mu\nu\rho}(x) = -G_{\mu\rho,\nu}(x) + G_{\nu\rho,\mu}(x),$$

where all derivatives are now with respect to  $x^{\mu}$ . Substituting these into the curvature constraints yield

$$G_{\mu\sigma\kappa\lambda}(x) = \partial_{\sigma}\partial_{\lambda}G_{\mu\kappa}(x) + \partial_{\mu}\partial_{\lambda}G_{\sigma\kappa}(x) + \partial_{\sigma}\partial_{\kappa}G_{\mu\lambda}(x) + \partial_{\mu}\partial_{\kappa}G_{\sigma\lambda}(x) - \partial_{\kappa}\partial_{\lambda}G_{\mu\sigma}(x) + O(\partial G, \partial B),$$
  
$$B_{\mu\sigma\kappa\lambda}(x) = \partial_{\sigma}\partial_{\lambda}B_{\mu\kappa}(x) - \partial_{\mu}\partial_{\lambda}B_{\sigma\kappa}(x) + \partial_{\sigma}\partial_{\kappa}B_{\mu\lambda}(x) - \partial_{\mu}\partial_{\kappa}B_{\sigma\lambda}(x) - \partial_{\kappa}\partial_{\lambda}B_{\mu\sigma}(x) + O(\partial G, \partial B),$$

where  $O(\partial G, \partial B)$  are terms of second order [14].

This is encouraging, but more work is needed to establish the exact connection between string theory effective actions and gravity on Hermitian manifolds and not only to second order. For this to happen, one must determine, unambiguously, the symmetry principle that restricts the continuous spectrum as function of the imaginary coordinates to a discrete one.

To summarize, the idea that complex dimensions play a role in physics is quite old [15]. So far it has provided a technical advantage in obtaining extensions and new solutions to the Einstein equations, or in providing elegant formulations of some field theories such as Yang-Mills theory in terms of twister spaces. At present, there is only circumstantial evidence, coming from the study of high-energy behavior of string scattering amplitudes, where it was observed that the imaginary parts of the string coordinates of the target manifold appear. The work presented here is an attempt to show that it might be possible to formulate geometrically the effective string theory for target manifolds with complex dimensions. In this picture the metric tensor and antisymmetric tensor of the effective theory are unified in one field, the metric tensor of the Hermitian manifold, an idea first put forward by Einstein.

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