# Five-branes and M-Theory On An Orbifold

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We relate Type IIB superstrings compactified to six dimensions on K3 to an elevendimensional theory compactified on  $(\mathbf{S}^1)^5/\mathbf{Z}_2$ . Eleven-dimensional five-branes enter the story in an interesting way.

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## 1. Introduction

By now, there is substantial evidence for the existence of an eleven-dimensional quantum theory with eleven-dimensional supergravity as its long wave-length limit. Moreover, the theory contains two-branes and five-branes at least macroscopically, and some of their properties are known; for instance, the  $\kappa$ -invariant Bergshoeff-Sezgin-Townsend action[1] describes the long wavelength excitations of a macroscopic two-brane.

The description by eleven-dimensional supergravity with two-branes and five-branes is expected to be valid when all characteristic length scales (of a space-time and the branes that it contains) are large compared to the Planck length. One also has some information about the behavior under certain conditions when some dimensions of space-time are *small* compared to the Planck scale. For instance, the eleven-dimensional "M-theory" (where M stands for magic, mystery, or membrane, according to taste) on  $X \times \mathbf{S}^1$ , with X any ten-manifold, is equivalent to Type IIA on X, with a Type IIA string coupling constant that becomes small when the radius of  $\mathbf{S}^1$  goes to zero. Likewise, the M-theory on  $Y \times \mathbf{K}^3$ , with Y a seven-manifold, is equivalent to the heterotic string on  $Y \times \mathbf{T}^3$ , and the M-theory on  $X \times \mathbf{S}^1/Z_2$ , with X a ten-manifold, is equivalent to the  $E_8 \times E_8$  heterotic string on X; in each case, the string coupling constant becomes small when the volume of the last factor goes to zero.

The evidence for the existence of the M-theory (beyond the consistency of the classical low energy theory) comes mainly from the success of statements deduced from the relations of the M-theory to strings. Even a few more similar examples might therefore significantly enrich the story. The purpose of the present paper is to add one more such example, by arguing that the M-theory on  $Z \times (\mathbf{S}^1)^5/\mathbf{Z}_2$  is equivalent to the Type IIB superstring on  $Z \times \mathrm{K3}$ . Here Z is an arbitrary six-manifold, but as usual in such arguments, by scaling up the metric of Z, one can reduce to the case that  $Z = \mathbf{R}^6$ . In fact, once an equivalence is established between the M-theory on  $Z \times (\mathbf{S}^1)^5/\mathbf{Z}_2$  and Type IIB on  $Z \times \mathrm{K3}$  when Z is large, it can be followed into the region of small Z.

The equivalence of the M-theory on  $\mathbf{R}^6 \times (\mathbf{S}^1)^5/\mathbf{Z}_2$  with Type IIB on  $\mathbf{R}^6 \times \mathrm{K3}$  was also conjectured recently by Dasgupta and Mukhi [2] who independently pointed out a problem – involving anomaly cancellation and the distribution of the twisted sectors among fixed points – that will be addressed below. Some general comments about Type IIB on K3 as an M-theory orbifold were also made recently by Hull [3].

# 2. The Low Energy Supergravity

Compactification of the Type IIB superstring on K3 gives a six-dimensional theory with a chiral supersymmetry which (upon toroidal compactification to four dimensions) is related to N=4 supersymmetry in D=4. We will call this six-dimensional chiral N=4 supersymmetry (though the number of supercharges is only twice the minimum possible number in D=6).

The supergravity multiplet of chiral N=4 supergravity contains, in addition to the graviton, five self-dual tensors (that is two-forms with self-dual field strength) plus gravitinos. The graviton in six dimensions has nine helicity states, while the self-dual tensor has three, so the total number of bosonic helicity states is  $9+5\cdot 3=24$ ; the gravitinos likewise have 24 helicity states. The supergravity multiplet has gravitational anomalies (which cannot be canceled by the Green-Schwarz mechanism alone), so any consistent theory with chiral N=4 supergravity in six dimensions must contain matter multiplets also.

There is actually only one possible matter multiplet in chiral N=4 supersymmetry. It is the tensor multiplet, which contains five spin zero bosons, an anti-self-dual antisymmetric tensor (that is a two-form field whose field strength is anti-self-dual) with three helicity states, and 5+3=8 helicity states of chiral fermions. Cancellation of gravitational anomalies requires that the number of tensor multiplets be precisely 21.

Using only the low energy supergravity, one can deduce (for a survey of such matters see [4]) that the moduli space  $\mathcal{M}$  of vacua is locally the homogeneous space  $SO(21,5)/SO(21) \times SO(5)$ . In the particular case of a chiral N=4 theory obtained by compactification of Type IIB on K3, the global structure is actually (as asserted in equation (4.16) of [5]; see [6] for a more precise justification)  $\mathcal{M} = SO(21,5; \mathbf{Z}) \setminus SO(21,5)/SO(21) \times SO(5)$ . This depends on knowledge of conformal field theory T-duality on K3 [7] together with the  $SL(2,\mathbf{Z})$  symmetry of ten-dimensional Type IIB superstring theory.

Note that since there is no scalar in the chiral N=4 supergravity multiplet, the dilaton is one of the  $5 \times 21 = 105$  scalars that come from the tensor multiplets. The  $SO(21,5; \mathbf{Z})$  discrete symmetry mixes up the dilaton with the other 104 scalars, relating some but not all of the "strong coupling" regimes to regions of weak coupling or large volume.

# 2.1. Five-branes And The Tensor Multiplet Anomaly

We will need some background about fivebranes and gravitational anomalies.

We want to consider a certain model of *global* chiral N=4 supergravity with the tensor multiplet. To do this, we begin in eleven-dimensional Minkowski space, with coordinates  $x^1, \ldots, x^{11}$  ( $x^1$  being the time), and gamma matrices  $\Gamma^1, \ldots, \Gamma^{11}$  which obey

$$\Gamma^1 \Gamma^2 \dots \Gamma^{11} = 1. \tag{2.1}$$

Now we introduce a five-brane with world-volume given by the equations

$$x^7 = \dots = x^{11} = 0. (2.2)$$

The presence of this five-brane breaks half of the 32 space-time supersymmetries. The 16 surviving supersymmetries are those that obey  $\Gamma^7 \dots \Gamma^{11} = 1$ , or equivalently, in view of  $(2.1), \Gamma^1 \dots \Gamma^6 = 1$ . Thus, the surviving supersymmetries are chiral in the six-dimensional sense; the world-volume theory of the five-brane has chiral N=4 supersymmetry. This is global supersymmetry since – as the graviton propagates in bulk – there is no massless graviton on the five-brane world-volume.

Therefore, the massless world-volume fields must make up a certain number of tensor multiplets, this being the only matter multiplet allowed by chiral N=4 supergravity. In fact, there is precisely one tensor multiplet. The five massless scalars are simply the fluctuations in  $x^7, \ldots, x^{11}$ ; the massless world-volume fermions are the Goldstone fermions associated with the supersymmetries under which the five-brane is not invariant; and the anti-self-dual tensor has an origin that was described semiclassically in [8]. The assertion that the massless world-volume excitations of the five-brane consist of precisely one tensor multiplet can also be checked by compactifying the  $x^{11}$  direction on a circle, and comparing to the structure of a Dirichlet four-brane of Type IIA [9]. (In compactifying the M-theory to Type IIA, the five-brane wrapped around  $x^{11}$  turns into a four-brane; the tensor multiplet of 5+1 dimensions reduces to a vector multiplet in 4+1 dimensions, which is the massless world-volume structure of the Dirichlet four-brane.)

Now we want to allow fluctuations in the position of the five-brane and compute the quantum behavior at long wavelengths. At once we run into the fact that the tensor multiplet on the five-brane world-volume has a gravitational anomaly. Without picking a coordinate system on the five-brane world-volume, how can one cancel the anomaly in the

one loop effective action of the massless world-volume fields (even at very long wavelengths where the one loop calculation is valid)?

This question was first discussed by Duff, Liu, and Minasian [10]; what follows is a sort of dual version of their resolution of the problem.<sup>2</sup> The tensor multiplet anomaly *cannot* be cancelled, as one might have hoped, by a world-volume Green-Schwarz mechanism. Instead one has to cancel a world-volume effect against a bulk contribution from the eleven-dimensional world, rather as in [11].

This theory has in the long-wavelength description a four-form F that is closed in the absence of five-branes, but which in the presence of five-branes obeys

$$dF = \delta_V \tag{2.3}$$

where  $\delta_V$  is a delta-function supported on the five-form world-volume V. There is here a key point in the terminology: given a codimension n submanifold W of space-time, the symbol  $\delta_W$  will denote not really a delta "function" but a closed n-form supported on W which integrates to one in the directions normal to W. For instance, in one dimension, if P is the origin on the x-axis, then  $\delta_P = \delta(x) dx$  where  $\delta(x)$  is the "Dirac delta function" and  $\delta(x) dx$  is, therefore, a closed one-form that vanishes away from the origin and whose integral over the x-axis (that is, the directions normal to P) is 1. With  $\delta_V$  thus understood as a closed five-form in the five-brane case, (2.3) is compatible with the Bianchi identity d(dF) = 0 and is, in fact, sometimes taken as a defining property of the five-brane as it asserts that the five-brane couples magnetically to F.

Now suppose that in the low energy expansion of the effective eleven-dimensional theory on a space-time M there is a term

$$\Delta L = \int_{M} F \wedge I_{7} \tag{2.4}$$

where  $I_7$  is a gravitational Chern-Simons seven-form. Exactly which Chern-Simons seven-form it should be will soon become clear. Under an infinitesimal diffeomorphism  $x^I \to x^I + \epsilon v^I$  ( $\epsilon$  being an infinitesimal parameter and v a vector field),  $I_7$  does not transform as a tensor, but rather  $I_7 \to I_7 + dJ_6$ , where  $J_6$  is a certain six-form (which depends upon v). The transformation of  $\Delta L$  under a diffeomorphism is therefore

$$\Delta L \to \Delta L + \int_M F \wedge dJ_6 = \Delta L - \int_M dF \wedge J_6.$$
 (2.5)

<sup>&</sup>lt;sup>2</sup> In the very similar case of ten-dimensional Type IIA five-branes, the dual version was worked out in unpublished work by J. Blum and J. A. Harvey.

Thus,  $\Delta L$  is generally covariant in the absence of five-branes, but in the presence of a five-brane, according to (2.3), one gets

$$\Delta L \to \Delta L - \int_V J_6.$$
 (2.6)

But gravitational anomalies in n dimensions involve precisely expressions  $\int J_n$  where  $J_n$  is as above (that is,  $J_n$  appears in the transformation law of a Chern-Simons n+1-form  $I_{n+1}$  by  $I_{n+1} \to I_{n+1} + dJ_n$ ; see [12] for an introduction to such matters.) Thus with the correct choice of  $I_7$ , the anomaly of  $\Delta L$  in the presence of a five-brane precisely cancels the world-volume anomaly of the tensor multiplet. This is thus a case in which an interaction in the bulk is needed to cancel on anomaly on the world-volume. Moreover (as explained in a dual language in [10]), the presence in eleven dimensions of the interaction  $\Delta L$  can be checked by noting that upon compactification on a circle, this interaction reduces to the  $H \wedge I_7$  term found in [13] for Type IIA superstrings; here H is the usual three-form field strength of the Type IIA theory.

What has been said to this point is sufficient for our purposes. However, I cannot resist a further comment that involves somewhat similar ideas. The seven-form F' dual to F does not obey dF' = 0 even in the absence of five-branes; from the eleven-dimensional supergravity one finds instead

$$dF' + \frac{1}{2}F \wedge F = 0. {(2.7)}$$

One may ask how this is compatible with the Bianchi identity d(dF') = 0 once – in the presence of five-branes – one encounters a situation with  $dF \neq 0$ . The answer involves the anti-self-dual three-form field strength T on the five-brane world-volume. According to equation (3.3) of [14], this field obeys not – as one might expect – dT = 0, but rather dT = F. If then in the presence of a five-brane, (2.7) becomes

$$dF' + \frac{1}{2}F \wedge F - T \wedge \delta_V = 0, \tag{2.8}$$

then the Bianchi identity still works even in the presence of the five-brane. The  $T \wedge \delta_V$  term in fact follows from the coupling in equation (3.3) of [14], which gives a fivebrane contribution to the equation of motion of the three-form A. Thus, we get a new derivation of the relevant coupling and in particular of the fact that dT = F.

# **3.** Type IIB On K3

We now come to the main focus of this paper. One would like to understand the "strong coupling behavior" of the Type IIB theory compactified on K3, or more precisely, the behavior as one goes to infinity in the moduli space  $\mathcal{M}$  of vacua. As explained above, this theory has a  $SO(21,5; \mathbf{Z})$  discrete symmetry, which gives many identifications of strong coupling or small volume with weak coupling or large volume, but there remain (as in section three of [5] or [6]) inequivalent limits in which one can go to infinity in  $\mathcal{M}$ .

Any limit can be reached by starting at a given point  $P \in \mathcal{M}$  and then considering the one-parameter family of vacua  $P_t = e^{tx}P$  where x is a generator of SO(21,5) and t is a positive real number. As  $t \to \infty$ , one approaches infinity in  $\mathcal{M}$  in a direction that depends upon x. In any such limit, by looking at the lightest states, one aims to find a description by an effective ten-dimensional string theory or eleven-dimensional field theory. The duality group visible (though mostly spontaneously broken, depending on the precise choice of P) in this effective theory will include the subgroup  $\Gamma$  of  $SO(21,5; \mathbf{Z})$  that commutes with x (and so preserves the particular direction in which one has gone to infinity).

As in [5,6], one really only needs to consider x's that lead to a maximal set of light states, and because of the discrete  $SO(21,5; \mathbf{Z})$ , there are only finitely many cases to consider. We will focus here on the one limit that seems to be related most directly to the M-theory.

Consider a subgroup  $SO(16) \times SO(5,5)$  of SO(21,5). Let x be a generator of SO(5,5) that commutes with an SL(5) subgroup. Then the subgroup of  $SO(21,5; \mathbb{Z})$  that commutes with x – and so is visible if one goes to infinity in the direction determined by x – contains  $SO(16) \times SL(5, \mathbb{Z})$ .

Since it will play a role later, let us discuss just how  $SL(5, \mathbf{Z})$  can be observed as a symmetry at infinity. Instead of making mathematical arguments, we will discuss another (not unrelated, as we will see) physical problem with  $SO(21, 5; \mathbf{Z})$  symmetry, namely the compactification on a five-torus of the SO(32) heterotic string to five dimensions, with  $SO(21, 5; \mathbf{Z})$  as the T-duality group. The region at infinity in moduli space in which there is a visible  $SL(5, \mathbf{Z})$  symmetry is simply the large volume limit, with the torus large in all directions. In what sense can  $SL(5, \mathbf{Z})$  be "observed"? It is spontaneously broken (to a finite subgroup, generically trivial) by the choice of a metric on the five-torus, but, if one is free to move around in the moduli space of large volume metrics (remaining at infinity in  $\mathcal{M}$ ) one can see that there is a spontaneously broken  $SL(5, \mathbf{Z})$ .

Now, actually, the relevant region at infinity in moduli space is parametrized by a large metric on the torus, a B-field, and a flat SO(32) bundle described by five commuting Wilson lines  $W_j$ . (For the moment we take the flat bundle to be topologically trivial, a point we return to in section 3.4.) If one is free to vary all of these, one can certainly observe the full  $SL(5, \mathbf{Z})$ . Suppose, though, that in some method of calculation, the Wilson lines are frozen at particular values, and one can only vary the metric and B-field. Then one will only observe the subgroup of  $SL(5, \mathbf{Z})$  that leaves the Wilson lines invariant.

For instance, if the Wilson lines are trivial – a rather special situation with unbroken SO(32) – one will see all of  $SL(5, \mathbf{Z})$ . Here is another case that will enter below though it will appear mysterious at the moment. As the  $W_j$  commute, they can be simultaneously diagonalized, with eigenvalues  $\lambda_j^a$ ,  $a=1,\ldots,32$ . Suppose that the  $\lambda_j^a$  are all  $\pm 1$ , and have the property that for each fixed a,  $\prod_j \lambda_j^a = -1$ . There are 16 collections of five  $\pm 1$ 's whose product is -1 (namely 1,1,1,1,-1 and four permutations of that sequence; 1,1,-1,-1, and nine permutations of that sequence; and -1,-1,-1,-1,-1. Let the  $\lambda_j^a$  be such that each such permutation appears exactly twice. This breaks  $SL(5,\mathbf{Z})$  to the finite index subgroup  $\Gamma$  consisting of  $SL(5,\mathbf{Z})$  matrices  $M^j{}_k$ ,  $j,k=1,\ldots,5$  such that  $\sum_j M^j{}_k$  is odd for each fixed k. If the Wilson lines are frozen at the stated values, it is only  $\Gamma$  and not all of  $SL(5,\mathbf{Z})$  that can be observed by varying the metric and B-field.

# 3.1. Interpretation Of The Symmetry

Let us go back to the Type IIB theory on K3 and the attempt to interpret the strong coupling limit that was described, the one with a visible  $SL(5, \mathbf{Z})$ . As in the example just discussed, the  $SL(5, \mathbf{Z})$  symmetry is strongly suggestive of the mapping class group of a five-torus. Thus, one is inclined to relate this particular limit of Type IIB on K3 to the M-theory on  $\mathbf{R}^6$  times a five-manifold built from  $(\mathbf{S}^1)^5$ . This cannot be  $(\mathbf{S}^1)^5$  itself, because the M-theory on  $\mathbf{R}^6 \times (\mathbf{S}^1)^5$  would have twice as much supersymmetry as we want. One is tempted instead to take an orbifold of  $(\mathbf{S}^1)^5$  in such a way as to break half of the supersymmetry while preserving the  $SL(5, \mathbf{Z})$ .

A natural way to break half the supersymmetries by orbifolding is to divide by a  $\mathbb{Z}_2$  that acts as -1 on all five circles. This is actually the only choice that breaks half the supersymmetry and gives a *chiral* N=4 supersymmetry in six dimensions. In fact, dividing by this  $\mathbb{Z}_2$  leaves precisely those supersymmetries whose generators obey  $\Gamma^7\Gamma^8 \dots \Gamma^{11}\epsilon = \epsilon$ . This condition was encountered in the discussion of the five-brane, and leaves the desired

chiral supersymmetry. So M-theory compactified on  $(\mathbf{S}^1)^5/\mathbf{Z}_2$  is our candidate for an eleven-dimensional interpretation of Type IIB superstrings on K3.

More precisely, the proposal is that M-theory on  $(\mathbf{S}^1)^5/\mathbf{Z}_2$  has the property that when any  $\mathbf{S}^1$  factor in  $(\mathbf{S}^1)^5/\mathbf{Z}_2$  goes to zero radius, the M-theory on this manifold goes over to a weakly coupled Type IIB superstring. This assertion should hold not just for one of the five circles in the definition of  $(\mathbf{S}^1)^5/\mathbf{Z}_2$ , but for any of infinitely many circles obtained from these by a suitable symmetry transformation.

#### 3.2. Anomalies

Let us work out the massless states of the theory, first (as in [15]) the "untwisted states," that is the states that come directly from massless eleven-dimensional fields, and then the "twisted states," that is, the states that in a macroscopic description appear to be supported at the classical singularities of  $(\mathbf{S}^1)^5/\mathbf{Z}_2$ .

The spectrum of untwisted states can be analyzed quickly by looking at antisymmetric tensors. The three-form A of the eleven-dimensional theory is odd under parity (because of the  $A \wedge F \wedge F$  supergravity interaction). Since the  $\mathbb{Z}_2$  by which we are dividing  $(\mathbf{S}^1)^5$  reverses orientation, A is odd under this transformation. The zero modes of A on  $(\mathbf{S}^1)^5$  therefore give, after the  $\mathbb{Z}_2$  projection, five two-forms (and ten scalars, but no vectors or three-forms) on  $\mathbb{R}^6$ . The self-dual parts of these tensors are the expected five self-dual tensors of the supergravity multiplet, and the anti-self-dual parts are part of five tensor multiplets. The number of tensor multiplets from the untwisted sector is therefore five.

Just as in [15], the untwisted spectrum is anomalous; there are five tensor multiplets, while 21 would be needed to cancel the gravitational anomalies. 16 additional tensor multiplets are needed from twisted sectors.

The problem, as independently raised in [2], is that there appear to be 32 identical twisted sectors, coming from the 32 fixed points of the  $\mathbb{Z}_2$  action on  $(\mathbb{S}^1)^5$ . How can one get 16 tensor multiplets from 32 fixed points? We will have to abandon the idea of finding a vacuum in which all fixed points enter symmetrically.

Even so, there seems to be a paradox. As explained in [15], since the eleven-dimensional theory has no gravitational anomaly on a smooth manifold, the gravitational anomaly of the eleven-dimensional massless fields on an orbifold is a sum of delta functions supported at the fixed points. In the case at hand, the anomaly can be canceled by 16 tensor multiplets (plus a Green-Schwarz mechanism), but there are 32 fixed points. Thus, each fixed point has an anomaly, coming from the massless eleven-dimensional fields,

that could be canceled by 16/32 = 1/2 tensor multiplets.<sup>3</sup> The paradox is that it is not enough to globally cancel the gravitational anomaly by adding sixteen tensor multiplets. One needs to cancel the anomaly locally in the eleven-dimensional world, somehow modifying the theory to add at each fixed point half the anomaly of a tensor multiplet. How can this be, given that the tensor multiplet is the only matter multiplet of chiral N = 4 supersymmetry, so that any matter system at a fixed point would be a (positive) integral number of tensor multiplets?

# 3.3. Resolution Of The Paradox

To resolve this paradox, the key point is that because the fixed points in  $(\mathbf{S}^1)^5/\mathbf{Z}_2$  have codimension five, just like the codimension of a five-brane world-volume, there is another way to cancel anomalies apart from including massless fields on the world-volume. We can assume that the fixed points are magnetic sources of the four-form F. In other words, we suppose that (even in the absence of conventional five-branes) dF is a sum of delta functions supported at the orbifold fixed points. If so, then the bulk interaction  $\Delta L = \int F \wedge I_7$  that was discussed earlier will give additional contributions to the anomalies supported on the fixed points.

Since a magnetic coupling of F to the five-brane cancels the anomaly of a tensor multiplet, if an orbifold fixed point has "magnetic charge" -1/2, this will cancel the anomaly from the eleven-dimensional massless fields (which otherwise could be canceled by 1/2 a tensor multiplet). If an orbifold fixed point has magnetic charge +1/2, this doubles the anomaly, so that it can be canceled if there is in addition a "twisted sector" tensor multiplet supported at that fixed point. Note that it is natural that a  $\mathbb{Z}_2$  orbifold point could have magnetic charge that is half-integral in units of the usual quantum of charge.

A constraint comes from the fact that the sum of the magnetic charges must vanish on the compact space  $(\mathbf{S}^1)^5/\mathbf{Z}_2$ . Another constraint comes from the fact that if we want to maintain supersymmetry, the charge for any fixed point cannot be less than -1/2. Indeed, a fixed point of charge less than -1/2 would have an anomaly that could not be canceled by tensor multiplets; a negative number of tensor multiplets or a positive number of wrong chirality tensor multiplets (violating supersymmetry) would be required. An

<sup>&</sup>lt;sup>3</sup> The eleven-dimensional massless fields by obvious symmetries contribute the same anomaly at each fixed point.

example of how to satisfy these constraints and ensure local cancellation of anomalies is to assign charge -1/2 to 16 of the fixed points, and charge +1/2 to the other 16. With one tensor multiplet supported at each of the last 16 fixed points, such a configuration has all anomalies locally cancelled in the eleven-dimensional sense.

Here is another way to cancel the anomalies locally. Assign magnetic charge -1/2 to each of the 32 fixed points, but include at each of 16 points on  $(\mathbf{S}^1)^5/\mathbf{Z}_2$  a conventional five-brane, of charge 1. The total magnetic charge vanishes (as 32(-1/2) + 16 = 0) and since both a fixed point of charge -1/2 and a conventional five-brane are anomaly-free, all anomalies are cancelled locally. Each five-brane supports one tensor multiplet; the scalars in the tensor multiplets determine the positions of the five-branes on  $(\mathbf{S}^1)^5/\mathbf{Z}_2$ .

I would like to suggest that this last anomaly-canceling mechanism is the general one, and that the case that the magnetic charge is all supported on the fixed points is just a special case in which the five-branes and fixed points coincide. In fact, if a five-brane happens to move around and meet a fixed point, the charge of that fixed point increases by 1. This gives a very natural interpretation of the "twisted sector" modes of a fixed point of charge 1/2. Such a fixed point supports a tensor multiplet, which contains five scalars; we interpret the scalars as representing a possible perturbation in the five-brane position away from the fixed point.

If we accept this interpretation, there is no issue of what is the "right" configuration of charges for the fixed points; any configuration obeying the constraints (total charge 0 and charge at least -1/2 for each fixed point) appears somewhere on the moduli space. The only issue is what configuration of charges has the most transparent relation to string theory.

Let us parametrize the five circles in  $(\mathbf{S}^1)^5$  by periodic variables  $x^j$ ,  $j=7,\ldots,11$ , of period 1, with  $\mathbf{Z}_2$  acting by  $x^j \to -x^j$  so that the fixed points have all coordinates 0 or 1/2. We take  $SL(5,\mathbf{Z})$  to act linearly, by  $x^j \to M^j{}_k x^k$ , with  $M^j{}_k$  an  $SL(5,\mathbf{Z})$  matrix. Thus  $SL(5,\mathbf{Z})$  leaves invariant one fixed point P, the "origin"  $x^j=0$ , and acts transitively on the other 31. The only  $SL(5,\mathbf{Z})$ -invariant configuration of charges obeying the constraints is to assign magnetic charge +31/2 to P and -1/2 to each of the others. Then each of the 16 tensor multiplets would be supported at the origin. This configuration cancels the anomalies and is  $SL(5,\mathbf{Z})$  invariant. However, it does not seem to be the configuration with the closest relation to string theory.

To see this, consider the limit in which one of the circles in  $(\mathbf{S}^1)^5$  becomes small. To an observer who does not detect this circle, one is then left with  $(\mathbf{S}^1)^4/\mathbf{Z}_2$ , which is a K3

orbifold. Our hypothesis about M-theory on  $(\mathbf{S}^1)^5/\mathbf{Z}_2$  says that this theory should go over to weakly coupled Type IIB on K3 when any circle shrinks. In  $(\mathbf{S}^1)^4/\mathbf{Z}_2$ , there are 16 fixed points; in quantization of Type IIB superstrings on this orbifold, one tensor multiplet comes from each of the 16 fixed points.

In M-theory on  $(\mathbf{S}^1)^5/\mathbf{Z}_2$ , there are 32 fixed points. When one of the circles is small, then – to an observer who does not resolve that circle – the 32 fixed points appear to coalesce pairwise to the 16 fixed points of the string theory on  $(\mathbf{S}^1)^4/\mathbf{Z}_2$ . To reproduce the string theory answer that one tensor multiplet comes from each singularity, we want to arrange the charges on  $(\mathbf{S}^1)^5/\mathbf{Z}_2$  in such a way that each pair of fixed points differing only in the values of one of the coordinates contributes one tensor multiplet.

This can be done by arranging the charges in the following "checkerboard" configuration. If a fixed point has  $\sum_j x^j$  integral, we give it charge -1/2. If  $\sum_j x^j$  is a half-integer, we give it charge +1/2. Then any two fixed points differing only by the value of the  $x^j$ coordinate – for any given j – have equal and opposite charge, and contribute a total of one tensor multiplet.

Moreover, the four-form field strength F of the M-theory reduces in ten dimensions to a three-form field strength H. This vanishes for string theory on K3, so one can ask how the string theory can be a limit of an eleven-dimensional theory in a vacuum with non-zero F. If we arrange the charges in the checkerboard fashion, this puzzle has a natural answer. In the limit in which the  $j^{th}$  circle shrinks to zero, equal and opposite charges are superposed and cancel, so the resulting ten-dimensional theory has zero H.

The checkerboard configuration is not invariant under all of  $SL(5, \mathbf{Z})$ , but only under the finite index subgroup  $\Gamma$  introduced just prior to section 3.1 (the subgroup consisting of matrices  $M^j{}_k$  such that  $\sum_j M^j{}_k$  is odd for each k). Thus the reduction to ten-dimensional string theory can work not only if one shrinks one of the five circles in the definition of  $(\mathbf{S}^1)^5/\mathbf{Z}_2$ , but also if one shrinks any of the (infinitely many) circles obtainable from these by a  $\Gamma$  transformation.

Just as in the discussion in which  $\Gamma$  was introduced, in the checkerboard vacuum, one cannot see the full  $SL(5, \mathbf{Z})$  if the only parameters one is free to vary are the metric and three-form A on  $(\mathbf{S}^1)^5/\mathbf{Z}_2$ . An  $SL(5, \mathbf{Z})$  transformation w not in  $\Gamma$  is a symmetry only if combined with a motion of the other moduli – in fact a motion of some five-branes to compensate for the action of w on the charges of fixed points.

# 3.4. Check By Comparison To Other Dualities

In the study of string theory dualities, once a conjecture is formulated that runs into no immediate contradiction, one of the main ways to test it is to try to see what implications it has when combined with other, better established dualities.

In the case at hand, we will (as was done independently by Dasgupta and Mukhi [2]) mainly compare our hypothesis about M-theory on  $(\mathbf{S}^1)^5/\mathbf{Z_2}$  to the assertion that M-theory on  $X \times \mathbf{S}^1$  is equivalent, for any five-manifold X, to Type IIA on X.

To combine the two assertions in an interesting way, we consider M-theory on  $\mathbf{R}^5 \times \mathbf{S}^1 \times (\mathbf{S}^1)^5/\mathbf{Z}_2$ . On the one hand, because of the  $\mathbf{S}^1$  factor, this should be equivalent to Type IIA on  $\mathbf{R}^5 \times (\mathbf{S}^1)^5/\mathbf{Z}_2$ , and on the other hand, because of the  $(\mathbf{S}^1)^5/\mathbf{Z}_2$  factor, it should be equivalent to Type IIB on  $\mathbf{R}^5 \times \mathbf{S}^1 \times \mathbf{K}3$ .

It is easy to see that, at least in general terms, we land on our feet. Type IIB on  $\mathbf{R}^5 \times \mathbf{S}^1 \times \mathbf{K}3$  is equivalent by T-duality to Type IIA on  $\mathbf{R}^5 \times \mathbf{S}^1 \times \mathbf{K}3$ , and the latter is equivalent by Type IIA - heterotic duality to the heterotic string on  $\mathbf{R}^5 \times \mathbf{S}^1 \times \mathbf{T}^4 = \mathbf{R}^5 \times \mathbf{T}^5$ , and thence by heterotic - Type I duality to Type I on  $\mathbf{R}^5 \times \mathbf{T}^5$ .

On the other hand, Type IIA on  $\mathbf{R}^5 \times (\mathbf{S}^1)^5/\mathbf{Z}_2$  is an orientifold which is equivalent by T-duality to Type I on  $\mathbf{R}^5 \times \mathbf{T}^5$  [16,17].

So the prediction from our hypothesis about M-theory on  $(\mathbf{S}^1)^5/\mathbf{Z}_2$  – that Type IIA on  $(\mathbf{S}^1)^5/\mathbf{Z}_2$  should be equivalent to Type IIB on  $\mathbf{S}^1 \times \mathrm{K3}$  – is correct. This is a powerful test.

# Components Of The Moduli Space

What remains to be said? The strangest part of our discussion about M-theory on  $(\mathbf{S}^1)^5/\mathbf{Z}_2$  was the absence of a vacuum with symmetry among the fixed points. We would like to find a counterpart of this at the string theory level, for Type IIA on  $(\mathbf{S}^1)^5/\mathbf{Z}_2$ .

The Type IIA orientifold on  $(\mathbf{S}^1)^5/\mathbf{Z}_2$  needs – to cancel anomalies – 32 *D*-branes located at 32 points in  $(\mathbf{S}^1)^5$ ; moreover, this configuration of 32 points must be invariant under  $\mathbf{Z}_2$ . It is perfectly possible to place one *D*-brane at each of the 32 fixed points, maintaining the symmetry between them. Does this not contradict what we found in eleven dimensions?

The resolution of this puzzle starts by observing that the D-branes that are not at fixed points are paired by the  $\mathbb{Z}_2$ . So as the D-branes move around in a  $\mathbb{Z}_2$ -invariant fashion, the number of D-branes at each fixed point is conserved modulo two (if a D-brane approaches a fixed point, its mirror image does also). Thus, there is a  $\mathbb{Z}_2$ -valued invariant

associated with each fixed point; allowing for the fact that the total number of D-branes is even, there are 31 independent  $\mathbb{Z}_2$ 's.

What does this correspond to on the Type I side? A configuration of 32 D-branes on  $(\mathbf{S}^1)^5/\mathbf{Z}_2$  is T-dual to a Type I theory compactified on  $(\mathbf{S}^1)^5$  with a flat SO(32) bundle. However, the moduli space of flat SO(32) connections on the five-torus is not connected – there are many components. One component contains the trivial connection and leads when one considers the deformations to the familiar Narain moduli space of the heterotic string on the five-torus. This actually corresponds to a D-brane configuration with an even number of D-branes at each fixed point. The Wilson lines  $W_j$  can be simultaneously block-diagonalized, with 16 two-by-two blocks. The  $a^{th}$  block in  $W_j$  is

$$\begin{pmatrix} \cos \theta_{j,a} & \sin \theta_{j,a} \\ -\sin \theta_{j,a} & \cos \theta_{j,a} \end{pmatrix}, \tag{3.1}$$

with  $\theta_{j,a}$ , j = 1, ..., 5 being angular variables that determine the position on  $(\mathbf{S}^1)^5$  of the  $a^{th}$  *D*-brane (which also has an image whose coordinates are  $-\theta_{j,a}$ ).

There are many other components of the moduli space of flat connections on the fivetorus, corresponding to the 32  $\mathbb{Z}_2$ 's noted above. Another component – in a sense at the opposite extreme from the component that contains the trivial connection – is the following. Consider a flat connection with the properties that the  $W_j$  can all be simultaneously diagonalized, with eigenvalues  $\lambda_{j,a}=\pm 1,\ a=1,\ldots,32$ . Since the positions of the Dbranes are the phases of the eigenvalues of the  $W_j$ , this corresponds to a situation in which all D-branes are at fixed points. Pick the  $\lambda_{i,a}$  such that each of the 32 possible sequences of five  $\pm 1$ 's arises as  $\lambda_{i,a}$  for some value of a. Then there is precisely one D-brane at each of the 32 fixed points. This flat bundle – call if F – cannot be deformed as a flat bundle to the flat bundle with trivial connection; that is clear from the fact that the number of D-branes at fixed points is odd. Therefore, F does not appear on the usual Narain moduli space of toroidal compactification of the heterotic string to five dimensions. However, it can be shown that the bundle F is topologically trivial so that the flat connection on it can be deformed (but not via flat connections) to the trivial connection.<sup>4</sup> Thus, compactification using the bundle F is continuously connected to the usual toroidal compactification, but only by going through configurations that are not classical solutions.

 $<sup>^4</sup>$  In a previous draft of this paper, it was erroneously claimed that the bundle F was topologically non-trivial, with non-vanishing Stieffel-Whitney classes. The error was pointed out by E. Sharpe and some topological details were clarified by D. Freed.

The fact that the configuration with one D-brane at each fixed point is not on the usual component of the moduli space leads to a solution to our puzzle. In reconciling the two string theory descriptions of M-theory on  $\mathbf{R}^5 \times \mathbf{S}^1 \times (\mathbf{S}^1)^5/\mathbf{Z}_2$ , a key step was Type IIA - heterotic string duality relating Type IIA on  $\mathbf{R}^5 \times \mathbf{S}^1 \times \mathbf{K}^3$  to the heterotic string on  $\mathbf{R}^6 \times \mathbf{S}^1 \times (\mathbf{S}^1)^4 = \mathbf{R}^5 \times \mathbf{T}^5$ . This duality holds with the *standard* component of the moduli space on  $\mathbf{T}^5$ , so even though the symmetrical D-brane configuration exists, it is not relevant to our problem because it is related to a different component of the moduli space of flat SO(32) bundles.

Working on the Type IIA orientifold on  $(S^1)^5/\mathbb{Z}_2$  which is T-dual to a flat SO(32) bundle on the usual component of the moduli space means that the number of D-branes at each fixed point is even. With 32 D-branes and 32 fixed points, it is then impossible to treat symmetrically all fixed points. One can, however, pick any 16 fixed points, and place two D-branes at each of those, and none at the others. In the quantization, one then gets one five-dimensional vector multiplet from each fixed point that is endowed with a D-brane and none from the others.<sup>5</sup> Recalling that the vector multiplet is the dimensional reduction of the tensor multiplet from six to five dimensions, this result agrees with what we had in eleven dimensions: given any 16 of the 32 fixed points, there is a point in moduli space such that each of those 16 contributes precisely one matter multiplet, and the others contribute none.

It is possible that the absence of a vacuum with symmetrical treatment of all fixed points means that these theories cannot be strictly understood as orbifolds, but in any event, whatever the appropriate description is in eleven dimensional M-theory, we have found a precisely analogous behavior in the ten-dimensional Type IIA orientifold.

#### Other Similar Checks

One might wonder about other similar checks of the claim about M-theory on  $(\mathbf{S}^1)^5/\mathbf{Z}_2$ . One idea is to look at M-theory on  $\mathbf{R}^5 \times \mathbf{S}^1/\mathbf{Z}_2 \times (\mathbf{S}^1)^5/\mathbf{Z}_2$ . The idea would be that this should turn into an  $E_8 \times E_8$  heterotic string upon taking the  $\mathbf{S}^1/\mathbf{Z}_2$  small, and into a Type IIB orientifold on  $\mathbf{S}^1/\mathbf{Z}_2 \times K3$  if one shrinks the  $(\mathbf{S}^1)^5/\mathbf{Z}_2$ . However, because

This is most easily seen by perturbing to a situation in which the pair of D-branes is near but not at the fixed point. For orientifolds, there are no twisted sector states from a fixed point that does not have D-branes. After the  $\mathbb{Z}_2$  projection, a pair of D-branes in the orientifold produces the same spectrum as a single D-brane in an unorientifolded Type IIA, and this is a single vector multiplet, as explained in detail in section 2 of [18].

the two  $\mathbb{Z}_2$ 's do not commute in acting on spinors, it is hard to make any sense of this orbifold.

A similar idea is to look at M-theory on  $\mathbf{R}^4 \times \mathrm{K3} \times (\mathbf{S}^1)^5/\mathbf{Z}_2$ . When the last factor shrinks, this should become Type IIB on  $\mathrm{K3} \times \mathrm{K3}$ , while if the K3 factor shrinks then (allowing, as in a discussion that will appear elsewhere [19], for how the  $\mathbf{Z}_2$  orbifolding acts on the homology of K3) one gets the heterotic string on  $(\mathbf{S}^1)^8/\mathbf{Z}_2$ . These should therefore be equivalent. But one does not immediately have tools to verify or disprove that equivalence.

# Relation To Extended Gauge Symmetry And Non-Critical Strings

A rather different kind of check can be made by looking at the behavior when some D-branes – or eleven-dimensional five-branes – coincide.

Type IIA on K3 gets an extended SU(2) gauge symmetry when the K3 develops an  $A_1$  singularity.<sup>6</sup> This is not possible for Type IIB on K3, which has a chiral N=4supersymmetry that forbids vector multiplets. Rather, the weakly coupled Type IIB theory on a K3 that is developing an  $A_1$  singularity develops [21] a non-critical string (that is, a string that propagates in flat Minkowski space and does not have the graviton as one of its modes) that couples to the anti-self-dual part of one of the antisymmetric tensor fields (the part that is in a tensor multiplet, not in the supergravity multiplet).

This six-dimensional non-critical string theory is a perhaps rather surprising example, apparently, of a non-trivial quantum theory in six-dimensional Minkowski space. Recently, it was argued by Strominger [22] that by considering almost coincident parallel five-branes in eleven dimensions, one gets on the world-volume an alternative realization of the same six-dimensional non-critical string theory.

We can now (as partly anticipated by Strominger's remarks) close the circle and deduce from the relation between M-theory on  $\mathbf{T}^5/\mathbf{Z}_2$  and Type IIB on K3 why Type IIB on a K3 with an  $A_1$  singularity gives the same unusual low energy dynamics as two nearby parallel five-branes in eleven dimensions. This follows from the fact that in the map from M-theory on  $\mathbf{T}^5/\mathbf{Z}_2$  to Type IIB on K3, a configuration on  $\mathbf{T}^5/\mathbf{Z}_2$  with two coincident five-branes is mapped to a K3 with an  $A_1$  singularity. To see that these configurations are mapped to each other, it is enough to note that upon compactification on an extra circle

<sup>&</sup>lt;sup>6</sup> We really mean a quantum  $A_1$  singularity including a condition on a certain world-sheet theta angle [20].

of generic radius, they are precisely the configurations that give an enhanced SU(2). This may be deduced as follows:

- (1) M-theory on  $\mathbf{R}^5 \times \mathbf{S}^1 \times \mathbf{T}^5/\mathbf{Z}_2$  is equivalent to Type IIA on  $\mathbf{R}^5 \times \mathbf{T}^5/\mathbf{Z}_2$ , with the five-branes replaced by D-branes, and gets an enhanced SU(2) gauge symmetry precisely when two five-branes, or D-branes, meet. Indeed, when two D-branes meet, their  $U(1) \times U(1)$  gauge symmetry (a U(1) for each D-brane) is enhanced to U(2) (from the Chan-Paton factors of two coincident D-branes), or equivalently a U(1) is enhanced to SU(2).
- (2) Type IIB on  $\mathbf{R}^5 \times \mathbf{S}^1 \times \mathbf{K}3$  is equivalent to Type IIA on  $\mathbf{R}^5 \times \mathbf{S}^1 \times \mathbf{K}3$  and therefore because of the behavior of Type IIA on K3 the condition on the K3 moduli that causes a U(1) to be extended to SU(2) is precisely that there should be an  $A_1$  singularity.

# Other Orbifolds

Dasgupta and Mukhi also discussed M-theory orbifolds  $\mathbf{R}^{11-n} \times (\mathbf{S}^1)^n/\mathbf{Z}_2$ . The  $\mathbf{Z}_2$  action on the fermions multiplies them by the matrix  $\widetilde{\Gamma} = \Gamma^{11-n+1}\Gamma^{11-n+2}\dots\Gamma^{11}$ , and the orbifold can therefore only be defined if  $\widetilde{\Gamma}^2 = 1$  (and not -1), which restricts us to n congruent to 0 or 1 modulo 4.

The case n = 1 was discussed in [15], n = 4 gives a K3 orbifold, and n = 5 has been the subject of the present paper. The next cases are n = 8, 9. For n = 8, as there are no anomalies, it would take a different approach to learn about the massless states from fixed points. For n = 9, Dasgupta and Mukhi pointed out the beautiful fact that the number of fixed points  $-2^9 = 512$  – equals the number of left-moving massless fermions needed to cancel anomalies, and suggested that one such fermion comes from each fixed point. Since the left-moving fermions are singlets under the (chiral, right-moving) supersymmetry, this scenario is entirely compatible with the supersymmetry and is very plausible.

## Reduced Rank

Finally, let us note the following interesting application of part of the discussion above. Toroidal compactification of the heterotic (or Type I) string on a flat SO(32) bundle that is not on the usual component of the moduli space (being T-dual to a configuration with an odd number of D-branes at fixed points) gives an interesting and simple way to reduce the rank of the gauge group while maintaining the full supersymmetry. Since 2n + 1 D-branes at a fixed point gives gauge group SO(2n + 1), one can in this way get gauge groups that are not simply laced. Models with these properties have been constructed via free fermions [23] and as asymmetric orbifolds [24].

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