TWO DIMENSIONAL GAUGE THEORIES REVISITED

EDWARD WITTEN[★]

School of Natural Sciences

Institute for Advanced Study

Olden Lane

Princeton, NJ 08540

ABSTRACT

Two dimensional quantum Yang-Mills theory is reexamined using a non-abelian version of the Duistermaat-Heckman integration formula to carry out the functional integral. This makes it possible to explain properties of the theory that are inaccessible to standard methods and to obtain general expressions for intersection pairings on moduli spaces of flat connections on a two dimensional surface. The latter expressions agree, for gauge group $SU(2)$, with formulas obtained recently by several methods.

---

1. Introduction

This paper will be devoted to a renewed study of two dimensional Yang-Mills theory without matter, a system which can be easily solved [1] and has been extensively studied [2–10]. Yet we will see that there is still much to say about this supposedly "trivial" system. To state our result in a nutshell, we will explain (in §3) a simple mapping from topological Yang-Mills theory to physical Yang-Mills theory in two dimensions – analogous to the far more mysterious equivalence of topological and physical gravity in two dimensions [11–13].

What can be learned from this? To begin with the physics, as in the case of any gauge theory, one can attempt to expand the partition function $Z(\epsilon)$ of two dimensional Yang-Mills theory in powers of the gauge coupling constant ϵ. In doing so (in a suitable topological sector), one finds a remarkable result: the perturbation series in ϵ stops after finitely many terms, yet $Z(\epsilon)$ is not a polynomial. $Z(\epsilon)$ contains exponentially small terms which can be identified as contributions of *unstable* classical solutions to the functional integral. (A solution with n unstable modes is weighted with a phase of i^n – it turns out that n is always even.) Conventional physical methods are quite inadequate for explaining such behavior. It turns out that this can be done using the relation that we will find between physical and topological gauge theories or differently put using a generalization to problems with non-abelian symmetries of the exact integration formula of Duistermaat and Heckman [15]. We will devote §2-3 to an explanation of the necessary ideas, first from a mathematical standpoint in §2, and then more physically in §3. The argument in §2 uses an idea similar to that in a proof by Bismut [16] of the DH formula. In §4, we will apply our integration formula to two dimensional gauge theories, explaining the peculiar properties of the function $Z(\epsilon)$.

The usual Duistermaat-Heckman (DH) formula can be applied to problems with non-abelian group action [20], and it may well be that in finite dimensions, most of the applications of our formula can be deduced from the DH formula. Even if this is so, the formulation we give is natural in infinite dimensions, as should be

clear in §3-4.

Mathematically, it has been known [8,9,10] that $Z(\epsilon)$ can be expressed at $\epsilon = 0$ in terms of the volumes of the moduli spaces of flat connections on a surface; this is essentially a consequence of old work by A. Schwarz [14] and was used in [9] to obtain precise formulas for these volumes. Yet a study of the function $Z(\epsilon)$ has suggested that the relation of this function to the topology of the moduli spaces is not limited to $\epsilon = 0$. Coming to grips with this phenomenon is the goal of the present paper from a mathematical standpoint. This requires the nonabelian integration formulas of §2-3 and their application to infinite dimensional functional integrals in §4-5. The upshot will be precise and general formulas for intersection pairings on moduli spaces of flat connection, which are presented in §5.

Readers of this paper may want to bear in mind that §2 and §3 cover much the same ground and are independent to a large extent. Some readers might prefer to focus on §3 after just glancing at §2.

We now turn to a more extensive introduction to our subject.

Integration Formulas

The DH formula, which has many fascinating applications, governs the following situation. Let X be a $2n$ dimensional compact symplectic manifold, with symplectic form ω. Suppose that the group $U(1)$ acts symplectically on X, the action being generated by a vector field V. The action is said to be Hamiltonian if there is a function H on X such that $\mathrm{d}H = -j_V\omega.^{\star}$ The partition function of classical statistical mechanics, with this phase space and Hamiltonian, is

$$\int_X \frac{\omega^n}{n!} e^{-\beta H}. \tag{1.1}$$

The DH formula asserts that this integral is given exactly by the semi-classical

---

$\star$ I will use the symbol j_V to denote contraction with a vector field V, so in local coordinates x^i, if $\omega = \frac{1}{2}\omega_{ij}dx^i \wedge dx^j$, then $j_V\omega = V^i\omega_{ij}dx^j$. This operator is more usually written as i_V, but I want to avoid confusion with $i = \sqrt{-1}$. For a one form λ, I also write $j_V(\lambda)$ as $\lambda(V)$.

approximation, provided that one sums over all critical points of H. If the critical points are isolated points P_i, then the formula is

$$\int_X \frac{\omega^n}{n!} e^{-\beta H} = \sum_i \frac{e^{-\beta H(P_i)}}{\beta^n e(P_i)}, \qquad (1.2)$$

where $e(P_i)$ is the product of the weights of the circle action in the tangent space at P_i; this factor can be interpreted as the determinant arising from a Gaussian integral near P_i. (The sign of $e(P_i)$ is $(-1)^{n_i/2}$, where n_i, which is even because of the circle action, is the Morse index of P_i – as if each unstable mode contributes a factor of i to the integral.)

The DH formula is usually stated for oscillatory integrals – imaginary β – in which case (1.2) is the assertion of exactness of the stationary phase approximation (summed over critical points). In our applications, real β is more natural. A simple example of the application of the DH formula is given in the appendix.

The DH formula has a cohomological interpretation [17] which shows that the basic principle is not stationary phase but localization at the fixed points of the $U(1)$ action; in this form (1.2) can be generalized to a larger class of integrals.

The Non-Abelian Case

Suppose we are given the action on X not of $U(1)$ but of a compact, connected Lie group G, with Lie algebra $\mathcal{G}$. The action of G is said to be Hamiltonian if it is induced from a homomorphism $\widetilde{\mu} : \mathcal{G} \to \text{Fun}(X)$, where $\text{Fun}(X)$ is the space of smooth functions on X, regarded as a Lie algebra via the Poisson bracket. This amounts to saying that for every element T_a of $\mathcal{G}$, represented by a vector field V_a on X, there is a corresponding Hamiltonian function μ_a, with $j_{V_a}(\omega) = -\mathrm{d}\mu_a$, and with the map $T_a \to \mu_a$ being a homomorphism. The μ_a can be assembled into a map $\mu : X \to \mathcal{G}^*$ ($\mathcal{G}^*$ is the dual of $\mathcal{G}$).

In particular, we do not have a single G invariant function μ, but a collection of them; so we must modify (1.1). To this end, we introduce an invariant quadratic

4

form $(\ ,\ )$ on $\mathcal{G}$ and consider the integral

$$Z = \int_M \frac{\omega^n}{n!}\ \exp\left(-\frac{\beta}{2}(\mu, \mu)\right). \tag{1.3}$$

The critical point set of the function $I = (\mu, \mu)$ that appears here is very special, since according to Atiyah and Bott [18] and Kirwan [19], I is an equivariantly perfect Morse function. In this paper, we will use the critical points of this function in another way. We will see that there is an analog of the DH formula expressing Z as a sum of contributions of critical points of I. The general statement is of the following form. Let S be the set of components of the critical point set of I. For every component X_α of the critical point set, there is a function $Z_\alpha(\beta)$, determined by the local behavior of ω and μ near X_α up to some finite order, such that

$$Z(\beta) = \sum_{\alpha \in S} Z_\alpha(\beta). \tag{1.4}$$

The $Z_\alpha(\beta)$ can be very complicated functions of β. Transcendental functions such as the error function arise even in the simple abelian example treated in the appendix.

However, a simple contribution arises in one important special case, which will be the basis for our applications. The absolute minimum of I – which will give the dominant contribution in the important limit of $\beta \to \infty$ – is $\mu^{-1}(0)$. The quotient $\mathcal{M} = \mu^{-1}(0)/G$ is called the reduced phase space or symplectic quotient of X by G. $\mu^{-1}(0)/G$ is naturally a symplectic manifold with a symplectic form that we will also call ω. If $\mu^{-1}(0)$ is a smooth manifold, on which G acts freely, then the contribution of $\mu^{-1}(0)$ to Z is given by a simple cohomological formula that we will explain, essentially

$$\frac{1}{\beta^{\frac{1}{2}\dim G}} \cdot \int_{\mathcal{M}} \exp\left(\omega + \frac{1}{2\beta}\Theta\right) \tag{1.5}$$

where Θ is a certain element of $H^4(\mathcal{M}, \mathbb{R})$ that will appear in due course. Thus, Z differs from (1.5) only by terms that vanish exponentially for $\beta \to \infty$.

(1.5) is analogous to a cohomological formula of DH for the pushforward of $\omega^n/n!$ by the moment map. Like the DH formula, our formula for Z comes from a localization principle which applies to a larger class of integrals; in fact we will need the generalization.

Application To Gauge Theories

Now let us describe the infinite dimensional setting in which we will apply these considerations.

Let H be a compact, connected (but perhaps not simply connected) Lie group with Lie algebra $\mathcal{H}$. We will assume H is simple. The extension to more general compact H does not involve essentially new ideas.

For $H = SU(N)$, introduce a quadratic form $(\ ,\ )$ on $\mathcal{H}$ by

$$(a, b) = -\operatorname{Tr} ab, \tag{1.6}$$

where Tr is the trace in the N dimensional representation. This has the property that the fundamental integer-valued characteristic number of a rank N vector bundle E with structure group $SU(N)$ on a closed four-manifold X is

$$-\frac{1}{8\pi^2} \int_X \operatorname{Tr} F \wedge F, \tag{1.7}$$

where F is the curvature of a connection on E. (This integrality insures integrality of the symplectic form introduced presently.) For any simple connected H, we define $-\operatorname{Tr}$ to be a positive definite quadratic form on $\mathcal{H}$ such that (1.7) is the fundamental characteristic number of an $\widetilde{H}$ bundle over a a four-manifold, $\widetilde{H}$ being the universal cover of H.

Let Σ be an oriented closed Riemann surface of genus g. Let E be an H bundle over Σ (one can think in terms of a principal bundle or a vector bundle with a reduction of the structure group to H). The adjoint vector bundle associated

6

with E will be called $\mathrm{ad}(E)$. Let $\mathcal{A}$ be the space of connections on E. The space of connections is an affine space whose tangent space can be identified with $\Omega^1(\Sigma, \mathrm{ad}(E))$ (that is, the space of $\mathrm{ad}(E)$-valued one-forms on Σ). This being so, a symplectic form on $\mathcal{A}$ can be defined by

$$\omega(a, b) = \frac{1}{4\pi^2} \int_{\Sigma} \mathrm{Tr}(a \wedge b). \tag{1.8}$$

Let G be the group of gauge transformations on E. The Lie algebra $\mathcal{G}$ of G is the space of $\mathrm{ad}(E)$-valued zero-forms; the dual of the Lie algebra consists of $\mathrm{ad}(E)$-valued two-forms. G acts symplectically on $\mathcal{A}$, with a moment map given [18] by the map

$$\mu(A) = -\frac{F}{4\pi^2}, \tag{1.9}$$

from the connection A to its $\mathrm{ad}(E)$-valued curvature two-form $F = \mathrm{d}A + A \wedge A$.

$\mu^{-1}(0)$ therefore consists of flat connections, and $\mu^{-1}(0)/G$ is the moduli space $\mathcal{M}$ of flat connections on E up to gauge transformation. $\mathcal{M}$ is a component of the moduli space of homomorphisms $\rho : \pi_1(\Sigma) \to H$, up to conjugation.

Endow Σ with a measure μ of total area 1. This determines a metric or quadratic form $(a, a) = -\int_{\Sigma} d\mu \cdot \mathrm{Tr}\, a^2$ on $\mathcal{G}$; hence it determines on the dual of $\mathcal{G}$ a quadratic form which we can write as $(F, F) = -\int_{\Sigma} d\mu \, \mathrm{Tr}\, f^2$ where $f = \star F$. (Here $\star$ is the Hodge star operator; we recall that in two dimensions the $\star$ operator between two-forms and zero-forms depends only on a measure, not a metric.) The partition function of two dimensional quantum Yang-Mills theory on the surface Σ is formally given by the Feynman path integral

$$Z(\epsilon) = \frac{1}{\mathrm{vol}(G)} \int_{\mathcal{A}} DA \, \exp\left(-\frac{1}{2\epsilon}(F, F)\right), \tag{1.10}$$

where ϵ is a real constant, DA is the symplectic measure on the infinite dimensional function space $\mathcal{A}$, and $\mathrm{vol}(G)$ is the volume of G (determined formally from the volume form on G associated with the metric on $\mathcal{G}$).

Local considerations due to Migdal [1] can be adapted in various ways [2–10] to give a rather direct computation of (1.10) (and various closely related integrals). On the other hand, since (F, F) is the norm of the moment map with respect to an invariant metric on the Lie algebra, the integral (1.10) is precisely of the form of the integrals (1.3) governed by the new localization principle that we will present. Comparing the known $Z(\epsilon)$ to predictions of the localization principle, we will find full agreement for all properties that can be computed on both sides, including some surprising properties of $Z(\epsilon)$ that we mentioned at the outset.

The Cohomology Ring Of The Moduli Space

Mathematically, the payoff comes by looking closely at the contribution of $\mu^{-1}(0)$ to the critical point formula (1.4). This contribution can be extracted from the small ϵ behavior of $Z(\epsilon)$; on the other hand, it has an interpretation in terms of the topology of $\mathcal{M}$ that emerges in the proof of the critical point formula. Comparing these will give our main topological conclusions.

The contribution of $\mu^{-1}(0)$ to (1.4) will turn out to have a simple cohomological interpretation when the gauge group and bundle are such that $\mu^{-1}(0)$ is non-singular, and acted on freely by G. In practice, this occurs in genus ≥ 1 for $H = SU(N)/\Gamma$, with Γ a subgroup of the center of $SU(N)$, and certain bundles E. In the rest of this introduction, we consider only such cases. However, even when there are singularities, the fixed point theorem gives in principle a more sophisticated formula, deserving of study, for the contribution of $\mu^{-1}(0)$.

Explicit generators x_i for $H^*(\mathcal{M}, \mathbb{R})$ are known [21,18]. We will recall their definition in §3.3. By comparing the direct evaluation of $Z(\epsilon)$ to the predictions of the critical point formula, we will obtain in the nonsingular case explicit expressions for all of the quantities

$$\int_{\mathcal{M}} x_1 \wedge x_2 \wedge \ldots \wedge x_n. \tag{1.11}$$

This is tantamount to a determination of the cohomology ring of $\mathcal{M}$.

The intersection numbers (1.11) have been the subject of recent papers [22–24] giving a complete answer for $H = SO(3)$ and E a bundle of non-zero w_2. Here is a brief synopsis of the methods and comparison to the approach here.

Thaddeus [22] uses the Verlinde formula of conformal field theory [25–30]. Since not all the x_i appear in the Verlinde formula, Thaddeus supplements this with a geometrical argument to eliminate the extra classes in the case of $SO(3)$. The analogous step in our calculation is the integration over ψ and change of variables to reduce (5.11) to (5.14), thereby eliminating (for any H) those x_i that do not appear in the Verlinde formula. It would be interesting to know how Thaddeus's geometrical argument could be extended to other H. As for the main part of Thaddeus's paper, which is the use of the Verlinde formula, the analog of this in the present paper is the explicit evaluation of the Yang-Mills path integral (5.14) by a gluing method. The origins of the Verlinde formula (and most approaches to proving it, though [28] is an exception) involve analogous but more sophisticated gluing arguments.

Kirwan [23] uses considerations involving the geometry of the space of connections starting with the fact that I is an equivariantly perfect Morse function. She proves for $SO(3)$ (*i.e.*, rank two bundles of odd degree) the completeness of the Mumford relations among products of the x_i; modulo a knowledge of the volumes of the moduli spaces, this is equivalent to evaluation of (1.11). Her considerations, which in principle apply for any H, probably underlie our nonabelian localization formula.

Donaldson [24] uses a topological gluing construction ingeniously extracted from work on the Verlinde formula. This makes it possible to evaluate the pairings (1.11) by pure differential topology.

2. Non-Abelian Localization

2.1. EQUIVARIANT INTEGRATION

In this section, I will explain the non-abelian localization principle.

Let X be a compact closed manifold acted on by a compact connected Lie group G, with Lie algebra $\mathcal{G}$.

We must recall the de Rham model for the G-equivariant cohomology of X (see [17, pp. 10-13] and [31] for explanations), which we will take with complex coefficients. Let $\Omega^*(X)$ be the de Rham complex of X. Let $\text{Fun}(\mathcal{G})$ be the algebra of polynomial functions on $\mathcal{G}$, graded so that an nth order homogeneous polynomial is considered to be of degree $2n$. We will later consider various completions of $\text{Fun}(\mathcal{G})$. The desired complex is $\Omega^*_G(X) = (\Omega^*(X) \otimes \text{Fun}(\mathcal{G}))^G$, where ^{G} denotes the G-invariant part. (For a description of the same thing in the notation of physicists, see the beginning of §3.) An element of this complex is called an equivariant differential form.

The G action on X is determined by a homomorphism from $\mathcal{G}$ to the Lie algebra $\text{Vect}(X)$ of vector fields on X. Let us denote the vector field on X corresponding to $\phi \in \mathcal{G}$ as $V(\phi)$. One endows $\Omega^*_G(X)$ with the differential[*]

$$D = \text{d} - i j_{V(\phi)}. \tag{2.1}$$

We have

$$D^2 = -i\mathcal{L}_{V(\phi)}, \tag{2.2}$$

with $\mathcal{L}_{V(\phi)} = \text{d}j_{V(\phi)} + j_{V(\phi)}\text{d}$ the Lie derivative with respect to V_a. Thus, $D^2 = 0$ precisely on the G-invariant subspace $\Omega^*_G(X)$ of $\Omega^*(X) \otimes \text{Fun}(\mathcal{G})$. The cohomology of the D operator is called the G-equivariant cohomology of X, $H^*_G(X)$.

---

[*] The factor of $i = \sqrt{-1}$ in the last term here is usually omitted in mathematical papers, and plays no essential role, since it can be removed by conjugation. I include it so that later formulas will agree with standard physics conventions.

Now we want to introduce a notion of integration of equivariant differential forms. The operation usually considered is the pushforward $\Omega_G^*(X) \to \Omega_G^*(\text{pt})$ obtained by integration over X:

$$\alpha \to \int_X \alpha. \tag{2.3}$$

To show that this descends to a map $H_G^*(X) \to H_G^*(\text{pt})$, one must show that one can integrate by parts:

$$0 = \int_X D\beta. \tag{2.4}$$

This is true since $\int_X \mathrm{d}\beta = 0$ by ordinary integration by parts, and $\int_X j_{V(\phi)}\beta = 0$ since $j_{V(\phi)}\beta$ does not have a component which is a differential form of top dimension.

This, however, is not quite the integration operation that we want. As a vector space, $\mathcal{G}$ has a natural translation invariant measure, unique up to a constant factor. To fix that factor, note that as $\mathcal{G}$ is naturally isomorphic to the tangent space to G at the identity, a choice of Haar measure on G determines a measure on $\mathcal{G}$. Picking an arbitrary Haar measure on G, with total volume $\mathrm{vol}(G)$, let $\phi_1, \phi_2, \ldots, \phi_s$ be Euclidean coordinates on $\mathcal{G}$ such that the measure $\mathrm{d}\phi_1\mathrm{d}\phi_2\ldots\mathrm{d}\phi_s$ on $\mathcal{G}$ coincides with the chosen Haar measure at the identity of G. Then

$$\frac{1}{\mathrm{vol(G)}}\mathrm{d}\phi_1\mathrm{d}\phi_2\ldots\mathrm{d}\phi_s \tag{2.5}$$

is a natural measure on $\mathcal{G}$, independent of the chosen Haar measure on G.

The integration operation that we want is now roughly speaking the map $\Omega_G^*(X) \to \mathbb{C}$ given by

$$\alpha \to \frac{1}{\mathrm{vol}(G)} \int_{\mathcal{G}\times X} \frac{\mathrm{d}\phi_1\mathrm{d}\phi_2\ldots\mathrm{d}\phi_s}{(2\pi)^s} \cdot \alpha. \tag{2.6}$$

This definition is, however, unsatisfactory as the integral does not generally converge. If we complete $\mathrm{Fun}(\mathcal{G})$ to permit functions that are not necessarily polyno-

11

mials in ϕ, then (2.6) converges for a suitable class of α's. Since we want to allow a somewhat larger class, we introduce a convergence factor in the definition. Let ϵ be a positive real number. Let (,) be the positive definite invariant quadratic form on $\mathcal{G}$ described in the introduction. The definition that we want is then

$$\oint \alpha = \frac{1}{\text{vol}(G)} \int\limits_{\mathcal{G} \times X} \frac{\mathrm{d}\phi_1 \ldots \mathrm{d}\phi_s}{(2\pi)^s} \, \alpha \cdot \exp\left(-\frac{\epsilon}{2}(\phi, \phi)\right). \tag{2.7}$$

This of course converges for forms with polynomial dependence on ϕ. Later, we will work not just with polynomial forms but with forms that are permitted to have an exponential growth for large ϕ; (2.7) also converges in this larger class. We will call the operation in (2.7) equivariant integration.

The same argument as above shows that $\oint D\beta = 0$ for any β, so $\oint$ descends to a map from $H_G^*(X)$ to $\mathbb{C}$. In fact, this map is just the composition of ordinary integration over X with the map $H_G^*(\text{pt}) \to \mathbb{C}$ given by

$$\gamma \to \frac{1}{\text{vol}(G)} \int \frac{\mathrm{d}\phi_1 \mathrm{d}\phi_2 \ldots \mathrm{d}\phi_s}{(2\pi)^s} \cdot \gamma. \tag{2.8}$$

By mapping from $H_G^*(\text{pt})$ to $\mathbb{C}$ we have, in a sense, discarded most of the information. The discarded information can be recovered by considering not just $\oint \alpha$ but also $\oint \alpha \cdot Q(\phi)$ with Q an arbitrary G invariant polynomial on $\mathcal{G}$ (that is, an arbitrary element of $\Omega_G^*(\text{pt})$). Happily, the localization principle to which we presently turn applies to all of these integrals.

<u>G Action On A Point</u>

In general, the $\oint$ operation diverges as $\epsilon \to 0$. For instance, in the basic case $X = $ a point, $\alpha = 1$, we have

$$\oint\limits_{\text{pt}} 1 = \frac{1}{\text{vol}(G) \cdot (2\pi\epsilon)^{s/2}}. \tag{2.9}$$

This is an important formula which is the simplest illustration of the relation of singularities of the G action to singularities of equivariant integrals as functions of ϵ.

12

2.2. The Localization Principle

Now we will explain the localization principle. Let α be any equivariantly closed form. Then for any real number t and any $\lambda \in \Omega_G^*(X)$ whose dependence on ϕ is sufficiently mild,

$$\oint_X \alpha = \oint_X \alpha \, e^{tD\lambda}, \tag{2.10}$$

since $\alpha \left(1 - e^{tD\lambda}\right)$ can be written as $D(\ldots)$ using $D\alpha = D^2 = 0$. A restriction on the ϕ dependence of λ must be imposed here to ensure the convergence of the right hand side of (2.10) and to justify the integration by parts that is involved in proving (2.10). We will consider only the case that λ is independent of ϕ. In fact, we will suppose that λ is a G-invariant one-form.

Pick an orthonormal basis T_a of $\mathcal{G}$, and write $V(\phi) = \sum_a \phi^a V_a$, where V_a is the vector field on M representing T_a, and the ϕ^a are linear functions on $\mathcal{G}$. Then (2.10) can be written out explicitly as

$$\oint_X \alpha = \frac{1}{\text{vol}(G)} \int \frac{d\phi_1 d\phi_2 \ldots d\phi_s}{(2\pi)^s} \alpha \cdot \exp\left(t \, d\lambda - it \sum_a \phi^a \lambda(V_a) - \frac{\epsilon}{2} \sum_a (\phi^a)^2\right). \tag{2.11}$$

If we suppose that α is independent of ϕ, then we can perform the Gaussian ϕ integral, to get

$$\oint_X \alpha = \frac{1}{\text{vol}(G) \cdot (2\pi\epsilon)^{s/2}} \int_X \alpha \cdot \exp\left(t \, d\lambda - \frac{t^2}{2\epsilon} \sum_a (\lambda(V_a))^2\right). \tag{2.12}$$

The crucial factor in (2.12) is the last exponential factor. Let X' be the subspace of X on which

$$\lambda(V_a) = 0, \quad a = 1, \ldots, s. \tag{2.13}$$

Write

$$X' = \cup_{\sigma \in S} X_\sigma, \tag{2.14}$$

13

where X_σ are the connected components of X', and S is the set of such components. Let W be a compact subset of X with $W \cap X' = \emptyset$. We know that (2.12) is independent of t. On the other hand, the integral over W of the right hand side of (2.12) vanishes for $t \to \infty$ as $\exp(-ct^2)$ for some positive constant c. Let Z_σ be the integral of the right hand side of (2.12) over a tubular neighborhood of X_σ, in the limit $t \to \infty$. Z_σ is determined by the local behavior of α and the G action near X_σ up to some finite order. (In the case that Z_σ is a point, this is a consequence of theorem 7.6 of [32].) And taking the large t limit of (2.12), we get an expression

$$\oint_X \alpha = \sum_{\sigma \in S} Z_\sigma. \tag{2.15}$$

for $\oint_X \alpha$ as a sum of local contributions.

So far we have assumed that α is independent of ϕ. It would not be different if α has a polynomial or even exponential dependence on ϕ. The ϕ integral in (2.11) would then contribute an extra polynomial or exponential t dependence on the right hand side of (2.12), too weak to affect the localization, which was determined by a factor $\exp(-ct^2)$. The detailed computation of Z_σ in the case that α has exponential ϕ dependence requires care and will be considered later in a special situation.

Stationary Phase

The argument leading to localization can actually be formulated without performing the ϕ integral. Looking back to (2.11), we see that apart from a polynomial in t that comes from the expansion of $\exp(t \, d\lambda)$, the t dependence appears entirely in a factor $\exp(-itK)$, where K is the function

$$K = \sum_a \phi^a \lambda(V_a) \tag{2.16}$$

on $X \times \mathcal{G}$. The method of stationary phase (expounded, for instance, in [32], chapter 7) enables one to compute the large t behavior of such an integral in terms

14

of local data on the critical point set of K. (The convergence factor $\exp(-\frac{\epsilon}{2}(\phi,\phi))$ in our integral ensures that there are no essential problems coming from the lack of compactness of $X \times \mathcal{G}$ and the possible lack of compactness of the critical point set.)

The critical point condition $dK = 0$ gives two conditions. Varying with respect to ϕ, we get the familiar equation

$$\lambda(V_a) = 0, \quad a = 1 \ldots s. \tag{2.17}$$

Varying with respect to the coordinates of X, we get

$$\phi^a \, d\left(\lambda(V_a)\right) = 0. \tag{2.18}$$

The ϕ^a take values in a vector space, which is contractible to the origin by scaling; and the equations are invariant under this scaling. So the homotopy type of the space of solutions would not be changed if we restrict to $\phi = 0$, and in particular the components $\widetilde{X}_\sigma$ of the critical set of K are in one to one correspondence with the components X_σ of solutions of (2.17). The contribution of $\widetilde{X}_\sigma$ to the large t evaluation of (2.11) coincides with the contribution of X_σ to the large t evaluation of (2.12), since it reduces to the latter upon performing the ϕ integral.

Because of the invariance under scaling of ϕ, $\widetilde{X}_\sigma$ is compact when and only when $\widetilde{X}_\sigma = X_\sigma$; this is so precisely when (2.18) implies that $\phi^a = 0$. When this is so, compactness of $\widetilde{X}_\sigma$ means that the convergence factor $\exp(-\frac{\epsilon}{2}(\phi,\phi))$ is not needed to make sense of the stationary phase integration; and therefore the contribution Z_σ of $\widetilde{X}_\sigma$ to the integral has a limit as $\epsilon \to 0$. In fact, general principles of stationary phase integration assert that Z_σ depends only on the behavior of the integrand in (2.11) up to finite order near the critical locus $\widetilde{X}_\sigma$, and so when $\widetilde{X}_\sigma$ is supported at $\phi^a = 0$, Z_σ is a polynomial in ϵ.

In case G acts freely on X_σ, one can make this much more explicit. Under this condition, $H_G^*(X_\sigma)$ is naturally isomorphic to $H^*(X_\sigma/G)$. Let Y be an equivariant

15

tubular neighborhood of X_σ in $X \times \mathcal{G}$. Y is equivariantly contractible to X_σ, so $H_G^*(Y) \cong H_G^*(X_\sigma) \cong H^*(X_\sigma/G)$. If $\pi : Y \to X_\sigma/G$ is the composite of an equivariant retraction $Y \to X_\sigma$ and the natural projection $X_\sigma \to X_\sigma/G$, then the natural isomorphism of $H^*(X_\sigma/G)$ with $H_G^*(Y)$ is simply the pullback π^* : $H^*(X_\sigma/G) \to H_G^*(Y)$. In particular, the element $-(\phi,\phi)/2 \in H_G^4(X)$, when restricted to Y, is $\pi^*(\Theta)$ for some $\Theta \in H^4(X_\sigma/G)$. (In fact, Θ is a characteristic class of $\mu^{-1}(0)$, regarded as a principal G bundle over $\mu^{-1}(0)/G$.) At the level of equivariant forms,

$$-\frac{(\phi,\phi)}{2} = \pi^*(\Theta) + Dw, \qquad (2.19)$$

for some $w \in \Omega_G^3(Y)$.

The fact that $X_\sigma = \widetilde{X}_\sigma$ means that the contribution of X_σ to $\oint \alpha$ can be evaluated by stationary phase evaluation of the integral

$$\frac{1}{\mathrm{vol}(G)} \int \frac{\mathrm{d}\phi_1 \mathrm{d}\phi_2 \ldots \mathrm{d}\phi_s}{(2\pi)^s} \alpha \cdot \exp\left(t \, \mathrm{d}\lambda - it \sum_a \phi^a \lambda(V_a) - \frac{\epsilon}{2} \sum_a (\phi^a)^2\right) \cdot u, \quad (2.20)$$

where u is a smooth G-invariant function that is 1 in an equivariant neighborhood of X_σ and zero outside Y. In using (2.19), one can integrate by parts and discard the $\mathrm{d}u$ term, since there are no critical points of K where $\mathrm{d}u \neq 0$. So in evaluating the contribution Z_σ of X_σ, we can make the substitution

$$\exp\left(-\frac{\epsilon}{2}(\phi,\phi)\right) \to \exp(\epsilon\Theta). \qquad (2.21)$$

Among other things, this makes it clear that Z_σ is a polynomial in ϵ of order at most $\frac{1}{4}\dim(X_\sigma/G)$.

Moreover, the isomorphism $H_G^*(Y) \cong H^*(X_\sigma/G)$ means that any $\alpha \in H_G^*(X)$ that we may be trying to integrate, when restricted to a neighborhood of X_σ, is

16

the pullback of some $\alpha' \in H^*(X_\sigma/G)$. So we can make the substitution

$$\alpha \rightarrow \alpha' \tag{2.22}$$

in the integral. We will later find an important situation to which these considerations apply.

Derivation Of The DH Formula

For clarity, we will now recall [16] how to obtain the DH formula in a similar fashion. So we assume $G = U(1)$. The differential in the de Rham model of $H_G^*(X)$ is now $D = \mathrm{d} - i\phi j_V$, where V is a vector field generating the G action, and ϕ is a linear function on the one dimensional Lie algebra of G. We "localize" algebraically by setting $\phi = i$. An equivariant form is then simply a differential form α obeying $(\mathrm{d} + j_V)\alpha = 0$. By integration of such a form we simply mean integration over X in the usual sense.

We have

$$\int_X \alpha = \int_X \alpha \cdot \exp(tD\lambda), \tag{2.23}$$

for any G-invariant λ. To pick a suitable λ, let g be a G-invariant Riemannian metric on X, and let λ be the one-form $\lambda = -g(V, \cdot)$. Then (2.23) amounts to

$$\int_X \alpha = \int_X \alpha \cdot \exp(-t \, \mathrm{d}\lambda - tg(V,V)). \tag{2.24}$$

So taking $t \rightarrow \infty$, we get a localization at the zeros of $g(V,V)$ or in other words at the zeros of V. At an isolated zero P of V, the Hessian of $g(V,V)$ is non-degenerate, so the large t limit can be evaluated by Gaussian integration. In this way, one gets the contribution of P to the generalized DH formula of [17, eqn. (3.8)].

17

2.3. The Symplectic Case

We now want to elucidate the meaning of the localization formula (2.13) in the following important case. We suppose that X is a symplectic manifold, with symplectic form ω, and that the G action on X has a moment map μ. We pick on X an almost complex structure J such that ω is of type $(1,1)$ and positive. Positivity means that the metric $g(\cdot, \cdot)$ defined by

$$g(u, v) = \omega(u, Jv) \tag{2.25}$$

is positive definite. Such a J always exists (and is unique up to homotopy) because the Siegel upper half plane is contractible. Set $I = (\mu, \mu)$ and

$$\lambda = \frac{1}{2} J(dI). \tag{2.26}$$

(So in components, $\lambda = \frac{1}{2} J^i{}_j \partial_i I dx^j$.)

Obviously, at the critical points of I, $\lambda = 0$ and hence $\lambda(V_a) = 0$. We want to prove the converse. Let $Y = \sum_a \mu_a V_a$. The moment equation $d\mu_a = -j_{V_a}(\omega)$ implies that $Y = \frac{1}{2}\omega^{-1} dI$, where ω^{-1} (which in components is the inverse matrix to ω, so $(\omega^{-1})^{ik}\omega_{kj} = \delta^i_j$) is regarded as a map $T^*X \to TX$. $\lambda(V_a) = 0$ implies $\lambda(Y) = 0$ or

$$\omega^{-1}(dI, JdI) = 0. \tag{2.27}$$

Positivity of (2.25) means that (2.27) holds only for $dI = 0$. Thus, in the case of symplectic manifolds, with our choice of λ, the localization principle is a reduction to the critical points of $I = (\mu, \mu)$.

G Action On T^*G

We now consider the basic case of $X = T^*G$, with the natural symplectic structure on T^*G and the G action on T^*G induced from the right action on G. The localization will be on $G \subset T^*G$. As T^*G is not compact, we must define the

$\oint$ operation by the right hand side of (2.12), insisting on $t \neq 0$. It is really the local behavior near $G \subset T^*G$ that matters, and the importance of the calculation is that it can be applied whenever the critical point set of I has a component that can be modeled on $G \subset T^*G$.

By going to a basis of right-invariant one forms, T^*G can be identified with $G \times \mathcal{G}$. In terms of $g \times \gamma \in G \times \mathcal{G}$, the symplectic form can be written

$$
\begin{aligned}
\omega &= (\mathrm{d}\gamma, \mathrm{d}gg^{-1}) + (\gamma, \mathrm{d}gg^{-1}\mathrm{d}gg^{-1}) \\
&= (\mathrm{d}\gamma + \frac{1}{2}[\gamma, \mathrm{d}gg^{-1}], \mathrm{d}gg^{-1}).
\end{aligned}
\tag{2.28}
$$

The vector field V associated with $a \in \mathcal{G}$ can be described by the formulas

$$
\delta g = -ga, \qquad \delta \gamma = 0. \tag{2.29}
$$

A small computation gives $j_V(\omega) = \mathrm{d}(\gamma, gag^{-1})$, so the moment map is $\mu(a) = -(\gamma, gag^{-1})$. The square of the moment map $I = (\mu, \mu)$ is therefore $I = (\gamma, \gamma)$. An almost complex structure on T^*G can be defined by the formulas $J(\theta) = \eta$, $J(\eta) = -\theta$, with

$$
\begin{aligned}
\theta &= \mathrm{d}\gamma + \frac{1}{2}[\gamma, \mathrm{d}gg^{-1}] \\
\eta &= \mathrm{d}gg^{-1}.
\end{aligned}
\tag{2.30}
$$

It is evident from the second description of ω in (2.28) that ω is positive and of type $(1,1)$. One now computes that $\lambda = \frac{1}{2}J(\mathrm{d}I)$ is

$$
\lambda = (\gamma, \mathrm{d}gg^{-1}). \tag{2.31}
$$

G acts freely on T^*G; the quotient can be identified with $\mathcal{G}$ via the projection $G \times \mathcal{G} \to \mathcal{G}$. So the equivariant cohomology of T^*G coincides with the ordinary cohomology of $\mathcal{G}$, and vanishes except in dimension zero, where it is represented

19

by the constants. The only integral that we really have to consider is therefore $\oint_{T^*G} 1$. This is

$$\oint_{T^*G} 1 = \frac{1}{\text{vol}(G)} \int \frac{d\phi_1 \ldots d\phi_s}{(2\pi)^s} \int_{T^*G} \exp\left(tD\lambda - \frac{\epsilon}{2}(\phi,\phi)\right). \qquad (2.32)$$

Since the integral is independent of t for $t \neq 0$, we can set $t = 1$. Working out the top form component of $D\lambda$ explicitly, and changing variables from ϕ to $g^{-1}\phi g$, we get

$$\oint_{T^*G} 1 = \frac{1}{\text{vol}(G)} \int \frac{d\phi_1 \ldots d\phi_s}{(2\pi)^s} \int_{T^*G} \exp\left(-i(\gamma,\phi) - \frac{\epsilon}{2}(\phi,\phi)\right) \frac{(d\gamma, dgg^{-1})^n}{n!}. \qquad (2.33)$$

(Note that ω^n coincides with $(d\gamma, dgg^{-1})^n$.) The γ integral can be done using

$$\int_{-\infty}^{\infty} du\, e^{-iuv} = 2\pi\delta(v). \qquad (2.34)$$

The ϕ integral can then be done using the delta functions. Notice that the ϵ dependence disappears at this point (a consequence of (2.21)). The integral over g cancels the factor of $\text{vol}(G)$. Assuming that the orientation of T^*G is chosen to agree with the one determined by the symplectic structure, the result is just

$$\oint_{T^*G} 1 = 1. \qquad (2.35)$$

That this is independent of ϵ is a consequence of the fact that for $X = T^*G$, (2.18) implies $\phi = 0$ (this is shown more generally in (2.45) below). Hence we could alternatively have proceeded as follows: set ϵ to zero in (2.33); evaluate the ϕ integral using (2.34); evaluate the γ integral using the resulting delta functions.

G action on $T^*(G/H)$

We will now briefly describe the extension of this to the case that $X = T^*(G/H)$, with H a subgroup of G of dimension r and Lie algebra $\mathcal{H} \subset \mathcal{G}$, and with the natural symplectic structure on $T^*(G/H)$.[★] The restriction of $(\, , \,)$ to $\mathcal{H}$ is an invariant quadratic form on $\mathcal{H}$ that we will denote by the same symbol. We have already carried out equivariant integration over $T^*(G/H)$ for $H = G$ and $H = \{1\}$ in (2.9) and (2.35) respectively. The general case turns out to be a natural combination of these.

$T^*(G/H)$ can be parametrized by pairs $g \times \gamma \in G \times \mathcal{G}$, with an equivalence relation $g \times \gamma \cong hg \times h\gamma h^{-1}$ for $h \in H$, and a constraint $(\gamma, b) = 0$, for $b \in \mathcal{H}$. The symplectic structure and the G action can still be described by (2.28) and (2.29). The moment map for $a \in \mathcal{G}$ is still $\mu = -(\gamma, gag^{-1})$, and its square is still $I = (\mu, \mu) = (\gamma, \gamma)$.

Let b_i be an orthonormal basis of $\mathcal{H}$, let Π be the orthogonal projection onto the complement $\mathcal{H}_\perp$ of $\mathcal{H}$, and introduce the $\mathcal{H}_\perp$-valued one-forms

$$\theta = \Pi \left(\mathrm{d}\gamma + \frac{1}{2}[\gamma, \mathrm{d}gg^{-1}] + \frac{1}{2}\sum_i [\gamma, b_i](b_i, \mathrm{d}gg^{-1}) \right) \tag{2.36}$$
$$\eta = \Pi \left(\mathrm{d}gg^{-1} \right).$$

An almost complex structure J such that ω is positive and of type $(1,1)$ can be defined as before by the formulas $J(\theta) = \eta, \ \ J(\eta) = -\theta$.

Now we introduce the usual G-invariant one-form

$$\lambda = \frac{1}{2}J\mathrm{d}\mu^2 = \frac{1}{2}J\mathrm{d}(\gamma, \gamma) = (\gamma, \mathrm{d}gg^{-1}). \tag{2.37}$$

So

$$D\lambda = (\mathrm{d}\gamma, \mathrm{d}gg^{-1}) + (\gamma, \mathrm{d}gg^{-1}\mathrm{d}gg^{-1}) + i(\gamma, g\phi g^{-1}) \tag{2.38}$$

Let α be an arbitrary element of $H_G^*(T^*(G/H))$. Since $T^*(G/H)$ has an equiv-

---

★ This case is generally less important in our applications, but will arise when we consider gauge theory in genus zero.

ariant retraction to G/H, we can represent α by the pullback of an element of $\Omega_G^*(G/H)$, which we will also call α. We want to compute

$$\oint_{T^*(G/H)} \alpha = \frac{1}{\text{vol}(G)} \int \frac{\mathrm{d}\phi_1 \ldots \mathrm{d}\phi_s}{(2\pi)^s} \int_{T^*(G/H)} \alpha \cdot \exp\left(tD\lambda - \frac{\epsilon}{2}(\phi,\phi)\right). \quad (2.39)$$

Using (2.38), setting t to 1, extracting the top form component, and replacing ϕ by $g^{-1}\phi g$, this is

$$\oint_{T^*(G/H)} \alpha = \frac{1}{\text{vol}(G)} \int \frac{\mathrm{d}\phi_1 \ldots \mathrm{d}\phi_s}{(2\pi)^s} \cdot \frac{(\mathrm{d}\gamma, \mathrm{d}gg^{-1})^s}{s!} \alpha \exp\left(i(\gamma,\phi) - \frac{\epsilon}{2}(\phi,\phi)\right). \quad (2.40)$$

α, being a pullback from G/H, is independent of γ. If, therefore, we write $\phi = \phi' + \phi_\perp$, with $\phi' \in \mathcal{H}$ and $\phi_\perp \in \mathcal{H}_\perp$, then the γ integral gives $(2\pi)^r \delta^r(\phi_\perp)$, with the aid of which the $\phi_\perp$ integral can be done. It remains to integrate over ϕ' and g.

Restricted to a given point in G/H, say the coset of $1 \in G$, α reduces to an H-invariant polynomial $\alpha'(\phi')$. (The map $\alpha \leftrightarrow \alpha'$ is a bijection in cohomology and is the natural isomorphism $H_G^*(G/H) \cong H_H^*(\text{pt})$.) The ϕ' integral gives a factor

$$\int \frac{\mathrm{d}\phi'_1 \ldots \mathrm{d}\phi'_{s-r}}{(2\pi)^{s-r}} \cdot \alpha'(\phi') \cdot \exp(-\frac{\epsilon}{2}(\phi',\phi')). \quad (2.41)$$

The g integral gives a factor of $\text{vol}(G/H) = \text{vol}(G)/\text{vol}(H)$. (The volumes are computed using the measures on G, H, G/H induced from the quadratic form $(\,,\,)$ on $\mathcal{G}$ and $\mathcal{H}$.) Combining the pieces, and assuming that $T^*(G/H)$ is given the orientation compatible with its symplectic structure, we get

$$\oint_{T^*(G/H)} \alpha = \frac{1}{\text{vol}(H)} \int \frac{\mathrm{d}\phi'_1 \ldots \mathrm{d}\phi'_{s-r}}{(2\pi)^{s-r}} \cdot \alpha'(\phi') \cdot \exp\left(-\frac{\epsilon}{2}(\phi',\phi')\right), \quad (2.42)$$

which shows that G-equivariant integration on $T^*(G/H)$ reduces to H-equivariant integration on a point.

22

2.4. More On The Symplectic Case

We now want to make a more intensive study of equivariant integration on a symplectic manifold X with symplectic form ω. We have seen that this reduces to a sum over critical points of the function $I = (\mu, \mu)$. The absolute minimum of that function is $\mu^{-1}(0)$. Assume first that $\mu^{-1}(0)$ is a smooth manifold, on which G acts freely, so that the quotient $\mathcal{M} = \mu^{-1}(0)/G$ is a smooth manifold with a naturally induced symplectic structure, which we will also call ω.

We want to compute the contribution of $\mu^{-1}(0)$ to

$$\oint_X \alpha = \frac{1}{\mathrm{vol}(G)} \int \frac{\mathrm{d}\phi_1 \dots \mathrm{d}\phi_s}{(2\pi)^s} \int_X \alpha \cdot \exp\left(tD\lambda - \frac{\epsilon}{2}(\phi, \phi)\right). \tag{2.43}$$

This reason that this is simple is that we can use (2.21) and (2.22). We recall that the criteria for the validity of (2.21) were that G should act freely on the critical point set, and that

$$\sum_a \phi^a \, \mathrm{d}\left(\lambda(V_a)\right) = 0 \tag{2.44}$$

should imply $\phi^a = 0$. A small computation shows that $\lambda(V_a) = \sum_b g(V_a, V_b) \cdot \mu_b$ where b runs over an orthonormal basis of $\mathcal{G}$, and $g(\cdot, \cdot)$ is the metric (2.25), which is positive definite by the choice of J. Wherever G acts freely, the V_a are linearly independent, and $g(V_a, V_b)$ is therefore a positive definite metric on $\mathcal{G}$ (but not necessarily G-invariant, of course, as we are not at a fixed point of G). At $\mu = 0$, (2.44) reduces to

$$\sum_a \phi^a \cdot \sum_b g(V_a, V_b) \cdot \mathrm{d}\mu_b = 0. \tag{2.45}$$

Since $g(V_a, V_b)$ is invertible, and the $\mathrm{d}\mu_b$ are linearly independent wherever G acts freely (since $\mathrm{d}\mu_b = -j_{V_b}\omega$), the coefficients of the ϕ^a in (2.45) are linearly independent on $\mu^{-1}(0)$, and hence also in a neighborhood thereof. So, as desired, (2.44) implies $\phi^a = 0$, and we can use (2.21) and (2.22).

Thus, let Y be an equivariant tubular neighborhood of $\mu^{-1}(0)$. Pick an equivariant retraction $Y \to \mu^{-1}(0)$. Composing this with the natural projection $\psi : \mu^{-1}(0) \to \mu^{-1}(0)/G$, one gets an equivariant projection $\pi : Y \to \mu^{-1}(0)/G$. The elements $-(\phi, \phi)/2$ and α of $H_G^*(Y)$ are the pullbacks by π of some classes Θ, α' in $H^*(\mu^{-1}(0)/G)$. (2.21) and (2.22) mean that in evaluating the contribution $Z(\mu^{-1}(0))$ of $\mu^{-1}(0)$ to $\oint_X \alpha$, we can write

$$Z(\mu^{-1}(0)) = \frac{1}{\mathrm{vol}(G)} \int \frac{\mathrm{d}\phi_1 \ldots \mathrm{d}\phi_s}{(2\pi)^s} \int_Y \alpha' \cdot \exp\left(tD\lambda + \epsilon\Theta\right). \qquad (2.46)$$

To study this integral, first integrate over the fibers of π. Everything in (2.46) is a pullback via π except $\exp(tD\lambda)$. Hence we must evaluate

$$\int \frac{\mathrm{d}\phi_1 \ldots \mathrm{d}\phi_s}{(2\pi)^s} \int_{\pi^{-1}(\mathrm{pt})} \exp\left(tD\lambda\right). \qquad (2.47)$$

Now, $\pi^{-1}(\mathrm{pt})$ is fibered over $G \cong \psi^{-1}(\mathrm{pt})$, and the G action on $\pi^{-1}(\mathrm{pt})$ can be modeled on a neighborhood of $G \subset T^*G$ (if Y and π are constructed as explained in [33, Theorem 39.2 and Proposition 40.1]). Hence, the large t limit of (2.47) is 1, using (2.35). For instance, one can do the calculation as in the comment after (2.35), integrating first over ϕ in (2.47) to produce a delta function supported on $\mu^{-1}(0)$. So integrating over the fibers of $\mu^{-1}(0) \to \mu^{-1}(0)/G$, we get the very simple result

$$Z(\mu^{-1}(0)) = \int_{\mu^{-1}(0)/G} \alpha' \cdot \exp(\epsilon\Theta). \qquad (2.48)$$

This formula is a major ingredient in our applications.

Higher Critical Points

We also want to say something about the contributions of the higher critical points of I in equivariant integration over X. At this point, we must specify what sort of equivariant forms we wish to integrate. Whenever one has a Hamiltonian

group action on a symplectic manifold, a basic equivariant differential form, exploited in [17], is the form of degree two $\overline{\varpi} = \omega - i \sum_a \phi^a \mu_a$. It can be regarded as an equivariant extension of ω. For α we will take

$$\alpha = \exp(\overline{\varpi}) \cdot \beta, \tag{2.49}$$

where we will require that β has only a polynomial dependence on ϕ. This means that the integrals we wish to calculate are of the general type (setting $\beta = 1$ for simplicity)

$$\oint \alpha = \frac{1}{\text{vol}(G)} \int \frac{d\phi_1 \ldots d\phi_s}{2\pi^s} \exp\left(\omega - i \sum_a \phi^a \mu_a - \frac{\epsilon}{2}(\phi, \phi)\right) \tag{2.50}$$

After performing the ϕ integral, this is

$$\oint \alpha = \frac{1}{\text{vol}(G) \cdot (2\pi\epsilon)^{s/2}} \cdot \int_X \frac{\omega^n}{n!} \exp\left(-\frac{I}{2\epsilon}\right), \tag{2.51}$$

with $I = (\mu, \mu)$. We note that, apart from some elementary factors, this is the integral (1.3) discussed in the introduction.

Let X_σ be a component of the critical set, and Y an equivariant tubular neighborhood of X_σ. We want to estimate, for small ϵ, the contribution $Z(X_\sigma)$ of X_σ to (2.51) (and its generalization with $\beta \neq 1$). In doing so we will assume that X_σ is a nondegenerate critical locus in the extended sense of Bott – that it is a smooth manifold and that the Hessian of I is invertible in the directions normal to X_σ. Naively, one would expect that, if there is any sort of representation of (2.51) as a sum over critical points, the contribution of X_σ for small ϵ should be of order

$$Z(X_\sigma) \sim \exp\left(-\frac{I(X_\sigma)}{2\epsilon}\right), \tag{2.52}$$

(where $I(X_\sigma)$ is the constant value of I on X_σ) up to a power of ϵ. This is so; we will see how this behavior emerges from the general localization procedure. To begin with, we consider the case $\beta = 1$.

$Z(X_\sigma)$ is the large t limit of

$$Z(X_\sigma; t) = \frac{1}{\text{vol}(G)} \int \frac{\mathrm{d}\phi_1 \ldots \mathrm{d}\phi_s}{(2\pi)^s} \int_Y \exp\left(\omega + t\,\mathrm{d}\lambda - i\sum_a \phi_a(\mu_a + t\lambda(V_a)) - \frac{\epsilon}{2}(\phi, \phi)\right).$$

(2.53)

Upon performing the Gaussian integral over ϕ, this becomes

$$\frac{1}{(2\pi\epsilon)^{\frac{s}{2}}} \int_Y \exp\left(\omega + t\,\mathrm{d}\lambda - \frac{1}{2\epsilon}W\right).$$

(2.54)

where

$$W = \sum_a (\mu_a + t\lambda(V_a))^2 = \sum_a (\mu_a + t\sum_b g(V_a, V_b)\mu_b)^2.$$

(2.55)

Restricted to the critical component X_σ of $I = (\mu, \mu)$, W is equal to the constant $I(X_\sigma)$, since $\sum_b g(V_a, V_b)\mu_b = 0$ on critical points. We will show momentarily that for $t >> 0$, X_σ is a local minimum of W (even though it may be an unstable critical set of I), and that the Hessian in the normal directions is positive definite and of order t. The integral in (2.54) is sharply peaked around this minimum, so the large t behavior of (2.54) is determined by local behavior near the X_σ.[★] The large t limit of (2.54) therefore vanishes exponentially for $\epsilon \to 0$ as $\exp\left(-W(X_\sigma)/2\epsilon\right) = \exp\left(-I(X_\sigma)/2\epsilon\right)$. This is the exponential behavior suggested intuitively in (2.52). The same exponential would arise if we consider not $\oint \exp(\overline{\omega})$, but $\oint \exp(\overline{\omega}) \cdot \beta$, with β an arbitrary equivariant form with a polynomial dependence on ϕ. The introduction of β would not modify the exponential factor in (2.54); it would merely produce a prefactor behaving for small ϵ as ϵ^{-n} for some n.

★ But a Gaussian approximation to the integral near X_σ is generally not valid, because the $t^2 W''$ term vanishes up to fourth order yet – because of the t^2 factor – cannot be ignored. For instance, the example treated in the appendix gives a two dimensional integral that is roughly $\int \mathrm{d}x_1\,\mathrm{d}x_2 \exp\left(-t(x_1^2 + x_2^2) - t^2(x_1^2 + x_2^2)^2\right)$, where x_i are local coordinates centered at one of the critical points. This integral cannot be approximated for large t by a Gaussian, and instead is easily seen to give the error function found in the appendix. The terms proportional to t and t^2 correspond to W' and W''.

It remains to show that for large enough t, X_σ is a local minimum of W, with a Hessian of order t. Let $\widehat{V} = \sum_a V_a \mu_a = \omega^{-1} dI/2$. We can write

$$W = I + W' + W'' \tag{2.56}$$

with

$$\begin{aligned} W' &= 2tg(\widehat{V}, \widehat{V}) \\ W'' &= t^2 \sum_a (g(V_a, \widehat{V}))^2. \end{aligned} \tag{2.57}$$

At a critical point of I, $\widehat{V} = 0$, so such a point is also a critical point of W' and W''. W' and W'' are positive semidefinite and vanish precisely at critical points of I.

At a critical point of I, the Hessian or matrix of second derivatives $\partial^2 I / \partial x^i \partial x^j$ may not be positive definite. However, it is dominated for $t \to +\infty$ by the Hessian of W', which turns out to be

$$\frac{\partial^2 W'}{\partial x^i \partial x^j} = \frac{t}{2} \frac{\partial^2 I}{\partial x^i \partial x^k} g^{kl} \frac{\partial^2 I}{\partial x^l \partial x^j} \tag{2.58}$$

and in particular is positive semidefinite (here g^{kl} are the matrix elements of the inverse of the metric $g(\cdot, \cdot)$). As critical points of I are certainly local minima of W'', this suffices to show that the Hessian of W is positive semidefinite (and proportional to t for large t) and that its kernel consists at most of the kernel of the Hessian of I. Since we have assumed that X_σ is a nondegenerate critical locus of I (in the sense that the Hessian is invertible in the normal directions), X_σ is a local minimum of W.

3. Rederivation In Quantum Field Theory Language

In this section, I will rederive some of the main results of the last section, using this time the language of quantum field theory. To begin with, I want to describe the nonabelian localization principle using more physical language, running through the relevant portions of §2 more quickly with the notation of physicists.

Notation is generally as in §2. In particular, X is once again a compact manifold acted on by a compact Lie group G, with Lie algebra $\mathcal{G}$; the G action on M is generated by vector fields V_a.

We let x^i be local coordinates on M, and let ψ^i be anticommuting variables tangent to M. Let ϕ^a be bosonic variables in the adjoint representation of G. Equivariant differential forms (elements of $\Omega_G^*(X)$) are just G-invariant functions of x, ψ, ϕ. Let

$$D = \sum_i \psi^i \frac{\partial}{\partial x^i} - i \sum_{i,a} \phi^a V_a^i \frac{\partial}{\partial \psi^i}. \tag{3.1}$$

This operator obeys $D^2 = 0$ when acting on G-invariant functions of x, ψ, ϕ. D is the standard differential in $\Omega_G^*(X)$, written in physical notation. Integration of an equivariant differential form α is defined by

$$\oint \alpha = \frac{1}{\text{vol}(G)} \int_X \mathrm{d}x^i \mathrm{d}\psi^i \int_{\mathcal{G}} \frac{\mathrm{d}\phi_1 \ldots \mathrm{d}\phi_s}{(2\pi)^s} \alpha \cdot \exp\left(-\frac{\epsilon}{2}(\phi, \phi)\right). \tag{3.2}$$

Note that although there is no natural measure for the x's or ψ's separately, there is a natural measure $\mathrm{d}x^i \mathrm{d}\psi^i$, since the Jacobian in a change of variables on M would cancel between bosons and fermions. (Integration with respect to this measure is what mathematicians call integration of differential forms.) As in §2, $\mathrm{d}\phi_1 \ldots \mathrm{d}\phi_s$ is an arbitrary measure on $\mathcal{G}$, and (identifying the tangent spaces of G with $\mathcal{G}$) the same measure is used in computing $\text{vol}(G)$. Obviously, by integrating by parts in

x and ψ,

$$\oint D\beta = 0, \tag{3.3}$$

for any β.

Nonabelian localization comes from the fact that – if $D\alpha = 0$ –

$$\oint \alpha = \oint \alpha \exp(tD\lambda), \tag{3.4}$$

for any equivariant form λ. This follows from (3.3), using $D\alpha = D^2 = 0$. We pick

$$\lambda = \sum_i \psi^i b_i, \tag{3.5}$$

where b_i is a function of the x's only. (So in mathematical terminology, $\lambda = \sum_i b_i \mathrm{d}x^i$ is the G-invariant one-form on M used in §2.) We insert this in (3.2), compute $D\lambda$, and perform the ϕ integral. If α is independent of ϕ, we get

$$\oint \alpha = \frac{1}{\mathrm{vol}(G) \cdot (2\pi\epsilon)^{s/2}} \int_X \mathrm{d}x^i \mathrm{d}\psi^i \ \alpha \exp\left(t \sum_{i,j} \psi^i \psi^j \partial_i b_j\right) \cdot \exp\left(-\frac{t^2}{2} \sum_a (V_a^i b_i)^2\right). \tag{3.6}$$

If α depends on ϕ in a sufficiently mild fashion, then (3.6) is replaced by a possibly more complicated formula with similar properties. The main point of (3.6) is that in the limit of $t \to \infty$, the integral becomes localized near the solutions of

$$V_a^i b_i = 0, \qquad a = 1 \ldots s, \tag{3.7}$$

and can be written as a sum of contributions that depend only on the local data near solutions of (3.7). This is the nonabelian localization of §2.

Now, in our more detailed applications, we wish to assume that X is a symplectic manifold, with symplectic form $\omega = \frac{1}{2}\omega_{ij}\psi^i\psi^j$. We also assume that the V_a

29

are derived via Poisson brackets from Hamiltonian functions μ_a, or in other words that

$$V_a^i = \omega^{ij}\partial_j\mu_a \tag{3.8}$$

(where $\omega^{ij}\omega_{jk} = \delta_k^i$). In this case, we set $I = \sum_a \mu_a^2$, and we let

$$b_j = \frac{1}{2}J^i{}_j\partial_i I, \tag{3.9}$$

with the matrix J restricted by requiring that $J^2 = -1$ and that the "metric"

$$g_{ij} = J^k{}_i\omega_{kj} \tag{3.10}$$

is symmetric and positive definite. This is usually described by saying that J is an almost complex structure on X for which ω is of type $(1,1)$ and positive. J's obeying these conditions always exist. In this situation, (3.7) can be written

$$0 = V_a^i\partial_i I. \tag{3.11}$$

This implies that

$$0 = \sum_a \phi^a V_a^i b_i = g^{ij}\partial_i I\partial_j I, \tag{3.12}$$

and hence that

$$0 = \partial_i I. \tag{3.13}$$

Since (3.13) obviously implies (3.11), we have learned that, with the particular choice of λ that we have made, the nonabelian localization for Hamiltonian actions on symplectic manifolds amounts to a formula involving a sum over the critical points of I, another main result of §2.

In §2, we then specialized further to the case that

$$\alpha = \exp\left(\frac{1}{2}\omega_{ij}\psi^i\psi^j - i\phi^a\mu_a\right) \cdot \beta, \tag{3.14}$$

where β has at most polynomial dependence on ϕ. For instance, suppose $\beta = 1$. In this case, it is convenient to perform the ψ integral to reduce the integration over x, ψ to an integration over x only. In fact, although in general there is no natural measure on a manifold X, by introducing the symplectic form ω as in (3.14) and performing the ψ integral we get such a measure:

$$dx^1 \wedge \ldots \wedge dx^n \int d\psi^1 \ldots d\psi^n \; \exp(\frac{1}{2}\omega_{ij}\psi^i\psi^j). \tag{3.15}$$

As the ψ integral gives $\sqrt{\det \omega}$, the measure on X obtained by performing this integral is simply the standard Liouville measure, which if ω is regarded as a two-form is usually written $\omega^n/n!$ In any event, up to a constant multiple, this is the only measure on X that can be constructed using ω alone.

After performing also the Gaussian integral over ϕ, we get

$$\oint \alpha = \frac{1}{\text{vol}(G) \cdot (2\pi\epsilon)^{s/2}} \int_X \frac{\omega^n}{n!} \exp\left(-\frac{1}{2\epsilon}(\mu,\mu)\right). \tag{3.16}$$

A more elaborate expression with the same essential properties arises if we permit $\beta \neq 1$ (with polynomial ϕ dependence).

(3.16) makes it clear that the dominant contribution for $\epsilon \to 0$ comes from the absolute minimum of $I = (\mu,\mu)$, that is the solutions of $\mu = 0$. One also expects heuristically that, in the evaluation via nonabelian localization, an arbitrary critical point P must make a contribution of order $\exp\left(-I(P)/2\epsilon\right)$ for $\epsilon \to 0$. This latter assertion was justified (under some assumptions) in §2.4, and will not be reconsidered here. What we want to do here is to reexamine from a physicist's point of view the relation found in §2 between the contribution of the minimum

31

at $\mu = 0$ and the cohomology of $\mathcal{M} = \mu^{-1}(0)/G$. This will be done by (in a field theory language) mapping a suitable "cohomological" field theory (for background see [34–40]) to a suitable "physical" field theory. The mapping between the two is essentially the proof of nonabelian localization, looked at in reverse. The mapping we will find has an analog, much less understood, in the relation [11–13] between physical and topological gravity in two dimensions.

3.1. The Cohomological Gauge Theory

At this stage, we will specialize to the case of two dimensional gauge theories. Thus, for the space X of the above discussion, we take the space $\mathcal{A}$ of connections on a vector bundle E, with compact structure group H, over a two dimensional surface Σ. For G we take the group of gauge transformations of E. The gauge field A plays the role of the x's in the above formulas; the ψ's are now an anticommuting one-form with values in the adjoint representation of H; and the ϕ's are a zero-form on Σ also with values in the adjoint representation.

The (A, ψ, ϕ) system is the basic multiplet of cohomological Yang-Mills theory. In physical notation the transformation laws are

$$
\begin{aligned}
\delta A_i &= i\epsilon \psi_i \\
\delta \psi_i &= -\epsilon D_i \phi = -\epsilon \left(\partial_i \phi + [A_i, \phi] \right) \\
\delta \phi &= 0,
\end{aligned}
\tag{3.17}
$$

with ϵ an anticommuting parameter. In terms of the operator D of equation (3.1), this can be written $\delta \Phi = iD\Phi$, for every Φ. It is also conventionally written $\delta \Phi = -i\{Q, \Phi\}$, where Q is the BRST operator (so $Q = -D$). As in the general discussion, $Q^2 = 0$ (or $D^2 = 0$) up to a gauge transformation. In fact, $Q^2 = -i\delta_\phi$, where δ_ϕ is the generator of a gauge transformation with infinitesimal parameter ϕ. We introduce a ghost number quantum number, with the ghost numbers of (A, ψ, ϕ) being $(0, 1, 2)$.

Additional multiplets, which typically are needed to write Lagrangians (and which are analogous to antighost multiplets in usual BRST quantization), can be introduced in the following standard way. One considers pairs (u, v) of opposite statistics and ghost numbers $(n, n + 1)$, for some n, and with

$$\delta u = i\epsilon v$$
$$\delta v = \epsilon[\phi, u]. \tag{3.18}$$

In two dimensional gauge theories, it is convenient to introduce two such pairs (λ, η) and $(\chi, -iH)$, with λ a commuting field of ghost number -2, and χ an anticommuting field of ghost number -1. So

$$\delta\lambda = i\epsilon\eta, \qquad \delta\eta = \epsilon[\phi, \lambda]$$
$$\delta\chi = \epsilon H \qquad \delta H = i\epsilon[\phi, \chi]. \tag{3.19}$$

Any expression $L = -i\{Q, V\}$, with V gauge invariant, will be Q-invariant, since $Q^2 = 0$ on gauge invariant functions. In writing a Lagrangian, we wish also to pick V so that all fields will have a nondegenerate kinetic energy. A suitable choice of V, which conserves ghost number, is

$$V = \frac{1}{h^2} \int_\Sigma \mathrm{d}\mu \, \mathrm{Tr} \left(\frac{1}{2}\chi(H - 2 \star F) + g^{ij}D_i\lambda\psi_j \right). \tag{3.20}$$

Here Σ is the Riemann surface on which we formulate the theory. It has been endowed with a metric g, and μ is the corresponding Riemannian measure. h is a real constant. $F = \mathrm{d}A + A \wedge A$ is the Yang-Mills field strength, and $\star$ is the Hodge star operator; we also set $f = \star F = \frac{1}{2}\epsilon^{ij}F_{ij}$. One finds that

$$L = -i\{Q, V\} = \frac{1}{h^2} \int_\Sigma \mathrm{d}\mu \, \mathrm{Tr} \left(\frac{1}{2}(H - f)^2 - \frac{1}{2}f^2 - i\chi \star D\psi + iD_i\eta\psi^i \right.$$
$$\left. + D_i\lambda D^i\phi + \frac{i}{2}\chi[\chi, \phi] + i[\psi_i, \lambda]\psi^i \right). \tag{3.21}$$

33

One can solve for the auxiliary field H by its equation of motion

$$H = f, \tag{3.22}$$

and delete the $(H - f)^2$ term from the Lagrangian.

The quantum field theory with Lagrangian (3.21) is a topological field theory, and independent of the choice of the coupling parameter h, because although the coupling and the metric g appear in L, they only appear in terms of the form $\{Q, \ldots\}$ (since L itself is of this form). By dropping the $(H - f)^2$ term and taking $h \to 0$, one sees that all computations can be performed by expanding around the minimum

$$F = 0 \tag{3.23}$$

of the gauge boson kinetic energy.

In the weak coupling limit, the $\phi - \lambda$ integral can be treated formally via one loop determinants and Feynman diagrams. If one wishes to treat this integral more honestly, one either considers ϕ to be complex and $\lambda = \overline{\phi}$, or (as was natural in [37]) one takes ϕ real and λ imaginary. For our purposes, such a choice need not be specified. To minimize the scalar kinetic energy one requires

$$0 = D_i \phi \tag{3.24}$$

(if $\lambda = \overline{\phi}$; in the other case this requirement comes from stationary phase). If A is an irreducible solution of (3.23) (in the sense that its holonomy group commutes only with the center of H), then (3.24) implies $\phi = 0$, since otherwise the holonomy group would have to commute with ϕ.

More fundamentally, (3.23) and (3.24) should be regarded as the conditions $\delta \chi = 0$ and $\delta \psi = 0$ for a BRST fixed point, as explained in [40, §3.1]. ($\delta \chi = 0$ coincides with (3.23) after using (3.22).) Let $\mathcal{U}$ be the space of solutions of (3.23) and (3.24).

(3.21) is a "standard" Lagrangian for the two dimensional analog of Donaldson theory. Because of the independence of h, all calculations can be performed in the weak coupling limit, where as just indicated they reduce to integrals over $\mathcal{U}$. For gauge groups H and bundles E such that reducible solutions of (3.23) do not exist, $\mathcal{U}$ is the same as the moduli space $\mathcal{M}$ of flat connections on E up to gauge transformation, and the correlation functions are intersection pairings on $\mathcal{M}$, as analyzed in detail by Baulieu and Singer [38] (in the analogous four dimensional theory). The principal difficulty in understanding the theory, as explained in the second paper in [34], comes from the zero modes of ϕ and of $\lambda = \overline{\phi}$ that can arise for reducible connections, and as a result of which $\mathcal{U}$ and $\mathcal{M}$ do not coincide in general. This motivated the search for the following method of eliminating these fields from the problem.

3.2. Mapping To The Physical Theory

The Strategy

If we replace V by $V + tV'$, with t a constant and V' some new gauge invariant operator, then the theory with Lagrangian

$$L(t) = -i\{Q, V + tV'\} \tag{3.25}$$

is independent of t as long as (i) V' is such that $L(t)$ has a nondegenerate kinetic energy for all $t^\star$; (ii) the perturbation by V' does not permit any new fixed points, that is solutions of $\delta\chi = \delta\psi = 0$, to flow in from infinity. The latter condition is needed because the space of fields over which one integrates (in performing the Feynman path integral) is not compact, and all statements about topological invariance require pinning down the behavior at infinity.

We will actually consider a choice of V' such that condition (ii) is not obeyed. When this is so, the fixed point equation $\delta\chi = \delta\psi = 0$, in addition to having

---

$\star$ This is a quantum field theory analog of requiring that a family of operators be elliptic.

the component $\mathcal{M}$ (or in general $\mathcal{U}$) discussed above[†] will acquire additional components $\mathcal{M}_\alpha$. The Feynman path integral will thus reduce to a *sum* over the contributions of $\mathcal{M}$ and $\mathcal{M}_\alpha$. The standard BRST arguments show that the contribution of $\mathcal{M}$ is independent of t, but as the path integral will give naturally a sum of the contributions from $\mathcal{M}$ and $\mathcal{M}_\alpha$, it might appear that there is no way from studying the (simpler) $L(t \neq 0)$ theory to recover the result of the (interesting but harder) theory $L(t = 0)$. To accomplish this, we will at a judicious moment introduce one further trick to disentangle the contributions of $\mathcal{M}$ and $\mathcal{M}_\alpha$.

<u>Elimination of λ</u>

We set

$$V' = -\frac{1}{h^2} \int_\Sigma d\mu \, \mathrm{Tr}\, \chi\lambda. \tag{3.26}$$

This will lead to a Lagrangian that does not conserve ghost number; the ghost number of V' is -3, so that of $\{Q, V'\}$ will be -2. We find that the analog of (3.21) is

$$L(t) = -i\{Q, V + tV'\} = \frac{1}{h^2} \int_\Sigma d\mu \, \mathrm{Tr}\, \left(\frac{1}{2}(H - \lambda t - f)^2 - \frac{1}{2}(\lambda t + f)^2 \right.$$
$$\left. + i\chi \star D\psi + iD_i\eta\psi^i - D_i\lambda D^i\phi + \frac{i}{2}\chi[\chi, \phi] + i[\psi_i, \lambda]\psi^i \right). \tag{3.27}$$

As before, H can be integrated out, simply setting $H - \lambda t - f = 0$. The benefit from perturbing L to $L(t)$ is that for $t \neq 0$, λ, χ, and η can also be integrated out, leaving a local Lagrangian:

$$L'(t) = \frac{1}{h^2} \int_\Sigma d\mu \, \mathrm{Tr}\, \left(\frac{1}{t} \left(D_i f D^i \phi + if[\psi_i, \psi^i] - iD_l\psi^l \epsilon^{ij} D_i\psi_j \right) \right.$$
$$\left. + \frac{1}{t^2} \left(\frac{i}{2} D_l\psi^l [D_k\psi^k, \phi] + \frac{1}{2} \left(-D_k D^k \phi + i[\psi_k, \psi^k] \right)^2 \right) \right). \tag{3.28}$$

---

† Which in general might undergo some perturbations when t is varied; but we will arrange to avoid this.

For instance, λ is integrated out by setting

$$\lambda = -\frac{f}{t}. \tag{3.29}$$

Already we can assert a major point: the standard "cohomological" Lagrangian (3.21), in which the correlation functions of BRST invariant operators have a known description in terms of cohomology of $\mathcal{M}$, can be deformed preserving the BRST symmetry to a Lagrangian written in terms of the minimal multiplet A, ψ, ϕ only, namely (3.28). (3.21) and (3.28) may not be equivalent, but the failure of such equivalence can only come from new components $\mathcal{M}_\alpha$ of moduli space that flow in from infinity for $t \neq 0$; the contribution of the "old" component $\mathcal{M}$ must be independent of t.

The terms of order $1/t$ in (3.28) are

$$\frac{i}{t}\{Q, \int_\Sigma d\mu \quad \mathrm{Tr}\, \psi^i D_i f\}. \tag{3.30}$$

The terms of order $1/t^2$ are similarly

$$\frac{i}{2t^2}\{Q, \int_\Sigma d\mu \quad \mathrm{Tr}\left(D_k D^k \phi \cdot D_l \psi^l - i[\psi_l, \psi^l] D_k \psi^k\right)\}. \tag{3.31}$$

We will study (3.28) in the limit of large imaginary t, and since the terms of order $1/t$ already give a nondegenerate kinetic energy, the terms of order $1/t^2$ can simply be dropped. Setting $t = -iu$, we reduce L' to

$$L''(u) = \frac{i}{h^2 u} \int_\Sigma d\mu\, \mathrm{Tr}\left(D_i f D^i \phi + if[\psi_i, \psi^i] - i D_l \psi^l \epsilon^{ij} D_i \psi_j\right). \tag{3.32}$$

Now, we want to ask what kind of "localization" there is in the path integral

$$\frac{1}{\mathrm{vol}(G)} \int DA\, D\psi\, D\phi \quad \exp(-L''(u)). \tag{3.33}$$

37

The main point is that the ϕ integral is

$$\int D\phi \, \exp\left(\frac{i}{h^2 u} \int_\Sigma d\mu \, \mathrm{Tr} \, \phi D_i D^i f\right) \sim \prod_{x\in\Sigma} \delta(D_i D^i f). \qquad (3.34)$$

The localization is therefore on the locus $D_i D^i f = 0$. This equation implies

$$0 = \int_\Sigma d\mu \, \mathrm{Tr} \, f D_i D^i f = -\int_\Sigma \mathrm{Tr}(D_i f)^2, \qquad (3.35)$$

and so it is equivalent to

$$0 = D_i f. \qquad (3.36)$$

These are the classical Yang-Mills equations, that is, the variational equations derived from the usual Yang-Mills action $I = -\int_\Sigma \mathrm{Tr} \, f^2$. The space of solutions of (3.36) has one component, $\mathcal{M}$, consisting of solutions of $f = 0$, and other components, $\mathcal{M}_\alpha$, consisting of higher critical points of the Yang-Mills action.

From (3.29), we see that the new components have $\lambda \sim 1/t$, and hence are absent at $t = 0$ and "flow in from infinity" when one perturbs to $t \neq 0$. This is the abstract scenario that we anticipated earlier for how a perturbation of the form $\{Q, \ldots\}$ might fail to leave the theory invariant.

Final Reduction

For any BRST invariant operator $\mathcal{O}$, let $\langle\mathcal{O}\rangle$ be the expectation value of $\mathcal{O}$ computed in the cohomological theory (3.21), and let $\langle\mathcal{O}\rangle'$ be the corresponding expectation value in the theory (3.32). In general $\langle\mathcal{O}\rangle \neq \langle\mathcal{O}\rangle'$, because of higher critical points contributing in (3.32). We will describe a class of $\mathcal{O}$'s such that the higher critical points do not contribute, and hence $\langle\mathcal{O}\rangle = \langle\mathcal{O}\rangle'$.

38

Two particular BRST invariant operators will play an important role. The first, related to the symplectic structure of $\mathcal{M}$, is

$$\omega = \frac{1}{4\pi^2} \int_\Sigma \mathrm{Tr}\left(i\phi F + \frac{1}{2}\psi \wedge \psi\right). \qquad (3.37)$$

The second is

$$\Theta = \frac{1}{8\pi^2} \int_\Sigma \mathrm{d}\mu\, \mathrm{Tr}\,\phi^2. \qquad (3.38)$$

We wish to compute

$$\langle \exp(\omega + \epsilon\Theta) \cdot \beta \rangle' \qquad (3.39)$$

with ϵ a positive real number, and β an arbitrary observable with at most a polynomial dependence on ϕ. This is

$$\langle \exp(\omega + \epsilon\Theta) \cdot \beta \rangle' = \frac{1}{\mathrm{vol}(G)} \int DA\, D\psi\, D\phi\ \cdot \beta \cdot \exp\left(\frac{1}{h^2 u}\{Q, \int_\Sigma \mathrm{d}\mu\ \psi^k D_k f\}\right.$$

$$\left. + \frac{1}{4\pi^2} \int_\Sigma \mathrm{Tr}\left(i\phi F + \frac{1}{2}\psi \wedge \psi\right) + \frac{\epsilon}{8\pi^2} \int_\Sigma \mathrm{d}\mu\, \mathrm{Tr}\,\phi^2\right). \qquad (3.40)$$

This is formally independent of u, and will really be independent of u as long as, in varying u, the Lagrangian remains nondegenerate and with a good behavior at infinity in field space. The particular choice of ω and Θ has been made to ensure that these conditions are obeyed (for $\epsilon \geq 0$) even at $u = \infty$. Thus, we can simply set $u = \infty$ in (3.40), discarding the terms of order $1/u$, and reducing to

$$\langle \exp(\omega + \epsilon\Theta) \cdot \beta \rangle' = \frac{1}{\mathrm{vol}(G)} \int DA\, D\psi\, D\phi\ \exp\left(\frac{1}{4\pi^2} \int_\Sigma \mathrm{Tr}\left(i\phi F + \frac{1}{2}\psi \wedge \psi\right)\right.$$

$$\left. + \frac{\epsilon}{8\pi^2} \int_\Sigma \mathrm{d}\mu\, \mathrm{Tr}\,\phi^2\right) \cdot \beta. \qquad (3.41)$$

This is the key step; we have passed from "cohomological" to "physical" Yang-Mills

theory.

First consider the special case of $\epsilon = 0$, from which all the topological information can be extracted. If we also set $\beta = 1$, then the ϕ integral gives a multiple of $\prod_{x \in \Sigma} \delta(F)$, and thus in this special case, the higher critical points with $F \neq 0$ do not contribute. Even for $\beta \neq 1$ (but still at $\epsilon = 0$), the ϕ integral gives a more complicated distribution supported at $F = 0$,[★] so still the higher critical points do not contribute. As (3.32) differs from (3.21) only by possible contributions of these higher critical points, the vanishing of these contributions means that

$$\langle \exp(\omega) \cdot \beta \rangle = \langle \exp(\omega) \cdot \beta \rangle', \tag{3.42}$$

for all β with polynomial ϕ dependence.

All information of topological interest can be extracted from (3.42). Knowledge of the left hand side of (3.42) for arbitrary β is enough to determine all expressions $\langle \beta \rangle$ for BRST invariant β. (In fact, this is so for a rather elementary reason. If β has definite ghost number, then $\langle \beta \rangle$ vanishes unless its ghost number is equal to the (real) dimension of $\mathcal{M}$, and in that case, since ω has ghost number 2, $\langle \beta \rangle = \langle \exp(\omega) \cdot \beta \rangle$.) On the other hand, the right hand side of (3.42) is effectively computable using (3.41), as will be clear in §4-5.

The exponent in (3.41) is a Lagrangian

$$L(A, \psi, \phi) = \int_{\Sigma} \mathrm{Tr} \left(-i\phi F - \frac{1}{2} \psi \wedge \psi \right) - \frac{\epsilon}{8\pi^2} \int_{\Sigma} \mathrm{d}\mu \, \mathrm{Tr} \, \phi^2 \tag{3.43}$$

that is entirely equivalent to conventional two-dimensional Yang-Mills theory. In fact, ψ is a decoupled field with only a mass term and Euler-Lagrange equation $\psi = 0$; its only role is to put things in the right theoretical context. (Including ψ in this way was originally suggested several years ago by S. Axelrod.) ϕ can also be integrated out, to put the Lagrangian in its conventional Yang-Mills form.

[★] This generalizes the fact that in one dimension, $f(x) = \int_{-\infty}^{\infty} \mathrm{d}\phi \, \exp(i\phi x) \cdot \phi^n$ is for any positive integer n a distribution supported at $x = 0$.

Interpretation Of The Measure

In quantum gauge theories in general, on a space-time manifold M, the path integral measure on the space $\mathcal{A}$ of connections is usually defined by first introducing a metric on M. This permits one to define on $\mathcal{A}$ a metric as follows: a tangent vector to $\mathcal{A}$ is an adjoint-valued one-form a, and one sets

$$|a|^2 = -\frac{1}{4\pi^2} \int_\Sigma \operatorname{Tr} a \wedge \star a. \tag{3.44}$$

From this metric one formally gets a measure, and this is the usual path integral measure for gauge fields.

However, in two dimensions, if M is orientable, there is another approach to defining a measure: this comes from the symplectic structure on $\mathcal{A}$, that we have noted in (1.8). Moreover, the measures on $\mathcal{A}$ defined by the metric or the symplectic structure agree, because the metric (3.44) is Kahler.

Look back to (3.41), assuming first that $\beta = 1$. In this case, the only ψ dependent factors are in

$$DA \; D\psi \;\; \exp(\frac{1}{4\pi^2} \int_\Sigma \operatorname{Tr} \psi \wedge \psi). \tag{3.45}$$

This should be compared to the integral in (3.15). As in the discussion of that equation, integrating out ψ will give the symplectic or Liouville measure on $\mathcal{A}$, which is the usual path integral measure. In keeping with convention (but somewhat inconsistently), we will call this measure DA.

Thus, if $\beta = 1$, the only role of ψ was to give a more sophisticated way to build in the standard measure on $\mathcal{A}$. Things are different if $\beta \neq 1$ and more specifically if β depends on ψ. In that case, integrating out ψ will replace β by some function of A and ϕ only. This will be a major step in the detailed computations in §4 and §5.

Elimination Of ϕ

The other main step in the calculations is closely related to the ability to eliminate ϕ. This is possible because derivatives of ϕ are absent in (3.41). (This is so even if $\beta \neq 1$; the BRST cohomology of the theory can be represented by operators that do not involve derivatives of ϕ.) At this point, let us generalize to $\epsilon \neq 0$, but for simplicity $\beta = 1$. In this case, by integrating out ϕ (after integrating out ψ as discussed above), we get

$$\langle \exp\left(\omega + \epsilon \Theta \right) \rangle' = \frac{1}{\text{vol}(G)} \int DA \ \exp\left(\frac{2\pi^2}{\epsilon} \int_\Sigma d\mu \, \text{Tr} \, f^2 \right). \qquad (3.46)$$

This is the path integral of conventional two dimensional Yang-Mills theory. Now, at $\epsilon \neq 0$, we cannot claim that the $\langle \ \ \rangle$ and $\langle \ \ \rangle'$ operations coincide, since the higher critical components $\mathcal{M}_\alpha$ contribute. However, their contributions are exponentially small, involving the relevant values of $I = -\int_\Sigma d\mu \, \text{Tr} \, f^2$. So we get

$$\langle \exp\left(\omega + \epsilon \Theta \right) \rangle = \frac{1}{\text{vol}(G)} \int DA \ \exp\left(\frac{2\pi^2}{\epsilon} \int_\Sigma d\mu \, \text{Tr} \, f^2 \right) + O(\exp(-2\pi^2 c/\epsilon),$$

$$(3.47)$$

where c is the smallest value of the Yang-Mills action I on one of the higher critical points.

For $\beta \neq 1$, the elimination of ϕ is more elaborate, but can still be carried out explicitly. A version of this will be done in our detailed calculation of intersection numbers in §5.

3.3. THE BRST COHOMOLOGY

Now we want to describe the BRST invariant observables of the theory.

Let T be a homogeneous invariant polynomial on the Lie algebra $\mathcal{H}$. Then a BRST invariant operator that cannot be written as $\{Q, \ldots\}$ is

$$\mathcal{O}_T^{(0)}(P) = T(\phi(P)), \tag{3.48}$$

with an arbitrary $P \in \Sigma$. However, for $P, P' \in \Sigma$, the difference $T(\phi(P)) - T(\phi(P'))$ is $\{Q, \ldots\}$; this follows from the formula

$$d\mathcal{O}_T^{(0)} = -i\{Q, \mathcal{O}_T^{(1)}\}, \tag{3.49}$$

with

$$\mathcal{O}_T^{(1)} = -\frac{\partial T}{\partial \phi^a}\psi^a. \tag{3.50}$$

One similarly has

$$d\mathcal{O}_T^{(1)} = -i\{Q, \mathcal{O}_T^{(2)}\}, \tag{3.51}$$

with

$$\mathcal{O}_T^{(2)} = \frac{1}{2}\frac{\partial^2 T}{\partial \phi^a \partial \phi^b}\psi^a \wedge \psi^b + i\frac{\partial T}{\partial \phi^a}F^a. \tag{3.52}$$

Here F^a are the components of the curvature two-form F, that is $F = \sum_a T_a F^a$.

The following BRST invariant observables can be formed from these $\mathcal{O}$'s. Since (3.51) asserts that $\mathcal{O}_T^{(2)}$ is annihilated by Q up to an exact form, we have the Q-invariant observable

$$T_{(2)} = \int_\Sigma \mathcal{O}_T^{(2)}. \tag{3.53}$$

Likewise, for every oriented circle $C \subset \Sigma$, we have

$$T_{(1)}(C) = \int_C \mathcal{O}_T^{(1)}. \tag{3.54}$$

(3.49) implies that this is Q-invariant, and (3.51) implies that, up to $\{Q, \ldots\}$, it

depends only on the homology class of C. Finally, we have the original operator

$$T_{(0)}(P) = \mathcal{O}^{(0)}(P). \qquad (3.55)$$

To unify the notation, note that for $j = 0, 1, 2$, and V a j-dimensional submanifold of Σ (so V is a point for $j = 0$, a circle for $j = 1$, and $V = \Sigma$ for $j = 2$), we have defined an operator $T_{(j)}(V)$. Notice that if A, ψ, ϕ are considered to have degree or ghost number $0, 1, 2$, and T is of order r, then $T_{(j)}$ is of ghost number $2r - j$.

In applications, it may be convenient to replace $T_{(0)}(P)$ by its averaged version

$$T_{(0)}(P) \to \int_{\Sigma} \mathrm{d}\mu \ T(\phi) \qquad (3.56)$$

which we will use in §4-5. Of course, the two are cohomologous.

Mathematical Counterpart

Now, mathematically, observables with properties analogous to these can be defined as follows. The moduli space $\mathcal{M}$ of flat connections on an H bundle E parametrizes a family of flat bundles on Σ. One can try to fit them together into a "universal bundle" $\mathcal{E}$ over $\mathcal{M} \times \Sigma$. The obstruction to existence of $\mathcal{E}$ comes from possible symmetries of flat connections. If one restricts to a dense open set in $\mathcal{M}$ parametrizing irreducible flat connections, the only symmetries are constant gauge transformations by elements of the center of H. In the adjoint representation, these act trivially, so the universal bundle exists at least as an adjoint bundle; this is good enough for defining the rational characteristic classes that we want.

So pick a connection B on $\mathcal{E}$, and let $\mathcal{F} = dB + B \wedge B$ be the curvature. Then, for T as above, the closed $2r$-form $T(\mathcal{F})$ defines an element of $H^{2r}(\mathcal{M} \times \Sigma, \mathbb{R})$. If V is a j-dimensional submanifold of Σ, then by restricting $T(\mathcal{F})$ to $\mathcal{M} \times V$, and integrating over the fibers of the projection $\mathcal{M} \times V \to \mathcal{M}$, one gets elements $\widehat{T}_{(j)}(V)$ in $H^{2r-j}(\mathcal{M}, \mathbb{R})$.

44

It is striking that the $\widehat{T}_{(j)}(V)$ have the same degree $2r - j$ and are determined by the same data as the $T_{(j)}$. In fact, it can be shown that in "cohomological gauge theories," as long as $\mathcal{M}$ is non-singular, there is a precise correspondence between the $T_{(j)}$'s and the $\widehat{T}_{(j)}$'s, in the sense that (as long as singularities of $\mathcal{M}$ can be neglected)

$$\langle \prod_{s=1}^{n} T_{j_s}(V_s) \rangle = \frac{1}{\#Z(H)} \int_{\mathcal{M}} \prod_{s=1}^{n} \widehat{T}_{(j_s)}(V_s). \tag{3.57}$$

(The factor of $1/\#Z(H)$ is explained in §2.2 of [9].) See [38] for an explanation of this formula in the context of four dimensional Donaldson theory. The discussion carries over without modification for the analogous two dimensional theory (3.21). Therefore, following through our derivations above, (3.57) holds for the contribution of $\mu^{-1}(0)$ if the left hand side of (3.57) is computed using the simplified cohomological Lagrangian (3.28), and holds modulo terms that are exponentially small for $\epsilon \to 0$ if one uses instead the physical Yang-Mills Lagrangian (3.43).

Reduction To Generators

Elementary arguments show that in case T can be factored as a product of invariant polynomials, say $T = UV$, then the operators $T_{(j)}$ can be expressed in terms of the $U_{(j)}$ and $V_{(j)}$. In fact, it is fairly obvious that up to $\{Q, \dots\}$,

$$T_{(0)} = U_{(0)} V_{(0)}, \tag{3.58}$$

and for any circle C

$$T_{(1)}(C) = U_{(0)} V_{(1)}(C) + U_{(1)}(C) V_{(0)}. \tag{3.59}$$

The corresponding factorization of $T_{(2)}$ is slightly more complicated. Let C_σ, $\sigma = 1 \dots 2g$ be circles representing a basis of $H_1(\Sigma, \mathbb{Z})$, with intersection form $\gamma_{\sigma\tau}$. Then

up to $\{Q, \ldots\}$,

$$T_{(2)} = U_{(2)}V_{(0)} + U_{(0)}V_{(2)} + \sum_{\sigma,\tau} \gamma_{\sigma\tau} U_{(1)}(C_\sigma)V_{(1)}(C_\tau). \tag{3.60}$$

These formulas show that it is sufficient to evaluate (3.57) with the T's taken from a set of generators of the ring of invariant polynomials on $\mathcal{H}$.

4. Localization And Yang-Mills Theory

Our goal in this section is to make a detailed comparison of the localization theorem with two dimensional Yang-Mills theory. We will also begin the computation of intersection numbers on moduli spaces of flat connections; these computations will be completed in the next section.

To begin with, we need some basic facts about the quantum gauge theory. I will here explain these facts from a continuum point of view. This discussion can be read in conjunction with §2.3 of [9] where many of the same facts are derived using a lattice regularization, with more detail on some points. Other approaches have been cited in the introduction.

Notation is generally as in earlier sections. Thus, A is a connection on an H-bundle E over a Riemann surface Σ. $F = \mathrm{d}A + A \wedge A$ is the curvature. ϕ is a zero-form with values in $\mathrm{ad}(E)$. We write $\phi = \sum_a \phi^a T_a$, with T_a an orthonormal basis of the Lie algebra $\mathcal{H}$ of H. At the outset, we assume that H is connected and simply connected. Later we relax the assumption of simple connectivity.

The Topological Field Theory

First we consider the topological field theory with Lagrangian

$$L = -\frac{i}{4\pi^2} \int_\Sigma \mathrm{Tr}\, \phi F, \tag{4.1}$$

which is related to Reidemeister-Ray-Singer torsion [14]. The partition function is

defined formally by

$$Z(\Sigma) = \frac{1}{\text{Vol}(G)} \int DA \; D\phi \quad \exp(-L). \tag{4.2}$$

Here if E is trivial, G is the group of maps of Σ to H; in general G is the group of gauge transformations.

(4.1) should really be considered to correspond to a one parameter family of topological field theories. Different methods of defining the path integral will differ by terms coming from a substitution

$$L \to L + v \int\limits_{\Sigma} \mathrm{d}\mu \; \frac{R}{4\pi}, \tag{4.3}$$

where v is an arbitrary constant, and R is the curvature of a metric on Σ (which might enter in regularizing and gauge fixing the theory; or if one uses a lattice regularization as in [9], a similar ambiguity arises in defining the local factors). Of course, the substitution (4.3) just multiplies the path integral on a surface Σ of Euler characteristic $\chi(\Sigma)$ by a "trivial" factor $\exp(-v\chi(\Sigma))$. We ultimately will fix v by requiring precise agreement (and not just agreement up to such a trivial factor) with the theory of Reidemeister-Ray-Singer torsion. In principle, with a careful calculation using a regulator in which the manipulations of §3 are valid, one should hopefully be able to determine *a priori* the necessary value of v.

Canonical Quantization

Canonical quantization of (4.1) shows that the canonical momentum to A is

$$\pi_A = \widetilde{\phi}, \tag{4.4}$$

where we set $\widetilde{\phi} = \phi/4\pi^2$. Hence in the quantum theory, if the A^a are taken to be multiplication operators, then $\widetilde{\phi}$ acts as follows:

$$\widetilde{\phi}_a = -i\frac{\delta}{\delta A^a(x)}. \tag{4.5}$$

If C is a circle in Σ – an initial value surface – then the Hilbert space $\mathcal{H}_C$

obtained by quantization on Σ can be considered with this representation of the canonical commutation relations to consist of gauge invariant functions $\Psi(A)$. Such a function must be a function only of the monodromy of A around C, which (picking a base point $x \in C$) we write as

$$U = P \exp \oint_x A. \tag{4.6}$$

To be more precise, Ψ must be a class function of U, invariant under conjugation, and so must have an expansion in characters:

$$\Psi(A) = \sum_\alpha c_\alpha \ \chi_\alpha(U). \tag{4.7}$$

The sum runs over all isomorphism classes of irreducible H representation α; $\chi_\alpha(U)$ is the trace of U in the representation α; c_α are complex numbers. Thus the functions $\chi_\alpha(U)$ give a basis of $\mathcal{H}_C$. The lattice regularization used in [9] makes it clear that this basis is orthonormal.

Casimir Operators

Let Q be an invariant polynomial on $\mathcal{H}$. Pick $y \in \Sigma$. We want to determine the quantum operator $\widehat{Q}$ on $\mathcal{H}_C$ corresponding to the classical observable $Q(\widetilde{\phi})$.

The result can be determined from (4.5), apart from a normal ordering ambiguity that will be treated later. We have

$$\widetilde{\phi}_a \cdot \chi_\alpha(U) = -i \, \mathrm{Tr}_\alpha \, T_a P \exp \oint_y A \tag{4.8}$$

since the right hand side is the first variation of $\chi_\alpha(U)$ with respect to A. Formally computing higher derivatives in the same way, we get

$$Q(\widetilde{\phi}) \to \widehat{Q} = Q(-iT). \tag{4.9}$$

The right hand side is just the Casimir operator determined by Q.

48

(4.9) is good enough for our purposes temporarily, but it has the following limitation. If one studies the theory (4.1) with an arbitrary regularization, then gauge invariant operators such as $Q(\widetilde{\phi})$ will "mix" under renormalization – by a normal ordering ambiguity – with similar operators determined by lower order polynomials. (4.9) is one natural choice, but we will ultimately have to modify it. For any given gauge group, there are only finitely many independent Casimir operators, and therefore the renormalization problem involves finitely many parameters. For instance, for $SU(2)$ or $SO(3)$, one such parameter will appear.

The Three-Holed Sphere

Every oriented Riemann surface can be built by gluing together three-holed spheres, so the path integral on a three-holed sphere Σ (figure (1)) is an important special case. Let C_i, $i = 1 \ldots 3$ be the three boundary circles. Let U_i be the holomony of the connection A about C_i (orientations on the C_i, used in computing the holonomy, are induced from Σ). The path integral on the three-holed sphere gives a vector $\Psi_{\{3\}} \in \otimes_{i=1}^{3} \mathcal{H}_{C_i}$, which must therefore be of the form

$$\Psi_{\{3\}} = \sum_{\alpha_1, \alpha_2, \alpha_3} c_{\alpha_1, \alpha_2, \alpha_3} \prod_{i=1}^{3} \chi_{\alpha_i}(C_i). \tag{4.10}$$

I now claim, however, that $c_{\alpha_1, \alpha_2, \alpha_3}$ vanishes unless $\alpha_1 = \alpha_2 = \alpha_3$. To see this, we consider the path integral of figure (1(b)) with the insertion of some operator $\mathcal{O} = Q(\widetilde{\phi})$ at some point $z \in \Sigma$. By factorizing in an appropriate channel, as in figure (1(c)), one can consider this operator to act on any of the three external states $\chi_{\alpha_i}(U_i)$, whence, according to (4.9), $\mathcal{O}$ can be replaced by the value of the Casimir operator $Q(-iT)$ for the representation α_i. Hence $c_{\alpha_1, \alpha_2, \alpha_3} = 0$ unless the α_i have the same values for all Casimirs. This implies that they must be isomorphic [41, §126]. (4.10) thus collapses to

$$\Psi_{\{3\}} = \sum_{\alpha} c_{\alpha} \prod_{i=1}^{3} \chi_{\alpha}(U_i). \tag{4.11}$$

49

The Two-Holed Sphere

We now consider the two-holed sphere of figure (2), with boundary components C_1, C_2 and monodromies about them U_1, U_2. Labeling the boundary components by representations α_i, the path integral gives a result of the general form $\Psi_{\{2\}} = \sum_{\alpha_1,\alpha_2} e_{\alpha_1,\alpha_2} \chi_{\alpha_1}(U_1) \chi_{\alpha_2}(U_2)$. However, $e_{\alpha_1,\alpha_2} = 0$ unless $\alpha_1 = \alpha_2$, by the argument in the last paragraph, so we can write $e_{\alpha_1,\alpha_2} = \delta_{\alpha_1,\alpha_2} e_{\alpha_1}$, for some e_α's.

The path integral of the two-holed sphere with labeled external states can be given a special interpretation. In fact, $e_{\alpha_1,\alpha_2} = \langle \chi_{\alpha_2} | \exp(-HT) | \chi_{\alpha_1} \rangle$, where H is the Hamiltonian and T is the elapsed time. As $H = 0$ for the topological field theory (4.1), we have merely $e_{\alpha_1,\alpha_2} = \delta_{\alpha_1,\alpha_2}$, so $e_\alpha = 1$ and

$$\Psi_{\{2\}} = \sum_\alpha \chi_\alpha(U_1) \chi_\alpha(U_2). \tag{4.12}$$

The One-Holed Sphere

The path integral of the disc or one-holed sphere (figure (3)) similarly gives a result of the general form

$$\Psi_{\{1\}}(U) = \sum_\alpha f_\alpha \chi_\alpha(U) \tag{4.13}$$

with U the monodromy about the boundary. Now, however, we can reason as follows. Writing down the path integral from (4.1)

$$\int DA \ D\phi \quad \exp\left(\frac{i}{4\pi^2} \int_\Sigma \operatorname{Tr} \phi F\right). \tag{4.14}$$

we see that the ϕ integral gives explicitly $\delta(F)$.

A connection on the disc with $F = 0$ necessarily has $U = 1$, so the function $\Psi_{\{1\}}(U)$ in (4.13) must be a delta function supported at $U = 1$. According to the

theory of compact Lie groups, we have $\delta(U-1) = \sum_\alpha \dim(\alpha)\chi_\alpha(U)$, with $\dim(\alpha)$ the dimension of the representation α. Thus

$$f_\alpha = w \cdot \dim(\alpha), \tag{4.15}$$

with w an unknown constant whose origin was explained in connection with (4.3).

We can now identify the unknown constants c_α in the path integral on a three-holed sphere. As in figure (4), we decompose a two-holed sphere as the union of a one-holed sphere and a three-holed sphere, glued on their common boundary. Factorizing the partition function of the three-holed sphere, we learn that

$$e_\alpha = c_\alpha f_\alpha. \tag{4.16}$$

Since we already know $e_\alpha = 1$, we get

$$c_\alpha = \frac{w^{-1}}{\dim(\alpha)}. \tag{4.17}$$

For a derivation of these results using a lattice regularization, see [9,§2.3].

<u>Combining The Pieces</u> Now we can combine the pieces and determine the partition function of the topological field theory (4.1) on an oriented Riemann surface Σ of genus g. Such a surface can be regarded, as in figure (5), as the union of $2g-2$ three-holed spheres, glued on $3g-3$ circles. Labeling each circle C_i by a representation α_i, and computing the path integral on each three holed sphere using (4.11), one finds that the sum over the α_i collapses to a single sum, because of the "diagonal" nature of the partition function of the three-holed sphere. The result for the partition function is thus $Z(\Sigma) = \sum_\alpha c_\alpha^{2g-2}$.

After adjusting w by comparing to Reidemeister-Ray-Singer torsion, as explained in [9], the partition function is finally

$$Z(\Sigma) = \left(\frac{\mathrm{Vol}(H)}{(2\pi)^{\dim H}} \right)^{2g-2} \cdot \sum_\alpha \frac{1}{(\dim\alpha)^{2g-2}}. \tag{4.18}$$

The moduli space $\mathcal{M}$ of flat connections on Σ has the symplectic structure ω of

equation (1.8), and hence a volume form $\omega^n/n!$ As explained in [9, §2], the partition function Z is related to the volume of $\mathcal{M}$ by $Z = \mathrm{Vol}(\mathcal{M})/\#Z(H)$. (One divides here by $\#Z(H)$, which is the number of elements of the center of H, essentially because the generic connection has $\#Z(H)$ symmetries.) Hence the volume of $\mathcal{M}$ is

$$\int_{\mathcal{M}} \exp(\omega) = \int_{\mathcal{M}} \frac{\omega^n}{n!} = \#Z(H) \cdot \left(\frac{\mathrm{Vol}(H)}{(2\pi)^{\dim H}} \right)^{2g-2} \cdot \sum_{\alpha} \frac{1}{(\dim \alpha)^{2g-2}}. \qquad (4.19)$$

This is then a first special case of evaluation of the intersection pairings on moduli space.

4.1. Groups With Non-Trivial π_1

So far we have assumed that the gauge group H is connected and simply connected. These conditions ensure that an H bundle E over a two dimensional surface Σ is trivial. We now wish to drop the condition of simple connectivity.

Let Γ be a subgroup of the center $Z(H)$. We take the gauge group to be $H' = H/\Gamma$, which of course is still connected but is not simply connected.

Classification Of Bundles

An H bundle on a two dimensional surface is necessarily trivial, but this is not so for H' bundles. The possible H' bundles have the following standard description. Let Σ be a closed oriented Riemann surface of genus g. Let E' be a principal H' bundle over Σ. Let P be a point in Σ. E' is necessarily trivial when restricted to $\Sigma - P$ (that is, Σ with P deleted). Thus, on $\Sigma - P$, E' can be lifted to a principal H bundle E.

A connection A' on E', when restricted to $\Sigma - P$, lifts to a connection A on E. However, A does not necessarily extend smoothly over P. The monodromy u of A about P is an element of H that projects to the identity in H' (since A' extends smoothly over P). Thus, u is an element of Γ. It is easy to see that u

is a topological invariant of E'. Conversely, it is standard that u is the only such invariant and can take arbitrary values. Thus, the possible H' bundles $E'(u)$ are classified by the arbitrary choice of $u \in \Gamma$.

The Non-Singular Cases

We will develop most of the story for general H' and $E'(u)$. However, the topological results that arise are easiest to understand in cases in which the space of flat connections on $E'(u)$ is smooth, and is acted on freely by the gauge group. This implies in particular that the moduli space $\mathcal{M}'(u)$ of flat connections on $E'(u)$ is smooth.

The main case in which this occurs is the following. Let Σ be a Riemann surface of genus ≥ 1. Let $H = SU(N)$, and let $H' = H/\Gamma$, with $\Gamma \cong \mathbf{Z}/N\mathbf{Z}$ the center of $SU(N)$. Finally, let u be a generator of Γ. In such cases, the topological conditions that we want are known to hold. (For $g = 1$, the moduli space of flat connections on $E'(u)$ is a single point, for $g > 1$ a smooth closed manifold of dimension $(N^2 - 1) \cdot (2g - 2)$.) The simplest example, which we use later for illustration, is $H = SU(2)$, $H' = SO(3)$, $u = -1$.

The Volume Of The Gauge Group

We want to perform the H' path integral

$$\frac{1}{\text{Vol}(G')} \int DA' \, D\phi \quad \exp(-L), \tag{4.20}$$

with G' the group of gauge transformations of $E'(u)$. We will do this by relating the integral over A' to an integral over its lift A. We must worry about two points: the comparison of volumes of G' and G, which we consider first; and the role of the singularity at P.

The group G of maps of Σ to the connected and simply connected group H is connected. This is not true for G'; the set of components of G' can be identified with the finite group $\text{Hom}(H_1(\Sigma, \mathbf{Z}), \Gamma)$. This group has $\#\Gamma^{2g}$ elements. Thus, if G_1' is the identity component of G', we have $\text{Vol}(G') = \#\Gamma^{2g}\text{Vol}(G_1')$.

On the other hand, applying pointwise the projection $H \to H' = H/\Gamma$ gives a natural map $G \to G_1'$ whose kernel consists of constant gauge transformations by elements of Γ. Hence $\mathrm{Vol}(G) = \#\Gamma \cdot \mathrm{Vol}(G_1')$.

Combining these formulas, we have

$$\mathrm{Vol}(G) = \#\Gamma^{1-2g}\mathrm{Vol}(G'). \tag{4.21}$$

For future use, we also note the following elementary facts:

$$\mathrm{Vol}(H) = \#\Gamma \cdot \mathrm{Vol}(H')$$
$$\#Z(H) = \#\Gamma \cdot \#Z(H') \tag{4.22}$$
$$\#\pi_1(H') = \#\Gamma.$$

The Singularity At P

We now come to the essence of the matter, which is the role of the singularity at P. This can be deduced by cutting out of Σ a disc D containing P – so as to factorize the computation on the Hilbert space $\mathcal{H}_C$ associated with $C = \partial D$. The path integral over ϕ is still localized – as in (4.14) – on connections with $F = 0$ away from P. Because the monodromy about P is prescribed to be $u \in \Gamma$, the monodromy U around C is u rather than 1. In the derivation of (4.15), we must replace $\delta(U - 1) = \sum_\alpha \dim(\alpha)\chi_\alpha(U)$ by $\delta(U - u) = \sum_\alpha \dim(\alpha)\chi_\alpha(U) \cdot \lambda_\alpha(u^{-1})$. Here $\lambda_\alpha(u^{-1})$ is the following. In the α representation of H, the element u^{-1} of the center of H is represented by a complex number of modulus one that we have called $\lambda_\alpha(u^{-1})$. (So $\lambda_\alpha(u^{-1}) = \chi_\alpha(u^{-1})/\dim(\alpha)$.)

Thus, the role of the singularity at P is to bring about a substitution

$$\sum_\alpha \dim(\alpha)\chi_\alpha(U) \to \sum_\alpha \dim(\alpha)\chi_\alpha(U) \cdot \lambda_\alpha(u^{-1}). \tag{4.23}$$

Upon gluing the disc D into the rest of Σ and carrying out the overall evaluation of the path integral, the only role of the singularity is to multiply the contribution

of a representation α by a factor of $\lambda_\alpha(u^{-1})$. (In [9,§3], the factor of $\lambda_\alpha(u^{-1})$ is extracted in the special case $H = SU(2)$, $H' = SO(3)$, $u = -1$ from the Verlinde formula. This can presumably be done in general.)

Evaluation Of The Twisted Partition Function

Now we want to calculate the H' partition function

$$\widetilde{Z}(\Sigma; u) = \frac{1}{\text{Vol}(G')} \int DA' \, D\phi \ \exp(-L). \tag{4.24}$$

First we calculate the corresponding H partition function for connections on $\Sigma - P$ with monodromy u around P. This is

$$Z(\Sigma; u) = \frac{1}{\text{Vol}(G)} \int DA \, D\phi \ \exp(-L). \tag{4.25}$$

A is the lift of A'. From what we have just said, this is given by the same formula as (4.18) but weighting each representation by an extra factor of $\lambda_\alpha(u^{-1})$. So

$$Z(\Sigma; u) = \left(\frac{\text{Vol}(H)}{(2\pi)^{\dim H}} \right)^{2g-2} \cdot \sum_\alpha \frac{\lambda_\alpha(u^{-1})}{(\dim \alpha)^{2g-2}}. \tag{4.26}$$

We now use (4.21) to relate $Z(\Sigma; u)$ to $\widetilde{Z}(\Sigma; u)$, and also (4.22) to express the result directly in terms of properties of H'. Using also (4.22), we get

$$\widetilde{Z}(\Sigma; u) = \frac{1}{\#\pi_1(H')} \left(\frac{\text{Vol}(H')}{(2\pi)^{\dim H'}} \right)^{2g-2} \cdot \sum_\alpha \frac{\lambda_\alpha(u^{-1})}{(\dim \alpha)^{2g-2}}. \tag{4.27}$$

Note that in this formula, the sum runs over all isomorphism classes of irreducible representations of the universal cover H of H'.

Here is a check. Note that $v \to \lambda_\alpha(v)$ is a character of the finite abelian group Γ. Irreducible H' modules are the same as irreducible H modules for which this

character is trivial. By the orthogonality of the characters, $\sum_{u \in \Gamma} \lambda_\alpha(u)$ vanishes unless λ_α is trivial, in which case of course it equals $\#\Gamma = \#\pi_1(H')$. Hence we get

$$\sum_{u \in \gamma} \widetilde{Z}(\Sigma; u) = \left(\frac{\text{Vol}(H')}{(2\pi)^{\dim H'}} \right)^{2g-2} {\sum_\alpha}' \frac{1}{(\dim \alpha)^{2g-2}}. \tag{4.28}$$

Here $\sum'$ is a sum over isomorphism classes of irreducible H' modules. On the left the sum over $u \in \Gamma$ should be interpreted as a sum over all isomorphism classes of H' bundles. In general, if u is construed to label these isomorphism classes, (4.28) is true, as explained in [9,§2.3], even if H' is not connected. If H' is connected and simply connected, there is only one isomorphism class of H' bundle on Σ, and (4.28) reduces back to (4.18).

Just as in our discussion of simply connected gauge groups, $\widetilde{Z}(\Sigma; u)$ can be interpreted as $1/\#Z(H')$ times the volume of the moduli space $\mathcal{M}'(u)$ of flat connections on the bundle $E'(u)$. ($\#Z(H')$ arises, again, as the number of symmetries of a generic H' connection A'.) So the volume of $\mathcal{M}'(u)$ is

$$\int_{\mathcal{M}'(u)} \exp(\omega) = \frac{\#Z(H')}{\#\pi_1(H')} \cdot \left(\frac{\text{Vol}(H')}{(2\pi)^{\dim H'}} \right)^{2g-2} \cdot \sum_\alpha \frac{\lambda_\alpha(u^{-1})}{(\dim \alpha)^{2g-2}}. \tag{4.29}$$

4.2. PHYSICAL YANG-MILLS THEORY

Now we leave the topological field theory (4.1), and turn to physical Yang-Mills theory. We introduce on Σ a measure $d\mu$ of total measure 1, and consider

$$L = -\frac{i}{4\pi^2} \int_\Sigma \text{Tr}\,\phi F - \frac{\epsilon}{8\pi^2} \int_\Sigma d\mu\,\text{Tr}\,\phi^2, \tag{4.30}$$

with ϵ a positive real number. We wish to evaluate the corresponding path integral

$$\int DA\,D\phi\,\exp\left(\frac{i}{4\pi^2} \int_\Sigma \text{Tr}\,\phi F + \frac{\epsilon}{8\pi^2} \int_\Sigma d\mu\,\text{Tr}\,\phi^2 \right). \tag{4.31}$$

Upon performing the Gaussian integral over ϕ (or eliminating ϕ by its classical

equations of motion), we see that the same theory could be defined by the Lagrangian

$$I = -\frac{1}{2\epsilon'} \int\limits_{\Sigma} \mathrm{d}\mu \, \mathrm{Tr} \, f^2, \tag{4.32}$$

with $f = \star F$ and $\epsilon' = 4\pi^2 \epsilon$. (The Yang-Mills Lagrangian is most often written in terms of ϵ', but the topological formulas are perhaps most naturally written in terms of ϵ.) For $\epsilon \neq 0$, the full diffeomorphism invariance of the topological field theory that we have discussed up to this point is reduced to invariance under the group of area-preserving diffeomorphisms.

We should now make a preliminary observation, analogous to the remark following (4.2). Different recipes for defining the path integral in (4.30) will differ (in addition to the ambiguity already cited in (4.3)) by

$$L \to L + t\epsilon' \int\limits_{\Sigma} \mathrm{d}\mu, \tag{4.33}$$

with t an arbitrary parameter. This term respects the invariance under area-preserving diffeomorphisms. Just as we fixed the ambiguity noted in (4.3) to agree with the theory of Reidemeister-Ray-Singer torsion, we will at a judicious moment adjust the value of t to agree with the topological theory to which we wish to compare. In principle, a suitable calculation using a regulator in which the manipulations of §3 are valid could probably be used to give an *a priori* computation of t. In any case, the extra term in (4.33) has a "trivial" effect on the partition function, multiplying it by $\exp(-t\epsilon')$.

The Classical Solutions

As we have recalled in the introduction, the space $\mathcal{A}$ of connections can be regarded as a symplectic manifold, acted on symplectically by the group G of gauge transformations; and I can be interpreted as the norm squared of the moment map μ, with respect to a certain invariant quadratic form. The critical points of I are the

classical solutions of two dimensional Yang-Mills theory. They have the following explicit description [17]. The Euler-Lagrange equation derived by varying I is

$$0 = Df, \tag{4.34}$$

with D the gauge-covariant extension of the exterior derivative. This is certainly obeyed for $f = 0$ – which corresponds to the zeros of the moment map and the absolute minimum of I. Higher critical points correspond to $f \neq 0$, in which case f, being covariantly constant, gives a reduction of the structure group of the connection to a subgroup H_0 that commutes with f. Solutions of (4.34) can therefore be described rather explicitly: they are flat H_0 connections twisted by constant curvature line bundles in the $U(1)$ subgroup generated by f.

However, to simplify things, in this part of our story we will just consider the case that the gauge group is $SU(2)$ or $SO(3)$. Then f, if not zero, reduces the structure group precisely to $U(1)$, so in the $SU(2)$ case, for instance, we get an $SU(2)$ bundle with a covariantly constant splitting as a sum of line bundles. From the classification of line bundles, it follows at once that the conjugacy class of f is given by

$$f = 2\pi m \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \tag{4.35}$$

with $m \in \mathbb{Z}$. The value of I at such a critical point is then directly computed to be

$$I_m = \frac{(2\pi m)^2}{\epsilon'}. \tag{4.36}$$

In the $SO(3)$ case, there are two isomorphism classes of bundle. The bundles that lift to $SU(2)$ bundles give the same result just described. The non-trivial $SO(3)$ bundles can be described as $SU(2)$ bundles on $\Sigma - P$ with monodromy

$u = -1$ about P. Allowing for this, (4.35) is just replaced by

$$f = 2\pi(m + 1/2) \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \tag{4.37}$$

with $m \in \mathbb{Z}$. The values of I at these critical points are now

$$I'_m = \frac{(2\pi(m + 1/2))^2}{\epsilon'}. \tag{4.38}$$

The Hamiltonian

Our next goal is to evaluate the partition function of physical Yang-Mills theory, with Lagrangian (4.30), on a Riemann surface of genus g.

The main thing that we need is to evaluate the Hamiltonian. To this aim, we pick an initial value circle $C \subset \Sigma$. We write the volume form of Σ in a neighborhood of C as $d\sigma \wedge d\tau$ where C is defined by $\tau = 0$, and σ is a periodic parameter on C, with $\oint_C d\sigma = 1$. The Hamiltonian operator H will be the generator of translations in τ.

Standard canonical quantization shows that

$$H = -\frac{\epsilon'}{2} \oint d\sigma \operatorname{Tr} \widetilde{\phi}^2 + t\epsilon'. \tag{4.39}$$

We have included the extra term from (4.33), which just adds a constant to H.

In view of (4.9), this can be described as follows. Let C_2 be the quadratic Casimir operator $C_2 = -\sum_a T_a^2$, with T_a an orthonormal basis of $\mathcal{H}$. Then

$$H = \frac{\epsilon'}{2} C_2 + t\epsilon'. \tag{4.40}$$

When $\epsilon \neq 0$, a measure μ must be introduced on every Riemann surface we consider. The simplest case is the two-holed sphere Σ of figure (2). Suppose that

the total measure is ρ. Then the area form of Σ can be represented by the two form $d\sigma \wedge d\tau$, with σ as above and $0 \leq \tau \leq \rho$. The path integral on Σ can be computed as the matrix element of $\exp(-\rho H)$ between external states on the boundary. The generalization of (4.12) to $\epsilon \neq 0$ is therefore simply

$$\Psi_{\{2\}} = \sum_\alpha \chi_\alpha(U_1)\chi_\alpha(U_2)\exp\left(-\epsilon'\rho\left(\frac{C_2(\alpha)}{2}+t\right)\right). \qquad (4.41)$$

The ρ dependence of the amplitude on any Riemann surface with an external line labeled by a representation α is given by the same factor $\exp\left(-\epsilon'\rho\left(\frac{C_2(\alpha)}{2}+t\right)\right)$ as in (4.41), since one can always increase the area of Σ by gluing a cylinder on to one of the external lines. Sewing together the external lines, it follows that also for a Riemann surface without boundary, the contribution of any representation α to the partition function has this universal ρ dependence.

Therefore, we can immediately write down the partition function, with gauge group H', for connections on a bundle $E'(u)$, generalizing (4.27) to $\epsilon \neq 0$. We get

$$\widetilde{Z}(\Sigma, \epsilon; u) = \frac{1}{\#\pi_1(H')} \cdot \left(\frac{\mathrm{Vol}(H')}{(2\pi)^{\mathrm{dimH'}}}\right)^{2g-2} \cdot \sum_\alpha \frac{\lambda_\alpha(u^{-1}) \cdot \exp\left(-\epsilon'\left(\frac{C_2(\alpha)}{2}+t\right)\right)}{(\dim\alpha)^{2g-2}}.$$
$$(4.42)$$

There is still one unknown parameter t.

4.3. Comparison With The Localization Formula

Finally, we can begin to enjoy the fruits of our labors – comparing (4.42) to the predictions of the localization formula of §2-3. We will do this in full detail – identifying the contributions of higher critical points – only for $SU(2)$ and $SO(3)$. For other groups, we will study only the contribution of $\mu^{-1}(0)$.

Up to isomorphism, $SU(2)$ has one irreducible representation α_n of dimension n for every positive integer n. The value of the quadratic Casimir for this representation is with our normalization $C_2(\alpha_n) = (n^2 - 1)/2$. We set $t = 1/4$, so that

60

the eigenvalues of the Hamiltonian are just $\epsilon'n^2/4$. As will be clear, this value is required for agreeing with the predictions of the topological theory. (Otherwise, the contribution of $\mu^{-1}(0)$ is not a polynomial in ϵ but has an extra exponential factor. Later, we will generalize this determination of t to general compact Lie groups and general Casimirs.)

First we consider the case of $H = SU(2)$. Then $\text{Vol}(SU(2)) = 2^{5/2}\pi^2$ with our conventions, and so

$$Z(\Sigma, \epsilon) = \frac{1}{(2\pi^2)^{g-1}} \sum_{n=1}^{\infty} \frac{\exp\left(-\epsilon'n^2/4\right)}{n^{2g-2}}. \tag{4.43}$$

On the other hand, for a non-trivial $SO(3)$ bundle with $u = -1$, we have $\lambda_n(u^{-1}) = (-1)^{n+1}$, $\#\pi_1(H') = 2$ and $\text{Vol}(SO(3)) = 2^{3/2}\pi^2$, so

$$\widetilde{Z}(\Sigma, \epsilon; -1) = \frac{1}{2 \cdot (8\pi^2)^{g-1}} \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \exp(-\epsilon'n^2/4)}{n^{2g-2}}. \tag{4.44}$$

Sum Over Critical Points

We will now show how (4.44) and (4.43) can be written as a sum over critical points. In doing so, we consider first the case of genus $g \geq 1$; then we will return to special features of $g = 0$.

First we consider the case of a non-trivial $SO(3)$ bundle. It is convenient to look at not $\widetilde{Z}$ but

$$\frac{\partial^{g-1}\widetilde{Z}}{\partial\epsilon'^{g-1}} = \frac{(-1)^g}{2 \cdot (32\pi^2)^{g-1}} \sum_{n=1}^{\infty} (-1)^n \exp(-\epsilon'n^2/4). \tag{4.45}$$

We write

$$\sum_{n=1}^{\infty}(-1)^n \exp(-\epsilon'n^2/4) = -\frac{1}{2} + \frac{1}{2}\sum_{n\in\mathbb{Z}}(-1)^n \exp(-\epsilon'n^2/4). \tag{4.46}$$

The sum on the right hand side of (4.46) is a theta function, and in the standard way we can use the Poisson summation formula to derive the Jacobi inversion

61

formula:

$$\sum_{n \in \mathbb{Z}} (-1)^n \exp(-\epsilon' n^2/4) = \sum_{m \in \mathbb{Z}} \int_{-\infty}^{\infty} dn \exp\left(2\pi i n m + i\pi n - \epsilon' n^2/4\right)$$

$$= \sqrt{\frac{4\pi}{\epsilon'}} \sum_{m \in \mathbb{Z}} \exp\left(-\frac{(2\pi(m+1/2))^2}{\epsilon'}\right). \tag{4.47}$$

Putting the pieces together,

$$\frac{\partial^{g-1} \widetilde{Z}}{\partial \epsilon'^{g-1}} = \frac{(-1)^g}{4 \cdot (32\pi^2)^{g-1}} \cdot \left(-1 + \sqrt{\frac{4\pi}{\epsilon'}} \sum_{m \in \mathbb{Z}} \exp\left(-\frac{(2\pi(m+1/2))^2}{\epsilon'}\right)\right). \tag{4.48}$$

Now let us compare this formula to the topological theory explained in §2. For a non-trivial $SO(3)$ bundle, $\mu^{-1}(0)$ is smooth and acted on freely by G, so we can apply the reasoning of §2.4. For $\epsilon \to 0$, the function $\widetilde{Z}(\epsilon)$ should be the sum of a polynomial in ϵ of degree at most $\dim \mathcal{M}/4$, plus exponentially small contributions of unstable critical points. Looking back to (4.38), where we determined the values of the square of the moment map at the unstable critical points, the mth critical point should make a contribution proportional for small ϵ to $\exp\left(-(2\pi(m+1/2))^2/\epsilon'\right)$, up to a power of ϵ. These are precisely the exponents on the right hand side of (4.48). (Integrating $g-1$ times with respect to ϵ to recover $\widetilde{Z}(\epsilon)$ from (4.48) will not change these exponents.)

(4.48) shows that $\partial^{g-1} \widetilde{Z}/\partial \epsilon'^{g-1}$ is a constant up to exponentially small terms, and hence $\widetilde{Z}(\epsilon)$ is a polynomial of degree $g-1$ up to exponentially small terms. The terms of order ϵ^k, $k \leq g-2$ that have been annihilated by differentiating $g-1$ times with respect to ϵ' are most easily computed by expanding (4.44) in powers of ϵ:

$$\widetilde{Z}(\epsilon) = \frac{1}{2(8\pi^2)^{g-1}} \sum_{k=0}^{g-2} \frac{(-\pi^2 \epsilon)^k}{k!} (1 - 2^{3-2g+2k}) \zeta(2g - 2 - 2k) + O(\epsilon^{g-1}). \tag{4.49}$$

Using Euler's formula expressing $\zeta(2n)$ for positive integral n in terms of the

Bernoulli number B_{2n},

$$\zeta(2n) = \frac{(2\pi)^{2n}(-1)^{n+1}B_{2n}}{2(2n)!},$$ (4.50)

(4.49) implies

$$\int_{\mathcal{M}'(-1)} \exp(\omega + \epsilon\Theta) = (-1)^{g+1} \sum_{k=0}^{g-1} \frac{\epsilon^k}{k!} \frac{(2^{2g-2-2k} - 2)B_{2g-2-2k}}{2^{3g-1}(2g-2-2k)!}.$$ (4.51)

(We have also used (4.48) and $B_0 = 1$ to get the term of order ϵ^{g-1}.) This agrees with the final equation (29) of [22] provided one notes that the relation between Thaddeus's classes and ours is $\alpha = 2\omega$, $\beta = 4\Theta$, and that his $\mathcal{N}_g$ (which is the moduli space of flat $SU(2)$ connections on a once-punctured surface with monodromy -1 around the puncture) is an unramified 2^{2g}-fold cover of our $\mathcal{M}'(-1)$ (which is the moduli space of flat $SO(3)$ connections on a non-trivial bundle).

In particular, (4.51) is a polynomial in ϵ of degree $g - 1$, while on dimensional grounds it might have been of degree $\dim(\mathcal{M})/4 = (3/2)(g - 1)$. The fact that the higher coefficients of the polynomial vanish is a reflection of the fact that $p_1(\mathcal{M})^g = 0$ (a conjecture of Gieseker proved by F. Kirwan [23]).

(4.51) is not yet a complete answer for the intersection pairings on $\mathcal{M}'(-1)$. It is necessary also to include certain non-algebraic cycles; we do this in §4.5.

Analogous Formulas For $SU(2)$

Now we consider the case of gauge group $SU(2)$. We start with

$$\frac{\partial^{g-1}Z(\Sigma, \epsilon)}{\partial\epsilon^{g-1}} = \frac{(-1)^g}{(8\pi^2)^{g-1}} \sum_{n=1}^{\infty} \exp\left(-\frac{\epsilon n^2}{4}\right) = \frac{(-1)^g}{2 \cdot (8\pi^2)^{g-1}} \left(-1 + \sum_{n\in\mathbb{Z}} \exp\left(-\frac{\epsilon n^2}{4}\right)\right).$$ (4.52)

Use of the Poisson summation formula now gives

$$\frac{\partial^{g-1}Z}{\partial\epsilon^{g-1}} = \frac{(-1)^g}{2 \cdot (8\pi^2)^{g-1}} \cdot \left(-1 + \sqrt{\frac{4\pi}{\epsilon}} \sum_{m\in\mathbb{Z}} \exp\left(-\frac{(2\pi m)^2}{\epsilon}\right)\right).$$ (4.53)

The exponents of the exponentially small terms in (4.53) are in agreement

with our expectations from (4.36). The novelty, compared to our discussion of $SO(3)$, is that the term on the right hand side of (4.53) with $m = 0$ does not vanish exponentially for small ϵ. As a result, the $(g-1)^{th}$ derivative of Z is not a constant for small ϵ, but proportional to $\epsilon^{-1/2}$. The general structure is thus

$$Z(\epsilon) = \sum_{k=0}^{g-2} a_k \epsilon^k + a_{g-3/2} \epsilon^{g-3/2} + \text{exponentially small terms.} \qquad (4.54)$$

The coefficients in this expansion can easily be worked out as in (4.49).

As always, the terms in (4.54) that do not vanish exponentially must be interpreted as the contribution of $\mu^{-1}(0)$ to the localization formula. The non-analyticity of the contribution of $\mu^{-1}(0)$ reflects the fact that, for gauge group $SU(2)$, $\mu^{-1}(0)$ is singular. By studying the predictions of the localization theory when $\mu^{-1}(0)$ is singular, it should be possible to interpret the exponent $g - 3/2$ of the singular term in terms of the singularities of the moduli space $\mathcal{M}$ of flat $SU(2)$ connections.

For non-trivial $SO(3)$ bundles, the contribution of $\mu^{-1}(0)$ to $\widetilde{Z}(\epsilon)$ is a polynomial, whose coefficients are elementary multiples of $\int_{\mathcal{M}'} \omega^{3g-3-2r} \Theta^r$. In the $SU(2)$ case, such an interpretation cannot hold as the contribution of $\mu^{-1}(0)$ is not a polynomial. In some instances, Donaldson has shown that intersection pairings (on singular four dimensional moduli spaces) analogous to $\int \omega^{3g-3-2r} \Theta^r$ are well-defined only for small enough r. Our considerations here perhaps give a new framework for this phenomenon: the function $Z(\epsilon)$ is defined in any case, but the extent to which it has an asymptotic expansion in integral powers of ϵ and the interpretation of the coefficients as intersection pairings depend on details of the classical geometry of the moduli space.

4.4. GENUS ZERO

Now we are going to look more closely at the behavior of the partition function for the case that Σ is a closed Riemann surface of genus zero. We consider arbitrary compact, connected (but not necessarily simply connected) gauge group H'. The general formula for the partition function, specialized to genus zero, is

$$\widetilde{Z}(\Sigma, \epsilon; u) = \frac{1}{\#\pi_1(H')} \cdot \left(\frac{(2\pi)^{\dim H'}}{\text{Vol}(H')} \right)^2 \cdot \sum_\alpha (\dim \alpha)^2 \lambda_\alpha(u^{-1}) \cdot \exp\left(-\epsilon \left(\frac{C_2(\alpha)}{2} + t \right) \right).$$
(4.55)

If one takes H' to be $SU(2)$ or $SO(3)$, then the right hand side of (4.55) is essentially the derivative of a theta function, rather than the $g-1$-fold integral of a theta function considered earlier. (4.55) can be expanded as a sum of contributions of critical points, using similar arguments to those we gave above for $g \geq 1$; this is left to the reader. Our intention here is to analyze closely the contribution of $\mu^{-1}(0)$. We will do this without restriction on H'. We actually will only look at the leading behavior for $\epsilon \to 0$, so we can set $t = 0$. (This is fortunate as we have not yet determined t for general H'.) The representation of $\widetilde{Z}$ that we will use appears in the work of Fine [6] and Forman [10].

Actually, $\mu^{-1}(0)$ is empty in genus zero unless $u = 1$, since a flat connection on a two sphere with one point deleted cannot have a non-trivial monodromy around the puncture. So (4.55) should vanish exponentially except for $u = 1$. We will verify this presently.

Let $\text{Fun}(H)$ be the space of functions on the H manifold, regarded as an $H_L \times H_R$ module (with $H_L \times H_R$ being two copies of H, acting on H by $h \to ahb^{-1}$). The decomposition of $\text{Fun}(H)$ in irreducible $H_L \times H_R$ modules is (by the Peter-Weyl theorem)

$$\text{Fun}(H) \cong \oplus_\alpha \alpha \otimes \overline{\alpha},$$
(4.56)

where the sum runs over isomorphism classes of irreducible H_L modules α, and $\overline{\alpha}$ is the complex conjugate H_R module.

We recall that we defined the quadratic Casimir operator C_2 by $C_2 = -\sum_a T_a^2$, where T_a runs over a basis of $\mathcal{H}$ orthonormal with respect to a certain invariant metric. The same metric determines a Laplace operator Δ on H. Since α and $\overline{\alpha}$ have the same value of C_2, it follows from (4.56) that the quadratic Casimirs of H_L and H_R coincide as operators on $\mathrm{Fun}(H)$; and moreover, both are equal to Δ.

If, therefore, u is in the center of H and $T(u^{-1})$ is the operator of left multiplication by $u^{-1} \in H$, then using the Peter-Weyl theorem

$$\mathrm{Tr}_{\mathrm{Fun}(H)}\, T(u^{-1}) \exp\left(\frac{-\epsilon\Delta}{2}\right) = \sum_\alpha (\dim \alpha)^2 \lambda_\alpha(u^{-1}) \exp\left(-\frac{\epsilon C_2(\alpha)}{2}\right). \quad (4.57)$$

We have used the fact that Δ acts on the representation α as $C_2(\alpha)$. The right hand side is, up to an elementary constant, the desired function $\widetilde{Z}(\Sigma, \epsilon; u)$.

We can now see the expected exponential vanishing of $\widetilde{Z}$ for $\epsilon \to 0$ with $u \neq 1$. The left hand side of (4.57) is essentially a matrix element of the heat kernel $\exp(-\epsilon\Delta/2)$. By the general theory of the short time behavior of the heat kernel, it vanishes exponentially for $\epsilon \to 0$ and any fixed $u \neq 1$, with an exponent determined by the length of the shortest geodesic on H from 1 to u.

It remains to consider the case $u = 1$. According to the general theory of the heat kernel, the left hand side of (4.57) is for $u = 1$ asymptotic for small ϵ to $\mathrm{Vol}(H)/(2\pi\epsilon)^{\dim(H)/2}$. Using this in (4.55) (and recalling that $\mathrm{Vol}(H) = \mathrm{Vol}(H') \cdot \#\pi_1(H')$), we get the small ϵ asymptotic behavior

$$\widetilde{Z}(\Sigma, \epsilon; u) \sim \frac{1}{\mathrm{Vol}(H')} \cdot \left(\frac{(2\pi)^3}{\epsilon}\right)^{\dim H'/2}. \quad (4.58)$$

Comparison To The Localization Formula

Now we compare to the localization formula. As always, $\mu^{-1}(0)$ consists of flat connections. In genus zero, every flat connection is gauge equivalent to the trivial connection. The group G' of gauge transformations does not act freely on

66

the space of flat connections; the trivial connection, for instance, is invariant under the finite dimensional group H' of constant gauge transformations, and every flat connection has a stabilizer isomorphic to this. So $\mu^{-1}(0)$ is a copy of G'/H'. A neighborhood of $\mu^{-1}(0)$ in the space $\mathcal{A}$ of connections can therefore be modeled on $T^*(G'/H')$. (By using the relation of connections and complex structures, one can even – after picking a complex structure on Σ – identify a dense open set in $\mathcal{A}$ with $T^*(G'/H')$.) Therefore, the problem of identifying the contribution of $\mu^{-1}(0)$ to $\widetilde{Z}$ is an infinite dimensional version of the problem that was solved in finite dimensions in equation (2.42).

Two-dimensional Yang-Mills theory is equivariant integration over $\mathcal{A}$ of $\exp(\overline{\omega})$, where $\overline{\omega} = \omega + i \sum_a \phi^a \mu_a$ is the equivariant extension of the symplectic form ω introduced in equation (1.8). Thus the form α of equation (2.42) can be identified with $\exp(\overline{\omega})$. Since ω restricts to zero on $\mu^{-1}(0)$, and the same is of course also true of μ, the reduced form α' of equation (2.42) is 1. Therefore (2.42) identifies the contribution of $\mu^{-1}(0)$ to the functional integral as

$$\frac{1}{\mathrm{Vol}(H')} \int \frac{\mathrm{d}\phi_1 \ldots \mathrm{d}\phi_{\dim H'}}{(2\pi)^{\dim H'}} \exp\left(\frac{\epsilon}{8\pi^2} \mathrm{Tr}\, \phi^2\right). \tag{4.59}$$

(The factor of $8\pi^2$ is inherited from the definition of the Yang-Mills Lagrangian in (4.30).) Upon performing explicitly the Gaussian integral in (4.59), we recover the asymptotic expression of (4.58), as desired.

4.5. Pairings Of Non-Algebraic Cycles For $SO(3)$

The cohomology of the smooth $SO(3)$ moduli space $\mathcal{M}'(-1)$ is known [21,18] to be generated by the classes ω and Θ, whose intersection pairings have been determined in equation (4.51) above, along with certain non-algebraic cycles, which we will now incorporate.

The basic formula that we will use is equation (3.41) from §3.2:

$$\langle \exp\left(\omega + \epsilon\Theta\right) \cdot \beta \rangle' = \frac{1}{\text{vol}(G)} \int DA \; D\psi \; D\phi \; \exp\left(\frac{1}{4\pi^2} \int_\Sigma \text{Tr}\left(i\phi F + \frac{1}{2}\psi \wedge \psi\right)\right.$$

$$\left.+\frac{\epsilon}{8\pi^2} \int_\Sigma d\mu \, \text{Tr}\,\phi^2\right) \cdot \beta.$$

$$(4.60)$$

We recall that $\langle \quad \rangle'$ coincides with integration over moduli space, up to terms that vanish exponentially for $\epsilon \to 0$.

Note that ψ is a free field, with a Gaussian measure, and the "trivial" propagator

$$\langle \psi_i^a(x)\psi_j^b(y) \rangle = -4\pi^2 \epsilon_{ij}\delta^{ab}\delta^2(x-y).$$

$$(4.61)$$

This will make life easy.

The new cycles that we must incorporate have the following description, from §3.3. For every circle $C \subset \Sigma$ there is a quantum field operator

$$V_C = \frac{1}{4\pi^2} \int_C \text{Tr}\,\phi\psi.$$

$$(4.62)$$

It represents a three dimensional class on moduli space; this class depends only on the homology class of C. As the algebraic cycles are even dimensional, non-zero intersection pairings are possible only with an even number of the V_C's. The first case is $\langle \exp(\omega + \epsilon\Theta) \cdot V_{C_1} V_{C_2} \rangle'$, with two oriented circles C_1, C_2 that we can suppose to intersect transversely in finitely many points. So we consider

$$\langle \exp\left(\omega + \epsilon\Theta\right) \cdot V_{C_1} V_{C_2} \rangle' = \frac{1}{\text{vol}(G)} \int DA \; D\psi \; D\phi \; \exp\left(\frac{1}{4\pi^2} \int_\Sigma \text{Tr}\left(i\phi F + \frac{1}{2}\psi \wedge \psi\right)\right.$$

$$\left.+\frac{\epsilon}{8\pi^2} \int_\Sigma d\mu \, \text{Tr}\,\phi^2\right) \cdot \frac{1}{4\pi^2} \int_{C_1} \text{Tr}\,\phi\psi \cdot \frac{1}{4\pi^2} \int_{C_2} \text{Tr}\,\phi\psi.$$

$$(4.63)$$

Upon performing the ψ integral, using (4.61), we see that this is equivalent to

$$\frac{1}{\text{vol}(G)} \int DA \, D\psi \, D\phi \, \exp \left(\frac{1}{4\pi^2} \int_\Sigma \text{Tr} \left(i\phi F + \frac{1}{2} \psi \wedge \psi \right) \right.$$

$$\left. + \frac{\epsilon}{8\pi^2} \int_\Sigma d\mu \, \text{Tr} \, \phi^2 \right) \cdot \sum_{P \in C_1 \cap C_2} \frac{-\sigma(P)}{4\pi^2} \text{Tr} \, \phi^2(P).$$

$$(4.64)$$

Here P runs over all intersection points of C_1 and C_2, and $\sigma(P) = \pm 1$ is the oriented intersection number of C_1 and C_2 at P. Since the cohomology class of $\text{Tr} \, \phi^2(P)$ is independent of P, and equal to that of $\int_\Sigma d\mu \, \text{Tr} \, \phi^2$, (4.64) implies

$$\langle \exp(\omega + \epsilon\Theta) \cdot V_{C_1} V_{C_2} \rangle' = \frac{1}{\text{vol}(G)} \int DA \, D\psi \, D\phi \, \exp \left(\frac{1}{4\pi^2} \int_\Sigma \text{Tr} \left(i\phi F + \frac{1}{2} \psi \wedge \psi \right) \right.$$

$$\left. + \frac{\epsilon}{8\pi^2} \int_\Sigma \text{Tr} \, \phi^2 \right) \cdot \left(-\frac{\#(C_1 \cap C_2)}{4\pi^2} \int_\Sigma \text{Tr} \, \phi^2 \right),$$

$$(4.65)$$

with $\#(C_1 \cap C_2) = \sum_P \sigma(P)$ the algebraic intersection number of C_1 and C_2. (4.65) is equivalent to

$$\langle \exp(\omega + \epsilon\Theta) V_{C_1} V_{C_2} \rangle' = -2\#(C_1 \cap C_2) \cdot \frac{\partial}{\partial\epsilon} \langle \exp(\omega + \epsilon\Theta) \rangle', \qquad (4.66)$$

which interpreted in terms of intersection numbers gives in particular

$$\int_{\mathcal{M}'(-1)} \exp(\omega + \epsilon\Theta) V_{C_1} V_{C_2} = -2\#(C_1 \cap C_2) \frac{\partial}{\partial\epsilon} \int_{\mathcal{M}'(-1)} \exp(\omega + \epsilon\Theta). \qquad (4.67)$$

Of course, the right hand side is known from (4.51).

The generalization to an arbitrary number of V's is almost immediate. Consider oriented circles C_σ, $\sigma = 1 \ldots 2g$, representing a basis of $H_1(\Sigma, \mathbb{Z})$. Let

69

$\gamma_{\sigma\tau} = \#(C_\sigma \cap C_\tau)$ be the matrix of intersection numbers. Introduce anticommuting parameters η_σ, $\sigma = 1 \ldots 2n$. I claim

$$\int_{\mathcal{M}'(-1)} \exp\left(\omega + \epsilon\Theta + \sum_{\sigma=1}^{2g} \eta_\sigma V_{C_\sigma}\right) = \int_{\mathcal{M}'(-1)} \exp\left(\omega + \widehat{\epsilon}\Theta\right), \qquad (4.68)$$

with

$$\widehat{\epsilon} = \epsilon - 2\sum_{\sigma<\tau} \eta_\sigma\eta_\tau\gamma_{\sigma\tau}. \qquad (4.69)$$

The computation leading to this formula is a minor variant of the one we have just done. The left hand side of (4.68) is equal (up to terms that vanish exponentially for $\epsilon \to 0$) to

$$\frac{1}{\mathrm{vol}(G)} \int DA \ D\psi \ D\phi \ \exp\left(\frac{1}{4\pi^2} \int_\Sigma \mathrm{Tr}\left(i\phi F + \frac{1}{2}\psi \wedge \psi\right)\right.$$
$$\left. + \frac{\epsilon}{8\pi^2} \int_\Sigma d\mu \, \mathrm{Tr} \, \phi^2 + \frac{1}{4\pi^2} \sum_{\sigma=1}^{2n} \eta_\sigma \int_{C_\sigma} \mathrm{Tr} \, \phi\psi\right). \qquad (4.70)$$

Shifting ψ to complete the square, and then performing the Gaussian integral over ψ, this becomes

$$\frac{1}{\mathrm{vol}(G)} \int DA \ D\psi \ D\phi \ \exp\left(\frac{1}{4\pi^2} \int_\Sigma \mathrm{Tr}\left(i\phi F + \frac{1}{2}\psi \wedge \psi\right)\right.$$
$$\left. + \frac{\widehat{\epsilon}}{8\pi^2} \int_\Sigma d\mu \, \mathrm{Tr} \, \phi^2\right). \qquad (4.71)$$

The polynomial part of this is the right hand side of (4.68).

The method of eliminating the non-algebraic cycles that we have just described is somewhat analogous to Proposition (26) of [22]. Our formulas (4.67) and (4.68) are equivalent to the formula given by Thaddeus in Proposition (26), modulo the evaluation (4.51) of intersection numbers of algebraic cycles and the fact that our V_C is the same as Thaddeus's ψ_C.

4.6. Casimir Operators For Arbitrary Lie Groups

Let $\widetilde{\phi} = \phi/4\pi^2$, and let $Q(\widetilde{\phi})$ be an invariant polynomial in ϕ, homogeneous of degree t. Q should correspond to an operator $\widehat{Q}$ in two-dimensional quantum Yang-Mills theory, and in (4.9), we showed that, with a particular definition of the quantum theory, this is just the Casimir operator $Q(-iT)$.

In general, the passage from classical to quantum mechanics is uniquely determined only up to a renormalization of the various operators and parameters. In a theory which is as strongly ultraviolet convergent as two-dimensional Yang-Mills theory, the only ambiguity is "normal ordering"; different ways of defining the quantum operator corresponding to $Q(\widetilde{\phi})$ will differ only by terms that can be considered to come from the addition to Q of invariant polynomials of lower degree. Thus, in general, with an arbitrary method of defining the theory, $\widehat{Q} = Q(-iT) + \text{lower order Casimir operators}$.

In principle, the topological regularization of §3 should uniquely determine the normal-ordering recipe; I will leave this as an interesting open problem. But part of the story is easy to discern. The topological regularization certainly preserves equation (3.58), so we should restrict ourselves to normal-ordering prescriptions compatible with this.* This means simply that $Q \to \widehat{Q}$ must be a ring homomorphism from invariant polynomials on the Lie algebra to quantum operators. Thus, it is sufficient to determine $\widehat{Q}$ with Q ranging over a set of generators of the ring of invariant polynomials.

For instance, for $SU(2)$, this ring is a polynomial ring with one generator, which we can take to be $Q(\widetilde{\phi}) = \text{Tr}\,\widetilde{\phi}^2$. In this case, (4.9) gives $\widehat{Q} = -\sum_a \text{Tr}\,T_a^2 = C_2$, while a lower order Casimir operator would have to be a constant. This one constant is the only normal-ordering ambiguity for $SU(2)$. It was called t in equation (4.40). Ideally, t should be determined by an *a priori* calculation using a regularization in which the relation of §3 between the physical and topological

---

$\star$ The extension of the formalism to incorporate (3.59) and (3.60) will be explained in §5.

theories is valid. We did so more pragmatically at the beginning of §4.3 to ensure a particular consequence of this relation (certain functions should be polynomials).

In general, for a compact Lie group H of rank r, the ring of invariant polynomials is a polynomial ring in r generators (see [41,§126]). For any given H, there are therefore finitely many analogs of the normal ordering constant t. We will now state a generalization of the prescription $t = 1/4$ to arbitrary Lie groups and arbitrary Casimirs.

First we give a convenient restatement of the situation for $SU(2)$. Let us call v the highest weight of the two dimensional representation of $SU(2)$. The highest weight of the n dimensional representation α_n is then $h = (n-1)v$. One half the sum of the positive roots of $SU(2)$ is $\delta = v$, so $h + \delta = nv$. The generator $Q = \mathrm{Tr}\,\widetilde{\phi}^2$ of the ring of invariants corresponds, if $t = 1/4$, to the quantum operator $\widehat{Q}(\alpha_n) = n^2/2$. Since the Weyl group of $SU(2)$ is $\mathbb{Z}/2\mathbb{Z}$, acting by $v \to -v$, this can be described as follows: for a representation of highest weight h, $\widehat{Q}$ is a homogeneous, Weyl-invariant polynomial in $h + \delta$, which agrees with the usual quadratic Casimir $Q(-iT)$ up to terms of lower order.

This formulation can be immediately generalized. Let $Q(\widetilde{\phi})$ be any homogeneous invariant polynomial of degree n on the Lie algebra of a compact simple Lie group H. According to [41,§126, Theorem 7], the corresponding Casimir $Q(-iT)$ is equal, on an irreducible representation α_h of highest weight h, to $Q'(h+\delta)$, where Q' is a Weyl-invariant polynomial of degree n. Moreover $Q'(h+\delta) = Q(h+\delta)+\ldots$, where the "..." are lower order Weyl-invariant polynomials, which coincide, by essentially the same theorem, with some Casimirs of lower order. Since the ability to add such terms is precisely the normal ordering ambiguity, we can pick a normal ordering recipe in which, acting on α_h,

$$\widehat{Q} = Q(h + \delta). \tag{4.72}$$

For the time being, this is an arbitrary ansatz. However, this choice will enter at the end of §5 in verifying consequences of the relation of §3 between the physical

and topological theories.

Generalization Of The Path Integral

We now want to evaluate the following generalization of the conventional Yang-Mills path integral (4.31):

$$\int DA \ D\phi \ \exp\left(\frac{i}{4\pi^2}\int_\Sigma \mathrm{Tr}\,\phi F + \int_\Sigma Q(\widetilde{\phi})\right), \tag{4.73}$$

with $Q(\widetilde{\phi})$ an arbitrary invariant polynomial on $\mathcal{H}$ (with some positivity properties to ensure convergence of the integral, or with the higher than quadratic terms in Q having nilpotent coefficients to avoid such questions). The path integral can be evaluated by summing over the same physical states as before. The only novelty is that the Hamiltonian is different: it is now $H = -\widehat{Q}$. With our normal-ordering recipe, the generalization of (4.42) is then

$$\widetilde{Z}(\Sigma, Q; u) = \frac{1}{\#\pi_1(H')} \cdot \left(\frac{\mathrm{Vol}(H')}{(2\pi)^{\dim\mathrm{H}'}}\right)^{2g-2} \cdot \sum_h \frac{\lambda_h(u^{-1})\cdot \exp\left(Q(h+\delta)\right)}{d(h)^{2g-2}}. \tag{4.74}$$

Here h runs over dominant weights of H (which are in one-to-one correspondence, of course, with isomorphism classes of irreducible H modules $\alpha(h)$), $d(h)$ is the dimension of $\alpha(h)$, and $\lambda_h = \lambda_{\alpha(h)}$.

5. The Intersection Ring Of The Moduli Space

In §4, we computed the intersection pairings on the moduli space of flat $SO(3)$ connections on a non-trivial bundle over a surface of genus ≥ 2. We now want to extend this computation to other groups. We consider an arbitrary compact connected gauge group H', with simply connected cover H. We work on an arbitrary H' bundle $E'(u)$ over a closed oriented surface Σ of genus g. We will evaluate the quantum field theory partition function in general, and then interpret it in terms of intersection pairings for the smooth cases.

The basic formula that we will use is our friend (3.41):

$$\langle \exp\left(\omega + \epsilon\Theta\right) \cdot \beta \rangle' = \frac{1}{\mathrm{vol}(G')} \int DA\, D\psi\, D\phi\, \exp\left(\frac{1}{4\pi^2} \int_\Sigma \mathrm{Tr}\left(i\phi F + \frac{1}{2}\psi \wedge \psi\right)\right.$$

$$\left. + \frac{\epsilon}{8\pi^2} \int_\Sigma \mathrm{d}\mu\, \mathrm{Tr}\,\phi^2 \right) \cdot \beta.$$

$$(5.1)$$

In (5.1), β is supposed to be an equivariant differential form with a polynomial dependence on ϕ.

Since exponentials will be much more convenient than polynomials, we will resort to the following device. We introduce bosonic and fermionic variables δ_i which are nilpotent with $\delta_i^{n_i} = 0$ for some unspecified n_i if δ_i is bosonic (or $n_i = 2$ if δ_i is fermionic). We call such variables formal variables. We write $\beta = \exp(\sum_i \delta_i \beta_i)$, where the β_i have only a polynomial dependence on ϕ and the δ_i are formal variables. Obviously, it is sufficient to study (5.1) for such β's. Taking the limit of $n_i \to \infty$ (for all i such that δ_i is bosonic), β becomes a formal power series.

The localization principle can be applied to (5.1) with such β, since, for any n_i, β is polynomial in ϕ. After performing the integral in (5.1), we get a function $g(\delta_i)$ of the formal variables which is, of course, a polynomial for any given n_i and becomes a formal power series for $n_i \to \infty$. Under certain conditions, the

localization theorem implies (and we will verify later) that these formal power series are really polynomials.

Let us recall from §3.3 what are the possible β_i. For any degree r invariant polynomial Q on $\mathcal{H}$, we have equivariant differential forms

$$Q_{(0)} = \int_\Sigma \mathrm{d}\mu \; Q(\phi) \tag{5.2}$$

and

$$Q_{(2)} = \int_\Sigma \left(i \frac{\partial Q}{\partial \phi^a} F^a + \frac{1}{2} \frac{\partial^2 Q}{\partial \phi^a \partial \phi^b} \psi^a \wedge \psi^b \right) \tag{5.3}$$

of degree $2r$ and $2r - 2$. Also, for any circle $C \subset \Sigma$, we have the form of degree $2r - 1$

$$Q_{(1)}(C) = - \int_C \frac{\partial Q}{\partial \phi_a} \psi^a. \tag{5.4}$$

Now, let $Q(\phi)$ be an invariant polynomial of the form

$$Q(\phi) = \frac{1}{8\pi^2} \operatorname{Tr} \phi^2 + \sum_i \delta_i q_i(\phi), \tag{5.5}$$

where the δ_i are formal variables, and the $q_i(\phi)$ are homogeneous of degree ≥ 3.

Let $T(\phi)$ be an invariant polynomial of the form

$$T(\phi) = \frac{\epsilon}{8\pi^2} \operatorname{Tr} \phi^2 + \sum_i \delta_i' t_i(\phi), \tag{5.6}$$

with the δ_i' formal variables and $t_i(\phi)$ homogeneous of degree ≥ 3.

Let $C_\rho \subset \Sigma$, $\rho = 1 \ldots 2g$ be oriented circles generating $H_1(\Sigma, \mathbb{Z})$, with intersection pairings $\gamma_{\sigma\tau} = \#(C_\sigma \cap C_\tau)$. For each ρ, pick an invariant polynomial

$$S^\rho(\phi) = \sum_i \eta_i^\rho s_i^\rho, \tag{5.7}$$

where the s_i^ρ are invariant homogeneous polynomials and the η_i^ρ are anticommuting (and so in particular nilpotent) parameters.

We aim to compute

$$\left\langle \exp\left(Q_{(2)} + T_{(0)} + \sum_\rho S^\rho_{(1)}(C_\rho) \right) \right\rangle'. \tag{5.8}$$

It is convenient to introduce

$$\widehat{\phi}_a = 4\pi^2 \frac{\partial Q}{\partial \phi^a}. \tag{5.9}$$

We will first evaluate (5.8) under the restriction

$$\det\left(\frac{\partial \widehat{\phi}^a}{\partial \phi^b} \right) = 1. \tag{5.10}$$

Modulo the elementary identities noted at the end of §3, there actually is no loss of information in evaluating (5.8) only for Q such that (5.10) holds. However, in any event, after carrying out the calculation assuming (5.10), we will then relax this requirement and consider the general case.

The Computation

The basic formula (5.1) equates (5.8) with the following path integral:

$$\frac{1}{\text{vol}(G')} \int DA \ D\psi \ D\phi \ \exp\left(\int_\Sigma \left(i\frac{\partial Q}{\partial \phi^a} F^a + \frac{1}{2}\frac{\partial^2 Q}{\partial \phi^a \partial \phi^b}\psi^a \wedge \psi^b \right) \right.$$
$$\left. - \sum_\sigma \oint_{C_\sigma} \frac{\partial S^\sigma}{\partial \phi^a}\psi^a + \int_\Sigma d\mu \ T(\phi) \right). \tag{5.11}$$

First we carry out the integral over ψ. Because of (5.10), the ψ determinant coincides with what it would be if $Q = \text{Tr}\,\phi^2/8\pi^2$. As we have discussed in connection with (3.45), this determinant just produces the standard symplectic measure on

76

the space $\mathcal{A}$ of connections; this measure we conventionally (but somewhat inconsistently) call DA. Let $(\partial^2 Q)^{-1}$ be the inverse matrix to the matrix $\partial^2 Q/\partial\phi^a\partial\phi^b$, and let

$$\widehat{T}(\phi) = T(\phi) - \sum_{\sigma<\tau}\gamma_{\sigma\tau}\frac{\partial S^\sigma}{\partial\phi^a}\frac{\partial S^\tau}{\partial\phi^b}(\partial^2 Q)^{-1}_{ab}. \tag{5.12}$$

The second term arises, as in the derivation of (4.65), in shifting ψ to complete the square in (5.11). Then integrating out ψ gives

$$\frac{1}{\mathrm{Vol}(G')}\int DA\ D\phi\ \exp\left(i\int_\Sigma\frac{\partial Q}{\partial\phi^a}F^a + \int_\Sigma d\mu\ \widehat{T}(\phi)\right). \tag{5.13}$$

Now change variables from ϕ to $\widehat{\phi}$, defined in (5.9). The Jacobian for this change of variables is 1 because of (5.10). Because the δ_i are nilpotent, the transformation is invertible; the inverse is given by some functions $\phi^a = W^a(\widehat{\phi})$. After the change of variables, (5.13) becomes

$$\frac{1}{\mathrm{Vol}(G')}\int DA\ D\widehat{\phi}\ \exp\left(\frac{i}{4\pi^2}\int_\Sigma \mathrm{Tr}\,\widehat{\phi}F + \int_\Sigma d\mu\ \widehat{T}\circ W(\widehat{\phi})\right). \tag{5.14}$$

This is a path integral of the type that we evaluated in equation (4.74).

We observed in §4 that in canonical quantization, $\widehat{\phi}^a/4\pi^2$ is identified with the group generator $-iT^a$. To avoid repeated factors of $4\pi^2$, define an invariant function V by $W(\widehat{\phi}) = V(\widehat{\phi}/4\pi^2)$. The invariant function $\widehat{T}\circ W(\widehat{\phi})$ corresponds in the quantum theory (using the normal-ordering prescription of §4.6) to the operator that on a representation of highest weight h is equal to $\widehat{T}\circ V(h+\delta)$, with δ equal to half the sum of the positive roots. Borrowing the result of (4.74), the explicit evaluation of (5.14) gives

$$\frac{1}{\#\pi_1(H')}\left(\frac{\mathrm{Vol}(H')}{(2\pi)^{\dim(H')}}\right)^{2g-2}\sum_h\frac{\lambda_h(u^{-1})\exp\left(\widehat{T}\circ V(h+\delta)\right)}{d(h)^{2g-2}}, \tag{5.15}$$

with h running over dominant weights and δ as above.

Now we want to relax the assumption of (5.10). We will do so somewhat informally, but the result could be justified using the regularization of §3.

(5.10) was used twice. The determinant in the ψ integral would be formally, if (5.10) is not assumed,

$$\prod_{x\in\Sigma}\det\left(\frac{\partial^2 Q'}{\partial\phi^a\partial\phi^b}\right),\tag{5.16}$$

times the determinant for $Q=\mathrm{Tr}\,\phi^2/8\pi^2$. We have set $Q'=4\pi^2 Q$. The factors in (5.16) are all equal up to coboundaries (since more generally, for any invariant function U on $\mathcal{H}$, $U(\phi(P))$ is cohomologous to $U(\phi(P'))$, for $P,P'\in\Sigma$, according to equation (3.49)). Of course, this infinite product of essentially equal factors diverges unless (5.10) is assumed. The Jacobian in the changes of variables from ϕ to $\widehat{\phi}$ is formally

$$\prod_{x\in\Sigma}\left(\det\left(\frac{\partial^2 Q'}{\partial\phi^a\partial\phi^b}\right)\right)^{-1}.\tag{5.17}$$

Formally, these two factors appear to cancel, but this cancellation should be taken to mean only that the result is finite, not that it equals one. The number of factors in (5.16) should be interpreted as $N_1/2$, half the dimension of the space of one-forms. The number of factors in (5.17) should be interpreted as N_0, the dimension of the space of zero-forms. The difference $N_1/2-N_0$ is $-1/2$ the Euler characteristic of Σ, or $g-1$. Thus the product of (5.16) and (5.17) should be interpreted as $\det\left(\partial^2 Q'/\partial\phi^a\partial\phi^b\right)^{g-1}$. A convenient function cohomologous to this is $\exp\left(\int_\Sigma(g-1)\ln\det\left(\partial^2 Q'/\partial\phi^a\partial\phi^b\right)\right)$.[⋆]

The sole result of relaxing (5.10) is accordingly that (5.14) becomes

$$\frac{1}{\mathrm{Vol}(G')}\int DA\ D\widehat{\phi}\ \exp\left(\frac{i}{4\pi^2}\int_\Sigma\mathrm{Tr}\,\widehat{\phi}F+\int_\Sigma\mathrm{d}\mu\ \widetilde{T}\circ W(\widehat{\phi})\right),\tag{5.18}$$

⋆ In a calculation using the regularization of §3, this factor would arise from the one loop chiral anomaly. It would appear in the form of a factor $\exp\left(-\int_\Sigma\mathrm{d}\mu\ (R/8\pi)\cdot\ln\det(\partial^2 Q'/\partial\phi^a\partial\phi^b)\right)$, with R the scalar curvature of the metric that enters in the cohomological Lagrangians of §3.

with

$$\widetilde{T} = \widehat{T} + (g-1)\ln\det\left(\frac{\partial^2 Q'}{\partial\phi^a\partial\phi^b}\right) = T - \sum_{\sigma<\tau}\gamma_{\sigma\tau}\frac{\partial S^\sigma}{\partial\phi^a}\frac{\partial S^\tau}{\partial\phi^b}(\partial^2 Q)^{-1}_{ab} + (g-1)\ln\det\left(\frac{\partial^2 Q'}{\partial\phi^a\partial\phi^b}\right).$$
(5.19)

The evaluation of the path integral therefore leaves in general not quite (5.15) but

$$\frac{1}{\#\pi_1(H')}\left(\frac{\mathrm{Vol}(H')}{(2\pi)^{\dim(H')}}\right)^{2g-2}\sum_h\frac{\lambda_h(u^{-1})\exp\left(\widetilde{T}\circ V(h+\delta)\right)}{d(h)^{2g-2}}.$$
(5.20)

The Topological Conclusion

For $\epsilon > 0$ and the other parameters nilpotent, the path integral can be evaluated as we have just done regardless of possible singularities of the moduli space $\mathcal{M}'(u)$ of flat connections on $E'(u)$. If H', u, and Σ are such that the space of flat connections is smooth, and the gauge group acts freely on it (at least modulo its center), then according to the theory of §2, (5.20) is a polynomial in ϵ and the formal variables, modulo terms that vanish exponentially for $\epsilon \to 0$. Moreover, this polynomial then has an interpretation in terms of intersection numbers on moduli space:

$$\int_{\mathcal{M}'(u)}\exp\left(Q_{(2)} + \sum_\sigma S^\sigma_{(1)}(C_\sigma) + T_{(0)}\right)$$

$$= \frac{\#Z(H')}{\#\pi_1(H')}\left(\frac{\mathrm{Vol}(H')}{(2\pi)^{\dim(H')}}\right)^{2g-2}\sum_h\frac{\lambda_h(u^{-1})\exp\left(\widetilde{T}\circ V(h+\delta)\right)}{d(h)^{2g-2}} + \dots,$$
(5.21)

where "..." are exponentially small terms. ($\#Z(H')$, the order of the isomorphism group of a generic connection, is the usual factor relating the path integral to the intersection theory.) This formula generalizes the computations we made in §4 for the case of $H = SU(2)$, $H' = SO(3)$, $u = -1$. It applies notably for $H = SU(N)$, with center $\Gamma \cong \mathbb{Z}/N\mathbb{Z}$, $H' = SU(N)/\Gamma$, and u a generator of Γ.

79

Even when H', u, and Σ are such that the space of flat connections is not smooth (or the gauge group does not act freely on it), certain intersection pairings on $\mathcal{M}'(u)$ are well-defined topologically. A more careful study of the contribution of $\mu^{-1}(0)$ to the localization formula will probably show that these are all given by the asymptotic expansion of (5.21).

An important check is that (5.21) is compatible with the general topological relations of equations (3.58)–(3.60). To write out these relations, let us abbreviate

$$Z(T, S^\sigma, Q) = \int_{\mathcal{M}'(u)} \exp\left(Q_{(2)} + \sum_\sigma S^\sigma_{(1)}(C_\sigma) + T_{(0)}\right). \qquad (5.22)$$

Then (3.58) amounts to the statement that for any invariant polynomials A, B

$$\left.\frac{\partial}{\partial\epsilon} Z(T + \epsilon AB, S^\sigma, Q)\right|_{\epsilon=0} = \left.\frac{\partial^2}{\partial\alpha\partial\beta} Z(T + \alpha A + \beta B, S^\sigma, Q)\right|_{\alpha=\beta=0}. \qquad (5.23)$$

This is easily verified. (3.59) amounts to the statement that for any A and B and any τ

$$\left.\frac{\partial}{\partial\epsilon} Z(T, S^\sigma + \delta^\sigma_\tau \epsilon AB, Q)\right|_{\epsilon=0} = \left.\left(\frac{\partial^2}{\partial\alpha\partial\beta} Z(T + \alpha A, S^\sigma + \delta^\sigma_\tau \beta B, Q) + A \leftrightarrow B\right)\right|_{\alpha=\beta=0}. \qquad (5.24)$$

Here α is commuting and β, ϵ are anticommuting. This is also easy to verify. The last relation (3.60) amounts to

$$\left.\frac{\partial}{\partial\epsilon} Z(T, S^\sigma, Q + \epsilon AB)\right|_{\epsilon=0} = \left.\left(\frac{\partial^2}{\partial\alpha\partial\beta} Z(T + \alpha A, S^\sigma, Q + \beta B) + A \leftrightarrow B\right)\right|_{\alpha=\beta=0}$$
$$+ \sum_{\tau,\nu=1}^{2g} \gamma_{\tau\nu} \frac{\partial^2}{\partial\alpha'\partial\beta'} Z(T, S^\sigma + \delta^\sigma_\tau \alpha' A + \delta^\sigma_\nu \beta' B, Q)|_{\alpha'=\beta'=0}. \qquad (5.25)$$

(Here α' and β' are anticommuting.) The verification is straightforward but a little longer. It is only here that the "anomaly" term – involving the determinant of $\partial^2 Q'/\partial\phi^2$ – plays a role.

<u>Polynomials</u>

Now we want to verify that, under the expected hypotheses, the formal power series on the right hand side of (5.21) is a polynomial in ϵ and the formal variables, modulo exponentially small terms. (A sharper bound than we will obtain on the degree of these polynomials is expected.)

The basic fact that we will use is that for w real and non-integral, P an arbitrary polynomial, and a real,

$$\sum_{n \in \mathbb{Z}} \exp\left(-\epsilon(n^2 + an) + 2\pi i w n\right) \cdot P(n) \qquad (5.26)$$

vanishes exponentially for $\epsilon \to 0$. This can be proved, for instance, by Poisson summation, as in the derivation of (4.47). There is also an obvious higher dimensional generalization of (5.26). Replace $\mathbb{Z}$ by a lattice $\Lambda \subset \mathbb{R}^n$, replace wn by a real-valued linear form $w(n)$ on Λ that is not integer-valued, replace n^2 by (n, n), with $(\ ,\ )$ a positive definite quadratic form on $\mathbb{R}^n$, and replace an by (a, n) with $a \in \mathbb{R}^n$. Then (5.26) becomes

$$\sum_{n \in \Lambda} \exp\left(-\epsilon\left((n, n) + (a, n)\right) + 2\pi i w(n)\right) \cdot P(n). \qquad (5.27)$$

That this vanishes exponentially for $\epsilon \to 0$ is proved the same way.

Now, let $\Lambda \subset \mathbb{R}^n$ be a lattice with an integer-valued non-degenerate quadratic form $(\ ,\ )$. Let e_i, $i = 1, \ldots, t$ be not necessarily distinct non-zero points in Λ. Let H_i be the sublattice of Λ defined by $(e_i, h) = 0$. Let us call a sublattice $\Lambda_0 \subset \Lambda$ distinguished if it is non-zero and can be written as an intersection of a subset of H_i's. Let w be a real-valued linear form on Λ which is not integer-valued when restricted to any distinguished Λ_0. Let P be a polynomial on Λ of degree $s > t$. Consider the sum

$$\sum_{h \in \Lambda'} \frac{\exp\left(-\epsilon(h, h) + 2\pi i w(h)\right) \cdot P(h)}{\prod_{j=1}^{t}(e_j, h)}, \qquad (5.28)$$

where $\Lambda' = \Lambda - \cup_j H_j$. I claim this sum vanishes exponentially for $\epsilon \to 0$.

81

Obviously, we can assume that P is a monomial, and thus a product of linear factors $P_1, \ldots, P_s$. If the e_j are linearly independent, then we can expand $P_1(h) = \sum_{j=1}^{t} c_j(e_j, h)$ with coefficients c_j. It is therefore enough in this case to show that (5.28) vanishes exponentially if P is replaced by $P' = (e_k, h) \cdot P_2 \cdot \ldots \cdot P_s$. The factor of (e_k, h) can be canceled with one factor in the denominator.

At this point, if e_k no longer appears in the list of remaining e's (recall that the e's may not be distinct), we want to extend the sum in (5.28) to run over $\Lambda'' = \Lambda - \cup_{j \neq k} H_j$. By induction on the dimension of Λ, we can assume that

$$\sum_{h \in H'_k} \frac{\exp\left(-\epsilon(h, h) + 2\pi i w(h)\right) \cdot P(h)}{\prod_{j \neq k}(e_j, h)} \tag{5.29}$$

vanishes exponentially for $\epsilon \to 0$, if $H'_k = H_k - \cup_{j \neq k}(H_j \cap H_k)$. As $\Lambda'' = \Lambda' \cup H'_k$, we can replace the sum over Λ' by a sum over Λ''.

After repeating this process, one reduces to the following situation. The remaining e_j, $j = 1 \ldots t'$ (if any) are not linearly independent, and the equations $(e_j, h) = 0$, $j = 1 \ldots t'$ define a non-zero lattice Λ_1. P is a polynomial on this lattice, of degree $s' > t' \geq 0$. By the hypothesis about w, w is not integer-valued when restricted to Λ_1. The e_j can be regarded as linear forms on a complementary lattice Λ_2 to Λ_1. The same Poisson summation used in proving (5.27), when applied to the sum over Λ_1, now shows that (5.28) vanishes exponentially for $\epsilon \to 0$.

To apply this to our problem, let Λ be the root lattice of the compact, connected, and simply-connected Lie group H. We recall from the theory of compact Lie groups that the dimension of a representation of highest weight h is $d(h) = \prod_i (e_i, h + \delta)$, where e_i, $i = 1 \ldots q$ are the positive roots of H, δ is one-half their sum, and (,) is the usual metric on Λ. To show that (5.21) is a polynomial in ϵ and the formal variables modulo exponentially small terms, it suffices to show that

$$\sum_h \frac{\lambda_h(u^{-1}) \exp\left(-\epsilon(h + \delta, h + \delta)\right) \cdot P(h + \delta)}{\prod_i(e_i, h + \delta)^{2g-2}} \tag{5.30}$$

vanishes exponentially for $\epsilon \to 0$, if $P(h + \delta)$ is a Weyl-invariant polynomial of

degree greater than $(2g-2)q$, and the sum runs over dominant weights h. Let H_i be the sublattice of Λ defined by $(e_i, h+\delta) = 0$. Changing variables from h to $h' = h + \delta$, and using the Weyl-invariance of the numerator and denominator in (5.30), we can replace this sum by

$$\frac{1}{\#W(H)} \sum_{h' \in \Lambda'} \frac{\lambda_h(u^{-1}) \exp\left(-\epsilon(h', h')\right) \cdot P(h')}{\prod_i (e_i, h')^{2g-2}}, \tag{5.31}$$

where $\Lambda' = \Lambda - \cup_i H_i$, and $W(H)$ is the Weyl group.

In (5.31), $\lambda_h(u^{-1})$ can be written $\exp(2\pi i w(h))$, where w is some linear form on Λ. To deduce the desired property of (5.31) from our earlier discussion, it suffices to show that $w(h)$ is not integer-valued when restricted to any sublattice $\Lambda_0 \subset \Lambda$ defined by vanishing of a subset of the linear forms (e_i, h').

At this point, we specialize to the case $H = SU(N)$ and u a generator of the center of H. The root lattice Λ can conveniently be taken to consist of N-tuples $(h_1, \ldots, h_N)$, with $\sum_\alpha h_\alpha = 0$, and $h_\alpha - h_\beta \in \mathbb{Z}$. The Weyl group is the group of permutations of the h's. The linear form w can be written as $w(h) = kh_1$, with k an integer depending on u. The hypothesis that u generates the center of $SU(N)$ is equivalent to $(k, N) = 1$. The linear forms (e_i, h) are $h_\alpha - h_\beta$, $1 \leq \alpha < \beta \leq N$. Any distinguished sublattice Λ_0 defined by equations $h_{\alpha_r} - h_{\beta_r} = 0$, $r = 1 \ldots m$, contains a sublattice Λ_0' equivalent (up to a Weyl transformation) to $h_1 = \ldots = h_m$, $h_{m+1} = \ldots = h_N$, with $0 < m < N$. This lattice contains the point $h_1 = \ldots = h_m = (N-m)/N$, $h_{m+1} = \ldots = h_N = -m/N$. If $(k, N) = 1$, then w is not an integer at this point, completing the proof.

This argument shows that (5.21) is a polynomial in ϵ and the formal variables in the expected cases, but (except in the special case $S^\rho = 0$, $Q(\phi) = \text{Tr } \phi^2/8\pi^2$), the bound on the degree of the polynomial so obtained is weaker than expected by dimension counting. It would be interesting to know how to obtain directly a sharper bound.

APPENDIX

This appendix is devoted to a simple illustration of the various localization formulas. We will consider a situation with $G = U(1)$ so that we can illustrate both the abelian DH formula and the new not necessarily abelian formula. (However, in the abelian case, the "new" formula is a consequence of the DH formula.) We endow the Lie algebra of G with the standard metric such that the volume of the group is 2π.

Let X be the two sphere

$$x^2 + y^2 + z^2 = 1 \tag{A.1}$$

and introduce the usual polar coordinates by $z = \cos\theta$, $x = \sin\theta\cos\psi$, $y = \sin\theta\sin\psi$. Thus $0 \le \theta \le \pi$ and $0 \le \psi \le 2\pi$. The usual symplectic volume form is $\omega = \mathrm{d}\cos\theta\,\mathrm{d}\psi$. We consider the $U(1)$ action $\psi \to \psi + \text{constant}$. The moment map is

$$\mu = \cos\theta + a, \tag{A.2}$$

where a is an arbitrary constant. The DH formula therefore applies to

$$Z = \int_X \omega\ e^{-\beta(\cos\theta+a)}. \tag{A.3}$$

Evaluating this explicitly, we have

$$
\begin{aligned}
Z &= \int_{-1}^{1} \mathrm{d}\cos\theta \int_0^{2\pi} \mathrm{d}\psi e^{-\beta(\cos\theta+a)} \\
&= \frac{2\pi}{\beta}\left(e^{\beta(1-a)} - e^{-\beta(1+a)}\right).
\end{aligned}
\tag{A.4}
$$

The two terms can be identified with the contributions from the critical points $P_\pm$ of μ at $\cos\theta = \pm 1$. The factors of $e^{-\beta(a\pm1)}$ are $e^{-\beta\mu(P_\pm)}$, while the one loop

determinants expanding around the $P_\pm$ give $\mp 2\pi/\beta$. A minus sign arises at P_+ because it is a local maximum of μ, unstable in two directions, each of which, heuristically, contributes a factor of i.

Now we want to illustrate the localization formula used in this paper. In so doing, we will assume $|a| < 1$, leaving the other (similar) cases to the reader. Using the formalism of §2, we consider the equivariantly closed form

$$\alpha = \exp(\omega + i\phi\mu) \tag{A.5}$$

where ϕ is a linear function on the one dimensional Lie algebra of G. Then

$$\oint_X \alpha = \frac{1}{\mathrm{vol}(U(1))} \int_{-\infty}^{\infty} \frac{\mathrm{d}\phi}{2\pi} \int_X \exp\left(\omega + i\phi(\cos\theta + a) - \frac{\epsilon}{2}\phi^2\right). \tag{A.6}$$

Integrating over ψ and ϕ and setting $x = \cos\theta$, and $\mathrm{vol}(U(1)) = 2\pi$, we get

$$\oint_X \alpha = \int_{-1}^{1} \frac{\mathrm{d}x}{(2\pi\epsilon)^{1/2}} \exp\left(-(x+a)^2/2\epsilon\right). \tag{A.7}$$

The integral would be 1 if the limits were extended from $-\infty$ to $+\infty$. As it is,

$$\oint_X \alpha = 1 - I_+ - I_-, \tag{A.8}$$

with

$$
\begin{aligned}
I_+ &= \int_{1}^{\infty} \frac{dx}{(2\pi\epsilon)^{1/2}} \exp\left(-(x+a)^2/2\epsilon\right) \\
I_- &= \int_{-\infty}^{-1} \frac{dx}{(2\pi\epsilon)^{1/2}} \exp\left(-(x+a)^2/2\epsilon\right).
\end{aligned}
\tag{A.9}
$$

The three terms correspond to the expected contributions from the critical points of $I = (\mu, \mu)$. The absolute minimum of I at $\cos\theta = -a$ can be locally

85

modeled on T^*G and hence contributes $+1$, as predicted from our general analysis in §2.3. The other critical points of I are the critical points $P_\pm$ of μ that already entered above. Their contributions are $-I_\pm$, which are transcendental (error) functions. The minus signs reflect the fact that the two points $P_\pm$ are both local maxima of I. The contributions of $P_\pm$ are asymptotic for $\epsilon \to 0$ to $\exp(-I(P_\pm)/2\epsilon)$, in agreement with the general theory. The complicated error functions contrast with the elementary functions that appear as local contributions in the DH formula. They arise because although $P_\pm$ are nondegenerate critical points of I, they are degenerate critical points of the function $(V, \lambda)^2$ that appears in the key formula (2.12). By examining (2.12), one can make quite explicit the fact that the error functions are entirely determined by the behavior of μ near $P_\pm$ up to second order.

REFERENCES

1. A. Migdal, Zh. Eksp. Teor. Fiz. **69** (1975) 810 (Sov. Phys. Jetp. **42** 413).

2. N. Bralic, Phys. Rev. **D22** (1980) 3090.

3. V. Kazakov and I. Kostov, Nucl. Phys. **B176** (1980) 199, Phys. Lett. **B105** (1981) 453, Nucl. Phys. **B179** (1981) 283.

4. L. Gross, C. King, and A. Sengupta, Ann. of Phys. **194** (1989) 65.

5. B. Rusakov, "Loop Averages And Partition Functions in $U(N)$ Gauge Theory On Two-Dimensional Manifolds," Mod. Phys. Lett. **A5** (1990) 693.

6. D. Fine, "Quantum Yang-Mills On The Two-Sphere," Commun. Math. Phys. **134** (1990) 273, "Quantum Yang-Mills On A Riemann Surface," to appear in Commun. Math. Phys.

7. A. Sengupta, "The Yang-Mills Measure For S^2," to appear in J. Funct. Anal., "Quantum Gauge Theory On Compact Surfaces," preprint.

8. M. Blau and G. Thompson, "Quantum Yang-Mills Theory On Arbitrary Surfaces," preprint NIKHEF-H/91-09, MZ-TH/91-17.

9. E. Witten, "On Quantum Gauge Theories In Two Dimensions," Commun. Math. Phys. **141** (1991) 153.

10. R. Forman, "Small Volume Limits of $2 - d$ Yang-Mills," preprint, Rice University (1991).

11. E. Witten, "On The Structure Of The Topological Phase Of Two Dimensional Gravity," Nucl. Phys. **B340** (1990) 281.

12. J. Distler, "$2 - D$ Quantum Gravity, Topological Field Theory, and the Multicritical Matrix Models," Nucl. Phys. **B342** (1990) 523.

13. M. Kontsevich, "Intersection Theory On The Moduli Space Of Curves And The Matrix Airy Function," preprint (1991).

14. A. Schwarz, "The Partition Function Of Degenerate Quadratic Functional And Ray-Singer Invariants," Lett. Math. Phys. **2** (1978) 247.

15. J. J. Duistermaat and G. J. Heckman, "On The Variation In The Cohomology In The Symplectic Form Of The Reduced Phase Space," Invent. Math. **69** (1982) 259.

16. J.-M. Bismut, "Localization Formulas, Superconnections, and The Index Theorem For Families," Comm. Math. Phys. **103** (1986) 127.

17. M. F. Atiyah and R. Bott, "The Moment Map And Equivariant Cohomology," Topology **23** (1984) 1.

18. M. F. Atiyah and R. Bott, "The Yang-Mills Equations Over Riemann Surfaces," Phil. Trans. Roy. Soc. London **A308** (1982) 523.

19. F. Kirwan, *Cohomology of Quotients In Symplectic And Algebraic Geometry,* Princeton University Press.

20. V. Guillemin and E. Prato, "Heckman, Kostant, and Steinberg Formulas for Symplectic Manifolds," Adv. Math. **82** (1990) 160.

21. P. E. Newstead, "Characteristic Classes Of Stable Bundles Over An Algebraic Curve," Trans. Am. Math. Soc. **169** (1972) 337.

22. M. Thaddeus, "Conformal Field Theory And The Cohomology Of The Moduli Space Of Stable Bundles," Oxford preprint (1991).

23. F. Kirwan, "The Cohomology Rings Of Moduli Spaces Of Vector Bundles Over Riemann Surfaces," Oxford preprint.

24. S. K. Donaldson, "Gluing Techniques In The Cohomology Of Moduli Spaces," Oxford preprint.

25. E. Verlinde, "Fusion Rules And Modular Transformations In 2d Conformal Field Theory," Nucl. Phys. **B300** (1988) 360.

26. A. Tsuchiya, K. Ueno, and Y. Yamada, "Conformal Field THeory On Universal Family Of Stable Curves With Gauge Symmetries," Adv. Studies in Pure Mathematics **19** (1989) 459.

27. G. Segal, "Two Dimensional Conformal Field Theories And Modular Functors," in *IXth International Conference On Mathematical Physics*, eds. B. Simon, A. Truman, and I. M. Davies (Adam Hilger, Bristol, 1989); "Fusion Of Representations Of Loop Groups," unpublished manuscript.

28. A. Szenes, "Hilbert Polynomials of Moduli Spaces of Rank 2 Vector Bundles, I" (Harvard preprint); A. Bertram and A. Szenes, "Hilbert Polynomials of Moduli Spaces of Rank 2 Vector Bundles, II" (Harvard preprint).

29. G. Daskalopoulos and R. Wentworth, "Geometric Quantization For The Moduli Space Of Vector Bundles With Parabolic Structure," "The Integrability Criterion in the $SU(2)$ Chern-Simons Gauge Theory," MIT preprints.

30. S. Chang, "Non-Abelian θ-Functions, Geometric Quantization, And A Mathematical Proof of Verlinde's Formula."

31. V. Mathai and D. Quillen, "Superconnections, Thom Classes, and Equivariant Differential Forms," Topology **25** (1986) 85.

32. V. I. Arnold, S. M. Gusein-Zade, and A. N. Varchenko, *Singularities Of Differentiable Maps*, Vol. II (Birkhauser, Boston, 1988).

33. V. Guillemin and S. Sternberg, *Symplectic Techniques In Physics,* (Cambridge University Press, 1984).

34. E. Witten, "Topological Quantum Field Theory," Commun. Math. Phys. **17** (1988) 353, "Introduction To Cohomological Field Theory," Int. J. Mod. Phys. **A6** (1991) 2775.

35. H. Kanno, "Weil Algebra Structure And Geometrical Meaning Of The BRST Transformation In Topological Quantum Field Theory," Zeitschrift fur Physik C - Particles And Fields **43** (1989) 477.

36. P. van Baal, "An Introduction To Topological Yang-Mills Theory," Acta. Phys. Polon. **B21** (1990) 73.

37. M. F. Atiyah and L. Jeffrey, "Topological Lagrangians And Cohomology," J. Geom. Phys. **7** (1990) 119.

38. L. Baulieu and I. M. Singer, "Topological Yang-Mills Symmetry," Nucl. Phys. (Proc. Supp.) **5B** (1988) 12.

39. D. Birmingham, M. Blau, M. Rakowski, and G. Thompson, "Topological Field Theory," Phys. Reports **209** (1991) 129.

40. E. Witten, "The N Matrix Model And Gauged WZW Models," IAS preprint, to appear in Nucl. Phys. B.

41. D. P. Zelobenko, "Compact Lie Groups And Their Representations," Translations of Mathematical Monographs, vol. 40 (American Mathematical Society, 1978).

FIGURE CAPTIONS

1) (a) A three-holed sphere, with the boundary components labeled by representations α_i. (b) The same path integral with an insertion of an operator $\mathcal{O}$. (c) One can consider this operator to act on any of the three external states.

2) A two-holed sphere with labeled boundaries.

3) A one-holed sphere with a labeled boundary.

4) Factoring the amplitude of a two-holed sphere in a suitable fashion.

5) A surface of genus g – in this case $g = 2$ – can be regarded as a union of $2g - 2$ three-holed spheres.

6) Factorizing the genus g amplitude on a disc D containing the point P.