Gott time machines cannot exist in an open

(2+1)-dimensional universe with timelike total momentum [*]

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We prove the title.

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I. INTRODUCTION

Absent some restriction on boundary conditions and energy sources, it is possible for the spacetime metric of general relativity to wreak havoc with our intuitive notions of "going forward in time." We can imagine metrics in which the worldline of a test particle, locally restricted to the interior of its forward light cone, can loop around to intersect itself — a closed timelike curve (CTC). Indeed, it is easy to construct solutions to Einstein's equations which exhibit such behavior [1,2].

Nonetheless, due to the causal paradoxes associated with such a time machine, it is tempting to believe that CTC's exist only in spacetimes that are in some way pathological. That is, we would expect that the laws of physics somehow act to prevent the occurrence of CTC's in the real universe. This expectation has been dubbed the "Chronology Protection Conjecture" [3].

Scientific interest in CTC's was recently invigorated by Gott's [4,5] construction of an extraordinarily simple solution to Einstein's equations that contains CTC's. Gott's solution represents two infinitely long parallel cosmic strings moving past each other at high velocity. The situation at early times is portrayed in Fig. 1, which shows the two strings approaching each other. Each string is associated with a deficit angle removed from the space, which we have oriented in a direction opposite to that of the motion of the string. Opposite sides of the excluded wedges are identified at equal times. Gott found that, as the strings approach each other, it becomes possible to traverse a closed timelike curve enclosing the strings in the sense opposite to their motion. Since many models of the early universe suggest that it was populated by cosmic strings moving at high velocities, the Gott solution raises the possibility that CTC's could occur in the evolution of the real world.

The Gott universe is topologically equivalent to Minkowski space,[2] free of singularities

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[2]In discussing the topology of the Gott universe, we are treating the strings as physical objects with a small but nonzero thickness. They are nonsingular configurations, and are not excised from

and event horizons, and does not violate the weak energy condition. Cutler [6] has shown that the spacetime contains regions free of CTC's, and indeed that there are complete spatial hypersurfaces both to the future and the past of the region containing CTC's. Cutler constructed such a hypersurface, but an even simpler example is provided by the constant-time slice portrayed in Fig. 1; the past light cone of any point on this surface extends through similar surfaces arbitrarily far into the past, implying that no timelike curve through such a point can be closed. As the two strings approach one another, each will ultimately collide with the trailing deficit angle of the other, at which time this coordinate system will fail; CTC's will then arise.

The Gott universe is known to contain CTC's of arbitrarily large spatial extent [7,6,8]. However, even if we consider arbitrarily large CTC's to be unrealistic, this fact by itself is no justification for rejecting the Gott time machine as a physical possibility. Since the universe does not consist of an isolated Gott time machine, the interesting question is whether a Gott time machine can arise within a universe populated by other objects as well. These other objects could affect the boundary conditions at spatial infinity, so the properties of the isolated solution at spatial infinity are not decisive.[3] Since the Cauchy problem of general

the spacetime.

[3]An analogy showing how one might be misled by considering a boundary condition at spatial infinity is the case of the 't Hooft-Polyakov magnetic monopole. A physicist studying the classical behavior of a grand unified theory might choose to impose a boundary condition requiring the Higgs field to approach a constant at infinity. This boundary condition would exclude a net magnetic charge, and hence the monopole solution. Nonetheless, monopole-antimonopole pair configurations are consistent with the boundary condition, and such configurations might evolve even if they are not included in the initial configuration. The monopoles arising in this way would only locally resemble the ideal 't Hooft-Polyakov solution, yet they would exhibit the same physical characteristics. Thus, although the stringent boundary condition can exclude the isolated monopole

relativity is formulated in terms of initial data on a complete spacelike hypersurface, our goal is to isolate the characteristics of those initial data that lead to CTC's.

We are thus led to study the conditions under which Gott time machines can arise. The issue of time machine construction in 3+1 dimensions has been addressed by Tipler [9] and Hawking [3], as we will further discuss in Sec. III. In this paper, however, we address the question in the context of (2+1)-dimensional gravity. The (2+1)-dimensional approach is based on the observation that a spacetime populated solely by infinitely long parallel cosmic strings is invariant with respect to boosts and translations along the direction of the strings. The dependence of the metric on this direction is trivial and can be disregarded, in which case the strings are equivalent to point masses in (2+1) dimensions. Each point particle is assigned a mass M equivalent to the mass per unit length of the cosmic string. This theory has been the object of extensive investigation [10,11]. It has been found that the metric in vacuum is necessarily flat, while in the presence of a single particle with mass M, the external metric is that of Minkowski space from which a wedge of angle $\alpha = 8\pi G M$ has been removed and opposite sides have been identified (G is Newton's constant). Solutions with several static particles are easily constructed by joining several one-particle solutions, in which case the space has a net deficit angle given by the sum of the deficit angles of the constituent particles. If the total deficit angle exceeds 2π the spatial sections of the spacetime must be closed, with topology S^2, and the total deficit angle is necessarily exactly 4π. By joining appropriately boosted single-particle spacetimes, the exact solutions with moving particles [11] and nontrivial decays and scatterings [12] may be constructed.

In their seminal paper on (2+1)-dimensional gravity, Deser, Jackiw, and 't Hooft [11] noted that a spinning point particle would give rise to CTC's. They added, however, that "such closed timelike contours are not possible in a space with n moving spinless particles, where angular momentum is purely orbital." When stated without qualification, this sen-

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solution, it cannot prevent physical monopoles from arising in the theory.

tence is apparently contradicted by the existence of the Gott solution. In this paper we find out what qualifications need to be added to this sentence in order to validate it. We will prove rigorously that there exists a class of initial conditions for (2+1)-dimensional universes from which a Gott time machine can never evolve.

The energy and momentum of a collection of particles can be conveniently characterized by the Lorentz transformation that a vector undergoes upon being parallel transported around the system [11,13]. The Lorentz transformation belongs to the three-dimensional Lorentz group, SO(2,1). Deser, Jackiw, and 't Hooft [8] have shown that the group element corresponding to the Gott time machine is boostlike — equivalent under similarity transformation to a pure boost. Each element of SO(2,1) can be identified with a 3-vector, which we will refer to as the energy-momentum vector of the system. The energy-momentum vector for a single particle is timelike, while that of the Gott two-particle system is spacelike (tachyonic), despite the fact that each particle is moving slower than c. (The notion of a tachyonic momentum is used somewhat loosely in this context; we will explain this more fully below.) This state of affairs can arise because the momentum of a system of particles is not the sum of the individual momenta. Thus (2+1) dimensional gravity offers the possibility of constructing tachyonic momenta from ordinary, subluminal matter.

We are accustomed to thinking of tachyons as "unphysical", but it does not follow as an immediate corollary that the Gott time machine cannot be built. The situation here is different from special relativity— in special relativity one can choose not to introduce tachyons as fundamental particles, and the linear rule for combining energy-momentum vectors then guarantees that no set of particles will have a tachyonic momentum. The corresponding relation in (2+1)-dimensional gravity, however, is nonlinear, allowing the construction of a tachyon from timelike particles. The isolated Gott time machine, of course, can be excluded by choosing to consider only universes for which the total momentum is timelike. However, if we want to consider the possibility of a Gott time machine existing in conjunction with other matter, then the question becomes more complicated. We would like to know whether a universe with a total momentum that is timelike can contain a subsystem

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of particles for which the combined momentum is spacelike. Note that in special relativity, spacelike momenta can be combined to form a timelike momentum. If this can also happen in (2+1)-dimensional gravity, then it is conceivable that a universe populated by slowly moving particles, in which there is no hint of a time machine at early times, could evolve by a series of decays and scatterings into a universe containing a Gott time machine.

In an open universe, such is not the case — we show in this paper that no collection of particles with net timelike momentum can contain a subset with spacelike momentum. In our previous paper [12] we presented the physical reason behind this result: there can never be enough mass in an open universe to accelerate two particles to sufficiently high velocity. In this paper we provide the formal proof. Specifically, we associate with every collection of particles an element of the universal covering group of SO(2,1), and show that an element corresponding to an open universe with timelike momentum is never the product of an element representing the Gott time machine and any number of elements representing massive particles. The proof can be constructed by algebraic manipulation; however, an elegant geometric demonstration is achieved by introducing an invariant metric on the parameter space of the group, in which case the group manifold becomes three-dimensional anti-de Sitter space.[4]

In a closed universe, this result no longer holds — we have found that it is possible to construct a spacetime in which a subset of particles with tachyonic momentum is created from initially static conditions (*e.g.*, by the decay of massive particles). However, 't Hooft [14] has shown that causal disaster is avoided, since the universe shrinks to zero volume before any CTC's can arise.[5] He goes on to argue that this phenomenon will always result

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[4]We are very grateful to Don Page and Alex Lyons, who pointed out to us the relationship between the (2+1)-dimensional Lorentz group and anti-de Sitter space.

[5]In a note added to our previous paper [12] we erroneously claimed that CTC's would arise. We thank G. 't Hooft for informing us of our mistake.

from an attempt to build a time machine in a closed universe. Therefore, neither closed nor open universes can evolve Gott time machines from initial conditions with slowly-moving particles. At the same time, the fact that a pair of Gott particles can be constructed from static conditions in a closed universe serves to emphasize the fact that a Gott pair is not intrinsically unphysical. It is therefore interesting to understand exactly under what conditions a Gott pair can and cannot be constructed.

II. THEOREM AND PROOF

Theorem. *Consider an open (2+1)-dimensional universe composed of point particles, each with a future-directed momentum that is either timelike or lightlike. If the total momentum of this universe is timelike, no subgroup of these particles can have a spacelike momentum.*

Since the Gott time machine consists of two particles with net spacelike momentum, it follows immediately that such a time machine can exist in an open universe only if the total momentum of the universe is spacelike.

A. Outline

The outline of the proof is as follows. We first review how to characterize the energy and momentum of a gravitating particle by an element of the Lorentz group SO(2,1). The element for a single particle in its rest frame is simply a rotation by its deficit angle α. The element for a moving particle is obtained by a Lorentz transformation, and the element for a collection of particles is the product of those associated with its constituents. We then endow the parameter space of SO(2,1) with an invariant metric, establishing the correspondence between the group manifold and anti-de Sitter space. We note that the curves in SO(2,1) defined by $T(\lambda) = \exp(-i\lambda X)$, where X is a fixed element of the Lie algebra, are geodesics of this metric passing through the identity. A particle with a future-directed timelike (light-like) momentum is associated with an element of SO(2,1) lying on a future-directed timelike

(lightlike) geodesic through the identity. The element associated with a collection of particles corresponds to a nonspacelike curve constructed from consecutive geodesic segments corresponding to the individual particles.

An open timelike universe (i.e., an open universe with a timelike total momentum) is associated (in its rest frame) with a rotation matrix $R(\theta_0)$, such that $\theta_0 \leq 2\pi$. We would like to show that there is no future-directed timelike or lightlike curve from the element associated with the Gott time machine to an element associated with an open timelike universe. However, a slight complication arises due to the periodicity of SO(2,1), in which a rotation by $\theta + 2\pi$ is equivalent to a rotation by θ. Starting from the Gott time machine, one can easily find future-directed timelike curves that use this periodicity to return to small values of θ, terminating at an open timelike universe. The total mass of a (2+1)-dimensional universe, however, is not periodic, and there is no way that particles with a total deficit angle of $2.2\,\pi$ can be put together to make an open universe. This complication is eliminated by replacing SO(2,1) with its universal covering group, within which the rotation angle θ can take any value from $-\infty$ to ∞ without identification. In the parameter space of the universal covering group, it can be shown that open timelike universes cannot be reached from the Gott time machine by future-directed timelike or lightlike curves. Consequently, one cannot construct an open universe with timelike total momentum that consists of a Gott time machine and any set of point particles with future-directed timelike or lightlike momenta.

We wish to comment that many of the tools we use are standard results in the theory of Lie groups and symmetric spaces [15]. Any Lie algebra has a natural metric, the Cartan-Killing form, given in components by

$$g_{\mu\nu} = c^\lambda{}_{\mu\sigma} c^\sigma{}_{\nu\lambda} \ , \tag{1}$$

where the $c^\alpha{}_{\beta\gamma}$ are the structure constants. For semisimple groups, this metric can be uniquely extended to a left- and right-invariant metric on the entire group manifold. The resulting space will be maximally symmetric, and paths from the identity defined by $T(\lambda) = \exp(-i\lambda X)$, where X is an element of the Lie algebra, will be geodesics of this metric.

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However, it is straightforward for us to derive these results for the case at hand, which we will do for clarity. (A similar discussion of the universal cover of SO(2,1) in a different context can be found in Ref. [16]. A more mathematical discussion can be found in Ref. [17].)

B. Energy and Momentum Conservation

We begin by recalling the characterization of energy and momentum in (2+1) dimensional gravity [11,13]. In any number of dimensions, spacetime curvature can be characterized by the Lorentz transformation (holonomy) that describes the result of parallel transporting a vector around a closed loop. In (2+1) dimensions this technique is especially convenient, since a closed curve unambiguously divides the space of worldlines into those which pass through the loop and those which do not. Further, since space is flat outside sources, any two loops enclosing the same matter distribution will yield the same transformation.

A vector parallel transported counterclockwise around a single particle of deficit angle α is transformed by a counterclockwise rotation matrix $R(\alpha)$. For a particle moving with velocity $\vec{v}$, the appropriate transformation can be obtained by boosting to the particle's rest frame, rotating, and boosting back. Thus, we associate with the moving particle a matrix

$$T = B(\vec{\xi})R(\alpha)B^{-1}(\vec{\xi}) \ , \tag{2}$$

where $\vec{\xi} = \hat{v}\tanh^{-1}|\vec{v}|$ is the rapidity of the particle and $B(\vec{\xi})$ is a boost bringing the rest vector to the velocity of the particle. A loop around a system of several particles can be deformed to a series of loops around the individual particles. Once an ordering is specified, the transformation associated with a system is therefore given by the product of the one-particle matrices:

$$T_{tot} = T_N T_{N-1} \dots T_1 \ . \tag{3}$$

As a system of N particles evolves via decay or scattering into a system of M particles, a loop around the system at one time can be deformed to a loop around the system at a

later time. Since the deformation carries the loop only through regions of flat spacetime, the resulting transformation matrix is not changed. Therefore, conservation of energy and momentum is expressed as the equality of Lorentz transformations at different times:

$$T_N T_{N-1} \ldots T_1 = T'_M T'_{M-1} \ldots T'_1 \ . \tag{4}$$

This rule was used in Ref. [12] in constructing the spacetime for one particle decaying into two.

We find it convenient to represent the transformations T by 2×2 matrices. The standard basis for the Lie algebra of SO(2,1) consists of the rotation generator J and two boost generators K_1 and K_2, with the commutation relations

$$[K_1, K_2] = -iJ$$

$$[J, K_1] = iK_2$$

$$[J, K_2] = -iK_1 \ . \tag{5}$$

Following the conventions used in Ref. [12], we take $J = \frac{1}{2}\sigma_3$ and $K_i = \frac{i}{2}\sigma_i$, where the σ's are the standard Pauli matrices. When exponentiated, these 2×2 matrices generate the group SU(1,1), which is a double cover of SO(2,1).

The generators J and K_i are components of an antisymmetric Lorentz tensor $M_{\mu\nu}$, with $J = M_{12}$ and $K_i = M_{i0}$. Since we are working in three spacetime dimensions, however, we can define a set of 3-vector group generators by using the Levi-Civita tensor:

$$\mathcal{J}_\mu = \frac{1}{2}\epsilon_\mu{}^{\lambda\sigma} M_{\lambda\sigma} \ , \tag{6}$$

where $\epsilon_{012} \equiv 1$, and indices are raised and lowered by the Lorentz metric $\eta_{\mu\nu} \equiv \text{diag}\,[-1, 1, 1]$. Explicitly, $\mathcal{J}_0 = J$, $\mathcal{J}_1 = -K_2$, and $\mathcal{J}_2 = K_1$.

A rotation by angle α is given by

$$R(\alpha) = e^{-i\alpha J} = \begin{pmatrix} e^{-i\alpha/2} & 0 \\ 0 & e^{i\alpha/2} \end{pmatrix} \tag{7}$$

and a boost is given by

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$$B(\vec{\xi}) = e^{-i\vec{\xi}\cdot\vec{K}} = \begin{pmatrix} \cosh\frac{\xi}{2} & e^{-i\phi}\sinh\frac{\xi}{2} \\ e^{i\phi}\sinh\frac{\xi}{2} & \cosh\frac{\xi}{2} \end{pmatrix}, \tag{8}$$

where $\xi = |\vec{\xi}|$ is the magnitude of the rapidity and ϕ is its polar angle. The matrix T associated with a single particle is found by evaluating Eq. (2):

$$T = \begin{pmatrix} \sqrt{1+p^2}\,e^{-i\alpha'/2} & ipe^{-i\phi} \\ -ipe^{i\phi} & \sqrt{1+p^2}\,e^{i\alpha'/2} \end{pmatrix}, \tag{9}$$

where

$$p \equiv \sinh\xi\sin\frac{\alpha}{2} \tag{10}$$

is a measure of the momentum of the particle, and α' is defined by

$$\tan\frac{\alpha'}{2} = \cosh\xi\tan\frac{\alpha}{2} . \tag{11}$$

Note that α' may be thought of as a boosted deficit angle, when the excised wedge is taken to lie along the direction of motion (as in Fig. 1).

C. The Metric on SO(2,1)

We now turn to the geometry of the parameter space of SO(2,1). For convenience we continue to use the 2×2 matrix representation. The space of 2×2 matrices is spanned by the group generators $\mathcal{J}_\mu$ ($\mu = 1, 2, 3$) and the identity matrix, so an arbitrary 2×2 matrix can be written as

$$T = w - 2i\chi^\mu \mathcal{J}_\mu = \begin{pmatrix} w - it & y + ix \\ y - ix & w + it \end{pmatrix}, \tag{12}$$

where $\chi^\mu \equiv (t, x, y)$, and $w, t, x,$ and y are complex. SU(1,1), which is the double cover of SO(2,1), consists of those matrices T satisfying

$$\det T = +1 \tag{13a}$$

and

$$T^\dagger \eta T = \eta , \tag{13b}$$

where

$$\eta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} . \tag{14}$$

It is shown in the Appendix that these conditions are obeyed if and only if (t, x, y, w) are real numbers satisfying

$$-t^2 + x^2 + y^2 - w^2 = -1 . \tag{15}$$

The form of this equation suggests that we consider the resulting three-dimensional space as embedded in a four-dimensional space with metric

$$ds^2 = -dt^2 + dx^2 + dy^2 - dw^2 . \tag{16}$$

It is natural to take the metric on the parameter space of SU(1,1) to be the metric induced by this embedding. Aficionados of de Sitter spaces will recognize this as three-dimensional anti-de Sitter space (see Ref. [2]). The group SO(2,2) which leaves this metric invariant will map the submanifold defined by Eq. (15) into itself, so the embedded three-manifold is maximally symmetric. Furthermore, it will be shown in the Appendix that the metric is group invariant— that is, either left- or right-translation by an element of SU(1,1) is an isometry of the metric.

To put coordinates on SU(1,1), note that we can decompose any element into a boost times a rotation:

$$
\begin{aligned}
T &= B(\vec{\zeta})R(\theta) \\
&= \begin{pmatrix} e^{-i\theta/2} \cosh \frac{\zeta}{2} & e^{i(\theta/2-\delta)} \sinh \frac{\zeta}{2} \\ e^{i(-\theta/2+\delta)} \sinh \frac{\zeta}{2} & e^{i\theta/2} \cosh \frac{\zeta}{2} \end{pmatrix} ,
\end{aligned} \tag{17}
$$

where ζ is the magnitude of the boost, δ is its polar angle, and θ is the angle of rotation. (Note that this is a different parameterization than the variables (α, ξ, ϕ) used in Eqs. (9-11).) Comparing (17) to Eq. (12) and defining $\psi = 2\delta - \theta$, we obtain

$$t = \sin\frac{\theta}{2}\,\cosh\frac{\zeta}{2}$$
$$x = \sin\frac{\psi}{2}\,\sinh\frac{\zeta}{2}$$
$$y = \cos\frac{\psi}{2}\,\sinh\frac{\zeta}{2}$$
$$w = \cos\frac{\theta}{2}\,\cosh\frac{\zeta}{2}\ . \tag{18}$$

The anti-de Sitter metric on SU(1,1) is the metric induced by Eq. (16) on the submanifold defined by Eq. (15). In the coordinates (θ, ζ, ψ), it is obtained by plugging the transformations (18) into Eq. (16), yielding

$$ds^2 = -\frac{1}{4}\cosh^2\frac{\zeta}{2}\,d\theta^2 + \frac{1}{4}d\zeta^2 + \frac{1}{4}\sinh^2\frac{\zeta}{2}\,d\psi^2\ . \tag{19}$$

Thus θ acts as a timelike coordinate.

D. The Energy-Momentum Vector

Elements of SU(1,1) sufficiently close to the identity may be written as exponentials of elements of the Lie algebra:

$$T = \exp\left(-i\phi^\mu \mathcal{J}_\mu\right)\ . \tag{20}$$

Since ϕ^μ describes a tangent vector at the identity of SU(1,1), we can compute its norm using the metric defined above. One could use Eq. (19), but it is easier to expand T to first order in ϕ^μ, and then to compare with Eq. (12) to determine the parameters (dt, dx, dy, dw) needed in Eq. (16). The result is

$$|\phi|^2 = \frac{1}{4}\eta_{\mu\nu}\,\phi^\mu\,\phi^\nu\ . \tag{21}$$

Not surprisingly, the group invariant metric in the vicinity of the identity coincides, up to a factor, with the usual (2+1)-dimensional Minkowski metric $\eta_{\mu\nu}$. If $|\phi|^2 < 0$ we will call the vector "timelike," keeping in mind the distinction between the timelike direction in the spacetime manifold and that on SU(1,1). Note that $|\phi|$ is actually the length of the

curve defined by $T(\lambda) = \exp(-i\lambda\phi^\mu \mathcal{J}_\mu)$, where λ varies from 0 to 1, as can be seen by first calculating the length of the segment from λ to $\lambda + d\lambda$.

To understand the properties of ϕ^μ, let us write the matrix describing parallel transport around a single particle in the form $T = \exp(-i\phi^\mu \mathcal{J}_\mu)$. Under a Lorentz transformation L in the physical spacetime, the group element $\exp(-i\phi^\mu \mathcal{J}_\mu)$ transforms as

$$L \exp(-i\phi^\mu \mathcal{J}_\mu) L^{-1} = \exp(-i\Lambda^\mu{}_\nu \phi^\nu \mathcal{J}_\mu) , \tag{22}$$

where L is an element of the 2×2 representation of SU(1,1) and Λ is the corresponding matrix in the 3×3 (adjoint) representation. In the rest frame of a single particle T is a pure rotation and ϕ^μ is equal to $(\alpha, 0, 0) = (8\pi GM, 0, 0)$. Using Eq. (22), one sees that in an arbitrary frame $\phi^\mu = 8\pi G(\gamma M, \gamma M\vec{v})$, in agreement (up to a factor) with the energy-momentum vector of special relativity. It follows immediately that any massive particle (moving slower than the speed of light) will be associated with a ϕ^μ that is timelike. If several particles are combined, however, then ϕ^μ_{tot} is not the sum of the individual momenta; rather, from Eq. (3),

$$\exp(-i\phi^\mu_{tot} \mathcal{J}_\mu) = \exp(-i\phi^\mu_N \mathcal{J}_\mu) \exp(-i\phi^\mu_{N-1} \mathcal{J}_\mu)$$
$$\times \ldots \exp(-i\phi^\mu_1 \mathcal{J}_\mu) . \tag{23}$$

In the $G \to 0$ limit, on the other hand, each exponential can be expanded to lowest order, and one finds that ϕ^μ_{tot} approaches the sum of the individual ϕ^μ_i. Since ϕ^μ_{tot} is a conserved Lorentz 3-vector which approaches the ordinary special relativistic energy-momentum vector in the $G \to 0$ limit, we will call it the energy-momentum vector. However, it should be recognized that it is a somewhat unconventional energy-momentum vector in at least two respects. First, and more importantly for our purposes, the energy-momentum vector for a group of particles is not the sum of the individual momenta. Second, it is constructed from the Lie algebra of the Lorentz group, rather than the tangent space of the spacetime. Nonetheless, in 2+1 dimensions the Lorentz generators can be rearranged to form a vector, as in Eq. (6). The expansion of a group element in terms of these generators gives rise to a three-tuple ϕ^μ,

which transforms according to Eq. (22) as a vector in the tangent space of the spacetime. The tangent space is constructed at the location of the base point of the closed loop used for parallel transport, which might be thought of as the location of the observer.[6]

For some purposes it will be convenient to write ϕ^μ as a parameter λ times a normalized vector n^μ; if ϕ^μ is timelike $|n|^2 = -1$, and for ϕ^μ spacelike $|n|^2 = +1$. For n^μ timelike, the explicit form for T is

$$
\begin{aligned}
e^{-i\lambda n^\mu \mathcal{J}_\mu} &= \cos\frac{\lambda}{2} - 2in^\mu \mathcal{J}_\mu \sin\frac{\lambda}{2} \\
&= \begin{pmatrix} \cos\frac{\lambda}{2} - in^0 \sin\frac{\lambda}{2} & (n^2 + in^1)\sin\frac{\lambda}{2} \\ (n^2 - in^1)\sin\frac{\lambda}{2} & \cos\frac{\lambda}{2} + in^0 \sin\frac{\lambda}{2} \end{pmatrix},
\end{aligned}
\tag{24}
$$

and for n^μ spacelike we obtain

$$
\begin{aligned}
e^{-i\lambda n^\mu \mathcal{J}_\mu} &= \cosh\frac{\lambda}{2} - 2in^\mu \mathcal{J}_\mu \sinh\frac{\lambda}{2} \\
&= \begin{pmatrix} \cosh\frac{\lambda}{2} - in^0 \sinh\frac{\lambda}{2} & (n^2 + in^1)\sinh\frac{\lambda}{2} \\ (n^2 - in^1)\sinh\frac{\lambda}{2} & \cosh\frac{\lambda}{2} + in^0 \sinh\frac{\lambda}{2} \end{pmatrix}.
\end{aligned}
\tag{25}
$$

Taking the trace of these two equations, we can see that a general matrix obtained by exponentiation satisfies

$$
\frac{1}{2}\,\mathrm{Tr}\left[e^{-i\lambda n^\mu \mathcal{J}_\mu}\right] = \begin{cases} \cos\frac{\lambda}{2} & \text{for } n^\mu \text{ timelike}, \\ \cosh\frac{\lambda}{2} & \text{for } n^\mu \text{ spacelike}. \end{cases}
\tag{26}
$$

It follows that

$$
\frac{1}{2}\,\mathrm{Tr}\left[e^{-i\lambda n^\mu \mathcal{J}_\mu}\right] \geq -1
\tag{27}
$$

for all cases.

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[6]If multiple observers are stretched along the loop, or along any deformation of the loop that intersects no other particles, then the observers will agree on the energy-momentum vector in the following sense: if the vector measured by one observer is parallel transported along the loop to another observer, it will agree with the vector measured by the second observer. We thank Ken Olum for explaining this to us.

From Eq. (27), it is easy to show that the SU(1,1) matrix corresponding to the Gott time machine is not the exponential of any generator. For simplicity, we take a configuration where the two particles approaching each other (as in Fig. 1) each have rest frame deficit angle α and rapidity ξ, with $\phi_1 = \pi$, $\phi_2 = 0$. Then we can use Eqs. (3) and (9) to write $T_G = T_2 T_1$ as

$$
T_G = \begin{pmatrix} (1+p^2)e^{-i\alpha'} - p^2 & -2p\sqrt{1+p^2}\sin\frac{\alpha'}{2} \\ -2p\sqrt{1+p^2}\sin\frac{\alpha'}{2} & (1+p^2)e^{i\alpha'} - p^2 \end{pmatrix} ,
\tag{28}
$$

where p and α' are given by Eqs. (10) and (11). The trace is then given by

$$
\frac{1}{2}\operatorname{Tr} T_G = (1+p^2)\cos\alpha' - p^2
$$
$$
= 1 - 2\cosh^2\xi\,\sin^2\frac{\alpha}{2} .
\tag{29}
$$

The condition that such a configuration contain closed timelike curves is [4]

$$
\cosh\xi\,\sin\frac{\alpha}{2} > 1 ,
\tag{30}
$$

or $\frac{1}{2}\operatorname{Tr} T_G < -1$. Thus, from Eq. (27) it follows that T_G cannot be written as an exponential.

However, SU(1,1) is a double cover of the Lorentz group, so the matrices $\pm T_G$ correspond to the same element of SO(2,1). We will see in the next section that $-T_G$ can be written as an exponential. Since $\frac{1}{2}\operatorname{Tr}(-T_G) > 1$, Eq. (26) implies that it is the exponential of a spacelike generator. The corresponding element of SO(2,1) can be obtained by exponentiating the corresponding generator, and thus the element of SO(2,1) associated with a Gott pair is spacelike or tachyonic (equivalent under similarity transformation to a pure boost). This is the sense in which we say that the Gott time machine has tachyonic momentum [8,12], even though $T_G \in$ SU(1,1) is not equivalent to the exponential of a spacelike generator — parallel transport of a spinor around a single tachyonic particle is not equivalent to parallel transport around the Gott two-particle system, although parallel transport of an SO(2,1) vector does not distinguish between the two cases.

E. The Proof Completed

We now return to the anti-de Sitter geometry of SU(1,1). For fixed n^μ, we may consider Eqs. (24) and (25) as defining curves parameterized by λ. The crucial observation is that these curves are geodesics in the metric (19), which can be checked directly. For example, by comparing Eqs. (17) and (24), one sees that the curve defined by (24) is equivalent to

$$\theta = 2\tan^{-1}\left(n^0 \tan\frac{\lambda}{2}\right)$$
$$\zeta = 2\sinh^{-1}\left[\sqrt{(n^0)^2 - 1}\sin\frac{\lambda}{2}\right]$$
$$\psi = -2\tan^{-1}\left(\frac{n^1}{n^2}\right) = \text{constant} . \tag{31}$$

It is straightforward to confirm that this solves the geodesic equation

$$\frac{d^2 x^\mu}{d\lambda^2} + \Gamma^\mu_{\rho\sigma}\frac{dx^\rho}{d\lambda}\frac{dx^\sigma}{d\lambda} = 0 , \tag{32}$$

for $x^\mu = (\theta, \zeta, \psi)$. This fact can be seen more directly by starting with a simple path, such as $T(\lambda) = \exp(-i\lambda\mathcal{J}_0)$, and verifying that this solves the geodesic equation. Then by Eq. (22) a Lorentz transformation $\Lambda^\mu{}_\nu$ will take this curve into another curve of the form $\exp(-i\lambda\phi^\mu\mathcal{J}_\mu)$, with $\phi^\mu = \Lambda^\mu{}_0$. Since the action of SU(1,1) is an isometry, the resulting curve must also be a geodesic. Finally, the isometry property also ensures that a curve of the form $T(\lambda) = T_0 \exp(-i\lambda n^\mu\mathcal{J}_\mu)$ $(T_0 \in$ SU(1,1)$)$ will be a geodesic through T_0.

A simple way to visualize anti-de Sitter space is in terms of its Penrose (conformal) diagram [2]. We define a new coordinate ζ' by

$$\zeta' = 4\tan^{-1}\left(e^{\zeta/2}\right) - \pi , \tag{33}$$

restricted to the range $0 \leq \zeta' < \pi$. The metric (19) becomes

$$ds^2 = \frac{1}{4\cos^2\frac{\zeta'}{2}}\left(-d\theta^2 + d\zeta'^2 + \sin^2\frac{\zeta'}{2}\,d\psi^2\right) . \tag{34}$$

The Penrose diagram is shown in Fig. 2; the angular coordinate ψ is suppressed. The light cones at each point are lines drawn at 45°. The right hand side of the rectangle is the

surface $\zeta' = \pi$, which represents spacelike and null infinity. The lower left corner is the origin, from which we have drawn typical spacelike and timelike geodesics. The lower and upper boundaries are the surfaces $\theta = 0$ and $\theta = 4\pi$, which are identified (the topology of SU(1,1) is thus $S^1 \times \mathbb{R}^2$). An important feature of this diagram is that timelike geodesics from the origin refocus at the point $(\theta = 2\pi, \zeta' = 0)$, as can be seen directly from Eq. (31) (note that $\zeta' = 0$ is equivalent to $\zeta = 0$). Therefore, points that are spacelike separated from $(\theta = 2\pi, \zeta' = 0)$ cannot be joined to the origin by a geodesic. These points correspond to the shaded region of the diagram.

Since every element of SU(1,1) that can be written as the exponential of a generator lies along a geodesic through the origin, points in the shaded region correspond to group elements that cannot be reached by exponentiation. The element T_G corresponding to the Gott time machine lies in this region. The element $-T_G$, on the other hand, can be obtained from T_G by subtracting 2π from θ, so $-T_G$ lies in the region that is spacelike separated from the origin. Thus $-T_G$ can be reached by exponentiation, as was claimed in the previous section.

The product of two elements $T_B = \exp(-i\phi_B^\mu \mathcal{J}_\mu)$ and $T_A = \exp(-i\phi_A^\mu \mathcal{J}_\mu)$ corresponds to traveling in the direction of ϕ_A^μ down a geodesic to T_A, then traveling along a different geodesic (not through the origin) to $T_B T_A$. As shown in the diagram, we can easily reach the shaded region, and hence the Gott time machine, in this manner. The parameter space of SO(2,1) can be visualized by cutting the diagram in half, identifying the surfaces $\theta = 0$ and $\theta = 2\pi$. Then the shaded region of Fig. 2 is mapped onto the wedge which is covered by spacelike geodesics emanating from the origin. This is consistent with our interpretation above of the tachyonic momentum of the Gott time machine.

Consider a system of particles represented by an element T_0 of SU(1,1). We divide this system into a subsystem represented by an element T_S and the remaining N individual particles represented by T_i:

$$T_0 = T_N \ldots T_1 T_S . \tag{35}$$

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This relation can be represented on the Penrose diagram by a future-directed nonspacelike curve from T_S to T_0, constructed from geodesic segments representing each of the T_i. It is clear that the periodicity in the θ direction allows any two points to be connected in this way — as far as SU(1,1) is concerned, any system of particles can contain a subset with arbitrary energy and momentum.

However, the identification $\theta \leftrightarrow \theta + 4\pi k$ obscures an important difference between physically distinct situations. To make this difference apparent, we must go to $\widetilde{SU}(1,1)$, the universal cover of SU(1,1). $\widetilde{SU}(1,1)$ is then also the universal cover of SO(2,1). In terms of the Penrose diagram, we no longer identify $\theta = 0$ with $\theta = 4\pi$, but instead we extend the picture infinitely far in the positive and negative θ direction (Fig. 3). The timelike geodesics from the origin will refocus at $\theta = 2\pi$, then continue onward, refocusing again at $\theta = 2\pi k$ for every integer k. Therefore the wedges of points that are spacelike separated from ($\theta = 2\pi k$, $\zeta' = 0$) for $k \neq 0$ cannot be reached from any geodesic through the origin. All of these points may be said to correspond to tachyonic momenta, since they map to elements of SO(2,1) lying on spacelike geodesics from the origin.

To describe a multiparticle system using $\widetilde{SU}(1,1)$, we express T_{tot} as a product as in Eq. (3), with each T_i representing a single particle. Any particle in its rest frame has a deficit angle $\alpha < 2\pi$, so one can uniquely define T_i in the universal covering group by using Eq. (2), interpreting $R(\alpha)$ as the element of the universal covering group described by ($\theta = \alpha, \zeta' = 0$), where $0 \leq \alpha < 2\pi$. Similarly $B(\vec{\xi})$ can be chosen to lie in the sector of the universal covering group that is spacelike separated from the identity. (In this case, however, any other choice would be equivalent. The ambiguity consists of any number of factors of the group element corresponding to a rotation by 2π, and this element commutes with all other elements. Therefore, in Eq. (2), the ambiguous factor would cancel between B and B^{-1}.)

Using this construction, a system of several particles, each with timelike momentum, can lead to a $T_{tot} \in \widetilde{SU}(1,1)$ in any region inside the future light cone of the origin, including the wedges that are spacelike separated from ($\theta = 2\pi k$, $\zeta' = 0$) for $k > 0$. (A two-particle

system, however, will only be able to reach the first such wedge).

We now possess the machinery necessary to prove the theorem. We consider an open universe, populated by particles with future-directed timelike or lightlike momenta, such that the total momentum is timelike. In such a universe we can always boost into the rest frame, where (by definition) parallel transport around all of the particles results in a rotation, $R(\theta_0)$. Since the universe is open, we must have $\theta_0 \leq 2\pi$. Therefore, our hypothetical universe lies on the straight line between the origin and $(\theta = 2\pi, \zeta' = 0)$ in Fig. 3. We now use the method of proof by contradiction. Suppose that some subset of particles in this universe has a net spacelike momentum, corresponding to an element T_G of $\widetilde{\mathrm{SU}}(1,1)$. We can then write

$$R(\theta_0) = T_N \ldots T_1 T_G \ , \tag{36}$$

where the T_i represent the individual particles comprising the rest of the universe. If we pick a point in the shaded region of Fig. 3 to represent T_G, then the right hand side of this expression must correspond to a point in the future light cone of this point, since each T_i can be represented by a segment of a future-directed timelike or null geodesic. However, it is clear that none of the points on the straight line between the origin and $(\theta = 2\pi, \zeta' = 0)$ lie in this light cone. Therefore, we have a contradiction, and our supposition must be false — in an open universe with timelike total momentum, no subset of particles can have a spacelike momentum. This proves the theorem; the statement in the title of our paper follows immediately.

III. DISCUSSION

We have proven that a system of particles with spacelike momentum can exist in an open (2+1)-dimensional universe only if the total momentum is spacelike. As the momentum of the Gott time machine is necessarily spacelike, this result precludes the existence (and hence the creation) of such a time machine in an open universe with timelike total momentum. In the process we have found that the anti-de Sitter geometry of $\mathrm{SU}(1,1)$ provides a useful tool for visualizing the momentum of a system of gravitating particles in 2+1 dimensions.

There is an ongoing discourse concerning the appearance and significance of CTC's in general relativity. The causal paradoxes which seem to inevitably accompany the existence of time machines have led to the general belief that CTC's are to be avoided — although some recent investigations have argued to the contrary [18]. If we adopt the conservative attitude that CTC's lead to unacceptably paradoxical consequences, we would hope that the laws of physics conspire to prevent their occurrence. While Einstein's equations permit a wide variety of solutions that include CTC's, most of these require either unrealistic energy-momentum tensors or global identifications that appear to be physically unattainable (for examples see Refs. [1] and [2]). We are therefore led to ask whether CTC's can arise in an "initially causal" universe.

Recent investigations of (3+1)-dimensional spacetimes with traversable wormholes [19] have shown that such configurations seem to easily develop CTC's. However, the mainte-nance of a traversable wormhole requires violation of the weak energy condition (WEC). While quantum field theory in curved spacetime might allow WEC violation, there is evi-dence that quantum fluctuations serve to destabilize the would-be time machine, preventing the appearance of CTC's [3,20] (for a competing view, see Ref. [21]). Boulware [22] has stud-ied the behavior of a scalar (2+1)-dimensional quantum field theory in the Gott spacetime, finding that if the mass of the quantum field is sufficiently large, then the renormalized stress tensor is regular in the vicinity of the Cauchy horizon. Given our incomplete understanding of quantum gravity, it is worthwhile to ask if classical general relativity (with the WEC enforced) can avoid CTC creation.

Tipler [9] has shown that creation of CTC's in a compact region of a (3+1)-dimensional spacetime must necessarily lead to the creation of a singularity, as long as the WEC holds and there are tidal forces on some point on the Cauchy horizon. (A similar theorem has been proven by Hawking in Ref. [3].) This theorem implies that an attempt to make a time machine from finite lengths of cosmic string would create a singularity, presumably (although not certainly) surrounded by an event horizon (as suggested in Ref. [4]). If the event horizon surrounds the CTC's, they would not interfere with the causal properties of the

external space. The hypothesis that CTC's in 3+1 dimensions are necessarily hidden behind horizons is reminiscent of the connection between the ability to reach Gott's condition in 2+1 dimensions and the closure of the universe.

Tipler's and Hawking's use of the (Newman-Penrose) optical scalar equations applies specifically to the (3+1)-dimensional case. However, analogous reasoning can be adapted to 2+1 dimensions, with the same result.[7] Thus, this method can be used to construct an alternative proof that Gott time machines cannot be created in a local region of an open (2+1)-dimensional universe. This result, therefore, agrees with the theorem in this paper and is significantly more general with regard to the construction of time machines.

Meanwhile, the theorem proven in Sec. II shows that a collection of particles with a tachyonic momentum cannot even exist in an open timelike universe, much less be created. This theorem, while limited to 2+1 dimensions, has no restriction analogous to the compact-region requirement of Hawking's theorem, and it applies to tachyons regardless of whether they are associated with CTC's. Nevertheless, we have not excluded the possibility that open timelike universes might contain CTC's distinct from the type proposed by Gott. Waelbroeck [23] has shown that a two-particle system with timelike momentum does not support CTC's, and Kabat [24] has presented arguments suggesting that this result is more general. A comprehensive proof (or counterexample) is worth searching for.

Of course, Gott's original universe is still with us. Does this solution represent the creation of a time machine? It is certainly not one we could build in the laboratory, as the CTC's do not originate in a local region — in the language of Hawking [3], the Cauchy horizon is not compactly generated. Nevertheless, there exist complete spacelike hypersurfaces which are not intersected by CTC's, on which initial data can presumably be specified [6]. Our work has shown that we can place a restriction on this data such that a Gott time machine can never arise. However, the restriction to timelike momenta is rather intimately

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[7]We are grateful to Ted Pyne for discussions on this point.

connected with the nature of (2+1)-dimensional gravity, so it is not clear how this could be promoted to a general principle applicable to the (3+1)-dimensional case.

The demonstration that Gott time machines cannot exist in open timelike (2+1)-dimensional universes leads naturally to the question of closed universes. In our earlier paper [12] we argued that the obstacle to building a time machine in an open universe was that there could never be enough mass-energy to accelerate particles to sufficiently high velocity; in other words, it is impossible to produce two particles satisfying the Gott condition by the decay of slowly-moving parent particles unless the total rest frame deficit angle exceeds 2π. In a closed universe, where the total deficit angle is 4π, this does not constitute an obstacle. It is easy to imagine a closed universe containing two particles, each with deficit angle between π and 2π, and a number of less massive spectator particles which bring the total deficit angle to 4π. Using the description of decays given in Ref. [12], we have found that the two massive particles can decay in such a way that each emits an offspring at sufficiently high velocity that the total momentum of the two fast-moving particles is tachyonic.[8] Thus, in a closed (2+1)-dimensional universe it is possible to "build a tachyon." However, as we mentioned in the introduction, 't Hooft [14] has shown that the size of the universe begins to shrink precipitously after the decays, leading to a crunch (zero volume) before any CTC's can arise. There is thus a sense in which general relativity is flexible enough to permit tachyons, but works very hard to prevent time travel.

ACKNOWLEDGMENTS

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[8]Our construction is described in Ref. [14].

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SOME PROPERTIES OF SU(1,1)

In this Appendix we demonstrate two technical properties concerning SU(1,1) and the embedding of its parameter space that is introduced in Sec. II-C. First, we prove the statement made in the text concerning the conditions under which a 2×2 matrix belongs to SU(1,1). Next we demonstrate the group invariance of the metric described in the text.

To derive the conditions under which a 2×2 matrix belongs to SU(1,1), we begin by parameterizing an arbitrary 2×2 matrix as

$$T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \tag{A.1}$$

where a, b, c, and d are all complex. As stated in the text, SU(1,1) consists of those matrices T satisfying

$$\det T = +1 \tag{A.2a}$$

and

$$T^\dagger \eta T = \eta , \tag{A.2b}$$

where

$$\eta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{A.3}$$

From Eqs. (A.2) one has immediately that

$$a^* a - c^* c = 1 \tag{A.4a}$$

$$b^* b - d^* d = -1 \tag{A.4b}$$

$$b^* a - d^* c = 0 \tag{A.4c}$$

and

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$$ad - bc = 1 \ . \tag{A.5}$$

If Eq. (A.4c) is solved for c and the result is inserted into (A.5), one finds

$$a(d^*d - b^*b) = d^* \ .$$

Using (A.4b), this reduces to

$$a = d^* \ . \tag{A.6}$$

Combining this result with Eq. (A.4c), one has

$$b^* = c \ . \tag{A.7}$$

Thus, T can be written as

$$T = \begin{pmatrix} a & b \\ b^* & a^* \end{pmatrix}, \tag{A.8}$$

where from Eq. (A.5) we have

$$a^*a - b^*b = 1 \ . \tag{A.9}$$

Comparing with the parameterization

$$T = \begin{pmatrix} w - it & y + ix \\ y - ix & w + it \end{pmatrix}, \tag{A.10}$$

used in the text, one sees that w, t, x, and y are all real and satisfy

$$- t^2 + x^2 + y^2 - w^2 = -1 \ . \tag{A.11}$$

Conversely, it is easily shown that if w, t, x, and y are all real and satisfy Eq. (A.11), then the matrix (A.10) belongs to SU(1,1).

Next, we wish to verify that the metric defined in Sec. II-C is group invariant. A group transformation on the group parameter space can be defined by mapping each element of the parameter space to the element obtained by multiplying on the left by a fixed element of the group, which we call $\tilde{T}$. Thus, the mapping is defined by

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$$\begin{pmatrix} w' - it' & y' + ix' \\ y' - ix' & w' + it' \end{pmatrix} = \tilde{T} \begin{pmatrix} w - it & y + ix \\ y - ix & w + it \end{pmatrix}. \tag{A.12}$$

Note that the metric of Eq. (16) can be written as

$$ds^2 = -dt^2 + dx^2 + dy^2 - dw^2$$

$$= \det(dT) , \tag{A.13}$$

where

$$dT = \begin{pmatrix} dw - idt & dy + idx \\ dy - idx & dw + idt \end{pmatrix}. \tag{A.14}$$

Since $\det \tilde{T} = 1$, it follows immediately that $\det(dT') = \det(dT)$, so the metric is invariant. It is similarly clear that the metric is invariant under multiplication by a fixed group element on the right.

(It is not needed in our derivation, but it is interesting to note that the full invariance group of the metric given by Eq. (16) is SO(2,2), for which the Lie algebra is identical to SU(1,1) × SU(1,1). One of the two SU(1,1) subgroups has generators that are self-dual (in the 4-dimensional w-t-x-y space), and the other has generators that are anti-self-dual. Transformations of the form described by Eq. (A.12) make up one of the SU(1,1) subgroups, while the other subgroup corresponds to multiplication on the right.)

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FIGURES

FIG. 1. *A spacelike slice through the Gott spacetime.* Two parallel cosmic strings perpendicular to the page, represented by dots, move past each other at high velocity. A deficit angle (shaded) is removed from the space around each string, with opposite sides identified at equal times. (If the deficit angles were oriented in any direction other than along the motion of the string, the identifications would be at unequal times.) Note that no CTC's pass through this spacelike surface, as explained in the text.

FIG. 2. *The conformal diagram of SU(1,1), the double cover of SO(2,1).* The group manifold of SU(1,1) with the invariant metric is shown. The ψ direction is suppressed; hence, each point away from the $\zeta' = 0$ line represents a circle. The top and bottom edges, $\theta = 4\pi$ and $\theta = 0$, are identified. The identity element is in the lower left hand corner; we have indicated some spacelike and timelike geodesics from this point. Elements of SU(1,1) which can be expressed as exponentials of generators lie on such geodesics. The product T_G of two such elements T_B and T_A is represented by a curve constructed from two consecutive geodesic segments, as shown. In this case the product lies in the shaded region, which represents elements which cannot be expressed as exponentials of generators. The Gott time machine lies in this region.

FIG. 3. *The universal cover of SU(1,1).* In the universal covering group, the identification of θ with $\theta + 4\pi$ is removed, so that θ takes values from $-\infty$ to ∞. Thus, the interior of the future light cone of T_G, which contains all universes consisting of the Gott time machine and a set of other massive particles, does not include any points on the $\zeta' = 0$ line between $\theta = 0$ and $\theta = 2\pi$. Since all open universes with timelike total momentum lie on this line (in their rest frame), no such universe can contain a Gott time machine.