The Super Period Matrix With Ramond Punctures

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Abstract

We generalize the super period matrix of a super Riemann surface to the case that Ramond punctures are present. For a super Riemann surface of genus g with 2r Ramond punctures, we define, modulo certain choices that generalize those in the classical theory (and assuming a certain generic condition is satisfied), a $g|r \times g|r$ period matrix that is symmetric in the \mathbb{Z}_2 -graded sense. As an application, we analyze the genus 2 vacuum amplitude in string theory compactifications to four dimensions that are supersymmetric at tree level. We find an explanation for a result that has been found in orbifold examples in explicit computations by D'Hoker and Phong: with their integration procedure, the genus 2 vacuum amplitude always vanishes "pointwise" after summing over spin structures, and hence is given entirely by a boundary contribution.

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1 Introduction

By analogy with the classical period matrix of an ordinary Riemann surface, one can define the super period matrix (sometimes just called the period matrix) of a super Riemann surface. If Σ is a super Riemann surface of genus g with even spin structure, then its super period matrix is a $g \times g$ symmetric matrix of positive imaginary part. Actually, the super period matrix is only defined for the case that Σ has an even spin structure, and even then it is only defined generically: it can acquire a pole, with nilpotent residue, when moduli of Σ are varied.

The first goal of the present paper is to extend the definition of the super period matrix to the case of a super Riemann surface with Ramond punctures. (A Neveu-Schwarz puncture, which is simply a marked point, does not affect the definition of the super period matrix.) It is conceivable that there is more than one reasonable definition. The definition we give here is motivated by an application that we will explain shortly. In this definition, the super period matrix of a super Riemann surface Σ of genus g with g Ramond punctures (the number of Ramond punctures is always even) is a $g|r \times g|r$ matrix, symmetric in the \mathbb{Z}_2 -graded sense, whose $g \times g$ bosonic block has positive definite imaginary part. (The super period matrix is in general not an arbitrary matrix of this sort, since in general there are

¹See [1–3] for original references, and section 8 of [4] for a review. The last reference also contains a general introduction to super Riemann surfaces.

Schottky relations.) Just as in the classical case, the definition of the period matrix depends on a choice of A-cycles; when (and only when) Ramond punctures are present, one has to define fermionic as well as bosonic A-cycles. If one changes the A-cycles that are used in defining it, the super period matrix is transformed by an element of an integral form of the supergroup OSp(2r|2g), generalizing the fact that in the classical case (or for a super Riemann surface without Ramond punctures), the period matrix is defined up to the action of an integral form of Sp(2g). Just as in the absence of Ramond punctures, the super period matrix is only generically defined, and can acquire singularities as moduli are varied.

The application we have in mind involves superstring perturbation theory in genus 2. Every 2×2 matrix of positive imaginary part is the period matrix of an ordinary Riemann surface of genus 2, unique up to isomorphism (Schottky relations only exist in genus ≥ 4). Hence, to a super Riemann surface Σ of genus 2 with even spin structure, we can associate an ordinary Riemann surface Σ_{red} of the same period matrix. Σ_{red} also inherits a spin structure from the spin structure of Σ , and the association $\Sigma \to \Sigma_{\text{red}}$ gives a natural holomorphic projection $\pi: \mathfrak{M}_{2,+} \to \mathcal{M}_{2,\text{spin}+}$ from the moduli space $\mathfrak{M}_{2,+}$ of super Riemann surfaces Σ of genus 2 with even spin structure to its reduced space $\mathcal{M}_{2,\text{spin}+}$ which parametrizes an ordinary Riemann surface Σ_{red} with even spin structure.²

By integrating over the fibers of π , one can map the two-loop vacuum amplitude of superstring theory, which is naturally a measure Υ on $\mathfrak{M}_{2,+}$, to a measure $\pi_*(\Upsilon)$ on $\mathcal{M}_{2,\mathrm{spin}+}$. This procedure was the starting point in the celebrated analysis of the two-loop vacuum amplitude by D'Hoker and Phong. (For a review with further references, see [5]. D'Hoker and Phong also went on to calculate scattering amplitudes in genus 2, a much more difficult computation that is beyond the scope of the present paper.)

To analyze the integral $\int_{\mathcal{M}_{2,\text{spin}+}} \pi_*(\Upsilon)$, it makes sense to first sum over even spin structures before performing any integration. In this way, one projects $\pi_*(\Upsilon)$ from $\mathcal{M}_{2,\text{spin}+}$ to \mathcal{M}_2 , with the spin structure forgotten. In their original work, D'Hoker and Phong analyzed this sum over spin structures for superstring theory on \mathbb{R}^{10} and for certain supersymmetric orbifold compactifications to six dimensions. They showed that the sum over spin structures vanishes in those models, analogous to the familiar GSO cancellation in genus 1.

Something new happens in general in the case of a compactification to four dimensions that at tree level has $\mathcal{N}=1$ supersymmetry. (The most simple examples are provided by compactification of the heterotic string on a Calabi-Yau three-fold.) In this case, it is possible for a 1-loop effect to generate a Fayet-Iliopoulos D-term, triggering the spontaneous breaking of supersymmetry [6–8]. When this happens, one expects the genus 2 vacuum amplitude to be non-zero and proportional to D^2 . How does this occur in the context of the

²The map π is everywhere defined (on $\mathfrak{M}_{2,+}$ as opposed to its Deligne-Mumford compactification), in part because for g=2 (unlike g>2) the super period matrix has no poles. Since the odd dimension of $\mathfrak{M}_{2,+}$ is 2, π is actually a splitting of $\mathfrak{M}_{2,+}$.

D'Hoker-Phong procedure for computing the genus 2 vacuum amplitude?

In general [9], the D'Hoker-Phong procedure must be supplemented with a boundary correction (a contribution supported on the divisor at infinity in the compactified moduli space). The boundary contribution to the genus 2 vacuum amplitude vanishes in supersymmetric compactifications above four dimensions, but in a four-dimensional model with $\mathcal{N}=1$ supersymmetry, it is proportional to D^2 .

This raises the possibility that in such models, the full answer comes from this boundary correction, and that the bulk contribution, computed with the procedure of D'Hoker and Phong, always vanishes. Something similar happens in the same models in one-loop computations of certain supersymmetry-violating mass splittings [7,8].

In fact, in examples of orbifold compactifications to four dimensions with $\mathcal{N}=1$ supersymmetry [10], the same behavior has been found that was found earlier in supersymmetric models above four dimensions: the bulk contribution $\pi_*(\Upsilon)$ to the genus 2 vacuum amplitude vanishes after summing over spin structures, without any integration over bosonic moduli. In the present paper, we will use the theory of the super period matrix with Ramond punctures to demonstrate that this very striking behavior will occur in all supersymmetric compactifications to four or more dimensions.

Perhaps we should remark that general arguments based on supersymmetric Ward identities (see for example section 4 of [9]) can be used to determine the *integrated* genus 2 vacuum amplitude, but do not explain the "pointwise" vanishing that occurs in the D'Hoker-Phong procedure. Our goal here is to explain this more detailed phenomenon. The arguments governing the integrated behavior are completely general and apply for all values of the genus. The procedure that leads to pointwise vanishing is defined only for genus ≤ 2 or at most (as we discuss in section 6) $g \leq 3$.

We define the super period matrix with Ramond punctures and explain some of its simplest properties in sections 2-5 of this paper. (A parallel treatment of some of these issues from the point of view of supergravity will appear elsewhere [11].) The application to the two-loop vacuum amplitude is in sections 6-9. The general strategy to constrain the vacuum amplitude via supersymmetry is familiar [12], and involves comparing the genus g vacuum amplitude to an amplitude computed on $\mathfrak{M}_{g,0,2}$. Since we specifically want to constrain the genus 2 vacuum amplitude computed with a procedure that uses the super period matrix, we have to begin with an understanding of the super period matrix of a genus 2 super Riemann surface with 2 Ramond punctures. The super period matrix in this

³In general, $\mathfrak{M}_{g,n,2r}$ will denote the moduli space of super Riemann surfaces of genus g with n Neveu-Schwarz punctures and 2r Ramond punctures. We write $\mathfrak{M}_{g,n,2r,\pm}$ if we wish to indicate the type of spin structure. Similarly, $\mathcal{M}_{g,n}$ is the moduli space of ordinary Riemann surfaces of genus g with n punctures, while $\mathcal{M}_{g,n,\mathrm{spin}\pm}$ is the corresponding moduli space with a choice of even or odd spin structure.

situation is only generically defined, with singularities on a certain locus in the moduli space. The trickiest part of our analysis is to show that these singularities do not ruin the argument; see section 9.

Some technical issues are treated in appendices. In Appendix A, we explain in detail why the super period matrix has a pole; in Appendix B, we give an alternative explanation of the fact that the D'Hoker-Phong procedure for integration over genus 2 supermoduli space requires a correction at infinity; and in Appendix C, we describe some properties of the genus 3 analog of the D'Hoker-Phong procedure.

2 Odd Periods

We will begin by recalling the definition of a super Riemann surface with or without Ramond punctures. (The reader may want to consult a more detailed reference such as [1] or [4].) Then we go on to discuss periods.

A super Riemann surface Σ is a complex supermanifold of dimension 1|1 whose tangent bundle $T\Sigma$ is endowed with a subbundle \mathcal{D} of rank 0|1 that is completely unintegrable. Complete unintegrability means that if s is a nonzero section of \mathcal{D} , then s and $\{s,s\}$ are everywhere linearly independent, so that $\{s,s\}$ generates $T\Sigma/\mathcal{D}$. Thus $T\Sigma$ fits in an exact sequence

$$0 \to \mathcal{D} \to T\Sigma \to \mathcal{D}^2 \to 0. \tag{2.1}$$

Dually, the cotangent bundle of Σ fits in an exact sequence

$$0 \to \mathcal{D}^{-2} \to T^* \Sigma \to \mathcal{D}^{-1} \to 0. \tag{2.2}$$

One can show that locally, one can pick coordinates $z|\theta$ on Σ such that \mathcal{D} is generated by

$$D_{\theta} = \frac{\partial}{\partial \theta} + \theta \frac{\partial}{\partial z}.$$
 (2.3)

Such coordinates are called local superconformal coordinates. Dually the subbundle \mathcal{D}^{-2} of $T^*\Sigma$ is generated by

$$\varpi = \mathrm{d}z - \theta \mathrm{d}\theta. \tag{2.4}$$

A super Riemann surface Σ with Ramond punctures is a complex supermanifold of dimension 1|1 whose tangent bundle is still endowed with a subbundle \mathcal{D} of rank 0|1, but now the condition of complete unintegrability fails along a certain divisor. The local behavior is

that \mathcal{D} is generated, in some coordinates $z|\theta$, by

$$D_{\theta}^* = \frac{\partial}{\partial \theta} + \theta z \frac{\partial}{\partial z}.$$
 (2.5)

Since $(D_{\theta}^*)^2 = z\partial_z$, we see that D_{θ}^* and $(D_{\theta}^*)^2$ fail to be linearly independent precisely along the divisor \mathcal{F} defined by z = 0. Thus, the exact sequence (2.1) is replaced by

$$0 \to \mathcal{D} \to T\Sigma \to \mathcal{D}^2(\mathcal{F}) \to 0. \tag{2.6}$$

Dually, one has

$$0 \to \mathcal{D}^{-2}(-\mathcal{F}) \to T^*\Sigma \to \mathcal{D}^{-1} \to 0, \tag{2.7}$$

with $\mathcal{D}^{-2}(-\mathcal{F})$ generated by

$$\varpi^* = \mathrm{d}z - z\theta\mathrm{d}\theta. \tag{2.8}$$

More globally, Σ may have many divisors \mathcal{F}_i along which the local behavior of the subbundle $\mathcal{D} \subset T\Sigma$ is as just described. We call the \mathcal{F}_i Ramond divisors. If Σ is compact, the number of Ramond divisors is always even. What appears in the exact sequences (2.6) and (2.7) is $\mathcal{F} = \sum_i \mathcal{F}_i$, which we might call the total Ramond divisor. Away from \mathcal{F} , Σ is an ordinary super Riemann surface. Along \mathcal{F} , Σ remains smooth, but there is a singularity in its superconformal structure.

A general holomorphic 1-form on Σ can be written locally as $f(z|\theta)dz + g(z|\theta)d\theta$. In contrast to the case of an ordinary Riemann surface, a holomorphic 1-form is not necessarily closed; if μ is a holomorphic 1-form, then $d\mu$ is a holomorphic 2-form, with an expansion $d\mu = a(z|\theta)dzd\theta + b(z|\theta)(d\theta)^2$. In defining differential forms on a supermanifold, we define the exterior derivative to be odd, so dz is odd and anticommutes with θ , while $d\theta$ is even and commutes with θ and dz.

We want to define periods of holomorphic 1-forms. Just as in ordinary geometry, periods, as topological invariants, are only defined for 1-forms that are closed.

The most obvious periods are the analogs of classical periods. Let μ be a 1-form on Σ . If S is an oriented circle and $\alpha: S \to \Sigma$ is any continuous map, then μ pulls back to an ordinary 1-form $\alpha^*(\mu)$ on S so we define the integral $\oint_S \alpha^*(\mu)$. Just as in the classical case, if μ is closed, then $\oint_S \alpha^*(\mu)$ only depends on the homology class determined by the map α . For our purposes, the case that α is an embedding is sufficient, so we can just think of S as a smooth submanifold⁴ of Σ , of real dimension 1 (or 1|0). In this case, we write just $\oint_S \mu$ rather than $\oint_S \alpha^*(\mu)$. An embedded circle in the reduced space $\Sigma_{\rm red}$ of Σ can be lifted (not canonically, but in a way that is unique up to homology) to an embedded circle in Σ . So,

⁴Though there is apparently not a useful notion of a smooth function on a complex supermanifold, there is a useful notion of a smooth submanifold of a complex supermanifold, or of a continuous map of a smooth submanifold to Σ . See section 5 of [13].

as in [1], one can define A-periods and B-periods for a closed holomorphic 1-form on Σ that correspond precisely to the familiar A-periods and B-periods of a holomorphic differential on $\Sigma_{\rm red}$.

Thus if μ is a closed holomorphic 1-form on Σ , it has the usual g A-periods and g B-periods. However, in the presence of Ramond punctures, such a μ also has what we might call odd periods. As explained above, near any Ramond divisor \mathcal{F}_0 , we can pick local coordinates $z|\theta$ in which \mathcal{F}_0 is defined by z=0 and in which the superconformal structure is defined by the distribution generated by

$$D_{\theta}^* = \frac{\partial}{\partial \theta} + \theta z \frac{\partial}{\partial z} \tag{2.9}$$

or dually by the subbundle of the contangent bundle generated by

$$\varpi^* = \mathrm{d}z - z\theta\mathrm{d}\theta. \tag{2.10}$$

In what follows, it is important that θ is uniquely defined up to

$$\theta \to \pm (\theta + c) \mod z,$$
 (2.11)

where c is an odd constant.⁵ In particular, at z = 0, we are not free to rescale θ except by ± 1 , since this would disturb the relation between the two terms in ϖ^* or in D_{θ}^* . Accordingly, the 1-form $d\theta$ on \mathcal{F}_0 is well-defined up to sign. We consider a choice of what we mean by $d\theta$ as opposed to $-d\theta$ to represent an "orientation" of \mathcal{F}_0 .

The odd periods of a closed holomorphic 1-form μ are now defined as follows. Since \mathcal{F}_0 is defined by z = 0, when restricted to \mathcal{F}_0 , we have

$$\mu = \frac{w}{\sqrt{2\pi\sqrt{-1}}} \,\mathrm{d}\theta \,\,\mathrm{mod}\,\,z. \tag{2.12}$$

Here w is a constant – independent of θ – since $d\mu = 0$. (The odd-looking factor $\sqrt{2\pi\sqrt{-1}}$ in the denominator will be convenient later.) We simply define w to be the odd period of μ associated to the Ramond divisor \mathcal{F}_0 . Thus, an odd "period" is not defined by an integral but by "evaluation" of a 1-form along a Ramond divisor. The sign of the odd period depends on the orientation of the Ramond divisor, somewhat analogously to the fact that a 1-manifold γ in an ordinary Riemann surface Σ_0 must be oriented if one wishes to define the sign of the period $\oint_{\gamma} \mu$ of a closed 1-form μ . With 2r Ramond divisors $\mathcal{F}_1, \ldots, \mathcal{F}_{2r}$, we define in this way 2r odd periods w_1, \ldots, w_{2r} .

⁵Since $z|\theta \to z| - \theta$ preserves ϖ^* , it is obvious that the superconformal structure determines θ only up to sign. To see that θ is also only defined up to a shift $\theta \to \theta + c$ with c an odd constant, one observes that the supergroup of dimension 0|1 that acts by $z|\theta \to z(1-c\theta)|\theta + c$ preserves the superconformal structure of Σ and acts transitively on \mathcal{F}_0 . One can verify that along \mathcal{F}_0 , any superconformal transformation is equivalent to $z|\theta \to z(1-c\theta)| \pm (\theta+c)$.

Thus in all, on a super Riemann surface Σ of genus g with 2r Ramond punctures, we define 2g even periods, just as in the classical theory, and 2r odd periods. What is the natural symmetry group acting on these periods? On the bosonic periods, we have the usual group $\operatorname{Sp}(2g;\mathbb{Z})$ of automorphisms of $H^1(\Sigma;\mathbb{Z})$ (which coincides with $H^1(\Sigma_{\operatorname{red}};\mathbb{Z})$). We can think of $\operatorname{Sp}(2g;\mathbb{Z})$ as the automorphism group of the skew form $\sum_{i=1}^g \operatorname{d} a^i \wedge \operatorname{d} b_i$ defined over \mathbb{Z} , where $a^i, b_j, i, j = 1, \ldots, g$ are even variables so $\operatorname{d} a^i, \operatorname{d} b_j$ are odd. The natural symmetries acting on the fermionic periods are as follows: we could permute the Ramond divisors, thus permuting the fermionic periods, or we could reverse the orientations of the Ramond divisors and thereby reverse the signs of the fermionic periods. The permutations and sign changes make up a finite group with $2^{2r}(2r)!$ elements. We can think of this group as a form of $\operatorname{O}(2r;\mathbb{Z})$, since it is the automorphism group of the quadratic form $\sum_{k=1}^{2r} \operatorname{d} x_k^2$, defined over \mathbb{Z} , where the $\operatorname{d} x_k$ are the even differentials of odd variables $x_k, k = 1, \ldots, 2r$. There are no obvious symmetries between even and odd periods. If we combine the two constructions, the full group $\operatorname{Sp}(2g;\mathbb{Z}) \times \operatorname{O}(2r;\mathbb{Z})$ that acts on the even and odd periods is the automorphism group of a form

$$\Theta = \sum_{i} da^{i} \wedge db_{i} - \sum_{k} dx_{k}^{2}$$
(2.13)

that is symmetric in the \mathbb{Z}_2 graded sense in 2g odd variables $\mathrm{d}a^i$, $\mathrm{d}b_j$, and 2r even variables $\mathrm{d}x_k$. (For now the sign of the $\sum_k \mathrm{d}x_k^2$ term is an arbitrary choice.) We think of this as a sort of superanalog of the intersection form of an ordinary Riemann surface. Over \mathbb{Z} , there are no symmetries that exchange even and odd variables, so $\mathrm{Sp}(2g;\mathbb{Z}) \times \mathrm{O}(2r;\mathbb{Z})$ can be interpreted as $\mathrm{OSp}(2r|2g;\mathbb{Z})$, that is, as a form over \mathbb{Z} of the orthosymplectic group. If we work over \mathbb{R} , of course the supergroup $\mathrm{OSp}(2r|2g;\mathbb{R})$ of symmetries of the form Θ does mix even and odd variables.

Typically, not all of $\mathrm{OSp}(2r|2g;\mathbb{Z})$ is realized as symmetries in string theory. A super Riemann surface has a spin structure, so usually one has to consider only the subgroup of $\mathrm{Sp}(2g;\mathbb{Z})$ that preserves a spin structure. Also it usually is more useful to consider the Ramond divisors to be labeled (or distinguishable), in which case one considers only the sign changes rather than permutations in $\mathrm{O}(2r;\mathbb{Z})$. Finally, in a certain sense, the space spanned by x_1,\ldots,x_{2r} has a natural orientation, as we explain in section 5.3, so one can replace $\mathrm{O}(2r;\mathbb{Z})$ by $\mathrm{SO}(2r;\mathbb{Z})$.

3 Closed Holomorphic Two-Forms

Let us recall how one proves the symmetry of the period matrix of an ordinary Riemann surface Σ_0 . If μ and μ' are closed 1-forms on Σ_0 , one has the topological fact

$$\int_{\Sigma_0} \mu \wedge \mu' = \sum_{i=1}^g \left(\oint_{A^i} \mu \oint_{B_i} \mu' - \oint_{A^i} \mu' \oint_{B_i} \mu \right). \tag{3.1}$$

If μ and μ' are holomorphic 1-forms, then $\mu \wedge \mu'$, which would be a holomorphic 2-form, vanishes identically. So the left hand side of (3.1) vanishes, and this leads to the symmetry of the period matrix.

To imitate this argument on a super Riemann surface Σ , we face two difficulties: (i) if we view Σ as a smooth supermanifold (this can be done, though not quite in a canonical way), then it has dimension 2|1, but a two-form $\mu \wedge \mu'$ can only be integrated on a manifold of dimension 2|0; (ii) on a super Riemann surface, a holomorphic 2-form is not necessarily 0.

The resolution of the first point is simply that, as explained in [1], if Σ is a super Riemann surface, with reduced space $\Sigma_{\rm red}$, then $\Sigma_{\rm red}$ can be embedded in Σ in a way that is not canonical (or holomorphic) but is unique up to homology. The image of the embedding is a smooth submanifold $\Sigma^* \subset \Sigma$ of dimension 2|0, and the proof of symmetry of the period matrix is made using integrals over Σ^* rather than Σ .

The resolution of the second point was also explained in [1] (see also section 8.2 of [4]). On a super Riemann surface (without Ramond punctures), a closed holomorphic 2-form is exact, so if μ and μ' are holomorphic 1-forms, then $\mu \wedge \mu' = d\lambda$ for some λ . This implies vanishing of $\int_{\Sigma^*} \mu \wedge \mu'$, just as if $\mu \wedge \mu'$ were 0, and leads to the proof of symmetry of the super period matrix.

Before describing these arguments, and explaining how they must be modified in the presence of Ramond punctures, we give a more elementary example. Let W be the supermanifold of dimension 1|1 defined as the quotient of $\mathbb{C}^{1|1}$ by

$$z|\theta \to z + 1|\theta + \alpha$$

 $z|\theta \to z + \sqrt{-1}|\theta.$ (3.2)

Here α is an odd constant. W is a complex supermanifold of dimension 1|1, but not a super Riemann surface since the identifications in eqn. (3.2) do not preserve a superconformal structure. The expression $dz d\theta$ defines a closed holomorphic 2-form on W that is globally-defined and is not exact (we can write it locally as $d(\theta dz)$, but θdz is not globally-defined,

since it is not invariant under $z|\theta \to z+1|\theta+\alpha$). The reduced space $W_{\rm red}$ of W is the ordinary Riemann surface of genus 1 defined by $z\cong z+1\cong z+\sqrt{-1}$. It can be embedded in W as the submanifold W^* defined by the equation $\theta=\alpha \operatorname{Re} z$. Then one finds that $\int_{W^*} \mathrm{d}z \,\mathrm{d}\theta = -\alpha \sqrt{-1} \neq 0$. The nonzero value of the integral is one way to prove that $\mathrm{d}z \,\mathrm{d}\theta$ is not exact. (Since $\mathrm{d}z \,\mathrm{d}\theta$ is closed, the value of this integral does not depend on the precise choice of W^* .)

On a super Riemann surface Σ without Ramond punctures, the proof that a closed holomorphic 2-form Ψ is exact proceeds as follows. In local superconformal coordinates $z|\theta$, Ψ can be expanded

$$\Psi = (d\theta)^2 p(z|\theta) + d\theta \,\varpi \,\rho(z|\theta), \tag{3.3}$$

where as usual $\varpi = dz - \theta d\theta$. The condition $d\Psi = 0$ gives

$$\rho = D_{\theta} p, \tag{3.4}$$

and then one finds that

$$\Psi = \mathrm{d}f, \quad f = -\varpi \, p(z|\theta).$$
 (3.5)

Even though we have computed in local superconformal coordinates, the object f that we have defined does not depend on this choice. This statement can be explained in the following way (as in footnote 34 of [4]), using the exact sequence $0 \to \mathcal{D}^{-2} \to T^*\Sigma \to \mathcal{D}^{-1}$. The projection $T^*\Sigma \to \mathcal{D}^{-1}$, tensored with itself, gives a holomorphic map $\wedge^2 T^*\Sigma \to \mathcal{D}^{-2}$. When this is composed with the inclusion $\mathcal{D}^{-2} \to T^*\Sigma$, we get a natural map $\wedge^2 T^*\Sigma \to T^*\Sigma$ which in local superconformal coordinates is the map $\Psi \to \varpi p = -f$.

Let us now see how these considerations are modified in the presence of a Ramond divisor. As usual, we consider a local model with coordinates $z|\theta$ and a superconformal structure defined by $\varpi^* = \mathrm{d}z - z\theta\mathrm{d}\theta$. The Ramond divisor $\mathcal F$ is defined by z = 0. A holomorphic 2-form Ψ can be expanded

$$\Psi = (d\theta)^2 p(z|\theta) + d\theta \varpi^* \rho(z|\theta). \tag{3.6}$$

The condition $d\Psi = 0$ implies that $p(z|\theta)$ is independent of θ at z = 0, so we define the constant

$$p = p(0|0). (3.7)$$

Note that p is completely well-defined, as it is not affected by the indeterminacy (2.11) of θ .

Away from z=0, we can replace $z|\theta$ by local superconformal coordinates $z|\widehat{\theta}=z|\theta z^{1/2}$ (these are superconformal coordinates, because in these coordinates the usual expression $\varpi^*=\mathrm{d}z-z\theta\mathrm{d}\theta$ near a Ramond puncture takes the standard superconformal form $\mathrm{d}z-\widehat{\theta}\mathrm{d}\widehat{\theta}$). In these coordinates, Ψ has a pole at z=0, with $\Psi\sim(\mathrm{d}\widehat{\theta})^2p/z$, and hence f as determined

in eqn. (3.5) behaves as

$$f \sim -\frac{p\overline{\omega}^*}{z} + \mathcal{O}(1), \quad z \to 0.$$
 (3.8)

Going back to the coordinates z, θ that behave well near the Ramond divisor, since $\varpi^* = dz - z\theta d\theta$, we have simply

$$f \sim -p \frac{\mathrm{d}z}{z} + \mathcal{O}(1), \quad z \to 0.$$
 (3.9)

Away from z=0, it is still true that $\Psi=\mathrm{d}f$. But because of the pole of f at z=0, there is actually a delta-function contribution to $\mathrm{d}f$ at z=0: $\mathrm{d}f=\Psi-2\pi\sqrt{-1}p\delta$, where δ is a two-form delta function that is Poincaré dual to the Ramond divisor \mathcal{F} at z=0. Including many Ramond divisors \mathcal{F}_{α} , we get

$$df = \Psi - 2\pi\sqrt{-1}\sum_{\alpha} p_{\alpha}\delta_{\alpha}, \qquad (3.10)$$

where δ_{α} is dual to \mathcal{F}_{α} and $-p_{\alpha}$ is the residue of the corresponding pole in f.

Integrating this formula over a submanifold $\Sigma^* \subset \Sigma$ that is isomorphic to $\Sigma_{\rm red}$, we get

$$\int_{\Sigma^*} \Psi = 2\pi \sqrt{-1} \sum_{\alpha} p_{\alpha}. \tag{3.11}$$

4 The Super Period Matrix

If $\Psi = \mu \wedge \mu'$, where μ and μ' are two closed holomorphic 1-forms on a super Riemann surface Σ with Ramond punctures, then we can combine (3.11) with the topological formula (3.1), with the result that

$$\sum_{i=1}^{g} \left(\oint_{A^i} \mu \oint_{B_i} \mu' - \oint_{A^i} \mu' \oint_{B_i} \mu \right) = 2\pi \sqrt{-1} \sum_{\alpha} p_{\alpha}. \tag{4.1}$$

To make use of this, we must express the constants p_{α} in terms of the odd periods w_{α} and w'_{α} of μ and μ' . Recalling that the odd period is simply the constant w in eqn. (2.12), we see that near the Ramond divisor \mathcal{F}_{α} , we have $\mu \sim w_{\alpha} \mathrm{d}\theta/\sqrt{2\pi\sqrt{-1}}$, $\mu' \sim w'_{\alpha} \mathrm{d}\theta/\sqrt{2\pi\sqrt{-1}}$, and hence $\mu \wedge \mu' \sim (\mathrm{d}\theta)^2 w_{\alpha} w'_{\alpha}/2\pi\sqrt{-1}$. Thus $2\pi\sqrt{-1}p_{\alpha} = w_{\alpha}w'_{\alpha}$.

If then we denote the A- and B-periods as $a_i = \oint_{A^i} \mu$, $b^i = \oint_{B_i} \mu$ and similarly $a'_i = \oint_{A^i} \mu'$, $b'^i = \oint_{B_i} \mu'$, then we arrive at the analog of the Riemann bilinear relations for a super

Riemann surface with Ramond punctures:

$$\sum_{i=1}^{g} \left(a_i b'^j - a'_i b^j \right) - \sum_{\alpha=1}^{2r} w_\alpha w'_\alpha = 0. \tag{4.2}$$

On the right hand side of eqn. (4.2), we see the "intersection form" Θ that was introduced in eqn. (2.13). Thus, it is natural to introduce a space $\Lambda \cong \mathbb{C}^{2g|2r}$ that is endowed with this quadratic form (tensored with \mathbb{C}). We denote the intersection form on Λ as \langle , \rangle . We combine the whole collection of even and odd periods of μ to a vector $\mu \in \Lambda$:

$$\mu = \{a_i, b^j | w_\alpha\}, \quad i, j = 1, \dots, g, \quad \alpha = 1, \dots, 2r.$$
(4.3)

Similarly, the periods of μ' combine to $\mu' = \{a'_i, b'^j | w'_{\alpha}\} \in \Lambda$. Eqn. (4.2) is equivalent to $\langle \mu, \mu' \rangle = 0$.

In other words, by analogy with the classical case, the bilinear relations assert that the subspace $\Lambda_0 \subset \Lambda$ that is spanned by the periods of holomorphic 1-forms is an isotropic subspace: the bilinear form $\langle \ , \ \rangle$ vanishes when restricted to Λ_0 . In section 5, we will show that generically, for r > 0 (or for r = 0 with even spin structure) the space of closed holomorphic 1-forms on Σ has dimension g|r. In this case, Λ_0 is middle-dimensional in Λ , and thus it is a maximal isotropic subspace of Λ , again by analogy with the classical case.

Just as in the more familiar case r=0, the information about a maximal isotropic subspace of Λ can generically be encoded by a super period matrix. The super period matrix will now be a $g|r \times g|r$ matrix that will be symmetric in the \mathbb{Z}_2 -graded sense, as described more concretely below. To define the classical period matrix of an ordinary Riemann surface Σ_0 , one starts by picking a set of A-periods. This amounts to picking a maximal isotropic subspace – of a particularly simple and convenient sort – for the intersection form on $H^1(\Sigma_0, \mathbb{Z})$. To generalize this for a super Riemann surface with Ramond punctures, we similarly must first pick a simple maximal isotropic subspace for the form Θ , which we regard as the superanalog of the classical intersection form. We again use a set of A-periods as a maximal set of even null vectors, but what is a natural set of odd null vectors? The simplest choice seems to be to order the fermionic periods as w_1, w_2, \ldots, w_{2r} and then form the complex linear combinations

$$w^{\zeta} = \frac{1}{\sqrt{2}}(w_{2\zeta-1} + \sqrt{-1}w_{2\zeta}), \quad \zeta = 1, \dots, r.$$
(4.4)

The complementary fermionic periods are

$$\widetilde{w}_{\zeta} = \frac{1}{\sqrt{2}} (w_{2\zeta-1} - \sqrt{-1}w_{2\zeta}), \quad \zeta = 1, \dots, r.$$
 (4.5)

The w^{ζ} and \widetilde{w}_{ζ} will be the fermionic analogs of A-periods and B-periods.

Now we define a basis of closed holomorphic 1-forms $\sigma_1, \ldots, \sigma_g | \nu_1, \ldots, \nu_r$ by requiring

$$a^i(\sigma_j) = \delta^i_j, \quad w^\eta(\sigma_j) = 0$$
 (4.6)

and

$$a^i(\nu_{\zeta}) = 0, \quad w^{\eta}(\nu_{\zeta}) = \delta^{\eta}_{\zeta}.$$
 (4.7)

For this definition to make sense, the choice of fermionic A-periods must be generic enough so that closed holomorphic 1-forms obeying the conditions (4.6) and (4.7) exist and are unique. The condition for this is that if Λ' is the subspace of Λ defined by $w^{\zeta} = 0 = a^{i}$, then we must have $\Lambda' \cap \Lambda_0 = 0$. We return to this condition in sections 5.2 and 5.3, and for now just remark that it places a non-trivial constraint on the choices of ordering and signs in the definition of the w^{ζ} .

Finally, we define the super period matrix $\widehat{\Omega}$ by specifying its matrix elements

$$\widehat{\Omega}_{ij} = \oint_{B^j} \sigma_i = b_j(\sigma_i)$$

$$\widehat{\Omega}_{i\eta} = \widetilde{w}_{\eta}(\sigma_i)$$

$$\widehat{\Omega}_{\eta j} = \oint_{B^j} \nu_{\eta} = b_j(\nu_{\eta})$$

$$\widehat{\Omega}_{\eta \zeta} = \widetilde{w}_{\zeta}(\nu_{\eta}).$$
(4.8)

As in the classical theory, the bilinear relations (4.2) imply that $\widehat{\Omega}$ is symmetric in the \mathbb{Z}_2 -graded sense: $\widehat{\Omega}_{ij} = \widehat{\Omega}_{ji}$, $\widehat{\Omega}_{i\eta} = \widehat{\Omega}_{\eta i}$, $\widehat{\Omega}_{\eta\zeta} = -\widehat{\Omega}_{\zeta\eta}$. If we write the super period matrix in blocks

$$\begin{pmatrix} g \times g & g \times r \\ r \times g & r \times r \end{pmatrix} \tag{4.9}$$

then the upper left $g \times g$ block, which we will call $\widehat{\Omega}_{g \times g}$, corresponds most closely to the ordinary period matrix in the classical theory of Riemann surfaces. We will call this the pseudoclassical block. If reduced modulo odd variables, it coincides with the ordinary period matrix of the reduced space $\Sigma_{\rm red}$. This will be clear in section 5.3. So in particular $\widehat{\Omega}_{g \times g}$ has positive-definite imaginary part. Note that the pseudoclassical block depends on the choice of fermionic A-periods (though this dependence disappears if we reduce modulo the odd variables), since the definition of the σ_i depends on that choice.

5 The Space Of Closed Holomorphic One-Forms

5.1 Counting Closed Holomorphic One-Forms

To count the closed holomorphic 1-forms on a genus g super Riemann surface Σ , first assume that Σ is split, with a reduced space $\Sigma_{\rm red}$ whose canonical bundle and tangent bundle we denote as K and T. To begin with, assume there are no Ramond punctures, Then Σ can be constructed as the total space of an odd line bundle $\Pi T^{1/2} \to \Sigma_{\rm red}$. Here $T^{1/2}$ is a square root of T corresponding to a choice of spin structure, the inverse of $T^{1/2}$ will be denoted $K^{1/2}$, and for a line bundle \mathcal{L} , $\Pi \mathcal{L}$ is \mathcal{L} with the fiber understood to be odd. There is a natural projection $\pi: \Sigma \to \Sigma_{\rm red}$ (and an embedding of $\Sigma_{\rm red}$ in Σ as the zero-section of $\Pi T^{1/2}$).

There is always a g-dimensional space of holomorphic 1-forms $b(z)\mathrm{d}z$ on Σ_{red} . These can be pulled back via π to closed holomorphic 1-forms on Σ . There actually are additional odd holomorphic 1-forms $b(z)\theta\mathrm{d}\theta$, but they are not closed. The situation for even closed holomorphic 1-forms is more interesting. An even holomorphic 1-form is in general $\lambda = a(z)\theta\mathrm{d}z + c(z)\mathrm{d}\theta$, but for λ to be closed, this expression must reduce to $\lambda = \mathrm{d}(c(z)\theta)$. Here in classical geometry c(z) represents a holomorphic section of $K^{1/2}$, that is, an element of $H^0(\Sigma_{\mathrm{red}}, K^{1/2})$. Generically, if the spin structure of Σ is even, $H^0(\Sigma_{\mathrm{red}}, K^{1/2}) = 0$. In this case the space of closed holomorphic 1-forms has dimension g|0 (we will reverse the parity in writing dimension formulas). Their periods are used to define a $g \times g$ super period matrix $\widehat{\Omega}_{ij}$, which is symmetric and has positive definite imaginary part, just as in the classical case.

Again assuming that Σ has even spin structure, in genus $g \geq 3$, there is a divisor $\mathfrak{D} \subset \mathcal{M}_{g,\mathrm{spin}+}$ along which $H^0(\Sigma_{\mathrm{red}},K^{1/2}) \neq 0$. The space of closed holomorphic 1-forms is then of dimension g|s, for some (even) s>0. There are more closed holomorphic 1-forms than periods so some closed holomorphic 1-forms must have vanishing periods. In fact, as explained in the last paragraph, the even closed holomorphic 1-forms are exact $(\lambda = \mathrm{d}(c(z)\theta))$, so their periods vanish. In defining a super period matrix, one can take the quotient of the space of closed holomorphic 1-forms by the subspace consisting of those whose periods vanish. For Σ split, the quotient space always has dimension g|0. So as long as Σ is split, the condition $H^0(\Sigma_{\mathrm{red}}, K^{1/2}) \neq 0$ does not lead to trouble in defining the super period matrix.

To define the super period matrix without assuming that Σ is split, we need to know that (away from \mathfrak{D}) the space of closed holomorphic 1-forms is still of dimension g|0 when the odd moduli of Σ are introduced. This is true but not completely trivial. One elegant proof uses the fact that there is a natural 1-1 correspondence between closed holomorphic 1-forms and holomorphic sections of $Ber(\Sigma)$, the Berezinian line bundle of Σ . The correspondence is given by an explicit formula; in local superconformal coordinates, a holomorphic section

⁶This proof was given in [1]. See also [4], section 8 and Appendix D, for a detailed explanation.

 $\phi(z|\theta)[\mathrm{d}z|\mathrm{d}\theta]$ of $Ber(\Sigma)$ corresponds to the closed holomorphic 1-form $\mu = \mathrm{d}\theta\phi + \varpi D_{\theta}\phi$. To show that the space of holomorphic 1-forms on Σ is still of dimension g|0 when Σ is not split, one must show the analogous statement for $H^0(\Sigma; Ber(\Sigma))$: its dimension should not jump when odd moduli are turned on. On general grounds, this is true if and only if $H^1(\Sigma, Ber(\Sigma))$ varies as the fiber of a locally-free sheaf (or vector bundle). But $H^1(\Sigma, Ber(\Sigma))$ is Serre-dual to $H^0(\Sigma, \mathcal{O})$, where \mathcal{O} is the sheaf of holomorphic sections on Σ . Away from the divisor \mathfrak{D} , one has $H^0(\Sigma, \mathcal{O}) \cong \mathbb{C}$, generated by the constant function 1. In particular, $H^0(\Sigma, \mathcal{O})$ is locally-free, and hence also are $H^1(\Sigma, Ber(\Sigma))$ and $H^0(\Sigma, Ber(\Sigma))$.

This reasoning fails along the divisor $\mathfrak{D} \subset \mathcal{M}_{g,\mathrm{spin}+}$, because given $c \in H^0(\Sigma_{\mathrm{red}}, K^{1/2})$, there is an odd holomorphic function $c(z)\theta$ on Σ . Thus for a split super Riemann surface Σ , vanishing of $H^0(\Sigma_{\mathrm{red}}, K^{1/2})$ is a necessary and sufficient condition for $H^0(\Sigma, \mathcal{O}) \cong \mathbb{C}$. What actually happens near \mathfrak{D} is that, although the super period matrix is well-defined and holomorphic as long as the odd moduli vanish, or in other words along the split locus $\mathcal{M}_{g,\mathrm{spin}+} \subset \mathfrak{M}_{g,+}$, it develops a pole (with nilpotent residue) as soon as one varies away from that locus. This follows from the formula of D'Hoker and Phong [2] for the dependence of the super period matrix on odd moduli. (See section 8.3 of [4], or Appendix A below.)

Now let us consider the case that Σ is a super Riemann surface with Ramond punctures. Again we start with the split case.

We pick in the reduced space Σ_{red} of Σ a collection of distinct points $p_1, \ldots, p_{2r} \in \Sigma_{\text{red}}$ that will represent the Ramond punctures, and a line bundle \mathcal{R} endowed with an isomorphism⁷

$$\mathcal{R}^2 \cong T \otimes \mathcal{O}(-p_1 - \dots - p_{2r}) \tag{5.1}$$

or equivalently

$$K \otimes \mathcal{R} \cong \mathcal{R}^{-1}(-p_1 - \dots - p_{2r}).$$
 (5.2)

The line bundle \mathcal{R} has degree 1-g-r. Σ is then defined to be the total space of the line bundle $\Pi\mathcal{R} \to \Sigma_{\text{red}}$. As before, there are projections $\pi: \Sigma \to \Sigma_{\text{red}}$ and an embedding $\Sigma_{\text{red}} \subset \Sigma$. Away from the points p_1, \ldots, p_{2r} , the line bundle \mathcal{R} is a square root of T and Σ is an ordinary super Riemann surface, which can be described by local superconformal coordinates $z|\theta$ and superconformal structure generated by $D_{\theta} = \partial_{\theta} + \theta \partial_{z}$. However, because the isomorphism $\mathcal{R}^2 \cong T$ is only valid away from the points p_i , the superconformal structure breaks down along the divisors $\mathcal{F}_{\alpha} = \pi^{-1}(p_{\alpha}) \subset \Sigma$. Those divisors are Ramond divisors, representing singularities in the superconformal structure of Σ .

⁷ Such a line bundle defines what we call a generalized spin structure. One is free to tensor \mathcal{R} with a line bundle of order 2, so for any r and any points p_1, \ldots, p_{2r} , there are 2^{2g} generalized spin structures. For r = 0, the choice of \mathcal{R} is tantamount to an ordinary spin structure on Σ; the spin structures on Σ can be naturally divided into odd and even ones. For r > 0, the 2^{2g} generalized spin structures are permuted transitively under monodromy of the points p_i , so there is no notion of an even or odd generalized spin structure.

Closed holomorphic 1-forms can be described as before. Holomorphic 1-forms on $\Sigma_{\rm red}$ can be pulled back to give a g|0-dimensional space of closed holomorphic 1-forms on Σ . As before, these are the only odd closed holomorphic 1-forms, and even ones are of the form $d(a(z)\theta)$, where now geometrically a(z) is a holomorphic section of $\mathcal{R}^{-1} \to \Sigma_{\rm red}$. This line bundle is of degree g-1+r, so generically $H^0(\Sigma_{\rm red}, \mathcal{R}^{-1})$ is of dimension r and $H^1(\Sigma_{\rm red}, \mathcal{R}^{-1}) = 0$. This fails on a locus \mathfrak{B} characterized by $H^1(\Sigma_{\rm red}, \mathcal{R}^{-1}) \neq 0$. This locus, which will be studied in section 5.6, is a rough analog of the theta-null divisor \mathfrak{D} for r=0.

Overall, away from \mathfrak{B} , the space of closed holomorphic 1-forms on a split super Riemann surface Σ has dimension q|r, as assumed in section 4 in defining the super period matrix. To show that this remains so if Σ is not assumed to be split, one can adapt the arguments that are used in the absence of Ramond punctures. In the presence of Ramond punctures, closed holomorphic 1-forms correspond (see Appendix D.1 of [4]) not to elements of $H^0(\Sigma, Ber(\Sigma))$, but to elements of $H^0(\Sigma, Ber'(\Sigma))$, where a section of $Ber'(\Sigma)$ is a section of $Ber(\Sigma)$ that is allowed to have a simple pole, with θ -independent residue, along a Ramond divisor. The space of closed holomorphic 1-forms is locally-free if $H^1(\Sigma, Ber'(\Sigma))$ is locally-free. By Serre duality, this is equivalent to $H^0(\Sigma, \mathcal{O}')$ being locally-free, where \mathcal{O}' is the sheaf of holomorphic functions on Σ that are constant when restricted to a Ramond divisor. Equivalently, $H^0(\Sigma, \mathcal{O}') = H^0(\Sigma', \mathcal{O})$, where Σ' is a complex supermanifold obtained from Σ by blowing down the Ramond divisors (this blowdown operation is discussed in [14], section 3.4.2). The locally-free condition $H^0(\Sigma', \mathcal{O}) = \mathbb{C}$ is equivalent to the familiar condition $H^1(\Sigma_{\rm red}, \mathcal{R}^{-1}) = 0$. The last claim is shown as follows. It suffices to assume that Σ is split, in which case Σ' is the total space of the line bundle $\Pi \mathcal{R}(p_1 + \cdots + p_{2r}) \to \Sigma_{\text{red}}$, so that an odd holomorphic function on Σ' (which would obstruct the claim that $H^0(\Sigma', \mathcal{O}) \cong \mathbb{C}$) corresponds to an element of $H^0(\Sigma_{\text{red}}, \mathcal{R}^{-1}(-p_1 - \cdots - p_{2r}))$. As a consequence of (5.2), we have

$$H^0(\Sigma_{\text{red}}, \mathcal{R}^{-1}(-p_1 - \dots - p_{2r})) \cong H^0(\Sigma_{\text{red}}, K \otimes \mathcal{R}).$$
 (5.3)

By Serre duality, this vanishes if and only if $H^1(\Sigma_{\text{red}}, \mathcal{R}^{-1}) = 0$. As long as this is true, the space of closed holomorphic 1-forms has the expected dimension..

5.2 Middle-Dimensionality Of The Periods

In defining the super period matrix in section 4, we assumed that the space Λ_0 of periods of closed holomorphic 1-forms is middle-dimensional in the space Λ of periods. We will now show that this is true for any split super Riemann surface Σ . As long as the locally-free condition $H^0(\Sigma, \mathcal{O}') \cong \mathbb{C}$ is satisfied, Λ_0 automatically remains middle-dimensional when Σ is deformed away from the split locus. (Since odd moduli are infinitesimal, turning them on will not cause a nonzero period to become zero, so it will not reduce the dimension of Λ_0 . On the other hand, the bilinear relations (4.2) ensure that the dimension of Λ_0 cannot increase.) For Σ split, the following analysis will show that Λ_0 is middle-dimensional even

for $H^1(\Sigma_{\text{red}}, \mathcal{R}^{-1}) \neq 0$, but in that case, we cannot say anything simple about what happens away from the split locus.

Middle-dimensionality of the even periods on a split super Riemann surface Σ just amounts to the classical fact that on an ordinary Riemann surface $\Sigma_{\rm red}$, the periods of holomorphic differentials are middle-dimensional in the space of all A- and B-periods. We will now show that the same is true for the odd periods.

For a line bundle $\mathcal{L} \to \Sigma_{\text{red}}$, we write $h^i(\mathcal{L})$ for the dimension of $H^i(\Sigma_{\text{red}}, \mathcal{L})$. Since \mathcal{R}^{-1} has degree g-1+r, the Riemann-Roch theorem gives $h^0(\mathcal{R}^{-1}) - h^1(\mathcal{R}^{-1}) = r$. Via Serre duality, this is equivalent to $h^0(\mathcal{R}^{-1}) - h^0(K \otimes \mathcal{R}) = r$. In view of (5.3), this is equivalent to

$$h^{0}(\mathcal{R}^{-1}) - h^{0}(\mathcal{R}^{-1}(-p_{1} - \dots - p_{2r})) = r.$$
 (5.4)

A section of $\mathcal{R}^{-1}(-p_1 - \cdots - p_{2r})$ is simply a section of \mathcal{R}^{-1} that vanishes at p_1, \ldots, p_{2r} , so $H^0(\Sigma_{\text{red}}, \mathcal{R}^{-1}(-p_1 - \cdots - p_{2r}))$ is a subspace of $H^0(\Sigma_{\text{red}}, \mathcal{R}^{-1})$. Eqn. (5.4) says that the quotient space has dimension r:

dim
$$(H^0(\Sigma_{\text{red}}, \mathcal{R}^{-1})/H^0(\Sigma_{\text{red}}, \mathcal{R}^{-1}(-p_1 - \dots - p_{2r}))) = r.$$
 (5.5)

Now, $H^0(\Sigma_{\text{red}}, \mathcal{R}^{-1})$ is the space of even closed holomorphic 1-forms, and $H^0(\Sigma_{\text{red}}, \mathcal{R}^{-1}(-p_1 - \cdots - p_{2r}))$ is its subspace consisting of those that vanish when restricted to Ramond divisors or in other words whose odd periods vanish. So eqn. (5.5) says that for any split super Riemann surface Σ , the space of even closed holomorphic 1-forms modulo those with vanishing periods is of dimension r.

In other words, for Σ split or for $H^0(\Sigma, \mathcal{O}') \cong \mathbb{C}$, the periods of closed holomorphic 1-forms always span a middle-dimensional subspace $\Lambda_0 \subset \Lambda$.

5.3 More On The Split Case

Next we will look more closely at the super period matrix of a split super Riemann surface Σ . Odd closed holomorphic 1-forms on Σ are simply pullbacks of holomorphic 1-forms on $\Sigma_{\rm red}$. Their periods are just the corresponding periods on $\Sigma_{\rm red}$. So the pseudoclassical block $\widehat{\Omega}_{g\times g}$ of the period matrix of Σ is just the classical period matrix of $\Sigma_{\rm red}$. A form on Σ that is a pullback from $\Sigma_{\rm red}$ has no $\mathrm{d}\theta$ component. So its odd periods vanish, and hence $\widehat{\Omega}_{g\times r}=\widehat{\Omega}_{r\times g}=0$. Thus the super period matrix of Σ is

$$\widehat{\Omega} = \begin{pmatrix} \widehat{\Omega}_{g \times g} & 0\\ 0 & \widehat{\Omega}_{r \times r} \end{pmatrix}, \tag{5.6}$$

where only $\widehat{\Omega}_{r \times r}$ remains to be understood.

For this, we first recall that an even holomorphic 1-form on Σ is exact, $\nu = \mathrm{d}(g(z)\theta)$ for some g(z), so its ordinary A- and B-periods – that is, its even periods – vanish. (This gives another explanation of why $\widehat{\Omega}_{g\times r} = \widehat{\Omega}_{r\times g} = 0$.) Now suppose that ν and ν' are two even closed holomorphic 1-forms on Σ , with respective odd periods w_{α} and w'_{α} . We combine the odd periods of ν and ν' into vectors \mathbf{v} , \mathbf{v}' , which take values in a vector space Λ of dimension 2r that has a basis corresponding to the oriented Ramond divisors \mathcal{F}_{α} . Specialized to the case that the even periods are zero, the bilinear relation of eqn. (4.2) reduces to

$$\sum_{\alpha} w_{\alpha} w_{\alpha}' = 0. \tag{5.7}$$

We can see very directly why this is true. Suppose that $\nu = d(g\theta)$, $\nu' = d(g'\theta)$, with $g, g' \in H^0(\Sigma_{\text{red}}, \mathcal{R}^{-1})$. The product gg' is then a section of \mathcal{R}^{-2} , but the isomorphism in (5.1) identifies this with $K \otimes \mathcal{O}(p_1 + \cdots + p_{2r})$. A section of $K \otimes \mathcal{O}(p_1 + \cdots + p_{2r})$ is a meromorphic 1-form that may have simple poles at the points p_1, \ldots, p_{2r} ; the residue of the pole of gg' at $z = z_{\alpha}$ is the product $w_{\alpha}w'_{\alpha}/2\pi\sqrt{-1}$. Thus eqn. (5.7) asserts the vanishing of the sum of residues of a certain meromorphic 1-form.

For an even more explicit example, we consider a super Riemann surface Σ of genus 0 with two Ramond punctures. We parametrize Σ – or more precisely, the complement of a divisor in Σ – by coordinates $z|\theta$ with superconformal structure defined by

$$\varpi^* = \mathrm{d}z - z\theta\mathrm{d}\theta. \tag{5.8}$$

The Ramond punctures are at z=0 and $z=\infty$. To understand what is happening at $z=\infty$, we introduce new coordinates via z=1/y, $\theta=\sqrt{-1}\psi$, whence

$$\varpi^* = -\frac{1}{y^2} \left(dy - y\psi d\psi \right). \tag{5.9}$$

The factor of $-1/y^2$ is not important here, since we only care about the subbundle of the cotangent bundle of Σ that is generated by ϖ^* . Thus the superconformal structure near y=0 is generated by $\mathrm{d} y - y \psi \mathrm{d} \psi$, showing that y=0 is a Ramond divisor and that the coordinate system $y|\psi$ puts the superconformal structure in a standard form near this divisor. Notice that the factor of $\sqrt{-1}$ in the formula $\theta = \sqrt{-1} \psi$ is necessary for this result. There are no nonzero odd closed holomorphic 1-forms on Σ , and the space of even ones is one-dimensional, generated by $\nu = \mathrm{d}\theta = \sqrt{-1} \mathrm{d}\psi$. The odd periods of ν are 1 at z=0 and $\sqrt{-1}$ at y=0, so the sum of squares of the odd periods is zero, as expected.

Of course, we could also describe the same example more globally in projective coordinates. For this, we simply take Σ to be a weighted projective superspace $\mathbb{WCP}^{1|1}(1,1|0)$

with even homogeneous coordinates u, v of weight 1 and an odd homogeneous coordinate θ of weight 0. The superconformal structure of Σ can be defined by the following section of $T^*\Sigma \otimes \mathcal{O}(2)$:

$$\widehat{\varpi} = u dv - v du - uv\theta d\theta. \tag{5.10}$$

More generally, we can describe a split genus 0 super Riemann surface Σ with 2r Ramond punctures in affine coordinates $z|\theta$ by the superconformal structure

$$\varpi^* = \mathrm{d}z - \prod_{k=1}^{2r} (z - e_k)\theta \mathrm{d}\theta. \tag{5.11}$$

Alternatively, Σ is the weighted projective space $\mathbb{WCP}^{1|1}(1,1|1-r)$ with homogeneous coordinates $u,v|\theta$ of weights 1,1|1-r and superconformal structure defined by $\widehat{\omega}=u\mathrm{d}v-v\mathrm{d}u-P(u,v)\theta\mathrm{d}\theta$, with P(u,v) a homogeneous polynomial of degree 2r.

It is instructive to consider this last example more carefully to see what is involved in orienting the Ramond divisors. To orient the Ramond divisor \mathcal{F}_k at, say, $z = e_k$, we should pick new coordinates $z|\hat{\theta}_k$ such that ϖ takes the standard form $dz - (z - e_k)\hat{\theta}_k d\hat{\theta}_k$ near $z = e_k$. Then we orient \mathcal{F}_k by choosing the differential $d\hat{\theta}_k$ along \mathcal{F}_k . Clearly, we need

$$\widehat{\theta}_k = \theta \cdot \sqrt{\prod_{k' \neq k} (e_k - e_{k'})}. \tag{5.12}$$

Now let us ask what happens to the orientations of the Ramond divisors \mathcal{F}_k when the moduli e_1, \ldots, e_k are varied. To keep things simple, we consider the Ramond divisors to be labeled (as is usually most natural in string theory), so we do not allow permutations of the e_k ; we consider only what happens when the e_k are braided around each other. Such a process is made by composing elementary moves in which one of the e's, say e_{k_1} , makes a small loop around another, say e_{k_2} . In the process, $\sqrt{e_{k_1} - e_{k_2}}$ changes sign, but there are no sign changes in $\sqrt{e_i - e_j}$ for any other pair. So the orientations of \mathcal{F}_{k_1} and \mathcal{F}_{k_2} are reversed, and no others. Combining any number of operations of this kind, we see that the only constraint is that there are an even number of orientation reversals. In other words, the only constraint is that the monodromies preserve the orientation of the space Λ that parametrizes the odd periods.

This has a simple explanation. Λ is an even-dimensional vector space with a non-degenerate quadratic form. In such a vector space, there are two families of maximal isotropic subspaces, associated to a choice of orientation. Λ has a distinguished middle-dimensional isotropic subspace, the space Λ_0 of odd periods of odd holomorphic differentials. So it has a preferred orientation that is preserved under any monodromies.

5.4 Two Ramond Punctures

Our application in the remainder of this paper will involve the case of a super Riemann surface Σ with precisely 2 Ramond punctures. So let us point out some particularly nice things that happen in this case.

With two Ramond punctures, there are precisely 2 odd periods, say w_1 and w_2 , so the space Λ of odd periods only has two null subspaces, generated by $w_1 \pm \sqrt{-1}w_2$. The choice of a fermionic A-period in this case (eqn. (4.4)) is particularly simple. There is only one fermionic A-period w, and up to an integer power of $\sqrt{-1}$ (which will arise if we exchange w_1 and w_2 , or reverse their orientations), it must be either $\frac{1}{\sqrt{2}}(w_1+\sqrt{-1}w_2)$ or $\frac{1}{\sqrt{2}}(w_1-\sqrt{-1}w_2)$.

However, only one of the two choices is viable in the definition of the super period matrix. According to eqn. (4.7), we are supposed to find an even closed holomorphic 1-form ν with $w(\nu) = 1$. However, for Σ split, ν is a null vector in Λ , so its periods obey $w_2 = \pm \sqrt{-1}w_1$, with one choice of the sign or the other. This means that if we choose the wrong sign in the definition of the fermionic Λ -period w, then we will get $w(\nu) = 0$ and will be unable to satisfy $w(\nu) = 1$.

So a unique definition of $w(\nu)$ is forced upon us, up to an integer power of $\sqrt{-1}$. Moreover, with this choice, as long as the space of closed holomorphic 1-forms has the expected dimension g!1, a unique set of forms obeying (4.6) and (4.7) always exists, Hence, the super period matrix is always defined away from the usual locus \mathfrak{B} along which $H^0(\Sigma, K \otimes \mathcal{R}^{-1}) \neq 0$.

If we do multiply w by $\sqrt{-1}^a$, for some integer a, what happens to $\widehat{\Omega}$? To compensate for the change in w, we will have to multiply \widetilde{w} and ν by $\sqrt{-1}^{-a}$. $\widehat{\Omega}_{r\times r}$ is multiplied by $(-1)^a$ and $\widehat{\Omega}_{r\times g}$ and $\widehat{\Omega}_{g\times r}$ are multiplied by $\sqrt{-1}^{-a}$. $\widehat{\Omega}_{g\times g}$ is unchanged.

With more than 2 Ramond punctures, there are more choices in defining the fermionic A-periods. The nondegeneracy condition in the definition of the super period matrix is more complicated, and the super period matrix has poles when this condition fails. We consider this next.

5.5 More Than Two Ramond Punctures

In the definition of the super period matrix, we needed to know that closed holomorphic 1-forms obeying the conditions (4.6) and (4.7) exist and are unique. Saying that a system of linear equations (with the same number of variables and unknowns) has a unique solution is an open condition, so it suffices to consider the case that Σ is split. Then the condition is simply that it should be possible to find an even differential ν with prescribed values of half

of its odd periods w^{η} , $\eta = 1, ..., r$ (and no condition on the other odd periods \widetilde{w}_{η}). (This is clearly equivalent to the existence of differentials ν_{ζ} with $w^{\eta}(\nu_{\zeta}) = \delta^{\eta}_{\zeta}$.) On dimensional grounds, an equivalent statement is the following. Let $\Lambda \cong \mathbb{C}^{2r}$ have a basis corresponding to oriented Ramond divisors, let Λ_0 be the middle-dimensional isotropic subspace of Λ that parametrizes odd periods of closed holomorphic 1-forms, and let Λ_1 be the middle-dimensional isotropic subspace of Λ characterized by $w^1 = \cdots = w^r = 0$. Then the desired condition is $\Lambda_0 \cap \Lambda_1 = 0$.

As remarked at the end of section 5.3, middle-dimensional isotropic subspaces of Λ come in two families associated with a choice of orientation of Λ . A necessary condition for $\Lambda_0 \cap \Lambda_1 = 0$ is that Λ_0 and Λ_1 should be oppositely oriented (that is, associated to opposite orientations of Λ) if r is odd, or oriented the same way if r is even. Conversely, if the orientations of two middle-dimensional isotropic subspaces Λ_0 and Λ_1 make this possible, then generically $\Lambda_0 \cap \Lambda_1 = 0$.

One may therefore expect that as long as the right orientation is used in defining the fermionic A-periods, the nondegeneracy condition $\Lambda_0 \cap \Lambda_1 = 0$ will be satisfied generically, on the complement of a divisor in the reduced space of $\mathfrak{M}_{g,0,2r}$. To show that this is true, it suffices to show it for g=0, since a super Riemann surface of any genus with 2r Ramond punctures can degenerate to several components one of which is a genus 0 surface containing all of the Ramond punctures. Using the explicit description (5.11) of a super Riemann surface of genus g with 2r Ramond punctures, one can show directly that, with the right definition of the fermionic A-periods, the nondegeneracy condition is obeyed generically. Pick r distinct points $f_1, \ldots, f_r \in \mathbb{C}$ and consider a limit with e_{2i-1}, e_{2i} near f_i for $i=1,\ldots,r$. Explicitly we find that in this limit, with fermionic A-periods w^i defined as in eqn. (4.4), the differentials ν_k that satisfy $w^i(\nu_k) = \delta_k^i$ are $\nu_k = \mathrm{d}(a_k(z)\theta)$ with

$$a_k(z) \sim \frac{\sqrt{e_{2k-1} - e_{2k}}}{\sqrt{4\pi\sqrt{-1}}} \prod_{j \neq k} (z - f_j).$$
 (5.13)

We do not know a useful characterization of the divisor on which $\Lambda_0 \cap \Lambda_1 \neq 0$, producing additional poles in the super period matrix.

5.6 The Bad Set

Here we will make some observations about the bad set \mathfrak{B} in moduli space along which $h^0(\mathcal{R}^{-1}) > r$.

Let $\mathcal{L} \to \Sigma_{\text{red}}$ be a line bundle of degree g - 1 + s, determined by a point in Jac_{g-1+s} , the component of the Jacobian that parametrizes line bundles of that degree. Generically,

the condition $h^0(\mathcal{L}) > s$ is satisfied only in codimension s+1 in Jac_{g-1+s} . For example, if s=g-1, then $h^0(\mathcal{L}) \geq s+1=g$ if and only if $\mathcal{L} \cong K$, which determines a unique point in Jac_{g-1+s} , of codimension g.

However, we are interested in a line bundle \mathcal{R}^{-1} with an isomorphism

$$\mathcal{R}^{-2} \cong K(p_1 + \dots + p_{2r}). \tag{5.14}$$

This case is somewhat exceptional for small r. We will examine this in detail, since our application later in this paper involves r = 1.

For r = 0, \mathcal{R}^{-1} is simply a square root of K. The codimension along which $h^0(\mathcal{R}^{-1}) \neq 0$ is 0 or 1 depending on whether \mathcal{R}^{-1} defines an odd or even spin structure.

The case r=1 is somewhat similar to r=0 with odd spin structure: the condition $h^0(\mathcal{R}^{-1}) \geq r+1=2$ is satisfied in codimension 1, not in the "expected" codimension r+1=2. Before explaining this in detail, we first consider some small genus cases that are relevant to our applications.

Our main application later in this paper involves genus 2. A genus 2 Riemann surface Σ_{red} is hyperellipic, and is a two-fold cover $\rho: \Sigma \to \mathbb{CP}^1$. For r=1, the line bundle \mathcal{R}^{-1} has degree 2. The Riemann-Roch formula gives $h^0(\mathcal{R}^{-1}) - h^1(\mathcal{R}^{-1}) = 1 - g + \deg \mathcal{R}^{-1} = 1$, and Serre duality gives $h^1(\mathcal{R}^{-1}) = h^0(K \otimes \mathcal{R})$. So $h^0(\mathcal{R}^{-1}) \geq 2$ is equivalent to $h^0(K \otimes \mathcal{R}) > 0$. But $K \otimes \mathcal{R}$ is of degree 0. A line bundle of degree 0 with a holomorphic section must be trivial, so $K \otimes \mathcal{R} \cong \mathcal{O}$ and $\mathcal{R}^{-1} \cong K$. Since also $\mathcal{R}^{-2} \cong K(p_1 + p_2)$, we must have $K \cong \mathcal{O}(p_1 + p_2)$. As explained in section 9.1, this is so if and only if the two points p_1, p_2 are exchanged by the hyperelliptic involution of Σ . There is a one-parameter family of such pairs. This family is of codimension 1 in the space of all pairs p_1, p_2 , so the exceptional set \mathfrak{B} is a divisor \mathfrak{D} in this case.

We will also consider in section 9.4 the case of genus 3. A generic genus 3 Riemann surface Σ_{red} , in affine coordinates, is described as a plane curve $P_4(x,y) = 0$, where P_4 is a quartic polynomial in two variables. A canonical divisor is the intersection of Σ_{red} with a line L in the plane. A line is of course defined by a linear equation $P_1(x,y) = 0$. Pick any point $w \in \Sigma_{\text{red}}$ and let L be the line tangent to Σ_{red} at w. The equations $P_1(x,y) = P_4(x,y) = 0$, which describe intersections of Σ and L, will be satisfied at four points in \mathbb{C}^2 , counted with multiplicity. The point w of tangency has multiplicity 2, so Σ_{red} and L intersect at two other points p_1, p_2 . Generically these are distinct points with multiplicity 1 each. So $K \cong \mathcal{O}(2w + p_1 + p_2)$. Thus the line bundle $\mathcal{R}^{-1} = \mathcal{O}(w + p_1 + p_2)$ admits an isomorphism $\mathcal{R}^{-2} \cong K(p_1 + p_2)$. The line bundle $\mathcal{R}^{-1}(-p_1 - p_2) \cong \mathcal{O}(w)$ has a non-zero holomorphic section (the section "1" that vanishes precisely at w). Since $\mathcal{O}(w)$ has degree 1, we have $h^0(\mathcal{O}(w)) = 1$ (a degree 1 line bundle \mathcal{L} over a curve of positive genus always has $h^0(\mathcal{L}) \leq 1$)

and the Riemann-Roch formula $h^0(\mathcal{O}(w)) - h^1(\mathcal{O}(w)) = 1 - g + 1 = -1$ implies that $h^1(\mathcal{O}(w)) = 2$. But by Serre duality $h^1(\mathcal{O}(w)) = h^0(K(-w)) = h^0(\mathcal{R}^{-1})$. So $h^0(\mathcal{R}^{-1}) = 2$. Thus we have found a 1-parameter family of pairs p_1, p_2 , parametrized by $w \in \Sigma$, such that $h^0(\mathcal{R}^{-1}) \geq r + 1 = 2$, showing that again the exceptional set \mathfrak{B} is of codimension 1.

Now let Σ_{red} have arbitrary genus g. For r=1, the line bundle $\mathcal{R}^{-1}(-p_1)$ has degree g-1, so one expects that there is a divisor \mathfrak{D} in the moduli space along which $h^0(\mathcal{R}^{-1}(-p_1)) > 0$. Along \mathfrak{D} , let s be a nonzero holomorphic section of $\mathcal{R}^{-1}(-p_1)$. Then s^2 is a holomorphic section of $\mathcal{R}^{-2}(-2p_1) \cong K(-p_1+p_2)$. In other words, s^2 is a meromorphic section of K that is holomorphic except possibly for a single pole at p_2 . Since the sum of residues of a meromorphic section of K must vanish, s^2 is actually holomorphic at p_2 . This means that s must vanish at p_2 , so s is a holomorphic section of $\mathcal{R}^{-1}(-p_1-p_2)$. Thus along \mathfrak{D} , $h^0(\mathcal{R}^{-1}(-p_1-p_2)) > 0$. Since $\mathcal{R}^{-1}(-p_1-p_2)$ has degree g-2, Riemann-Roch implies that $h^0(\mathcal{R}^{-1}(-p_1-p_2)) - h^1(\mathcal{R}^{-1}(-p_1-p_2)) = -1$, so $h^1(\mathcal{R}^{-1}(-p_1-p_2)) \geq 2$. By Serre duality, this is equivalent to $h^0(K \otimes \mathcal{R}(p_1+p_2)) \geq 2$. Finally, using the isomorphism (5.14), this is equivalent to $h^0(\mathcal{R}^{-1}) \geq 2$. In short, along the divisor \mathfrak{D} , one has $h^0(\mathcal{R}^{-1}) \geq 2$. Along this divisor, the super period matrix has a pole with nilpotent residue, as will be described in Appendix A.

For r > 1, it is likely that the exceptional set \mathfrak{B} has codimension greater than 1, but we will not analyze this case in detail. We should note that although the choice of fermionic A-periods is essentially unique for r = 1, as described in section 5.4, for r > 1, one requires a somewhat arbitrary choice of fermionic A-periods and this introduces artificial singularities in codimension 1. Thus describing the periods by a period matrix is less natural for r > 1 than it is for r = 1. It is perhaps more natural for r > 1 to simply study the Lagrangian submanifold spanned by the periods, rather than to define a period matrix.

6 Low Genus

In the rest of this paper, we navigate toward an application of the super period matrix with Ramond punctures that was described in the introduction. The application mainly involves super Riemann surfaces of genus 2, so we begin by explaining some special facts about the super period matrix for small values of the genus.

In genus $g \leq 3$, any $g \times g$ complex symmetric matrix with positive imaginary part is the period matrix of an ordinary Riemann surface Σ_0 , which is unique up to isomorphism. (This is not true for g > 3; for a symmetric matrix of positive imaginary part to be a period matrix, it must obey the Schottky relations. That is why the following construction is limited to $g \leq 3$.) If Σ_0 is a Riemann surface of genus ≤ 2 with even spin structure, then $H^0(\Sigma_0, K^{1/2})$

is zero always⁸ (and not just generically, as is the case for $g \geq 3$), so any super Riemann surface of genus 2 with even spin structure has a super period matrix $\widehat{\Omega}$. By mapping a super Riemann surface Σ to the ordinary Riemann surface with the same period matrix, we get for g = 1, 2 a natural holomorphic map⁹ $\pi : \mathfrak{M}_{g,+} \to \mathcal{M}_{g,\text{spin}+}$, where $\mathfrak{M}_{g,+}$ parametrizes super Riemann surfaces of genus g with even spin structure, and its reduced space $\mathcal{M}_{g,\text{spin}+}$ parametrizes an ordinary Riemann surface of genus g also with even spin structure.

For g=1, this construction is trivial, as a genus 1 super Riemann surface with even spin structure has no odd moduli and $\mathfrak{M}_{1,+}=\mathcal{M}_{1,\mathrm{spin}+}$. However, for g=2, the statement is non-trivial and has been exploited by D'Hoker and Phong in computing superstring scattering amplitudes, as summarized in [5]. We will ultimately focus mostly on this case. For g=3, since every 3×3 complex matrix of positive definite imaginary part is a period matrix, we can use the same construction to define a meromorphic projection $\pi:\mathfrak{M}_{3,+}\to\mathcal{M}_{3,\mathrm{spin}+}$, but now π has poles (with nilpotent residue). As explained in Appendix C, π has for genus 3 a fairly obvious pole along the divisor \mathfrak{D} where the super period matrix has a pole, and a somewhat less obvious pole along a second divisor \mathfrak{D}' .

We can do something somewhat similar for $\mathfrak{M}_{g,0,2r}$, which parametrizes a genus g super Riemann surface Σ with 2r Ramond punctures and no NS punctures. Once we pick a set of fermionic A-periods, we can define a super period matrix $\widehat{\Omega}$, which in particular has the $g \times g$ pseudoclassical block $\widehat{\Omega}_{g \times g}$. For $g \leq 3$, mapping Σ to an ordinary Riemann surface Σ_0 whose period matrix Ω coincides with $\widehat{\Omega}_{g \times g}$ gives a map

$$\begin{array}{ccc}
\mathcal{X} & \to & \mathfrak{M}_{g,0,2r} \\
& \downarrow \pi \\
& \mathcal{M}_g.
\end{array} (6.1)$$

The fiber \mathcal{X} parametrizes all moduli of Σ other than its super period matrix. Note that in (6.1), the base space is simply \mathcal{M}_g , with no memory of the generalized spin structure. In the absence of Ramond punctures, the analogous projection $\pi: \mathfrak{M}_{g,\pm} \to \mathcal{M}_{g,\mathrm{spin}\pm}$ can be defined to remember the spin structure. We cannot do something analogous in the case of a projection from $\mathfrak{M}_{g,0,2r}$ to \mathcal{M}_g ; since the definition of a generalized spin structure (a line bundle \mathcal{R} with the isomorphism in eqn. (5.1)) depends on the positions of the Ramond punctures, there is no way to forget the Ramond punctures while remembering the generalized spin structure. That is why anything that one can deduce from the fibration π for r > 0 – notably the vanishing under certain conditions of the dilaton tadpole – will involve a sum over spin structures.

⁸ This statement is true for a smooth Riemann surface of genus 2. However, a smooth curve of genus 2 with even spin structure can degenerate to a pair of genus 1 components each with odd spin structure, meeting at a point; for such a singular curve, the appropriate analog of $H^0(\Sigma_0, K^{1/2})$ is non-zero. Thus the divisor \mathfrak{D} has a component at infinity in the Deligne-Mumford compactification of $\mathcal{M}_{2,\text{spin}+}$.

⁹For $g \leq 3$, we generically use the symbol π to denote a projection defined using the period matrix, or its pseudoclassical block in the presence of Ramond punctures.

The nicest case of the fibration $\pi: \mathfrak{M}_{g,0,2r} \to \mathcal{M}_g$ – and the case that we will use in our application – is for r=1, for then the choice of a fermionic A-period is essentially unique (and $\widehat{\Omega}_{g\times g}$ is entirely unique) as we saw in section 5.4. For r>1, the projection π does depend on the choice of fermionic A-periods and moreover it has unphysical singularities that depend on that choice.

Keeping $g \leq 3$, there is no trouble to include NS punctures in this picture, since an NS puncture is just a marked point. There is therefore a forgetful map for NS punctures, and an NS puncture does not affect the definition of the super period matrix. Composing the map $\mathfrak{M}_{g,n,2r} \to \mathfrak{M}_{g,0,2r}$ which forgets the NS punctures with the projection π , we get a projection

$$\begin{array}{ccc}
\mathcal{Y} & \to & \mathfrak{M}_{g,n,2r} \\
& \downarrow \pi' \\
& \mathcal{M}_g,
\end{array} (6.2)$$

where now the fiber \mathcal{Y} parametrizes also the positions of the NS punctures.

Our application will involve the case r=1 (precisely 2 Ramond punctures) and n=0 (no NS punctures). We will also encounter the case r=0, n=1. The projection π : $\mathfrak{M}_{2,0,2} \to \mathcal{M}_2$ that we will use has poles associated to poles of the super period matrix. The complications associated to those poles can be overcome, at least for certain purposes. We will analyze this in section 9, but there are a number of things to explain first.

All statements in this section have been made for the uncompactified moduli spaces. Some care is required to extend these statements over the corresponding Deligne-Mumford compactifications. For example, in the absence of Ramond punctures, the projection $\mathfrak{M}_{2,+} \to \mathcal{M}_{2,\mathrm{spin}+}$ used by d'Hoker and Phong develops a pole if one attempts to extend it over the Deligne-Mumford compactifications of these spaces. This has been explained in [9] (the underlying reason was explained in footnote 8 above), and is important in understanding the behavior near infinity of the measure on $\mathcal{M}_{2,\mathrm{spin}+}$ computed by d'Hoker and Phong.

7 Ward Identities

A certain subtlety of superstring theory is important in our application. Before explaining it, we begin by recalling what happens in bosonic string theory.

Consider a closed oriented bosonic string vacuum such that at tree level the matter system has a continuous symmetry, ¹⁰ with a conserved charge J associated to a conserved

 $^{^{10}}D$ -branes and/or an orientifold projection can be included in the following discussion if the D-brane boundary condition and/or the orientifold projection are chosen to preserve the symmetry.

worldsheet current J.

Such a symmetry constrains the worldsheet correlation functions on any worldsheet. Let V_1, \ldots, V_n be an arbitrary set of matter vertex operators that are eigenstates of J (and are conformal primaries of the appropriate dimension). Then the correlation function $\langle V_1 V_2 \ldots V_n \rangle$ on a worldsheet Σ_0 of any genus g will vanish unless the sum of the charges of the operators V_i vanishes. Thus, the existence of a conserved worldsheet current leads automatically to a conservation law for genus g amplitudes, for any g. As a result, in closed oriented bosonic string theory, a continuous symmetry at tree level remains valid as a continuous symmetry to all orders of perturbation theory. This result holds for any given Riemann surface Σ_0 ; it does not in any way involve integration over moduli space.

The same argument holds for continuous symmetries of superstring theory that arise from the NS sector.¹¹ Thus, loop corrections do not trigger spontaneous breakdown of a continuous global symmetry that arises in closed oriented bosonic string theory or the NS sector of superstring theory.

However, we cannot make such a simple argument for spacetime supersymmetry, which comes from the Ramond sector. The analog of the conserved current J is the usual fermionic vertex operator S_A (here A is a spinor index and S_A is the combined spin field of matter and ghosts). The considerations of this paper apply equally to heterotic or Type II superstrings, but for simplicity, and also with a view toward our eventual application, we consider the heterotic string, in which spacetime supersymmetry comes from the holomorphic part of the worldsheet theory only. Then S_A is a holomorphic object, but it cannot be understood as a conserved current on a fixed super Riemann surface Σ ; indeed, it is inserted at a Ramond puncture, which is a singularity in the superconformal structure of Σ , and there is no notion of moving a Ramond puncture while otherwise leaving Σ unchanged.

Instead, the proof of a supersymmetric Ward identity proceeds essentially by constructing a conserved current on the super moduli space rather than on the super Riemann surface Σ . To do this, we combine the spin field with the ordinary c ghost to make a dimension (0,0) superconformal primary $S_A = cS_A$. We similarly combine the matter vertex operators V_i with ghosts in the usual way to make superconformal primaries V_i of dimension (0,0). (For example, if V_i is a vertex operator of the NS sector, then we set $V_i = \tilde{c}c\delta(\gamma)V_i$, where c, γ are the holomorphic superconformal ghost fields and \tilde{c} is the antiholomorphic ghost field.) The correlation function

$$F_{\mathcal{S}_A \mathcal{V}_1 \dots \mathcal{V}_n} = \langle \mathcal{S}_A \mathcal{V}_1 \dots \mathcal{V}_n \rangle \tag{7.1}$$

does not define a measure on supermoduli space – or more precisely 12 on the appropriate

¹¹If (as is typical) the continuous symmetry is associated at tree level to a massless gauge particle, the gauge field may gain mass in perturbation theory but the symmetry is unbroken as a global symmetry to all orders of perturbation theory. See [9] for further detail.

¹²As explained in [9] and more fully in section 5 of [13] (see also [15], pp. 94-5), the usual notion of

integration cycle Γ of superstring perturbation theory – because the operator S_A has ghost number less by 1 than the ghost number of a standard Ramond-sector vertex operator. As a result, $F_{S_A \mathcal{V}_1 \dots \mathcal{V}_n}$ represents not a measure on Γ that could be integrated to compute a scattering amplitude, but an integral form of codimension 1 – the supermanifold analog of a conserved current (see for example [15] or [13]). BRST-invariance of S_A and of the V_i implies that this form is closed, $dF_{S_A \mathcal{V}_1 \dots \mathcal{V}_n} = 0$. Using this fact and the supermanifold version of Stokes's theorem, we derive a supersymmetric Ward identity:

$$0 = \int_{\Gamma} dF_{\mathcal{S}_A \mathcal{V}_1 \dots \mathcal{V}_n} = \int_{\partial \Gamma} F_{\mathcal{S}_A \mathcal{V}_1 \dots \mathcal{V}_n}.$$
 (7.2)

On the right hand side, $\partial \Gamma$ is a sum of components "at infinity" in Γ ; they correspond to different ways that Σ might degenerate.

Although in general Σ has various separating and nonseparating degenerations, there are, as explained in [9] (and more fully in section 8 of [16]), only two types of degeneration that contribute to the Ward identity. One such degeneration is the case that Σ degenerates to a union of two components, one of which contains S_A and just one of the V_i . Such a degeneration is sketched in fig. 1. As is explained in [9], if only such components of $\partial \Gamma$ contribute on the right hand side of (7.2), then (7.2) becomes a standard Ward identity of unbroken supersymmetry and the genus g contribution to the S-matrix is spacetime supersymmetric. In general, there may also be a Goldstone fermion contribution to the Ward identity; it can arise from a degeneration in which Σ splits off a positive genus contribution Σ_{ℓ} that contains S_A but no other vertex operator, as sketched in fig. 2. Just as in field theory, if the genus g Ward identity receives a Goldstone fermion contribution, then the genus g contribution to the S-matrix is not spacetime supersymmetric.

The importance of the fibrations that were described in section 6 is that in favorable cases, they can be used to establish supersymmetric Ward identities in which one knows a priori that there can be no Goldstone fermion contribution. Rather than try to explain abstractly how that can happen, we will first describe the problem that we have in mind.

integration over supermoduli space is an approximation to a more precise notion of integration over a certain cycle Γ in the product of holomorphic and antiholomorphic moduli spaces (in the case of the heterotic string, the holomorphic moduli space is the super moduli space $\mathfrak{M}_{g,n,2r}$ and the antiholomorphic space is its reduced space with complex structure reversed; one can take the reduced space of Γ to coincide with the reduced space of $\mathfrak{M}_{g,n,2r}$). The distinction between $\mathfrak{M}_{g,n,2r}$ and Γ will not be very important in the present paper, but we express our statements in terms of integration over Γ since this is more accurate.

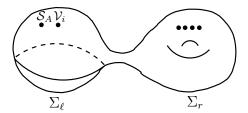


Figure 1: The Ward identity always receives contributions from separating degenerations of this kind in which one component Σ_{ℓ} contains a supersymmetry generator S and precisely one more vertex operator. If these are the only contributions, then the Ward identity expresses the invariance of the S-matrix under spacetime supersymmetry. The usual case, as sketched here, is that Σ_{ℓ} has genus 0. This leads to the familiar tree-level expressions for the supercharges.

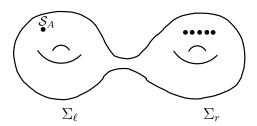


Figure 2: One other degeneration may contribute to the Ward identity. This is the Goldstone fermion contribution. It represents spontaneous breaking of spacetime supersymmetry. This contribution can exist only when the genus of Σ_{ℓ} is positive.

8 The Two-Loop Vacuum Amplitude

We turn to the two-loop vacuum amplitude of heterotic string theory. Potential contributions come from worldsheets with even spin structure only (worldsheets with odd spin structure contribute to parity-violating amplitudes), so the vacuum amplitude is found by integration over $\mathfrak{M}_{2,+}$. An effective procedure [5, 10] has been to first integrate over the fibers of the projection $\pi:\mathfrak{M}_{2,+}\to\mathcal{M}_{2,\mathrm{spin}+}$, so as to reduce the vacuum amplitude to an integral over the bosonic moduli space $\mathcal{M}_{2,\mathrm{spin}+}$. At that point, it makes sense to sum over spin structures without integrating over any additional moduli. (This does not make sense before reducing to the bosonic moduli space, since there is no notion of changing the spin structure of a super Riemann surface while otherwise leaving the surface unchanged.) Quite a few supersymmetric compactifications to ≥ 4 dimensions have been studied this way. In each case, after reducing to $\mathcal{M}_{2,\mathrm{spin}+}$, the two-loop vacuum amplitude vanishes upon summing over spin structures, even without any integral over the bosonic moduli. Our goal here is to explain this and show that it is true in general.

As we recalled in the introduction, in general, in a supersymmetric compactification to four dimensions, the procedure of integrating over $\mathcal{M}_{2,\mathrm{spin}+}$ by integrating first over the fibers of the projection $\pi:\mathfrak{M}_{2,+}\to\mathcal{M}_{2,+}$ misses a contribution at infinity. So our result means that in such compactifications, the full two-loop vacuum amplitude is given by the contribution at infinity. In supersymmetric compactifications above four dimensions, there is no such contribution at infinity, and the whole two-loop vacuum amplitude vanishes.

Before discussing the two-loop case, let us recall one way to understand what happens in genus 1. On a superstring worldsheet Σ of genus 1, we consider a two point function

$$F_{\mathcal{S}_A \mathcal{V}^B} = \langle \mathcal{S}_A(z) \mathcal{V}^B(\widetilde{y}; y) \rangle. \tag{8.1}$$

Here as before $S_A = cS_A$ is the spacetime supersymmetry generator with the ghost field c included. On the other hand, we take V^B to be the vertex operator for the dilatino – the spin 1/2 partner of the dilaton – at zero momentum. For superstring theory in \mathbb{R}^{10} ,

$$\mathcal{V}^B = \widetilde{c}c\delta(\gamma)\partial_{\widetilde{z}}X^I\Gamma_I^{BC}S_C, \tag{8.2}$$

where X^I , I = 1, ..., 10 are matter superfields representing the motion of the strings in \mathbb{R}^{10} , S_C is once again the holomorphic spin field, and Γ_I^{BC} , I = 1, ..., 10, B, C = 1, ... 16 are gamma matrices. In compactified models, the definition of \mathcal{V}^B is changed slightly, but the details will not be important. The notation $\mathcal{S}_A(z)$ and $\mathcal{V}^B(\widetilde{y};y)$ is just meant to remind us that \mathcal{S}_A is holomorphic while \mathcal{V}^B is neither holomorphic nor antiholomorphic.

These two vertex operators are both in the Ramond sector, so Σ is a genus 1 surface with 2 Ramond punctures. Such a surface has precisely 1 odd modulus. The characteristic

subtleties of superstring perturbation theory result from the possibility of changes of variables such as $m \to m + \eta \eta'$, where m is an even modulus and η, η' are odd moduli. So they do not come into play until there are at least two odd moduli. Accordingly, none of these subtleties are relevant to the example under discussion. Except for one important detail, we can think of z and y as points on the ordinary genus 1 Riemann surface $\Sigma_{\rm red}$, and use the translation symmetries of $\Sigma_{\rm red}$ to set $y = \tilde{y} = 0$. The detail in question is as follows: the generalized spin structure of $\Sigma_{\rm red}$ changes when z moves around a non-contractible loop in $\Sigma_{\rm red}$, and hence we should think of z as a point in a $2^{2g} = 4$ -fold unramified cover of $\Sigma_{\rm red}$, which we will call $\Sigma'_{\rm red}$, that parametrizes a point in $\Sigma_{\rm red}$ together with a generalized spin structure. The condition z = 0 defines a single point on $\Sigma_{\rm red}$, but on $\Sigma'_{\rm red}$ it corresponds to four possible points p_1, \ldots, p_4 , which are labeled by the four possible spin structures on Σ or equivalently on $\Sigma_{\rm red}$. (The generalized spin structure of Σ reduces to an ordinary spin structure for $z \to 0$, where the two Ramond punctures coincide.) The correlation function $\langle S_A(z) \mathcal{V}^B(0;0) \rangle$ is a holomorphic 1-form on $\Sigma'_{\rm red}$ that has poles only at the points $^{13}p_i$. Because of the operator product relation

$$S_A(z)\mathcal{V}^B(0;0) \sim \frac{\delta_A^B}{z} \mathcal{V}_{\text{Dil}}(0;0),$$
 (8.3)

where \mathcal{V}_{Dil} is the dilaton vertex operator at zero momentum, the residue of each pole is the dilaton tadpole corresponding to the given spin structure. The vanishing of the sum of the residues of the holomorphic 1-form $\langle S_A(z) \mathcal{V}^B(0;0) \rangle$ on Σ'_{red} means that the dilaton tadpole vanishes, after summing over spin structures and before integration over bosonic moduli.

A standard argument shows that in genus g, the dilaton tadpole is 2g times the vacuum amplitude. So the genus 1 vacuum amplitude likewise vanishes after summing over spin structures, but without integration over bosonic moduli.

The same argument does not work in genus g > 1, because there is more than 1 odd modulus, and the subtleties of superstring perturbation theory do come into play. We cannot think of the fermion vertex operator S_A as a conserved current on a fixed worldsheet; instead, as was summarized in section 7, a proper general argument proceeds by applying the identity (7.2) to the correlation function $F_{S_AV^B}$ and analyzing all possible boundary contributions, including contributions that involve degenerations of Σ . By such reasoning one can show that (in a supersymmetric compactification with no Goldstone fermion contribution), the genus g vacuum amplitude vanishes after integration over all moduli. There is in general no simpler version of this statement that involves integrating or summing over only some of the moduli.

However, for g = 2 we can in fact imitate the classical genus 1 argument, with some more care, using the map $\pi : \mathfrak{M}_{2,0,2} \to \mathcal{M}_2$ described in eqn. (6.1). Of course, we will have to take into account the fact that this map is only generically defined, but this turns out to cause

¹³The residue of the pole at the point corresponding to the odd spin structure on Σ vanishes because of fermion zero modes in the matter system, so there are really only three poles, not four.

no problem. The obvious singularities of the correlation function $F_{S_A V^B} = \langle S_A(z) V^B(0;0) \rangle$ arise from a collision of the two vertex operators; such a collision gives a pole whose residue is the amplitude on $\mathfrak{M}_{2,1,+}$ associated to the dilaton tadpole. We will reduce from $\mathfrak{M}_{2,0,2}$ to $\mathfrak{M}_{2,1,+}$ by extracting this residue. By adapting the classical argument that was explained above in genus 1, we will show that if one follows the procedure of d'Hoker and Phong to compute genus 2 amplitudes, then the bulk contribution to the dilaton tadpole – and hence to the vacuum amplitude – vanishes after integrating over the fiber of π' : $\mathfrak{M}_{2,1,0} \to \mathcal{M}_2$, even before integrating over \mathcal{M}_2 . Concretely, integrating over the fiber of π' means integrating over the position at which the dilaton vertex operator is inserted, and summing over spin structures, but not integrating over the moduli of the underlying genus 2 surface Σ_{red} . The bulk contribution to the genus 2 vacuum amplitude, computed with the d'Hoker-Phong procedure, vanishes before that last integration.

To make the analysis, we write Ξ for a generic fiber of the projection $\pi:\mathfrak{M}_{2,0,2}\to\mathcal{M}_2$, and we look at the behavior of $F_{S_AV^B}$ near Ξ . We recall that, as discussed in section 7, $F_{S_AV^B}$ is not a volume form on $\mathfrak{M}_{2,0,2}$ but a form of codimension 1 (technically an integral form of codimension 1). Near Ξ , we can factorize $F_{\mathcal{S}_A\mathcal{V}^B}$ as $\pi^*(\mu) \cdot F_{\mathcal{S}_A\mathcal{V}^B}^*$, where μ is a volume form on \mathcal{M}_2 and $F_{\mathcal{S}_A\mathcal{V}^B}^*$ is a form of codimension 1 along the fibers of π . The reason that this factorization exists is that the missing "index" of the codimension 1 form $F_{\mathcal{S}_A\mathcal{V}^B}$ is tangent to the fibers of π , not the base. This index is missing because \mathcal{S}_A is missing a factor of the antiholomorphic ghost field \tilde{c} . From an antiholomorphic point of view, the map π is just the forgetful map $\mathcal{M}_{2,2} \to \mathcal{M}_2$, and \tilde{c} represents a 1-form dual to the motion of the operator \mathcal{S}_A along the fiber of this map. So the missing index is tangent to the "fiber" of the fibration π , not to the base, and that is why $F_{\mathcal{S}_A\mathcal{V}^B}$ can be factored as the product of the pullback $\pi^*(\mu)$ of a full volume form μ on the base times a form $F^*_{\mathcal{S}_A\mathcal{V}^B}$ of codimension 1 in the fiber direction. This factorization is not completely canonical, since μ could be multiplied by a nonzero function on the base of the fibration, but once we restrict to a particular fiber Ξ , $F_{\mathcal{S}_A\mathcal{V}^B}^*$ depends on the choice of μ only by an overall multiplicative constant, which will not affect what follows.

We can now make the same argument as in (7.2), but with the fiber Ξ replacing the full integration cycle Γ , and $F_{\mathcal{S}_A\mathcal{V}^B}^*$ replacing $F_{\mathcal{S}_A\mathcal{V}^B}$. Since $\mathrm{d}F_{\mathcal{S}_A\mathcal{V}^B}^*=0$, we have

$$0 = \int_{\Xi'} dF_{\mathcal{S}_A \mathcal{V}^B}^* = \int_{\partial \Xi'} F_{\mathcal{S}_A \mathcal{V}^B}^*. \tag{8.4}$$

Here Ξ' is defined by throwing away a small neighborhood of the singularities of $F_{S_AV^B}^*$ in Ξ , and $\partial\Xi'$ is a union of the boundaries of those small neighborhoods.

The obvious singularities of $F_{S_AV^B}^*$ come from a collision of the two vertex operators S_A and V^B . Via the operator product (8.3), such a collision gives a pole whose residue is a dilaton tadpole evaluated on the surface Σ , endowed with one of its possible spin structures.

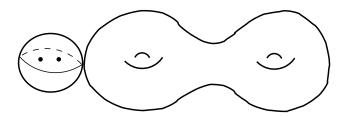


Figure 3: The obvious contributions to the identity (8.4) come from collisions of the two vertex operators. From a conformal point of view, these involve the splitting of Σ into two components, of which one is a genus 0 surface with 2 Ramond punctures, one NS punctures, and no even or odd moduli, and the second is a genus 2 surface with 1 NS puncture. The contribution of such a splitting to the identity is the bulk contribution to the genus 2 dilaton tadpole, computed with the procedure of d'Hoker and Phong. (The dilaton vertex operator is inserted on the genus 2 component at the point where it meets the second component.)

All possible spin structures occur because the fiber Ξ parametrizes among other things the generalized spin structure of the super Riemann surface Σ (just as in genus 1, the possible choices of this generalized spin structure are permuted when S_A or \mathcal{V}^B is taken around a noncontractible loop in Σ , and the generalized spin structure reduces to an ordinary one when the two operators meet). The locus in $\mathfrak{M}_{2,0,2}$ along which the two vertex operators meet is a divisor \mathfrak{D} that is a copy of $\mathfrak{M}_{2,1,0}$ (fig. 3). The projection $\pi: \mathfrak{M}_{2,0,2} \to \mathcal{M}_2$ which we use to formulate eqn. (8.4) restricts along \mathfrak{D} to the projection $\pi': \mathfrak{M}_{2,1,0} \to \mathcal{M}_2$ that d'Hoker and Phong would use to compute the bulk contribution to the dilaton tadpole. So if the singularities that correspond to dilaton tadpoles are the only ones that contribute in the identity (8.4), then we learn that with the d'Hoker-Phong procedure, the sum over spin structures of the genus 2 dilaton tadpole vanishes. (The full answer therefore comes from the correction at infinity that is described in [9].)

It remains to consider the possibility of additional singularities contributing to the identity (8.4), resulting from the fact that the super period matrix is only generically defined. We will show in section 9 that although there is indeed a locus on Ξ on which the super period matrix has a pole (with nilpotent residue), this leads to no contribution in the identity (8.4) because the fermions of the matter system acquire zero-modes on just the dangerous locus.

9 What Happens At Poles Of The Super Period Matrix?

In this section, we complete the argument of section 8, by analyzing the singularities associated to poles of the super period matrix, and showing that they do not affect the vanishing of the bulk contribution to the two-loop vacuum amplitude.

9.1 The Locus Of Spurious Singularities

As described in section 5.6, in the case of a genus 2 surface Σ with Ramond punctures p_1, p_2 , the condition $h^0(\mathcal{R}^{-1}) > 1$ is equivalent to $\mathcal{O}(p_1 + p_2) \cong K$. Moreover, this is also equivalent to $\mathcal{O}(p_1 + p_2) \cong \mathcal{R}^{-1}$. To make this condition more explicit, we recall that a genus 2 Riemann surface Σ_{red} is hyperelliptic; it admits a holomorphic map $\rho : \Sigma_{\text{red}} \to \mathbb{CP}^1$ that is a double covering, branched over 6 points. Σ_{red} has a \mathbb{Z}_2 symmetry group, generated by a symmetry $\tau : \Sigma_{\text{red}} \to \Sigma_{\text{red}}$ that exchanges the two sheets of the covering. Concretely, Σ_{red} can be described by a hyperelliptic equation

$$y^2 = P_6(u, v), (9.1)$$

where P_6 is a homogeneous polynomial of degree 6 in homogeneous coordinates u, v of \mathbb{CP}^1 . In this description, ρ is defined by forgetting y and τ acts by $y \to -y$. For q a point in Σ_{red} , we sometimes write q' for $\tau(q)$ and say that q and q' are conjugate. The condition that $\mathcal{O}(p_1 + p_2) \cong K$ means that there is a holomorphic differential on Σ_{red} whose zeroes are p_1 and p_2 . A holomorphic differential on Σ_{red} has the form $\omega = (au + bv)(udv - vdu)/y$, with constants a, b. It vanishes precisely when au + bv = 0, a condition that defines a single point in \mathbb{CP}^1 , but a conjugate pair of points in Σ_{red} . Since p_1 can be chosen arbitrarily and p_2 is then determined to be $\tau(p_1)$, the condition that p_1 and p_2 are conjugate defines a divisor \mathfrak{D} in the space of all pairs p_1, p_2 . For the case g = 2, r = 1, this is the bad set in the definition of the super period matrix. As explained in detail in Appendix A, the super period matrix has a pole with nilpotent residue along \mathfrak{D} .

This pole does not lead to additional contributions in the Ward identity (8.4) because when the Ramond divisors are associated to conjugate points p_1, p_2 , the fermions of the matter system have many zero-modes that cancel the potential singularity due to the pole in the period matrix. To understand this, we must recall some facts about the superstring spin field S_A and its coupling to the matter fermions of the RNS model.

9.2 Fermion Zero Modes

First we consider the case of uncompactified ten-dimensional spacetime. The rotation group is SO(10), and its maximal torus is $U(1)^5$. The weights of the spinor representation take the form $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_5$, where ε_i is the charge under the i^{th} U(1) and each ε_i is $\pm 1/2$. For a spinor of positive chirality, the number of weights -1/2 is even; we will assume that the spin field S_A is a spinor of positive chirality.

The matter fermions of the heterotic string can be grouped as five complex fermions ψ_i and their charge conjugates $\widehat{\psi}_i$, all of which are holomorphic fields. ψ_i and $\widehat{\psi}_i$ have charge 1 and -1, respectively for the i^{th} U(1) in the maximal torus, and zero charge for the other U(1)'s. The fields ψ_i and $\widehat{\psi}_i$ appear in a Dirac action $\int_{\Sigma_{\text{red}}} d^2z \, \widehat{\psi}_i \widetilde{\partial} \psi_i$, and this tells us that if ψ_i is a section of a line bundle \mathcal{L} , then $\widehat{\psi}_i$ is a section of a conjugate line bundle $K \otimes \mathcal{L}^{-1}$.

Suppose that these fermions interact with spin fields placed at points $p_{\alpha} \in \Sigma_{\text{red}}$, $\alpha = 1, \ldots, 2r$, and let the weights of the spin field at p_{α} be $\varepsilon_{i,\alpha} = \pm 1/2$, $i = 1, \ldots, 5$. In the most traditional formulation, one says that as $z \to p_{\alpha}$, $\psi_i(z)$ has a half-order zero or pole at $z = p_{\alpha}$, depending on $\varepsilon_{i,\alpha}$, with $\psi_i(z) \sim (z - p_{\alpha})^{-\varepsilon_{i,\alpha}}$. In terms of complex geometry, this means roughly that ψ_i is a section of $K^{1/2} \otimes_{\alpha=1}^{2r} \mathcal{O}(p_{\alpha})^{\varepsilon_{i,\alpha}}$. But what precisely is meant by the half-integral powers $\mathcal{O}(p_{\alpha})^{\pm 1/2}$? If (for some i) the $\varepsilon_{i,\alpha}$ are all $\pm 1/2$, the meaning is simply that ψ_i is a section of \mathcal{R}^{-1} (which has an isomorphism $(\mathcal{R}^{-1})^2 \cong K \otimes_{\alpha} \mathcal{O}(p_{\alpha})$, so it is informally $K^{1/2} \otimes_{\alpha} \mathcal{O}(p_{\alpha})^{1/2}$). In general, ψ_i is a section of $\mathcal{L}_i = \mathcal{R}^{-1} \otimes_{\alpha|\varepsilon_{i,\alpha}=-1/2} \mathcal{O}(-p_{\alpha})$, and dually $\widehat{\psi}_i$ is a section of $K \otimes \mathcal{L}^{-1} \cong K \otimes \mathcal{R} \otimes_{\alpha|\varepsilon_{i,\alpha}=-1/2} \mathcal{O}(p_{\alpha})$. These statements incorporate the usual assertions about zeroes and poles of half-order, but in a way that is more natural in algebraic geometry. By including in \mathcal{L}_i a factor of $\mathcal{O}(-p_{\alpha})$ whenever $\varepsilon_{i,\alpha} = -1/2$, we ensure that in this case (in the more informal language), ψ_i has a half-order zero at p_{α} rather than a half-order pole.

In our application, there are just two Ramond insertions – the operator S_A at one point p and the operator $V^B = \tilde{\partial} X^I \Gamma_I^{BC} S_C$ at another point q. (To reduce clutter, in the following discussion we write p and q rather than p_1 and p_2 for the points with Ramond insertions.) We first discuss the case of superstring theory in \mathbb{R}^{10} . It is convenient to pick S_A to be the particular spin field whose weights are all 1/2, and to pick V^B to have all weights -1/2. This choice ensures that the dilaton vertex operator V_{dil} does appear as a pole in the product $S_A V^B$. Acting with a matrix Γ_I (for any value of I) will always flip precisely one weight. So (looking at the definition (8.2) of V^B) the component of V^B with all weights -1/2 is a sum of terms, each proportional to a spin field S_C that has four weights -1/2 and one weight +1/2. As long as we are on \mathbb{R}^{10} , for counting fermion zero-modes, the different components are equivalent. We may as well look at the term with $\varepsilon_{1,q} = 1/2$ and $\varepsilon_{i,q} = -1/2$, i > 1.

With those weights, ψ_1 is a section of \mathbb{R}^{-1} . As we have seen above, the critical case

that might give an unwanted contribution to the identity (8.4) is that q = p' (that is, q is conjugate to p), and moreover in this case, $\mathcal{R}^{-1} \cong K \cong \mathcal{O}(p+q)$. So in the critical case, ψ_1 is a section of $\mathcal{R}^{-1} \cong K$ and dually $\widehat{\psi}_1$ is a section of $K \otimes \mathcal{R} \cong \mathcal{O}$. On the other hand, ψ_i for i > 1 is a section of $\mathcal{R}^{-1} \otimes \mathcal{O}(-q) \cong \mathcal{O}(p)$. Dually, $\widehat{\psi}_i$ for i > 1 is a section of $K \otimes \mathcal{O}(p)^{-1} \cong \mathcal{O}(q)$.

Now we can count fermion zero modes. The line bundle K has a two-dimensional space of holomorphic sections, while \mathcal{O} , $\mathcal{O}(p)$, and $\mathcal{O}(p')$ all have one-dimensional spaces of holomorphic sections. So in the critical case q = p', all ψ_i and $\widehat{\psi}_j$ have 1 zero-mode, except that ψ_1 has 2, making a total of 11 fermion zero-modes in all.

This is far too many zero modes for the locus q = p' to be dangerous. Before discussing what happens to the identity (8.4), let us simply discuss how the correlation function $F_{\mathcal{S}_A \mathcal{V}^B}$, understood as a form on $\mathfrak{M}_{2,0,2}$ of codimension 1, behaves for $q \to p'$. $\mathfrak{M}_{2,0,2}$ has odd dimension 3, so in conventional language, to compute $F_{\mathcal{S}_A \mathcal{V}^B}$, one inserts three picture-changing operators on the worldsheet Σ . Each picture-changing operator (PCO) contains a factor of the worldsheet supercurrent, which (for strings in \mathbb{R}^{10}) is linear in the RNS fermions ψ_i and $\hat{\psi}_j$ and so can absorb one zero-mode. So the PCO's remove 3 zero-modes, leaving 8. Because of the 8 remaining zero-modes, $F_{\mathcal{S}_A \mathcal{V}^B}$ vanishes on the divisor q = p'. To determine the order of its vanishing, we let u be a local parameter on $\mathfrak{M}_{2,0,2}$ with a simple zero at q = p'. At u = 0, there are eight fermion zero-modes that are not lifted by the PCO's. All of these modes are lifted away from zero for $u \neq 0$ and generically they are lifted to first order in u. So if we write ζ_1, \ldots, ζ_8 for the relevant modes that become zero-modes at u = 0, the integral over those modes near u = 0 looks like

$$\int d^8 \zeta \, \exp(u m_{ij} \zeta_i \zeta_j) \sim u^4 \, \text{Pfaff}(m), \tag{9.2}$$

where m is an antisymmetric form that generically is nondegenerate and Pfaff(m) is its Pfaffian. Thus $F_{S_A \mathcal{V}^B} \sim u^4$ for $u \to 0$.

The analysis of the identity (8.4) is more tricky than this, because in defining $F_{S_AV^B}^*$, we remove a factor $\pi^*(\mu)$ that has a pole at u=0. As we explain in section 9.3, the effect of this is to cancel one power of u, so that $F_{S_AV^B}^* \sim u^3$ for $u \to 0$. In particular, $F_{S_AV^B}^*$ has no singularity that would contribute to the identity (8.4).

Now let us consider compactification to four dimensions. The main difference is that the fields ψ_j , $\widehat{\psi}_j$, j=3,4,5 are not free fields on the worldsheet; we cannot talk about their zero-modes or use them in a simple way to predict a zero of an amplitude. Also, we need to slightly modify the definition of the dilatino vertex operator \mathcal{V}^B that has all weights -1/2; it is now the sum of two terms, each proportional to a spin field S_C whose weights ε_i are all -1/2 except for either ε_1 or ε_2 (this ensures that the part of \mathcal{V}^B in the internal

compact manifold is actually a holomorphic primary field of the appropriate dimension). In the counting of zero-modes, we should consider only ψ_i and $\widehat{\psi}_i$ for i=1,2, and the number of such modes, counted the same way as before, is now 5 instead of 11. Repeating the previous reasoning but with the smaller number of zero-modes, we find that $F_{\mathcal{S}_A\mathcal{V}^B}$ is of order u for $u \to 0$ and $F_{\mathcal{S}_A\mathcal{V}^B}^*$ is of order 1. In particular, $F_{\mathcal{S}_A\mathcal{V}^B}^*$ has no singularity at u=0 that could contribute in the identity (8.4). Hence in general, with the d'Hoker-Phong procedure, the "bulk" contribution to the superstring measure vanishes in genus 2 after integrating over odd moduli and summing over spin structures, but before any bosonic integrations.

9.3 Behavior of the Correlation Function

Our goal here is to justify some of these statements by describing the behavior of $F_{\mathcal{S}_A\mathcal{V}^B}^*$ along the divisor $\mathfrak{D} \subset \mathfrak{M}_{2,0,2}$ where the super period matrix has a pole. It is helpful first to recall (following [17], and using the language explained in section 3.6 of [16]) the origin of the picture-changing operators in worldsheet path integrals. For each odd variable η , it is convenient to introduce also its differential $\mathrm{d}\eta$. Then the insertion of a PCO at a point $p \in \Sigma_{\mathrm{red}}$ comes from an integral

$$\int \mathcal{D}(\eta, d\eta) \exp(d\eta \beta(p) + \eta S_{z\theta}(p)), \qquad (9.3)$$

where β is the usual commuting antighost field of superstring perturbation theory and $S_{z\theta}$ is the worldsheet supercurrent. The integral over $d\eta$ gives a factor $\delta(\beta(p))$, and naively the integral over η gives a factor of $S_{z\theta}$. The product is the PCO $\delta(\beta(p))S_{z\theta}(p)$. However, although this treatment of the integral over $d\eta$ is correct, there is more subtlety in the integral over η . Because of the lack of a natural separation between even and odd variables on supermoduli space, there are in general, depending on precisely how one parametrizes supermoduli space, additional η -dependent contributions hidden in other parts of the worldsheet path integral, beyond those written in (9.3). We can expand $\exp(\eta S_{z\theta}(p)) = 1 + \eta S_{z\theta}(p)$. The $\eta S_{z\theta}(p)$ term leads after integrating over η to an insertion at p of the usual PCO $\delta(\beta(p))S_{z\theta}(p)$, but the "1" term leads to insertion at p of the operator $\delta(\beta(p))$, which we might call an incomplete PCO. Locally, one can parametrize supermoduli space in such a way that the incomplete PCO's can be ignored, but in general, globally either one cannot do this or doing this introduces other complications. So it is best to keep track of the contributions involving incomplete PCO's. The only general constraint is that the worldsheet path integral with insertion of an odd number of incomplete PCO's vanishes because of fermi statistics, so one can always assume the number of incomplete PCO's to be even.

Now we consider our problem of the influence of the poles of the super period matrix on $F_{S_AV^B}^*$. Those poles will only affect the dependence of $F_{S_AV^B}^*$ on holomorphic variables, and in what follows we may as well ignore the antiholomorphic variables. $\mathfrak{M}_{2,0,2}$ is a smooth

supermanifold (or rather orbifold) of dimension 5|3, say with even and odd coordinates $h_1, \ldots, h_5 | \eta_1, \ldots, \eta_3$. $F_{\mathcal{S}_A \mathcal{V}^B}$ is, from a holomorphic point of view, a smooth top form, so ignoring its dependence on antiholomorphic variables, it is

$$F_{\mathcal{S}_A \mathcal{V}^B} = f(h_i | \eta_i) [\mathrm{d}h_1 \dots \mathrm{d}h_5 | \mathrm{d}\eta_1 \dots \mathrm{d}\eta_3]. \tag{9.4}$$

The function $f(h_i|\eta_j)$ can be expanded in powers of the η_i . When $F_{S_AV^B}$ is computed in superconformal field theory, there is a contribution, proportional to $\eta_1\eta_2\eta_3$, that can be evaluated using complete PCO's only. As there are 3 odd moduli, this contribution involves 3 insertions of $S_{z\theta}$. These insertions can absorb 3 of the 11 fermion zero-modes (for strings in \mathbb{R}^{10}) described in section 9.2, leading to a contribution to f that, as explained there, is of order u^4 near u=0. $F_{S_AV^B}$ also has a contribution that is linear in the η_i , arising from a contribution evaluated with 2 incomplete PCO's and only one insertion of $S_{z\theta}$. This insertion lifts only one fermion zero-mode, giving a contribution to $F_{S_AV^B}$ that is of order u^5 . We can thus rewrite (9.4) in the more detailed form

$$F_{\mathcal{S}_A \mathcal{V}^B} = \left(f_0(h_i) \eta_1 \eta_2 \eta_3 + \sum_{j=1}^3 f_j(h_i) \eta_j \right) [dh_1 \dots dh_5 | d\eta_1 \dots d\eta_3], \tag{9.5}$$

with

$$f_0 \sim u^4, \quad f_j \sim u^5, \quad j > 0.$$
 (9.6)

Naively, the term in (9.5) that is linear in the η 's is irrelevant both because it vanishes more rapidly for $u \to 0$ and because, with some of the η 's missing, one might think that this term would not contribute in the integration over odd moduli that goes into evaluating the boundary contributions in eqn. (8.4).

However, this reasoning is not valid, for reasons that are related in part to some of the usual subtleties of superstring perturbation theory. To define $F_{S_AV^B}^*$, we are supposed to split off from $F_{S_AV^B}$ the pullback of a volume form on \mathcal{M}_2 . Concretely, let m_1, m_2, m_3 be the three independent matrix elements of the pseudoclassical block of the super period matrix. Then we can define $F_{S_AV^B}^*$ by

$$F_{\mathcal{S}_A \mathcal{V}^B} = [\mathrm{d} m_1 \mathrm{d} m_2 \mathrm{d} m_3] \cdot F_{\mathcal{S}_A \mathcal{V}^B}^*, \tag{9.7}$$

where $F_{S_A V^B}^*$ is a relative volume form along the fibers of $\pi: \mathfrak{M}_{2,0,2} \to \mathcal{M}_2$. The reason that pulling out the factor of $[\mathrm{d}m_1\mathrm{d}m_2\mathrm{d}m_3]$ changes the order of vanishing at u=0 is that the m_i have poles at u=0. With only three odd variables η_i , since the m_i are even, the residue of such a pole can only be a bilinear expression in the η_i , and the products of the residues of the poles in different m_i would vanish. Accordingly, we lose nothing essential if we assume that only m_1 has a pole, and that its residue is proportional to $\eta_1\eta_2$. We can pick coordinates so that when the η_i vanish, the m_i coincide with h_i , $i=1,\ldots,3$. Then the

general form of the pole is

$$m_1 = h_1 + \frac{\eta_1 \eta_2 g(h_1, \dots, h_5)}{u}$$
 (9.8)

for some function q. So we see that

$$dm_1 = dh_1 \left(1 - \frac{\eta_1 \eta_2 g}{u^2} \frac{\partial u}{\partial h_1} \right) + \dots,$$

$$(9.9)$$

where we write only terms proportional to dh_1 on the right hand side; we have also dropped a term $\partial_{h_1}g\eta_1\eta_2/u$, as it is less singular than the $\eta_1\eta_2/u^2$ term that we have kept. If in defining $F_{\mathcal{S}_A\mathcal{V}^B}^*$ in eqn. (9.7), we were to split off a factor of $[dh_1dh_2dh_3]$, then $F_{\mathcal{S}_A\mathcal{V}^B}^*$ would have the same behavior for $u \to 0$ as $F_{\mathcal{S}_A\mathcal{V}^B}$. However, instead we are supposed to split off a factor of $[dm_1dm_2dm_3] \sim [dh_1dh_2dh_3](1-\eta_1\eta_2g\partial_{h_1}u/u^2+\dots)$, and this gives

$$F_{\mathcal{S}_{A}\mathcal{V}^{B}}^{*} \sim \left(f_{0}(h_{i})\eta_{1}\eta_{2}\eta_{3} + f_{3}(h_{i})\eta_{3} \left(1 + \frac{\eta_{1}\eta_{2}g\partial_{h_{1}}u}{u^{2}} \right) + \dots \right) [\mathrm{d}h_{4}\mathrm{d}h_{5}|\eta_{1}\eta_{2}\eta_{3}], \tag{9.10}$$

where we omit some terms that do not affect the argument. Since $f_3 \sim u^5$, it follows that $F_{S_A V^B}^*$ has a contribution proportional to $u^3 \eta_1 \eta_2 \eta_3$. Since this term is proportional to the product of all three η 's, it can contribute to the surface integrals on the right hand side of eqn. (8.4). The effect of the poles of the super period matrix together with the use of incomplete PCO's has been to reduce by 1 the expected order of vanishing of $F_{S_A V^B}^*$ near u = 0: it vanishes as u^3 rather than u^4 .

With more odd variables, we can repeat this process. For every pair of odd variables, the order of vanishing of $F_{S_AV^B}^*$ along u=0 is reduced by 1 (or the order of a pole is increased by 1) compared to what one would expect from counting fermion zero-modes while ignoring the pole of the super period matrix.

9.4 Extension To Genus Three

Much less is known about superstring scattering amplitudes for g > 2. However, for g = 3, one can still define a meromorphic projection $\pi : \mathfrak{M}_{3,+} \to \mathcal{M}_{3,\mathrm{spin}+}$ using the super period matrix. This map behaves badly along hyperelliptic divisors in $\mathcal{M}_{3,\mathrm{spin}+}$ that are described in Appendix C, and this may ultimately prevent it from being very useful. In the following, we simply avoid these issues by assuming that Σ_{red} (which is kept fixed in the whole analysis) is not hyperelliptic.

The natural superstring measure on $\mathfrak{M}_{3,+}$ can be pushed forward to a measure on $\mathcal{M}_{3,\text{spin}+}$ by integrating over the fibers of π . Here we will show, generalizing the above arguments for g=2, that in the case of superstring theory in \mathbb{R}^{10} , the resulting measure on $\mathcal{M}_{3,\text{spin}+}$

vanishes when summed over spin structures, without any integration over bosonic moduli.

For g = 3, the condition on a pair of Ramond insertions such that $h^0(\mathcal{R}^{-1}) > 1$ was determined in section 5.6. In particular, this condition defines a divisor \mathfrak{D} in the space of all pairs $p, q \in \Sigma_{\text{red}}$.

It actually turns out that the counting of fermion zero-modes can be done without using explicit knowledge of \mathfrak{D} . For strings in \mathbb{R}^{10} , with the same configuration of spin fields as in section 9.1, ψ_1 is a section of \mathcal{R}^{-1} , $\widehat{\psi}_1$ is a section of $K \otimes \mathcal{R}$, ψ_i for i > 1 is a section of $\mathcal{R}^{-1} \otimes \mathcal{O}(-q)$, and $\widehat{\psi}_i$ for i > 1 is a section of $K \otimes \mathcal{R} \otimes \mathcal{O}(q)$. We have to count the number of zero-modes of these fermion fields.

The Riemann-Roch theorem and Serre duality tell us that $h^0(\mathcal{R}^{-1}) - h^0(\Sigma, K \otimes \mathcal{R}) = 1 - g + \deg \mathcal{R}^{-1} = 1$. Along the divisor \mathfrak{D} , $h^0(R^{-1})$ is at least 1, so $h^0(K \otimes \mathcal{R})$ is at least 1. Hence along \mathfrak{D} , the fields ψ_1 and $\widetilde{\psi}_1$ have together at least 2+1=3 zero-modes. For the fields ψ_i , i>1, we have to replace \mathcal{R}^{-1} by $\mathcal{R}^{-1}\otimes\mathcal{O}(-q)$. This replacement imposes 1 condition on a holomorphic section of \mathcal{R}^{-1} (it must vanish at q), so it reduces the number of zero-modes by at most 1 (and generically by 1). So along \mathfrak{D} , the number of ψ_i zero-modes for i>1 is always at least 1. For \mathcal{R}^{-1} of degree g=3, so that $\mathcal{R}^{-1}\otimes\mathcal{O}(-q)$ has degree 2, the Riemann-Roch theorem and Serre duality imply that $h^0(\mathcal{R}^{-1}\otimes\mathcal{O}(-q)) - h^0(K\otimes\mathcal{R}\otimes\mathcal{O}(q)) = 1 - g + 2 = 0$, or $h^0(\mathcal{R}^{-1}\otimes\mathcal{O}(-q)) = h^0(K\otimes\mathcal{R}\otimes\mathcal{O}(q))$, so that $h^0(K\otimes\mathcal{R}\otimes\mathcal{O}(q))$, which is the number of $\widehat{\psi}_i$ zero modes for j>1, is also at least 1.

In sum, along \mathfrak{D} , we always have for g=3 at least the same 11 fermion zero-modes that we found for g=2 (and generically the number is precisely 11). We can then use the same reasoning as in section 9.1 to analyze the behavior of $F_{\mathcal{S}_A\mathcal{V}^B}$ and $F_{\mathcal{S}_A\mathcal{V}^B}^*$ along \mathfrak{D} . The only difference is that the number of odd moduli of $\mathfrak{M}_{3,0,2}$ is 5 instead of 3. So to compute $F_{\mathcal{S}_A\mathcal{V}^B}$, we would use 5 PCO's instead of 3, with the result that $F_{\mathcal{S}_A\mathcal{V}^B}$ has a zero of order 3 along \mathfrak{D} , rather than the zero of order 4 that we found for genus 2. In going from $F_{\mathcal{S}_A\mathcal{V}^B}$ to $F_{\mathcal{S}_A\mathcal{V}^B}^*$, we can now lose two orders of vanishing (one for each pair of incomplete PCO's), so $F_{\mathcal{S}_A\mathcal{V}^B}^*$ has a simple zero along \mathfrak{D} .

As before, supersymmetric compactification to four dimensions reduces the number of zero-modes from 11 to 5. But now, with 5 PCO's, $F_{S_AV^B}$ need not vanish along \mathfrak{D} , and $F_{S_AV^B}^*$ may have a double pole. So it appears that in a general supersymmetric compactification to four dimensions, the bulk contribution to the genus 3 vacuum amplitude, defined using the projection $\mathfrak{M}_{3,+} \to \mathcal{M}_{3,\text{spin}+}$ derived from the super period matrix, does not necessarily vanish pointwise after summing over spin structures.

For any g, with precisely 2 Ramond punctures, Riemann-Roch and Serre duality can be

¹⁴Here and in what follows, the minimum values are also the generic values.

used in the same way just described, to show that if $h^0(\mathcal{R}^{-1}) \geq 2$, then the RNS fermions associated to strings in \mathbb{R}^{10} have at least 11 zero-modes. But for g > 3, this cannot be used in any obvious way to study the superstring measure. The most basic problem is that for g > 3, the super period matrix generically does not obey the Schottky relations that are satisfied by the period matrix of an ordinary Riemann surface, and consequently cannot be used to define a projection $\mathfrak{M}_{g,+} \to \mathcal{M}_{g,\text{spin}+}$. Also, as g increases, the odd dimension of $\mathfrak{M}_{g,0,2}$ increases. For $g \geq 4$, even if we did have a holomorphic projection $\mathfrak{M}_{g,+} \to \mathcal{M}_{g,\text{spin}+}$ with the properties that we have exploited, the above reasoning would allow a pole of $F_{\mathcal{S}_A\mathcal{V}^B}^*$ along \mathfrak{D} , leading to no simple conclusion from the identity (8.4).

A The Pole Of The Super Period Matrix For r = 1

For the case of two Ramond punctures, that is r=1, we have found in section 5.6 that the exceptional set along which $h^0(\mathcal{R}^{-1}) \geq 2$ is of codimension 1, and thus defines a divisor \mathfrak{D} in the reduced space of $\mathfrak{M}_{g,0,2}$. In section 9.3, it was important to know how the super period matrix $\widehat{\Omega}$ behaves along \mathfrak{D} . We claim that the pseudoclassical block $\widehat{\Omega}_{g\times g}$ of the super period matrix has a pole along \mathfrak{D} with nilpotent residue. To be more precise, the claim is that if u is a local parameter on $\mathfrak{M}_{g,0,2,\mathrm{red}}$ with a simple zero along \mathfrak{D} , and η_1,\ldots,η_s are the odd moduli, then $\widehat{\Omega}_{g\times g}$ has an expansion in which the leading term at $\eta=0$ is the classical period matrix Ω and the corrections are a series in $\eta_a\eta_b/u$:

$$\widehat{\Omega} = \Omega + \frac{w_{ab}^{(2)} \eta_a \eta_b}{u} + \frac{w_{abcd}^{(4)} \eta_a \eta_b \eta_c \eta_d}{u^2} + \dots$$
(A.1)

Here $w_{ab}^{(2)}$, $w_{abcd}^{(4)}$,... are functions on \mathfrak{D} , and less singular terms are omitted. To deduce this formula, we simply use the D'Hoker-Phong expansion of the super period matrix as a function of odd variables [2], slightly adapted to take into account the presence of Ramond punctures. For each pair of odd variables, the D'Hoker-Phong formula contains a fermion propagator (called S(z, z') below) that behaves as 1/u because of the presence of zero-modes at u = 0. This accounts for the form of the expansion in eqn. (A.1).

We will assume that the reader is familiar with the derivation of the D'Hoker-Phong expansion given in section 8.3 of [4]. We will essentially repeat this derivation, with some minor modifications to account for Ramond punctures.

Starting with a split super Riemann surface Σ , we want to give a smooth model for its deformations associated to odd moduli. In the absence of Ramond punctures, this is done as follows (eqn. (8.17) of [4]). Locally Σ can be described by holomorphic superconformal coordinates $z|\theta$ and a local antiholomorphic coordinate \widetilde{z} ; in the case that Σ is split, we can take \widetilde{z} to be the complex conjugate of z. A holomorphic function on Σ is a function

annihilated by $\partial_{\tilde{z}}$. To deform the complex structure of Σ , we replace $\partial_{\tilde{z}}$ by 15

$$\partial_{\widetilde{z}}' = \partial_{\widetilde{z}} + \chi_{\widetilde{z}}^{\theta} (\partial_{\theta} - \theta \partial_{z}), \tag{A.2}$$

where $\chi_{\widetilde{z}}^{\theta}$ is a (0,1)-form on Σ_{red} valued in $T^{1/2}$. (T and K will denote the holomorphic tangent and cotangent bundles of Σ_{red} .) The expression $\chi_{\widetilde{z}}(\partial_{\theta} - \theta \partial_{z})$ is an odd (0,1)-form on Σ valued in odd superconformal vector fields.

We need to find an analogous construction in the presence of Ramond punctures. The first step is to reinterpret $\chi_{\tilde{z}}^{\theta}$ as a (0,1)-form on Σ_{red} that is valued in \mathcal{R} , which we characterize as in eqn. (5.1) as a line bundle on Σ_{red} with an isomorphism $\mathcal{R}^2 \cong T(-p_1 - \cdots - p_{2r})$, where p_1, \ldots, p_{2r} are the locations of Ramond punctures. Equivalently, there is an isomorphism $\mathcal{R} \cong \mathcal{R}^{-1} \otimes T(-p_1 - \cdots - p_{2r})$. Yet another equivalent statement is that there is a homomorphism

$$\xi: \mathcal{R} \to \mathcal{R}^{-1} \otimes T \tag{A.3}$$

which maps a section s of \mathcal{R} to a section $\xi(s)$ of $\mathcal{R}^{-1} \otimes T$ that vanishes at p_1, \ldots, p_{2r} .

If s is a section of $\mathcal{R} \to \Sigma_{\rm red}$, then $s\partial_{\theta} - \xi(s)\theta\partial_{z}$ is an odd superconformal vector field on Σ . (If the superconformal structure of Σ is defined locally by $\varpi^{*} = \mathrm{d}z - z\theta\mathrm{d}\theta$, then $s\partial_{\theta} - \xi(s)\theta\partial_{z} = s(\partial_{\theta} - \theta z\partial_{z})$.) With s replaced by a (0,1)-form $\chi_{\overline{z}}^{\theta}$ valued in \mathcal{R} , we get the appropriate generalization of eqn. (A.2) in the presence of Ramond punctures:

$$\partial_{\widetilde{z}}' = \partial_{\widetilde{z}} + \chi_{\widetilde{z}}^{\theta} \partial_{\theta} - \xi(\chi_{\widetilde{z}}^{\theta}) \theta \partial_{z}. \tag{A.4}$$

We expand $\chi_{\tilde{z}}^{\theta}$ in a basis $f_{a\tilde{z}}^{\theta}$ of $H^1(\Sigma_{\text{red}}, \mathcal{R})$, with coefficients the odd moduli η_a :

$$\chi_{\widetilde{z}}^{\theta} = \sum_{a=1}^{h^1(\mathcal{R})} \eta_a f_{a\widetilde{z}}^{\theta}. \tag{A.5}$$

In this way, the odd parameters η_a , $a=1,\ldots,h^1(\mathcal{R})$ are used to deform the complex structure of Σ .

With this description of the perturbation that we are trying to make, it is actually straightforward to repeat the derivation in section 8.3 of [4]. One basically just needs to replace $\chi_{\tilde{z}}^{\theta}$ in some places by $\xi(\chi_{\tilde{z}}^{\theta})$.

A holomorphic 1-form b(z)dz on Σ_{red} corresponds to a section $\sigma = b(z)\theta[dz|d\theta]$ of $Ber(\Sigma)$. If Σ is split, the pseudoclassical block of its super period matrix coincides with the classical period matrix of Σ_{red} . To compute the dependence of the super period matrix on odd moduli,

¹⁵In a more complete treatment, we would include even deformations by adding to the right hand side an additional term $h_{\tilde{z}}^z \partial_z + \frac{1}{2} \partial_z h_{\tilde{z}}^z \theta \partial_\theta$, where $h_{\tilde{z}}^z$ is a (0,1)-form on $\Sigma_{\rm red}$ valued in T. This is not necessary for extracting the singular behavior claimed in eqn. (A.1).

we have to analyze how a section σ of $Ber(\Sigma)$ changes when the odd moduli are turned on.

By analogy with eqn. (8.19) of [4], a general section of $Ber(\Sigma)$ in the presence of the perturbation is

 $\widehat{\sigma} = \widehat{\phi}(\widetilde{z}; z | \theta) \left[dz + \xi(\chi_{\widetilde{z}}^{\theta}) \theta d\widetilde{z} | d\theta + \chi_{\widetilde{z}}^{\theta} d\widetilde{z} \right]. \tag{A.6}$

If $\widehat{\sigma}$ is understood as an integral form, then as in eqn. (8.22) of [4], the condition for $\widehat{\sigma}$ to be holomorphic is $0 = d\widehat{\sigma}$, where

$$d\widehat{\sigma} = -d\widetilde{z}dz\delta(d\theta) \left(\partial_{\widetilde{z}}\widehat{\phi} - \partial_{z}(\widehat{\phi}\,\xi(\chi_{\widetilde{z}}^{\theta})\theta) + \partial_{\theta}(\widehat{\phi}\chi_{\widetilde{z}}^{\theta}) \right). \tag{A.7}$$

We expand

$$\widehat{\phi}(\widetilde{z}; z|\theta) = \widehat{\alpha}(\widetilde{z}; z) + \theta \widehat{b}(\widetilde{z}; z). \tag{A.8}$$

For $\widehat{\phi}$ to be a section of $Ber(\Sigma)$, $\widehat{\alpha}$ should be a section of $K \otimes \mathcal{R} \cong \mathcal{R}^{-1}(-p_1 - \cdots - p_{2r})$, and \widehat{b} should be a section of K. The condition $d\widehat{\sigma} = 0$ becomes a pair of equations, generalizing eqn. (8.23) of [4]:

$$\partial_{\widetilde{z}}\widehat{\alpha} + \widehat{b}\chi_{\widetilde{z}}^{\theta} = 0$$

$$\partial_{\widetilde{z}}\widehat{b} - \partial_{z}\left(\widehat{\alpha}\,\xi(\chi_{\widetilde{z}}^{\theta})\right) = 0. \tag{A.9}$$

Define an ordinary 1-form on Σ_{red} :

$$\widehat{\rho} = \widehat{b} \, \mathrm{d}z + \widehat{\alpha} \, \xi(\chi_z^\theta) \, \mathrm{d}\widetilde{z}. \tag{A.10}$$

The second equation in (A.9) says that $d\hat{\rho} = 0$. The A- and B-periods of the closed holomorphic 1-form μ on Σ that corresponds to $\hat{\sigma}$ are simply the ordinary A- and B-periods of the ordinary 1-form $\hat{\rho}$. (The proof of this statement can be found in section 8.3 of [4] and is unaffected by the existence of Ramond punctures.) So to compute the super period matrix of Σ as a function of odd moduli, we simply have to compute the periods of $\hat{\rho}$.

To do this, we have to solve the equations (A.9) as a function of the odd parameters η_a . On considering the first of these equations, we immediately run into a problem. Generically, this equation has no solution. The obstruction lies in $H^1(\Sigma_{\rm red}, K \otimes \mathcal{R})$, which generically (in the presence of 2r Ramond punctures) is of dimension r. This obstruction reflects something that was explained in Appendix D.1 of [4], and that was important in section 5.1 above. In the presence of Ramond punctures, closed holomorphic 1-forms on Σ correspond to sections not of $Ber(\Sigma)$, but of $Ber'(\Sigma)$, where $Ber'(\Sigma)$ is the sheaf whose sections are sections of $Ber(\Sigma)$ that may have poles, with zero residue, along a Ramond divisor. For our purposes, this means that in solving eqn. (A.9), we should allow $\widehat{\alpha}$ to have simple poles at p_1, \ldots, p_{2r} . Thus, it is a section not of $K \otimes \mathcal{R}$ but of $K \otimes \mathcal{R}(p_1 + \cdots + p_{2r}) \cong \mathcal{R}^{-1}$.

With $\widehat{\alpha}$ understood in this way, the first of eqns. (A.9) can be solved, but the solu-

tion is not unique: it is unique only modulo the possibility of adding to $\widehat{\alpha}$ an element of $H^0(\Sigma_{\mathrm{red}}, \mathcal{R}^{-1})$, which generically is of dimension r. This non-uniqueness was to be expected; it reflects the fact that $Ber'(\Sigma)$ has both odd and even holomorphic sections, with the odd sections having the form $\widehat{\alpha}[\mathrm{d}z|\mathrm{d}\theta]$ for $\widehat{\alpha} \in H^0(\Sigma_{\mathrm{red}}, \mathcal{R}^{-1})$. When we deform a section σ of $Ber'(\Sigma)$ as a function of the odd moduli η_a , we are free to add to σ a linear combination of the odd sections with η_a -dependent coefficients.

The procedure in defining the super period matrix is to consider closed holomorphic 1-forms, or equivalently sections of $Ber'(\Sigma)$, that have specified A-periods. We start at $\eta_a = 0$ with holomorphic 1-forms $\mu_j = b_j(z) \mathrm{d}z$ such that, on the ordinary Riemann surface Σ_{red} , $\oint_{A_{\mathrm{red}}^i} \mu_j = \delta_j^i$. The section $\sigma_j = \theta b_j(z)[\mathrm{d}z|\mathrm{d}\theta]$ of $Ber'(\Sigma)$ then automatically has vanishing fermionic A- and B-periods. We deform σ_j as a function of the odd moduli η_a to get a section $\widehat{\sigma}_j$ of $Ber'(\Sigma)$ such that $\oint_{A^i} \widehat{\sigma}_j = \delta_j^i$, and $\widehat{\sigma}_j$ has vanishing fermionic A-periods. As in the classical theory, it is not possible to constrain the bosonic or fermionic B-periods of $\widehat{\sigma}_j$; these make up the $g \times g$ and $g \times r$ blocks of the super period matrix.

In particular, the vanishing of the fermionic A-periods of $\widehat{\sigma}_j$ will make the solution for $\widehat{\alpha}$ unique. This condition means that $\widehat{\alpha}$ is not an arbitrary section of \mathcal{R}^{-1} but (in the notation of eqn. (4.4)) obeys

$$\widehat{\alpha}(w_{2\zeta-1}) + \sqrt{-1}\,\widehat{\alpha}(w_{2\zeta}) = 0, \quad \zeta = 1, \dots, r. \tag{A.11}$$

Away from a bad locus in the reduced space of $\mathfrak{M}_{g,0,2r}$ on which the super period matrix develops a singularity, there is no holomorphic section of \mathcal{R}^{-1} that satisfies (A.11), but there is such a section if we allow a pole at some point $z' \in \Sigma_{\text{red}} \setminus \{p_1, \ldots, p_{2r}\}$. This means that there is a unique solution S(z, z') of the equation

$$\partial_{\tilde{z}}S(z,z') = 2\pi\delta^2(z,z'). \tag{A.12}$$

(The delta function is defined by $\int d^2z \, \delta^2(z, z') = 1$, where $d^2z = -id\tilde{z} \, dz$.) S plays the role in the presence of Ramond punctures that the ordinary Dirac propagator plays in their absence. We will describe it more precisely momentarily. We can express $\hat{\alpha}$ in terms of \hat{b} :

$$\widehat{\alpha}(\widetilde{z};z) = -\frac{1}{2\pi} \int_{\Sigma'_{\text{red}}} S(z,z') \chi^{\theta}_{\widetilde{z}'}(\widetilde{z}';z'), \widehat{b}(\widetilde{z}';z') d^2 z'. \tag{A.13}$$

And hence we can write an equation for \hat{b} only:

$$\partial_{\widetilde{z}}\widehat{b}(\widetilde{z};z) = -\frac{1}{2\pi}\partial_{z}\int_{\Sigma'_{red}} \xi(\chi^{\theta}_{\widetilde{z}}(\widetilde{z};z))S(z,z')\chi^{\theta}_{\widetilde{z}'}(\widetilde{z}';z')\,\widehat{b}(\widetilde{z}';z')\mathrm{d}^{2}z' \tag{A.14}$$

However, we should explain better what sort of geometric object is S(z,z'). In its de-

pendence on z, it is a section of \mathcal{R}^{-1} that is constrained by eqn. (A.11), but what about its dependence on z'? Since the answer is a little tricky and involves a slightly exotic use of duality, we first give an example. We take Σ to be the complex z-plane with Ramond divisors at z = 0 and $z = \infty$, as in eqn. (5.8). If as in that discussion we write $d\theta$ for a section of \mathcal{R}^{-1} that has fermionic periods 1 and $\sqrt{-1}$ at z = 0 and $z = \infty$, then

$$S(z, z') = d\theta \boxtimes d\theta' \frac{z + z'}{2(z - z')}.$$
(A.15)

(The symbol \boxtimes just denotes a tensor product of forms on two different factors of Σ , one parametrized by $z|\theta$ and one by $z'|\theta'$.) The point of this formula is that the function (z+z')/2(z-z') equals $-\frac{1}{2}$ or $+\frac{1}{2}$ at z=0 or $z=\infty$, and hence S(z,z') has fermionic periods $-\frac{1}{2}$ and $+\frac{1}{2}\sqrt{-1}$ at z=0 and $z=\infty$. The relative minus sign means that while $d\theta$ has a vanishing fermionic B-period, S(z,z') has a vanishing fermionic A-period in its dependence on z, as desired.

We note from eqn. (A.15) that S(z,z') is odd under $z\leftrightarrow z'$, and in particular that it is the same sort of geometric object in each variable. This is what we want to explain in general. We will use a physical language. If we are given a pair of fermi fields ψ , $\widetilde{\psi}$ that are sections of Serre dual line bundles \mathcal{L} and $K\otimes\mathcal{L}^{-1}$, where \mathcal{L} has degree g-1 and $h^0(\mathcal{L})=0$, then we would have a Dirac action $I=\frac{1}{2\pi}\int\psi\partial_{\widetilde{z}}\widetilde{\psi}$, with a fermion propagator S(z,z') obeying eqn. (A.12). It would be a section of $\mathcal{L}\boxtimes K\otimes\mathcal{L}^{-1}\to\Sigma\times\Sigma$ (with a simple pole on the diagonal of residue 1). In our case, S(z,z') as a function of z is a section of the line bundle \mathcal{R}^{-1} of degree g-1+r, but it obeys the r constraints (A.11). To write an action describing this situation, we take ψ and $\widetilde{\psi}$ to be sections of \mathcal{R}^{-1} and $K\otimes\mathcal{R}$, respectively, but with Lagrange multipliers that enforce the constraints:

$$I = \frac{1}{2\pi} \int \psi \partial_{\widetilde{z}} \widetilde{\psi} + \sum_{\zeta=1}^{r} c_{\zeta} \left(\psi(p_{2\zeta-1}) + \sqrt{-1} \psi(p_{2\zeta}) \right). \tag{A.16}$$

Varying with respect to ψ , the equation of motion for $\widetilde{\psi}$ is

$$\frac{1}{2\pi}\partial_{\widetilde{z}}\widetilde{\psi}(z) = \sum_{\zeta=1}^{r} c_{\zeta} \left(\delta^{2}(z, z_{2\zeta-1}) + \sqrt{-1}\delta^{2}(z, z_{2\zeta})\right). \tag{A.17}$$

Thus, $\widetilde{\psi}$ has poles at p_1, \ldots, p_{2r} , and we can view it as a section of $K \otimes \mathcal{R}(p_1 + \cdots + p_{2r}) \cong \mathcal{R}^{-1}$. But the 2r poles are not independent; their residues are determined by the r Lagrange multipliers c_{ζ} . The resulting relations between the residues mean precisely that as a section of \mathcal{R}^{-1} , $\widetilde{\psi}$ obeys the constraints (A.11) and thus that it is the same sort of geometric object as ψ . Thus we can view S(z, z') in eqn. (A.12) as a section of $\mathcal{R}^{-1} \boxtimes \mathcal{R}^{-1}$ that obeys the conditions (A.11) in each variable and has a simple pole on the diagonal. S(z', z) obeys all

of the same conditions, but with an opposite residue for the pole on the diagonal, so

$$S(z', z) = -S(z, z').$$
 (A.18)

This is important in the symmetry of the formula (A.27) below for the first correction to the super period matrix.

However, if we want to view S(z, z') as a section of $\mathcal{R}^{-1} \boxtimes \mathcal{R}^{-1}$ rather than of $\mathcal{R}^{-1} \boxtimes K \otimes \mathcal{R}(p_1 + \cdots + p_{2r})$, we have to replace $\chi_{\tilde{z}'}^{\theta}$ in eqns. (A.13) and (A.14) with $\xi(\chi_{\tilde{z}'}^{\theta})$, to give

$$\widehat{\alpha}(\widetilde{z};z) = -\frac{1}{2\pi} \int_{\Sigma'_{\text{red}}} S(z,z') \xi(\chi^{\theta}_{\widetilde{z}'}(\widetilde{z}';z')) \,\widehat{b}(\widetilde{z}';z') d^2 z' \tag{A.19}$$

and

$$\partial_{\widetilde{z}}\widehat{b}(\widetilde{z};z) = -\frac{1}{2\pi}\partial_{z}\int_{\Sigma'_{z,z}} \xi(\chi^{\theta}_{\widetilde{z}}(\widetilde{z};z))S(z,z')\xi(\chi^{\theta}_{\widetilde{z}'}(\widetilde{z}';z'))\,\widehat{b}(\widetilde{z}';z')\mathrm{d}^{2}z'. \tag{A.20}$$

Starting with a choice of \hat{b} at $\eta_a = 0$, eqn. (A.20) can be solved iteratively to determine \hat{b} as a function of the η_a . To make the solution unique, we need a condition on the A-periods of the ordinary 1-form $\hat{\rho}$. To find the right condition, recall that to compute the super period matrix, we start with holomorphic 1-forms $\mu_j = b_j(z) dz$ on Σ_{red} with canonical A-periods, $\oint_{A_{\text{red}}} b_j(z) dz = \delta_j^i$. We promote them to η_a -dependent holomorphic sections $\hat{\sigma}_j$ on Σ , such that, with $\hat{\rho}_j$ defined by eqns. (A.6), (A.8) and (A.10),

$$\oint_{A_{\text{red}}^i} \widehat{\rho}_j = \delta_j^i. \tag{A.21}$$

This condition on bosonic A-periods makes the solution of eqn. (A.20) unique.

The final computation of the super period matrix $\widehat{\Omega}$ proceeds rather as in section 8.3 of [4]. The $g \times g$ block of $\widehat{\Omega}$ is defined as

$$\widehat{\Omega}_{ij} = \oint_{B_j} \widehat{\sigma}_i = \oint_{B_{j,\text{red}}} \widehat{\rho}_i. \tag{A.22}$$

The difference between $\widehat{\Omega}_{ij}$ and the classical period matrix Ω_{ij} is

$$\widehat{\Omega}_{ij} - \Omega_{ij} = \oint_{B_{j,\text{red}}} (\widehat{\rho}_i - \mu_i) = \oint_{B_{j,\text{red}}} \widehat{\rho}'_i, \tag{A.23}$$

where $\hat{\rho}_i' = \hat{\rho}_i - \mu_i$. Riemann's bilinear relations say that if κ, λ are closed 1-forms on the

ordinary Riemann surface $\Sigma_{\rm red}$, then

$$\int_{\Sigma_{\text{red}}} \kappa \wedge \lambda = \sum_{i} \left(\oint_{A_{\text{red}}^{i}} \kappa \oint_{B_{i,\text{red}}} \lambda - \oint_{B_{i,\text{red}}} \lambda \oint_{A_{\text{red}}^{i}} \kappa \right). \tag{A.24}$$

Taking $\kappa = \mu_j = b_j(z) dz$, $\lambda = \widehat{\rho}_i$, and remembering that μ_j is of type (1,0) on the ordinary Riemann surface $\Sigma_{\rm red}$, so that the (1,0) part of $\widehat{\rho}_i$ does not contribute to $\mu_j \wedge \widehat{\rho}_i$, we learn that

$$\widehat{\Omega}_{ij} - \Omega_{ij} = \int_{\Sigma_{\text{red}}} \mu_j \wedge \widehat{\rho}'_i = \int_{\Sigma_{\text{red}}} \mu_j \wedge \widehat{\alpha}_i \xi(\chi_{\widetilde{z}}^{\theta}) \, d\widetilde{z}.$$
(A.25)

In turn, we can use (A.19) to eliminate $\widehat{\alpha}_i$:

$$\widehat{\Omega}_{ij} - \Omega_{ij} = -\frac{1}{2\pi} \int_{\Sigma_{\text{red}} \times \Sigma'_{\text{red}}} \mu_j(z) \xi(\chi_{\widetilde{z}}^{\theta}(\widetilde{z}; z)) d\widetilde{z} S(z, z') \xi(\chi_{\widetilde{z}}^{\theta}(\widetilde{z}'; z')) \widehat{b}_i(\widetilde{z}'; z') d^2 z'.$$
 (A.26)

When $\widehat{\Omega}_{ij} - \Omega_{ij}$ is expanded in powers of the η_a 's, the lowest order term, which we call $\widehat{\Omega}_{ij}^{(2)}$, is quadratic. It can found by just replacing $\widehat{b}_i(\widetilde{z}';z')$ in the last formula by b_i and writing $b_i d^2 z = -i d\widetilde{z} \mu_i$:

$$\widehat{\Omega}_{ij}^{(2)} = \frac{i}{2\pi} \int_{\Sigma_{\text{red}} \times \Sigma'_{\text{red}}} \mu_j(z) \xi(\chi_{\widetilde{z}}^{\theta}(\widetilde{z}; z)) d\widetilde{z} S(z, z') \xi(\chi_{\widetilde{z}'}^{\theta}(z')) d\widetilde{z}' \, \mu_i(z'). \tag{A.27}$$

This is the analog of the D'Hoker-Phong formula in the absence of Ramond punctures. Higher order terms can be evaluated by using (A.20) to express \hat{b}_i as a polynomial in the η 's.

Now we can explain the form of the expansion in eqn. (A.1). The kernel S(z, z') develops a pole as a function of the moduli of Σ when the $\partial_{\tilde{z}}$ operator, acting on sections of \mathcal{R}^{-1} that satisfy eqn. (A.11), has a non-trivial kernel. This is the bad set discussed in section 5.6. If u is a local parameter with a simple zero on the bad set, then $S(z, z') \sim 1/u$ near u = 0. When $\widehat{\Omega}_{ij} - \Omega_{ij}$ is evaluated by solving eqn. (A.20) iteratively for b and substituting in eqn. (A.26), each pair of odd moduli is accompanied by a factor of S(z, z'). This leads to the behavior claimed in eqn. (A.1).

The meaning of the poles at u=0 is, however, quite different for r=1 or r>1. For r=1, the choice of fermionic A-periods is essentially unique, as explained in section 5.4; the condition $h^0(\mathcal{R}^{-1}) > r$ defines a divisor \mathfrak{D} ; and the poles of $\widehat{\Omega}_{ij}$ along this divisor do not depend on any arbitrary choices. By contrast, for r>1, the definition of $\widehat{\Omega}_{ij}$ does depend on a somewhat arbitrary choice of fermionic A-periods, and $\widehat{\Omega}_{ij}$ develops poles along a divisor on which this choice breaks down. These are the poles described in the above formula. The condition $h^0(\mathcal{R}^{-1}) > r$ for r>1 defines a set of complex codimension greater than 1, and is not the locus of poles in $\widehat{\Omega}_{ij}$.

B A Note On The Behavior At Infinity In Genus Two

As explained in the introduction, in general the procedure of D'Hoker and Phong to compute the two-loop vacuum amplitude must be supplemented by a correction at infinity.

The explanation given in [9] for the need for this correction was the following. The correction is associated to the splitting of a genus 2 super Riemann surface Σ with even spin structure to two genus 1 components each with even spin structure. So consider a partial compactification¹⁶ in which one allows this type of degeneration, thus adding a divisor \mathfrak{T} to $\mathfrak{M}_{2,+}$. Although the holomorphic splitting $\pi:\mathfrak{M}_{2,+}\to\mathcal{M}_{2,\mathrm{spin}+}$ used by D'Hoker and Phong does extend to a holomorphic map between the corresponding partially compactified spaces, the extended map does not restrict to a splitting of the divisor \mathfrak{T} . This leads to a slight mismatch between what one gets by first integrating over the fibers of π and the general formalism of superstring perturbation theory.

We will here give an alternative explanation in the language of picture-changing operators rather than super Riemann surface theory. In Appendix B.1, we recall some elementary facts about Riemann surfaces of genus 1. Then in Appendix B.2, we recall what the D'Hoker-Phong procedure means when expressed in terms of picture-changing operators (PCO's) and analyze how this procedure behaves near the relevant separating degeneration.

B.1 Spin Structures In Genus 1

Let Σ_1 be a Riemann surface of genus 1. Picking a point $p \in \Sigma_1$ as the "origin," Σ_1 becomes an elliptic curve. Although the canonical bundle K of Σ_1 is trivial, a square root $K^{1/2}$ of K may not be trivial. An even spin structure on Σ_1 corresponds to the case that $K^{1/2}$ is a non-trivial line bundle of order 2, namely $K^{1/2} = \mathcal{O}(-p+q)$, where q is one of the three non-zero points of order 2 on Σ_1 .

The Dirac propagator S(y,z) is a section of $K^{1/2} \boxtimes K^{1/2} \to \Sigma_1 \times \Sigma_1$ with a simple pole of residue 1 on the diagonal. We will need to understand the propagator for the case that y=p. S(p,z) is, as a function of z, a nonvanishing section of $K^{1/2}=\mathcal{O}(-p+q)$ that has a simple pole at p (and no other singularities). Differently put, it is a nonzero section of $\mathcal{O}(q)$. But a nonzero section of $\mathcal{O}(q)$ vanishes at q and nowhere else. Thus, S(p,z) vanishes precisely at z=q.

¹⁶We introduce this partial compactification because π does have a pole at infinity if one tries to extend it over the full Deligne-Mumford compactification of $\mathfrak{M}_{2,+}$. The pole arises on the degeneration in which Σ decomposes to two genus 1 components each with odd spin structure. Because of fermion zero-modes, this pole in π does not lead to any correction to the D'Hoker-Phong procedure for analyzing the vacuum amplitude.

Suppose that Σ_1 is the reduced space of a super Riemann surface Σ with one NS puncture. In the picture-changing formalism, we represent an NS puncture on Σ as a point in Σ_1 , which because of the translation symmetries of Σ_1 we may as well take to be p, and we represent the odd modulus of Σ by including a PCO at some point $u \in \Sigma$. The only constraint on u is that to avoid a "spurious singularity," we want $H^0(\Sigma_1, T^{1/2}(-p) \otimes \mathcal{O}(u)) = 0$. (Here $T^{1/2}(-p)$ is the sheaf of odd superconformal vector fields that vanish at p, and the condition to avoid a spurious singularity is that this sheaf has no section whose only singularity is a simple pole at the position u of the PCO.) But $T^{1/2}(-p) \cong \mathcal{O}(-q)$ so we want $H^0(\Sigma_1, \mathcal{O}(-q+u)) = 0$. This is true if and only if $u \neq q$. In other words, we may place the PCO anywhere except at q without running into a spurious singularity.

B.2 The D'Hoker-Phong Procedure

Now let Σ_2 be a Riemann surface of genus 2 with even spin structure. We view it as the reduced space of a genus 2 super Riemann surface Σ . A family of Σ 's depending on a full set of odd parameters – namey 2q-2=2 of them – can be constructed by inserting PCO's at two points $u, v \in \Sigma_2$. In this language, the D'Hoker-Phong procedure amounts to the following: one should choose the points u, v so that the Dirac propagator connecting them vanishes: S(u,v)=0. (To obey S(u,v)=0, we may pick any u and for given u there are then two choices of v.) The D'Hoker-Phong formula for the dependence of the super period matrix on odd moduli (see [2] and also [4], section 8.3, or eqn. (A.27) above) shows that we get a family of super Riemann surfaces whose super period matrix does not depend on the odd moduli if we include odd moduli by placing PCO's at any points u, v satisfying S(u, v) = 0, and avoiding spurious singularities. (Generic points satisfying S(u,v)=0 do avoid spurious singularities.) This means in particular that as long as we require S(u,v)=0 and avoid spurious singularities, the PCO formalism will give an answer that does not depend on the specific choice of u and v. This is the D'Hoker-Phong procedure expressed in terms of PCO's. Following this procedure will certainly give a unique, globally-defined answer. There cannot be any global obstruction to following this procedure, because u and v do not have to vary continuously; we can use different pairs u, v in different regions of moduli space.

Now consider what happens when Σ_2 degenerates to a pair of genus 1 components Σ_1 and Σ'_1 , each with even spin structure. Σ_2 is constructed by gluing a point $p \in \Sigma_1$ to a point $p' \in \Sigma'_1$ (fig. 4). The spin bundle of Σ_1 is $\mathcal{O}(-p+q)$ for some $q \in \Sigma_1$, and the spin bundle of Σ'_1 is $\mathcal{O}(-p'+q')$ for some $q' \in \Sigma'_1$.

The general formalism of superstring perturbation theory, when expressed in terms of PCO's, tells us that in the limit that Σ_2 degenerates as described in the last paragraph, we must ensure that the PCO's are in opposite branches, say $u \in \Sigma_1$, $v \in \Sigma'_1$. (For example, see section 6.3.6 of [16].) Moreover, to avoid a spurious singularity, we need $u \neq q$ and $v \neq q'$.

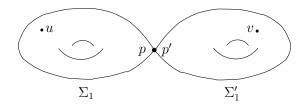


Figure 4: A singular genus 2 surface is made by gluing $p \in \Sigma_1$ to $p' \in \Sigma'_1$. PCO's are inserted at $u \in \Sigma_1$ and $v \in \Sigma'_1$.

We will now see that there is a conflict between the D'Hoker-Phong procedure and the conditions stated in the last paragraph. That is why the D'Hoker-Phong procedure requires a correction at infinity.

If Σ_2 is a singular surface made by gluing together Σ_1 and Σ'_1 , then the Dirac propagator S(u,v) is identically 0 for $u \in \Sigma_1$, $v \in \Sigma'_1$. However, we really want to know what happens if we impose the D'Hoker-Phong condition S(u,v)=0 away from the degeneration limit, and then let Σ_2 degenerate. If we deform a singular surface Σ_2 made by gluing at p and p' into a smooth surface with a small deformation parameter ε , then to lowest order in ε , the propagator S(u,v) with u and v on opposite branches is

$$S(u, v) \sim \varepsilon S(u, p) S(p', v) + \mathcal{O}(\varepsilon^2).$$
 (B.1)

(For example, see eqn. (3.49) of [9].)

As long as Σ_2 is smooth, there is no difficulty in ensuring that S(u,v)=0 and avoiding spurious singularities. Now let us consider when happens as $\varepsilon \to 0$. Trying to avoid a spurious singularity, we take u to not approach the point $q \in \Sigma_1$ as $\varepsilon \to 0$. But then as we know from Appendix B.1, $S(u,p) \not\to 0$ for $\varepsilon \to 0$. Eqn. (B.1) then implies that to make S(u,v) identically 0 for all ε , we will have to have $v \to q'$ for $\varepsilon \to 0$ in order to make S(p',v) vanish. (Otherwise, S(u,v) is of order ε for small ε and cannot vanish identically.) But this means that for $\varepsilon \to 0$, we will meet a spurious singularity after all.

In short, if we follow the D'Hoker-Phong procedure, then the divisor in which Σ splits to a pair of genus 1 components with even spin structure is a locus of spurious singularities. That is why in general this procedure needs to be supplemented by adding a correction term supported on this divisor.

In this analysis, since we were not able to avoid a spurious singularity at infinity, we did not gain much by placing u and v on opposite branches of Σ_2 in the degeneration limit. We could just as well have chosen u and v to be both in, say, Σ_1 , in the limit that Σ_2 degenerates. For this, we could take u to be a generic point in Σ_1 and then pick $v \in \Sigma_1$ such

that S(u, v) = 0. The correction at infinity can be computed by comparing this procedure to the general formalism of superstring perturbation theory. This can possibly be the basis for justifying the procedure of [18].

C The Hyperelliptic Locus In Genus Three

C.1 The Purely Bosonic Case

A hyperelliptic Riemann surface Σ_3 of genus 3 can be described in affine coordinates by an equation

$$y^2 = \prod_{a=1}^{8} (x - e_a). \tag{C.1}$$

Thus Σ_3 is a double cover of \mathbb{CP}^1 (parametrized by x with a point at infinity added) with 8 branch points e_1, \ldots, e_8 . The hyperelliptic involution acts by $\tau : y \to -y$, and acts as -1 on the three linearly independent holomorphic differentials

$$\omega_t = \frac{\mathrm{d}x}{y} x^t, \quad t = 0, 1, 2. \tag{C.2}$$

Dually, τ acts as -1 on $H_1(\Sigma_3, \mathbb{Z})$ and hence on all A- and B-periods. Accordingly the 3×3 period matrix is even under τ .

Modulo the action of $SL(2,\mathbb{C})$ on \mathbb{CP}^1 , this family of genus 3 hyperelliptic curves depends on 5 complex parameters (the action of $SL(2,\mathbb{C})$ can be used to fix 3 of the 8 branch points e_a). These 5 moduli are clearly τ -invariant. A genus 3 Riemann surface has altogether 3g-3=6 moduli, so there is 1 additional modulus ε that is odd under τ . We can confirm this by examining the quadratic differentials, which are dual to the infinitesimal deformations of Σ_3 . There are 5 even ones

$$\left(\frac{\mathrm{d}x}{y}\right)^2 x^t, \quad 0 \le t \le 4,\tag{C.3}$$

and 1 odd one

$$\frac{(\mathrm{d}x)^2}{y}.\tag{C.4}$$

A genus 3 Riemann surface is parametrized locally by its 3×3 symmetric period matrix Ω_{ij} . Since hyperelliptic curves of genus 3 are a five-parameter family, they span a codimension 1 subspace of period matrices. Let q be a holomorphic function of the matrix elements Ω_{ij} with a simple zero along the subspace that parametrizes hyperelliptic Riemann surfaces.

Since all matrix elements of Ω_{ij} are τ -invariant, q in particular is an even function of ε , and we can normalize q and/or ε so that the relation between them is simply

$$q = \varepsilon^2$$
. (C.5)

C.2 Spin Structures

Now suppose that Σ_3 is the reduced space $\Sigma_{\rm red}$ of a split super Riemann surface Σ , and in particular is endowed with a spin structure. A genus 3 surface has $\frac{1}{2}(2^{2g}-2^g)=28$ odd spin structures and $\frac{1}{2}(2^{2g}+2^g)=36$ even ones. A spin structure is a line bundle \mathcal{L} with an isomorphism $\psi:\mathcal{L}\otimes\mathcal{L}\cong K$. Such a line bundle can be characterized by specifying $\psi(s\otimes s)$ for some meromorphic section s of \mathcal{L} . Here $\psi(s\otimes s)$ will have precisely the same zeroes and poles as s, but with twice the multiplicity, so in particular it has zeroes and poles of even multiplicity only. Often we write $K^{1/2}$ for \mathcal{L} .

For example, to define an odd spin structure, we pick a pair of branch points e_a , e_b , and characterize \mathcal{L} by saying that it has a section s with

$$\psi(s \otimes s) = \frac{\mathrm{d}x}{y}(x - e_a)(x - e_b). \tag{C.6}$$

The differential $(dx/y)(x-e_a)(x-e_b)$ has double zeroes at $x=e_a$, y=0 and at $x=e_b$, y=0, and no other zeroes or poles. So s has simple zeroes at those two points and no poles. In particular, s is holomorphic. We can describe this more intuitively by writing

$$s = \sqrt{\frac{\mathrm{d}x}{y}(x - e_a)(x - e_b)}.$$
 (C.7)

Up to a constant multiple, s is the only holomorphic section of \mathcal{L} . So $h^0(\mathcal{L}) = 1$ and \mathcal{L} is an odd spin structure. The 28 odd spin structures on Σ_{red} are all obtained by this construction, with the $8 \cdot 7/2 = 28$ possible choices of the pair e_a , e_b . Obviously, these 28 choices are permuted transitively by permutations of the branch points e_a . So in the moduli space $\mathcal{M}_{3,\text{spin-}}$ that parametrizes a genus 3 surface with an odd spin structure, the hyperelliptic surfaces form an irreducible divisor \mathfrak{D}_{-} .

By contrast, there are two essentially different types of even spin structure on $\Sigma_{\rm red}$, corresponding to whether $h^0(\mathcal{L})$ equals 0 or 2. There is only 1 spin structure with $h^0(\mathcal{L}) = 2$. The space $H^0(\Sigma_3, \mathcal{L})$ is spanned by sections s, s' that obey $\psi(s \otimes s) = \mathrm{d}x/y, s' = xs$ (s has simple zeroes at two points in $\Sigma_{\rm red}$ lying above $x = \infty$ and s' has simple zeroes at two points

lying above x = 0). More informally, we write

$$s = \sqrt{\frac{\mathrm{d}x}{y}}.\tag{C.8}$$

Since there is just one spin structure with $h^0(\mathcal{L}) = 2$, genus 3 surfaces with such a spin structure are parametrized by an irreducible divisor $\mathfrak{D} \subset \mathcal{M}_{3,\text{spin}+}$.

An example of an even spin structure with $h^0(\mathcal{L}) = 0$ is given by assuming that \mathcal{L} has a meromorphic section s with

$$\psi(s \otimes s) = \frac{\mathrm{d}x}{y} \frac{(x - e_1)(x - e_2)(x - e_3)}{(x - e_4)}.$$
 (C.9)

Informally,

$$s = \sqrt{\frac{\mathrm{d}x}{y} \frac{(x - e_1)(x - e_2)(x - e_3)}{(x - e_4)}}.$$
 (C.10)

Such an \mathcal{L} has no holomorphic section. How many choices are there of such \mathcal{L} 's? We get an isomorphic line bundle if we exchange e_4 with, say, e_3 , since this can be accomplished by replacing s by $s' = s(x - e_4)/(x - e_3)$:

$$\psi(s' \otimes s') = \frac{\mathrm{d}x}{y} \frac{(x - e_1)(x - e_2)(x - e_4)}{x - e_3}.$$
 (C.11)

Likewise, we do not get anything new if we replace e_1, e_2, e_3, e_4 by e_5, e_6, e_7, e_8 , since this exchange is equivalent to replacing s by $s'' = s(x - e_5)(x - e_6)(x - e_7)(x - e_4)/y$. So the choice of \mathcal{L} depends only on the partition of the set $\{e_1, \ldots, e_8\}$ as the union of two subsets $\{e_1, \ldots, e_4\}$ and $\{e_5, \ldots, e_8\}$, each with 4 elements. There are 35 such partitions, and thus 35 even spin structures with $h^0(\mathcal{L}) = 0$. They are obviously permuted transitively by permutations of the e_a . So genus 3 surfaces with an even spin structure of this type are parametrized by an irreducible divisor $\mathfrak{D}' \subset \mathcal{M}_{3,\text{spin}+}$.

C.3 Period Matrix Near \mathfrak{D} and Near \mathfrak{D}'

As described in section 6, mapping a super Riemann surface to the Riemann surface with the same period matrix gives a meromorphic projection $\pi: \mathfrak{M}_{3,+} \to \mathcal{M}_{3,\mathrm{spin}+}$. It is fairly obvious that π has a pole along \mathfrak{D} , since the super period matrix has a pole there. It is less obvious that π has a pole along \mathfrak{D}' ; this was essentially explained to the author by P. Deligne. In the remainder of this appendix, we have two goals. The first is to describe the origin of the pole of π along \mathfrak{D}' . The second goal involves the following application to

superstring perturbation theory. Let Υ be the holomorphic measure on $\mathfrak{M}_{3,+}$ determined by superstring theory on \mathbb{R}^{10} . Integration over odd moduli of Σ keeping fixing its super period matrix $\widehat{\Omega}$ generates a natural measure $\pi_*(\Upsilon)$ on $\mathcal{M}_{3,\text{spin}+}$. Our second goal is to describe the behavior of $\pi_*(\Upsilon)$ along \mathfrak{D} and along \mathfrak{D}' .

We start by considering \mathfrak{D} . We will need to understand how the odd moduli of Σ transform under the hyperelliptic "involution" τ . We have put the words "involution" in quotes because in acting on fermions, $\tau^2 = -1$, not +1. This is clear from the action of τ on $s = \sqrt{\mathrm{d}x/y}$. Since τ acts by $x, y \to x, -y$, it multiplies $\sqrt{\mathrm{d}x/y}$ by $\pm \sqrt{-1}$, where the choice of sign is arbitrary: it is up to us with which sign we want to take τ to act on the spin bundle. Let us make a choice of $\sqrt{-1}$ and declare that $\tau s = \sqrt{-1}s$. Having made this choice, it is now meaningful to ask how τ acts on the odd moduli of Σ .

It is convenient to write $K^{1/2}$ for what we have called \mathcal{L} in Appendix C.2 and $T^{1/2}$ for \mathcal{L}^{-1} . We want to know how τ acts on $H^1(\Sigma_{\rm red}, T^{1/2})$, which parametrizes the odd moduli along the split locus. It is slightly more convenient to determine the action of τ on the dual space $H^0(\Sigma_{\rm red}, K^{3/2})$.

Along \mathfrak{D} , the four sections of $K^{3/2}$ are $(\mathrm{d}x/y)^{3/2}x^t$, $t=0,\ldots,3$. Differently put, a general element of $H^0(\Sigma,K^{3/2})$ is $s^3P_3(x)$ where $s=\sqrt{\mathrm{d}x/y}\in H^0(\Sigma_{\mathrm{red}},K^{1/2})$ and $P_3(x)$ is a cubic polynomial. τ acts as $-\sqrt{-1}$ on $s^3P_3(x)$, so dually it acts as $+\sqrt{-1}$ on $H^1(\Sigma_{\mathrm{red}},T^{1/2})$. In other words, Σ has four odd moduli α_i , $i=1,\ldots,4$ all transforming as $\sqrt{-1}$ under τ .

Along \mathfrak{D}' , the four sections of $K^{3/2}$ are

$$\frac{\mathrm{d}x}{y}\sqrt{\frac{\mathrm{d}x}{y}(x-e_1)(x-e_2)(x-e_3)(x-e_4)} \cdot x^t, \quad t = 0, 1,$$
(C.12)

transforming under τ as $-\sqrt{-1}$, and

$$\frac{\mathrm{d}x}{y}\sqrt{\frac{\mathrm{d}x}{y}(x-e_1)(x-e_2)(x-e_3)(x-e_4)} \cdot \frac{y}{(x-e_1)(x-e_2)(x-e_3)(x-e_4)} \cdot x^t, \quad t = 0, 1,$$
(C.13)

transforming as $+\sqrt{-1}$. Here $\sqrt{\frac{\mathrm{d}x}{y}}(x-e_1)(x-e_2)(x-e_3)(x-e_4)$ is an abbreviation for $s(x-e_4)$, where s was characterized in eqns. (C.9) and (C.10) and is assumed to transform as $\sqrt{-1}$. (Note that $s(x-e_4)$ has simple zeroes at e_1,\ldots,e_4 , while $y/(x-e_1)(x-e_2)(x-e_3)(x-e_4)$ has simple poles at those points, so that the two sections listed in eqn. (C.13) are regular and nonzero at e_1,\ldots,e_4 , while the sections in (C.12) have simple zeroes there. The behavior is reversed at e_5,\ldots,e_8 . Also, $\mathrm{d}x/y$ is of order $1/x^2$ at infinity, while $s(x-e_4)$ is of order x at infinity and $y/(x-e_1)(x-e_2)(x-e_3)(x-e_4)$ is nonzero and bounded there;

this is why we take $t \leq 1$ in eqns. (C.12) and (C.13).) Dual to this, along \mathfrak{D}' , Σ has two odd moduli α_1, α_2 that transform as $\sqrt{-1}$ under τ and two odd moduli β_1, β_2 that transform as $-\sqrt{-1}$.

As we essentially learned in Appendix C.1, near either \mathfrak{D} or \mathfrak{D}' , $\mathfrak{M}_{3,+}$ has 5 bosonic moduli m_1, \ldots, m_5 that are even under τ and 1 bosonic modulus $\widehat{\varepsilon}$ that is odd. We define $\widehat{\varepsilon}$ so that on the reduced space of $\mathfrak{M}_{3,+}$, it restricts to the parameter called ε in Appendix C.1. If we write q for the same function of a 3×3 period matrix that was introduced in section C.1, then along the reduced space of $\mathfrak{M}_{3,+}$, we have $q = \widehat{\varepsilon}^2$, precisely in parallel with the purely bosonic formula $q = \varepsilon^2$. However, the relation $q = \widehat{\varepsilon}^2$ has corrections when the odd moduli are turned on. Corrections to this relation that vanish at $\widehat{\varepsilon} = 0$ could be eliminated by redefining $\widehat{\varepsilon}$, so we are only interested in corrections that are nonvanishing at $\widehat{\varepsilon} = 0$, or even have a pole there.

First we consider the behavior along \mathfrak{D}' . Since q is a matrix element of the super period matrix, we can use the D'Hoker-Phong formula for the dependence of the super period matrix on odd moduli to compute its dependence on the α_i and β_j . The leading correction to q due to the odd moduli is a function bilinear in odd moduli. This function is constrained by τ -invariance, but is otherwise fairly generic, and in particular has no reason to vanish at $\hat{\varepsilon} = 0$. Thus, the bilinear correction to q is $w_{ij}\alpha_i\beta_j$ for some 2×2 matrix-valued function w_{ij} . Generically along \mathfrak{D}' , this matrix is nondegenerate and we can pick the odd moduli α_1, α_2 and β_1, β_2 so that

$$q = \hat{\varepsilon}^2 + \alpha_1 \beta_1 + \alpha_2 \beta_2 + \mathcal{O}(\alpha_1 \alpha_2 \beta_1 \beta_2), \tag{C.14}$$

where the $\alpha_1\alpha_2\beta_1\beta_2$ term comes from the terms in the D'Hoker-Phong formula for the super period matrix that are quartic in the odd variables. Since we have only explained what we mean by $\hat{\varepsilon}$ along the reduced space of $\mathfrak{M}_{3,+}$, we are free to redefine $\hat{\varepsilon}$ by adding holomorphic terms that are bilinear in the odd variables α_i , β_j . But it is not possible in this way to eliminate the nilpotent terms on the right hand side of eqn. (C.14).

Along \mathfrak{D} , we have $H^0(\Sigma_{\rm red}, K^{1/2}) \neq 0$, and this means that the super period matrix has a pole at $\widehat{\varepsilon} = 0$; the Dirac propagator, which enters the D'Hoker-Phong formula, is proportional to $\widehat{\varepsilon}^{-1}$. So the analog of eqn. (C.14) is

$$q = \hat{\varepsilon}^2 + \frac{\alpha_1 \alpha_2 + \alpha_3 \alpha_4}{\hat{\varepsilon}} + \mathcal{O}\left(\frac{\alpha_1 \alpha_2 \alpha_3 \alpha_4}{\hat{\varepsilon}^2}\right). \tag{C.15}$$

(In the D'Hoker-Phong expansion of the super period matrix, the term quartic in odd variables multiplies the product of two Dirac propagators and so is $\mathcal{O}(\widehat{\varepsilon}^{-2})$.)

Eqn. (C.15) makes clear that $\pi: \mathfrak{M}_{3,+} \to \mathcal{M}_{3,\text{spin}+}$ has a pole along \mathfrak{D} . Less obviously, however, this is also true along \mathfrak{D}' . This is because the natural variable that we use in expanding a string or superstring theory measure about the hyperelliptic locus \mathfrak{D} or \mathfrak{D}' is

not the matrix element q of the period matrix, but its square root ε , which is associated to a deformation of the complex structure of $\Sigma_{\rm red}$ (or its metric); that is, it is associated to an element of $H^1(\Sigma_{\rm red}, T)$. Rather than give an abstract explanation of this statement, we refer the reader to Appendix C.4, where the point will hopefully become clear. Regarding ε rather than $q = \varepsilon^2$ as a local parameter along $\mathfrak{D}' \subset \mathcal{M}_{3,{\rm spin}+}$ amounts to treating $\mathcal{M}_{3,{\rm spin}+}$ as a "moduli stack" rather than a "moduli space."

Let us see what happens if eqn. (C.14) is written in terms of not q but $\varepsilon = q^{1/2}$. We get

$$\varepsilon = \sqrt{\widehat{\varepsilon}^2 + \alpha_1 \beta_1 + \alpha_2 \beta_2} + \dots = \widehat{\varepsilon} + \frac{\alpha_1 \beta_1 + \alpha_2 \beta_2}{2\widehat{\varepsilon}} + \dots$$
 (C.16)

The ellipses in eqn. (C.16) comes both from expanding the square root to higher orders and from the term quartic in odd variables that was omitted in eqn. (C.14). We should interpret this formula as giving the pullback of the function ε on $\mathcal{M}_{3,\text{spin}+}$ to $\mathfrak{M}_{3,+}$:

$$\pi^*(\varepsilon) = \widehat{\varepsilon} + \frac{\alpha_1 \beta_1 + \alpha_2 \beta_2}{2\widehat{\varepsilon}} + \dots$$
 (C.17)

Clearly, this pullback has a pole. If we are supposed to take ε seriously as a function on $\mathcal{M}_{3,\mathrm{spin}+}$, then the fact that it pulls back under π to a function on $\mathfrak{M}_{3,+}$ with a pole along \mathfrak{D}' shows that the projection $\pi:\mathfrak{M}_{3,+}\to\mathcal{M}_{3,\mathrm{spin}+}$ has a pole along \mathfrak{D}' .

The analog of (C.16) along \mathfrak{D} is

$$\varepsilon = \sqrt{\widehat{\varepsilon}^2 + \frac{\alpha_1 \alpha_2 + \alpha_3 \alpha_4}{\widehat{\varepsilon}}} = \widehat{\varepsilon} + \frac{\alpha_1 \alpha_2 + \alpha_3 \alpha_4}{2\widehat{\varepsilon}^2} + \dots$$
 (C.18)

C.4 Behavior of $\pi_*(\Upsilon)$ Near $\mathfrak D$ and Near $\mathfrak D'$

To describe the superstring measure Υ near \mathfrak{D}' , we use τ -invariant local bosonic coordinates $\widehat{m}_i = \pi^*(m_i)$, along with bosonic and fermionic coordinates $\widehat{\varepsilon}$ and $\alpha_1, \alpha_2, \beta_1, \beta_2$ that are not τ -invariant. The super Mumford isomorphism (see [19–22] for original references and [23] for an introduction) says that¹⁷

$$\Upsilon = \frac{F(\widehat{\varepsilon}, \widehat{m}_1, \dots, \widehat{m}_5 | \alpha_1, \alpha_2, \beta_1, \beta_2) [d\widehat{\varepsilon} d\widehat{m}_1 \cdots d\widehat{m}_5 | d\alpha_1 d\alpha_2 d\beta_1 d\beta_2]}{\left(\frac{dx}{y} \wedge x \frac{dx}{y} \wedge x^2 \frac{dx}{y}\right)^5}, \quad (C.19)$$

¹⁷The denominator in this formula represents a section of the fifth power of the Berezinian of the cohomology of Σ. To make eqn. (C.19) simple and concrete, we have written in the denominator a section of this line bundle over $\mathcal{M}_{3,\text{spin}+}$. To write an accurate formula, one should replace $\omega_t = x^t \, \mathrm{d}x/y$, t = 0, 1, 2, with corresponding differentials on the super Riemann surface Σ. The choice of these differentials will affect the function F, but has no essential bearing on our discussion below.

where the function F is holomorphic, nonzero, and τ -invariant. Notice that Υ is τ -invariant: $d\varepsilon$ is odd under τ , the denominator $\left(\frac{dx}{y} \wedge x \frac{dx}{y} \wedge x^2 \frac{dx}{y}\right)^5$ is also odd, and the rest of the formula is τ -invariant. It is crucial that, despite being τ -invariant, Υ cannot be written in terms of $\widehat{\varepsilon}^2$ and $d(\widehat{\varepsilon}^2)$; as we will see momentarily, this fact leads to a pole in $\pi_*(\Upsilon)$ along \mathfrak{D}' .

To compute the measure $\pi_*(\Upsilon)$ on the reduced space $\mathcal{M}_{3,\text{spin}+}$, the first step is to reexpress Υ in terms of ε rather than $\widehat{\varepsilon}$. We do this by solving eqn. (C.16) for $\widehat{\varepsilon}$:

$$\widehat{\varepsilon} = \varepsilon - \frac{\alpha_1 \beta_1 + \alpha_2 \beta_2}{2\varepsilon} + k(m_1, \dots, m_5) \frac{\alpha_1 \alpha_2 \beta_1 \beta_2}{\varepsilon^3}.$$
 (C.20)

The function $k(m_1, ..., m_5)$ receives a contribution from the quartic terms that were omitted in eqn. (C.14), and also from the expansion of the square root in eqn. (C.16). From (C.20), we have

$$d\widehat{\varepsilon} = -3k \frac{d\varepsilon}{\varepsilon^4} \alpha_1 \alpha_2 \beta_1 \beta_2 + \dots, \tag{C.21}$$

where we have written only the most singular term for $\varepsilon \to 0$. Making this substitution in (C.19) and integrating over the odd variables, we find the singular behavior of $\pi_*(\Upsilon)$ along \mathfrak{D}' :

$$\pi_*(\Upsilon) \sim \frac{-3kF}{\varepsilon^4} \frac{\mathrm{d}\varepsilon \,\mathrm{d}m_1 \dots \mathrm{d}m_5}{\left(\frac{\mathrm{d}x}{y} \wedge x \,\frac{\mathrm{d}x}{y} \wedge x^2 \,\frac{\mathrm{d}x}{y}\right)^5}$$
 (C.22)

Thus $\pi_*(\Upsilon)$ is of order $1/\varepsilon^4$ or $1/q^2$ along \mathfrak{D}' .

The literature actually contains a proposal [24] for a holomorphic measure on $\mathcal{M}_{3,\text{spin}+}$ that is supposed to arise by integrating over the odd variables in some fashion that has not been specified. This formula is holomorphic along \mathfrak{D}' , so it does not coincide with $\pi_*(\Upsilon)$.

We can similarly determine the behavior of $\pi_*(\Upsilon)$ along \mathfrak{D} . The difference between \mathfrak{D} and \mathfrak{D}' is that, if Σ is split, each of the 10 RNS fermions has a pair of zero-modes along \mathfrak{D} and hence the path integral of the RNS fermions is proportional to ε^{10} . Along the split locus, therefore, one has near $\varepsilon = 0$

$$\Upsilon = \varepsilon^{10} \frac{G(\varepsilon, m_1, \dots, m_5) [d\widehat{\varepsilon} d\widehat{m}_1 \cdots d\widehat{m}_5 | d\alpha_1 d\alpha_2 d\alpha_3 d\alpha_4]}{\left(\frac{dx}{y} \wedge x \frac{dx}{y} \wedge x^2 \frac{dx}{y}\right)^5},$$
(C.23)

with G non-zero.¹⁸ To generalize this formula away from the split locus, we have to take

¹⁸At $\varepsilon = 0$, the cohomology $H^0(\Sigma, Ber(\Sigma))$ jumps, since $H^0(\Sigma_{red}, K^{1/2})$ is nonzero along \mathfrak{D} . The denominator in (C.23) therefore trivializes the appropriate line bundle only for $\varepsilon \neq 0$. As a result the formula (C.23), which vanishes at $\varepsilon = 0$, does not exhibit the super Mumford isomorphism at $\varepsilon = 0$. To do so, one would have to write the formula in a more sophisticated way, taking into account the jumping of the cohomology,

into account that, in conventional language, PCO insertions can absorb some zero-modes of the matter fermions, as discussed in section 9.3. However, the most singular behavior comes from expressing $d\hat{\varepsilon}$ in terms of $d\varepsilon$. The analog of eqn. (C.20) is

$$\widehat{\varepsilon} = \varepsilon - \frac{\alpha_1 \alpha_2 + \alpha_3 \alpha_4}{2\varepsilon^2} + \frac{\alpha_1 \alpha_2 \alpha_3 \alpha_4}{4\varepsilon^5}.$$
 (C.24)

(The $\alpha_1\alpha_2\alpha_3\alpha_4$ term comes from solving eqn. (C.18) for $\hat{\varepsilon}$ and does not depend on the $\mathcal{O}(\alpha_1\alpha_2\alpha_3\alpha_4/\hat{\varepsilon}^2)$ term in eqn. (C.14).) Hence

$$d\hat{\varepsilon} \sim -\frac{5}{4}\alpha_1\alpha_2\alpha_3\alpha_4\frac{d\varepsilon}{\varepsilon^6} + \dots,$$
 (C.25)

where less singular terms are omitted. Combining this factor of ε^{-6} with the ε^{10} in eqn. (C.23), and integrating over the odd variables, we expect $\pi_*(\Upsilon)$ to vanish as $\varepsilon^4 = q^2$ along \mathfrak{D} .

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but this is not necessary for our purposes, basically because there is no jumping in the cohomology of $\Sigma_{\rm red}$.

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