VECTOR BUNDLES OVER ELLIPTIC FIBRATIONS

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Introduction

Let $\pi\colon Z\to B$ be an elliptic fibration with a section. The goal of this paper is to study holomorphic vector bundles over Z. We are mainly concerned with vector bundles V with trivial determinant, or more generally such that $\det V$ has trivial restriction to each fiber, so that $\det V$ is the pullback of a line bundle on B. (The case where $\det V$ has nonzero degree on every fiber is in a certain sense simpler, since it usually reduces to the case considered here for a bundle of smaller rank.) We give two constructions of vector bundles, one based on the idea of a spectral cover of B and the other based on the idea of extensions of certain fixed bundles over the elliptic manifold Z. Each of these constructions has advantages and the combination of the two seems to give the most comprehensive information.

Vector bundles over a single elliptic curve were first classified by Atiyah [1]; however, he did not attempt to construct universal bundles or work in families. The case of rank two bundles over an elliptic surface was studied in [3], [6], [4] with a view toward making computations in Donaldson theory. The motivation for this paper and the more general study of the moduli of principal G-bundles over families of elliptic curves (which will be treated in another paper) grew out of questions arising in the recent study of F-theory by physicists. The explanation of these connections was given in [7]. For these applications Z is assumed to be a Calabi-Yau manifold, usually of dimension two or three. However, most of the results on vector bundles and more generally G-bundles are true with no assumptions on Z. The case of a general simple and simply connected complex Lie group G involves a fair amount of algebraic group theory and will be treated elsewhere, but the case $G = SL_n(\mathbb{C})$ can be done in a quite explicit and concrete way, and that is the subject of this paper.

For both mathematical and physical reasons, we shall be primarily interested in constructing stable vector bundles on Z. Of course, stability must be defined with respect to a suitable ample divisor. Following well-established principles, the natural ample divisors to work with are those of the form $H_0 + N\pi^*H$ for $N \gg 0$, where H_0 is some fixed ample divisor on Z and H is an ample divisor on B. If V is stable with respect to such a divisor, then V|f is semistable with respect to almost all fibers f. (However the converse is not necessarily true.) One special feature of vector bundles V with trivial determinant on an elliptic curve is that, if the rank of V is at least two, then V is never properly stable. Moreover, if V has rank n > 1,

The first author was partially supported by NSF grant DMS-96-22681. The second author was partially supported by NSF grant DMS-94-02988. The third author was partially supported by NSF grant PHY-95-13835.

then V is never simple; in fact, the endomorphism algebra of V has dimension at least n. But there is still a relative coarse moduli space $\mathcal{M}_{Z/B}$, which turns out to be a \mathbb{P}^{n-1} -bundle over B. A stable vector bundle on Z defines a rational section of $\mathcal{M}_{Z/B}$. Conversely, a regular section of $\mathcal{M}_{Z/B}$ defines a vector bundle over Z, and in fact it defines many such bundles. Our goal will be to describe all such bundles, to see how the properties of the section are reflected in the properties of these bundles, and to find sufficient conditions for the bundles in question to be stable.

In the first three sections we consider a single (generalized) elliptic curve E. In Section 1 we construct a coarse moduli space for S-equivalence classes of semistable $SL_n(\mathbb{C})$ -bundles over E. It is a projective space \mathbb{P}^{n-1} , in fact it is the projective space of the complete linear system $|np_0|$ where $p_0 \in E$ is the origin of the group law. It turns out that each S-equivalence class of semistable bundles has a "best" representative, the so-called regular representative. The defining property of these bundles, at least when E is smooth, is that their automorphism groups are of the smallest possible dimension, namely n. We view them as analogues of regular elements in the group $SL_n(\mathbb{C})$. The moduli space we construct is also the coarse moduli space for isomorphism classes of regular semistable $SL_n(\mathbb{C})$ -bundles over E. As we shall see, the regular bundles are the bundles which arise if we try to fit together the S-equivalence classes in order to find universal holomorphic bundles over $\mathbb{P}^{n-1} \times E$.

In Section 2, assuming that E is smooth, we construct a tautological bundle U over $\mathbb{P}^{n-1} \times E$ which is regular semistable and with trivial determinant on each slice $\{x\} \times E$ and such that $U|\{x\} \times E$ corresponds to the regular bundle over E whose S-equivalence class is x. There is not a unique such bundle over $\mathbb{P}^{n-1} \times E$, and we proceed to construct all such. The idea is that there is an n-sheeted covering $T \to \mathbb{P}^{n-1}$ called the spectral cover, such that U is obtained by pushing down a Poincaré line bundle $P \to T \times E$ under the covering map. It turns out that every bundle over $\mathbb{P}^{n-1} \times E$ which is of the correct isomorphism class on each slice $\{x\} \times E$ is obtained by pushing down $P \otimes p_1^* M$ for some line bundle M on T. There is a generalization of this result to cover the case of families of regular semistable bundles on E parameterized by arbitrary spaces S.

In Section 3 we turn to a different construction of "universal" bundles over $\mathbb{P}^{n-1} \times E$. Here we consider the space of extensions of two fixed bundles with determinants $\mathcal{O}_E(\pm p_0)$. For a fixed rank d, there is a unique stable bundle W_d of rank d such that $\det W_d \cong \mathcal{O}_E(p_0)$. For $1 \leq d \leq n-1$, we consider the space of all nonsplit extensions V of the form

$$0 \to W_d^{\vee} \to V \to W_{n-d} \to 0.$$

The moduli space of all such extensions is simply $\mathbb{P}H^1(W_{n-d}^{\vee} \otimes W_d^{\vee}) \cong \mathbb{P}^{n-1}$. Over $\mathbb{P}^{n-1} \times E$ there is a universal extension whose restriction to each fiber is regular semistable. There is thus an induced map from the \mathbb{P}^{n-1} of extensions to the coarse moduli space defined in Section 1, which is $|np_0| \cong \mathbb{P}^{n-1}$. By a direct analysis we show that this map is an isomorphism. Actually, there are n-1 different versions of this construction, depending on the choice of the integer d, but the projective spaces that they produce are all canonically identified. On the other hand, the universal extensions associated with different versions of the construction are non-isomorphic universal bundles. Finally, we relate these families of bundles to the

ones arising from the spectral cover construction, which we can then extend to the case where E is singular. We remark here that we can interpret the construction of Section 3 as parametrizing those bundles whose structure group can be reduced to a maximal parabolic subgroup P of SL_n , such that the induced bundle on the Levi factor is required to be $W_d^{\vee} \oplus W_{n-d}$ in the obvious sense. This interpretation can then be generalized to other complex simple groups [8].

In Section 4 we generalize the results of the first three sections to a family $\pi\colon Z\to B$ of elliptic curves with a section σ . By taking cohomology along the fibers of π , we produce a vector bundle over the base, namely $\pi_* \mathcal{O}_Z(n\sigma) = \mathcal{V}_n$, which globalizes $H^0(E, \mathcal{O}_E(np_0))$. The associated projective bundle $\mathbb{P}_{\pi_*}\mathcal{O}_Z(n\sigma) = \mathbb{P}\mathcal{V}_n$ then becomes the appropriate relative coarse moduli space. We show that $\pi_*\mathcal{O}_Z(n\sigma)$ has a natural splitting as a direct sum of line bundles. This decomposition is closely related to the fact that the coefficients of the characteristic polynomial of an element in \mathfrak{sl}_n are a polynomial basis for the algebra of polynomial functions on \mathfrak{sl}_n invariant under the adjoint action. Having constructed the relative coarse moduli space, we give a relative version of the constructions of Sections 2 and 3 to produce bundles over $\mathbb{P}\mathcal{V}_n \times_B Z$. The extension construction generalizes easily. The bundles we used over a single elliptic curve have natural extensions to any elliptic fibration. We form the relative extension bundle and the universal relative extension in direct analogy with the case of a single elliptic curve. Relative versions of results from Section 3 show that the relative extension space is identified with $\mathbb{P}\mathcal{V}_n$. Following the pattern of Section 3, we use the extension picture to define a universal spectral cover of $\mathbb{P}\mathcal{V}_n$, and in turn use this spectral cover to construct new universal vector bundles. Finally, we calculate the Chern classes of the universal bundles we have constructed.

In Section 5, using the theory developed in the first four sections, we study vector bundles V over an elliptic fibration $\pi \colon Z \to B$ such that the restriction of V to every fiber is regular and semistable. To such a bundle V, we associate a section A(V) of $\mathbb{P}\mathcal{V}_n$ and a cover $C_A \to B$ of degree n, the spectral cover of B determined by V. Conversely, V is determined by A and by the choice of a line bundle on C_A . After computing some determinants and Chern classes, we discuss the possible line bundles which can exist on the spectral cover. Then we turn to specific types of bundles. After describing symmetric bundles, which are interesting from the point of view of F-theory, we turn to bundles corresponding to a degenerate section. First we consider the most degenerate case, and then we consider reducible sections where the restriction of V to every fiber has a section. Finally, we relate reducible sections to the existence of certain subbundles of V.

In Section 6, we consider bundles V whose restriction to a generic fiber is regular and semistable, but such that there exist fibers E_b where $V|E_b$ is either unstable or it is semistable but not regular. If V fails to be regular or semistable in codimension one, it can be improved by elementary modifications to a reflexive sheaf whose restriction to every fiber outside a codimension two set is regular and semistable. We describe this process and, as an illustration, analyze the tangent bundle to an elliptic surface. On the other hand, if the locus of bad fibers has codimension at least two, no procedure exists for improving V, and we must analyze it directly. The case of instability in codimension two or higher corresponds to the case where the rational section A determined by V does not actually define a regular section (this case can also lead to reflexive but non-locally free sheaves). The case where V has irregular restriction to certain fibers in codimension at least two corresponds to

singular spectral covers. We give some examples of such behavior, without trying to be definitive. Our construction can be viewed as a generalization of the method of Section 3 to certain non-maximal parabolic subgroups of SL_n .

Finally, we turn in Section 7 to the problem of deciding when the bundles V constructed by our methods are stable. This is the most interesting case for both mathematical and physical reasons. While we do not try to give necessary and sufficient conditions, we show that, in case the spectral cover C_A of B determined by V is irreducible, then V is stable with respect to all ample divisors of the form $H_0 + N\pi^*H$, where H_0 is an ample divisor on Z and H is an ample divisor on B, and $N \gg 0$. We are only able to give an effective bound on N in case dim B = 1, i.e. Z is an elliptic surface, but it seems likely that such an effective bound exists in general.

We will have to deal systematically with singular fibers of $Z \to B$, and the price that must be paid for analyzing this case is a heavy dose of commutative algebra. In an attempt to make the paper more readable, we have tried to isolate these arguments where possible. We collect here some preliminary definitions and technical results. While these results are well-known, we could not find an adequate reference for many of them.

Notation and conventions.

All schemes are assumed to be separated and of finite type over \mathbb{C} . A sheaf is always a coherent sheaf. We will identify a vector bundle with its locally free sheaf of sections, covariantly. If V is a vector bundle, then $\mathbb{P}V$ is the projective space bundle whose associated sheaf of graded algebras is $\bigoplus_{k\geq 0} \operatorname{Sym}^k V^\vee$; thus these conventions are opposite to those of EGA or [10]. Given sheaves $\mathcal{S}, \mathcal{S}'$, we denote by $\operatorname{Hom}(\mathcal{S}, \mathcal{S}')$ the sheaf of homomorphisms from \mathcal{S} to \mathcal{S}' and by $\operatorname{Hom}(\mathcal{S}, \mathcal{S}') = H^0(\operatorname{Hom}(\mathcal{S}, \mathcal{S}'))$ the group of all such homomorphisms. Likewise $\operatorname{Ext}^k(\mathcal{S}, \mathcal{S}')$ is the Ext sheaf and $\operatorname{Ext}^k(\mathcal{S}, \mathcal{S}')$ is the global Ext group (related to the local Ext groups by the local to global spectral sequence).

0.1. Elliptic curves and elliptic fibrations.

Recall that a Weierstrass equation is a homogeneous cubic equation of the form

$$(0.1) Y^2 Z = 4X^3 - g_2 X Z^2 - g_3 Z^3,$$

with g_2, g_3 constants. We will refer to the curve E in \mathbb{P}^2 defined by such an equation, together with the marked point $p_0 = [0, 1, 0]$ at infinity, as a Weierstrass cubic. Setting

$$\Delta(g_2, g_3) = g_2^3 - 27g_3^2,$$

if $\Delta(g_2, g_3) \neq 0$, then (0.1) defines a smooth cubic curve in \mathbb{P}^2 with the marked point [0, 1, 0], i.e., defines the structure of an elliptic curve. If $\Delta(g_2, g_3) = 0$, then the corresponding plane cubic E is a singular curve with arithmetic genus $p_a(E) = 1$. If (g_2, g_3) is a smooth point of the locus $\Delta(g_2, g_3) = 0$, then the corresponding plane cubic curve is a rational curve with a single node. The smooth points of such a curve form a group isomorphic to \mathbb{C}^* with identity element p_0 . The point $g_2 = g_3 = 0$ is the unique singular point of $\Delta(g_2, g_3) = 0$ and the corresponding plane curve is a rational curve with a single cusp. Once again its smooth points form a group, isomorphic to \mathbb{C} , with identity element p_0 . These are all possible reduced and irreducible curves of arithmetic genus one.

Next we consider the relative version of a Weierstrass equation. Let $\pi\colon Z\to B$ be a flat morphism of relative dimension one, such that the general fiber is a smooth elliptic curve and all fibers are isomorphic to reduced irreducible plane cubics. Here we will assume that B is a smooth variety (although the case of a complex manifold is similar). We shall always suppose that π has a section σ , i.e. there exists a divisor σ contained in the smooth points of Z such that $\pi|\sigma$ is an isomorphism. Let $L=R^1\pi_*\mathcal{O}_Z\cong\mathcal{O}_Z(-\sigma)|\sigma$, viewed as a line bundle on B. Then there are sections $G_2\in H^0(B;L^{\otimes 4})$ and $G_3\in H^0(B;L^{\otimes 6})$ such that $\Delta(G_2,G_3)\neq 0$ as a section of $L^{\otimes 12}$, and Z is isomorphic to the subvariety of $\mathbb{P}(\mathcal{O}_B\oplus L^2\oplus L^3)$ defined by the Weierstrass equation $Y^2Z=4X^3-G_2XZ^2-G_3Z^3$. Conversely, given the line bundle L on B and sections $G_2\in H^0(B;L^{\otimes 4})$, $G_3\in H^0(B;L^{\otimes 6})$ such that $\Delta(G_2,G_3)\neq 0$, the equation $Y^2Z=4X^3-G_2XZ^2-G_3Z^3$ defines a hypersurface Z in $\mathbb{P}(\mathcal{O}_B\oplus L^2\oplus L^3)$, such that the projection to Z is a flat morphism whose fibers are reduced irreducible plane curves, generically smooth. We will not need to assume that Z is smooth; it is always Gorenstein and the relative dualizing sheaf $\omega_{Z/B}$ is isomorphic to Z. Thus, the dualizing sheaf Z is isomorphic to Z.

Let us describe explicitly the case where the divisors associated to G_2 and G_3 are smooth and meet transversally. This means in particular that if G_2 and G_3 are chosen generically, then $G_2^3 - 27G_3^2$ defines a section of L^{12} . We shall denote by $\overline{\Gamma}$ the zero set of this section. Then $\overline{\Gamma}$ is smooth except where $G_2 = G_3 = 0$, where it has singularities which are locally trivial families of cusps. The fiber of π over a smooth point of $\overline{\Gamma}$ is a nodal plane cubic, and over a point where $G_2 = G_3 = 0$ the fibers of π are cusps. Let Γ be the locus of points where π is singular. Thus Γ maps bijectively onto $\overline{\Gamma}$. There are local analytic coordinates on B so that, near a cuspidal fiber Z has the local equation $y^2 = x^3 + sx + t$. Here x, y are a set of fiber coordinates for $\mathbb{P}(\mathcal{O}_B \oplus L^2 \oplus L^3)$ away from the line at infinity and x, y, s, t form part of a set of local coordinates for $\mathbb{P}(\mathcal{O}_B \oplus L^2 \oplus L^3)$. Thus x, y, s are coordinates for Z. The local equation for $\overline{\Gamma}$ is $z \in \mathbb{P}(\mathcal{O}_B \oplus L^2 \oplus L^3)$. Thus $z \in \mathbb{P}(\mathcal{O}_B \oplus L^2 \oplus L^3)$ is given locally by $z \in \mathbb{P}(\mathcal{O}_B \oplus L^2 \oplus L^3)$. The morphism from $z \in \mathbb{P}(\mathcal{O}_B \oplus L^2 \oplus L^3)$ is given locally by $z \in \mathbb{P}(\mathcal{O}_B \oplus L^2 \oplus L^3)$.

0.2. Rank one torsion free sheaves.

Let E be a singular Weierstrass cubic and let E_{reg} be the set of smooth points of E. The arithmetic genus $p_a(E)$ is one. We let $n \colon \tilde{E} \to E$ be the normalization map. The generalized Jacobian J(E) is the group of line bundles of degree zero on E, and (as in the smooth case) is isomorphic to E_{reg} via the map $e \in E_{\text{reg}} \mapsto \mathcal{O}_E(e - p_0)$. Just as we can compactify E_{reg} to E by adding the singular point, we can compactify J(E) to the compactified generalized Jacobian $\bar{J}(E)$, by adding the unique rank one torsion free sheaf which is not locally free. Here a sheaf S over E is torsion-free if it has no nonzero sections which are supported on a proper closed subset (i.e. a finite set). In particular, the restriction of S to the smooth points of E is a vector bundle, and so has a well-defined rank, which we also call the rank of \mathcal{S} . If \mathcal{S} is a torsion-free sheaf on E we let $\deg S = \chi(S) + (p_a(E) - 1)(\operatorname{rank} S) = \chi(S)$. (This agrees with the usual Riemann-Roch formula in case E is smooth.) Thus the degree of such sheaves is additive in exact sequences, and if $\mathcal{S}' \subseteq \mathcal{S}$ such that the quotient is supported at a finite set of points, then $\deg S' \leq \deg S$ with equality if and only if S' = S. If S is torsion free and V is locally free, then $\deg(V \otimes S) = (\deg V)(\operatorname{rank} S) + (\deg S)(\operatorname{rank} V)$. To see this, first use the fact that

there is a filtration of V by subbundles whose successive quotients are line bundles, so by the additivity of degree we can reduce to the case where V is a line bundle. In this case, we may write $V = \mathcal{O}_E(d_1 - d_2)$, where d_1 and d_2 are effective divisors supported on the smooth points of E, and then use the exact sequences

$$0 \to \mathcal{S} \otimes \mathcal{O}_E(-d_2) \to \mathcal{S} \to \mathcal{S} \otimes \mathcal{O}_{d_2} \to 0$$

and

$$0 \to \mathcal{S} \otimes \mathcal{O}_E(-d_2) \to \mathcal{S} \otimes \mathcal{O}_E(d_1 - d_2) \to \mathcal{S} \otimes \mathcal{O}_{d_1} \to 0$$
,

together with the usual properties, to conclude that $\deg(V \otimes S) = (\deg V)(\operatorname{rank} S) + (\deg S)$ in case V is a line bundle. Thus we have established the formula in general.

Next let us show that there is a unique torsion free rank one sheaf \mathcal{F} which compactifies the generalized Jacobian.

Lemma 0.2. There is a unique rank one torsion free sheaf \mathcal{F} on E of degree zero which is not locally free. It satisfies:

- (i) $Hom(\mathcal{F}, \mathcal{F}) = n_* \mathcal{O}_{\tilde{E}}$.
- (ii) $\mathcal{F}^{\vee} \cong \mathcal{F}$.
- (iii) For all line bundles λ of degree zero, $Hom(\lambda, \mathcal{F}) = Hom(\mathcal{F}, \lambda) = \mathcal{F}$ and $Hom(\lambda, \mathcal{F}) = Hom(\mathcal{F}, \lambda) = 0$. Likewise $Ext^1(\mathcal{F}, \lambda) = Ext^1(\lambda, \mathcal{F}) = 0$.

Proof. The first statement is essentially a local result. Let R be the local ring of E at the singular point and let \tilde{R} be the normalization of R. If locally \mathcal{F} corresponds to the R-module M, let \tilde{M} be the \tilde{R} -module $M \otimes_R \tilde{R}$ modulo torsion. Then by construction \tilde{M} is a torsion free rank one \tilde{R} -module, so that we may choose a \tilde{R} -module isomorphism from \tilde{M} to \tilde{R} . Since M is torsion free, the natural map from M to $\tilde{M} \cong \tilde{R}$ is injective, identifying M as an R-submodule of \tilde{R} which generates \tilde{R} as a \tilde{R} -module. Thus M contains a unit of \tilde{R} , which after a change of basis we may assume to be 1, and furthermore M contains $R \cdot 1 = R \subseteq \tilde{R}$. But since the singularity of E is a node or a cusp, $\ell(\tilde{R}/R) = 1$, and so either M = R or $M = \tilde{R}$. Note that there are two isomorphic non-locally free R-modules of rank one: \tilde{R} and m, where m is the maximal ideal of R. The ideal m is the conductor of the extension \tilde{R} of R, and $\operatorname{Hom}_R(\tilde{R},R) \cong m$, where the isomorphism is canonical.

By the above, every rank one torsion free sheaf on E is either a line bundle or of the form n_*L , where L is a line bundle on \tilde{E} . Now $\tilde{E} \cong \mathbb{P}^1$, and $\deg n_*\mathcal{O}_{\mathbb{P}^1}(a) = a+1$. Thus $n_*\mathcal{O}_{\mathbb{P}^1}(-1)$ is the unique rank one torsion free sheaf on E of degree zero which is not locally free. Note that, if \mathfrak{m}_x is the ideal sheaf of the singular point $x \in E$, then $\deg \mathfrak{m}_x = -1$, by using the exact sequence

$$0 \to \mathfrak{m}_x \to \mathcal{O}_E \to \mathbb{C}_x \to 0.$$

Thus $\mathfrak{m}_x = n_* \mathcal{O}_{\tilde{E}}(-2)$.

To see (i), note that

$$Hom(n_*\mathcal{O}_{\mathbb{P}^1}(-1), n_*\mathcal{O}_{\mathbb{P}^1}(-1)) = n_*Hom(n^*n_*\mathcal{O}_{\mathbb{P}^1}(-1), \mathcal{O}_{\mathbb{P}^1}(-1)).$$

Since n is finite, the natural map $n^*n_*\mathcal{O}_{\mathbb{P}^1}(-1) \to \mathcal{O}_{\mathbb{P}^1}(-1)$ is surjective, and its kernel is torsion. Thus

$$Hom(n^*n_*\mathcal{O}_{\mathbb{P}^1}(-1),\mathcal{O}_{\mathbb{P}^1}(-1)) \cong Hom(\mathcal{O}_{\mathbb{P}^1}(-1),\mathcal{O}_{\mathbb{P}^1}(-1)) = \mathcal{O}_{\tilde{E}},$$

proving (i). To see (ii), we have invariantly that

$$Hom(n_*\mathcal{O}_{\tilde{E}}, \mathcal{O}_E) = \mathfrak{m}_x = n_*\mathcal{O}_{\tilde{E}}(-2).$$

Thus tensoring with $\mathcal{O}_E(-p_0)$ and using $n_*\mathcal{O}_{\tilde{E}}\otimes\mathcal{O}_E(-p_0)=n_*n^*\mathcal{O}_E(-p_0)=n_*\mathcal{O}_{\tilde{E}}(-1)$ gives

$$Hom(n_*\mathcal{O}_{\tilde{E}}(-1),\mathcal{O}_E) = Hom(n_*\mathcal{O}_{\tilde{E}} \otimes \mathcal{O}_E(-p_0),\mathcal{O}_E) =$$

$$= n_*\mathcal{O}_{\tilde{E}}(-2) \otimes \mathcal{O}_E(p_0) = n_*\mathcal{O}_{\tilde{E}}(-1),$$

which is the statement that $\mathcal{F}^{\vee} \cong \mathcal{F}$. To see (iii), if λ is a line bundle of degree zero, then $Hom(\lambda, \mathcal{F}) = \lambda^{-1} \otimes \mathcal{F}$ is a nonlocally free sheaf of degree zero, and hence it is isomorphic to \mathcal{F} by uniqueness. Likewise

$$Hom(\mathcal{F}, \lambda) \cong \lambda \otimes \mathcal{F}^{\vee} \cong \lambda \otimes \mathcal{F} \cong \mathcal{F}.$$

Moreover, $\operatorname{Hom}(\lambda,\mathcal{F})=H^0(\mathcal{F})=0$, since by degree considerations a nonzero map $\lambda^{-1}\to\mathcal{F}$ would have to be an isomorphism, contradicting the fact that \mathcal{F} is not locally free. The proof that $\operatorname{Hom}(\mathcal{F},\lambda)=0$ is similar. Now $\operatorname{Ext}^1(\mathcal{F},\lambda)$ is Serre dual to $\operatorname{Hom}(\lambda,\mathcal{F})=0$, since λ is locally free. Also, $\operatorname{Ext}^1(\lambda,\mathcal{F})=H^1(\lambda^{-1}\otimes\mathcal{F})=H^1(\mathcal{F})=0$, since $h^0(\mathcal{F})=\deg\mathcal{F}=0$. \square

Remark. In case E is nodal, $\operatorname{Ext}^1(\mathcal{F},\mathcal{F})$ is not Serre dual to $\operatorname{Hom}(\mathcal{F},\mathcal{F})$, and in fact $\operatorname{Ext}^1(\mathcal{F},\mathcal{F}) \cong H^0(\operatorname{Ext}^1(\mathcal{F},\mathcal{F}))$ has dimension two. In this case $\mathbb{P}\operatorname{Ext}^1(\mathcal{F},\mathcal{F}) \cong \mathbb{P}^1$ can be identified with the normalization of E. The preimages $\{x_1,x_2\}$ of the singular point give two different non-locally free extensions, and the remaining locally free extensions V of \mathcal{F} by \mathcal{F} are parametrized by $\mathbb{P}^1 - \{x_1, x_2\} \cong \mathbb{C}^*$. The set of such V is in 1-1 correspondence with $J(E) \cong E$ via the determinant.

Next we define the compactified generalized Jacobian of E. Let Δ_0 be the diagonal in $E \times E$ and let I_{Δ_0} be its ideal sheaf. We let $\mathcal{O}_{E \times E}(\Delta_0) = I_{\Delta_0}^{\vee}$ and

$$\mathcal{P}_0 = \mathcal{O}_{E \times E}(\Delta_0 - (E \times \{p_0\})) = I_{\Delta_0}^{\vee} \otimes \pi_2^* \mathcal{O}_E(-p_0).$$

Lemma 0.3. In the above notation,

- (i) \mathcal{P}_0 is flat over both factors of $E \times E$, and \mathcal{P}_0^{\vee} is locally isomorphic to I_{Δ_0} .
- (ii) If e is a smooth point of E, the restriction of \mathcal{P}_0 to the slice $\{e\} \times E$ is $\mathcal{O}_E(e-p_0)$. If x is the singular point of E, the restriction of \mathcal{P}_0 to the slice $\{x\} \times E$ is \mathcal{F} .
- (iii) Suppose that S is a scheme and that \mathcal{L} is a coherent sheaf on $S \times E$, flat over S, such that for every slice $\{s\} \times E$, the restriction of \mathcal{L} to $\{s\} \times E$ is a rank one torsion free sheaf on E of degree zero. Then there exists a unique morphism $f: S \to E$ and a line bundle M on S such that $\mathcal{L} = (f \times \mathrm{Id})^* \mathcal{P}_0 \otimes \pi_1^* M$.

Proof. We shall just outline the proof of this essentially standard result. The proofs of (i) and (ii) in case \mathcal{P}_0 is replaced by I_{Δ_0} , with the necessary changes in (ii), are easy: From the exact sequence

$$0 \to I_{\Delta_0} \to \mathcal{O}_{E \times E} \to \mathcal{O}_{\Delta_0} \to 0$$
,

and the fact that both $E \times E$ and Δ_0 are flat over each factor, we see that I_{Δ_0} is flat over both factors, and the restriction of I_{Δ_0} to the slice $\{e\} \times E$ is $\mathcal{O}_E(-e)$, if $e \neq x$, and is \mathfrak{m}_x in case e = x. To handle the case of \mathcal{P}_0 , the main point is to check that $\mathcal{O}_{E \times E}(\Delta_0) = I_{\Delta_0}^{\vee}$ is locally isomorphic to I_{Δ_0} , and that the inclusion $I_{\Delta_0} \to \mathcal{O}_{E \times E}$ dualizes to give an exact sequence

$$0 \to \mathcal{O}_{E \times E} \to \mathcal{O}_{E \times E}(\Delta_0) \to \mathcal{O}_{\Delta_0} \to 0.$$

This may be checked by hand, by working out a local resolution of I_{Δ_0} . We omit the details.

Another, less concrete, proof which generalizes to a flat family $\pi\colon Z\to B$ is as follows. Dualizing the inclusion of $I_{\Delta_0}\to \mathcal{O}_{E\times E}$ gives an exact sequence

$$0 \to \mathcal{O}_{E \times E} \to I_{\Delta_0}^{\vee} \to Ext^1(\mathcal{O}_{\Delta_0}, \mathcal{O}_{E \times E}) \to 0.$$

To check flatness of $I_{\Delta_0}^{\vee}$ and the remaining statements of (ii), it suffices to show that, locally, $Ext^1(\mathcal{O}_{\Delta_0}, \mathcal{O}_{E\times E})\cong \mathcal{O}_{\Delta_0}$. Clearly $Ext^1(\mathcal{O}_{\Delta_0}, \mathcal{O}_{E\times E})$ is a sheaf of \mathcal{O}_{Δ_0} -modules and thus it is identified with a sheaf on E via the first projection. If $\pi_1, \pi_2 \colon E \times E \to E$ are the projections, we have the relative Ext sheaves $Ext^i_{\pi_1}(\mathcal{O}_{\Delta_0}, \mathcal{O}_{E\times E})$. (See for example [2] for properties of these sheaves.) The curve E is Gorenstein and thus $Ext^1(\mathbb{C}_x, \mathcal{O}_E) \cong \mathbb{C}$ for all $x \in E$. By base change, $Ext^1_{\pi_1}(\mathcal{O}_{\Delta_0}, \mathcal{O}_{E\times E})$ is a line bundle on E. On the other hand, by the local to global spectral sequence,

$$Ext^1_{\pi_1}(\mathcal{O}_{\Delta_0}, \mathcal{O}_{E\times E}) = \pi_{1*}Ext^1(\mathcal{O}_{\Delta_0}, \mathcal{O}_{E\times E}).$$

Thus $Ext^1(\mathcal{O}_{\Delta_0}, \mathcal{O}_{E\times E})$ can be identified with a line bundle on Δ_0 , and so it is locally isomorphic to \mathcal{O}_{Δ_0} . Dualizing this argument gives an exact sequence (locally)

$$0 \to \mathcal{P}_0^{\vee} \to \mathcal{O}_{E \times E} \to Ext^1(\mathcal{O}_{\Delta_0}, \mathcal{O}_{E \times E}) \to 0,$$

and so (locally again) $\mathcal{P}_0^{\vee} = (I_{\Delta_0})^{\vee\vee} \cong I_{\Delta_0}$. In particular \mathcal{P}_0^{\vee} is also flat over E. To see (iii), suppose that S and \mathcal{L} are as in (iii). By base change, $\pi_{1*}(\mathcal{L} \otimes \pi_2^*\mathcal{O}_E(p_0)) = M^{-1}$ is a line bundle on S, and the morphism

$$\pi_1^* \pi_{1*}(\mathcal{L} \otimes \pi_2^* \mathcal{O}_E(p_0)) = \pi_1^* M^{-1} \to \mathcal{L} \otimes \pi_2^* \mathcal{O}_E(p_0)$$

vanishes along a subscheme \mathcal{Z} of $S \times E$, flat over S and of degree one on every slice. Thus \mathcal{Z} corresponds to a morphism $f \colon S \to E = \operatorname{Hilb}^1 E$, such that \mathcal{Z} is the pullback of $\Delta_0 \subset E \times E$ by $(f \times \operatorname{Id})^*$. This proves (iii). \square

A very similar argument proves the corresponding result for the dual of the ideal of the diagonal in $Z \times_B Z$, where $\pi \colon Z \to B$ is a flat family of Weierstrass cubics. In this case, we let Δ_0 be the ideal sheaf of the diagonal in $Z \times_B Z$, and set $\mathcal{P}_0 = I_{\Delta_0}^{\vee} \otimes \pi_2^* \mathcal{O}_Z(-\sigma)$, where σ is the section. Then \mathcal{P}_0 is flat over both factors Z, and has the properties (i)–(iii) of (0.3). We leave the details of the formulation and the proof to the reader.

Finally we discuss a local result which will be needed to handle semistable sheaves on a singular E. (In the application, R is the local ring of E at a singular point.)

Lemma 0.4. Let R be a local Cohen-Macaulay domain of dimension one and let Q be a finitely generated torsion free R-module. Then $\operatorname{Ext}^1_R(Q,R)=0$.

Proof. By a standard argument, if Q has rank n there exists an inclusion $Q \subseteq R^n$. Thus necessarily the quotient R^n/Q is a torsion R-module T. Now $\operatorname{Ext}^1_R(Q,R) \cong \operatorname{Ext}^2_R(T,R)$. Since R is Cohen-Macaulay, if $\mathfrak m$ is the maximal ideal of R, then $\operatorname{Ext}^2_R(R/\mathfrak m,R)=0$. An induction on the length of T then shows that $\operatorname{Ext}^2_R(T,R)=0$ for all R-modules T of finite length. Hence $\operatorname{Ext}^1_R(Q,R)=0$. \square

0.3. Semistable bundles and sheaves on singular curves.

Let E be a Weierstrass cubic and let S be a torsion free sheaf on E. The normalized degree or slope $\mu(S)$ of S is defined to be deg S/ rank S. A torsion free sheaf S is semistable if, for every subsheaf S' of S with $0 < \operatorname{rank} S' < \operatorname{rank} S$, then we have $\mu(S') \leq \mu(S)$, and it is unstable if it is not semistable. Equivalently, S is semistable if, for all surjections $S \to S''$, where S'' is torsion free and nonzero, we have $\mu(S'') \geq \mu(S)$. A torsion free rank one sheaf is semistable. Given an exact sequence

$$0 \to \mathcal{S}' \to \mathcal{S} \to \mathcal{S}'' \to 0$$
.

with $\mu(S') = \mu(S) = \mu(S'')$, S is semistable if and only if both S' and S'' are semistable. If S is a torsion free semistable sheaf of negative degree, then (for E of arithmetic genus one) $h^0(S) = 0$ and hence $h^1(S) = -\deg S$, and if S is a torsion free semistable sheaf of strictly positive degree, then since $h^1(S)$ is dual to $\operatorname{Hom}(S, \mathcal{O}_E)$, it follows that $h^1(S) = 0$ and that $h^0(S) = \deg S$. Every torsion free sheaf S has a canonical Harder-Narasimhan filtration, in other words a filtration by subsheaves $F^0 \subset F^1 \subset \cdots$ such that F^{i+1}/F^i is torsion free and semistable and $\mu(F^i/F^{i-1}) > \mu(F^{i+1}/F^i)$ for all $i \geq 1$.

Definition 0.5. Let V and V' be two semistable torsion free sheaves on E. We say that V and V' are S-equivalent if there exists a connected scheme S and a coherent sheaf V on $S \times E$, flat over S, and a point $s' \in S$ such that $V \cong V | \{s\} \times E$ if $s \neq s'$ and $V' \cong V | \{s'\} \times E$. We define S-equivalence to be the equivalence relation on semistable torsion free sheaves generated by the above relation. Suppose that V and V' are two semistable bundles on E. We say that V and V' are restricted S-equivalent if there exists a connected scheme S, a vector bundle V on $S \times E$, and a point $s' \in S$ such that $V \cong V | \{s\} \times E$ if $s \neq s'$ and $V' \cong V | \{s'\} \times E$. We define restricted S-equivalence to be the equivalence relation on semistable bundles generated by the above relation.

1. A coarse moduli space for semistable bundles over a Weierstrass cubic.

Fix a Weierstrass cubic E with an origin p_0 and consider semistable vector bundles of rank n and trivial determinant over E. Our goal in this section will be to construct a coarse moduli space of such bundles, which we will identify with the linear system $|np_0|$. Given a vector bundle V, we associate to V a point $\zeta(V)$ in the projective space $|np_0|$ associated to the linear system $\mathcal{O}_E(np_0)$ on E. In case E is smooth, $\zeta(V)$ records the unordered set of degree zero line bundles that occur as Jordan-Hölder quotients of any maximal filtration of V. More generally, if $V \to S \times E$ is an algebraic (or holomorphic) family of bundles of the above type on

E, then the function $\Phi: S \to |np_0|$ defined by $\Phi(s) = \zeta(\mathcal{V}|\{s\} \times E)$ is a morphism. If E is smooth, two semistable bundles V and V' are are S-equivalent if and only $\zeta(V) = \zeta(V')$. This identifies $|np_0|$ as a (coarse) moduli space of S-equivalence classes of semistable rank n bundles with trivial determinant on E. A similar result holds if E is cuspidal. In case E is nodal, however, there exist S-equivalent bundles V and V' such that $\zeta(V) \neq \zeta(V')$. It seems likely that, in case E is nodal, $\zeta(V) = \zeta(V')$ if and only if V and V' are restricted S-equivalent (0.5).

The moduli space $|np_0|$ is not a fine moduli space, for two reasons. One problem is the issue of S-equivalence versus isomorphism. To deal with this problem, we will attempt to choose a "best" representative for each S-equivalence class, the regular representative. In case E is smooth, a regular bundle V is one whose automorphism group has dimension equal to its rank, the minimum possible dimension. Even after choosing the regular representative, however, $|np_0|$ fails to be a fine moduli space because the bundles V are never simple. This allows us to twist universal bundles by line bundles on an n-sheeted cover of $|np_0|$, the spectral cover. This construction will be described in Section 2.

1.1. The Jordan-Hölder constituents of a semistable bundle.

The two main results of this section are the following:

Theorem 1.1. Let V be a semistable torsion free sheaf of rank n and degree zero over E. Then V has a Jordan-Hölder filtration

$$0 \subset F^0 \subset F^1 \subset \cdots \subset F^n = V$$

so that each quotient F^i/F^{i-1} is a rank one torsion free sheaf of degree zero. For λ a rank one torsion free sheaf of degree zero, define $V(\lambda)$ to be the sum of all the subsheaves of V which have a filtration such that all of the successive quotients are isomorphic to λ . Then $V = \bigoplus_{\lambda} V(\lambda)$. In particular, if V is locally free, then $V(\lambda)$ is locally free for every λ .

Theorem 1.2. Let V be a semistable torsion free sheaf of rank n and degree zero on E. Then

(i) $h^0(V \otimes \mathcal{O}_E(p_0)) = n$ and the natural evaluation map

$$ev: H^0(V \otimes \mathcal{O}_E(p_0)) \otimes_{\mathbb{C}} \mathcal{O}_E \to V \otimes \mathcal{O}_E(p_0)$$

is an isomorphism over the generic point of E.

(ii) Suppose that V is locally free with $\det V = \mathcal{O}_E(e - p_0)$. The induced map on determinants defines a map

$$\wedge^n ev \colon \det H^0(V \otimes \mathcal{O}_E(p_0)) \otimes_{\mathbb{C}} \mathcal{O}_E \cong \mathcal{O}_E \to \det (V \otimes \mathcal{O}_E(p_0)) \cong \mathcal{O}_E((n-1)p_0 + e).$$

Thus $\wedge^n ev$ defines a non-zero section of $\mathcal{O}_E((n-1)p_0+e)$ up to a nonzero scalar multiple, i.e. a point of $|(n-1)p_0+e|$. We denote this element by $\zeta(V)$. In particular, if $e=p_0$, then $\zeta(V) \in |np_0|$.

Proof of Theorem 1.2. Let V be a semistable sheaf of degree zero and rank n on E. The degree of $V \otimes \mathcal{O}_E(p_0)$ is n. By definition, $h^0(V \otimes \mathcal{O}_E(p_0)) - h^1(V \otimes \mathcal{O}_E(p_0)) = n$. By Serre duality, $h^1(V \otimes \mathcal{O}_E(p_0)) = \dim \operatorname{Hom}(V, \mathcal{O}_E(-p_0))$. Since V is semistable, $\operatorname{Hom}(V, \mathcal{O}_E(-p_0)) = 0$. Thus $h^0(V \otimes \mathcal{O}_E(p_0)) = n$. Next we claim that the induced map $ev: H^0(V \otimes \mathcal{O}_E(p_0)) \otimes_{\mathbb{C}} \mathcal{O}_E \to V \otimes \mathcal{O}_E(p_0)$ is an isomorphism over the generic point of E; equivalently, its image $I \subset V \otimes \mathcal{O}_E(p_0)$ has rank n. To prove this, we use the following lemma.

Lemma 1.3. Let E be a Weierstrass cubic, let I be a torsion free sheaf on E and let $\mu_0(I)$ be the maximal value of $\mu(J)$ as J runs over all torsion free subsheaves of I. Then

$$h^0(I) \le \max(\mu_0(I), 1) \operatorname{rank} I.$$

Proof. If $0 \subset F^0 \subset \cdots \subset F^k = I$ is the Harder-Narasimhan filtration of I, then $\mu_0(I) = \mu(F^0)$, F^{i+1}/F^i is semistable, and $\mu(F^{i+1}/F^i) < \mu_0(I)$ for all $i \geq 1$. Furthermore

$$h^0(I) \le \sum_i h^0(F^{i+1}/F^i).$$

Now if $\mu(F^{i+1}/F^i) > 0$, then since $h^1(F^{i+1}/F^i) = \dim \operatorname{Hom}(F^{i+1}/F^i, \mathcal{O}_E) = 0$, it follows that $h^0(F^{i+1}/F^i) = \deg(F^{i+1}/F^i) = \mu(F^{i+1}/F^i) \cdot \operatorname{rank}(F^{i+1}/F^i) \le \mu_0(I) \operatorname{rank}(F^{i+1}/F^i)$. If $\mu(F^{i+1}/F^i) < 0$, then

$$h^0(F^{i+1}/F^i) = 0 \le \operatorname{rank}(F^{i+1}/F^i).$$

There remains the case that $\mu(F^{i+1}/F^i) = 0$. In this case, we claim that

$$h^0(F^{i+1}/F^i) \le \operatorname{rank}(F^{i+1}/F^i).$$

In fact since F^{i+1}/F^i is semistable, this follows from the next claim.

Lemma 1.4. If V is a semistable torsion free sheaf on E with $\mu(V) = 0$, then $h^0(V) \leq \operatorname{rank} V$.

Proof. Argue by induction on rank V. If rank V=1 and $h^0(V) \geq 1$, then there exists a nonzero map $\mathcal{O}_E \to V$, and since $\mu(\mathcal{O}_E) = \mu(V)$, this map must be an isomorphism. Thus $h^0(V) = 1$. In general, if rank V = n+1 and $h^0(V) \neq 0$, choose a nonzero map $\mathcal{O}_E \to V$. Since V is semistable, the cokernel Q of this map is torsion free and thus is also semistable, with $\mu(Q) = 0$. Since the rank of Q is n, by induction we have $h^0(V) \leq 1 + h^0(Q) \leq n+1$. \square

Returning to the proof of (1.3), we see that in all cases

$$h^0(F^{i+1}/F^i) \le \max(\mu_0(I), 1) \operatorname{rank}(F^{i+1}/F^i).$$

Summing over i gives the statement of (1.3). \square

We continue with the proof of Theorem 1.2. There is the map

$$ev: H^0(V \otimes \mathcal{O}_E(p_0)) \otimes_{\mathbb{C}} \mathcal{O}_E \to V \otimes \mathcal{O}_E(p_0).$$

Let I be its image. By construction I is a subsheaf of a locally free sheaf and hence is torsion free. Also, by construction the map $H^0(I) \to H^0(V \otimes \mathcal{O}_E(p_0))$ is an isomorphism, and thus $h^0(I) = n$. Since $V \otimes \mathcal{O}_E(p_0)$ is semistable and $\mu(V \otimes \mathcal{O}_E(p_0)) = 1$, we have $\mu_0(I) \leq 1$. Thus, by (1.3), $n = h^0(I) \leq \text{rank } I \leq n$, and so rank I = n. Equivalently, the image of ev is equal to $V \otimes \mathcal{O}_E(p_0)$ at the generic point. From this, the remaining statements in Theorem 1.2 are clear. \square

Proof of Theorem 1.1. Let us first show that V has a Jordan-Hölder filtration as described. The proof is by induction on the rank n of V. If n = 1, there is nothing to prove. For arbitrary n, we shall show that there exists a nonzero map $\lambda \to V$,

where λ is a rank one torsion free sheaf of degree at least zero. By semistability, the degree of λ is exactly zero and V/λ is torsion free. We can then apply induction to V/λ .

The proof of Theorem 1.2 above shows that, if V is a semistable torsion free sheaf of rank n and degree zero, then there is an injective map $\mathcal{O}_E^{\oplus n} \to V \otimes \mathcal{O}_E(p_0)$ whose image has rank n. Thus there is a map $\mathcal{O}_E(-p_0)^{\oplus n} \to V$ whose image has rank n. The cokernel of this map must be a torsion sheaf τ . Note that, in case V is locally free, τ is supported exactly at the points in the support of $\zeta(V)$. Since $\deg V = 0$, $\tau \neq 0$. Choose a point x in the support of τ . If R is the local ring of E at x and m is the maximal ideal of x, then τ_x is annihilated by some power of m. Let k be such that $\mathfrak{m}^k \tau \neq 0$ but $\mathfrak{m}^{k+1} \tau = 0$. Choosing a section of $\mathfrak{m}^k \tau$ produces a subsheaf τ_0 of τ which is isomorphic to \mathbb{C}_x , in other words is isomorphic to R/\mathfrak{m} as an R-module.

Let $V_0 \subseteq V$ be the inverse image of τ_0 . Then V_0 corresponds to an extension of \mathbb{C}_x by $\mathcal{O}_E(-p_0)^{\oplus n}$, and hence to an extension class in

$$\operatorname{Ext}^1(\mathbb{C}_x, \mathcal{O}_E(-p_0)^{\oplus n}) \cong H^0(\operatorname{Ext}^1(\mathbb{C}_x, \mathcal{O}_E(-p_0)^{\oplus n}) \cong \operatorname{Ext}^1_R(R/\mathfrak{m}, R^n).$$

The ring R is a Gorenstein local ring of dimension one, and so $\operatorname{Ext}^1_R(R/\mathfrak{m},R) \cong \mathbb{C}$. (Of course, this could be verified directly for the local rings R under consideration.) In fact, if x is a smooth point of E and t is a local parameter at x, then the unique nontrivial extension of R/\mathfrak{m} by R corresponds to the exact sequence

$$0 \to R \xrightarrow{\times t} R \to R/\mathfrak{m} \to 0,$$

whereas if x is a singular point then the nontrivial extension is given by

$$0 \to R \to \tilde{R} \to R/\mathfrak{m} \to 0.$$

Let ξ be the extension class corresponding to V_0 in

$$\operatorname{Ext}^1(\mathbb{C}_x, \mathcal{O}_E(-p_0)^{\oplus n}) \cong \operatorname{Ext}^1_R(R/\mathfrak{m}, R^n) \cong \mathbb{C}^n.$$

In the local setting, let M be the R-module corresponding to V_0 , and suppose that we are given an extension

$$0 \to R \to N \to R/\mathfrak{m} \to 0$$
,

with a corresponding extension class $\eta \in \operatorname{Ext}^1_R(R/\mathfrak{m},R)$ and a homomorphism $f\colon R \to R^n$ such that $f_*(\eta) = \xi$. By a standard result, there is a homomorphism $N \to M$ lifting f, viewed as a homomorphism $R \to M$. In particular, this says that the image of R in M is contained in a strictly larger rank one torsion free R-module.

Returning to the global situation, let λ be the unique nontrivial extension of \mathbb{C}_x by $\mathcal{O}_E(-p_0)$, and let η be the corresponding extension class, well-defined up to a nonzero scalar. Thus λ is a rank one torsion free sheaf of degree zero. Since $Hom(\mathcal{O}_E(-p_0), \mathcal{O}_E(-p_0)^{\oplus n})$ is generated by its global sections, there exists a homomorphism $f \colon \mathcal{O}_E(-p_0) \to \mathcal{O}_E(-p_0)^{\oplus n}$ such that the image of η under f_* in $\operatorname{Ext}^1(\mathbb{C}_x, \mathcal{O}_E(-p_0)^{\oplus n})$ is ξ . Then the inclusion $\mathcal{O}_E(-p_0) \to \mathcal{O}_E(-p_0)^{\oplus n} \to V_0 \to V$ factors through a nonzero map $\lambda \to V$, necessarily an inclusion with torsion free

cokernel. Thus we have proved the existence of the Jordan-Hölder filtration by induction.

By using the fact that $\operatorname{Ext}^1(\lambda,\lambda')=0$ if $\lambda\neq\lambda'$, an easy argument left to the reader shows that $V(\lambda)\neq0$ if and only if $\operatorname{Hom}(V,\lambda)\neq0$ if and only if $\operatorname{Hom}(\lambda,V)\neq0$. Thus we can always arrange that, if λ is a sheaf appearing as one of the quotients in Theorem 1.1, then there exists a filtration for which $\lambda=F^0$ is the first such sheaf which appears, and also one for which $\lambda=F^n/F^{n-1}$ is the last such sheaf which appears.

Fix a rank one torsion free sheaf λ of degree zero, and let $V'(\lambda)$ be the sum of all subsheaves of V which have a filtration by rank one torsion free sheaves of degree zero which are not isomorphic to λ . Let $V(\lambda) = V/V'(\lambda)$. Clearly $V(\lambda)$ is a torsion free semistable sheaf, such that all of the quotients in a Jordan-Hölder filtration of V are isomorphic to λ . Again using $\operatorname{Ext}^1(\lambda, \lambda') = 0$ if $\lambda \neq \lambda'$, one checks that $\operatorname{Ext}^1(V(\lambda), V'(\lambda)) = 0$. Thus, by induction on the rank, V is isomorphic to the direct sum of the $V(\lambda)$. This concludes the proof of Theorem 1.1. \square

The construction of Theorem 1.2 works well in families.

Theorem 1.5. Let E be a Weierstrass cubic, and let S be a scheme or analytic space. Let V be a rank n vector bundle over $S \times E$ such that on each slice $\{s\} \times E$, V restricts to a semistable vector bundle V_s of trivial determinant. Then there exists a morphism $\Phi \colon S \to |np_0| = \mathbb{P}^{n-1}$ such that, for all $s \in S$, we have $\Phi(s) = \zeta(V_s)$. In particular, if V and V' are restricted S-equivalent, then $\zeta(V) = \zeta(V')$.

Proof. Let p_1, p_2 be the projections from $S \times E$ to S and E. To construct a morphism from S to $|np_0|$ we shall construct a homomorphism $\Psi: p_1^*L_0 \to p_1^*L_1 \otimes p_2^*\mathcal{O}_E(np_0)$, where L_0, L_1 are line bundles on S, with the property that the restriction of Ψ to each slice $\{s\} \times E$ determines a nonzero section of $\mathcal{O}_E(np_0)$ (which is thus well-defined mod scalars), agreeing with $\wedge^n ev$. The map Ψ is defined in the next lemma.

Lemma 1.6. The sheaf $p_{1*}(\mathcal{V} \otimes p_2^*\mathcal{O}_E(p_0))$ is a locally free sheaf of rank n on S. Let L_0 be its determinant line bundle. If $\hat{\Psi}: p_1^*p_{1*}(\mathcal{V} \otimes p_2^*\mathcal{O}_E(p_0)) \to \mathcal{V} \otimes p_2^*\mathcal{O}_E(p_0)$ is the natural evaluation map, then its restriction to each slice $\{s\} \times E$ is generically an isomorphism, agreeing with ev. Thus

$$\Psi = \det \hat{\Psi} : p_1^* L_0 \to \det \mathcal{V} \otimes p_2^* \mathcal{O}_E(np_0)$$

has the property that its restriction to each slice $\{s\} \times E$ is is nonzero and agrees with $\wedge^n ev$.

Proof. It follows from Theorem 1.2 that, if \mathcal{V}_s is the restriction of \mathcal{V} to the slice $\{s\} \times E$, then $h^0(\mathcal{V}_s \otimes \mathcal{O}_E(p_0)) = n$ is independent of s. Standard base change arguments [10, Theorem 12.11, pp. 290–291] show that, even if S is nonreduced, $p_{1*}(\mathcal{V} \otimes p_2^*\mathcal{O}_E(p_0))$ is a locally free sheaf of rank n on S, and the natural map $p_{1*}(\mathcal{V} \otimes p_2^*\mathcal{O}_E(p_0))_s \to H^0(V_s \otimes \mathcal{O}_E(p_0))$ is an isomorphism for every $s \in S$. Thus the induced morphism $\hat{\Psi} \colon p_1^*p_{1*}(\mathcal{V} \otimes p_2^*\mathcal{O}_E(p_0)) \to \mathcal{V} \otimes p_2^*\mathcal{O}_E(p_0)$ is a morphism between two vector bundles of rank n. Let V_s be the restriction of \mathcal{V} to the slice $\{s\} \times E$. Again by base change, the natural map $p_{1*}(\mathcal{V} \otimes p_2^*\mathcal{O}_E(p_0))_s \to H^0(V_s \otimes \mathcal{O}_E(p_0)) \otimes_{\mathbb{C}} \mathcal{O}_E$ is surjective. The result is now immediate from Theorem 1.2. \square

Next notice that since for every $s \in S$, $\det \mathcal{V}|(\{s\} \times E)$ is trivial, it follows that $\det \mathcal{V}$ is isomorphic to $p_1^*L_1$ for some line bundle L_1 on S. To complete the proof

of Theorem 1.5 we need to check that the section $\Psi(s) = \zeta(V_s)$ for all $s \in S$. This is immediate from the corresponding statement in Theorem 1.2. \square

In fancier terms, Theorem 1.5 says that there is a morphism of functors from the deformation functor of semistable vector bundles of rank n and trivial determinant on E to the functor represented by the scheme $|np_0|$. In general, this morphism is far from smooth; for example, at the trivial bundle \mathcal{O}_E^n , the derivative of the morphism is identically zero. However, if we restrict to regular semistable bundles (to be defined in §1.2 below), then it will follow from (v) in Theorem 3.2 that the derivative is always an isomorphism.

The sheaves λ which appear as successive quotients of V in Theorem 1.1 are the Jordan-Hölder quotients or Jordan-Hölder constituents of V. They appear with multiplicities and the multiplicity of λ in V is independent of the choice of the filtration. The summands $V(\lambda)$ of V are canonically defined. It is easy to see from the construction that $\zeta(V) = \sum_{\lambda} \zeta(V(\lambda))$. More generally, ζ is additive over exact sequences of semistable vector bundles of degree zero. Also, if $\det V = \mathcal{O}_E(e-p_0)$ and e' is a smooth point of E, then e' lies in the support of $\zeta(V)$ as a divisor in $|(n-1)p_0+e|$ if and only if $\lambda = \mathcal{O}_E(e'-p_0)$ is a Jordan-Hölder constituent of V. Thus, if the rank of $V(\lambda)$ is d_{λ} , then

$$\zeta(V) = \sum_{\lambda \neq \mathcal{F}} d_{\lambda} e_{\lambda} + e_{\mathcal{F}},$$

where $\lambda \cong \mathcal{O}_E(e_{\lambda} - p_0)$ and $e_{\mathcal{F}}$ is a divisor of degree $d_{\mathcal{F}}$ supported at the singular point of E. In this way we can associate a point of the n^{th} symmetric product of E with such a V: namely

$$\sum_{\lambda \neq \mathcal{F}} \operatorname{rank}(V(\lambda)) e_{\lambda} + d_{\mathcal{F}} \cdot s$$

where $\lambda \cong \mathcal{O}_E(e_{\lambda} - p_0)$ and $s \in E$ is the singular point. Note that there is a morphism $|np_0| \to \operatorname{Sym}^n E$, which is a closed embedding if E is smooth, or more generally away from the elements of $|np_0|$ whose support meets the singular point of E.

Suppose that E is smooth. Since a degree zero line bundle on E is identified with a point of E via the correspondence $\lambda \mapsto q$ if $\lambda \cong \mathcal{O}_E(q-p_0)$, the map which assigns to a semistable bundle V the unordered n-tuple of its Jordan-Hölder quotients, including multiplicities, is the same as the map assigning to V an unordered n-tuple $\zeta(V)$ of points of E, i.e., a point

$$\zeta(V) \in \underbrace{(E \times \cdots \times E)}_{n \text{ times}} / \mathfrak{S}_n,$$

where \mathfrak{S}_n is the symmetric group on n letters. If $\zeta(V)=(e_1,\ldots,e_n)$, then the condition that the determinant of V is trivial means that $\sum_{i=1}^n e_i=0$ in the group law of E, or equivalently that the divsior $\sum_{i=1}^n e_i$ is linearly equivalent to np_0 . Thus, the unordered n-tuple (e_1,\ldots,e_n) associated to V can be identified with a point in the complete linear system $|np_0|$, and this point is exactly $\zeta(V)$.

An important difference in case E is singular is that, while a point of $|np_0|$ determines a point on the symmetric n-fold product of E, in general it contains

more information at the singular point than just its multiplicity. Thus, the function Φ should be viewed not as a point in the n-fold symmetric product but as a point in the linear system $|np_0|$. For example, if E is nodal and n>2, then an element of $|np_0|$ supported entirely at the singular point corresponds to a hyperplane in \mathbb{P}^{n-1} meeting the image of E embedded by the complete linear system $|np_0|$ just at the singular point. As such, it is specified by two positive integers a and b with a+b=n, the orders of contact of the hyperplane with the two branches of E at the node.

1.2. Regular bundles over a Weierstrass cubic.

Let E be a Weierstrass cubic. Every semistable bundle is of the form

$$V \cong \bigoplus_{\lambda} V(\lambda)$$

where λ ranges over the isomorphism classes of rank one torsion free sheaves on E of degree zero. Let us first analyze $V(\lambda)$ in case λ is a line bundle. If $V(\lambda)$ is a semistable bundle with the property that all Jordan-Hölder quotients of V are isomorphic to λ , or in other words $H^0(\lambda' \otimes V(\lambda)) = 0$ for all $\lambda' \neq \lambda^{-1}$, then the associated graded to every Jordan-Hölder filtration of $V(\lambda)$ is a direct sum of line bundles isomorphic to λ . Of course, one possibility for $V(\lambda)$ is the split one:

$$V(\lambda) \cong \lambda^{\oplus r}$$
.

At the other extreme we have the maximally non-split case:

Lemma 1.7. Let E be a Weierstrass cubic, possibly singular. For each natural number r > 0 and each line bundle λ of degree zero there is a unique bundle $I_r(\lambda)$ up to isomorphism with the following properties:

- (i) the rank of $I_r(\lambda)$ is r.
- (ii) all the Jordan-Hölder quotients of $I_r(\lambda)$ are isomorphic to λ .
- (iii) $I_r(\lambda)$ is indecomposable under direct sum.

Furthermore, for all r > 0 and all line bundles λ , $I_r(\lambda)$ is semistable, $I_r(\lambda)^{\vee} = I_r(\lambda^{-1})$, det $I_r(\lambda) = \lambda^r$, and dim $\text{Hom}(I_r(\lambda), \lambda) = \text{dim Hom}(\lambda, I_r(\lambda)) = 1$.

Proof. We first construct the bundle $I_r = I_r(\mathcal{O}_E)$ by induction on r. For r = 1 we set $I_r = \mathcal{O}_E$. Suppose inductively that we have constructed I_{r-1} with the properties given in the lemma. Suppose in addition that $H^0(I_{r-1}) \cong \mathbb{C}$. Since the degree of I_{r-1} is zero, it follows that $H^1(I_{r-1}) \cong \mathbb{C}$ and hence there is exactly one non-trivial extension, up to scalar multiples, of the form

$$0 \to I_{r-1} \to I_r \to \mathcal{O}_E \to 0.$$

One checks easily all the inductive hypotheses for the total space of this extension. This proves the existence of I_r for all r > 0. Uniqueness is easy and is left to the reader.

We define $I_r(\lambda) = I_r \otimes \lambda$. The statements of (1.7) are then clear. \square

The bundle $I_r(\lambda)$ has an increasing filtration by subbundles isomorphic to $I_k(\lambda)$, $k \leq r$. We denote this filtration by

$$\{0\} \subset F_1I_r(\lambda) \subset \cdots \subset F_rI_r(\lambda) = I_r(\lambda),$$

and refer to $F_iI_r(\lambda)$ as the i^{th} filtrant of $I_r(\lambda)$. When the bundle is clear from the context, we denote the subbundles in this filtration by F_i . Notice that $F_t \cong I_t(\lambda)$ and that $I_r(\lambda)/F_{r-t} \cong I_t(\lambda)$.

Let us note some of the basic properties of the bundles $I_r(\lambda)$.

Lemma 1.8. Let J be a proper degree zero subsheaf of $I_r(\lambda)$. Then J is contained in F_{r-1} . In fact, $J = F_t$ for some t < r.

Proof. By the semistability of $I_r(\lambda)$, $I_r(\lambda)/J$ is a nonzero semistable torsion free sheaf of degree zero. Clearly all of its Jordan-Hölder quotients are isomorphic to λ . In particular there is a nonzero map $I_r(\lambda)/J \to \lambda$. The composition $I_r(\lambda) \to I_r(\lambda)/J \to \lambda$ defines a nonzero map from $I_r(\lambda)$ to λ containing J in its kernel. By (1.7), there is a unique such nonzero map mod scalars, and its kernel is F_{r-1} . Thus $J \subseteq F_{r-1}$. Applying induction to the inclusion $J \subseteq F_{r-1} \cong I_{r-1}(\lambda)$, we see that $J = F_t$ for some t < r. \square

Our next result is that the filtration is canonical, i.e., invariant under any automorphism of $I_r(\lambda)$.

Corollary 1.9. If $\varphi: I_r(\lambda) \to I_r(\lambda)$ is a homomorphism, then, for all $i \leq r$,

$$\varphi(F_i) \subset F_i$$
.

It follows that if φ is an automorphism, then for all i we have $\varphi(F_i) = F_i$. More generally, if $\varphi: I_r(\lambda) \to I_t(\lambda)$ is a homomorphism, then

$$\varphi(F_s(I_r(\lambda))) \subseteq F_s(I_t(\lambda)).$$

Proof. It suffices to prove the last statement. By the semistability of $I_r(\lambda)$ and $I_t(\lambda)$, $\varphi(F_s(I_r(\lambda)))$ is a degree zero subsheaf of $I_t(\lambda)$ of rank at most s. Thus it is contained in $F_s(I_t(\lambda))$. \square

Lemma 1.10. Fix $0 \le t \le r$ and let $q_{r,t}: I_r(\lambda) \to I_t(\lambda)$ be the natural quotient map. Then $q_{r,t}$ induces a surjective homomorphism from the endomorphism algebra of $I_r(\lambda)$ to that of $I_t(\lambda)$. A similar statement holds for the automorphism groups. Finally, as a \mathbb{C} -algebra, $\operatorname{Hom}(I_r(\lambda), I_r(\lambda)) \cong \mathbb{C}[t]/(t^r)$.

Proof. That $q_{r,t}$ induces a map on endomorphism algebras is immediate from (1.9). Let us show that it is surjective. We might as well assume that t=r-1>0 since the other cases will then follow by induction. Let $A: I_{r-1}(\lambda) \to I_{r-1}(\lambda)$ be an endomorphism. Since the map $q_{r,r-1}: I_r(\lambda) \to I_{r-1}(\lambda)$ induces the zero map on $\operatorname{Hom}(\lambda,\cdot)$, it follows by duality that the map $q_{r,r-1}^*: \operatorname{Ext}^1(I_{r-1}(\lambda),\lambda) \to \operatorname{Ext}^1(I_r(\lambda),\lambda)$ is zero. Hence the composition $A \circ q_{r,r-1}: I_r(\lambda) \to I_{r-1}(\lambda)$ lifts to a map $\hat{A}: I_r(\lambda) \to I_r(\lambda)$. Thus the restriction map on endomorphism algebras is surjective. To see the statement on automorphism groups, suppose that A is an isomorphism. We wish to show that \hat{A} is an isomorphism. To see this, perform the construction for A^{-1} as well, obtaining a map $\widehat{A^{-1}}: I_r(\lambda) \to I_r(\lambda)$. The composition $B = \widehat{A^{-1}} \circ \hat{A}: I_r(\lambda) \to I_r(\lambda)$ projects to the identity on $I_{r-1}(\lambda)$. This means that $B - \operatorname{Id}: I_r(\lambda) \to F_1(I_\lambda)$. Since r > 1, this map is nilpotent, and hence $B = \operatorname{Id} + (B - \operatorname{Id})$ is an isomorphism.

Finally we prove the last statement. Let $A_r: I_r(\lambda) \to I_r(\lambda)$ be any endomorphism defined by a composition

$$I_r(\lambda) \twoheadrightarrow I_{r-1}(\lambda) \hookrightarrow I_r(\lambda).$$

Note that $A_r^r = 0$ and that the restriction of A_r to $I_r(\lambda)/F_1 \cong I_{r-1}(\lambda)$ is of the form A_{r-1} . Suppose by induction that $\operatorname{Hom}(I_{r-1}(\lambda), I_{r-1}(\lambda)) = \mathbb{C}[A_{r-1}]$. Then every

endomorphism T of $I_r(\lambda)$ is of the form $T = p(A_r) + T'$, where p is a polynomial of degree at most r-2 in A_r and T' induces the zero map on $I_r(\lambda)/F_1$. In this case T' is given by a map from $I_r(\lambda)$ to F_1 , necessarily zero on F_{r-1} , and it is easy to check that T' must in fact be a multiple of A_r^{r-1} . Thus $\operatorname{Hom}(I_r(\lambda), I_r(\lambda)) = \mathbb{C}[A_r] \cong \mathbb{C}[t]/(t^r)$. \square

We need to define an analogue of $I_r(\lambda)$ in case the Jordan-Hölder quotients are all isomorphic to the non-locally free sheaf \mathcal{F} . We say that a semistable degree zero bundle $I(\mathcal{F})$ concentrated at the singular point of E is strongly indecomposable if $Hom(I(\mathcal{F}),\mathcal{F})\cong\mathbb{C}$. Notice that since $Hom(V(\mathcal{F}),\mathcal{F})\neq 0$ for any non-trivial semistable bundle $V(\mathcal{F})$ concentrated at the singular point, it follows that if $I(\mathcal{F})$ is strongly indecomposable, then it is indecomposable as a vector bundle in the usual sense. However, the converse is not true: there exist indecomposable vector bundles which are not strongly indecomposable. It is natural to ask if every vector bundle supported at \mathcal{F} is an extension of strongly indecomposable bundles. Unlike the smooth case, it is also not true that $I(\mathcal{F})$ is determined up to isomorphism by its rank and the fact that it is strongly indecomposable. Nor is it true that $I(\mathcal{F})$ always has a unique filtration with successive quotients isomorphic to \mathcal{F} . As we shall show in Section 3, $I(\mathcal{F})$ is determined up to isomorphism by its rank and the point $\zeta(I(\mathcal{F}))$.

There is the following analogue for $I(\mathcal{F})$ of (1.8):

Lemma 1.11. Suppose that $I(\mathcal{F})$ is strongly indecomposable. Let $\rho: I(\mathcal{F}) \to \mathcal{F}$ be a nonzero homomorphism, unique up to scalar multiples, and let $X = \operatorname{Ker} \rho$. If J is a proper degree zero subsheaf of $I(\mathcal{F})$, then J is contained in X.

Proof. Let $J \subset I(\mathcal{F})$ be a subsheaf of degree zero. The quotient $Q = I(\mathcal{F})/J$ must be torsion-free, for otherwise J would be contained in a larger subsheaf \hat{J} of the same rank and bigger degree, contradicting the semistability of $I(\mathcal{F})$. This means that Q is semistable of degree zero. Clearly, it is concentrated at the singular point. Thus, there is a nontrivial map $Q \to \mathcal{F}$. By the strong indecomposability of $I(\mathcal{F})$, the composition $I(\mathcal{F}) \to Q \to \mathcal{F}$ is some nonzero multiple of ρ . In particular, the kernel of this composition is X. This proves that $J \subset X$. \square

Definition 1.12. Let V be a semistable bundle with trivial determinant over a Weierstrass cubic. We say that V is regular or maximally nonsplit if,

$$V \cong \bigoplus_{i} I_{r_i}(\lambda_i) \oplus I(\mathcal{F})$$

where the λ_i are pairwise distinct line bundles and $I(\mathcal{F})$ is a strongly indecomposable bundle concentrated at the singular point.

For E smooth, Atiyah proved [1] that every vector bundle V, all of whose Jordan-Hölder quotients are isomorphic to λ , can be written as a direct sum $\bigoplus_i I_{r_i}(\lambda)$. The argument carries over to the case where E is singular, provided that λ is a line bundle $\mathcal{O}_E(e-p_0)$. Thus, in this case there is a unique regular bundle V of rank r such that the support of $\zeta(V)$ is e. More generally, given a divisor $e_1+\cdots+e_n\in |np_0|$ supported on the smooth points, there is a unique regular semistable rank n vector bundle V of trivial determinant over E such that $\zeta(V)=(e_1,\ldots,e_n)$. An analogue of Atiyah's theorem for the singular points has been established by T. Teodorescu [12]. In this paper, we shall show in Section 3 that, given a Cartier divisor D in

 $|np_0|$ whose support is the singular point, then there is a unique regular semistable rank n vector bundle V of trivial determinant such that $\zeta(V) = D$.

Regular bundles have an extremely nice property: Their automorphism groups have minimal possible dimension. We shall show this for smooth E in the next lemma. To put this property in context, let us consider first the centralizers of elements in $GL_n(\mathbb{C})$. The centralizer of any element has dimension at least n. Elements in $GL_n(\mathbb{C})$ whose centralizers have dimension exactly n are said to be regular elements. Every element in $GL_n(\mathbb{C})$ is S-equivalent to a unique regular element up to conjugation. Here two elements $A, B \in GL_n(\mathbb{C})$ are said to be S-equivalent if every algebraic function on $GL_n(\mathbb{C})$ which is invariant under conjugation takes the same value on A and B. Said another way, A and B are S-equivalent if there is an element $C \in GL_n(\mathbb{C})$ which is in the closure of the orbits of both A and B under the conjugation action of $GL_n(\mathbb{C})$ on itself. From our point of view regular bundles are the analogue of regular elements. In fact, for a smooth elliptic curve E, one way to construct a holomorphic vector bundle over E is to fix an element u in the Lie algebra of $SL_n(\mathbb{C})$. Define a holomorphic connection on the trivial bundle

$$\overline{\partial}_u = \overline{\partial} + ud\overline{z}$$

where $\overline{\partial}$ is the usual operator on the trivial bundle. If u is close to the origin in the Lie algebra, then the automorphism group of this new holomorphic bundle will be the centralizer of u in $GL_n(\mathbb{C})$. In particular, this bundle will be regular and have trivial determinant if and only if $U = \exp(u)$ is a regular element in $SL_n(\mathbb{C})$. For example, if U is a regular semisimple element of $SL_n(\mathbb{C})$ then the corresponding vector bundle over E will be a sum of distinct line bundles of degree zero. More generally, the decomposition of U into its generalized eigenspaces will correspond to the decomposition of V into its components $V(\lambda)$. Clearly, S-equivalent elements of $GL_n(\mathbb{C})$ yield S-equivalent bundles.

Here is the analogue of the dimension statements for vector bundles over a smooth elliptic curve.

Lemma 1.13. Let V be a semistable rank n vector bundle over a smooth elliptic curve E.

- (i) dim Hom $(V, V) \ge n$.
- (ii) V is regular if and only if $\dim \operatorname{Hom}(V,V) = n$. In this case, if $V = \bigoplus_i I_{d_i}(\lambda_i)$, then the \mathbb{C} -algebra $\operatorname{Hom}(V,V)$ is isomorphic to $\bigoplus_i \mathbb{C}[t]/(t^{d_i})$. In particular, $\operatorname{Hom}(V,V)$ is an abelian \mathbb{C} -algebra.
- (iii) V is regular if and only if, for all line bundles λ of degree zero on E, $h^0(V \otimes \lambda^{-1}) \leq 1$.

Proof. It is easy to check that $\operatorname{Hom}(V(\lambda), V(\lambda')) \neq 0$ if and only if $\lambda = \lambda'$, and (using Corollary 1.9 and Lemma 1.10) that $\operatorname{Hom}(I_d, I_d) \cong \mathbb{C}[t]/(t^d)$. The statements (i) and (ii) follow easily from this and from Atiyah's theorem. To see (iii), note that, for a line bundle μ of degree zero, $V(\mu)$ is regular if and only if $h^0(V(\mu) \otimes \mu^{-1}) = 1$, which implies that $h^0(V \otimes \lambda^{-1}) \leq 1$ for all λ since $h^0(V(\mu) \otimes \lambda^{-1}) = 0$ if $\lambda \neq \mu$. \square

We will prove a partial analogue of (ii) in Lemma 1.13 for singular curves in Section 3.

Very similar arguments show:

Lemma 1.14. Let E be a Weierstrass cubic and let V be a semistable rank n vector bundle over E. Then V is regular if and only if, for every rank one torsion free sheaf λ of degree zero on E, dim $\operatorname{Hom}(V,\lambda) \leq 1$. Moreover, suppose that V is regular and that S is a semistable torsion free sheaf of degree zero on E. Then $\operatorname{dim} \operatorname{Hom}(V,S) \leq \operatorname{rank} S$. \square

2. The spectral cover construction.

In this section we shall construct families of regular semistable bundles over a smooth elliptic curve E. The main result is Theorem 2.1, which gives the basic construction of a universal bundle over $|np_0| \times E$, where $|np_0| \cong \mathbb{P}^{n-1}$ is the coarse moduli space of the last section. We prove that the restriction of the universal bundle to every slice is in fact regular, and that every regular bundle occurs in this way. By twisting by a line bundle on the spectral cover, we construct all possible families of universal bundles (Theorem 2.4) and show how they are all related by elementary modifications. In Theorem 2.8, we generalize this result to families of regular semistable bundles parametrized by an arbitrary base scheme. In case E is singular, we establish slightly weaker versions of these results. Most of this material will be redone from a different perspective in the next section. Finally, we return to the smooth case and give the formulas for the Chern classes of the universal bundles

Throughout this section, unless otherwise noted, E denotes a smooth elliptic curve with origin p_0 .

2.1. The spectral cover of $|np_0|$.

Let E^{n-1} be embedded in E^n as the set of n-tuples (e_1,\ldots,e_n) such that $\sum_i e_i = 0$ in the group law on E, or equivalently, such that the divisor $\sum_i e_i$ on E is linearly equivalent to np_0 . The natural action of the symmetric group \mathfrak{S}_n on E^n thus induces an action of \mathfrak{S}_n on E^{n-1} . As we have seen, the quotient E^{n-1}/\mathfrak{S}_n is naturally the projective space $|np_0| \cong \mathbb{P}^{n-1}$. View \mathfrak{S}_{n-1} as the subgroup of \mathfrak{S}_n fixing n, and let $T = E^{n-1}/\mathfrak{S}_{n-1}$. Corresponding to the inclusion $\mathfrak{S}_{n-1} \subset \mathfrak{S}_n$ there is a morphism $\nu \colon T \to \mathbb{P}^{n-1}$ which realizes T as an n-sheeted cover of \mathbb{P}^{n-1} . Here ν is unbranched over $e_1 + \cdots + e_n \in |np_0|$ if and only if the e_i are distinct. The branch locus of ν in \mathbb{P}^{n-1} is naturally the dual hypersurface to the elliptic normal curve defined by the embedding of E in the dual projective space (except in case n=2, where it corresponds to the four branch points of the map from E to \mathbb{P}^1). The map $\nu \colon T \to |np_0|$ is called the spectral cover of $|np_0|$. We will discuss the reason for this name later.

The sum map $(e_1, \ldots, e_n) \mapsto -\sum_{i=1}^{n-1} e_i$ is a surjective homomorphism from E^n to E, and its restriction to E^{n-1} is again surjective, with fibers invariant under \mathfrak{S}_{n-1} . Thus there is an induced morphism $r: T \to E$. In fact, $r(e_1, \ldots, e_n) = e_n$ and

$$F_e = r^{-1}(e) = \{ (e_1, \dots, e_{n-1}, e) : \sum_{i=1}^{n-1} e_i + e = np_0 \},$$

modulo the obvious \mathfrak{S}_{n-1} -action. Thus the fiber of r over e is the projective space $|np_0 - e|$, of dimension n-2. Globally, T is the projectivization of the rank n-1 bundle \mathcal{E} over E defined by the exact sequence

$$0 \to \mathcal{E} \to H^0(E; \mathcal{O}_E(np_0)) \otimes_{\mathbb{C}} \mathcal{O}_E \to \mathcal{O}_E(np_0) \to 0,$$

where the last map is evaluation and is surjective since $\mathcal{O}_E(np_0)$ is generated by its global sections. The fiber of r over a point $e \in E$ consists of those sections of $\mathcal{O}_E(np_0)$ vanishing at e, and the corresponding projective space is just $|np_0 - e|$. We see that there is an induced morphism on projective bundles

$$g: \mathbb{P}\mathcal{E} \to \mathbb{P}\left(H^0(E; \mathcal{O}_E(np_0)) \otimes_{\mathbb{C}} \mathcal{O}_E\right) = |np_0| \times E \cong \mathbb{P}^{n-1} \times E,$$

such that g is a closed embedding of T onto the incidence divisor in $|np_0| \times E$, and that r is just the composition of this morphism with the projection $|np_0| \times E \to E$. Clearly ν is the composition of the morphism $g \colon \mathbb{P}\mathcal{E} \to |np_0| \times E$ with projection to the first factor, or equivalently $g = (\nu, r)$. Given $e \in E$, let $F_e = r^*e$ be the fiber over e and let ζ be the divisor class corresponding to $c_1(\mathcal{O}_T(1))$, viewing T as $\mathbb{P}\mathcal{E}$. Since \mathcal{E} sits inside the trivial bundle, it follows that $\zeta = g^*\pi_1^*h$, where $h = c_1(\mathcal{O}_{\mathbb{P}^{n-1}}(1))$, and thus $\zeta = \nu^*h$. Note also that each fiber F_e of $T = \mathbb{P}\mathcal{E} \to E$ is mapped linearly into the corresponding hyperplane $H_e = |np_0 - e|$ of $\mathbb{P}^{n-1} = |np_0|$ consisting of divisors containing e in their support. Thus as divisor classes $\nu_*[F_e] = h$.

There is a special point $\mathbf{o} = \mathbf{o}_E = np_0 \in |np_0|$. (In terms of regular semistable bundles, \mathbf{o} corresponds to I_n .) It is one of the n^2 points of ramification of order n for the map $T \to |np_0|$, corresponding to the n-torsion points of E.

2.2. A universal family of regular semistable bundles.

Next we turn to the construction of a universal family of regular semistable bundles E. It will be given by a bundle U_0 over $|np_0| \times E$. Over $E^{n-1} \times E$, we have the diagonal divisor

$$\{(e_1,\ldots,e_n,e): e=e_n, \sum_{i=1}^n e_i=0\},\$$

which is invariant under the \mathfrak{S}_{n-1} -action and so descends to a divisor Δ on $T \times E$, which is the graph of the map $r: T \to E$. Note that $\Delta \cong T$ and that

$$\Delta = (r \times \mathrm{Id})^* \Delta_0,$$

where Δ_0 is the diagonal $\{(e, e) : e \in E\}$. Let $G = T \times \{p_0\}$. Then the divisor $\Delta - G$ has the property that its restriction to a slice $\{(e_1, \dots, e_{n-1}, e_n)\} \times E$ can be identified with the line bundle $\mathcal{O}_E(e_n - p_0)$. We define $\mathcal{L}_0 \to T \times E$ to be the line bundle $\mathcal{O}_{T \times E}(\Delta - G)$, and we set

$$U_0 = (\nu \times \mathrm{Id})_* \mathcal{L}_0.$$

Theorem 2.1. Let E be a smooth elliptic curve. The sheaf U_0 over $|np_0| \times E$ constructed above is a vector bundle of rank n. For each $x \in |np_0|$ the restriction of U_0 to $\{x\} \times E$ is a regular semistable bundle V_x with trivial determinant and with the property that $\zeta(V_x) = x$.

Proof. Since $\nu \times \text{Id}$ is an *n*-sheeted covering of smooth varieties, it is a finite flat morphism and hence U_0 is a vector bundle of rank n. If $\{(e_1, \ldots, e_{n-1}, e_n)\}$ is not a branch point of ν , or in other words if the e_i are pairwise distinct, then

 $U_0 = (\nu \times \mathrm{Id})_* \mathcal{O}_{T \times E}(\Delta - G)$ restricts over the slice $\{(e_1, \dots, e_{n-1}, e_n)\} \times E$ to a bundle isomorphic to the direct sum

$$\bigoplus_{e_i} \mathcal{L}_0|\{e_i\} \times E$$

which is clearly isomorphic to

$$\mathcal{O}_E(e_1-p_0)\oplus\cdots\oplus\mathcal{O}_E(e_n-p_0).$$

This shows that for a generic point $s \in |np_0|$ the restriction $U_0|\{s\} \times E$ is as claimed: it is the unique regular semistable bundle with the given Jordan-Hölder quotients.

In general, consider a point $x \in |np_0|$ of the form $\sum_{i=1}^{\ell} r_i e_i$, where the $e_i \in E$ and the r_i are positive integers with $\sum_i r_i = n$. We claim that the Jordan-Hölder quotients of the corresponding bundle are $\mathcal{O}_E(e_i - p_0)$, with multiplicity r_i . The preimage of x in T consists of ℓ points y_1, \ldots, y_{ℓ} , each of multiplicity r_i . Viewing T as the incidence correspondence in $\mathbb{P}^{n-1} \times E$, the point y_i corresponds to $\left(\sum_{i=1}^{\ell} r_i e_i, e_i\right)$. If $R = \mathcal{O}_T/\mathfrak{m}_x \mathcal{O}_T$ is the coordinate ring of the fiber over x, then R is the product of ℓ local rings R_i of lengths r_1, \ldots, r_{ℓ} . It clearly suffices to prove the following claim.

Claim 2.2. In the above notation, $\mathcal{L}_0 \otimes R_i$ has a filtration all of whose successive quotients are isomorphic to λ_i where $\lambda_i \cong \mathcal{O}_E(e_i - p_0)$. In particular, the restriction of U_0 to this slice is semistable and has determinant λ^{r_i} .

Proof. The ring R_i has dimension r_i and is filtered by ideals whose successive quotients are isomorphic to \mathbb{C}_{y_i} . Thus $\mathcal{L}_0 \otimes R_i$ is filtered by subbundles whose quotients are all isomorphic to the line bundle $\mathcal{L}_0|\{y_i\} \times E$. But by construction this restriction is $\mathcal{O}_E(e_i - p_0)$. \square

At this point, we have seen that U_0 is a family of semistable bundles on E whose restriction to every fiber has trivial determinant and with the "correct" Jordan-Hölder quotients. It remains to show that U_0 is a family of regular bundles over E.

Claim 2.3. The restriction of U_0 to every slice $\{e\} \times E$ is regular.

Proof of the claim. To see that the restriction to each slice is regular, note that a semistable V of degree 0 on E is regular if and only if, for all line bundles λ on E of degree zero, $h^0(V \otimes \lambda^{-1}) \leq 1$. By Riemann-Roch on E, $h^0(V \otimes \lambda^{-1}) = h^1(V \otimes \lambda^{-1})$. Thus we must show that $h^1(V \otimes \lambda^{-1}) \leq 1$.

First we calculate $R^1\pi_{1*}(U_0 \otimes \pi_2^*\lambda^{-1})$, where $\pi_1: \mathbb{P}^{n-1} \times E \to \mathbb{P}^{n-1}$ is the projection to the first factor. Let $q_1: T \times E \to T$ be the first projection. Consider the diagram

$$T \times E \xrightarrow{\nu \times \mathrm{Id}} \mathbb{P}^{N-1} \times E$$

$$\downarrow^{q_1} \qquad \qquad \downarrow^{\pi_1}$$

$$T \xrightarrow{\nu} \mathbb{P}^{n-1}.$$

Since ν and $\nu \times \mathrm{Id}$ are affine, we obtain

$$R^1\pi_{1*}\left[(\nu\times\mathrm{Id})_*\mathcal{O}_{T\times E}(\Delta-G)\otimes\pi_2^*\lambda^{-1}\right]=\nu_*R^1q_{1*}\left[\mathcal{O}_{T\times E}(\Delta-G)\otimes q_2^*\lambda^{-1}\right].$$

Now apply flat base change to the Cartesian diagram

$$T \times E \xrightarrow{r \times \mathrm{Id}} E \times E$$

$$\downarrow^{q_1} \qquad \qquad \downarrow^{p_1}$$

$$T \xrightarrow{r} E.$$

We have $\mathcal{O}_{T\times E}(\Delta-G)=(r\times \mathrm{Id})^*\mathcal{O}_{E\times E}(\Delta_0-(E\times \{p_0\}))$, and thus the sheaf $R^1q_{1*}\left[\mathcal{O}_{T\times E}(\Delta-G)\otimes q_2^*\lambda^{-1}\right]$ is isomorphic to

$$r^*R^1p_{1*} \left[\mathcal{O}_{E \times E}(\Delta_0 - (E \times \{p_0\})) \otimes p_2^*\lambda^{-1} \right].$$

Restricting to the slice $\{e\} \times E$, we see that

$$R^1 p_{1*} \left(\mathcal{O}_{E \times E} (\Delta_0 - (E \times \{p_0\})) \otimes p_2^* \lambda^{-1} \right)$$

is supported at the point e of E corresponding to the line bundle λ (i.e. $\lambda = \mathcal{O}_E(e-p_0)$), and the calculation of [6], Lemma 1.19 of Chapter 7, shows that the length at this point is one. Thus taking r^* gives the sheaf \mathcal{O}_{F_e} , and $\nu_*\mathcal{O}_{F_e} = \mathcal{O}_{H_e}$, where H_e is a reduced hyperplane in \mathbb{P}^{n-1} . Thus we have seen that $R^1\pi_{1*}(U\otimes\pi_2^*\lambda^{-1})$ is (up to twisting by a line bundle) \mathcal{O}_{H_e} , where H_e is the hyperplane in \mathbb{P}^{n-1} corresponding to $|np_0-e|$. Since π_1 has relative dimension one, $R^2\pi_{1*}(U\otimes\pi_2^*\lambda^{-1})=0$. It follows by the theorem on cohomology and base change [10] Theorem 12.11(b) that the map $R^1\pi_{1*}(U\otimes\pi_2^*\lambda^{-1})\to H^1(V\otimes\lambda^{-1})$ is surjective, and thus $h^1(V\otimes\lambda^{-1})\leq 1$ as desired. \square

2.3. All universal families of regular semistable bundles.

We have constructed a bundle U_0 over $|np_0| \times E$ with given restriction to each slice. Our next goal is to understand all such bundles.

Theorem 2.4. Let E be a smooth elliptic curve. Let $\pi_1: |np_0| \times E \to |np_0|$ be the projection onto the first factor, and let U_0 be the bundle constructed in Theorem 2.1. Then:

- (i) The sheaf $\pi_{1*}Hom(U_0, U_0)$ is a locally free sheaf of algebras of rank n over $|np_0|$ which is isomorphic to $\nu_*\mathcal{O}_T$.
- (ii) Let U' be a rank n vector bundle over $|np_0| \times E$ with the following property. For each $x \in |np_0|$ the restriction of U' to $\{x\} \times E$ is isomorphic to the restriction of U_0 to $\{x\} \times E$. Then $U' = (\nu \times \mathrm{Id})_* [\mathcal{O}_{T \times E}(\Delta G) \otimes q_1^* L]$ for a unique line bundle L on T.

Proof. In view of Claim 2.3 and the definition of a regular bundle, $\pi_{1*}Hom(U_0, U_0)$ is a locally free sheaf of algebras of rank n over $|np_0|$. To see that it is isomorphic to $\nu_*\mathcal{O}_T$, note that multiplication by functions defines a homomorphism $\nu_*\mathcal{O}_T \to \pi_{1*}Hom(U_0, U_0)$ which is clearly an inclusion of algebras. Since both sheaves of algebras are rank n vector bundles over \mathbb{P}^{n-1} , they agree at the generic point of $|np_0|$. Thus, over every affine open susbet of $|np_0|$ the rings corresponding to $\pi_{1*}Hom(U_0, U_0)$ and $\nu_*\mathcal{O}_T$ are two integral domains with the same quotient fields. Since T is normal and $\pi_{1*}Hom(U_0, U_0)$ is finite over $\nu_*\mathcal{O}_T$ (since it is finite over $\mathcal{O}_{\mathbb{P}^{n-1}}$), the two sheaves of algebras must coincide. This proves (i).

Now suppose that U' satisfies the hypotheses (ii) of (2.4). By base change $\pi_{1*}Hom(U',U_0)$ is a locally free rank n sheaf over $|np_0|$. Composition of homomorphisms induces the structure of a $\pi_{1*}Hom(U_0,U_0)$ -module on $\pi_{1*}Hom(U',U_0)$. Thus $\pi_{1*}Hom(U',U_0)$ corresponds to a $\nu_*\mathcal{O}_T$ -module. We claim that, as an \mathcal{O}_T -module, $\pi_{1*}Hom(U',U_0)$ is locally free rank of rank one. To see this, fix a point x in $|np_0|$ and let V',V be the vector bundles corresponding to the restrictions of U',U_0 to the slice $\{x\} \times E$. Of course, by hypothesis V' and V are isomorphic. Choose an isomorphism $s\colon V' \to V$ and extend it to a local section of $\pi_{1*}Hom(U',U_0)$ in a neighborhood of x, also denoted s. The map $\pi_{1*}Hom(U_0,U_0) \to \pi_{1*}Hom(U',U_0)$ defined by multiplying against the section s is then surjective at s, and hence in a neighborhood. Viewing both sides as locally free rank s sheaves over $|np_0|$, the map is then a local isomorphism. But this exactly says that $\pi_{1*}Hom(U',U_0)$ is a locally free $\pi_{1*}Hom(U_0,U_0)$ -module of rank one. Thus $\pi_{1*}Hom(U',U_0)$ corresponds to a line bundle on T, which we denote by L^{-1} . Of course, for any line bundle M on T, setting

$$U_0[M] = (\nu \times \mathrm{Id})_* \left[\mathcal{O}_{T \times E}(\Delta - G) \otimes q_1^* M \right].$$

we have

$$\pi_{1*}Hom(U', U_0[M]) = \pi_{1*}(\nu \times \operatorname{Id})_*Hom((\nu \times \operatorname{Id})^*U', \mathcal{O}_{T \times E}(\Delta - G) \otimes q_1^*M))$$

$$= \nu_*q_{1*} \left[q_1^*M \otimes Hom((\nu \times \operatorname{Id})^*U', \mathcal{O}_{T \times E}(\Delta - G))\right]$$

$$= \nu_* \left[M \otimes q_{1*}Hom((\nu \times \operatorname{Id})^*U', \mathcal{O}_{T \times E}(\Delta - G))\right],$$

The case $M = \mathcal{O}_T$ tells us that, as $\nu_* \mathcal{O}_T$ -modules,

$$\nu_* L^{-1} = \pi_{1*} Hom(U', U_0) = \nu_* \left[q_{1*} Hom((\nu \times \mathrm{Id})^* U', \mathcal{O}_{T \times E}(\Delta - G)) \right].$$

Hence, we have

$$\pi_{1*}Hom(U', U_0[M]) = \nu_*(M \otimes L^{-1}).$$

Taking M = L we have

$$\pi_{1*}Hom(U', U_0[L]) \cong \nu_*\mathcal{O}_T.$$

Via this identification, the section $1 \in H^0(\mathcal{O}_T)$ then defines an isomorphism from U' to $U_0(L)$, as claimed. \square

In view of the previous result, we need to describe all line bundles on T. Since T is a \mathbb{P}^{n-2} bundle over E, we have:

Lemma 2.5. The projection mapping $r: T \to E$ induces an injection

$$r^* \operatorname{Pic} E \to \operatorname{Pic} T$$
.

If n=2, r is an isomorphism and thus $\operatorname{Pic} T \cong \operatorname{Pic} E$. For n>2, since T is included in $\mathbb{P}\mathcal{E} \subset \mathbb{P}^{n-1} \times E$, we can define by restriction the line bundle $\mathcal{O}_{\mathbb{P}^{n-1}}(1)|T=\mathcal{O}_T(1)$ on T. Then

$$\operatorname{Pic} T = r^* \operatorname{Pic} E \oplus \mathbb{Z}[\mathcal{O}_T(1)].$$

In view of Lemma 2.5, we make the following definition. For $p \in E$, let $F_p \subset T$ be the divisor which is the preimage of p.

Definition 2.6. For every integer a, let $U_a = (\nu \times \mathrm{Id})_* \mathcal{O}_{T \times E}(\Delta - G - a(F_{p_0} \times E))$. More generally, given $e \in E$, define

$$U_a[e] = (\nu \times \mathrm{Id})_* \mathcal{O}_{T \times E}(\Delta - G - (a+1)(F_{p_0} \times E) + (F_e \times E)).$$

Thus $U_a[p_0] = U_a$. By Lemma 2.5, every vector bundle obtained from U_0 by twisting by a line bundle on the spectral cover is of the form $U_a[e] \otimes \pi_1^* \mathcal{O}_{\mathbb{P}^{n-1}}(b)$ for some $b \in \mathbb{Z}$ and $e \in E$. (For n = 2, we have the relation $\nu^* \mathcal{O}_{\mathbb{P}^1}(1) = \mathcal{O}_E(2p_0)$, and thus $U_a \otimes \pi_1^* \mathcal{O}_{\mathbb{P}^1}(b) \cong U_{a-2b}$.)

The next lemma says that the U_a are all elementary modifications of each other:

Lemma 2.7. Let $H = \nu(F_{p_0})$ be the hyperplane in $\mathbb{P}^{n-1} = |np_0|$ of divisors whose support contains p_0 , and let $i: H \to \mathbb{P}^{n-1}$ be the inclusion. Then there is an exact sequence

$$0 \to U_a \to U_{a-1} \to (i \times \mathrm{Id})_* \mathcal{O}_{H \times E} \to 0.$$

Moreover dim Hom $(U_{a-1}|H \times E, \mathcal{O}_{H \times E}) = 1$, so that the above exact sequence is the unique elementary modification of this type. Likewise, $U_a[e]$ is given as an elementary modification:

$$0 \to U_{a+1} \to U_a[e] \to (i \times \mathrm{Id})_* \mathcal{O}_{H_e \times E} \otimes \pi_2^* \mathcal{O}_E(e-p_0) \to 0.$$

Proof. Consider the exact sequence

$$0 \to \mathcal{O}_{T \times E}(\Delta - G - a(F_{p_0} \times E)) \to$$
$$\to \mathcal{O}_{T \times E}(\Delta - G - (a - 1)(F_{p_0} \times E)) \to \mathcal{O}_{F_{p_0} \times E}(\Delta - G - (a - 1)(F_{p_0} \times E)) \to 0.$$

Clearly the restriction of the line bundle $\mathcal{O}_{T\times E}(F_{p_0}\times E)$ to $F_{p_0}\times E$ is trivial, and G and Δ both restrict to the divisor $F_{p_0}\times \{p_0\}\subset F_{p_0}\times E$. Hence the last term in the above sequence is $\mathcal{O}_{F_{p_0}\times E}$. Applying $(\nu\times \mathrm{Id})_*$ to the sequence gives the exact sequence of (2.7). For V a bundle corresponding to a point of H, $\dim\mathrm{Hom}(V,\mathcal{O}_E)=1$. Thus $\pi_{1*}Hom(U_a|H\times E,\mathcal{O}_{H\times E})$ is a line bundle on H. The given map $U_a|H\times E\to \mathcal{O}_{H\times E}$ constructed above is an everywhere generating section of this line bundle, so that $\pi_{1*}Hom(U_a|H\times E,\mathcal{O}_{H\times E})$ is trivial and $\dim\mathrm{Hom}(U_a|H\times E,\mathcal{O}_{H\times E})=1$.

The proof of the exact sequence relating U_{a+1} and $U_a[e]$ is similar. \square

In fact, suppose that we have an elementary modification

$$0 \to U' \to U_a \to \mathcal{O}_{D \times E} \otimes \pi_2^* \lambda \to 0$$
,

where D is a hypersurface in $|np_0|$ and λ is a line bundle of degree zero on E. Then it is easy to check that necessarily $D = H_e$ for some e and $\lambda = \mathcal{O}_E(e - p_0)$. Of course, it is also possible to make elementary modifications along certain hyperplanes corresponding to taking higher rank quotients of U_a .

2.4. Families of bundles over more general parameter spaces.

Now let us examine in what sense the bundles $U \to |np_0| \times E$ that we have constructed are universal.

Theorem 2.8. Let E be a smooth elliptic curve and let S be a scheme or analytic space. Suppose that $U \to S \times E$ is a rank n holomorphic vector bundle whose restriction to each slice $\{s\} \times E$ is a regular semistable bundle with trivial determinant. Let $\Phi: S \to |np_0|$ be the morphism constructed in Theorem 1.5. Let $\nu_S: \tilde{S} \to S$ be the pullback via Φ of the spectral covering $T \to |np_0|$:

$$\tilde{S} = S \times_{|np_0|} T$$
,

and let $\tilde{\Phi}: \tilde{S} \to T$ be the map covering Φ . Let $q_1: \tilde{S} \times E \to \tilde{S}$ be the projection onto the first factor. Then there is a line bundle $\mathcal{M} \to \tilde{S}$ and an isomorphism of \mathcal{U} with

$$(\nu_S \times \mathrm{Id})_* \left((\tilde{\Phi} \times \mathrm{Id})^* (\mathcal{O}_{T \times E}(\Delta - G)) \otimes q_1^* \mathcal{M} \right).$$

Proof. By construction the bundle

$$(\nu_S \times \mathrm{Id})_* (\tilde{\Phi} \times \mathrm{Id})^* (\mathcal{O}_{T \times E} (\Delta - G))$$

is a family of regular semistable bundles with trivial determinant E, which fiber by fiber have the same Jordan-Hölder quotients as the family \mathcal{U} . But regular semistable bundles are determined up to isomorphism by their Jordan-Hölder quotients. This means that the two families are isomorphic on each slice $\{s\} \times E$. Now the argument in the proof of Theorem 2.4 applies to establish the existence of the line bundle \mathcal{M} on the spectral covering \tilde{S} as required. \square

We can also construct the spectral cover \tilde{S} of S directly. This construction will also the explain the origin of the name spectral cover. If p_1, p_2 are the projections of $S \times E$ to the first and second factors, then by standard base change results $p_{1*}Hom(\mathcal{U},\mathcal{U})$ is a locally free sheaf of coherent S-algebras. Moreover, by the classification of regular semistable bundles, it is commutative. Thus there is a well-defined space $\tilde{S} = \mathbf{Spec} \, p_{1*}Hom(\mathcal{U},\mathcal{U})$ and a morphism $\nu \colon \tilde{S} \to S$ such that $\mathcal{O}_{\tilde{S}} = p_{1*}Hom(\mathcal{U},\mathcal{U})$. It is easy to check directly that $\tilde{S} = S \times_{|np_0|} T$. By construction, there is an action of $\mathcal{O}_{\tilde{S}}$ on \mathcal{U} that commutes with the action of \mathcal{O}_E , and thus \mathcal{U} corresponds to a coherent sheaf \mathcal{L} on $\tilde{S} \times E$. Again by the classification of regular semistable bundles, it is straightforward to check directly that \mathcal{L} is locally free of rank one. Clearly, $(\nu \times \mathrm{Id})_*\mathcal{L} = \mathcal{U}$.

We can view Theorem 2.8 as allowing us to replace a family of possibly non-regular, semistable bundles with trivial determinant on E with a family of regular semistable bundles without changing the Jordan-Hölder quotients on any slice. Suppose that $\mathcal{V} \to S \times E$ is any family of semistable bundles with trivial determinant over E. We have the map $\Phi: S \to |np_0|$ of Theorem 1.5, and $(\Phi \times \mathrm{Id})^*U_0 \to S \times E$ is a family of regular semistable bundles with the same Jordan-Hölder quotients as \mathcal{V} along each slice $\{s\} \times E$. Of course, the new bundle will not be isomorphic to \mathcal{V} (even after twisting with a line bundle on the spectral cover) unless the original family is a family of regular bundles.

2.5. The case of singular curves.

There is an analogue of these constructions for singular curves. Let E be a Weierstrass cubic. The constuction given at the beginning of this section is valid in this context and produces a \mathbb{P}^{n-2} -bundle $T = \mathbb{P}\mathcal{E}$ over E and an n-fold covering

map $\nu: T \to |np_0|$. By the description of T as $\mathbb{P}\mathcal{E}$, the projection $T \to \mathbb{P}^{n-1}$ is a finite flat morphism.

Let $\Omega \subset |np_0|$ be the Zariski open subset of all divisors whose support does not contain the singular point of E, and let $T_{\Omega} \subset T$ be $\nu^{-1}(\Omega)$. We denote by ν_{Ω} the restriction of ν to T_{Ω} . It is a finite surjective morphism of degree n between smooth varieties. As before, we have the divisor $\Delta \subset T \times E$. We denote by $\Delta_{\Omega} \subset T_{\Omega} \times E$ the restriction of Δ to this open subset. We form the line bundle

$$\mathcal{L}_0^{\Omega} = \mathcal{O}_{T_{\Omega} \times E}(\Delta_{\Omega} - G),$$

where G is the divisor $T_{\Omega} \times \{p_0\}$. Let U_0^{Ω} be the sheaf $(\nu_{\Omega} \times \mathrm{Id})_*(\mathcal{L}_0^{\Omega})$ over $\Omega \times E$. It is a vector bundle of rank n. The arguments in the proof of Claim 2.3 apply in this context and show that U_0^{Ω} is a family of regular bundles on E parametrized by Ω .

The arguments in the proof of Theorem 2.4 apply to yield the following result.

Proposition 2.9. Let E be a Weierstrass cubic. Let $\Omega \subset |np_0|$ be the Zariski open subset defined in the previous paragraph. Let $\pi_1^{\Omega}: \Omega \times E \to \Omega$ be the projection onto the first factor, and let $U_0^{\Omega} = (\nu_{\Omega} \times \mathrm{Id})_* \mathcal{L}_0^{\Omega}$ be the bundle over $\Omega \times E$ constructed in the previous paragraph. Then:

- (i) The sheaf $(\pi_1^{\Omega})_* Hom(U_0^{\Omega}, U_0^{\Omega})$ is a locally free sheaf of algebras of rank n over $\Omega \subset |np_0|$ which is isomorphic to $\nu_*\mathcal{O}_{T_{\Omega}}$.
- (ii) Let U' be a rank n vector bundle over $\Omega \times E$ with the following property. For each $x \in \Omega$ the restriction of U' to $\{x\} \times E$ is isomorphic to the restriction of U_0 to $\{x\} \times E$. Then $U' = (\nu_{\Omega} \times \mathrm{Id})_* [\mathcal{O}_{T_{\Omega} \times E}(\Delta_{\Omega} - G) \otimes q_1^*L]$ for a unique line bundle L on T_{Ω} .

In Section 3, we shall show how to extend this construction over the singular points of E.

2.6. Chern classes.

Finally, we return to the case where E is smooth and give the Chern classes of the various bundles over $|np_0| \times E$ in case E is smooth. The proof of (2.10) will be given in the next section, and we will prove the remaining results assuming (2.10).

Proposition 2.10. Identify $h \in H^2(\mathbb{P}^{n-1})$ with its pullback to $\mathbb{P}^{n-1} \times E$. Then the total Chern class $c(U_0)$ and the Chern character $ch(U_0)$ of U_0 are given by the formulas:

$$c(U_0) = (1 - h + \pi_2^*[p_0] \cdot h)(1 - h)^{n-2}$$

$$ch U_0 = ne^{-h} + (1 - \pi_2^*[p_0])(1 - e^{-h}).$$

Once we have (2.10), we can calculate the Chern classes of all the universal

Proposition 2.11. Let U_a and $U_a[e]$ be defined as in (2.6). Let h be the class of ahyperplane in $\operatorname{Pic} \mathbb{P}^{n-1}$, which we also view by pullback as an element of $\operatorname{Pic}(\mathbb{P}^{n-1} \times \mathbb{P}^{n-1})$

- $\begin{array}{l} \text{(i)} \ c(U_a) = (1-h+\pi_2^*[p_0]\cdot h)(1-h)^{a+n-2}.\\ \text{(ii)} \ \operatorname{ch}(U_a\otimes\pi_1^*\mathcal{O}_{\mathbb{P}^{n-1}}(b)) = ne^{(b-1)h} + (1-a-\pi_2^*[p_0])(e^{bh}-e^{(b-1)h}). \end{array}$

- (iii) $\det U_a[e] = -(a+n-1)h$.
- (iv) Let \tilde{c}_2 denote the refined Chern class of a vector bundle in the Chow group $A^2(\mathbb{P}^{n-1} \times E)$. Let $A_0^2(\mathbb{P}^{n-1} \times E)$ be the subgroup of $A^2(\mathbb{P}^{n-1} \times E)$ of all cycles homologous to zero, so that $A_0^2(\mathbb{P}^{n-1} \times E) \cong E$. Then

$$\tilde{c}_2(U_a[e] \otimes \pi_1^* \mathcal{O}_{\mathbb{P}^{n-1}}(b)) - \tilde{c}_2(U_a \otimes \pi_1^* \mathcal{O}_{\mathbb{P}^{n-1}}(b)) = e$$

as an element of $A_0^2(\mathbb{P}^{n-1} \times E) \cong E$.

Proof. By (2.7),

$$c(U_a) = c(U_{a-1})c((i \times \mathrm{Id})_* \mathcal{O}_{H \times E})^{-1}$$

and likewise

$$\operatorname{ch} U_a = \operatorname{ch} U_{a-1} - \operatorname{ch}((i \times \operatorname{Id})_* \mathcal{O}_{H \times E}).$$

Using the exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^{n-1} \times E}(-H \times E) \to \mathcal{O}_{\mathbb{P}^{n-1} \times E} \to (i \times \mathrm{Id})_* \mathcal{O}_{H \times E} \to 0,$$

we have

$$c((i \times \mathrm{Id})_* \mathcal{O}_{H \times E}) = (1 - h)^{-1};$$

$$\mathrm{ch}((i \times \mathrm{Id})_* \mathcal{O}_{H \times E}) = 1 - e^{-h}.$$

A little manipulation, starting with (2.10), gives (i) and (ii). To see (iii), note that by construction $\det U_a[e]$ is the pullback of a class in $\operatorname{Pic} \mathbb{P}^{n-1}$. Moreover, it is independent of the choice of $e \in E$. Thus we may as well take $e = p_0$, in which case $U_a[p_0] = U_a$. In this case, the result is immediate from (i). (iv) follows by using the elementary modification relating $U_a[e]$ and U_{a+1} . \square

Note that

$$c_1\left(U_a\otimes\pi_1^*\mathcal{O}_{\mathbb{P}^{n-1}}(b)\right))=0$$

if and only if a-1=n(b-1). A natural solution to this equation is a=b=1. The bundle $U=U_1\otimes\pi_1^*\mathcal{O}_{\mathbb{P}^{n-1}}(b)=(\nu\times\mathrm{Id})_*\mathcal{O}_{T\times E}(\Delta-G-(F_{p_0}\times E))$ is singled out in this way as $(\nu\times\mathrm{Id})_*\mathcal{P}$, where \mathcal{P} is the pullback to $T\times E$ of the symmetric line bundle $\mathcal{O}_{E\times E}(\Delta_0-\{p_0\}\times E-f\times\{p_0\})$, which is a Poincaré line bundle for $E\times E$. In this case $\mathrm{ch}\,U=n+\pi_2^*[p_0](1-e^h)$. Moreover, one can check that $c_1(U)=0$ and $c_k(U)=(-1)^kh^{k-1}\pi_2^*[p_0]$ for $k\geq 2$.

It is easy to check that, for n > 2, $U_a \otimes \pi_1^* \mathcal{O}_{\mathbb{P}^{n-1}}(b) = U_{a'} \otimes \pi_1^* \mathcal{O}_{\mathbb{P}^{n-1}}(b')$ if and only if a = a' and b = b'. It is also possible to vary aF within its algebraic equivalence class, which is a family isomorphic to E, and this difference is detectable by looking at $c_2(U_a \otimes \pi_1^* \mathcal{O}_{\mathbb{P}^{n-1}}(b))$ in the Chow group $A^2(\mathbb{P}^{n-1} \times E)$. More precisely, we have the following:

Proposition 2.12. Given two vector bundles $U' = (\nu \times \operatorname{Id})_* (\mathcal{O}_{T \times E}(\Delta - G) \otimes M')$ and $U'' = (\nu \times \operatorname{Id})_* (\mathcal{O}_{T \times E}(\Delta - G) \otimes M'')$, where M' and M'' are line bundles on E, then U' and U'' are isomorphic if and only if they have the same Chern classes as elements of $A^*(\mathbb{P}^{n-1} \times E)$.

Proof. For simplicity, we shall just consider the case n > 2. Using the notation of (2.6) and the description of Pic T, it suffices to show that, if the Chern classes of $U_a[e] \otimes \pi_1^* \mathcal{O}_{\mathbb{P}^{n-1}}(b)$ and of $U_{a'}[e'] \otimes \pi_1^* \mathcal{O}_{\mathbb{P}^{n-1}}(b')$ are equal in the Chow ring, then a = a', b = b', and e = e'. Following the above remarks, the Chern classes of $U_a[e] \otimes \pi_1^* \mathcal{O}_{\mathbb{P}^{n-1}}(b)$ in rational cohomology, which are of course the same as those of $U_a \otimes \pi_1^* \mathcal{O}_{\mathbb{P}^{n-1}}(b)$, determine a and b (use c_3 to find b and c_1 to find a). By (iii) of (2.11), the class \tilde{c}_2 then determines e. \square

3. Moduli spaces via extensions.

In this section we shall describe a completely different approach to constructing universal bundles over $\mathbb{P}^{n-1} \times E$. The idea here is to consider the space of extensions of fixed (and carefully chosen) bundles over E. From this point of view the projective space is the projective space of the relevant extension group, which is a priori a very different animal from $|np_0|$. We shall show however (Proposition 3.13) that this projective space is naturally identified with $|np_0|$. There are several reasons for considering this alternative approach. First of all it works as well for singular curves as for smooth ones, so that the restrictions of the last section to smooth curves or to bundles concentrated away from the singularities of a singular curve can be removed. Also, this method works well for a family of elliptic curves, not just a single elliptic curve. Lastly, this approach has a natural generalization to all holomorphic principal bundles with structure group an arbitrary complex simple group G, something which so far is not clear for the spectral cover approach. The generalization to G-bundles is discussed in [8]. The disadvantange of the approach of this section is that it constructs some but not all of the families that the spectral cover approach gives. The reason is that from this point of view one cannot see directly the analogue of twisting by a general line bundle on the spectral cover to produce the general family of regular bundles.

The main results of this section are as follows. In Theorem 3.2, we consider the set of relevant extensions and show that every such extension is a regular semistable bundle with trivial determinant. Conversely, every regular semistable bundle with trivial determinant arises as such an extension. In the construction of bundles of rank n over E we must choose an integer d with $1 \le d < n$. We show that constructions for different d are related to one another (Proposition 3.11 and Theorem 3.12). Next, we compare the extension moduli space, which is a \mathbb{P}^{n-1} , to the coarse moduli space which is $|np_0|$. We find a natural cohomological identification of these two projective spaces (Theorem 3.13) and check that it corresponds to the morphism Φ of Section 1 (Proposition 3.16). Next we show how the universal bundles defined via the extension approach lead to the spectral covers of Section 2 (Theorem 3.21). In this way, we can both identify the universal bundles constructed here with those constructed via spectral covers (Theorem 3.23 and Corollary 3.24), and extend the spectral cover construction to the case of a singular E.

Throughout this section, E denotes a Weierstrass cubic with origin p_0 .

3.1. The basic extensions.

We begin by recalling a result, essentially due to Atiyah, which produces the basic bundles for our extensions:

Lemma 3.1. For each $d \geq 1$, there is a stable bundle W_d of rank d on E whose determinant is isomorphic to $\mathcal{O}_E(p_0)$. It is unique up to isomorphism. For every rank one torsion free sheaf λ of degree zero, $h^0(W_d \otimes \lambda) = 1$ and $h^1(W_d \otimes \lambda) = 0$.

Proof. We briefly outline the proof. An inductive construction of W_d is as follows: set $W_1 = \mathcal{O}_E(p_0)$. Assume inductively that W_{d-1} has been constructed and that $h^0(W_{d-1}) = 1$. It then follows by Riemann-Roch that $h^1(W_{d-1}) = 0$, and thus that $h^0(W_{d-1}^{\vee}) = 0$, $h^1(W_{d-1}^{\vee}) = 1$. We then define W_d by taking the unique nonsplit extension

$$0 \to \mathcal{O}_E \to W_d \to W_{d-1} \to 0.$$

By construction W_d has a filtration whose successive quotients, in increasing order, are $\mathcal{O}_E, \ldots, \mathcal{O}_E, \mathcal{O}_E(p_0)$, and such that all of the intermediate extensions are not split. It is the unique bundle with this property. An easy induction shows that W_d is stable. To see this, note that W_d is stable if and only if every proper subsheaf J of W_d has degree at most zero. But if J is a proper subsheaf of W_d of positive degree, then the image of J in W_{d-1} also has positive degree, and hence $J \to W_{d-1}$ is surjective. But since the rank of J is at most d-1, the projection of J to W_{d-1} is an isomorphism. This says that W_d is a split extension of W_{d-1} by \mathcal{O}_E , a contradiction. Thus W_d is stable.

The uniqueness statement is clear in the case of rank one. Now assume inductively that we have showed that, for d < n, every stable bundle of rank d whose determinant is isomorphic to $\mathcal{O}_E(p_0)$ is isomorphic to W_d . Let W be a stable bundle of rank n such that det $W = \mathcal{O}_E(p_0)$. By stability, $h^1(W) = \dim \operatorname{Hom}(W, \mathcal{O}_E) = 0$, and so $h^0(W) = 1$. If $\mathcal{O}_E \to W$ is the map corresponding to a nonzero section, then by stability the cokernel Q is torsion free. An argument as in the proof that W_d is stable shows that Q is stable. If E is smooth, then Q is automatically locally free. When E is singular, Lemma 0.4 implies that W is locally isomorphic to $Q \oplus \mathcal{O}_E$. Thus, if W is locally free, then Q is locally free as well. Once we know that Q is locally free, we are done by induction.

To see the final statement, first note that, since $deg(W_d \otimes \lambda) = 1$, we have by definition that

$$h^0(W_d \otimes \lambda) - h^1(W_d \otimes \lambda) = 1.$$

It will thus suffice to show that $h^1(W_d \otimes \lambda) = 0$. By Serre duality,

$$h^1(W_d \otimes \lambda) = \dim \operatorname{Hom}(W_d \otimes \lambda, \mathcal{O}_E) = \dim \operatorname{Hom}(W_d, \lambda^{\vee}).$$

Since λ^{\vee} is also a rank one torsion free sheaf of degree zero, $\operatorname{Hom}(W_d, \lambda^{\vee}) = 0$ by stability. \square

Exercise. We have defined \mathcal{E} in the previous section as the rank n-1 vector bundle which is the kernel of the evaluation map $H^0(\mathcal{O}_E(np_0)) \otimes \mathcal{O}_E \to \mathcal{O}_E(np_0)$. Show that

$$\mathcal{E} \cong W_{n-1}^{\vee} \otimes \mathcal{O}_E(-p_0).$$

Now we are ready to see how extensions of the W_d can be used to make regular semistable bundles.

Theorem 3.2. Let V be an extension of the form

$$0 \to W_d^{\vee} \to V \to W_{n-d} \to 0.$$

Then:

- (i) V has trivial determinant
- (ii) V is semistable if and only if the above extension is not split. In this case V is regular.
- (iii) Suppose that V is semistable, i.e. that the above extension is not split. Then $\dim \operatorname{Hom}(V,V) = n$ and $\operatorname{Hom}(V,V)$ is an abelian $\mathbb C$ -algebra. Moreover, every homomorphism $W_d^{\vee} \to V$ is of the form $\phi \circ \iota$, where $\phi \in \operatorname{Hom}(V,V)$ and ι is the given inclusion $W_d^{\vee} \to V$. If V and V' are given as extensions

as above, then V and V' are isomorphic if and only if their extension classes in $\operatorname{Ext}^1(W_{n-d}, W_d^{\vee})$ are multiples of each other.

- (iv) If V is a regular semistable vector bundle of rank n > 1 with trivial determinant, then V can be written as an extension as above.
- (v) If V is a nontrivial extension of W_{n-d} by W_d^{\vee} , and ad(V) is the sheaf of trace free endomorphisms of V, then $H^0(ad(V)) \cong \operatorname{Ker} \{\operatorname{Hom}(W_d^{\vee}, W_{n-d}) \rightarrow$ $H^1(Hom(W_d^{\vee}, W_d^{\vee})) \cong \mathbb{C} \}$ and $H^1(ad(V)) \cong \operatorname{Ext}^1(W_{n-d}, W_d^{\vee})/\mathbb{C}\xi$, where ξ is the extension class corresponding to V.

Proof. (i) This is clear since det $W_d \cong \det W_{n-d}$.

(ii) If V is unstable, let W be the maximal destabilizing subsheaf. Then W is stable of positive degree and rank r for some r < n. Since $\operatorname{Hom}(W, W_d^{\vee}) = 0$, the induced map $W \to W_{n-d}$ is nonzero. Now it is easy to see by the stability of W_s that if there is a nonzero map $W \to W_s$, where W has positive degree and rank r, then $r \geq s$, and every nonzero such map is surjective. (From this it follows in particular that, for $r \geq s$, $\operatorname{Hom}(W_r, W_s) \cong \operatorname{Hom}(W_s, W_s) = \mathbb{C}$.) If r > n - d, the kernel of the map $W \to W_{n-d}$ is a subsheaf of degree at least zero of W_d^{\vee} , and since W_d^{\vee} is a stable bundle of degree -1, the kernel is zero. Hence $W \cong W_{n-d}$, which means that the extension is split. Conversely, if the extension is split then V is unstable.

Next we show that V is regular. Since W_{n-d} is a stable bundle of degree 1, $\operatorname{Hom}(W_{n-d},\lambda)=0$ for every rank one torsion free sheaf λ of degree zero. Moreover, with λ as above, $h^0((W_d \otimes \lambda) = \dim \operatorname{Hom}(W_d^{\vee}, \lambda) = 1$ by the last sentence in Lemma 3.1. Thus dim $\operatorname{Hom}(V,\lambda) \leq 1$ for every λ of degree zero, so that, by (1.14), V is regular.

(iii) Consider the exact sequence

$$\operatorname{Hom}(W_{n-d}, W_d^{\vee}) \to \operatorname{Hom}(W_{n-d}, V) \to \operatorname{Hom}(W_{n-d}, W_{n-d}) \to \operatorname{Ext}^1(W_{n-d}, W_d^{\vee}).$$

Note that $\operatorname{Hom}(W_{n-d}, W_d^{\vee}) = 0$ by stability. Since W_{n-d} is stable, it is simple, and so $\operatorname{Hom}(W_{n-d},W_{n-d})=\mathbb{C}\cdot\operatorname{Id}$. But the image of Id in $\operatorname{Ext}^1(W_{n-d},W_d^{\vee})$ is the extension class. Since this class is nonzero, $\operatorname{Hom}(W_{n-d},V)=0$ as well.

Next consider the exact sequence

$$0 = \operatorname{Hom}(W_{n-d}, V) \to \operatorname{Hom}(V, V) \to \operatorname{Hom}(W_d^{\vee}, V) \to H^1(W_{n-d}^{\vee} \otimes V).$$

Since $\operatorname{Hom}(W_{n-d},V)=0$, the map $\operatorname{Hom}(V,V)\to\operatorname{Hom}(W_d^\vee,V)$ is an injection. Moreover, we have a commutative diagram

$$\operatorname{Hom}(W_d^{\vee}, W_d^{\vee}) \longrightarrow \operatorname{Hom}(W_d^{\vee}, V) \longrightarrow \operatorname{Hom}(W_d^{\vee}, W_{n-d}) \longrightarrow H^1(\operatorname{Hom}(W_d^{\vee}, W_d^{\vee}))$$

Thus we see that $H^0(ad(V)) \cong \operatorname{Ker} \{ \operatorname{Hom}(W_d^{\vee}, W_{n-d}) \to H^1(\operatorname{Hom}(W_d^{\vee}, W_d^{\vee})) \cong \operatorname{Hom}(W_d^{\vee}, W_d^{\vee}) \}$ \mathbb{C} and by duality $H^1(ad(V)) \cong \operatorname{Ext}^1(W_{n-d}, W_d^{\vee})/\mathbb{C}\xi$, where ξ is the extension class corresponding to V. This proves (v).

Let us assume that dim $\operatorname{Hom}(V,V) \geq n$, which we have already checked in case E is smooth. (We will establish this for singular curves after proving Part (iv), as well as checking the fact that Hom(V,V) is abelian. These results are not used in the proof of Part (iv) of the theorem.) If we can show that $\dim \operatorname{Hom}(W_d^{\vee}, V) = n$, then $\operatorname{Hom}(V, V) \to \operatorname{Hom}(W_d^{\vee}, V)$ is an isomorphism, and in particular $\dim \operatorname{Hom}(V, V) = n$ as well.

We compute the dimension of $\operatorname{Hom}(W_d^{\vee}, V)$. Consider the exact sequence

$$0 \to \operatorname{Hom}(W_d^\vee, W_d^\vee) \to \operatorname{Hom}(W_d^\vee, V) \to \operatorname{Hom}(W_d^\vee, W_{n-d}) \to H^1(\operatorname{Hom}(W_d^\vee, W_d^\vee))$$

Since W_d^{\vee} is stable, dim $\operatorname{Hom}(W_d^{\vee}, W_d^{\vee}) = 1$. Next we claim that

$$\dim \operatorname{Hom}(W_d^{\vee}, W_{n-d}) = h^0(W_d \otimes W_{n-d}) = n.$$

Claim 3.3. If E is a Weierstrass cubic, then $h^0(W_d \otimes W_{n-d}) = n$ and $h^1(W_d \otimes W_{n-d}) = 0$. Dually, $h^0(W_d^{\vee} \otimes W_{n-d}^{\vee}) = 0$ and $h^1(W_d^{\vee} \otimes W_{n-d}^{\vee}) = n$.

Proof. If d = 1, this follows from the exact sequence

$$0 \to W_{n-1} \to W_{n-1} \otimes \mathcal{O}_E(p_0) \to (\mathbb{C}_{p_0})^{n-1} \to 0,$$

together with the fact that $h^1(W_{n-1}) = 0$ by stability. The general case follows by induction on n, by tensoring the exact sequence

$$0 \to \mathcal{O}_E \to W_d \to W_{d-1} \to 0$$

by W_{n-d} . \square

By Riemann-Roch, $h^1(Hom(W_d^\vee, W_d^\vee)) = 1$. Thus by counting dimensions, to show that $\dim \operatorname{Hom}(W_d^\vee, V) = n$ it will suffice to show that $\operatorname{Hom}(W_d^\vee, W_{n-d}) \to H^1(Hom(W_d^\vee, W_d^\vee))$ is surjective. Equivalently we must show that the map from $H^1(Hom(W_d^\vee, W_d^\vee)) \to H^1(Hom(W_d^\vee, V))$ is zero. But this map is dual to the map $\operatorname{Hom}(V, W_d^\vee) \to \operatorname{Hom}(W_d^\vee, W_d^\vee) = \mathbb{C} \cdot \operatorname{Id}$. A lifting of Id to a homomorphism $V \to W_d^\vee$ would split the exact sequence, contrary to assumption. This completes the proof of all of Part (iii) except for the last sentence.

We turn to the last statement in Part (iii). If V is a split extension, then it is unstable and so V' is unstable and therefore a split extension as well. Thus we may suppose that V and V' are nontrivial extensions of the given type and that $\psi \colon V' \to V$ is an isomorphism. Using ψ to identify V and V', suppose that we are given two inclusions $\iota_1, \iota_2 \colon W_d^\vee \to V$ such that both quotients are isomorphic to W_{n-d} . By the first part of (iii), there is an endomorphism A of V such that $A \circ \iota_1 = \iota_2$. Since W_{n-d} is simple, the induced map on the quotient W_{n-d} factors must be a multiple $\alpha \in \mathbb{C}$ of the identity. This multiple α cannot be zero, since otherwise A would define a splitting of the extension corresponding to ι_2 . In particular, A is an automorphism of V. Furthermore, we see that the extension class for V' is α times the extension class for V. This completes the proof of (iii).

To prove Part (iv) of the theorem, given a semistable V, we seek subbundles of V isomorphic to W_d^{\vee} such that the quotient is isomorphic to W_{n-d} .

Lemma 3.4. Fix d > 0. For any r > 0 and any line bundle λ of degree zero there is a map

$$W_d^{\vee} \to I_r(\lambda)$$

whose image is not contained in a proper degree zero subsheaf of $I_r(\lambda)$. Likewise, for any strongly indecomposable, degree zero, semistable bundle $I(\mathcal{F})$ concentrated

at the singular point of a singular curve, there is a map $W_d^{\vee} \to I(\mathcal{F})$ whose image is not contained in a proper degree zero subsheaf.

Proof. We consider case of $I_r(\lambda)$ first. It suffices by (1.8) to show that there is a map $W_d^{\vee} \to I_r(\lambda)$ whose image is not contained in $F_{r-1} \cong I_{r-1}(\lambda)$. Tensoring the exact sequence

$$0 \to I_{r-1}(\lambda) \to I_r(\lambda) \to \lambda \to 0$$

with W_d , we see that there is an exact sequence

$$0 \to \operatorname{Hom}(W_d^{\vee}, I_{r-1}(\lambda)) \to \operatorname{Hom}(W_d^{\vee}, I_r(\lambda)) \to \operatorname{Hom}(W_d^{\vee}, \lambda) \to 0$$

By the last statement in (3.1), there is a nonzero element of $\operatorname{Hom}(W_d^{\vee}, \lambda)$, and by induction on r, $H^1(W_d \otimes I_{r-1}(\lambda)) = 0$. Thus there is a map $W_d^{\vee} \to I_r(\lambda)$ not in the image of a homomorphism into $I_{r-1}(\lambda)$.

Now let us consider the case of a strongly indecomposable bundle $I(\mathcal{F})$. Since every semistable bundle concentrated at the singular point is filtered with associated gradeds isomorphic to \mathcal{F} , we have a short exact sequence

$$0 \to X \to I(\mathcal{F}) \to \mathcal{F} \to 0.$$

Direct cohomology computations as above show that there is a map $W_d^{\vee} \to I(\mathcal{F})$ which has nontrivial image in the quotient \mathcal{F} . Clearly, the image of this map is not contained in X. But, by (1.11), every proper degree zero subsheaf of $I(\mathcal{F})$ is contained in X, proving the result in this case as well. \square

We can generalize (3.4) to every regular semistable bundle V.

Corollary 3.5. Let V be a regular semistable bundle and let d be a positive integer. Then there is a map $W_d^{\vee} \to V$ whose image is not contained in any proper degree zero subsheaf of V.

Proof. This is immediate from the previous result and the fact that V decomposes uniquely as a direct sum $\bigoplus_i I_{r_i}(\lambda_i) \oplus I(\mathcal{F})$, where the λ_i are pairwise distinct line bundles of degree zero and $I(\mathcal{F})$ is a strongly indecomposable bundle concentrated at the node. Since the λ_i are pairwise distinct, any degree zero subsheaf of V is a direct sum of subsheaves of the factors. Thus, for each summand $I_{r_i}(\lambda_i)$ or $I(\mathcal{F})$, choose a map W_d^{\vee} to the corresponding summand whose image is not contained in any proper degree zero subbundle of the summand. The induced map of W_d^{\vee} into the direct sum is as desired. \square

Note that, if instead $\lambda_i = \lambda_j$ for some $i \neq j$, then there would exist degree zero subsheaves of the direct sum which were not a direct sum of subsheaves of the summands, and in fact (3.5) always fails to hold in this case.

Now let us show that the quotient of a map satisfying the conclusions of (3.5) is W_{n-d} .

Proposition 3.6. Let V be a semistable regular bundle of rank n with trivial determinant and let $\iota \colon W_d^{\vee} \to V$ be a map whose image is not contained in any proper degree zero subsheaf of V. If the rank of V is strictly greater than d, then ι is an inclusion and the quotient V/W_d^{\vee} is isomorphic to W_{n-d} . Conversely, if ι is the

inclusion of W_d^{\vee} in V so that the quotient V/W_d^{\vee} is isomorphic to W_{n-d} , then the image of ι is not contained in a proper subsheaf of degree zero.

Proof. Let V have rank $n \geq d+1$, and suppose that we have a map $\iota \colon W_d^\vee \to V$ whose image is not contained in a proper degree zero subsheaf of V. In particular, ι is nontrivial. If ι is not injective, then by the stability of W_d^\vee , the image of ι is a subsheaf of V of rank $\leq d-1$ and degree ≥ 0 , and hence is a proper subsheaf of V of degree zero, contrary to assumption. Likewise, if the cokernel of ι is not torsion free, then the image of ι is contained in a proper subsheaf of V whose degree is strictly larger than -1, and thus the degree is at least zero. This again contradicts our assumption about the map and the fact that the rank of V is at least d+1. Thus ι is injective and its cokernel W is torsion free. Using (0.4), W is locally a direct summand of V, and thus W is locally free. It follows that W is a rank n-d vector bundle whose determinant is $\mathcal{O}_E(p_0)$.

To conclude that the quotient W is isomorphic to W_{n-d} , it suffices to show that W is stable. If W is not stable, then there is a proper subsheaf U of W with degree at least one. Let $U'' \subset V$ be the preimage of U. The degree of U'' is at least zero, and hence, by the semistability of V is of degree zero. Clearly, U'' contains the image of ι . Hence by our hypothesis on ι , U'' = V, and consequently, U = W. This is a contradiction, so that W is stable.

Finally we must show that, if V is written as an extension of W_{n-d} by W_d^{\vee} , then the subbundle W_d^{\vee} cannot be contained in a proper subsheaf U of V of degree zero. If U is a such a subsheaf, then U/W_d^{\vee} would be a proper subsheaf of W_{n-d} of degree at least one, contradicting the stability of W_{n-d} . \square

Corollary 3.5 and Proposition 3.6 show that any regular semistable bundle over E can be written as an extension of W_{n-d} by W_d^{\vee} . This completes the proof of Part (iv).

Now let us return to the point in the proof of (iii) where it is claimed that $\dim \operatorname{Hom}(V,V) \geq n$ for all V which are given as a nonsplit extension of W_d by W_{n-d}^{\vee} . In order to establish this result, we first describe the space of all such extensions, which is an immediate consequence of (3.3):

Lemma 3.7. The space $\operatorname{Ext}^1(W_{n-d},W_d^\vee)=H^1(W_{n-d}^\vee\otimes W_d^\vee)$ has dimension n, and thus the associated projective space is a \mathbb{P}^{n-1} . \square

By general properties, there is a universal extension $\mathbf{U}(d;n)$ over $\mathbb{P}_d^{n-1} \times E$ of the form

$$0 \to \pi_2^* W_d^{\vee} \otimes \pi_1^* \mathcal{O}_{\mathbb{P}_d^{n-1}}(1) \to \mathbf{U}(d;n) \to \pi_2^* W_{n-d} \to 0,$$

with the restriction of $\mathbf{U}(d;n)$ restricted to any slice $\{x\} \times E$ being isomorphic the bundle V given by the line $\mathbb{C} \cdot x \subset \operatorname{Ext}^1(W_{n-d}, W_d^{\vee})$. When n is clear from the context, we shall abbreviate $\mathbf{U}(d;n)$ by $\mathbf{U}(d)$.

Next we claim that there is a nonempty open subset of \mathbb{P}_d^{n-1} such that for V a vector bundle corresponding to a point of this subset, dim $\operatorname{Hom}(V,V)=n$. In fact, suppose that $V=\bigoplus_{i=1}^n \lambda_i$, where the λ_i are distinct line bundles of degree zero whose product is trivial. By Part (iv) of the theorem, V can be written as a nonsplit extension of W_d by W_{n-d}^{\vee} , and we have seen that dim $\operatorname{Hom}(V,V)=n$. A straightforward argument by counting dimensions shows that the set of such V is an open subset of \mathbb{P}_d^{n-1} ; indeed, we will identify this set more precisely in (3.17) below as corresponding to the set of all sections in $|np_0|$ consisting of n smooth points on

E. Thus there is a nonempty open subset of bundles V such that $\dim \operatorname{Hom}(V,V) = n$. By upper semicontinuity applied to the bundle $\operatorname{Hom}(\mathbf{U}(d;n),\mathbf{U}(d;n))$ over $\mathbb{P}_d^{n-1} \times E$, it follows that $\dim \operatorname{Hom}(V,V) \geq n$ for all bundles V corresponding to a point of \mathbb{P}_d^{n-1} .

Finally, we must show that $\operatorname{Hom}(V,V)$ is abelian. Using the universal extension $\mathbf{U}(d)$ as above, we can fit together the $\operatorname{Hom}(V,V)$ to a rank n vector bundle $\pi_{1*}\operatorname{Hom}(\mathbf{U}(d),\mathbf{U}(d))$, which is a coherent sheaf of algebras over \mathbb{P}^{n-1} . Consider the map

$$\pi_{1*}Hom(\mathbf{U}(d),\mathbf{U}(d))\otimes \pi_{1*}Hom(\mathbf{U}(d),\mathbf{U}(d)) \to \pi_{1*}Hom(\mathbf{U}(d),\mathbf{U}(d))$$

defined by $(A, B) \mapsto AB - BA$. Since $\operatorname{Hom}(V, V)$ is abelian for V in a Zariski open subset of \mathbb{P}_d^{n-1} , namely those V which are a direct sum of n distinct line bundles of degree zero, this map is identically zero. By base change, the fiber of $\pi_{1*}\operatorname{Hom}(\mathbf{U}(d),\mathbf{U}(d))$ at a point corresponding to V is $\operatorname{Hom}(V,V)$. Thus $\operatorname{Hom}(V,V)$ is abelian. \square

The following was checked directly in Lemma 1.13 if E is smooth, but is by no means obvious in the singular case:

Corollary 3.8. Let V be a regular semistable bundle of rank n over a Weierstrass cubic. Then:

- (i) $\operatorname{Hom}(V, V)$ is an abelian \mathbb{C} -algebra of rank n.
- (ii) The dual bundle V^{\vee} is a regular semistable bundle.
- (iii) For all rank one torsion free sheaves λ of rank zero on E, dim $\operatorname{Hom}(\lambda, V) \leq 1$.

Proof. The first part is immediate from Parts (iv) and (iii) of Theorem 3.2. (ii) is clear since if V is a nonsplit extension of W_{n-d} by W_d^{\vee} , then V^{\vee} is a nonsplit extension of W_d by W_{n-d}^{\vee} . (iii) follows from (ii), since $\mathcal{F}^{\vee} \cong \mathcal{F}$ and $\operatorname{Hom}(\lambda, V) \cong \operatorname{Hom}(V^{\vee}, \lambda^{\vee})$ for all rank one torsion free sheaves λ . \square

Question. For if V is a semistable bundle of degree zero whose support is concentrated at a smooth point of E, then V is regular if and only if dim Hom(V, V) = rank V. Does this continue to hold at the singular point of a singular curve? For V strongly indecomposable, what is the structure of the algebra Hom(V, V)?

3.2. Relationship between the constructions for various d.

For each d with $1 \leq d < n$ we have a family of regular semistable bundles parametrized by the projective space $\mathbb{P}_d^{n-1} = \mathbb{P}(\operatorname{Ext}^1(W_{n-d}, W_d^{\vee}))$, and given as a universal extension

$$0 \to \pi_2^* W_d^{\vee} \otimes \pi_1^* \mathcal{O}_{\mathbb{P}_d^{n-1}}(1) \to \mathbf{U}(d) \to \pi_2^* W_{n-d} \to 0.$$

In this section we shall identify the \mathbb{P}_d^{n-1} for the various d, although under this identification the bundles $\mathbf{U}(d)$ are different for different d. Using the universal bundle $\mathbf{U}(d)$ and Theorem 1.5, there is a morphism $\mathbb{P}_d^{n-1} \to |np_0|$, which is easily checked to be of degree one and thus an isomorphism. Thus all of the \mathbb{P}_d^{n-1} are identified with $|np_0|$ and hence with each other, but we want to find a direct identification here so as to be able to compare universal bundles.

Lemma 3.9. Let $d, n - d \ge 1$. The natural injection

$$W_d^{\vee} \otimes W_{n-d}^{\vee} \to W_{d+1}^{\vee} \otimes W_{n-d}^{\vee}$$

induces an injective map on H^1 . The image of this map on H^1 is the kernel of the map induced by the tensor products of the projections

$$H^1(W_{d+1}^{\vee} \otimes W_{n-d}^{\vee}) \to H^1(\mathcal{O}_E \otimes \mathcal{O}_E) = H^1(\mathcal{O}_E).$$

The extensions X of W_{n-d} by W_{d+1}^{\vee} which are in the image of the above map are exactly the extensions X such that $\operatorname{Hom}(X, \mathcal{O}_E) \neq 0$.

Proof. We have a short exact sequence

$$0 \to W_d^\vee \otimes W_{n-d}^\vee \to W_{d+1}^\vee \otimes W_{n-d}^\vee \to \mathcal{O}_E \otimes W_{n-d}^\vee \to 0.$$

By (3.3), all the H^0 terms vanish. Thus, the injectivity of the map on H^1 is immediate. Furthermore, the image is identified with the kernel of the map

$$H^1(W_{d+1}^{\vee} \otimes W_{n-d}^{\vee}) \to H^1(\mathcal{O}_E \otimes W_{n-d}^{\vee}).$$

The last term is one-dimensional and the projection $\mathcal{O}_E \otimes W_{n-d}^{\vee} \to \mathcal{O}_E \otimes \mathcal{O}_E$ induces an isomorphism on H^1 .

Finally, a bundle X corresponds to an extension in the image of the map on H^1 's if and only if X is the pushout of an extension of W_{n-d} by W_d^{\vee} under the inclusion $W_d^{\vee} \to W_{d+1}^{\vee}$. Thus, if X is the image of an extension V, the quotient of the inclusion $V \to X$ is \mathcal{O}_E . Conversely, if there is a nontrivial map $X \to \mathcal{O}_E$, then the induced map $W_{d+1}^{\vee} \to \mathcal{O}_E$ is nonzero and thus surjective, and the kernel of $X \to \mathcal{O}_E$ is then an extension V of W_{n-d} by W_d^{\vee} such that X is the pushout of V. \square

The symmetry of the situation with respect to the two factors leads immediately to the following.

Corollary 3.10. If $n - d \ge 2$, then the natural inclusions of bundles induce the maps

$$H^1(W_d^\vee \otimes W_{n-d}^\vee) \to H^1(W_{d+1}^\vee \otimes W_{n-d}^\vee)$$

and

$$H^1(W_{d+1}^{\vee} \otimes W_{n-d-1}^{\vee}) \to H^1(W_{d+1}^{\vee} \otimes W_{n-d}^{\vee})$$

which are injections with the same images. In particular, this produces a natural identification of $\operatorname{Ext}^1(W_{n-d},W_d^\vee)$ with $\operatorname{Ext}^1(W_{n-d-1},W_{d+1}^\vee)$, and hence of the projective spaces $\mathbb{P}_d^{n-1} \cong \mathbb{P}_{d+1}^{n-1}$. Finally, an extension X of W_{n-d} by W_{d+1}^\vee is obtained from an extension V of W_{n-d-1} by W_{d+1}^\vee via pullback if and only if $\operatorname{Hom}(X,\mathcal{O}_E) \neq 0$. \square

Now let us see how the bundles described by extensions which are identified under this isomorphism are related. **Proposition 3.11.** Let $\epsilon_d \in \operatorname{Ext}^1(W_{n-d}, W_d^{\vee})$ and $\epsilon_{d+1} \in \operatorname{Ext}^1(W_{n-d-1}, W_{d+1}^{\vee})$ be nonzero classes that correspond under the identification given in Corollary 3.10. Let V and V', respectively, be the total spaces of these extensions. Then V and V' are isomorphic bundles.

Proof. Let X be the bundle of rank n+1 obtained by pushing out the extension V by the map $W_d^{\vee} \to W_{d+1}^{\vee}$. Clearly, we have a short exact sequence

$$0 \to V \to X \to \mathcal{O}_E \to 0.$$

Similarly, let X' be the rank n+1-bundle obtained by pulling back the extension V' along the map $W_{n-d} \to W_{n-d-1}$. Dually, we have an exact sequence

$$0 \to \mathcal{O}_E \to X' \to V' \to 0.$$

It follows easily from Theorem 3.2 that writing $V = Y \oplus I_r(\mathcal{O}_E)$ with $H^0(Y) = 0$, we have $X \cong Y \oplus I_{r+1}(\mathcal{O}_E)$. Similarly, writing $V' = Y' \oplus I_s(\mathcal{O}_E)$ we have $X' \cong Y' \oplus I_{s+1}(\mathcal{O}_E)$.

The fact that ϵ_d and ϵ_{d+1} are identified means that the extensions for X and X' are isomorphic. In particular, X and X' are isomorphic bundles. This implies that r=s and that Y and Y' are isomorphic. But then V and V' are isomorphic as well. \square

Notice that the isomorphism produced by the previous result is canonical on $Y \subseteq V$ but is not canonical on the $I_r(\mathcal{O}_E)$ factor. We shall see later that the families of bundles $\mathbf{U}(d)$ and $\mathbf{U}(d+1)$ are not isomorphic, which means that there cannot be a canonical isomorphism in general between corresponding bundles. In practice, this means the following: suppose that V is given as an extension of W_{n-d} by W_d^\vee , with n-d>1. Then W_{n-d} has the distinguished subbundle isomorphic to \mathcal{O}_E . Let W' be the preimage in V of this bundle, so that W' is an extension of \mathcal{O}_E by W_d^\vee . Then $W'\cong W_{d+1}^\vee$ if and only if $h^0(V)=0$, if and only if the support of V does not contain \mathcal{O}_E , but otherwise $W'\cong W_d^\vee \oplus \mathcal{O}_E$.

The direct comparison of the extension classes given above leads to a comparison of universal bundles.

Theorem 3.12. Let H be the divisor in \mathbb{P}_d^{n-1} such that, if $x \in H$ and V is the corresponding extension, then $h^0(V) = 1$. Let $i: H \to \mathbb{P}_d^{n-1}$ be the inclusion. Then there is an exact sequence

$$0 \to \mathbf{U}(d-1) \to \mathbf{U}(d) \to (i \times \mathrm{Id})_* \mathcal{O}_{H \times E}(1) \to 0,$$

which expresses U(d-1;n) as an elementary modification of U(d;n). Moreover, this elementary modification is unique in an appropriate sense.

Proof. Let H' be the hyperplane in \mathbb{P}_d^n corresponding to the set of extensions X of W_{n-d+1} by W_d^{\vee} such that $\operatorname{Hom}(X,\mathcal{O}_E) \neq 0$. By the last statements of (3.9) and (3.10), H' is the image of \mathbb{P}_d^{n-1} in \mathbb{P}_d^n as well as the image of \mathbb{P}_{d+1}^{n-1} . By base change, $\pi_{1*}Hom(\mathbf{U}(d;n+1)|H'\times E,\mathcal{O}_{H'\times E})$ is a line bundle on H'. By looking at the exact sequence

$$0 \to Hom(\pi_2^* W_{n-d+1}, \mathcal{O}_{\mathbb{P}_d^n \times E}) \to Hom(\mathbf{U}(d; n+1) | H' \times E, \mathcal{O}_{\mathbb{P}_d^n \times E}) \to \\ \to Hom(\pi_2^* W_d^{\vee} \otimes \pi_1^* \mathcal{O}_{\mathbb{P}_d^n}(1), \mathcal{O}_{\mathbb{P}_d^n \times E}) \to 0,$$

and restricting to $H' \times E$, we see that this line bundle is $\mathcal{O}_{H' \times E}(-1)$. Thus there is a surjection

$$\mathbf{U}(d; n+1)|H' \times E \to \mathcal{O}_{H' \times E}(1).$$

We claim that the kernel of this surjection is identified with $\mathbf{U}(d-1;n)$. In fact, if \mathbf{U}' denotes the kernel, we have a commutative diagram with exact rows and columns

$$0 \longrightarrow \pi_2^* W_{d-1}^{\vee} \otimes \mathcal{O}_{H' \times E}(1) \longrightarrow \mathbf{U}' \longrightarrow \pi_2^* W_{n-d+1} \longrightarrow 0$$

$$\downarrow \qquad \qquad \downarrow \qquad \qquad \parallel$$

$$0 \longrightarrow \pi_2^* W_d^{\vee} \otimes \mathcal{O}_{H' \times E}(1) \longrightarrow \mathbf{U}(d; n+1) | H' \times E \longrightarrow \pi_2^* W_{n-d+1} \longrightarrow 0$$

$$\downarrow \qquad \qquad \downarrow \qquad \qquad \downarrow$$

$$\mathcal{O}_{H' \times E}(1) \longrightarrow \mathcal{O}_{H' \times E}(1)$$

$$\downarrow \qquad \qquad \downarrow$$

$$0 \longrightarrow 0$$
and tracing through the diagram identifies \mathbf{U}' with $\mathbf{U}(d-1,n)$ connectibly with

and tracing through the diagram identifies \mathbf{U}' with $\mathbf{U}(d-1;n)$, compatibly with the identification of H' with \mathbb{P}_d^{n-1} .

Now we can also consider the line bundle $\pi_{1*}(\mathbf{U}(d;n+1)|H'\times E)$. A very similar argument shows that this line bundle is isomorphic to $\mathcal{O}_{H'}$, and that the quotient \mathbf{U}'' of $\mathbf{U}(d;n+1)|H'\times E$ via the natural map

$$\mathcal{O}_{H'\times E} = \pi^*\pi_{1*}(\mathbf{U}(d;n+1)|H'\times E) \to \mathbf{U}(d;n+1)|H'\times E$$

is isomorphic to $\mathbf{U}(d;n)$. Putting these two constructions together, we see that we have found

$$\mathbf{U}(d-1;n) \to \mathbf{U}(d;n+1)|H' \times E \to \mathbf{U}(d;n).$$

Away from H, which is the image of \mathbb{P}_d^{n-2} in \mathbb{P}_d^{n-1} , the inclusion $\mathbf{U}(d-1;n) \to \mathbf{U}(d;n)$ is an equality. To summarize, then, there is a commutative diagram

The map $\mathcal{O}_{H'\times E} \to \mathcal{O}_{H'\times E}(1)$ can only vanish along H. Thus it vanishes exactly along H, and the quotient of $\mathbf{U}(d;n)$ by $\mathbf{U}(d-1;n)$ is a line bundle supported on $H\times E$. By what we showed above for $\mathbf{U}(d;n+1)$, this line bundle is necessarily $\mathcal{O}_{H\times E}(1)$. Hence we have found an exact sequence

$$0 \to \mathbf{U}(d-1;n) \to \mathbf{U}(d;n) \to (i \times \mathrm{Id})_* \mathcal{O}_{H \times E}(1) \to 0,$$

realizing $\mathbf{U}(d-1;n)$ as an elementary modification of $\mathbf{U}(d;n)$. The uniqueness is straightforward and left to the reader. This completes the proof of Theorem 3.12. \square

3.3. Comparison of coarse moduli spaces.

We have succeeded in identifying the \mathbb{P}_d^{n-1} for the various d, $1 \leq d < n$, in a purely cohomological way and in showing that extension classes in different groups which are identified produce isomorphic vector bundles. Next we wish to identify these projective spaces with the projective space $|np_0|$ which is the parameter space of regular semistable rank n bundles with trivial determinant in the spectral cover construction of these bundles. Of course, the existence of the bundle $\mathbf{U}(d)$ and Theorem 1.5 give us one such identification. However, although we shall not need this in what follows, we want to find a direct cohomological comparison between $\mathrm{Ext}^1(W_{n-d},W_d^\vee)$ and $H^0(\mathcal{O}_E(np_0))$. We have two identifications: one purely cohomological and the other using the bundles to identify the spaces. We shall show that these identifications agree.

Let us begin with the purely cohomological identification. (We will be pedantic here about identifying various one-dimensional vector spaces with $\mathbb C$ in order to carry out the discussion in families in the next section.)

Proposition 3.13. Let $H_{n-1}^0 = H^0(\mathcal{O}_E(p_0) \otimes W_{n-1})$. It is an n-dimensional vector space. Let $D = H^1(\mathcal{O}_E)$ be the dualizing line. Let

$$I: H^1(\mathcal{O}_E(-p_0) \otimes W_{n-1}^{\vee}) \to H^0(\mathcal{O}_E(np_0)) \otimes \det(H_{n-1}^0)^{-1} \otimes D$$

be the composition

$$H^{1}(\mathcal{O}_{E}(-p_{0}) \otimes W_{n-1}^{\vee}) \xrightarrow{S} (H_{n-1}^{0})^{*} \otimes D \xrightarrow{A} \bigwedge^{n-1} H_{n-1}^{0} \otimes \det(H_{n-1}^{0})^{-1} \otimes D \rightarrow \underbrace{\stackrel{ev \otimes \operatorname{Id} \otimes \operatorname{Id}}{\longrightarrow}} H^{0}(\det(\mathcal{O}_{E}(p_{0}) \otimes W_{n-1})) \otimes \det(H_{n-1}^{0})^{-1} \otimes D$$

$$= H^{0}(\mathcal{O}_{E}(np_{0})) \otimes \det(H_{n-1}^{0})^{-1} \otimes D,$$

where S is Serre duality, A is the map induced by taking adjoints from the natural pairing

$$H_{n-1}^0\otimes \bigwedge^{n-1}H_{n-1}^0\to \det(H_{n-1}^0),$$

ev is the map

$$ev: \bigwedge^{n-1} H^0(\mathcal{O}_E(p_0) \otimes W_{n-1}) \to H^0(\bigwedge^{n-1} (\mathcal{O}_E(p_0) \otimes W_{n-1})).$$

Then I is an isomorphism.

Proof. On general principles S is an isomorphism. Since H_{n-1}^0 is n-dimensional, the adjoint map A is clearly an isomorphism. What is less obvious that the map ev is an isomorphism, which follows from the next claim.

Claim 3.14. The evaluation map

$$ev: \bigwedge^{n-1} H^0(\mathcal{O}_E(p_0) \otimes W_{n-1}) \to H^0(\bigwedge^{n-1} (\mathcal{O}_E(p_0) \otimes W_{n-1}))$$

is an isomorphism

Proof. We prove this by induction. The case n=2 is clear since ev is the identity. Assume the result for $n-1 \ge 2$. There is a short exact sequence

(*)
$$0 \to \mathcal{O}_E(p_0) \to \mathcal{O}_E(p_0) \otimes W_{n-1} \to \mathcal{O}_E(p_0) \otimes W_{n-2} \to 0$$

leading to a short exact sequence

$$0 \to H^0(\mathcal{O}_E(p_0)) \to H^0(\mathcal{O}_E(p_0) \otimes W_{n-1}) \to H^0(\mathcal{O}_E(p_0) \otimes W_{n-2}) \to 0,$$

since by (3.3) the H^1 terms vanish. Since the first term has dimension one, we have a short exact sequence

$$0 \to H^0(\mathcal{O}_E(p_0)) \otimes \bigwedge^{n-2} H^0_{n-2} \to \bigwedge^{n-1} H^0_{n-1} \to \bigwedge^{n-1} H^0_{n-2} \to 0.$$

Taking determinants in (*) yields an isomorphism

$$\bigwedge^{n-1} (\mathcal{O}_E(p_0) \otimes W_{n-1}) \cong \mathcal{O}_E(p_0) \otimes \bigwedge^{n-2} (\mathcal{O}_E(p_0) \otimes W_{n-2}).$$

Tensoring the inclusion $\mathcal{O}_E \to \mathcal{O}_E(p_0)$ with $\bigwedge^{n-2}(\mathcal{O}_E(p_0) \otimes W_{n-2})$ and using the above isomorphism leads to a short exact sequence

$$0 \to \bigwedge^{n-2} (\mathcal{O}_E(p_0) \otimes W_{n-2}) \to \bigwedge^{n-1} (\mathcal{O}_E(p_0) \otimes W_{n-1}) \to \bigwedge^{n-1} (\mathcal{O}_E(p_0) \otimes W_{n-1})|_{p_0} \to 0.$$

Unraveling the definitions one sees that the map ev induces a commutative diagram, with exact columns,

and that the restriction of ev to the first term is simply the tensor product of the identity on $H^0(\mathcal{O}_E(p_0))$ and the evaluation map, with n-2 replacing n-1. By induction, the top horizontal map is an isomorphism.

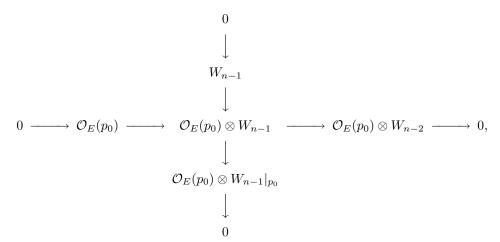
To finish the proof of (3.14), and thus of (3.13), it suffices by the 5-lemma to show that e is an isomorphism. Now e is the (n-1)-fold wedge product of a map

$$\overline{e}: H^0(\mathcal{O}_E(p_0) \otimes W_{n-2}) \to (\mathcal{O}_E(p_0) \otimes W_{n-1})|_{p_0}$$

defined as follows. For any section ψ of $\mathcal{O}_E(p_0) \otimes W_{n-2}$, lift to a section $\tilde{\psi}$ of $\mathcal{O}_E(p_0) \otimes W_{n-1}$, and then restrict $\tilde{\psi}$ to p_0 . Thus, it suffices to prove:

Claim 3.15. The map \overline{e} described above is an isomorphism.

Proof. First notice that if $\tilde{\psi}$ and $\tilde{\psi}'$ are lifts of ψ to sections of $\mathcal{O}_E(p_0) \otimes W_{n-1}$, then they differ by a section of $\mathcal{O}_E(p_0) \subset \mathcal{O}_E(p_0) \otimes W_{n-1}$. But any section of $\mathcal{O}_E(p_0)$ vanishes at p_0 so that $\tilde{\psi}$ and $\tilde{\psi}'$ have the same restriction to p_0 . This shows that \overline{e} is well-defined. From the diagram



the fact that all the H^1 terms vanish, and the fact that both $H^0(\mathcal{O}_E(p_0))$ and $H^0(W_{n-1})$ are one dimensional, the claim comes down to the statement that the images in $H^0(\mathcal{O}_E(p_0) \otimes W_{n-1})$ of $H^0(\mathcal{O}_E(p_0))$ and of $H^0(W_{n-1})$ are equal. But we also have a commutative square

$$\mathcal{O}_E \longrightarrow \mathcal{O}_E \otimes W_{n-1}$$

$$\downarrow \qquad \qquad \downarrow$$

$$\mathcal{O}_E(p_0) \longrightarrow \mathcal{O}_E(p_0) \otimes W_{n-1}$$

with the top arrow and the left arrow inducing isomorphisms on H^0 . Claim 3.14 and hence (3.13) now follow. \square

Proposition 3.13 and Corollary 3.10 have produced cohomological isomorphisms between the extension groups $\operatorname{Ext}^1(W_{n-d},W_d^\vee)$ and $H^0(\mathcal{O}_E(np_0))$. On the other hand, as remarked previously, from the existence of the bundle $\mathbf{U}(d) \to \mathbb{P}_d^{n-1} \times E$, Theorem 1.5 produces isomorphisms

$$\Phi_d : \mathbb{P}_d^{n-1} \to |np_0|$$

sending $x \in \mathbb{P}_d^{n-1}$ to the point $\zeta(V_x)$, where V is the the extension determined by the point x. By Proposition 3.11 the identification of \mathbb{P}_d^{n-1} with \mathbb{P}_{d+1}^{n-1} given in Corollary 3.10 identifies Φ_d with Φ_{d+1} . Still, it remains to compare the map Φ_1 with the projectivization of the map produced by Proposition 3.13.

Proposition 3.16. The map $\Phi_1: \mathbb{P}_1^{n-1} \to |np_0|$ is the projectivization of the identification

$$I: H^1(\mathcal{O}_E(-p_0) \otimes W_{n-1}^{\vee}) \to H^0(\mathcal{O}_E(np_0)) \otimes M,$$

where M is the line $\det(H^0(\mathcal{O}_E(p_0) \otimes W_{n-1}))^{-1} \otimes D$. In other words, if V is a nontrivial extension corresponding to $\alpha \in \operatorname{Ext}^1(W_{n-1}, \mathcal{O}_E(-p_0))$, then the point $\zeta(V) \in |np_0|$ corresponds to the line

$$\mathbb{C} \cdot I(\alpha) \subset H^0(\mathcal{O}_E(np_0)) \otimes M.$$

In particular, Φ_1 is an isomorphism, and hence so is Φ_d for every $1 \leq d < n$.

Proof. Let $\bar{I}: \mathbb{P}_1^{n-1} \to |np_0|$ be the projectivization of I. We begin by determining when a line bundle λ of degree zero is in the support of V.

Claim 3.17. Let V be given by an extension class $\alpha \in H^1(\mathcal{O}_E(-p_0) \otimes W_{n-1}^{\vee})$. Then $\operatorname{Hom}(V, \lambda) \neq 0$ if and only if the image of α in $H^1(\lambda \otimes W_{n-1}^{\vee})$ under the map induced by the inclusion $\mathcal{O}_E(-p_0) \to \lambda$ is zero.

Proof. There is a nonzero map $\mathcal{O}_E(-p_0) \to \lambda$, and it is unique up to a scalar. Let V' be the pushout of the extension V by the map $\mathcal{O}_E(-p_0) \to \lambda$. Then the pushout extension is trivial, i.e. the image of α in $H^1(\lambda \otimes W_{n-1}^{\vee})$ is zero, if and only if there is a map $V' \to \lambda$ splitting the inclusion of λ into V'. Such a map is equivalent to a map $V \to \lambda$ so that the composition $\mathcal{O}_E(-p_0) \to V \to \lambda$ is the inclusion. By (3.6), if V has a nontrivial map to λ , then this map restricts to $\mathcal{O}_E(-p_0)$ to be the inclusion (since otherwise the image of $\mathcal{O}_E(-p_0)$ would be contained in a proper subbundle of degree zero). Thus, there is such a map if and only if the λ component of V is nonzero, which is equivalent to the existence of a nonzero map $V \to \lambda$. \square

Claim 3.18. Suppose that $\lambda \cong \mathcal{O}_E(q-p_0)$ for some $q \in E$. Let V be given by an extension class $\alpha \in H^1(\mathcal{O}_E(-p_0) \otimes W_{n-1}^{\vee})$. Then λ is in the support of V if and only if q is in the support of $\bar{I}(\alpha) \in |np_0|$.

Proof. Applying Serre duality S and the adjoint map A to the previous claim, and tracing through the identifications, we see that $\lambda = \mathcal{O}_E(q - p_0)$ is in the support of V if and only if the corresponding map

$$\bigwedge^{n-1} H_{n-1}^0 \otimes L \to \bigwedge^{n-1} (\mathcal{O}_E(p) \otimes W_{n-1} \otimes L)|_q$$

vanishes, if and only if the section giving $I(\alpha)$ in $\det(\mathcal{O}_E(p_0) \otimes W_{n-1}) \otimes L$ vanishes at q, if and only if q is in the support of $\bar{I}(\alpha)$. \square

Now we can prove Proposition 3.16. We have two maps \bar{I} , the projectivization of the linear map I, and Φ_1 , mapping $\mathbb{P}_1^{n-1} \to |np_0|$. We wish to show \bar{I} and Φ_1 are equal. We know that \bar{I} is an isomorphism. Thus, it suffices to show that, for an open dense subset U of $|np_0|$, $\Phi_1(x) = \bar{I}(x)$ for all $x \in \Phi_1^{-1}(U)$. Choose U to be the open subset of divisors in $|np_0|$ whose support is n distinct smooth points of E. Let $x = \sum_i e_i \in U$. The extension determined by $\alpha = \Phi_1^{-1}(x)$ is a semistable bundle V which is written as $\bigoplus_i \lambda_i$ for n distinct line bundles λ_i , where $\lambda_i = \mathcal{O}_E(e_i - p_0)$. According to Claim 3.18, $\bar{I}(\alpha)$ contains e_i , $1 \le i \le n$ in its support, and hence $\bar{I} \circ \Phi_1^{-1}(x) = x$. This completes the proof of Proposition 3.16. \square

3.4. From universal bundles to spectral covers.

We now take another look at the spectral cover construction, and generalize it to singular curves. Fix an integer d, $1 \le d \le n-1$, and consider the sheaf

 $\mathbf{A} = \pi_{1*}Hom(\mathbf{U}(d), \mathbf{U}(d))$. It is a locally free rank n sheaf of commutative algebras over \mathbb{P}^{n-1} , and in case E is smooth we have identified this sheaf with $\nu_*\mathcal{O}_T$ in (2.4). (There is nothing special about taking $\mathbf{U}(d)$; we could replace $\mathbf{U}(d)$ by any "universal" bundle, once we know how to construct one.) We propose to reverse this procedure: starting with \mathbf{A} , define T to be the corresponding space $\mathbf{Spec} \mathbf{A}$. In particular, this gives a definition of T in case E is singular.

Lemma 3.19. Let E be a Weierstrass cubic. With T as defined above, there is a finite flat morphism $\nu \colon T \to \mathbb{P}^{n-1}$. Moreover, T is reduced.

Proof. By construction, there is a finite morphism $\nu: T \to \mathbb{P}^{n-1}$. In fact, since **A** is locally free, T is flat over \mathbb{P}^{n-1} of degree n.

The fact that T is reduced follows from the fact that \mathbb{P}^{n-1} is reduced and that T is generically reduced, and as such is a general fact concerning finite flat morphisms. Cover \mathbb{P}^{n-1} by affine open sets Spec R such that $\nu^{-1}(\operatorname{Spec} R) = \operatorname{Spec} R'$, where R' is a free rank n R-module. It will suffice to show that R' is reduced for every such R. If $f \in R'$, then f does not vanish on a Zariski open set, since R' is locally free and R is reduced. Thus the restriction of f to a general fiber of ν , consisting of n distinct (reduced) points, is nonzero. It follows that $f^k \neq 0$ for every k > 0. Thus R' is reduced, and so T is reduced. \square

In case E is smooth, Theorem 2.4 shows that the T defined above is the same as the spectral cover T defined in Section 2, although even in this case it will be useful to define T as we have above in order to compare $\mathbf{U}(d)$ with the bundles U_a .

The points of T are by definition in one-to-one correspondence with pairs (V, \mathfrak{m}) , where V is a regular semistable bundle with trivial determinant and \mathfrak{m} is a maximal ideal in $\operatorname{Hom}(V, V)$. Let us describe such maximal ideals:

Lemma 3.20. If V is a regular semistable bundle of rank n, then the maximal ideals \mathfrak{m} of $\operatorname{Hom}(V,V)$ are in one-to-one correspondence with nonzero homomorphisms $\rho\colon V\to \lambda$ mod scalars, where λ is a torsion free rank one sheaf of degree zero. The correspondence is as follows: given ρ , we set

$$\mathfrak{m} = \{ A \in \operatorname{Hom}(V, V) : \rho \circ A = 0 \},$$

and given a maximal ideal \mathfrak{m} , we define $\lambda = V/\mathfrak{m} \cdot V$ and take ρ to be the obvious projection.

Proof. If $V = (\bigoplus_i I_{r_i}(\lambda_i)) \oplus I(\mathcal{F})$, then $\operatorname{Hom}(V, V)$ is a direct sum

$$\left(\bigoplus_{i}\operatorname{Hom}(I_{r_{i}}(\lambda_{i}),I_{r_{i}}(\lambda_{i}))\right)\oplus\operatorname{Hom}(I(\mathcal{F}),I(\mathcal{F})),$$

and it will clearly suffice to consider the case where V is either $I_{r_i}(\lambda_i)$ or $I(\mathcal{F})$. For simplicity, we assume that $V = I(\mathcal{F})$. Thus there is a unique ρ mod scalars by definition. If we set $\mathfrak{m} = \{ A \in \operatorname{Hom}(V, V) : \rho \circ A = 0 \}$, then \mathfrak{m} is an ideal of $\operatorname{Hom}(V, V)$. In fact, there is an induced homomorphism $\operatorname{Hom}(V, V) \to \operatorname{Hom}(\mathcal{F}, \mathcal{F}) = H^0(\mathcal{O}_{\tilde{E}}) = \mathbb{C}$, and \mathfrak{m} is the kernel of this homomorphism. Thus \mathfrak{m} is a maximal ideal.

Next we claim that \mathfrak{m} is the unique maximal ideal in $\operatorname{Hom}(V,V)$. It suffices to show that \mathfrak{m} contains every non-invertible element of $\operatorname{Hom}(V,V)$. If $A \in \operatorname{Hom}(V,V)$ is not invertible, then $\operatorname{Im} A$ is a proper torsion free subsheaf of V of rank smaller

than n and degree at least zero. It follows that deg Im A=0. But then, by Lemma 1.11, Im $A \subseteq \operatorname{Ker} \rho$. It follows that $\rho \circ A=0$, so that by definition $A \in \mathfrak{m}$. Thus \mathfrak{m} is the unique maximal ideal of $\operatorname{Hom}(V,V)$.

Finally we claim that $V/\mathfrak{m} \cdot V \cong \mathcal{F}$. By definition, the surjection $\rho \colon V \to \mathcal{F}$ factors through the quotient $V/\mathfrak{m} \cdot V$, so that $\mathfrak{m} \cdot V \subseteq \operatorname{Ker} \rho$. Choosing a basis A_1, \dots, A_{n-1} for \mathfrak{m} , we see that $\mathfrak{m} \cdot V$ is of the form $A_1(V) + \dots + A_{n-1}(V)$, and thus it is a subsheaf of V of degree at least zero. Hence it has degree exactly zero, and thus it is filtered by subsheaves whose quotients are isomorphic to \mathcal{F} . If $\mathfrak{m} \cdot V$ has rank r, it follows by Lemma 1.14 that $\dim \operatorname{Hom}(V,\mathfrak{m} \cdot V) \leq r$. But clearly $\operatorname{Hom}(V,V) = \operatorname{Hom}(V,\mathfrak{m} \cdot V) \oplus \mathbb{C} \operatorname{Id}$, and since $\dim \operatorname{Hom}(V,V) = n$, we must have r = n - 1. Since both $\mathfrak{m} \cdot V$ and $\operatorname{Ker} \rho$ have degree zero and rank n - 1, and $\mathfrak{m} \cdot V \subseteq \operatorname{Ker} \rho$, $\mathfrak{m} \cdot V = \operatorname{Ker} \rho$. Thus $V/\mathfrak{m} \cdot V \cong \mathcal{F}$. \square

Next, given the spectral cover T, by construction $\mathbf{U}(d)$ is a module over $\mathcal{O}_T = \pi_{1*}Hom(\mathbf{U}(d),\mathbf{U}(d))$, and thus $\mathbf{U}(d)$ corresponds to a coherent sheaf \mathcal{L}_d over $T\times E$. By construction, $(\nu\times\mathrm{Id})_*\mathcal{L}_d=\mathbf{U}(d)$. In case E is smooth, or more generally in case (V,\mathfrak{m}) is a point of T such that the support of V does not contain the singular point of E, then it is easy to check directly that \mathcal{L}_d is a line bundle near $(V,\mathfrak{m})\times E$.

We can now summarize our description of T as follows:

Theorem 3.21. There is an isomorphism (ν, r) of T onto the incidence correspondence in $|np_0| \times E$ with the following property: Let Δ_0 be the diagonal in $E \times E$, with ideal sheaf I_{Δ_0} , and let \mathcal{P}_0 be the sheaf on $E \times E$ defined by $Hom(I_{\Delta_0}, \mathcal{O}_{E \times E}(-E \times \{p_0\}))$. Then there exists a line bundle M on T such that $\mathcal{L}_d = (r \times \mathrm{Id})^* \mathcal{P}_0 \otimes \pi_1^* M$.

Proof. We have shown in (0.3) that \mathcal{P}_0 is flat over the first factor in the product $E \times E$ and identifies the first factor with $\bar{J}(E)$, the compactified generalized Jacobian of E. Let T' be the incidence correspondence in $|np_0| \times E$. Note that T' is irreducible; in fact, projection onto the second factor makes T' a \mathbb{P}^{n-2} -bundle over E, namely $T' = \mathbb{P}\mathcal{E}$ as in Section 2. We will first find a morphism from T' to T which is a bijection as a set-valued function. Let $\nu' : T' \to \mathbb{P}^{n-1}$ and $r' : T' \to E$ be the projections to the first and second factors.

By construction, for a point $(x, e) \in T'$, if V is the vector bundle over E corresponding to x and λ is the rank one torsion free sheaf of degree zero corresponding to e, dim $\text{Hom}(V, \lambda) = 1$. Thus, by base change, with $\pi_1 : T' \times E \to T'$ the projection,

$$\pi_{1*}[(\nu' \times \mathrm{Id})^* \mathbf{U}(d)^{\vee} \otimes (r' \times \mathrm{Id})^* \mathcal{P}_0] = M$$

is a line bundle on T'. After replacing $(r' \times \mathrm{Id})^* \mathcal{P}_0$ by $(r' \times \mathrm{Id})^* \mathcal{P}_0 \otimes M^{-1} = \mathcal{P}'$, we can assume that there is a surjection

$$\rho \colon (\nu' \times \mathrm{Id})^* \mathbf{U}(d) \to \mathcal{P}'.$$

On every fiber, the homomorphism $V \to \lambda$ is preserved up to scalars by every endomorphism of V. Thus, there is an induced homomorphism

$$\pi_{1*}Hom((\nu' \times \mathrm{Id})^*\mathbf{U}(d), (\nu' \times \mathrm{Id})^*\mathbf{U}(d)) \to \pi_{1*}Hom(\mathcal{P}', \mathcal{P}').$$

Now by the flatness of \mathcal{P}_0 , it is easy to check that $Hom(\mathcal{P}', \mathcal{P}')$ is flat over T' and that the natural multiplication map $\mathcal{O}_{T'} \to \pi_{1*}Hom(\mathcal{P}', \mathcal{P}')$ is an isomorphism. By base change, the first term $\pi_{1*}Hom((\nu' \times \mathrm{Id})^*\mathbf{U}(d), (\nu' \times \mathrm{Id})^*\mathbf{U}(d))$

in the above homomorphism is the pullback to T' of the sheaf of algebras $\mathbf{A} = \pi_{1*}Hom(\mathbf{U}(d),\mathbf{U}(d))$ over \mathbb{P}^{n-1} , and hence it is just the structure sheaf $\mathcal{O}_{T'\times_{\mathbb{P}^{n-1}}T}$ of the fiber product $T'\times_{\mathbb{P}^{n-1}}T$. The homomorphism $\mathcal{O}_{T'\times_{\mathbb{P}^{n-1}}T}\to \mathcal{O}_{T'}$ corresponds to a morphism $T'\to T'\times_{\mathbb{P}^{n-1}}T$, which is a section of the natural projection $T'\times_{\mathbb{P}^{n-1}}T\to T'$. Such a section is the same thing as a morphism $T'\to T$ (covering the given maps to \mathbb{P}^{n-1}). On the level of points, this morphism is as follows: take an element D in $|np_0|$ and a point e in the support of D. Pass to the corresponding vector bundle V and the morphism $V\to\lambda$, where λ is the rank one degree zero torsion free sheaf corresponding to e, and then set m to be the maximal ideal corresponding to $V\to\lambda$. It is then clear that $T'\to T$, as constructed above, is a bijection of sets. In particular, T is irreducible.

Now we want to construct a morphism which is the inverse of the morphism $T' \to T$. It suffices to find the morphism $r \colon T \to E$. Viewing E as isomorphic to the compactified generalized Jacobian of E, we can find such a morphism once we know that the sheaf \mathcal{L}_d is flat over E:

Lemma 3.22. The sheaf \mathcal{L}_d is flat over T. If $t \in T$ corresponds to the the pair (V, \mathfrak{m}) , and λ is the rank one torsion free sheaf of degree zero given by $V/\mathfrak{m} \cdot V$, then the restriction of \mathcal{L}_d to the slice $\{t\} \times E$ is λ . Thus \mathcal{L}_d is a flat family of rank one torsion free sheaves on $T \times E$.

Proof. First let us show that, in the above notation, the restriction of \mathcal{L}_d to the slice $\{t\} \times E$ is λ . In fact, suppose that t corresponds to the pair (V, \mathfrak{m}) and view V as a rank one module over $\operatorname{Hom}(V,V)$. Then the restriction of V to $\{t\} \times E$ is given by $V/\mathfrak{m} \cdot V = \lambda$. Now the Hilbert polynomial $P_{\lambda}(n) = \chi(E; \lambda \otimes \mathcal{O}_E(np_0))$ is independent of the choice of λ . As we have proved above, T is irreducible since it is the image of T', and thus, since it is reduced, it is integral. The proof of Theorem 9.9 on p. 261 of [10] then shows that \mathcal{L}_d is flat over T. The last statement is then clear. \square

By (0.3), as \mathcal{L}_d is flat over T, it defines a morphism $r\colon T\to E$ (viewing E as $\bar{J}(E)$). Thus we also have the product morphism $(\nu,r)\colon T\to \mathbb{P}^{n-1}\times E$, whose image is T'. Clearly, on the level of sets, the morphism $T\to T'$ is the inverse of the morphism $T'\to T$ constructed above. Since both T and T' are reduced, in fact the two maps are inverses as morphisms. By the functorial properties of the compactified Jacobian (0.3), $\mathcal{L}_d=(r\times \mathrm{Id})^*\mathcal{P}\otimes\pi_1^*M$. This then concludes the proof of (3.21). \square

We have now lined up the spectral covers, and proceed to identify the bundles $\mathbf{U}(d)$ in terms of T. It suffices to identify the bundle π_1^*M in (3.21). We do this first for d=1. In order to do so, we first make the following preliminary remarks. Let $\mathcal{O}_{T\times E}(\Delta)$ denote the rank one torsion free sheaf $(r\times \mathrm{Id})^*Hom(I_{\Delta_0},\mathcal{O}_{E\times E})$. Suppose that \mathcal{L} is any flat family of rank one torsion free sheaves on $T\times E$ such that there exists an injection $\mathcal{O}_{T\times E}\to \mathcal{L}$, with the cokernel exactly supported along Δ and with multiplicity one at a nonempty Zariski open subset of the smooth points. We claim that in this case $\mathcal{L}=\mathcal{O}_{T\times E}(\Delta)$. First, the universal property of the compactified Jacobian implies that $\mathcal{L}=(\alpha\times\mathrm{Id})^*Hom(I_{\Delta_0},\mathcal{O}_{E\times E})\otimes\pi_1^*M$ for some morphism $\alpha\colon T\to E$ and line bundle N on T. By hypothesis, $\alpha=r$ on a Zariski open dense subset of T, and thus everywhere. Next, since $\mathcal{O}_T\to\pi_{1*}\mathcal{O}_{T\times E}(\Delta)$ is an isomorphism, $H^0(N)=H^0(\pi_{1*}\mathcal{O}_{T\times E}(\Delta)\otimes N)=H^0(\mathcal{L})$, and every section of \mathcal{L} is given by multiplying the natural section of $\mathcal{O}_{T\times E}(\Delta)$ by a section s of N. In this

case, the cokernel is supported at $\Delta \cup \pi_1^{-1}(D)$, where D is the divisor of zeroes of s. Thus, if the support of the cokernel is Δ , then N must have a nowhere vanishing section, and so is trivial. We may thus conclude that $\mathcal{L} = \mathcal{O}_{T \times E}(\Delta)$.

Theorem 3.23.
$$U(1) = (\nu \times \mathrm{Id})_* \mathcal{O}_{T \times E}(\Delta - G) \otimes \pi_1^* \mathcal{O}_{\mathbb{P}^{n-1}}(1)$$
.

Proof. An equivalent formulation is:

$$(\nu \times \mathrm{Id})_* \mathcal{O}_{T \times E}(\Delta) \cong \mathbf{U}(1) \otimes \pi_2^* \mathcal{O}_E(p_0) \otimes \pi_1^* \mathcal{O}_{\mathbb{P}^{n-1}}(-1).$$

We will find a section of the torsion free rank one sheaf \mathcal{L} on $T \times E$ corresponding to $\mathbf{U}(1) \otimes \pi_2^* \mathcal{O}_E(p_0) \otimes \pi_1^* \mathcal{O}_{\mathbb{P}^{n-1}}(-1)$ which vanishes to order one along Δ . By the remarks before the proof, this will imply that $\mathcal{L} = \mathcal{O}_{T \times E}(\Delta)$. Now there is an inclusion

$$\mathcal{O}_{\mathbb{P}^{n-1}\times E} \to \mathbf{U}(1)\otimes \pi_2^*\mathcal{O}_E(p_0)\otimes \pi_1^*\mathcal{O}_{\mathbb{P}^{n-1}}(-1)$$

whose cokernel is $\pi_2^*(\mathcal{O}_E(p_0) \otimes W_{n-1}) \otimes \pi_1^*\mathcal{O}_{\mathbb{P}^{n-1}}(-1)$. Thus $h^0(\mathcal{L}) = h^0(\mathbf{U}(1) \otimes \pi_2^*\mathcal{O}_E(p_0) \otimes \pi_1^*\mathcal{O}_{\mathbb{P}^{n-1}}(-1)) = 1$. To see where the unique section of \mathcal{L} vanishes, fix a point $x \in \mathbb{P}^{n-1}$ corresponding to a regular semistable V, and consider where the corresponding section of $V \otimes \mathcal{O}_E(p_0)$ vanishes. This section arises from a homomorphism $\mathcal{O}_E(-p_0) \to V$ constructed in (3.4) and (3.5). For example, if $V = \bigoplus_i \mathcal{O}_E(e_i - p_0)$, then, up to the action of $(\mathbb{C}^*)^n$, the map is the direct sum of the natural inclusions $\mathcal{O}_E(-p_0) \to \mathcal{O}_E(e_i - p_0)$. At each fiber $\{(V, e_i)\} \times E$ of $T \times E$ lying over $\{V\}$, the section therefore vanishes simply at $((V, e_i), e_i)$. For a general point $t = (V, \mathfrak{m})$ of T, the restriction of the section of \mathcal{L} to the fiber $\{t\} \times E$ vanishes at the point of E where the corresponding section of the composite map

$$\mathcal{O}_E(-p_0) \to V \to V/\mathfrak{m} \cdot V \cong \lambda$$

vanishes. By the construction of (3.4), the composite map $\mathcal{O}_E(-p_0) \to V \to \lambda$ is not identically zero, and hence vanishes exactly at the point e of E corresponding to λ . Thus the section of \mathcal{L} vanishes exactly along Δ , with multiplicity one on a Zariski open and dense subset, proving (3.23). \square

Corollary 3.24. Suppose that E is smooth. For all $d \in \mathbb{Z}$ with $1 \leq d \leq n-1$, $\mathbf{U}(d) = U_{1-d} \otimes \pi_1^* \mathcal{O}_{\mathbb{P}^{n-1}}(1)$.

Proof. This follows by writing both sides as successive elementary modifications of U(1), resp. $U_0 \otimes \pi_1^* \mathcal{O}_{\mathbb{P}^{n-1}}(1)$, and applying (3.23). \square

3.5. The general spectral cover construction.

For every Weierstrass cubic E, we have now constructed a finite cover $T \to \mathbb{P}^{n-1}$ and a torsion free rank one sheaf $\mathcal{L}_0 = \mathcal{O}_{T \times E}(\Delta) \otimes \pi_2^* \mathcal{O}_E(-p_0)$. The proof of Theorem 2.4 goes over word-for-word to show:

Theorem 3.25. Let E be a Weierstrass cubic and let U' be a rank n vector bundle over $|np_0| \times E$ with the following property. For each $x \in |np_0|$ the restriction of U' to $\{x\} \times E$ is isomorphic to the restriction of U_0 to $\{x\} \times E$. Then $U' = (\nu \times \mathrm{Id})_* [\mathcal{O}_{T \times E}(\Delta - G) \otimes q_1^* L]$ for a unique line bundle L on T. \square

We may define U_a and, for a smooth point $e \in E$, $U_a[e]$ exactly as in Definition 2.6, and the proof of Lemma 2.7 shows that U_a is an elementary modification of

 U_{a-1} , and similarly for $U_a[e]$. Since $\operatorname{Pic} T \cong r^*\operatorname{Pic} E \oplus \mathbb{Z}$, every bundle U' as described in Theorem 3.25 is of the form $U_a[e] \otimes \mathcal{O}_{\mathbb{P}^{n-1}}(b)$ for integers a,b and a smooth point $e \in E$.

Question. In case E is singular, T is singular as well. Is there an analogue of twisting by Weil divisors on T which are not Cartier, which produces bundles which are not regular, or perhaps sheaves which are not locally free, over points of \mathbb{P}^{n-1} corresponding to the singular points of T? See Section 6 for a related construction in the smooth case.

The following is proved as in the proof of Theorem 2.8.

Theorem 3.26. Let E be a Weierstrass cubic and let S be a scheme or analytic space. Suppose that $\mathcal{U} \to S \times E$ is a rank n holomorphic vector bundle whose restriction to each slice $\{s\} \times E$ is a regular semistable bundle with trivial determinant. Let $\Phi: S \to |np_0|$ be the morphism constructed in Theorem 1.5. Let $\nu_S: \tilde{S} \to S$ be the pullback via Φ of the spectral covering $T \to |np_0|$:

$$\tilde{S} = S \times_{|np_0|} T,$$

and let $\tilde{\Phi}: \tilde{S} \to T$ be the map covering Φ . Let $q_1: \tilde{S} \times E \to \tilde{S}$ be the projection onto the first factor. Then there is a line bundle $\mathcal{M} \to \tilde{S}$ and an isomorphism of \mathcal{U} with

$$(\nu_S \times \mathrm{Id})_* \left((\tilde{\Phi} \times \mathrm{Id})^* (\mathcal{O}_{T \times E}(\Delta - G)) \otimes q_1^* \mathcal{M} \right).$$

3.6. Chern classes.

Theorem 3.27. For all d with $1 \le d \le n-1$, the total Chern class and the Chern character of $\mathbf{U}(d)$ are given by:

$$c(\mathbf{U}(d)) = (1 + h + \pi_2^*[p_0])(1 + h)^{d-1};$$

$$\operatorname{ch} \mathbf{U}(d) = (d - \pi_2^*[p_0])e^h + (n - d) + [p_0].$$

Thus
$$c(U_0) = (1 - h + \pi_2^*[p_0] \cdot h)(1 - h)^{n-2}$$
 and $\operatorname{ch} U_0 = ne^{-h} + (1 - \pi_2^*[p_0])(1 - e^{-h})$. \square

Proof. In K-theory, W_d is the same as the direct sum of d-1 trivial bundles and the line bundle $\mathcal{O}_E(p_0)$. Thus

$$c(\mathbf{U}(d)) = (1 - \pi_2^*[p_0] + h)(1+h)^{d-1}(1 + \pi_2^*[p_0])$$

= $(1 + h + \pi_2^*[p_0])(1+h)^{d-1}$,

and

$$\operatorname{ch} \mathbf{U}(d) = \pi_2^* \operatorname{ch}(W_d^{\vee}) \cdot \pi_1^* \operatorname{ch}(\mathcal{O}_{\mathbb{P}^{n-1}}(1)) + \pi_2^* \operatorname{ch} W_{n-d}$$
$$= (d - 1 + \pi_2^* e^{-[p_0]}) e^h + (n - d - 1 + \pi_2^* e^{[p_0]})$$
$$= (d - \pi_2^* [p_0]) e^h + (n - d) + [p_0],$$

since $e^{-[p_0]} = 1 - [p_0]$ and similarly for $e^{[p_0]}$. The formulas for $c(U_0)$ and $ch U_0$ then follow from (3.23). \square

4. A relative moduli space for elliptic fibrations.

Our goal in this section is to do the constructions of the last three sections in the relative setting of a family $\pi\colon Z\to B$ of elliptic curves (possibly with singular fibers), in order to produce families of bundles whose restriction to every fiber of π is regular semistable and with trivial determinant. First we identify the relative coarse moduli space as a projective bundle over Z. The extension picture generalizes in a straightforward way to give n-1 "universal" bundles $\mathbf{U}(d), 1 \leq d \leq n-1$, and they are related via elementary modifications. Using these bundles, we can generalize the spectral covers picture as well. Finally, we compute the Chern classes of the universal bundles.

Let $\pi \colon Z \to B$ be an elliptic fibration with a section σ . Following the notational conventions of the introduction, we shall always let $L^{-1} = R^1 \pi_* \mathcal{O}_Z$, which we can also identify with the normal bundle $\mathcal{O}_Z(\sigma)|\sigma$.

4.1. A relative coarse moduli space.

Our first task is to find a relative version of $|np_0|$ for a single elliptic curve. The relative version of the vector space $H^0(E; \mathcal{O}_E(np_0))$ is just the rank n vector bundle $\pi_*\mathcal{O}_Z(n\sigma) = \mathcal{V}_n$, and the relative moduli space will then be the associated projective bundle. From the exact sequence

$$0 \to \mathcal{O}_Z((n-1)\sigma) \to \mathcal{O}_Z(n\sigma) \to \pi^*L^{-n}|\sigma \to 0,$$

we obtain for $n \geq 2$ an exact sequence

$$0 \to \mathcal{V}_{n-1} \to \mathcal{V}_n \to L^{-n} \to 0.$$

(For n=1 the corresponding sequence identifies $\pi_*\mathcal{O}_Z(\sigma)$ with \mathcal{O}_B and shows that there is an isomorphism $\mathcal{O}_Z(\sigma)|\sigma\to R^1\pi_*\mathcal{O}_Z=L^{-1}$.) Thus \mathcal{V}_n is naturally filtered by subbundles such that the successive quotients are decreasing powers of L. The following well-known lemma shows that this filtration is split:

Lemma 4.1. $\pi_*\mathcal{O}_Z(\sigma) = \mathcal{O}_B$, and, for $n \geq 2$,

$$\pi_* \mathcal{O}_Z(n\sigma) = \mathcal{V}_n = \mathcal{O}_B \oplus L^{-2} \oplus \cdots \oplus L^{-n}.$$

Proof. Since $h^0(E; \mathcal{O}_E(np_0)) = n$ for all the fibers f of π , it follows from base change that $\pi_*\mathcal{O}_Z(n\sigma)$ is a vector bundle of rank n. Furthermore, the local sections of this bundle over an open subset $U \subset B$ are simply the meromorphic functions on $\pi^{-1}(U)$ which have poles of order at most n along $\sigma \cap U$. For U sufficiently small, there are functions X with a pole of order 2 along σ and Y with a pole of order 3. Moreover, if we require that X and Y satisfy a Weierstrass equation, then X and Y are unique up to nowhere vanishing functions in U and transform as sections of L^{-2}, L^{-3} respectively. We can also use the defining equation of Z to write Y^2 as a cubic polynomial in X. Now every section of $\pi_*\mathcal{O}_Z(n\sigma)$ can be written uniquely as

$$(\alpha_0 + \alpha_1 X + \dots + \alpha_k X^k) + Y(\beta_0 + \beta_1 X + \dots + \beta_\ell X^\ell)$$

where the α_i are holomorphic sections of L^{-2i} and the β_j are holomorphic sections of L^{-2j-3} and $2k \leq n$ and $2\ell + 3 \leq n$. The α_i, β_j determine the isomorphism claimed in the statement of the lemma. \square

Notice that the inclusion $\mathcal{V}_{n-1} \subset \mathcal{V}_n$ corresponds to the natural inclusion

$$\mathcal{O}_B \oplus L^{-2} \oplus \cdots \oplus L^{-(n-1)} \subset \mathcal{O}_B \oplus L^{-2} \oplus \cdots \oplus L^{-n}.$$

In particular, the distinguished points $\mathbf{o}_E = np_0 \in |np_0|$ corresponding to the bundles with all Jordan-Hölder quotients trivial fit together to make a section \mathbf{o}_Z of $\mathbb{P}\mathcal{V}_n$. This section is the projectivization $\mathbb{P}\mathcal{O}_B$ of the first factor \mathcal{O}_B in the above decomposition.

We call the above splitting the X-Y splitting of $\pi_*\mathcal{O}_Z(n\sigma)$. While this decomposition of $\pi_*\mathcal{O}_Z(n\sigma)$ is natural it is not the only possible decomposition, even having the property described in the previous paragraph. For example, another splitting was suggested to us by P. Deligne. There is a global holomorphic differential ω on E which is given on a Zariski open subset of E by dX/Y. There is a local complex coordinate ζ for E centered at p_0 with the property that on the open set on which this local coordinate is defined we have $\omega = d\zeta$. Of course, there is a homomorphism $\mathbb{C} \to E$ which pulls ζ back to the usual coordinate on \mathbb{C} . Every meromorphic function on E with a pole of order at most n at p_0 can be expanded as a Laurent series in ζ :

$$f = \sum_{i=-n}^{\infty} b_i \zeta^i.$$

The coefficient b_i in this expansion is a section of L^i . We can then use the coefficients $b_{-n}, \ldots, b_{-2}, b_0$ to define a splitting of $\pi_* \mathcal{O}_Z(n\sigma)$. (If f is a meromorphic function on E whose only pole is at p_0 then b_{-1} is determined by the b_{-i} for $-n \le -i \le -2$.) This splitting is different from the X-Y splitting, but both splittings induce the same filtration on $\pi_* \mathcal{O}_Z(n\sigma)$.

In Theorem 1.2 we showed how a semistable bundle of rank n and trivial determinant on a smooth elliptic curve E determines a point of the linear series $\mathcal{O}_E(np_0)$. This works well for bundles over families of elliptic curves.

Lemma 4.2. Let $p: \mathbb{P}V_n \to B$ be the projection. Thus, the fiber of p over $b \in B$ is the complete linear system $|np_0|$, where $E_b = \pi^{-1}(b)$ and p_0 is the smooth point $\sigma \cap E_b$. If $V \to Z$ is a rank n vector bundle whose restriction to each fiber of $Z \to B$ is a semistable bundle with trivial determinant, then V determines a section

$$A(V): B \to \mathbb{P}\mathcal{V}_n$$

with the property that, for each $b \in B$,

$$A(V)(b) = \zeta(V|E_b).$$

Proof. Arguing as in (1.6), there is an induced morphism

$$\Psi \colon \pi^* \pi_* (V \otimes \mathcal{O}_Z(\sigma)) \to V \otimes \mathcal{O}_Z(\sigma).$$

The determinant of this morphism is a section of $\pi^*M \otimes \mathcal{O}_Z(n\sigma)$, for some line bundle M on B, and it gives a well-defined section A(V) of $\mathbb{P}\mathcal{V}_n$ over B. \square

We note that the proof of (4.2) does not require that B be smooth, or even reduced.

There is also an analogue for families of elliptic curves of Theorem 1.5.

Lemma 4.3. Let S be a scheme or analytic space over B and let V be a rank n vector bundle over $S \times_B Z$, such that the restriction of V to every fiber $p_1^{-1}(s) \cong \pi^{-1}(b)$ is semistable with trivial determinant, where p_1, p_2 are the projections of $S \times_B Z$ to the first and second factors and b is the point of B lying under b. Then there is an induced morphism $a : S \to \mathbb{P}V_n$ of spaces over b, which agrees over each $b \in B$ with the morphism defined in $b \in B$ with the morphism defined in $b \in B$.

Proof. Let $\hat{Z} = S \times_B Z$, with $\hat{\pi} \colon \hat{Z} \to S$ the first projection. Then \hat{Z} is an elliptic scheme over S which maps naturally to Z covering the map of $S \to B$. Let $\hat{\sigma}$ be the induced section. Set $\hat{\mathcal{V}}_n = \hat{\pi}_* \mathcal{O}_{\hat{Z}}(n\hat{\sigma})$. Clearly $\mathbb{P}\hat{\mathcal{V}}_n$ is identified with the pullback of $\mathbb{P}\mathcal{V}_n$. Now apply the above result to this elliptic scheme to produce a section $S \to \mathbb{P}\hat{\mathcal{V}}_n$ which when composed with the natural map $\mathbb{P}\hat{\mathcal{V}}_n \to \mathbb{P}\mathcal{V}_n$ is the morphism Φ of the proposition. \square

4.2. Construction of bundles via extensions.

Our goal for the remainder of this section is to construct various "universal" bundles over $\mathbb{P}\mathcal{V}_n \times_B Z$. The first and easiest construction of the universal moduli space is via the extension approach, generalizing what we did in Section 3 for a single elliptic curve.

In order to make the extension construction in families, we first need to extend the basic bundle W_k over E to bundles over the elliptic scheme Z.

Proposition 4.4. There is a vector bundle W_d on Z such that W_d is filtered, with successive quotients π^*L^{d-1} , π^*L^{d-2} , ..., $\mathcal{O}_Z(\sigma)$, and such that on every fiber W_d restricts to W_d . Moreover, W_d is uniquely specified by the above properties. In fact, if W is a vector bundle on Z such that W restricts to W_d on every fiber, then there exists a line bundle M on B such that $W = W_d \otimes \pi^*M$. Finally, $R^0\pi_*W_d = L^{(d-1)}$ and $R^1\pi_*(W_d^{\vee}) = L^{-d}$.

Proof. In case d=1, take $\mathcal{W}_1=\mathcal{O}_Z(\sigma)$. Now suppose inductively that \mathcal{W}_{d-1} has been defined, and that $R^1\pi_*(\mathcal{W}_{d-1}^\vee)=L^{-(d-1)}$. We seek an extension of \mathcal{W}_{d-1} by a line bundle trivial on ever fiber of π , and thus of the form π^*M for some line bundle M on B, and such that $H^0(R^1\pi_*(\mathcal{W}_{d-1}^\vee\otimes\pi^*M))$ has an everywhere generating section. Now

$$R^1\pi_*(\mathcal{W}_{d-1}^{\vee} \otimes \pi^*M) = R^1\pi_*(\mathcal{W}_{d-1}^{\vee}) \otimes M = L^{-(d-1)} \otimes M.$$

Thus we must have $M=L^{d-1}$. With this choice, noting that $R^0\pi_*(\mathcal{W}_{d-1}^{\vee}\otimes\pi^*L^{d-1})=0$ since W_d^{\vee} has no sections, the Leray spectral sequence gives an isomorphism

$$H^1(\mathcal{W}_{d-1}^{\vee} \otimes \pi^* L^{d-1}) \cong H^0(R^1 \pi_* (\mathcal{W}_{d-1}^{\vee} \otimes \pi^* L^{d-1})) = H^0(\mathcal{O}_B)$$

and thus a global extension of W_{d-1} by π^*L^{d-1} restricting to W_d on every fiber. Since the unique section of W_d is given by the inclusion of the canonical subbundle $\mathcal{O}_f \to W_d$, we must have $R^0\pi_*\mathcal{W}_d = L^{(d-1)}$, and a similar argument (or relative duality) evaluates $R^1\pi_*(\mathcal{W}_d^{\vee})$.

Finally suppose that W is another bundle on Z restricting to W_d on every fiber. Then since W_d is simple, $\pi_* \operatorname{Hom}(W_d, W)$ is a line bundle M on B, and thus $\pi_* \operatorname{Hom}(\mathcal{W}_d \otimes \pi^*M, \mathcal{W}) \cong \mathcal{O}_B$. The element $1 \in H^0(\mathcal{O}_B)$ then defines an isomorphism from $\mathcal{W}_d \otimes \pi^*M$ to \mathcal{W} . \square

Note that the formation of W_d is compatible with base change, in the following sense. Given a morphism $g: B' \to B$, let $Z' = Z \times_B B'$, with $f: Z' \to Z$ the induced morphism, and let σ' be the induced section of $\pi': Z' \to B'$. Then the bundle W'_d constructed for $\pi': Z' \to B'$ and the section σ' is f^*W_d .

Next we construct a universal bundle via extensions. First we identify the relevant bundles to use as the parameter space of the extension:

Lemma 4.5. For $1 \leq d \leq n-1$, the sheaves $R^1\pi_*(\mathcal{W}_{n-d}^{\vee} \otimes \mathcal{W}_d^{\vee}) = \mathcal{V}_{n,d}$ are locally free of rank n over B, and are all canonically identified.

Proof. The local freeness and the rank statement follow from Claim 3.3 and base change. The canonical identifications follow from Corollary 3.10. \Box

Let $\mathcal{V}_{n,d} = R^1 \pi_* (\mathcal{W}_{n-d}^{\vee} \otimes \mathcal{W}_d^{\vee})$ as above, and let $\mathcal{P}_{n-1,d}$ be the associated projective space bundle $\mathbb{P}(\mathcal{V}_{n,d}) \to B$. By the general properties of extensions, there is a universal extension over $\mathcal{P}_{n-1,d} \times_B Z$ of the form

$$0 \to \pi_2^* \mathcal{W}_d^{\vee} \otimes \pi_2^* \mathcal{O}_{\mathcal{P}_{n-1,d}}(1) \to \mathbf{U}(d) \to \pi_2^* \mathcal{W}_{n-d} \to 0.$$

Applying Lemma 4.3 to these bundles produces bundle maps over B

$$\Phi_d: \mathcal{P}_{n-1,d} \to \mathbb{P}\mathcal{V}_n.$$

The projective space bundles $\mathcal{P}_{n-1,d}$ over B are all canonically isomorphic. Under these isomorphisms, the universal bundles $\mathbf{U}(d)$ are all distinct. Nevertheless, the result in Proposition 3.10 shows that there is an isomorphism I which identifies $R^1\pi_*(\mathcal{O}_Z(-\sigma)\otimes \mathcal{W}_{n-1}^{\vee})$ with

$$R^0\pi_*(\det(\mathcal{O}_Z(\sigma)\otimes\mathcal{W}_{n-1}))\otimes\det(R^0\pi_*(\mathcal{O}_Z(\sigma)\otimes\mathcal{W}_{n-1}))^{-1}\otimes R^1\pi_*\mathcal{O}_Z.$$

Let us identify the various factors on the right-hand-side of this expression. First of all, it is straightforward given the inductive definition of the W_{n-1} to show that.

$$\det(\mathcal{O}_Z(\sigma)\otimes\mathcal{W}_{n-1})\cong\mathcal{O}_Z(n\sigma)\otimes\pi^*L^{(n-1)(n-2)/2}.$$

It follows that $R^0\pi_*(\det(\mathcal{O}_Z(\sigma)\otimes\mathcal{W}_{n-1}))\cong R^0\pi_*\mathcal{O}_Z(n\sigma)\otimes L^{(n-1)(n-2)/2}$. Next, we have exact sequences

$$0 \to R^0 \pi_* (\mathcal{O}_Z(\sigma) \otimes \pi^* L^{n-2}) \to R^0 \pi_* (\mathcal{O}_Z(\sigma) \otimes \mathcal{W}_{n-1}) \to R^0 \pi_* (\mathcal{O}_Z(\sigma) \otimes \mathcal{W}_{n-2}) \to 0.$$

Since by Proposition 4.4 we have $R^0\pi_*(\pi^*L^a\otimes\mathcal{O}_Z(\sigma))\cong L^a$, and since

$$R^0\pi_*(\mathcal{O}_Z(2\sigma))\cong L^{-2}\oplus \mathcal{O}_B,$$

an easy inductive argument shows that

$$\det(R^0\pi_*(\mathcal{O}_Z(\sigma)\otimes\mathcal{W}_{n-1}))\cong L^{((n-1)(n-2)/2)-2}$$

Lastly, $R^1\pi_*\mathcal{O}_Z\cong L^{-1}$.

Putting all this together, we get:

Theorem 4.6. There is an isomorphism of vector bundles over B

$$I: R^1\pi_*(\mathcal{O}_Z(-\sigma) \otimes \mathcal{W}_{n-1}) \cong R^0\pi_*\mathcal{O}_Z(n\sigma) \otimes L,$$

which fiber by fiber agrees with the map I of Proposition 3.13. In other words,

$$\mathcal{V}_{n,1} \cong \mathcal{V}_n \otimes L$$
.

Furthermore, the map induced by projectivizing I agrees with the map Φ_1 produced by applying Lemma 4.3 to the family $\mathbf{U}(1)$ over $\mathcal{P}_{n-1,1} \times_B Z$. Let Φ_d be the map $\mathcal{P}_{n-1,d} \to \mathbb{P}\mathcal{V}_n$ obtained by applying Lemma 4.3 to the family $\mathbf{U}(d)$. Then, the maps Φ_d for $1 \leq d < n$ are compatible with the identifications coming from Corollary 3.10, and hence each of these maps is an isomorphism of projective bundles over B.

Note that, while the \mathbb{P}^{n-1} -bundles $\mathcal{P}_{n-1,d}$ and $\mathbb{P}\mathcal{V}_n$ are isomorphic, the tautological bundles $\mathcal{O}_{\mathcal{P}_{n-1},d}(1)$ and $\mathcal{O}_{\mathbb{P}\mathcal{V}_n}(1)$ differ by a twist by p^*L . We shall use \mathcal{P}_{n-1} to denote the bundle $\mathbb{P}\mathcal{V}_n$ together with its tautological line bundle. If $\zeta = c_1(\mathcal{O}_{\mathbb{P}\mathcal{V}_n}(1))$ and $\zeta' = c_1(\mathcal{O}_{\mathcal{P}_{n-1},d}(1))$, then $\zeta = \zeta' + L$.

Corollary 4.7. Via the isomorphism of (4.6) and (4.6),

$$\mathcal{V}_{n,d} = R^1 \pi_* (\mathcal{W}_{n-d}^{\vee} \otimes \mathcal{W}_d^{\vee}) \cong L \oplus L^{-1} \oplus \cdots \oplus L^{-(n-1)}.$$

This splitting is compatible with the inclusion of $\mathcal{V}_{n-1,d}$ in $\mathcal{V}_{n,d}$ as well as that of $\mathcal{V}_{n-1,d-1}$ in $\mathcal{V}_{n,d}$. \square

Corollary 4.8. Under the isomorphism $\pi | \sigma \colon \sigma \cong B$, there is a natural splitting

$$\mathcal{W}_n|\sigma \cong L^{n-1} \oplus L^{n-2} \oplus \cdots \oplus L \oplus L^{-1}.$$

In fact, the extension

$$0 \to \pi^* L^{n-1} \to \mathcal{W}_n \to \mathcal{W}_{n-1} \to 0$$

restricts to the split extension over σ .

Proof. Let us first show that the restriction of \mathcal{W}_n^{\vee} to σ is split. Begin with the exact sequence

$$0 \to \mathcal{O}_Z(-\sigma) \otimes \mathcal{W}_n^{\vee} \to \mathcal{W}_n^{\vee} \to \mathcal{W}_n^{\vee} | \sigma \to 0,$$

and apply $R^i\pi_*$. We get an exact sequence

$$0 \longrightarrow \pi_*(\mathcal{W}_n^{\vee}|\sigma) \longrightarrow R^1\pi_*(\mathcal{O}_Z(-\sigma) \otimes \mathcal{W}_n^{\vee}) \longrightarrow R^1\pi_*\mathcal{W}_n^{\vee} \longrightarrow 0$$

$$\parallel \qquad \qquad \parallel \qquad \qquad \parallel$$

$$0 \longrightarrow \pi_*\mathcal{W}_n^{\vee}|\sigma \longrightarrow L \oplus L^{-1} \oplus \cdots \oplus L^{-n} \longrightarrow L^{-n} \longrightarrow 0.$$

Tracing through the identifications shows that the map $R^1\pi_*(\mathcal{O}_Z(-\sigma)\otimes\mathcal{W}_n^{\vee})\to R^1\pi_*\mathcal{W}_n^{\vee}$ is the same as the map

$$R^1\pi_*(\mathcal{O}_Z(-\sigma)\otimes\mathcal{W}_n^\vee)\to R^1\pi_*\mathcal{O}_Z(-\sigma)\otimes L^{-n+1}=L^{-1}\otimes L^{-n+1}=L^{-n}$$

coming from the long exact sequence associated to

$$0 \to \mathcal{O}_Z(-\sigma) \otimes \mathcal{W}_{n-1}^{\vee} \to \mathcal{O}_Z(-\sigma) \otimes \mathcal{W}_n^{\vee} \to \mathcal{O}_Z(-\sigma) \otimes \pi^*L^{-n+1} \to 0.$$

This identifies the map $L \oplus L^{-1} \oplus \cdots \oplus L^{-n} \to L^{-n}$ with projection onto the last factor. Hence $\mathcal{W}_n^{\vee}|\sigma$ is identified with $L \oplus L^{-1} \oplus \cdots \oplus L^{-n+1}$. Dualizing gives the splitting of $\mathcal{W}_n|\sigma$. The splitting of the extension

$$0 \to \pi^* L^{n-1} \to \mathcal{W}_n \to \mathcal{W}_{n-1} \to 0$$

is similar. \square

Now let us relate the bundles U(d) via elementary modifications.

Proposition 4.9. Let \mathcal{H} be the smooth divisor which is the image of $\mathbb{P}\mathcal{V}_{n-1} = \mathcal{P}_{n-2}$ in \mathcal{P}_{n-1} under the natural inclusion $\pi_*\mathcal{O}_Z((n-1)\sigma) \subset \pi_*\mathcal{O}_Z(n\sigma)$, and let $i: \mathcal{H} \to \mathcal{P}_{n-1}$ be the inclusion. Then there is an exact sequence

$$0 \to \mathbf{U}(d) \to \mathbf{U}(d+1) \to (i \times \mathrm{Id})_* \mathcal{O}_{\mathcal{H} \times_B Z}(1) \otimes \pi^* L^{-d} \to 0,$$

where $\mathcal{O}_{\mathcal{H}\times_B Z}(1)$ denotes the restriction of $\mathcal{O}_{\mathcal{P}_{n-1,d}}(1) = \mathcal{O}_{\mathbb{P}\mathcal{V}_{n,d}}(1)$ to $\mathcal{H}\times_B Z$. Thus $\mathbf{U}(d)$ is an elementary modification of $\mathbf{U}(d+1)$, and it is the only possible such modification along $\mathcal{H}\times_B Z$.

Proof. The construction of the proof of Theorem 3.12 gives an inclusion $\mathbf{U}(d) \to \mathbf{U}(d+1)$ whose cokernel is the direct image of a line bundle supported along $\mathcal{H} \times_B Z$. As in the first paragraph of the proof of (3.12), this line bundle is the inverse of $\pi_{1*}Hom(\mathbf{U}(d+1)|\mathcal{H} \times_B Z, \mathcal{O}_{\mathcal{H} \times_B Z})$ (where for the rest of the proof we let π_1 be the first projection $\mathcal{H} \times_B Z \to \mathcal{H}$). From the defining exact sequence for $\mathbf{U}(d+1)$,

$$\pi_{1*}Hom(\mathbf{U}(d+1)|\mathcal{H}\times_B Z, \mathcal{O}_{\mathcal{H}\times_B Z}) \cong \pi_{1*}Hom(\pi_2^*\mathcal{W}_{d+1}^{\vee}\otimes \mathcal{O}_{\mathcal{H}\times_B Z}(1), \mathcal{O}_{\mathcal{H}\times_B Z})$$
$$= \pi_{1*}\pi_2^*\mathcal{W}_{d+1}\otimes \mathcal{O}_{\mathcal{H}\times_B Z}(-1).$$

Here by base change $\pi_{1*}\pi_{2}^{*}\mathcal{W}_{d+1}$ is a line bundle on \mathcal{H} whose restriction to every fiber is the nonzero section of W_{d+1} on that fiber. Now W_{d+1} is filtered by subbundles with successive quotients $\pi^{*}L^{d}$, $\pi^{*}L^{d-1}$,..., $\mathcal{O}_{Z}(\sigma)$, and the inclusion of $\pi^{*}L^{d}$ in W_{d+1} defines a map $L^{d} \to \pi_{1*}\pi_{2}^{*}\mathcal{W}_{d+1}$ which restricts to the nonzero section on every fiber. Thus $\pi_{1*}\pi_{2}^{*}\mathcal{W}_{d+1} \cong L^{d}$. Hence

$$\pi_{1*}Hom(\mathbf{U}(d+1)|\mathcal{H}\times_B Z, \mathcal{O}_{\mathcal{H}\times_B Z}) \cong L^d \otimes \mathcal{O}_{\mathcal{H}\times_B Z}(-1),$$

and thus the cokernel of the map $\mathbf{U}(d) \to \mathbf{U}(d+1)$ is as claimed. The uniqueness is clear. \square

This completes the construction of "universal" bundles over $\mathbb{P}\mathcal{V}_n \times_B Z$. However, we have constructed only n-1 bundles $\mathbf{U}(d)$ for $1 \leq d < n$.

Note that the formation of the universal bundles U(d) over $\mathcal{P}_{n-1,d} \times_B Z$ is also compatible with base change $B' \to B$ in the obvious sense.

4.3. The spectral cover construction.

Now we turn to the generalization of the spectral covering construction. First let us define the analogues of E^{n-1}, T , and ν . By repeating the construction on each smooth fiber, we could take the (n-1)-fold fiber product $Z \times_B Z \times_B \cdots \times_B Z$ and its quotient under \mathfrak{S}_n and \mathfrak{S}_{n-1} . However this construction runs into trouble at the singular fibers, reflecting the difference between the n-fold symmetric product of E and the linear system $|np_0|$ for a singular fiber. Instead, we construct the spectral cover in families as follows: Let \mathcal{E} be defined by the exact sequence of vector bundles over Z

$$0 \to \mathcal{E} \to \pi^* \pi_* \mathcal{O}_Z(n\sigma) \to \mathcal{O}_Z(n\sigma) \to 0,$$

where the last map is the natural evaluation map and is surjective. We set $\mathcal{T} = \mathbb{P}\mathcal{E}$, with $r \colon \mathcal{T} \to Z$ the projection. By construction \mathcal{T} is a \mathbb{P}^{n-2} -bundle over Z. There is an inclusion

$$\mathcal{T} \to \mathbb{P}(\pi^*\pi_*\mathcal{O}_Z(n\sigma)) = \mathbb{P}\mathcal{V}_n \times_B Z,$$

and we let ν be the composition of this morphism with the projection $q_1 : \mathbb{P}\mathcal{V}_n \times_B Z \to \mathbb{P}\mathcal{V}_n$. It is easy to see that $\nu : \mathcal{T} \to \mathbb{P}\mathcal{V}_n$ is an *n*-sheeted covering, which restricts to the spectral cover described in Section 2 on each smooth fiber of $Z \to B$. By analogy with the case of a single elliptic curve, we would like to consider the sheaf

$$\mathcal{U}_0 = (\nu \times \mathrm{Id})_* \mathcal{O}_{\mathcal{T} \times_B Z} (\Delta - \mathcal{G}),$$

where $\Delta = (r \times \mathrm{Id})^*(\Delta_0)$, for Δ_0 the diagonal in $Z \times_B Z$, and $\mathcal{G} = (r \times \mathrm{Id})^* p_2^* \sigma$ for p_1, p_2 the projections of $Z \times_B Z$ to the first and second factors. Here we can define $\mathcal{O}_{Z \times_B Z}(\Delta_0)$ to be the dual of the ideal sheaf of Δ_0 in $Z \times_B Z$. It is an invertible sheaf away from the singularities of $Z \times_B Z$. The proof of (0.4) shows that $\mathcal{O}_{Z \times_B Z}(\Delta_0)$ is flat over both factors of $Z \times_B Z$, and identifies the first factor, say, with the relative compactified generalized Jacobian. As we shall see, \mathcal{U}_0 is indeed a vector bundle of rank n, and that its restriction over each smooth fiber is the bundle U_0 described in Proposition 2.9. In particular, \mathcal{U}_0 is a family of regular, semistable bundles with trivial determinant over the family $Z \to B$ of elliptic curves.

Although we shall not need this in what follows, for concreteness sake let us describe the singularities of $Z \times_B Z$ and \mathcal{T} explicitly the case where the divisors associated to G_2 and G_3 are smooth and meet transversally as in the introduction. In this case Z has the local equation $y^2 = x^3 + sx + t$. The morphism to B is given locally by (s,t), where $t = y^2 - x^3 - sx$. Using x,y,s as part of a set of local coordinates for Z, the fiber product has local coordinates x,y,s,x',y',\ldots and a local equation

$$y^2 - x^3 - sx = (y')^2 - (x')^3 - s(x').$$

Rewrite this equation as

$$y^{2} - (y')^{2} = (x - x')(s + x^{2} + xx' + (x')^{2}) = h_{1}h_{2},$$

say, where a local calculation shows that $h_1(s, x, x')$ and $h_2(s, x, x')$ define two smooth hypersurfaces meeting transversally along $\Gamma \times_B \Gamma$. It follows that the total singularity in $Z \times_B Z$ is a locally trivial fibration of ordinary threefold double points, and Δ is a smooth divisor which fails to be Cartier at the singularities.

We return now to the case of a general Z. Fix a d with $1 \le d \le n-1$, and consider the sheaf of algebras $\pi_{1*}Hom(\mathbf{U}(d),\mathbf{U}(d)) = \mathbf{A}$ over \mathcal{P}_{n-1} . Arguing as in Lemma 3.19, the space $\mathbf{Spec} \mathbf{A}$ is reduced and there is a finite flat morphism $\nu \colon \mathbf{Spec} \mathbf{A} \to \mathcal{P}_{n-1}$ restricts over each fiber to give $\nu \colon T \to \mathbb{P}^{n-1}$. Moreover, $\mathbf{U}(d) = (\nu \times \mathrm{Id})_* \mathcal{L}_d$ for some sheaf \mathcal{L}_d on $\mathbf{Spec} \mathbf{A} \times_B Z$. The method of proof of Theorem 3.21 then shows:

Theorem 4.10. There is an isomorphism from Spec A to \mathcal{T} . Under this isomorphism, there is a line bundle \mathcal{M} over \mathcal{T} such that $\mathbf{U}(d) = (\nu \times \mathrm{Id})_* \mathcal{O}_{\mathcal{T} \times_B Z}(\Delta - \mathcal{G}) \otimes \pi_1^* \mathcal{M}$. \square

Since $U(1) \otimes \mathcal{O}_{\mathbb{P}\mathcal{V}_{n,d}}(-1) \otimes \pi_2^* \mathcal{O}_Z(\sigma)$ has a section vanishing exactly along Δ , the proof of 3.23 identifies this line bundle in case d=1:

Theorem 4.11. In the above notation,

$$\mathbf{U}(1) \cong (\nu \times \mathrm{Id})_* \mathcal{O}_{\mathcal{T} \times_B Z}(\Delta - \mathcal{G}) \otimes \mathcal{O}_{\mathbb{P}\mathcal{V}_{n,d}}(1)$$
$$= (\nu \times \mathrm{Id})_* \mathcal{O}_{\mathcal{T} \times_B Z}(\Delta - \mathcal{G}) \otimes \mathcal{O}_{\mathbb{P}\mathcal{V}_n}(1) \otimes L^{-1}. \quad \Box$$

For every $a \in \mathbb{Z}$, we can then define $\mathcal{U}_a = (\nu \times \mathrm{Id})_* \mathcal{O}_{\mathcal{T} \times_B Z} (\Delta - \mathcal{G} - a(r^* \sigma \times_B Z))$. It follows that \mathcal{U}_a is a vector bundle for every $a \in \mathbb{Z}$.

Theorem 4.12. With U_a defined as above, there is an exact sequence

$$0 \to \mathcal{U}_a \to \mathcal{U}_{a-1} \to (i \times \mathrm{Id})_* \mathcal{O}_{\mathcal{H} \times_B Z} \otimes \pi^* L^{a-1} \to 0$$

which realizes \mathcal{U}_a as an elementary modification of \mathcal{U}_{a-1} . Thus, for $1 \leq d \leq n-1$,

$$\mathbf{U}(d) \cong \mathcal{U}_{1-d} \otimes \pi_1^* \mathcal{O}_{\mathbb{P} \mathcal{V}_n}(1) \otimes L^{-1}.$$

Proof. From the definition of \mathcal{U}_a , there is an exact sequence

$$0 \to (\nu \times \operatorname{Id})_* \mathcal{O}_{\mathcal{T} \times_B Z} (\Delta - \mathcal{G} - a(r^* \sigma \times_B Z))$$

 $\to (\nu \times \operatorname{Id})_* \mathcal{O}_{\mathcal{T} \times_B Z} (\Delta - \mathcal{G} - (a - 1)(r^* \sigma \times_B Z))$
 $\to (\nu \times \operatorname{Id})_* \mathcal{O}_{r^* \sigma \times_B Z} (\Delta - \mathcal{G} - (a - 1)(r^* \sigma \times_B Z)) \to 0.$

The divisor $r^*\sigma$ is a \mathbb{P}^{n-2} -bundle over B which intersects each fiber of $\mathcal{T} \to B$ in the \mathbb{P}^{n-2} fiber $r^{-1}(p_0)$. This fiber is mapped linearly via ν to the hyperplane H_{p_0} in $|np_0|$. Thus, $\nu_*r^*\sigma = \mathcal{H}$. Now both Δ and \mathcal{G} have the same restriction to $r^*\sigma \times_B Z$, namely $r^*\sigma \times_B \sigma$. Also, $\mathcal{O}_{r^*\sigma \times_B Z}(r^*\sigma \times_B Z)$ is the pullback of the line bundle $\mathcal{O}_Z(\sigma)|\sigma=L^{-1}$. It follows that the quotient of \mathcal{U}_{a-1} by the image of \mathcal{U}_a is exactly the direct image of $\mathcal{O}_{\mathcal{H}\times_B Z}\otimes \pi^*L^{a-1}$, as claimed. The final statement in (4.12) then follows by comparing elementary modifications. \square

Finally we shall need the analogue of Proposition 2.4 for a single elliptic curve. It is proved exactly as in (2.4).

Theorem 4.13. Let \mathcal{U}' be a rank n vector bundle over $\mathbb{P}\mathcal{V}_n \times_B Z$ such that, for all $x \in \mathbb{P}\mathcal{V}_n$, $\mathcal{U}'|q_1^{-1}(x) \cong \mathcal{U}_0|q_1^{-1}(x)$. Then there is a unique line bundle \mathcal{M} on \mathcal{T} such that, if $\pi_1 \colon \mathcal{T} \times_B Z \to \mathcal{T}$ is projection onto the first factor, then $\mathcal{U}' \cong (\nu \times \mathrm{Id})_* (\mathcal{O}_{\mathcal{T} \times_B Z}(\Delta - \mathcal{G}) \otimes \pi_1^* \mathcal{M})$. \square

4.4. Chern class calculations.

Recall that we let $\zeta = c_1(\mathcal{O}_{\mathcal{P}_{n-1}}(1))$, viewed as a class in $H^2(\mathcal{P}_{n-1})$. By pullback, we can also view ζ as an element of $H^2(\mathcal{P}_{n-1} \times_B Z)$. We also have the line bundle $\mathcal{O}_{\mathcal{P}_{n-1},d}(1)$), and its first Chern class ζ' is given by $\zeta' = \zeta - L$ (where we identify L with its first Chern class in $H^2(B)$ and then by pullback in any of the relevant spaces).

Theorem 4.14. The Chern characters of the bundles U(d) and U_a are given by:

$$\operatorname{ch} \mathbf{U}(d) = (e^{-\sigma} + e^{-L} + \dots + e^{-(d-1)L})e^{\zeta - L} + (e^{\sigma} + e^{L} + \dots + e^{(n-d-1)L});$$

$$\operatorname{ch} \mathcal{U}_{a} = e^{-\zeta} \left(\frac{1 - e^{(a+n)L}}{1 - e^{L}} \right) - \frac{1 - e^{aL}}{1 - e^{L}} + e^{-\sigma} (1 - e^{-\zeta}).$$

Proof. The first statement is clear from the filtration on the W_k and the definition of ζ . To see the second, we use (4.12) for $1 \le d \le n-1$ and calculate

$$\operatorname{ch}(\mathbf{U}(d) \otimes \mathcal{O}_{\mathbb{P}\mathcal{V}_n}(-1) \otimes L) = \operatorname{ch} \mathbf{U}(d) \cdot e^{-\zeta + L} =$$

$$(e^{-\sigma} + e^{-L} + \dots + e^{-(d-1)L}) + (e^{\sigma + L} + e^{2L} + \dots + e^{(n-d)L})e^{-\zeta}$$

$$= (e^{-\sigma} - 1) + (1 + e^{-L} + \dots + e^{-(d-1)L}) +$$

$$+ (e^{\sigma + L - \zeta} - e^{L - \zeta}) + (e^{L} + e^{2L} + \dots + e^{(n-d)L})e^{-\zeta}.$$

Let a = 1 - d. A little manipulation shows that we can write:

$$1 + e^{-L} + \dots + e^{-(d-1)L} = -\frac{e^L - e^{aL}}{1 - e^L};$$

$$e^L + e^{2L} + \dots + e^{(n-d)L} = \frac{e^L - e^{(a+n)L}}{1 - e^L};$$

$$(e^{-\sigma} - 1) + (e^{\sigma + L - \zeta} - e^{L - \zeta}) = -(1 - e^{-\sigma})(1 - e^{\sigma + L - \zeta}).$$

In the last term, note that $1 - e^{-\sigma}$ is a power series without constant term in σ and thus annihilates every power series without constant term in $\sigma + L$, since $\sigma^2 = -L \cdot \sigma$. Thus we can replace the last term by $-(1 - e^{-\sigma})(1 - e^{-\zeta})$. It follows that

$$\begin{split} \operatorname{ch} \mathbf{U}(d) \cdot e^{-\zeta + L} &= e^{-\zeta} \bigg(\frac{e^L - e^{(a+n)L}}{1 - e^L} \bigg) - \frac{e^L - e^{aL}}{1 - e^L} - (1 - e^{-\sigma})(1 - e^{-\zeta}) \\ &= e^{-\zeta} \bigg(\frac{1 - e^{(a+n)L}}{1 - e^L} \bigg) - \frac{1 - e^{aL}}{1 - e^L} + e^{-\sigma}(1 - e^{-\zeta}). \end{split}$$

In particular, we have established the formula in (4.14) for $\operatorname{ch} \mathcal{U}_a$ provided a = 1 - d with $1 \leq d \leq n - 1$. On the other hand, the formula for \mathcal{U}_a as an elementary modification shows that

$$\operatorname{ch} \mathcal{U}_a = \operatorname{ch} \mathcal{U}_{a-1} - \operatorname{ch} (\mathcal{O}_{\mathcal{H} \times_B Z} \otimes L^{a-1}).$$

Now from the exact sequence

$$0 \to \mathcal{O}_{\mathcal{P}_{n-1} \times_B Z}(-\mathcal{H} \times_B Z) \to \mathcal{O}_{\mathcal{P}_{n-1} \times_B Z} \to \mathcal{O}_{\mathcal{H} \times_B Z} \to 0,$$

we see that

$$\operatorname{ch}(\mathcal{O}_{\mathcal{H}\times_B Z} \otimes L^{a-1}) = \operatorname{ch}(\mathcal{O}_{\mathcal{H}\times_B Z}) \cdot e^{(a-1)L}$$
$$= e^{(a-1)L}(1 - e^{-\mathcal{H}}).$$

Next we claim:

Lemma 4.15. $[\mathcal{H}] = \zeta - nL$.

Proof. We have identified \mathcal{H} with the image of $\mathbb{P}\mathcal{V}_{n-1}$ in $\mathbb{P}\mathcal{V}_n$. The lemma now follows from the more general statement below, whose proof is left to the reader:

Lemma 4.16. Let V be a vector bundle over a scheme B, and suppose that there is an exact sequence

$$0 \to \mathcal{V}' \to \mathcal{V} \to M \to 0$$
.

where M is a line bundle on B. Let \mathcal{H} be the Cartier divisor $\mathbb{P}(\mathcal{V}') \subset \mathbb{P}(\mathcal{V})$. Then, if $p \colon \mathbb{P}(\mathcal{V}) \to B$ is the projection,

$$\mathcal{O}_{\mathbb{P}(\mathcal{V})}(\mathcal{H}) = \mathcal{O}_{\mathbb{P}(\mathcal{V})}(1) \otimes p^*M.$$

Plugging in the expression for $[\mathcal{H}]$, we see that

$$\operatorname{ch} \mathcal{U}_a - \operatorname{ch} \mathcal{U}_{a-1} = -e^{(a-1)L} (1 - e^{-(\zeta - nL)}).$$

Comparing this difference with the formula of (4.14) shows that (4.14) holds for one value of a if and only if it holds for all values of a. Since we have already checked it for a=0, we are done. \square

Similar computations give the Chern class of $\mathbf{U}(d)$ and \mathcal{U}_a . We leave the calculations to the reader.

Theorem 4.17. The total Chern class of U(d) is given by the formula:

$$c(\mathbf{U}(d)) = (1 + \zeta - L + \zeta \cdot \sigma) \prod_{r=1}^{d-1} (1 - (r+1)L + \zeta) \prod_{s=1}^{n-d-1} (1 + sL).$$

If $a \geq 0$, then

$$c(\mathcal{U}_a) = (1 - \zeta + L + \zeta \cdot \sigma) \prod_{s=1}^{n+a-2} (1 + (s+1)L - \zeta) \prod_{r=1}^{a-1} (1 + rL)^{-1}.$$

If $-(n-1) \le a < 0$, then

$$c(\mathcal{U}_a) = (1 - \zeta + L + \zeta \cdot \sigma) \prod_{s=1}^{n+a-2} (1 + (s+1)L - \zeta) \prod_{r=1}^{-a} (1 - rL).$$

If a < -(n-1), then

$$c(\mathcal{U}_a) = (1 - \zeta + L + \zeta \cdot \sigma) \prod_{s=0}^{1-n-a} (1 - (s-1)L - \zeta)^{-1} \prod_{r=1}^{-a} (1 - rL). \quad \Box$$

Let us work out explicitly the first two Chern classes of \mathcal{U}_a . First,

$$c_1(\mathcal{U}_a) = \left[an + \left(\frac{n^2 - n}{2}\right)\right]L - (n + a - 1)\zeta.$$

To give $c_2(\mathcal{U}_a)$, write

$$\frac{1 - e^{cx}}{1 - e^x} = c + \left(\frac{c^2 - c}{2}\right)x + P(c)x^2 + \cdots,$$

where

$$P(c) = \frac{c(2c-1)(c-1)}{12} = \frac{2c^3 - 3c^2 + c}{12}$$

(if c is a positive integer then $P(c) = \frac{1}{2} \sum_{i=1}^{c-1} i^2$). A little manipulation shows that $c_2(\mathcal{U}_a)$ is equal to

$$\frac{(a+n-1)(a+n-2)}{2}\zeta^{2} - (n^{2} + 2an - 2n - a)\left(\frac{a+n-1}{2}\right)\zeta \cdot L + \left[\frac{1}{2}\left(an + \frac{n^{2} - n}{2}\right)^{2} - P(a+n) + P(a)\right]L^{2} + (\sigma \cdot \zeta).$$

Finally, we remark that it is possible to work out the first two terms in $\operatorname{ch} \mathcal{U}_a$ by applying the Grothendieck-Riemann-Roch theorem directly to the description of \mathcal{U}_a as $(\nu \times \operatorname{Id})_* \mathcal{O}_{\mathcal{T} \times_B Z}(\Delta - \mathcal{G} - a(r^*\sigma \times_B Z))$. This calculation is somewhat long and painful, and does not give the full calculation of $\operatorname{ch} \mathcal{U}_a$ because Δ is not a Cartier divisor.

5. Bundles which are regular and semistable on every fiber.

So far in this paper we have been working universally with the moduli space of all regular semistable bundles with trivial determinant over an elliptic curve or an elliptic fibration. In this section we wish to study bundles V over an elliptic fibration $\pi\colon Z\to B$ with the property that the restriction of V to every fiber is a regular semistable bundle with trivial determinant.

5.1. Sections and spectral covers.

Suppose that $V \to Z$ is a vector bundle of rank n whose restriction to each fiber is a regular semistable bundle with trivial determinant. Then for each $b \in B$ the bundle $V|_{E_b}$ determines a point in the fiber of \mathcal{P}_{n-1} over b. This means that V determines a section $A(V) = A: B \to \mathcal{P}_{n-1}$, as follows from (4.2). We shall usually identify A with the image A(B) of A in \mathcal{P}_{n-1} . Conversely, given a section A of \mathcal{P}_{n-1} we can construct a bundle V over Z which is regular semistable with trivial determinant on each fiber and such that the section determined by V is A. There are many bundles with this property and we shall analyze all such.

We first begin by describing all sections of \mathcal{P}_{n-1} .

Lemma 5.1. A section $A: B \to \mathcal{P}_{n-1}$ is equivalent to a line bundle $M \to B$ and an inclusion of M^{-1} into \mathcal{V}_n , or equivalently to sections of $M \otimes L^{-i}$ for $i = 0, 2, 3, \ldots, n$ which do not all vanish at any point of B, modulo the diagonal action of \mathbb{C}^* . Under this correspondence, the normal bundle of A in \mathcal{P}_{n-1} is isomorphic to $(\mathcal{V}_n \otimes M)/\mathcal{O}_B$, where the inclusion of \mathcal{O}_B in $\mathcal{V}_n \otimes M$ corresponds to the inclusion of M^{-1} into \mathcal{V}_n . Finally, if either $h^1(\mathcal{O}_B) = 0$ or $h^1(\mathcal{V}_n \otimes M) = 0$, then the deformations of A in \mathcal{P}_{n-1} are unobstructed.

Proof. Let A be a section, which we identify with its image in \mathcal{P}_{n-1} . Of course, $A \cong B$ via the projection $p: \mathcal{P}_{n-1} \to B$. We have the inclusion of $\mathcal{O}_{\mathbb{P}\mathcal{V}_n}(-1)$ in $p^*\mathcal{V}_n$. Pulling back via A, we set $M = \mathcal{O}_{\mathbb{P}\mathcal{V}_n}(1)|A$, which is a line bundle such that M^{-1} is a subbundle of $p^*\mathcal{V}_n|A = \mathcal{V}_n$. An inclusion

$$M^{-1} \to \mathcal{V}_n = \mathcal{O}_B \oplus L^{-2} \oplus \cdots \oplus L^{-n}$$

is given by a nowhere vanishing section of $(M \otimes \mathcal{O}_B) \oplus (M \otimes L^{-2}) \oplus \cdots \oplus (M \otimes L^{-n})$, or equivalently by sections of the bundles $(M \otimes \mathcal{O}_B)$, $(M \otimes L^{-2})$, ..., $(M \otimes L^{-n})$ which do not all vanish simultaneously, and these sections are well-defined modulo the diagonal \mathbb{C}^* action. Conversely, a nowhere vanishing section of $\mathcal{V}_n \otimes M$ defines an inclusion $M^{-1} \to \mathcal{V}_n$ and thus a section of \mathcal{P}_{n-1} , and the two constructions are inverse to each other.

The normal bundle $N_{A/\mathcal{P}_{n-1}}$ to A in \mathcal{P}_{n-1} is just the restriction to A of the relative tangent bundle $T_{\mathcal{P}_{n-1}/B}$, and thus it is isomorphic to $(\mathcal{V}_n \otimes M)/\mathcal{O}_B$. The deformations of the subvariety A are unobstructed if every element of $H^0(N_{A/\mathcal{P}_{n-1}})$ corresponds to an actual deformation of A. From the exact sequence

$$0 \to H^0(\mathcal{O}_B) \to H^0(\mathcal{V}_n \otimes M) \to H^0(N_{A/\mathcal{P}_{n-1}}) \to H^1(\mathcal{O}_B) \to H^1(\mathcal{V}_n \otimes M),$$

we see that, if $H^1(\mathcal{O}_B) = 0$, then every section of the normal bundle lifts to a section of $\mathcal{V}_n \otimes M$, unique mod the image of $H^0(\mathcal{O}_B) = \mathbb{C}$, and thus gives an actual deformation of A. If $H^1(\mathcal{V}_n \otimes M) = 0$, then viewing the deformations of M as parametrized by Pic B, if M' is sufficiently close to M in Pic B, then $H^1(\mathcal{V}_n \otimes M) = 0$.

M')=0 as well and by standard base change results the groups $H^0(\mathcal{V}_n\otimes M')$ fit together to give a vector bundle over a neighborhood of M in Pic B. The associated projective space bundle then gives a smooth family of deformations of A such that the associated Kodaira-Spencer map is an isomorphism onto $H^0(N_{A/\mathcal{P}_{n-1}})$. Thus A is unobstructed in this case as well. \square

Definition 5.2. Let $A: B \to \mathcal{P}_{n-1}$ be a section, and let (A, Id) be the corresponding section of $\mathcal{P}_{n-1} \times_B Z \to Z$. For all $a \in \mathbb{Z}$, let

$$V_{A,a} = (A, \mathrm{Id})^* \mathcal{U}_a.$$

For every pair (A, a), the bundle $V_{A,a}$ is of rank n and the restriction of $V_{A,a}$ to every fiber of π is regular and semistable with trivial determinant. Furthermore, for all $a \in \mathbb{Z}$, the section determined by $V_{A,a}$ is A.

More generally, we could take any bundle \mathcal{U} over $\mathcal{P}_{n-1} \times_B Z$ obtained by twisting \mathcal{U}_a by a line bundle on the universal spectral cover \mathcal{T} over \mathcal{P}_{n-1} , and form $V_{A,\mathcal{U}} = (A, \mathrm{Id})^*\mathcal{U}$ to produce a bundle with these properties. However, these will not exhaust all the possibilities in general. To describe all possible bundles V corresponding to A, we shall need to define the spectral cover associated to A.

Definition 5.3. Let $A \subseteq \mathcal{P}_{n-1}$ be a section. The scheme-theoretic inverse image ν^*A of A in \mathcal{T} is a subscheme C_A of \mathcal{T} , not necessarily reduced or irreducible. The morphism $g_A = \nu | A \colon C_A \to A \cong B$ is finite and flat of degree n. We call C_A the spectral cover associated to the section A.

In the notation of (5.1), we shall show below that C_A is smooth for M sufficiently ample and for a general section corresponding to M. In general, however, no matter how bad the singularities of C_A , we have the following:

Lemma 5.4. The restriction of r to C_A embeds C_A as a subscheme of Z which is a Cartier divisor. In fact, if V is a vector bundle with semistable restriction to every fiber and A is the associated section, then C_A is the scheme of zeroes of $\det \Psi$, where

$$\Psi \colon \pi^* \pi_* (V \otimes \mathcal{O}_Z(\sigma)) \to V \otimes \mathcal{O}_Z(\sigma)$$

is the natural map. The line bundle $\mathcal{O}_Z(C_A)$ corresponding to C_A is isomorphic to $\mathcal{O}_Z(n\sigma)\otimes\pi^*M$, where M is the line bundle corresponding to the section A. Moreover, the image of C_A in Z determines A. Finally, if $C\subset Z$ is the zero locus of a section of $\mathcal{O}_Z(n\sigma)\otimes\pi^*M$ and the induced morphism from C to B is finite, then $C=C_A$ for a unique section A of \mathcal{P}_{n-1} .

Proof. Let $i: C_A \to \mathcal{T}$ be the natural embedding. We claim that $r \circ i: C_A \to Z$ is a scheme-theoretic embedding. To see this, recall that we have $\mathcal{T} \subset \mathcal{P}_{n-1} \times_B Z$ via (ν, r) . In fact, from the defining exact sequence

$$0 \to \mathcal{E} \to \pi^* \pi_* \mathcal{O}_Z(n\sigma) \to \mathcal{O}_Z(n\sigma) \to 0,$$

we see that $\mathcal{T} = \mathbb{P}\mathcal{E}$ is a Cartier divisor in $\mathbb{P}(\pi^*\pi_*\mathcal{O}_Z(n\sigma)) = \mathcal{P}_{n-1} \times_B Z$ defined by the vanishing of a section of $\pi_2^*\mathcal{O}_Z(n\sigma) \otimes \pi_1^*\mathcal{O}_{\mathcal{P}_{n-1}}(1)$. Clearly, the image of $i(C_A)$ under the map $C_A \to \mathcal{T} \to \mathcal{P}_{n-1} \times_B Z$ is an embedding of C_A in $A \times_B Z \cong Z$. Thus $r \circ i$ is an embedding of C_A into Z. Moreover, C_A is the restriction of $\mathcal{T} \subset \mathcal{P}_{n-1} \times_B Z$ to $A \times_B Z$, and thus C_A is a Cartier divisor in Z. Essentially by

definition, C_A is defined by the vanishing of det Ψ (since this holds on every fiber E_b). Moreover, $\mathcal{O}_Z(C_A)$ is the restriction to $A \times_B Z$ of $\pi_2^* \mathcal{O}_Z(n\sigma) \otimes \pi_1^* \mathcal{O}_{\mathcal{P}_{n-1}}(1)$, namely $\mathcal{O}_Z(n\sigma) \otimes \pi^* M$.

Since the hypersurface $\mathcal{T} \subset \mathcal{P}_{n-1} \times_B Z$ is the incidence correspondence, the line bundle $\mathcal{O}_{\mathcal{P}_{n-1} \times_B Z}(\mathcal{T})$ restricts on every fiber E of π to $\mathcal{O}_E(np_0)$, and the effective divisor \mathcal{T} restricts to the tautological divisor in $|np_0| \times E$ corresponding to the inclusion $\mathcal{T} \subset \mathcal{P}_{n-1} \times_B Z$. Thus, by restriction, if $\mathcal{O}_Z(C_A)$ is the line bundle in Z corresponding to the Cartier divisor C_A , then for every fiber $E = E_b$ of π , $\mathcal{O}_Z(C_A)|E \cong \mathcal{O}_E(np_0)$, and the section of $\mathcal{O}_E(np_0)$ defined by C_A is A(b). Thus the image of C_A in Z determines A.

Finally, let C be the zero locus of a section of $\mathcal{O}_Z(n\sigma)\otimes\pi^*M$. Note that

$$H^0(Z; \mathcal{O}_Z(n\sigma) \otimes \pi^*M) = H^0(B; \pi_*(\mathcal{O}_Z(n\sigma) \otimes \pi^*M)) = H^0(B; \mathcal{V}_n \otimes M),$$

so that sections s of $\mathcal{O}_Z(n\sigma) \otimes \pi^*M$ mod \mathbb{C}^* correspond to sections s' of $\mathcal{V}_n \otimes M$. Under this correspondence, s' vanishes at a point of B if and only if s vanishes along the complete fiber $\pi^{-1}(b)$. Thus we see that the subschemes C mapping finitely onto B are in 1-1 correspondence with sections A of \mathcal{P}_{n-1} whose associated line bundle is M. \square

We define $T_A = C_A \times_B Z \subseteq \mathcal{T} \times_B Z$, and let $\rho_A \colon T_A \to C_A$ be the natural map. There is an induced map $\nu_A \colon T_A \to Z$ such that the following diagram is Cartesian:

$$T_{A} \xrightarrow{\nu_{A}} Z$$

$$\rho_{A} \downarrow \qquad \qquad \downarrow \pi$$

$$C_{A} \xrightarrow{g_{A}} B.$$

Thus, T_A is an elliptic scheme over C_A pulled back from the elliptic scheme $Z \to B$ via the natural projection mapping $C_A \to B$. Even if C_A is smooth, however, T_A is singular along the intersection of $C_A \times_B Z$ with $\Gamma \times_B \Gamma \subset Z \times_B Z$, at points corresponding to $\Gamma \cap C_A \subset Z$. If dim B=1, the generic section A will be such that $C_A \cap \Gamma = \emptyset$. However, if dim $B \geq 2$ and A is sufficiently ample, $C_A \cap \Gamma$ is nonempty. In the generic situation described in the last section, where G_2 and G_3 are smooth and meet transversally, the singularities of T_A are locally trivial families of threefold double points. In general, if no component of Γ is contained in C_A , the codimension of $C_A \cap \Gamma$ in C_A is two and the codimension of the corresponding subset of T_A is three. If a component of Γ is contained in C_A , then the codimension of $C_A \cap \Gamma$ in C_A is one and the codimension of the corresponding subset of T_A is two. Note that Δ is a Cartier divisor in the complement of the subset of T_A consisting of singular points of singular fibers lying over $C_A \cap \Gamma$.

Let us examine the pullback to $T_A = C_A \times_B Z$ of the divisors in \mathcal{T} . The section $\sigma \subset Z$ pulls back via ν_A^* to a section Σ_A of the elliptic fibration $\nu_A \colon T_A \to C_A$. Clearly $\Sigma_A = \nu_A^* \sigma = \mathcal{G}|T_A$, where as in the last section \mathcal{G} is the pullback to $\mathcal{T} \times_B Z$ of $\sigma \subset Z$ by the second projection. The diagonal Δ_0 in $Z \times_B Z$ pulls back to a hypersurface in T_A , which is the restriction of $\Delta \subset \mathcal{T} \times_B Z$ to $C_A \times_B Z = T_A$. We shall continue to denote this subvariety by Δ . However Δ is not a Cartier divisor along the singular set of T_A . On the other hand, the restriction of ρ_A to Δ is an isomorphism from Δ to C_A , so that in a formal sense Δ is a section. There is also the class ζ , which is obtained as follows: take the class ζ on \mathcal{P}_{n-1} , pull it back to \mathcal{T} ,

and then restrict to C_A . In the notation of (5.1), this class is just $\alpha = c_1(M)$, pulled back from B. The remaining "extra" class $r^*\sigma \times_B Z|T_A$ corresponds to $\sigma \cdot C_A = F$ in Z, and in particular it is pulled back from a class on C_A . Note that F maps isomorphically to its image in B. Using $\nu_*r^*\sigma = \mathcal{H}$, we see that the image of F in B corresponds to $A \cap \mathcal{H}$. If D is the divisor in B corresponding to $A \cap \mathcal{H}$ and V is a bundle with semistable restriction to every fiber whose associated section A(V) is A, then $V|E_b$ has \mathcal{O}_E as a Jordan-Hölder quotient if and only if $b \in D$. The above classes, together with the pullbacks of classes from B, are the only divisor classes that exist "universally" on $C_A \times_B Z = T_A$ for all sections A.

Using these classes, let us realize the bundles $V_{A,a}$ as pushforwards from T_A . Note that, from the definition, it is not a priori clear that $(\nu_A)_*\mathcal{O}_{T_A}(\Delta - \Sigma_A)$ is locally free, since Δ need not be Cartier.

Lemma 5.5. For every section A of \mathcal{P}_{n-1} and for every $a \in \mathbb{Z}$, we have

$$V_{A,a} = (\nu_A)_* \mathcal{O}_{T_A} (\Delta - \Sigma_A - aF).$$

Proof. There is a commutative diagram, which is in fact a Cartesian square:

$$T_A \longrightarrow T \times_B Z$$

$$\downarrow^{\nu_A} \downarrow \qquad \qquad \downarrow^{\nu \times \mathrm{Id}}$$

$$Z \xrightarrow{(A,\mathrm{Id})} \mathcal{P}_{n-1} \times_B Z.$$

Moreover, by definition $V_{A,a} = (A, \mathrm{Id})^* (\nu \times \mathrm{Id})_* \mathcal{O}_{\mathcal{T} \times_B Z}(\Delta - \mathcal{G} - aF)$. The morphism $\nu \times \mathrm{Id}$ is finite. Pulling back by the top horizontal arrow, the sheaf $\mathcal{O}_{\mathcal{T} \times_B Z}(\Delta - \mathcal{G} - aF)$ restricts to $\mathcal{O}_{T_A}(\Delta - \Sigma_A - aF)$. Thus, (5.5) is a consequence of the following general result:

Lemma 5.6. Let

$$X' \xrightarrow{f} X$$

$$\pi' \downarrow \qquad \qquad \downarrow \pi$$

$$Y' \xrightarrow{g} Y$$

be a Cartesian diagram of schemes, with π a finite morphism. Let S be a sheaf on X. Then the natural map $g^*\pi_*S \to (\pi')_*f^*S$ is an isomorphism.

Proof. The question is local in Y and Y', so that we may assume that $Y = \operatorname{Spec} R$ and $Y' = \operatorname{Spec} R'$ are affine. Since π and π' are finite, and thus affine, we may thus assume that $X = \operatorname{Spec} S$ and $X' = \operatorname{Spec} S'$, with $S' = S \otimes_R R'$. Suppose that S corresponds to the S-module M. Let M_R be the S-module M, viewed as an R-module. The assertion of the lemma is the statement that

$$(M_R) \otimes_R R' \cong (M \otimes_S S')_{R'}$$
.

But $M \otimes_S S' = M \otimes_S (S \otimes_R R')$, and a standard argument now identifies $(M \otimes_S (S \otimes_R R'))_{R'}$ with $(M_R) \otimes_R R'$. This proves the lemma. \square

Once we know that the sheaf $\mathcal{O}_{T_A}(\Delta - \Sigma_A - aF)$ pushes down to a vector bundle on Z, the same will be true for the twist of this sheaf by any line bundle on C_A . Conversely, we have the following:

Proposition 5.7. Let V be a vector bundle of rank n on Z such that $V|E_b$ is a regular semistable bundle with trivial determinant for every fiber E_b . Let A = A(V) be the section determined by V and let $C_A \to A$ be the induced spectral cover. Then there is a unique bundle N on C_A , such that $V \cong (\nu_A)_* [\mathcal{O}_{T_A}(\Delta - \Sigma_A) \otimes \rho_A^* N]$. \square

The proof of this result is similar to the proof of Part (ii) of Theorem 2.4 and will be omitted.

Next we look at the deformation theory of V.

Proposition 5.8. applying the Leray spectral sequence for $\pi: Z \to B$ to compute $H^1(Z; Hom(V, V))$, there is an exact sequence

$$0 \to H^1(B; \pi_*Hom(V, V)) \to H^1(Z; Hom(V, V)) \to H^0(B; R^1\pi_*Hom(V, V)).$$

- (i) The first term is $H^1(\mathcal{O}_{C_A})$ and corresponds to first order deformations of a line bundle on the spectral cover C_A ;
- (ii) If L is not trivial, then $H^0(B; R^1\pi_*Hom(V, V))$ is the tangent space to A in the space of all sections of \mathcal{P}_{n-1} , and the restriction map

$$H^1(Z; Hom(V, V)) \rightarrow H^0(B; R^1\pi_*Hom(V, V))$$

is the natural one which associates to a first order deformation of V a first order deformation of the section A(V).

(iii) Suppose that L is nontrivial and that C_A is smooth, or more generally that $h^1(\mathcal{O}_{C_A})$ is constant in a neighborhood of A. Suppose also either that $h^1(\mathcal{O}_B) = 0$ or that $h^1(\mathcal{V}_n \otimes M) = 0$, which will hold as soon as M is sufficiently ample. Then the local moduli space of deformations of V is smooth of dimension equal to $h^1(Z; Hom(V, V))$. In other words, all first order deformations of V are unobstructed.

Proof. By construction $\pi_*Hom(V,V) = (g_A)_*\mathcal{O}_{C_A}$, and we leave to the reader the check that the inclusion $H^1(B;\pi_*Hom(V,V)) \to H^1(Z;Hom(V,V))$ corresponds to deforming the line bundle on C_A . Next, let us fix for a moment a regular semistable bundle V over a single Weierstrass cubic E. Applying (1.5) with $S = \mathbb{C}[\epsilon]$, the dual numbers, for every deformation of V over S, there is an induced morphism $S \to |np_0|$ which restricts over S_{red} to $\zeta(V)$. Thus there is an intrinsic homomorphism from $H^1(ad(V))$ to the tangent space $H^0(\mathcal{O}_E(np_0))/\mathbb{C} \cdot \zeta(V)$ of $|np_0|$ at $\zeta(V)$. By (v) of Theorem 3.2, if V is a regular semistable bundle, then there is an exact sequence

$$0 \to \mathbb{C} \to H^1(W_{n-d}^{\vee} \otimes W_d^{\vee}) \to H^1(ad(V)) \to 0.$$

which identifies $H^1(ad(V))$ with the tangent space to $|np_0|$ at $\zeta(V)$. Using the parametrized version of this construction (Lemma 4.3, with S equal to $\mathbb{C}[\epsilon] \times B$), there is an induced morphism from $H^0(R^1\pi_*ad(V))$ to $\operatorname{Hom}(\mathbb{C}[\epsilon] \times B, \mathcal{P}_{n-1}; A)$, the space of morphisms from $\mathbb{C}[\epsilon] \times B$ to \mathcal{P}_{n-1} extending the section A. This gives an isomorphism from $R^1\pi_*ad(V)$ to the relative tangent bundle $T_{\mathcal{P}_{n-1}/B}$ restricted to A. As we have seen in Lemma 5.1, this restriction is just the normal bundle $N_{A/\mathcal{P}_{n-1}}$ to A in \mathcal{P}_{n-1} . Clearly the map $H^0(R^1\pi_*ad(V)) \to H^0(N_{A/\mathcal{P}_{n-1}})$ is the natural map from the tangent space of deformations of V to the tangent space to

deformations of the section A in \mathcal{P}_{n-1} . Now $Hom(V,V) = ad(V) \oplus \mathcal{O}_Z$, and so $R^1\pi_*Hom(V,V) = R^1\pi_*ad(V) \oplus L^{-1}$. Either L^4 or L^6 has a nonzero section, so that L^{-1} has a nonzero section if and only if L is trivial. Thus, if L is not trivial, then $H^0(L^{-1}) = 0$, and so

$$H^{0}(B; R^{1}\pi_{*}Hom(V, V)) = H^{0}(R^{1}\pi_{*}ad(V))$$

as claimed in (ii).

To prove (iii), begin by using Lemma 5.1 to find a smooth space Y parametrizing small deformations of the section A, of dimension $h^0(N_{A/\mathcal{P}_{n-1}})$. If $A \to Y$ is the total space of this family, there is an induced family of spectral covers $\mathcal{C} \to Y$. By assumption, the relative Picard scheme $\operatorname{Pic}(\mathcal{C}/Y)$ is smooth in a neighborhood of the fiber over A. Use this smooth space of dimension $h^1(\mathcal{O}_{C_A}) + h^0(N_{A/\mathcal{P}_{n-1}})$ to find a family of bundles parametrized by a smooth scheme S, which is an open subset of $\operatorname{Pic}(\mathcal{C}/Y)$ and thus is fibered over the open subset Y of sections of \mathcal{P}_{n-1} . This implies that the Kodaira-Spencer map of this family, followed by the map from $H^1(Z; Hom(V, V))$ to $H^0(B; R^1\pi_*Hom(V, V))$ is onto, and then that the Kodaira-Spencer map is an isomorphism onto $H^1(Z; Hom(V, V))$. Thus, the first order deformations of V are unobstructed. \square

5.2. Relationship to the extension point of view.

Next we relate the description of bundles constructed out of sections A of \mathcal{P}_{n-1} with the point of view of extensions. As usual, this will enable us to construct some of the bundles previously constructed via spectral covers, but not all.

We have already constructed the bundles W_k over Z as well as the universal extension $\mathbf{U}(d)$, $1 \le d < n$, which sits in an exact sequence

$$0 \to \pi_2^* \mathcal{W}_d^{\vee} \otimes \pi_1^* \mathcal{O}_{\mathcal{P}_{n-1,d}}(1) \to \mathbf{U}(d) \to \pi_2^* \mathcal{W}_{n-d} \to 0.$$

Here the projective space $\mathcal{P}_{n-1,d}$ of the vector space of extensions is identified with \mathcal{P}_{n-1} , but, by Theorem 4.6, under this identification

$$\mathcal{O}_{\mathcal{P}_{n-1,d}}(1) \otimes \pi^* L = \mathcal{O}_{\mathcal{P}_{n-1}}(1).$$

Finally, we have

$$\mathbf{U}(d) = \mathcal{U}_{1-d} \otimes \pi_1^* \mathcal{O}_{\mathcal{P}_{n-1}}(1) \otimes L^{-1}.$$

Thus there is an exact sequence

$$0 \to \pi_2^* \mathcal{W}_d^{\vee} \to \mathcal{U}_{1-d} \to \pi_2^* \mathcal{W}_{n-d} \otimes \pi_1^* \mathcal{O}_{\mathcal{P}_{n-1}}(-1) \otimes L \to 0.$$

Given a section A of $\mathcal{P}_{n-1,d} = \mathcal{P}_{n-1}$ such that $\mathcal{O}_{\mathcal{P}_{n-1,d}}(1)|A = M'$, we can pull back the defining extension for $\mathbf{U}(d)$ to obtain an extension

$$0 \to \mathcal{W}_d^{\vee} \otimes \pi^* M' \to U_A \to \mathcal{W}_{n-d} \to 0.$$

(Of course, M' is $M \otimes L^{-1}$.) Conversely, suppose that we are given an extension of W_{n-d} by $W_d^{\vee} \otimes \pi^* M'$, where M' is a line bundle on B which we can write as $M \otimes L^{-1}$. In this case, by the Leray spectral sequence

$$H^{1}(\mathcal{W}_{n-d}^{\vee} \otimes \mathcal{W}_{d}^{\vee} \otimes \pi^{*}M') \cong H^{0}(R^{1}\pi_{*}(\mathcal{W}_{n-d}^{\vee} \otimes \mathcal{W}_{d}^{\vee}) \otimes M')$$
$$= H^{0}(\mathcal{V}_{n,d} \otimes M \otimes L^{-1}) = H^{0}(\mathcal{V}_{n} \otimes M).$$

Thus nontrivial extensions of W_{n-d} by $W_d^{\vee} \otimes \pi^* M'$ which restrict to nontrivial extensions on every fiber can be identified with sections of $\mathcal{P}_{n-1,d}$ corresponding to the line bundle M. Finally, we see that, for $1 \leq d \leq n-1$, we can write $V_{A,1-d}$ as an extension

$$0 \to \mathcal{W}_d^{\vee} \to V_{A,1-d} \to \mathcal{W}_{n-d} \otimes \pi^*(M^{-1} \otimes L) \to 0.$$

We can also relate the deformation theory of U_A above to the bundles \mathcal{W}_d and \mathcal{W}_{n-d} . Thus, the tangent space to $\operatorname{Ker}\{(g_a)_*\colon \operatorname{Pic} C_A \to \operatorname{Pic} B\}$ is $H^1(B;\pi_*(\mathcal{W}_d\otimes \mathcal{W}_{n-d})\otimes M^{-1}\otimes L)$, and the tangent space to deformations of the section A is $H^0(B;R^1\pi_*(\mathcal{W}_d^\vee\otimes \mathcal{W}_{n-d}^\vee)\otimes M\otimes L^{-1})$, provided that L is not trivial.

5.3. Chern classes and determinants.

Let A be a section of \mathcal{P}_{n-1} . Corresponding to A, there is the line bundle M on B which is the restriction to A of $\mathcal{O}_{\mathcal{P}_{n-1}}(1)$. We denote by α the class $c_1(M) \in H^2(B;\mathbb{Z})$. Our goal is to express the Chern classes of $V_{A,a}$ in terms of α and the standard classes on Z. We will also consider more general bundles arising from twisting by a line bundle on the spectral cover.

First we shall determine the Chern classes of $V_{A,a}$. We begin with the following lemma:

Lemma 5.9. Let A be a section of \mathcal{P}_{n-1} corresponding to the inclusion of a line bundle M^{-1} in \mathcal{V}_n . Then, for $k \geq 0$, we have $p_*([A] \cdot \zeta^k) = \alpha^k \in H^{2k}(B; \mathbb{Z})$.

Proof. Note that by definition $\zeta|_A = c_1(M) = \alpha$ when we identify A and B in the obvious way. It follows that $\zeta^k|_A = \alpha^k$. This means that $p_*([A] \cdot \zeta^k) = \alpha^k$. \square

Using (5.9), we can compute the Chern classes $c_i(V_{A,a})$ by taking the formula for $c_i(\mathcal{U}_a)$ and replacing ζ^i by α^i . Thus

Theorem 5.10. Suppose that A is a section of \mathcal{P}_{n-1} such that the corresponding line bundle M has $c_1(M) = \alpha \in H^2(B)$ (or Pic B). Then

$$\operatorname{ch}(V_{A,a}) = e^{-\alpha} \left(\frac{1 - e^{(a+n)L}}{1 - e^L} \right) - \frac{1 - e^{aL}}{1 - e^L} + e^{-\sigma} (1 - e^{-\alpha}).$$

Moreover, in $\pi^* \operatorname{Pic} B \subset \operatorname{Pic} Z$,

$$\det(V_{A,a}) = -(n+a-1)\alpha + \left[an + \left(\frac{n^2 - n}{2}\right)\right]L. \quad \Box$$

There is also a formula for $c(V_{A,a})$ which follows similarly from the formula for $c(\mathcal{U}_a)$.

Now let us consider the effect of twisting by a line bundle on the spectral cover. If N is a line bundle on the spectral cover C_A associated to A, let

$$V_{A,0}[N] = (\nu_A)_* \left[\mathcal{O}_{T_A}(\Delta - \Sigma_A) \otimes \rho_A^* N \right].$$

For example, suppose that N is of the form $\mathcal{O}_{C_A}(-aF)\otimes g_A^*N_0$, where N_0 is a line bundle on B. Then

$$V_{A,0}[N] = V_{A,a} \otimes \pi^* N_0.$$

In particular, we see that if $N = \mathcal{O}_{C_A}(-aF) \otimes g_A^* N_0$, for some line bundle N_0 on B and some integer a, then

$$\operatorname{ch}(V_{A,0}[N]) = \left[e^{-\alpha} \left(\frac{1 - e^{(a+n)L}}{1 - e^L} \right) - \frac{1 - e^{aL}}{1 - e^L} + e^{-\sigma} (1 - e^{-\alpha}) \right] \cdot e^{c_1(N_0)}.$$

For more general line bundles N on C_A , we can calculate the determinant of $V_{A,0}[N]$. In what follows, we identify $\operatorname{Pic} B$ with a subgroup of $\operatorname{Pic} Z$ via π^* and write the group law additively.

Lemma 5.11. With $V_{A,0}[N]$ as defined above, the following formula holds in Pic B:

$$c_1(V_{A,0}[N]) = -(n-1)\alpha + \left(\frac{n^2 - n}{2}\right)L + (g_A)_*c_1(N).$$

Thus, for a fixed section A of \mathcal{P}_{n-1} and a fixed line bundle \mathcal{N} on B, the set of bundles V on Z which are regular semistable on every fiber, with A(V) = A and $\det V = \pi^* \mathcal{N}$ is a principal homogeneous space over $\ker\{g_{A*} \colon \operatorname{Pic} C_A \to \operatorname{Pic} B\}$, which is a generalized abelian variety times a finitely generated abelian group.

Proof. Since it is enough to compute the determinant in the complement of a set of codimension two, we may restrict attention to the open subset of T_A where Δ is a Cartier divisor. Now it is a general formula that, for a Cartier divisor D on T_A ,

$$c_1[(\nu_A)_*\mathcal{O}_{T_A}(D)] = c_1[(\nu_A)_*\mathcal{O}_{T_A}] + (\nu_A)_*D.$$

Thus, applying this formula to $\mathcal{O}_{T_A}(\Delta - \Sigma_A)$ and $\mathcal{O}_{T_A}(\Delta - \Sigma_A) \otimes \rho_A^* N$, we see that

$$c_1(V_{A,0}[N]) = c_1(V_{A,0}) + (\nu_A)_* \rho_A^* c_1(N).$$

But we have calculated $c_1(V_{A,0}) = -(n-1)\alpha + \left(\frac{n^2 - n}{2}\right)L$, and $(\nu_A)_*\rho_A^*c_1(N) = \pi^*(g_A)_*c_1(N)$ since $T_A = C_A \times_B Z$. Putting these together gives the formula in (5.11). \square

If dim $B \geq 2$ and M is sufficiently ample, we we will see in the next subsection that the generalized abelian variety $\operatorname{Ker}\{g_{A*}\colon \operatorname{Pic} C_A \to \operatorname{Pic} B\}$ is in fact a finitely generated abelian group, with no component of positive dimension.

Using (5.11), let us consider the following problem: Given the section A, when can we find a line bundle N such that $V_{A,0}[N]$ actually has trivial determinant? We are now in position to answer this question in this case if we consider twisting only by line bundles which exist universally for all spectral covers.

Proposition 5.12. Given a section A, suppose that $N = \mathcal{O}_{C_A}(-aF) \otimes g_A^* N_0$ for a line bundle N_0 on B and an integer a. Then $V_{A,0}[N]$ has trivial determinant for some choice of an N as above if at least one of the following conditions holds:

- (i) n is odd,
- (ii) L is divisible by 2 in Pic B, or
- (iii) $\alpha \equiv L \mod 2$ in Pic B.

Proof. It suffices to show that there exists an $a \in \mathbb{Z}$ such that $\det(V_{A,a})$ is divisible by n. For then, for an appropriate line bundle N_0 on B, we can arrange that $V = V_{A,a} \otimes N$ has trivial determinant. By (5.10), we must have

$$(a-1)\alpha \equiv \frac{n(n-1)}{2}L \bmod n.$$

In the first two cases we simply take $a \equiv 1 \mod n$. Lastly, let us suppose that n is even and that L is not divisible by 2. Then the condition $(a-1)\alpha \equiv \frac{n(n-1)}{2}L \mod n$ is a nontrivial condition on α . It is satisfied for the appropriate a if $\alpha \equiv L \mod 2$ in Pic B. \square

We leave it to the reader to write out necessary and sufficient conditions for the equation $(a-1)\alpha \equiv \frac{n(n-1)}{2}L \mod n$ to have a solution in general.

For a general line bundle N on C_A , we can use the Grothendieck-Riemann-Roch theorem to calculate the higher Chern classes of $\operatorname{ch}(V_{A,0}[N])$, but only in the range where Δ is a Cartier divisor. Thus, we are essentially only able to compute c_2 by this method for a general line bundle N:

Proposition 5.13. Suppose that no component of Γ is contained in C_A . Let ch_2 be the degree two component of the Chern character. Then

$$\operatorname{ch}_{2}(V_{A,0}[N]) - \operatorname{ch}_{2}(V_{A,0}) = (\nu_{A})_{*} \left(\left(\Delta - \Sigma_{A} + \frac{1}{2} (\nu_{A}^{*} K_{Z} - K_{T_{A}}) \right) \cdot (\rho_{A})^{*}(N) \right) + (\pi_{A})^{*} (g_{A})_{*} \frac{(N)^{2}}{2}.$$

Proof. Working where Δ is Cartier, we can apply the Grothendieck-Riemann-Roch theorem to the local complete intersection morphism $\nu_A : T_A \to Z$ to determine the Chern character of $V_{A,0} = (\nu_A)_* \mathcal{O}_{T_A}(\Delta - \Sigma_A)$:

$$\operatorname{ch}(V_{A,a}) = (\nu_A)_* \left(e^{\Delta - \Sigma} \operatorname{Todd}(T_A/Z) \right),\,$$

valid under our assumptions through terms of degree two. Applying the same method to calculate the Chern character of $V_{A,0}[N]$, we find that, at least through degree two,

$$\operatorname{ch}(V_{A,0}[N]) - \operatorname{ch}(V_{A,0}) = (\nu_A)_* \left((e^N - 1)(e^{\Delta - \Sigma} \operatorname{Todd}(T_A/Z) \right).$$

Expanding this out gives (5.13). \square

5.4. Line bundles on the spectral cover.

In this section, we look at the problem of finding extra line bundles on the spectral cover C_A , under the assumption that C_A is smooth and that M is sufficiently ample. As we shall see, the discussion falls naturally into three cases: dim B = 1, dim B = 2, dim $B \ge 3$.

First let us consider the case that B is a curve, with M arbitrary but C_A assumed to be smooth, or more generally reduced. Let A correspond to the line bundle M

on B. Given $V = V_{A,0}[N]$, we seek det V and $c_2(V)$. First, by (5.11), working in Pic B written additively,

$$\det V_{A,0}[N] = -(n-1)M + \left(\frac{n^2 - n}{2}\right)L + (g_A)_*N.$$

Since g_{A*} : Pic $C_A \to \text{Pic } B$ is surjective in case C_A is reduced, we can arrange that the determinant is in fact trivial, and then the line bundle $\mathcal{O}_{C_A}(D)$ is determined up to the subgroup $\text{Ker}\{g_{A*}\colon \text{Pic } C_A \to \text{Pic } B\}$. If C_A is smooth, then this subgroup is the product of an abelian variety and a finite group

We may summarize this discussion as follows:

Theorem 5.14. Suppose that dim B=1. Given a section A of \mathcal{P}_{n-1} such that C_A is reduced, the set of bundles V with trivial determinant such that A(V)=A is a nonempty principal homogeneous space over $\operatorname{Ker}\{g_{A*}\colon\operatorname{Pic} C_A\to\operatorname{Pic} B\}$. The same statement holds if we replace the condition that V has trivial determinant by the condition that the determinant of V is $\pi^*\lambda$ for some fixed line bundle λ on B. \square

The remaining Chern class is $c_2(V)$. In this case, in $H^4(Z; \mathbb{Z})$, with no assumptions on Γ , we have (as computed in [3] in case n = 2):

Proposition 5.15. For every line bundle N on C_A ,

$$c_2(V_{A,0}[N]) = c_2(V) = \sigma \cdot \alpha = \deg M.$$

Proof. First assume that C_A is reduced. Write $N \cong \mathcal{O}_{C_A}(\sum_i p_i)$, where the p_i are points in the smooth locus of C_A which lie under smooth fibers. Thus $\rho_A^{-1}(p_i) = f_i$ is a smooth fiber of T_A . In this case, we can obtain $V_{A,0}[N]$ as a sequence of elementary modifications of the form

$$0 \to V_{A,0}[N_i] \to V_{A,0}[N_{i+1}] \to (i_i)_* \lambda_i \to 0,$$

where E_j is the fiber on Z corresponding to $f_j \subset T_A$, $i_j : E_j \to Z$ is the inclusion, and $\lambda_j = \mathcal{O}_{T_A}(\Delta - \Sigma_A)|f_j$ is a line bundle of degree zero. By standard calculations,

$$c_2(V_{A,0}[N_i]) = c_2(V_{A,0}[N_{i+1}])$$

and so $c_2(V_{A,0}[N]) = c_2(V_{A,0}) = \sigma \cdot \alpha$.

In case C_A is not reduced, a similar argument applies, where we replace p_i by a Cartier divisor whose support is contained in the smooth locus of $(C_A)_{\text{red}}$ and E_j by a thickened fiber. \square

Remark. On the level of Chow groups, the refined Chern class $\tilde{c}_2(V_{A,0}[N])$ essentially records the extra information coming from the natural map $\operatorname{Pic} C_A \to A^2(Z)$.

Next we consider the case where dim B > 1. First we have the following result, with no assumption on C_A , concerning the connected component of Pic C_A .

Lemma 5.16. Suppose that dim $B \geq 2$ and that M is sufficiently ample. More precisely, suppose that

$$H^{i}(B; L^{-1} \otimes M^{-1}) = H^{i}(B; L \otimes M^{-1}) = \dots = H^{i}(B; L^{n-1} \otimes M^{-1}) = 0$$

for i = 0, 1. Then the natural map from $H^1(Z; \mathcal{O}_Z)$ to $H^1(C_A; \mathcal{O}_{C_A})$ is an isomorphism. Finally, if in addition L is not trivial, then the norm map from $\operatorname{Pic}^0 C_A$ to $\operatorname{Pic}^0 B$ is surjective with finite kernel. Thus $\operatorname{Ker}\{g_{A*}\colon \operatorname{Pic} C_A \to \operatorname{Pic} B\}$ is a finitely generated abelian group.

Proof. From the exact sequence

$$0 \to \mathcal{O}_Z(-n\sigma) \otimes \pi^* M^{-1} \to \mathcal{O}_Z \to \mathcal{O}_{C_A} \to 0,$$

we see that there is a long exact sequence

$$H^1(\mathcal{O}_Z(-n\sigma)\otimes\pi^*M^{-1})\to H^1(\mathcal{O}_Z)\to H^1(\mathcal{O}_{C_A})\to H^2(\mathcal{O}_Z(-n\sigma)\otimes\pi^*M^{-1}).$$

Applying the Leray spectral sequence to $\mathcal{O}_Z(-n\sigma) \otimes \pi^*M^{-1}$, we have that

$$H^i(\mathcal{O}_Z(-n\sigma)\otimes\pi^*M^{-1})=H^{i-1}(R^1\pi_*\left[\mathcal{O}_Z(-n\sigma)\otimes\pi^*M^{-1}\right]).$$

Now, by duality,

$$R^{1}\pi_{*} \left[\mathcal{O}_{Z}(-n\sigma) \otimes \pi^{*}M^{-1} \right] = R^{1}\pi_{*}\mathcal{O}_{Z}(-n\sigma) \otimes M^{-1}$$
$$= \left(L^{-1} \oplus L \oplus \cdots \oplus L^{n-1} \right) \otimes M^{-1}.$$

Thus by our assumptions the map $H^1\mathcal{O}_Z) \to H^1(\mathcal{O}_{C_A})$ is an isomorphism. By applying the Leray spectral sequence to \mathcal{O}_Z , we see that there is an exact sequence $0 \to H^1(\mathcal{O}_B) \to H^1\mathcal{O}_Z) \to H^0(L^{-1})$. As we saw in the proof of (ii) of (5.8), if L is not trivial, then $H^0(L^{-1}) = 0$ and the pullback map $H^1(\mathcal{O}_B) \to H^1\mathcal{O}_Z)$ is an isomorphism. The last statement of the lemma is then clear. \square

Lemma 5.17. If M is sufficiently ample on B, then C_A is an ample divisor in Z.

Proof. Equivalently, we must show that for M sufficiently ample on B, $\pi^*M \otimes \mathcal{O}_Z(n\sigma)$ is ample. But $\mathcal{O}_Z(n\sigma)$ is relatively ample, and thus by a standard result $\pi^*M \otimes \mathcal{O}_Z(n\sigma)$ is ample for M sufficiently ample (compare [10, p. 161. (7.10)(b)] for the case where $\mathcal{O}_Z(n\sigma)$ is relatively very ample). \square

Corollary 5.18. If dim $B \geq 3$, M is sufficiently ample, and Z and C_A are smooth, then $\operatorname{Pic} Z \cong \operatorname{Pic} C_A$. If dim B = 2, M is sufficiently ample, and Z and C_A are smooth, then the restriction mapping $\operatorname{Pic} Z \to \operatorname{Pic} C_A$ is injective.

Proof. This is immediate from the Lefschetz theorem and (5.17). \square

Remark. If dim B=2 and M is sufficiently ample, it is natural to expect an analogue of the Noether-Lefschetz theorem to hold: for generic sections C_A of $\pi^*M\otimes \mathcal{O}_Z(n\sigma)$, $\operatorname{Pic} Z\cong \operatorname{Pic} C_A$. However, in the next section, we will see how to construct sections A such that the spectral cover C_A is smooth but has larger Picard number than expected.

5.5. Symmetric bundles.

Next we turn to bundles with a special invariance property.

Definition 5.19. Let $\iota: Z \to Z$ be the involution which is -1 in every fiber. A bundle V is symmetric if $\iota^*V \cong V^{\vee}$.

We shall now analyze when a bundle V is symmetric. We fix a section A, corresponding to the class α and denote C_A , ν_A , T_A , $T_$

Proposition 5.20. For a suitable choice of $N \in \text{Pic } C$ the bundle $V_{A,0}[N]$ is symmetric if and only if $g^*(L + \alpha) + nF$ is divisible by 2 in Pic C. In this case, for a fixed section A, the set of all symmetric bundles whose section is A is a principal homogeneous space over the 2-torsion in Pic C.

Proof. Suppose that $V = V_{A,0}[N] = \nu_* [\mathcal{O}_T(\Delta - \Sigma_A) \otimes \rho^* \mathcal{O}_C(N)]$, where N is a divisor on C. For our purposes, since both $\iota^* V$ and V^{\vee} are bundles, they are isomorphic if and only if they are isomorphic outside the complement of a set of codimension two in Z. Thus, we shall work as if Δ is a Cartier divisor.

There is an induced involution on T, also denoted by ι , for which ν is equivariant. Thus

$$\iota^* V = \iota^* \nu_* \left[\mathcal{O}_T(\Delta - \Sigma_A) \otimes \rho^* \mathcal{O}_C(N) \right]$$
$$= \nu_* \iota^* \left[\mathcal{O}_T(\Delta - \Sigma_A) \otimes \rho^* \mathcal{O}_C(N) \right].$$

Now $\iota^*\Sigma_A = \Sigma_A$ and $\iota^*\rho^*\mathcal{O}_C(N) = \rho^*\mathcal{O}_C(N)$. One the other hand, $\iota^*\Delta$ is linearly equivalent to $2\Sigma_A - \Delta$ on a generic fiber. This says that

$$\iota^* \Delta = 2\Sigma_A - \Delta + \rho^* D$$

for some divisor D on C. To determine D, restrict both sides above to Σ_A where ι acts trivially. We find that $D = 2\Delta \cdot \Sigma_A - 2\Sigma_A^2$, viewed in the obvious way as a divisor class on C. Thus

$$\iota^* \Delta = 2\Sigma_A - \Delta + 2\rho^* D_0$$

where D_0 is the fixed divisor class $\Delta \cdot \Sigma_A - \Sigma_A^2$, viewed as a divisor on C. Here the main point will be the factor of 2. However we note that $\Sigma_A^2 = -[L']$, where $L' = g^*L$ is the line bundle for the elliptic scheme T, and

$$\Delta \cdot \Sigma_A = \Delta \cdot \nu^* \sigma = \nu^* (\nu_* \Delta) \cdot \sigma = \nu^* (C \cdot \sigma),$$

which after pullback corresponds to the divisor class F on T. (Here $\Delta = C \times_B C \subset C \times_B Z$, and so $\nu_* \Delta = C$ since ν is just the natural projection of $T = C \times_B Z$ to Z.)

Next we calculate V^{\vee} . Relative duality for the finite flat morphism ν says that, for every Cartier divisor D on T, $[\nu_*\mathcal{O}_T(D)]^{\vee} = \nu_* [\mathcal{O}_T(-D) \otimes K_{T/Z}]$, where $K_{T/Z} = K_T \otimes \nu^* K_Z^{-1}$ is the relative dualizing sheaf of the morphism ν . Thus we must have

$$\Sigma_A - \Delta - \rho^* N + K_T - \nu^* K_Z = \Sigma_A - \Delta + 2\rho^* D_0 + \rho^* N.$$

Equivalently, we must have

$$K_T - \nu^* K_Z = 2\rho^* N + 2\rho^* D_0 = 2\rho^* (N + [L'] + F).$$

Conversely, given that the above equality holds, the corresponding vector bundles will be symmetric. To see if this equality holds for the appropriate choice of N, we must calculate $K_T - \nu^* K_Z$. Since Z is an elliptic fibration, $K_Z = \pi^* (K_B + L)$, and likewise $K_T = \rho^* (K_C + L')$, where $L' = g^* L$. Thus $K_T - \nu^* K_Z = \rho^* (K_C - g^* K_B)$. To calculate K_C , we use (5.4), which says that $K_C = K_Z + C|C = K_Z + \pi^* L + n\sigma|C$. On the other hand, $K_Z - \pi^* K_B = \pi^* L$. Restricting to C gives:

$$K_C - g^* K_B = g^* (L + \alpha) + nF.$$

Putting this together, we see that, if V is symmetric, then we must have $\rho^*(g^*(L + \alpha) + nF)$ divisible by 2 in $\rho^* \operatorname{Pic} C$, and conversely.

Next we claim that ρ^* : Pic $C \to \text{Pic } T$ is injective. It suffices to show that $\rho_*\mathcal{O}_T = \mathcal{O}_C$, for then $\rho_*\rho^*N = N$ for every line bundle N on C. But by flat base change $g^*\pi_*\mathcal{O}_Z = \rho_*\nu^*\mathcal{O}_Z = \rho_*\mathcal{O}_T$. Since $\pi_*\mathcal{O}_Z = \mathcal{O}_B$, we have that $g^*\pi_*\mathcal{O}_Z = g^*\mathcal{O}_B = \mathcal{O}_C = \rho_*\mathcal{O}_T$. Hence $\rho_*\mathcal{O}_T = \mathcal{O}_C$, and so ρ^* is injective.

Thus, V is symmetric if and only if $g^*(L + \alpha) + nF$ divisible by 2 in Pic C. Moreover the set of possible line bundles N for which $V_{A,0}[N]$ is symmetric is a principal homogeneous space over the 2-torsion in Pic C, as claimed. This concludes the proof of (5.20). \square

If dim $B \geq 3$, Z and C are smooth, and M is sufficiently ample, then $g^*(L + \alpha) + nF$ is divisible by 2 in Pic C if and only if $\pi^*(L + \alpha) + n\sigma$ is divisible by 2 in Pic Z. This can only happen if n is even and $\alpha \equiv L \mod 2$. A similar statement is likely to hold if dim B = 2 and A is also assumed to be general.

We can see the conditions n is even and $\alpha \equiv L \mod 2$ clearly in terms of extensions. In this case n = 2d, and we can write $V_{A,1-d}$ as an extension

$$0 \to \mathcal{W}_d^{\vee} \to V_{A,1-d} \to \mathcal{W}_d \otimes M^{-1} \otimes L \to 0.$$

Under the assumption that $M^{-1} \otimes L = M_0^{\otimes 2}$ for some line bundle M_0 , we can write $V_{A,1-d} \otimes M_0^{-1}$ as an extension of $\mathcal{W}_d \otimes M_0$ by the dual bundle $\mathcal{W}_d^{\vee} \otimes M_0^{-1}$, and then check directly that the corresponding bundles are symmetric.

5.6. The case of the trivial section.

We turn to bundles which have reducible or non-reduced spectral covers. We begin with the extreme case of the trivial section $\mathbf{o} = \mathbf{o}_Z = \mathbb{P}\mathcal{O}_B \subset \mathcal{P}_{n-1}$. To construct this section we take $M = \mathcal{O}_B$ and take a nowhere vanishing section of \mathcal{O}_B and the zero section of L^{-a} for all a > 0. Since $M = \mathcal{O}_B$, the class α is zero. The spectral cover $C = C_{\mathbf{o}} \subset Z$ is simply the nonreduced scheme $n\sigma$, and the associated reduced subscheme C_{red} is identified with B. The bundles associated to this section have the property that their restrictions to each fiber of Z are isomorphic to $I_n(\mathcal{O})$. Conversely, if we have such a bundle V over Z, then the section it determines is \mathbf{o} .

By our general existence theorem we immediately conclude:

Corollary 5.21. For every $n \geq 1$ there is a vector bundle $V \to Z$ whose restriction to each fiber $E_b \subset Z$ is isomorphic to $I_n(\mathcal{O}_{E_b})$. \square

The structure sheaf \mathcal{O}_C is filtered by subsheaves with successive quotients

$$L^{n-1}, L^{n-2}, \ldots, \mathcal{O}_B.$$

The restriction of $\mathcal{O}_{C\times_B Z}(\Delta - \Sigma_A - aF)$ to $C_{\text{red}} \times_B Z \cong Z$, is isomorphic to

$$\mathcal{O}_{C \times_B Z}(\Delta - \Sigma_A - aF) | (C_{\text{red}} \times_B Z) \cong \mathcal{O}_Z(\sigma - \sigma) \otimes L^a) = L^a.$$

From this it follows that $V_{\mathbf{o},a}$ has a filtration by subbundles with successive quotients $L^{a+n-1}, L^{a+n-2}, \ldots, L^a$. Consequently,

$$ch(V_{\mathbf{o},a}) = \frac{e^{aL} - e^{(a+n)L}}{1 - e^L},$$

which agrees with the formula in Theorem 5.10 since $\alpha = 0$.

We have the inclusion $B = C_{\text{red}} \subset C$ and the projection $C \to B$ so that \mathcal{O}_C splits as a module over \mathcal{O}_B into $\mathcal{S} \oplus \mathcal{O}_B$ with \mathcal{S} a locally free sheaf of rank n-1 over \mathcal{O}_B . From the filtration of \mathcal{O}_C as an \mathcal{O}_B -module, we see that \mathcal{S} has a filtration with successive quotients $L^{n-1}, L^{n-2}, \ldots, L$. Thus, $\operatorname{Pic} C \cong \operatorname{Pic} B \oplus H^1(\mathcal{S})$, and $H^1(\mathcal{S})$ is a vector group. In particular, as far as Chern classes are concerned, we may as well just twist by line bundles N on C which are pulled back from B. Even if the line bundle N on C is not pulled back from B, if N_0 is the restriction of N to $C_{\text{red}} \cong B$, it is still clear that $V_{(\mathbf{o},0)}[N]$ has a filtration with successive quotients $L^{n-1} \otimes N_0, L^{n-2} \otimes N_0, \ldots, L \otimes N_0$. We have

$$\operatorname{ch}(V_{\mathbf{o},0}[N]) = \frac{1 - e^{nL}}{1 - e^{L}} \cdot e^{N_0}.$$

Remark. (1) Note that, unless L is a torsion line bundle, the bundles $V_{\mathbf{o},0}[N]$ are unstable with respect to every ample divisor.

(2) By contrast with (5.14), even if dim B=1, we cannot always arrange trivial determinant for $V_{\mathbf{o},\mathbf{0}}[N]$.

If instead we try to construct $V_{\mathbf{o},a}$ directly as a sequence of global extensions on Z, we run into the following type of question. Suppose for simplicity that n=2 and that a=0. In this case we try to find a bundle on Z which restricts over every fiber f of Z to be the nontrivial extension of \mathcal{O}_f by \mathcal{O}_f , in other words to I_2 . We may as well try to write it as an extension of \mathcal{O}_Z by the pullback of a line bundle N on B. To do this we need a class $H^1(\pi^*N)$ whose restriction to every fiber is non-trivial. That is to say, we need an element in $H^1(\pi^*N)$ whose image under the natural map ψ in the Leray spectral sequence (which is an exact sequence in this case)

$$H^1(\pi_*\pi^*N) \to H^1(\pi^*N) \xrightarrow{\psi} H^0(R^1\pi_*\pi^*N) \to H^2(\pi_*\pi^*N)$$

is a nowhere zero section of $R^1\pi_*\pi^*N$. Of course, $\pi_*\pi^*N\cong N$ and $R^1\pi_*\pi^*N\cong N\otimes R^1\pi_*\mathcal{O}_Z=N\otimes L^{-1}$. Thus if there is to exist a nowhere vanishing section of $H^0(R^1\pi_*\pi^*N)$, it must be the case that N=L. But we also need the condition that the map $H^1(\pi^*L)\to H^0(L\otimes L^{-1})=H^0(\mathcal{O}_B)$ is surjective. This is not immediately obvious from the spectral sequence since there is no reason for $H^2(B;L)$ to vanish. Nevertheless, it follows from our construction of $V_{\mathbf{o},0}$ that the map ψ is onto in the case N=L. Finally, the set of possible extensions is a principal homogeneous space over $H^1(B;L)$, which is identified with the kernel of the natural map $\operatorname{Pic}(2\sigma)\to\operatorname{Pic} B$.

5.7. Deformation to a reducible spectral cover.

For every choice of a rank $n > \dim B$ and for all sections A of \mathcal{P}_{n-1} which correspond to a sufficiently ample line bundle, we have constructed vector bundles $V_{A,a} = V_{A,a}(n)$. In this subsection, we try to relate the $V_{A,a}(n)$ for various choices of n. To this end, let $\mathcal{H} = \mathcal{P}^{n-2} = \mathbb{P}(\mathcal{O}_B \oplus L^{-2} \oplus \cdots \oplus L^{-n+1}) \subset \mathcal{P}_{n-1}$. We begin by considering what happens when the section A lies in the subbundle \mathcal{H} , but is otherwise generic. To insure that there are actually sections of \mathcal{H} as opposed to just rational sections, it is reasonable to assume that $n \geq \dim B + 2$. A section A of \mathcal{H} is given by a line bundle M and by n sections $\sigma_0, \ldots, \sigma_{n-1}$ of $M, M \otimes$ $L^{-2}, \ldots, M \otimes L^{-(n-1)}$ which have no common zeroes. If M is sufficiently ample, the section $A = A_0$ will then move in a family A_t of sections of \mathcal{P}_{n-1} , by choosing a nonzero section σ_n of $M \otimes L^{-n}$ and considering the family defined by the sections $A_t = (\sigma_0, \dots, \sigma_{n-1}, t\sigma_n)$. Roughly speaking, $V_{A_0,a}(n)$ is obtained from the bundle V' of rank n-1 corresponding to A_0 , viewed as a section of \mathcal{P}_{n-2} . Along each fiber f we add a trivial \mathcal{O}_f factor to the restriction of V'. This statement is correct as long as the restriction of V' to the fiber does not itself contain an \mathcal{O}_f factor, or more generally a summand of the form $I_d(\mathcal{O}_f)$ for some $d \leq n$. The simplest possibility would be that $V_{A_0,a}(n)$ is a deformation of $V_{A,a}(n-1) \oplus \mathcal{O}_Z$, but a calculation with Chern classes rules this out. Likewise, $V_{A_0,a}(n)$ is not a deformation of $V_{A,a}(n-1) \oplus \pi^*N$ for any line bundle N on B. Instead, we shall see that $V_{A_0,a}(n)$ is a deformation of a suitable elementary modification of $V_{A,a}(n-1) \oplus \pi^* L^a$. Finally, we shall use the construction to check the Chern class calculations.

To make this construction, it is best to begin by working universally again. We have the n-to-1 map $\nu \colon \mathcal{T} \to \mathcal{P}_{n-1}$. Inside \mathcal{P}_{n-1} , there is the smooth divisor $\mathcal{H} = \mathcal{P}_{n-2}$. Now in $\mathcal{T} = \mathcal{T}_{n-1}$ there is a smooth divisor $\mathcal{T}' \cong \mathcal{T}_{n-2}$ defined by the diagram

$$0 \longrightarrow \mathcal{E} \longrightarrow \pi^* \pi_* \mathcal{O}_Z(n\sigma) \longrightarrow \mathcal{O}_Z(n\sigma) \longrightarrow 0$$

$$\uparrow \qquad \qquad \uparrow \qquad \qquad \uparrow$$

$$0 \longrightarrow \mathcal{E}' \longrightarrow \pi^* \pi_* \mathcal{O}_Z((n-1)\sigma) \longrightarrow \mathcal{O}_Z((n-1)\sigma) \longrightarrow 0.$$

We take $\mathcal{T}' = \mathbb{P}(\mathcal{E}') \subset \mathbb{P}(\mathcal{E}) = \mathcal{T}$. The restriction of ν to \mathcal{T}' defines the corresponding map $\mathcal{T}_{n-2} \to \mathcal{P}_{n-2}$, and in particular $\nu | \mathcal{T}'$ has degree n-1. Clearly, we have an equality of smooth divisors in \mathcal{T} :

$$\nu^*\mathcal{H}=\mathcal{T}'+r^*\sigma.$$

The intersection $\mathcal{T}' \cap r^*\sigma$ is clearly the smooth divisor $\mathcal{P}_{n-3} \subset r^*\sigma \cong \mathcal{P}_{n-2}$; it lies over \mathcal{P}_{n-3} . A local calculation shows that \mathcal{T}' and $r^*\sigma$ meet transversally at the generic point of \mathcal{P}_{n-3} and thus everywhere. Note that $\mathcal{T}_1 \cong Z$, $r: \mathcal{T}_1 \to Z$ is the identity, and the intersection of \mathcal{T}_1 and $r^*\sigma$ in \mathcal{T}_2 is $\sigma \subset \mathcal{T}_1$. This is compatible with the convention $\mathbf{o} \cong \mathcal{P}_0 \cong B$.

Let $\mathcal{D} = \mathcal{T}' \times_B Z$ and, as usual, let $F = r^* \sigma \times_B Z$. Then F is a smooth divisor and \mathcal{D} is smooth away from the singularities of $\mathcal{T} \times_B Z$. The divisors \mathcal{D} and F meet in a reduced divisor $\mathcal{P}_{n-3} \times_B Z$. We thus have an exact sequence:

$$0 \to \mathcal{O}_{\mathcal{D}+F} \to \mathcal{O}_{\mathcal{D}} \oplus \mathcal{O}_F \to \mathcal{O}_{\mathcal{D}\cap F} \to 0.$$

Tensoring the above exact sequence by the sheaf $\mathcal{O}_{\mathcal{T}\times_B Z}(\Delta - \Sigma_A - aF)$, using the fact that $\Delta \cap F = \Sigma_A \cap F$, gives a new exact sequence

$$0 \to \mathcal{O}_{\mathcal{D}+F}(\Delta - \Sigma_A - aF) \to \mathcal{O}_{\mathcal{D}}(\Delta - \Sigma_A - aF) \oplus \mathcal{O}_F(-aF) \to \mathcal{O}_{\mathcal{D}\cap F}(-aF) \to 0.$$

(In a neighborhood of F, Δ is Cartier, and so the above sequence is still exact.) Of course, F|F = -L|F. Now apply $(\nu \times \mathrm{Id})_*$ to the above exact sequence. To keep track of the ranks, we shall write $\mathcal{U}_a(n)$ when we want to denote the appropriate vector bundle of rank n, and similarly for $V_{A,a}(n)$. (However, in the notation, $V_{A,a}(n-1)$ will be a general rank (n-1)-bundle but $V_{A,a}(n)$ will be the special rank n bundle corresponding to a reducible section. Of course, this will not affect Chern class calculations.) We have:

$$0 \to \mathcal{U}_a(n)|\mathcal{P}_{n-2} \times_B Z \to \mathcal{U}_a(n-1) \oplus (L^a|\mathcal{P}_{n-2} \times_B Z) \to L^a|\mathcal{P}_{n-3} \times_B Z \to 0.$$

Let A be a section of \mathcal{P}_{n-1} lying in \mathcal{P}_{n-2} and otherwise general. Pulling back the above exact sequence via A, we get an exact sequence relating the special rank n bundle $V_{A,a}(n)$ with a general rank (n-1)-bundle $V_{A,a}(n-1)$ obtained by viewing A as a section of \mathcal{P}_{n-2} :

$$0 \to V_{A,a}(n) \to V_{A,a}(n-1) \oplus \pi^* L^a \to (\pi^* L^a)|D \to 0,$$

where D is the divisor in Z corresponding to $\mathcal{P}_{n-3} \cap A$. In particular D is pulled back from $B \cong A$. Thus we have realized the special bundle $V_{A,a}(n)$ as an elementary modification of $V_{A,a}(n-1) \oplus \pi^*L^a$ along the divisor D.

To calculate the cohomology class of D, note that the class of \mathcal{P}_{n-3} in \mathcal{P}_{n-2} is given by $\zeta - (n-1)L$ (by applying (4.15) with n replaced by n-1), and so the class of D is given by $p_*([A] \cdot (\zeta - (n-1)L))$. By (5.9),

$$[D] = \alpha - (n-1)L.$$

For M sufficiently ample and A general, D is a smooth divisor, and we get $V_{A,a}(n)$ by an elementary modification of the direct sum $V_{A,a}(n-1) \oplus \pi^*L^a$ along D. Here, of course, the surjection from $V_{A,a}(n-1)$ to $\pi^*L^a|D$ arises because on every fiber f over a point of D, $V_{A,a}(n-1)$ has a trivial quotient \mathcal{O}_f .

Note that, assuming we are the range where the calculations are correct, we obtain an inductive formula for $\operatorname{ch} V_{A,a}(n)$:

$$\operatorname{ch} V_{A,a}(n) = \operatorname{ch} V_{A,a}(n-1) + \operatorname{ch}(L^a) - \operatorname{ch}(L^a|D).$$

Now from the exact sequence

$$0 \to L^a \otimes \mathcal{O}_Z(-D) \to L^a \to L^a|D \to 0$$
,

we see that $\operatorname{ch}(L^a|D) = \operatorname{ch}(L^a) - \operatorname{ch}(L^a \otimes \mathcal{O}_Z(-D))$, and thus using (5.22)

$$\operatorname{ch} V_{A,a}(n) = \operatorname{ch} V_{A,a}(n-1) + e^{(a+n-1)L+\alpha}.$$

Note that this is consistent with the formula given in (5.10) for ch $V_{A,a}$.

This inductive picture must be modified for small values of n. For example, in case dim B=3, a general section in \mathcal{P}_2 degenerates to a rational section of \mathcal{P}_1 plus some exceptional fibers, and there is a further problem in the passage from \mathcal{P}_1 to $\mathcal{P}_0 = \mathbf{o}$. However, we will not discuss these matters further.

5.8. Subsheaves of V and reducible spectral covers.

Proposition 5.23. Let V be a rank n bundle on Z whose restriction to every fiber is regular and semistable with trivial determinant. Then the spectral cover $C = C_A$ associated to V is reduced and irreducible if and only if there is no subsheaf $V' \subset V$ whose restriction to the generic fiber is a semistable bundle of degree zero and rank r with 0 < r < n, if and only if there is no quotient sheaf V'' of V which is torsion free and whose restriction to the generic fiber is a semistable bundle of degree zero and rank r with 0 < r < n.

Proof. Clearly, V has a subsheaf V' as in the statement of the proposition if and only if it has a quotient sheaf V'' as described above.

If C is not reduced and irreducible, then there is a proper closed subvariety $C' \subset C$ which maps surjectively onto B and is finite of degree r, 0 < r < n over B. We may assume that C' is reduced. Let $T' = T \times_B C'$ be the corresponding subscheme of $T = T_A$. The surjection $\mathcal{O}_T \to \mathcal{O}_{T'}$ and the fact that $\nu = \nu_A$ is finite leads to a surjection

$$V = (\nu \times \mathrm{Id})_* \left[\mathcal{O}_T(\Delta - \Sigma_A) \otimes \rho^* N \right] \twoheadrightarrow (\nu \times \mathrm{Id})_* \left[\mathcal{O}_{T'}(\Delta - \Sigma_A) \otimes \rho^* N \right] = V''.$$

By construction, V'' is a torsion free sheaf on Z of rank r with 0 < r < n. Restrict to a generic smooth fiber $\pi^{-1}(b) = E_b$ of π such that the fiber of the projection $C' \to B$ has r distinct points $e_1, \ldots, e_r \in E_b$ over b. By Lemma 5.6, the restriction of V'' to E_b is a direct sum of the r line bundles $\mathcal{O}_{E_b}(e_i - p_0)$, and in particular it is semistable (and in fact regular).

Conversely, suppose that there is an exact sequence

$$0 \to V' \to V \to V'' \to 0$$
.

where both V' and V'' are nonzero torsion free sheaves whose restrictions to a generic fiber are semistable. Let r' be the rank of V' and r'' be the rank of V''. After restricting to a nonempty Zariski open subset of Z, we may assume that V' and V'' are locally free. Consider now the commutative diagram

By definition, C is the Cartier divisor which is the scheme of zeroes of $\det \Psi$. On the other hand, we clearly have $\det \Psi = \det \Psi' \cdot \det \Psi''$. If C' is the scheme of zeroes of $\det \Psi'$, and C'' is the scheme of zeroes of $\det \Psi''$, then C = C' + C'' on a nonempty Zariski open subset of Z. Furthermore, C' maps to B with degree F' and F' maps to F' with degree F', so that neither of F' is trivial. It follows that the restriction of F' to a nonempty Zariski open subset of F' is either nonreduced or reducible, and so the same is true for F' as well. F'

Finally, let us remark that if V is merely assumed to be regular and semistable on a generic fiber, so that A(V) is just a rational section, the above proof still goes through.

6. Bundles which are not regular and semistable on every fiber.

Let $\pi: Z \to B$ be an elliptic fibration with dim B = d, and let $E_b = \pi^{-1}(b)$. In this section, we consider some examples of bundles V, such that $\det V$ has trivial restriction to each fiber, which fail to be regular or semistable on every fiber E_b . From the general principles mentioned in the introduction, it is reasonable to consider only those bundles whose restriction to the generic fiber is semistable. We shall further assume here that the restriction to the generic fiber is regular (this will exclude, for example, the tangent bundle of an elliptic fibration whose base B has dimension at least two). Thus, we shall consider bundles V such that, for a nonempty proper closed subset Y of B and for all $y \in Y$, either $V|E_y$ is unstable or it is semistable but not regular. There is an important difference between the case dim Y = d - 1 and dim Y < d - 1. In the first case, V is not determined by its restriction to $\pi^{-1}(B-Y)$ and can be obtained via elementary modifications from a "better" bundle (or reflexive sheaf). In this case, there is a lot of freedom in creating such V where $V|E_y$ is unstable along a hypersurface. By contrast, it is more difficult to arrange that $V|E_y$ is semistable but not regular along a hypersurface. If $\dim Y < d-1$, then, since V is a vector bundle, it is determined by its restriction to $\pi^{-1}(B-Y)$ and the behavior of V is much more tightly controlled by the rational section A(V) of \mathcal{P}_{n-1} . Here the case where $V|E_y$ is unstable for $y \in Y$ (as well as the case where V is reflexive but not locally free) corresponds to the case where A(V) is just a quasisection, i.e. where the projection $A(V) \to B$ has degree one but is not an isomorphism. The case where $V|E_y$ is semistable but not regular for $y \in Y$ corresponds to the case where there are singularities in the spectral cover C_A , and V is obtained by twisting by a line bundle on $C_A|B-Y$ which does not extend to a line bundle on B. As will be clear from the examples, a wide variety of behavior is possible, and we shall not try to give an exhaustive discussion of all that can occur.

6.1. Codimension one phenomena and elementary modifications.

First we shall discuss the phenomena which occur in codimension one, and which amount to generalized elementary modifications. As will be clear, when we make the most general elementary modifications, we lose control in codimension two on B. Thus for example many of the constructions lead to reflexive sheaves which are not locally free. For this reason, we shall concentrate to a certain extent on the case $\dim B = 1$, which will suffice for the generic behavior in codimension one when $\dim B$ is arbitrary.

The first very general lemma says that, locally, every possible bundle with a given restriction to the generic fiber arises as an elementary modification.

Lemma 6.1. Let V be a vector bundle on Z whose restriction to every fiber E_b is semistable and whose restriction to the generic fiber is regular. Suppose that A(V) = A is the section of \mathcal{P}_{n-1} corresponding to V. Let

$$Y = \{ b \in B : V | E_b \text{ is not regular } \}.$$

Then Y is a Zariski closed subset of B. For every $y \in Y$, there exists a Zariski neighborhood Ω of y in B and a morphism $\varphi \colon V_{A,0}|\pi^{-1}(\Omega) \to V|\pi^{-1}(\Omega)$ which is an isomorphism over a nonempty Zariski open subset of Ω . Moreover, we can choose a φ which extends to a homomorphism $V_{A,0} \otimes \pi^* M^{-1} \to V$, where M is a sufficiently ample line bundle on B.

More generally, suppose that V is merely assumed to have regular semistable restriction to the generic fiber, so that $V|E_b$ may be unstable for some fibers. Then there exists a closed subset X of B of codimension at least two such that the section A(V) extends over B-X and, with Y as above, for every $y \in Y-X$, there exists a Zariski neighborhood Ω of Y in Y and a morphism Y is an isomorphism over a nonempty Zariski open subset of Y is Y in Y which is an isomorphism to a homomorphism Y is a sufficiently ample line bundle on Y.

Proof. Let us first consider the case where the restriction of V to every fiber is semistable. In this case the section A = A(V) is defined over all of B. Consider the sheaf $\pi_*Hom(V_{A,0},V)$. On B-Y, this sheaf is locally free of rank n. On a sufficiently small open set Ω , we can thus find a section φ of $\pi_*Hom(V_{A,0},V)|\Omega$ which restricts to an isomorphism on a general fiber. Since this is an open condition, the set of points $b \in \Omega$ such that φ fails to be an isomorphism on E_b is a proper Zariski closed subset of Ω , as claimed. Finally, if M is sufficiently ample, then $\pi_*Hom(V_{A,0},V)\otimes M$ is generated by its global sections. Choosing such a section which restricts to an isomorphism from $V_{A,0}\otimes\pi^*M^{-1}|E_b$ to $V|E_b$ for a fiber E_b defines a map φ which extends to a homomorphism $V_{A,0}\otimes\pi^*M^{-1}\to V$, as claimed.

In case V has unstable restriction to some fibers, the above proof goes through as long as we are able to define the section A(V). Now the rational section of \mathcal{P}_{n-1} defined by V extends to a closed irreducible subvariety of \mathcal{P}_{n-1} , which we shall also denote by A(V) = A. The morphism $p|A: A \to B$ is birational, and thus over the complement of a codimension two set X in B it is an isomorphism. Thus A is a well-defined section over B - X, and so defines a bundle $V_{A,0}$ over $\pi^{-1}(B - X)$. We may then apply the first part of the proof. \square

Let V be a vector bundle on Z whose restriction to the generic fiber E_b is semistable. Let

$$Y = \{ b \in B : V | E_b \text{ is not semistable } \}.$$

Then Y is a Zariski closed subset of B. Suppose that $W = \pi^{-1}(Y) \subset Z$. We can restrict V to the elliptic fibration $W \to Y$. For simplicity, we shall assume that W is irreducible (otherwise we would need to work one irreducible component at a time). By general theory, there exists a torsion free sheaf S over W and a surjection $V|W \to S$, such that at a generic point w of W, the map $V|E_w \to S|E_w$ is the maximal destabilizing quotient of $V|E_w$. Let $i\colon W \to Z$ be the inclusion and let V' be the kernel of the surjection $V \to i_*S$. If W is a hypersurface in Z, i.e. if Y is a hypersurface in B, then V' is a reflexive sheaf. However, if W has codimension greater than one, V' fails to be reflexive, and in fact $(V')^{\vee\vee} = V$.

For example, if dim B=1, W is a finite set of points. Choosing one such point w, we have that $V|E_w$ is unstable. Let Q be the maximal destabilizing quotient sheaf for $V|E_w$, and suppose that deg Q=e<0. Then V' fits into an exact sequence

$$0 \to V' \to V \to i_*Q \to 0$$
,

where i is the inclusion of the fiber E_w in Z. Such elementary modifications of V are allowable in the terminology of [4], [5]. As opposed to the general construction of (6.1), allowable elementary modifications are canonical, subject to a choice of an irreducible component of W. For the above allowable elementary modification over

an elliptic surface, we have

$$c_2(V') = c_2(V) + e < c_2(V).$$

Thus an allowable elementary modification always decreases c_2 .

Lemma 6.2. A sequence of allowable elementary modifications terminates. The end result is a torsion free reflexive sheaf V' such that the set

$$\{b \in B : V' | E_b \text{ is not semistable } \}$$

has codimension at least two.

Proof. We shall just write out the proof in the case dim B=1. In this case, by (6.1), we can fix a bundle $V_0=V_{A,0}\otimes\pi^*M^{-1}$ for some section A, together with a morphism $\varphi\colon V_0\to V$ which is an isomorphism over a general fiber. Thus det φ defines an effective Cartier divisor, not necessarily reduced, supported on a union of fibers of π . Denote this divisor by D. Clearly D is the pullback of a divisor \mathbf{d} on B, and thus has a well-defined length ℓ , namely the degree of \mathbf{d} . We claim that every sequence of allowable elementary modifications has length at most ℓ . This is clearly true if $\ell=0$, since then $V_0\to V$ is an isomorphism and every fiber of V is already semistable. Since a sequence of allowable elementary modifications will stop only when the restriction of V to every fiber is semistable, we will get the desired conclusion.

Let V' be an allowable elementary modification of V at the fiber E_w . We claim that φ factors through the map $V' \to V$. In this case, it follows that E_w is in the support of D. Thus, if $\varphi' \colon V_0 \to V'$ is the induced map, then $(\det \varphi') = D - E_w$, which has length $\ell-1$, and we will be done by induction on the length ℓ . It suffices to prove that the induced map $V_0 \to i_*Q$ is zero in the above notation. Equivalently, we must show that the induced map $V_0|E_w \to Q$ is zero. But $V_0|E_w$ is semistable and $\deg Q < 0$, and so we are done. \square

As a corollary, we have the following Bogomolov type inequality:

Corollary 6.3. Let V be a vector bundle on Z such that the restriction of V to a generic fiber E_b is regular and semistable. Suppose that $\dim B = d$. Then, for every ample divisor H on B, $c_2(V) \cdot \pi^* H^{d-1} \geq 0$. Moreover, equality holds if and only if V is semistable in codimension one and the line bundle M corresponding to the rational section A(V) is a torsion line bundle. Finally, M is a torsion line bundle if and only if either the rational section $A(V) = \mathbf{o}$ or L is a torsion line bundle and M is a power of L.

Proof. We may assume that H is very ample. By choosing a general curve which is a complete intersection of d-1 divisors linearly equivalent to H, we can further assume that dim B=1, and must show that $c_2(V) \geq 0$. Since an allowable elementary modification strictly decreases c_2 , we can further assume that the restriction of V to every fiber is semistable. Choose a nonzero map $V_0 \to V$, where V_0 is regular semistable on every fiber. Defining Q by the exact sequence

$$0 \to V_0 \to V \to Q \to 0,$$

Q is a torsion sheaf supported on some (possibly nonreduced) fibers whose restriction to a $b \in B$ has a filtration by degree zero sheaves on E_b . It then follows

that $c_2(V)=c_2(V_0)$. Now if M is the line bundle corresponding to the section A(V) of \mathcal{P}_{n-1} , then by (5.15) $c_2(V_0)=\deg M$. On the other hand, at least one of $M, M\otimes L^{-2},\ldots,M\otimes L^{-n}$ has a nonzero section. Thus, for some $i=0,2,\ldots,n$, $\deg M\geq i\deg L$. Now $\deg L\geq 0$, and $\deg L=0$ only if L is a torsion line bundle. Thus, $\deg M\geq 0$, and $\deg M=0$ only if i=0, in which case M is trivial, or L is torsion and there is a nowhere vanishing section of $M\otimes L^{-i}$. In all cases M is a torsion line bundle and we have proved the statements of the lemma. \square

Remark. (1) If $c_2(V) \cdot \pi^* H^{d-1} = 0$ above, in other words we have equality, it follows that the rational section A(V) is actually a section.

(2) If A is a rational section and $A \neq \mathbf{o}$, we get better inequalities along the lines of

$$c_2(V) \cdot \pi^* H^{d-1} \ge 2L \cdot H^{d-1}$$

since we must have nonzero sections of $M \otimes L^{-i}$, i = 0, 2, ..., n for at least two values of i. If A is a section, then except for a small number of exceptional cases we will actually have $c_2(V) \cdot \pi^* H^{d-1} \geq (d+1)L \cdot H^{d-1}$.

The process of taking allowable elementary modifications is in a certain sense reversible: we can begin with a bundle V_0 such that the restriction of V_0 to every fiber is semistable and introduce instability by making elementary modifications. Let us first consider the case where dim B=1. At the first stage, fixing a fiber E_b and a stable sheaf Q on E_b of positive degree, we seek a surjection $V_0|E_b \to Q$. To analyze when such surjections exist is beyond the scope of this paper. However, in case $V_0|E_b$ is regular and $Q=W_k$, then we have seen in Section 3 that such a surjection always exists; indeed, the set of all surjections is an open subset in $\operatorname{Hom}(V,W_k)$ which has dimension n. Note however that while allowable elementary modifications are canonical, their inverses are not. To be able to continue to make elementary modifications along the same fiber, we would also have to analyze when there exist surjections from $V|E_b$ to Q, where V is a rank n bundle on E_b of degree zero, Q is a torsion free sheaf of rank r < n on E_b , and $\mu(Q)$ is larger than the maximum of $\mu(\mathcal{S})$ as \mathcal{S} ranges over all proper torsion free subsheaves of $V|E_b$.

In case dim B > 1, further complications can ensue in codimension two. For example, suppose that V_0 has regular semistable restriction to every fiber of π . Let D be a divisor in B and let $W = \pi^{-1}(D)$, with $\pi' = \pi | W$. Even though we can find a surjection $V|E_b \to W_k$ for every $b \in D$, we can only find a global surjection $V|W \to \mathcal{W}_k \otimes (\pi')^*N$, for some line bundle N on D, under special circumstances. We can find a nonzero such map in general, but it will vanish in general in codimension two, leading to a reflexive but not locally free sheaf.

We turn next to the issue of bundles which are semistable on every fiber, but which are not regular in codimension one. It turns out that we do not have the freedom that we did before in introducing instability on a fiber; there is a condition on the spectral cover in order to be able to make a bundle not be regular. (See [3], [6] for the rank two case.) We shall just state the result in the case where $\dim B = 1$. The result is that, if the spectral cover is smooth, it is not possible to create a non-regular but semistable bundle over any fiber.

Proposition 6.4. Let dim B=1 and let V be a vector bundle over Z whose restriction to every fiber is semistable and whose restriction to the generic fiber is regular. Let A=A(V) be the corresponding section and $C=C_A$ be the spectral cover. If $b \in B$ and C is smooth at all points lying over $b \in B$, then $V|E_b$ is regular.

Proof. Using Lemma 6.1, write V as a generalized elementary modification

$$0 \to V_0 \to V \to Q \to 0$$
,

where V_0 is regular and semistable on every fiber, and Q is a torsion sheaf supported on fibers. Looking just at the part of Q which is supported on E_b , this sheaf (as a sheaf on Z) has a filtration whose successive quotients are direct images of torsion free rank one sheaves of degree zero on E_b . By induction on the length of Q, as in the proof of Lemma 6.2, it will suffice to show the following: if V_0 has regular semistable restriction to E_b , if $i: E_b \to Z$ is the inclusion, and if C is smooth over all points lying over b, then for every exact sequence

$$0 \to V_0 \to V \to i_* \lambda \to 0$$
,

where λ is a rank one torsion free sheaf on E_b of degree zero, $V|E_b$ is again regular. After shrinking B, we can assume that V_0 is regular everywhere and that $V_0 \to V$ is an isomorphism away from b.

It will suffice to show that $V^{\vee}|E_b$ is regular. There is the dual exact sequence

$$0 \to V^{\vee} \to V_0^{\vee} \to i_* \lambda^{-1} \to 0.$$

By assumption, $\dim \operatorname{Hom}(V_0^{\vee}, i_*\lambda^{-1}) = \dim \operatorname{Hom}(V_0^{\vee}|E_b, \lambda^{-1}) = 1$. Thus there is a unique possible elementary modification. On the other hand, there is a unique point $b' \in C$ lying above b and corresponding to the surjection $V_0^{\vee}|E_b \to \lambda^{-1}$. Since by assumption b' is a smooth point of C, it is a Cartier divisor, and the ideal sheaf of b' is the line bundle $\mathcal{O}_C(-b')$. Now we know that V_0^{\vee} is of the form $(\nu \times \operatorname{Id})_* [\mathcal{O}_C(\Delta - \Sigma) \otimes \rho^* N] = V_{A,0}[N]$ for a line bundle N on C. Let i' be the inclusion of the fiber over b' (which is just E_b) into T. Applying $(\nu \times \operatorname{Id})_*$ to the exact sequence

$$0 \to \mathcal{O}_C(\Delta - \Sigma) \otimes \rho^*(N \otimes \mathcal{O}_C(-b')) \to \mathcal{O}_C(\Delta - \Sigma) \otimes \rho^*N \to (i')_*\lambda^{-1} \to 0,$$

we get an exact sequence

$$0 \to V_{A,0}[N \otimes \mathcal{O}_C(-b')] \to V_0^{\vee} \to i_* \lambda^{-1} \to 0.$$

By the uniqueness of the map $V_0^{\vee} \to i_* \lambda^{-1}$, it then follows that

$$V^{\vee} = V_{A,0}[N \otimes \mathcal{O}_C(-b')]$$

and in particular it is regular. Thus the same is true for V. \square

Remark. (1) Of course, Proposition 6.4 gives conditions in case dim B > 1 as well. (2) The condition that C is singular at the point corresponding to b and $V_0 \to \lambda$ is not a sufficient condition for there to exist an elementary modification such that the result is not regular over E_b .

6.2. The tangent bundle of an elliptic surface.

As an example of the preceding discussion, we analyze the tangent bundle of an elliptic surface. Let $\pi\colon Z\to B$ be an elliptic surface over the smooth curve B, with g(B)=g. We suppose that Z is generic in the following sense: Z is smooth, the

line bundle L has positive degree d, so that the Euler characteristic of Z is 12d, all the singular fibers of π are nodal curves (and thus there are 12d such curves), and the j-function $B \to \mathbb{P}^1$ has generic branching behavior in the sense of [6, p. 63]. The assumption of generic branching behavior implies that the Kodaira-Spencer map associated to the deformation Z of the fibers of π is an isomorphism at the curves with $j=0,1728,\infty$ and that the Kodaira-Spencer map vanishes simply where it fails to be an isomorphism. By the Riemann-Hurwitz formula, if b is equal to the number of points where the Kodaira-Spencer map is not an isomorphism, then b=10d+2g-2 [6, p. 68].

Quite generally, we have the following lemma:

Lemma 6.5. Let $\pi: Z \to B$ be a smooth elliptic surface, and suppose that V is a a vector bundle on Z whose restriction to a general fiber is I_2 . Then there is an exact sequence

$$0 \to \pi^* M_1 \to V \to \pi^* M_2 \otimes I_X \to 0,$$

where M_1 and M_2 are line bundles on B and X is a zero-dimensional local complete intersection subscheme of Z. Here $\det V = \pi^*(M_1 \otimes M_2)$ and $c_2(V) = \ell(X)$.

Proof. By assumption, $\pi_*V = M_1$ is a rank one torsion free sheaf on B, and thus it is a line bundle. We have the natural map $\psi \colon \pi^*\pi_*V = \pi^*M_1 \to V$. If this map were to vanish along a divisor, the divisor would have to be a union of fibers. But this is impossible since the induced map

$$\pi_*\pi^*M_1 = M_1 \to \pi_*V = M_1$$

is the identity. Thus ψ only vanishes in codimension two. The remaining statements are clear. \square

Of course, in the case of the tangent bundle, we can identify this sequence precisely as follows:

Lemma 6.6. With $\pi\colon Z\to B$ a smooth elliptic surface as before, there is an exact sequence

$$0 \to T_{Z/B} \to T_Z \to \pi^* T_B \otimes I_X \to 0.$$

Here $T_{Z/B} = L^{-1}$ is the sheaf of relative tangent vectors and I_X is the ideal sheaf of the 12d singular points of the singular fibers.

Proof. Begin with the natural map $T_Z \to \pi^* T_B$. This map is surjective except at a singular point of a singular fiber, where it has the local form

$$h_1 \frac{\partial}{\partial z_1} + h_2 \frac{\partial}{\partial z_2} \mapsto (z_2 h_1 + z_1 h_2) \frac{\partial}{\partial t}.$$

Thus the image of T_Z in π^*T_B is exactly $\pi^*T_B \otimes I_X$. The kernel of the map $T_Z \to \pi^*T_B$ is by definition $T_{Z/B}$, which can be checked directly to be a line bundle in local coordinates. Moreover, $T_{Z/B}$ is dual to $K_{Z/B} = L$, and thus $T_{Z/B} = L^{-1}$. \square

Corollary 6.7. If E is a singular fiber of π , there is an exact sequence

$$0 \to n_* \mathcal{O}_{\tilde{E}} \to T_Z | E \to \mathfrak{m}_x \to 0$$
,

where $n \colon \tilde{E} \to E$ is the normalization and x is the singular point of E. In particular $T_Z|E$ is unstable. If E is a smooth fiber where the Kodaira-Spencer map is zero, then $T_Z|E \cong \mathcal{O}_E \oplus \mathcal{O}_E$. For all other fibers E, $T_Z|E \cong I_2$.

Proof. If E is a singular fiber, then by restriction we have a surjection $T_Z|E \to \mathfrak{m}_x$. The kernel must be a non-locally free rank one torsion free sheaf of degree one, and thus it is isomorphic to $n_*\mathcal{O}_{\tilde{E}}$. For a smooth fiber E, restricting the tangent bundle sequence to E gives an exact sequence

$$0 \to \mathcal{O}_E \to T_Z | E \to \mathcal{O}_E \to 0$$
,

such that the coboundary map

$$\theta \colon H^0(\mathcal{O}_E) = H^0(N_{E/Z}) \to H^1(\mathcal{O}_E) = H^1(T_E)$$

is the Kodaira-Spencer map. This map is nonzero, then, if and only if $T_Z|E \cong I_2$, and it is zero if and only if $T_Z|E \cong \mathcal{O}_E \oplus \mathcal{O}_E$. \square

To go from T_Z to one of our standard bundles, begin by making the allowable elementary modifications along the singular fibers, by taking V' to be the kernel of the induced map $T_Z \to \bigoplus_x (i_x)_* \mathfrak{m}_x$. Here the sum is over the the singular points, i.e. the $x \in X$, and i_x is the inclusion of the singular fiber containing x in Z. Note that $c_2(V') = 0$, so that no further allowable elementary modifications are possible, and the restriction of V' to every fiber is semistable. Let F be the union of the singular fibers. Thus as a divisor on Z, $F = \pi^* \mathbf{f}$, where \mathbf{f} is a divisor on B of degree 12d which is a section of L^{12} . If I_F is the ideal of F, then there is an inclusion $I_F \subset I_X$ and thus an inclusion $\pi^* T_B \otimes I_F \subset \pi^* T_B \otimes I_X$. Clearly V' is the result of pulling back the extension T_Z of $\pi^* T_B \otimes I_X$ by L^{-1} via the inclusion $\pi^* T_B \otimes I_F \subset \pi^* T_B \otimes I_X$. Thus there is an exact sequence

$$0 \to \pi^* L^{-1} \to V' \to \pi^* (T_B \otimes \mathcal{O}_B(-\mathbf{f})) \to 0.$$

Taking the map

$$\operatorname{Ext}^{1}(\pi^{*}T_{B} \otimes I_{F}, L^{-1}) = H^{1}(\pi^{*}T_{B}^{-1} \otimes \mathcal{O}_{Z}(F) \otimes L^{-1})$$

$$\to H^{0}(R^{1}\pi_{*}(\pi^{*}T_{B}^{-1} \otimes \mathcal{O}_{Z}(F) \otimes L^{-1})) = H^{0}(B; K_{B} \otimes L^{-2} \otimes \mathcal{O}_{B}(\mathbf{f})),$$

and using the fact that $\mathcal{O}_B(\mathbf{f}) \cong L^{12}$, we see that the extension restricts to the trivial extension over a section of $K_B \otimes L^{10}$, and thus at 10d + 2g - 2 points, confirming the numerology above. Note that the passage from T_Z to V' was canonical.

Next we want to go from V' to a bundle V_0 which is regular semistable on every fiber, and thus is isomorphic to I_2 on every fiber. We claim that a further elementary modification of V' will give us back a bundle which restricts to I_2 on every fiber. Quite generally, suppose that V' is given as an extension

$$0 \to \pi^* L_1 \to V' \to \pi^* L_2 \to 0$$
,

where the image of the extension class in $H^0(B; L_2^{-1} \otimes L_1 \otimes L^{-1})$ vanishes simply at k points x_1, \ldots, x_k . After twisting V' by the line bundle $\pi^* L_2^{-1}$, we may assume

that L_2 is trivial. Thus in the case where we began with the tangent bundle, and after relabeling V', we wind up with a bundle V' which fits into an exact sequence

$$0 \to \pi^* L_1 \to V' \to \mathcal{O}_Z \to 0$$
,

where $L_1 = K_B \otimes L^{11}$. The extension class for V' defines an element of $H^1(Z; \pi^*L_1)$. Via the Leray spectral sequence, there is a homomorphism from $H^1(Z; \pi^*L_1)$ to $H^0(B; R^1\pi_*\mathcal{O}_Z \otimes L_1) = H^0(B; L_1 \otimes L^{-1})$. Thus there is a section of $L_1 \otimes L^{-1}$, well-defined up to a nonzero scalar, and it defines a homomorphism $\pi^*L \to \pi^*L_1$ and thus a homomorphism $H^1(\pi^*L) \to H^1(\pi^*L_1)$. Consider the commutative diagram

The induced map $H^0(\mathcal{O}_B) \to H^0(L_1 \otimes L^{-1})$ is just the given section of $L_1 \otimes L^{-1}$. We have seen in §5.6 that there is a class $\xi_0 \in H^1(\pi^*L)$ mapping to $1 \in H^0(\mathcal{O}_B)$. Since the map $H^1(B;L) \to H^1(B;L_1)$ is surjective, we can modify ξ_0 by an element in $H^1(B;L)$ so that its image in $H^1(\pi^*L_1)$ is the same as the extension class for V', and the resulting element ξ of $H^1(\pi^*L)$ is unique up to adding an element of the kernel of the map $H^1(B;L) \to H^1(B;L_1)$. Let V_0 be the extension of \mathcal{O}_Z by π^*L corresponding to ξ . Thus V_0 is some bundle of the form $V_{\mathbf{o},0}[N]$. There is an induced map of extensions

$$0 \longrightarrow \pi^*L_1 \longrightarrow V' \longrightarrow \mathcal{O}_Z \longrightarrow 0$$

$$\downarrow \qquad \qquad \downarrow$$

$$0 \longrightarrow \pi^*L_1 \longrightarrow V_0 \otimes \pi^*(L_1 \otimes L^{-1}) \longrightarrow \pi^*(L_1 \otimes L^{-1}) \longrightarrow 0.$$

Thus there is an exact sequence

$$0 \to V' \to V_0 \otimes \pi^*(L_1 \otimes L^{-1}) \to \bigoplus_i \mathcal{O}_{E_{x_i}} \to 0,$$

and we have realized the tangent bundle as obtained from V_0 by elementary modification and twisting.

Of course, we can construct many other bundles this way, starting from V_0 , not just the tangent bundle. Begin with V_0 which has restriction I_2 to every fiber. Normalize so that there is an exact sequence

$$0 \to \pi^* L \to V_0 \to \mathcal{O}_Z \to 0$$

as in §5.6. Here $L = \pi_* V_0$. The bundle $\pi^* L$ is destabilizing. Choose r fibers E_{x_i} lying over $x_i \in B$, where we make elementary modifications by taking the unique quotient $\mathcal{O}_{E_{x_i}}$ of $V_0|E_{x_i}$. The result is a new bundle V'. The subbundle $\pi^* L$ still maps into V', in fact we continue to have $L = \pi_* V'$, and the quotient is $\pi^* \mathcal{O}_B(-\mathbf{r})$, where \mathbf{r} is the divisor $\sum_i x_i$ of degree r on B. The bundle V' is the pullback of the extension V_0 by the morphism $\pi^* \mathcal{O}_B(-\mathbf{r}) \to \pi^* \mathcal{O}_B$. In particular, by reversing the arguments above, we see that the restriction of the extension to E_{x_i} becomes split.

Thus $V'|E_{x_i} \cong \mathcal{O}_{E_{x_i}} \oplus \mathcal{O}_{E_{x_i}}$ and the restriction of V' to all other fibers is I_2 . Note that π^*L continues to destabilize V'.

Choose s fibers lying over points $y_j \in B$ distinct from the x_i , and let s be the divisor $\sum_j y_j$. Choose rank one torsion free sheaves μ_j on E_{y_j} of degree $d_j > 0$ and surjections from I_2 to μ_j . (Such surjections always exist.) Take the bundle V defined to be the kernel of the given surjection $V' \to \bigoplus_j \mu_j$. Now det V = (d-r-s)f and $c_2(V) = \sum_j \deg \mu_j$. The bundle π^*L no longer maps into V, since the composed morphism $\pi^*L|E_{y_j} \to \mu_j$ is nontrivial for every j. In fact, $\pi^*(L \otimes \mathcal{O}_B(-\mathbf{s}))$ maps to V, and $\pi_*V = L \otimes \mathcal{O}_B(-\mathbf{s})$. Note that this subbundle fails to be destabilizing exactly when 2(d-s) < d-r-s, or equivalently d+r < s. In this case, for a suitable ample divisor H as defined in [6], V is H-stable.

6.3. Quasisections and unstable fibers.

For the rest of this section, we shall assume that V is regular and semistable in codimension one and consider the phenomena that arise in higher codimension. Over a Zariski open subset of B, we have defined A(V), and it extends to a subvariety of \mathcal{P}_{n-1} mapping birationally to B, in other words to a quasisection of \mathcal{P}_{n-1} . Of course, if the restriction of V to every fiber is semistable, then A(V) is a section.

Question. Suppose that V is a vector bundle over Z and that there exists a closed subset Y of B of codimension at least two such that, for all $b \notin Y$, $V|E_b$ is semistable. Suppose further that A(V) is actually a section. Does it then follow that $V|E_b$ is semistable for all $b \in B$?

For the remainder of this subsection, we shall assume that A(V) is an honest quasisection, in other words that the morphism $A(V) \to B$ is not an isomorphism, and see what kind of behavior is forced on V. For example, if $n \leq \dim B$, then with a few trivial exceptions there are no honest sections of \mathcal{P}_{n-1} and we are forced to consider quasisections. We will analyze the case where $\dim B = 2$ and see that two kinds of behavior are possible: either V has unstable restriction to some fibers or V fails to be locally free at finitely many points Z. For example, suppose that $1 \leq d \leq n-1$ and consider $V_{A,1-d}$, defined over the complement of a set of codimension 2 in B. Then as we have seen in §5.2, $V_{A,1-d}$ is given as an extension of $W_{n-d} \otimes \pi^*(M^{-1} \otimes L)$ by \mathcal{W}_d^{\vee} . This extension extends over B, but it induces the split extension of W_{n-d} by W_d^{\vee} wherever the section of $\mathcal{V}_n \otimes M$ vanishes.

Assume that dim B=2 and let s be a section of $\mathcal{V}_n\otimes M$ which vanishes simply at finitely many points, but which is otherwise generic. The corresponding quasi-section A=A(V) will contain a line inside the full fiber of \mathcal{P}_{n-1} at these points, which is a \mathbb{P}^{n-1} , and will simply be the blowup of B over the corresponding points. Pulling back the \mathbb{P}^{n-1} -bundle \mathcal{P}_{n-1} by the morphism $A\to B$, we get an honest section over A. Let $\tilde{Z}=Z\times_BA$. Clearly \tilde{Z} is the blowup of Z along the fibers over the exceptional points of B, and the exceptional divisors of $\tilde{Z}\to Z$ are of the form $\mathbb{P}^1\times E_b$, where the \mathbb{P}^1 is linearly embedded in the \mathbb{P}^{n-1} fiber. The section A of $\mathcal{P}_{n-1}\times_BA$ defines a vector bundle $\tilde{\mathcal{U}}_a\to \tilde{Z}$ for every $a\in\mathbb{Z}$. To decide what happens over the exceptional points of B, we need to understand the restriction of $\tilde{\mathcal{U}}_a$ to the exceptional fibers $\mathbb{P}^1\times E_b$. Of course, this is just the restriction of the universal bundle U_a defined over $\mathbb{P}^{n-1}\times E_b$ to the subvariety $\mathbb{P}^1\times E_b$. Thus we need to know the restriction of U_a to $\mathbb{P}^1\times \{e\}$. We shall be able to find this restriction in case $-(n-2)\leq a\leq 1$, but for arbitrary a we shall further need to assume that the \mathbb{P}^1 is a generic line in \mathbb{P}^{n-1} .

Proposition 6.8. Let E be a smooth elliptic curve and let $e \in E$. Suppose that $-(n-2) \le a \le 1$. Then

$$U_a|\mathbb{P}^{n-1} \times \{e\} \cong \left\{ \begin{array}{ll} \mathcal{O}_{\mathbb{P}^{n-1}}^{1-a} \oplus \mathcal{O}_{\mathbb{P}^{n-1}}(-1)^{n-1+a}, & \text{if } a \neq 1 \text{ or } e \neq p_0; \\ \mathcal{O}_{\mathbb{P}^{n-1}} \oplus \Omega_{\mathbb{P}^{n-1}}^1, & \text{if } a = 1 \text{ and } e = p_0, \end{array} \right.$$

where $\Omega^1_{\mathbb{P}^{n-1}}$ is the cotangent bundle of \mathbb{P}^{n-1} .

Proof. Let i_e be the inclusion of \mathbb{P}^{n-1} in $\mathbb{P}^{n-1} \times E$ via the slice $\mathbb{P}^{n-1} \times \{e\}$. Then

$$i_e^*(\nu \times \mathrm{Id})_*(\Delta - G - aF) = \nu_* \mathcal{O}_T(F_e - aF_{p_0}) = \nu_* r^* \mathcal{O}_E(e - ap_0).$$

Set d = 1 - a, then $U_a = \mathbf{U}(d) \otimes \pi_1^* \mathcal{O}_{\mathbb{P}^{n-1}}(-1)$. Thus, for $e \in E$, the restriction of $\pi_2^* W_d^{\vee}$ to $\mathbb{P}^{n-1} \times \{e\}$ is trivial, and similarly for $\pi_2^* W_{n-d}$, and the defining exact sequence

$$0 \to \pi_2^* W_d^{\vee} \to U_a \to \pi_2^* W_{n-d} \otimes \pi_1^* \mathcal{O}_{\mathbb{P}^{n-1}}(-1) \to 0$$

restricts to the exact sequence

$$0 \to \mathcal{O}^d_{\mathbb{P}^{n-1}} \to U_a | \mathbb{P}^{n-1} \times \{e\} \to \mathcal{O}_{\mathbb{P}^{n-1}} (-1)^{n-d} \to 0.$$

Since $\operatorname{Ext}^1(\mathcal{O}_{\mathbb{P}^{n-1}}(-1)^{n-d},\mathcal{O}_{\mathbb{P}^{n-1}}^d)=H^1(\mathcal{O}_{\mathbb{P}^{n-1}}(1))^{d(n-d)}=0$, this extension splits and we see that

$$U_a|\mathbb{P}^{n-1} \times \{e\} \cong \mathcal{O}^d_{\mathbb{P}^{n-1}} \oplus \mathcal{O}_{\mathbb{P}^{n-1}}(-1)^{n-d}$$
$$\cong \mathcal{O}^{1-a}_{\mathbb{P}^{n-1}} \oplus \mathcal{O}_{\mathbb{P}^{n-1}}(-1)^{n+a-1}.$$

Now suppose that a=1. In this case $U_1|\mathbb{P}^{n-1}\times\{e\}=\nu_*r^*\mathcal{O}_E(e-p_0)$, and thus $h^0(U_1|\mathbb{P}^{n-1}\times\{e\})$ is zero if $e\neq p_0$ and one if $e=p_0$. We have the elementary modification

$$0 \to U_1 | \mathbb{P}^{n-1} \times \{e\} \to U_0 | \mathbb{P}^{n-1} \times \{e\} \to \mathcal{O}_H \to 0,$$

where H is a hyperplane in \mathbb{P}^{n-1} . Thus we may write

$$0 \to U_1 | \mathbb{P}^{n-1} \times \{e\} \to \mathcal{O}_{\mathbb{P}^{n-1}} \oplus \mathcal{O}_{\mathbb{P}^{n-1}}(-1)^{n-1} \to \mathcal{O}_H \to 0.$$

Clearly $h^0(U_1|\mathbb{P}^{n-1} \times \{e\}) = 0$ if and only if the induced map $\mathcal{O}_{\mathbb{P}^{n-1}} \to \mathcal{O}_H$ is nonzero, or equivalently onto. In this case, we can choose a summand $\mathcal{O}_{\mathbb{P}^{n-1}}$ of $\mathcal{O}_{\mathbb{P}^{n-1}} \oplus \mathcal{O}_{\mathbb{P}^{n-1}}(-1)^{n-1}$ such that the map $\mathcal{O}_{\mathbb{P}^{n-1}} \oplus \mathcal{O}_{\mathbb{P}^{n-1}}(-1)^{n-1} \to \mathcal{O}_H$ is zero on the factor $\mathcal{O}_{\mathbb{P}^{n-1}}(-1)^{n-1}$ and is the obvious map on the first factor. Thus the kernel is $\mathcal{O}_{\mathbb{P}^{n-1}}(-1)^n$.

In the remaining case, corresponding to $e = p_0$ and a = 1, the map $\mathcal{O}_{\mathbb{P}^{n-1}} \oplus \mathcal{O}_{\mathbb{P}^{n-1}}(-1)^{n-1} \to \mathcal{O}_H$ is zero on the first factor. Now $H \cong \mathbb{P}^{n-2}$, and modulo automorphisms of $\mathcal{O}_{\mathbb{P}^{n-1}}(-1)^{n-1}$ there is a unique surjection $\mathcal{O}_{\mathbb{P}^{n-1}}(-1)^{n-1} \to \mathcal{O}_H$. We must therefore identify the kernel of this surjection with $\Omega^1_{\mathbb{P}^{n-1}}$. Begin with the Euler sequence

$$0 \to \Omega^1_{\mathbb{P}^{n-1}} \to \bigoplus_{i=1}^n \mathcal{O}_{\mathbb{P}^{n-1}}(-1) \to \mathcal{O}_{\mathbb{P}^{n-1}} \to 0.$$

After a change of basis in the direct sum, we can assume that the right hand map restricted to the n^{th} factor vanishes along H. Thus there is an induced surjection

$$\bigoplus_{i=1}^{n-1} \mathcal{O}_{\mathbb{P}^{n-1}}(-1) \to \mathcal{O}_{\mathbb{P}^{n-1}}/\mathcal{O}_{\mathbb{P}^{n-1}}(-1) = \mathcal{O}_H$$

whose kernel is $\Omega^1_{\mathbb{P}^{n-1}}$, as claimed. \square

We remark that, in case dim B is arbitrary, a=1 and A is a quasisection corresponding to a simple blowup of B, then one can show directly from (6.8) that $V_{A,1}$ does not extend to a vector bundle over Z.

When we are not in the range $-(n-2) \le a \le 1$, we do not identify explicitly the bundle $U_a|\mathbb{P}^{n-1} \times \{e\}$, except in case n=2. However, the next result identifies its restriction to a generic line.

Proposition 6.9. Let $\ell \cong \mathbb{P}^1$ be a line in \mathbb{P}^{n-1} , and suppose that ℓ is not contained in any of the one-dimensional family of hyperplanes H_e . Write a = a' + nk, where $-(n-2) \leq a' \leq 1$. Then

$$U_a|\ell \times E \cong (U_{a'} \otimes \pi_1^* \mathcal{O}_{\mathbb{P}^1}(-k))|\mathbb{P}^1 \times E.$$

In particular

$$U_{a}|\ell \times \{e\} \cong \begin{cases} \mathcal{O}_{\mathbb{P}^{1}}(-k)^{1-a'} \oplus \mathcal{O}_{\mathbb{P}^{1}}(-k-1)^{n-1+a'}, & \text{if } a' \neq 1 \text{ or } e \neq p_{0}; \\ \mathcal{O}_{\mathbb{P}^{1}}^{n-2}(-k-1) \oplus \mathcal{O}_{\mathbb{P}^{1}}(-k) \oplus \mathcal{O}_{\mathbb{P}^{1}}(-k-2), & \text{if } a' = 1 \text{ and } e = p_{0}. \end{cases}$$

Proof. Let C be the preimage of ℓ in T. If ℓ is not contained in any of the hyperplanes H_e , then it will meet each H_e in exactly one point. Thus the map $r|C:C\to E$ has degree one, and $F_{p_0}\cdot C=p_0$. We claim that, under the morphism $\nu:E\to \mathbb{P}^1$, $\mathcal{O}_{\mathbb{P}^1}(1)$ pulls back to $\mathcal{O}_E(np_0)=nF_{p_0}|C$. To see this, let $\nu^*\mathcal{O}_{\mathbb{P}^{n-1}}(1)=\zeta\in \operatorname{Pic} T$. Then the class of $\nu^*\ell$ lies in ζ^{n-1} . Now $T=\mathbb{P}\mathcal{E}$ with $c_1(\mathcal{E})=-np_0$. Thus, in $A^{n-1}(T)$,

$$\zeta^{n-1} = r^*(np_0) \cdot \zeta^{n-2}.$$

Hence
$$\zeta | C = r^*(np_0) | C = nF_{p_0} | C$$
.
Write $a = a' + nk$ with $-(n-2) \le a' \le 1$. Then
$$(\nu \times \operatorname{Id})_* \mathcal{O}_{C \times E} (\Delta - G - aF_{p_0}) =$$

$$= (\nu \times \operatorname{Id})_* (\mathcal{O}_{C \times E} (\Delta - G - a'F_{p_0}) \otimes \pi_1^* \mathcal{O}_E (-nkp_0))$$

$$= (\nu \times \operatorname{Id})_* (\mathcal{O}_{C \times E} (\Delta - G - a'F_{p_0}) \otimes (\nu \times \operatorname{Id})^* \mathcal{O}_{\mathbb{P}^1} (-k))$$

$$= (\nu \times \mathrm{Id})_* \mathcal{O}_{C \times E}(\Delta - G - a' F_{p_0}) \otimes \pi_1^* \mathcal{O}_{\mathbb{P}^1}(-k),$$

proving the first claim. The second statement follows from the special case $-(n-2) \le a \le 1$ proved in (6.8), and the well-known fact (which follows from the conormal sequence) that $\Omega^1_{\mathbb{P}^{n-1}}|\mathbb{P}^1 \cong \mathcal{O}_{\mathbb{P}^1}(-1)^{n-2} \oplus \mathcal{O}_{\mathbb{P}^1}(-2)$. \square

Now we can analyze what happens to $V_{A,a}$ when dim B=2 and A is a quasi-section, under a slight genericity condition on A, generalizing the case (for dim B arbitrary) where $-(n-2) \le a \le 0$:

Theorem 6.10. Suppose that dim B = 2. Let A be a quasisection of \mathcal{P}_{n-1} , and suppose that $a \not\equiv 1 \mod n$. Suppose that A is smooth and is the blowup of B at a finite number of points b_1, \ldots, b_r , and that the image of the exceptional \mathbb{P}^1 is a generic line in the fiber \mathbb{P}^{n-1} as in (6.9), in other words it is not contained in one of the hyperplanes H_e . Then the rank n bundle $V_{A,a}$, which is defined on $Z - \bigcup_i E_{b_i}$, extends to a vector bundle over Z, which we continue to denote by $V_{A,a}$. The restriction of $V_{A,a}$ to a fiber E_{b_i} is the unstable bundle $W_d^{\vee} \oplus W_{n-d}$, where a = a' + nk with $-(n-2) \leq a' \leq 1$, and d = 1 - a'.

Proof. By assumption, A is the blowup of B at a finite number of points b_1, \ldots, b_r , where the quasisection A contains a \mathbb{P}^1 lying in the \mathbb{P}^{n-1} -fiber of $p \colon \mathcal{P}_{n-1} \to B$. As we have defined earlier, let $\tilde{Z} = Z \times_B A$, so that \tilde{Z} is a blowup of Z at the fibers E_{b_i} . Let $D_i \cong \mathbb{P}^1 \times E_{b_i}$ be the exceptional divisor of the blowup $q \colon \tilde{Z} \to Z$ over E_{b_i} . There is a section of $\tilde{B} \to A$ corresponding to the inclusion of A in \mathcal{P}_{n-1} , and hence by pulling back \mathcal{U}_a there is a bundle corresponding to A, which we shall denote by \tilde{V} . Using (6.8) and (6.9), the restriction of \tilde{V} to an exceptional divisor $D_i = \mathbb{P}^1 \times E_{b_i}$, which is the same as the restriction of \mathcal{U}_a , namely U_a , fits into an exact sequence

$$0 \to \pi_2^* W_d^{\vee} \otimes \pi_1^* \mathcal{O}_{\mathbb{P}^1}(-k+1) \to \tilde{V}|D_i \to \pi_2^* W_{n-d} \otimes \pi_1^* \mathcal{O}_{\mathbb{P}^1}(-k) \to 0.$$

Make the elementary modification along the divisor D_i corresponding to the surjection $\tilde{V}|D_i \to \pi_2^* W_{n-d} \otimes \pi_1^* \mathcal{O}_{\mathbb{P}^1}(-k)$. The result is a new bundle V' over \tilde{Z} , such that over D_i we have an exact sequence

$$0 \to \pi_2^* W_{n-d} \otimes \pi_1^* \mathcal{O}_{\mathbb{P}^1}(-k+1) \to V'|D_i \to \pi_2^* W_d^{\vee} \otimes \pi_1^* \mathcal{O}_{\mathbb{P}^1}(-k+1) \to 0.$$

Now, since $H^1(E_{b_i}; W_d \otimes W_{n-d}) = H^1(\mathcal{O}_{\mathbb{P}^1}) = 0$, it follows from the Künneth formula that

$$\operatorname{Ext}^{1}(\pi_{2}^{*}W_{d}^{\vee}\otimes\pi_{1}^{*}\mathcal{O}_{\mathbb{P}^{1}}(-k+1),\pi_{2}^{*}W_{n-d}\otimes\pi_{1}^{*}\mathcal{O}_{\mathbb{P}^{1}}(-k+1))=0.$$

Thus $V' \otimes \mathcal{O}_{\tilde{Z}}(-(k+1)D_i)|D_i = \pi_2^*(W_d^{\vee} \oplus W_{n-d})$. It follows by standard blowup results that $q_*V' \otimes \mathcal{O}_{\tilde{Z}}(-(k+1)\sum_i D_i)$ is locally free on Z and its restriction to each fiber E_{b_i} is $W_d^{\vee} \oplus W_{n-d}$. This completes the proof. \square

Finally we must deal with the case $a \equiv 1 \mod n$.

Theorem 6.11. Suppose that dim B = 2. Let A be a quasisection of \mathcal{P}_{n-1} , and suppose that $a \equiv 1 \mod n$. Suppose that A is smooth and is the blowup of B at a finite number of points b_1, \ldots, b_r , and that the image of the exceptional \mathbb{P}^1 is a generic line in the fiber \mathbb{P}^{n-1} as in (6.9), in other words it is not contained in one of the hyperplanes H_e . Then the rank n bundle $V_{A,a}$, which is defined on $Z - \bigcup_i E_{b_i}$, extends to a reflexive non-locally free sheaf on Z, which we continue to denote by $V_{A,a}$. The sheaf $V_{A,a}$ is locally free except at the points $\sigma \cap E_{b_i}$. Near such points, $V_{A,a}$ has the local form

$$R^{n-2} \oplus M$$
.

where $R = \mathbb{C}\{z_1, z_2, z_3\}$, and M is the standard rank two reflexive non-locally free sheaf given by the exact sequence

$$0 \to R \to R^3 \to M \to 0$$
,

where the map $R \to R^3$ is given by $1 \mapsto (z_1, z_2, z_3)$.

Proof. We shall just work near a single fiber $E_b = E_{b_i}$ for some i. Thus let \tilde{Z} be the blowup of Z along E_b , with exceptional divisor $D \cong \mathbb{P}^1 \times E_b$. The basic birational picture to keep in mind is the following: if we blow up the subvariety $\mathbb{P}^1 \times \{p_0\} \subset D$, we get a new exceptional divisor D_1 in $Z_1 = \mathrm{Bl}_{\mathbb{P}^1 \times \{p_0\}} \tilde{Z}$. Here $D_1 \cong \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1))$, and so D_1 is isomorphic to the blowup \mathbb{F}_1 of \mathbb{P}^2 at one point. The proper transform D' of D in Z_1 meets D_1 along the exceptional divisor in D_1 , and can be contracted in Z_1 . The result is a new manifold Z_2 , isomorphic to the blowup of Z at the point $\sigma \cap E_b$, where D_1 blows down to the exceptional divisor P in Z_2 .

The quasisection A defines a section of the pullback of \mathcal{P}_{n-1} to B, and thus a bundle \tilde{V} over \tilde{Z} , which we can then pull back to Z_1 . The next step is to show that, after appropriate elementary modifications, \tilde{V} corresponds to a bundle over Z_2 whose restriction to P is just $(T_P \otimes \mathcal{O}_P(-1)) \oplus \mathcal{O}_P^{n-2}$, where T_P is the tangent bundle to P. Finally, a local lemma shows that every such bundle has a direct image on Z which has the local form $M \oplus R^{n-2}$. Since each of these steps is somewhat involved, we divide the proof into three parts. First we describe the basic geometry of the blowups involved.

Let \tilde{Z} be the blowup of Z along E_b , with exceptional divisor $D \cong \mathbb{P}^1 \times E_b$. Let Z_1 be the blowup of \tilde{Z} along $\mathbb{P}^1 \times \{p_0\} \subset D$, with exceptional divisor D_1 . Let D' be the proper transform of D in Z_1 . The divisor $D_1 = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1))$ is isomorphic to \mathbb{F}_1 . Let $j: D_1 \to Z_1$ be the inclusion and $q: D_1 \to \mathbb{P}^1$ be the morphism induced by projection from a point. Let $\ell = \mathbb{P}^1 \times \{p_0\} = D' \cap D_1$, so that ℓ is the exceptional divisor in D_1 viewed as the blowup of \mathbb{P}^2 . Finally we let $s: D_1 \to \mathbb{P}^2$ be the blowup map. On a fiber $\mathbb{P}^1 \times \{e\}$ with $e \neq p_0$, $\tilde{V} \otimes \mathcal{O}_{\tilde{Z}}(-D')$ restricts to $\mathcal{O}_{\mathbb{P}^1}^n$, whereas it restricts on $\mathbb{P}^1 \times \{p_0\}$ to $\mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}^{n-2}$. Thus, if V_0 is the pullback to Z_1 of $\tilde{V} \otimes \mathcal{O}_{\tilde{Z}}(-D')$, then V_0 restricts on D_1 to $q^* [\mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}^{n-2}]$.

Claim 1. Let V_0 be the pullback to Z_1 of $\tilde{V} \otimes \mathcal{O}_{\tilde{Z}}(-D')$. Make the elementary modification

$$0 \to V' \to V_0 \to j_* q^* \mathcal{O}_{\mathbb{P}^1}(-1) \to 0.$$

Then V' restricted to ℓ is the trivial bundle $\mathcal{O}_{\mathbb{P}^1}^n$. It follows that V'|D' is pulled back from the factor E_b .

Proof. We have an exact sequence

$$0 \to V'|D' \to V_0|D' \to j_*\mathcal{O}_{\mathbb{P}^1}(-1) \to 0,$$

where we write j also for the inclusion of the fiber $\ell = \mathbb{P}^1 \times \{p_0\}$ in the ruled surface $D' \cong \mathbb{P}^1 \times E_b$. By standard formulas for elementary modifications, it is straightforward to compute that $c_2(V'|D') = c_2(V_0|D') - 1$. But $c_2(V_0|D') = h\pi_2^*[p_0] = 1$ by the formulas of §2.6. Thus $c_2(V'|D') = 0$. Now by a sequence of allowable elementary modifications $V_0|D',V'|D'=V_1,\ldots,V_r$, we can reach a vector bundle V_r over D' whose restriction to every fiber $\mathbb{P}^1 \times \{e\}$ is semistable and thus trivial; this happens if and only if V_r is pulled back from the base, and so has $c_2 = 0$. But each allowable elementary modification along the fiber $\mathbb{P}^1 \times \{p_0\}$ drops c_2 by a positive integer. Since V'|D' already has $c_2 = 0$, no further elementary modifications are possible. Hence $V'|\ell$ is already semistable and therefore trivial, and thus V'|D' is pulled back from E_b as claimed. \square

By construction, $V'|\ell$ is given as an extension

$$0 \to \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}^{n-2} \to V'|\ell \to \mathcal{O}_{\mathbb{P}^1}(1) \to 0.$$

Now $\operatorname{Ext}^1(\mathcal{O}_{\mathbb{P}^1}(1), \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}^{n-2}) \cong H^1(\mathcal{O}_{\mathbb{P}^1}(-2)) \cong \mathbb{C}$, so there is a unique nonsplit extension of this type, which is clearly the trivial bundle $\mathcal{O}_{\mathbb{P}^1}^n$.

Claim 2. With V' as in Claim 1, the restriction of V' to D_1 is the pullback $s^*(T_{\mathbb{P}^2}(-1) \oplus \mathcal{O}_{\mathbb{P}^2}^{n-2})$.

Proof. By definition, there is an exact sequence

$$0 \to q^*[\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}^{n-2}] \otimes \mathcal{O}_{D_1}(-D_1) \to V'|D_1 \to q^*\mathcal{O}_{\mathbb{P}^1}(1) \to 0.$$

Next, a straightforward calculation shows that $\mathcal{O}_{D_1}(-D_1) = \mathcal{O}_{D_1}(\ell) \otimes q^*\mathcal{O}_{\mathbb{P}^1}(1)$. Thus the extensions of $q^*\mathcal{O}_{\mathbb{P}^1}(1)$ by $q^*[\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}^{n-2}] \otimes \mathcal{O}_{D_1}(-D_1)$) are classified by

$$H^1(D_1; q^*[\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}^{n-2}] \otimes \mathcal{O}_{D_1}(\ell)).$$

It is easy to check that $H^1(D_1; \mathcal{O}_{D_1}(\ell)) = 0$ and that $h^1(\mathcal{O}_{D_1}(\ell) \otimes q^*\mathcal{O}_{\mathbb{P}^1}(-1)) = 1$. Thus the dimension of the Ext group in question is one, so that there just one nontrivial extension up to isomorphism. Note that $V'|D_1$ is itself such an extension: it cannot be the split extension since the restriction of $V'|D_1$ to ℓ is trivial. Thus, to complete the proof of Claim 2, it will suffice to show that $s^*(T_{\mathbb{P}^2}(-1) \oplus \mathcal{O}_{\mathbb{P}^2}^{n-2})$ is also given as an extension of $q^*\mathcal{O}_{\mathbb{P}^1}(1)$ by $q^*[\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}^{n-2}] \otimes \mathcal{O}_{D_1}(-D_1)$. It clearly suffices to do the case n=2, i.e. show that $s^*T_{\mathbb{P}^2}(-1)$ is an extension of $q^*\mathcal{O}_{\mathbb{P}^1}(1)$ by $q^*\mathcal{O}_{\mathbb{P}^1}(-1)$, necessarily nonsplit since the restriction to ℓ is trivial. To see this, note that $T_{\mathbb{P}^2}(-1)$ has restriction $\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1)$ to every line. Thus by the standard construction (cf. [11], p. 60) there is an exact sequence

$$0 \to \mathcal{O}_{\mathbb{F}_1}(\ell) \otimes q^* \mathcal{O}_{\mathbb{P}^1}(t) \to s^* T_{\mathbb{P}^2}(-1) \to q^* \mathcal{O}_{\mathbb{P}^1}(1-t) \to 0$$

for some integer t. By looking at c_2 , we must have t=0 and thus $s^*T_{\mathbb{P}^2}(-1)$ is an extension of $q^*\mathcal{O}_{\mathbb{P}^1}(1)$ by $\mathcal{O}_{\mathbb{F}_1}(\ell)$, which is nonsplit because its restriction to ℓ is trivial. Thus we have identified $V'|D_1$ with $s^*(T_{\mathbb{P}^2}(-1) \oplus \mathcal{O}_{\mathbb{P}^2}^{n-2})$. \square

Let Z_2 be the result of contracting D' in Z_1 . This has the effect of contracting $\ell \subset D_1$ to a point, so that the image of D_1 in Z_2 is an exceptional \mathbb{P}^2 , which we denote by P. Moreover, by the above claims V' induces a vector bundle on Z_2 whose restriction to P is identified with $T_{\mathbb{P}^2}(-1) \oplus \mathcal{O}_{\mathbb{P}^2}^{n-2}$. Thus, the proof of (6.11) will be complete once we prove the following:

Claim 3. Let X be a manifold of dimension 3 and let \tilde{X} be the blowup of X at a point x, with exceptional divisor $P \cong \mathbb{P}^2$. Suppose that W is a vector bundle on \tilde{X} such that $W|P \cong T_{\mathbb{P}^2}(-1) \oplus \mathcal{O}_{\mathbb{P}^2}^{n-2}$. Let $\rho \colon \tilde{X} \to X$ be the blowup map. Then ρ_*W is locally isomorphic to $M \oplus R^{n-2}$ in the notation above. In particular, ρ_*W is reflexive but not locally free.

Proof. We shall just do the case n=2, the other cases being similar. By the formal functions theorem, the completion of the stalk of the direct image ρ_*W at x is $M' = \varprojlim H^0(W \otimes \mathcal{O}_{nP})$. Now from the exact sequences

$$0 \to \mathcal{O}_{\mathbb{P}^2}(-1) \to \mathcal{O}_P^3 \to W|P \to 0$$

and the sequence

$$0 \to W \otimes \mathcal{O}_{\tilde{X}}(-(n+1)P) \to W \otimes \mathcal{O}_{(n+1)P} \to W \otimes \mathcal{O}_{nP} \to 0,$$

it is easy to check that the three sections of W|P lift to give three generators of M' as an R-module. Hence there is a surjection $\mathcal{O}_{\tilde{X}}^3 \to \rho^* \rho_* W \to W$, and by checking determinants the kernel is $\mathcal{O}_{\tilde{X}}(P)$. Now up to an change of coordinates in \mathbb{C}^3 the only injective homomorphism from $\mathcal{O}_{\tilde{X}}(P)$ to $\mathcal{O}_{\tilde{X}}^3$ is given by the three generators of the maximal ideal of \mathbb{C}^3 at the origin. Taking direct images of the exact sequence

$$0 \to \mathcal{O}_{\tilde{X}}(P) \to \mathcal{O}_{\tilde{X}}^3 \to W \to 0$$

and using the vanishing for the first direct image of $\mathcal{O}_{\tilde{X}}(P)$ gives $M' \cong M$ as previously defined. So we have established Claim 3, and hence (6.11). \square

We give a brief and inconclusive discussion of how the above constructions begave in families, assuming dim B=2 for simplicity. Let D be the unit disk in \mathbb{C} . Suppose that we are given a general family of nowhere vanishing sections s_t of \mathcal{V}_n which at a special point t=0 acquires a simple zero at $b\in B$. We can view the family $s=\{s_t\}$ as a section of the pullback of \mathcal{V}_n to $B\times D$, where it has a simple zero at (b,0). Thus, for an integer a, there is a bundle $\mathcal{V}_{s,a}$ over $Z\times D-\{(b,0)\}$, which completes uniquely to a reflexive sheaf over $Z\times D$, which we continue to denote by $\mathcal{V}_{s,a}$. For example, if $-(n-2)\leq a\leq 0$, then it is easy to see that $\mathcal{V}_{s,a}$ is a bundle over $Z\times D$, whose restriction to $Z\times\{0\}$ is everywhere regular semistable except over E_b where it restricts to $W_d^\vee\oplus W_{n-d}$ for the appropriate d. One can ask if this holds for all $a\not\equiv 1$ mod n. Note that, if we consider the relative deformation theory of the unstable bundle $W_d^\vee\oplus W_{n-d}$ over the base B, for n=2 the codimension of the locus of unstable bundles forces every deformation of V to have unstable restriction to some fibers, whereas for n>2 we expect that in the general deformation V_t we can arrange that the restriction of V_t to every fiber is semistable.

If $a \equiv 1 \mod n$, then $\mathcal{V}_{s,a}$ is a flat family of coherent sheaves. However, there is no reason a priori why $\mathcal{V}_{s,a}|Z \times \{0\}$ is reflexive. In fact, preliminary calculations suggest that, for a=1, the restriction $\mathcal{V}_{s,a}|Z \times \{0\}$ has the local form $M \oplus \mathfrak{m}^{n-2}$, where \mathfrak{m} is the maximal ideal of the point $\sigma \cap E_b$. Note that the R-module M is not smoothable, even locally, but that $R^k \oplus M$ is smoothable to a free R-module for all $k \geq 1$. One can also show that the more complicated R-module $\mathfrak{m}^k \oplus M$ is smoothable to a free R-module for all $k \geq 1$. This agrees with the picture for sections of the bundle \mathcal{V}_n : for n=2, if a section has a simple isolated zero, that zero must remain under deformation, but for n>2 we expect in general that we can deform to an everywhere nonzero section in general.

6.4. Bundles which are not regular in high codimension.

In this subsection we consider bundles V such that $V|E_b$ is semistable for all b, and $Y = \{b \in B : V|E_b \text{ is not regular}\}$ has codimension at least 2 in B. The first lemma shows that, if the spectral cover C_A is smooth, then V is in fact everywhere regular.

Lemma 6.12. Let V be a vector bundle over Z such that $V|E_b$ is semistable for all b, and $Y = \{b \in B : V|E_b \text{ is not regular}\}$ has codimension at least 2 in B. Suppose that the associated spectral cover C_A is smooth. Then $V|E_b$ is regular for

all $b \in B$. More generally, suppose that V is a vector bundle over Z such that $Y = \{b \in B : V | E_b \text{ is either not semistable or not regular}\}$ has codimension at least 2 in B, that the section A defined by V over B - Y extends to a section over all of B, and that the associated spectral cover C_A of B is smooth. Then $V | E_b$ is semistable and regular for all $b \in B$.

Proof. We have seen in (5.7) that there is a line bundle N on $C_A - g_A^{-1}(Y)$ such that $V|Z - \pi^{-1}(Y) \cong V_{A,0}[N]$. Since C_A is smooth, and $g_A^{-1}(Y)$ has codimension at least two in C_A , the line bundle N on $C_A - g_A^{-1}(Y)$ extends to a line bundle over C_A , which we continue to denote by N. We now have two vector bundles on Z, namely V and $V_{A,0}[N]$, which are isomorphic over $Z - \pi^{-1}(Y)$. Since the codimension of $\pi^{-1}(Y)$ in Z is at least two, V and $V_{A,0}[N]$ are isomorphic. But $V_{A,0}[N]$ restricts to a regular bundle on every fiber, and so the same must be true for V. \square

We turn to methods for constructing bundles which are semistable on every fiber but which are not regular in codimension two. Of course, by the above lemma, the corresponding spectral covers will not be smooth. The idea is to find such bundles by using a three step filtration, as opposed to the two-step extensions which have used from Section 3 onwards in our constructions. Such constructions correspond to nonmaximal parabolic subgroups in SL_n .

Consider first the case of a single Weierstrass cubic E. We seek bundles of rank n+1 which have a filtration $0 \subset F^0 \subset F^1 \subset V$, where $F^0 \cong W_k^{\vee}, F^1/F^0 \cong \mathcal{O}_E$, and $V/F^1 \cong W_{n-k}$. Such extensions can be described by a nonabelian cohomology group as in [8]. However, it is also easy to describe them directly. Note that a fixed F^1 is described by an extension class α_0 in $\operatorname{Ext}^1(\mathcal{O}_E, W_k^{\vee}) \cong H^1(W_k^{\vee}) \cong \mathbb{C}$. If $\alpha_0 = 0$, then $F^1 = W_k^{\vee} \oplus \mathcal{O}_E$, and if $\alpha_0 \neq 0$ then $F^1 \cong W_{k+1}^{\vee}$. Having determined F^1 , the extension F^2 corresponds to a class in $\operatorname{Ext}^1(W_{n-k}, F^1)$. Since $\operatorname{Hom}(W_{n-k}, \mathcal{O}_E) = \operatorname{Ext}^2(W_{n-k}, W_k^{\vee}) = 0$, there is a short exact sequence

$$0 \to \operatorname{Ext}^1(W_{n-k}, W_k^{\vee}) \to \operatorname{Ext}^1(W_{n-k}, F^1) \to \operatorname{Ext}^1(W_{n-k}, \mathcal{O}_E) \to 0,$$

and so dim $\operatorname{Ext}^1(W_{n-k}, F^1) = n+1$. Thus roughly speaking the moduli space of filtrations as above is an affine space \mathbb{C}^{n+2} . In fact, by general construction techniques there is a universal bundle \mathcal{F}^1 over $\operatorname{Ext}^1(\mathcal{O}_E, W_k^{\vee}) \times E = \mathbb{C} \times E$. We can then form the relative Ext sheaf

$$Ext_{\pi_1}^1(\pi_2^*W_{n-k}, \mathcal{F}^1) = R^1\pi_{1*}(\pi_2^*W_{n-k}^{\vee} \otimes \mathcal{F}^1).$$

It is a vector bundle of rank n+1 over \mathbb{C} , which is necessarily trivial, and thus the total space of this vector bundle is \mathbb{C}^{n+2} . There is a universal extension of $\pi_2^*W_{n-k}$ by \mathcal{F}^1 defined over $\mathbb{C}^{n+2}\times E$. It follows that the set of filtrations is indeed parametrized by a moduli space isomorphic to \mathbb{C}^{n+2} , although there is not a canonical linear structure. What is canonical is the exact sequence

$$0 \to \operatorname{Ext}^1(W_{n-k}, W_k^{\vee}) \to \mathbb{C}^{n+2} \to \operatorname{Ext}^1(\mathcal{O}_E, W_k^{\vee}) \oplus \operatorname{Ext}^1(W_{n-k}, \mathcal{O}_E) \to 0.$$

We understand this sequence to mean that the first term, which is a vector space, acts on the middle term, which is just an affine space, via affine translations, and the quotient is the last term, which is again a vector space. Here the projection

to $\operatorname{Ext}^1(\mathcal{O}_E,W_k^\vee) \oplus \operatorname{Ext}^1(W_{n-k},\mathcal{O}_E)$ measures the extensions F^1 of \mathcal{O}_E by W_k^\vee and V/F^0 of W_{n-k} by \mathcal{O}_E . We denote the image of $\xi \in \mathbb{C}^{n+2}$ in $\operatorname{Ext}^1(\mathcal{O}_E,W_k^\vee) \oplus \operatorname{Ext}^1(W_{n-k},\mathcal{O}_E) \cong \mathbb{C} \oplus \mathbb{C}$ by (α_0,α_1) . Here $\alpha_0 \neq 0$ if and only if $F^1 \cong W_{k+1}^\vee$ and $\alpha_1 \neq 0$ if and only if $V/F^0 \cong W_{n-k+1}$. In case $\alpha_0 = 0$, say, $F^1 \cong W_k^\vee \oplus \mathcal{O}_E$, and $\operatorname{Ext}^1(W_{n-k},F^1)$ naturally splits as $\operatorname{Ext}^1(W_{n-k},W_k^\vee) \oplus \operatorname{Ext}^1(W_{n-k},\mathcal{O}_E)$. In this case, both the class α_1 and the class $e \in \operatorname{Ext}^1(W_{n-k},W_k^\vee)$ are well-defined. A similar statement holds if $\alpha_1 = 0$. Note that the affine space \mathbb{C}^{n+2} parametrizes filtrations F^i together with fixed isomorphisms $F^0 \to W_k^\vee$, $F^1/F^0 \to \mathcal{O}_E$, $V/F^1 \to W_{n-k}$.

The subspace $\operatorname{Ext}^1(W_{n-k}, W_k^{\vee})$, namely where both α_0 and α_1 vanish, corresponds to those V of the form $V' \oplus \mathcal{O}_E$, where V' is an extension of W_{n-k} by W_k^{\vee} . There is a hyperplane H in $\operatorname{Ext}^1(W_{n-k}, W_k^{\vee})$ where such V' contain a Jordan-Hölder quotient isomorphic to \mathcal{O}_E , and thus over the locus $\alpha_0 = \alpha_1 = 0, e \in H$, $V = V' \oplus \mathcal{O}_E$ has a subbundle of the form $\mathcal{O}_E \oplus \mathcal{O}_E$. Hence, over a affine subspace of \mathbb{C}^{n+2} of codimension three, the V we have constructed are not regular.

Lemma 6.13. Suppose that V corresponds to a class $\xi \in \mathbb{C}^{n+2}$, and that α_0, α_1 are as above.

- (i) V is unstable if and only $\alpha_0 \alpha_1 = 0$ and e = 0 (this statement is well-defined by the above remarks).
- (ii) If $\alpha_0 \alpha_1 \neq 0$, then $h^0(V) = 0$ and conversely.

Proof. (i) Let us assume for example that $\alpha_0=e=0$. Then $F^1=W_k^\vee\oplus\mathcal{O}_E$ and V is isomorphic either to $W_k^\vee\oplus W_{n-k+1}$ or to $W_k^\vee\oplus\mathcal{O}_E\oplus W_{n-k}$, and in either case it is unstable. Conversely, if V is unstable, then it has a maximal destabilizing subsheaf W of positive degree, which is stable and which must map nontrivially onto W_{n-k} . Thus deg W=1. Now if $W\cap W_k^\vee\neq 0$, then $W\cap W_k^\vee$ has degree ≤ -1 and is contained in the kernel of the map $W\to W_{n-k}$. This would force the image of W to have degree at most zero, which is impossible. So $W\cap W_k^\vee=0$ and thus the map $W\to V/F^0$ is injective. Now either $V/F^0\cong W_{n-k+1}$ or $V/F^0\cong W_{n-k}\oplus\mathcal{O}_E$. In the first case, $W\cong W_{n-k+1}$ by the stability of W_{n-k+1} and $V\cong W_k^\vee\oplus W_{n-k+1}$. In this case $\alpha_0=e=0$. In the remaining case, $V/F^0\cong W_{n-k}\oplus\mathcal{O}_E$ and $W\cong W_{n-k}$. In this case $\alpha_1=e=0$. In both cases we must have $\alpha_0\alpha_1=0$ and e=0.

(ii) First suppose that $\alpha_0\alpha_1\neq 0$. Since $\alpha_0\neq 0,\ F^1\cong W_{k+1}^\vee$. From the exact sequence

$$0 \to F^1 \to V \to W_{n-k} \to 0$$
,

and the fact that $H^0(F^1) = 0$, there is an exact sequence $H^0(V) \to H^0(W_{n-k}) \to H^1(F^1)$. If we compose the map $H^0(W_{n-k}) \to H^1(F^1)$ with the natural map $H^1(F^1) \to H^1(\mathcal{O}_E)$, the result is α_1 up to a nonzero scalar. Thus, if $\alpha_1 \neq 0$, the map $H^0(W_{n-k}) \to H^1(F^1)$ is injective and so $H^0(V) = 0$. Conversely, suppose that either α_0 or α_1 is zero. If for example $\alpha_1 = 0$, then $V/F^0 \cong W_{n-k} \oplus \mathcal{O}_E$, so that $h^0(V/F^0) = 2$. Since $H^0(V)$ is the kernel of the map $H^0(V/F^0) \to H^1(F^0) = H^1(W_k^V) \cong \mathbb{C}$, $H^0(V) \neq 0$. The case $\alpha_0 \neq 0$ is similar and simpler. Thus, if $h^0(V) = 0$, then $\alpha_0 \alpha_1 \neq 0$. \square

The group $\mathbb{C}^* \times \mathbb{C}^*$ (or more precisely \mathbb{C}^3/\mathbb{C}) acts on the affine space \mathbb{C}^{n+2} by acting on the identifications of the quotients F^{i+1}/F^i with the standard bundles. The quotient by this action (which is not in fact separated) is the set of bundles V of rank n+1, together with filtrations on V with the appropriate graded object.

The action of $\mathbb{C}^* \times \mathbb{C}^*$ is compatible with the projection to \mathbb{C}^2 . If we normalize the action so that $(\lambda, \mu) \cdot (\alpha_0, \alpha_1) = (\lambda \alpha_0, \mu \alpha_1)$, then (λ, μ) acts on the distinguished subspace $\operatorname{Ext}^1(W_{n-k}, W_k^\vee) \cong \mathbb{C}^n$ by $e \mapsto \lambda \mu e$. Clearly, the action is free over the set $\alpha_0 \alpha_1 \neq 0$. The quotient of the points where $\alpha_0 \neq 0$, $\alpha_1 = 0$, $e \neq 0$ is a \mathbb{P}^{n-1} , and this \mathbb{P}^{n-1} is identified with the corresponding \mathbb{P}^{n-1} where $\alpha_1 \neq 0$, $\alpha_0 = 0$, $e \neq 0$; in fact, the $\mathbb{C}^* \times \mathbb{C}^*$ -orbits intersect along the subspace where $\alpha_0 = \alpha_1 = 0$, $e \neq 0$. The points $\alpha_0 \alpha_1 = e = 0$ are unstable points and do not appear in a GIT quotient for the action. We also have the map $(1.5) \Phi \colon \mathbb{C}^{n+2} - (\mathbb{C} \cup \mathbb{C})$ to the coarse moduli space \mathbb{P}^n of semistable bundles of rank n+1 on E. By Lemma 6.13, the image of the two subsets $\{\alpha_0 = 0, e \neq 0\}$ and $\{\alpha_1 = 0, e \neq 0\}$ is exactly the hyperplane in \mathbb{P}^n corresponding to bundles V such that $h^0(V) \neq 0$, or in other words such that V has \mathcal{O}_E as a Jordan-Hölder quotient.

Lemma 6.14. The map $\Phi: \mathbb{C}^{n+2} - (\mathbb{C} \cup \mathbb{C}) \to \mathbb{P}^n$ is the geometric invariant theory quotient of $\mathbb{C}^{n+2} - (\mathbb{C} \cup \mathbb{C})$ by the action of $\mathbb{C}^* \times \mathbb{C}^*$.

Proof. First suppose that the point $x \in \mathbb{C}^{n+2} - (\mathbb{C} \cup \mathbb{C})$ lies in the open dense subset $\alpha_0\alpha_1 \neq 0$ where $\mathbb{C}^* \times \mathbb{C}^*$ acts freely. Thus if V is the vector bundle corresponding to x, then $h^0(V) = 0$; equivalently, V has no Jordan-Hölder quotient equal to \mathcal{O}_E , and V is a regular semistable bundle. If $\Phi(x) = \Phi(x')$, then x' also lies in the set $\alpha_0\alpha_1 \neq 0$, and the bundle V' corresponding to x' is also regular and semistable. Thus $V \cong V'$, and we must determine if the filtration F^i on V exists is unique up to isomorphism. First, if V is a regular semistable bundle of rank n+1, then it is an extension of W_{n-k} by W_{k+1}^{\vee} , where the subbundle W_{k+1}^{\vee} of V is unique modulo automorphisms of V, and taking the further filtration of W_{k+1}^{\vee} by the subbundle W_k^{\vee} , with quotient \mathcal{O}_E . Thus $V = \Phi(x)$ for some x. Conversely, if V has on it a filtration F^i with $\alpha_0\alpha_1 \neq 0$, then $F^1 \cong W_{k+1}^{\vee}$. Moreover, if $H^0(V) = 0$, then every subbundle of V isomorphic to W_k^{\vee} is contained in a subbundle isomorphic to W_{k+1}^{\vee} (whereas if $H^0(V) \neq 0$, this is no longer the case; cf. §3.2). Thus the filtration F^i is unique up to automorphisms of V. The above argument shows that Φ induces an isomorphism

$$\left(\mathbb{C}^{n+2} - \left\{ \alpha_0 \alpha_1 = 0 \right\} \right) / \mathbb{C}^* \times \mathbb{C}^* \to \mathbb{P}^n - H.$$

In case x lies in the set $\alpha_0 = 0, \alpha_1 \neq 0, e \neq 0$, a straightforward argument identifies the quotient by $\mathbb{C}^* \times \mathbb{C}^*$ with $H \subset \mathbb{P}^n$, and likewise for $\alpha_0 \neq 0, \alpha_1 = 0, e \neq 0, \alpha_0 = \alpha_1 = 0, e \neq 0$. \square

The coarse moduli space \mathbb{P}^n has its associated spectral cover T, which is an (n+1)-sheeted cover of \mathbb{P}^n . Let $\tilde{T} \to \mathbb{C}^{n+2} - (\mathbb{C} \cup \mathbb{C})$ be the pulled back cover of $\mathbb{C}^{n+2} - (\mathbb{C} \cup \mathbb{C})$ via the morphism Φ . Using Lemma 6.14, we can see directly that \tilde{T} is singular, with the generic singularities a locally trivial family of threefold double points. In fact, the inverse image of H in T is of the form $H \cup T'$, where T' is the spectral cover of $H \cong \mathbb{P}^{n-1}$. The intersection of H and T' is transverse (see §5.7), and $H \cap T'$, viewed as a subset of $H \subset \mathbb{P}^n$, corresponds to those bundles which have \mathcal{O}_E as a Jordan-Hölder quotient with multiplicity at least two. If t is a local equation for H in \mathbb{P}^n near a generic point of $H \cap T'$, there are local coordinates on T for which t = uv, since H splits into $H \cup T'$. Thus the local equation for \tilde{T} is $\alpha_0\alpha_1 = uv$, which is the equation for a family of threefold double points.

We can also do the above constructions in families $\pi\colon Z\to B$. We could take the point of view of [8] and realize the relative nonabelian cohomology groups as a

bundle of affine spaces over B. However, it is also possible to proceed directly as in §5.2. We seek vector bundles V which have a filtration $0 \subset F^0 \subset F^1 \subset V$, where $F^0 \cong \mathcal{W}_k^{\vee} \otimes \pi^* M_0$, $F^1/F^0 \cong \pi^* M_1$, and $V/F^1 \cong \mathcal{W}_{n-k} \otimes \pi^* M_2$ for line bundles M_0, M_1, M_2 on B. Of course, we can normalize by twisting V so that one of the M_i is trivial. The analysis of such extensions parallels the analysis for a single E. We begin by constructing F^0 . It is described by an extension class in

$$H^{1}(Z; \pi^{*}M_{1}^{-1} \otimes \mathcal{W}_{k}^{\vee} \otimes \pi^{*}M_{0}) \cong H^{0}(B; R^{1}\pi_{*}(\mathcal{W}_{k}^{\vee}) \otimes M_{1}^{-1} \otimes M_{0})$$

= $H^{0}(B; L^{-k} \otimes M_{1}^{-1} \otimes M_{0}).$

If the difference line bundle $M_1^{-1} \otimes M_0$ is sufficiently ample, then there will be nonzero sections α_0 of $L^{-k} \otimes M_1^{-1} \otimes M_0$ vanishing along a divisor D_0 in B. Next, we seek extensions of F^1 by $\mathcal{W}_{n-k} \otimes \pi^* M_2$. Now $H^0(\mathcal{W}_{n-k}^{\vee} \otimes \pi^* M_2^{-1} \otimes \pi^* M_1) = 0$, and by the Leray spectral sequence

$$H^2(\mathcal{W}_{n-k}^{\vee} \otimes \pi^* M_2^{-1} \otimes \mathcal{W}_k^{\vee} \otimes \pi^* M_0) \cong H^1(B; R^1 \pi_* (\mathcal{W}_{n-k}^{\vee} \otimes \mathcal{W}_k^{\vee}) \otimes M_2^{-1} \otimes M_0).$$

We assume that $M_2^{-1} \otimes M_0$ is so ample that the above group is zero. In this case there is an exact sequence

$$0 \to H^1(\mathcal{W}_{n-k}^{\vee} \otimes \pi^* M_2^{-1} \otimes \mathcal{W}_k^{\vee} \otimes \pi^* M_0) \to \operatorname{Ext}^1(\mathcal{W}_{n-k} \otimes \pi^* M_2, F^1) \to H^1(\mathcal{W}_{n-k}^{\vee} \otimes \pi^* M_2^{-1} \otimes \pi^* M_1) \to 0.$$

The left-hand group is $H^0(R^1\pi_*(\mathcal{W}_{n-k}^{\vee}\otimes\mathcal{W}_k^{\vee})\otimes M_2^{-1}\otimes M_0)$, and the right-hand group is $H^0(L^{n-k}\otimes M_2^{-1}\otimes M_1)$. Thus, for $M_2^{-1}\otimes M_1$ sufficiently ample, there will exist sections α_1 of $L^{n-k}\otimes M_2^{-1}\otimes M_1$, vanishing along a divisor D_1 in B, and we will be able to lift these sections to extension classes in $\operatorname{Ext}^1(\mathcal{W}_{n-k}\otimes\pi^*M_2,F^1)$. Moreover, if we restrict, say, to the divisor $D_0=0$, then there is also a well-defined class e in $H^0(R^1\pi_*(\mathcal{W}_{n-k}^{\vee}\otimes\mathcal{W}_k^{\vee})\otimes M_2^{-1}\otimes M_0)$. There is a divisor D on B corresponding to such extensions which have a factor \mathcal{O}_{E_b} for $b\in D$. In fact, if $\alpha=c_1(L\otimes M_2^{-1}\otimes M_0)$, then it follows from (4.15) and (5.9) that $[D]=\alpha-nL$. (Compare also (5.21).) As long as $M_2^{-1}\otimes M_0$ is also sufficiently ample, we can assume that the divisors D_0,D_1 and D are smooth and intersect transversally in a subvariety of B of codimension three. Along this subvariety, V fails to be regular.

Note that the V constructed above are a deformation of $V' \oplus \mathcal{O}_Z$, where V' is a twist of a bundle of the form $V_{A,1-k}$; it suffices for example to take $\alpha_0 = 0$ and $e \neq 0$.

For generic choices, the spectral cover C_A will acquire singularities in codimension three, which will generically be families of threefold double points. In particular, there are Weil divisors on C_A which do not extend to Cartier divisors, as predicted by Lemma 6.12. It is also amusing to look at the case dim B=2, where for generic choices the spectral cover will be smooth. The construction then deforms $V' \oplus \mathcal{O}_Z$ to a bundle V which has regular semistable restriction to every fiber. Starting with a generic $V' = V_{A,a}(n)$ of rank n, we cannot in general deform $V' \oplus \mathcal{O}_Z$ to a standard bundle $V_{A,b}(n+1) \otimes \pi^* N_0$. Instead, the spectral cover C_A has Picard number larger than expected. In fact, we have the divisor $F = (r^*\sigma \times_B Z) \cap C_A$, which is mapped isomorphically onto its image in B, and

this image is the same as $A \cap \mathcal{H} \subset \mathcal{P}_{n-1}$. Now $A \cap \mathcal{H}$ corresponds to the bundles V such that $h^0(V) \neq 0$, and thus by Lemma 6.13 this locus is just $D_0 \cup D_1$. Thus in C_A the divisor F splits into a sum of two divisors, which we continue to denote by D_0 and D_1 . Using these extra divisors, we can construct more vector bundles over Z, of the form $V_{A,0}[N]$ for some extra line bundle N, which enable us to deform $V' \oplus \mathcal{O}_Z$ to a bundle which is everywhere regular and semistable.

Let us just give the details in a symmetric case. Let M be a sufficiently ample line bundle on B. There exist bundles V on Z which have regular semistable restriction to every fiber and also have a filtration $F^0 \subset F^1 \subset V$, with

$$V/F^1 \cong \mathcal{W}_k \otimes \pi^* M^{-1}; \qquad F^1/F^0 \cong \mathcal{O}_Z; \qquad F^0 \cong \mathcal{W}_k^{\vee} \otimes \pi^* M.$$

The bundle V is a deformation of a bundle of the form $V' \oplus \mathcal{O}_Z$. Thus, there must exist a line bundle N on the spectral cover C_A such that $V_{A,0}[N]$ has the same Chern classes as V. Direct calculation with the Grothendieck-Riemann-Roch theorem shows that this happens for

$$N = M \otimes \mathcal{O}_{C_A}(-F + (k+1)D_0 + kD_1)$$

as well as for

$$N = M \otimes \mathcal{O}_{C_A}(-F + kD_0 + (k+1)D_1),$$

and that these are the only two "universal" choices for N.

Question. Suppose that dim B=3 and consider spectral covers which have an ordinary threefold double point singularity. The local Picard group of the singularity is \mathbb{Z} . Given $a\in\mathbb{Z}$, we can twist by a line bundle over the complement of the singularity which correspond to the element $a\in\mathbb{Z}$. The result is a vector bundle on Z, defined in the complement of finitely many fibers, and thus the direct image is a coherent reflexive sheaf on Z. What is the relationship of local behavior of this sheaf at the finitely many fibers to the integer a?

7. Stability.

Our goal in this final section will be to find sufficient conditions for $V_{A,a}$, or more general bundles constructed in the previous two sections, to be stable with respect to a suitable ample divisor. Here suitable means in general a divisor of the form $H_0 + N\pi^*H$, where H_0 is an ample divisor on Z and H is an ample divisor on B, and $N \gg 0$. As we have already see in §5.6, for $A = \mathbf{o}$, the bundle $V_{\mathbf{o},a}$ is essentially always unstable with respect to every ample divisor. Likewise, suppose that A is a section lying in \mathcal{H} as in §5.7, so that $V_{A,a}$ has a surjection to π^*L^a . If the line bundle corresponding to A is sufficiently ample, it is easy to see that for appropriate choices of a we can always arrange $\mu_H(\pi^*L^a) < \mu_H(V_{A,a})$, so that $V_{A,a}$ is unstable. Thus, we shall have to make some assumptions about A. More generally, let Vbe a bundle whose restriction to the generic fiber is regular and semistable, and let A be the associated quasisection. It turns out that, if the spectral cover C_A is irreducible, then V is stable with respect to all divisors of the form $H_0 + N\pi^*H$, provided that $N \gg 0$. A similar result holds in families. However, we are only able to give an effective estimate for N in case dim B=1. In particular, whether there is an effective bound for N which depends only on Z, H_0 , H, $c_1(V)$, and $c_2(V)$ is open in case $\dim B > 1$. We believe that such a bound should exist, and can

give such an explicit bound for a general B in the rank two case for an irreducible quasisection A. (Of course, when $\dim B > 2$, an irreducible quasisection A will almost never be an actual section.) However, we shall not give the details in this paper.

7.1. The case of a general Z.

Let $\pi\colon Z\to B$ be a flat family of Weierstrass cubics with a section. We suppose in fact that Z is smooth of dimension d+1. Fix an ample divisor H_0 on Z and an ample divisor H on B, which we will often identify with π^*H on Z.

Theorem 7.1. Let V be a vector bundle of rank n over Z whose restriction to the generic fiber is regular and semistable, and such that the spectral cover of the quasisection corresponding to V is irreducible. Then there exists an $\epsilon_0 > 0$, depending on V, H_0, H , such that V is is stable with respect to $\epsilon H_0 + H$ for all $0 < \epsilon < \epsilon_0$.

Proof. Let W be a subsheaf of V with $0 < \operatorname{rank} W < r$. The semistability assumption on V|f, for a generic f, and the fact that $W|f \to V|f$ is injective for a generic f imply that $c_1(W) \cdot f \leq 0$. If however $c_1(W) \cdot f = 0$, then W and V/W are also semistable on the generic fiber. By Proposition 5.22, the spectral cover corresponding to V would then be reducible (the proof in (5.22) needed only that V has regular semistable restriction to the generic fiber), contrary to hypothesis. Thus in fact $c_1(W) \cdot f < 0$. Equivalently, $c_1(W) \cdot H^d < 0$.

For a torsion free sheaf W, define $\mu_H(W) = \frac{c_1(W) \cdot H^d}{\operatorname{rank} W}$, by analogy with an ample H_0 . If W is a subsheaf of V such that $0 < \operatorname{rank} W < n$, then $\mu_H(W)$ is a strictly negative rational number with denominator bounded by n-1.

Lemma 7.2. There is a constant A, depending only on V, H_0, H , such that

$$\frac{c_1(W) \cdot H^i \cdot H_0^{d-i}}{\operatorname{rank} W} \le A$$

for all i with $0 \le i \le d$ and all nonzero subsheaves W of V.

Proof. There exists a filtration

$$0 \subset F^0 \subset F^1 \subset \dots \subset F^{n-1} = V$$

such that F^j/F^{j-1} is a torsion free rank one sheaf, and thus is of the form $L_j \otimes I_{X_j}$ for L_j a line bundle on Z and X_j a subscheme of codimension at least two (possibly empty). Suppose that W has rank one. Then there is a nonzero map from W to $L_j \otimes I_{X_j}$ for some j, and thus W is of the form $L_j \otimes \mathcal{O}_Z(-D) \otimes I_X$ for some effective divisor D on Z and subscheme X of codimension at least two (possibly empty). Thus

$$c_1(W) \cdot H^i \cdot H_0^{d-i} \le c_1(L_j) \cdot H^i \cdot H_0^{d-i}.$$

Thus these numbers are bounded independently of W. In case W has arbitrary rank r, $1 \le r \le n-1$, find a similar filtration of the bundle $\bigwedge^r V$ by subsheaves whose successive quotients are rank one torsion free sheaves, and use the existence of a nonzero map $\bigwedge^r W \to \bigwedge^r V$ to argue as before. \square

Returning to the proof of Theorem 7.1, if W is a subsheaf of V such that $0 < \operatorname{rank} W < n$, it follows that

$$\mu_{\epsilon H_0 + H}(W) = \frac{c_1(W) \cdot (\epsilon H_0 + H)^d}{\operatorname{rank} W} \le -\frac{1}{n - 1} + O(\epsilon).$$

On the other hand, since det V is pulled back from $B, c_1(V) \cdot H^d = 0$ and so

$$\mu_{\epsilon H_0 + H}(V) = \frac{c_1(V) \cdot (\epsilon H_0 + H)^d}{n} = O(\epsilon).$$

Thus, for ϵ sufficiently small, for every subsheaf W of V with $0 < \operatorname{rank} W < n$,

$$\mu_{\epsilon H_0 + H}(W) < \mu_{\epsilon H_0 + H}(V).$$

In other words, V is stable with respect to $\epsilon H_0 + H$. \square

Corollary 7.3. Let V be a family of vector bundles over $S \times Z$, such that, for each $s \in S$, the restriction $V_s = \mathcal{V}|\{s\} \times Z$ has regular semistable restriction to the generic fiber of π and the corresponding spectral cover is irreducible. Then there exists an $\epsilon_0 > 0$, depending on \mathcal{V}, H_0, H , such that, for every $s \in S$, V_s is is stable with respect to $\epsilon H_0 + H$ for all $0 < \epsilon < \epsilon_0$.

Proof. We may assume that S is irreducible. The proof of Theorem 7.1 goes through as before as long as we can uniformly bound the integers $c_1(W) \cdot H^i \cdot H_0^{d-i}$ as W ranges over subsheaves of V_s over all $s \in S$. But there exists a filtration

$$0 \subset F^0 \subset F^1 \subset \cdots \subset F^{n-1} = \mathcal{V}$$

such that F^j/F^{j-1} is a torsion free rank one sheaf on $S\times Z$, and thus is of the form $\mathcal{L}_j\otimes I_{\mathcal{X}_j}$ for \mathcal{L}_j a line bundle on $S\times Z$ and \mathcal{X}_j a subscheme of $S\times Z$ of codimension at least two, such that, at a generic point s of S, $(\{s\}\times Z)\cap \mathcal{X}_j$ has codimension at least two in Z. On a nonempty Zariski open subset Ω of S, the filtration restricts to a filtration of V_s of the form used in the proof of Lemma 7.2, and $c_1(\mathcal{L}_j|\{s\}\times Z)$ is independent of s. Similar filtrations exist for the exterior powers $\bigwedge^r \mathcal{V}$. This bounds $c_1(W)\cdot H^i\cdot H_0^{d-i}$ as W ranges over subsheaves of V_s over all s in a nonempty Zariski open subset of S. By applying the same construction to the components of $S-\Omega$ and induction on dim S, we can find the desired bound for all $s\in S$. \square

7.2. The case of an elliptic surface.

In case $\dim B = 1$, there is a more precise result.

Theorem 7.4. Let $\pi: Z \to B$ be an elliptic surface and let H_0 be an ample divisor on Z. Let f be the numerical equivalence class of a fiber. Let V be a vector bundle of rank n on Z which is regular and semistable on the generic fiber, with $\det V$ the pullback of a line bundle on B, and with $c_2(V) = c$, and such that the spectral cover of V is irreducible. Then for all $t \geq t_0 = \frac{n^3}{4}c_2(V)$, V is stable with respect to $H_0 + tf = H_t$.

Proof. If V is H_{t_0} -stable, then as it is f-stable (here stability is defined with respect to the nef divisor f) it is stable with respect to every convex combination of H_{t_0} and f and thus for every divisor H_t with $t \geq t_0$. Thus we may assume that V is not H_{t_0} -stable for some $t_0 \geq 0$.

Lemma 7.5. Suppose that V is not H_{t_0} -stable for some $t_0 \ge 0$. Then there exists $a \ t_1 \ge t_0$ and $a \ divisor \ D$ such that $D \cdot H_{t_1} = 0$ and

$$-\frac{n^3}{2}c_2(V) \le D^2 < 0.$$

Proof. By Theorem 7.1, for all $t \gg 0$, V is H_t -stable. Let t_1 be the greatest lower bound of the t such that, for all $t' \geq t$, V is $H_{t'}$ -stable. Thus $t_1 \geq 0$. The condition that V is H_t -unstable is clearly an open condition on t. It follows that V is strictly H_{t_1} -semistable, so that there is an exact sequence

$$0 \to V' \to V \to V'' \to 0$$
.

with both V', V'' torsion free and of strictly smaller rank than V, and with $\mu_{H_{t_1}}(V') = \mu_{H_{t_1}}(V'') = \mu_{H_{t_1}}(V)$. Thus, both V' and V'' are H_{t_1} -semistable. Let $D = r'c_1(V'') - r''c_1(V')$. Then the equality $\mu_{H_{t_1}}(V') = \mu_{H_{t_1}}(V'')$ is equivalent to $D \cdot H_{t_1} = 0$. Note that D is not numerically trivial, for otherwise V would be strictly H_t -semistable for all t, contradicting the fact that it is H_t -stable for $t \gg 0$. Thus, by the Hodge index theorem, $D^2 < 0$. Now, for a torsion free sheaf of rank r, define the Bogomolov number (or discriminant) of W by

$$B(W) = 2rc_2(W) - (r-1)c_1(W)^2.$$

If W is semistable with respect to some ample divisor, then $B(W) \ge 0$. Finally, we have the identity ([5], Chapter 9, ex. 4):

$$B(V) = 2nc_2(V) = \frac{n}{r'}B(V') + \frac{n}{r''}B(V'') - \frac{D^2}{r'r''}$$

and thus, as $B(V') \geq 0$ and $B(V'') \geq 0$ by Bogomolov's inequality,

$$D^2 \ge -(r'r'')2nc_2(V).$$

Since r' + r'' = n, $r'r'' \le n^2/4$, and plugging this in to the above inequality proves the lemma. \square

Returning to the proof of Theorem 7.4, the proof of Lemma 1.2 in Chapter 7 of [6] (see also [5], Chapter 6, Lemma 3) shows that, if $t \geq t_0 = \frac{n^3}{4}c_2(V)$, then for every divisor D such that $D^2 \geq -\frac{n^3}{2}c_2(V)$ and $D \cdot f > 0$, we have $D \cdot H_t > 0$. Now, if V is not H_{t_0} -stable, we would be able to find a $t_1 \geq t_0$ and an exact sequence $0 \to V' \to V \to V'' \to 0$ as above, with $\mu_{H_{t_1}}(V') = \mu_{H_{t_1}}(V'')$. Now setting $D = r'c_1(V'') - r''c_1(V')$ as before, we have

$$0 < \mu_f(V) - \mu_f(V') = \frac{c_1(V') \cdot f + c_1(V'') \cdot f}{n} - \frac{c_1(V') \cdot f}{r'}$$
$$= \frac{(r'c_1(V'') - r''c_1(V')) \cdot f}{r'n} = \frac{D \cdot f}{r'n},$$

so that $D \cdot f > 0$, and likewise $D \cdot H_{t_1} = 0$. Thus $D \cdot H_{t_0} < 0$, contradicting the choice of t_0 . It follows that, for all $t \geq t_0 = \frac{n^3}{4}c_2(V)$, V is H_t -stable. This completes the proof of (7.5). \square

As a final comment, the difficulty in finding an effective bound in case dim B>1 is the following: For a torsion free sheaf W, we can define B(W) as before, but it is an element of $H^4(Z)$, not an integer. In the notation of the proof of Lemma 7.5, Bogomolov's inequality can be used to give a bound for $B(V') \cdot H_t^{n-2}$ and $B(V'') \cdot H_t^{n-2}$ for some (unknown) value of t, and thus there is a lower bound for $D^2 \cdot H_t^{n-2}$, again for one unknown value of t. However this does not seem to give enough information to complete the proof of the theorem, except in the rank two case where the lower bound can be made explicit for all t.

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