MIRROR SYMMETRY, HITCHIN'S EQUATIONS, AND LANGLANDS DUALITY

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ABSTRACT. Geometric Langlands duality can be understood from statements of mirror symmetry that can be formulated in purely topological terms for an oriented two-manifold C. But understanding these statements is extremely difficult without picking a complex structure on C and using Hitchin's equations. We sketch the essential statements both for the "unramified" case that C is a compact oriented two-manifold without boundary, and the "ramified" case that one allows punctures. We also give a few indications of why a more precise description requires a starting point in four-dimensional gauge theory.

1. The A-Model and the B-Model

Let G be a compact Lie group and let $G_{\mathbb{C}}$ be its complexification. And let C be a compact oriented two-manifold without boundary. We write $\mathcal{Y}(G,C)$ (or simply $\mathcal{Y}(G)$ or \mathcal{Y} if the context is clear) for the moduli space of flat $G_{\mathbb{C}}$ bundles $E \to C$, modulo gauge transformations. Equivalently, $\mathcal{Y}(G,C)$ parametrizes homomorphisms¹ of the fundamental group of C to $G_{\mathbb{C}}$.

 $\mathcal{Y}(G,C)$ is in a natural way a complex symplectic manifold, that is a complex manifold with a nondegenerate holomorphic two-form. The complex structure comes simply from the complex structure of $G_{\mathbb{C}}$, and the symplectic form, which we call Ω , comes from the intersection pairing² on $H^1(C, \mathrm{ad}(E))$, where $\mathrm{ad}(E)$ is the adjoint bundle associated to a flat bundle E. Since $\mathcal{Y}(G,C)$ is a complex symplectic manifold, in particular it follows that its canonical line bundle is naturally trivial.

Geometric Langlands duality is concerned with certain topological field theories associated with $\mathcal{Y}(G,C)$. The most basic of these are the *B*-model that is defined by viewing $\mathcal{Y}(G,C)$ as a complex manifold with trivial canonical bundle, and the *A*-model that is defined by viewing it as a real symplectic manifold with symplectic form³ $\omega = \text{Im }\Omega$.

Date: February, 2007.

Supported in part by NSF Grant Phy-0503584.

¹Actually, it is best to define $\mathcal{Y}(G,C)$ as a geometric invariant theory quotient that parametrizes stable homomorphisms plus equivalence classes of semi-stable ones. This refinement will not concern us here. See section 6.1.

²The definition of this intersection pairing depends on the choice of an invariant quadratic form on the Lie algebra of G. It can be shown using Hitchin's \mathbb{C}^* action on the moduli space of Higgs bundles that the A-model that we define shortly is independent of this choice, up to a natural isomorphism. The geometric Langlands duality that one ultimately defines likewise does not depend on this choice.

³The usual definition of Ω is such that Im Ω is cohomologically trivial, while Re Ω is not. The fact that $\omega = \operatorname{Im} \Omega$ is cohomologically trivial is a partial explanation of the fact, mentioned in the last footnote, that the A-model of $\mathcal Y$ is invariant under scaling of ω .

These are the topological field theories that are relevant to the most basic form of geometric Langlands duality. However, there is also a generalization that is relevant to what is sometimes called quantum geometric Langlands. From the A-model side, it is obvious that a generalization is possible, since we could use a more general linear combination of Re Ω and Im Ω in defining the A-model. What is less evident is that the B-model can actually be deformed, as a topological field theory, into this family of A-models. This rather surprising fact is natural from the point of view of generalized complex geometry, see Hitchin (2003), and has been explained from that point of view in section 4.6 of Gualtieri (2003), as a general statement about complex symplectic manifolds. In Kapustin & Witten (2007), sections 5.2 and 11.3, it was shown that quantum geometric Langlands is naturally understood in precisely this setting.

Here, however, to keep things simple, we will focus on the most basic B-model and A-model that were just described.

2. Mirror Symmetry And Hitchin's Equations

The next ingredient we need is Langlands or Goddard-Nuyts-Olive duality. To every compact Lie group G is naturally associated its dual group LG . For example, the dual of SU(N) is $PSU(N) = SU(N)/\mathbb{Z}_N$, the dual of E_8 is E_8 , and so on. And we must also recall the concept of mirror symmetry between A-models and B-models (for example, see Hori et al. (2003)). This is a quantum symmetry of two-dimensional nonlinear sigma models whose most basic role is to transform questions of complex geometry into questions of symplectic geometry. The geometric Langlands correspondence does not appear at first sight to be an example of mirror symmetry, but it turns out that it is.

With a little bit of hindsight (the question was first addressed in Hausel & Thaddeus (2002), following earlier work by Bershadsky et al. (1995), and Harvey et al. (1995)), we may ask whether the B-model of $\mathcal{Y}(^LG,C)$ may be mirror to the A-model of $\mathcal{Y}(G,C)$. Even once this question is asked, it is difficult to answer it without some additional structure. The additional structure that comes in handy is provided by Hitchin's equations, see Hitchin (1987a). Until this point, C has simply been an oriented two-manifold (compact and without boundary). But now we pick a complex structure and view C as a complex Riemann surface. Hitchin's equations with gauge group G are equations for a pair (A,ϕ) . Here A is a connection on a G-bundle $E \to C$ (we stress that the structure group of E is now the compact group G), and ϕ is a one-form on C with values in $\mathrm{ad}(E)$. Hitchin's equations, which are elliptic modulo the gauge group, are the system:

(2.1)
$$F - \phi \wedge \phi = 0$$
$$D\phi = D \star \phi = 0.$$

Here \star is the Hodge star operator determined by the complex structure on C. The role of the complex structure of C is that it enables us to write the last of these equations.

A solution of Hitchin's equations has two interpretations. On the one hand, given such a solution, we can define the complex-valued connection $\mathcal{A}=A+i\phi$. Hitchin's equations imply that the corresponding curvature $\mathcal{F}=d\mathcal{A}+\mathcal{A}\wedge\mathcal{A}$ vanishes, so a

solution of Hitchin's equations defines a complex-valued flat connection, and thus a point in $\mathcal{Y}(G,C)$.

On the other hand, the (0,1) part of the connection A determines a $\bar{\partial}$ operator on the bundle E (or rather its complexification, which we also call E). There is no integrability condition on $\bar{\partial}$ operators in complex dimension 1, so this $\bar{\partial}$ operator endows E with a complex structure; it becomes a holomorphic $G_{\mathbb{C}}$ bundle over C. Moreover, let us write $\phi = \varphi + \bar{\varphi}$, where φ and $\bar{\varphi}$ are the (1,0) and (0,1) parts of φ . Then Hitchin's equations imply that φ , regarded as a section of $K \otimes \mathrm{ad}(E)$ (with K the canonical line bundle of C), is holomorphic. The pair (E, φ) , where $E \to C$ is a holomorphic $G_{\mathbb{C}}$ bundle and $\varphi \in H^0(C, K \otimes \mathrm{ad}(E))$, is known as a Higgs bundle.

We write \mathcal{M}_H for the moduli space of solutions of Hitchin's equations, modulo a gauge transformation. The fact that a solution of these equations can be interpreted in two different ways means that \mathcal{M}_H is endowed with two different natural complex structures. In one complex structure, which has been called I, \mathcal{M}_H parametrizes isomorphism classes of semistable Higgs bundles (E,φ) . In another complex structure, J, it parametrizes equivalence classes of flat $G_{\mathbb{C}}$ -bundles or in other words homomorphisms $\rho:\pi_1(C)\to G_{\mathbb{C}}$. I, J, and K=IJ fit together to a natural hyper-Kahler structure on \mathcal{M}_H , as described in Hitchin (1987a). In particular, there are holomophic two-forms $\Omega_I,\Omega_J,\Omega_K$ and Kahler forms $\omega_I,\omega_J,\omega_K$. These are all related by $\Omega_I=\omega_J+i\omega_K$, and cyclic permutations of this statement, as is usual in hyper-Kahler geometry.

In complex structure J, \mathcal{M}_H is the same as the variety \mathcal{Y} that we described earlier. The natural holomorphic symplectic form Ω of \mathcal{Y} is the same as $i\Omega_J$. And the real symplectic form $\omega = \operatorname{Im} \Omega$ used in defining the A-model coincides with ω_K . Complex structure J and the holomorphic symplectic form $\Omega_J = \omega_K + i\omega_I$ do not depend on the chosen complex structure on C, in contrast to the rest of the hyper-Kahler structure of \mathcal{M}_H .

Remark 2.1. As an aside, one may ask how closely related ϕ , known in the present context as the Higgs field, is to the Higgs fields of particle physics. Thus, to what extent is the terminology that was introduced in Hitchin (1987a) actually justified? The main difference is that Higgs fields in particle physics are scalar fields, while ϕ is a one-form on C (valued in each case in some representation of the gauge group). However, although Hitchin's equations were first written down and studied directly, they can be obtained from $\mathcal{N}=4$ supersymmetric gauge theory via a sort of twisting procedure (similar to the procedure that leads from $\mathcal{N}=2$ supersymmetric gauge theory to Donaldson theory). In this twisting procedure, some of the Higgs-like scalar fields of $\mathcal{N}=4$ super Yang-Mills theory are indeed converted into the Higgs field that enters in Hitchin's equations. This gives a reasonable justification for the terminology.

As we will explain next, it is possible, with the aid of Hitchin's equations, to answer the question of whether the B-model of $\mathcal{Y}(^L\!G,C)$ is mirror to the A-model of $\mathcal{Y}(G,C)$. This in fact was first pointed out in Hausel & Thaddeus (2002), and used in Kapustin & Witten (2007) as a key ingredient in understanding the geometric Langlands correspondence.

3. The Hitchin Fibration

We will have to use the Hitchin fibration. This is the map, holomorphic in complex structure I, that takes a Higgs bundle (E,φ) to the characteristic polynomial

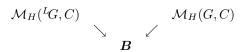
of φ . For example, for G = SU(2), (E, φ) is mapped simply to the quadratic differential det φ . The target of the Hitchin fibration is thus in this case the space $\mathbf{B} = H^0(C, K^2)$ that parametrizes quadratic differentials. This has a natural analog for any G. From the standpoint of complex structure I, the generic fiber of the map $\pi : \mathcal{M}_H \to \mathbf{B}$ is a complex abelian variety (or to be slightly more precise, in general a torsor for one). The fibers are Lagrangian from the standpoint of the holomorphic symplectic form Ω_I . Such a fibration by complex Lagrangian tori turns \mathcal{M}_H into a completely integrable Hamiltonian system in the complex sense Hitchin (1987b).

There is, however, another way to look at the Hitchin fibration, as first described in Hausel & Thaddeus (2002). Let us go back to the A-model defined with the real symplectic structure ω . Since the fibers of $\pi: \mathcal{M}_H \to \mathbf{B}$ are Lagrangian for $\Omega_I = \omega_J + i\omega_K$, they are in particular Lagrangian for $\omega = \omega_K$. Moreover, being holomorphic in complex structure I, these fibers are actually area-minimizing in their homology class – here areas are computed using the hyper-Kahler metric on \mathcal{M}_H . So the Hitchin fibration, from the standpoint of the A-model, is actually a fibration of \mathcal{M}_H by special Lagrangian tori.

Mirror symmetry is believed to arise from T-duality on the fibers of a special Lagrangian fibration, see Strominger et al. (1996). Generally, it is very difficult to explicitly exhibit a non-trivial special Lagrangian fibration. The present example is one of the few instances in which this can actually be done, with the aid of the hyper-Kahler structure of \mathcal{M}_H and its integrable nature. Non-trivial special Lagrangian fibrations are hard to understand because it is difficult to elucidate the structure of the singularities. In the hyper-Kahler context, the fact that the fibers are holomorphic in a different complex structure makes everything far more accessible.

Once we actually find a special Lagrangian fibration, what we are supposed to do with it, in order to give an example of mirror symmetry, is to construct the dual special Lagrangian fibration, which should be mirror to the original one. The mirror map exchanges the symplectic structure on one side with the complex structure on the other side.

In the present context, there is a very beautiful description of the dual fibration: it is, as first shown in Hausel & Thaddeus (2002), simply the Hitchin fibration of the dual group. Thus one considers $\mathcal{M}_H(^L\!G,C)$, the moduli space of solutions of Hitchin's equation for the dual group $^L\!G$. It turns out that the bases of the Hitchin fibrations for G and $^L\!G$ can be identified in a natural way. The resulting picture is something like this:



In complex structure I, the fibers over a generic point $b \in \mathbf{B}$ are, roughly speaking, dual abelian varieties (more precisely, they are torsors for dual abelian varieties).

Alternatively, the fibers are special Lagrangian submanifolds in the symplectic structure $\omega = \omega_K$. From this second point of view, the same picture leads to a mirror symmetry between the *B*-model of $\mathcal{M}_H(^L\!G,C)$ in complex structure J and the A-model of $\mathcal{M}_H(G,C)$ with symplectic structure ω_K .

As we have just explained, the tools that make this mirror symmetry visible are the hyper-Kahler structure of \mathcal{M}_H and its Hitchin fibration. Those structures depend on the choice of a complex structure on C, but in fact, the resulting mirror symmetry does not really depend on this choice. This was shown in Kapustin & Witten (2007) in the process of deriving this example of mirror symmetry from a four-dimensional topological field theory. The topological field theory in question is obtained by twisting of $\mathcal{N}=4$ super Yang-Mills theory.

3.1. A Few Hints. There are a few obstacles to overcome to go from this instance of mirror symmetry to the usual formulation of geometric Langlands duality. Unfortunately, it will not be practical here to give more than a few hints.

One key point is that in the usual formulation, the dual of a B-brane on $\mathcal{M}_H(^LG,C)$ is supposed to be not an A-brane on $\mathcal{M}_H(G,C)$ — which is what we most naturally get from the above construction — but a sheaf of \mathcal{D} -modules on $\mathcal{M}(G,C)$, the moduli space of G-bundles over C (a sheaf of \mathcal{D} -modules is by definition a sheaf of modules for the sheaf \mathcal{D} of differential operators on $\mathcal{M}(G,C)$). The link between the two statements is explained in Kapustin & Witten (2007), section 11, using the existence of a special A-brane on $\mathcal{M}_H(G,C)$ that is intimately related to differential operators on $\mathcal{M}(G,C)$. This relation is possible because, as explained in Hitchin (1987a), $\mathcal{M}_H(G,C)$ contains $T^*\mathcal{M}^{st}(G,C)$ as a Zariski open set; here $\mathcal{M}^{st}(G,C)$ is the subspace of $\mathcal{M}(G,C)$ parametrizing strictly stable bundles.

Another key point is the following. A central role in the usual formulation is played by the geometric Hecke operators, which act on holomorphic G-bundles over C and therefore also on \mathcal{D} -modules on $\mathcal{M}(G,C)$. They have a natural role in the present story, but this is one place that one misses something if one attempts to express this subject just in terms of two-dimensional sigma models and mirror symmetry. This particular instance of mirror symmetry actually originates from a duality in an underlying four-dimensional gauge theory. Once this is understood, basic facts about the Wilson and 't Hooft line operators of gauge theory lead to the usual statements about Hecke eigensheaves, as explained in sections 9 and 10 of Kapustin & Witten (2007). The geometric Hecke operators are naturally reinterpreted in this context in terms of the Bogomolny equations of three-dimensional gauge theory, which are of great geometrical as well as physical interest and have been much studied, for example in Atiyah & Hitchin (1988).

A proper formulation of some of these statements leads to another important role for four dimensions. The usual formulation of geometric Langlands involves \mathcal{D} -modules not on the moduli space of semistable G-bundles but on the moduli stack of all G-bundles. The main reason for this is that one cannot see the action of the Hecke operators if one considers only semistable bundles. As we will explain in section 6, the role of stacks in the standard description can be understood as a strong clue for an alternative approach that starts in four-dimensional gauge theory.

4. Ramification

Before getting back to stacks, however, I want to give an idea of what is called "ramification" in the context of geometric Langlands.

A simple generalization of what we have said so far is to consider flat bundles not on a closed oriented two-manifold C but on a punctured two-manifold $C' = C \setminus p$; that is, C' is C with a point p omitted.

We pick a conjugacy class $\mathfrak{C} \subset G_{\mathbb{C}}$, and we let $\mathcal{Y}(G,C';\mathfrak{C})$ denote the moduli space of homomorphisms $\rho: \pi_1(C') \to G_{\mathbb{C}}$, up to conjugation, such that the monodromy around p is in the conjugacy class \mathfrak{C} .

Many statements that we made before have natural analogs in this punctured case. In particular, $\mathcal{Y}(G,C';\mathfrak{C})$ has a natural structure of a complex symplectic manifold. It has a natural complex structure and holomorphic symplectic form Ω . Just as in the unpunctured case, we can define a B-model of $\mathcal{Y}(G,C';\mathfrak{C})$. Also, viewing $\mathcal{Y}(G,C';\mathfrak{C})$ as a real symplectic manifold with symplectic form $\omega=\operatorname{Im}\Omega$, we can define an A-model. The B-model and the A-model are both completely independent of the complex structure of C'.

Next, introduce the dual group ${}^L\!G$ and let ${}^L\!\mathcal{C}$ denote a conjugacy class in its complexification. Again, the space $\mathcal{Y}({}^L\!G,C';{}^L\!\mathcal{C})$ has a natural B-model and A-model.

Based on what we have said so far, one may wonder if, for some map between \mathcal{C} and $^L\mathcal{C}$, there might be a mirror symmetry between $\mathcal{Y}(G,C';\mathcal{C})$ and $\mathcal{Y}(^LG,C';^L\mathcal{C})$. The answer to this question is "not quite," for a number of reasons. One problem is that there is no natural correspondence between conjugacy classes in $G_{\mathbb{C}}$ and in $^LG_{\mathbb{C}}$. A more fundamental problem is that the B-model of $\mathcal{Y}(G,C';\mathcal{C})$ varies holomorphically with the conjugacy class \mathcal{C} , but the A-model of the same space does not. To find a version of the statement that has a chance of being right, we have to add additional parameters to find a mirror-symmetric set.

In any event, regardless of what parameters one adds, it is very difficult to answer the question about mirror symmetry if C' is viewed simply as an oriented two-manifold with a puncture. To make progress, just as in the unramified case (that is, the case without punctures), it is very helpful to endow C' with a complex structure and to use Hitchin's equations. This actually also helps us in finding the right parameters, because an improved set of parameters appears just in trying to give a natural formulation of Hitchin's equations on a punctured surface. Let z be a local parameter near the puncture and write $z = re^{i\theta}$. In the punctured case, it is natural, see Simpson (1990), to introduce variables α, β, γ taking values in the Lie algebra t of a maximal torus $T \subset G$, and consider solutions of Hitchin's equations on C' whose behavior near z = 0 is as follows:

$$(4.1) A = \alpha \, d\theta + \dots$$

$$\phi = \beta \, \frac{dr}{r} - \gamma \, d\theta + \dots$$

The ellipses refer to terms that are less singular near z = 0.

All the usual statements about Hitchin's equations have close analogs in this situation. The moduli space of solutions of Hitchin's equations with this sort of singularity is a hyper-Kahler manifold $\mathcal{M}_H(G,C;\alpha,\beta,\gamma)$. In one complex structure, usually called J, it coincides with $\mathcal{Y}(G,C;\mathfrak{C})$, where \mathfrak{C} is the conjugacy class that contains⁴ $U=\exp(-2\pi(\alpha-i\gamma))$. In another complex structure, often called I, $\mathcal{M}_H(G,C;\alpha,\beta,\gamma)$ parametrizes Higgs bundles (E,φ) , where $\varphi\in H^0(C',K\otimes \mathrm{ad}(E))$ has a pole at z=0, with $\varphi\sim\frac{1}{2}(\beta+i\gamma)(dz/z)$. Moreover, there is a Hitchin fibration, and most of the usual statements about the unramified case – those that we have explained and those that we have omitted here – have close analogs. For

 $^{^4}$ For simplicity, we assume that U is regular. The more involved statement that holds in general is explained in Gukov & Witten (2006).

a much more detailed explanation, and references to the original literature, see Gukov & Witten (2006).

The variables α, β, γ are a natural set of parameters for the classical geometry. However, quantum mechanically, there is one more natural variable η (analogous to the usual θ -angles of gauge theory), as described in section 2.3 of Gukov & Witten (2006). With the complete set of parameters $(\alpha, \beta, \gamma, \eta)$ at hand, it is possible to formulate a natural duality statement, according to which $\mathcal{M}_H(^LG, C; ^L\alpha, ^L\beta, ^L\gamma, ^L\eta)$ is mirror to $\mathcal{M}_H(G, C; \alpha, \beta, \gamma, \eta)$, with a certain map between the parameters, described in section 2.4 of Gukov & Witten (2006). The main point of this map is that $(\alpha, \eta) = (^L\eta, ^{-L}\alpha)$. Since the monodromy U depends on $^L\alpha$, this shows that the dual of the monodromy involves the quantum parameter η that is invisible in the classical geometry. In the A-model, η becomes the imaginary part of the complexified Kahler class.

This duality statement leads, after again mapping A-branes to \mathcal{D} -modules, to a statement of geometric Langlands duality for this situation similar to what has been obtained via algebraic geometry and two-dimensional conformal field theory in Frenkel & Gaitsgory (2005).

Remark 4.1. We pause here to explain one very elementary fact about the classical geometry that will be helpful as background for section 5. In complex structure J, a solution of Hitchin's equations with the singularity of eqn. (4.1) describes a flat $G_{\mathbb{C}}$ bundle $E \to C'$ with monodromy around the puncture p. E can be extended over p as a holomorphic bundle, though of course not as a flat one, and moreover from a holomorphic point of view, E can be trivialized near p. The flat connection on $E \to C'$ is then represented, in this gauge, by a holomorphic (1,0)-form on C' (valued in the Lie algebra of $G_{\mathbb{C}}$) with a simple pole at p:

(4.3)
$$A = dz \left(\frac{\alpha - i\gamma}{iz} + \dots \right),$$

where the omitted terms are regular at z=0. The singularity of the connection at z=0 is a simple pole because the ansatz (4.1) for Hitchin's equations only allows a singularity of order 1/|z|. A holomorphic connection with such a simple pole is said to have a regular singularity.

In geometric Langlands, what is usually called tame ramification is, roughly speaking, the case that we have just arrived at: a holomorphic bundle $E \to C$ that has a holomorphic connection form with a regular singularity. Actually, the phrase "tame ramification" is sometimes taken to refer to the case that the residue of the simple pole is nilpotent, while in eqn. (4.3) we seem to be in the opposite case of semi-simple residue. In Gukov & Witten (2006), it is explained that, with some care, mirror symmetry for $\mathcal{M}_H(G,C';\alpha,\beta,\gamma,\eta)$ is actually a sufficient framework to understand geometric Langlands for a connection with a simple pole of any residue. For example, the case of a nilpotent residue can be understood by setting $L_{\alpha} = L_{\gamma} = 0$ (or $\gamma = \eta = 0$ in the dual description).

5. WILD RAMIFICATION

Based on an analogy with number theory, geometric Langlands is usually formulated not only for the case of tame ramification. One goes on to inquire about an analogous duality statement involving a holomorphic bundle $E \to C$ with a holomorphic connection that has a pole of any order. In other words, after trivializing

the holomorphic structure of E near a point $p \in C$, the connection looks like

(5.1)
$$\mathcal{A} = dz \left(\frac{T_n}{z^n} + \frac{T_{n-1}}{z^{n-1}} + \dots + \frac{T_1}{z} + \dots \right),$$

where regular terms are omitted. A meromorphic connection with a pole of degree greater than 1 is said to have an irregular singularity.

Trying to formulate a duality statement for this situation poses, at first sight, a severe challenge for the approach to geometric Langlands described here. Our basic point of view is that the fundamental duality statements depend on C only as an oriented two-manifold. A complex structure on C is introduced only as a tool to answer certain natural questions that can be asked without introducing the complex structure.

From this point of view, tame ramification is natural because a simple pole in this sense has a clear topological meaning. A meromorphic connection with a simple pole at a point $p \in C$ is a natural way to encode the monodromy about p of a flat connection on $C' = C \setminus p$. And this monodromy, of course, is a purely topological notion. But what could possibly be the topological meaning of a connection with a pole of degree greater than 1?

A closely related observation is that T_1 is the residue of the pole in \mathcal{A} at z=0, and so is independent of the choice of local coordinate z. However, the coefficients T_2, \ldots, T_n of the higher order poles most definitely do depend on the choice of a local coordinate. How can we hope to include them in a theory that is supposed to depend on C only as an oriented two-manifold?

Moreover, if the plan is to formulate a duality conjecture of a topological nature and then prove it using Hitchin's equations, we face the question of whether Hitchin's equations are compatible with an irregular singularity. Hitchin's equations for a pair $\Phi=(A,\phi)$ are schematically of the form $d\Phi+\Phi^2=0$. If near z=0, we have a singularity with $|\Phi|\sim 1/|z|^n$, then $|d\Phi|\sim 1/|z|^{n+1}$ and $|\Phi|^2\sim 1/|z|^{2n}$. For n=1, $d\Phi$ and $|\Phi|^2$ are comparable in magnitude, and therefore Hitchin's equations look reasonable. However, for n>1, we have $|\Phi|^2>>|d\Phi|$, and it looks like the nonlinear term in Hitchin's equations will be too strong.

Both questions, however, have natural answers. The answer to the first question is that, despite appearances, one actually can associate to a connection with irregular singularity something that goes beyond the ordinary monodromy and has a purely topological meaning. The appropriate concept is an extended monodromy that includes Stokes matrices as well as the ordinary monodromy. Stokes matrices are part of the classical theory of ordinary differential equations with irregular singularity; for example, see Wasow (1965).

Assuming for brevity that the leading coefficient T_n of the singular part of the connection is regular semi-simple, one can make a gauge transformation to conjugate T_1, \ldots, T_n to the maximal torus. Then one defines a moduli space $\mathcal{Y}(G, C; T_1, \ldots, T_n)$ that parametrizes, up to a gauge transformation, pairs consisting of a holomorphic $G_{\mathbb{C}}$ -bundle over C and a connection with an irregular singularity of the form described in eqn. (5.1). As shown in Boalch (2001), it turns out that this space $\mathcal{Y}(G, C; T_1, \ldots, T_n)$ is in a natural way a complex symplectic manifold, with a complex symplectic structure that depends on C only as an oriented two-manifold. This can be proved by adapting to the present setting the gauge theory definition of the symplectic structure, formulated in Atiyah & Bott (1982).

Moreover the complex symplectic structure of $\mathcal{Y}(G, C; T_1, \ldots, T_n)$ is independent of T_2, \ldots, T_n (as long as T_n remains semi-simple).

At this point the important concept of isomonodromic deformation, introduced by Jimbo et al. (1981), comes into play. There is a natural way to vary the parameters T_2, \ldots, T_n , without changing the generalized monodromy data that is parametrized by $\mathcal{Y}(G,C;T_1,\ldots,T_n)$. Moreover, as has been proved quite recently in Boalch (2001), the complex symplectic structure of the space of generalized monodromy data is invariant under isomonodromic deformation. Thus, roughly speaking, one can define a complex symplectic manifold $\mathcal{Y}(G,C;T_1,n)$ that depends only on T_1 and the integer $n \geq 1$.

The fact that the parameters T_2, \ldots, T_n turn out to be, in the sense just described, inessential, is certainly welcome, since as we have already observed, they have no evident topological meaning. Now we are in a situation very similar to what we had in the unramified and tamely ramified cases. Given $\mathcal{Y}(G, C; T_1, n)$ as a complex symplectic manifold, with complex symplectic form Ω , we can define its B-model, or its A-model using the real symplectic form $\omega = \text{Im }\Omega$. Of course, we can do the same for the dual group, defining another complex symplectic manifold $\mathcal{Y}(^LG, C; ^LT_1, n)$, with its own B-model and A-model. And, just as in the unramified case, we can ask if these two models are mirror to each other.

Even before trying to answer this question, we should refine it slightly. Because of the constraint that T_n should be regular semi-simple, it is not quite correct to simply forget about T_n . There can be monodromies as T_n varies. We think of T_n as taking values in $\mathfrak{t}_{\mathbb{C}}^{\mathrm{reg}} \otimes K_p^{n-1}$, with notation as follows: $\mathfrak{t}_{\mathbb{C}}$ is the Lie algebra of a maximal torus in $G_{\mathbb{C}}$, $\mathfrak{t}_{\mathbb{C}}^{\mathrm{reg}}$ is its subspace consisting of regular elements, and K_p is the fiber at p of the cotangent bundle to C. The fundamental group of $\mathfrak{t}_{\mathbb{C}}^{\mathrm{reg}}$ is known as the braid group of G; we call it B(G). Because of the monodromies, one really needs to choose a basepoint $* \in \mathfrak{t}^{\text{reg}}_{\mathbb{C}}$ to define $\mathcal{Y}({}^{L}\!G,C;{}^{L}T_{1},n);$ to be more precise, we can denote this space as $\mathcal{Y}({}^{L}G,C;{}^{L}T_{1},n,*)$. The group B(G) acts via monodromies on both the B-model and the A-model of $\mathcal{Y}(G,C;T_1,n,*)$. Dually, the corresponding braid group $B(^LG)$ acts on the B-model and the A-model of $\mathcal{Y}(^LG,C;^LT_1,n,*)$. However, the two groups B(G) and $B({}^{L}G)$ are naturally isomorphic; indeed, modulo a choice of an invariant quadratic form, there is a natural map from $\mathfrak{t}_{\mathbb{C}}^{\text{reg}}$ to ${}^{L}\mathfrak{t}_{\mathbb{C}}^{\text{reg}}$, so the two spaces have the same fundamental group and a choice of basepoint in one determines a basepoint in the other, up to homotopy. A better (but still not yet precise) question is whether there is a mirror symmetry between $\mathcal{Y}(G,C;T_1,n,*)$ and $\mathcal{Y}({}^{L}G, C; {}^{L}T_{1}, n, *)$ that commutes with the braid group.

We expect as well that this mirror symmetry depends on C only as an oriented two-manifold, and so commutes with the mapping class group. We can think of the mapping class group of C and the braid group as playing quite parallel roles. In fact, because of the appearance of the fiber K_p of the canonical bundle in the last paragraph, these two groups do not simply commute with each other; the group that acts is an extension of the mapping class group by B(G).

Just as in the tamely ramified case, to get the right mirror symmetry conjecture, we need to extend the parameters slightly to get a mirror-symmetric set. But we also face the fundamental question of whether Hitchin's equations are compatible with wild ramification. As explained above, the nonlinearity of Hitchin's equations makes this appear doubtful at first sight. But happily, it turns out that all is well, as shown in Biquard & Boalch (2004). The key point is that, again with T_n assumed

to be regular semi-simple, we can assume that the singular part of the connection is abelian. Though Hitchin's equations are nonlinear, they become linear in the abelian case, and once abelianized, they are compatible with a singularity of any order. Using this as a starting point, it turns out that, for any n, one can develop a theory of Hitchin's equations with irregular singularity that is quite parallel to the more familiar story in the unramified case. For example, the moduli space \mathcal{M}_H of solutions of the equations is hyper-Kahler. In one complex structure, \mathcal{M}_H parametrizes flat connections with a singularity similar to that in eqn. (5.1); in another complex structure, it parametrizes Higgs bundles (E,φ) in which φ has an analogous pole of order n. There is a Hitchin fibration, and all the usual properties have close analogs.

All this gives precisely the right ingredients to use Hitchin's equations to establish the desired mirror symmetry between the two moduli spaces. See Witten (2007) for a detailed explanation in which this classical geometry is embedded in four-dimensional gauge theory. Many of the arguments are quite similar to those given in the tame case in Gukov & Witten (2006). The construction makes it apparent that the duality commutes with isomonodromic deformation.

Finally, one might worry that the assumption that T_n is regular semi-simple may have simplified things in some unrealistic way. This is actually not the case. For one thing, the analysis in Biquard & Boalch (2004) requires only that T_2, \ldots, T_n should be simultaneously diagonalizable (in some gauge), and in particular semi-simple, but not that T_n is regular. But even if these coefficients are not semi-simple, there is no essential problem. In the classical theory of ordinary differential equations, it is shown that given any such equation with an irregular singularity at z=0, after possibly passing to a finite cover of the punctured z-plane and changing the extension of a holomorphic bundle over the puncture at z=0, one can reduce to the case that the irregular part of the singularity has the properties assumed in Biquard & Boalch (2004). Given this, one can adapt all the relevant arguments concerning geometric Langlands duality to the more general case, as is explained in section 6 of Witten (2007).

6. Four-Dimensional Gauge Theory And Stacks

To a physicist, it is natural, in studying dualities involving gauge theory, to begin in four dimensions, which is often found to be the natural setting for gauge theory duality. There is a simple reason for this. The curvature, which is one of the most fundamental concepts in gauge theory, is a 2-form. In d dimensions, the dual of a 2-form is a (d-2)-form, so it is only a 2-form if d=4. This suggests that d=4 is the most natural dimension in which the dual of a gauge theory might be another gauge theory.

Moreover, $\mathcal{N}=4$ supersymmetric Yang-Mills theory, originally constructed in Brink et al. (1977), is a natural place to start, as it has the maximal possible supersymmetry, and has the celebrated duality whose origins go back to the early work of Montonen & Olive (1977). It indeed turns out that geometric Langlands has a natural origin in a twisted version of $\mathcal{N}=4$ super Yang-Mills theory in four dimensions. The twisting is quite analogous to the twisting of $\mathcal{N}=2$ super Yang-Mills theory that leads to Donaldson theory.

That particular motivation may seem opaque to some, and instead I will adopt here a different approach in explaining why it is natural to begin in four dimensions for understanding geometric Langlands, instead of relying only on the B-model and A-model of $\mathcal{M}_H(G,C)$.

First of all, the B-model and the A-model of any space X are both twisted versions of a quantum sigma model that governs maps $\Phi: \Sigma \to X$, where Σ is a two-manifold (or better, a supermanifold of bosonic dimension two). Since the A-model involves in its most elementary form a counting of holomorphic maps $\Phi: \Sigma \to X$ that obey appropriate conditions, the roles of Σ and Φ are clear in the A-model. Mirror symmetry indicates that it must be correct to also formulate the B-model in terms of maps $\Phi: \Sigma \to X$, and this is done in the usual formulation by physicists.

In the present case, we are interested, roughly speaking, in the B- and A-models of $\mathcal{M}_H(G,C)$, for some compact Lie group G and two-manifold C. Therefore, roughly speaking, we want to study a sigma model of maps $\Phi: \Sigma \to \mathcal{M}_H(G,C)$, where as before Σ is an auxiliary two-manifold.

The reason that this description is rough is that $\mathcal{M}_H(G,C)$ has singularities,⁵ and the sigma model of target $\mathcal{M}_H(G,C)$ is therefore not really well-defined. Therefore a complete description cannot be made purely in terms of a sigma model in which the target space is $\mathcal{M}_H(G,C)$, viewed as an abstract manifold. We need a more complete description that will tell us how to treat the singularities. What might this be?

By definition, a point in $\mathcal{M}_H(G, C)$ determines up to gauge-equivalence a pair (A, ϕ) obeying Hitchin's equations. A and ϕ are fields defined on C, so let us write them more explicitly as $(A(y), \phi(y))$, where y is a coordinate on C.

Now suppose that we have a map $\Phi: \Sigma \to \mathcal{M}_H(G,C)$, where Σ is a Riemann surface with a local coordinate x. Such a map is described by a pair $(A(y), \phi(y))$ that also depends on x. So we can describe the map Φ via fields $(A(x,y),\phi(x,y))$ that depend on both x and y. We would like to interpret these fields as fields on the four-manifold $M = \Sigma \times C$. The pair $(A(x,y),\phi(x,y))$ is not quite a natural set of fields on M but can be naturally completed to one. For example, A(x,y) is locally a one-form tangent to the second factor in $M = \Sigma \times C$; to get a four-dimensional gauge field, we should relax the condition that A is tangent to the second factor. Similarly, we can extend ϕ to an adjoint-valued one-form on $\Sigma \times C$. $\mathcal{N} = 4$ super Yang-Mills theory, or rather its twisted version that is related to geometric Langlands, is obtained by completing this set of fields to a supersymmetric combination in a minimal fashion.

In $\mathcal{N}=4$ super Yang-Mills theory, there are no singularities analogous to the singularities of $\mathcal{M}_H(G,C)$. The space of gauge fields, for example, is an affine space, and the other fields (such as ϕ) take values in linear spaces. The problems with singularities that make it difficult to define a sigma model of maps $\Phi: \Sigma \to \mathcal{M}_H(G,C)$ have no analog in defining gauge theory on $M=\Sigma\times C$ (or any other four-manifold). The relation between the two is that the two-dimensional sigma model is an approximation to the four-dimensional gauge theory. The approximation breaks down when one runs into the singularities of $\mathcal{M}_H(G,C)$. Any question that involves those singularities should be addressed in the underlying four-dimensional gauge theory.

 $^{^5}$ Moreover, these singularities are worse than orbifold singularities. Orbifold singularities would cause no difficulty. See Frenkel & Witten (2007) for a discussion of orbifold singularities in geometric Langlands.

But away from singularities, it suffices to consider only the smaller set of fields that describe a map $\Phi: \Sigma \to \mathcal{M}_H(G,C)$. Many questions do not depend on the singularities and for these questions the description via two-dimensional sigma models and mirror symmetry is adequate.

6.1. **Stacks.** To conclude, we will make contact with the counterpart of this discussion in the usual mathematical theory. We start with bundles rather than Higgs bundles because this case will be easier to explain.

In the usual mathematical theory, the right hand side of the geometric Langlands correspondence is described in terms of \mathcal{D} -modules on, roughly speaking, the moduli space of all holomorphic $G_{\mathbb{C}}$ bundles on the Riemann surface C.

However, instead of the moduli space $\mathcal{M}(G,C)$ of semi-stable holomorphic $G_{\mathbb{C}}$ bundles $E \to C$, one considers \mathcal{D} -modules on the "stack" $\operatorname{Bun}_G(C)$ of all such bundles. The main reason for this is that to define the action of Hecke operators, it is necessary to allow unstable bundles. Unstable bundles are related to the non-orbifold singularities of $\mathcal{M}(G,C)$.

What is a stack? Roughly, it is a space that can everywhere be locally described as a quotient. The trivial case is a stack that can actually be described globally as a quotient. Interpreting $\operatorname{Bun}_G(C)$ as a global quotient would mean finding a pair $(Y, W_{\mathbb{C}})$, consisting of a smooth algebraic variety Y and a complex Lie group $W_{\mathbb{C}}$ acting on Y, with the following properties. Isomorphism classes of holomorphic $G_{\mathbb{C}}$ bundles $E \to C$ should be in 1-1 correspondence with $W_{\mathbb{C}}$ orbits on Y, and for every $E \to C$, its automorphism group should be isomorphic to the subgroup of $W_{\mathbb{C}}$ leaving fixed the corresponding point in Y.

A pair $(Y, W_{\mathbb{C}})$ representing in this way the stack $\operatorname{Bun}_G(C)$ does not exist if Y and $W_{\mathbb{C}}$ are supposed to be finite-dimensional. Indeed, the $G_{\mathbb{C}}$ -bundle $E \to C$ can be arbitrarily unstable, so there is no upper bound on the dimension of its automorphism group. So no finite-dimensional $W_{\mathbb{C}}$ can contain all such automorphism groups as subgroups.

However as shown in Atiyah & Bott (1982), taking G to be of adjoint type for simplicity, there is a natural infinite-dimensional pair $(Y, W_{\mathbb{C}})$. One simply takes Y to be the space of all connections on a given G-bundle $E \to C$ which initially is defined only topologically. One defines W to be the group of all gauge transformations of the bundle E; thus, if E is topologically trivial, we can identify W as the group $\mathrm{Maps}(C,G)$. Then we take $W_{\mathbb{C}}$ to be the complexification of W, or in other words $\mathrm{Maps}(C,G_{\mathbb{C}})$. (This complexification acts on Y as follows. We associate to a connection A the corresponding $\bar{\partial}$ operator $\bar{\partial}_A$. Then a complex-valued gauge transformation acts by $\bar{\partial}_A \to g\bar{\partial}_A g^{-1}$.)

Suppose then that we were presented with the problem of making sense of the supersymmetric sigma model of maps $\Phi: \Sigma \to \mathcal{M}(G,C)$, given the singularities of $\mathcal{M}(G,C)$. (This is a practice case for our actual problem, which involves $\mathcal{M}_H(G,C)$ rather than $\mathcal{M}(G,C)$.) Our friends in algebraic geometry would tell us to replace $\mathcal{M}(G,C)$ by the stack $\operatorname{Bun}_G(C)$. We interpret this stack as the pair $(Y,W_{\mathbb{C}})$, where Y is the space of all connections on a G-bundle $E \to C$ and $W_{\mathbb{C}}$ is the complexified group of gauge transformations. The connected components of the stack correspond to the topological choices for E.

By a supersymmetric sigma model with target a pair $(Y, W_{\mathbb{C}})$, with $W_{\mathbb{C}}$ a complex Lie group acting on a complex manifold Y, we mean⁶ in the finite-dimensional case a gauge-invariant supersymmetric sigma model in which the gauge group is W (a maximal compact subgroup of $W_{\mathbb{C}}$) and the target is Y. Actually, to define this sigma model, we want Y to be a Kahler manifold with an W-invariant (but of course not $W_{\mathbb{C}}$ -invariant) Kahler structure. The sigma model action contains a term which is the square of the moment map for the action of W. This term is minimized precisely when the moment map vanishes. The combined operation of setting the moment map to zero and dividing by W is equivalent classically to dividing by $W_{\mathbb{C}}$.

To write down the term in the action that involves the square of the moment map (and in fact, to write down the kinetic energy of the gauge fields) one needs an invariant and positive definite quadratic form on the Lie algebra of W. If W is finite-dimensional, existence of such a form is equivalent to W being compact. However, the appropriate quadratic form also exists in the infinite-dimensional case that $W = \operatorname{Maps}(C,G)$ for some space C. (An element of the Lie algebra of W is a \mathfrak{g} -valued function ϵ on C, and the quadratic form is defined by $\int_C d\mu \left(\epsilon,\epsilon\right)$, where (,) is an invariant positive-definite quadratic form on \mathfrak{g} , and μ is a suitable measure on C.)

Now, suppose we construct the two-dimensional sigma model of maps from a Riemann surface Σ to the stack $\operatorname{Bun}_G(C)$, understood as above. What is the group of gauge transformations of the sigma model? In general, in a gauge theory on any space Σ with gauge group W, the group of gauge transformations (of a topologically trivial W-bundle, for simplicity) is the group of maps from Σ to W, or $\operatorname{Maps}(\Sigma,W)$. In our case, W is in turn $\operatorname{Maps}(C,G)$. So $\operatorname{Maps}(\Sigma,W)$ is the same as $\operatorname{Maps}(M,G)$, where $M=\Sigma\times C$. But this is simply the group of gauge transformations in gauge theory on M, with gauge group G. In the present case, Σ and C are two-manifolds and M is a four-manifold. We have arrived at four-dimensional gauge theory. If we chase through the definitions a little more, we learn that the supersymmetric sigma model of maps $\Phi: \Sigma \to \operatorname{Bun}_G(C)$ should be understood as four-dimensional $\mathcal{N}=2$ supersymmetric gauge theory, with gauge group G, on the four-manifold $M=\Sigma\times C$. (This is the theory that after twisting is related to Donaldson theory.)

Now let us return to the original problem. Geometric Langlands duality is a statement about the B-model and A-model not of $\mathcal{M}(G,C)$ but of $\mathcal{M}_H(G,C)$, the corresponding moduli space of Higgs bundles, and its analog for the dual group LG . To deal with the singularities, we want to "stackify" this situation. We are now in a hyper-Kahler context and the appropriate concept of a stack should incorporate this. (What algebraic geometers would call the stack of Higgs bundles does not quite do justice to the situation, since it emphasizes one complex structure too much.) Since quaternionic Lie groups do not exist, we cannot ask to construct $\mathcal{M}_H(G,C)$ as the quotient of a smooth space by a quaternionic Lie group. However, the notion of a symplectic quotient does have a good analog in the hyper-Kahler world, namely the hyper-Kahler quotient, described in Hitchin et al. (1987). The analog of what we explained for $\mathcal{M}(G,C)$ is to realize $\mathcal{M}_H(G,C)$ as the hyper-Kahler quotient of a smooth space Y by a group W. It may be impossible to do this with finite-dimensional Y and W, but in the infinite-dimensional world, this problem has a natural solution described in Hitchin's original paper on the Hitchin equations,

⁶For a discussion of this construction in relation to stacks, see Pantev & Sharpe (2006).

Hitchin (1987a). (Y is the space of pairs (A, ϕ) on C, and W = Maps(C, G).) Taking this as input and interpreting what it should mean to have a sigma model whose target is the hyper-Kahler stack corresponding to $\mathcal{M}_H(G, C)$, one arrives at the twisted version of $\mathcal{N} = 4$ super Yang-Mills theory that was the starting point in Kapustin & Witten (2007).

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