# Non-Perturbative Superpotentials In String Theory

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The non-perturbative superpotential can be effectively calculated in M-theory compactification to three dimensions on a Calabi-Yau four-fold X. For certain X, the superpotential is identically zero, while for other X, a non-perturbative superpotential is generated. Using F-theory, these results carry over to certain Type IIB and heterotic string compactifications to four dimensions with N=1 supersymmetry. In the heterotic string case, the non-perturbative superpotential can be interpreted as coming from space-time and world-sheet instantons; in many simple cases contributions come only from finitely many values of the instanton numbers.

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#### 1. Introduction

Surprisingly much of the dynamics of supersymmetric field theories and string theories has proved to be knowable, leading one to wonder how much farther one can get using techniques that are more or less already available. In particular, one would very much like to obtain non-perturbative information about the superpotential of N=1 compactifications to four dimensions.

In this paper, we take some steps in this direction. We consider in section two the compactification of eleven-dimensional M-theory to three dimensions on a manifold X of SU(4) holonomy. This gives a model with N=2 supersymmetry in three dimensions, which is roughly comparable to N=1 in four dimensions. We argue that superpotentials in this model are generated entirely by instantons obtained by wrapping a five-brane over a complex divisor D in X. Moreover, only very special D's can contribute; for a given X one can effectively find all of the relevant D's, which in many simple examples are finite in number, and obtain a fairly precise formula (which depends on one loop determinants that are hard to make explicit) for the superpotential. For many X's – such as complete intersections in projective spaces – there are no D's with the right properties, and the superpotential is identically zero; supersymmetry is thus unbroken in these models. Other simple examples generate non-trivial superpotentials.

Some of these models can be directly related to four-dimensional models via F-theory [1]. If X admits an elliptic fibration, that is if there is a holomorphic map  $X \to B$  with the generic fiber being an elliptic curve, then M-theory on X goes over, in a certain limit, to Type IIB superstring theory on B, which is a four-dimensional theory with N=1 supersymmetry. Thus, by taking suitable limits of the formulas of section two, one gets, as we discuss in section three, exact superpotentials for this class of four-dimensional N=1 models.

Moreover, if B in turn is "rationally ruled," that is if there is a holomorphic fibration  $B \to B'$  with the fibers being  $\mathbf{P}^1$ 's, then Type IIB on B is equivalent to the heterotic string on a Calabi-Yau threefold Z that is elliptically fibered over B'. This conclusion follows upon fiberwise application of the equivalence [1] of the heterotic string on a two-torus with Type IIB on a two-sphere, and (for B of complex dimension two) has been used to study the heterotic string on K3 [2]. So, as we discuss in section four, the results of section three imply exact, non-perturbative formulas for the superpotentials in certain Calabi-Yau compactifications of the heterotic string. As might be expected, these superpotentials

can be interpreted as sums of contributions from world-sheet instantons and space-time instantons. Given the results mentioned above in connection with M-theory, it turns out that, in many simple cases, non-vanishing contributions to the world-sheet and space-time instanton numbers arise only for *finitely* many values of the instanton numbers.

Ironically, these non-perturbative results give new information about the heterotic string even at the *perturbative* level. It has long been known [9] that in principle world-sheet instantons should generate a superpotential that would spoil the conformal invariance of (0, 2) sigma models. Yet a concrete example in which this actually occurs has not been found. Indeed, global holomorphy can sometimes be used [10] to show that cancellations must occur between different instantons of the same instanton number, preventing the generation of a superpotential, and it has actually appeared difficult to see how such cancellations could be avoided. One mechanism for avoiding such cancellation and actually generating a world-sheet instanton superpotential will become clear in this paper.

The goal of understanding the superpotential is, of course, to understand supersymmetry breaking and the vanishing of the cosmological constant. The main clue we get in that direction may be that the special divisors from which the superpotential is generated are often the ones whose collapse can sometimes lead (at least in similar problems above four dimensions) to novel infrared physics with, roughly, tensionless strings [3-8] and an abrupt "end" of the moduli space. Since the vanishing of the cosmological constant seems to defy understanding in terms of conventional infrared physics, the fact that the mechanism (or at least the divisor) that generates a superpotential may also generate novel infrared physics may be encouraging.

# 2. M-Theory On A Calabi-Yau Four-Fold

We consider compactification of M-theory from eleven to three dimensions on a manifold X of holonomy SU(4). This gives a theory with three-dimensional N=2 supersymmetry (which has a structure very similar to N=1 in four dimensions) in which there are three kinds of supermultiplets that contain scalar fields:

- (1) Fields derived from the complex structure of X are the scalar components of chiral multiplets.
- (2) Fields obtained by integrating the three-form potential C of eleven-dimensional supergravity over three-cycles in X are likewise scalar components of chiral multiplets.

(3) There is also one slightly more exotic case analogous to the "linear multiplet" in four dimensions [11-13]. Let r be the dimension of  $H^2(X, \mathbf{R})$  (which coincides with the dimension of  $H^{1,1}(X)$ , since  $H^{2,0}$  vanishes for manifolds of holonomy SU(N)). Then there are r multiplets, known as three-dimensional linear multiplets, with the following structure. (Such multiplets have been discussed in [14].) The bosonic fields in such a multiplet consist of one real scalar that is obtained by integrating the Kahler form  $\omega$  of X over a two-cycle E, along with a three-dimensional vector field A that is obtained by integrating the three-form C over E. Note that in three dimensions, a vector is dual to a scalar, and if one performs a duality transformation to convert A into a scalar  $\phi$ , one gets a conventional chiral supermultiplet with two real scalar fields. We will call the  $\phi$ 's obtained this way "dual scalars."

Now, there are no terms in the superpotential that are independent of the linear multiplets. In fact, such terms, being independent of the Kahler class of X, could be computed by scaling up the metric of X; but as the metric is scaled up, M-theory reduces to eleven-dimensional supergravity, which has  $\mathbf{R}^3 \times X$  as an exact solution (since X obeys the Einstein equations), showing that there is no superpotential in this limit.

At first sight, it also seems impossible to have a superpotential interaction that depends on the linear multiplets. In fact, it appears that the gauge field A can only have derivative couplings, through the gauge invariant field F = dA; in that case, the scalar  $\phi$ , introduced by  $d\phi = *F$ , likewise only has derivative couplings, so that a superpotential depending on  $\phi$  is impossible.

However, there is a fallacy in the last claim. In a situation such as this, interactions that are not invariant under  $\phi \to \phi + \text{constant}$  are indeed absent perturbatively, but can be generated [15] by certain kinds of instanton, namely those that look like magnetic monopoles for the F-field. In other words, in the relevant instanton field, F has a non-zero integral over a large sphere at infinity in  $\mathbb{R}^3$ , and therefore decays at infinity as  $1/r^2$ , with r the distance to the origin. Dually, this means that in the instanton field,  $\phi$  falls off as 1/r, corresponding to the effects of a source at the origin that is linear in  $\phi$  and not invariant under addition to  $\phi$  of a constant. The interactions generated by such an instanton are proportional to

$$e^{-i\gamma\phi},$$
 (2.1)

where the constant  $\gamma$  is proportional to the magnetic charge of the instanton.

So far, we have not assumed supersymmetry. If we do incorporate three-dimensional N=2 supersymmetry, these interactions will be superpotential terms precisely if the

instanton is invariant under two of the four supersymmetry charges [16]. This ensures that the instanton generates a superpotential term  $\int d^2\theta$  ... and not a generic coupling  $\int d^4\theta$  ....

In the present context, A originates as a mode of C, so the requisite instanton is a magnetic source for C. In general, the magnetic source (as reviewed in [17]) is the eleven-dimensional five-brane. So roughly as in [18] the relevant instantons must be made by wrapping the six-dimensional world-volume of the five-brane over a six-cycle D in X, giving what in three dimensions looks like an instanton. This instanton is invariant under some supersymmetry precisely if D is a complex divisor, that is, a complex submanifold of X. We will actually see that D must obey an additional condition that eliminates most divisors.

The amplitude of such an instanton is proportional to  $e^{-V_D}$ , where  $V_D$  is the volume of D, measured in units of the five-brane tension. There is an additional factor  $e^{-i\phi_D}$ , where as in (2.1),  $\phi_D$  is a linear combination of the dual scalars. These factors thus combine to a factor of  $e^{-(V_D+i\phi_D)}$ , strongly suggesting that  $V_D$  and  $\phi_D$  are the real and imaginary parts of a chiral supermultiplet. This can indeed be verified directly in eleven-dimensional supergravity. (Beyond the supergravity approximation,  $\phi_D$  – being dual to a vector, which is not subject to nonlinear change of variable – is naturally defined up to an additive constant in the exact M-theory; that is not so for  $V_D$ , which in the exact theory one can simply define to be the superpartner of  $\phi_D$ .) The factor  $e^{-(V_D+i\phi_D)}$  thus has the holomorphy expected of a superpotential.

The  $\phi_D$  dependence is here exactly fixed by the magnetic charge of the instanton, so the  $V_D$  dependence is in turn exactly fixed by holomorphy. This then means that – apart from the known factor  $e^{-(V_D+i\phi_D)}$  – the superpotential generated by the instanton is independent of the Kahler class of X, and so can be computed by scaling up the metric of X. The instanton amplitude is computed by multiplying a classical factor  $e^{-(V_D+i\phi_D)}$  times a one-loop determinant of world-volume fields – which is invariant under scaling the metric of X. Higher loop corrections to the world-volume computation would be proportional to inverse powers of the Kahler class and so in fact vanish by holomorphy. The one-loop approximation to the instanton amplitude is thus exact, for the purposes of computing the superpotential. This is a situation often found, for somewhat analogous reasons, in computations of superpotential generation by space-time [19] or world-sheet [9] instantons.

The one-loop amplitude must also depend holomorphically on the other variables – and notably on the complex structure of X, which enters via the complex structure of D. It would be of interest to analyze this dependence, which presumably involves something similar to Ray-Singer analytic torsion of complex manifolds. What makes it difficult to immediately identify the one-loop determinants in this problem with anything known is that the world-volume theory on D is somewhat exotic, because of the presence of a two-form with self-dual field strength.

But one important property of the one-loop amplitude is easy to extract: the contribution to the superpotential vanishes, because of an anomaly, unless D obeys a certain rather strong condition. This result will be obtained by matching instanton quantum numbers with quantum numbers needed for a superpotential in a way roughly familiar [19] from field theory analyses of instanton-generated superpotentials, though there will be some unusual details in the present case.

In the analysis, I will assume that D is smooth. Since D is of complex codimension one in X, the normal bundle to D in X is a complex line bundle N. The fact that the canonical bundle of X is trivial means that N is isomorphic to the canonical bundle  $K_D$  of D. Locally, near D, X looks like the total space of the normal bundle; this approximation becomes better as the metric is scaled up, and is exact for the world-volume theory in the linearized approximation. If z is a local coordinate in the normal direction (vanishing along D), then one can make the U(1) transformation  $z \to e^{i\theta}z$ . This, roughly, is the symmetry whose possible anomaly we want to analyze.

Let us recall that the positive chirality spinor bundle  $S_+$  of the six-manifold D has rank four. The normal bundle  $\widehat{N}$  to D in  $\mathbb{R}^3 \times X$  has rank five, and (since the spinor representation of SO(5) has dimension four) the spinor bundle  $\widetilde{S}$  constructed from  $\widehat{N}$  has rank four. On the five-brane world-volume D propagate sixteen fermi fields  $\psi$  transforming as a section of  $S_+ \otimes \widetilde{S}$ . The world-volume action of the five-brane is not completely understood, but the part quadratic in  $\psi$  and not involving the two-form is simply the Dirac action for chiral spinors coupled to the metric of D and to the SO(5) structure group of  $\widehat{N}$ :

$$L_f = \int_D d^6 x \ \psi \mathcal{D} \psi. \tag{2.2}$$

Here  $\mathcal{D}$  is the Dirac operator coupled to the metric and SO(5) gauge fields.

For a generic six-dimensional submanifold D of an eleven-manifold, the Dirac action (2.2) has no global symmetries except mod two conservation of the number of fermions.

We are here, however, in a special situation in which  $\widehat{N}$  is simply  $N \oplus T\mathbf{R}^3$ , where N is the normal bundle to D in X (now regarded as a rank two real bundle), and  $T\mathbf{R}^3$  is the (trivial) tangent bundle to  $\mathbf{R}^3$ . As a result, the SO(5) structure group of the normal bundle reduces to an SO(2) (which acts only on N). The subgroup of SO(5) that commutes with SO(2) is  $SO(2) \times SO(3)$  (the two factors correspond to rotations of N and of  $T\mathbf{R}^3$ , respectively), and  $SO(2) \times SO(3)$  is therefore a symmetry group of the classical action (2.2). We call the SO(2) generator W. It is actually W whose anomaly will control the generation of a superpotential.

It is important to know whether W is an exact symmetry of the five-brane action, or only a symmetry in the approximation of (2.2). It is somewhat hard to answer this question definitively because the five-brane action is not fully known. However, one may note that arbitrary couplings of  $\psi$  to fields defined on the five-brane will automatically be W-invariant (with W understood as acting trivially on fields other than  $\psi$ ). To violate W would require couplings of  $\psi$  to the normal derivatives (that is, normal to D) of some of the eleven-dimensional fields, for instance a coupling  $\psi \psi R$  with R the eleven-dimensional Riemann tensor. It seems very likely that such couplings are absent in the minimal supersymmetric five-brane action, would vanish when X is scaled up, and would not contribute to the superpotential. At any rate, this assumption will be made in the present paper.

Because  $\widehat{N} = N \oplus T\mathbf{R}^3$ , the spinor bundle  $\widetilde{S}$  of  $\widehat{N}$  is simply the tensor product of a rank two spin bundle S' derived from N with a constant rank two bundle S'' of spinors of  $T\mathbf{R}^3$ . The fermions on D are thus simply two copies of spinors with values in  $S_+ \otimes S'$ , with the extra two-valued index (which comes from tensoring with S'') transforming as spin one-half under rotations of  $\mathbf{R}^3$ .

Because of the relation of N to the canonical bundle of D, the spin bundle S' derived from N is isomorphic to  $S' = K^{1/2} \oplus K^{-1/2}$ , where  $K^{1/2}$  is a square root of K. (It is not essential whether such a square root exists, since the square roots will cancel out when we construct  $S_+ \otimes S'$ .) If we want to keep track of the transformation law under W, then we can write this as  $S' = K_{1/2}^{1/2} \oplus K_{-1/2}^{-1/2}$ , where now the subscript is the W charge.

On the other hand, if  $\Omega^{0,n}$  is the bundle of complex-valued (0,n)-forms on D, then (with an appropriate matching of complex structure and orientation), the positive and negative chirality spin bundles of D are

$$S_{+} = K^{1/2} \oplus \left(K^{1/2} \otimes \Omega^{0,2}\right)$$

$$S_{-} = \left(K^{1/2} \otimes \Omega^{0,1}\right) \oplus \left(K^{1/2} \otimes \Omega^{0,3}\right).$$

$$(2.3)$$

(Rotations of the normal bundle act trivially on the tangent bundle to D and hence on  $S_+$  and  $S_-$ .) So the fermions on D take values in

$$S_{+} \otimes S' = \mathcal{O}_{-1/2} \oplus \Omega^{0,2}_{-1/2} \oplus K_{1/2} \oplus (\Omega^{0,2} \otimes K_{1/2}).$$
 (2.4)

Here  $\mathcal{O}$  (which is the same as  $\Omega^{0,0}$ ) is a trivial line bundle.

Now let  $h_k$  be the dimension of the cohomology group  $H^{0,k}(D)$ , or equivalently the dimension of  $H^{k,0}(D)$ , the space of holomorphic k-forms on D. The number of fermion zero modes with values in  $\Omega^{0,n}$  is  $h_n$ , and (by Serre duality) this also equals the number of zero modes with values in  $K \otimes \Omega^{0,3-n}$ . So, looking at (2.3), we see that the number of W = -1/2 zero modes is  $h_0 + h_2$ , and the number of W = 1/2 zero modes is  $h_1 + h_3$ . Allowing also for the doubling of the spectrum from tensoring with the spinors of  $T\mathbf{R}^3$ , the total violation of W because of the fermion zero modes is

$$\Delta W = \chi(D, \mathcal{O}_D) = \sum_{n=0}^{3} (-1)^n h_n, \tag{2.5}$$

where  $\chi(D, \mathcal{O}_D)$  is known as the arithmetic genus of D.  $(\chi(D, \mathcal{O}_D)$  is sometimes abbreviated as  $\chi(D)$ , but this notation can cause confusion with the topological Euler characteristic of D.) More generally, it will be convenient, for any holomorphic line bundle  $\mathcal{L}$  on a complex manifold Y, to define

$$\chi(Y, \mathcal{L}) = \sum_{i=0}^{\dim_{\mathbf{C}} Y} (-1)^i \dim H^i(Y, \mathcal{L}). \tag{2.6}$$

With  $\mathcal{O}_D$  defined to be a trivial line bundle on D, the definition (2.6) for Y = D and  $\mathcal{L} = \mathcal{O}_D$  reduces to  $\chi(D, \mathcal{O}_D)$  as defined in (2.5).

Before proceeding, we might ask how to interpret this violation of W. After all, under a favorable condition (if X is the total space of a line bundle over a divisor D), W is simply the generator of a diffeomorphism, and the theory is supposed to be exactly invariant under diffeomorphisms! The answer to this question [20,21] is that, when one interprets W as generating a diffeomorphism, in addition to the violation of W by one-loop world-volume effects, there is also a "classical" violation that comes from a term  $C \wedge I_8(R)$  in the low energy expansion of M-theory; here C is the massless three-form and  $I_8(R)$  is a homogeneous quartic polynomial in the Riemann tensor. The  $C \wedge I_8(R)$  term has a diffeomorphism anomaly that cancels the one-loop anomaly of the five-brane world-volume fields for arbitrary diffeomorphisms and so in particular for W.

Since the  $C \wedge I_8(R)$  term is a "classical" effect, we should expect to see it in the classical factor  $e^{-(V_D+i\phi_D)}$  that was obtained above. This factor should have an anomaly  $-\chi(D,\mathcal{O}_D)$  under W, canceling the anomaly of the one-loop factor. This must mean that under a rotation of the normal bundle,  $\phi_D$  is shifted by a constant times  $\chi(D,\mathcal{O}_D)$ . I will not derive this explicitly, but note that since the additive constant in  $\phi_D$  is rather subtle to define, there is room for such an effect.

Now let us look at the fermion zero modes in this problem a little more closely. Two fermion zero modes are present universally. These are generated by the two supersymmetries that are unbroken in compactification on X, but broken by wrapping a five-brane on D. The two supersymmetries, being broken by the five-brane, generate zero modes in the world-volume theory along D. The wave function of those zero modes is the constant section 1 of  $\mathcal{O}$ . There are two of them once one tensors with the spinors of  $T\mathbb{R}^3$ .

Certain other fermi zero modes have a particularly simple interpretation. A deformation of the complex divisor D comes from a holomorphic section of the normal bundle N, so the space of such deformations, in first order, is  $H^0(D, N)$ . But the relation  $N = K_D$ means that this space is just  $H^{3,0}(D)$ . So  $h_3$  measures the number of possible first order motions of the divisor D.

Now, here is a simple situation in which a superpotential is generated: the case in which  $h_1 = h_2 = h_3 = 0$ . The effect of the two fermion zero modes that come from supersymmetry is simply that the classical factor  $e^{-(V_D+i\phi_D)}$  becomes a superpotential  $\int d^2\theta \ e^{-(V_D+i\phi_D)}$ . Having  $h_1 = h_2 = h_3 = 0$  means that there are no extra fermion zero modes that could cause the superpotential to vanish. There are also, because  $h_3 = 0$ , no moduli in the position of D; the presence of such moduli would be dangerous for generating a superpotential, since the integration over the moduli space might give a cancellation.<sup>2</sup> So when  $h_1 = h_2 = h_3 = 0$ , one gets a superpotential interaction

$$\int d^2\theta \ e^{-(V_D + i\phi_D)} T(m_\alpha), \tag{2.7}$$

where  $T(m_{\alpha})$  is a holomorphic function of other moduli  $m_{\alpha}$  (such as the complex moduli of X) which comes from the determinant for the *non-zero* modes and so is everywhere *non-zero*.

When  $h_3 = 0$ , the only bosonic moduli in the world-volume theory are the zero modes of the self-dual two-form field, but these zero modes decouple because of the two-form gauge invariance, so one gets no cancellation from integration over the torus  $H^2(X, \mathbf{R})/H^2(X, \mathbf{Z})$ , to which these modes are tangent.

Now the factor  $e^{-(V_D+i\phi_D)}$  carries charge  $W=-\chi(D,\mathcal{O}_D)$ , as we described earlier. When  $h_1=h_2=h_3=0$ ,  $\chi(D,\mathcal{O}_D)=1$ , and in this case  $e^{-(V_D+i\phi_D)}$  has W=-1. It must then be that the measure  $d^2\theta$  has charge W=1, to make it possible to generate the interaction (2.7) under these conditions. In fact, the measure  $d^2\theta$  always carries W=1; this is clear from the fact that the supersymmetries broken by the five-brane create the two "universal" fermion zero modes that are sections of  $\mathcal{O}_{-1/2}$ . While the measure always has W=1, the factor  $e^{-(V_D+i\phi_D)}$  has  $W=-\chi(D,\mathcal{O}_D)$ . Therefore, a superpotential can only be generated by wrapping a five-brane on D if  $\chi(D,\mathcal{O}_D)=1$ . We know a partial converse from the last paragraph: any divisor with  $h_1=h_2=h_3=0$  does contribute to the superpotential. When there are several such divisors, or when  $h_3\neq 0$  so that there is a positive-dimensional moduli space of divisors to integrate over, cancellations are conceivable.

By a standard argument in complex geometry,  $\chi(D, \mathcal{O}_D)$  only depends on the cohomology class of D. Tautologically, one defines a line bundle  $\mathcal{O}_X(D)$  on X that admits a holomorphic section s that vanishes precisely on D, and looks at the exact sequence of sheaves

$$0 \to \mathcal{O}_X(-D) \to \mathcal{O}_X \to \mathcal{O}_D \to 0, \tag{2.8}$$

where  $\mathcal{O}_X$  and  $\mathcal{O}_D$  are the trivial bundles (or "structure sheaves") defined on X and D, respectively, and  $\mathcal{O}_X(-D)$  is the inverse of  $\mathcal{O}_X(D)$ ; the first map in (2.8) is multiplication by s, and the second is restriction to D. The exact cohomology sequence derived from (2.8) then implies that  $\chi(D, \mathcal{O}_D) = -\chi(X, \mathcal{O}_X(-D)) + \chi(X, \mathcal{O}_X)$  (where  $\chi(X, \mathcal{L})$  was defined in (2.6)). From the index theorem, one deduces therefore that

$$\chi(D, \mathcal{O}_D) = \int_X \left( 1 - e^{-[D]} \right) \operatorname{Td}(X), \tag{2.9}$$

where  $[D] = c_1(\mathcal{O}_X(D))$  is the cohomology class dual to D, and  $\mathrm{Td}(X)$  is the Todd class. This gives an explicit formula for  $\chi(D, \mathcal{O}_D)$  in terms of the cohomology class of D, and severely limits the possible D's with  $\chi(D, \mathcal{O}_D) = 1$ .

In the examples that follow, we either show that a superpotential is not generated by instantons by showing that any divisor D on X has  $\chi(D, \mathcal{O}_D) \neq 1$ , or we show that a superpotential is generated by showing that for some choice of the cohomology class there is precisely one complex divisor D, which moreover has  $h_1 = h_2 = h_3 = 0$ .

In four-dimensional supersymmetric field theory, it sometimes happens [19] that a superpotential cannot be generated by instantons but is generated by non-perturbative strong infrared dynamics.<sup>3</sup> This apparently does not, however, happen in N=2 theories in three dimensions, where (as the vector multiplet contains a scalar that comes by dimensional reduction from a vector in four dimensions) one can always go to a Coulomb branch with the gauge group broken to an abelian subgroup, and turn off the strong gauge dynamics. So in such three-dimensional theories, including the string theories that we have been studying, it seems plausible that the instanton-induced superpotential is exact.

In section three, we will consider partial decompactification to four dimensions, but only in examples in which this occurs without restoring a non-abelian gauge symmetry and producing strong infrared dynamics. Actually, in field theory, generation of a superpotential by strong infrared dynamics (rather than instantons) happens when symmetry violation by instantons has the right sign but is too large to generate a superpotential; then, sometimes, there is spontaneous symmetry breaking that liberates effective "fractional instantons" with the right quantum numbers. In our problem, this might correspond to a situation in which there are divisors D with  $\chi(D, \mathcal{O}_D)$  positive, but the smallest positive value is some n > 1, so that to generate a superpotential one would formally need to wrap 1/n five-branes over D. We will not actually find such a situation (possibly because the models we consider all have generically abelian unbroken gauge groups even after decompactification to four dimensions); in the models we consider, either a superpotential is generated or  $\chi(D, \mathcal{O}_D)$  is always negative. In field theory, when symmetry violation by instantons has the wrong sign, there is no superpotential from instantons or otherwise. I find it plausible that that is also true in string theory.

#### 2.1. Examples

For our first example, we consider a Calabi-Yau manifold X built as an intersection in  $\mathbf{P}^{4+k}$  of k hypersurfaces whose degree adds up to 5+k. Thus, X is defined by equations  $g_1 = \ldots = g_k = 0$ , where the  $g_j$  are homogeneous polynomials of degree  $a_j$  in the homogeneous coordinates of  $\mathbf{P}^{4+k}$ , and  $\sum_{j=1}^k a_j = 5+k$ . For instance, we can take k=1 and consider a degree six equation in  $\mathbf{P}^5$ .

To find divisors in X, we first classify the possible line bundles – since every divisor D is associated with a line bundle  $\mathcal{O}_X(D)$ . Such line bundles are classified by their first Chern class in  $H^2(X, \mathbf{Z})$ . To determine  $H^2(X, \mathbf{Z})$ , one uses the Lefschetz theorem (see for instance [22,23]) which states that if Y is a complex manifold, and s is a section of

<sup>&</sup>lt;sup>3</sup> The author was reminded of this by N. Seiberg.

a positive line bundle over Y, then the hypersurface Z defined by s=0 has the same (integral) cohomology as Y, up to the middle dimension. (Thus, restriction from Y to Z gives an isomorphism from  $H^r(Y, \mathbf{Z})$  to  $H^r(Z, \mathbf{Z})$  for  $r < \dim_{\mathbf{C}}(Z)$ ; this isomorphism is compatible with the Hodge decomposition.) In our problem, since the  $g_j$  are all sections of positive line bundles, repeated application of this theorem – successively imposing one after another of the equations  $g_j = 0$  – implies that  $H^2(X, \mathbf{Z})$  coincides (under restriction or pull-back) with  $H^2(\mathbf{P}^{4+k}, \mathbf{Z})$ , so that the line bundles on X are simply pull-backs of line bundles on  $\mathbf{P}^{4+k}$ .

This means that the divisor D is simply defined by an equation f = 0, with f a homogeneous polynomial of some degree n > 0 in the homogeneous coordinates of  $\mathbf{P}^{4+k}$ . D is therefore defined by the equations  $f = g_1 = \ldots = g_k = 0$  in  $\mathbf{P}^{4+k}$ . Repeated application of the Lefschetz theorem therefore implies that the cohomology of D coincides with that of  $\mathbf{P}^{4+k}$  up to the middle dimension, which is three, so that in particular  $h_j(D) = h_j(\mathbf{P}^{4+k})$  for j = 1, 2. Now  $H^{n,m}(\mathbf{P}^r) = 0$  except for n = m, so that  $h_n(\mathbf{P}^r) = 0$  for n > 0. Hence  $h_1(D) = h_2(D) = 0$ . On the other hand,  $h_3(D)$  is the number of complex deformations of D as a hypersurface in X, and this number, which is the number of adjustable coefficients in the polynomial f, is strictly positive. So D has arithmetic genus less than one (and in fact negative). Therefore, in M-theory compactification on X, no superpotential is generated and supersymmetry is unbroken.

Now to give a simple example in which a superpotential is generated, let Y be any Calabi-Yau manifold with an isolated singularity that looks locally like the quotient of  $\mathbb{C}^4$  by the  $\mathbb{Z}_4$  group generated by  $(x_1, x_2, x_3, x_4) \to (ix_1, ix_2, ix_3, ix_4)$ . Such a singularity can be resolved by blowing up the origin, replacing it by a divisor D that is a copy of  $\mathbb{P}^3$ . Since  $h_n(\mathbb{P}^r) = 0$  for n > 0, such a divisor D has  $h_1 = h_2 = h_3 = 0$ . Moreover, any such divisor is the unique divisor in its cohomology class, so cancellations are not possible. So on any smooth Calabi-Yau manifold X obtained in this way, a superpotential is generated.

For example, a singular Calabi-Yau manifold Y with such orbifold singularities can be constructed as a hypersurface of degree 12 in the weighted projective space  $\mathbf{P}_{1,1,1,1,4,4}^5$ , the subscripts being the weights. There are three  $\mathbf{Z}_4$  orbifold singularities in Y, at points at which the first four homogeneous coordinates vanish, and their local structure is as described in the last paragraph. So a superpotential is generated in compactification on the smooth Calabi-Yau manifold X obtained by blowing up these singularities. (X can be constructed as a hypersurface in a  $\mathbf{P}^2$  bundle over  $\mathbf{P}^3$ , and so is related to other examples below.)

Now to consider a slightly more difficult example in which a superpotential is not generated, consider the case of a hypersurface X of degree (n+1, m+1) in  $\mathbf{P}^n \times \mathbf{P}^m$ , with n+m=5. Such a hypersurface is defined by an equation g=0, with g being a polynomial homogeneous of degree n+1 in the homogeneous coordinates of  $\mathbf{P}^n$  and of degree m+1 in the homogeneous coordinates of  $\mathbf{P}^m$ . Since g is a section of a positive line bundle over  $\mathbf{P}^n \times \mathbf{P}^m$ , the Lefschetz theorem implies that any divisor D on X is given by an equation f=0, where f is a homogeneous polynomial of degree (a,b) in the homogeneous coordinates of  $\mathbf{P}^n \times \mathbf{P}^m$ , for some integers a,b, which moreover must be non-negative (and not both zero) for D to exist.

If a and b are both positive, another application of the Lefschetz theorem says that  $h_1(D) = h_2(D) = 0$ . Since  $h_3(D)$  is positive (equalling the number of variable parameters in f), such a divisor does not contribute to the superpotential. It remains to consider the case that a or b is 0; there is no essential loss in generality to suppose that b = 0. In this case, it is helpful to first look at the hypersurface Y defined by the equation f = 0 in  $\mathbf{P}^n$ . The Lefschetz theorem asserts that Y has  $h_1 = h_2 = 0$  if  $n \geq 4$ , and  $h_1 = 0$  if n = 3. For n = 2, Y is a curve, which necessarily has  $h_2 = 0$ , and for n = 1, Y is a finite set of points, with  $h_1 = h_2 = 0$ . Now,  $Y \times \mathbf{P}^m$  has the same  $h_j$  as Y, and the Lefschetz theorem says that D (which can be defined by an equation g = 0 in  $Y \times \mathbf{P}^m$ , where g is a section of a positive line bundle) has the same  $h_1$  and  $h_2$  as  $Y \times \mathbf{P}^m$ . In particular,  $h_2(D) = 0$  unless n = 3. Since also  $h_3(D) > 0$ , if  $h_2(D) = 0$ ,  $\chi(D, \mathcal{O}_D) < 1$  and there is no contribution to the superpotential.

It remains then to look at the case n=3, for which  $h_1(D)=0$ , but  $h_2(D)$  and  $h_3(D)$  are both non-zero. Explicitly, we are here dealing with a Calabi-Yau hypersurface X in a product  $\mathbf{P}^3 \times \mathbf{P}^2$ , with homogeneous coordinates  $(x_1, \ldots, x_4)$  and  $(y_1, \ldots, y_3)$ , respectively; X is defined by an equation g=0, with g of degree (4,3). In the case not settled above, the divisor D is given by an equation  $f(x_1, \ldots, x_4)=0$ , homogeneous of degree a>0 in the  $x_i$ . A holomorphic two-form on D is of the form

$$\omega_2 = F(x_1, \dots, x_4) \frac{(x_1 dx_2 \wedge dx_3 + \text{cyclic permutations})}{\partial f / \partial x_4},$$
 (2.10)

with F homogeneous of degree n-4. Therefore,  $h_2(D)$  is the dimension of the space of polynomials that are homogeneous of this degree. On the other hand, a holomorphic three-form on D is of the form

$$\omega_3 = G(x_1, \dots, x_4) \frac{(x_1 dx_2 \wedge dx_3 + \text{cyclic permutations})}{\partial f / \partial x_4} \frac{(y_1 dy_2 - y_2 dy_1)}{\partial g / \partial y_3}, \qquad (2.11)$$

with G a polynomial homogeneous of degree n (modulo the relation  $G \to G + \lambda g$ , since g = 0 on D). Evidently, there are many more G's than F's, so  $0 < h_2(D) < h_3(D)$ , and in fact  $\chi(D, \mathcal{O}_D) < 0$ .

Thus, there is no superpotential in compactification on a Calabi-Yau hypersurface in  $\mathbf{P}^n \times \mathbf{P}^m$ . Essentially the same arguments can be used to show that there is also no superpotential in compactification on a Calabi-Yau hypersurface in a product  $\mathbf{P}^{n_1} \times \ldots \times \mathbf{P}^{n_k}$  of any number of projective spaces.

There can, however, be a superpotential if one replaces  $\mathbf{P}^n \times \mathbf{P}^m$  by a  $\mathbf{P}^m$  bundle over  $\mathbf{P}^n$ , still with m+n=5. To describe such a bundle Z for which there is a superpotential, introduce coordinates  $x_1, \ldots, x_{n+1}$  and  $y_1, \ldots, y_{m+1}$ , and divide by  $\mathbf{C}^* \times \mathbf{C}^*$  that acts by

$$(x_1, \dots, x_{n+1}, y_1, \dots, y_{m+1}) \to (\lambda x_1, \dots, \lambda x_{n+1}, \mu y_1, \lambda \mu y_2, \dots, \lambda \mu y_{m+1})$$
 (2.12)

with  $\lambda, \mu \in \mathbf{C}^*$ . In other words, the  $x_j$  transform with degree (1,0) in  $\lambda, \mu$ , while  $y_1$  transforms with degree (0,1), and the other  $y_k$  with degree (1,1). Let X be a Calabi-Yau manifold defined by an equation g=0 in Z, where g is homogeneous of degree (n+m+1,m+1). I claim that a superpotential is generated in compactification on X.

In fact,  $y_1$  is the unique monomial of degree (0,1), so the divisor D defined by  $y_1 = 0$  is the unique divisor in its cohomology class. Hence  $h_3(D) = 0$ . But since D is defined by an equation g = 0 in  $\mathbf{P}^n \times \mathbf{P}^{m-1}$ , the Lefschetz theorem implies that  $h_1(D) = h_2(D) = 0$ . So a superpotential is generated. An interesting feature of this example is that the divisor D on which the five-brane wraps to give a superpotential is somewhat more general than in the previous examples.

One can similarly work out other examples of hypersurfaces, and intersections of hypersurfaces, in other toric varieties, for which a superpotential is or is not generated. It would be attractive to understand a systematic approach.

# 3. Application To F-Theory On A Calabi-Yau Four-Fold

Now suppose that the Calabi-Yau four-fold X can be elliptically fibered, that is that there is a holomorphic map  $\pi: X \to B$  where B is a complex three-fold and the generic fibers are two-tori E. Suppose moreover that  $\pi$  has a holomorphic section. <sup>4</sup> Then M-theory on X is closely related [1] to Type IIB superstring theory on B. The relation is

<sup>&</sup>lt;sup>4</sup> One can actually proceed even if  $\pi$  does not have such a section; then M-theory on X is equivalent to Type IIB on  $\mathbf{R}^3 \times \mathbf{S}^1 \times B$  with non-zero three-form field strengths on  $\mathbf{S}^1 \times B$ , as explained below. But I will here consider examples in which  $\pi$  has a section and the three-forms vanish in the Type IIB description.

made as follows. Let  $\epsilon$  be the area of E. As  $\epsilon \to 0$ , using fiber-wise the relation of M-theory on  $\mathbf{R}^9 \times \mathbf{T}^2$  with Type IIB on  $\mathbf{R}^9 \times \mathbf{S}^1$ , one replaces the two-torus fibers E with a fixed  $\mathbf{S}^1$ , also replacing M-theory with Type IIB. So M-theory on  $\mathbf{R}^3 \times X$  is Type IIB on  $\mathbf{R}^3 \times \mathbf{S}^1 \times B$ . (This is [1] an unconventional Type IIB perturbative vacuum; the Type IIB coupling varies with the position on B. Type IIB vacua of this kind are also called F-theory vacua.) The radius of the  $\mathbf{S}^1$  varies as an inverse power of  $\epsilon$ , and so for  $\epsilon \to 0$  one gets Type IIB on  $\mathbf{R}^4 \times B$ .

We want to study superpotential generation by instantons in Type IIB compactified on B. This can be done simply by using the conclusions of the last section, taking the limit as  $\epsilon \to 0$ .

For the present purposes, we should distinguish two kinds of divisor D on X, depending on whether the complex manifold  $\pi(D)$ , which is a submanifold of B, is all of B or a proper submanifold.

- (a) In the first case, D is either a "section" of  $\pi$ , or a "multisection" (obtained by mapping holomorphically to X a branched m-sheeted cover of B, for some m > 1).
- (b) Alternatively, one might take a divisor C on B, and set  $D = \pi^{-1}(C)$  (or a component of  $\pi^{-1}(C)$  in the exceptional case in which  $\pi^{-1}(C)$  has several components).

We will see that the divisors of type (b) are the ones that lead to instanton generation of a superpotential in Type IIB compactification on B. Let us start with an M-theory five-brane wrapped on a divisor D of type (b) and see what it corresponds to in Type IIB theory on B. Locally, when the fibers E are small,  $\mathbf{R}^3 \times X$  looks like  $W \times \mathbf{S}^1 \times \mathbf{S}^1$  where  $E = \mathbf{S}^1 \times \mathbf{S}^1$  and W is a nine-manifold; locally along W, the divisor D is  $C \times \mathbf{S}^1 \times \mathbf{S}^1$  where C is a four-cycle in W. We first go to Type IIA by shrinking the second  $\mathbf{S}^1$  in  $\mathbf{S}^1 \times \mathbf{S}^1$ . We get locally Type IIA on  $W \times \mathbf{S}^1$ , with the five-brane wrapped on  $D = C \times \mathbf{S}^1 \times \mathbf{S}^1$  turning into a four-brane wrapped on  $C \times \mathbf{S}^1$ . Now we do T-duality on the remaining  $\mathbf{S}^1$ , going over to Type IIB on  $W \times \mathbf{S}^1$ ; this turns the four-brane into a three-brane wrapped on C.

Though the intermediate steps here were local, the final answer holds globally: M-theory instantons of type (b) correspond in F-theory to three-branes whose fourdimensional world-volume is wrapped on the divisor  $C \subset B$ . The action for such an instanton is therefore of order  $V_C$ , the volume of C. In particular, this action is of order one if we take  $\epsilon \to 0$  while keeping the Type IIB or F-theory geometry fixed.

It is now evident that instantons of type (a) do not survive when we take  $\epsilon \to 0$ . In fact, the volume of a divisor of type (a) is of order  $1/\epsilon$  compared to that of a divisor of

type (b) (if we take  $\epsilon$  to 0 keeping fixed the geometry of B), so divisors of type (a) have action of order  $1/\epsilon$  in F-theory units.

On the other hand, the divisors of type (a) obviously have finite action for non-zero  $\epsilon$ , and thus can contribute for Type IIB on  $\mathbf{S}^1 \times B$  (which after all is the same as M-theory on X).

Now, as we will see presently, there may or may not be a superpotential in Type IIB compactification on B. But there is always a superpotential (vanishing exponentially in the radius of the  $\mathbf{S}^1$ ) in Type IIB on  $\mathbf{S}^1 \times B$  (with vanishing three-forms). In fact, there is always a divisor of type (a), unique in its cohomology class, with  $h_1 = h_2 = h_3 = 0$ . Such a divisor D is the section of  $\pi: X \to B$  which is part of the defining data of F-theory. Indeed, this D is isomorphic to B, but B always has  $h_1 = h_2 = h_3 = 0$ . The reason is that a holomorphic k-form on B would pull back under  $\pi$  to a holomorphic k-form on X; but the existence of such a holomorphic k-form on X for k = 1, 2, or 3 would contradict the Calabi-Yau property. If the existence of a superpotential that vanishes when the  $\mathbf{S}^1$  becomes large is undesireable (as may be the case [24]), one can at least sometimes eliminate it by turning on an H-field and replacing  $X \to B$  with a map that does not have a section, a situation discussed later.

In what follows, we consider only Type IIB compactification to four dimensions on B, so we are interested only in divisors of type (b), that is, divisors C in B. To get a superpotential, it is necessary for C to have the property that  $D = \pi^{-1}(C)$  has arithmetic genus 1, and sufficient to have  $h_j(D) = 0$  for j = 1, 2, 3.

#### Examples

First, we consider some simple examples in which a superpotential is not generated.

We begin with  $B = \mathbf{P}^3$ . A simple way to construct a Calabi-Yau four-fold  $X_0$  that is elliptically fibered over  $\mathbf{P}^3$  is to take a hypersurface of degree (4,3) in  $\mathbf{P}^3 \times \mathbf{P}^2$ . The map  $\pi_0 : X_0 \to \mathbf{P}^3$  consists of forgetting the  $\mathbf{P}^2$  factor; the fibers are two-tori since a degree three equation in  $\mathbf{P}^2$  defines a curve of genus one. We considered this example in the last section in the context of M-theory and showed that no superpotential is generated because every divisor D has  $\chi(D, \mathcal{O}_D) < 1$ . For F-theory (with H = 0), one cannot use  $X_0$  because  $\pi_0$  does not have a section. Type IIB on  $\mathbf{P}^3$  is constructed instead using a Calabi-Yau four-fold X constructed as a hypersurface in a certain  $\mathbf{P}^2$  bundle over  $\mathbf{P}^3$  (and not simply in the product  $\mathbf{P}^3 \times \mathbf{P}^2$ ). We call the total space of this  $\mathbf{P}^2$  bundle W and let  $\sigma: W \to \mathbf{P}^3$  be the projection.

We now must pick a divisor C in  $B = \mathbf{P}^3$ , and compute the arithmetic genus of  $D = \sigma^{-1}(C) \cap X$ . To compute the invariants  $h_j(D)$  requires a method somewhat more powerful than used in the last section. One can conveniently use the spectral sequence for the projection  $\sigma : \sigma^{-1}(C) \to C$  as in Proposition 2.2 in [25].<sup>5</sup> The conclusion is similar to what we found in the last section for a divisor of degree (a,0) in  $X_0$ :  $h_1(D) = 0$ ,  $0 < h_2(D) < h_3(D)$ , so  $\chi(D, \mathcal{O}_D) < 1$ . In fact, in using Proposition 2.2 of [25], it does not matter whether the  $\mathbf{P}^2$  bundle over  $\mathbf{P}^3$  is trivial or not; in either case, one gets an explicit description of holomorphic forms on D along the lines of (2.10) and (2.11), with an obvious counting showing that  $h_2(D) < h_3(D)$ . So there is no instanton-generated superpotential in F-theory on  $\mathbf{P}^3$ . (The fact that in the computation it does not matter whether the map  $X \to \mathbf{P}^3$  has a section has a physical explanation given below.)

The same conclusion can be reached in the same way if  $\mathbf{P}^3$  is replaced by  $\mathbf{P}^2 \times \mathbf{P}^1$  or  $(\mathbf{P}^1)^3$ , or, roughly, any example in which the normal bundle to a divisor always has enough positivity. These latter examples have some interest because (by forgetting one of the  $\mathbf{P}^1$  factors) they are fibered over  $\mathbf{P}^2$  or  $\mathbf{P}^1 \times \mathbf{P}^1$ , with fiber  $\mathbf{P}^1$ ; they thus have interpretations in terms of the heterotic string, as we discuss in the next section.

For a rather different example, suppose that B is obtained from another surface, such as  $\mathbf{P}^3$ , by blowing up a point x, an operation that replaces x by a divisor C isomorphic to  $\mathbf{P}^2$ , with normal bundle  $\mathcal{O}(-1)$ . Then given an elliptic fibration  $\pi: X \to B$ , let  $D = \pi^{-1}(C)$ . One can compute, for example by again using Proposition 2.2 of [25], that  $h_j(D) = 0$  for j = 1, 2, 3, so that the wrapping of a Type IIB three-brane over such an "exceptional divisor" D does generate a superpotential. <sup>6</sup>

This last example has an interpretation in terms of the heterotic string, since  $\mathbf{P}^3$  with a point blown up can be fibered over  $\mathbf{P}^2$  with fibers  $\mathbf{P}^1$ . To see this, consider  $\mathbf{C}^5$  with coordinates  $(x_1, x_2, x_3, u, v)$ . Define a three-fold B to be the quotient of  $\mathbf{C}^5$  (with the points with u = v = 0 or  $x_1 = x_2 = x_3 = u = 0$  deleted) by a  $\mathbf{C}^* \times \mathbf{C}^*$  action  $(x_1, x_2, x_3, u, v) \rightarrow (\lambda x_1, \lambda x_2, \lambda x_3, \mu u, \lambda \mu v)$ . Then B is fibered over  $\mathbf{P}^2$  by forgetting u, v; the fibers are  $\mathbf{P}^1$ 's, obtained by projectivizing u, v. The divisor C with u = 0 is a section of the fibration

<sup>&</sup>lt;sup>5</sup> This reference was pointed out by M. Gross, who also showed that the condition on normal crossings of the discriminant can be replaced by the fact that D lies in a  $\mathbf{P}^2$  bundle over C.

<sup>&</sup>lt;sup>6</sup> There is actually a puzzle here, because one would expect as in [8] and the second paper in [2] to see a phase transition from Type IIB on  $\mathbf{P}^3$  with a point blown up to Type IIB on  $\mathbf{P}^3$ ; this seems to be a transition from a phase with broken supersymmetry to a phase with unbroken supersymmetry, something that one would not usually expect.

 $B \to \mathbf{P}^2$ . C is isomorphic to  $\mathbf{P}^2$ ; once u is set to zero, the scaling by  $\mu$  can be used to eliminate v, and then  $x_1, x_2, x_3$  are interpreted as homogeneous coordinates for  $C \cong \mathbf{P}^2$ . The normal bundle to C in B is  $\mathcal{O}(-1)$  (in scaling by  $\lambda$ , to preserve a "gauge condition" v = 1 by which  $\mu$  and v were eliminated, one sets  $\mu = \lambda^{-1}$ , so that the normal coordinate u to C scales as  $\lambda^{-1}$ ). One maps B to  $\mathbf{P}^3$  by  $(x_1, x_2, x_3, u, v) \to (ux_1, ux_2, ux_3, v)$ . This is an isomorphism away from u = 0, and "blows down" the divisor C to the point (0, 0, 0, 1).

Clearly, the three-fold B just considered is rather similar to the Hirzebruch surface  $\mathbf{F}_1$ , used in Type IIB compactification in [2]. This surface is isomorphic to  $\mathbf{P}^2$  with a point blown up, or to a  $\mathbf{P}^1$  bundle over  $\mathbf{P}^1$ , with a section, E, of self-intersection -1. For our final example, take  $B = G \times \mathbf{F}_1$ , with G another copy of  $\mathbf{P}^1$ . Note that B can be given two different structures of  $\mathbf{P}^1$  fibration: one has  $\tau : B \to \mathbf{F}_1$  by forgetting G, or  $\tau' : B \to G \times \mathbf{P}^1$  by taking the product of the identity map on G with the projection  $\mathbf{F}_1 \to \mathbf{P}^1$ . This will lead to two different identifications with the heterotic string rather as the existence of two different K3 fibrations has been exploited [26,2].

Now, let C be the divisor  $G \times E$  in  $G \times \mathbf{F}_1$ . Let X be a Calabi-Yau four-fold with an elliptic fibration  $\sigma: X \to B$ . Using proposition 2.2 of [25], the divisor  $D = \sigma^{-1}(C)$  in X can be shown to have  $h_j = 0$ , j = 1, 2, 3, so therefore a superpotential is generated in wrapping a Type IIB three-brane on C. Note that C is a section of  $\tau'$ , but not of  $\tau$ ; rather,  $C = \tau^{-1}(C')$  where C' = E is a curve in  $\tau(B) = \mathbf{F}_1$ . The consequences of these statements for the heterotic string will be clear in the next section.

## Elliptic Fibrations Without A Section

Finally, let us briefly consider M-theory compactification on an elliptically fibered four-fold X that does not have a section.

As the fibers shrink,  $\mathbf{R}^3 \times X$  looks locally like  $W \times \mathbf{S}^1 \times \mathbf{S}^1$  with a nine-manifold W. Absence of a section means that, calling the last two coordinates  $x^{10}$  and  $x^{11}$ , there are terms in the metric  $g_{i\,10}$  and  $g_{i\,11}$  (with  $i=1,\ldots,9$ ) that cannot be eliminated by shifting  $x^{10}$  and  $x^{11}$  by functions of the first nine coordinates. After shrinking the second circle, one locally along W gets Type IIA on  $W \times \mathbf{S}^1$  with a non-trivial  $g_{i\,10}$ ; the T-duality that maps this to Type IIB on  $W \times \mathbf{S}^1$  (which is the description that makes sense globally) turns this into a non-zero  $B_{i\,10}$ , with B the massless Neveu-Schwarz two-form of the theory; absence of a section in the original description means that H = dB is non-zero. The Ramond two-form  $\widetilde{B}$  of the Type IIB theory is also non-zero, since it arises from  $g_{i\,11}$  in the same

chain of dualities. Of course, they are related to each other by  $SL(2, \mathbf{Z})$ , so in F-theory one could not have one without the other.

The H-fields obtained this way have topologically normalized periods, and therefore vanish as forms in the limit as the radius of the  $S^1$  is scaled up and one goes to four dimensions. This gives a physical explanation for the fact (which was exhibited above as a consequence of Proposition 2.2 of [25]) that to analyze the part of the superpotential that survives when one gets to four dimensions, it does not matter whether the morphism  $X \to B$  has a section.

# 4. Application To The Heterotic String On A Calabi-Yau Three-Fold

According to [1], Type IIB on  $\mathbf{P}^1$  is the same as the heterotic string on  $\mathbf{T}^2$ . Therefore, if the complex three-fold B is fibered over a two-fold B' with fibers  $\mathbf{P}^1$ , by a holomorphic map  $\tau: B \to B'$  with  $\mathbf{P}^1$  fibers, then Type IIB on B is equivalent to the heterotic string on a Calabi-Yau three-fold Z that is fibered over B' with the  $\mathbf{P}^1$  fibers replaced by  $\mathbf{T}^2$ 's. For the analogous case with B and B' of complex dimension two and one, this construction has been used [2] to study the heterotic string on K3.

The equivalence of certain heterotic string models to Type IIB compactifications makes it possible to control the superpotential by studying divisors, as in the last section. Some of the examples given in the last section admit such  $\mathbf{P}^1$  fibrations  $B \to B'$ , and we need not repeat the examples here. I will not try in this paper to be explicit about the precise heterotic string models that these Type IIB compactifications correspond to. (Many relevant facts are in the second paper in [2].) But I will compare the qualitative results to what is expected of heterotic string physics.

We classify divisors  $C \subset B$  according to whether  $\tau(C)$  is all of B' or a submanifold  $C' \subset B'$ :

- (a') In the first case, C is a section or multi-section of  $\tau: B \to C$ .
- (b') In the second case, we start with a Riemann surface  $C' \subset B'$ , and  $C = \tau^{-1}(C')$  (or possibly a component thereof).

Unlike the corresponding situation in F-theory, divisors of either type may contribute to the superpotential, since we are not interested in taking any particular limit on the area of the  $\mathbf{P}^1$ . In the heterotic string, one expects at least two weak coupling mechanisms for generating a superpotential, namely:

(a'') Space-time instantons.

# (b'') World-sheet instantons.

I claim that contributions from divisors of type (a') correspond to space-time instanton effects, and contributions from divisors of type (b') correspond to world-sheet instanton effects.

A preliminary check is as follows. Note that space-time instantons on  $\mathbb{R}^4 \times Z$  are localized in  $\mathbb{R}^4$  but spread out over Z. By contrast, world-sheet instantons are localized on a Riemann surface  $F \subset Z$ . In the Type IIB description on  $\mathbb{R}^4 \times B$ , we cannot conveniently see Z, but we can conveniently see the four-dimensional space B' that Z maps to. It is clear that divisors of type (a'), which map to all of B', cannot be localized on a submanifold of Z of real dimension less than four, and so could not correspond to world-sheet instantons. However, divisors of type (b') are localized in two dimensions on B', and so might possibly be localized on a two-dimensional submanifold of Z and correspond to world-sheet instantons.

For more precise information, begin with Type IIB theory compactified to eight dimensions on a  $\mathbf{P}^1$  of volume V. The action for the massless graviton and gauge fields on  $\mathbf{R}^8$  is qualitatively

$$L = \int_{\mathbf{R}^8} d^8 x \sqrt{g} \left( VR + \text{tr} F^2 \right), \tag{4.1}$$

with g the metric in the Type IIB description, R the Ricci scalar, and F the Yang-Mills field strength. (There is no dilaton in the formula since the coupling varies on  $\mathbf{P}^1$  in a way uniquely determined by the vector moduli, which have been suppressed.) The point here is that the gravitational action has a factor of V from integration over  $\mathbf{P}^1$ , but the gauge fields are supported at special points on  $\mathbf{P}^1$  (related to singularities of the F-theory fibration) and have no such factor of V. To go to a heterotic string description, one introduces the heterotic string metric  $g_h$  by  $g = V^{-1}g_h$ , whereupon one gets

$$L = \int_{\mathbf{R}^8} d^8 x \sqrt{g_h} V^{-2} \left( R_h + \text{tr} F^2 \right). \tag{4.2}$$

 $(R_h \text{ is the Ricci scalar constructed from } g_h.)$  From (4.2) we see that the heterotic string coupling is  $\lambda_h = V$  [1].

Note that in the heterotic string description, the  $\mathbf{P}^1$  is replaced by a  $\mathbf{T}^2$  whose volume, at a generic point in Narain moduli space, is of order

$$V_h = 1. (4.3)$$

Now we want to consider what happens when one compactifies to four dimensions on a manifold B with a  $\mathbf{P}^1$  fibration. For simplicity, we consider a product  $B = \mathbf{P}^1 \times B'$ , rather than a fibration. Let V' be the volume of the four-manifold B', in Type IIB units. Because of the relation  $g = V^{-1}g_h$ , the volume of B' in heterotic string units is then  $V'_h = V^2V'$ . Because of (4.3), the volume of  $V_Z$  of Z in heterotic string units is also of order  $V'_h$ . The action  $I_{ST}$  of a space-time instanton of the heterotic string is of order  $V_Z/\lambda_h^2$ ; combining the above formulas we get

$$I_{ST} = V'. (4.4)$$

But V' is the action, in the Type IIB description, of an instanton of type (a'), which corresponds simply to a three-brane that wraps over B', whose volume is V'.

Now consider a divisor of type (b'), coming from a Riemann surface  $C' \subset B'$  whose area in Type IIB units is A. The volume of the divisor  $B' = \tau^{-1}(C')$  is then

$$Vol(B') = VA, \tag{4.5}$$

and this is the action in the Type IIB description of a three-brane wrapping over this divisor. In the heterotic string description, if an instanton of type (b') is going to correspond to a world-sheet instanton, then this instanton will have to correspond to a Riemann surface  $C'' \subset Z$  which is mapped to C' by the projection  $Z \to B'$ . The action  $I_{WS}$  of such an instanton is of order the area  $A_h$  of C' in the heterotic string description; because of the Weyl transformation between the two metrics,  $A_h = VA$ . So comparing to (4.5), we get the desired relation

$$I_{WS} = Vol(B'). (4.6)$$

From the examples of section (3), one sees instantons of either kind contributing to heterotic string superpotentials. Curiously, as was explained in the introduction, there is some novelty in this even for the world-sheet instantons.

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