A Derivation of
K-Theory from M-Theory

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We show how some aspects of the K-theory classification of RR fluxes follow from a careful analysis of the phase of the M-theory action. This is a shortened and simplified companion paper to "E_8 Gauge Theory, and a Derivation of K-Theory from M-Theory."

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1. Introduction

In the past few years, we have learned that D-brane charges should be thought of in the framework of K-theory [1,2,3]. More recently, it has been realized that the topological classification of RR fluxes in Type II string theory is also K-theoretic [4,5,6]. Other developments of the past few years, such as Matrix Theory, and the AdS/CFT correspondence, have shown that D-branes play an important role in the search for a more fundamental formulation of M-theory. It is natural, therefore, to ask how the K-theoretic formulation of RR charges and fluxes can be formulated in terms of M-theory.

In hindsight, the K-theoretic interpretation of RR fluxes and charges is almost inevitable, given the existence of Chan-Paton vector bundles on D-branes. But M-branes do not carry Chan-Paton bundles or vector fields, so the M-theoretic origin of K-theory is not manifest. In this letter, we will outline how some aspects of the K-theoretic formulation of RR charges and fluxes can in fact be derived from M-theory. In brief, when M-theory is formulated on an eleven-manifold of the form $Y = S^1 \times X$, we can derive what might be called "integral equations of motion" for the topological class of the four-form flux of M-theory. These equations state that (on-shell) the four-form flux of M-theory is in fact K-theoretic, in a sense we will make more precise below. In order to keep technical complications to a minimum, we will make several simplifying assumptions on the topology of X. Further details, without the simplifying assumptions, can be found in [7].

Let X be a compact spin 10-manifold with a fixed Riemannian metric $g_{\mu\nu}$. Consider Type IIA superstring theory on X with metric $tg_{\mu\nu}$. We will study the partition function of the theory in the limit $g_s \to 0$, where g_s is the string coupling, and then $t \to \infty$. We will then consider M-theory on $Y = X \times S^1$, with metric

$$ds_M^2 = g_s^{4/3}(dx^{11})^2 + tg_s^{-2/3}g_{\mu\nu}dx^\mu dx^\nu. \tag{1.1}$$

In M-theory, we will study the partition function in the limit $t \to \infty$ and then $g_s \to 0$. Finally, we will show that the leading terms in the partition function of M theory and Type IIA theory are in agreement.

One might at first think the agreement between the two expressions is trivial, since eleven-dimensional supergravity reduces on a circle to ten-dimensional supergravity. However, things are not so simple because the K-theoretic nature of RR fluxes implies that the sum over RR field-strengths is *not* simply a sum over all harmonic forms that obey conventional Dirac quantization. The quantization law is more subtle, and, as we will see,

there are subtle phases in the action which are not present in the standard treatments of supergravity Lagrangians. The goal will be to derive these subtleties of the Type IIA theory starting from M-theory. In section 6, we will describe three results, of independent interest, which are corollaries of our analysis.

2. The Type IIA Partition Function

In the weak coupling limit described above, we consider the NS fields fixed at some classical (not necessarily on-shell) values. In this background, we will study the one-loop quantum mechanics of fermions and the free-field quantum mechanics of the RR fields. The partition function is accordingly

$$Z_{IIA} \sim \exp\left(-\frac{1}{g_s^2}S_{NS}\right)\frac{\Theta_{IIA}}{\Delta} \tag{2.1}$$

where S_{NS} is the action for the NS-sector fields, Δ is a product of bose and fermi determinants, and Θ_{IIA}, which is the factor that we will really focus on, is a theta function which arises from summing over the fluxes of the RR p-form fields.

A more complete treatment of the problem would involve integrating over the moduli of the NS fields – notably the metric and the B-field – but in the present discussion we hold these fixed and in particular set $B = 0$.

A complete description of how to construct Θ_{IIA} can be found in [8,4,5]. We consider the full RR field-strength $G = G_0 + G_2 + \cdots + G_{10}$. This is a sum of real differential forms on X of even degree. In IIA supergravity, there is a Bianchi identity $dG = 0$, so G has a characteristic cohomology class that one can regard as an element of the even degree cohomology $H^{\text{even}}(X; \mathbf{R})$. Because D-branes exist, G obeys a Dirac-like quantization condition, but this condition does not merely state, as one might guess, that the periods of $G/2\pi$ are integral or in other words that $G/2\pi$ is associated with an element of the integral cohomology $H^{\text{even}}(X; \mathbf{Z})$. Instead, from the existence of D-branes and their couplings to RR fields, one can deduce [4,5,6] that the topological sectors of RR fields are classified by an element $x \in K(X)$ of the K-theory of X. (In Type IIB one would use $x \in K^1(X)$ in a similar way.) For given x, the cohomology class of G is

$$G(x) = \text{ch}(x)\sqrt{\hat{A}(X)}. \tag{2.2}$$

This is the K-theory version of Dirac quantization.

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For our purposes, to construct the partition function Θ_{IIA}, we will be summing over on shell RR fields. For each $x \in K(X)$, there is a unique harmonic form in the cohomology class of $G(x)$; we use this in the sum over RR fields.

The second subtlety is that the total field-strength G should be considered to be self-dual, $G = *G$, an equation which at the classical level is hardly compatible with (2.2) for general metrics. The resolution of this paradox is that [8,4] one interprets $G = *G$ as a statement in the quantum theory. One sums over "half" the fluxes, and these are indeed quantized by (2.2). More precisely $\Gamma = K(X)/K(X)_{tors}$ (where $K(X)_{tors}$ is the torsion subgroup) is a lattice with an integral symplectic form given by the index of the Dirac operator coupled to $x \otimes \overline{y}$. If we let $I(z)$ denote the index of the Dirac operator coupled to a bundle z, and we use the Atiyah-Singer formula for the index, then the symplectic form $\omega(x, y)$ is defined by

$$\omega(x, y) = I(x \otimes \overline{y}) = \int_X \text{ch}(x \otimes \overline{y}) \hat{A}. \tag{2.3}$$

Poincaré duality in K-theory implies that this form is *unimodular*. One sums over "half the fluxes" by summing over fluxes associated with K-theory classes in a maximal Lagrangian sublattice $\Gamma_1 \subset \Gamma$. This sum is the theta function for the quantum self-dual RR field.

To sketch in somewhat more detail the definition of the theta function, we use the symplectic structure (2.3) to give the torus $\mathcal{P}_K(X) = (K(X) \otimes \mathbf{R})/\Gamma$ the structure of a compact phase space. Moreover, this phase space has a metric

$$\| x \|^2 = \int G(x) \wedge *G(x). \tag{2.4}$$

There is therefore a unique translation invariant complex structure J on $\mathcal{P}_K(X)$ such that the metric (2.4) is of type $(1, 1)$. Explicitly, $G_{2p}(Jx) = (-1)^{p+1} * (G_{10-2p}(x))$. Coherent state quantization with respect to this complex structure leads to a unique quantum state, since the symplectic volume of $\mathcal{P}_K(X)$ is one. Θ_{IIA} is the wavefunction of this quantum state. To write it more explicitly, we now choose a complementary Lagrangian sublattice Γ_2 so that $\Gamma = \Gamma_1 \oplus \Gamma_2$. The lattice vectors in Γ_1, Γ_2 define "a-cycles" and "b-cycles" in $\mathcal{P}_K(X)$, and with respect to this decomposition we have a period matrix τ, which is a quadratic form on $\Gamma_1 \otimes \mathbf{R}$ with positive imaginary part. Finally, we must define the characteristics of the theta function. This is the subtlest part of the quantization procedure. We introduce a function $\Omega : \Gamma \to \mathbf{Z}_2$ such that

$$\Omega(x + y) = \Omega(x)\Omega(y)e^{i\pi\omega(x,y)}. \tag{2.5}$$

Then, we may define the characteristics $\theta \in \Gamma_1, \phi \in \Gamma_2$ by

$$\Omega(x) = (-1)^{\omega(x,\phi)} \text{ for } x \in \Gamma_1 \qquad \Omega(x) = (-1)^{\omega(x,\theta)} \text{ for } x \in \Gamma_2. \tag{2.6}$$

For an explanation of why the Ω function is needed and the rationale for the definition of the characteristics, see [8,4]. Finally, we may write the explicit formula for the theta function:

$$\Theta_{IIA} = e^{-i\pi \text{Re } (\tau(\theta/2))} \sum_{x \in \Gamma_1} e^{i\pi\tau(x + \frac{1}{2}\theta)} \Omega(x). \tag{2.7}$$

It remains to identify Ω. There is a (presumably unique) T-duality invariant choice [8,4] given in terms of the mod two index of Atiyah and Singer [9]. If V is a real vector bundle on X, then we define $q(V)$ to be the number, modulo two, of chiral zero modes of the Dirac operator $\slashed{D}_V$ coupled to V. For X of dimension $8k + 2$, $q(V)$ is a topological invariant. The definition of [4] is:

$$\Omega(x) = (-1)^{q(x \otimes \overline{x})}. \tag{2.8}$$

It is often useful to regard x as the charge of a D-brane in Type IIB theory. Then, by a Born-Oppenheimer argument [5], $q(x \otimes \overline{x})$ counts the number modulo two of fermion zero modes in the Ramond sector for open strings with boundary conditions defined by x at each end. This makes the T-duality invariance of Ω manifest.

A few facts about the mod two index will prove useful below. In general, the mod two index is not just the reduction modulo two of an ordinary index (which is, after all, simply zero in 10 dimensions for V real). It is true that if the real bundle V can (after complexification) be written as $V = x \oplus \overline{x}$ where $x \in K(X)$ and $\overline{x}$ is the complex conjugate of x, then $q(V)$ equals the mod two reduction of $I(x)$. This fact is used [4] in showing that Ω satisfies the cocycle relation (2.5).

There are many different choices of sublattices Γ_1. Up to an overall normalization, different choices lead to different descriptions of the same partition function. In the problem discussed in this letter, there is a very natural choice of polarization. To motivate it, consider the behavior of the kinetic energy of a non-self-dual field G as we scale the metric $g_{\mu\nu} \to t g_{\mu\nu}$. The kinetic energies scale as:

$$t^5 \parallel G_0 \parallel^2 + t^3 \parallel G_2 \parallel^2 + t \parallel G_4 \parallel^2 + t^{-1} \parallel G_6 \parallel^2 + t^{-3} \parallel G_8 \parallel^2 + t^{-5} \parallel G_{10} \parallel^2 . \tag{2.9}$$

We would like to choose a polarization so that only positive powers of t show up in the exponential. Otherwise the sum over fluxes becomes less and less convergent as $t \to \infty$,

and the terms in the sum do not accurately reflect the long-distance physics. We will have only positive powers of t if we take Γ_2 to be the set of K-theory classes x with $\mathrm{ch}_0(x) = \mathrm{ch}_1(x) = \mathrm{ch}_2(x) = 0$, and then take Γ_1 to be a complementary Lagrangian sublattice. Working through the definitions of the quantization procedure one finds:

$$\Theta_{IIA} = e^{-i\pi \mathrm{Re}\ (\tau(\theta/2))} \sum_{x \in \Gamma_1} e^{-\pi(t^5\|G_0\|^2 + t^3\|G_2\|^2 + t\|G_4\|^2)}$$
$$e^{i\pi \int (G_0 G_{10} - G_2 G_8 + G_4 G_6)} \Omega(x) \tag{2.10}$$

where G is understood to be given by (2.2) evaluated for $x + \frac{1}{2}\theta$. It might look at first sight like this is just the standard recipe for computing the RR partition function as a sum over fluxes of G_0, G_2, and G_4, with the higher RR fields eliminated using self-duality. However, the allowed values of the G_0, G_2, and G_4 fluxes differ from what one would conventionally guess. Moreover, in addition to a factor from the standard kinetic energy of G_0, G_2, and G_4, the action contains nonstandard phase factors. The factor $e^{i\pi \int (G_0 G_{10} - G_2 G_8 + G_4 G_6)}$ (which arises by computing the real part of the τ function of the lattice) is, after imposing (2.2) (which constrains the G_{2p} of $p > 2$ in terms of those of $p \leq 2$) a 120th root of unity that is given by a complicated cohomological formula and is not part of the standard supergravity formalism. The sign factor $\Omega(x)$ is not given by any cohomological formula.

As we have stressed in the introduction, we want to focus on the behavior for $t \to \infty$. The dominant contributions come from K-theory classes $x \in \Gamma_1$ such that $G_0(x) = G_2(x) = 0$. A glance at (2.2) shows that these are classes of virtual dimension zero such that $c_1(x) = 0$. Denoting by Γ_1' the sublattice of such classes, the leading term in the partition function may be simplified to

$$\Theta_{IIA} = e^{-i\pi \mathrm{Re}\ (\tau(\theta/2))} \sum_{x \in \Gamma_1'} e^{-\pi t\|G_4\|^2} e^{i\pi \int G_4 G_6} \Omega(x). \tag{2.11}$$

It is also important to include G_2 for a more complete comparison to M-theory (G_0 has no known origin in M-theory, at least for general backgrounds), but for simplicity, in the present letter we consider only G_4.

3. The M-Theory Partition Function

The partition function for M-theory in the large volume limit is given by

$$Z_M \sim \exp\left(-\int_Y \sqrt{g}\mathcal{R}\right) \frac{1}{\Delta_M} \Theta_M \tag{3.1}$$

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where the leading term is the Einstein action of a fixed Riemannian metric (1.1) on a spin 11-manifold Y, Δ_M are one-loop determinants (which we take to be positive, absorbing the sign in Θ_M), and Θ_M is the sum over the classical on-shell configurations of the C-field.

As in Type IIA, there is a subtle quantization law on $G = dC$ as well as a subtle phase-factor in the path integral [10]. The topological quantization of 4-form field-strengths is given by choosing any element $a \in H^4(Y; \mathbf{Z})$ and taking $G(a)$ to be a certain de Rham representative satisfying

$$\frac{G(a)}{2\pi} = a - \frac{1}{2}\lambda. \tag{3.2}$$

Here λ is the degree four class represented at the level of differential forms by $-\frac{\mathrm{tr}\,R \wedge R}{16\pi^2}$. It is an integral class on a spin manifold and satisfies $2\lambda = p_1$. The contribution of a field C in the topological sector a to the M-theory partition function is

$$e^{-\|G(a)\|^2}\Omega_M(C), \tag{3.3}$$

where $G(a)$ is the on-shell field configuration.

The phase factor $\Omega_M(C)$ is a globally well-defined version of the familiar supergravity interaction $\sim \int_Y CGG + \cdots$. Since $G(a)$ is a nontrivial cohomology class, C is not globally well-defined as a three-form, and the proper formulation of the phase is tricky [10]. We first find a 12-manifold Z such that $\partial Z = Y$ and a extends to $\widetilde{a} \in H^4(Z; \mathbf{Z})$. The existence of the pair $(Z, \widetilde{a})$ is highly nontrivial, but guaranteed by a result of Stong [11]. We then define the phase by

$$\Omega_M(C) = \epsilon \exp\left[2\pi i \int_Z \left(\frac{1}{6}(\widetilde{a} - \frac{1}{2}\lambda)^3 + (\widetilde{a} - \frac{1}{2}\lambda)\frac{\lambda^2 - p_2}{48}\right)\right], \tag{3.4}$$

where ϵ is the sign of the Pfaffian of the gravitino operator. In a topologically trivial situation we may identify $\widetilde{a} - \frac{1}{2}\lambda = \widetilde{G} = d\widetilde{C}$ and apply Stokes' theorem to make contact with the more standard supergravity expressions.

The expression (3.4) is not manifestly well-defined since the choice of Z is not unique. It was shown in [10] that this difficulty is most elegantly eliminated by using E_8 index theory. We will also find the connection to E_8 useful below. Therefore, let us recall that on manifolds $\mathcal{M}$ of dimension less than 16, cohomology classes $b \in H^4(\mathcal{M}; \mathbf{Z})$ are in 1-1 correspondence with topological classes of E_8 vector bundles $V(b)$ on $\mathcal{M}$ [12]. We hence consider the E_8 bundle $V(\widetilde{a})$ on Z in the adjoint representation and choose a connection A on $V(\widetilde{a})$ such that on Y

$$C = \frac{1}{16\pi^2}\frac{1}{30}\mathrm{Tr}_{\mathbf{248}}(AdA + \frac{2}{3}A^3) + \frac{1}{32\pi^2}\mathrm{Tr}(\omega d\omega + \frac{2}{3}\omega^3). \tag{3.5}$$

In other words, we interpret C as a Chern-Simons three form of E_8 gauge theory plus gravity. (It is not obvious that given a C-field a corresponding gauge field A always exists. A slightly longer argument can be made if a connection A making (3.5) hold on the nose does not exist.) We then can evaluate the phase in terms of the η invariants of the Dirac operator $\not{D}_{V(a)}$ and the gauge-fixed Rarita-Schwinger operator $D_{RS} = \not{D}_{TX-3\mathcal{O}}$ on Y. Here $\mathcal{O}$ is the trivial real line bundle. Using the APS index theorem [13], and the fact that the index of $\not{D}_{V(a)}$ is even in 12 dimensions, we can rewrite the phase as

$$\Omega_M(C) = \exp\left[2\pi i\left(\frac{\eta(\not{D}_{V(a)}) + h(\not{D}_{V(a)})}{4} + \frac{\eta(D_{RS}) + h(D_{RS})}{8}\right)\right]. \qquad (3.6)$$

Here h denotes the number of zero modes of the operator in question on Y. (The sign ϵ in Ω_M is absent as it cancels against a term that comes from the APS theorem.)

In order to compare to IIA superstrings, we will now restrict attention to $Y = X \times \mathbf{S}^1$. We restrict to fields invariant under rotations of $\mathbf{S}^1$, and, since we have taken $B = 0$ on the Type IIA side, we assume that the C-field is a pullback from X. Under these conditions, $\Omega_M(C)$ is a topological invariant that depends only on the characteristic class a, and not on C, so we will denote it as $\Omega_M(a)$. Moreover, the η invariants vanish for $X \times \mathbf{S}^1$ because of the reflection symmetry of $\mathbf{S}^1$, and only the contributions from the number h of zero modes survive. The a-dependent factor is then simply

$$\Omega_M(a) = \exp[i\pi h(\not{D}_{V(a)})/2]. \qquad (3.7)$$

Using the standard relation between the radius R of S^1 and the IIA string coupling we find:

$$\Theta_M = \sum_{a \in H^4(X;\mathbf{Z})} e^{-\|G(a)\|^2} \Omega_M(a) + \mathcal{O}(e^{-1/g_s^2}). \qquad (3.8)$$

where $G(a)$ is the harmonic form in the cohomology class $a - \frac{1}{2}\lambda$ and the corrections correspond to field-strengths which have an index tangent to the M-theory circle or are not invariant under rotations of the circle. [1]

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[1] In fact, the supergravity equation is $\kappa d * G = \frac{1}{2} G \wedge G + (\lambda^2 - p_2)/48$ for a suitable constant κ. The components of this equation which are pulled back from X are enforced by subsequent integration over the B-field, which is held fixed (and zero) in this letter. The difference from the harmonic form is cohomologically trivial.

4. K-Theory vs. Cohomology

We would now like to compare (2.11) to (3.8). These are, *a priori*, rather different expressions. One involves a sum over a certain part of $K(X)$ that is related to $H^4(X; \mathbf{Z})$ and the other involves a sum over $H^4(X; \mathbf{Z})$. If we included the other RR fields, then on the Type IIA side, we would be summing over the lattice $\Gamma_1 \subset K(X)$, and on the M-theory side we would be summing over degree two and degree four cohomology classes of X.

In general, $K(X)$ and $H^{\mathrm{even}}(X; \mathbf{Z})$ (the sum of the even degree integral cohomology groups of X) are closely related groups. If one tensors with the real numbers, they become isomorphic (by the map that takes an element of $K(X)$ to its Chern character). However the integral structures (which determine the Dirac quantization conditions) are different, and the torsion subgroups can be very different. For example, $H^{\mathrm{even}}(RP^{2n+1}; \mathbf{Z}) \cong \mathbf{Z} \oplus \mathbf{Z}_2^n$ while $K(RP^{2n+1}; \mathbf{Z}) \cong \mathbf{Z} \oplus \mathbf{Z}_{2^n}$. We now describe with some more precision the relation of K and H^{even} at the integral level. The reader will find more detail in [7].

Let us first describe how integral cohomology arises from a K-theory class x. For every K-theory class x, there is a smallest integer i such that x can be represented as the class of a $(2i-1)$-brane, wrapped on a $2i$-dimensional submanifold Q of X. The Poincaré dual of Q is a $(10-2i)$-dimensional cohomology class α associated with x.

The map from a K-theory class x to an associated cohomology class $\alpha(x)$ is the first step in a systematic procedure, known as the Atiyah-Hirzebruch spectral sequence, for comparing K-theory to cohomology. This map is relevant to us, because the Type IIA formula (2.11) involves a sum over K-theory classes x for which $\alpha(x)$ is an element of $H^4(X; \mathbf{Z})$ (modulo those for which $\alpha(x)$ is of degree six or higher), while the M-theory formula (3.8) is a sum over the characteristic class $a \in H^4(X; \mathbf{Z})$. We will compare the M-theory sum over a to the Type IIA sum over $\alpha(x)$.

Is it the case that every $a \in H^4(X; \mathbf{Z})$ is $\alpha(x)$ for some $x \in K(X)$? The answer to this question is "no." In ten dimensions, a necessary and sufficient condition for x to exist is that

$$Sq^3(a) = 0, \tag{4.1}$$

where Sq^3 is a certain cohomology operation, known as a Steenrod square. If x exists, we call it a "K-theory lift" of a. Such an x has virtual dimension zero, $c_1(x) = 0$, and $c_2(x) = -a$.

An introduction to the Steenrod squares Sq^i is given in [7]. In brief, if $a \in H^k(X; \mathbf{Z})$ then $Sq^3(a) \in H^{k+3}(X; \mathbf{Z})$ may be defined as follows. Let $Q(a)$ be a submanifold that is

Poincaré dual to a in X. Then the normal bundle $N(Q)$ of Q has integral characteristic classes $W_i(N(Q)) \in H^i(Q; \mathbf{Z})$ for i odd. These can be pushed into a tubular neighborhood of Q, allowing us to define

$$Sq^3(a) = W_3(N(Q)) \cup a. \tag{4.2}$$

Similarly, one defines the mod two Steenrod squares for all i (not necessarily odd) as follows. For $\overline{a} \in H^k(X; \mathbf{Z}/2)$, we set $Sq^i(\overline{a}) = w_i(N(Q)) \cup \overline{a} \in H^{k+i}(X; \mathbf{Z}/2)$. The Steenrod squares obey many identities; the ones we need are as follows (for sketches of proofs see [7]). First, the integral and mod two squares are related. In fact, $Sq^3(a) = 0$ if and only if $Sq^2(a)$ has an integral lift, that is, if and only if there is an integral class b whose mod two reduction is $Sq^2(a)$. Second, it is possible to "integrate by parts" with Steenrod squares. That is, for any a, b, $\int_X a \cup Sq^2 b = \int_X Sq^2 a \cup b$. Closely related to the criterion (4.1) for a K-theory lift to exist is the fact that if x is a K-theory class with $c_1(x) = 0$, then the higher Chern classes of x obey

$$c_3(x) = Sq^2 c_2(x) \bmod 2. \tag{4.3}$$

Finally, $Sq^3 Sq^3 = 0$, so we may take its cohomology, namely, the kernel of Sq^3 acting on $H^{\mathrm{even}}(X, \mathbf{Z})$ modulo the image of Sq^3 acting on $H^{\mathrm{odd}}(X, \mathbf{Z})$. In our situation, the cohomology of Sq^3 is, essentially, $K(X)$.

While the equation $Sq^3(a) = 0$ might seem somewhat exotic, it is a close cousin of a condition that has already appeared in the physics literature on D-branes. In particular, if we think of x as determining the D-brane *charge* (in IIB string theory) of a brane wrapped on Q, then cancellation of worldsheet global anomalies implies that $W_3(N(Q)) = 0$ [14]. Thus, by (4.2) it follows that $Sq^3(a) = 0$ if a is Poincaré dual to Q.

Let us now apply these remarks to study (2.11). We would like to convert the sum over the sublattice Γ'_1 in K-theory to a sum over cohomology elements, namely to a sum over classes $a \in H^4(X; \mathbf{Z})$ such that a has a K-theory lift. At this point we run into an apparent difficulty. A K-theory lift of a, if it exists, is not unique because given one lift x it is always possible to add a class y to x where y is any element of the lattice Γ_2 introduced earlier (thus $G_{2p}(y) = 0$ for $p \leq 2$). The quantities $G_6(x)$ and $\Omega(x)$ in (2.11) definitely depend on the choice of K-theory lift, but the product

$$e^{i\pi \int G_4(x + \frac{1}{2}\theta) G_6(x + \frac{1}{2}\theta)} \Omega(x) \tag{4.4}$$

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does not. This can be demonstrated using the facts noted above. Using (2.2), (2.6), and integration by parts, we first rewrite (4.4) as

$$\exp[-i\frac{\pi}{4}\int c_2(\theta)c_3(\theta)]\exp[i\frac{\pi}{2}\int (c_2(x)+c_2(\theta))c_3(x)]\Omega(x). \tag{4.5}$$

If we change the K-theory lift by $x \to x + y$ with y as above then using the cocycle condition (2.5), the definition (2.6) of θ, and the index theorem one can verify that (4.5) is unchanged. At this point, we have shown that the Type IIA partition function can be written as a sum over G_4 fluxes (along with G_2 and G_0 if one chooses to include them), as one would naively expect, but with subtle shifts in the Dirac quantization condition and an exotic sign factor in the sum over fluxes.

Finally, it remains to translate the characteristic θ into cohomology. Again, it is useful to regard x as the D-brane charge of a brane in IIB theory wrapping some worldvolume Q of smallest possible dimension. As we noted above, the mod two index $q(x \otimes \overline{x})$ is given by the number mod two of the fermion zero modes of a singly-wrapped brane on Q. The classes $x \in \Gamma_2$ correspond to $D(-1)$, $D1$ and $D3$ instantons in X. In the first two cases, the number of zero modes is easily seen to be even. On the other hand, for a $D3$ instanton, the number of fermion zero modes is given by the index theorem to be $\int_Q \lambda$ mod 2. We conclude that $\mathrm{ch}(\theta) = -\lambda + \cdots$. Thus, we can simplify (2.11) to

$$\Theta_{IIA} = \sum_{a \in H^4(X;\mathbf{Z}):\ Sq^3(a)=0} e^{-\pi t\|a-\frac{1}{2}\lambda\|^2} e^{i\frac{\pi}{2}\int (c_2(x(a))+\lambda)c_3(x(a))}\Omega(x(a)) \tag{4.6}$$

where $x(a)$ is any K-theory lift of a. We have now expressed the K-theory sum in terms of cohomology. It is now time to re-examine the M-theory sum (3.8).

5. The Integral Equation of Motion in M-Theory

In the previous section we reduced the K-theory partition function to a sum over a subgroup of $H^4(X;\mathbf{Z})$. This subgroup is of finite index, since for any a, $Sq^3(a)$ is of order 2, and hence $Sq^3(2a) = 0$. By contrast, the M-theory partition function is a sum over the full group $H^4(X;\mathbf{Z})$. To show that the two expressions for the partition functions agree, we will argue that the M-theory phase $\Omega_M(a)$ leads to an "integral equation of motion" $Sq^3(a) = 0$ on the topological sectors in M-theory.

In this letter we will, for simplicity, show agreement of (4.6) and (3.8) under the assumption that $\Omega_M(c) = 1$ for torsion c, and that $Sq^3(c) = 0$ for all torsion elements c.

Suppose $a \in H^4(X; \mathbf{Z})$ is any class and $c \in H^4(X; \mathbf{Z})_{tors}$. The kinetic energy of G in the topological sector $a+c$ is identical to that in the sector a because the field-strength defined by (3.2) is a real differential form and hence $G(a + c) = G(a)$. Since the torsion subgroup is finite we may equally well write (3.8) as

$$\Theta_M = \sum_{a \in H^4(X; \mathbf{Z})} e^{-\|G(a)\|^2} \Omega_M^{\mathrm{av}}(a) \tag{5.1}$$

with

$$\Omega_M^{\mathrm{av}}(a) = \frac{1}{|H^4(X; \mathbf{Z})_{tors}|} \sum_{c \in H^4(X; \mathbf{Z})_{tors}} \Omega_M(a + c). \tag{5.2}$$

This is useful because the M-theory phase $\Omega_M(a)$ is not independent of the shift $a \to a+c$. Indeed, the bundles $V(a + c)$ and $V(a)$ are definitely not isomorphic, and, as we will demonstrate below, Ω_M^{av} is in fact a projection operator. Under a simplifying topological assumption (described below) this operator is:

$$\begin{aligned}
\Omega_M^{\mathrm{av}}(a) &= 0 && \text{if} \quad Sq^3(a) \neq 0 \\
&= \Omega_M(a) && \text{if} \quad Sq^3(a) = 0.
\end{aligned} \tag{5.3}$$

Moreover, when $Sq^3(a) = 0$, so that a has a K-theory lift $x \in K(X)$, we can compare M-theory and K-theory phases. We will show that they agree

$$\Omega_M(a) = \Omega(x) e^{\frac{i\pi}{2} \int_X (c_2(x) + \lambda) c_3(x)}. \tag{5.4}$$

The agreement of (4.6) with (3.8) immediately follows from the above pair of results. We will now sketch how they are derived, beginning with the proof of (5.4). It is here that the interpretation of the M-theory phase in terms of E_8 gauge theory is particularly effective. We are interested in $Y = X \times S^1$ with supersymmetric spin structure on the M-theory circle. In evaluating (3.7), we use the fact that a zero mode of $\slashed{D}_{V(a)}$ is constant along the M-circle so that the phase just depends on the number of zero modes on X. These may be expressed in terms of the number of chiral zero modes in 10 dimensions via $h(\slashed{D}_{\pi^* V(a)}) = h^+(\slashed{D}_{V(a)}) + h^-(\slashed{D}_{V(a)}) = 2h^+(\slashed{D}_{V(a)})$. We conclude that the phase is expressed in terms of a mod two index, $f(a) = q(V(a))$:

$$\Omega_M(a) = (-1)^{f(a)}. \tag{5.5}$$

The next step is to relate the E_8 bundle $V(a)$ to a K-theory class x. In general, K-theory classes are differences of vector bundles $x = E_1 - E_2$ where the structure group of

E_i must be taken to be $U(N)$ for some large N. However, we are working in 10-dimensions, and this dimension is sufficiently small that *all K-theory classes on X with $c_1(x) = 0$ can be realized using $SU(5)$ bundles.* The reason for this is that the classification of bundles on a ten-manifold depends only on the homotopy groups $\pi_i(SU(N))$ for $i < 10$. (See [12] for a description of this approach.) In 10 dimensions, $SU(5)$ is in the stable range: $\pi_i(SU(5)) = \pi_i(SU(\infty))$, $i < 10$. We can therefore take our K-theory lift to be $x = E - F$ where F is a trivial rank 5 bundle and E is an $SU(5)$ bundle with $\mathrm{ch}(E) = 5 + a + \cdots$. We can construct an E_8 bundle $V(a)$ with characteristic class a using the "embedding" of $SU(5) \times SU(5)$ in E_8, taking the two $SU(5)$ bundles to be respectively E and F. Using the decomposition of the adjoint representation of E_8 under $SU(5) \times SU(5)$ and the fact that F is trivial, one finds that for mod 2 index theory (throwing away representations that appear an even number of times), the **248** is equivalent to $E \otimes \overline{E} - \mathcal{O} + \wedge^2 E + \wedge^2 \overline{E}$, where $\mathcal{O}$ is a trivial line bundle and $\wedge^2$ denotes the second antisymmetric product. Using the properties of the mod 2 index described above we now learn that

$$q(V(a)) = q(E \otimes \overline{E} - \mathcal{O}) + I(\Lambda^2(E)) = q(x \otimes \overline{x}) + I(\Lambda^2(E)) + I(E) \bmod 2. \quad (5.6)$$

The formula (5.6) leads directly to (5.4). Indeed, the first term on the RHS of (5.6) corresponds to $\Omega(x)$, while by an easy application of the index theorem, the second term is $\frac{1}{2} \int (c_2(x) + \lambda) c_3(x) \bmod 2$.

It remains to show (5.3). This is based on an analog of (2.5) for the E_8 mod two index $f(a)$. Namely, f satisfies the bilinear identity

$$f(a + a') = f(a) + f(a') + \int_X a \cup Sq^2 a'. \quad (5.7)$$

Unfortunately, there does not appear to be an elementary proof of (5.7). A proof using "cobordism theory" can be found in section 3.2 of [7]. Granted this, we are now ready to complete the proof of (5.3). The argument simplifies considerably if we assume that $f(c) = f_0$ is independent of c for *torsion* classes c. In this case we can write

$$\Omega_M^{\mathrm{av}}(a) = (-1)^{f(a) + f_0} \frac{1}{|H^4(X; \mathbf{Z})_{tors}|} \sum_{c \in H^4(X; \mathbf{Z})_{tors}} e^{i\pi \int c \cup Sq^2 a}. \quad (5.8)$$

Now, for any $b \in H^6(X; \mathbf{Z})$ and any torsion c it is always true that $\int b \cup c = \frac{1}{n} \int b \cup (nc) = 0$. Using Poincaré duality, one can prove the converse: if $\overline{b} \in H^6(X; \mathbf{Z}_2)$ satisfies $\int c \cup \overline{b} = 0$ for all $c \in H^4(X; \mathbf{Z})_{tors}$, then $\overline{b}$ is the reduction of some integral class. Therefore $\Omega_M^{\mathrm{av}}(a)$ projects onto the set of classes a such that $Sq^2 a$ has an integral lift. This is equivalent to $Sq^3(a) = 0$, i.e., to the statement that a has a K-theory lift x. Indeed $Sq^2 a$ is the reduction modulo two of $c_3(x)$. This completes the proof of (5.3), and therefore establishes the equivalence of (2.11) and (3.8).

6. Three applications

The K-theory/E_8-formalism described above leads to three interesting physical effects which we will sketch very briefly here.

First, an easy consequence of (5.7) leads to a new topological consistency condition on string backgrounds. By (5.7) we have $f(a + 2c) = f(a) + f(2c)$, and moreover $f(2c) = \int c \cup Sq^2 c$. By a result of Stong [11] $\int c \cup Sq^2 c = \int c \cup Sq^2 \lambda$. Now, the reasoning below (5.8) shows that $\Omega_M^{av}(a) = 0$, and hence $\Theta_M = 0$ if $Sq^3 \lambda \neq 0$. In algebraic topology one shows that a certain characteristic class $W_7(X)$ of X is $Sq^3(\lambda)$. Thus $W_7(X) = 0$ is a necessary condition for a consistent background. Unfortunately, we do not know an intuitive interpretation of this condition.

Second, it turns out that the parity symmetry of M theory on $X \times S^1$ (coming from reflection of the S^1) is anomaly-free, but this depends on a surprising anomaly cancellation between bosons and fermions. In IIA theory, this symmetry is $(-1)^{F_L}$, and in IIB it is related to strong/weak coupling duality. By counting fermion zero-modes one can show that the gravitino measure μ transforms under parity as $\mu \to (-1)^{q(TX)}\mu$. The mod-two index $q(TX)$ is nonvanishing for certain 10-folds, such as $X = T^2 \times \mathbf{HP}^2$. The fermion anomaly is cancelled by the nontrivial transformation law of the G-flux partition function (3.8). Indeed, parity acts as $G \to -G$ and by (3.2) $a \to \lambda - a$. Using the bilinear identity (5.7) and $\int a \cup Sq^2 a = \int a \cup Sq^2 \lambda$, one finds that $\Theta_M \to (-1)^{f(\lambda)}\Theta_M$. It follows that the total parity anomaly is $(-1)^{q(TX)+f(\lambda)}$. On the other hand, the fermion measure of the *heterotic string* on X transforms under $(-1)^F$ by the same factor $(-1)^{q(TX)+f(\lambda)}$. It is shown in [12] that the heterotic string measure is well-defined, so we conclude that $q(TX) + f(\lambda) = 0$ mod 2, and hence that parity is a good symmetry of M-theory.

Our third application concerns the instability of some Type IIB branes wrapping homologically nontrivial cycles. The K-theory interpretation of D-branes means that D-branes cannot be wrapped on certain cycles; it also means that D-branes wrapped on certain cycles are unstable even though the cycles are nontrivial in homology. Let Q be a cycle Poincaré dual to an integral class $a \in H^{\text{even}}(X, \mathbf{Z})$. If $Sq^3(a) \neq 0$ then, as we have mentioned, we cannot wrap a D-brane on Q. However, as stressed near (4.3), $K(X)$ is, essentially, the *cohomology* of Sq^3. Thus, if a is "closed," that is $Sq^3(a) = 0$, then a can be lifted to a K-theory class x, but if a is "exact," that is $a = Sq^3(a_0)$ for some a_0, then one can take $x = 0$. A D-brane whose lowest RR charge is given by such an a can in fact be unstable, even though the class a is nonzero in cohomology. Annihilation of such a D-brane occurs via nucleation and subsequent annihilation of $D9 - \overline{D9}$ pairs. This follows from the K-theoretic interpretation [2] of the work of Sen on brane-antibrane annihilation.

7. Conclusions, Further Results, and Open Problems

The matching described above and in [7] between the M-theory formalism based on E_8 and the Type IIA formalism based on K-theory gives considerable added confidence in both. In particular, we gain added confidence that not only D-brane charges, but also RR fluxes, should be classified by K-theory. This is an important conceptual change from the K-theoretic classification of D-brane charge; among other things, it suggests that RR fields, and not just D-branes, should somehow be associated with vector bundles.

We have focused here on the simplest case of the computation of [7] in order to illustrate some of the central ideas. The simplifying topological assumptions we have made are relaxed in [7]. Also, in [7] we extend the computation sketched above to include $G_2 \neq 0$ in Type IIA theory; in M-theory, this corresponds letting Y be a circle bundle over X with Euler class $G_2/2\pi$. After a lengthier analog of the above computation with some additional ingredients added, the phases turn out to agree.

One interesting general lesson that emerges from [7] is that when one takes torsion into account there is no direct relation between the flux G_4 in IIA theory and the four-form G in M-theory. They have different underlying integral quantizations, and there is no 1-1 correspondence of the terms in the M-theory and the IIA theta functions. It is only after applying the "integral equation of motion," $Sq^3(a) = 0$ that one can compare results.

As for future directions, it should be very interesting to compare the absolute normalization of the M-theory and Type IIA partition functions; this depends on the one-loop determinants as well as some other overall normalization constants which arise when Sq^3 does not annihilate the torsion. One would like to extend the computation to include D-brane and M-brane instanton effects. Another, more difficult, open problem concerns the proper interpretation of nonzero values of G_0. While it is straightforward to include the effects of G_0 in the IIA partition function, comparing the results to M-theory presents an interesting and unsolved challenge.

Our computation confirms the utility of relating the C-field of M-theory to E_8 gauge theory as in [10]. Other clues of a possible role of E_8 in the formulation of M-theory include the possibility of writing eleven-dimensional supergravity in terms of gauge fields of a noncompact form of E_8 [15], evidence for propagating E_8 gauge fields in M-theory on a manifold with boundary [16], and further issues considered in [17].

Finally, we mention that these considerations lead to an unresolved question in the case of Type IIB superstring theory. The problem is to reconcile the $SL(2, \mathbf{Z})$ symmetry

of this theory with the K-theoretic interpretation of RR charges and fluxes. Although we have found some nontrivial partial results relevant to this problem, the main puzzle remains unsolved. Nevertheless, we hope that the clarification of the relation of M-theory and K-theory will play some role in the resolution.

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