Three-Dimensional Gravity Reconsidered

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We consider the problem of identifying the CFT's that may be dual to pure gravity in three dimensions with negative cosmological constant. The c-theorem indicates that three-dimensional pure gravity is consistent only at certain values of the coupling constant, and the relation to Chern-Simons gauge theory hints that these may be the values at which the dual CFT can be holomorphically factorized. If so, and one takes at face value the minimum mass of a BTZ black hole, then the energy spectrum of three-dimensional gravity with negative cosmological constant can be determined exactly. At the most negative possible value of the cosmological constant, the dual CFT is very likely the monster theory of Frenkel, Lepowsky, and Meurman. The monster theory may be the first in a discrete series of CFT's that are dual to three-dimensional gravity. The partition function of the second theory in the sequence can be determined on a hyperelliptic Riemann surface of any genus. We also make a similar analysis of supergravity.

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1. Introduction

Three-dimensional pure quantum gravity, with the Einstein-Hilbert action

$$ I = \frac{1}{16\pi G} \int d^3x \sqrt{g} \left(R + \frac{2}{\ell^2} \right), \qquad (1.1) $$

has been studied from many points of view (see [1-3] for some early developments and [4] for a recent review and references), but its status is fundamentally unclear. The present paper is devoted to a tentative attempt to reconsider it. Before giving an overview of the paper, we begin with an introduction to the problem.

The first thought about this theory is that at the classical level it is "trivial," in the sense that there are no gravitational waves, and any two solutions are equivalent locally. So perhaps it might be tractable quantum mechanically.

A second thought is that despite being "trivial," the theory actually is unrenormalizable by power counting, since the gravitational constant G has dimensions of length. So perhaps the quantum theory does not exist.

The claim about unrenormalizability, however, is fallacious, precisely because the classical theory is trivial. In three dimensions, the Riemann tensor R_{ijkl} can be expressed in terms of the Ricci tensor R_{ij}. In the case of pure gravity, the equations of motion express the Ricci tensor as a constant times the metric. So finally, any possible counterterm can be reduced to a multiple of $\int d^3x \sqrt{g}$ and is equivalent on-shell to a renormalization of the cosmological constant, which is parametrized in (1.1) via the parameter ℓ^2. A counterterm that vanishes on shell can be removed by a local redefinition of the metric tensor g (of the general form $g_{ij} \to g_{ij} + aR_{ij} + \ldots$, where a is a constant and the ellipses refer to local terms of higher order). So a more precise statement is that any divergences in perturbation theory can be removed by a field redefinition and a renormalization of ℓ^2.

1.1. Relation To Gauge Theory

The claim just made is valid regardless of how one formulates perturbation theory. But actually, there is a natural formulation in which no field redefinition or renormalization is needed. This comes from the fact that classically, $2 + 1$-dimensional pure gravity can be expressed in terms of gauge theory. The spin connection ω is an $SO(2,1)$ gauge field (or an $SO(3)$ gauge field in the case of Euclidean signature). It can be combined with the "vierbein" e to make a gauge field of the group $SO(2,2)$ if the cosmological constant is negative (and a similar group if the cosmological constant is zero or positive). We simply combine ω and e to a 4×4 matrix A of one-forms:

$$A = \begin{pmatrix} \omega & e/\ell \\ -e/\ell & 0 \end{pmatrix}. \tag{1.2}$$

Here ω fills out a 3×3 block, while e occupies the last row and column. As long as e is invertible, the usual transformations of e and ω under infinitesimal local Lorentz transformations and diffeomorphisms combine together into gauge transformations of A. This statement actually has a close analog in any spacetime dimension d, with $SO(d-1,2)$ replacing $SO(2,2)$. What is special in $d = 3$ is that [5,6] it is also possible to write the action in a gauge-invariant form. Indeed the usual Einstein-Hilbert action (1.1) is equivalent to a Chern-Simons Lagrangian[1] for the gauge field A:

$$I = \frac{k}{4\pi} \int \mathrm{tr}^* \left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right). \tag{1.3}$$

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[1] Here tr^* denotes an invariant quadratic form on the Lie algebra of $SO(2,2)$, defined by $\mathrm{tr}^* ab = \mathrm{tr}\, a \star b$, where tr is the trace in the four-dimensional representation and $\star$ is the Hodge star, $(\star b)_{ij} = \frac{1}{2}\epsilon_{ijkl}b^{kl}$.

In dimensions other than three, it is not possible to similarly replace the Einstein-Hilbert action with a gauge-invariant action for gauge fields.[2]

In the gauge theory description, perturbation theory is renormalizable by power counting, and is actually finite, because there are no possible local counterterms. The Chern-Simons functional itself is the only gauge-invariant action that can be written in terms of A alone without a metric tensor; as it is not the integral of a gauge-invariant local density, it will not appear as a counterterm in perturbation theory. The cosmological constant cannot be renormalized, since in the gauge theory description, it is a structure constant of the gauge group.

As we have already remarked, the Chern-Simons description of three-dimensional gravity is valid when the vierbein is invertible. This is so for a classical solution, so it is true if one is sufficiently close to a classical solution. Perturbation theory, starting with a classical solution, will not take us out of the region in which the vierbein is invertible, so the Chern-Simons description of three-dimensional gravity is valid perturbatively. The fact that, in this formulation, the perturbation expansion of three-dimensional gravity is actually finite can reasonably be taken as a hint that the corresponding quantum theory really does exist.

However, nonperturbatively, the relation between three-dimensional gravity and Chern-Simons gauge theory is unclear. For one thing, in Chern-Simons theory, nonperturbatively the vierbein may cease to be invertible. For example, there is a classical solution with $A = \omega = e = 0$. The viewpoint in [6] was that such non-geometrical configurations must be included to make sense of three-dimensional quantum gravity nonperturbatively. But it has has been pointed out (notably by N. Seiberg) that when we do know how to make sense of quantum gravity, we take the invertibility of the vierbein seriously. For example, in perturbative string theory, understood as a model of quantum gravity in two spacetime dimensions, the integration over moduli space of Riemann surfaces that leads to a sensible theory is derived assuming that the metric should be non-degenerate.

There are other possible problems in the nonperturbative relation between three-dimensional gravity and Chern-Simons theory. The equivalence between diffeomorphisms and gauge transformations is limited to diffeomorphisms that are continuously connected to the identity. However, in gravity, we believe that more general diffeomorphisms (such as

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[2] In $d = 4$, it is possible to write the Hamiltonian constraints of General Relativity in terms of gauge fields [7]. This has been taken as the starting point of loop quantum gravity.

modular transformations in perturbative string theory) play an important role. These are not naturally incorporated in the Chern-Simons description. One can by hand supplement the gauge theory description by imposing invariance under disconnected diffeomorphisms, but it is not clear how natural this is.

Similarly, in quantum gravity, one expects that it is necessary to sum over the different topologies of spacetime. Nothing in the Chern-Simons description requires us to make such a sum. We can supplement the Chern-Simons action with an instruction to sum over three-manifolds, but it is not clear why we should do this.

From the point of view of the Chern-Simons description, it seems natural to fix a particular Riemann surface Σ, say of genus g, and construct a quantum Hilbert space by quantizing the Chern-Simons gauge fields on Σ. (Indeed, there has been remarkable progress in learning how to do this and to relate the results to Liouville theory [8-11].) In quantum gravity, we expect topology-changing processes, such that it might not be possible to associate a Hilbert space with a particular spatial manifold.

Regardless of one's opinion of questions such as these, there is a more serious problem with the idea that gravity and gauge theory are equivalent non-perturbatively in three dimensions. Some years after the gauge/gravity relation was suggested, it was discovered by Bañados, Teitelboim, and Zanelli [12] that in three-dimensional gravity with negative cosmological constant, there are black hole solutions. The existence of these objects, generally called BTZ black holes, is surprising given that the classical theory is "trivial." Subsequent work [13,14] has made it clear that three-dimensional black holes should be taken seriously, particularly in the context of the AdS/CFT correspondence [15].

The BTZ black hole has a horizon of positive length and a corresponding Bekenstein-Hawking entropy. If, therefore, three-dimensional gravity does correspond to a quantum theory, this theory should have a huge degeneracy of black hole states. It seems unlikely that this degeneracy can be understood in Chern-Simons gauge theory, because this essentially topological theory has too few degrees of freedom. However, some interesting attempts have been made; for a review, see [4].

The existence of the BTZ black hole makes three-dimensional gravity a much more exciting problem. This might be our best chance for a solvable model with quantum black holes. Surely in 3+1 dimensions, the existence of gravitational waves with their nonlinear interactions means that one cannot hope for an exact solution of any system that includes

quantum gravity.[3] There might be an exact solution of a 1+1-dimensional model with black holes (interesting attempts have been made [16]), but such a model is likely to be much less realistic than three-dimensional pure gravity. For example, in 1+1 dimensions, the horizon of a black hole just consists of two points, so there is no good analog of the area of the black hole horizon.

1.2. What To Aim For

So we would like to solve three-dimensional pure quantum gravity. But what would it mean to solve it?

First of all, we will only consider the case that the cosmological constant Λ is negative. This is the only case in which we know what it would mean to solve the theory.

Currently, there is some suspicion (for example, see [17]) that quantum gravity with $\Lambda > 0$ does not exist nonperturbatively, in any dimension. One reason is that it does not appear possible with $\Lambda > 0$ to define precise observables, at least none [18] that can be measured by an observer in the spacetime.[4] This is natural if a world with positive Λ (like the one we may be living in) is always at best metastable – as is indeed the case for known embeddings of de Sitter space in string theory [20]. If that is so, then pure gravity with $\Lambda > 0$ does not really make sense as an exact theory in its own right but (like an unstable particle) must be studied as part of a larger system. There may be many choices of the larger system (for example, many embeddings in string theory) and it may be unrealistic to expect any of them to be soluble.

Whether that is the right interpretation or not, we cannot in this paper attempt to solve three-dimensional gravity with $\Lambda > 0$, since, not knowing how to define any mathematically precise observables, we do not know what to try to calculate.

For $\Lambda = 0$, above three dimensions there is a precise observable in quantum gravity, the S-matrix. However, in the three-dimensional case, there is no S-matrix in the usual sense, since in any state with nonzero energy, the spacetime is only locally asymptotic to Minkowski space at infinity [3]. More relevantly for our purposes, in three-dimensional pure gravity with $\Lambda = 0$, there is no S-matrix since there are no particles that can be

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[3] An exact solution or at least an illuminating description of the appropriate Hamiltonian for near-extremal black holes interacting with external massless particles is conceivable.

[4] At least perturbatively, the de Sitter/CFT correspondence gives observables that can be measured by an observer who looks at the whole universe from the outside [19,18]. These observables characterize the wavefunction of the ground state.

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scattered. There are no gravitons in three dimensions, and there are also no black holes unless $\Lambda < 0$. So again, we do not have a clear picture of what we would aim for to solve three-dimensional gravity with zero cosmological constant.

With negative cosmological constant, there is an analog, and in fact a much richer analog, of the S-matrix, namely the dual conformal field theory (CFT). This is of course a *two*-dimensional CFT, defined on the asymptotic boundary of spacetime. Not only does AdS/CFT duality make sense in three dimensions, but in fact one of the precursors of the AdS/CFT correspondence was the discovery by Brown and Henneaux [21] of an asymptotic Virasoro algebra in three-dimensional gravity. They considered three-dimensional gravity with negative cosmological constant possibly coupled to additional fields. The action is

$$I = \frac{1}{16\pi G} \int d^3x \sqrt{g} \left(R + \frac{2}{\ell^2} + \dots \right), \tag{1.4}$$

where the ellipses reflect the contributions of other fields. Their main result is that the physical Hilbert space obtained in quantizing this theory (in an asymptotically Anti de Sitter or AdS spacetime) has an action of left- and right-moving Virasoro algebras with $c_L = c_R = 3\ell/2G$. In our modern understanding [15], this is part of a much richer structure – the boundary conformal field theory.

What it means to solve pure quantum gravity with $\Lambda < 0$ is to find this dual conformal field theory. We focus on the case $\Lambda < 0$ because this is the only case in which we would know what it means to solve the theory. Luckily, and perhaps not coincidentally, this is also the case that has black holes.

1.3. A Non-Classical Restriction

This formulation of what we aim to do makes it clear that we must anticipate a restriction that is rather surprising from a classical point of view. In contemplating the classical action (1.4), it appears that the dimensionless ratio ℓ/G is a free parameter. But the formula for the central charge $c_L = c_R = 3\ell/2G$ shows that this cannot be the case. According to the Zamolodchikov c-theorem [22], in any continuously varying family of conformal field theories in $1+1$ dimensions, the central charge c is constant. More generally, the same is true for the left- and right-moving central charges c_L and c_R.

So the central charges of the dual CFT cannot depend on a continuously variable parameter ℓ/G. It must be [18] that the theory only makes sense for specific values of ℓ/G.

Of course, the c-theorem has an important technical assumption: the theory must have a normalizable and $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$-invariant ground state. (The two factors of $SL(2,\mathbb{R})$ are for left- and right-moving boundary excitations.) This condition is obeyed by three-dimensional gravity, with Anti de Sitter space being the classical approximation to the vacuum. The desired $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$ symmetry is simply the classical $SO(2,2)$ symmetry of three-dimensional AdS space.

The statement that ℓ/G cannot be continuously varied is not limited to pure gravity. It holds for the same reason in any theory of three-dimensional gravity plus matter that has a sensible AdS vacuum. For example, in the string theory models whose CFT duals are known, ℓ/G is expressed in terms of integer-valued fluxes; this gives a direct explanation of why it cannot be varied continuously.

1.4. Plan Of This Paper

Now we can describe the plan of this paper.

We aim to solve three-dimensional gravity with negative Λ, at some distinguished values of ℓ/G at which it makes sense.

We do not have any rigorous way to determine the right values. However, in section 2, taking at face value the Chern-Simons description of three-dimensional gravity, we will use it to motivate certain values of ℓ/G. The values that emerge – with the help of a small sleight of hand in the choice of the gauge group for the Chern-Simons theory – are interesting. They are the values at which c_L and c_R are integer multiples of 24, and complete holomorphic factorization of the dual CFT is conceivable.

Not wishing to look a gift horse in the mouth, we will assume that these are the right values to consider. Relying on holomorphic factorization, to describe the solution of the theory, we must describe a sequence of holomorphic CFT's with $c = 24k$, $k = 1, 2, 3, \ldots$.

For $k = 1$, it is believed [23] that there are precisely 71 holomorphic CFT's with the relevant central charge $c = 24$. Of these theories, 70 have some form of Kac-Moody or current algebra symmetry extending the conformal symmetry. In the AdS/CFT correspondence, these theories are dual to three-dimensional theories describing gravity plus additional gauge fields (with Chern-Simons interactions). To describe pure gravity, we need a holomorphic CFT with $c = 24$ and no Kac-Moody symmetry.

Such a model was constructed nearly twenty-five years ago by Frenkel, Lepowsky, and Meurman [24], who also conjectured its uniqueness. The motivation for constructing the model was that it admits as a group of symmetries the Fischer-Griess monster group $\mathbb{M}$ –

the largest of the sporadic finite groups. (The link between the monster and conformal field theory was suggested by developments springing from an observation by McKay relating the monster to the j-function, as we explain more fully in section 3.1.) Arguably, the FLM model is the most natural known structure with $\mathbb{M}$ symmetry. A detailed and elegant description is in the book [25]; for short summaries, see [26,27], and for subsequent developments and surveys, see [28-30]. Assuming the (unproved) uniqueness conjecture of FLM, we propose that their model must give the CFT that is dual to three-dimensional gravity at $c = 24$.

For $c = 24k$, $k > 1$, we need an analog of requiring that there is no Kac-Moody symmetry. A plausible analog, expressing the idea that we aim to describe pure gravity, is that there should be no primary fields of low dimension other than the identity. A small calculation shows that at $c = 24k$, the lowest dimension of a primary other than the identity cannot be greater than $k + 1$, and if we assume that this dimension is precisely $k + 1$, then the partition function is uniquely determined. Conformal field theories with this property were first investigated by Höhn in [31,32] and have been called extremal CFT's; see also [33].

It is not known[5] if extremal CFT's exist for $k > 1$. If such a CFT does exist, it is an attractive candidate for the dual of three-dimensional gravity at the appropriate value of the cosmological constant. The primaries of dimension $k + 1$ and above would be interpreted as operators that create black holes. The dimension $k + 1$ agrees well with the minimum mass of a BTZ black hole. This statement may sound like magic, since the value $k + 1$ is determined from holomorphy and modular invariance without mentioning black holes; but the result is not so surprising if one is familiar with previous results on the AdS/CFT correspondence in three dimensions [34].

Section 3 of this paper is devoted to describing the partition function of an extremal CFT and discussing how such a theory could be related to three-dimensional gravity. In both sections 2 and 3, we consider also the case of three-dimensional supergravity. More precisely, we consider only minimal supergravity, corresponding to $N = 1$ superconformal symmetry for the boundary CFT. In this case, holomorphic factorization is conceivable at $c = 12k^*$, $k^* = 1, 2, 3, \ldots$. Here there is a little ambiguity about exactly what we

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[5] Höhn's definition of an extremal CFT allowed holomorphic factorization up to a phase, so that c may be a multiple of 8, not 24. As a result, he discussed several examples of extremal theories that we will not consider here. These examples have k non-integral and less than 2.

should mean by an extremal superconformal field theory (SCFT), but pragmatically, there are good candidates at $k^* = 1, 2$. The $k^* = 1$ theory was constructed by Frenkel, Lepowsky, and Meurman, who also conjectured its uniqueness. Its discrete symmetries were understood only recently in work by Duncan [35]. For $k^* = 2$, the extremal SCFT was constructed by Dixon, Ginsparg, and Harvey [26], by modifying the orbifold projection that had been used [24] in constructing the $k = 1$ extremal CFT. Interestingly, the extremal SCFT's with $k^* = 1, 2$ both admit[6] an action of very large discrete groups related to the Conway group. This is a further indication that unusual discrete groups are relevant to three-dimensional gravity and supergravity. In fact, we find some hints that supergravity may have monster symmetry at $k^* = 4$ and baby monster symmetry at $k^* = 6$.

Regrettably, we do not know how to construct new examples of extremal conformal or superconformal field theories. Section 4 is devoted to a calculation that aims to give modest support to the idea that new extremal theories do exist. We consider an extremal CFT with $k = 2$ and show that its partition function can be uniquely determined on a hyperelliptic Riemann surface of any genus (including, for example, any Riemann surface of genus 2). The fact that a partition function with the right properties exists and is unique for any genus is hopefully a hint that an extremal $k = 2$ CFT does exist.

We make at each stage the most optimistic possible assumption. Decisive arguments in favor of the proposals made here are still lacking. The literature on three-dimensional gravity is filled with claims (including some by the present author [6]) that in hindsight seem less than fully satisfactory. Hopefully, future work will clarify things.

2. Gauge Theory And The Value Of c

The goal of the present section is to determine what values of the cosmological constant, or equivalently of the central charge c of the boundary CFT, are suggested by the relation between three-dimensional gravity and Chern-Simons gauge theory.

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[6] The fact that they have essentially the same symmetry group was pointed out by J. Duncan, who also suggested the identification of the $k^* = 2$ theory.

Before proceeding to any calculation, we will dispose of a few preliminary points. The first is that [36], as long as the three-dimensional spacetime is oriented, as we will assume in this paper, three-dimensional gravity can be generalized to include an additional interaction, the Chern-Simons functional of the spin connection ω:

$$\Delta_0 I = \frac{k'}{4\pi} \int_W \operatorname{tr} \left(\omega \wedge d\omega + \frac{2}{3} \omega \wedge \omega \wedge \omega \right). \tag{2.1}$$

Here we think of ω as an $SO(2,1)$ gauge field (or an $SO(3)$ gauge field in the case of Euclidean signature). Also, tr is the trace in the three-dimensional representation of $SO(2,1)$, and k' is quantized for topological reasons (the precise normalization depends on some assumptions and is discussed in sections 2.1 and 2.4). Equivalently, instead of ω, we could use the $SO(2,2)$ gauge field A introduced in eqn. (1.2), and add to the action a term of the form

$$\Delta I = \frac{k'}{4\pi} \int_M \operatorname{tr} \left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right), \tag{2.2}$$

where now tr is the trace in the four-dimensional representation of $SO(2,2)$. Provided that the conventional Einstein action (1.1) is also present, it does not matter which form of the gravitational Chern-Simons interaction we use, since they lead to equivalent theories. If one adds[7] to ω a multiple of e, the Einstein action (1.1) transforms in a way that cancels the e-dependent part of (2.2), reducing it to (2.1) (while modifying the parameters in the Einstein action).

For our purposes, the $SO(2,2)$-invariant form (2.2) is more useful. This way of writing the Chern-Simons functional places it precisely in parallel with the Einstein-Hilbert action, which as in (1.3) can similarly be expressed as a Chern-Simons interaction, defined with a different quadratic form. We will use the fact that all interactions can be written as Chern-Simons interactions to constrain the proper quantization of all dimensionless parameters, including ℓ/G.

We start with the fact that the group $SO(2,2)$ is locally equivalent to $SO(2,1) \times SO(2,1)$. Moreover, we will in performing the computation assume to start with that $SO(2,1) \times SO(2,1)$ is the right global form of the gauge group. (Then we consider covering

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[7] This operation is invariant under diffeomorphisms and local Lorentz transformations because ω and e transform in the same way under local Lorentz transformations – a statement that holds precisely in three spacetime dimensions.

groups[8] that are only locally isomorphic to $SO(2,1) \times SO(2,1)$.) Thus, by taking suitable linear combinations of ω and e, we will obtain a pair of $SO(2,1)$ gauge fields A_L and A_R. These have Chern-Simons interactions

$$I = \frac{k_L}{4\pi} \int \mathrm{tr} \left(A_L \wedge dA_L + \frac{2}{3} A_L \wedge A_L \wedge A_L \right) - \frac{k_R}{4\pi} \int \mathrm{tr} \left(A_R \wedge dA_R + \frac{2}{3} A_R \wedge A_R \wedge A_R \right).$$
(2.3)

Both k_L and k_R are integers for topological reasons, and this will lead to a quantization of the ratio G/ℓ that appears in the Einstein-Hilbert action, as well as the gravitational Chern-Simons coupling (2.2). The minus sign multiplying the last term in (2.3) is convenient; it will ensure that k_L and k_R are both positive.

2.1. Quantization Of Parameters

For completeness, we begin by reviewing the quantization of the Chern-Simons coupling in gauge theory. The basic case to consider is that the gauge group is $U(1)$. The gauge field A is a connection on a complex line bundle $\mathcal{L}$ over a three-manifold W, which for simplicity we will assume to have no boundary. Naively speaking, the Chern-Simons action is

$$I = \frac{k}{2\pi} \int_W A \wedge dA$$
(2.4)

with some coefficient k. If the line bundle $\mathcal{L}$ is trivial, then we can interpret A as a one-form, and I is well-defined as a real-valued functional. If this were the general situation, there would be no need to quantize k.

However, in general $\mathcal{L}$ is non-trivial, A has Dirac string singularities, and the formula (2.4) is not really well-defined as written. To do better, we pick a four-manifold M of boundary W and such that $\mathcal{L}$ extends over M. Such an M always exists. Then we pick an extension of $\mathcal{L}$ and A over M, and replace the definition (2.4) with

$$I_M = \frac{k}{2\pi} \int_M F \wedge F,$$
(2.5)

where $F = dA$ is the curvature. Now there is no Dirac string singularity, and the definition of I_M makes sense. But I_M does depend on M (and on the chosen extension of $\mathcal{L}$, though we do not indicate this in the notation). To quantify the dependence on M, we consider

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[8] For an early treatment of coverings in the context of Chern-Simons theory with compact gauge group, see [37]

two different four-manifolds M and M' with boundary W and chosen extensions of $\mathcal{L}$. We can build a four-manifold X with no boundary by gluing together M and M' along W, with opposite orientation for M' so that they fit smoothly along their common boundary. Then we get

$$I_M - I_{M'} = \frac{k}{2\pi} \int_X F \wedge F. \tag{2.6}$$

Now, on the closed four-manifold X, the quantity $\int_X F \wedge F / (2\pi)^2$ represents $\int_X c_1(\mathcal{L})^2$ (here c_1 is the first Chern class) and so is an integer. In quantum mechanics, the action function I should be defined modulo 2π (so that $\exp(iI)$, which appears in the path integral, is single-valued). Requiring $I_M - I_{M'}$ to be an integer multiple of 2π, we learn that k must be an integer. This is the quantization of the Chern-Simons coupling for $U(1)$ gauge theory.[9]

Now let us move on to the case of gauge group $SO(2,1)$. The group $SO(2,1)$ is contractible onto its maximal compact subgroup $SO(2)$, which is isomorphic to $U(1)$. So quantization of the Chern-Simons coupling for an $SO(2,1)$ gauge field can be deduced immediately from the result for $U(1)$. Let A be an $SO(2,1)$ gauge field and define the Chern-Simons coupling

$$I = \frac{k}{4\pi} \int_W \mathrm{tr}\left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A\right), \tag{2.7}$$

where tr is the trace in the three-dimensional representation of $SO(2,1)$. Then, in order for I to be part of the action of a quantum theory, k must be an integer. The reason that the factor $1/2\pi$ in (2.5) has been replaced by $1/4\pi$ in (2.7) is simply that, when we identify $U(1)$ with $SO(2)$ and then embed it in $SO(2,1)$, the trace gives a factor of 2.

Coverings

So we have obtained the appropriate quantization of the Chern-Simons coupling for gauge group $SO(2,1)$. However, this is not quite the whole story, because $SO(2,1)$ is not simply-connected. As it is contractible to $SO(2) \cong U(1)$, it has the same fundamental group as $U(1)$, namely $\mathbb{Z}$. Hence it is possible, for every positive integer n, to take an

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[9] There is a refinement if the three-manifold W is endowed with a spin structure. In this case, k can be a half-integer, as explained in [38]. This refinement is physically realized in the quantum Hall effect with filling fraction 1. That effect can be described by an electromagnetic Chern-Simons coupling with $k = 1/2$; the half-integral value is consistent because the microscopic theory has fermions and so requires a spin structure.

n-fold cover of $SO(2,1)$. The most familiar of these is the two-fold cover, $SL(2,\mathbb{R})$. In addition, $SO(2,1)$ has a simply-connected universal cover.

We want to work out the quantization of the Chern-Simons interaction if $SO(2,1)$ is replaced by one of these covering groups. Again, it is convenient to start with $U(1)$. To say that the gauge group of an abelian gauge theory is $U(1)$ rather than $\mathbb{R}$ means precisely that the possible electric charges form a lattice, generated by a fundamental charge that we call "charge 1." Dually, the magnetic fluxes are quantized, with $\int_C F/2\pi \in \mathbb{Z}$ for any two-cycle C. Replacing $U(1)$ by an n-fold cover means that the electric charges take values in $n^{-1}\mathbb{Z}$, and dually, the magnetic fluxes are divisible by n, $\int_C F/2\pi \in n\mathbb{Z}$. As a result, for a four-manifold X, we have $\int_C F \wedge F/(2\pi)^2 \in n^2\mathbb{Z}$. So, in requiring that the Chern-Simons function (2.4) should be well-defined modulo 2π, we require that $k \in n^{-2}\mathbb{Z}$. This is the appropriate result for an n-fold cover. In the case of the universal cover, with $U(1)$ replaced by $\mathbb{R}$, the magnetic fluxes vanish and there is no topological restriction on k.

These statements carry over immediately to covers of $SO(2,1)$, whose covers are all contractible to corresponding covers of $U(1)$. So for an n-fold cover of $SO(2,1)$, we require

$$k \in n^{-2}\mathbb{Z}, \tag{2.8}$$

and for the universal cover of $SO(2,1)$, k is arbitrary and can vary continuously.

Diagonal Covers

There is more to say, because three-dimensional gravity is actually related to $SO(2,1) \times SO(2,1)$ gauge theory, not just to gauge theory with a single $SO(2,1)$. So we should consider covers of $SO(2,1) \times SO(2,1)$ that do not necessarily come from separate covers of the two factors.

As $SO(2,1) \times SO(2,1)$ is contractible to $SO(2) \times SO(2) = U(1) \times U(1)$, we can proceed by first analyzing the $U(1) \times U(1)$ case. We consider a $U(1) \times U(1)$ gauge theory with gauge fields A, B and a Chern-Simons action

$$I = \frac{k_L}{2\pi} \int_W A \wedge dA - \frac{k_R}{2\pi} \int_W B \wedge dB. \tag{2.9}$$

To define I in the topologically non-trivial case, we pick a four-manifold M over which everything extends and define

$$I_M = \int_M \left(\frac{k_L}{2\pi} F_A \wedge F_A - \frac{k_R}{2\pi} F_B \wedge F_B \right), \tag{2.10}$$

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where F_A and F_B are the two curvatures. This is well-defined mod 2π if

$$I_X = \int_X \left(\frac{k_L}{2\pi} F_A \wedge F_A - \frac{k_R}{2\pi} F_B \wedge F_B \right) \tag{2.11}$$

is a multiple of 2π for any $U(1) \times U(1)$ gauge field over a closed four-manifold X.

In $U(1) \times U(1)$ gauge theory, the charge lattice is generated by charges $(1,0)$ and $(0,1)$, and the cohomology classes $x = F_A/2\pi$ and $y = F_B/2\pi$ are integral. For a cover of $U(1) \times U(1)$, we want to extend the charge lattice. To keep things simple, we will consider only the case that will actually be important in our application: a diagonal cover, in which one adds the charge vector $(1/n, 1/n)$ for some integer n. In this case, x and y are still integral, and their difference is divisible by n: $x = y + nz$ where n is an integral class. We have

$$I_X = 2\pi(k_L - k_R) \int_X y^2 + 2\pi k_L \int_X (n^2 z^2 + 2nyz). \tag{2.12}$$

The condition that this is a multiple of 2π for any X and any integral classes y, z is that

$$k_L \in \begin{cases} n^{-1}\mathbb{Z} & \text{if } n \text{ is odd} \\ (2n)^{-1}\mathbb{Z} & \text{if } n \text{ is even} \end{cases} \tag{2.13}$$
$$k_L - k_R \in \mathbb{Z}.$$

These are also the restrictions on k_L and k_R if the gauge group is a diagonal cover of $SO(2,1) \times SO(2,1)$ with action (2.3). For example, the group $SO(2,2)$ is a double cover of $SO(2,1) \times SO(2,1)$, and this cover corresponds to the case $n = 2$ of the above discussion. So for $SO(2,2)$ gauge theory, the appropriate restriction on the Chern-Simons levels is

$$k_L \in \frac{1}{4}\mathbb{Z} \tag{2.14}$$
$$k_L - k_R \in \mathbb{Z}.$$

For a general n-fold diagonal cover, one should use (2.13).

It is also possible to form a "universal diagonal cover," corresponding roughly to the limit $n \to \infty$ in the above formulas. With this gauge group, there is no restriction on k_L, but $k_L - k_R$ is an integer. In terms of three-dimensional gravity, to which we return next, this corresponds to letting ℓ/G be a freely variable parameter, while the gravitational Chern-Simons coupling k' defined in eqn. (2.2) is an integer. As explained in section 1.3, although this is the state of affairs classically in three-dimensional gravity, it cannot be the correct answer quantum mechanically.

2.2. Comparison To Three-Dimensional Gravity

So far, we have understood the appropriate gauge theory normalizations for the Chern-Simons action

$$
\begin{aligned}
I =& k_L I_L + k_R I_R \\
=& \frac{k_L}{4\pi} \int \mathrm{tr} \left(A_L \wedge dA_L + \frac{2}{3} A_L \wedge A_L \wedge A_L \right) - \frac{k_R}{4\pi} \int \mathrm{tr} \left(A_R \wedge dA_R + \frac{2}{3} A_R \wedge A_R \wedge A_R \right).
\end{aligned}
\tag{2.15}
$$

Our next step will be to express A_L and A_R, which are gauge fields of $SO(2,1) \times SO(2,1)$ (or a covering group) in terms of gravitational variables, and thereby determine the constraints on the gravitational couplings. We have

$$
I = \frac{k_L + k_R}{2} (I_L - I_R) + (k_L - k_R) \frac{(I_L + I_R)}{2}.
\tag{2.16}
$$

The term in (2.16) proportional to $I_L - I_R$ will gave the Einstein-Hilbert action (1.1), while the term proportional to $(I_L + I_R)/2$ is equivalent to the gravitational Chern-Simons coupling (2.2) with coefficient $k' = k_L - k_R$.

The spin connection $\omega^{ab} = \sum_i dx^i \omega_i^{ab}$ is a one-form with values in antisymmetric 3×3 matrices. The vierbein is conventionally a one-form valued in Lorentz vectors, $e^a = \sum_i dx^i e_i^a$. The metric is expressed in terms of e in the usual way, $g_{ij} dx^i \otimes dx^j = \sum_{ab} \eta_{ab} e^a \otimes e^b$, where $\eta = \mathrm{diag}(-1, 1, 1)$ is the Lorentz metric; and the Riemannian volume form is $d^3x \sqrt{g} = \frac{1}{6} \epsilon_{abc} e^a \wedge e^b \wedge e^c$, where ϵ_{abc} is the antisymmetric tensor with, say, $\epsilon_{012} = 1$. All this has an obvious analog in any dimension. However, in three dimensions, a Lorentz vector is equivalent to an antisymmetric tensor; this is the fact that makes it possible to relate gravity and gauge theory. It is convenient to introduce ${}^*e_{ab} = \epsilon_{abc} e^c$, which is a one-form valued in antisymmetric matrices, just like ω. We raise and lower local Lorentz indices with the Lorentz metric η, so $\frac{1}{2} \epsilon^{abc} \epsilon_{bcd} = -\delta_d^a$, and $e^c = -\frac{1}{2} \epsilon^{abc} {}^*e_{bc}$.

We can combine ω and ${}^*e$ and set $A_L = \omega - {}^*e/\ell$, $A_R = \omega + {}^*e/\ell$. A small computation gives

$$
I_L - I_R = -\frac{1}{\pi \ell} \int \mathrm{tr}\, {}^*e (d\omega + \omega \wedge \omega) - \frac{1}{3\pi \ell^3} \int \mathrm{tr}\, ({}^*e \wedge {}^*e \wedge {}^*e).
\tag{2.17}
$$

In terms of the matrix-valued curvature two-form $R^{ab} = (d\omega + \omega \wedge \omega)^{ab} = \frac{1}{2} \sum_{ij} dx^i \wedge dx_j R_{ij}^{ab}$, where R_{ij}^{ab} is the Riemann tensor, and the metric tensor g, this is equivalent to

$$
I_L - I_R = \frac{1}{\pi \ell} \int d^3x \sqrt{g} \left(R + \frac{2}{\ell^2} \right).
\tag{2.18}
$$

Remembering the factor of $(k_L + k_R)/2$ in (2.16), we see that this agrees with the Einstein-Hilbert action (1.1) precisely if

$$k_L + k_R = \frac{\ell}{8G}. \tag{2.19}$$

The central charge of the boundary conformal field theory was originally computed by Brown and Henneaux [21] for the case that the gravitational Chern-Simons coupling $k' = k_L - k_R$ vanishes. In this case, we set $k = k_L = k_R = \ell/16G$. The formula for the central charge is $c = 3\ell/2G$, and this leads to $c = 24k$. For the case $k' = 0$, the boundary CFT is left-right symmetric, with $c_L = c_R$, so in fact $c_L = c_R = 24k$.

In general, the boundary CFT has left- and right-moving Virasoro algebras that can be interpreted (for a suitable orientation of the boundary) as boundary excitations associated with A_L and A_R respectively. So the central charges c_L and c_R are functions only of k_L and k_R, respectively. Hence the generalization of the result obtained in the last paragraph is

$$(c_L, c_R) = (24k_L, 24k_R). \tag{2.20}$$

2.3. Holomorphic Factorization

In conformal field theory in two dimensions, the ground state energy is $-c/24$. More generally, if there are separate left and right central charges c_L and c_R, the ground state energies for left- and right-movers are $(-c_L/24, -c_R/24)$. Modular invariance says that the difference between the left- and right-moving ground state energies must be an integer. In the above calculation, $(c_L - c_R)/24 = k_L - k_R$. According to (2.20), this is an integer provided that the gravitational Chern-Simons coupling $k' = k_L - k_R$ is integral.

Holomorphic factorization requires that the left- and right-moving ground state energies should be separately integral, so that there is modular invariance separately for left-moving and right-moving modes of the CFT. Thus, for holomorphic factorization, c_L and c_R must both be integer multiples of 24. This is the only constraint, since holomorphic CFT's with $c = 24$ do exist (and have been classified [23] modulo a conjecture mentioned in section 1.4).

According to (2.20), the condition for c_L and c_R to be multiples of 24 is precisely that k_L and k_R must be integers. As in our discussion of eqn. (2.7), this is the right condition if the gauge group is precisely $SO(2,1) \times SO(2,1)$ rather than a covering group.

It is possible to give an intuitive explanation of why this is the right gauge group if the boundary CFT is supposed to be holomorphically factorized. The Virasoro algebra is, of

course, infinite-dimensional, but it has a finite-dimensional subalgebra, generated by $L_{\pm 1}$ and L_0, that is a symmetry of the vacuum. It is customary to refer to the corresponding symmetry group of the vacuum as $SL(2,\mathbb{R})$, but in fact, in a holomorphic CFT, in which all energies are integers, the group that acts faithfully is really $SO(2,1)$. So a holomorphically factorized CFT has symmetry group $SO(2,1) \times SO(2,1)$, and it is natural that this is the right gauge group in a gauge theory description of (aspects of) the dual gravitational theory.

Now let us consider some other possible gauge groups. One possibility is to take a double cover of each factor of $SO(2,1) \times SO(2,1)$, taking the gauge group to be $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$. The appropriate restriction on the gauge theory couplings was determined in (2.8) (where we should set $n = 2$) and is that k_L and k_R take values in $\frac{1}{4}\mathbb{Z}$. Hence the central charges c_L and c_R are multiples of 6. In particular, $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$ gauge theory would allow us to consider values of the couplings that contradict modular invariance of the boundary CFT.

There is perhaps a more intuitive argument suggesting that $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$ is not the right group to consider. If the gauge group is $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$, then upon taking the two-dimensional representation of one of the $SL(2,\mathbb{R})$'s, we get a two-dimensional real vector bundle over W which generalizes what in classical geometry is the spin bundle. Thus, this would be a theory in which, in the classical limit, W is a spin manifold, endowed with a distinguished spin structure (or even two of them). That is appropriate in a theory with fermions, but not, presumably, in a theory of pure gravity.

We similarly lose modular invariance and have difficult to interpret geometric structures if we consider other non-diagonal covers of $SO(2,1) \times SO(2,1)$. So let us discuss the diagonal covers of $SO(2,1) \times SO(2,1)$ that were considered at the end of section 2.1. We know that the universal diagonal cover is not right, since then ℓ/G could vary continuously. This leaves the possibility of an n-fold diagonal cover for some n. The best-motivated example is perhaps the two-fold cover $SO(2,2)$, which is the symmetry of Anti de Sitter spacetime (as opposed to a cover of that spacetime). In this case, according to (2.14), k_L and k_R can take values in $\mathbb{Z}/4$, as long as $k' = k_L - k_R$ is integral. For the boundary CFT, this means that c_L and c_R can be multiples of 6 (with their difference a multiple of 24). For example, let us consider a hypothetical CFT of $(c_L, c_R) = (6, 6)$. The ground state energies are $(-1/4, -1/4)$. As these values are not integers, the symmetry group of the ground state is not $SO(2,1) \times SO(2,1)$, but a four-fold diagonal cover, which is a double

cover of the gauge group $SO(2,2)$ that was assumed. Such a CFT cannot be holomorphically factorized, since the left- and right-moving ground state energies are not integers; it cannot even be holomorphically factorized up to a phase.[10] The structure is considerably more complicated than in the holomorphically factorized case. Similar remarks apply to other diagonal covers.

Pragmatically, the most important argument against trying to describe three-dimensional gravity via a cover of $SO(2,1) \times SO(2,1)$ may be simply the fact that no good candidates are known. For example, no especially interesting bosonic CFT (as opposed to a superconformal field theory) seems to be known at $c = 6, 12$, or 18, which are values that we would expect if we use the gauge group $SO(2,2)$. By contrast, at $c = 24$, which is natural for $SO(2,1) \times SO(2,1)$, the Frenkel-Lepowsky-Meurman monster theory is a distinguished candidate, as we mentioned in section 1.4 and will explain in more detail in section 3.

The simplest possible hypothesis is that the right gauge group to consider in studying three-dimensional pure gravity is $SO(2,1) \times SO(2,1)$, with integral k_L and k_R and holomorphic factorization of the boundary CFT. It would be highly unnatural to overlook the fact that $SO(2,1) \times SO(2,1)$ gauge theory leads to precisely the values of the central charge at which the drastic simplification of the boundary CFT known as holomorphic factorization is conceivable. Moreover, the fact that the classical action (2.15) of the gauge theory is a sum of decoupled actions for A_L and A_R – related respectively to left- and right-moving modes of the boundary CFT – is a hint of holomorphic factorization of the boundary theory.

At any rate, regardless of whether they give the whole story, it does seem well-motivated to look for holomorphically factorized CFT's with central charges multiples of 24 that are dual to three-dimensional pure gravity at special values of the cosmological constant. That will be our main focus in the rest of this paper.

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[10] The left-moving partition function would have to be $\Phi/\Delta^{1/4}$, where $\Delta = \eta(q)^{24}$ is the discriminant, η being the Dedekind eta function, and Φ is a modular form of weight 3. The power of Δ was determined to get the right ground state energy; modular invariance implies that the modular weight of Φ must equal that of $\Delta^{1/4}$. As there is no modular form of weight 3, such a theory does not exist. Even at $(c_L, c_R) = (12, 12)$, it is not possible to have holomorphic factorization up to a phase. To get modular invariance and the right ground state energy, the left-moving partition function would have to be $E_6/\Delta^{1/2}$, where E_6 is the Eisenstein series of weight 6. However, the coefficients in the q-expansion of this function are not positive.

2.4. Interpretation

A few further words of interpretation seem called for.

We do *not* claim that three-dimensional gravity is equivalent, nonperturbatively, to Chern-Simons gauge theory. Some objections to this idea were described in section 1.1. We know that Chern-Simons gauge theory is useful for perturbation theory, as was explained in that section, and we hope that it is useful for understanding some nonperturbative questions. We used the gauge theory approach to get some hints about the right values of the cosmological constant (or equivalently of the central charge) simply because it was the only tool available. We certainly do not claim to have a solid argument that the values of (c_L, c_R) suggested by $SO(2,1) \times SO(2,1)$ gauge theory are the only relevant ones.

Implicit in the gauge theory approach to quantization of the dimensionless parameter ℓ/G is an assumption that at least some of the non-geometrical states that can be described in gauge theory make sense. Indeed, when understood in classical geometry with the metric assumed to be smooth and nondegenerate, the Einstein-Hilbert action (1.1) is well-defined as a real-valued function. The topological problems that cause it to be multivalued when interpreted in gauge theory as a Chern-Simons function, and lead to quantization of ℓ/G, depend on allowing certain configurations that are natural in gauge theory, but singular in geometry because the vierbein is not invertible.

Furthermore, holomorphic factorization most likely is possible only if the path integral includes a sum over some non-geometrical configurations. In quantum gravity, we at least expect to sum over all topologies of a three-manifold W, perhaps with some fixed asymptotic behavior. The choice of topology should be expected to affect both the left- and right-movers of the boundary CFT. To achieve holomorphic factorization, there must presumably be some sort of separate topological sum for left-movers and right-movers. As this does not occur in classical geometry, it must depend on some sort of non-geometric contributions to the path integral – though these contributions may be exponentially small in many circumstances.

The Gravitational Chern-Simons Action Reconsidered

Our discussion of the quantization of the gravitational Chern-Simons coupling (2.1)

$$\Delta_0 I = \frac{k'}{4\pi} \int_W \text{tr} \left(\omega \wedge d\omega + \frac{2}{3} \omega \wedge \omega \wedge \omega \right) \tag{2.21}$$

has been based entirely on gauge theory. We now want to describe the similar but slightly different answer that would come from classical differential geometry. As always, the

differences reflect the fact that the gauge theory analysis allows some non-geometrical configurations.

We first briefly restate the gauge theory analysis. ω is a connection on an $SO(2,1)$ or (in Euclidean signature) $SO(3)$ bundle over a three-manifold W. As usual, to define $\Delta_0 I$ more precisely, we pick an oriented four-manifold M of boundary W with an extension of ω over W. Then we define

$$I_M = \frac{k'}{4\pi} \int_M \mathrm{tr}\, F \wedge F. \tag{2.22}$$

If M is replaced by some other four-manifold M', and $X = M - M'$ is a four-manifold without boundary obtained by gluing together M and M', then

$$I_M - I_{M'} = \frac{k'}{4\pi} \int_X \mathrm{tr}\, F \wedge F = 2\pi k' \int_X p_1(F), \tag{2.23}$$

where $p_1(F) = (1/8\pi^2)\mathrm{tr}\, F \wedge F$ is the first Pontryagin form. In general, $\int_X p_1(F)$ can be any integer, so the condition that the indeterminacy in I_M is an integer multiple of 2π means simply that k' is an integer.

This is the right answer if the right thing to do is to simply think of ω as an $SO(3)$ or $SO(2,1)$ gauge field, ignoring its classical relation to gravity. However, in classical gravity, one can get a better answer. In classical gravity, ω is a connection on the tangent bundle TW of W. Now replace TW by $TW \oplus \epsilon$, where ϵ is a trivial real line bundle. Then ω can be regarded as a connection on $TW \oplus \epsilon$ in an obvious way, and $TW \oplus \epsilon$ extends over M as the tangent bundle of M. With this choice, (2.22) becomes

$$I_M = \frac{k'}{4\pi} \int_M \mathrm{tr}\, R \wedge R, \tag{2.24}$$

where R is the curvature form of M, and (2.23) becomes

$$I_M - I_{M'} = 2\pi k' \int_X p_1(R). \tag{2.25}$$

The effect of this is that instead of the first Pontryagin number of a general bundle over X, as in (2.23), we have here the first Pontryagin number $p_1(TX)$ of the tangent bundle of X. This number is divisible by 3, because of the signature theorem, which says that for a four-manifold X, $p_1(TX)/3$ is an integer, the signature of X. Hence, in the gravitational interpretation, the condition on k' is

$$k' \in \frac{1}{3}\mathbb{Z}. \tag{2.26}$$

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There is also a variant of this. If W is a spin manifold and we are willing to define the gravitational Chern-Simons coupling (2.21) in a way that depends on the spin structure of W (this does not seem natural in ordinary gravity, but it may be natural in supergravity, which we come to in section 3.2), the condition on k' can be further relaxed. In this case, we can select M so that the chosen spin structure on W extends over M. If M' is another choice with the same property, then $X = M - M'$ is a spin manifold. But (as follows from the Atiyah-Singer index theorem for the Dirac equation) the signature of a four-dimensional spin manifold is divisible by 16. So in this situation $p_1(TX)$ is a multiple of 48, and the result for k' under these assumptions is

$$k' \in \frac{1}{48}\mathbb{Z}. \tag{2.27}$$

The Physical Hilbert Space

Now we shall discuss the meaning for gravity of the physical Hilbert space of Chern-Simons gauge theory.

In three-dimensional Chern-Simons gauge theory, one can fix a Riemann surface C and construct a Hilbert space $\mathcal{H}_C$ of physical states obtained by quantizing the given theory on C. This Hilbert space depends on the Chern-Simons couplings, so when we want to be more precise, we will call it $\mathcal{H}_C(k_L, k_R)$. In [6], it was proposed that the physical Hilbert space of three-dimensional gravity on a Riemann surface C should be obtained in essentially this way, using $SO(2,1) \times SO(2,1)$ gauge theory (or something similar, depending on whether the cosmological constant is positive, negative, or zero). Some serious objections to the claim that pure gravity and Chern-Simons gauge theory are equivalent in three dimensions were noted in section 1.1. Yet remarkable progress has made made [8,11] in understanding the quantization of $SO(2,1)$ gauge theory, and even more remarkably, in relating this quantization to Liouville theory. It therefore seems likely that the Hilbert space obtained by quantizing Chern-Simons gauge theory of $SO(2,1) \times SO(2,1)$ means something for three-dimensional gravity even if the proposal in [6] was premature. The rest of this subsection will be devoted to an attempt (not used in the rest of the paper) to reconcile the different points of view. See [39] for some useful background.

We may start by asking what we mean by the physical Hilbert space $\mathcal{H}_C$ obtained in quantum gravity by quantizing on a closed manifold C. What type of question is this space supposed to answer? Quantization on, for example, an asymptotically flat spacetime leads to a Hilbert space that can be interpreted in a relatively straightforward way, but the physical meaning of a Hilbert space obtained by quantizing on a compact spatial manifold (if such a Hilbert space can be defined at all) is not clear.
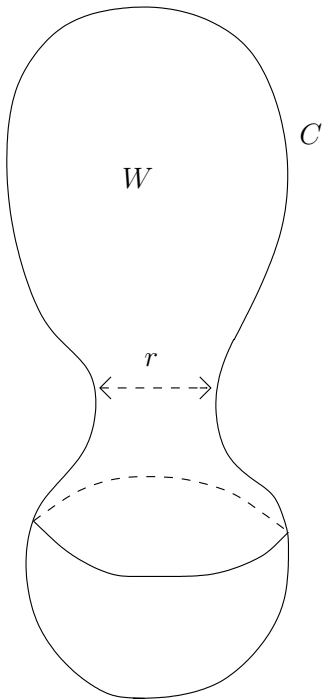
Fig. 1:

The Hartle-Hawking wavefunction Ψ is computed by integrating over three-manifolds W with a given boundary C. The dotted line labeled r denotes a path in W connecting two points in C. It is possible to vary W so that the length of this path goes to zero with no change in the geometry of C. In this limit, W becomes singular even though C is smooth. This gives a "source term" in the Wheeler-de Witt equation, because of which the Hartle-Hawking wavefunction does not obey this equation.

One line of thought that is relatively close to working is to consider the Hartle-Hawking wavefunction [40] and claim that it is a vector in $\mathcal{H}_C$. (The obvious idea that quantum mechanical probabilities would be calculated in terms of inner products of vectors in a Hilbert space $\mathcal{H}_C$ of physical states is afflicted with similar but more serious problems.) The Hartle-Hawking wavefunction is a functional of metrics on C. For every metric h on C, we define $\Psi(h)$ as the result of performing a path integral over three-manifolds W whose boundary is C and whose metric g coincides with h on the boundary. Formally, one can try to argue that $\Psi(h)$ obeys the Wheeler-de Witt equation and thus is a vector in a Hilbert space $\mathcal{H}_C$ of solutions of this equation. Moreover, one can formally match the Wheeler-de Witt equations of gravity with the conditions for a physical state in Chern-Simons gauge theory. Though many steps in these arguments work nicely, one runs into

trouble because a Riemann surface can be immersed, rather than embedded, in a three-manifold, and hence it is possible for W to degenerate without C degenerating (fig. 1). As a result, the Hartle-Hawking wavefunction does not obey the Wheeler-de Witt equation and is not a vector in $\mathcal{H}_C$.

In the case of negative cosmological constant, the boundary CFT gives a sort of cure for the problem with the Hartle-Hawking wavefunction. Instead of thinking of C as an ordinary boundary of W, we think of it as a conformal boundary at infinity. The partition function $\widehat{\Psi}(h)$ of the boundary CFT is defined by performing the path integral over all choices of W with C as conformal boundary. This is well-behaved, because, with C at conformal infinity, it is definitely embedded rather than immersed. Moreover, $\widehat{\Psi}(h)$ is a sort of limiting value of the Hartle-Hawking wavefunction. Indeed, let ϕ be a positive function on C. Then $\widehat{\Psi}(h)$ is essentially[11] the limiting value of $\Psi(e^\phi h)$ as $\phi \to \infty$.

This suggests that we should be able to think of $\widehat{\Psi}(h)$ as a vector in the Hilbert space $\mathcal{H}_C$ associated with three-dimensional gravity and a two-manifold C. This viewpoint can be explained more fully. The phase space of $SO(2,1) \times SO(2,1)$ Chern-Simons theory on C is the space of $SO(2,1) \times SO(2,1)$ flat connections on C. The space of $SO(2,1)$ flat connections on C has several topological components (labeled by the first Chern class, which comes in because $SO(2,1)$ is contractible to $SO(2) \cong U(1)$). One of these components, the only one that can be simply interpreted in terms of classical gravity with negative cosmological constant,[12] is isomorphic to Teichmuller space $\mathcal{T}$. Thus, this component of the classical phase space $\mathcal{M}$ is a product of two copies of $\mathcal{T}$, parametrized by a pair of points $\tau, \tau' \in \mathcal{T}$. One can quantize $\mathcal{M}$ (or at least this component of it) naively by using the standard holomorphic structure of $\mathcal{T}$. If we do this, the wavefunction of a physical state is a "function" of τ and τ' that is holomorphic in τ and antiholomorphic in τ'. (Antiholomorphy in one variable reflects the relative minus sign in the Chern-Simons action (2.15); we assume that k_L and k_R are positive.) Actually, a physical state wavefunction $\Psi(\tau, \overline{\tau}')$ is not quite a function of τ and $\overline{\tau}'$ in the usual sense, but is a form, of weights determined by k_L and k_R. So it takes values in a Hilbert space $\mathcal{H}_C(k_L, k_R)$ that depends on the Chern-Simons couplings. Such a wavefunction $\Psi(\tau, \overline{\tau}')$ is determined by its restriction

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[11] One needs some renormalization in this limit; the necessary renormalization reflects the conformal anomaly. Because of this anomaly, $\widehat{\Psi}(h)$ is not quite a function only of the conformal structure of C but a function of the metric that transforms with a certain weight under conformal rescalings. As a result, the precise choice of ϕ matters, but only in a rather simple way.

[12] In the approach to quantization developed in [8,11], this is the only component considered.

to the diagonal subspace $\tau = \tau'$. Moreover, if we want to make a relation to gravity, it is natural to require that Ψ should be invariant under the diagonal action of the mapping class group on τ and τ'; this condition is compatible with restricting to $\tau = \tau'$.

Similarly, the partition function of a CFT on the Riemann surface C is a not necessarily holomorphic "function" (actually a form of appropriate weights) $\Psi(\tau, \overline{\tau})$. Being real analytic, Ψ can be analytically continued to a function $\Psi(\tau, \overline{\tau}')$ with τ' at least slightly away from τ. It does not seem to be a standard fact[13] that Ψ analytically continues to a holomorphic function on $\mathcal{T} \times \mathcal{T}$ (with invariance only under one diagonal copy of the mapping class group). However, this is true in genus 1, since the partition function can be defined as $\mathrm{Tr}\, q^{L_0} \overline{q}'^{\overline{L}_0}$, where we can take q and q' to be independent complex variables of modulus less than 1. It seems very plausible that the statement is actually true for all values of the genus, since one can move on Teichmuller space by "cutting" on a circle and inserting $q^{L_0} \overline{q}'^{\overline{L}_0}$. If so, the partition function of the CFT can always be interpreted as a vector[14] in the Chern-Simons Hilbert space $\mathcal{H}_C(k_L, k_R)$.

If we are given a theory of three-dimensional gravity, possibly coupled to other fields, the partition function of the dual CFT is a wavefunction $\Psi(\tau, \overline{\tau}')$ which, according to the conjecture just stated, is a vector in $\mathcal{H}_C(k_L, k_R)$. Any gravitational theory of the same central charges leads to another vector in the same space.

From this point of view, it seems that we should not claim, as was done in [6], that $\mathcal{H}_C(k_L, k_R)$ is a space of physical states that are physically meaningful in pure three-dimensional gravity. Rather, a particular bulk gravitational theory, such as pure gravity, gives rise to a particular dual CFT whose partition function gives a definite vector in $\mathcal{H}_C(k_L, k_R)$. Another gravitational theory, perhaps with matter fields, whose dual CFT has the same values of the central charges, will lead to a dual partition function that is another vector in the same space. Thus, $\mathcal{H}_C(k_L, k_R)$ is in a sense a universal target for gravitational theories – with arbitrary matter fields – of given central charges.

We have formulated this for a particular Riemann surface C, but in either the gravitational theory or the dual CFT, C can vary and there is a nice behavior when C degenerates. So it is more natural to think of this as a structure that is defined for all Riemann surfaces. In conformal theory, this perspective is described in [41].

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[13] However, G. Segal has obtained results in this direction.

[14] It may be necessary here to extend $\mathcal{H}_C(k_L, k_R)$ to a space of forms on $\mathcal{T} \times \mathcal{T}$ that are invariant under the action of the mapping class group but not necessarily square-integrable.

2.5. Analog For Supergravity

Here we will discuss the extension of some of these ideas to three-dimensional supergravity.

We consider primarily minimal supergravity, corresponding to the case that the boundary CFT has $\mathcal{N} = 1$ supersymmetry for left-movers or right-movers or perhaps both. Thus, for, say, the left-movers, the Virasoro algebra is replace by an $\mathcal{N} = 1$ super-Virasoro algebra. The symmetry algebra generated by $L_{\pm 1}$ and L_0 is extended (in the Neveu-Schwarz sector) to a superalgebra that also includes the fermionic generators $\mathcal{G}_{\pm 1/2}$. This is the Lie superalgebra of the supergroup $OSp(1|2)$, whose bosonic part is $Sp(2, \mathbb{R})$, or equivalently $SL(2, \mathbb{R})$. In particular, since the operators $\mathcal{G}_{\pm 1/2}$ transform in the two-dimensional representation of $SL(2, \mathbb{R})$, the relevant group is definitely $SL(2, \mathbb{R})$ (or possibly a covering of it), not its quotient $SO(2, 1)$.

For definiteness, consider a two-dimensional CFT with $(0, 1)$ supersymmetry, that is, with $\mathcal{N} = 1$ supersymmetry for right-movers and none for left-movers. Then left-movers have an ordinary Virasoro symmetry and right-movers have an $\mathcal{N} = 1$ super-Virasoro symmetry. Such a theory can be dual to a three-dimensional supergravity theory, which classically can be described by a Chern-Simons gauge theory in which the gauge supergroup is $SO(2, 1) \times OSp(1|2)$, or possibly a cover thereof. For brevity, we will here assume that the gauge group is precisely $SO(2, 1) \times OSp(1|2)$. The action is the obvious analog of (2.15):

$$
\begin{aligned}
I =& k_L I_L + k_R I_R \\
=& \frac{k_L}{4\pi} \int \mathrm{tr} \left(A_L \wedge dA_L + \frac{2}{3} A_L \wedge A_L \wedge A_L \right) - \frac{k_R}{4\pi} \int \mathrm{str} \left(A_R \wedge dA_R + \frac{2}{3} A_R \wedge A_R \wedge A_R \right).
\end{aligned}
$$

(2.28)

Here A_L is an $SO(2, 1)$ gauge field, A_R is an $OSp(1|2)$ gauge field, and str is the supertrace in the adjoint representation of $OSp(1|2)$.

We want to generalize the analysis of section 2.1 to determine the allowed values of k_L and k_R. There is actually almost nothing to do. A_L is simply an $SO(2, 1)$ gauge field, so k_L must be an integer. As for A_R, we can for topological purposes replace the supergroup $OSp(1|2)$ by its bosonic reduction $SL(2, \mathbb{R})$, since the fermionic directions are infinitesimal and carry no topology. So we can borrow the result of (2.8):

$$
\begin{aligned}
k_L &\in \mathbb{Z} \\
k_R &\in \frac{1}{4} \mathbb{Z}.
\end{aligned}
$$

(2.29)

We still have $(c_L, c_R) = (24k_L, 24k_R)$, since the Brown-Henneaux computation of the central charge depends only on the bosonic part of the action. So c_L must be a multiple of 24, as before, but now it seems that c_R should be a multiple of 6.

This is not the result that one might hope for, because holomorphic factorization in $\mathcal{N} = 1$ superconformal field theories requires that c_R should be a multiple of 12, not 6. So half of the seemingly allowed values of c_R are difficult to interpret in the spirit of this paper. Replacing $SO(2,1) \times OSp(1|2)$ by a covering group would only make things worse, as in the bosonic case.

There are many conceivable ways to interpret this result, including the possibility that our assumptions have been too optimistic. However, one additional possibility seems worthy of mention here. Part of the structure of a superconformal field theory is that there are Ramond-sector vertex operators. They introduce a "twist" in the supercurrent, which has a monodromy of -1 around a point at which a Ramond vertex operator is inserted. Let us assume that the superconformal dual of three-dimensional supergravity should be a theory in which Ramond vertex operators make sense. What is the gravitational dual of an insertion of such a vertex operator?

As in fig. 2, points in the conformal boundary of spacetime at which a Ramond vertex operator is inserted must be connected in the bulk by lines – such as the line labeled L in the figure – around which the fermionic fields of $OSp(1|2)$ (that is, the gravitinos) have a monodromy -1. It is plausible to interpret these lines, which we will call Ramond lines, as world-lines of Ramond-sector black holes. A spacetime history containing a black hole trajectory is of course more complicated than a spacetime with a simple line drawn in it. But for our present purposes, which are purely topological, the difference may not be important. There can also be Neveu-Schwarz sector black holes (and black holes in purely bosonic gravity), but we are about to discuss an effect which seems special to Ramond-sector black holes.

So now we have a new problem. We want to define

$$I_R = \frac{1}{4\pi} \int_W \text{str} \left(A_R \wedge dA_R + \frac{2}{3} A_R \wedge A_R \wedge A_R \right) \tag{2.30}$$

in the presence of a Ramond world-line L on W. In the absence of the Ramond line, we know that I_R in $OSp(1|2)$ gauge theory is defined modulo $4 \cdot 2\pi$, which is why as stated in (2.29), k_R can be a multiple of $1/4$. What happens in the presence of the Ramond line?

Rather than solving a new topological problem, we can take the following shortcut. Let W' be a new three-manifold obtained by taking a double cover of W branched over the
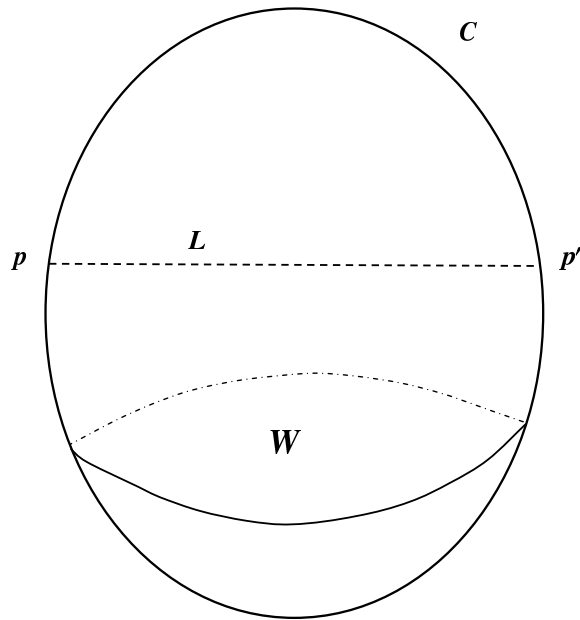
Fig. 2:

    C is a Riemann surface with a pair of insertions of Ramond-sector vertex operators at points labeled p and p'. They are connected by a line L that runs in a three-manifold W whose boundary is C. Supergravity on W contributes to conformal field theory on C. The fermion fields of $OSp(1|2)$ receive a minus sign in monodromy around L.

line L. We let $I_R(W)$ be the action (2.30) and $I_R(W')$ be the corresponding action for the gauge field A_R pulled back to W'. When pulled back to W', the singularity of A_R along the Ramond line disappears, so $I_R(W')$ is defined modulo $4 \cdot 2\pi$. There is no better way to define $I_R(W)$ in the presence of a Ramond line than to say that $I_R(W) = I_R(W')/2$. So $I_R(W)$ is defined modulo $2 \cdot 2\pi$.

    This means that k_R should be a multiple of $1/2$, not $1/4$. So in other words, if including Ramond lines is the right thing to do, we get

$$
\begin{aligned}
k_L &\in \mathbb{Z} \\
k_R &\in \frac{1}{2}\mathbb{Z},
\end{aligned}
\tag{2.31}
$$

and hence c_L and c_R are multiples of 24 and 12, respectively.

    Unfortunately, the above "derivation" is little more than a scenario to try to justify the answer that we hoped for. However, a good pragmatic reason to focus on the case

that c_R is a multiple of 12 is that there are interesting candidate superconformal field theories (SCFT's) in that case, as we discuss in section 3. There are no obvious interesting candidates at $c_R = 6, 18$, etc.

In the supersymmetric case it is convenient to express the Chern-Simons coupling k as $k = k^*/2$, where we will focus on the case that k^* is an integer. In terms of k^*, the central charge is $c = 12k^*$.

Some Generalizations

We conclude this section by briefly mentioning some simple generalizations.

First of all, $(1,1)$ supersymmetry in two dimensions, with $\mathcal{N} = 1$ super-Virasoro symmetry for both left-movers and right-movers, is dual to three-dimensional supergravity theories related to $OSp(1|2) \times OSp(1|2)$ Chern-Simons gauge theory. If one wants the left- or right-movers to have more than $\mathcal{N} = 1$ supersymmetry, one simply replaces $OSp(1|2)$ by an appropriate supergroup with more fermionic generators. For example, $OSp(2|2)$ is related to $\mathcal{N} = 2$ supersymmetry, $PSU(2|2)$ is related to what is usually called $\mathcal{N} = 4$ supersymmetry (with the "small" $\mathcal{N} = 4$ superconformal algebra), and $OSp(r|2)$ with $r > 2$ is related to theories with "large" $\mathcal{N} = k$ superconformal algebras. (The most widely studied case of these algebras is $r = 4$; for example, see [42].)

It is also possible to consider other extended chiral algebras, apart from superconformal algebras. For example, one can start in three dimensions with an $SL(3, \mathbb{R})$ Chern-Simons gauge theory, which plausibly may be related to some sort of three-dimensional W_3 gravity theory in much the same (not fully understood) way that $SL(2, \mathbb{R})$ Chern-Simons theory is related to ordinary three-dimensional gravity. A dual CFT would very likely then have a W_3 chiral algebra. An analogous statement plausibly holds for many extended chiral algebras.

3. Partition Functions

In this section, we will determine what we propose to be the exact spectrum of physical states of three-dimensional gravity or supergravity with negative cosmological constant, in a spacetime asymptotic at infinity to Anti de Sitter space. Equivalently, we will determine the genus one partition function of the dual CFT.

In all cases, we work at the values of ℓ/G at which holomorphic factorization is possible. These values were related to gauge theory in section 2. We assume holomorphic

factorization, and mainly consider only the holomorphic sector of the theory. The full partition function is the product of the function we determine and a similar antiholomorphic function. These partition functions have been studied before [31,32], with different motivation.

3.1. The Bosonic Case

We begin with the bosonic case. What are the physical states of pure gravity in a spacetime asymptotic at infinity to AdS$_3$?

Since there are no gravitational waves in the theory, the only state that is obvious at first sight is the vacuum, corresponding in the classical limit to Anti de Sitter space. In a conformal field theory with central charge $c = 24k$, the ground state energy is $L_0 = -c/24 = -k$. The contribution of the ground state $|\Omega\rangle$ to the partition function $Z(q) = \mathrm{Tr}\, q^{L_0}$ is therefore q^{-k}.

Of course, there is more to the theory than just the ground state. According to Brown and Henneaux [21], a proper treatment of the behavior at infinity leads to the construction of a Virasoro algebra that acts on the physical Hilbert space. The Virasoro generators L_n, $n \geq -1$ annihilate $|\Omega\rangle$, but by acting with $L_{-2}, L_{-3}, \ldots$, we can make new states of the general form $\prod_{n=2}^{\infty} L_{-n}^{s_n} |\Omega\rangle$, with energy $-k + \sum_n n s_n$. (We assume that all but finitely many of the s_n vanish.) If these are the only states to consider, then the partition function would be

$$Z_0(q) = q^{-k} \prod_{n=2}^{\infty} \frac{1}{1 - q^n}. \tag{3.1}$$

This cannot be the complete answer, because the function $Z_0(q)$ is not modular-invariant. There must be additional states such that $Z_0(q)$ is completed to a modular-invariant function.

Additional states are expected, because the theory also has BTZ black holes. The main reason for writing the present paper, after all, is to understand the role of the BTZ black holes in the quantum theory. We will assume that black holes account for the difference between the naive partition function $Z_0(q)$ and the exact one $Z(q)$. To use this assumption to determine $Z(q)$, we need to know something about the black holes.

The classical BTZ black hole is characterized by its mass M and angular momentum J. In terms of the Virasoro generators,

$$\begin{aligned} M &= \frac{1}{\ell}(L_0 + \overline{L}_0) \\ J &= (L_0 - \overline{L}_0), \end{aligned} \tag{3.2}$$

so $L_0 = (\ell M + J)/2$, $\overline{L}_0 = (\ell M - J)/2$. The classical BTZ black hole obeys $M\ell \geq |J|$, or $L_0, \overline{L}_0 \geq 0$. The BTZ black hole is usually studied in the absence of the gravitational Chern-Simons coupling, that is for $k_L = k_R = k$. Its entropy is $S = \pi(\ell/2G)^{1/2}\left(\sqrt{M\ell - J} + \sqrt{M\ell + J}\right)$. (This entropy was first expressed in terms of two-dimensional conformal field theory in [13].) With $\ell/G = 16k$ as in (2.19), this is equivalent to $S = 4\pi\sqrt{k}\left(\sqrt{L_0} + \sqrt{\overline{L}_0}\right)$. For the holomorphic sector, the entropy is therefore

$$S_L = 4\pi\sqrt{k_L L_0}, \tag{3.3}$$

and similarly for the antiholomorphic sector.

There is no classical BTZ black hole with $L_0 < 0$, and the entropy of such a black hole is zero if $L_0 = 0$. We will take this as a suggestion that quantum states corresponding to black holes exist only if $L_0 > 0$, that is $L_0 \geq 1$. This means that the exact partition function $Z(q)$ should differ from the function $Z_0(q)$ in (3.1) by terms of order q:

$$Z(q) = q^{-k} \prod_{n=2}^{\infty} \frac{1}{1 - q^n} + \mathcal{O}(q). \tag{3.4}$$

Admittedly, we are here trying to squeeze more information from the classical result than is justified. But it turns out that a modular-invariant partition function of this form exists and is unique. This result (which is due to Höhn [31]) follows from the fact that the moduli space $\mathcal{M}_1$ of Riemann surfaces of genus 1 is itself a Riemann surface of genus 0, in fact parametrized by the j-function. If E_4 and E_6 are the usual Eisenstein series of weights 4 and 6, then $j = 1728E_4^3/(E_4^3 - E_6^2)$. Its expansion in powers of q is

$$j(q) = q^{-1} + 744 + 196884q + 21493760q^2 + 864299970q^3 + 20245856256q^4 + \ldots. \tag{3.5}$$

Actually, it is more convenient to use the function

$$J(q) = j(q) - 744 = q^{-1} + 196884q + 21493760q^2 + \ldots, \tag{3.6}$$

which likewise parametrizes the moduli space.

The J-function has a pole at $q = 0$ and no other poles. The statement that J parametrizes the moduli space means precisely that any modular-invariant function can be written as a function of J. The partition function $Z(q)$ has a pole at $q = 0$, that is at $J = \infty$. This pole is of order k, and the partition function has no other poles. Any holomorphic function of J that has no singularity except at $J = \infty$ is a polynomial in J.

In the present case, as the pole in $Z(q)$ at $q = 0$ is of order k, Z must be a polynomial in J of degree k. Thus

$$Z(q) = \sum_{r=0}^{k} f_r J^r, \tag{3.7}$$

with some coefficients f_r. These $k + 1$ coefficients can be adjusted in a unique fashion to ensure that $Z(q)$ takes the form (3.4), or in other words to ensure that the terms in $Z(q)$ of order q^{-n}, $n = 0, \ldots, k$, coincide with the naive function $Z_0(q)$.

When we do this, we get a function that we will call $Z_k(q)$, $k = 1, 2, 3, \ldots$. This function is our candidate for the generating function that counts the quantum states of three-dimensional gravity in a spacetime asymptotic to AdS$_3$. For example, for $k = 1$ we have simply $Z_1(q) = J(q)$, and the next few examples are

$$
\begin{aligned}
Z_2(q) =& J(q)^2 - 393767 \\
=& q^{-2} + 1 + 42987520q + 40491909396q^2 + \ldots \\
Z_3(q) =& J(q)^3 - 590651J(q) - 64481279 \\
=& q^{-3} + q^{-1} + 1 + 2593096794q + 12756091394048q^2 + \ldots \\
Z_4(q) =& J(q)^4 - 787535J(q)^2 - 8597555039J(q) - 644481279 \\
=& q^{-4} + q^{-2} + q^{-1} + 2 + 81026609428q + 1604671292452452276q^2 + \ldots.
\end{aligned}
\tag{3.8}
$$

Following [31], we refer to a holomorphic CFT with $c = 24k$ and partition function $Z_k(q)$ as an extremal CFT. According to our proposals, the dual of three-dimensional gravity should be an extremal CFT.

As was already noted in the introduction, Frenkel, Lepowsky, and Meurman constructed [24,25] an extremal CFT with $k = 1$, that is, a holomorphic CFT with $c = 24$ and partition function $J(q) = Z_1(q)$. They also conjectured its uniqueness. If that conjecture as well as the ideas in the present paper are correct, then the FLM theory must be the dual to quantum gravity for $k = 1$. Unfortunately, as also noted in the introduction, for $k > 1$, extremal CFT's are not known, though their possible existence has been discussed in the literature [31-33] for reasons not related to three-dimensional gravity. Our reasoning in this paper suggests that such theories should exist and be unique for each k.

The main point of the FLM construction was that their theory has as a group of symmetries the Fischer-Griess monster group $\mathbb{M}$, the largest of the sporadic finite groups. Arguably, the FLM theory is the most natural known structure with $\mathbb{M}$ symmetry. The coefficients in the q-expansion of the J-function are integers for number-theoretic reasons,

but the FLM construction gave a new perspective on why they are positive; this is a property of the partition function of any CFT. It also gave a new perspective on why these coefficients are so large; this follows from $\mathbb{M}$ symmetry, since $\mathbb{M}$ does not have small representations.

Indeed, the original clue to the FLM construction was the observation by J. McKay that the first non-trivial coefficient 196884 of the function $J(q)$ is nearly equal to the smallest dimension 196883 of a non-trivial representation of $\mathbb{M}$. (This observation was later greatly generalized [43,44].) The FLM interpretation is that 196884 is the number of operators of dimension 2 in their theory. One of these operators is the stress tensor, while the other 196883 are primary fields transforming in the smallest non-trivial representation of $\mathbb{M}$.

In our interpretation, the 196883 primaries are operators that (when combined with suitable anti-holomorphic factors) create black holes. It is illuminating to compare the number 196883 to the Bekenstein-Hawking formula. An exact quantum degeneracy of 196883 corresponds to an entropy of $\ln 196883 \cong 12.19$. By contrast, the Bekenstein-Hawking entropy at $k = 1$ and $L_0 = 1$ is $4\pi \cong 12.57$. We should not expect perfect agreement, because the Bekenstein-Hawking formula is derived in a semiclassical approximation which is valid for large k.

Agreement improves rapidly if one increases k. For example, at $k = 4$, and again taking $L_0 = 1$, the exact quantum degeneracy of primary states is 81026609426, according to eqn. (3.8). (Two of the states at this level are descendants.) This corresponds to an entropy $\ln 81026609426 \cong 25.12$, compared to the Bekenstein-Hawking entropy $8\pi \cong 25.13$. Shortly we will compute the entropy in the large k limit.

Our interpretation is that a primary state $|\Lambda\rangle$ represents a black hole, while a descendant $\prod_{n=1}^{\infty} L_{-n}^{s_n} |\Lambda\rangle$ describes a black hole embellished by boundary excitations. These boundary excitations are the closest we can come in $2 + 1$ dimensions to the gravitational waves that a black hole can interact with in a larger number of dimensions. If this interpretation is correct, then for an exact count of black hole states, we should count primaries only. However, as the above examples illustrate, in practice this issue has only a very slight effect on the black hole degeneracies. The boundary excitations contribute to the entropy an amount that is independent of k and thus negligible in the regime where we compare to the Bekenstein-Hawking entropy, and also, numerically, negligible even for small k. The separation between the black hole and the boundary excitations is best-motivated for large k.

Alternative Formula

Now we will present an alternative formula for the partition functions $Z_k(q)$. This formula avoids the large coefficients involved in writing the Z_k as a polynomial in J, and will make some properties of the Z_k more manifest.

Let $q = \exp(2\pi i \tau)$ with τ in the upper half plane. If p is a prime number, then the Hecke operators acting on functions $F(\tau)$ are defined essentially by

$$T'_p F(\tau) = F(p\tau) + \sum_{b=0}^{p-1} F((\tau + b)/p). \tag{3.9}$$

(What we call T'_p is p times the Hecke operator T_p as usually defined. See [45] for an introduction to Hecke operators and [29,30] for their use in the present subject.) More generally, for any positive integer t, we define

$$T'_t F(\tau) = \sum_{d|t} \sum_{b=0}^{d-1} F((t\tau + bd)/d^2). \tag{3.10}$$

For $t = 0$, we define $T'_0 F(\tau) = 1$. If $F(\tau)$ is modular-invariant, then so is $T'_t F(\tau)$. The definition of Hecke operators generalizes naturally to modular forms, but we will not need this.

An immediate consequence of the definition is that, for any $F(\tau)$ that is invariant under $\tau \to \tau + 1$, and so has a Laurent expansion in powers of $q = \exp(2\pi i \tau)$, the coefficients in the q-expansion of $T'_n F(\tau)$ are linear combinations with positive integer coefficients of the q-expansion coefficients of $F(\tau)$. The FLM construction shows that the q-expansion coefficients of the J-function are non-negative integers and the coefficients of positive powers of q are dimensions of non-trivial (reducible) monster representations. Hence the same is true of $T'_n J$.

An important special case of how the Hecke operators act on q-expansion coefficients is that if $F(\tau) = q^{-1} + \mathcal{O}(q)$, then

$$T'_n F(\tau) = q^{-n} + \mathcal{O}(q). \tag{3.11}$$

Now we can give an alternative description of the partition functions $Z_k(\tau)$. We recall that these functions are uniquely determined by being modular-invariant and agreeing up to order q with the function $Z_0 = q^{-k} \prod_{n=2}^{\infty} (1 - q^n)^{-1}$. To find a function with these properties, we simply expand

$$Z_0 = \sum_{r=-k}^{\infty} a_r q^r, \tag{3.12}$$

and then we let

$$Z_k(\tau) = \sum_{r=0}^{k} a_{-r} T_r' J(\tau). \tag{3.13}$$

For example,

$$Z_2(\tau) = (T_2' + T_0') J(\tau) = J(2\tau) + J(\tau/2) + J((\tau+1)/2) + 1. \tag{3.14}$$

This representation makes it clear that, just like the $T_r' J$, the functions Z_k share some properties with the J-function: the q-expansion coefficients are non-negative, and the coefficients of positive powers of q are dimensions of non-trivial monster representations.

This makes it tempting to speculate that the CFT's dual to three-dimensional gravity have $\mathbb{M}$ symmetry for all k. If so, the $\mathbb{M}$ symmetry is invisible in classical General Relativity and acts on the microstates of black holes. (However, the analogous conjecture for supergravity, which would involve the Conway group, appears to be untrue, as we will see in section 3.3.)

As an illustration of the usefulness of the expression for the partition function in terms of Hecke operators, we will use it to compare with the Bekenstein-Hawking entropy. We consider the semiclassical limit $k, L_0 \to \infty$, with $r = L_0/k$ positive and fixed. We take $r = p/q$ to be a rational number and assume that k is divisible by q, so that L_0 is an integer n. If we write

$$J(q) = \sum_{m=-1}^{\infty} c_m q^m, \tag{3.15}$$

then a formula of Petersson and Rademacher gives the asymptotic behavior

$$\ln c_m \sim 4\pi\sqrt{m} - \frac{3}{4}\ln m - \frac{1}{2}\ln 2 + \ldots. \tag{3.16}$$

Let

$$Z_k(\tau) = \sum_{n=-k}^{\infty} b_{k,n} q^n. \tag{3.17}$$

We want to determine the behavior of $b_{k,n}$ for large k and n with $r = n/k$ fixed. To evaluate the partition function using the formula (3.13), let us first look at the contribution from $r = k$, that is $T_k' J(\tau)$. In evaluating $T_k' J$ from the definition (3.10), the dominant term (for large k and n) is the term with $d = k$. This contribution to $b_{k,n}$ is $b_{k,n}^0 \sim k c_{kn}$, and hence, using the Petersson-Rademacher formula,

$$\ln b_{k,n}^0 \sim 4\pi\sqrt{kn} + \frac{1}{4}\ln k - \frac{3}{4}\ln n - \frac{1}{2}\ln 2 + \ldots. \tag{3.18}$$

34

The first term is the Bekenstein-Hawking entropy (3.3). The additional terms in (3.18), as well as the remaining contributions in (3.13) with $r < k$ that we have omitted in deriving (3.18), do not modify the Bekenstein-Hawking formula in the limit of large k, fixed n/k, but give interesting subleading corrections that will be studied elsewhere [46].

Another method to get similar results (expressing black hole degeneracies in terms of coefficients of the singular part of the partition function) is the Farey tail expansion [34]. See also [47].

The FLM Construction

Consider a holomorphic CFT with $c = 24$. If the number of primary fields of dimension 1 is s, then the expansion of the partition function near $q = 0$ begins $Z(q) = q^{-1} + s + \mathcal{O}(q)$, and modular invariance implies that the genus 1 partition function is precisely $Z(q) = J(q) + s = q^{-1} + s + 196884q + \ldots$. Of the 196884 fields of dimension 2, one is a descendant of the identity and s are descendants of dimension 1 primaries. Hence there are $196883 - s$ primary fields of dimension 2, too few to furnish a non-trivial representation of the monster group unless $s = 0$.

For this reason, Frenkel, Lepowsky, and Meurman [24] aimed to construct a holomorphic CFT of $c = 24$ with no primary fields of dimension 1, hoping that it would have monster symmetry. Their construction was made as follows. The Leech lattice is an even, unimodular lattice of rank 24 with no vector of length squared less than 4. One can construct a holomorphic theory of $c = 24$ by compactifying 24 chiral bosons X_i, $i = 1, \ldots, 24$ using any even unimodular lattice. If one uses the Leech lattice, then because it contains no vector of length squared less then 4, a primary field of the form $\exp(ip \cdot X)$ has dimension at least 2, as desired. However, this lattice theory has 24 primary fields ∂X_i of dimension 1. Aiming to eliminate those fields, FLM considered an orbifold[15] by the $\mathbb{Z}_2$ symmetry $X_i \to -X_i$ which eliminates the dimension 1 primaries. As it turns out, this particular orbifold preserves modular invariance, does not add dimension 1 primaries in twisted sectors, and gives the hoped-for theory with monster symmetry.

FLM conjectured that the model they constructed is the unique holomorphic CFT with partition function J. If this conjecture is correct, it implies that many other constructions give the same theory. There are [49] many orbifolds of the Leech lattice theory that give

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[15] Their work preceded the general study of orbifolds in string theory [48], though some related stringy constructions, such as the Ramond and Neveu-Schwarz sectors of string theory, were already known.

holomorphic CFT's with no primary of dimension 1. According to the FLM uniqueness conjecture, all of these theories are isomorphic to their monster theory. Moreover, the Leech lattice theory itself can be obtained as an orbifold theory starting from another even unimodular lattice of rank 24, so other lattices can be used as starting points as well.

Aiming to imitate the FLM construction for $k = 2$, one might start with an even unimodular rank 48 lattice with no vector of length squared less than 6. Three such lattices are known; two are described in [50] and a third has been constructed more recently [51]. (There is no reason to believe that these are the only three rank 48 lattices with this property; there may be a vast number of them.) In the theory of 48 free chiral bosons X_i compactified using such a lattice, a primary field $\exp(ip \cdot X)$ has dimension at least 3, but there are dimension 1 primaries ∂X_i and dimension 2 primaries $\partial X_i \partial X_j - \frac{1}{48}\delta_{ij}\partial X \cdot \partial X$. A simple $\mathbb{Z}_2$ orbifold eliminates the dimension 1 primaries, as in the FLM case, but not the dimension 2 primaries. One may attempt to find a more complicated orbifolding construction to remove the unwanted primaries. Though all three lattices have interesting discrete symmetry groups, it appears that there is no anomaly-free subgroup that removes all dimension 2 primaries.[16] Many other orbifolds can be considered, such as the symmetric product of k copies of the $k = 1$ monster theory, but it appears difficult to remove all primaries of low dimension.

Optimistically speaking, this situation might be compared to current algebra of a simply-laced compact Lie group G. At level 1, one has the Frenkel-Kac-Segal construction of current algebra of G via free bosons. At higher integer level, the theory still exists, but generically has no equally straightforward realization in terms of free fields. Perhaps the situation is somewhat similar for extremal CFT's.

Further Remarks

We have emphasized here the partition function, because it is what we can determine for general k. However, if one can actually describe the relevant CFT – as conjecturally we can at $k = 1$ via the FLM construction – this gives much more than a partition function. In this case, by computing matrix elements of primary fields, we get a detailed description of the black hole quantum mechanics.

Above 2+1 dimensions, a black hole can form out of radiation and ordinary matter. In pure gravity in 2+1 dimensions, the only thing that a black hole can form from is, roughly

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[16] I am grateful for assistance from G. Nebe in investigating this question.

speaking, smaller black holes. To be more precise, let Φ_i, $i = 1, \ldots, w$ be primary fields of dimension not much greater than k that individually could, in acting on the vacuum, create black holes of fairly small mass. A product of such fields acting on the vacuum at prescribed points

$$\prod_{i=1}^{w} \Phi_i(z_i)|\Psi\rangle \tag{3.19}$$

may create a black hole of large mass, that is, a primary state of large energy or a descendant of such a state. Evaluating such matrix elements is the closest analog we can find, in the present model, to describing the formation of a large mass black hole from matter. Unfortunately, at the moment, we can perform such computations only for $k = 1$, since for other values of k we do not know the CFT. Ideally, one would like to describe the CFT for all k and investigate the semiclassical limit of large k.

If our hypotheses are correct, the following may be the most significant difference between three-dimensional pure gravity and a more realistic theory of black holes in $3 + 1$ dimensions. In the present model, as the holomorphic fields are all conserved currents, and the energy levels are integers, the dynamics is integrable and periodic on a short time scale. This is certainly not expected for black holes in general. An embedding of three-dimensional gravity in a larger system, such as a string theory, would be expected to give a non-integrable deformation of the model.

A Conundrum

We have here considered the holomorphic sector of a CFT, but the CFT dual of three-dimensional gravity has both holomorphic and antiholomorphic degrees of freedom. Let us therefore discuss a puzzle (stressed by J. Maldacena) that arises when one combines the holomorphic and antiholomorphic degrees of freedom. We let $|\Omega\rangle$, $|\widetilde{\Omega}\rangle$ be the ground states in the holomorphic and antiholomorphic sectors, and let $|\Phi\rangle$ and $|\widetilde{\Phi}\rangle$ denote primary states other than the ground state. The state $|\Omega\rangle \otimes |\widetilde{\Omega}\rangle$ and its descendants correspond, according to our picture, to Anti de Sitter space and its boundary excitations. A state $|\Phi\rangle \otimes |\widetilde{\Phi}\rangle$, or a descendant thereof, corresponds to a BTZ black hole, perhaps with boundary excitations. But what do we make of states $|\Omega\rangle \otimes |\widetilde{\Phi}\rangle$ or $|\Phi\rangle \otimes |\widetilde{\Omega}\rangle$, and their descendants? Such states are trying to be Anti de Sitter space for holomorphic variables and black holes for antiholomorphic variables, or vice-versa.

In classical three-dimensional gravity, there is no satisfactory solution that has this interpretation. We can, however, see what is involved in trying to make one. First of all,

three-dimensional Anti de Sitter space AdS_3 is simply the universal cover of the $SL(2,\mathbb{R})$ group manifold, with the symmetry $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$ (or rather a cover thereof) coming from the left and right action of $SL(2,\mathbb{R})$ on itself. The conformal boundary of the universal cover of AdS_3 is a cylinder $\mathbb{R} \times S^1$. We can parametrize it by $-\infty < \tau < \infty$, $0 \leq \sigma \leq 2\pi$, with conformal structure described by the metric $ds^2 = d\tau^2 - d\sigma^2$. The BTZ black hole is the quotient of (a cover of) $SL(2,\mathbb{R})$ by a subgroup $\mathbb{Z} \subset SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$ that is not simply a subgroup of one factor or the other. This quotient has conformal boundary that is again a cylinder, conformally equivalent to the boundary of Anti de Sitter space itself. That is why the BTZ black hole can be regarded as an excitation of Anti de Sitter space.

To describe a state $|\Omega\rangle \otimes |\Phi\rangle$ (or its parity conjugate) at the classical level, we want a solution of Einstein's equations that is invariant under precisely one of the two factors of the $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$ symmetry. This will reflect the fact that $|\Omega\rangle$ is $SL(2,\mathbb{R})$-invariant and $|\Phi\rangle$ is not. A space with just one of the $SL(2,\mathbb{R})$ symmetries is $\mathrm{AdS}_3/\mathbb{Z}$, where $\mathbb{Z}$ is a subgroup of one $SL(2,\mathbb{R})$ factor. The quotient $\mathrm{AdS}_3/\mathbb{Z}$ is a perfectly good solution of Einstein's equations in bulk, but its asymptotic behavior at infinity does not quite agree with that of AdS_3. The boundary at infinity is a cylinder, but the compact direction in the cylinder is lightlike rather than spacelike. For our approach to three-dimensional gravity in the present paper to be correct, something like this quotient $\mathrm{AdS}_3/\mathbb{Z}$ has to make sense at the quantum level, perhaps because of a small quantum correction that makes the compact direction in the cylinder effectively spacelike. Note that the metric $ds^2_\epsilon = d\tau^2 - \epsilon d\sigma^2$ induces on the cylinder a conformal structure that is independent of ϵ up to isomorphism as long as $\epsilon > 0$.

What we have described is an analog for black holes of a statement made in section 2.4: to ensure holomorphic factorization, the sum over topologies probably must be extended to include configurations that are difficult to interpret classically. In the present case, the analog of the sum over topologies is the sum over states with or without the presence of a black hole.

3.2. The Supersymmetric Case

Now we consider the analog for supergravity. The analysis is a little more complicated and in some ways the results are less satisfactory.

A rather similar problem has been treated by Höhn [31]. He considered not a superconformal field theory, but a more general holomorphic theory with bosonic operators of integral dimension and fermionic operators of half-integral dimension. In that case, the

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partition function can have a pole at the Ramond cusp, and is given by a Laurent series in $j_\theta = K - 24$ rather than a polynomial.

We continue to assume holomorphic factorization, and we assume that the holomorphic part of the boundary CFT is an $\mathcal{N} = 1$ superconformal field theory (SCFT). This means (in whatever not well understood sense gravity is related to gauge theory) that the gauge group of the bulk theory has a factor $OSp(1|2)$. From a classical point of view, the $OSp(1|2)$ bundle over a three-manifold W endows W with a spin structure. In the AdS/CFT correspondence, W has for conformal boundary a Riemann surface C, and the spin structure on W determines one on C. One is instructed in the AdS/CFT correspondence to specify C, including its spin structure, and sum over all choices of W. (Depending on the theory, the antiholomorphic degrees of freedom may themselves be supersymmetric and endowed with a choice of spin structure.) We will assume that it is physically appropriate to specify the spin structure on C rather than summing over it (just as one specifies the complex structure of C), though at the end of section 3.3, we discuss what happens if one sums over spin structures.
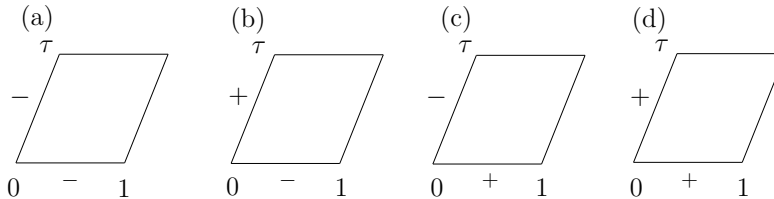

Fig. 3:

We represent a genus 1 Riemann surface C as the quotient of the complex plane by the lattice generated by complex numbers 1 and τ, where τ lies in the upper half plane $\mathcal{H}$. There are four possible spin structures, classified by whether the fermions are periodic or antiperiodic around the "horizontal" and "vertical" cycles. This is indicated by labeling the horizontal and vertical directions with a $+$ sign for periodicity or a $-$ sign for antiperiodicity.

We focus on the case that C has genus 1, in which case there are four possible spin structures (fig. 3). The path integrals for these four spin structures can be interpreted in terms of traces in two Hilbert spaces, known as the Neveu-Schwarz (NS) and Ramond (R) Hilbert spaces $\mathcal{H}_{\mathrm{NS}}$ and $\mathcal{H}_{\mathrm{R}}$, respectively.

Traces in $\mathcal{H}_{\text{NS}}$ are constructed using the spin structures, shown in (a) and (b) in the figure, in which fermions are antiperiodic in the horizontal direction. If additionally the fermions are antiperiodic in the vertical direction, as in (a), then the path integral computes

$$F(\tau) = \text{Tr}_{\mathcal{H}_{\text{NS}}} q^{L_0}, \tag{3.20}$$

where $q = \exp(2\pi i \tau)$. We call this the NS partition function. If they are periodic in the vertical direction, as in (b), the path integral computes instead

$$G(\tau) = \text{Tr}_{\mathcal{H}_{\text{NS}}} (-1)^F q^{L_0}, \tag{3.21}$$

where $(-1)^F$ is the operator that is $+1$ on bosonic states, including the ground state of the NS sector, and -1 on fermionic states.

Traces in $\mathcal{H}_{\text{R}}$ are computed using spin structures, shown in (c) and (d) in the figure, in which fermions are periodic in the horizontal direction. If the fermions are antiperiodic in the vertical direction, the path integral computes an ordinary trace

$$H(\tau) = \text{Tr}_{\mathcal{H}_{\text{R}}} q^{L_0}. \tag{3.22}$$

If fermions are periodic in both directions, the path integral gives

$$\chi = \text{Tr}_{\mathcal{H}_{\text{R}}} (-1)^F q^{L_0} = \text{Tr}\,(-1)^F. \tag{3.23}$$

Both $H(\tau)$ and χ are severely constrained by the fact that in the R sector, one of the superconformal generators, which we will $\mathcal{G}_0$, commutes with L_0 and obeys $\mathcal{G}_0^2 = L_0$. This implies first of all that $L_0 \geq 0$ in the R sector, so $H(\tau)$ has no pole at $q = 0$. Further, the action of $\mathcal{G}_0$ implies that states of $L_0 > 0$ are paired between bosons and fermions. They cancel out of χ and make equal contributions to $H(\tau)$. It follows that χ is simply an integer, the trace of the operator $(-1)^F$ in the subspace with $L_0 = 0$ (or alternatively the index of the operator $\mathcal{G}_0$ mapping from bosonic states to fermionic ones). Moreover, if we expand $H(\tau)$ in powers of q

$$H(\tau) = \sum_{n=0}^{\infty} h_n q^n, \tag{3.24}$$

then all coefficients h_n are even except possibly h_0. In this expansion, only integral powers of q appear, since all fields obey integral boundary conditions in the R sector.

In the NS sector, the ground state energy is $-c/24$. In the holomorphically factorized case that we focus on, $c = 12k^*$ for an integer k^*, so the ground state energy is $-k^*/2$.

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Since fermions in the NS sector are antiperiodic in the spatial direction, NS excitations may have either integer or half-integer energy above the ground state. The partition function therefore takes the general form

$$F(\tau) = q^{-k^*/2}(1 + aq^{1/2} + bq + cq^{3/2} + \ldots) = \sum_{m \in \mathbb{Z}/2,\, m \geq -k^*/2} f_m q^m. \tag{3.25}$$

In this expansion, the states in which $m + k/2$ is an integer are bosons, while the states in which $m + k/2$ is half-integral are fermions. That is so because, in the NS sector, bosonic fields are periodic and have integral excitation energies, while fermionic fields are half-integral and have half-integral excitation energies. $G(\tau)$ can therefore very explicitly be expressed in terms of the same coefficients:

$$G(\tau) = \sum_{m \in \mathbb{Z}/2,\, m \geq -k^*/2} (-1)^{2m+k^*} f_m q^m. \tag{3.26}$$

We have simply included a factor that is $+1$ on bosons and -1 on fermions. A more succinct way to say the same thing is that

$$G(\tau) = (-1)^{k^*} F(\tau + 1). \tag{3.27}$$

Thus, G is simply determined in terms of F. The same is true for H. Consider the modular transformation $\tau \to -1/\tau$, which has the effect of exchanging the horizontal and vertical directions in fig. 3 (with a reversal of orientation for one). This exchanges spin structures (b) and (c), as a result of which

$$H(\tau) = G(-1/\tau). \tag{3.28}$$

We can combine (3.27) and (3.28) to get

$$H(\tau) = (-1)^{k^*} F(-1/\tau + 1). \tag{3.29}$$

An important special case of this is

$$\lim_{\tau \to i\infty} H(\tau) = (-1)^{k^*} F(1). \tag{3.30}$$

The limit on the left hand side exists, because H has no pole at $q = 0$. The limit is just the coefficient h_0 in (3.24):

$$h_0 = (-1)^{k^*} F(1). \tag{3.31}$$

h_0 is the number of Ramond states of zero energy.

According to the above formulas, everything (except an integer χ, the supersymmetric index) can be expressed in terms of F. So it is useful to understand the modular properties of F. F is not invariant under $\tau \to \tau + 1$, as we have already seen. But, since all energies take values in $\mathbb{Z}/2$, F is invariant under $\tau \to \tau + 2$. In addition, since the modular transformation $\tau \to -1/\tau$ maps spin structure (a) itself, F is invariant under $\tau \to -1/\tau$. The two transformations $\tau \to \tau + 2$ and $\tau \to -1/\tau$ generate a subgroup of $SL(2, \mathbb{Z})$ that consists of 2×2 integral unimodular matrices

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{3.32}$$

that are congruent mod 2 to one of

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{3.33}$$

These matrices of course act on τ in the usual fashion, $\tau \to (a\tau + b)/(c\tau + d)$. This group is sometimes called Γ_θ; it is conjugate to the group $\Gamma_0(2)$ characterized by requiring b to be even. (A third conjugate group is characterized by the condition that c should be even. Each of these three groups can be defined as the subgroup of $SL(2, \mathbb{Z})$ that leaves fixed one of the even spin structures, that is, one of the first three shown in fig. 3.)

The quotient of the upper half plane $\mathcal{H}$ by $SL(2, \mathbb{Z})$ is a curve of genus zero with one point missing. This can be seen by studying the fundamental domain (fig. 4(a)). The missing point is called the "cusp" at $\tau = i\infty$. It is because the quotient has genus zero that Riemann surfaces of genus 1 can be parametrized by a single holomorphic function, namely $J(\tau)$. As we have seen in section 3.1, this is useful because it means that the constraints of modular invariance are equivalent to the statement that the partition function is a function of J. Similarly, the quotient of the upper half plane $\mathcal{H}$ by Γ_θ is again a curve of genus zero, this time with two missing points or cusps, as explained in fig. 4(b). This means that there is a function $K(\tau)$ such that any Γ_θ-invariant function F can be expressed as a function of K.

The two cusps of $\mathcal{H}/\Gamma_\theta$ are at $\tau \to i\infty$ and $\tau = 1$, and correspond respectively to the NS sector and the R sector. For $\tau \to i\infty$, the NS partition function $F(\tau)$ has an expansion in terms of L_0 eigenvalues in the NS sector. Likewise, the Ramond partition function $H(\tau)$ can be expanded for $\tau \to i\infty$ in L_0 eigenvalues in the Ramond sector; but according to (3.29), the behavior of $H(\tau)$ for τ near $i\infty$ is the same as the behavior of $F(\tau)$ for τ near 1.
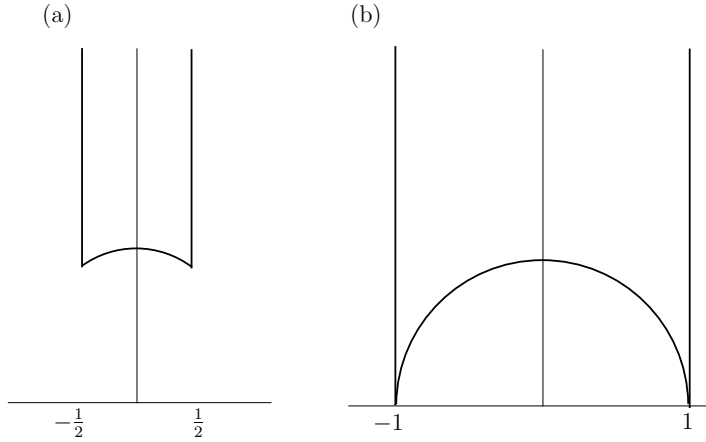
Fig. 4:

      (a) A fundamental domain for the action of $SL(2,\mathbb{Z})$ on the upper half plane. By the action of $\tau \to \tau + 1$, one can take $|\mathrm{Re}\,\tau| \leq 1/2$, and by the action of $\tau \to -1/\tau$, one can take $|\tau| \geq 1$. The $SL(2,\mathbb{Z})$ action identifies the left and right hand halves of the boundary of the fundamental domain, making what topologically is a Riemann surface of genus zero with one missing point at $\tau = i\infty$. This point is called the cusp. (b) The analogous fundamental domain for the action of Γ_θ. Here the symmetry $\tau \to \tau + 2$ lets us reduce to $|\mathrm{Re}\,\tau| \leq 1$, and $\tau \to -1/\tau$ still lets us reduce to $|\tau| \geq 1$. The group action still identifies the left and right hand halves of the boundary of the fundamental domain, and the result is a Riemann surface of genus zero, now with two points omitted. The missing points are the Neveu-Schwarz cusp at $\tau = i\infty$ and the Ramond cusp at $\tau = \pm 1$ (those two points are equivalent under $\tau \to \tau + 2$). The Ramond cusp is missing in the quotient of the upper half plane by Γ_θ because the points $\tau = \pm 1$ are not in the upper half plane, but on its boundary.

So in terms of the function F, the behavior at the two cusps is determined by the low-lying spectrum in the NS and Ramond sectors, respectively.

      We therefore call these the NS and R cusps. The natural uniformizing parameter at the NS cusp is $q^{1/2} = \exp(i\pi\tau)$, where τ is the argument of the function F in (3.25). The right parameter is $\exp(i\pi\tau)$ since the symmetry is $\tau \to \tau + 2$. At the R cusp, the natural parameter is $q = \exp(2\pi i\tau)$, where now τ is the argument of H.

      To get a clear picture of the nature of a holomorphic function F on $\mathcal{H}/\Gamma_\theta$, it is necessary to describe the behavior near the cusps, or equivalently to describe what happens when one tries to extend F to a function on the compactification of $\mathcal{H}/\Gamma_\theta$ that is obtained by adding the cusps. This compactification is a space $\mathcal{Y} \cong \mathbb{CP}^1$.

We can explicitly describe a function K that parametrizes $\mathcal{H}/\Gamma_\theta$ and has a pole only at the NS cusp. This can be done in several ways. One formula is

$$K(\tau) = \frac{\Delta(\tau)^2}{\Delta(2\tau)\Delta(\tau/2)} - 24, \qquad (3.34)$$

where $\Delta = q\prod_{n=1}^{24}(1-q^n)^{24}$ is the discriminant, a modular form of weight 24. In (3.34), we have subtracted the constant 24 so that the expansion of $K(\tau)$ in powers of $q^{1/2}$ has no constant term:

$$K(\tau) = q^{-1/2} + 276q^{1/2} + 2048q + 11202q^{3/2} + 49152q^2 + 184024q^{5/2} + 614400q^3$$
$$+ 1881471q^{7/2} + 5373952q^4 + 14478180q^{9/2} + \ldots.$$
$$(3.35)$$

This is analogous to the definition of the J-function without a constant term. Another formula for K is

$$K = \frac{q^{-1/2}}{2}\left(\prod_{n=1}^\infty(1+q^{n-1/2})^{24} + \prod_{n=1}^\infty(1-q^{n-1/2})^{24}\right) + 2048q\prod_{n=1}^\infty(1+q^n)^{24}. \qquad (3.36)$$

The product formula in (3.34), since it converges for all $|q| < 1$, shows that K is non-singular as a function on $\mathcal{H}$. As for the behavior at the cusps, either formula shows that K has a simple pole at the NS cusp, that is, it behaves for $q \to 0$ as $q^{-1/2}$. K is regular at the Ramond cusp; in fact

$$K(\tau = 1) = -24. \qquad (3.37)$$

This statement is equivalent to the statement that $K + 24 = \Delta(\tau)^2/\Delta(2\tau)\Delta(\tau/2)$ vanishes at $\tau = 1$. Indeed, that function has a pole at the NS cusp, so it must have a zero somewhere. Its representation as a convergent infinite product shows that it is nonzero for $0 < |q| < 1$, so the zero is at the Ramond cusp. We give another explanation of (3.37) in section 3.3 in analyzing the $k^* = 1$ model. The fact that the holomorphic function K on $\mathcal{Y}$ has only one pole, which is of first order, gives another way to prove that $\mathcal{Y}$ is of genus zero.

Now let us consider the Neveu-Schwarz partition function F of a holomorphic SCFT. Any Γ_θ-invariant function F on $\mathcal{H}$ can be written as a function of K. The function F arising in a holomorphic SCFT is actually *polynomial* in K. Indeed, since the definition of F as $\mathrm{Tr}\, q^{L_0}$ is convergent for $0 < |q| < 1$, F is regular as a function on $\mathcal{H}$. So the only poles of F are at cusps, but the formula (3.31), which reflects the fact that the ground state energy in the Ramond sector is zero, says that F has no pole at the Ramond cusp.

So the only pole of F is at the Neveu-Schwarz cusp, that is at $K = \infty$. Consequently, in any holomorphic SCFT, the Neveu-Schwarz partition function F is a polynomial in K.

The degree of this polynomial is precisely k^*, since $F \sim q^{-k^*/2}$ for $q \to 0$. So

$$F = \sum_{r=0}^{k^*} f_r K^r. \tag{3.38}$$

Thus, F depends on $k^* + 1$ coefficients.

Now, for the case of an SCFT that is dual to three-dimensional supergravity, what would we expect F to be? To follow the same logic as in the bosonic case, we would argue that since the minimum energy of a BTZ black hole is $L_0 = 0$, and the entropy vanishes at $L_0 = 0$, primary states corresponding to black holes should be absent for $L_0 \leq 0$. Equivalently, primary fields other than the identity should be absent for dimension less than $(k^* + 1)/2$. Either statement would mean that up to terms of order $q^{1/2}$, F would coincide with the naive function

$$F_0(k^*) = q^{-k^*/2} \prod_{n=2}^{\infty} \frac{1 + q^{n-1/2}}{1 - q^n} \tag{3.39}$$

that counts superconformal descendants of the identity.

For each positive integer k^*, there is a uniquely determined function F_{k^*} that is a polynomial in K and coincides with $F_0(k^*)$ up to order $q^{1/2}$. This is a natural analog of the partition function Z_k that we defined for an extremal CFT without supersymmetry.

However, the same logic that makes us think that there should be no NS primaries of $L_0 \leq 0$ also would make us believe that there should be no Ramond primaries of $L_0 = 0$. (There are automatically no Ramond states of $L_0 < 0$, since $L_0 = \mathcal{G}_0^2$.) In either the NS or Ramond sector, the classical black hole entropy vanishes for $L_0 \leq 0$.

The number of Ramond primaries of $L_0 = 0$ in a theory with Neveu-Schwarz partition function F_{k^*} is, according to (3.31), $h_0 = (-1)^{k^*} F_{k^*}(1)$. Let us call this number β_{k^*}. β_{k^*} is uniquely determined for each k^*. A practical way to determine it, using (3.37), is to write

$$\beta_{k^*} = (-1)^{k^*} F_{k^*}(1) = (-1)^{k^*} \sum_{r=0}^{k^*} f_r K(1)^r = (-1)^{k^*} \sum_{r=0}^{k^*} f_r (-24)^r. \tag{3.40}$$

So one first evaluates the coefficients f_r in (3.38) to ensure that there are no NS primary states with $L_0 \leq 0$, and then one evaluates the sum in (3.40).

The first ten values of β_{k^*} are given in Table 1. The first observation is that β_{k^*} never vanishes in this range. Consequently, it is impossible to assume that there are no primary operators other than the identity of dimension less than $(k^* + 1)/2$. If we assume that there are no such primaries in the NS sector, then there are primaries of dimension $k^*/2$ in the Ramond sector.

k^*	β_{k^*}
1	24
2	24
3	95
4	1
5	143
6	1
7	262
8	-213
9	453
10	-261

Table 1. Values of the index β_{k^*} for $k^* = 1, \ldots, 10$.

The good news is that the numbers in the table are rather small compared to black hole multiplicities that arise in the classically allowed region $L_0 > 0$. So an optimistic view is to interpret the numbers in the table as quantum corrections. For example, at $k^* = 9$, where β takes the relatively large value 453, the multiplicity of the lowest mass classically allowed black hole, namely $L_0 = 1/2$ in the NS sector, turns out to be 135149371 if the partition function F_9 can be trusted. (As usual, we count only primaries, although this involves only a small correction.) The lowest classically allowed black hole in the Ramond sector, at $L_0 = 1$, has multiplicity 381161020987.

However, it may not seem logical to assume that there are no NS primaries at dimension $k^*/2$ and allow Ramond primaries of that dimension. Moreover, it really does not make sense to do this, since some values of β in the table are negative.

So we retreat from claiming that there are no NS primary fields of dimension less than $(k^* + 1)/2$, and instead we consider the hypothesis that there are no such primaries

of dimension less than $k^*/2$. In this case, we are free to add an integer s to F_{k^*}. If we do so, the number of NS primaries of dimension $k^*/2$ becomes $f_0 = s$, and the number h_0 of Ramond primaries of that dimension changes by $(-1)^{k^*}s$. The combination

$$\beta_{k^*} = h_0 - (-1)^{k^*} f_0 \qquad (3.41)$$

is unchanged, and so is still given by the numbers in the table.

Unfortunately, then, we have no way to determine the numbers f_0 and h_0 separately, only the "index" β. We also have little insight about the more conventional index χ defined in (3.23). Both of these quantities characterize the primary states of $L_0 = 0$.

As in the bosonic case, one can construct useful formulas for the functions F_{k^*} in terms of Hecke operators acting on F_1. However, we will omit this.

Optimistic Conjecture

We define an extremal SCFT to be one with no primary fields other than the identity of dimension less than $k^*/2$. Equivalently, the NS partition function is equal up to an additive integer to the function F_{k^*} that was defined above.

Generalizing what we have said about the bosonic case, the most optimistic conjecture we can propose is that an extremal SCFT exists and is unique for every positive integer k^*, and is dual to three-dimensional supergravity. The best evidence we can offer is that extremal SCFT's in this sense do exist for $k^* = 1, 2$ and there are results about uniqueness [31,35] at least for $k^* = 1$.

In addition, as we will see, the functions F_{k^*} with $k^* = 3, 4$ have interesting properties suggestive of the existence of actual theories.

These matters are discussed in section 3.3. In addition, in the appendix, we describe the functions F_{k^*} for $5 \leq k^* \leq 10$.

3.3. Extremal SCFT's With Small k^*

$k^*=1$

The first construction of an extremal SCFT was made by FLM at $k^* = 1$, that is $c = 12$. They considered eight free bosons X_i compactified using the E_8 root lattice, combined with eight free fermions ψ_i to achieve superconformal symmetry. This theory has NS primary fields of dimension $1/2$, namely the ψ_i. To eliminate these fields, they considered a $\mathbb{Z}_2$ orbifold, dividing by the operation that acts as -1 on all X_i and ψ_i.

They conjectured that this construction gave the unique SCFT that has $c = 12$ and no NS primary of dimension $1/2$.

While the construction is simple, it has the drawback of not making manifest the global symmetry of the model. This was remedied much more recently by Duncan [35], who described the same model in another way. In this construction, one begins with 24 free fermions λ_i, $i = 1, \ldots, 24$, forming again a system with $c = 12$. In quantization, one can require the λ_i to be either antiperiodic or periodic around the spatial direction, giving what we will call NS_0 and R_0 sectors. (We reserve the name NS and R for another construction that will appear shortly.)

With 24 fermions, the ground state energy in the NS_0 sector is $-1/2$, while in the R_0 sector, the ground state energy is $+1$. The minimum dimension of a spin field is the difference between these two energies, or $3/2$.

The model has an $O(24)$ symmetry that rotates the fields λ_i. The ground state in the R_0 sector is highly degenerate, because of zero modes of the fields λ_i. It consists of $2^{12} = 4096$ states, transforming in the spinor representation of $O(24)$ (or rather its double cover).

The spin fields of lowest dimension are therefore 4096 fields W_α of dimension $3/2$ transforming as spinors. Dimension $3/2$ is the correct dimension for a supercurrent, but a generic linear combination $W = \sum_\alpha \epsilon^\alpha W_\alpha$, with c-number coefficients ϵ^α, does not generate a superconformal algebra. Schematically, and without worrying about the precise coefficients, the operator product $W \cdot W$ gives

$$W(x)W(0) \sim \frac{\overline{\epsilon}\epsilon}{x^3} + \frac{\overline{\epsilon}\epsilon\,T}{x} + \frac{\overline{\epsilon}\Gamma^{ij}\epsilon\,\lambda_i\lambda_j}{x^2} + \frac{\overline{\epsilon}\Gamma^{ijkl}\epsilon\,\lambda_i\lambda_j\lambda_k\lambda_l}{x} + \text{regular}. \tag{3.42}$$

(Here T is the stress tensor and Γ_i are the gamma matrices of $O(24)$.) The condition for W to generate a superconformal algebra is that the last two terms should be absent. This is equivalent to

$$\overline{\epsilon}\Gamma^{ij}\epsilon = \overline{\epsilon}\Gamma^{ijkl}\epsilon = 0. \tag{3.43}$$

It is shown in [35] that a spinor ϵ obeying these conditions exists and, if normalized to have $\overline{\epsilon}\epsilon = 1$, is unique up to an $O(24)$ transformation. Another important fact is that any such ϵ has one definite $SO(24)$ chirality or the other.

Any choice of a solution of (3.43) turns the theory into an $\mathcal{N} = 1$ superconformal field theory, with W as the supercurrent. However, as ϵ is not $O(24)$-invariant, the theory regarded as an $\mathcal{N} = 1$ SCFT does not have $O(24)$ symmetry. Rather, the choice of ϵ breaks

$O(24)$ symmetry to a group that is known as the Conway group Co_0. It is the symmetry group of the Leech lattice, and is a double cover of a sporadic finite group Co_1. Thus, as a superconformal field theory, this model has symmetry Co_0, not $O(24)$.

With this choice of superconformal algebra, we need to understand what are the Neveu-Schwarz and Ramond vertex operators. A Ramond vertex operator $\mathcal{O}$ has a square root singularity in the presence of the supercurrent W:

$$W(x)\mathcal{O}(x') \sim \frac{\mathcal{O}'}{(x-x')^{n-1/2}} \tag{3.44}$$

for some integer n and some operator $\mathcal{O}'$. With W understood as a spin operator with respect to the original fermions λ, those fermions have precisely this property. So they are Ramond fields. This enables us to analyze all of the states in the original NS_0 sector. The ground state corresponds to the identity operator, whose OPE with W of course has no branch cut, so it is an NS rather than Ramond operator. The operators in the NS_0 sector that have no branch cut with W are those that are products of an even number of λ's and their derivatives. The partition function that counts the corresponding states is

$$\frac{q^{-1/2}}{2}\left(\prod_{n=1}^{\infty}(1+q^{n-1/2})^{24} + \prod_{n=1}^{\infty}(1-q^{n-1/2})^{24}\right). \tag{3.45}$$

We recognize this as part of the formula (3.36) for the function K. In the R_0 sector, of the 4096 ground states, half have one chirality or fermion number and half have the other. Any excitation at all can be combined with a ground state of properly chosen chirality to get an operator that either does or does not have a branch cut with W, as desired. So for each L_0 eigenvalue in the R_0 sector, precisely half the states contribute to the NS sector and half to the R sector. The contribution of R_0 states to the NS sector is therefore

$$2048q\prod_{n=1}^{\infty}(1+q^n)^{24}. \tag{3.46}$$

Adding up (3.45) and (3.46), we see that the total partition function F_1 of the NS sector in this model is precisely what we have called K:

$$F_1 = K = \frac{q^{-1/2}}{2}\left(\prod_{n=1}^{\infty}(1+q^{n-1/2})^{24} + \prod_{n=1}^{\infty}(1-q^{n-1/2})^{24}\right) + 2048q\prod_{n=1}^{\infty}(1+q^n)^{24}. \tag{3.47}$$

We can similarly compute the Ramond partition function H_1 of this model. The contribution of the NS_0 sector is obtained from (3.45) by changing a sign so as to project

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onto states of odd fermion number, rather than even fermion number. And the contribution of the R_0 sector is the same as (3.46). So

$$H_1 = \frac{q^{-1/2}}{2} \left(\prod_{n=1}^{\infty} (1 + q^{n-1/2})^{24} - \prod_{n=1}^{\infty} (1 - q^{n-1/2})^{24} \right) + 2048q \prod_{n=1}^{\infty} (1 + q^n)^{24} \tag{3.48}$$
$$= 24 + 4096q + 98304q^2 + 1228800q^3 + 10747904q^4 + \ldots.$$

Except for states of $L_0 = 0$, the global supercharge $\mathcal{G}_0$ of the Ramond sector exchanges the part of the Ramond sector coming from NS_0 with the part coming from R_0. This implies that we can alternatively write

$$H_1 = 24 + 4096q \prod_{n=1}^{\infty} (1 + q^n)^{24}, \tag{3.49}$$

where we have removed the NS_0 contribution except for the ground states, and doubled the R_0 contribution. The equivalence of these formulas is not very obvious.

A consequence of the above formulas is that h_0, the number of Ramond states of $L_0 = 0$, is equal to 24. This gives another explanation for (3.37). Another consequence of (3.49) and (3.47) is that for $L_0 > 0$, the coefficients h_n in the q-expansion of H_1 are precisely twice the corresponding coefficients f_n in the expansion of F_1:

$$h_n = 2f_n. \tag{3.50}$$

(Here h_n is defined for integer n, while f_n is defined for integer or half-integer n, so this formula involves all coefficients h_n but only half of the f_n.) This is highly exceptional for SCFT's and reflects the particular way that this one was constructed. The relation (3.50) is equivalent to

$$0 = F_1(\tau) + F_1(\tau + 1) - H_1(\tau) = F_1(\tau) - G_1(\tau + 1) - H_1(\tau). \tag{3.51}$$

Although disguised in our way of presenting it, this is a standard relation in string theory. In the FLM construction of the model via 8 free bosons and 8 free fermions, (3.51) amounts to the statement of Gliozzi, Olive, and Scherk [52] that the Ramond-Neveu-Schwarz model has equal partition function in the Ramond and Neveu-Schwarz sectors, because of space-time supersymmetry.

The factor of 2 in (3.50) is consistent with asymptotic equality between the numbers of NS and Ramond black holes because it only involves half of the coefficients of F_1. H_1 has only half as many coefficients as F_1, but they are twice as big.

As we have explained, the NS partition function F_{k^*} of an extremal SCFT with any k^* is a polynomial in K or equivalently in F_1,

$$F_{k^*} = f(F_1), \tag{3.52}$$

for some polynomial f. From (3.29), it follows that the Ramond partition function H_{k^*} of an extremal SCFT can be obtained from H_1 using the same polynomial:

$$H_{k^*} = (-1)^{k^*} f(-H_1). \tag{3.53}$$

A Model With $k^* = 2$

What in our language is an extremal SCFT of $k^* = 2$ was constructed [26] by Dixon, Ginsparg, and Harvey (DGH), relatively soon after the work of FLM, by twisting the FLM monster construction in a somewhat similar fashion. Note that an extremal SCFT of $k^* = 2$ has $c = 24$, like the FLM monster theory.

We recall that the starting point of the FLM construction is 24 free bosons X_i that are compactified via the Leech lattice. Then one performs a $\mathbb{Z}_2$ orbifold, dividing by the symmetry $X_i \to -X_i$. To construct an orbifold, one first constructs untwisted and twisted sectors (by quantizing fields $X_i(\sigma)$ that are assumed to be periodic or antiperiodic functions of σ) and then one projects both sectors onto their $\mathbb{Z}_2$-invariant subspaces.

DGH observed that the ground state energy in the untwisted sector is -1, while the ground state energy in the twisted sector is $+1/2$. So a twist operator of lowest energy has dimension $3/2$, the right dimension for a supercurrent. They went on to show that it is possible to pick a twist operator $\mathcal{S}$ that generates a superconformal algebra.

In the FLM construction, the field $\mathcal{S}$ is not present, since it is odd rather than even under $\mathbb{Z}_2$, and is projected out when one forms the orbifold theory. However, DGH showed that it is possible to obtain an $\mathcal{N} = 1$ SCFT by modifying the usual orbifold projection. They defined the NS sector to consist of fields that do not have a cut in their OPE with $\mathcal{S}$, while the Ramond sector consists of fields that do have such a cut. Concretely, the NS sector consists of the $\mathbb{Z}_2$-even part of the untwisted sector of the orbifold plus the part of the twisted sector that transforms under $\mathbb{Z}_2$ as does $\mathcal{S}$ (so that $\mathcal{S}$, in particular, is an NS field), while the Ramond sector consists of the $\mathbb{Z}_2$-odd part of the untwisted sector plus the $\mathbb{Z}_2$-even part of the twisted sector.

From this information, one can of course compute the NS and Ramond partition functions F_2 and H_2. From our point of view, of course, F_2 is a polynomial in $F_1 = K$,

chosen so that $F_2 = q^{-1} + \mathcal{O}(q^{1/2})$. This gives $F_2 = K^2 - 552$, and similarly, in view of (3.53), $H_2 = H_1^2 - 552$. From this, we get

$$F_2 = q^{-1} + 4096q^{1/2} + 98580q + 1228800q^{3/2} + 10745856q^2 + 74244096q^{5/2} + 432155586q^3 + \dots$$

$$H_2 = 24 + 196608q + 21495808q^2 + 864288768q^3 + \dots.$$

$$(3.54)$$

The number of states of $L_0 = 0$ is 24 in the Ramond sector and 0 in the NS sector.

The global symmetry of the DGH model is closely related to the symmetry group Co_0 of the Leech lattice (it is an extension of this by a finite abelian group that involves the momenta of the Leech lattice). Since the $k^* = 1$ model also has Co_0 symmetry, one might optimistically conjecture that this group is relevant to supergravity at all k^*. However, results below about $k^* = 3$ indicate that this is not the case.

Partition Function With $k^* = 3$

Unfortunately, this is the last case for which a suitable SCFT is known, but we can of course determine the appropriate NS and Ramond partition functions for all k^*, at least modulo an additive integer. We consider $k^* = 3, 4$ here because they turn out to have unusual properties, and relegate the further cases $k^* = 5, 6, \dots, 10$ to an appendix.

For $k^* = 3$, we have

$$F_3 = K^3 - 828K - 6143. \qquad (3.55)$$

This leads to

$$F_3 = q^{-3/2} + 1 + 33606q^{1/2} + 1843200q + 43434816q^{3/2} + 648216576q^2$$
$$+ 7171304841q^{5/2} + 63903727616q^3 + \dots \qquad (3.56)$$
$$H_3 = 95 + 3686400q + 1296433152q^2 + 127807455232q^3 + \dots.$$

Here one finds an identity just like (3.50): the expansion coefficients h_n of H_3 are related for $n > 0$ to analogous coefficients f_n of F_3 by

$$h_n = 2f_n. \qquad (3.57)$$

This expresses the fact that

$$F_1^3(\tau) + F_1^3(\tau + 1) = H_1^3(\tau) - 1536, \qquad (3.58)$$

along with the similar linear relation (3.51).

It seems that the identity $h_n = 2f_n$, $n > 0$, does not hold for any values of k^* except 1 and 3. In general, the differences $h_n - 2f_n$ are relatively small, reflecting the fact that asymptotically an NS or Ramond black hole has the same entropy, but they are not zero. The fact that these degeneracies are actually equal for $k^* = 3$ may be a clue to the construction of this model.

One important point about the $k^* = 3$ model is that it seems very unlikely to have Co_0 or Co_1 symmetry. Indeed the leading coefficient 95 of H_3 is not in any economical way the dimension of a representation of Co_0. (The dimensions of the first few irreducible representations of this group are 1, 24, 276, 299, and 1771; for Co_1, one must omit the number 24 from this list.) Adding an integer n to F_3 and therefore subtracting n from H_3, does not help; we cannot pick n so that n and $95 - n$ are both dimensions of Co_0 representations in an economical fashion. So it appears that, despite the tempting evidence from the cases $k^* = 1, 2$, the group Co_0 is probably not a general symmetry of three-dimensional supergravity.

Monster Symmetry For $k^* = 4$?

However, in the next case, $k^* = 4$, there may well be a much larger discrete symmetry group, namely the monster group $\mathbb{M}$.

For $k^* = 4$, we have

$$F_4 = K^4 - 1104K^2 - 8191K + 107545. \tag{3.59}$$

This leads to the NS and Ramond partition functions

$$
\begin{aligned}
F_4 = {}& q^{-2} + q^{-1/2} + 1 + 196884q^{1/2} + 21493760q + 864299970q^{3/2} + 20246053140q^2 \\
& + 333202640600q^{5/2} + 4252023300096q^3 + 44656994071935q^{7/2} + 401490908149760q^4 + \dots \\
H_4 = {}& 1 + 42987520q + 40491712512q^2 + 8504046600192q^3 + 802981773312000q^4 + \dots.
\end{aligned}
\tag{3.60}
$$

The first non-trivial coefficient in F_4 is the famous number $196884 = 196883 + 1$ whose appearance in the J function gave the original hint of a connection between $\mathbb{M}$ and modular functions and ultimately conformal field theory. Moreover, the leading coefficient in H_4 is 1, which is compatible with any assumption about an automorphism group, since it is the dimension of an irreducible representation, namely the trivial one. The contrasts with what we found for $k^* = 3$, where the leading coefficient, namely 95, makes it difficult to postulate a large discrete symmetry.

To explore further the hypothesis of $\mathbb{M}$ symmetry, we attempt to express the coefficients in F_4 and H_4 in terms of dimensions of representations of $\mathbb{M}$. The dimensions $d_1, d_2, \ldots, d_{12}$ of the first 12 monster representations, which we call R_i, $i = 1, \ldots, 12$, are given in the table.

d_1	1
d_2	196883
d_3	21296876
d_4	842609326
d_5	18538750076
d_6	19360062527
d_7	293553734298
d_8	3879214937598
d_9	36173193327999
d_{10}	125510727015275
d_{11}	190292345709543
d_{12}	222879856734249

Table 2. Presented here from [53] are the dimensions d_i of the ith irreducible representation of the monster group $\mathbb{M}$, for $i = 1, \ldots, 12$. We denote as R_i the representation of dimension d_i.

A little experimentation soon shows that the nontrivial coefficients in the NS partition function F_4 can be nicely written in terms of the d_i. The first 8 nontrivial coefficients $f_{1/2}, f_1, \ldots, f_4$ are

$$196884 = d_1 + d_2$$

$$21493760 = d_1 + d_2 + d_3$$

$$864299970 = 2d_1 + 2d_2 + d_3 + d_4$$

$$20246053140 = 3d_1 + 4d_2 + 2d_3 + d_4 + d_6$$

$$333202640600 = 4d_1 + 5d_2 + 3d_3 + 2d_4 + d_5 + d_6 + d_7$$

$$4252023300096 = 5d_1 + 7d_2 + 4d_3 + 4d_4 + 2d_5 + 2d_6 + d_7 + d_8$$

$$44656994071935 = 7d_1 + 11d_2 + 7d_3 + 6d_4 + 3d_5 + 4d_6 + 2d_7 + 2d_8 + d_9$$

$$401490908149760 = 10d_1 + 16d_2 + 12d_3 + 9d_4 + 5d_5 + 7d_6 + 4d_7 + 4d_8 + d_9 + d_{10} + d_{12}.$$

$$(3.61)$$

The coefficients on the right hand side are rather small given the numbers involved, and as we will explain shortly, they do not grow much faster than is required by superconformal symmetry. The first three numbers here equal the first three non-trivial coefficients of the J function, but afterwards the two series diverge.

The coefficients h_n in the q-expansion of H_4 can similarly be expressed in terms of dimensions of monster representations. In doing so, however, the following is a useful shortcut and also possibly a clue to constructing the model. First of all, in contrast to the case of $k^* = 1, 3$, it is not true that $h_n - 2f_n = 0$ for $k^* = 4$ and all $n > 0$. But some of these differences vanish, and they are all relatively small. As we have already expressed the f_n for small n in terms of monster dimensions in (3.61), it suffices now to find similar expressions for $h_n - 2f_n$. Actually, as the h_n are all even for $n > 0$, we prefer to divide by 2, and we find

$$
\begin{aligned}
\frac{h_1}{2} - f_1 &= 0 \\
\frac{h_2}{2} - f_2 &= -f_{1/2} \\
\frac{h_3}{2} - f_3 &= 0 \\
\frac{h_4}{2} - f_4 &= -f_1 \\
\frac{h_5}{2} - f_5 &= 0.
\end{aligned}
\tag{3.62}
$$

When this is combined with (3.61), we see that the $h_i/2$, $i \le 5$ are linear combinations of the d_i with small positive integer coefficients. So the initial coefficients of the Ramond partition function are compatible with $\mathbb{M}$ symmetry, and in addition many of the differences $h_n/2 - f_n$ vanish.

T. Gannon has given a simple explanation of these results by showing that one can express F_4 and H_4 in terms of the J-function via $F_4(\tau) = J(2\tau) + J(\tau/2) + 1$, $H_4(\tau) = J(\tau/2) + J((\tau+1)/2) + 1$. Hence F_4 and H_4 inherit a relation to the monster from J. The fact that the formulas have such a direct explanation may lessen the case for a new SCFT with monster symmetry.

Given that the above formulas exist, they are not quite unique, because there are

some linear relations[17] among the d_i with small coefficients:

$$d_1 + d_4 + d_5 = d_3 + d_6$$
$$d_3 + d_8 + d_{12} = d_2 + d_7 + d_9 + d_{11}. \tag{3.63}$$

We have made some choices to ensure that the above formulas are compatible with super-conformal symmetry, in the following sense. The first formula in (3.61) indicates that at $L_0 = 1/2$, the NS sector has 196883 primary states $|\rho_i\rangle$ transforming in the representation R_2. It follows that at $L_0 = 1$, there are descendants $\mathcal{G}_{-1/2}|\rho_i\rangle$ transforming in the same representation. This continues at higher levels; the number of copies of R_2 appearing as descendants of the primary states $|\rho_i\rangle$ at $L_0 = s/2$ is $1, 1, 1, 2, 3, 4, 5, 7$ for $s = 1, 2, \ldots, 8$. If we count only primary states, we find, assuming that (3.61) is the right decomposition, that primary states in the representation R_2 occur at $L_0 = 1/2, 3/2, 2, 7/2, 4$, each time with multiplicity 1. Similar remarks apply to other representations. A special case is that, for the range of L_0 considered in (3.61), most of the states that transform in the trivial representation R_1 are actually descendants of the identity. If (3.61) is the right decomposition, then the first primary field that is $\mathbb{M}$-invariant is at $L_0 = 5/2$. If we were to rewrite (3.61) in terms of primary states only, the coefficients would be even smaller (but still nonnegative), strengthening the case for monster symmetry.

In this particular example, we cannot add an integer to F_4 without spoiling the hypothesis of monster symmetry. If there were an NS primary with $L_0 = 2$, it would have a descendant at $L_0 = 5/2$, and the number of primaries at $L_0 = 5/2$ would be less than 196883.

In the appendix, we present somewhat similar though less extensive evidence for baby monster symmetry at $k^* = 6$.

Sum Over Spin Structures

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[17] These relations can be explained as follows. The symmetric part of $R_2 \otimes R_2$ decomposes as $\text{Sym}^2 R_2 \cong R_1 \oplus R_2 \oplus R_4 \oplus R_5$, and the antisymmetric part decomposes as $\wedge^2 R_2 \cong R_3 \oplus R_6$. On the other hand, $\dim(\text{Sym}^2 R_2) - \dim(\wedge^2 R_2) = \dim R_2 = d_2$. Taken together, these facts imply the first relation in (3.63), and the second follows similarly by considering the decomposition of $R_3 \otimes R_3$.

Given an SCFT with c an integer multiple[18] of 24, by summing over spin structures, one can make an ordinary bosonic CFT. The sum over spin structures projects both the NS and Ramond sectors onto bosonic states of integer dimension, as is appropriate for a bosonic CFT.

For c to be a multiple of 24, k^* must be even. It was already shown by DGH that the sum over spin structures, applied to their model which in our language is $k^* = 2$, gives the FLM model at $k = 1$. A natural question (raised by J. Duncan) is what this operation does at higher even values of k^*.

The sum over spin structures removes fermionic operators such as the supercurrent $\mathcal{S}$, of dimension 3/2. However, it leaves operators such as $\mathcal{S}\partial\mathcal{S}$ that are even in $\mathcal{S}$. This operator has dimension 4, and is a Virasoro primary (though, of course, it is a descendant in the $\mathcal{N} = 1$ super-Virasoro algebra). A bosonic extremal CFT of $k \geq 4$ should not have a primary of dimension 4. Hence, the "sum over spin structures" operation applied to an extremal SCFT of $k^* \geq 8$ does not give an extremal CFT. However, it does give an extremal CFT if applied to a theory of $k^* = 4$ or 6.

Hence, if extremal SCFT's exist with $k^* = 4, 6$, they can be used to generate extremal CFT's of $k = 2, 3$. It is interesting that $k^* = 4, 6$ are precisely the values at which extremal SCFT's may have monster or baby monster symmetry. (The baby monster group is the centralizer of an involution in the monster, and hence it is conceivable for an SCFT at $k^* = 6$ to have baby monster symmetry while the CFT at $k = 3$ has monster symmetry. The framework for this is described in [26].)

Part of what we have said can be understood in another way. Supergravity and gravity are different in the semiclassical limit, and one cannot be obtained from the other by merely summing over spin structures. So whatever is the SCFT dual of supergravity (even if our assumptions are too optimistic and it is not extremal), the sum over spin structures applied to this SCFT cannot give, for arbitrarily large k^*, the CFT that is dual to gravity.

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[18] If c is an odd multiple of 12, the sum over spin structures is modular-invariant but projects out the NS ground state and does not give a CFT in the usual sense. This is actually important in superstring theory, where, with $c = 12$ in the light-cone treatment, the NS ground state is a "tachyon" and is removed in the sum over spin structures.

4. $k = 2$ **Partition Function On A Hyperelliptic Riemann Surface**

The most important question raised by this paper is certainly whether appropriate CFT's and SCFT's exist, beyond the few examples that are known.

In this section, we will perform a small computation that aims to give a hint that this is the case, at least for the bosonic theory with $k = 2$ and hence $c = 48$. We will show that the partition function of such a theory on a hyperelliptic Riemann surface of any genus can be determined in a unique and consistent way. This includes, for example, any Riemann surface of genus 2. The fact that we get a unique and consistent result in this situation gives some encouragement for believing that the $k = 2$ model does exist and is unique.

In this paper, we merely demonstrate the consistency of an algorithm for determining the partition function. Hopefully it will be possible in future work to get explicit formulas, at least for genus 2. Our method also works for $k = 1$, though here the ability in principle to determine the partition function comes as no surprise, since the model has been constructed explicitly [24] and in fact the genus 2 partition function has been computed [54] by a quite different method.

Perhaps it would help orient the reader to compare what we will do to another possible approach to determining the partition function. Genus 2 partition functions were determined in [54] for a variety of holomorphic CFT's with $c = 24$. The basic method was to express the partition function in terms of a Siegel modular form, which depends on only finitely many coefficients, and determine the coefficients by considering the behavior when the Riemann surface C degenerates. There are two types of degeneration (C can break up into two genus 1 curves joined at a point, or can reduce to a single genus 1 curve with two points glued together). The partition function can be determined from the behavior at just one degeneration, and there is a problem of consistency to show that one gets the same result either way. One could attempt to demonstrate the consistency by studying the appropriate Siegel modular forms, but we prefer to prove consistency by establishing the associativity of a certain operator product algebra. One advantage is that this method can be used to determine the partition function on a hyperelliptic Riemann surface of any genus. On the other hand, if one could overcome the technicalities in genus 2 (see [55] for one approach), one could possibly compute the genus 2 partition function of an extremal CFT for all k.

4.1. Twist Fields

A hyperelliptic Riemann surface C is a double cover of the complex plane, for example a double cover of the complex x-plane, which we will call C_0, described by an equation

$$y^2 = \prod_{i=1}^{2g+2} (x - e_i). \tag{4.1}$$

The $2:1$ cover $C \to C_0$ is branched at the points $e_1, \ldots, e_{2g+2}$. C is smooth if the e_i are distinct, and has genus g if the number of branch points is precisely $2g + 2$.

Our approach to determining the partition function on such a Riemann surface is based on an old idea [56-59]. The partition function of a conformal field theory $\mathcal{W}$ on the hyperelliptic Riemann surface C can be determined by computing, in a doubled theory, the correlation function of $2g+2$ copies of a "twist field" $\mathcal{E}$, inserted at the points $e_1, \ldots, e_{2g-2}$ in C_0.

Consider any CFT $\mathcal{W}$ of central charge c. Away from branch points, the theory $\mathcal{W}$ on the double cover C looks locally like the theory $\mathcal{W} \times \mathcal{W}$ on C_0. Here we have one copy of $\mathcal{W}$ for each of the two branches of $C \to C_0$.

In going around a branch point, the two copies of $\mathcal{W}$ are exchanged. So a more complete description is to say that the theory on C_0 is $\mathrm{Sym}^2\mathcal{W}$, the symmetric product of two copies of $\mathcal{W}$. The symmetric product theory is an orbifold of the product theory $\mathcal{W} \times \mathcal{W}$ in which one divides by the $\mathbb{Z}_2$ symmetry that exchanges the two branches.

Let us describe what kind of operators the symmetric product theory has. First, there are "untwisted" operators. These are simply operators of the theory $\mathcal{W} \times \mathcal{W}$ that are invariant under the exchange of the two factors. For example, let T^+ and T^- be the stress tensors of the two factors. Any local operator constructed as a polynomial in T^+ and T^- and their derivatives and invariant under the exchange $T^+ \leftrightarrow T^-$ gives an operator in the symmetric product theory. Operators of this particular type generate a chiral algebra that we will call $\mathrm{Sym}^2\mathcal{V}$, where $\mathcal{V}$ denotes the chiral algebra generated by a single stress tensor.

In addition, there are "twisted sector" operators. These operators correspond (in the operator/state correspondence of CFT) to states obtained by quantizing the theory $\mathcal{W} \times \mathcal{W}$ on a circle S^1, in such a way that the two copies of $\mathcal{W}$ are exchanged in monodromy around the circle. These states are the states of a *single* copy of $\mathcal{W}$ on a circle of twice the circumference. The energies of the states in the twisted sector of the $\mathrm{Sym}^2\mathcal{W}$ theory are therefore precisely one-half the energies of the original $\mathcal{W}$ theory (in conformal field theory, doubling the circumference of the circle divides the Hamiltonian by two).

Operators related to states in the twisted sector are called twist fields. The dimension of a twist field is the same as the difference in energy between the corresponding twisted sector state and the untwisted ground state. The ground state energy in the untwisted sector is $-2 \cdot c/24$ (where the 2 comes from the two copies of $\mathcal{W}$) and the ground state energy in the twisted sector is $-(1/2) \cdot c/24$ (where the factor of $1/2$ was explained in the last paragraph). The difference is $d_{\mathcal{E}} = (c/24)(-1/2 - (-2)) = (3/2)c/24$, and this is the dimension of the twist field $\mathcal{E}$ of lowest energy. For our application, we are interested in the case that $c/24$ is an integer k, and hence $d_{\mathcal{E}} = 3k/2$.

Now we can explain how the partition function of the theory $\mathcal{W}$ on the hyperelliptic Riemann surface C defined in eqn. (4.1) can be interpreted as a genus zero correlation function in the theory $\mathrm{Sym}^2\mathcal{W}$. From the standpoint of the $\mathrm{Sym}^2\mathcal{W}$ theory on the x-plane, the role of the branch points is just to exchange the two copies of $\mathcal{W}$, via a twist field. On the double cover, there is no operator insertion at the branch points except the identity; the identity operator at a branch point corresponds in the downstairs description to the ground state in the twisted sector and hence to the twist operator $\mathcal{E}$ of lowest dimension. So the partition function on the double cover can be expressed in terms of the correlation function $\langle \mathcal{E}(e_1)\mathcal{E}(e_2)\ldots\mathcal{E}(e_{2g+2}) \rangle$ on the x-plane. (The precise statement involves the conformal anomaly; this is deferred to section 4.5.)

To compute this correlation function, we need to understand the chiral algebra that is obtained by extending the symmetric product $\mathrm{Sym}^2\mathcal{V}$ of two Virasoro algebras by the field $\mathcal{E}$. For this, we need to know what primary fields (of $\mathrm{Sym}^2\mathcal{V}$) appear in the operator product $\mathcal{E} \cdot \mathcal{E}$. To calculate the product $\mathcal{E} \cdot \mathcal{E}$, we need to look at the behavior when a pair of branch points approach each other. This situation is described locally by an equation

$$y^2 = (x - e)(x - e'), \tag{4.2}$$

and we are interested in the behavior for $e \to e'$. In that limit, the equation becomes $y^2 = (x - e)^2$, and the surface breaks up into two branches $C_{\pm} : y = \pm(x - e)$. Each branch is a copy of the complex plane. We can express the product $\mathcal{E}(e) \cdot \mathcal{E}(e')$ for $e \to e'$ as a sum of operators of the form $\mathcal{U}_+ \otimes \mathcal{U}_-$, where $\mathcal{U}_{\pm}$ is an operator in the original CFT $\mathcal{W}$ on the branch $C_{\pm}$. Moreover, the $\mathcal{U}_{\pm}$ are descendants of some Virasoro primary fields $\mathcal{O}_{\pm}$.
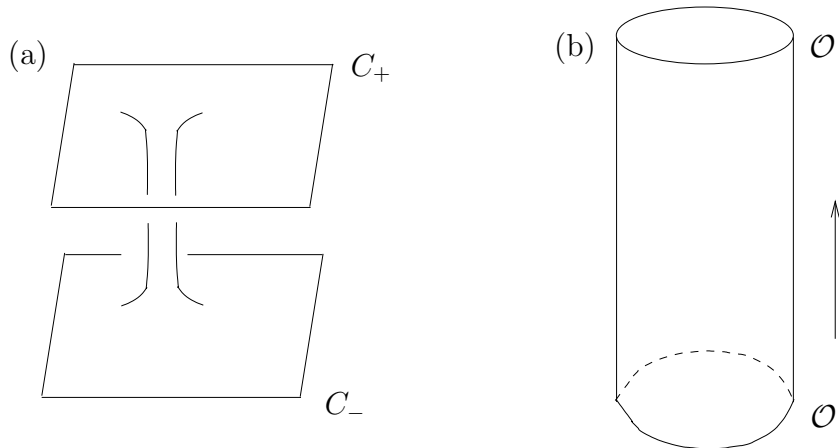
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Fig. 5:

(a) The equation $y^2 = (x - e)(x - e')$ describes two branches connected by a tube. The tube collapses to a point as $e \to e'$. (b) Topologically, the tube is equivalent to a cylinder. Propagation along the tube multiplies any energy eigenstate by a c-number factor. If a primary field $\mathcal{O}$ flows in from the past, the same field flows out in the future. Alternatively, if we think of all fields as flowing out, then the two fields emerging at the two ends of the cylinder are conjugate.

The decomposition of $\mathcal{E}(e) \cdot \mathcal{E}(e')$ as a sum of descendants of operators $\mathcal{O}_+ \otimes \mathcal{O}_-$ can be made in such a way that the operator $\mathcal{O}_+$ is conjugate to $\mathcal{O}_-$, meaning that the two-point function $\langle \mathcal{O}_+ \mathcal{O}_- \rangle$ (in the original theory $\mathcal{W}$) is nonzero in genus zero. To see this, go back to the smooth double cover with $e \neq e'$. For e near e', the two branches are smoothly connected by a tube (fig. 5a). The surface thus has the topology of a cylinder (fig. 5b). Propagation along this cylinder is the "identity operator" on primary fields. So if a given primary state flows in at the bottom, it or flows out at the top. When we write $\mathcal{E}(e) \cdot \mathcal{E}(e')$ as a sum of descendants of operators $\mathcal{O}_+ \otimes \mathcal{O}_-$, the convention is that both $\mathcal{O}_+$ and $\mathcal{O}_-$ are outgoing, so they are conjugate, rather than being equal. At any rate, the important conclusion for us is that $\mathcal{O}_+$ and $\mathcal{O}_-$ have the same dimension.

According to our interpretation of three-dimensional gravity, it is described by a conformal field theory with $c = 24k$ and no primary field other than the identity of dimension less than $k + 1$. Hence, a primary operator $\mathcal{O}_+ \otimes \mathcal{O}_-$ of the symmetric product theory, where $\mathcal{O}_+$ and $\mathcal{O}_-$ have the same dimension, is either the identity operator or has dimension at least $2(k + 1)$.

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To understand the chiral algebra or superalgebra[19] generated by $\mathcal{E}$ together with $\mathrm{Sym}^2\mathcal{V}$, we need to know all primary fields that appear with singular coefficients in the operator product $\mathcal{E}(e)\cdot\mathcal{E}(e')$. Since $\mathcal{E}$ has dimension $3k/2$, an operator that will contribute a singularity must have dimension less than $3k$ and hence at most $3k-1$. On the other hand, we have just seen that all operators appearing in the OPE are either descendants of the identity or have dimension at least $2(k+1)$.

For $k=1$ or 2, we have $3k-1 < 2(k+1)$. This leads to a drastic simplification, which is the reason that we will restrict ourselves here to $k=1,2$: singularities in the product $\mathcal{E}(e)\cdot\mathcal{E}(e')$ come only from the identity operator and its $\mathrm{Sym}^2\mathcal{V}$ descendants. Therefore, to understand correlation functions of $\mathcal{E}$, we need only understand the chiral algebra that is obtained by adding to $\mathrm{Sym}^2\mathcal{V}$ the primary field $\mathcal{E}$ of dimension $3k/2$ with the OPE schematically

$$\mathcal{E}\cdot\mathcal{E}\sim 1 + \text{descendants}. \tag{4.3}$$

This is a chiral algebra that can be described in closed form and explicitly proved to obey the Jacobi identity. The only $\mathrm{Sym}^2\mathcal{V}$ primaries are 1 and $\mathcal{E}$. The only genus zero three point functions of primaries are $\langle 1 \cdot 1 \cdot 1\rangle$, which is trivial, and $\langle 1\cdot\mathcal{E}\cdot\mathcal{E}\rangle$, which is almost trivial in the sense that it reduces to a two-point function. To understand correlation functions of descendants, it suffices to understand the module for $\mathrm{Sym}^2\mathcal{V}$ consisting of $\mathcal{E}$ and its descendants. One way to construct this module explicitly is to observe that on the double cover of the x-plane, $\mathcal{E}$ simply corresponds to the identity operator and $\mathrm{Sym}^2\mathcal{V}$ to the ordinary Virasoro algebra; so this module for $\mathrm{Sym}^2\mathcal{V}$ can be deduced from the identity module for $\mathcal{V}$.

To get beyond the three-point function, the key step is to show that the chiral algebra obtained by adding $\mathcal{E}$ to $\mathrm{Sym}^2\mathcal{V}$ is consistent, that is, that the Jacobi identity is obeyed. If so, this chiral algebra determines all genus zero correlation functions of $\mathcal{E}$, and hence determines the partition function of our CFT on a general hyperelliptic Riemann surface.

In such a problem of trying to extend a known chiral algebra $\mathrm{Sym}^2\mathcal{V}$ by additional primary fields $\mathcal{E}$ (of integer or half-integer dimension, in which case one will get an extended chiral algebra or a superalgebra), the Jacobi identity is equivalent [60] to the statement that there exists a four-point function of the additional primary fields that has the appropriate singularities determined by the operator product expansion in all channels. (Such a

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[19] As $\mathcal{E}$ has dimension $3k/2$, it is a fermionic operator if k is odd.

function automatically has the right symmetries.) In the case at hand, since the only primary field that we are trying to add is $\mathcal{E}$, the only non-trivial primary four-point function that we have to consider is $\langle\mathcal{E}(e_1)\mathcal{E}(e_2)\mathcal{E}(e_3)\mathcal{E}(e_4)\rangle$. In section 4.3, we explicitly construct this four-point function and show that it behaves correctly in all channels.

In the following sense, this result is not at all surprising. The four-point function $\langle\mathcal{E}(e_1)\mathcal{E}(e_2)\mathcal{E}(e_3)\mathcal{E}(e_4)\rangle$ is essentially the partition function of the underlying CFT on the hyperelliptic Riemann surface $y^2 = (x-e_1)(x-e_2)(x-e_3)(x-e_4)$. That surface has genus 1, and we already know from section 3 that for every k there is a unique, natural genus 1 partition function with the properties we want. In section 4.4, we show explicitly, for $k = 1, 2$, that the four-point function $\langle\mathcal{E}(e_1)\mathcal{E}(e_2)\mathcal{E}(e_3)\mathcal{E}(e_4)\rangle$ determined from the OPE's agrees with the genus 1 partition function as described in section 3.

4.2. The $\mathcal{E}\cdot\mathcal{E}$ Operator Product

As a first step in that direction, we will calculate the details of the $\mathcal{E}\cdot\mathcal{E}$ operator product. For this, we start with a double cover C of the x-plane branched at e and $e' = -e$, and so described by an equation $y^2 = x^2 - e^2$. If $u = x + y$, $v = x - y$, then the equation is $uv = e^2$. The two branches C_+ and C_- correspond respectively to $u \to \infty$, $v \to 0$ and $u \to 0$, $v \to \infty$.

The path integral over C gives a quantum state Ψ in the theory $\mathcal{W}\times\mathcal{W}$, that is, one copy of $\mathcal{W}$ for each branch. This state is invariant under exchange of the two branches by the symmetry $y \to -y$, so it is really a state in the symmetric product theory $\mathrm{Sym}^2\mathcal{W}$. We are really only interested in the part of Ψ proportional to the vacuum state and its descendants. As we have seen, this part suffices to describe the desired chiral algebra if $k \le 2$.

We will determine Ψ by using the fact that certain elements in the product $\mathcal{V}\times\mathcal{V}$ of two Virasoro algebras annihilate Ψ. This is so because there are globally-defined holomorphic vector fields on C, of the form $V_n = 2^{-n}u^{n+1}d/du = -2^{-n}(e^2/v)^n vd/dv$. Let $\mathcal{S}$ be a contour on the surface C that wraps once around the "hole." If T is the stress tensor, the contour integral $\int_{\mathcal{S}} V_nT$ can be regarded, for any n, as an operator acting on the state Ψ. This operator is invariant under deformation of the contour. It can be deformed (fig. 6) to a contour $\mathcal{S}_+$ in the upper branch C_+ or a contour $\mathcal{S}_-$ in the lower branch C_-. So we have for all n

$$\left(\int_{\mathcal{S}_+} V_nT - \int_{\mathcal{S}_-} V_nT\right)\Psi = 0. \tag{4.4}$$
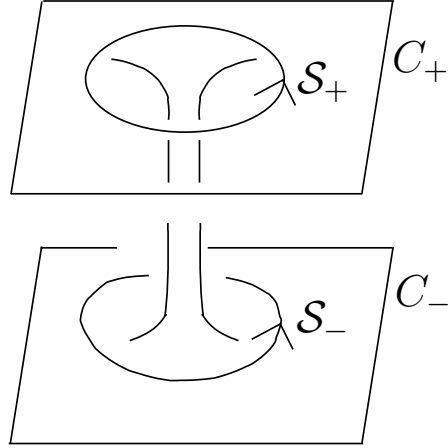
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Fig. 6:

A contour $\mathcal{S}$ that wraps once around the hole can be deformed to the contour $\mathcal{S}_+$ in the upper branch or to the contour $\mathcal{S}_-$ in the lower branch.

We want to express the two contour integrals that appear here in terms of Virasoro generators on the two branches. To do this, we simply express V_n as a vector field on the branch C_+ or C_-, either of which we identify with the x-plane. For example, on the branch C_+, we write explicitly $y = \sqrt{1 - e^2/x^2} = 1 - e^2/2x^2 - e^4/8x^4 + \mathcal{O}(e^6)$. We have carried the expansion far enough to determine (for $k \leq 2$) all singular terms in the product $\mathcal{E}(e) \cdot \mathcal{E}(-e)$. So

$$V_n = x^{n+1}\left(1 - (n+2)\frac{e^2}{4x^2} + \frac{e^4}{x^4}\left(\frac{n^2 + n - 4}{32}\right) + \mathcal{O}(e^6)\right)\frac{d}{dx}. \qquad (4.5)$$

This means, if we ignore the conformal anomaly for the moment, that $\int_{\mathcal{S}_+} V_n T$ corresponds, on the branch C_+, to the operator

$$Q_n^+ = L_n^+ - \frac{n+2}{4}e^2 L_{n-2}^+ + \left(\frac{n^2 + n - 4}{32}\right)e^4 L_{n-4}^+ + \mathcal{O}(e^6). \qquad (4.6)$$

As a check, one can verify that $[Q_n^+, Q_m^+] = (n-m)Q_{n+m}^+$. Similarly, $\int_{\mathcal{S}_-} V_n T$ corresponds on the branch C_- to the operator

$$Q_n^- = \left(\frac{e^2}{4}\right)^n\left(L_{-n}^- - e^2\left(\frac{-n+2}{4}\right)L_{-n-2}^- + e^4\left(\frac{n^2 - n - 4}{32}\right)L_{-n-4}^- + \mathcal{O}(e^6)\right). \qquad (4.7)$$

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The state Ψ is determined for each value of e (up to multiplication by a complex scalar) by the condition that $\widehat{Q}_n \Psi = 0$, where $\widehat{Q}_n = Q_n^+ - Q_n^-$. However, because of the Virasoro anomaly some c-number terms must be added to the above formulas, reflecting the conformal anomaly in the mapping from u to x. There is no such c-number contribution to $\widehat{Q}_0$, since it cancels between the two branches. The c-numbers in $\widehat{Q}_n$ for other n can be conveniently determined by requiring that $[\widehat{Q}_n, \widehat{Q}_m] = (n - m)\widehat{Q}_{n+m}$. For our purposes, the only formulas we need are

$$\widehat{Q}_0 = \left(L_0^+ - \frac{e^2}{2}L_{-2}^+ - \frac{e^4}{8}L_{-4}^-\right) - \left(L_0^+ - \frac{e^2}{2}L_{-2}^- - \frac{e^4}{8}L_{-4}^-\right) + \dots$$

$$\widehat{Q}_1 = \left(L_1^+ - \frac{3e^2}{4}L_{-1}^+ - \frac{e^4}{16}L_{-3}^+\right) - \frac{e^2}{4}\left(L_{-1}^- - \frac{e^2}{4}L_{-3}^-\right) + \dots \tag{4.8}$$

$$\widehat{Q}_2 = L_2^+ - e^2 L_0^+ - 3ke^2 + \frac{e^4}{16}L_{-2}^+ - \frac{e^4}{16}L_{-2}^- + \dots.$$

Terms of order e^6 have been omitted. The constant in $\widehat{Q}_2$ was obtained from $[\widehat{Q}_0, \widehat{Q}_2] = -2\widehat{Q}_2$.

By requiring that $\widehat{Q}_m \Psi = 0$ for $m = 0, 1, 2$, and that Ψ converges to the Fock vacuum $|\Omega\rangle$ for $e \to 0$, we now find Ψ to be

$$\Psi(e) = \left(1 + \frac{e^2}{4}(L_{-2}^+ + L_{-2}^-) + \frac{e^4}{32}(L_{-4}^+ + L_{-4}^-)\right.$$
$$\left. + \frac{e^4}{32}(L_{-2}^+ + L_{-2}^-)^2 + \frac{e^4}{192k}L_{-2}^+ L_{-2}^- + \dots\right) |\Omega\rangle. \tag{4.9}$$

We can immediately use this formula to determine the singular part of the $\mathcal{E} \cdot \mathcal{E}$ OPE. We normalize $\mathcal{E}$ so that the most singular term is $\mathcal{E}(x/2)\mathcal{E}(-x/2) \sim 1/x^{3k}$. Then, setting $e = x/2$, we get

$$\mathcal{E}(x/2)\mathcal{E}(-x/2) \sim \frac{1}{x^{3k}}\left(1 + \frac{x^2}{16}T + \frac{x^4}{2^{10}}\partial^2 T\right.$$
$$\left. + \frac{x^4}{2^9}T \star T + \frac{x^4}{3k \cdot 2^{10}}T^+ T^-\right) + \mathcal{O}(x^{-3k+6}). \tag{4.10}$$

All we have done is to write, on the right hand side, the operator that corresponds to the state $\Psi(e)/x^{3k}$, for $e = x/2$. (The factor of $1/x^{3k}$ must be supplied by hand, since in defining $\Psi(e)$, we just normalized it so that the coefficient of the Fock vacuum is 1.) Also, T^+ and T^- are the stress tensors on the branches C_+ and C_-, respectively. $T = T^+ + T^-$ is the diagonal or total stress tensor, and similarly $L_n = L_n^+ + L_n^-$. In addition, we use the fact that in the correspondence between operators and states, the states $L_{-2}^\pm|\Omega\rangle$ correspond to the operators $T^\pm$, while $L_{-4}^\pm|\Omega\rangle$ correspond to $\frac{1}{2}\partial^2 T^\pm$. Finally, we have written $T \star T$ for the operator corresponding to the state $L_{-2}^2|\Omega\rangle$. $T \star T(0)$ can be obtained as the term of order x^0 in the operator product $T(x) \cdot T(0)$.

4.3. Determining The Four-Point Function

Our goal is to use the explicit formula (4.10) to show that the chiral algebra obtained by adjoining $\mathcal{E}$ to $\mathrm{Sym}^2\mathcal{V}$ obeys the Jacobi identity. The Jacobi identity is equivalent to the existence of a four-point function for primary fields with the right symmetry properties and the right OPE singularities in all channels. It is enough to consider primary fields with respect to the $\mathrm{Sym}^2\mathcal{V}$ algebra, which we already know to exist. The only non-trivial case is the four-point function of $\mathcal{E}$.

For $k = 1$, the singular part of the $\mathcal{E} \cdot \mathcal{E}$ operator product only involves the diagonal stress tensor $T = T^+ + T^-$. This is a substantial simplification; the extended chiral super-algebra obtained by incorporating $\mathcal{E}$ can be viewed an extension of an ordinary Virasoro algebra $\mathcal{V}$, rather than a symmetric product $\mathrm{Sym}^2\mathcal{V}$. In fact, this algebra is a familiar one, an $\mathcal{N} = 1$ super Virasoro algebra:

$$\mathcal{E}(x/2)\mathcal{E}(-x/2) \sim \frac{1}{x^3}\left(1 + \frac{x^2}{16}T\right) + \mathcal{O}(x). \tag{4.11}$$

The stress tensor T has $c = 48$ (as it is the diagonal stress tensor in the symmetric product of two copies of a $c = 24$ theory). A superconformal field theory with this central charge can be constructed from 32 chiral multiplets consisting of bosonic and fermionic fields ϕ_i and ψ_i, where $\langle \phi_i(x)\phi_j(0)\rangle = -\delta_{ij}\ln(x)$, and $\langle \psi_i(x)\psi_j(0)\rangle = \delta_{ij}/x$, for $i, j = 1, \ldots, 32$. In terms of these fields, we can take

$$\mathcal{E} = \frac{i}{4\sqrt{2}}\sum_{k=1}^{32}\psi_k\partial\phi_k. \tag{4.12}$$

With this free field realization, all correlation functions of $\mathcal{E}$ can be explicitly computed. For example, the four-point function is

$$\langle\mathcal{E}(e_1)\mathcal{E}(e_2)\mathcal{E}(e_3)\mathcal{E}(e_4)\rangle = \prod_{i=1}^{4}(de_i)^{3/2}\left(\left(\frac{1}{(e_1 - e_2)^3(e_3 - e_4)^3} + \mathrm{cyclic}\right)\right.$$
$$\left. + \frac{3}{32}\frac{1}{\prod_{i<j}(e_i - e_j)}\right), \tag{4.13}$$

where additional terms obtained by cyclic permutations of e_2, e_3, e_4 are to be added in the first term on the right. We will eventually compare this formula to the known genus 1 partition function of the monster theory of $k = 1$.

In (4.13), we included a factor of $(de)^{3/2}$ for each operator $\mathcal{E}(e)$ to reflect the fact that $\mathcal{E}$ has dimension $3/2$. Thus the correlation "function" $\langle \mathcal{E}(e_1)\mathcal{E}(e_2)\mathcal{E}(e_3)\mathcal{E}(e_4)\rangle$ is most naturally understood as a $3/2$-differential in each variable. This factor is often omitted in writing such formulas, and we will do so as an abbreviation except when the factor is important.

For $k = 2$, the singular part of $\mathcal{E} \cdot \mathcal{E}$ cannot be expressed in terms of the diagonal stress tensor T only, and there is no way to avoid using $\mathrm{Sym}^2\mathcal{V}$. Correspondingly, the chiral algebra for $k = 2$ seems to be unfamiliar, and it is considerably harder to get the analog of (4.13) for $k = 2$. Since we do not have an explicit realization of the algebra analogous to (4.12), we cannot compute the four-point function directly, and instead we will determine it by requiring the appropriate OPE singularities.

The function $\langle \mathcal{E}(e_1)\mathcal{E}(e_2)\mathcal{E}(e_3)\mathcal{E}(e_4)\rangle$, with $e_1, \ldots, e_4 \in C_0 \cong \mathbb{CP}^1$, must be symmetric in all arguments and must have the appropriate OPE singularities in all channels. In addition, it is highly constrained by invariance under the action of $SL(2, \mathbb{C})$ on C_0. The most general "function," actually a cubic differential in each variable, that is symmetric, $SL(2, \mathbb{C})$-invariant, and consistent with at least the leading OPE singularity, is

$$\langle \mathcal{E}(e_1)\mathcal{E}(e_2)\mathcal{E}(e_3)\mathcal{E}(e_4)\rangle = \prod_{i=1}^{4}(de_i)^3 \left(\left(\frac{1}{(e_1-e_2)^6(e_3-e_4)^6} + \text{cyclic}\right)\right.$$
$$+ A\left(\frac{1}{(e_1-e_2)^4(e_3-e_4)^4}\prod_{i=1,2,\,j=3,4}\frac{1}{e_i-e_j} + \text{cyclic}\right) \quad (4.14)$$
$$\left. + B\frac{1}{\prod_{i<j}(e_i-e_j)^2}\cdot\right).$$

Cyclic permutations of e_2, e_3, e_4 are to be added as shown, and the coefficients A and B must be determined to ensure that the OPE singularities are correct.

After a detailed calculation, one finds that the right values are

$$A = \frac{3}{16}, \ B = \frac{12k+1}{2^{16}} \quad (4.15)$$

where of course $k = 2$. To get these formulas, one may for instance set $e_1 = -e_2 = e$, and use (4.10) to determine the singular behavior for $e \to 0$. In this way, one expresses the singular behavior of the four-point function in terms of three-point functions $\langle \mathcal{X}(0)\mathcal{E}(e_3)\mathcal{E}(e_4)\rangle$, where $\mathcal{X}$ is one of the descendants of the identity that appear in (4.10).

The coefficient A appears in the coefficient of $1/e^4$ for $e \to 0$, and in determining it, the only important choices of $\mathcal{X}$ are 1 and T. The three-point functions that we need are

$$\langle 1 \cdot \mathcal{E}(z)\mathcal{E}(w) \rangle = \frac{1}{(z-w)^6}$$

$$\langle T(x)\mathcal{E}(z)\mathcal{E}(w) \rangle = \frac{3}{(z-x)^2(w-x)^2(z-w)^4}. \tag{4.16}$$

The first is immediate from the way we have normalized $\mathcal{E}$. The second, since the three fields involved are quasi-primary fields (they transform as primaries under $SL(2,\mathbb{C})$) is determined by $SL(2,\mathbb{C})$ invariance up to a constant multiple. The constant can be determined using the $T \cdot \mathcal{E}$ OPE (which reflects the fact that $\mathcal{E}$ is a primary of dimension $3 = 3k/2$) or the $\mathcal{E} \cdot \mathcal{E}$ OPE.

To determine B, we need three more three-point functions, namely

$$\langle \partial^2 T(0)\mathcal{E}(z)\mathcal{E}(w) \rangle = \left(\frac{1}{z^2} + \frac{1}{w^2} \right) \frac{18}{z^2 w^2 (z-w)^4} + \frac{24}{z^3 w^3 (z-w)^4}$$

$$\langle T \star T(0)\mathcal{E}(z)\mathcal{E}(w) \rangle = \left(\frac{5}{z^2} + \frac{5}{w^2} - \frac{4}{zw} \right) \frac{3}{z^2 w^2 (z-w)^4} \tag{4.17}$$

$$\langle T^+ T^-(0)\mathcal{E}(z)\mathcal{E}(w) \rangle = \left(\frac{9k^2}{16} + \frac{3k}{64} \right) \frac{1}{z^4 w^4 (z-w)^2}.$$

The first of these results is obtained simply by differentiating the second formula in (4.16).

To get the second, we use the fact that

$$T \star T(0) = \operatorname{Res}_{x=0} \frac{dx\, T(x)}{x} T(0). \tag{4.18}$$

Now consider the expression

$$\left\langle \frac{dx\, T(x)}{x} T(0)\mathcal{E}(z)\mathcal{E}(w) \right\rangle. \tag{4.19}$$

We view this as a differential form on the complex x-plane, with z and w kept fixed. As such, it has poles precisely at $x = 0, z, w$. The residue at $x = 0$ is the desired three-point function $\langle T \star T(0)\mathcal{E}(z)\mathcal{E}(w) \rangle$. The residues at at $x = z, w$ can be computed in terms of the correlator $\langle T(0)\mathcal{E}(z)\mathcal{E}(w) \rangle$ by using the fact that $\mathcal{E}$ is a primary of dimension 3, which determines the singular behavior of the product $T \cdot \mathcal{E}$. The sum of the residues must vanish, and this gives our result.

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Finally, to get the last formula in (4.17), we note first that since the operators T^+T^- and $\mathcal{E}$ are quasiprimary fields of dimension 4 and 3, we have

$$\langle T^+T^-(x)\mathcal{E}(z)\mathcal{E}(w)\rangle = \frac{\theta}{(x-z)^4(x-w)^4(z-w)^2},\tag{4.20}$$

for some constant θ. We have

$$\lim_{w\to\infty} \frac{\langle T^+T^-(x)\mathcal{E}(0)\mathcal{E}(w)\rangle}{\langle \mathcal{E}(0)\mathcal{E}(w)\rangle} = \frac{\theta}{x^4}.\tag{4.21}$$

This is supposed to be computed in the theory $\mathrm{Sym}^2\mathcal{W}$ on the x-plane, with insertions of $\mathcal{E}$ at $0, w$. This is equivalent to a computation in the theory $\mathcal{W}$ on a double cover of the x-plane branched at $0, w$. In the limit $w \to \infty$, the double cover is described by the equation $y^2 = x$, and it is convenient to transfer the computation to the y-plane.

We will now write the stress tensor explicitly as T_{xx} or T_{yy} depending on which local parameter is used in defining it. Allowing for the conformal anomaly, the relation between the stress tensor T_{xx} on the x-plane and the corresponding stress tensor T_{yy} on the y-plane is

$$\left(\frac{\partial y}{\partial x}\right)^2 T_{yy} = T_{xx} - \frac{c}{12}\{y, x\},\tag{4.22}$$

where $\{y, x\}$ denotes the Schwarzian derivative of these two functions. With $y^2 = x$, we get $\{y, x\} = 3/8x^2$. In the case of interest, $c = 24k$ so

$$T_{xx} = \frac{1}{4x}T_{yy} + \frac{3k}{4x^2}.\tag{4.23}$$

This formula enables us to express T^+ in terms of $T_{yy}|_{y=\sqrt{x}}$ and T^- in terms of $T_{yy}|_{y=-\sqrt{x}}$. The limit in (4.21) becomes

$$\left\langle \left(\frac{1}{4x}T_{yy}|_{y=\sqrt{x}} + \frac{3k}{4x^2}\right)\left(\frac{1}{4x}T_{yy}|_{y=-\sqrt{x}} + \frac{3k}{4x^2}\right)\right\rangle.\tag{4.24}$$

(The idea is that passing to the y-plane eliminates the twist operators $\mathcal{E}$ from the formalism, and replaces $T^\pm(x)$ by a stress tensor evaluated at $y = \pm\sqrt{x}$.) Using the fact that in conformal field theory on the y-plane, $\langle T_{yy}\rangle = 0$ and $\langle T_{yy}(y_1)T_{yy}(y_2)\rangle = c/2(y_1 - y_2)^4$ with $c = 24k$, the correlation function in (4.24) can be evaluated to give θ/x^4 with $\theta = 9k^2/16 + 3k/64$. This gives the last formula in (4.17).

With the aid of the three-point functions in (4.16) and (4.17) and the operator product formula (4.10), one can verify that the candidate four-point function (4.14) has the right singularities precisely if the coefficients A and B are as claimed in (4.15).

Our derivation has in fact been somewhat redundant, since the formulas for three-point functions can largely be deduced from the formula for the $\mathcal{E} \cdot \mathcal{E}$ operator product, and vice-versa.

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4.4. The Genus One Partition Function Revisited

We will now explicitly use our formulas for the four-point function of the twist field $\mathcal{E}$ to recover the partition functions of a $k = 1$ or $k = 2$ extremal CFT on a Riemann surface of genus 1.

First we must describe the necessary formalism more precisely. The partition function $Z_C(e_1, e_2, \ldots, e_{2g+2})$ of any conformal field theory $\mathcal{W}$ on the hyperelliptic Riemann surface C defined by

$$y^2 = \prod_{i=1}^{2g+2} (x - e_i) \tag{4.25}$$

can indeed be expressed in terms of the genus zero twist field correlation function

$$\langle \mathcal{E}(e_1) \mathcal{E}(e_2) \ldots \mathcal{E}(e_{2g+2}) \rangle. \tag{4.26}$$

However, to do so requires an important detail that we have not yet described.

There are at first sight several strange features in relating Z_C to the twist field correlation function:

(1) The dimension of the twist field $\mathcal{E}$ seems to be wrong. The Riemann surface C degenerates as $e_i \to e_j$ for any i and j. As the complex structure of C is invariant under the exchange $e_i \leftrightarrow e_j$, a natural local parameter near the degeneration locus on the moduli space is actually $w = (e_i - e_j)^2$. (For example, this statement can be made precise using eqn. (4.30) below.) One expects Z_C to behave for $e_i \to e_j$ as w^{-k}, since the ground state energy is $-k$. However, as $\mathcal{E}$ has dimension $3k/2$, the behavior of the twist field correlation function as $e_i \to e_j$ is actually $(e_i - e_j)^{-3k} \sim w^{-3k/2}$.

(2) This brings in relief the fact that if k is odd, $\mathcal{E}$ actually has half-integral dimension and so is fermionic. For odd k, the twist field correlation function (4.26) actually changes sign under exchange of e_i and e_j, while Z_C is invariant under this exchange.

(3) The twist field correlation function is most naturally understood as a $3k/2$-differential in each variable. We have written the formulas (4.13) and (4.14) in such a way as to emphasize this. It may not be immediately clear what this means for the partition function Z_C.

The answer to all of these questions has to do with the proper treatment of the conformal anomaly in the holomorphic context. We postpone a full explanation to section

4.5 and for now we merely write down the precise relation between the genus 1 partition function and the twist field correlation function:

$$Z_C(e_1, e_2, \ldots, e_4) = 2^{8k} \left(\prod_{1 \leq i < j \leq 4} (e_i - e_j)^k \right) \langle \mathcal{E}(e_1)\mathcal{E}(e_2)\ldots\mathcal{E}(e_4) \rangle. \qquad (4.27)$$

The factor of 2^{8k} just reflects the way we have normalized $\mathcal{E}$. The factor of $\prod_{i<j}(e_i - e_j)^k$ resolves problems (1) and (2) above. In section 4.5, we will explain the meaning of that factor and its relation to problem (3) (as well as the generalization to higher genus). For now, we merely mention that in eqn. (4.27), the twist field correlation function should be understood as a function, without the factor of $\prod_i (de_i)^{3k/2}$.

Now we will use the general relation (4.27) along with our previous formulas for the twist field correlation functions to analyze the genus 1 partition function for $k = 1, 2$. We make an $SL(2, \mathbb{C})$ transformation to set $e_4 = \infty$ and to fix

$$e_1 + e_2 + e_3 = 0. \qquad (4.28)$$

The genus 1 partition function at $k = 1$ is then, using (4.13),

$$Z_C(k = 1) = \frac{2^8}{\prod_{i<j}(e_i - e_j)^2} \left((e_1 - e_3)^3 (e_2 - e_3)^3 + \text{cyclic} \right) + 24. \qquad (4.29)$$

We want to express this in terms of $q = \exp(2\pi i \tau)$, the usual parameter on the genus 1 moduli space, and to show explicitly that Z_C has the expected behavior near $q = 0$. The classical formulas are

$$e_1 - e_2 = \theta_3^4(0, \tau)$$
$$e_3 - e_2 = \theta_1^4(0, \tau) \qquad (4.30)$$
$$e_1 - e_3 = \theta_2^4(0, \tau),$$

where

$$\theta_1(0, \tau) = \sum_{n \in \mathbb{Z}} q^{(n+1/2)^2/2}$$
$$\theta_2(0, \tau) = \sum_{n \in \mathbb{Z}} (-1)^n q^{n^2/2} \qquad (4.31)$$
$$\theta_3(0, \tau) = \sum_{n \in \mathbb{Z}} q^{n^2/2}.$$

Expanding (4.29) near $q = 0$, one can readily verify the expected behavior $Z_C(k = 1) = q^{-1} + \mathcal{O}(q)$. Since Z_C is manifestly modular-invariant, this implies that Z_C must equal

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J. This can also be verified directly by comparing (4.29) to the classical formula for the j-function

$$j(q) = 2^5 \frac{\left((e_1 - e_2)^2 + (e_2 - e_3)^2 + (e_3 - e_1)^2\right)^3}{\prod_{i<j}(e_i - e_j)^2} \tag{4.32}$$

and recalling that $J = j - 744$.

The case $k = 2$ can be treated similarly. Our previous formula (4.14) for the twist field four-point function implies, via (4.27), a formula for $Z_C(k = 2)$. If we take $e_4 \to \infty$ and $e_1 + e_2 + e_3 = 0$, this formula becomes

$$\begin{aligned}
Z_C(k = 2) =& 2^{16}\left(\frac{(e_1 - e_3)^2(e_2 - e_3)^2}{(e_1 - e_2)^4} + \text{cyclic}\right) \\
&+ 3 \cdot 2^{12}\left(\frac{(e_1 - e_3)(e_2 - e_3)}{(e_1 - e_2)^2} + \text{cyclic}\right) + (12k + 1).
\end{aligned} \tag{4.33}$$

One can expand this near $q = 0$ and verify the expected behavior $Z_C(k = 2) = q^{-2} + 1 + \mathcal{O}(q)$. Together with modular invariance, this implies that $Z_C(k = 2) = J^2 - 393767$, as one can also verify directly using (4.32).

4.5. Role Of The Conformal Anomaly

Naively speaking, the partition "function" of a conformal field theory on a Riemann surface C depends only on the complex or conformal structure of C.

Actually, because of the conformal anomaly, things are more subtle. The usual way to describe the situation is to endow C with a metric h, and to think of the partition function as a function $Z(h)$ that transforms in a certain prescribed way (depending on the central charge) under conformal rescaling $h \to e^\phi h$ of the metric.

This description is natural in differential geometry. However, it has a few limitations. One is that it really only gives an adequate framework if the left and right central charges c_L and c_R are equal. A second limitation, more crucial for our purposes, is that it is not well-adapted to holomorphic factorization. In a holomorphic CFT, everything will be much simpler if we can incorporate the conformal anomaly in purely holomorphic terms.

An alternative approach is to think of the partition function not as a function but as a section of a suitable line bundle $\mathcal{R}$ over $\mathcal{M}$, the moduli space of complex Riemann surfaces. This approach is developed in detail in [61]. Holomorphic factorization is incorporated naturally in this approach by saying that in a holomorphic CFT, $\mathcal{R}$ is a holomorphic line bundle over $\mathcal{M}$.

More specifically, $\mathcal{R} \cong \mathcal{L}^{c/2}$, where c is the central charge and $\mathcal{L}$ is a fundamental line bundle over $\mathcal{M}$ that is known as the determinant line bundle. $\mathcal{L}$ is defined as follows. Let p be a point in $\mathcal{M}$ corresponding to a Riemann surface C. Then $\mathcal{L}_p$, the fiber of $\mathcal{L}$ at p, is the top exterior power of the vector space $H^0(C, K_C)$ of holomorphic differentials on C. So if C has genus g and $\omega_1, \ldots, \omega_g$ are holomorphic differentials on C, then the expression $\omega_1 \wedge \omega_2 \wedge \ldots \wedge \omega_g$ defines a vector in $\mathcal{L}_p$. In our problem, $c = 24k$, and the partition function is a section of $\mathcal{L}^{12k}$.

On a hyperelliptic Riemann surface

$$y^2 = \prod_{i=1}^{2g+2} (x - e_i), \tag{4.34}$$

we can make this completely explicit. A basis of the space of holomorphic differentials is given by

$$\frac{dx}{y}, \ \frac{x\, dx}{y}, \ \frac{x^2\, dx}{y}, \ldots, \ \frac{x^{g-1}\, dx}{y}. \tag{4.35}$$

A section of $\mathcal{L}^{12k}$ over the space of hyperelliptic equations is hence an expression of the form

$$\Theta = \left(\frac{dx}{y} \wedge \frac{x\, dx}{y} \wedge \frac{x^2\, dx}{y} \wedge \ldots \wedge \frac{x^{g-1}\, dx}{y} \right)^{12k} F(e_1, \ldots, e_{2g+2}). \tag{4.36}$$

To get a section of $\mathcal{L}^{12k}$ over the moduli space of hyperelliptic Riemann surfaces, Θ must be invariant under the action of $SL(2, \mathbb{C})$ on the space of hyperelliptic equations. This action takes the form

$$\begin{aligned} e_i &\to \frac{ae_i + b}{ce_i + d} \\ x &\to \frac{ax + b}{cx + d} \\ y &\to \frac{y}{(cx + d)^{g+1} \prod_{i=1}^{2g+2} (ce_i + d)^{1/2}}, \end{aligned} \tag{4.37}$$

and the condition for Θ to be invariant is that F transforms by

$$F(e_1, \ldots, e_{2g+2}) \to F(e_1, \ldots, e_{2g+2}) \prod_{i=1}^{2g+2} (ce_i + d)^{-6kg}. \tag{4.38}$$

The twist field correlation function, on the other hand, is of the form

$$\langle \mathcal{E}(e_1) \mathcal{E}(e_2) \ldots \mathcal{E}(e_{2g+2}) \rangle = \prod_{i=1}^{2g+2} (de_i)^{3k/2} G(e_1, \ldots, e_{2g+2}) \tag{4.39}$$

for some "function" G. $SL(2, \mathbb{C})$ invariance of the correlation function means that G transforms by

$$G(e_1, \ldots, e_{2g+2}) \to G(e_1, \ldots, e_{2g+2}) \prod_{i=1}^{2g+2} (ce_i + d)^{3k}. \tag{4.40}$$

Comparing (4.38) and (4.40), we see that the two transformation laws are consistent if the relation between F and G is

$$F(e_1, \ldots, e_{2g+2}) = A_g\, G(e_1, \ldots, e_{2g+2}) \prod_{1 \le i < j \le 2g+2} (e_i - e_j)^{3k} \tag{4.41}$$

with some constant A_g. Moreover, this is the most general transformation between F and G that is holomorphic, invariant under permutations of the e_i, and an isomorphism as long as the e_i are all distinct. So it must be correct with some choice of the constant. (The relation between F and G is actually part of a general theory that is briefly indicated below.)

This is not the whole story. The partition function has another important property in genus 1: it can be regarded as an ordinary function, rather than a section of a line bundle. Indeed, the genus 1 partition function can be defined as a trace, $Z(q) = \mathrm{Tr}\, q^{L_0}$, and in this form it manifestly is equal to an ordinary function of q. Yet in the above presentation, the genus 1 partition function is a section of $\mathcal{L}^{12k}$. What reconciles the two points of view is that the line bundle $\mathcal{L}^{12}$ is trivial in genus 1. If we describe a genus 1 Riemann surface C by the hyperelliptic equation $y^2 = \prod_{i=1}^{4}(x - e_i)$, then as explained in [61], $\mathcal{L}^{12}$ is trivialized by the section

$$s = \left(\frac{dx}{y}\right)^{12} \prod_{1 \le i < j \le 4} (e_i - e_j)^2. \tag{4.42}$$

The point of this formula is that s is holomorphic, $SL(2, \mathbb{C})$-invariant, invariant under permutations of the e_i, and nonzero as long as the e_i are all distinct (so that C is smooth). A power $\mathcal{L}^n$ of the line bundle $\mathcal{L}$ can be trivialized in this fashion precisely if n is an integer multiple of 12 (and this gives one way to understand the fact that holomorphic factorization is possible only if c is an integer multiple of 24). The genus 1 partition function Z understood as an ordinary function is obtained from the section Θ of $\mathcal{L}^{12k}$ by dividing by s^k. Thus

$$Z = \frac{F}{\prod_{1 \le i < j \le 4}(e_i - e_j)^{2k}}. \tag{4.43}$$

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Combining this with (4.41), we get the relation (4.27) between the genus 1 partition function, understood as a trace, and the correlation function of twist fields:

$$Z = A_1 \left(\prod_{1 \leq i < j \leq 4} (e_i - e_j) \right) \langle \mathcal{E}(e_1)\mathcal{E}(e_2)\mathcal{E}(e_3)\mathcal{E}(e_4) \rangle. \tag{4.44}$$

The numerical value of the constant A_1 depends on how the twist field $\mathcal{E}$ has been normalized.

What about $g > 1$? What is the best formula depends on how one prefers to treat the conformal anomaly, and there seems to be in the physics literature no standard recipe for doing so in the holomorphic context. To make contact with the genus 2 formulas of Tuite [54], we may proceed as follows. A genus 2 Riemann surface is always a hyperelliptic curve, $y^2 = \prod_{i=1}^{6} (x - e_i)$. In genus 2, we cannot trivialize the line bundle[20] $\mathcal{L}^{12}$, but we can trivialize $\mathcal{L}^{10}$ by an obvious analog of (4.42):

$$\widetilde{s} = \left(\frac{dx}{y} \wedge \frac{x\,dx}{y} \right)^{10} \prod_{1 \leq i < j \leq 6} (e_i - e_j)^2. \tag{4.45}$$

Hence, instead of regarding the genus 2 partition function as a section Θ of $\mathcal{L}^{12k}$, we can regard it as a section $Z = \Theta/\widetilde{s}^k$ of $\mathcal{L}^{2k}$. If we do this, then the relation between Z and the twist field correlation function is the obvious analog of (4.44):

$$Z = A_2 \left(\prod_{1 \leq i < j \leq 6} (e_i - e_j) \right) \langle \mathcal{E}(e_1)\mathcal{E}(e_2)\ldots\mathcal{E}(e_6) \rangle. \tag{4.46}$$

It can be shown that, in genus 2, a section of $\mathcal{L}^r$ is the same as a Siegel modular function of weight r. So the formula (4.46) is the one we should use if we wish to regard the genus 2 partition function at $c = 24k$ as a Siegel modular function of weight $2k$, as was done in [54]. This probably gives the most intuitively appealing formulas in genus 2. For genus greater than 2, we leave the choice of the most useful formalism to the reader.

We conclude by briefly sketching the theoretical context for the formula (4.41). In doing so, we will not limit ourselves to the case of a hyperelliptic Riemann surface, but will consider an arbitrary base curve C_0 and a two-fold cover $C \to C_0$ branched at points $e_1, \ldots, e_s$. Let K_i be the cotangent bundle to C_0 at the point e_i. Let $\mathcal{L}_{C_0}$ and $\mathcal{L}_C$ be the determinant lines of C_0 and C respectively. Then as has been shown by P. Deligne, using results in [62], there is a natural isomorphism $\mathcal{L}_C^8 \cong \mathcal{L}_{C_0}^{16} \otimes_i K_{C_i}$. If C_0 is of genus zero, then $\mathcal{L}_{C_0}$ is naturally trivial, and the $3k/2$ power of this isomorphism is the map in (4.41). (The isomorphism has a natural square root, relevant if k is odd.)

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[20] This fact is consistent with holomorphic factorization at $c = 24k$, since the genus 2 partition function cannot be written as a trace and so need not be definable as an ordinary function.

Appendix A. More Extremal Partition Functions

F In section 3.3, we have already described the extremal superconformal partition functions F_{k^*} for $k^* \leq 4$. The purpose of this appendix is to briefly describe these functions, and the corresponding Ramond partition functions H_{k^*}, for $5 \leq k^* \leq 10$.

For $k^* = 5$, we have

$$
\begin{aligned}
F_5 &= K^5 - 1380K^3 - 10239K^2 + 324871K + 2579929 \\
&= q^{-5/2} + q^{-1} + q^{-1/2} + 1 + 924492q^{1/2} + 185710868q + 11953414737q^{3/2} \quad \text{(A.1)} \\
&\quad + 416513865728q^2 + 9727092442216q^{5/2} + \ldots.
\end{aligned}
$$

The corresponding Ramond partition function is

$$
H_5 = 143 + 371027968q + 832984743936q^2 + 340658459983872q^3 + 56418463131631616q^4 + \ldots.
$$
(A.2)

The coefficients in these series cannot be expressed as linear combinations of dimensions of monster representations with reasonably small coefficients. But strangely, at least one difference can be so represented; the difference between the coefficient of q in F_5 and half the corresponding coefficient in H_5 is $196884 = 1 + 196883$. (The coefficients in H_5, except the first one, are even because of supersymmetry, so it is natural to divide by 2.) Also, the first coefficient in H_5, namely 143, is the dimension of the smallest nontrivial representation of a sporadic finite group known as the Suzuki group. Unfortunately, the other coefficients are too large, and the representations of the Suzuki group too small, for it to be easy to get convincing evidence of whether the Suzuki group is a symmetry of supergravity at $k^* = 5$.

For $k^* = 6$, the partition functions are

$$
\begin{aligned}
F_6 &= K^6 - 1656K^4 - 12287K^3 + 618373K^2 + 6487237K - 12026567 \\
&= q^{-3} + q^{-3/2} + q^{-1} + q^{-1/2} + 1 + 3724378q^{1/2} + 1298410586q \\
&\quad + 127852130050q^{3/2} + 6378693040128q^2 + 204402839559265q^{5/2} + \ldots \quad \text{(A.3)} \\
H_6 &= 1 + 2589372416q + 12754796707840q^2 \\
&\quad + 9529193701720064q^3 + 2622934801904828416q^4 + \ldots.
\end{aligned}
$$

The fact that the leading coefficient in the Ramond sector is 1 suggests that the model may be invariant under a very large discrete symmetry group. The monster $\mathbb{M}$ does not work, since the first non-trivial NS coefficient 3724378 cannot be expressed nicely in terms of dimensions of monster representations. It turns out that another large sporadic group,

the baby monster $\mathbb{B}$, is a better candidate. The dimensions $r_1, \ldots, r_{12}$ of the first 12 irreducible representations of $\mathbb{B}$ are listed in Table 3. In terms of those dimensions, we find that the first two non-trivial NS coefficients in the partition functions can be expressed as follows

$$3724378 = 7r_1 + 4r_2 + 3r_3 + 3r_4$$
$$1298410586 = 14r_1 + 16r_2 + 7r_3 + 8r_4 + 4r_6 + 3r_7 + r_8 + 2r_9. \tag{A.4}$$

These formulas are not unique, as the r_i obey linear relations with small coefficients, for example $r_1 + r_3 + r_5 + r_8 = r_2 + r_4 + r_9$. If we let h_1 denote the first non-trivial coefficient in H_6, then it turns out that $h_1/2 = f_1 - f_{1/2} = 1298410586 - 3724378$. So by subtracting the formulas in (A.4), $h_1/2$ can also be expressed as a positive linear combination of the r_i with fairly small coefficients. If is also true that $h_2/2 = f_2 - f_1 + f_{1/2}$, and so is consistent with $\mathbb{B}$ symmetry if f_2 can be suitably expressed in terms of the r_i.

These results are suggestive of baby monster symmetry at $k^* = 6$, though they are perhaps a little less striking than the evidence presented in section 3.3 for monster symmetry at $k^* = 4$, since the required coefficients are larger and the dimensions involved are smaller. Also, it is harder to continue this analysis beyond the first few coefficients, as many representations of $\mathbb{B}$ come in.

r_1	1
r_2	4371
r_3	96255
r_4	1139374
r_5	9458750
r_6	9550635
r_7	63532485
r_8	347643114
r_9	356054375
r_{10}	1407126890
r_{11}	3214743741
r_{12}	4221380670

Table 3. Presented here from [53] are the dimensions r_i of the ith irreducible representation of the baby monster group $\mathbb{B}$, for $i = 1, \ldots, 12$.

We present the remaining cases with little comment. For $k^* = 7$, we have

$$F_7 = K^7 - 1932K^5 - 14335K^4 + 988051K^3 + 11525041K2 - 75824563K - 840705550$$

$$= q^{-7/2} + q^{-2} + q^{-3/2} + q^{-1} + q^{-1/2} + 2 + 13404883q^{1/2}$$

$$+ 7740446996q + 1126452195714q^{3/2} + 78170641348884q^2 + \dots.$$

$$H_7 = 262 + 15394525184q + 156260255891456q^2 + 203203950584774656q^3 + \dots.$$

$$\text{(A.5)}$$

For $k^* = 8$, the analogous formulas are

$$F_8 = K^8 - 2208K^6 - 16383K^5 + 1433905K^4 + 17693341K^3 - 213343055K^2$$

$$- 3164679732K - 2780845557$$

$$= q^{-4} + q^{-5/2} + q^{-2} + q^{-3/2} + q^{-1} + 2q^{-1/2} + 3 + 44146598q^{1/2}$$

$$+ 40700662036q + 8516908978515q^{3/2} + \dots$$

$$H_8 = -213 + 80651894784q + 1605169778655232q^2 + 34969225973865551296q^3 + \dots.$$

$$\text{(A.6)}$$

This is the first case in which the leading coefficient in H is negative, so that it is necessary to add an integer to F and H.

For $k^* = 9$, we find

$$F_9 = K^9 - 2484K^7 - 18431K^6 + 1955935K^5 + 24992137K^4 - 445606619K^3$$

$$- 7443672266K^2 + 1774761946K + 223898812203$$

$$= q^{-9/2} + q^{-3} + q^{-5/2} + q^{-2} + q^{-3/2} + 2q^{-1} + 3q^{-1/2} + 3 + 135149374q^{1/2}$$

$$+ 193216791918q + 56847816152503q^{3/2} + \dots$$

$$H_9 = 453 + 381161021440q + 14282018665201664q^2 + 5051928839165624300q^3 + \dots$$

$$\text{(A.7)}$$

Finally, for $k^* = 10$, we find

$$F_{10} = K^{10} - 2760K^8 - 20479K^7 + 2554141K^6 + 33421429K^5 - 793639831K^4$$

$$- 14145708496K^3 + 30831695165K^2 + 1166011724825K + 2482063616019$$

$$= q^{-5} + q^{-7/2} + q^{-3} + q^{-5/2} + q^{-2} + 2q^{-3/2} + 3q^{-1} + 3q^{-1/2} + 3$$

$$+ 389274233q^{1/2} + 842231630010q + 341925580784341q^{3/2} + \dots$$

$$H_{10} = -261 + 1652836102144q + 112692628289650688q^2 + 630520566901614002176q^3 + \dots.$$

$$\text{(A.8)}$$

This gives a second example in which the leading coefficient of H is negative, so that it is necessary to add an integer.

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