# ON THE LANDAU-GINZBURG DESCRIPTION
# OF $N = 2$ MINIMAL MODELS

EDWARD WITTEN[*]

*School of Natural Sciences*
*Institute for Advanced Study*
*Olden Lane*
*Princeton, NJ 08540*

## ABSTRACT

The conjecture that $N = 2$ minimal models in two dimensions are critical points of a super-renormalizable Landau-Ginzburg model can be tested by computing the path integral of the Landau-Ginzburg model with certain twisted boundary conditions. This leads to simple expressions for certain characters of the $N = 2$ models which can be verified at least at low levels. An $N = 2$ superconformal algebra can in fact be found directly in the *noncritical* Landau-Ginzburg system, giving further support for the conjecture.

# 1. Introduction

$N = 2$ supersymmetry in two dimensions has several distinguished classes of simple realizations. In the present paper, we will focus on two of these, and the conjectured relationship between them.

On the one hand, there is a discrete series of representations of the $N = 2$ superconformal algebra [1–3] with $\widehat{c} < 1$, in fact with

$$\widehat{c} = 1 - \frac{2}{k+2}, \quad k = 1, 2, 3, \ldots \tag{1.1}$$

Based on these representations, one can construct families of quantum field theories known as $N = 2$ minimal models. Actually, these models have an $\mathbf{A} - \mathbf{D} - \mathbf{E}$ classification [4–6]; in this paper we consider only the simplest $\mathbf{A}$ series.

On the other hand, there are super-renormalizable Landau-Ginzburg models, constructed from chiral superfields. Such a superfield has an expansion

$$\Phi(x, \theta) = \phi(y) + \sqrt{2}\theta^\alpha \psi_\alpha(y) + \theta^\alpha \theta_\alpha F(y), \tag{1.2}$$

where $x, \theta$ are coordinates of two dimensional $N = 2$ superspace, and $y^m = x^m + i\theta^\alpha \sigma^m_{\alpha\dot\alpha}\overline{\theta}^{\dot\alpha}$; all conventions are as in [7,8]. In the simplest case of interest, there is only one chiral superfield, and the superspace Lagrangian is

$$L = \int \mathrm{d}^2x\, \mathrm{d}^4\theta\, \overline{\Phi}\Phi - \int \mathrm{d}^2x\, \mathrm{d}^2\theta\, \frac{\Phi^{k+2}}{k+2} - \int \mathrm{d}^2x\, \mathrm{d}^2\overline{\theta}\, \frac{\overline{\Phi}^{k+2}}{k+2}. \tag{1.3}$$

The function $\Phi^{k+2}/(k+2)$ that appears here is called the superpotential; it is a homogeneous function, a fact that ensures the existence of $R$-symmetries that will play a crucial role later.

The non-renormalization theorems for the superpotential of $N = 2$ models (see for instance [9, p. 358]) strongly suggest that the $\Phi$ superfield is exactly massless in the Landau-Ginzburg model with this superpotential. It has been conjectured

that, in the infrared limit, the interactions of this massless superfield are precisely governed by the **A** series of $N = 2$ minimal models at level $k$. This conjecture, which followed a somewhat analogous discussion for $N = 0$ by Zamolodchikov [10] and was motivated in part by results of Gepner [6], was formulated and tested in [11–18]. The goal of the present paper is to exploit and shed light on this conjecture.

Any observable that is effectively computable in both the Landau-Ginzburg and minimal models can of course serve as the basis for a test of their conjectured relation. However, generic correlation functions of the Landau-Ginzburg model are not effectively computable, especially in the infrared where the comparison must be made. In practice, evidence for the conjecture has come largely from heuristic renormalization group arguments, comparison of renormalization group flows for large $k$ [11], a $2 + \epsilon$ expansion [15] (with $\epsilon \sim 1/k$), a heuristic computation of the central charge [13], comparison of chiral rings and other properties of the chiral primary fields [14,16,17], and studies of properties such as soliton scattering and spectrum in integrable deformations [18].

As we will see, an interesting and particularly rich comparison between Landau-Ginzburg and minimal models can be made by consideration of the elliptic genus. This is simply a genus one path integral with certain twisted boundary conditions [19,20]. It has many convenient properties. The elliptic genus of a supersymmetric model is conformally invariant even if the underlying model is not conformally invariant. There is, therefore, no difficulty in taking the infrared limit, which simply coincides with the ultraviolet limit. Moreover, the elliptic genus is usually effectively computable, and this is so for both Landau-Ginzburg and minimal models. This is related to the fact that the elliptic genus (of a sigma model, for instance) is a topological invariant of the target space.

In §2, we will compute the elliptic genus of the $N = 2$ minimal models and of the Landau-Ginzburg models introduced above. By equating the formulas, we will obtain simple free field formulas for certain characters of the $N = 2$ algebra that

follow from the conjectured relation between these models. We will verify the first few terms of these formulas. A general verification of them has been proposed by N. Warner.

In §3, we study the mechanism that underlies the character formula proposed in §2. In fact, we will see that an $N = 2$ algebra, with the expected central charge, and acting on a module with the character found in §2, can be constructed directly in the non-critical Landau-Ginzburg model. This leads naturally to a free field realization of the $N = 2$ algebra together with a construction of a screening charge. (We will not get a proof of the formula of §2 since we will not prove that the module is irreducible.) The considerations of §3 are a special case of a general procedure for extracting a chiral algebra (in the sense of rational conformal field theory) from any $N = 2$ supersymmetric model with $R$ symmetry. The resulting chiral algebras seem worthy of further study.

In sum, we will identify directly in the non-critical Landau-Ginzburg model an $N = 2$ algebra with the expected central charge and a formula for certain of the $N = 2$ characters.

This paper is dedicated to Professor C. N. Yang on the occasion of his 70th birthday. Symmetries and interactions of elementary particles have, of course, always been foremost in his work. He has also made lasting contributions to statistical mechanics and many-body physics, including but not limited to the study of exactly soluble models in $1 + 1$ dimensions. The $N = 2$ superconformal algebra in two dimensions is a relatively new symmetry structure, linked on the one hand to soluble models in $1 + 1$ dimensions, and on the other hand to novel constructions in string theory of models of elementary particle physics. So I hope that the modest contribution to understanding this algebra that I make in the present paper is appropriate on this occasion.

# 2. The Character Formula

## 2.1. THE ELLIPTIC GENUS OF THE MINIMAL MODELS

The characters of the $N = 2$ superconformal algebra in two dimensions are functions of two variables, since the algebra contains a $U(1)$ charge $J_0$ that commutes with the Hamiltonian $H = L_0$. The supercurrents $S_\pm$ have eigenvalues $\pm 1$ under $J_0$, in the sense that $[J_0, S_\pm] = \pm S_\pm$. It follows that

$$\exp(i\pi J_0)S_\pm \exp(-i\pi J_0) = -S_\pm, \tag{2.1}$$

while $\exp(i\pi J_0)$ commutes with the bosonic generators of the $N = 2$ algebra, namely the current $J$ and stress tensor $T$:

$$\exp(i\pi J_0)\begin{Bmatrix} J \\ T \end{Bmatrix}\exp(-i\pi J_0) = \begin{Bmatrix} J \\ T \end{Bmatrix}. \tag{2.2}$$

Relations (2.1) and (2.2) are also valid if $\exp(i\pi J_0)$ is replaced by the operator $(-1)^F$ that is $+1$ for bosons and $-1$ for fermions. It follows that the product $(-1)^F \exp(i\pi J_0)$ is central, and so is a complex constant in any irreducible representation of the $N = 2$ algebra.

We will take the definition of the character of a representation $R$ of the $N = 2$ algebra to be

$$\chi_R(q, \gamma) = \mathrm{Tr}_R(-1)^F q^H e^{i\gamma J_0}. \tag{2.3}$$

The factor of $(-1)^F$ could be replaced by $\exp(i\pi J_0)$ in view of the above remarks.

Of course, in physical terms, the $N = 2$ algebra, in its irreducible highest weight representations, acts in a Hilbert space of only left-moving (or only right-moving) degrees of freedom. The $N = 2$ minimal models are quantum field theories obtained by combining left- and right-movers. There are several ways to do this, with an $\mathbf{A} - \mathbf{D} - \mathbf{E}$ classification; for simplicity, we will consider only the $\mathbf{A}$ series. This

is constructed as follows: if $R_\alpha$ are the irreducible representations of a left-moving $N = 2$ algebra, and $\overline{R}_\alpha$ are the complex conjugate representations of a right-moving $N = 2$ algebra, then the Hilbert space of the **A** series of $N = 2$ minimal models is $\mathcal{H} = \oplus_\alpha R_\alpha \otimes \overline{R}_\alpha$. Nothing essential is lost if we consider only the case that both left-movers and right-movers are in the Ramond sector; corresponding results for the Neveu-Schwarz sector can be obtained by spectral flow.

Let $H_L$ and $H_R$ $(= \overline{L}_0, L_0)$ be the Hamiltonians of left-movers and right-movers; and let $J_{0,L}$ and $J_{0,R}$ be the $U(1)$ charges of the left-movers and right-movers. The natural generalization of (2.3) to include both left- and right-movers is

$$
\begin{aligned}
Z(q, \gamma_L, \gamma_R) &= \mathrm{Tr}_\mathcal{H}(-1)^F q^{H_L} \overline{q}^{H_R} \exp(i\gamma_L J_{0,L} + i\gamma_R J_{0,R}) \\
&= \sum_\alpha \mathrm{Tr}_{R_\alpha}(-1)^{F_L} q^{H_L} \exp(i\gamma_L J_{0,L}) \; \mathrm{Tr}_{\overline{R}_\alpha}(-1)^{F_R} \overline{q}^{H_R} \exp(i\gamma_R J_{0,R}).
\end{aligned}
$$

$$(2.4)$$

Here we have factored $(-1)^F$ as $(-1)^F = (-1)^{F_L}(-1)^{F_R}$, where $(-1)^{F_L}$ and $(-1)^{F_R}$ act in left- and right-moving Hilbert spaces.

The partition function $Z(q, \gamma_L, \gamma_R)$ is effectively computable for the $N = 2$ minimal models by algebraic methods. However, our goal is really to compare the minimal models to Landau-Ginzburg models, and the full partition function is not effectively computable in the Landau-Ginzburg representation. The situation changes markedly if we consider not the full character, but the elliptic genus, which is simply the partition function restricted to $\gamma_R = 0$. This specialization of the partition function is effectively computable in the Landau-Ginzburg representation, as we will see later. For the moment, we work out the elliptic genus of the minimal models. The right-moving factor in (2.4) is simply $\mathrm{Tr}_{\overline{R}_\alpha}(-1)^{F_R} \overline{q}^{H_R}$. As is usual in index theory, because of bose-fermi cancellation, this expression can be evaluated just by counting states of $H_R = 0$. It equals $+1$ for $\alpha$ such that the ground state of the representation $R_\alpha$ has $H_R = 0$, and 0 for other $\alpha$. The elliptic genus is hence

$$
Z(q, \gamma_L, 0) = {\sum_\alpha}' \mathrm{Tr}_{R_\alpha}(-1)^{F_L} q^{H_L} \exp(i\gamma_L J_{0,L}) = {\sum_\alpha}' \chi_\alpha(q, \gamma_L), \qquad (2.5)
$$

where $\sum_\alpha'$ is a sum restricted to $\alpha$ such that the vacuum vector of $R_\alpha$ has $H_R = 0$. These are precisely the representations which upon spectral flow to the Neveu-Schwarz sector correspond to the chiral primary states.

From (2.5), it may appear that the elliptic genus determines not individual characters of the $N = 2$ algebra, but only certain linear combinations of them. However, by using some information about the spectrum of $J_{0,L}$, it is possible to invert this relation to express (certain) characters of the $N = 2$ algebra in terms of the elliptic genus. To begin with, in the Ramond sector, the $U(1)$ charges of the chiral primary states are $n/(k + 2)$, with $n = 0, 1, 2, \ldots, k$. (These states are represented in the Landau-Ginzburg language as $\Phi^n$, where $\Phi$ has charge $1/(k+2)$.) Under spectral flow, these states become Ramond sector ground states with $J_{0,L}$ eigenvalue

$$q_n = -\frac{\widehat{c}}{2} + \frac{n}{k + 2}, \tag{2.6}$$

where for these particular models the central charge is

$$\widehat{c} = 1 - 2\alpha \tag{2.7}$$

with

$$\alpha = \frac{1}{k + 2}. \tag{2.8}$$

Let $R_n$, $n = 0, \ldots, k$ be the Ramond sector representation of the $N = 2$ algebra containing a ground state of $H = 0$ and $J_0 = q_n$. Let

$$\chi_n(q, \gamma) = \mathrm{Tr}_{R_n}(-1)^F q^H \exp(i\gamma J_0) \tag{2.9}$$

for $0 \leq n \leq k$, and $\chi_{k+1}(q, \gamma) = 0$. The eigenvalues of $J_{0,L}$ in the representation $R_n$ are congruent to $q_n$ modulo $\mathbf{Z}$, so

$$\frac{1}{k + 2} \sum_{m=0}^{k+1} \chi_n(q, \gamma + 2\pi m) \exp(\pi i \widehat{c} m - 2\pi i s m \alpha) = \delta_{s,n} \chi_n(q, \gamma). \tag{2.10}$$

Upon summing over $n$ and using (2.5), this gives

$$\chi_s(q, \gamma) = \frac{1}{k+2} \sum_{m=0}^{k+1} Z(q, \gamma + 2\pi m, 0) \exp\left(\pi i \hat{c} m - 2\pi i \alpha m s\right). \tag{2.11}$$

## 2.2. THE ELLIPTIC GENUS OF A LANDAU-GINZBURG MODEL

We now wish to compute the function $Z(q, \gamma, 0)$ in the Landau-Ginzburg model. The first step is to identify the left- and right-moving $U(1)$ charges. The supersymmetry transformation laws of the Landau-Ginzburg model, after eliminating the auxiliary field $F$ by its equation of motion, are (with conventions as in [7,8])

$$\begin{aligned}
\delta\phi &= \sqrt{2}\left(-\epsilon_-\psi_+ + \epsilon_+\psi_-\right) \\
\delta\psi_+ &= i\sqrt{2}\left(\partial_0 + \partial_1\right)\phi\bar{\epsilon}_- + \sqrt{2}\epsilon_+\bar{\phi}^{k+1} \\
\delta\psi_- &= -i\sqrt{2}\left(\partial_0 - \partial_1\right)\phi\bar{\epsilon}_+ + \sqrt{2}\epsilon_-\bar{\phi}^{k+1}.
\end{aligned} \tag{2.12}$$

The Lagrangian of the model, after integrating out fermionic coordinates and eliminating the auxiliary field, is

$$\begin{aligned}
L = \int d^2x \Big( &-\partial_\alpha\bar{\phi}\partial^\alpha\phi + i\bar{\psi}_-(\partial_0 + \partial_1)\psi_- + i\bar{\psi}_+(\partial_0 - \partial_1)\psi_+ \\
&- (\bar{\phi}\phi)^{k+1} - (k+1)\phi^k\psi_-\psi_+ - (k+1)\bar{\phi}^k\bar{\psi}_+\bar{\psi}_- \Big).
\end{aligned} \tag{2.13}$$

The left-moving $U(1)$ charge $J_{0,L}$ generates a symmetry of the Lagrangian (2.13) under which the supersymmetry generators transform as follows: the generator $\epsilon^+ = -\epsilon_-$ (which generates a symmetry that in the massless limit couples only to right-movers) should be invariant, while $\epsilon^- = \epsilon_+$ should have charge 1. These conditions uniquely identify the symmetry generated by $J_{0,L}$ to be

$$\begin{aligned}
\phi &\to \exp\left(i\gamma\alpha\right)\phi \\
\psi_+ &\to \exp\left(i\gamma\alpha\right)\psi_+ \\
\psi_- &\to \exp\left(-i(k+1)\gamma\alpha\right)\psi_-.
\end{aligned} \tag{2.14}$$

Thus, if the conjectured relation between the Landau-Ginzburg model and the

$N = 2$ minimal model is correct, then the left-moving $U(1)$ symmetry of the minimal model must correspond under this relation to the symmetry group (2.14).

## The Elliptic Genus

We wish to consider path integrals on a genus one Riemann surface,[*] say the torus $\Sigma$ obtained by dividing the $x^1 - x^2$ plane by the equivalence relation $x^1 \to x^1 + m$, $x^2 \to x^2 + n$, with $m, n \in \mathbf{Z}$. Given a particular physical model, various path integrals can be defined on $\Sigma$, depending on the boundary conditions that one chooses for the fields. For example, the simplest supersymmetric index $\mathrm{Tr}(-1)^F$ corresponds to using untwisted boundary conditions for all bosons and fermions in both the $x^1$ and $x^2$ direction. The elliptic genus is obtained, instead, by twisting the fields by a left-moving $R$ symmetry, that is a symmetry that commutes with the right-moving supersymmetries but not with the left-moving ones. In a model with $N = 1$ supersymmetry, the $R$ symmetry group (if any) is $\mathbf{Z}_2$, and the usual elliptic genus is obtained by twisting by the non-trivial element of $\mathbf{Z}_2$. With $N = 2$ supersymmetry, the $R$ symmetry group can be $U(1)$ (as in the models considered in this paper), and then one can twist by arbitrary elements of $U(1)$.

The elliptic genus $Z(q, \gamma, 0)$ of the $N = 2$ minimal models was defined by the formula

$$Z(q, \gamma, 0) = \mathrm{Tr}_{\mathcal{H}}(-1)^F q^{H_L} \overline{q}^{H_R} \exp(i\gamma J_{0,L}). \qquad (2.15)$$

Here $\mathcal{H}$ was an ordinary, untwisted Hilbert space, so the path integral representation of this quantity involves untwisted boundary conditions in the "space" direction, which we can take to be the $x^1$ direction. The factor of $\exp(i\gamma J_{0,L})$ means that in the "time" direction, which we can take to be the $x^2$ direction, the fields must be twisted by $\exp(i\gamma J_{0,L})$. In the Landau-Ginzburg model, this means by

---

[*] We take this surface to be of positive signature. The relation to the Lorentz signature formulas used elsewhere in this paper is by a standard Wick rotation $x^0 = -ix^2$.

virtue of (2.14) that the fields obey

$$\phi(x^1, x^2 + 1) = \exp\left(i\gamma\alpha\right)\phi(x^1, x^2)$$
$$\psi_+(x^1, x^2 + 1) = \exp\left(i\gamma\alpha\right)\psi_+(x^1, x^2) \qquad (2.16)$$
$$\psi_-(x^1, x^2 + 1) = \exp\left(-i(k+1)\gamma\alpha\right)\psi_-(x^1, x^2).$$

And of course they are invariant under $x^1 \to x^1 + 1$.

Can the Landau-Ginzburg path integral with these boundary conditions be effectively evaluated? Landau-Ginzburg path integrals are intractable in general, but here the topological invariance of the elliptic genus comes to our aid. The elliptic genus has an interpretation (which we recall at the beginning of §3) as an index of a right-moving supercharge; this ensures that it is invariant under continuous variations of the Lagrangian of a supersymmetric system – as long as one considers only systems with a good behavior for large values of the fields, so that low energy states cannot appear or disappear at infinity in field space. As a result, the elliptic genus would be unchanged if one replaces the superpotential $W(\Phi) = \Phi^{k+2}/(k+2)$ by $\widetilde{W}(\Phi) = \epsilon\Phi^{k+2}/(k+2)$ for any non-zero $\epsilon$. One is tempted to try to take the limit $\epsilon \to 0$ to get a free field theory. This is dangerous because precisely at $\epsilon = 0$ the $\phi$ field can be arbitrarily large at no cost in energy.

For instance, the supersymmetric index $\mathrm{Tr}(-1)^F$ is not continuous at $\epsilon = 0$; it equals $k + 1$ for $\epsilon \neq 0$, and is ill-defined at $\epsilon = 0$ because in finite volume the zero mode of the $\phi$ field gives the Hamiltonian of the system a continuous spectrum going down to zero energy. The path integral representation of $\mathrm{Tr}(-1)^F$ involves untwisted boundary conditions for $\phi$ (and all other fields) in the $x^1$ and $x^2$ directions, and is ill-behaved at $\epsilon = 0$ because one loses control over the zero mode of the $\phi$ field.

For the elliptic genus, the situation is completely different. The twisted boundary conditions (2.14) remove the zero mode of $\phi$, and ensure that the path integral is convergent even at $\epsilon = 0$ – and that there is no problem in formal arguments

showing that $Z(q, \gamma, 0)$ is independent of $\epsilon$. (This generalizes the fact that supersymmetric ground states in twisted sectors of Landau-Ginzburg models can be computed via free field theory [21].) So we can simply set $\epsilon = 0$, and we get a free field representation of the elliptic genus of the Landau-Ginzburg model.

If the Landau-Ginzburg model has the conjectured relation to the $N = 2$ minimal model, this will give us a free field representation of the elliptic genus of the minimal model and therefore (according to (2.11)) of certain $N = 2$ characters.

We could of course generalize the definition of the function $Z(q, \gamma, 0)$ to allow for twists by the left-moving $U(1)$ symmetry in the $x^1$ as well as $x^2$ direction. This has one theoretical advantage: it eliminates the bosonic zero mode from the Hamiltonian as well as the path integral formulation.

Free Field Computation

The free field computation of the twisted partition function that gives the elliptic genus can perhaps be performed most conveniently in a Hilbert space approach.

Let us work out the contributions of the fermionic and bosonic zero and non-zero modes to the elliptic genus defined as in (2.15). We begin with the fermionic zero modes. The zero modes of $\psi_-$ and $\overline{\psi}_-$ – call them $\psi_{-,0}$ and $\overline{\psi}_{-,0}$ – obey $\{\psi_{-,0}, \overline{\psi}_{-,0}\} = 1$. So they are represented in a space of two states $|\downarrow\rangle$ and $|\uparrow\rangle$, with

$$\psi_{-,0}|\downarrow\rangle = |\uparrow\rangle, \quad \overline{\psi}_{-,0}|\uparrow\rangle = |\downarrow\rangle \tag{2.17}$$

and other matrix elements zero. In view of the quantum numbers of $\psi_{-,0}$ and $\overline{\psi}_{-,0}$, one of the states $|\uparrow\rangle$, $|\downarrow\rangle$ is bosonic and one is fermionic; and they transform under the $U(1)$ symmetry (2.14) as $\exp(\mp i\gamma(k+1)\alpha/2)$. The contribution of these two states to (2.15) is therefore (up to an overall sign that can be absorbed in the definition of the operator $(-1)^F$) a factor of

$$e^{-i\gamma(k+1)\alpha/2} - e^{i\gamma(k+1)\alpha/2}. \tag{2.18}$$

11

Similarly, the zero modes of $\psi_+, \overline{\psi}_+$ contribute a factor of

$$e^{i\gamma\alpha/2} - e^{-i\gamma\alpha/2}. \tag{2.19}$$

The overall factor coming from fermi zero modes is the product of these factors; this can be written

$$e^{-i\gamma k\alpha/2} \cdot \left(1 - e^{i\gamma(k+1)\alpha}\right) \cdot \left(1 - e^{-i\gamma\alpha}\right). \tag{2.20}$$

The non-zero modes of left- and right-moving fermions contribute a factor

$$\prod_{n=1}^{\infty} \left(1 - q^n e^{-i(k+1)\gamma\alpha}\right) \left(1 - q^n e^{i(k+1)\gamma\alpha}\right) \left(1 - \overline{q}^n e^{i\gamma\alpha}\right) \left(1 - \overline{q}^n e^{-i\gamma\alpha}\right) \tag{2.21}$$

coming from a trace over the fermion Fock space. The first two factors in (2.21) come from left-movers and the last two from right-movers. The non-zero modes of bosons contribute an analogous factor

$$\prod_{n=1}^{\infty} \frac{1}{1 - q^n e^{i\gamma\alpha}} \cdot \frac{1}{1 - q^n e^{-i\gamma\alpha}} \cdot \frac{1}{1 - \overline{q}^n e^{i\gamma\alpha}} \cdot \frac{1}{1 - \overline{q}^n e^{-i\gamma\alpha}}, \tag{2.22}$$

from a trace over the bosonic Fock space. The contribution of the bosonic zero modes is actually given by a factor of the same structure, namely

$$\frac{1}{1 - e^{i\gamma\alpha}} \cdot \frac{1}{1 - e^{-i\gamma\alpha}}, \tag{2.23}$$

even though the Hilbert space of the bosonic zero modes (which is naturally represented as $\mathbf{L}^2(\mathbf{C})$, where $\mathbf{C}$ is a copy of the complex plane parametrized by the zero mode of the $\phi$ field) has no natural representation as a Fock space. One way to justify (2.23) is to perturb the definition of $Z(q, \gamma, 0)$ to introduce a small twist by the left-moving $U(1)$ charge in the $x^1$ direction. This has the effect of displacing the bosonic zero mode of the Hamiltonian description, giving it a non-zero

energy, so that the Hilbert space has a standard Fock space representation, and the would-be zero mode can manifestly be treated in the same way as the other bosonic modes. Upon removing the twist in the $x^1$ direction, one gets (2.23) as the zero mode contribution. (Alternatively, one can get (2.23) from the path integral, which because of the twist in the $x^2$ direction has no problem with zero modes.)

So putting the factors together, we find for the elliptic genus of the Landau-Ginzburg model

$$Z(q,\gamma,0) = e^{-i\gamma k\alpha/2} \cdot \frac{1 - e^{i\gamma(k+1)\alpha}}{1 - e^{i\gamma\alpha}} \cdot \prod_{n=1}^{\infty} \frac{(1 - q^n e^{i\gamma(k+1)\alpha})(1 - q^n e^{-i\gamma(k+1)\alpha})}{(1 - q^n e^{i\gamma\alpha})(1 - q^n e^{-i\gamma\alpha})}.$$
(2.24)

If Landau-Ginzburg models are related to $N = 2$ minimal models in the conjectured fashion, then via (2.11), we get from this an explicit formula for certain characters of the $N = 2$ superconformal algebra. Indeed, the Ramond sector character $\chi_s(q,\gamma)$ corresponding to the chiral primary field $\Phi^s$ must be

$$\chi_s(q,\gamma) = \frac{1}{k+2} \sum_{m=0}^{k+1} Z(q,\gamma + 2\pi m, 0) \exp\left(i\pi\widehat{c}m - 2\pi ims\alpha\right),$$
(2.25)

with $Z(q,\gamma,0)$ as in (2.24).

It is straightforward to expand the above formula for the $\chi_s$ in powers of $q$ and verify the first few terms. For instance, verifying the formulas for the first two levels is straightforward using the formulas

$$\begin{aligned}
\chi_0 &= e^{-i\gamma k\alpha/2}\left(1 + q(1 - e^{i\gamma}) + O(q^2)\right) \\
\chi_n &= e^{-i\gamma k\alpha/2}e^{i\gamma n\alpha}\left(1 + q(2 - e^{i\gamma} - e^{-i\gamma}) + O(q^2)\right), \quad \text{for } 0 < n < k \\
\chi_k &= e^{i\gamma k\alpha/2}\left(1 + q(1 - e^{-i\gamma}) + O(q^2)\right) \\
\chi_{k+1} &= 0
\end{aligned}$$
(2.26)

which follow easily from the $N = 2$ algebra. Apart from the vanishing of $\chi_{k+1}$, which reflects the fact that $\Phi^{k+1}$ is not a primary field, these formulas express the

fact that all the $\chi_s$ have highest weight states of zero energy, and $\chi_0$ and $\chi_k$ have additional null vectors of level one, because they are related under spectral flow to the representation containing the identity operator.

A general proof of the above formulas for $\chi_s$ has been proposed by N. Warner (private communication), by comparing them to previously known character formulas [22–27]. I believe that it should be possible to obtain a more conceptual proof along the following lines. The **A** series of $N = 2$ minimal models is known to have a representation as a gauged WZW model of $SU(2)/U(1)$. It should be possible to compute the elliptic genus of a gauged WZW model along the same lines as the above, and in the special case of $SU(2)/U(1)$, this is likely to give back the above formulas.

In the rest of this paper, I will pursue a somewhat different approach to better understanding of (2.25). I will extract from the Landau-Ginzburg model a free field representation of the $N = 2$ superconformal algebra, in a module with character (2.25).

## 3. The Landau-Ginzburg Model And The $N = 2$ Algebra

### 3.1. $N = 1$ INDEX AND $N = 2$ COHOMOLOGY

First we recall the interpretation of the elliptic genus as an index. For this, it suffices to have global $N = 1$ supersymmetry, with a $\mathbf{Z}_2$ $R$ symmetry. Such a symmetry is a factorization

$$(-1)^F = (-1)^{F_L}(-1)^{F_R}, \tag{3.1}$$

where $(-1)^{F_R}$ anticommutes with the right-moving supercharge $Q_R$ and commutes with the left-moving supercharge $Q_L$, and vice-versa for $(-1)^{F_L}$. In string theory, such a factorization is used in performing the GSO projection.

The right-moving part of the $N = 1$ supersymmetry algebra is $Q_R{}^2 = H_R$, where $H_R = L_0 = (H + P)/2$, with $H$ and $P$ being the Hamiltonian and the momentum. The index of $Q_R$, regarded as an operator from states of $(-1)^{F_R} = 1$ to states of $(-1)^{F_R} = -1$, is by a standard argument the difference between the number of bosonic and fermionic states of $H_+ = 0$, assuming these numbers are finite. One can write it as

$$\text{Tr}(-1)^{F_R}\overline{q}^{H_R}, \tag{3.2}$$

if that trace converges. Standard arguments show that states of $H_R \neq 0$ cancel out in pairs in (3.2), so if the spectrum is such that (3.2) is convergent, this expression is independent of $\overline{q}$ and is a topological invariant.

The index of $Q_R$ in that naive sense is not defined in most quantum field theories, because we have included no convergence factor for the left-movers. In general, it is essential to consider a twisted or character-valued version of the index, a quantity such as the elliptic genus

$$\text{Tr}(-1)^{F_R}\widetilde{q}^{H_R}q^{H_L} \cdot X \tag{3.3}$$

with $|q|, |\widetilde{q}| < 1$, $H_L = \overline{L}_0$, and $X$ any operator that commutes with $Q_R$. (In $N = 2$ models with continuous $R$ symmetry, it is convenient to take $X = (-1)^{F_L}\exp(i\gamma J_{0,L})$ as in §2.) Using the fact that $q^{H_L}X$ commutes with $Q_R$, one can show that states of $H_R \neq 0$ cancel out in pairs from this trace. Hence if (as in most interesting field theories), the $H_L$ eigenvalues of states of $H_R = 0$ grow fast enough that the trace in (3.3) converges, this trace is independent of $\widetilde{q}$ (and so is holomorphic in $q$ if, as usual, one sets $\widetilde{q} = \overline{q}$).

So far, we have used only $N = 1$ supersymmetry and discrete $R$ invariance. With $N = 2$ supersymmetry, there are two right-moving supercharges, $Q_+$ and $\overline{Q}_+$, with $Q_+{}^2 = \overline{Q}_+{}^2 = 0$, $\{Q_+, \overline{Q}_+\} = 2H_R$. The fundamental novelty is that, using the fact that $\overline{Q}_+{}^2 = 0$, one can define the *cohomology* of $\overline{Q}_+$. By standard arguments of Hodge theory, the cohomology of $\overline{Q}_+$ can be identified with the

15

kernel of $H_R$. Hence, any trace such as (3.3) can be regarded as a trace in the cohomology of $\overline{Q}_+$. However, by considering the cohomology of $\overline{Q}_+$ one has much more structure than if one considers only the graded traces in this cohomology. The cohomology is a graded vector space, and one can perform operations in this space other than taking cohomology.

For instance, one can look for operators that commute with $\overline{Q}_+$ and so act on the cohomology of this operator. Operators of the form $\{\overline{Q}_+, \ldots\}$ will act trivially on the cohomology of $\overline{Q}_+$, so the natural problem is to consider operators that commute with $\overline{Q}_+$ modulo operators of the form $\{\overline{Q}_+, \ldots\}$; that is, we are interested in cohomology classes of operators that commute with $\overline{Q}_+$. If $\{\overline{Q}_+, \mathcal{O}\} = \{\overline{Q}_+, \mathcal{O}'\} = 0$, then $\{\overline{Q}_+, \mathcal{O}\mathcal{O}'\} = 0$, and if $\{\overline{Q}_+, \mathcal{O}\} = 0$, then $\mathcal{O}\{\overline{Q}_+, X\} = \{\overline{Q}_+, \mathcal{O}X\}$. These two statements mean that the cohomology classes of operators that commute with $\overline{Q}_+$ form a closed (and well-defined) algebra under operator products.

Moreover, if $\{\overline{Q}_+, \mathcal{O}\} = 0$, then $[H_R, \mathcal{O}] = \{\overline{Q}_+, \{Q_+, \mathcal{O}\}\}/2$, or more briefly $[H_R, \mathcal{O}] = \{\overline{Q}_+, \ldots\}$. Hence, operators invariant under $\overline{Q}_+$ commute with $H_R$ up to $\{\overline{Q}_+, \ldots\}$. Thus, $H_R$ annihilates cohomology classes of $\overline{Q}_+$-invariant operators. An operator annihilated by $H_R$ varies holomorphically on the world-sheet. All this can be summarized by saying that the cohomology of $\overline{Q}_+$ has the structure of a closed algebra of operators that vary holomorphically. This structure is a (not necessarily unitary) chiral algebra in the sense of rational conformal field theory. This has been noted earlier (see question 7 in [28]).

Along with WZW models and their derivatives – which have been extensively studied – this construction seems to give one of the few known sources of such chiral algebras.

In what follows, we will study the chiral algebras acting on the cohomology of an $N = 2$ Landau-Ginzburg model with one chiral superfield $\Phi$ and superpotential $W(\Phi) = \Phi^{k+2}/(k+2)$. We will show that the chiral algebra derived from this theory is an $N = 2$ superconformal algebra with the central charge of the $N = 2$

minimal models. Since this algebra acts naturally in the cohomology of $\overline{Q}_+$, which is a graded vector space whose character is the candidate (2.25) for characters of the $N = 2$ algebra, this gives a natural explanation of (2.25) (but not yet a proof as we will not prove irreducibility of the action of the $N = 2$ algebra on the cohomology).

There are several notable features about the $N = 2$ algebra that acts on the cohomology of the Landau-Ginzburg model. (1) This is a *superconformal* algebra even though the underlying Landau-Ginzburg model had global supersymmetry only. Hopefully, its occurrence is a harbinger of the conjectured Landau-Ginzburg/minimal model connection. (2) Once the $N = 2$ algebra that acts on the cohomology is found, its definition continues to make sense if the superpotential of the $N = 2$ model is set to zero. We get then a *free field realization* of the $N = 2$ superconformal algebra; such representations are known [2,25,29–36]. The one we obtain is not the one that has been most frequently seen, but it can be obtained from that of [36] by a simple transformation (regarding $\phi$ and $\partial\bar\phi$ as independent variables $\beta$ and $\gamma$). The considerations of [36] are based on a simple model that in contrast to the Landau-Ginzburg model has a manifest $N = 2$ superconformal symmetry (classically and quantum mechanically) but no manifest properties of unitarity. (3) The "screening charge" associated with this free field representation (which was also introduced in [36]) can be simply identified by further study of $\overline{Q}_+$, making this one of the few cases in which such a screening charge has a simple conceptual explanation.

## 3.2. The Free Field Realization

We now want to find cohomology classes of the $\overline{Q}_+$ operator of the Landau-Ginzburg model. It is convenient to first transform the problem as follows.

We recall that supersymmetry is realized in $N = 2$ superspace by the differen-

tial operators

$$Q_\alpha = \frac{\partial}{\partial \theta^\alpha} - i\sigma^m_{\alpha\dot\alpha}\overline\theta^{\dot\alpha}\frac{\partial}{\partial x^m}$$
$$\overline Q_{\dot\alpha} = -\frac{\partial}{\partial \overline\theta^{\dot\alpha}} + i\sigma^m_{\alpha\dot\alpha}\theta^\alpha\frac{\partial}{\partial x^m}. \tag{3.4}$$

These operators commute with the operators

$$D_\alpha = \frac{\partial}{\partial \theta^\alpha} + i\sigma^m_{\alpha\dot\alpha}\overline\theta^{\dot\alpha}\frac{\partial}{\partial x^m}$$
$$\overline D_{\dot\alpha} = -\frac{\partial}{\partial \overline\theta^{\dot\alpha}} - i\sigma^m_{\alpha\dot\alpha}\theta^\alpha\frac{\partial}{\partial x^m}. \tag{3.5}$$

The $Q$'s and $D$'s are related by formulas such as

$$\overline Q_{\dot\alpha} = \exp\left(-2i\sigma^m_{\alpha\dot\alpha}\theta^\alpha\overline\theta^{\dot\alpha}\frac{\partial}{\partial x^m}\right)\overline D_{\dot\alpha}\exp\left(2i\sigma^m_{\alpha\dot\alpha}\theta^\alpha\overline\theta^{\dot\alpha}\frac{\partial}{\partial x^m}\right). \tag{3.6}$$

Setting $\dot\alpha = +$, this formula means that the operators $\overline Q_+$ and $\overline D_+$ are conjugate; hence instead of computing the cohomology of $\overline Q_+$, we can compute the cohomology of $\overline D_+$. This is a more convenient formulation.

The superspace equations of motion derived from the superspace Landau-Ginzburg Lagrangian (1.3) are

$$2\overline D_+\overline D_-\overline\Phi - \Phi^{k+1} = 0. \tag{3.7}$$

A short computation shows that if

$$\mathcal{J} = \frac{1}{2}(1-\alpha)D_-\Phi\overline D_-\overline\Phi - i\alpha\Phi(\partial_0 - \partial_1)\overline\Phi, \tag{3.8}$$

then $\overline D_+\mathcal{J} = 0$, and $\mathcal{J}$ cannot be written as $\{\overline D_+, \ldots\}$. $\mathcal{J}$ can be expanded in components to give various operators that all represent non-trivial $\overline D_+$ cohomology

18

classes:

$$\mathcal{J}|_{\theta=\overline{\theta}=0} = (1-\alpha)\psi_-\overline{\psi}_- - i\alpha\phi(\partial_0 - \partial_1)\overline{\phi}$$

$$D_-\mathcal{J}|_{\theta=\overline{\theta}=0} = -i\sqrt{2}\psi_-(\partial_0 - \partial_1)\overline{\phi}$$

$$\overline{D}_-\mathcal{J}|_{\theta=\overline{\theta}=0} = i\sqrt{2}(1-\alpha)(\partial_0 - \partial_1)\phi\cdot\overline{\psi}_- - i\sqrt{2}\alpha\phi(\partial_0 - \partial_1)\overline{\psi}_-$$

$$\frac{1}{2}(\overline{D}_-D_- - D_-\overline{D}_-)\mathcal{J}\bigg|_{\theta=\overline{\theta}=0} = 2(\partial_0 - \partial_1)\phi\cdot(\partial_0 - \partial_1)\overline{\phi} + i\left(\psi_-(\partial_0 - \partial_1)\overline{\psi}_- - (\partial_0 - \partial_1)\psi_-\cdot\overline{\psi}_-\right)$$

$$+ \alpha(\partial_0 - \partial_1)\left(i\psi_-\overline{\psi}_- - \phi(\partial_0 - \partial_1)\overline{\phi}\right).$$

$$(3.9)$$

As we will see, these components are respectively the $U(1)$ current, the two supercurrents, and the stress tensor of an $N = 2$ superconformal algebra with $\hat{c} = 1 - 2\alpha = 1 - 2/(k+2)$.

We will verify this by computing the singular terms in the operator products $\mathcal{J}(x,\theta)\mathcal{J}(x',\theta')$. In doing so, we can ignore the part of the Lagrangian coming from the superpotential because these terms are too "soft" to affect the singularities of the $\mathcal{J}\mathcal{J}$ operator product. Therefore, we can study the operator products of the $\mathcal{J}$'s in free field theory. Thus, our computation will amount to a demonstration that the formulas (3.9) give a free field representation of the $N = 2$ algebra at level $k$.

According to [7, eqn. 9.14], the free field propagator of the superfield $\Phi$ is

$$\langle\Phi(x,\theta)\overline{\Phi}(x',\theta')\rangle = -\frac{1}{4\pi}\ln(\widetilde{x}^m\widetilde{x}_m), \tag{3.10}$$

where

$$\widetilde{x}^m = (x - x')^m + i\theta\sigma^m\overline{\theta} + i\theta'\sigma^m\overline{\theta}' - 2i\theta\sigma^m\overline{\theta}'. \tag{3.11}$$

Using this propagator, it is straightforward but slightly lengthy to compute the singular part of the operator products,

$$(4\pi)^2\mathcal{J}(x,\theta)\mathcal{J}(0,0) \sim -\frac{8\theta^-\overline{\theta}^-}{(x^0 - x^1)^2}\mathcal{J} - \frac{2i\theta^-}{x^0 - x^1}D_-\mathcal{J} - \frac{2i\overline{\theta}^-}{x^0 - x^1}\overline{D}_-\mathcal{J}$$

$$- \frac{4\theta^-\overline{\theta}^-}{x^0 - x^1}(\partial_0 - \partial_1)\mathcal{J} - \frac{4(1-2\alpha)}{(x^0 - x^1)^2} + \dots. \tag{3.12}$$

19

This is a closed operator algebra, as expected; indeed, it is an $N = 2$ superconformal algebra with the expected central charge $\widehat{c} = 1 - 2\alpha$ (for instance, compare to equation (10) of [1]).

I will not try to prove that the operator algebra generated by $\mathcal{J}$ is the *full* chiral algebra acting on the cohomology of the $\overline{Q}_+$ operator in the Landau-Ginzburg model. However, this would be a consequence of our other conjecture, which was that the $N = 2$ algebra generated by $\mathcal{J}$ acts irreducibly on the $\overline{Q}_+$ cohomology.

Many Superfields

Though we have focussed in this paper on Landau-Ginzburg models with just one chiral superfield, most of our considerations carry over to more general models. Consider a Landau-Ginzburg model with several chiral superfields $\Phi_i$, $i = 1 \ldots n$ and superspace Lagrangian

$$L = \int \mathrm{d}^2x \, \mathrm{d}^4\theta \sum_i \overline{\Phi}_i \Phi_i - \int \mathrm{d}^2x \, \mathrm{d}^2\theta \, W(\Phi_i) - \int \mathrm{d}^2x \, \mathrm{d}^2\overline{\theta} \, \overline{W}(\overline{\Phi}_i). \qquad (3.13)$$

The equations of motion are

$$2\overline{D}_+\overline{D}_-\overline{\Phi}_i = \frac{\partial W}{\partial \Phi_i}. \qquad (3.14)$$

The superpotential is said to be quasi-homogeneous if for some real numbers $\alpha_i$, the Euler equation

$$W = \sum_i \alpha_i \Phi_i \frac{\partial W}{\partial \Phi_i} \qquad (3.15)$$

is obeyed. If $W$ is quasi-homogeneous, a small calculation shows that

$$\mathcal{J} = \sum_i \left( \frac{1 - \alpha_i}{2} D_-\Phi_i \overline{D}_-\overline{\Phi}_i - i\alpha_i \Phi_i(\partial_0 - \partial_1)\overline{\Phi}_i \right) \qquad (3.16)$$

obeys $\overline{D}_+\mathcal{J} = 0$.

In computing the singular part of the $\mathcal{J}\mathcal{J}$ operator products, the superpotential can be dropped, just as in the one variable case. The $\Phi_i$ can hence be treated as decoupled free fields. Formula (3.12) from the case of one superfield therefore carries over at once to show that $\mathcal{J}$ generates an $N = 2$ superconformal algebra; but the central charge receives contributions from each of the $\Phi_i$ and so is now

$$\widehat{c} = \sum_i (1 - 2\alpha_i). \tag{3.17}$$

This agrees with a formula of [12–14] for the central charge of the conformal field theory arising in the infrared from a Landau-Ginzburg model with quasi-homogeneous superpotential.

One important difference from the one superfield case is that in general one cannot expect $\mathcal{J}$ to generate the full chiral algebra acting on the $\overline{Q}_+$ cohomology, only a sub-algebra. It would be interesting to learn more about the full chiral algebra that acts on the cohomology of a Landau-Ginzburg model.

### 3.3. THE SCREENING CHARGE

The $\overline{Q}_+$ operator can be written explicitly in components

$$\overline{Q}_+ = \int \mathrm{d}x^1 \left( i\overline{\psi}_+ (\partial_0 + \partial_1)\phi + \phi^{k+1}\psi_- \right). \tag{3.18}$$

We can write this $\overline{Q}_+ = \overline{Q}_{+,L} + \overline{Q}_{+,R}$, where

$$\begin{aligned} \overline{Q}_{+,R} &= \int \mathrm{d}x^1 \ i\overline{\psi}_+(\partial_0 + \partial_1)\phi \\ \overline{Q}_{+,L} &= \int \mathrm{d}x^1 \ \phi^{k+1}\psi_-. \end{aligned} \tag{3.19}$$

These obey $\overline{Q}_{+,R}^2 = \overline{Q}_{+,L}^2 = \{\overline{Q}_{+,R}, \overline{Q}_{+,L}\} = 0$.

Because the Landau-Ginzburg theory is superrenormalizable and in fact entirely free of divergences, the Hilbert space $\mathcal{H}$ of the interacting theory can be taken to coincide with the Hilbert space of the free theory with vanishing superpotential. So it is natural to consider $\overline{Q}_+$ as an operator on the free field Fock space. In so doing, one must specify how to treat the zero modes $\phi_0, \overline{\phi}_0$ of $\phi, \overline{\phi}$; we will simply work in a Fock space of states $|n, m\rangle = \phi_0{}^n \overline{\phi}_0{}^m |\Omega\rangle.^\star$

Now let $\mathcal{H}_R$ to be a Fock space of right-moving modes of $\phi, \overline{\phi}, \psi_+$, and $\overline{\psi}_+$, and zero modes of $\overline{\phi}, \psi_+, \overline{\psi}_+$; and let $\mathcal{H}_L$ to be a Fock space of left-moving modes of $\phi, \overline{\phi}, \psi_-$, and $\overline{\psi}_-$, and zero modes of $\phi, \psi_-, \overline{\psi}_-$. Then $\mathcal{H} = \mathcal{H}_L \otimes \mathcal{H}_R$. Moreover, $\overline{Q}_L$ acts in $\mathcal{H}_L$, and $\overline{Q}_R$ acts in $\mathcal{H}_R$. So the cohomology of $\overline{Q}_+$ acting on $\mathcal{H}$ is simply the tensor product of the cohomology of $\overline{Q}_{+,L}$ acting on $\mathcal{H}_L$ and that of $\overline{Q}_{+,R}$ acting on $\mathcal{H}_R$.

But the cohomology of $\overline{Q}_{+,R}$ acting on $\mathcal{H}_R$ is one dimensional. Indeed, as $\overline{Q}_{+,R}$ is quadratic in the fields, its cohomology can be conveniently computed by going to a basis of Fourier modes. The part of $\overline{Q}_{+,R}$ involving the $n^{th}$ Fourier mode of $\phi, \psi_+$ (and so the $-n^{th}$ mode of $\overline{\phi}, \overline{\psi}_+$) is of the form

$$S_n = \begin{pmatrix} 0 & A \\ 0 & 0 \end{pmatrix}, \tag{3.20}$$

where $A$ is a bosonic creation or annihilation operator. The cohomology of $S_n$ is one dimensional, and $\overline{Q}_{+,R}$ is simply an infinite sum of operators of this form, each acting on its own Hilbert space; and so $\overline{Q}_R$ has one dimensional cohomology.

---

$\star$ This Fock space is precisely what one would have if one "twists" the theory slightly in the $x^1$ direction by $J_{0,L}$, thus shifting the zero modes away from zero and reducing everything to a standard Fock space. This is a convenient artifice; however, in a more natural formulation of the problem, one permits the zero mode part of the quantum wave-function to be an arbitrary $\mathbf{L}^2$ function $\chi(\phi_0)$, with no natural description as a Fock space. In this case, it is still true that the cohomology of $\overline{Q}_+$ coincides with the cohomology of the operator $\overline{Q}_{+,L}$ acting on the space $\mathcal{H}_L$ defined in the text; but the argument required to prove this is a little more elaborate. The argument can be made by a very simple case of a spectral sequence, filtering the Hilbert space $\mathcal{H}$ by an operator that commutes with $\overline{Q}_{+,R}$ and under which $\overline{Q}_{+,L}$ has positive degree; then one computes the cohomology of $\overline{Q}_+$ by computing first the cohomology of $\overline{Q}_{+,R}$.

Hence the cohomology of $\overline{Q}_+$ is the same as the cohomology of $\overline{Q}_{+,L}$ acting on $\mathcal{H}_L$.

Now, just like $\overline{Q}_{+,L}$, the $N = 2$ generator $\mathcal{J}$ acts in $\mathcal{H}_L$ (the only non-trivial point is that the $\phi$ zero mode but not the $\overline{\phi}$ zero mode enters in the formula for $\mathcal{J}$). The fact that $\mathcal{J}$ commutes with $\overline{Q}_+$ therefore means that $\mathcal{J}$ commutes with $\overline{Q}_{+,L}$. Hence, the $N = 2$ algebra generated by $\mathcal{J}$ actually acts on the cohomology of $\overline{Q}_{+,L}$. The Landau-Ginzburg/minimal model correspondence strongly suggests that this action is unitary and irreducible.

In the language usually used in describing free field representations of chiral algebras, the operator $\overline{Q}_{+,L}$ is the "screening charge" of the free field representation of the $N = 2$ algebra that we have extracted from the Landau-Ginzburg model. Like the construction of $\mathcal{J}$, the construction of $\overline{Q}_{+,L}$ can be generalized directly to Landau-Ginzburg theories with several superfields $\Phi_i$ and an arbitrary superpotential $W$ (which in this case need not even be quasi-homogeneous). The general formula is simply

$$\overline{Q}_{+,L} = \int dx^1 \, \frac{\partial W}{\partial \phi_i} \psi_{-,i}. \tag{3.21}$$

The chiral algebra of the $N = 2$ model, which at a fundamental level consists of fields that commute with $\overline{Q}_+$, is equivalent to the algebra of local functions of left-moving free fields that commute with the screening charge (3.21).

# REFERENCES

1. P. DiVecchia, J. L. Peterson, and H. B. Zheng, "$N = 2$ Superconformal Theory In Two Dimensions," Phys. Lett. **162B** (1985) 327

2. P. DiVecchia, J. L. Petersen and M. Yu, "On The Unitary Representations of $N = 2$ Superconformal Theory," Phys. Lett. **B172** (1986) 211; P. DiVecchia, J. L. Petersen, M. Yu, and H. B. Zheng, "Explicit Construction Of Unitary Representations Of The $N = 2$ Superconformal Algebra," Phys. Lett. **B174** (1986) 280.

3. W. Boucher, D. Friedan, and A. Kent, "Determinant Formulae and Unitarity For the $N = 2$ Superconformal Algebra In Two Dimensions, Or Exact Results On String Compactification," Phys. Let. **B172** (1986) 316.

4. A. Cappelli, C. Itzykson, and J.-B. Zuber, "Modular Invariant Partition Functions In Two Dimensions," Nucl. Phys. **B280** [**FS 18**] (1987) 445; A. Cappelli, "Modular Invariant Partition Functions Of Superconformal Theories," Phys. Lett. **B185** (1987) 82.

5. D. Gepner and Z. Qiu, "Modular Invariant Partition Functions For Parafermionic Field Theories," Nucl. Phys. **B285** [**FS 19**] (1987) 423.

6. D. Gepner, "Space-Time Supersymmetry In Compactified String Theory And Superconformal Models," Nucl. Phys. **B296** (1988) 757.

7. J. Wess and J. Bagger, *Supersymmetry And Supergravity*, Princeton University Press (second edition, 1992).

8. E. Witten, "Phases Of $N = 2$ Models In Two Dimensions," IASSNS-HEP-93/3, to appear in Nucl. Phys. B.

9. S. J. Gates, Jr., M. T. Grisaru, M. Rocek, and W. Siegel, *Superspace, or One Thousand And One Lessons In Supersymmetry* (Benjamin-Cummings, 1983).

10. A. Zamolodchikov, "Conformal Symmetry And Multicritical Points In Two Dimensional Quantum Field Theory," Sov. J. Nucl. Phys. **44** (1986) 529.

11. D. Kastor, E. Martinec, and S. Shenker, "RG Flow In $N = 1$ Discrete Series," Nucl. Phys. **B316** (1989) 590.

12. E. Martinec, "Criticality, Catastrophes, and Compactifications," in *Physics And Mathematics Of Strings*, ed. L. Brink, D. Friedan, and A. M. Polyakov (World Scientific, 1990).

13. C. Vafa and N. Warner, "Catastrophes And The Classification Of Conformal Field Theories," Phys. Lett. **218B** (1989) 51.

14. W. Lerche, C. Vafa, and N. Warner, "Chiral Rings In $N = 2$ Superconformal Theory," Nucl. Phys. **B324** (1989) 427.

15. P. S. Howe and P. C. West, "Chiral Correlators In Landau-Ginzburg Models And $N = 2$ Superconformal Models," Phys. Lett. **B227** (1989) 397, "Fixed Points In Multi-Field Landau-Ginzburg Models," Phys. Lett. **B244** (1990) 270.

16. S. Cecotti, L. Girardello, and A. Pasquinucci, "Non-perturbative Aspects And Exact Results For The $N = 2$ Landau-Ginzburg Models," Nucl. Phys. **B238** (1989) 701, "Singularity Theory And $N = 2$ Supersymmetry," Int. J. Mod. Phys. **A6** (1991) 2427.

17. S. Cecotti, "$N = 2$ Landau-Ginzburg vs. Calabi-Yau Sigma Models: Non-perturbative Aspects," Int. J. Mod. Phys. **A6** (1991) 1749.

18. P. Fendley and K. Intriligator, "Scattering And Thermodynamics Of Fractionally Charged Supersymmetric Solitons," Nucl. Phys. **B372** (1992) 533, "Scattering And Thermodynamics In Integrable $N = 2$ Theories," Nucl. Phys. **B380** (1992) 385; S. Cecotti, P. Fendley, K. Intriligator, and C. Vafa, "A New Supersymmetric Index," Nucl. Phys. **B386** (1992) 405.

19. A. Schellekens and N. Warner, "Anomalies And Modular Invariance In String Theory," Phys. Lett. **177B** (1986) 317, "Anomaly Cancellation And Self-Dual Lattices," Phys. Lett. **181B** (1986) 339; K. Pilch, A. Schellekens, and N. Warner, "Anomalies, Characters, And Strings," Nucl. Phys. **B287** (1987) 317.

20. E. Witten, "Elliptic Genera And Quantum Field THeory," Commun. Math. Phys. **109** (1987) 525, "The Index Of The Dirac Operator In Loop Space," in P. Landwebber, ed., *Elliptic Curves And Modular Forms In Algebraic Topology* (Springer-Verlag, 1988).

21. C. Vafa, "String Theory And Orbifoldized LG Models," Mod. Phys. Lett. **A4** (1989) 1169.

22. V. K. Dobrev, "Characters Of The Unitarizable Highest Weight Modules Over The $N = 2$ Superconformal Algebra," Phys. Lett. **B186** (1987) 43.

23. F. Ravanini and S.-K. Yang, "Modular Invariance In $N = 2$ Superconformal Field Theory," Phys. Lett. **B195** (1987) 202.

24. Z. Qiu, "Modular Invariant Partition Functions For $N = 2$ Superconformal Field Theory," Phys. Lett. **B198** (1987) 497.

25. J. Distler and Z. Qiu, "BRS Cohomology And A Feigin-Fuchs Representation of Kac-Moody and Parafermionic Theories," Nucl. Phys. **B336** (1990) 533.

26. T. Eguchi, T. Kawai, S. Mizoguchi, and S.-K. Yang, "Character Formulas For Coset $N = 2$ Superconformal Theories," IEK Preprint 91-111.

27. R. Kedem, T. R. Klassen, B. M. McCoy, and E. Melzer, "Fermionic Sum Representations For Conformal Field Theory Characters," Stonybrook preprint ITP-SB-93-0.

28. C. Vafa, "Superstring Vacua," in the proceedings of the Trieste School on High Energy Physics, 1989.

29. A. B. Zamolodchikov and V. A. Fateev, "Disorder Fields In Two-Dimensional Conformal Quantum-Field Theory And $N = 2$ Extended Supersymmetry," Sov. Phys. JETP **63** (1986) 913.

30. S. Nam, "The Kac Formula For The $N = 1$ and $N = 2$ Superconformal Algebras," Phys. Lett. **B172** (1986) 323.

31. G. Mussardo, "Fusion Rules, Four Point Functions, and Discrete Symmetries Of $N = 2$ Superconformal Models," Phys. Lett. **B128** (1989) 191, "$N = 2$ Superconformal Minimal Models," Int. J. Mod. Phys. **A4** (1989) 1135.

32. N. Ohta and H. Suzuki, "$N = 2$ Superconformal Models And Their Free Field Realizations," Nucl. Phys. **B332** (1990) 146.

33. K. Ito, "Quantum Hamiltonian Reduction And $N = 2$ Coset Models," Phys. Lett. **B259** (1991) 73.

34. D. Nemeschansky and S. Yankielowicz, "$N = 2$ $W$-algebras, Kazama-Suzuki Models, and Drinfeld-Sokolov Reduction," USC preprint USC-007-91 (1991).

35. V. Sadov, "Free Field Resolution For Nonunitary Representations Of $N = 2$ SuperVirasoro," Harvard preprint HUTP-92/A070.

36. P. Fré, F. Gliozzi, M. Monteiro, and A. Piras, "A Moduli-Dependent Lagrangian For $(2, 2)$ Theories On Calabi-Yau $n$-Folds," Class. Quant. Grav. **8** (1991) 1455; P. Fré, L. Girardello, A. Lerda, and P. Soriani, "Topological First-Order Systems With Landau-Ginzburg Interactions," Nucl. Phys. **B387** (1992) 333.