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Optimal Universal Uncertainty Relations

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We study universal uncertainty relations and present a method called joint probability distribution diagram to improve the majorization bounds constructed independently in [Phys. Rev. Lett. 111, 230401 (2013)] and [J. Phys. A. 46, 272002 (2013)]. The results give rise to state independent uncertainty relations satisfied by any nonnegative Schur-concave functions. On the other hand, a remarkable recent result of entropic uncertainty relation is the direct-sum majorization relation. In this paper, we illustrate our bounds by showing how they provide a complement to that in [Phys. Rev. A. 89, 052115 (2014)].

Uncertainty relations[1] are of profound significance in quantum mechanics and quantum information theory. Various important applications of uncertainty relations have been discovered such as entanglement detection[2], steering inequalities[3] and quantum cryptography[4–6]. The well-known form of the Heisenberg's uncertainty relations, given by Robertson[7], says that the standard deviations of the observables ΔA and ΔB satisfy the following inequality,

$$\Delta A \cdot \Delta B \geq |\langle \psi | [A, B] | \psi \rangle| / 2. \tag{1}$$

As a consequence of the uncertainty relations, it is impossible to determine the exact values of the two incompatible observables simultaneously. However, the lower bound in the above uncertainty inequality may become trivial if the measured state $|\psi\rangle$ belongs to the nullspace of the commutator $[A, B]$.

In fact, the uncertainty relations provide a limitation on how much information one can obtain by measuring a physical system, and can be characterized in terms of the probability distributions of the measurement outcomes. In order to overcome the drawback in the product form of variance base uncertainty relations, Deutsch[8] introduced the entropic uncertainty relations, which were later improved by Maassen and Uffink[9]: $H(A) + H(B) \geq -2 \log c(A, B)$, where H is the Shannon entropy, $c(A, B) = \max_{m,n} |\langle a_m | b_n \rangle|$ is maximum overlap between the basis elements $\{|a_m\rangle\}$ and $\{|b_n\rangle\}$ of the eigenbases of A and B, respectively. Recently, the Maassen-Uffink bound has been surprisingly improved by Coles and Piani[10], Rudnicki, Puchała and Życzkowski[11], for a review on entopic uncertainty relations see refs 12 and 13.

Friedland, Gheorghiu and Gour[14] proposed a new concept called "universal uncertainty relations" which are not limited to considering only the well-known entropic functions such as Shannon entropy, Renyi entropy and Tsallis entropy, but also any nonnegative Schur-concave functions. On the other hand, Puchała, Rudnicki and Życzkowski[15] independently used majorization technique to establish entropic uncertainty relations similar to "universal uncertainty relations". Let $|a_m\rangle_{m=1}^d$ and $|b_n\rangle_{n=1}^d$ be orthonormal bases of a d-dimensional Hilbert space H. Denote by $p_m(\rho) = \langle a_m | \rho | a_m \rangle$ and $q_n(\rho) = \langle b_n | \rho | b_n \rangle$ the probability distributions obtained by measuring the state ρ with respect to these bases, which constitute two probability vectors $\mathbf{p}(\rho) = (p_1, p_2, \ldots, p_d)$ and $\mathbf{q}(\rho) = (q_1, q_2, \ldots, q_d)$, respectively. It has been shown that the tensor product of the two probability vectors $\mathbf{p}(\rho)$ and $\mathbf{q}(\rho)$ is majored by a vector ω independent from the state ρ,

$$\mathbf{p}(\rho) \otimes \mathbf{q}(\rho) \prec \omega, \quad \text{for any } \rho, \tag{2}$$

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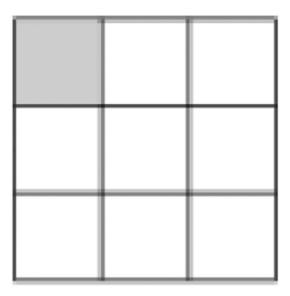
Figure 1. The left-top shadow of the block box specifies the entry $p_1 q_1$.

where "$\prec$" stands for "majorization": $\mathbf{x} \prec \mathbf{y}$ in R^d if $\sum_{j=1}^{k} x_k^{\downarrow} \leq \sum_{j=1}^{k} y_k^{\downarrow}$ for all $1 \leq k \leq d-1$ and $\sum_{j=1}^{d} x_k^{\downarrow} = \sum_{j=1}^{d} y_k^{\downarrow}$. The down-arrow vector $\mathbf{x}^{\downarrow}$ denotes that the components of $\mathbf{x}$ are rearranged in descending order, $x_1^{\downarrow} \geq x_2^{\downarrow} \geq \cdots \geq x_d^{\downarrow}$. The d^2-dimensional vector ω is given by

$$\omega = (\Omega_1, \Omega_2 - \Omega_1, \ldots, \Omega_d - \Omega_{d-1}, 0, \ldots, 0), \tag{3}$$

where

$$\Omega_k = \max_{I_k} \max_{\rho} \sum_{(m,n) \in I_k} p_m(\rho) q_n(\rho) \tag{4}$$

with $I_k \subset [d] \times [d]$ being a subset of k distinct pairs of indices (m, n) and $[d]$ is the set of the natural numbers from 1 to d. The outer maximum is over all subsets I_k with cardinality k and the inner maximum runs over all density matrices.

Equation (2) is called a universal uncertainty relation, as for any uncertainty measure Φ, a nonnegative Schur-concave function, one has that

$$\Phi[p(\rho) \otimes q(\rho)] \geq \Phi(\omega), \quad \textit{for any } \rho. \tag{5}$$

The universal uncertainty relation (UUR) (2) generates infinitely many uncertainty relations, for each Φ, in which the right hand side provides a single lower bound.

In relation (2), the state independent vector ω decided by Ω_k in Eq. (4) is too hard to evaluate explicitly in general, as it is involved with a highly nontrivial optimization problem. For this reason, only an approximation $\tilde{\Omega}_k$ of Ω_k has been presented[14,15] to construct a weaker majorization vector $\tilde{\omega}$. Naturally, how to find a stronger approximation than previous works becomes an interesting open question.

Results

We first introduce a scheme called "joint probability distribution diagram" (JPDD) to consider the optimization problem involved in calculating Ω_k. Next, we present a stronger approximation by proposing an analytical formula for Ω_k. To facilitate presentation, we denote our stronger approximation as Ω_k without ambiguity. All uncertainty relations considered in the paper will be in the absence of quantum side information.

To construct the joint probability distribution diagram, we associate each summand $p_i(\rho) q_j(\rho)$ in Ω_k to a box located at the position (i, j). Then the summation in Ω_k corresponds to certain region of boxes (or rather lattice points) in the first quadrant. We configure the region in a combinatorial way. Suppose that $p_1 \geq p_2 \geq \cdots \geq p_d$, $q_1 \geq q_2 \geq \cdots \geq q_d$. Consider the following $d \times d$-matrix

$$\begin{pmatrix} p_1 q_1 & p_1 q_2 & \cdots & p_1 q_d \\ p_2 q_1 & p_2 q_2 & \cdots & p_2 q_d \\ \vdots & \vdots & \ddots & \vdots \\ p_d q_1 & p_d q_2 & \cdots & p_d q_d \end{pmatrix}, \tag{6}$$

where the entries descend along the rows and columns by assumption. Now, we use a box $\square$ to represent an entry of the matrix. A *shadow* or *grey box* in the JPDD means the corresponding entry in the matrix. For example, the
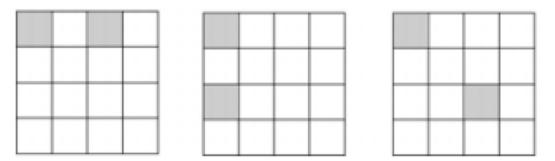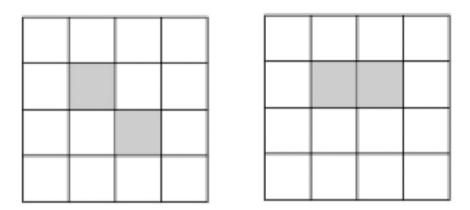
Figure 2. Different boxes.

Figure 3. Connectedness.

top left shadow of the block box specifies the entry p_1q_1, see Fig. 1. Thus the region corresponding to the summation in $\tilde{\Omega}_k$ will be a special region of the rectangular matrix.

Our scheme, JPDD, provides a combinatorial method to compute the special region with respect to the Ω_k. First, it is easy to see that the top upper left box in JPDD is the maximal element, i.e. Ω_1, since $p_1q_1 - p_iq_j \geq p_iq_1 - p_iq_j = p_i(q_1 - q_j) \geq 0$. The main idea is that each exact solution of Ω_k corresponds to a particular region in this matrix.

Suppose that the k-th region is found, i.e. Ω_k is obtained, then the next $(k+1)$-th region is obtained from the k-th region by adding a special box, which must be "connected" with certain boundary of the k-th region. This iterative procedure enables us to compute all Ω_k. Before proving the statement rigorously, we first introduce some terminologies.

[Definition 1] (Different boxes) Two boxes (matrix elements) p_iq_j and p_kq_l are said to be *different* if they occupy different positions in JPDD, namely, $i \neq k$ or $j \neq l$. The Fig. 2 shows three examples of different boxes. Note that it may happen that even if the numerical values of p_iq_j and p_kq_l are the same, but graphically they are treated as different boxes. "different" and "same" do not imply their quantitative relation. For example, p_1q_1 may equal to p_1q_3 in general.

[Definition 2] (Connectedness). Two boxes p_iq_j and p_kq_l in JPDD are *connected* if there does not exist any box p_mq_n, different from both p_iq_j and p_kq_l, such that $\min\{p_iq_j, p_kq_l\} < p_mq_n < \max\{p_iq_j, p_kq_l\}$. For example, different boxes p_iq_j and p_kq_l are connected if $p_iq_j = p_kq_l$. For $d = 4$, if $p_1 > p_2 > p_3 > p_4, q_1 > q_2 > q_3 > q_4, p_iq_j \neq p_kq_l, p_mq_n \neq p_iq_j, p_mq_n \neq p_kq_l$, for any $i, j, k, l \in \{1, 2, 3, 4\}$, then p_2q_2 and p_3q_3 are not connected while p_2q_3 and p_2q_2 are connected, see Fig. 3.

[Definition 3] (Connected region). A set of different boxes is called a *region*, denoted by $\mathfrak{A}$. A region $\mathfrak{A}$ is *connected* if $\forall\ p_iq_j \in \max \mathfrak{A}$, then either $p_{i-1}q_j \in \mathfrak{A}$ or $p_iq_{j-1} \in \mathfrak{A}$, where $\max \mathfrak{A}$ is the maximal value of all the elements in $\mathfrak{A}$. Note that the region of boxes corresponding to a Ω_k must contain the top-left element p_1q_1 in JPDD as its largest element.

For any probability vector $\mathbf{p}(\rho)$ on a d-dimensional Hilbert space with $p_i = \langle a_i|\rho|a_i \rangle$, let $p_1' = \max_i\{p_i\}$, $p_2' \max_{p_i \neq p_j'}\{p_i\}$, $\ldots$, $p_k' = \max_{p_i \neq p_j'}\{p_i\}$. Similarly, $q_1', \cdots, q_k'$ are defined similarly for another probability
$j=1$ $j=1,2\cdots k-1$
vector $\mathbf{q}(\rho)$ on the same Hilbert space. For any sequence $k_1, \cdots, k_n, 1 \leq n \leq d$, we define $\Omega_{k_1,k_2,\cdots,k_n}^d = p_1'(q_1' + q_2' + \cdots q_{k_1}') + p_2'(q_1' + q_2' + \cdots q_{k_2}') + \cdots + p_n'(q_1' + q_2' + \cdots q_{k_n}')$. In particular, if $p_1 \geq \cdots \geq p_d, q_1 \geq \cdots \geq q_d$, then $\Omega_{k_1,k_2,\cdots,k_n}^d = p_1(q_1 + q_2 + \cdots q_{k_1}) + p_2(q_1 + q_2 + \cdots q_{k_2}) + \cdots + p_n(q_1 + q_2 + \cdots q_{k_n})$, which can be configured by the Fig. 4. In a JPDD when the first k boxes are chosen, the next
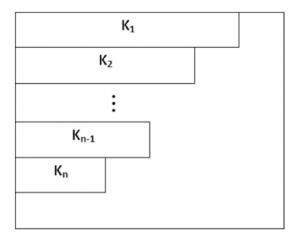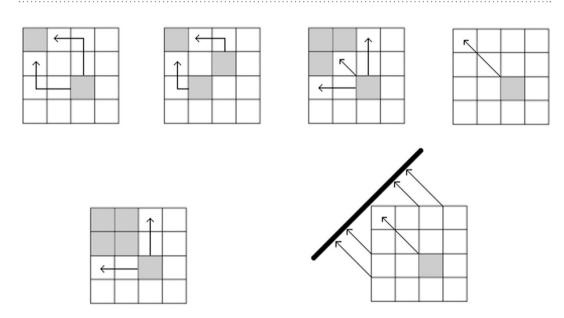
Figure 4. $\Omega^d_{k_1,k_2,\cdots,k_n}$ **related joint probability distribution diagram.**

Figure 5. The (k + 1)-th box is fixed by Ω_k. The arrows show the position of the (k + 1)-th box.

(maximal) $(k+1)$-th box must appear at the top left corner in the unoccupied region, we give this as follow lemma:

[Lemma]: The maximal k boxes for Ω_k in the JPDD can be selected to form a connected region. ∎

Lemma gives a way to get Ω_{k+1} from Ω_k in a JPDD. As an example, we show how to get Ω_3 from Ω_2. Set $c_{11} = \max_{m,n}|\langle a_m|b_n\rangle|$, $c_{21} = \max_{m,n'\neq n}\sqrt{|\langle a_m|b_n\rangle|^2 + |\langle a_m|b_{n'}\rangle|^2}$, $c_{22} = \max_{n,m'\neq m}\sqrt{|\langle a_m|b_n\rangle|^2 + |\langle a_{m'}|b_n\rangle|^2}$. If $c_{21} \geq c_{22}$, then $\Omega_3 = \max\{\Omega^d_3, \Omega^d_{2,1}\}$. If $c_{21} \leq c_{22}$, then $\Omega_3 = \max\{\Omega^d_{1,1,1}, \Omega^d_{2,1}\}$. That is, if $\Omega_2 = \Omega^d_2$, then $\Omega_3 = \max\{\Omega^d_3, \Omega^d_{2,1}\}$. If $\Omega_2 = \Omega^d_{1,1}$, then $\Omega_3 = \max\{\Omega^d_{1,1,1}, \Omega^d_{2,1}\}$. Namely, $\Omega_3 = \max\{\Omega^d_3, \Omega^d_{2,1}, \Omega^d_{1,1,1}\}$. Thus Ω_3 is determined by Ω_2.

In general, $\Omega_{k-1} = \Omega^d_{k_1,k_2,\cdots,k_n}$, subjecting to $\sum_{i=1}^n k_i = k - 1$. By Lemma it follows that

$$\begin{aligned}\Omega_k &= \max\left\{p_1 q_{k_1+1}, \; p_2 q_{k_2+1}, \; \cdots, \; p_n q_{k_n+1}, \; p_{n+1}q_1\right\} + \Omega_{k-1}\\ &= \max\left\{\Omega^d_{k_1+1,k_2,\cdots,k_n}, \; \Omega^d_{k_1,k_2+1,\cdots,k_n}, \; \cdots, \; \Omega^d_{k_1,k_2,\cdots,k_n+1}\right\},\end{aligned} \quad (7)$$

which gives an iterative formula of Ω_k in terms of Ω^d_k's.

We list in Figs 5 and 6 all the possible Ω_k for $k = 1, 2, \ldots, 4$. The above example to get Ω_3 from Ω_2 corresponds to move from the second row to the third. Now we are ready to show the main result.
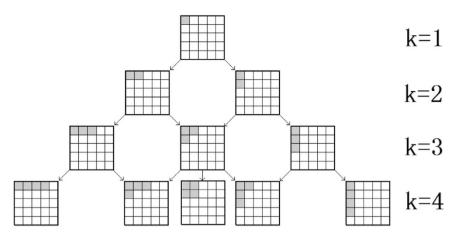
[Theorem] The quantities Ω_k are given by

Figure 6. The tree diagrams to get the JPDDs in the k-th row by adding one shadow box to the JPDDs in the $k-1$ row.

$$\Omega_k = \frac{1}{4} \sum_{\substack{i=1 \\ k_1 \geq \ldots \geq k_n}}^{\max_n} \max_{\substack{k_i = k}} \left\{ \left(\left\| \max_{\substack{|R|=n \\ |S|=k_n}} \left\| \sum_{m \in R} |a_m\rangle\langle a_m| + \sum_{n \in S} |b_n\rangle\langle b_n| \right\|_\infty^2 - \max_{\substack{|R|=n-1 \\ |S|=k_n}} \left\| \sum_{m \in R} |a_m\rangle\langle a_m| + \sum_{n \in S} |b_n\rangle\langle b_n| \right\|_\infty^2 \right) \right. \right.$$

$$+ \left(\max_{\substack{|R|=n-1 \\ |S|=k_{n-1}}} \left\| \sum_{m \in R} |a_m\rangle\langle a_m| + \sum_{n \in S} |b_n\rangle\langle b_n| \right\|_\infty^2 - \max_{\substack{|R|=n-2 \\ |S|=k_{n-1}}} \left\| \sum_{m \in R} |a_m\rangle\langle a_m| + \sum_{n \in S} |b_n\rangle\langle b_n| \right\|_\infty^2 \right)$$

$$+ \cdots + \left(\max_{\substack{|R|=2 \\ |S|=k_2}} \left\| \sum_{m \in R} |a_m\rangle\langle a_m| + \sum_{n \in S} |b_n\rangle\langle b_n| \right\|_\infty^2 - \max_{\substack{|R|=1 \\ |S|=k_2}} \left\| \sum_{m \in R} |a_m\rangle\langle a_m| + \sum_{n \in S} |b_n\rangle\langle b_n| \right\|_\infty^2 \right)$$

$$\left. \left. + \max_{\substack{|R|=1 \\ |S|=k_1}} \left\| \sum_{m \in R} |a_m\rangle\langle a_m| + \sum_{n \in S} |b_n\rangle\langle b_n| \right\|_\infty^2 \right\} . \qquad k = 1, 2, \ldots, d. \quad \blacksquare \right. \tag{8}$$

The solution given in Eq. (8) can be explained as follows. First, for $k = 1, 2$, they are solved simply as

$$\Omega_1 = \frac{1}{4}[1 + c]^2, \quad \Omega_2 = \frac{1}{4}[1 + c']^2,$$

where $c = \max_{m,n}|\langle a_m|b_n\rangle|$, $c' = \max \sqrt{|\langle a_m|b_n\rangle|^2 + |\langle a_{m'}|b_{n'}\rangle|^2}$, the maximum is taken over all indices $m \neq m'$, $n \neq n'$, and over all $n = n', m \neq m'$. Then, for $k = 3$ in JPDD, $\Omega_3 = \max_{I_3} \max_{\rho} \sum_{(M,N) \in I_3} p_m(\rho) q_n(\rho) = \max\{\Omega_3^d, \Omega_{2,1}^d, \Omega_{1,1,1}^d\}$. Furthermore,

$\Omega_3^d = \max_{\substack{|R|=1 \\ |S|=3}} \max_{\rho} (\sum_{m \in R} p_m(\rho))(\sum_{n \in S} q_n(\rho)) = \frac{1}{4} \max_{\substack{|R|=1 \\ |S|=3}} \|\sum_{m \in R} |a_m\rangle\langle a_m| + \sum_{n \in S} |b_n\rangle\langle b_n|\|_\infty^2$ **and** $\Omega_{1,1,1}^d = \frac{1}{4} \max_{\substack{|R|=3 \\ |S|=1}} \|\sum_{m \in R} |a_m\rangle\langle a_m| + \sum_{n \in S} |b_n\rangle\langle b_n|\|_\infty^2$.

Since $\Omega_{2,1}^d = (\Omega_{1,1}^d - \Omega_1) + \Omega_2^d$, we get $\Omega_2^d = \frac{1}{4} \max_{\substack{|R|=1 \\ |S|=2}} \|\sum_{m \in R} |a_m\rangle\langle a_m| + \sum_{n \in S} |b_n\rangle\langle b_n|\|_\infty^2$ and $\Omega_{1,1}^d = \frac{1}{4} \max_{\substack{|R|=2 \\ |S|=1}} \|\sum_{m \in R} |a_m\rangle\langle a_m| + \sum_{n \in S} |b_n\rangle\langle b_n|\|_\infty^2$.

We have shown how to calculate Ω_1, Ω_2 and Ω_3. For the cases $k \geq 4$, interested readers can calculate Ω_k using a similar method and we sketch the details in the Methods.

The above theory enables us to formulate a series of Ω_k, based on all quantities we obtain a tighter majorization vector ω. Note that our method is valid when all the maximums are taken over the same quantum state, otherwise our bounds will fail to hold. Even so, our results can outperform B_{Maj2}[11] to some extent.

Our results enable us to strengthen the bounds on the sum of two Shannon entropies by $\mathbf{B}_{\mathbf{JPDD}} = H(\omega)$, where ω is given by the improved Ω_k in Eq. (8) and H is the Shannon entropy. To see this phenomenon, let us first consider a 4-dimensional system with incompatible observables $\{|a_m\rangle, |b_n\rangle\}_{m,n=1}^4$

$$\left\{ \begin{aligned} &\left(\sqrt{\tfrac{2}{3}}\cos\theta, \sqrt{\tfrac{2}{3}}\sin\theta, \sqrt{\tfrac{1}{3}}\cos\theta, \sqrt{\tfrac{1}{3}}\sin\theta\right), \ (0.68, 0.72, 0.12, -0.071), \ \left(-\sqrt{\tfrac{2}{3}}\sin\theta, \sqrt{\tfrac{2}{3}}\cos\theta, -\sqrt{\tfrac{1}{3}}\sin\theta, \sqrt{\tfrac{1}{3}}\cos\theta\right), \ (0.32, -0.47, 0.71, -0.42), \\ &\left(-\sqrt{\tfrac{1}{3}}\cos\theta, -\sqrt{\tfrac{2}{3}}\sin\theta, -\sqrt{\tfrac{2}{3}}\cos\theta, \sqrt{\tfrac{2}{3}}\sin\theta\right), \ (-0.61, 0.48, 0.63, 0.064), \ \left(\sqrt{\tfrac{1}{3}}\sin\theta, -\sqrt{\tfrac{1}{3}}\cos\theta, -\sqrt{\tfrac{2}{3}}\sin\theta, \sqrt{\tfrac{2}{3}}\cos\theta\right), \ (-0.24, 0.19, -0.29, -0.9) \end{aligned} \right\},$$

and the unitary transformation U ($U_{mn} = \langle a_m|b_n\rangle$) between them

$$\begin{pmatrix} 0.63\cos\theta + 0.54\sin\theta & 0.67\cos\theta - 0.62\sin\theta & -0.13\cos\theta + 0.43\sin\theta & -0.37\cos\theta - 0.36\sin\theta \\ 0.54\cos\theta - 0.63\sin\theta & -0.62\cos\theta - 0.67\sin\theta & 0.43\cos\theta + 0.13\sin\theta & 0.43\cos\theta + 0.13\sin\theta \\ -0.30\cos\theta - 0.47\sin\theta & 0.4\cos\theta - 0.072\sin\theta & 0.86\cos\theta - 0.23\sin\theta & -0.098\cos\theta - 0.85\sin\theta \\ -0.47\cos\theta + 0.30\sin\theta & -0.072\cos\theta - 0.4\sin\theta & -0.23\cos\theta - 0.86\sin\theta & -0.85\cos\theta + 0.098\sin\theta \end{pmatrix}.$$
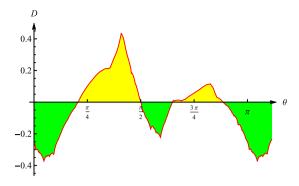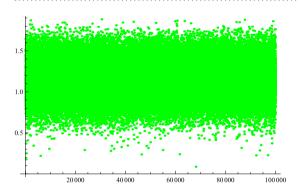
Figure 7. Difference between B_{JPDD} and B_{Maj2}.

Figure 8. The vertical coordinate is $G(c) - F(c)$. The horizontal coordinate is for random runs. It can be seen that our bound outperforms the bound[16] 100% of the time, while a bound given by Friedland[14] outperforms the bound[16] around 90% of the time.

In Fig. 7, we plot the difference between $\mathbf{B_{JPDD}}$ and B_{Maj2}[11], i.e. $B_{JPDD} - B_{Maj2}$ (the red line). Clearly, our bound B_{JPDD} is tighter than B_{Maj2} to some extent. Note that, the entropies are defined with base 2 in general. But in our figures, in order to make them more readable, we take the natural logarithm instead.

To appreciate the stronger vector ω in the improved UUR, we can consider Shannon entropy in the uncertainty relations to obtain a tighter bound than the previous work[16]. Namely if we Φ in Eq. (5) as the Shannon entropy H, then we have

$$H(A) + H(B) \geq \begin{cases} -2\log c, & \text{if} \quad 0 < c \leq 1/\sqrt{2}, \\ H_1(c), & \text{if} \quad 1/\sqrt{2} \leq c \leq c^*, \\ G(c), & \text{if} \quad c^* \leq c \leq 1, \end{cases} \tag{9}$$

where $c^* \approx 0.834$, the bound $H_1(c)$ is the same as the one given in Vincente and Ruiz's work[16], and $G(c) = H(\omega)$ with ω given by our formula Eq. (3). The bound $G(c)$ outperforms the Vincente and Ruiz's bound $F(c)$[16] in the interval $[c^*, 1]$. For further details, see Fig. 8.

Conclusion

In conclusion, we have presented a method called joint probability distribution diagram to strengthen the bounds for the universal uncertainty relations. As an example, we consider the bounds on the sum of the Shannon entropies. As the universal uncertainty relations capture the essence of uncertainty in quantum theory, it is unnecessary to quantify them by particular measures of uncertainty such as Shannon or Renyi entropies. Our results give a way to resolve some important cases in this direction, and is shown to offer a better bound for any uncertainty relations given by the nonnegative Schur-concave functions. Furthermore, how to extend this method to the case of multiple measurements are interesting, which requires further studies.

Methods

Proof of the Lamma. The case of $k = 1$ is obvious since the maximal element is $p_1 q_1$. Assume that the statement holds for the case of $k - 1$: $\Omega_{k-1} = \Omega^d_{k_1, k_2, \cdots k_n}$ is connected with $k_1 > \ldots > k_n > 0$. Suppose on the contrary that the next maximum $\Omega_k = \Omega_{k-1} + p_i q_j$ is not connected. Then there are two possibilities: (i) $i > n$ or $j > k_1$, thus we can replace $p_i q_j$ by $p_i q_1$ or $p_1 q_j$ and move further to $p_{n+1} q_1$ or $p_1 q_{k_1+1}$ to get a possible bigger value for Ω_k. (ii) $i \leq n$ and $j > k_i$, in this case we can also replace the box $p_i q_j$ by a connected box $p_{i'} q_{j'}$ to the region of Ω_{k-1} by sliding it leftward or upward. Hence the statement is true by induction. ∎

Proof of the Theorem. To calculate $\Omega_k = \Omega^d_{k_1, k_2, \cdots k_n}$, $k_1 \geq \ldots \geq k_n$ and $k_1 + \cdots + k_n = k$, we note that

$$\Omega_{k_1}^d = \frac{1}{4}\max_{\substack{|R|=1 \\ |S|=k_1}}\left\|\sum_{m\in R}|a_m\rangle\langle a_m| + \sum_{n\in S}|b_n\rangle\langle b_n|\right\|_\infty^2,$$

$$\Omega_{k_2,k_2}^d = \frac{1}{4}\max_{\substack{|R|=2 \\ |S|=k_2}}\left\|\sum_{m\in R}|a_m\rangle\langle a_m| + \sum_{n\in S}|b_n\rangle\langle b_n|\right\|_\infty^2.$$

where R and S are subsets of distinct indices from $[d]$, $|R|$ is the cardinality of R, and $\|\cdot\|_\infty$ is the infinity operator norm which coincides with the maximum eigenvalue of the positive operator. For a given k, there exist sets of $k_1 \geq \ldots \geq k_n$ such that $\sum_{i=1}^n k_i = k$ for some n. For any such given $k_1 \geq \ldots \geq k_n$, the quantity in $[.]$ in Eq. (8) can be calculated. The outer max picks up the largest quantity for all such possible $k_1, \ldots, k_n$. Then $\Omega_{k_1,k_2}^d = \Omega_{k_2,k_2}^d + (\Omega_{k_1}^d - \Omega_{k_2}^d)$ and $\Omega_{k_1,k_2,k_3}^d = \Omega_{k_3,k_3,k_3}^d + (\Omega_{k_1,k_2}^d - \Omega_{k_3,k_3}^d)$. Continuing in this way we have

$$\begin{aligned}
\Omega_{k_1,\cdots,k_n}^d &= \Omega_{\underbrace{k_n,\cdots,k_n}_{n}}^d + \left(\Omega_{k_1,\cdots,k_{n-1}}^d - \Omega_{\underbrace{k_n,\cdots,k_n}_{n-1}}^d\right) \\
&= \Omega_{\underbrace{k_n,\cdots,k_n}_{n}}^d - \Omega_{\underbrace{k_n,\cdots,k_n}_{n-1}}^d + \Omega_{k_1,\cdots,k_{n-1}}^d \\
&= \left(\Omega_{\underbrace{k_n,\cdots,k_n}_{n}}^d - \Omega_{\underbrace{k_n,\cdots,k_n}_{n-1}}^d\right) + \left(\Omega_{\underbrace{k_{n-1},\cdots,k_{n-1}}_{n-1}}^d - \Omega_{\underbrace{k_{n-1},\cdots,k_{n-1}}_{n-2}}^d\right) \\
&\quad + \cdots + \left(\Omega_{k_2,k_2}^d - \Omega_{k_2}^d\right) + \Omega_{k_1}^d \\
&= \frac{1}{4}\left[\left(\max_{\substack{|R|=n \\ |S|=k_n}}\left\|\sum_{m\in R}|a_m\rangle\langle a_m| + \sum_{n\in S}|b_n\rangle\langle b_n|\right\|_\infty^2 - \max_{\substack{|R|=n-1 \\ |S|=k_n}}\left\|\sum_{m\in R}|a_m\rangle\langle a_m| + \sum_{n\in S}|b_n\rangle\langle b_n|\right\|_\infty^2\right)\right. \\
&\quad + \left(\max_{\substack{|R|=n-1 \\ |S|=k_{n-1}}}\left\|\sum_{m\in R}|a_m\rangle\langle a_m| + \sum_{n\in S}|b_n\rangle\langle b_n|\right\|_\infty^2 - \max_{\substack{|R|=n-2 \\ |S|=k_{n-1}}}\left\|\sum_{m\in R}|a_m\rangle\langle a_m| + \sum_{n\in S}|b_n\rangle\langle b_n|\right\|_\infty^2\right) \\
&\quad + \cdots + \left(\max_{\substack{|R|=2 \\ |S|=k_2}}\left\|\sum_{m\in R}|a_m\rangle\langle a_m| + \sum_{n\in S}|b_n\rangle\langle b_n|\right\|_\infty^2 - \max_{\substack{|R|=1 \\ |S|=k_2}}\left\|\sum_{m\in R}|a_m\rangle\langle a_m| + \sum_{n\in S}|b_n\rangle\langle b_n|\right\|_\infty^2\right) \\
&\quad \left. + \max_{\substack{|R|=1 \\ |S|=k_1}}\left\|\sum_{m\in R}|a_m\rangle\langle a_m| + \sum_{n\in S}|b_n\rangle\langle b_n|\right\|_\infty^2\right],
\end{aligned}$$

which gives the improved values of $\Omega_k = \max \Omega_{k_1,k_2,\cdots k_n}^d$, where max is taken over $\sum_{i=1}^n k_i = k$ with $k_1 \geq \ldots \geq k_n$, and hence $\omega = (\Omega_1, \Omega_2 - \Omega_1, \ldots, \Omega_d - \Omega_{d-1}, 0, \ldots, 0)$ for any ρ. ∎

References

1. Heisenberg, W. Über den anschaulichen Inhalt der quantentheoretischen Kinematik und Mechanik. *Zeitschrift für Physik* **43(3–4),** 172–198 (1927).
2. Gühne, O. Characterizing entanglement via uncertainty relations. *Phys. Rev. Lett.* **92(11),** 117903 (2004).
3. Schneeloch, J., Broadbent, C. J. & Howell, J. C. Cryptography from Noisy Storage. *Phys. Lett. A* **378,** 766 (2014).
4. Koashi, M. Simple security proof of quantum key distribution based on complementarity. *New J. Phys.* **11(4),** 045018 (2009).
5. Renes, J. M. & Boileau, J. C. Physical underpinnings of privacy. *Phys. Rev. A* **78(3),** 032335 (2008).
6. Tomamichel, M., Lim, C. C. W. & Gisin, N. *et al.* Tight finite-key analysis for quantum cryptography. *Nature commun.* **3,** 634 (2012).
7. Robertson, H. P. The uncertainty principle. *Phys. Rev.* **34(1),** 163 (1929).
8. Deutsch, D. Uncertainty in quantum measurements. *Phys. Rev. Lett.* **50(9),** 631 (1983).
9. Maassen, H. & Uffink, J. B. M. Generalized entropic uncertainty relations. *Phys. Rev. Lett.* **60(12),** 1103 (1988).
10. Coles, P. J. & Piani, M. Improved entropic uncertainty relations and information exclusion relations. *Phys. Rev. A* **89(2),** 022112 (2014).
11. Rudnicki, Ł. Puchała, Z. & Życzkowski, K. Strong majorization entropic uncertainty relations. *Phys. Rev. A* **89(5),** 052115 (2014).
12. Wehner, S. & Winter, A. Entropic uncertainty relations – a survey. *New J. Phys.* **12(2),** 025009 (2010).
13. Coles, P. J., Berta, M. & Tomamichel, M. *et al.* Entropic Uncertainty Relations and their Applications. *arXiv*, 1511.04857 (2015)
14. Friedland, S., Gheorghiu, V. & Gour, G. Universal uncertainty relations. *Phys. Rev. Lett.* **111(23),** 230401 (2013).
15. Puchała, Z., Rudnicki, Ł. & Życzkowski, K. Majorization entropic uncertainty relations. *J. Phys. A*: Math. and Theor. **46(27),** 272002 (2013).
16. de Vicente, J. I. & Sánchez-Ruiz, J. Improved bounds on entropic uncertainty relations. *Phys. Rev. A* **77(4),** 042110 (2008).

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Additional Information