

OPEN Entropy exchange for infinitedimensional systems

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Received: 26 July 2016 Accepted: 29 December 2016 Published: 06 February 2017 In this paper the entropy exchange for channels and states in infinite-dimensional systems are defined and studied. It is shown that, this entropy exchange depends only on the given channel and the state. An explicit expression of the entropy exchange in terms of the state and the channel is proposed. The generalized Klein's inequality, the subadditivity and the triangle inequality about the entropy including infinite entropy for the infinite-dimensional systems are established, and then, applied to compare the entropy exchange with the entropy change.

In quantum mechanics a quantum system is associated with a separable complex Hilbert space H. A quantum state ρ is a density operator, that is, $\rho \in \mathcal{T}(H) \subset \mathcal{B}(H)$ which is positive and has trace 1, where $\mathcal{B}(H)$ and $\mathcal{T}(H)$ denote the von Neumann algebras of all bounded linear operators and the space of all trace-class operators with $||T||_1 = \text{Tr}(T^{\dagger}T) < \infty$, respectively. Let us denote by $\mathcal{S}(H)$ the set of all states in the quantum system associated with *H*. A state ρ is called a pure state if $\rho^2 = \rho$; otherwise, ρ is called a mixed state.

Consider two quantum systems associated with Hilbert spaces H and K respectively. Recall that a quantum channel between these two systems is a trace-preserving completely positive linear map from T(H) into T(K). It is known¹⁻⁴ that every channel $\Phi: \mathcal{T}(H) \to \mathcal{T}(K)$ has an operator-sum representation

$$\Phi(\rho) = \sum_{k=1}^{N} E_k \rho E_k^{\dagger},\tag{1}$$

where $1 \le N \le \infty$ and $\{E_k\}_{k=1}^N \subseteq \mathcal{B}(H, K)$ is a sequence of bounded linear operators from H into K with $\sum_{k=1}^{N} E_k^{\dagger} E_k I$. $E_k s$ are called the operation elements or Kraus operators of the quantum channel Φ . The representation of Φ in Eq. (1) is not unique. If both H and K are finite-dimensional, it is well known that $N \le \dim H \dim K < \infty$ and the sequences $\{E_k\}_{k=1}^N$ and $\{F_l\}_{l=1}^M$ of operation elements of any two representations of Φ are connected by a unitary matrix, $U=(u_{kl})\in M_{\max\{N,M\}}$ such that $F_l=\sum_{k=1}^N u_{kl}E_k, l=1,2,...,M$. This fact is so-called the unitary freedom in the operator-sum representation for quantum channels. However, unitary freedom is no longer valid for infinite-dimensional systems⁵. In fact, what we have is so-called the bi-contractive freedom, which asserts that, if a channel $\Phi: \mathcal{T}(H) \to \mathcal{T}(K)$ has two operator-sum representations $\Phi(X) = \sum_i A_i X A_i^{\dagger} = \sum_j B_j X B_j^{\dagger}$ $\forall X \in \mathcal{T}(H)$, then there exist contractive matrices $\Omega = (\omega_{ij})$ and $\Gamma = (\gamma_{ji})$ such that $A_i = \sum_j \omega_{ij} B_j$ for each i and $B_j = \sum_i \gamma_{ji} A_i$ for each j. The converse is also true. Particularly, if $\Omega = (\omega_{ij})$ is an isometry so that $A_i = \sum_j \omega_{ij} B_j$ for each *i*, then $\sum_{i} A_{i} X A_{i}^{\dagger} = \sum_{i} B_{i} X B_{i}^{\dagger}$ holds for any *X*.

Let R and Q be two quantum systems described by Hilbert spaces H_R and H_Q , respectively. Suppose that the joint system RQ is prepared in a pure entangled state $|\Psi^{RQ}\rangle$ and the initial state of system Q is $\rho^{Q} = \text{Tr}_{R}|\Psi^{RQ}\rangle\langle\Psi^{RQ}|$. The system R is dynamically isolated and has a zero internal Hamiltonian, while the system Q undergoes some evolution that possibly involves interaction with the environment E. The final state of RQ is possibly mixed and is described by the density operator $\rho^{RQ'}$. Thus, if the dynamical evolution that Q is subjected to is described by Φ^Q , then the final state is $\rho^{RQ'} = I^R \otimes \Phi^Q(|\Psi^{RQ}\rangle\langle\Psi^{RQ}|)$ and the entanglement fidelity is refs 6–9

$$F_e = {\rm Tr} \; |\Psi^{RQ}\rangle \langle \Psi^{RQ}|\rho^{RQ'} = \langle \Psi^{RQ}|\rho^{RQ'}|\Psi^{RQ}\rangle. \eqno(2)$$

The value of F_e is independent of the choice $|\Psi^{RQ}\rangle$ of purification of ρ^Q . In fact, it was shown^{5,7,10,11} that for any $\rho \in \mathcal{S}(H)$ with dim $H \leq \infty$ and any quantum channel Φ with operation elements $\{E_i\}$, we have $F_e = \sum_i |\operatorname{Tr}(E_i \rho)|^2$.

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For finite-dimensional systems there is another quantity concerning channels and states that is intrinsic to subsystem Q. This quantity is called the entropy exchange. For a given state ρ^Q and a given channel Φ^Q in a finite-dimensional system Q, recall that the entropy exchange S_e is defined by refs 1, 6 and 12–14

$$S_{e} = S(\rho^{RQ'}) = - \operatorname{Tr} \rho^{RQ'} \log \rho^{RQ'}, \tag{3}$$

where $\rho^{RQ'}=I^R\otimes\Phi^Q(|\Psi^{RQ}\rangle\langle\Psi^{RQ}|)$ and $|\Psi^{RQ}\rangle$ is a purification of ρ^Q . It was shown^{1,6} that the entropy exchange S_e is independent of the choice of purification $|\Psi^{RQ}\rangle$ of the state ρ^Q . It was also shown¹ that the entropy exchange S_e has another explicit formulation

$$S_e = \min \left\{ -\sum_{\mu} p_{\mu} \log p_{\mu} \right\}, \tag{4}$$

where $p_{\mu}={
m Tr}~A_{\mu}^{Q}\rho^{Q}A_{\mu}^{Q\dagger}$ with $\{A_{\mu}^{Q}\}$ the sequence of the Kraus operators of an operator-sum representation of Φ and the minimum is taken over all operator-sum representations of Φ^{Q} .

It is clear that Eq. (3) can be naturally generalized to infinite-dimensional case to give a definition of the entropy exchange for channels and states in infinite-dimensional systems. In continuous variable systems, Chen and Qiu¹⁵ studied the coherent information $I_e = S(\rho^Q) - S_e$ of the thermal radiation signal ρ^Q transmitted over the thermal radiation noise channel, one of the most essential quantum Gaussian channels, and derived an analytical expression for computation of the value of it. However, as the von Neumann entropy $S(\rho)$ of a non-Gaussian state in an infinite-dimensional system may be $+\infty^{16}$, we may have $S_e = +\infty$. In this paper we consider general states and channels and show that the definition Eq. (3) does not depend on the choice of the purification of the state either, and Eq. (4) is still true for infinite-dimensional systems.

For finite-dimensional systems, it is known⁶ that the entropy exchange is larger than or equal to the change of the entropy, that is,

$$S_{\rho} \ge |S(\rho^{Q}) - S(\rho^{Q'})|,\tag{5}$$

where $\rho^{Q'} = \Phi^Q(\rho^Q)$. The second purpose of the present paper is to compare the entropy exchange with the change of the entropy and to check whether or not the inequality (5) is still valid in infinite-dimensional systems. We show that, for infinite-dimensional case, what we can have are the following three inequalities: $S_e \leq S(\rho^Q) + S(\rho^{Q'}), S_e + S(\rho^Q) \geq S(\rho^{Q'})$ and $S_e + S(\rho^{Q'}) \geq S(\rho^Q)$. Thus, if both $S(\rho^Q)$ and $S(\rho^Q)$ are finite, we still have $S_e \geq |S(\rho^Q) - S(\rho^Q)|$. To prove the above inequalities, we need the subadditivity and the triangle inequality of von Neumann entropies for infinite-dimensional quantum systems. These two inequalities were established in a more general frame of von Neumann algebras for normal states with finite entropy¹⁷. However, for the convenience of readers, we present some elementary proofs including the case of infinite von Neumann entropy here by establishing the generalized Klein's inequality for infinite-dimensional case. We also give some examples which illustrates that the entropy exchange is different from the change of entropy.

Entropy exchange for infinite-dimensional systems

In this section, we mainly give some properties of the entropy exchange for infinite-dimensional systems. In fact, the results in this section hold for both finite- and infinite-dimensional cases.

Recall that a linear operator U from a Hilbert space into another is called an isometry if $U^{\dagger}U=I$; a coisometry if $UU^{\dagger}=I$. Obviously, if the spaces are finite-dimensional with the same dimension, isometries and coisometries are unitary operators.

Lemma 1. Suppose $|\phi\rangle$ and $|\psi\rangle$ are two pure states of an infinite-dimensional composite system with subsystems R and Q. If they have identical Schmidt coefficients, then there are isometries or coisometries U on system R and V on system Q such that $|\phi\rangle = (U \otimes V)|\psi\rangle$.

Proof. By the assumption, $|\phi\rangle$ and $|\psi\rangle$ have respectively the Schmidt decompositions $|\phi\rangle = \sum_{i=1}^N \lambda_i |i^R\rangle |i^Q\rangle$ and $|\psi\rangle = \sum_{i=1}^N \lambda_i |i^R\rangle |i^Q\rangle$, where $\{|i^R\rangle\}_{i=1}^N$ and $\{|i'^R\rangle\}_{i=1}^N$ are two orthonormal sets for system R, $\{|i^Q\rangle\}$ and $\{|i'^Q\rangle\}$ are two orthonormal sets for system Q, $\lambda_i > 0$ with $\sum_{i=1}^N \lambda_i^2 = 1$. Extend $\{|i^R\rangle\}$ to an orthonormal basis $\{|j^R\rangle\}\{|i^R\rangle\}$, $\{|j^R\rangle\}$ and $\{|j'^R\rangle\}$ to an orthonormal basis $\{|i^Q\rangle\}$, and $\{|i'^Q\rangle\}$ to an orthonormal basis $\{|i^Q\rangle\}$, and $\{|i'^Q\rangle\}$ to an orthonormal basis $\{|i^Q\rangle\}$, $|i^Q\rangle\}$, and $\{|i^Q\rangle\}$ to an orthonormal basis $\{|i^Q\rangle\}$, $|i^Q\rangle\}$ of the system Q. Denote the cardinal number of a set S by \overline{S} . Let $\{|j^{\overline{R}}\rangle\} = d_1$, $\{|j^{\overline{I}}\rangle\} = d_2$, $\{|j^{\overline{Q}}\rangle\} = d_3$ and $\{|i^{\overline{Q}}\rangle\} = d_4$. Clearly, we have 9 possible cases.

Case 1. $d_1 = d_2$ and $d_3 = d_4$. Let unitary operators U on system R and V on system Q be defined respectively by $U|i^R\rangle = |i'^R\rangle$ for $1 \le i \le N$ and $U|j^R\rangle = |j'^R\rangle$ for $1 \le j \le d_1 = d_2$; $V|i^Q\rangle = |i'^Q\rangle$ for $1 \le i \le N$ and $V|l^Q\rangle = |l'^Q\rangle$ for $1 \le l \le d_3 = d_4$. Then $|\phi\rangle = (U \otimes V)|\psi\rangle$.

Case 2. $d_1 = d_2$ and $d_3 < d_4$. Let U be defined as in Case 1 and V be defined by $V|i^Q\rangle = |i'^Q\rangle$ for $1 \le i \le N$ and $V|l^Q\rangle = |l'^Q\rangle$ for $1 \le l \le d_3 < d_4$. Then U is a unitary operator on system R and V is an isometry V on system Q satisfying $|\phi\rangle = (U \otimes V)|\psi\rangle$.

Case 3. $d_1 = d_2$ and $d_3 > d_4$. Define U on system R as in Case 1 and define V on system Q by $V|i^Q\rangle = |i'^Q\rangle$ for $1 \le i \le N$, and $V|I^Q\rangle = |I'^Q\rangle$ for $1 \le l \le d_4$ and $V|I^Q\rangle = 0$ for $d_4 < l \le d_3$. Then U is unitary and V is coisometric so that $|\phi\rangle = (U \otimes V)|\psi\rangle$.

In a similar way, it is obvious to see that

Case 4. $d_1 < d_2$ and $d_3 = d_4$. There is an isometry U on system R and a unitary V on system Q such that $|\phi\rangle = (U \otimes V)|\psi\rangle$.

Case 5. $d_1 < d_2$ and $d_3 < d_4$. There are isometries U on system R and V on system Q such that $|\phi\rangle = (U \otimes V)|\psi\rangle$.

Case 6. $d_1 < d_2$ and $d_3 > d_4$. There is an isometry U on system R and a coisometry V on system Q such that $|\phi\rangle = (U \otimes V)|\psi\rangle$.

Case 7. $d_1 > d_2$ and $d_3 = d_4$. There is a coisometry U on system R and a unitary V on system Q such that $|\phi\rangle = (U \otimes V)|\psi\rangle$.

Case 8. $d_1 > d_2$ and $d_3 < d_4$. There is a coisometry U on system R and an isometry V on system Q such that $|\phi\rangle = (U \otimes V)|\psi\rangle$.

Case 9. $d_1 > d_2$ and $d_3 > d_4$, there are coisometries U on system R and V on system Q such that $|\phi\rangle = (U \otimes V)|\psi\rangle$.

Lemma 2. If $|\Psi_1^{RQ}\rangle$ and $|\Psi_2^{RQ}\rangle$ are purifications of a state ρ^Q to a composite system RQ, then there exists an isometry V^R on system R such that either $|\Psi_1^{RQ}\rangle=(V^R\otimes I^Q)|\Psi_2^{RQ}\rangle$ or $|\Psi_2^{RQ}\rangle=(V^R\otimes I^Q)|\Psi_1^{RQ}\rangle$.

Proof. Let $\rho^Q = \sum_i \lambda_i |i^Q\rangle \langle i^Q|$ be the spectral decomposition of ρ^Q with $\lambda_i \geq \lambda_{i+1}$. Since both $|\Psi_1^{RQ}\rangle$ and $|\Psi_2^{RQ}\rangle$ are purifications of ρ^Q , their Schmidt decompositions have the form $|\Psi_1^{RQ}\rangle = \sum_i \sqrt{\lambda_i} |i^R\rangle |i^Q\rangle$ and $|\Psi_2^{RQ}\rangle = \sum_i \sqrt{\lambda_i} |i'^R\rangle |i^Q\rangle$, where $\{|i^R\rangle\}$ and $\{|j'^R\rangle\}$ are two orthonormal sets for system R. Hence $|\Psi_1^{RQ}\rangle$ and $|\Psi_2^{RQ}\rangle$ have identical Schmidt coefficients. Making use of lemma 1, there is an isometry or a coisometry U^R on system R such that $|\Psi_1^{RQ}\rangle = (U^R\otimes I^Q)|\Psi_2^{RQ}\rangle$. If U^R is already an isometry, we have done. If U^R is a coisometry, by the proof of Lemma 1 we see that there is an isometry V^R such that $|\Psi_2^{RQ}\rangle = (V^R\otimes I^Q)|\Psi_1^{RQ}\rangle$ and $|\Psi_1^{RQ}\rangle = ((V^R)^\dagger\otimes I^Q)|\Psi_2^{RQ}\rangle$.

Lemma 3. Assume that $|\Psi_1^{RQ}\rangle$ and $|\Psi_2^{RQ}\rangle$ are two purifications of a state ρ^Q to a composite system RQ, and each is subjected to the same evolution superoperator $I^R\otimes\Phi^Q$ with the resulting states respectively $\rho_1^{RQ'}$ and $\rho_2^{RQ'}$, i.e., $\rho_1^{RQ'}=I^R\otimes\Phi^Q(|\Psi_1^{RQ}\rangle\langle\Psi_1^{RQ}|)$ and $\rho_2^{RQ'}=I^R\otimes\Phi^Q(|\Psi_2^{RQ}\rangle\langle\Psi_2^{RQ}|)$. Then there exists an isometry V^R on system R such that either $\rho_2^{RQ'}=(V^R\otimes I^{Q'})\rho_1^{RQ'}(V^R\otimes I^{Q'})^{\dagger}$ or $\rho_1^{RQ'}=(V^R\otimes I^{Q'})\rho_2^{RQ'}(V^R\otimes I^{Q'})^{\dagger}$.

Proof. By lemma 2, there exists an isometry transformation V^R acting on system R such that either $|\Psi_1^{RQ}\rangle=(V^R\otimes I^Q)|\Psi_2^{RQ}\rangle$ or $|\Psi_2^{RQ}\rangle=(V^R\otimes I^Q)|\Psi_1^{RQ}\rangle$. Without loss of generality, assume that $|\Psi_2^{RQ}\rangle=(V_R\otimes I_Q)|\Psi_1^{RQ}\rangle$. Let $\Phi^Q(\cdot)=\sum_\mu A_\mu(\cdot)A_\mu^\dagger$ be an operator-sum representation of Φ^Q . Then

$$\rho_{2}^{RQ'} = I^{R} \otimes \Phi^{Q}(|\Psi_{2}^{RQ}\rangle\langle\Psi_{2}^{RQ}|)
= \sum_{\mu} (I^{R} \otimes A_{\mu}^{Q})|\Psi_{2}^{RQ}\rangle\langle\Psi_{2}^{RQ}|(I^{R} \otimes A_{\mu}^{Q})^{\dagger}
= \sum_{\mu} (I^{R} \otimes A_{\mu}^{Q})(V^{R} \otimes I^{Q})|\Psi_{1}^{RQ}\rangle\langle\Psi_{1}^{RQ}|(V^{R} \otimes I^{Q})^{\dagger}(I^{R} \otimes A_{\mu}^{Q})^{\dagger}
= (V^{R} \otimes I^{Q'}) \left\{ \sum_{\mu} (I^{R} \otimes A_{\mu}^{Q})|\Psi_{1}^{RQ}\rangle\langle\Psi_{1}^{RQ}|(I^{R} \otimes A_{\mu}^{Q})^{\dagger} \right\}(V^{R} \otimes I^{Q'})^{\dagger}
= (V^{R} \otimes I^{Q'})\rho_{1}^{RQ'}(V^{R} \otimes I^{Q'})^{\dagger}.$$
(6)

Similarly, if $|\Psi_1^{RQ}\rangle = (V^R \otimes I^Q) |\Psi_2^{RQ}\rangle$ holds, then we have

$$\rho_1^{RQ'} = (V^R \otimes I^{Q'})\rho_2^{RQ'}(V^R \otimes I^{Q'})^{\dagger}$$

$$\tag{7}$$

Lemma 4. If A is a bounded self-adjoint operator on a complex Hilbert space and f is a continuous function on $\sigma(A)$, the spectrum of A, then, for any isometric operator V, we have $f(VAV^{\dagger}) = Vf(A)V^{\dagger}$.

Proof. As A is a bounded self-adjoint operator, $\sigma(A) \subset (-\infty, +\infty)$ is a bounded closed set. Because f is a continuous function on $\sigma(A)$, we can apply the Weierstrass theorem to find a sequence of polynomials $\{P_n\}$ such that $P_n \rightarrow f$ uniformly on $\sigma(A)$. Write $P_n(t) = \sum \alpha_i^{(n)} t^i$. It is clear that $P_n(VAV^\dagger) = \sum \alpha_i^{(n)} (VAV^\dagger)^i = VP_n(A)V^\dagger$ since V is an isometric operator. Let $n \rightarrow \infty$, we see that $f(VAV^\dagger) = \lim_{n \rightarrow \infty} P_n(VAV^\dagger) = \lim_{n \rightarrow \infty} VP_n(A)V^\dagger = Vf(A)V^\dagger$.

The following result reveals that, for infinite-dimensional systems, similar to the entanglement fidelity⁵, the value of entropy exchange is also independent of the choice of purifications of the initial state.

Theorem 5. The entropy exchange of a channel Φ^Q and a state ρ^Q is independent of the choice of purifications of the state ρ^Q .

Proof. Let $|\Psi_1^{RQ}\rangle$ and $|\Psi_2^{RQ}\rangle$ be two purifications of the state ρ^Q in composite system RQ, and denote $\rho_1^{RQ'}=I^R\otimes\Phi^Q(|\Psi_1^{RQ}\rangle\langle\Psi_1^{RQ}|)$ and $\rho_2^{RQ'}=I^R\otimes\Phi^Q(|\Psi_2^{RQ}\rangle\langle\Psi_2^{RQ}|)$. By the definition Eq. (3), we have to show that

$$S(\rho_2^{RQ'}) = S(\rho_1^{RQ'}).$$
 (8)

By lemma 3, there is an isometry V^R so that the resulting states $\rho_1^{RQ'}$ and $\rho_2^{RQ'}$ satisfy either $\rho_{2_{RQ'}}^{RQ'} = (V^R \otimes I^{Q'}) \rho_{1_{RQ'}}^{RQ'}(V^R \otimes I^{Q'})^{\dagger}$ or $\rho_1^{RQ'} = (V^R \otimes I^{Q'}) \rho_2^{RQ'}(V^R \otimes I^{Q'})^{\dagger}$. Without loss of generality, suppose $\rho_2^{RQ'} = (V^R \otimes I^{Q'}) \rho_1^{RQ'}(V^R \otimes I^{Q'})^{\dagger}$. Note that $f(x) = x \log x$ is a continuous function on $\sigma(\rho_1^{RQ})$. Then, by lemma 4,

$$\begin{split} S(\rho_{2}^{RQ'}) &= - \mathrm{Tr} \; \rho_{2}^{RQ'} \log \, \rho_{2}^{RQ'} \\ &= - \mathrm{Tr} \{ (V^{R} \otimes I^{Q'}) \rho_{1}^{RQ'} (V^{R} \otimes I^{Q'})^{\dagger} [\log(V^{R} \otimes I^{Q'}) \rho_{1}^{RQ'} (V^{R} \otimes I^{Q'})^{\dagger}] \} \\ &= - \mathrm{Tr} \{ (V^{R} \otimes I^{Q'}) \rho_{1}^{RQ'} (V^{R} \otimes I^{Q'})^{\dagger} (V^{R} \otimes I^{Q'}) [\log \rho_{1}^{RQ'}] (V^{R} \otimes I^{Q'})^{\dagger} \} \\ &= - \mathrm{Tr} \{ \rho_{1}^{RQ'} [\log \, \rho_{1}^{RQ'}] (V^{R} \otimes I^{Q'})^{\dagger} (V^{R} \otimes I^{Q'}) \} \\ &= - \mathrm{Tr} \; \rho_{1}^{RQ'} \log \, \rho_{1}^{RQ'} \\ &= S(\rho_{1}^{RQ'}), \end{split}$$
(9)

as desired.

In the sequel, analogue to Eq. (4) for finite-dimensional systems, we derive an explicit expression for S_e in terms of ρ^Q and Φ^Q for infinite-dimensional systems.

To do this, we need some more lemmas.

Lemma 6. Let $\rho \in \mathcal{S}(H_A \otimes H_B)$ with $\dim H_A \otimes H_B \leq +\infty$. For any $A \in \mathcal{B}(H_A)$ and $B \in \mathcal{B}(H_B)$, we have $\operatorname{Tr}_B[(A \otimes B)\rho] = A \operatorname{Tr}_B[(B \otimes B)\rho]$ and $\operatorname{Tr}_B[\rho(A \otimes B)] = \{\operatorname{Tr}_B[\rho(B \otimes B)]\}A$.

Proof. Fix an orthonormal basis $\{|i\rangle\}$ of H_B . Then B can be written in a matrix $B=(b_{ij})$, and $A\otimes B$ and ρ can be written in operator matrices $A\otimes B=(b_{ij}A)$ and $\rho=(\rho_{ij})$, respectively. Thus we have $(A\otimes B)\rho=(b_{ii}A)(\rho_{ii})=(\sum_k b_{ik}A\rho_{ki})$, and then

$$\operatorname{Tr}_{B}[(A \otimes B)\rho] = \operatorname{Tr}_{B}\left(\sum_{k} b_{ik} A \rho_{kj}\right)$$

$$= A\left(\sum_{i} \sum_{k} b_{ik} \rho_{ki}\right)$$

$$= A \operatorname{Tr}_{B}[(I \otimes B)\rho]. \tag{10}$$

Similarly, we can drive that

$$\operatorname{Tr}_{B}[\rho(A \otimes B)] = \{\operatorname{Tr}_{B}[\rho(I \otimes B)]\}A. \tag{11}$$

Lemma 7. Let $\rho \in \mathcal{S}(H_A \otimes H_B)$ with $\dim H_A \otimes H_B \leq +\infty$. Then, for any $A \in \mathcal{B}(H_A)$ and $B \in \mathcal{B}(H_B)$, we have

$$\operatorname{Tr}_{B}[(A \otimes B)\rho(A \otimes B)^{\dagger}] = A\{\operatorname{Tr}_{B}[\rho(I \otimes B^{\dagger}B)]\}A^{\dagger} = A\{\operatorname{Tr}_{B}[(I \otimes B^{\dagger}B)\rho]\}A^{\dagger}. \tag{12}$$

Proof. By lemma 6 and with the same symbols as in the proof of lemma 6, we have

$$\operatorname{Tr}_{B}[(A \otimes B)\rho(A \otimes B)^{\dagger}] = A\{\operatorname{Tr}_{B}[(I \otimes B)\rho(I \otimes B^{\dagger})]\}A^{\dagger}$$

$$= A\left\{\operatorname{Tr}_{B}\left[\left(\sum_{l}\sum_{k}b_{ik}\rho_{kl}b_{li}^{*}\right)\right]\right\}A^{\dagger}$$

$$= A\left\{\operatorname{Tr}_{B}\left(\sum_{l}\sum_{k}\rho_{kl}b_{ik}b_{li}^{*}\right)\right\}A^{\dagger}$$

$$= A\{\operatorname{Tr}_{B}[\rho(I \otimes B^{\dagger}B)]\}A^{\dagger}$$

$$= A\{\operatorname{Tr}_{B}[(I \otimes B^{\dagger}B)\rho]\}A^{\dagger}. \tag{13}$$

Let Φ^Q be a channel from system Q into system Q'. Suppose $\Phi^Q(\cdot) = \sum_{\mu=1}^M A_\mu^Q(\cdot) A_\mu^{Q^\dagger}$ $(M \le \infty)$ is an operator-sum representation for the channel Φ^Q . If ρ^Q is a state of system Q and $|\Psi^{RQ}\rangle$ is a purification of ρ^Q into composite system RQ, then, for any μ , let $|\tilde{\Phi}_\mu^{RQ'}\rangle = (I^R \otimes A_\mu^Q) |\Psi^{RQ}\rangle$. Thus the resulting state $\rho^{RQ'}$ can be written in

$$\rho^{RQ'} = \sum_{\mu} (I^R \otimes A_{\mu}^Q) |\Psi^{RQ}\rangle \langle \Psi^{RQ}| (I^R \otimes A_{\mu}^Q)^{\dagger}
= \sum_{\mu} |\tilde{\Phi}_{\mu}^{RQ'}\rangle \langle \tilde{\Phi}_{\mu}^{RQ'}|.$$
(14)

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Therefore $\{\||\widetilde{\Phi}_{\mu}^{RQ'}\rangle\|^{-1}|\widetilde{\Phi}_{\mu}^{RQ'}\rangle\}$ is a pure state ensemble for $\rho^{RQ'}$. Let us adjoin a system E with Hilbert space H_E , where $\dim H_E = M$. Then, for any orthonormal basis $\{|\mu^E\rangle\} \subset H_E$, the state $|\Upsilon^{RQ'E}\rangle = \sum_{\mu} |\widetilde{\Phi}_{\mu}^{RQ'}\rangle \otimes |\mu^E\rangle$ is a purification of $\rho^{RQ'}$. With these symbols, we have

Lemma 8. Let $\rho^E = \operatorname{Tr}_{RO'}|\Upsilon^{RQ'E}\rangle\langle\Upsilon^{RQ'E}|$. Then we have $S_e = S(\rho^E)$.

Proof. Since the state $|\Upsilon^{RQ'E}\rangle$ is a pure state, the reduced states $\rho^{RQ'} = \mathrm{Tr}_E |\Upsilon^{RQ'E}\rangle \langle \Upsilon^{RQ'E}|$ and $\rho^E = \mathrm{Tr}_{RQ'} |\Upsilon^{RQ'E}\rangle \langle \Upsilon^{RQ'E}|$ have the same von Neumann entropy. Therefore, by the definition of the exchange entropy, we get $S_e = S(\rho^{RQ'}) = S(\rho^E)$.

Furthermore, let us write down the density operator ρ^E in matrix form. Clearly,

$$\rho^{E} = \operatorname{Tr}_{RQ'} |\Upsilon^{RQ'E}\rangle \langle \Upsilon^{RQ'E}|$$

$$= \sum_{\mu\nu} \langle \widetilde{\Phi}_{\nu}^{RQ'} |\widetilde{\Phi}_{\mu}^{RQ'}\rangle |\mu^{E}\rangle \langle \nu^{E}| = \sum_{\mu\nu} W_{\mu\nu} |\mu^{E}\rangle \langle \nu^{E}|$$
(15)

with $W_{\mu\nu} = \langle \widetilde{\Phi}_{\nu}^{RQ'} | \widetilde{\Phi}_{\mu}^{RQ'} \rangle$. By lemmas 6 and 7, we see that

$$\begin{split} W_{\mu\nu} &= \langle \widetilde{\Phi}_{\nu}^{RQ'} | \widetilde{\Phi}_{\mu}^{RQ'} \rangle \\ &= \operatorname{Tr} | \widetilde{\Phi}_{\mu}^{RQ'} \rangle \langle \widetilde{\Phi}_{\nu}^{RQ'} | \\ &= \operatorname{Tr} (I^R \otimes A_{\mu}^Q) | \Psi^{RQ} \rangle \langle \Psi^{RQ} | (I^R \otimes A_{\nu}^Q)^{\dagger} \\ &= \operatorname{Tr}_Q \operatorname{Tr}_R \{ | \Psi^{RQ} \rangle \langle \Psi^{RQ} | (I^R \otimes A_{\nu}^{Q\dagger} A_{\mu}^Q) \} \\ &= \operatorname{Tr}_Q \operatorname{Tr}_R \{ | \Psi^{RQ} \rangle \langle \Psi^{RQ} | (I^R \otimes I^Q) \} A_{\nu}^{Q\dagger} A_{\mu}^Q \\ &= \operatorname{Tr}_Q \{ A_{\mu}^Q [\operatorname{Tr}_R | \Psi^{RQ} \rangle \langle \Psi^{RQ} |] A_{\nu}^{Q\dagger} \} \\ &= \operatorname{Tr}_Q A_{\mu}^Q \rho^Q A_{\nu}^{Q\dagger}. \end{split}$$

$$\tag{16}$$

Let W be the density operator with components $W_{\mu\nu}=\mathrm{Tr}_QA_\mu^Q\rho^QA_\nu^{Q\dagger}$. Then, by lemma 8, $S_e=S(W)$. Now, let $\vec{P}=\{P_\mu\}$ with $P_\mu=W_{\mu\mu}$. Thus \vec{P} is a probabilities which is given by the state W from a complete measurement using the basis that yields the matrix elements $W_{\mu\nu}$. Therefore we have $H(\vec{P})\geq S(W)$ as measurements increasing the entropy.

Now, we are at a position to give an explicit formula for the entropy exchange based upon the operator-sum representation for quantum channel Φ^Q and the initial state ρ^Q for an infinite-dimensional system.

Theorem 9. Let $\rho^Q \in \mathcal{S}(H_Q)$ be a state with $\dim H_Q \leq \infty$ and $\Phi^Q \colon \mathcal{S}(H_Q) \to \mathcal{S}(H_{Q'})$ a channel. Then the entropy exchange

$$S_e = \min_{\left\{A_{\mu}^{Q}\right\}} \left\{ -\sum_{\mu} \operatorname{Tr} A_{\mu}^{Q} \rho^{Q} A_{\mu}^{Q\dagger} \log \operatorname{Tr} A_{\mu}^{Q} \rho^{Q} A_{\mu}^{Q\dagger} \right\}, \tag{17}$$

where $\{A_{\mu}^{Q}\}$ is a sequence of Kraus operators of an operator-sum representation of Φ^{Q} , that is, $\Phi^{Q}(\cdot) = \sum_{\mu} A_{\mu}^{Q}(\cdot) A_{\mu}^{Q^{\dagger}}$, and the minimum is taken over all operator-sum representations of Φ^{Q} .

Proof. For given state ρ^Q and quantum channel Φ^Q , if $\{A_\mu\}$ is the sequence of Kraus operators of an operator-sum representation of Φ^Q , then by lemma 8 and the discussion previous theorem 9, $S_e = S(\rho^E) \leq -\sum_\mu P_\mu \log P_\mu$, where, $P_\mu = \operatorname{Tr} A_\mu \rho^Q A_\mu^\dagger, \rho^E = \sum_{\mu\nu} \operatorname{Tr}(A_\mu \rho A_\nu^\dagger) |\mu^E\rangle \langle \nu^E|$ for some orthonormal basis $\{|\mu^E\rangle\}$ for the environment system E. Hence we have $S_e \leq \inf\{-\sum_\mu P_\mu \log P_\mu\}$. In the sequel we show that $S_e = -\sum_\mu P_\mu \log P_\mu$ for some suitable choice of operator-sum representation of Φ^Q . In fact, for a given sequence $\{A_\mu\}$ of Kraus operators for an operator-sum representation of Φ^Q , $\operatorname{Tr}(A_\mu \rho^Q A_\nu^\dagger)$ s are the matrix elements of ρ^E in the orthonormal basis $\{|\mu^E\rangle\}$. Let W be the associated matrix with entries $W_{\mu\nu} \equiv \operatorname{Tr}(A_\mu \rho^Q A_\nu^\dagger)$, that is, W is the matrix of ρ^E in an appropriate basis; then $S_e = S(W)$. Since W is a matrix representation of the environmental density operator, it may be diagonalized by a unitary matrix $U = (u_{\mu\nu})$, i.e., $UWU^\dagger = W_0$, where $W_0 = \operatorname{diag}(p_1, p_2, \dots, p_\mu, \dots)$ is a diagonal matrix. Letting $|\mu'|^E = U |\mu|^E = U |\mu|^E$

$$S_e = \min_{\left\{A_\mu^Q\right\}} \left\{ -\sum_\mu P_\mu \log P_\mu \right\},\tag{18}$$

where $P_{\mu} = \text{Tr } A_{\mu}^{Q} \rho^{Q} A_{\mu}^{Q\dagger}$ and the minimum is taken over all operator-sum representations of Φ^{Q} .

Comparison with entropy change

The entropy exchange S_e simply characterizes the information exchange between the system Q and the external world during the evolution given by Φ^Q . It is interesting to explore the relationship between the entropy exchange and the entropy change during the same evolution. Such a question was studied for finite-dimensional systems and the inequality (5) was established. However, the inequality (5) does not always valid in infinite-dimensional case. To solve the question for infinite-dimensional systems, we need the subadditivity and the triangle inequality of von Neumann entropies for infinite-dimensional systems which was established in the textbook 17 for normal states with finite entropy in a more general frame of von Neumann algebras. However, we have to deal with the states with infinite entropy. Here we present somewhat elementary proofs for these two inequalities by generalizing the generalized Klein's inequality from finite-dimensional systems to the infinite-dimensional systems and clarify when the inequalities are still valid for states with infinite entropy.

Let $f: \mathcal{D}_f \subseteq \mathbb{R} \to \mathbb{R}$ be a function. The following lemma 10 and 11 are obvious¹⁸.

Lemma 10. *If f is a convex (concave) function, then f is continuous.*

Lemma 11. *If f is a convex (concave) function, then f(y)* $- f(x) \ge (\le)(y-x)f'(x)$.

Lemma 12. Suppose f is a convex (concave) function and A is a bounded self-adjoint operator on a Hilbert space H with $\sigma(A) \subseteq \mathcal{D}_f$. If $|\phi\rangle \in H$ is an unit vector, then $\langle \phi | f(A) | \phi \rangle \geq (\leq) f(\langle \phi | A | \phi \rangle)$.

Proof. By lemma 10, f is continuous. Let $A = \int_{\sigma(A)} \lambda dE_{\lambda}$ be the spectral decomposition of the self-adjoint operator A. Assume that f is convex. For any unit vector $|\phi\rangle \in H$, denote by μ the probability measure defined by $\mu(\Delta) = \langle \phi | E(\Delta) | \phi \rangle$ for any Borel set Δ . With $\{\Delta_k\}$ any finite Borel partition of $\sigma(A)$ and $\lambda_k \in \Delta_k$, we have

$$\langle \phi | f(A) | \phi \rangle = \int_{\sigma(A)} f(\lambda) d\mu_{\lambda} = \lim_{\{\Delta_k\}} \int_{k} f(\lambda_k) \mu(\Delta_k)$$

$$\geq \lim_{\{\Delta_k\}} \left(\sum_{k} \lambda_k \mu(\Delta_k) \right) = f\left(\int_{\sigma(A)} \lambda d\mu_{\lambda} \right) = f(\langle \phi | A | \phi \rangle). \tag{19}$$

Similarly, if f is concave, then one gets

$$\langle \phi | f(A) | \phi \rangle \le f(\langle \phi | A | \phi \rangle).$$
 (20)

Lemma 13. Suppose f is a convex (concave) function. If A, B are two positive operators acting on a Hilbert space B and B is of trace-class, then

$$Tr(f(B) - f(A)) \ge (\le) Tr((B - A)f'(A)). \tag{21}$$

Proof. As *A* is a positive operator of trace-class, by spectral theorem, there exists an orthnormal basis $\{|\phi_i\rangle\}$ of *H* and nonnegative numbers λ_i such that $A = \sum_i \lambda_i |\phi_i\rangle \langle \phi_i|$. If *f* is convex, then by lemma 12 and lemma 11 we have

$$\operatorname{Tr}[f(B) - f(A)] = \sum_{i} [\langle \phi_{i} | f(B) | \phi_{i} \rangle - f(\lambda_{i})]$$

$$\geq \sum_{i} [f(\langle \phi_{i} | B | \phi_{i} \rangle) - f(\lambda_{i})]$$

$$\geq \sum_{i} [(\langle \phi_{i} | B | \phi_{i} \rangle - \lambda_{i}) f'(\lambda_{i})]$$

$$= \operatorname{Tr}[(B - A) f'(A)]. \tag{22}$$

Similarly, if *f* is concave, then

$$Tr(f(B) - f(A)) \le Tr((B - A)f'(A)). \tag{23}$$

In finite-dimensional case, the following result is valid and is called the generalized Klein's inequality. We generalize it to infinite-dimensional case.

Lemma 14. (Generalized Klein's inequality) *Let A, B be two positive operators of trace-class on a Hilbert space H. If* $Tr(A \log A) < \infty$, *then*

$$Tr(-B \log B) \le Tr(-B \log A - B + A). \tag{24}$$

Proof. Take f so that $f(x) = -x \log x$ for x > 0 and f(0) = 0. Then f(x) is a concave function with $f'(x) = -\log x - \frac{1}{\ln 2}$ and $f''(x) = -\frac{1}{\ln 2} \frac{1}{x} < 0$ for x > 0. By lemma 13, we have

$$Tr[-B \log B + A \log A] \le Tr[(B - A)(-\log A - 1)] = Tr[-B \log A + A \log A - B + A].$$
 (25)

Since $\operatorname{Tr} A \log A < \infty$, we get $\operatorname{Tr} (-B \log B) \leq \operatorname{Tr} (-B \log A - B + A)$, as desired.

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Making use of this result, we see that the relative entropy $S(\rho, \sigma) = \text{Tr}(\rho(\log \rho - \log \sigma))$ is also non-negative for the infinite-dimensional quantum systems whenever $S(\sigma) < \infty$.

Corollary 15. For any two density operators $\rho, \sigma \in \mathcal{S}(H)$, if $\text{Tr}(\sigma \log \sigma) < \infty$, then

$$\operatorname{Tr}(-\rho \log \rho) \le \operatorname{Tr}(-\rho \log \sigma).$$
 (26)

Proof. Since ρ , σ are two density operators, $\operatorname{Tr} \rho = \operatorname{Tr} \sigma = 1$. Substituting these in the inequality (24), we have $\operatorname{Tr}(-\rho \log \rho) \leq \operatorname{Tr}(-\rho \log \sigma)$.

Next, we apply the corollary 15 to prove the subadditivity inequality (27) and the triangle inequalities (29) and (30) for Von Neumann entropy.

Lemma 16. Let $\rho^{AB} \in \mathcal{S}(H_A \otimes H_B)$ be a state with $\dim H_A \otimes H_B \leq +\infty$. Then

$$S(\rho^{AB}) \le S(\rho^A) + S(\rho^B) \tag{27}$$

where $\rho^A = \text{Tr}_B \rho^{AB}$ and $\rho^B = \text{Tr}_A \rho^{AB}$.

Proof. Let $\rho = \rho^{AB}$ and $\sigma = \rho^A \otimes \rho^B$. Then, $S(\sigma) = S(\rho^A \otimes \rho^B) = S(\rho^A) + S(\rho^B)$. Note that

$$\begin{aligned}
-\text{Tr}(\rho \log \sigma) &= -\text{Tr}[\rho^{AB}(\log \rho^{A} \otimes I^{B} + I^{A} \otimes \log \rho^{B})] \\
&= -\text{Tr}[\rho^{AB}(\log \rho^{A} \otimes I^{B})] - \text{Tr}[\rho^{AB}(I^{A} \otimes \log \rho^{B})] \\
&= -\text{Tr}_{A} \text{Tr}_{B}[\rho^{AB}(\log \rho^{A} \otimes I^{B})] - \text{Tr}_{B} \text{Tr}_{A}[\rho^{AB}(I^{A} \otimes \log \rho^{B})] \\
&= -\text{Tr}_{A}(\rho^{A} \log \rho^{A}) - \text{Tr}_{B}(\rho^{B} \log \rho^{B}) \\
&= S(\rho^{A}) + S(\rho^{B}).
\end{aligned}$$
(28)

If $S(\sigma) < \infty$, corollary 15 and the above equations imply $S(\rho) \le -\text{Tr}(\rho \log \sigma) = S(\rho^A) + S(\rho^B)$. If $S(\sigma) = \infty$, then $S(\rho^A) + S(\rho^B) = \infty$, and obviously $S(\rho^{AB}) \le S(\rho^A) + S(\rho^B)$ holds. \square In finite-dimensional case, the inequalities $S(\rho^{AB}) \ge |S(\rho^A) - S(\rho^B)|$ holds for any bipartite states and is

In finite-dimensional case, the inequalities $S(\rho^{AB}) \ge |S(\rho^A) - S(\rho^B)|$ holds for any bipartite states and is called the triangle inequality. In infinite-dimensional case, this inequality may be not valid except the case when both $S(\rho^A)$, $S(\rho^B)$ are finite. What we can have is the triangle inequalities of the following kind.

Lemma 17. Let $\rho^{AB} \in \mathcal{S}(H_A \otimes H_B)$ with $\dim H_A \otimes H_B = +\infty$. Then

$$S(\rho^A) + S(\rho^{AB}) \ge S(\rho^B) \tag{29}$$

and

$$S(\rho^B) + S(\rho^{AB}) \ge S(\rho^A) \tag{30}$$

where $\rho^A = \operatorname{Tr}_B \rho^{AB}$, and $\rho^B = \operatorname{Tr}_A \rho^{AB}$.

Proof. To prove the inequality (29), we introduce a system C which purifies the system AB. Let $|\Psi^{ABC}\rangle$ be a purification of ρ^{AB} ; then

$$\operatorname{Tr}_{C}|\Psi^{ABC}\rangle\langle\Psi^{ABC}|=\rho^{AB}, \quad \operatorname{Tr}_{BC}|\Psi^{ABC}\rangle\langle\Psi^{ABC}|=\rho^{A}$$
 (31)

and

$$\operatorname{Tr}_{AB}|\Psi^{ABC}\rangle\langle\Psi^{ABC}|=\rho^{C}.\tag{32}$$

Applying the subadditivity, that is, lemma 16, we have

$$S(\rho^A) + S(\rho^C) \ge S(\rho^{AC}). \tag{33}$$

Since $|\Psi^{ABC}\rangle$ is a pure state, $S(\rho^{AB}) = S(\rho^C)$ and $S(\rho^{AC}) = S(\rho^B)$. Hence the previous inequality is the same as $S(\rho^A) + S(\rho^{AB}) \ge S(\rho^B)$.

By symmetry between the systems A and B one sees that $S(\rho^B) + S(\rho^{AB}) > S(\rho^A)$ is also true.

Now, we relate the entropy exchange to change in the entropy of the system Q for infinite-dimensional quantum systems.

Theorem 18. For any evolution Φ^Q and initial state ρ^Q in an infinite-dimensional system Q, with $\rho^Q = \Phi^Q(\rho^Q)$, the following inequalities are true.

$$S_e + S(\rho^Q) \ge S(\rho^{Q'}),\tag{34}$$

$$S_{\rho} + S(\rho^{Q'}) \ge S(\rho^{Q}) \tag{35}$$

and

$$S(\rho^{Q}) + S(\rho^{Q'}) \ge S_{e}. \tag{36}$$

Proof. The evolution Φ^Q in fact is due to a unitary evolution of a larger system that includes an environment E with a pure initial state $|0^E\rangle$ and the joint initial state $\rho^{QE}=\rho^Q\otimes|0^E\rangle\langle 0^E|$. Obviously, we have $S(\rho^{QE})=S(\rho^Q)$. Since the joint system QE evolves unitarily, say $\rho^{Q'E}=U\rho^{QE}U^\dagger$, one sees that $\rho^Q'=\Phi^Q(\rho^Q)=\mathrm{Tr}_E\rho^{Q'E}$ and the entropy of the joint state remains unchanged. Thus have $S(\rho^{Q'E})=S(\rho^Q)$. Let $|\Psi^{RQ}\rangle$ be a purification of ρ^Q to a larger system RQ; then $\rho^{RQ'}=(I\otimes\Phi^Q)|\Psi^{RQ}\rangle\langle \Psi^{RQ}|=\mathrm{Tr}_E(I\otimes U)(|\Psi^{RQ}\rangle|0^E\rangle\langle \Psi^{RQ}|\langle 0^E|)(I\otimes U^\dagger)$. This means that $|\Upsilon^{RQ'E}\rangle=(I\otimes U)(|\Psi^{RQ}\rangle|0^E\rangle\rangle$ is a purification of $\rho^{RQ'}$. Let $\rho^E=\mathrm{Tr}_{RQ'}(I\otimes U)(|\Psi^{RQ}\rangle|0^E\rangle\langle \Psi^{RQ}|\langle 0^E|)(I\otimes U^\dagger)$. Then by the lemma B, the entropy exchange $B_{e}=B_{e}$ is B_{e} . Using the inequality (27), one gets B_{e} is B_{e} . Which gives B_{e} 0 is B_{e} 1 inequality (29) implies that B_{e} 1 inequality B_{e} 2 implies that B_{e} 3 in B_{e} 4 inequality B_{e} 4 inequality B_{e} 5 in B_{e} 5 in B_{e} 5 in B_{e} 5 in B_{e} 6 in B_{e} 7 in B_{e} 8 in B_{e} 8 in B_{e} 8 in B_{e} 9 in B_{e} 9

By theorem 18 we known that $S_e \leq S(\rho^Q) + S(\rho^{Q'})$ is always true. And, if both $S(\rho^Q)$, $S(\rho^{Q'})$ are finite, then, as in finite-dimensional case, we have $S_e \geq |S(\rho^Q) - S(\rho^{Q'})|$, which means that the entropy exchange is not less than the change in entropy of the system Q. In general, the entropy exchange is different from the change in entropy of the system Q, that is, $S_e > |S(\rho^Q) - S(\rho^{Q'})|$ holds for some channels and states.

Examples. The following is an example for finite-dimensional case.

Example 1. Let $\rho^Q = \frac{1}{2}I \in \mathcal{S}(H_Q)$ with dim $H_Q = 2$. The bit flip channel Φ^Q flips the state of a qubit from $|0\rangle$ to $|1\rangle$ with probability 1-p. It has operation elements

$$E_0 = \sqrt{p} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \ E_1 = \sqrt{1 - p} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$
 (37)

After some calculation, $\rho^{Q'} = \Phi^Q(\rho^Q) = E_0 \rho^Q E_0 + E_1 \rho^Q E_1 = \frac{1}{2}I$, thus $S(\rho^{Q'}) - S(\rho^Q) = 1 - 1 = 0$. On the other hand, note that $|\Psi^{RQ}\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$ is a purifications of ρ^Q to a composite system RQ, where $\dim H_R = 2$. Thus

$$\rho^{RQ'} = I^R \otimes \Phi^Q(|\Psi^{RQ}\rangle \langle \Psi^{RQ}|)
= I^R \otimes \Phi^Q\{\frac{1}{2}[|0\rangle\langle 0| \otimes |1\rangle\langle 1| + |0\rangle\langle 1| \otimes |1\rangle\langle 0| + |1\rangle\langle 0| \otimes |0\rangle\langle 1|
+ |1\rangle\langle 1| \otimes |0\rangle\langle 0|]\}
= \frac{1}{2}[|0\rangle\langle 0| \otimes \Phi^Q(|1\rangle\langle 1|) + |0\rangle\langle 1| \otimes \Phi^Q(|1\rangle\langle 0|)
+ |1\rangle\langle 0| \otimes \Phi^Q(|0\rangle\langle 1|) + |1\rangle\langle 1| \otimes \Phi^Q(|0\rangle\langle 0|)]
= \frac{1}{2}\begin{bmatrix} 1 - p & 0 & 0 & 1 - p \\ 0 & p & p & 0 \\ 0 & p & p & 0 \\ 1 - p & 0 & 0 & 1 - p \end{bmatrix}.$$
(38)

Obviously, the nonzero eigenvalues of $\rho^{RQ'}$ are p and 1-p, and thus, $Se=S(\rho^{RQ'})=-p\log p-(1-p)\log (1-p)$. Hence we have $S_e>|S(\rho^Q)-S(\rho^{Q'})|$ whenever 0< p<1. Next we give an example for infinite-dimensional case.

Example 2. Consider the thermal radiation signal ρ^Q on a Gaussian system Q, which has Glauber's P representation $\rho^Q = (\pi N)^{-1} \int \exp(-|\alpha|^2/N) |\alpha\rangle \langle \alpha| d^2\alpha$. Here N is the average number of photons of ρ^Q , $|\alpha\rangle$ is the coherent state and is an eigenstate of the annihilation operator a for each complex number α . Let Φ^Q be the thermal radiation noise channel, $\Phi^Q(\rho^Q) = (\pi N_n)^{-1} \int \exp(-|\alpha|^2/N_n) D(\alpha) \rho^Q D^\dagger(\alpha) d^2\alpha$, where $D(\alpha) = \exp(\alpha a^\dagger - \alpha^* a)$ is the displacement operator, and N_n is the average photon number of the output state if the input is the vacuum. If the input state ρ^Q is a thermal noise signal with its average photon number N_s , then the output state ρ^Q will be a thermal noise signal with its average photon number $N_s + N_n^{19}$. We know that the entropy of any Gaussian state ρ is finite and is formulated by $S(\rho) = g(N)$, where $g(x) = (x+1)\ln(x+1) - x\ln x$ is a monotonically increasing convex function and N is the average number of photons of the Gaussian state ρ . Thus, we can get $S(\rho^{Q'}) - S(\rho^Q) = g(N_s + N_n) - g(N_s)$. Now, we introduce a reference system R, initially, the joint system RQ is prepared in a pure entangled states $|\Psi^{RQ}\rangle$ with $\text{Tr}_R|\Psi^{RQ}\rangle \langle \Psi^{RQ}| = \rho^Q$, i.e., the pure state $|\Psi^{RQ}\rangle$ is a purification of the state ρ^Q . The system R is dynamically isolated and has a zero internal Hamiltonian, while the system Q undergoes an internal with above thermal noise channel Φ^Q . The final state of RQ is described by the state $\rho^{RQ'}$. Then the entropy exchange $S_e = S(\rho^{RQ'}) = g(N_1) + g(N_2)$, where $N_1 = \frac{v_s v_n}{u - y_s v_n}$, $N_2 = \frac{v_n}{u - v_n}$, $v_s = \frac{N_s}{N_s + 1}$, $v_n = \frac{N_s}{N_s + 1}$ and u is the positive root of the equation $u^2 - (1 - v_s + 2v_n v_s)u + v_n^2 v_s = 0$.

(1) If $N_s=0$, i.e., the input state $\rho^Q=|0\rangle\langle 0|$, then we can easily derive $S(\rho^{Q'})-S(\rho^Q)=g(N_s+N_n)-g(N_s)=g(N_n)$. On the other hand, as $\nu_s=0$ and u=1, we see that $N_1=0$, $N_2=\frac{\nu_n}{1-\nu_n}=N_n$ and $S_e=g(N_1)+g(N_2)=g(N_n)$. Thus it follows that $S_e=|S(\rho^Q)-S(\rho^{Q'})|$ in this case.

(2) If $\rho^Q \neq |0\rangle \langle 0|$, we can set $N_s = 1$ and $N_n = 1$. Then, $v_s = \frac{1}{2}$ and $v_n = \frac{1}{2}$. In this case we can derive $N_1 = \frac{12}{29}$ and $N_2 = \frac{24}{17}$. Then it is easily checked that $S(\rho^{Q'}) - S(\rho^Q) = g(N_s + N_n) - g(N_s) = g(2) - g(1) = 0.5232$ and $S_e = g(N_1) + g(N_2) = g\left(\frac{12}{29}\right) + g\left(\frac{24}{17}\right) = 2.4911$. Hence we have $S_e > |S(\rho^Q) - S(\rho^{Q'})|$ whenever ρ^Q .

Discussion

The notion of entropy exchange can be introduced in infinite-dimensional quantum systems with the same form as that in finite-dimensional systems if we allow it may take infinity value. Thus, for a state ρ^Q and a channel Φ^Q in an infinite-dimensional system Q, the entropy exchange S_e is defined as $S_e = S(\rho^{RQ'})$, where $\rho^{RQ'} = (I^R \otimes \Phi^Q) |\Psi^{RQ}\rangle \langle \Psi^{RQ}|$ and $|\Psi^{RQ}\rangle$ is a purification of ρ^Q in a larger system RQ. This quantity does not depend on the choice of purifications of the state ρ^Q and characterizes the information exchange between the system Q and the external world during the evolution given by Φ^Q . An explicit expression for S_e in terms of ρ^Q and Φ^Q is established, which asserts that $S_e = \min\{-\sum_\mu P_\mu \log P_\mu\}$, where $P_\mu = \operatorname{Tr} A_\mu^Q \rho^Q A_\mu^{Q\dagger}$ with $\{A_\mu^Q\}$ the sequence of Kraus operators in an operator-sum representation of Φ^Q , and the minimum is taken over all operator-sum representations of Φ^Q . In general, the entropy exchange is not equal to the change in entropy $|S(\rho^Q) - S(\rho^Q')|$ of the system Q, where $\rho^Q = \Phi^Q(\rho^Q)$. But we have $S_e \leq S(\rho^Q) + S(\rho^{Q'})$, $S_e + S(\rho^Q) \geq S(\rho^Q)$ and $S_e + S(\rho^Q') \geq S(\rho^Q)$. Thus, if $S(\rho^Q)$, $S(\rho^Q)$ are both finite, then $S_e \geq |S(\rho^Q) - S(\rho^Q')|$. We also give some examples which illustrates that the entropy exchange is different from the change of entropy. In general the entropy exchange is larger than the change of entropy.

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