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# Entanglement of formation and monogamy of multi-party quantum entanglement

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We provide a sufficient condition for the monogamy inequality of multi-party quantum entanglement of arbitrary dimensions in terms of entanglement of formation. Based on the classical-classical-quantum(ccq) states whose quantum parts are obtained from the two-party reduced density matrices of a three-party quantum state, we show the additivity of the mutual information of the ccq states guarantees the monogamy inequality of the three-party pure state in terms of EoF. After illustrating the result with some examples, we generalize our result of three-party systems into any multi-party systems of arbitrary dimensions.

Quantum entanglement is a non-classical nature of quantum mechanics, which is a useful resource in many quantum information processing tasks such as quantum teleportation, dense coding and quantum cryptography<sup>1-3</sup>. Because of its important roles in the field of quantum information and computation theory, there has been a significant amount of research focused on quantification of entanglement in bipartite quantum systems. *Entanglement of formation*(EoF) is the most well-known bipartite entanglement measure with an operational meaning that asymptotically quantifies how many bell states are needed to prepare the given state using local quantum operations and classical communications<sup>4</sup>. Although EoF is defined in any bipartite quantum systems of arbitrary dimension, its definition for mixed states is based on 'convex-roof extension', which takes the minimum average over all pure-state decompositions of the given state. As such an optimization is hard to deal with, analytic evaluation of EoF is known only in two-qubit systems<sup>5</sup> and some restricted cases of higher-dimensional systems so far.

In multi-party quantum systems, entanglement shows a distinct behavior that does not have any classical counterpart; if a pair of parties in a multi-party quantum system is maximally entangled, then they cannot be entanglement, not even classically correlated, with the rest parties. This restriction of sharing entanglement in multi-party quantum systems is known as the *monogamy of entanglement* (MoE)<sup>6,7</sup>. MoE plays an important role such as the security proof of quantum key distribution in quantum cryptography<sup>2,8</sup> and the *N*-representability problem for fermions in condensed-matter physics<sup>9</sup>.

Mathematically, MoE can be characterized using *monogamy inequality*; for a three-party quantum state  $\rho_{ABC}$  and its two-party reduced density matrices  $\rho_{AB}$  and  $\rho_{AC}$ ,

$$E(\rho_{A(BC)}) \ge E(\rho_{AB}) + E(\rho_{AC}) \tag{1}$$

where  $E(\rho_{XY})$  is an entanglement measure quantifying the amount of entanglement between subsystems X and Y of the bipartite quantum state  $\rho_{XY}$ . Inequality (1) shows the mutually exclusive nature of bipartite entanglement  $E(\rho_{AB})$  and  $E(\rho_{AC})$  shared in three-party quantum systems so that their summation cannot exceeds the total entanglement  $E(\rho_{A(BC)})$ .

Using tangle<sup>10</sup> as the bipartite entanglement measure, Inequality (1) was first shown to be true for all three-qubit states, and generalized for multi-qubit systems as well as some cases of higher-dimensional quantum systems<sup>11,12</sup>. However, not all bipartite entanglement measures can characterize MoE in forms of Inequality (1), but only few measures are known so far satisfying such monogamy inequality<sup>13–15</sup>. Although EoF is the most natural bipartite entanglement measure with the operational meaning in quantum state preparation, EoF is known to fail in characterizing MoE as the monogamy inequality in (1) even in three-qubit systems; there exists quantum states in three-qubit systems violating Inequality (1) if EoF is used as the bipartite entanglement measure. Thus, a natural question we can ask is 'On what condition does the monogamy inequality hold in terms of the given bipartite entanglement measure?'.

Here, we provide a sufficient condition that monogamy inequality of quantum entanglement holds in terms of EoF in multi-party, arbitrary dimensional quantum systems. For a three-party quantum state, we first consider

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the classical-classical-quantum (ccq) states whose quantum parts are obtained from the two-party reduced density matrices of the three-party state. By evaluating quantum mutual information of the ccq states as well as their reduced density matrices, we show that the additivity of the mutual information of the ccq states guarantees the monogamy inequality of the three-party quantum state in terms of EoF. We provide some examples of three-party pure state to illustrate our result, and we generalize our result of three-party systems into any multi-party systems of arbitrary dimensions.

This paper is organized as follows. First we briefly review the definitions of classical and quantum correlations in bipartite quantum systems and recall their trade-off relation in three-party quantum systems. After providing the definition of ccq states as well as their mutual information between classical and quantum parts, we establish the monogamy inequality of three-party quantum entanglement in arbitrary dimensional quantum systems in terms of EoF conditioned on the additivity of the mutual information for the ccq states. We also illustrate our result of monogamy inequality in three-party quantum systems with some examples, and we generalize our result of entanglement monogamy inequality into multi-party quantum systems of arbitrary dimensions. Finally, we summarize our results.

### Results

**Correlations in bipartite quantum systems.** For a bipartite pure state  $|\psi\rangle_{AB}$ , its *entanglement of formation* (EoF) is defined by the entropy of a subsystem,  $E_f(|\psi\rangle_{AB}) = S(\rho_A)$  where  $\rho_A = \operatorname{tr}_B |\psi\rangle_{AB} \langle \psi|$  is the reduced density matrix of  $|\psi\rangle_{AB}$  on subsystem A, and  $S(\rho) = -\operatorname{tr}\rho \ln \rho$  is the von Neumann entropy of the quantum state  $\rho$ . For a bipartite mixed state  $\rho_{AB}$ , its EoF is defined by the minimum average entanglement

$$E_{\mathbf{f}}(\rho_{AB}) = \min \sum_{i} p_{i} E_{\mathbf{f}}(|\psi_{i}\rangle_{AB}), \tag{2}$$

over all possible pure state decompositions of  $\rho_{AB} = \sum_i p_i |\psi_i\rangle_{AB} \langle \psi_i|$ .

For a probability ensemble  $\mathscr{E}=\{p_i,\rho_i\}$  realizing a quantum state  $\rho$  such that  $\rho=\sum_i p_i\rho_i$ , its *Holevo quantity* is defined as

$$\chi\left(\mathscr{E}\right) = S(\rho) - \sum_{i} p_{i} S(\rho_{i}). \tag{3}$$

Given a bipartite quantum state  $\rho_{AB}$ , each measurement  $\{M_B^x\}$  applied on subsystem B induces a probability ensemble  $\mathscr{E} = \{p_x, \rho_A^x\}$  of the reduced density matrix  $\rho_A = \operatorname{tr}_A \rho_{AB}$  in the way that  $p_x = \operatorname{tr}[(I_A \otimes M_B^x)\rho_{AB}]$  is the probability of the outcome x and  $\rho_A^x = \operatorname{tr}_B[(I_A \otimes M_B^x)\rho_{AB}]/p_x$  is the state of system A when the outcome was x. The one-way classical correlation (CC)<sup>16</sup> of a bipartite state  $\rho_{AB}$  is defined by the maximum Holevo quantity

$$\mathscr{J} \leftarrow (\rho_{AB}) = \max_{\mathscr{E}} \chi(\mathscr{E})$$
 (4)

over all possible ensemble representations  $\mathscr{E}$  of  $\rho_A$  induced by measurements on subsystem B.

The following proposition shows a trade-off relation between classical correlation and quantum entanglement (measured by CC and EoF, respectively) distributed in three-party quantum systems.

**Proposition 1.** <sup>17</sup> For a three-party pure state  $|\psi\rangle_{ABC}$  with reduced density matrices  $\rho_{AB}=\mathrm{tr}_C|\psi\rangle_{ABC}\langle\psi|$ ,  $\rho_{AC}=\mathrm{tr}_B|\psi\rangle_{ABC}\langle\psi|$  and  $\rho_A=\mathrm{tr}_{BC}|\psi\rangle_{ABC}\langle\psi|$ , we have

$$S(\rho_A) = \mathscr{J} \leftarrow (\rho_{AB}) + E_f(\rho_{AC}). \tag{5}$$

**Classical–classical–quantum (CCQ) states.** In this section, we consider a four-party ccq states obtained from a bipartite state  $\rho_{AB}$ , and provide detail evaluations of their mutual information. Without loss of generality, we assume that any bipartite state as a two-qudit state by taking d as the dimension of larger dimensional subsystem.

For a two-qudit state  $\rho_{AB}$ , let us consider the reduced density matrix  $\rho_B = \operatorname{tr}_A \rho_{AB}$  and its spectral decomposition

$$\rho_B = \sum_{i=0}^{d-1} \lambda_i |e_i\rangle_B \langle e_i|. \tag{6}$$

Let

$$\mathscr{E}_0 = \{\lambda_i, \sigma_A^i\}_i \tag{7}$$

be the probability ensemble of  $\rho_A=\mathrm{tr}_B\rho_{AB}$  from the measurement  $\{|e_i\rangle_B\langle e_i|\}_{i=1}^{d-1}$  on subsystem B of  $\rho_{AB}$ , in a way that

$$\lambda_i = \operatorname{tr}[(I_A \otimes |e_i\rangle_B \langle e_i|)\rho_{AB}] \tag{8}$$

and

$$\sigma_A^i = \frac{1}{\lambda_i} \operatorname{tr}_B[(I_A \otimes |e_i\rangle_B \langle e_i|)\rho_{AB}]. \tag{9}$$

Based on the eigenvectors  $\{|e_j\rangle_B\}$  of  $\rho_B$ , we also consider the d-dimensional Fourier basis elements  $|\tilde{e}_j\rangle_B = \frac{1}{\sqrt{d}} \sum_{k=0}^{d-1} \omega_d^{jk} |e_k\rangle_B$  for each  $j=0,\ldots,d-1$ , where  $\omega_d = e^{\frac{2\pi i}{d}}$  is the dth-root of unity. Let

$$\mathscr{E}_1 = \left\{ \frac{1}{d}, \tau_A^j \right\}_j,\tag{10}$$

be the probability ensemble of  $\rho_A = \operatorname{tr}_B \rho_{AB}$  obtained by measuring subsystem B in terms of the Fourier basis  $\{|\tilde{e}_j\rangle_B \langle \tilde{e}_j|\}_{j=1}^{d-1}$  where

$$\frac{1}{d} = \operatorname{tr}\left[ (I_A \otimes |\tilde{e}_j\rangle_B \langle \tilde{e}_j|) \rho_{AB} \right] \tag{11}$$

and

$$\tau_A^j = d \operatorname{tr}_B \left[ (I_A \otimes |\tilde{e}_i\rangle_B \langle \tilde{e}_i|) \rho_{AB} \right]. \tag{12}$$

Now we define the generalized d-dimensional Pauli operators based on the eigenvectors of  $\rho_B$  as

$$Z = \sum_{i=0}^{d-1} \omega_d^i |e_j\rangle_B \langle e_j|, \ X = \sum_{i=0}^{d-1} |e_{j+1}\rangle_B \langle e_j| = \sum_{i=0}^{d-1} \omega_d^{-i} |\tilde{e}_j\rangle_B \langle \tilde{e}_j|, \tag{13}$$

and consider a four-qudit ccq state  $\Gamma_{XYAB}$ 

$$\Gamma_{XYAB} = \frac{1}{d^2} \sum_{x,y=0}^{d-1} |x\rangle_X \langle x| \otimes |y\rangle_Y \langle y| \otimes \left[ (I_A \otimes X_B^x Z_B^y) \rho_{AB} (I_A \otimes Z_B^{-y} X_B^{-x}) \right], \tag{14}$$

for some *d*-dimensional orthonormal bases  $\{|x\rangle_X\}$  and  $\{|y\rangle_Y\}$  of the subsystems *X* and *Y*, respectively. From Eqs. (8), (9), (11), (12) and (13), the reduced density matrices of  $\Gamma_{XYAB}$  are obtained as

$$\Gamma_{XAB} = \frac{1}{d} \sum_{x=0}^{d-1} |x\rangle_X \langle x| \otimes \left[ \left( I_A \otimes X_B^x \right) \left( \sum_{i=0}^{d-1} \sigma_A^i \otimes \lambda_i |e_i\rangle_B \langle e_i| \right) \left( I_A \otimes X_B^{-x} \right) \right], \tag{15}$$

$$\Gamma_{YAB} = \frac{1}{d} \sum_{y=0}^{d-1} |y\rangle_{Y} \langle y| \otimes \left[ \left( I_{A} \otimes Z_{B}^{y} \right) \left( \sum_{j=0}^{d-1} \tau_{A}^{j} \otimes \frac{1}{d} |\tilde{e}_{j}\rangle_{B} \langle \tilde{e}_{j} | \right) \left( I_{A} \otimes Z_{B}^{-y} \right) \right], \tag{16}$$

and

$$\Gamma_{AB} = \rho_A \otimes \frac{I_B}{d},\tag{17}$$

where  $I_A$  and  $I_B$  are d-dimensional identity operators of subsystems A and B, respectively.

Here we note that the mutual information between the classical and quantum parts of the ccq state in Eq. (14) as well as its reduced density matrices in Eqs. (15) and (16) are

$$I(\Gamma_{XY:AB}) = \ln d + S(\rho_A) - S(\rho_{AB}), \tag{18}$$

$$I(\Gamma_{X:AB}) = \ln d - S(\rho_B) + \chi(\mathscr{E}_0), \tag{19}$$

and

$$I(\Gamma_{Y:AB}) = \chi(\mathscr{E}_1),\tag{20}$$

where the detail calculation can be found in "Methods" section.

**Monogamy inequality of multi-party entanglement in terms of EoF.** It is known that quantum mutual information is superadditive for any ccq state of the form

$$\Xi_{XYAB} = \frac{1}{d^2} \sum_{x,y=0}^{d-1} |x\rangle_X \langle x| \otimes |y\rangle_Y \langle y| \otimes \sigma_{AB}^{xy}, \tag{21}$$

that is,  $I(\Xi_{XY:AB}) \ge I(\Xi_{X:AB}) + I(\Xi_{Y:AB})^{18}$ . The following theorem shows that the additivity of quantum mutual information for ccq states guarantees the monogamy inequality of three-party quantum entanglement in therms of EoF.

**Figure 1.** The entanglement between A and BC quantified by  $E_{\mathbf{f}}(|\psi\rangle_{A(BC)})((\mathbf{a})$  in the figure) bounds the summation of the entanglement between A and B quantified by  $E_{\mathbf{f}}(\rho_{AB})((\mathbf{b})$  in the figure) and the entanglement between A and C quantified by  $E_{\mathbf{f}}(\rho_{AC})((\mathbf{c})$  in the figure).

**Theorem 1.** For any three-party pure state  $|\psi\rangle_{ABC}$  with its two-qudit reduced density matrices  ${\rm tr}_C |\psi\rangle_{ABC} \langle \psi| = \rho_{AB}$  and  ${\rm tr}_B |\psi\rangle_{ABC} \langle \psi| = \rho_{AC}$ , we have

$$E_{\mathbf{f}}(|\psi\rangle_{A(BC)}) \ge E_{\mathbf{f}}(\rho_{AB}) + E_{\mathbf{f}}(\rho_{AC}),$$
 (22)

conditioned on the additivity of quantum mutual information

$$I(\Gamma_{XY:AB}) = I(\Gamma_{X:AB}) + I(\Gamma_{Y:AB})$$
(23)

and

$$I(\Gamma_{XY:AC}) = I(\Gamma_{X:AC}) + I(\Gamma_{Y:AC})$$
(24)

where  $\Gamma_{XYAB}$  and  $\Gamma_{XYAC}$  are the ccq states of the form in Eq. (14) obtained by  $\rho_{AB}$  and  $\rho_{AC}$ , respectively.

Conditioned on the additivity of quantum mutual information for ccq states, Theorem 1 shows that EoF can characterize the monogamous nature of bipartite entanglement shared in three-party quantum systems, which is illustrateed in Fig. 1.

Example 1. Let us consider three-qubit GHZ state<sup>19</sup>,

$$|GHZ\rangle_{ABC} = \frac{1}{\sqrt{2}}(|000\rangle_{ABC} + |111\rangle_{ABC}), \tag{25}$$

with its reduced density matrices

$$\rho_{AB} = \frac{1}{2}(|00\rangle_{AB}\langle 00| + |11\rangle_{AB}\langle 11|), \ \rho_{A} = \frac{1}{2}(|0\rangle_{A}\langle 0| + |1\rangle_{A}\langle 1|), \ \rho_{B} = \frac{1}{2}(|0\rangle_{B}\langle 0| + |1\rangle_{B}\langle 1|). \tag{26}$$

The eigenvalues of  $\rho_B$  are  $\lambda_0 = \lambda_1 = \frac{1}{2}$  with corresponding eigenvectors  $|e_0\rangle_B = |0\rangle_B$  and  $|e_1\rangle_B = |1\rangle_B$  respectively. Thus the ensemble of  $\rho_A$  induced by measuring subsystem B of  $\rho_{AB}$  in terms of the eigenvectors of  $\rho_B$ , that is,  $\{M_B^0 = |0\rangle_B \langle 0|, M_B^1 = |1\rangle_B \langle 1|\}$  is

$$\mathscr{E}_0 = \{ \lambda_0 = \frac{1}{2}, \sigma_A^0 = |0\rangle_A \langle 0|, \lambda_1 = \frac{1}{2}, \sigma_A^1 = |1\rangle_A \langle 1| \}.$$
 (27)

Because the Fourier basis elements of subsystem B with respect to the eigenvectors of  $\rho_B$  are

$$|\tilde{e}_0\rangle_B = \frac{1}{2}(|0\rangle_B + |1\rangle_B), \ |\tilde{e}_1\rangle_B = \frac{1}{2}(|0\rangle_B - |1\rangle_B),$$
 (28)

the ensemble of  $\rho_A$  induced by measuring subsystem B of  $\rho_{AB}$  in terms of the Fourier basis in Eq. (28) is

$$\mathscr{E}_1 = \{\frac{1}{2}, \tau_A^0 = \frac{1}{2}I_A, \frac{1}{2}, \tau_A^1 = \frac{1}{2}I_A\}. \tag{29}$$

Now we consider the additivity of mutual information of the ccq state  $\Gamma_{XYAB}$  obtained from  $\rho_{AB}$  in Eq. (26). Due to Eq. (18), the mutual information of  $\Gamma_{XYAB}$  between XY and AB is

$$I(\Gamma_{XY:AB}) = \ln 2 + S(\rho_A) - S(\rho_{AB}) = \ln 2$$
 (30)

because  $S(\rho_A) = S(\rho_{AB}) = \ln 2$  from Eqs. (26). For the mutual information of  $\Gamma_{XAB}$  between X and AB, Eq. (19) leads us to

$$I(\Gamma_{X:AB}) = \ln 2 - S(\rho_B) + \chi(\mathscr{E}_0) = \ln 2$$
(31)

where the second equality is from  $S(\rho_B) = \ln 2$  and

$$\chi(\mathscr{E}_0) = S(\rho_A) - \frac{1}{2}S(|0\rangle_A\langle 0|) - \frac{1}{2}S(|1\rangle_A\langle 1|) = \ln 2,$$
(32)

for the ensemble  $\mathscr{E}_0$  in Eq. (27). For the mutual information of  $\Gamma_{YAB}$  between Y and AB, Eq. (20) leads us to

$$I(\Gamma_{Y:AB}) = \chi(\mathscr{E}_1) = S(\rho_A) - \frac{1}{2}S(\frac{1}{2}I_A) - \frac{1}{2}S(\frac{1}{2}I_A) = 0$$
(33)

where the second equality is due to the ensemble  $\mathcal{E}_1$  in Eq. (29).

From Eqs. (30), (31) and (33), we note that the mutual information of the ccq state  $\Gamma_{XYAB}$  obtained from  $\rho_{AB}$  is additive as in Eq. (23). Moreover, the symmetry of GHZ state assures that the same is also true for the reduced density matrix  $\rho_{AC} = \operatorname{tr}_B |GHZ\rangle_{ABC} \langle GHZ|$ . Thus Theorem 1 guarantees the monogamy inequality of the three-qubit GHZ state in Eq. (25) in terms of EoF. In fact, we have  $E_{\mathbf{f}}(|GHZ\rangle_{A(BC)}) = S(\rho_A) = \ln 2$ , whereas the two-qubit reduced density matrices  $\rho_{AB}$  and  $\rho_{AC}$  are separeble. Thus  $E_{\mathbf{f}}(\rho_{AB}) = E_{\mathbf{f}}(\rho_{AC}) = 0$  and this implies the monogamy inequality in (22).

Let us consider another example of three-qubit state.

**Example 2.** Three-qubit W-state is defined as<sup>20</sup>

$$|W\rangle_{ABC} = \frac{1}{\sqrt{3}}(|100\rangle_{ABC} + |010\rangle_{ABC} + |001\rangle_{ABC}).$$
 (34)

The two-qubit reduced density matrix of  $|W\rangle_{ABC}$  on subsystem AB is obtained as

$$\rho_{AB} = \frac{2}{3} |\psi^{+}\rangle_{AB} \langle \psi^{+}| + \frac{1}{3} |00\rangle_{AB} \langle 00| \tag{35}$$

where  $\left|\psi^{+}\right\rangle_{AB}=\frac{1}{\sqrt{2}}(|01\rangle_{AB}+|10\rangle_{AB})$  is the two-qubit Bell state. The one-qubit reduced density matrices of  $\rho_{AB}$  are

$$\rho_A = \frac{2}{3} |0\rangle_A \langle 0| + \frac{1}{3} |1\rangle_A \langle 1|, \ \rho_B = \frac{2}{3} |0\rangle_B \langle 0| + \frac{1}{3} |1\rangle_B \langle 1|.$$
 (36)

From to the spectral decomposition of  $\rho_B$  in Eq. (36) with the eigenvalues  $\lambda_0 = \frac{2}{3}$ ,  $\lambda_1 = \frac{1}{3}$  and corresponding eigenvectors  $|e_0\rangle_B = |0\rangle_B$  and  $|e_1\rangle_B = |1\rangle_B$ , respectively, it is straightforward to check that the ensemble of  $\rho_A$  induced from measuring subsystem B of  $\rho_{AB}$  by the eigenvectors of  $\rho_B$  is

$$\mathscr{E}_0 = \{ \lambda_0 = \frac{2}{3}, \sigma_A^0 = \frac{1}{2} I_A, \lambda_1 = \frac{1}{3}, \sigma_A^1 = |0\rangle_A \langle 0| \}.$$
 (37)

Because the Fourier basis of subsystem A is the same as Eq. (28), it is also straightforward to obtain the ensemble of  $\rho_A$  induced by measuring subsystem B of  $\rho_{AB}$  in terms of the Fourier basis,

$$\mathscr{E}_1 = \{\frac{1}{2}, \tau_A^j\}_{j=1,2},\tag{38}$$

where

$$\tau_{A}^{0} = \frac{1}{3}(2|0\rangle_{A}\langle 0| + |0\rangle_{A}\langle 1| + |1\rangle_{A}\langle 0| + |1\rangle_{A}\langle 1|), \ \tau_{A}^{1} = \frac{1}{3}(2|0\rangle_{A}\langle 0| - |0\rangle_{A}\langle 1| - |1\rangle_{A}\langle 0| + |1\rangle_{A}\langle 1|). \ \ (39)$$

For the mutual information of the ccq state  $\Gamma_{XYAB}$  obtained from  $\rho_{AB}$  in Eq. (35), Eq. (18) together with Eqs. (35) and (36) lead us to

$$I(\Gamma_{XY:AB}) = \ln 2 + S(\rho_A) - S(\rho_{AB}) = \ln 2.$$
 (40)

For the mutual information of  $\Gamma_{XAB}$  between X and AB, Eq. (19) leads us to

$$I(\Gamma_{X:AB}) = \ln 2 - S(\rho_B) + \chi(\mathcal{E}_0), \tag{41}$$

where Eq. (37) impies

$$\chi(\mathscr{E}_0) = S(\rho_A) - \frac{2}{3}S\left(\frac{1}{2}I_A\right) - \frac{1}{3}S(|0\rangle_A\langle 0|) = S(\rho_A) - \frac{2}{3}\ln 2.$$
 (42)

Due to Eq. (36), we have  $S(\rho_A) = S(\rho_B)$ , therefore Eqs. (41) and (42) lead us to

$$I(\Gamma_{X:AB}) = \frac{1}{3} \ln 2. \tag{43}$$

For the mutual information of  $\Gamma_{YAB}$  between Y and AB, Eq. (20) leads us to

$$I(\Gamma_{Y:AB}) = \chi(\mathscr{E}_1) = S(\rho_A) - \frac{1}{2}S(\tau_A^0) - \frac{1}{2}S(\tau_A^1), \tag{44}$$

where the second equality is from the ensemble  $\mathscr{E}_1$  in Eq. (38). Here we note that  $\tau_A^0$  and  $\tau_A^1$  in Eq. (39) have the same eigenvalues, that is  $\mu_0 = \frac{3+\sqrt{5}}{6}$  and  $\mu_0 = \frac{3-\sqrt{5}}{6}$ , therefore we have  $S(\tau_A^0) = S(\tau_A^1)$ . From the spectral decomposition of  $\rho_A$  in Eq. (36), we have  $S(\rho_A) = \ln 3 - \frac{2}{3} \ln 2$ , and this turns Eq. (44) into

$$I(\Gamma_{Y:AB}) = \ln 3 - \frac{2}{3} \ln 2 - S(\tau_A^0). \tag{45}$$

From Eqs. (40), (43) and (45), we have

$$I(\Gamma_{XY:AB}) - I(\Gamma_{X:AB}) - I(\Gamma_{Y:AB}) = \frac{4}{3} \ln 2 - \ln 3 + S(\tau_A^0), \tag{46}$$

where  $\ln 2 \approx 0.693147$ ,  $\ln 3 \approx 1.098612$  and  $S(\tau_A^0) \approx 0.381264$ . Thus we have

$$I(\Gamma_{XY:AB}) - I(\Gamma_{X:AB}) - I(\Gamma_{Y:AB}) \approx 0.206848 > 0,$$
 (47)

which implies the nonadditivity of mutual information for the ccq state  $\Gamma_{XYAB}$  obtained from  $\rho_{AB}$  in Eq. (35). We also note that the symmetry of W state in Eq. (34) would imply the nonadditivity of mutual information for the ccq state  $\Gamma_{XYAC}$  obtained from the two-qubit reduced density matrix  $\rho_{AC}$  of W state in Eq. (34).

As the additivity of mutual information in Theorem 1 is only a sufficient condition for monogamy inequality in terms of EoF, nonadditivity does not directly imply violation of Inequality (22) for the W state in Eq. (34). However, we note that  $\rho_{AB}$  in Eq. (35) is a two-qubit state, therefore its EoF can be analytically evaluated as<sup>5</sup>

$$E_{\mathbf{f}}(\rho_{AB}) \approx 0.3812. \tag{48}$$

Moreover, the symmetry of the W state assures that the EoF of  $\rho_{AC} = \text{tr}_B |W\rangle_{ABC} \langle W|$  is the same,

$$E_{\mathbf{f}}(\rho_{AC}) \approx 0.3812,\tag{49}$$

whereas

$$E_{\mathbf{f}}(|W\rangle_{A(BC)}) = S(\rho_A) \approx 0.6365. \tag{50}$$

As Eqs. (48), (49) and (50) imply the violation of Inequality (22), W state in Eq. (34) can be considered as an example for the contraposition of Theorem 1; violation of monogamy inequality in (22) implies nonadditivity of quantum mutual information for the ccq state.

Now, we generalize Theorem 1 for multi-party quantum states of arbitrary dimension.

**Theorem 2.** For any multi-party quantum state  $\rho_{A_1A_2\cdots A_n}$  with two-party reduced density matrices  $\rho_{A_1A_i}$  for  $i=2,\cdots,n$ , we have

$$E_{\mathbf{f}}(\rho_{A_1(A_2\cdots A_n)}) \ge \sum_{i=2}^n E_{\mathbf{f}}(\rho_{A_1A_i}),\tag{51}$$

conditioned on the additivity of quantum mutual information

$$I(\Gamma_{XY:A_1A_i}) = I(\Gamma_{X:A_1A_i}) + I(\Gamma_{Y:A_1A_i})$$
(52)

where  $\Gamma_{XYA_1A_i}$  is the ccq state of the form in Eq. (14) obtained by  $\rho_{A_1A_i}$  for  $i=2,\ldots,n$ .

### Discussion

We have considered possible conditions for monogamy inequality of multi-party quantum entanglement in terms of EoF, and shown that the additivity of mutual information of the ccq states implies the monogamy inequality of three-party quantum entanglement in terms of EoF. We have also provided examples of three-qubit GHZ and W states to illustrate our result in three-party case, and generalized our result into any multi-party systems of arbitrary dimensions.

Most monogamy inequalities of quantum entanglement deal with bipartite entanglement measures based on the minimization over all possible pure state ensembles. As analytic evaluation of such entanglement measure is generally hard especially in higher dimensional quantum systems more than qubits, the situation becomes far more difficult in investigating and establishing entanglement monogamy of multi-party quantum systems of arbitrary dimensions. The sufficient condition provided here deals with the quantum mutual information of the ccq states to guarantee the monogamy inequality of entanglement in terms of EoF in arbitrary dimensions. As the sufficient condition is not involved with any minimization process, our result can provide a useful methodology to understand the monogamy nature of multi-party quantum entanglement in arbitrary dimensions. We finally remark that it would be an interesting future task to investigate if the condition provided here is also necessary.

### Methods

**Evaluation for the quantum mutual information of the ccq states.** Here we evaluate the mutual information of the ccq state in Eq. (14) as well as the reduced density matrices in Eqs. (15) and (16); the classical parts of the four-qudit ccq state  $\Gamma_{XYAB}$  in Eq. (14) is  $\Gamma_{XY} = \frac{1}{d^2} \sum_{x,y=0}^{d-1} |x\rangle_X \langle x| \otimes |y\rangle_Y \langle y|$ , which is the maximally mixed state in  $d^2$ -dimensional quantum system, therefore its von Neumann entropy is

$$S(\Gamma_{XY}) = -\sum_{x,y=0}^{d-1} \frac{1}{d^2} \ln\left(\frac{1}{d^2}\right) = 2 \ln d.$$
 (53)

We also note that Eq. (17) leads us to

$$S(\Gamma_{AB}) = S(\rho_A) + \ln d. \tag{54}$$

From the *joint entropy theorem*<sup>21,22</sup>, we have

$$S(\Gamma_{XYAB}) = 2\ln d + \frac{1}{d^2} \sum_{x,y=0}^{d-1} S\left( (I_A \otimes X_B^x Z_B^y) \rho_{AB} (I_A \otimes Z_B^{-y} X_B^{-x}) \right) = 2\ln d + S(\rho_{AB}), \tag{55}$$

where the second equality is due to the unitary invariance of von Neumann entropy. Thus Eqs. (53), (54) and (55) give us the mutual information of the four-qudit ccq state  $\Gamma_{XYAB}$  with respect to the bipartition between XY and AB in Eq. (18).

For the von Neumann entropy of  $\Gamma_{XAB}$  in Eq. (15), we have

$$S(\Gamma_{XAB}) = \ln d + \frac{1}{d} \sum_{x=0}^{d-1} S\left(\left(I_A \otimes X_B^x\right) \left(\sum_{i=0}^{d-1} \sigma_A^i \otimes \lambda_i | e_i \rangle_B \langle e_i |\right) \left(I_A \otimes X_B^{-x}\right)\right)$$

$$= \ln d + S\left(\sum_{i=0}^{d-1} \sigma_A^i \otimes \lambda_i | e_i \rangle_B \langle e_i |\right)$$

$$= \ln d + H(\Lambda) + \sum_{i=1}^{d-1} \lambda_i S\left(\sigma_A^i\right),$$
(56)

where the first equality is from the joint entropy theorem, the second equality is due to the unitary invariance of von Neumann entropy and the last equality is due to the joint entropy theorem together with  $H(\Lambda) = -\sum_i \lambda_i \ln \lambda_i$  that is the shannon entropy of the spectrum  $\Lambda = \{\lambda_i\}$  of  $\rho_B$  in Eq. (6). Thus we can rewrite the von Neumann entropy of  $\Gamma_{XAB}$  as

$$S(\Gamma_{XAB}) = \ln d + S(\rho_B) + \sum_{i=1}^{d-1} \lambda_i S(\sigma_A^i).$$
 (57)

Because the classical parts of  $\Gamma_{XAB}$  is the d-dimensional maximally mixed state  $\Gamma_X = \frac{1}{d} \sum_{x=0}^{d-1} |x\rangle_X \langle x|$ , we have the mutual information of  $\Gamma_{XAB}$  with respect to the bipartition between X and AB as

$$I(\Gamma_{X:AB}) = S(\Gamma_X) + S(\Gamma_{AB}) - S(\Gamma_{XAB}) = \ln d - S(\rho_B) + \chi(\mathscr{E}_0). \tag{58}$$

For the von Neumann entropy of  $\Gamma_{YAB}$  in Eq. (16), we have

$$S(\Gamma_{XAB}) = \ln d + \frac{1}{d} \sum_{y=0}^{d-1} S\left(\left(I_A \otimes Z_B^y\right) \left(\sum_{j=0}^{d-1} \tau_A^j \otimes \frac{1}{d} |\tilde{e}_j\rangle_B \langle \tilde{e}_j|\right) \left(I_A \otimes Z_B^{-y}\right)\right)$$

$$= \ln d + S\left(\sum_{j=0}^{d-1} \tau_A^j \otimes \frac{1}{d} |\tilde{e}_j\rangle_B \langle \tilde{e}_j|\right)$$

$$= 2 \ln d + \frac{1}{d} \sum_{i=1}^{d-1} S\left(\tau_A^j\right),$$
(59)

where the first and third equalities are due to the joint entropy theorem and the second equality is from the unitary invariance of von Neumann entropy. Thus the mutual information of  $\Gamma_{YAB}$  with respect to the bipartition between Y and AB is

$$I(\Gamma_{Y:AB}) = S(\Gamma_Y) + S(\Gamma_{AB}) - S(\Gamma_{YAB})$$

$$= \ln d + S(\rho_A) + \ln d - 2\ln d - \frac{1}{d} \sum_{j=1}^{d-1} S(\tau_A^j)$$

$$= \chi(\mathcal{E}_1).$$
(60)

**Proof of Theorem 1.** Let us first consider the four-qudit ccq state  $\Gamma_{XYAB}$  of the form in Eq. (14) obtained by the two-qudit reduced density matrix  $\rho_{AB}$  of  $|\psi\rangle_{ABC}$ . From Eqs. (18), (19) and (20), the additivity condition of quantum mutual information for  $\Gamma_{XYAB}$  in Eq. (23) can be rewritten as

$$\chi(\mathscr{E}_0) + \chi(\mathscr{E}_1) = S(\rho_A) + S(\rho_B) - S(\rho_{AB}) = I(\rho_{AB}), \tag{61}$$

where  $\mathscr{E}_0$  and  $\mathscr{E}_1$  are the probability ensembles of  $\rho_A$  in Eqs. (7) and (10), respectively.

Because  $\mathscr{E}_0$  and  $\mathscr{E}_1$  can be obtained from measuring subsystem B of  $\rho_{AB}$  by the rank-1 measurement  $\{|e_i\rangle_B\langle e_i|\}_{i=1}^{d-1}$  and  $\{|\tilde{e}_j\rangle_B\langle \tilde{e}_j|\}_{j=1}^{d-1}$ , respectively, the definition of CC in Eq. (4) leads us to  $\mathscr{J} \leftarrow (\rho_{AB}) \geq \chi(\mathscr{E}_0)$  and  $\mathscr{J}^{\leftarrow}(\rho_{AB}) \geq \chi(\mathscr{E}_1)$ , therefore

$$\mathscr{J} \leftarrow (\rho_{AB}) \ge \frac{1}{2} \left( \chi(\mathscr{E}_0) + \chi(\mathscr{E}_1) \right) = \frac{1}{2} I(\rho_{AB}). \tag{62}$$

By considering the ccq state  $\Gamma_{XYAC}$  obtained by  $\rho_{AC}$  as well as Eq. (24), we can analogously have

$$\mathcal{J} \leftarrow (\rho_{AC}) \ge \frac{1}{2} I(\rho_{AC}). \tag{63}$$

As the trade-off relation of Eq. (5) in Proposition 1 is universal with respect to the subsystems, we also have  $S(\rho_A) = \mathscr{J} \leftarrow (\rho_{AC}) + E_{\mathbf{f}}(\rho_{AB})$  for the given two-qudit state  $|\psi\rangle_{ABC}$ , therefore

$$E_{\mathbf{f}}(\rho_{AB}) + E_{\mathbf{f}}(\rho_{AC}) = 2S(\rho_A) - (\mathscr{J} \leftarrow (\rho_{AB}) + \mathscr{J} \leftarrow (\rho_{AC})). \tag{64}$$

Now Inequalities (62), (63) as well as Eq. (64) lead us to

$$E_{\mathbf{f}}(\rho_{AB}) + E_{\mathbf{f}}(\rho_{AC}) \leq 2S(\rho_{A}) - \frac{1}{2}(I(\rho_{AB}) + I(\rho_{AC}))$$

$$= 2S(\rho_{A}) - \frac{1}{2}(S(\rho_{A}) + S(\rho_{B}) - S(\rho_{AB}) + S(\rho_{A}) + S(\rho_{C}) - S(\rho_{AC}))$$

$$= S(\rho_{A})$$

$$= E_{\mathbf{f}}(|\psi\rangle_{A(BC)}),$$
(65)

where the second equality is due to  $\rho_{AC} = \rho_B$  and  $\rho_{AB} = \rho_C$  for three-party pure state  $|\psi\rangle_{ABC}$ .

**Proof of Theorem 2.** We first prove the theorem for any three-party mixed state  $\rho_{ABC}$ , and inductively show the validity of the theorem for any *n*-party quantum state  $\rho_{A_1A_2\cdots A_n}$ . For a three-party mixed state  $\rho_{ABC}$ , let us consider an optimal decomposition of  $\rho_{ABC}$  realizing EoF with respect to the bipartition between A and BC, that is,

$$\rho_{ABC} = \sum_{i} p_i |\psi_i\rangle_{ABC} \langle \psi_i|, \tag{66}$$

with  $E_{\mathbf{f}}\left(\rho_{A(BC)}\right) = \sum_{i} p_{i} E_{\mathbf{f}}\left(|\psi_{i}\rangle_{A(BC)}\right)$ . From Theorem 1, each pure state  $|\psi_{i}\rangle_{ABC}$  of the decomposition (66) satisfies  $E_{\mathbf{f}}\left(|\psi_{i}\rangle_{A(BC)}\right) \geq E_{\mathbf{f}}\left(\rho_{AB}^{i}\right) + E_{\mathbf{f}}\left(\rho_{AC}^{i}\right)$  with  $\rho_{AB}^{i} = \operatorname{tr}_{C}|\psi_{i}\rangle_{ABC}\langle\psi_{i}|$  and  $\rho_{AC}^{i} = \operatorname{tr}_{B}|\psi_{i}\rangle_{ABC}\langle\psi_{i}|$ , therefore,

$$E_{\mathbf{f}}(\rho_{A(BC)}) = \sum_{i} p_{i} E_{\mathbf{f}}(|\psi_{i}\rangle_{A(BC)}) \ge \sum_{i} p_{i} E_{\mathbf{f}}(\rho_{AB}^{i}) + \sum_{i} p_{i} E_{\mathbf{f}}(\rho_{AC}^{i}).$$

$$(67)$$

For each i and the two-party reduced density matrices  $\rho_{AB}^i$ , let us consider its optimal decomposition  $\rho_{AB}^{i} = \sum_{j} r_{ij} \left| \mu_{j}^{i} \right\rangle_{AB} \left\langle \mu_{j}^{i} \right|$  realizing EoF, that is,  $E_{\mathbf{f}}(\rho_{AB}^{i}) = \sum_{j} r_{ij} E_{\mathbf{f}}(\left| \mu_{j}^{i} \right\rangle_{AB})$ . Now we have

$$\sum_{i} p_{i} E_{\mathbf{f}} \left( \rho_{AB}^{i} \right) = \sum_{i,j} p_{i} r_{ij} E_{\mathbf{f}} \left( \left| \mu_{j}^{i} \right\rangle_{AB} \right) \ge E_{\mathbf{f}} \left( \left| \rho_{AB} \right\rangle_{AB} \right), \tag{68}$$

where the inequality is due to  $\rho_{AB} = \sum_{i} p_{i} \rho_{AB}^{i} = \sum_{i,j} p_{i} r_{ij} \left| \mu_{j}^{i} \right\rangle_{AB} \left\langle \mu_{j}^{i} \right|$  and the definition of EoF.

For each i, we also consider an optimal decomposition  $\rho_{AC}^{i} = \sum_{l} s_{il} \left| \nu_{l}^{i} \right\rangle_{AC} \left\langle \nu_{l}^{i} \right|$  such that  $E_{\mathbf{f}}(\rho_{AC}^{i}) = \sum_{l} s_{il} E_{\mathbf{f}}(|\nu_{l}^{i}\rangle_{AC})$ . We can analogously have

$$\sum_{i} p_{i} E_{\mathbf{f}} \left( \rho_{AC}^{i} \right) \ge E_{\mathbf{f}} \left( \rho_{AC} \right), \tag{69}$$

due to  $\rho_{AC} = \sum_{i} p_{i} \rho_{AC}^{i} = \sum_{i,l} p_{i} s_{il} \left| v_{l}^{i} \right\rangle_{AC} \left\langle v_{l}^{i} \right|$  and the definition of EoF in Eq. (2). From Inequalities (67), (68)

$$E_{\mathbf{f}}(\rho_{A(BC)}) \ge E_{\mathbf{f}}(\rho_{AB}) + E_{\mathbf{f}}(\rho_{AC}),$$
 (70)

which proves the theorem for three-party mixed states.

For general multi-party quantum system, we use the mathematical induction on the number of parties n; let us assume Inequality (51) is true for any k-party quantum state, and consider an k+1-party quantum state  $\rho_{A_1A_2\cdots A_{k+1}}$  for  $k\geq 3$ . By considering  $\rho_{A_1A_2\cdots A_{k+1}}$  as a three-party state with respect to the tripartition  $A_1,A_2\cdots A_k$ and  $A_{k+1}$ , Inequality (70) leads us to

$$E_{\mathbf{f}}(\rho_{A_1(A_2\cdots A_{k+1})}) \ge E_{\mathbf{f}}(\rho_{A_1(A_2\cdots A_k)}) + E_{\mathbf{f}}(\rho_{A_1A_{k+1}}). \tag{71}$$

As  $\rho_{A_1A_2\cdots A_k}$  in Inequality (71) is a k-party quantum state, the induction hypothesis assures that

$$E_{\mathbf{f}}(\rho_{A_1(A_2\cdots A_k)}) \ge E_{\mathbf{f}}(\rho_{A_1A_2}) + \cdots + E_{\mathbf{f}}(\rho_{A_1A_k}). \tag{72}$$

Now Inequalities (71) and (72) lead us to the monogamy inequality in (51), which completes the proof.

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## **Author contributions**

J.S.K. conceived the idea, performed the calculations and the proofs, interpreted the results, and wrote down the manuscript.

### Competing interests

The author declares no competing interests.

### Additional information

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