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Minimal number of runs and the sequential scheme for local discrimination between special unitary operations

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It has been shown that any two different multipartite unitary operations are perfectly distinguishable by local operations and classical communication with a finite number of runs. Meanwhile, two open questions were left. One is how to determine the minimal number of runs needed for the local discrimination, and the other is whether a perfect local discrimination can be achieved by merely a sequential scheme. In this paper, we answer the two questions for some unitary operations U_1 and U_2 with $U_1^\dagger U_2$ locally unitary equivalent to a diagonal unitary matrix in a product basis. Specifically, we give the minimal number of runs needed for the local discrimination, which is the same with that needed for the global discrimination. In this sense, the local operation works the same with the global one. Moreover, when adding the local property to U_1 or U_2, we present that the perfect local discrimination can be also realized by merely a sequential scheme with the minimal number of runs. Both results contribute to saving the resources used for the discrimination.

Quantum operations, which include unitary operations, quantum measurements, and quantum channels, are an important subject in the fields of quantum control and quantum information theory[1–3]. Recently, the discrimination of quantum operations[4–9], especially the discrimination of unitary operations[10–20], has received extensive attention. Indeed, the study itself of the distinguishability for unitary operations is a fundamental problem in quantum information theory, and after successfully discriminating unitary operations, we can further employ them to accomplish many other quantum information processing tasks. It should be noted that we only need to discuss the discrimination between two unitary operations, which is due to the fact that the discrimination of multiple unitary operations has been reduced to that of two unitary operations[11].

Two different unitary operations are said to be perfectly distinguishable, if there exists at least an input state such that two corresponding output states, generated by the two unitary operations acting on the input state respectively, are orthogonal. That is to say, the orthogonality of the two output states implies the distinguishability of the two unitary operations. Thus, the issue of discrimination of unitary operations can be simplified to the study of orthogonality of the corresponding output states. Generally speaking, there are two kinds of distinguishing schemes, or to say, generating orthogonal output states schemes, *i.e.* the parallel scheme and the sequential scheme. Suppose U_1 and U_2 are two different unitary operations to be distinguished. In the parallel scheme, there exists a finite number N and an input state $|\psi\rangle$ such that two output states $U_1^{\otimes N}|\psi\rangle$ and $U_2^{\otimes N}|\psi\rangle$ are orthogonal[13]. In the sequential scheme, there exist auxiliary unitary operations $X_1, \cdots, X_N$ and an input state $|\psi\rangle$ such that the output states $U_1X_NU_1 \cdots X_1U_1|\psi\rangle$ and $U_2X_NU_2 \cdots X_1U_2|\psi\rangle$ are orthogonal[13]. Furthermore, the minimal number of runs is the minimal number of times we apply the unknown unitary operation to make them perfectly distinguishable[11].

In addition, the discrimination of unitary operations has been classified into two scenarios, *i.e.* the global one and the local one. In the global scenario, the unknown unitary operation is under the complete control of a single party who can perform any allowable physical operations[10–15]. It has been shown that any two different unitary operations can be perfectly distinguished, no matter by the parallel scheme[11,12] or the sequential scheme[13], when

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a finite number of runs are allowed. Besides, in both schemes, the minimal numbers of runs needed for the perfect discrimination are the same[11,13]. In the local scenario, which is the main point we shall discuss in this paper, the unknown unitary operation is shared by several physically distant parties. So local operations and classical communication (LOCC) are natural requirements for each party when they try to accomplish the discrimination. Such a restriction makes the discrimination of unitary operations more complicated[16–20]. Interestingly, Zhou *et al.*[16] presented that any two different multipartite unitary operations are perfectly distinguished by LOCC with multiple runs. Later, Duan *et al.*[17] independently proved the same result by introducing the theory of local numerical range. Specifically, they pointed out that any two different unitary operations can be locally distinguished by the parallel scheme or first sequential then parallel scheme after a finite amount of runs.

As is shown in ref. 13, the minimal number of runs can save the temporal resources, and the sequential scheme can save the spatial resources due to the fact that in the sequential scheme no entanglement or joint quantum operations are needed. So in order to save resources as much as possible, it is natural to ask the following two questions: What is the minimal number of runs needed for the above parallel scheme, even for any distinguishing scheme, no matter parallel or sequential? Can the perfect local discrimination be completed by merely a sequential scheme? Both questions, also referred to in ref.17, need further considerations. Yet until now there has been no relevant progress about the research, even for a special class of unitary operations.

In this paper we answer the two questions for some unitary operations. In detail, suppose U_1 and U_2 are any two different unitary operations on the $s\otimes t(s, t\geq 2)$ quantum system such that $U_1^{\dagger}U_2$ is local unitary equivalent to $\mathrm{diag}(e^{\mathbf{i}\phi_{11}}, \cdots, e^{\mathbf{i}\phi_{1t}}, \cdots, e^{\mathbf{i}\phi_{s1}}, \cdots, e^{\mathbf{i}\phi_{st}}) \triangleq V$, where $\phi_{ij}\in[0, 2\pi)$ and $\Theta(V)\in(0, \pi]$. If the endpoints of $\Theta(V)$ are ϕ_{ij} and ϕ_{il}, or ϕ_{ij} and ϕ_{hj}, then the minimal number of runs needed for the local discrimination equals that needed for the global scenario, *i.e.* $\lceil\pi/\Theta(V)\rceil$, in which $\lceil x\rceil$ denotes the smallest integer that is not less than x, and $\Theta(V)$ represents the length of the smallest arc containing all the eigenvalues of V on the unit circle[11]. In this sense the local operation achieves the same function with the global one. Furthermore, when adding the local property to U_1 or U_2, by merely a sequential scheme the perfect local discrimination can be also accomplished with the minimal number of runs. Finally, the above results can be generalized to multipartite unitary operations.

Results

Local numerical range. We first introduce some definitions and results.

Consider a quantum system associated with a finite dimensional state space H. We denote the set of linear operations acting on H by $L(H)$. In particular, $u(H)$ is the set of unitary operations acting on H. Two unitary operations $U_1, U_2\in u(H)$ are said to be different if U_1 is not the form $e^{\mathbf{i}\theta}U_2$ for any real number θ. Let us introduce the notion of numerical range.

Definition 1. For $A\in L(H)$, the numerical range of A is a subset of complex numbers defined as

$$W(A) = \{\langle\psi|A|\psi\rangle : \langle\psi|\psi\rangle = 1\}. \tag{1}$$

Suppose now we are concerned with a multipartite quantum system consisting of m parties, say, $M=\{A_1, \cdots, A_m\}$. Assume that the party A_k has a state space H_k with dimension d_k. Then the whole state space is given by $H = \otimes_{k=1}^{m} H_k$ with total dimension $d=d_1\cdots d_m$.

Definition 2[17]. $U\in u(H)$ is said to be local or decomposable if $U = \otimes_{k=1}^{m} U_k$ such that $U_k\in u(H_k)$. Otherwise U is nonlocal or entangled.

Definition 3[17]. The local numerical range of A is

$$W^{\mathrm{local}}(A) = \{\langle\psi|A|\psi\rangle : |\psi\rangle = \otimes_{k=1}^{m} |\psi_k\rangle\}, \tag{2}$$

where $|\psi_k\rangle\in H_k$ and $\langle\psi_k|\psi_k\rangle = 1$.

Let U and V be two matrices on the $s\otimes t$ space. U and V are called local unitary equivalent if there exist $U_1\in u(s)$ and $U_2\in u(t)$ such that $U=(U_1\otimes U_2)V(U_1\otimes U_2)^{\dagger}$. Moreover, when U is local unitary equivalent to V, we can obtain that $U^{\otimes p}$ and $V^{\otimes p}$ are local unitary equivalent, and $W^{\mathrm{local}}(U^{\otimes p}) = W^{\mathrm{local}}(V^{\otimes p})$ for any $p\in N$[21].

Next two relevant lemmas about the local discrimination of unitary operations will be presented.

Lemma 1[17]. Two different unitary operations U_1 and U_2 are perfectly distinguishable by LOCC in the single-run scenario if and only if $0 \in W^{\mathrm{local}}(U_1^{\dagger}U_2)$.

Lemma 2[17]. Suppose two different multipartite unitary operations U_1 and U_2 satisfy that $U_1^{\dagger}U_2$ is non-Hermitian (up to some phase factor), then there exists a finite N such that $0 \in W^{\mathrm{local}}((U_1^{\dagger}U_2)^{\otimes N})$.

Lemma 2 gives the existence of a finite number needed for the perfect discrimination in the local scenario.

For simplicity, in what follows we shall only consider the case in which U_1 and U_2 are both bipartite unitary operations acting on the $s\otimes t(s, t\geq 2)$ space, and the multipartite case can be similarly discussed.

Minimal number of runs for the local distinguishability. In this section, we mainly discuss the minimal number of runs needed for a perfect discrimination between two bipartite unitary operations in the LOCC scenario.

First, two different unitary operations U_1 and U_2 such that $U_1^{\dagger}U_2$ is local unitary equivalent to V are considered, where V is a diagonal unitary matrix in a product basis. According to that $\Theta(U)=\Theta(XUX^{\dagger})$ for any $X\in u(H)$ and the local unitary transformations do not alter the product state nature of the basis in general, we have the theorem.

Theorem 1. *Let U_1 and U_2 be any two different bipartite unitary operations on the $s\otimes t(s, t\geq 2)$ space such that $U_1^{\dagger}U_2$ is local unitary equivalent to*

$$diag\,(e^{\mathbf{i}\phi_{11}},\,\cdots,\,e^{\mathbf{i}\phi_{1t}},\,\cdots,\,e^{\mathbf{i}\phi_{s1}},\,\cdots,\,e^{\mathbf{i}\phi_{st}})\triangleq V, \tag{3}$$

where $\phi_{ij}\in[0, 2\pi)$ *and* $\Theta(V)\in(0, \pi]$. *If the endpoints of* $\Theta(V)$ *are* ϕ_{ij} *and* ϕ_{il}, *or* ϕ_{ij} *and* ϕ_{hj}, *then* $\lceil\pi/\Theta(V)\rceil$ *is the minimal number of runs needed for distinguishing* U_1 *and* U_2 *locally with certainty.*

Proof. By the conditions, $\lceil\pi/\Theta(V)\rceil\triangleq n$ is the minimal number of runs needed for globally distinguishing U_1 and U_2 with certainty. Thus, if they are perfectly distinguished by LOCC, the minimal number of runs cannot be less than n. In the following we will illustrate that n is just the minimal number of runs by finding a parallel scheme to distinguish them locally.

Without loss of generality, suppose U_1 and U_2 are unitary operations consisting of two parties A and B, where A is s-dimensional, and B is t-dimensional. Let the endpoints of $\Theta(V)$ be ϕ_{ij} and ϕ_{il}, where $j<l$.

In fact, we can find a bipartite product state

$$|\psi\rangle = ae^{\mathbf{i}r}|i-1, j-1\rangle_{AB}^{\otimes n} + be^{\mathbf{i}\delta}|i-1, j-1\rangle_{AB}\otimes|i-1, l-1\rangle_{AB}^{\otimes(n-1)} + ce^{\mathbf{i}\kappa}|i-1, l-1\rangle_{AB}^{\otimes n}, \tag{4}$$

where $r, \delta, \kappa\in[0, 2\pi)$, and

$$\begin{aligned} a &= \sqrt{\frac{\sin(\phi_{ij}-\phi_{il})}{\sin(\phi_{ij}-\phi_{il})+\sin\left[(n-1)(\phi_{ij}-\phi_{il})\right]-\sin n(\phi_{ij}-\phi_{il})}},\\ b &= \sqrt{\frac{-\sin n(\phi_{ij}-\phi_{il})}{\sin(\phi_{ij}-\phi_{il})+\sin\left[(n-1)(\phi_{ij}-\phi_{il})\right]-\sin n(\phi_{ij}-\phi_{il})}},\\ c &= \sqrt{\frac{\sin\left[(n-1)(\phi_{ij}-\phi_{il})\right]}{\sin(\phi_{ij}-\phi_{il})+\sin\left[(n-1)(\phi_{ij}-\phi_{il})\right]-\sin n(\phi_{ij}-\phi_{il})}}, \end{aligned} \tag{5}$$

such that

$$\langle\psi|V^{\otimes n}|\psi\rangle = a^2e^{\mathbf{i}n\phi_{ij}} + b^2e^{\mathbf{i}[\phi_{ij}+(n-1)\phi_{il}]} + c^2e^{\mathbf{i}n\phi_{il}} = 0, \tag{6}$$

which means that when U_1 and U_2 are applied n times in parallel, they can be locally distinguished by Lemma 2.

To sum up, we can claim that $\lceil\pi/\Theta(V)\rceil$ is the minimal number of runs needed for the perfect discrimination between U_1 and U_2 in the LOCC scenario.

From the above proof, it is clear that the minimal number of runs needed for the local discrimination is the same with that needed for the global scenario. The fact reveals a counterintuitive result: For the perfect discrimination of two unitary operations as in Theorem 1, the global operation has no advantages over the local one. As an illustrative example of Theorem 1, consider a special case where U_1 and U_2 are two $2\otimes 3$ unitary operations satisfying that

$$\begin{aligned} U_1^\dagger U_2 &= |00\rangle\langle 00| + e^{\mathbf{i}\frac{\pi}{4}}|01\rangle\langle 01| + e^{\mathbf{i}\frac{\pi}{3}}|02\rangle\langle 02| + e^{\mathbf{i}\frac{\pi}{6}}|10\rangle\langle 10|\\ &\quad + e^{\mathbf{i}\frac{\pi}{5}}|11\rangle\langle 11| + e^{\mathbf{i}\frac{\pi}{4}}|12\rangle\langle 12|. \end{aligned} \tag{7}$$

One can directly see that $\Theta(U_1^\dagger U_2)=\pi/3$, and the endpoints of $\Theta(U_1^\dagger U_2)$ are $\phi_{11}=0$ and $\phi_{13}=\pi/3$. As we can find a bipartite product state $|\psi\rangle = \frac{\sqrt{2}}{2}e^{\mathbf{i}r}|00\rangle_{AB}^{\otimes 3} + \frac{\sqrt{2}}{2}e^{\mathbf{i}\kappa}|02\rangle_{AB}^{\otimes 3}$, where $r, \kappa\in[0, 2\pi)$, such that $\langle\psi|(U_1^\dagger U_2)^{\otimes 3}|\psi\rangle = 0$, then $\lceil\pi/\Theta(U_1^\dagger U_2)\rceil = 3$ is the minimal number of runs needed for the perfect discrimination of U_1 and U_2 in the LOCC scenario.

From Theorem 1, one can see that the endpoints of $\Theta(V)$ being ϕ_{ij} and ϕ_{il}, or ϕ_{ij} and ϕ_{hj} are just the sufficient conditions. So when will they be also the necessary conditions? The following corollary will give an answer.

Corollary *Let* U_1 *and* U_2 *be any two different bipartite unitary operations on the* $s\otimes t(s, t\geq 2)$ *space such that* $U_1^\dagger U_2$ *is local unitary equivalent to*

$$diag\,(e^{\mathbf{i}\phi_{11}},\,\cdots,\,e^{\mathbf{i}\phi_{1t}},\,\cdots,\,e^{\mathbf{i}\phi_{s1}},\,\cdots,\,e^{\mathbf{i}\phi_{st}})\triangleq V, \tag{8}$$

where $\phi_{ij}\in[0, 2\pi)$ *and* $\Theta(V)=\pi$. *They are perfectly distinguished by LOCC in the single-run scenario if and only if the endpoints of* $\Theta(V)$ *are* ϕ_{ij} *and* ϕ_{il}, *or* ϕ_{ij} *and* ϕ_{hj}.

Proof. It suffices to show the necessity.

Without loss of generality, suppose the endpoints of $\Theta(V)$ are ϕ_{ij} and ϕ_{hl}, where $i<h$, $j\neq l$ and $\phi_{ij}<\phi_{hl}$. We have $\phi_{hl}=\phi_{ij}+\pi$.

According to that U_1 and U_2 are locally distinguished with a single run, there must be an $s\otimes t$ product state

$$\begin{aligned} |\psi\rangle &= (a_1e^{\mathbf{i}\lambda_1}|0\rangle + a_2e^{\mathbf{i}\lambda_2}|1\rangle + \cdots + a_se^{\mathbf{i}\lambda_s}|s-1\rangle)_s\\ &\quad\otimes(b_1e^{\mathbf{i}\delta_1}|0\rangle + b_2e^{\mathbf{i}\delta_2}|1\rangle + \cdots + b_te^{\mathbf{i}\delta_t}|t-1\rangle)_t, \end{aligned} \tag{9}$$

where $a_k\geq 0$, $b_p\geq 0$, $\sum_{k=1}^{s}a_k^2 = 1$, $\sum_{p=1}^{t}b_p^2 = 1$, and $\lambda_k, \delta_p\in[0, 2\pi)$, such that $\langle\psi|V|\psi\rangle = 0$, which is equivalent to

$$a_1^2 b_1^2 e^{\mathbf{i}\phi_{11}} + \cdots + a_i^2 b_j^2 e^{\mathbf{i}\phi_{ij}} + \cdots + a_h^2 b_l^2 e^{\mathbf{i}\phi_{hl}} + \cdots + a_s^2 b_t^2 e^{\mathbf{i}\phi_{st}} = 0. \tag{10}$$

Let $\phi_{ij} = 0$ and other $\phi_{mn} \in (0, \pi)$. We have $\phi_{hl} = \pi$. Further $a_i^2 + a_h^2 = 1$, $b_j^2 + b_l^2 = 1$, and $a_i^2 b_j^2 = a_h^2 b_l^2$. A routine calculation shows that $a_i = b_l$ and $a_h = b_j$. By $j \neq l$, we can get $a_i^2 b_l^2 = 0$ and $a_h^2 b_j^2 = 0$. Thus, $b_j = b_l = 0$, which is a contradiction.

Corollary indicates that for any two different unitary operations constrained as above, our result is more practical and efficient than Lemma 1 in determining their local distinguishability in the single-run scenario. Because we do not need to compute the local numerical range which itself is generally difficult to calculate[22]. To see this, take U_1 and U_2 such that

$$U_1^\dagger U_2 = |00\rangle\langle 00| + e^{\mathbf{i}\theta_1}|01\rangle\langle 01| + e^{\mathbf{i}\theta_2}|10\rangle\langle 10| - |11\rangle\langle 11| \tag{11}$$

for $0 < \theta_1, \theta_2 < \pi$. It is clear that $\Theta(U_1^\dagger U_2) = \pi$, and the endpoints of $\Theta(U_1^\dagger U_2)$ are $\phi_{11} = 0$ and $\phi_{22} = \pi$. By Corollary, we can immediately claim that U_1 and U_2 cannot be locally distinguished in the single-run scenario, without complicated calculations to demonstrate $0 \notin W^{\text{local}}(U_1^\dagger U_2)$ as in ref. 17.

Sequential scheme of the local distinguishability. In this part, we primarily focus on the question that whether a perfect local discrimination can be achieved by merely a sequential scheme. The solution to the question is helpful to save the spatial resources as no entanglement or joint quantum operations are needed in the sequential scheme. Fortunately, a positive answer will be made.

Theorem 2 *Let U_1 and U_2 be any two different $s \otimes t (s, t \geq 2)$ unitary operations such that $U_1^\dagger U_2$ is local unitary equivalent to*

$$diag\,(e^{\mathbf{i}\phi_{11}}, \cdots, e^{\mathbf{i}\phi_{1t}}, \cdots, e^{\mathbf{i}\phi_{s1}}, \cdots, e^{\mathbf{i}\phi_{st}}) \triangleq V, \tag{12}$$

where $\phi_{ij} \in [0, 2\pi)$ and $\Theta(V) \in (0, \pi]$. Suppose one of them is local and $n = \lceil \pi/\Theta(V) \rceil$. If the endpoints of $\Theta(V)$ are ϕ_{ij} and ϕ_{il}, or ϕ_{ij} and ϕ_{hj}, then there exist local unitary operations $X_1, X_2, \cdots, X_{n-1}$ and an $s \otimes t$ product state $|\phi\rangle$ such that

$$U_1 X_{n-1} U_1 \cdots X_1 U_1 |\phi\rangle \perp U_2 X_{n-1} U_2 \cdots X_1 U_2 |\phi\rangle. \tag{13}$$

Proof. By Theorem 1, n is not only the minimal number of runs needed for a perfect local discrimination, but also can be achieved in the parallel scheme.

Here we will illustrate that when U_1 and U_2 are sequentially applied n runs, they can be locally distinguished. Therefore, the minimal number of runs n is also realized when U_1 and U_2 are perfectly distinguished by LOCC with the sequential scheme. The method in the proving process is inspired by Duan *et al.*[13].

Without loss of generality, let

$$U_1^\dagger U_2 = (V_1 \otimes V_2) V (V_1 \otimes V_2)^\dagger \triangleq \overline{V}, \tag{14}$$

where V_1 is an $s \times s$ unitary matrix and V_2 is $t \times t$. Suppose U_1 is local, and the endpoints of $\Theta(V)$ are ϕ_{ij} and ϕ_{il}, where $j < l$.

First, we consider the case $U_1 = I$ and $U_2 = \overline{V}$. Subsequently the general case can be reduced to this special one.

Next we will prove that there always exists a local unitary operation $\overline{X}'$ and an $s \otimes t$ product state $|\varphi'\rangle$ such that $\overline{X}'|\varphi'\rangle$ and $\bar{V}\bar{X}'\bar{V}^{n-1}|\varphi'\rangle$ are orthogonal. In other words, through being sequentially applied n times, I and $\overline{V}$ can be locally discriminated.

Let $V' = \text{diag}(e^{\mathbf{i}\phi_{ij}}, e^{\mathbf{i}\phi_{il}})$. Suppose

$$X = \begin{pmatrix} \cos\beta & \sin\beta \\ -\sin\beta & \cos\beta \end{pmatrix} \tag{15}$$

is the general form of real unitary matrices on the 2-dimensional space. We will find β satisfying $\text{tr}(X^\dagger V' X V'^{n-1}) = 0$ as follows.

A routine calculation shows that

$$\cos^2\beta e^{\mathbf{i}n\phi_{ij}} + \sin^2\beta e^{\mathbf{i}[\phi_{il}+(n-1)\phi_{ij}]} + \sin^2\beta e^{\mathbf{i}[\phi_{ij}+(n-1)\phi_{il}]} + \cos^2\beta e^{\mathbf{i}n\phi_{il}} = 0. \tag{16}$$

Combining with the sum-to-product identities and $\sin\frac{n(\phi_{ij}+\phi_{il})}{2} \neq 0$, or $\cos\frac{n(\phi_{ij}+\phi_{il})}{2} \neq 0$, we can get that

$$\sin^2\beta\,\cos\frac{(n-2)(\phi_{ij}-\phi_{il})}{2} + \cos^2\beta\,\cos\frac{n(\phi_{ij}-\phi_{il})}{2} = 0. \tag{17}$$

Further,

$$\beta = \arctan\sqrt{-\frac{\cos\dfrac{n(\phi_{ij}-\phi_{il})}{2}}{\cos\dfrac{(n-2)(\phi_{ij}-\phi_{il})}{2}}}. \tag{18}$$

Thus, for above β, $X^\dagger V'XV'^{n-1}$ has two opposite eigenvalues. Further we can assume the spectral decomposition as

$$X^\dagger V'XV'^{n-1} = e^{\mathbf{i}\delta}|\varphi_1\rangle\langle\varphi_1| - e^{\mathbf{i}\delta}|\varphi_2\rangle\langle\varphi_2|. \tag{19}$$

Let

$$|\varphi\rangle = (1/\sqrt{2})(|\varphi_1\rangle + |\varphi_2\rangle) \triangleq (\alpha_1 \;\; \alpha_2)^T, \tag{20}$$

where T denotes the matrix transposition. Thus, we have found β, X and $|\varphi\rangle$ such that $\langle\varphi|X^\dagger V'XV'^{n-1}|\varphi\rangle = 0$.

Suppose

$$\overline{X}_t = \begin{pmatrix} I_{j-1} & & & & \\ & \cos\beta & 0 & \sin\beta & \\ & 0 & I_{l-j-1} & 0 & \\ & -\sin\beta & 0 & \cos\beta & \\ & & & & I_{t-l} \end{pmatrix}, \tag{21}$$

$$\overline{X}' = (V_1 \otimes V_2)(I_s \otimes \overline{X}_t)(V_1 \otimes V_2)^\dagger, \tag{22}$$

and

$$|\varphi'\rangle = (V_1 \otimes V_2)\Big[|i-1\rangle_s \otimes (\alpha_1|j-1\rangle_t + \alpha_2|l-1\rangle_t)\Big], \tag{23}$$

where I_s represents the $s \times s$ identity matrix. We can conclude that

$$\langle\varphi'|\overline{X}'^\dagger\overline{V}\bar{X}'\overline{V}^{n-1}|\varphi'\rangle = 0. \tag{24}$$

Second, for general U_1 and U_2 satisfying $U_1^\dagger U_2 = \overline{V}$, there always exists the local unitary operation $\overline{X}'$ and the $s\otimes t$ product state $|\varphi'\rangle$ as above such that

$$\langle\varphi'|\overline{X}'^\dagger U_1^\dagger U_2\overline{X}'(U_1^\dagger U_2)^{n-1}|\varphi'\rangle = 0. \tag{25}$$

Finally, suppose $X_1 = \cdots = X_{n-2} = U_1^\dagger$, $X_{n-1} = \overline{X}'U_1^\dagger$, and $|\phi\rangle = |\varphi'\rangle$ in the theorem. Then

$$U_1X_{n-1}U_1\cdots X_1U_1|\phi\rangle \perp U_2X_{n-1}U_2\cdots X_1U_2|\phi\rangle. \tag{26}$$

It can be seen from Theorem 2 that there indeed exist some unitary operations such that their local discrimination can be completed by merely a sequential scheme, and meanwhile we present an explicit protocol of the local discrimination without any entanglement or joint quantum operations. Interestingly, the minimal number of runs is also n, which is the same with that in the global scenario. All these make the local discrimination actually feasible in experiment.

As an application of Theorem 2, consider a particular case where U_1 and U_2 are two-qubit unitary operations satisfying that U_1 is local and

$$U_1^\dagger U_2 = |00\rangle\langle 00| + e^{\mathbf{i}\frac{\pi}{3}}|01\rangle\langle 01| + e^{\mathbf{i}\frac{\pi}{4}}|10\rangle\langle 10| + e^{\mathbf{i}\frac{2\pi}{7}}|11\rangle\langle 11|. \tag{27}$$

One can directly see that $\Theta(U_1^\dagger U_2) = \pi/3$, $n=3$, and the endpoints of $\Theta(U_1^\dagger U_2)$ are $\phi_{11}=0$ and $\phi_{12}=\pi/3$. Let $V' = \mathrm{diag}(1, e^{\mathbf{i}\frac{\pi}{3}})$. We can find $\beta=0$, $X=I_2$, and $|\varphi\rangle = \left(\frac{\sqrt{2}}{2} \;\; \frac{\sqrt{2}}{2}\right)^T$, where T denotes the matrix transposition, such that $\langle\varphi|X^\dagger V'XV'^2|\varphi\rangle = 0$. Suppose $X_1 = U_1^\dagger$, $X_2 = (I_2 \otimes I_2)U_1^\dagger$, and $|\phi\rangle = |0\rangle_2 \otimes \left(\frac{\sqrt{2}}{2}|0\rangle_2 + \frac{\sqrt{2}}{2}|1\rangle_2\right)$, we have $U_1X_2U_1X_1U_1|\phi\rangle \perp U_2X_2U_2X_1U_2|\phi\rangle$. Therefore, it has been shown that the perfect local discrimination of U_1 and U_2 can be achieved by merely a sequential scheme.

Discussion

The local discrimination of two unitary operations U_1 and U_2 discussed in refs 19 and 20 is in the single-shot scenario. Bae[19] mainly investigated the relations between the discrimination and the entangling capabilities of given U_1 and U_2, and drew the conclusion that there exist non-entangling unitary operations being perfectly distinguishable only for global operations. While Cao *et al.*[20] presented a necessary and sufficient condition, which can be employed to efficiently determine the perfect local distinguishability of U_1 and U_2 satisfying $U_1^\dagger U_2 = V$ with V being a two-qubit diagonal unitary matrix.

Compared to the results in refs 19 and 20, our study on the local discrimination of unitary operations in this paper is primarily in the multiple-runs scenario. We have determined the minimal number of runs and put forward the sequential scheme for the local discrimination of some unitary operations. Concretely, for any two different unitary operations with certain limitations, we show that the minimal number of runs needed for the local discrimination is equal to that needed for the global scenario, which means that the local operation achieves the same function with the global one. Furthermore, when one more condition is restricted to the two unitary operations, we present that by merely a sequential scheme the perfect local discrimination can be also completed with the minimal number of runs. Both results are benefit for saving the resources, temporal or spatial, which are crucial in practice.

Despite the above research progress, we have yet neither determined the minimal number of runs, nor given the existence of an effective merely-sequential scheme for the local discrimination of two general unitary operations. But we believe that the results about the special unitary operations can provide new insight into the study of the two questions and help us to make a further research.

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Author Contributions

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Additional Information

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