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MATHEMATICAL GAMES

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MATHEMATICAL GAMES

Sim, Chomp and Race Track: new games for the intellect (and not for Lady Luck)

by Martin Gardner

New mathematical games of a competitive type, demanding more intellectual skill than luck, continue to proliferate both in the U.S. and abroad. In Britain they have become so popular that a monthly periodical called *Games and Puzzles* was started in 1972 just to keep devotees informed. (Interested readers can contact the publisher at P.O. Box 4, London N6 4DF.) *Strategy and Tactics* (a bimonthly with offices at 44 East 23rd Street, New York, N.Y. 10010) is primarily concerned with games that simulate political or military conflicts, but a column in the publication by Sidney Sackson reports on new mathematical games of all kinds. Sackson's book *A Gamut of Games* (1969) has a bibliography of more than 200 of the best mathematical board games now on the market.

Simulation games are games that model some aspect of human conflict: war, population growth, pollution, marriage, sex, the stock market, elections, racism, gangsterism—almost anything at all. They are being used as teaching devices, and some notion of how widely can be gained from the fact that a 1970 catalogue, *The Guide to Simulation Games for Education and Training,* by David W. Zuckerman and Robert E. Horn, runs to 334 pages. (This valuable reference is available from Information Resources, Inc., 1675 Massachusetts Avenue, Cambridge, Mass. 02138.)

This month we take a look at three unusual new mathematical games. None requires a special board or equipment; all that is needed are pencil and paper (graph paper for the first game) and (for the third) a supply of counters.

Race Track, virtually unknown in this country, is a truly remarkable simulation of automobile racing. I do not know who invented it. It was called to my attention by Jurg Nievergelt, a computer scientist at the University of Illinois who picked it up on a recent trip to Switzerland.

The game is played on graph paper. A racetrack wide enough to accommodate a car for each player is drawn on the sheet. The track may be of any length or shape, but to make the game interesting it should be strongly curved [*see illustration on opposite page*]. Each contestant should have a pencil or pen of a different color. To line up the cars each player draws a tiny box just below a grid point on the starting line. In the example illustrated the track will take three cars, but to simplify things a race of two cars is shown. Lots can be drawn to decide the order of moving. In the sample game, provided by Nievergelt, Black moves first.

You might suppose that a randomizing device now comes into play to determine how the cars move, but such is not the case. At each turn a player simply moves his car ahead along the track to a new grid point, subject to the following three rules:

1. The new grid point and the straight line segment joining it to the preceding grid point must lie entirely within the track.

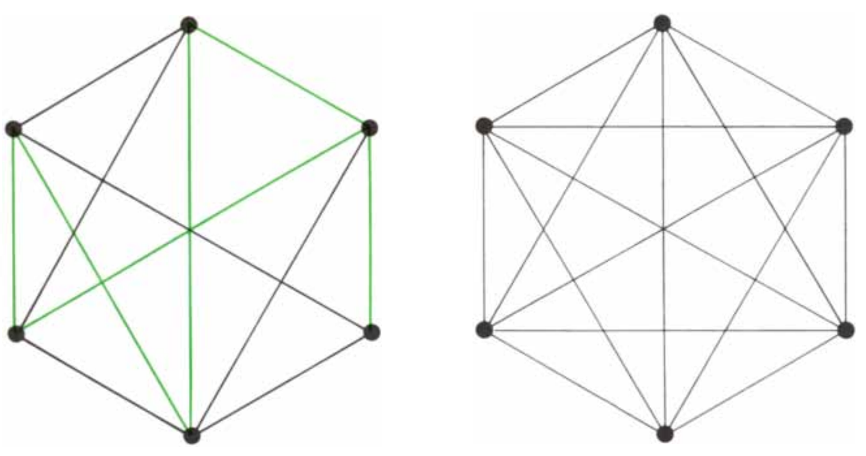
2. No two cars may simultaneously occupy the same grid point. In other words, no collisions are allowed. For instance, consider move 22. Green, the second player, would probably have preferred to go to the spot taken by Black on his 22nd move, but the no-collision rule prevented it.

3. Acceleration and deceleration are simulated in the following ingenious way. Assume that your previous move was k units vertically and m units horizontally and that your present move is k' vertically and m' horizontally. The absolute difference between k and k' must be either 0 or 1, and the absolute difference between m and m' must be either 0 or 1. In effect, a car can maintain its speed in either direction or it can change its speed by only one unit distance per move. The first move, following this rule, is one unit horizontally or vertically, or both.

The first car to cross the finish line wins. A car that collides with another car or leaves the track is out of the race. In the sample game Green slows too late to make the first turn efficiently. He narrowly avoids a crash, and the bad turn forces him to fall behind in the middle of the race. He takes the last curve superbly, however, and he wins by crossing the finish line one move ahead of Black.

Nievergelt programmed Race Track for the University of Illinois's Plato IV computer-assisted instruction system, which uses a new type of graphic display called a plasma panel. Two or three people can play against one another or one person can play alone. The game became so popular that the authorities made it inaccessible for a week to prevent students from wasting too much time on it.

Our second pencil-and-paper game is called Sim after Gustavus J. Simmons, a physicist at the Sandia Corporation laboratories in Albuquerque, who invented it when he was working on his Ph.D. thesis on graph theory. He was not the first to think of it (the idea occurred independently to a number of mathematicians) but he was the first to publish it and to analyze it completely with a com-



The game of Sim

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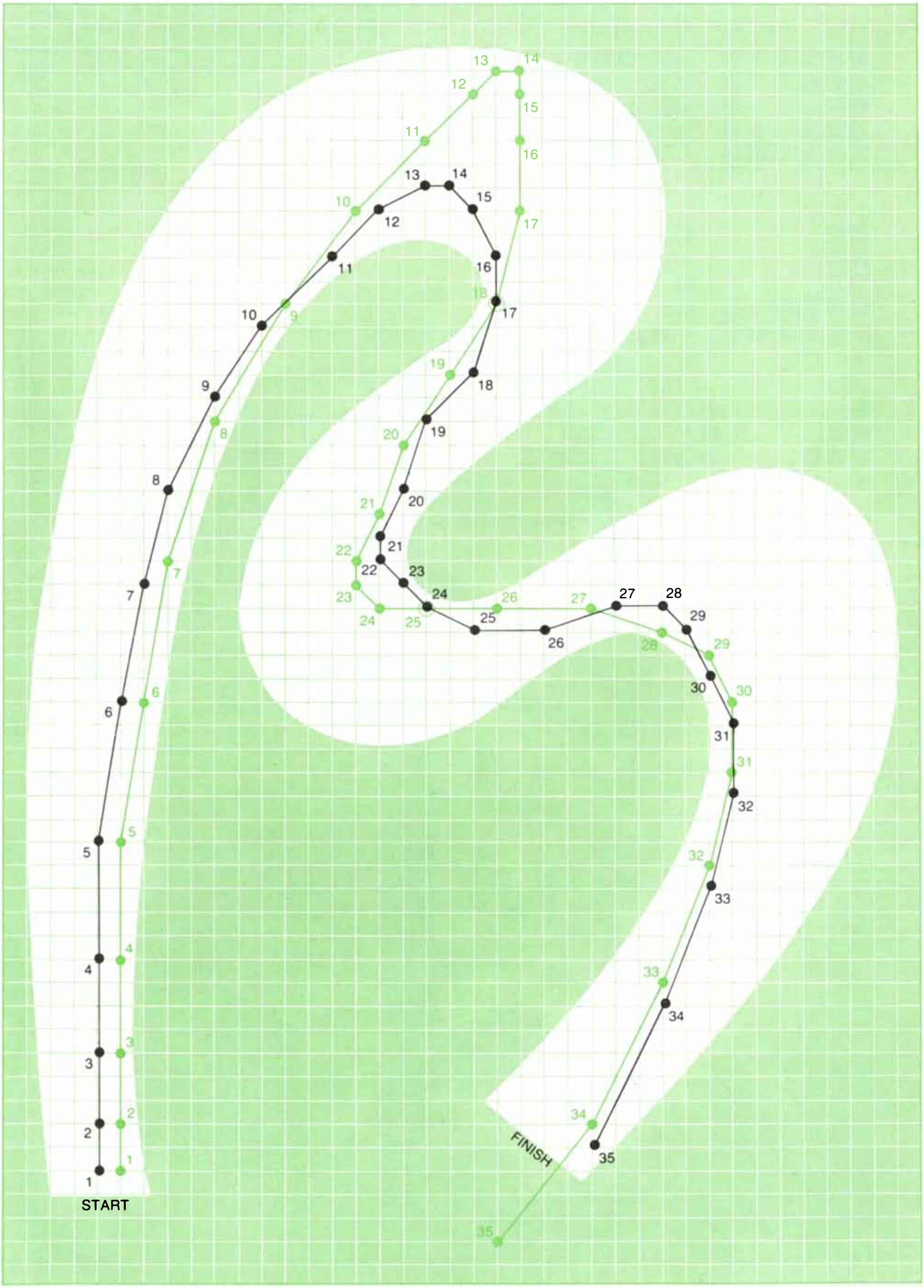

The Race Track game

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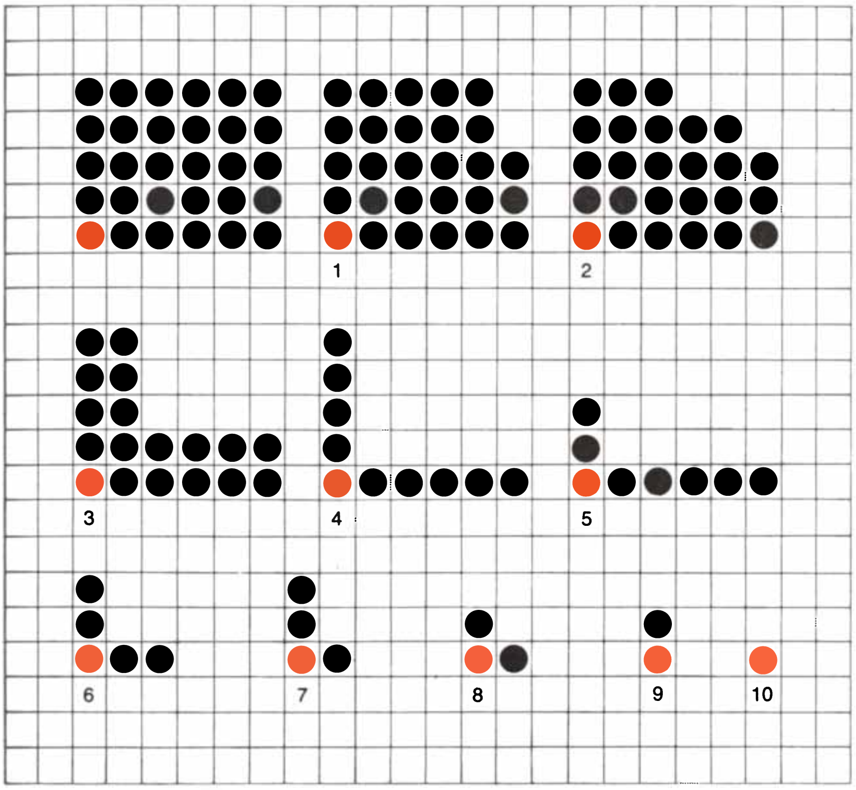


Chomp on a 5-by-6 field

puter program. In his note on "The Game of Sim" (*Journal of Recreational Mathematics,* Vol. 2, April, 1969, page 66) he says that one of his colleagues picked the name as short for SIMple SIMmons, and because the game resembles the familiar game of nim.

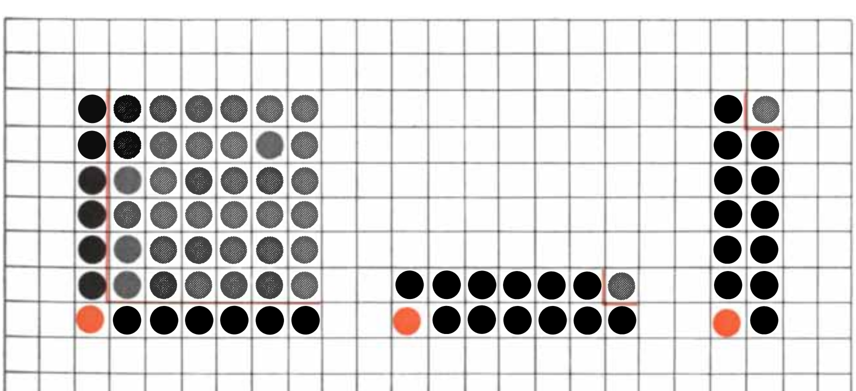
Six points are placed on a sheet of paper to mark the vertexes of a regular hexagon. There are 15 ways to draw straight lines connecting a pair of points, producing what is called the complete graph for six points [*see illustration on page 108*]. Two Sim players take turns drawing one of the 15 edges of the graph, each using a different color. The first player to be forced to form a triangle of his own color (only triangles whose vertexes are among the six starting points count) is the loser.

If only two colors are used for the edges of a chromatic graph, it is not hard to prove that six is the smallest number of points whose complete chromatic graph is certain to contain a triangle with sides all the same color. Simmons gives the proof as follows: "Consider any vertex in a completely filled-in game. Since five lines originate there, at least three must be the same color—say blue. No one of the three lines joining the end points of these lines can be blue if the player is not to form a blue triangle, but then the three interconnecting lines form a red triangle. Hence at least one monochromatic (all one color) triangle must exist, and a drawn game is impossible."

With a bit more work a stronger theorem can be established. There must be at least *two* monochromatic triangles. A detailed proof of this is given by Frank Harary, a University of Michigan graph theorist, in his paper "The Two-Triangle Case of the Acquaintance Graph" in *Mathematics Magazine* (Vol. 45, May, 1972, pages 130–135). Harary calls it an acquaintance graph because it provides the solution to an old brainteaser: Of any six people, prove that at least three are mutual acquaintances or at least three are mutual strangers. Harary not only proves that there are at least two such sets but also shows that if there are exactly two, they are of opposite types (colors on the graph) if and only if the two sets have just one person (point) in common.

Because Sim cannot be a draw, it follows that either the first or the second player can always win if he plays correctly. When Simmons wrote his note in 1969, he did not know which player had the win, and in actual play among equally skillful players wins are about equally divided. Later he made an exhaustive computer analysis showing that the second player could always win. Because of symmetry all first moves are alike. The computer results showed that the second player could respond by coloring any of the remaining 14 edges and still guarantee himself a win. (Actually, for symmetry reasons, there are only two fundamentally different second moves: one that connects with the first move and one that does not.) After the first player has made his second move exactly half of the remaining plays lead to a sure win for the second player and half to a sure loss, assuming of course that both sides play rationally. If 14 moves are made without a win, the last move, by the first player, will always produce two monochromatic triangles of his color. This 14-move pattern is unique in the sense that all such patterns are topologically the same. Can you find a way of coloring 14 edges of the Sim graph, seven in one color and seven in another, so that there is no monochromatic triangle on the field? A solution will be given next month.

The most interesting unanswered question about Sim is whether there is a relatively simple strategy by which the second player can win without having to memorize all the correct responses. Even if he has at hand a computer printout of the total game tree, it is of little practical use because of the enormous difficulty of locating on the



*Winning first bites on square field, 2-by-*n *field and* n*-by-2 field*

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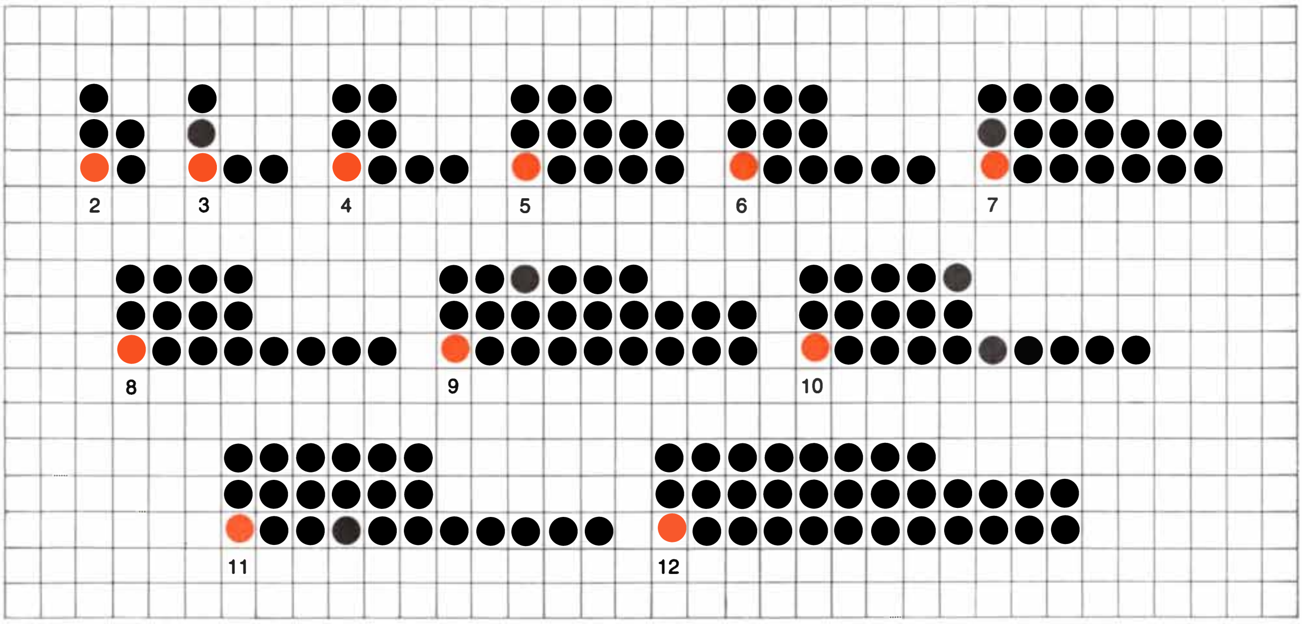

*Winning first bites on 3-by-*n *fields*

printout a position isomorphic to the one on the board. Simmons' computer results have been verified by programs written by Michael Beeler at the Artificial Intelligence Laboratory of the Massachusetts Institute of Technology and more recently by Jesse W. Croach, Jr., of West Grove, Pa., but no one has been able to extract from the game tree a useful mnemonic for the second player.

Sim can of course be played on other graphs. On complete graphs for three and four points the game is trivial, and for more than six points it becomes too complicated. The pentagonal five-point graph, however, is playable. Although draws are possible, I am not aware of any proof that a draw is inevitable if both sides make their best moves.

Our third game, which I call Chomp, is a nim-type game invented by David Gale, a mathematician and economist at the University of California at Berkeley. Gale is the inventor of Bridg-it, a popular topological board game still on the market. (It was first described in this department in October, 1958, and a winning strategy was first disclosed in July, 1961.) What follows is based entirely on results recently provided by Gale.

Chomp can be played with a supply of counters [*see top illustration on opposite page*] or with *O*'s or *X*'s on a sheet of paper. The counters are arranged in a rectangular formation. Two players take turns removing counters as follows. Any counter is selected. Imagine that this counter is inside the vertex of a right angle through the field, the base of the angle extending east below the counter's row and its other side extending vertically north along the left side of the counter's column. All counters inside the right angle are removed. This constitutes a move. It is as though the field were a cracker and a right-angled bite were taken from it by jaws approaching the cracker from the northeast.

The object of the game is to force your opponent to chomp the poison counter at the lower left corner of the array [*colored counter*]. The reverse form of Chomp—winning by taking this counter—is trivial because the first player can always win on his first move by swallowing the entire rectangle.

What is known about this game? First, we dispose of two special cases for which winning strategies have been found.

1. When the field is square, the first player wins by taking a square bite whose side is one less than that of the original square. This leaves one column and one row, with the poison piece at the vertex [*see bottom illustration on opposite page*]. From now on the first player "symmetrizes." Whatever his opponent takes from either line, he takes equally from the other. Eventually the second player must take the poison piece.

2. When the field is 2 by *n*, the first player can always win by taking the counter at top right [*see bottom illustration on opposite page*]. Removing that counter leaves a pattern in which the bottom row has one more counter than the top row. From now on the first player always plays to restore this situation. One can easily see that it can always be done and that it ensures a win. The same strategy applies to fields of width 2, except now the first player always makes sure that the left column has one more counter than the right column.

With the exception of these two trivial cases, no general strategy for Chomp is known. There is, however, and this is what makes Chomp so interesting, a simple proof that the first player can always win. Like similar proofs that apply to Bridg-it, Hex, generalized ticktacktoe and many other games, the proof is nonconstructive in that it is of no use in finding a winning line of play. It only tells you that such a line exists. The proof hinges on taking the single counter at the upper right corner in the opening move. There are two possibilities: (1) It is a winning first move; (2) it is a losing first move. If it is a losing one, the second player can respond with a winning move. Put another way, he can take a bite that leaves a position that is a sure loss for the first player. But no matter how the second player bites, it leaves a position that the first player could have left if his first bite had been bigger. Therefore if the second player has a winning response to the opening move of taking the counter at top right, the first player could have won by a different opening move that left exactly the same pattern.

In short, either the first player can always win by taking the counter at top right or he can always win by some other first move.

"We normally think of nonconstructive proofs in mathematics as being proofs by contradiction," Gale writes. "Note that the above proof is not of that

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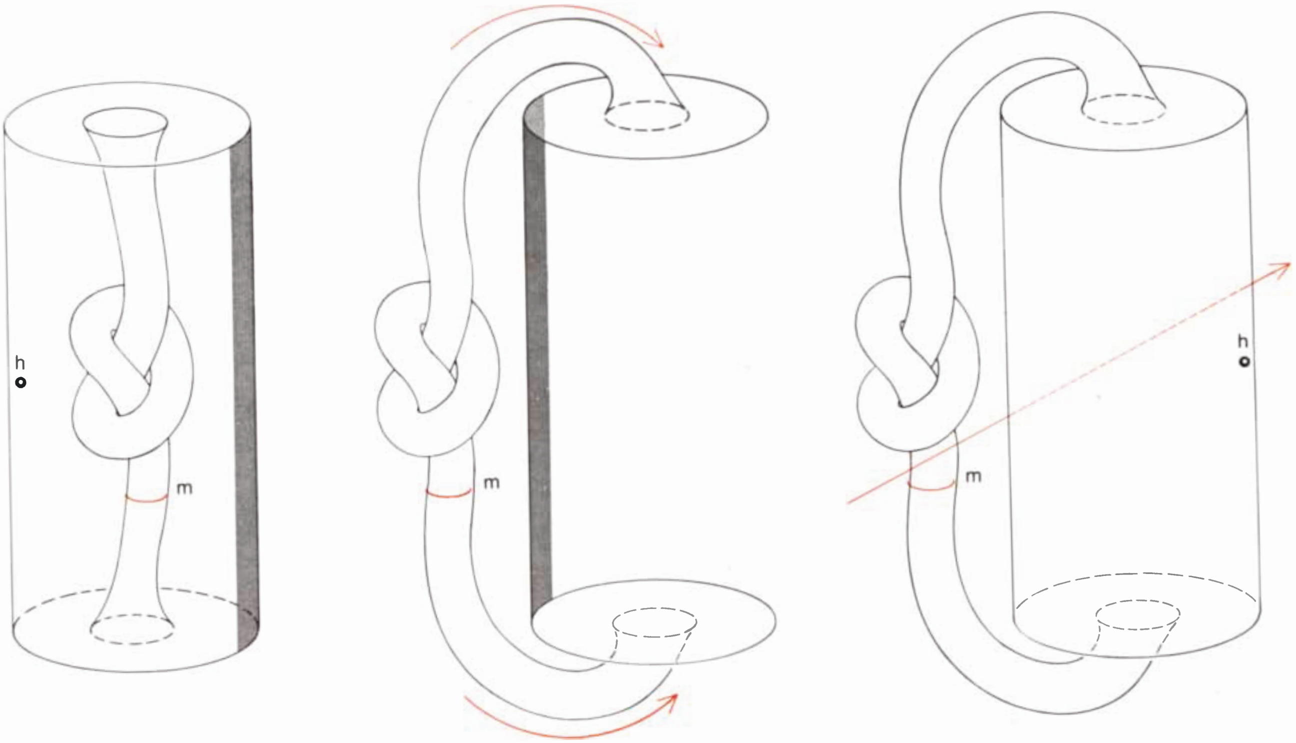
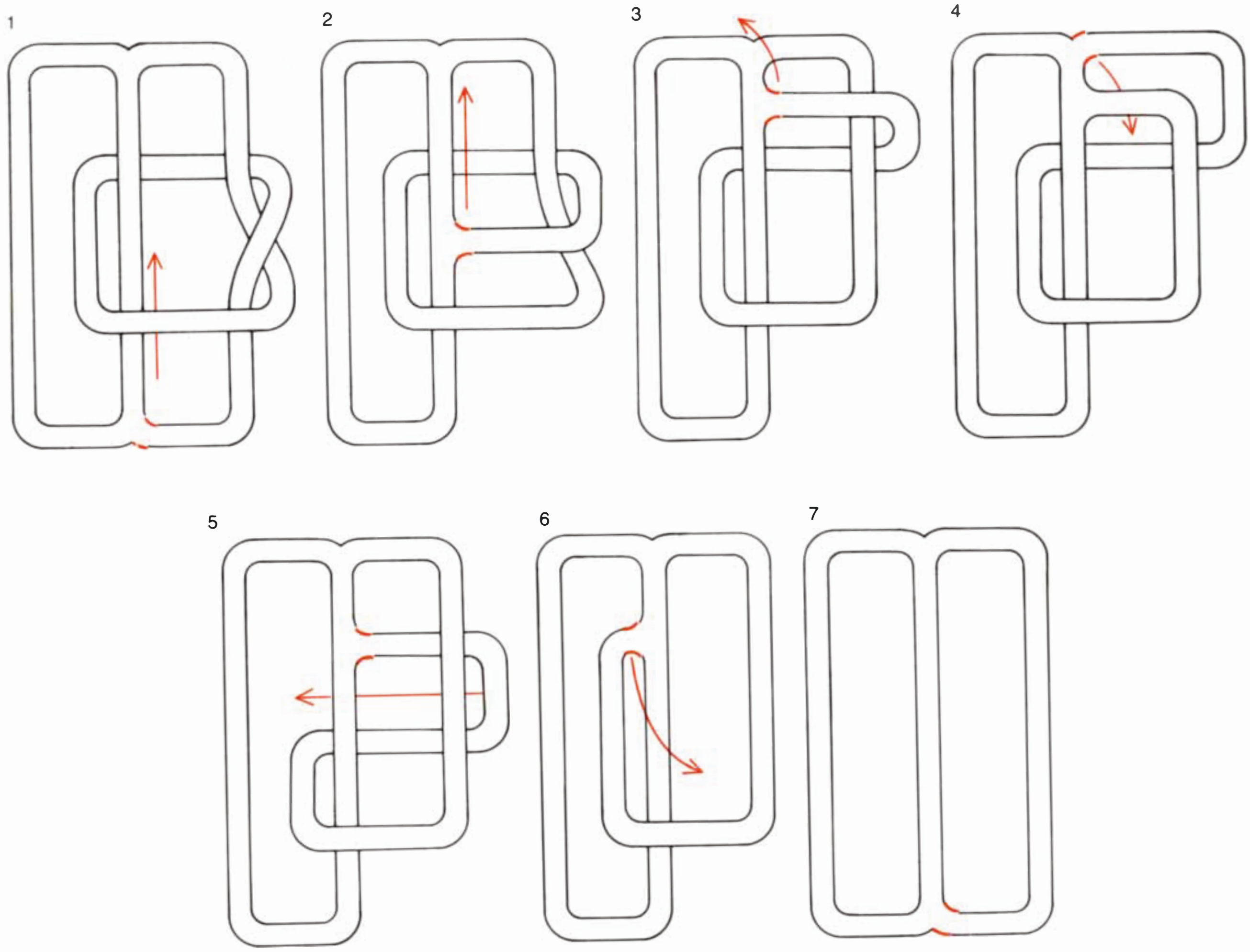


Solution to torus-reversal problem



Unknotting a two-hole torus

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type. We did not start by assuming that the game was a loss for the first player and then obtain a contradiction. We showed directly that there was a winning strategy for the first player. The word 'not' was never used in the argument. Of course we used implicitly the fact that any game of this kind is a win for either the first or the second player, but even the proof of this fact can be given by a simple inductive argument that does not use any law of the excluded middle."

This is essentially all that is known about Chomp except for some curious empirical results Gale obtained from a complete computer analysis of the 3-by-n game for all n's equal to or less than 100. In every case it turned out that the winning first move is unique. The illustration on page 111 shows the winning moves for 3-high fields of widths two through 12. Rotating and reflecting these patterns give winning moves on 3-wide fields of height two through 12, because any m-by-n game is symmetrically the same as the n-by-m game.

A winning first move on a 3-high field must be one or two rows deep. (A 3-deep bite would leave a smaller rectangle and thus throw the win to the second player.) Roughly 58 percent of the winning first moves are two rows deep and 42 percent are one row deep. Note that the one-row moves either stay the same or increase in width as n increases, and the same is true of the two-row moves. A partial analysis of all 3-high fields with widths less than 171 showed that the sole exception to this rule occurs when n is 88. The winning first move on the 3-by-88 rectangle is 2 by 36, which is one unit less wide than the winning 2-by-37 move on the 3-by-87 field. "Phenomena like this," Gale writes, "lead one to believe that a simple formula for the winning strategy might be quite hard to come by."

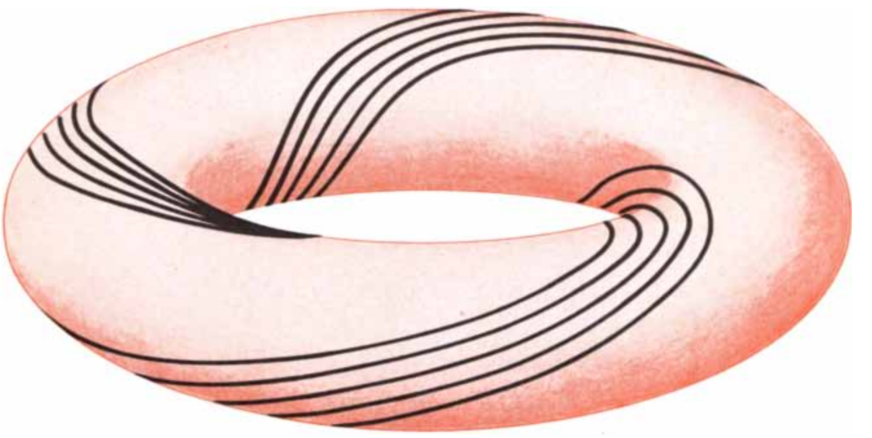
There are two outstanding unproved conjectures:

1. There is only one winning first move on all fields.

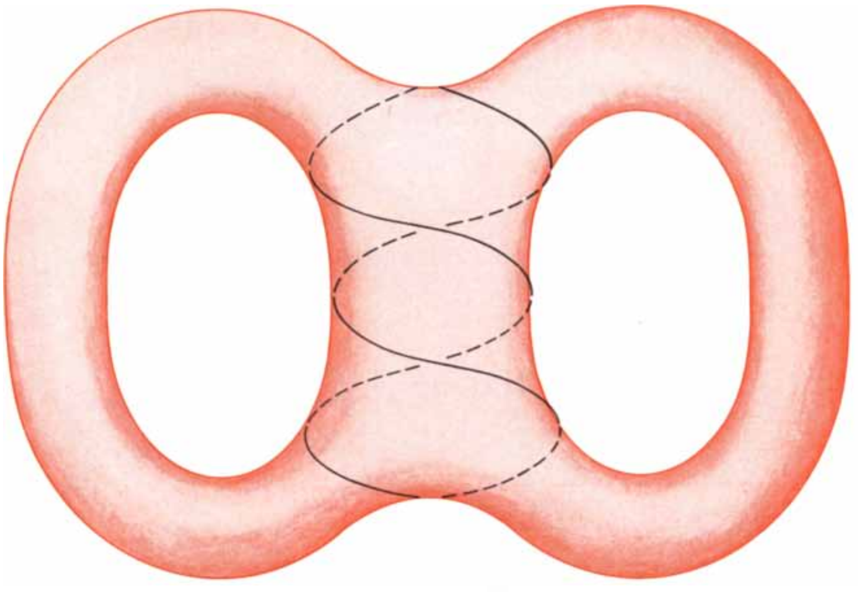
2. Taking the counter at the top right corner always loses except on 2-by-n (or n-by-2) fields.

The second conjecture has been established only for fields with widths or heights of 3. As a problem to be answered next month, readers are invited to discover the unique winning openings on 4-by-5 and 4-by-6 rectangles.

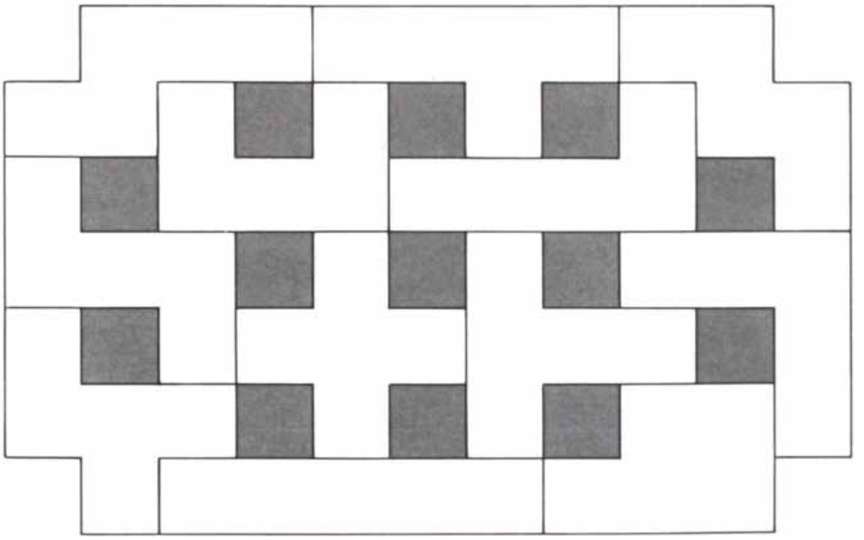
Here are answers to last month's problems. R. H. Bing shows how an internally knotted torus can be reversed through a hole to produce an externally



Knotted, nonintersecting curves on a torus



Rotating slice through a two-hole torus



Symmetrical solution to pentomino problem

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knotted torus [*see top illustration on page 112*]. A small hole, h, is enlarged to cover almost the entire side of the cylinder, leaving only the shaded strip on the right. The top and bottom disks of the cylinder are flipped over and then the hole is shrunk to its original size.

As in reversing the unknotted torus through a hole, the deformation interchanges meridians and parallels. You might not at first think so because the colored circle, m, appears the same in all three pictures. The fact is, however, that initially it is a parallel circling the torus's elongated hole whereas after the reversal it has become a meridian. Moreover, after the reversal the torus's original hole is no longer through the knotted tube, which is now closed at both ends. As is indicated by the arrow, the hole is now surrounded by the knotted tube.

Piet Hein's two-hole torus, with an internal knot passing through an external one, is easily shown to be the same as a two-holer with only an external knot. Simply slide one end of the inside knot around the outside knot (in the manner explained last month) and back to its starting point. This unties the internal knot. Piet Hein's two-holer, with the external knot going through a hole, can be unknotted by the deformation shown in the bottom illustration on page 112.

Answers to the final three toroidal questions are:

1. An infinity of noncrossing closed curves, each knotted with the same handedness, can be drawn on a torus [*see top illustration on preceding page*]. If a torus surface is cut along any of these curves, the result is a two-sided, knotted band.

2. Two closed curves on a torus, knotted with opposite handedness, will intersect each other at least 12 times.

3. A rotating slice through a solid two-hole doughnut is used to produce a solid that is topologically equivalent to a solid, knotted torus [*see middle illustration on preceding page*]. Think of a short blade as moving downward and rotating one and a half turns as it descends. If the blade does not turn at all, the result is two solid toruses. A half-turn produces one solid, unknotted torus. One turn produces two solid, unknotted linked toruses. Readers may enjoy investigating the general case of n half-turns.

Wade E. Philpott of Lima, Ohio, was the only reader who sent all 13 solutions to the Diabolical cube, whose six pieces were shown in last September's column. Before this column appeared I had occasion to show the puzzle to John Horton Conway of the University of Cambridge. He mentally labeled the pieces with a

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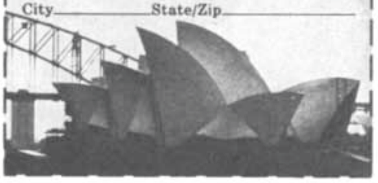

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checkerboard coloring, then began testing the pieces rapidly, talking out loud and occasionally scribbling a note. It was like watching Bobby Fischer play blitzkrieg chess. About 15 minutes later he announced that there were just 13 solutions. To distinguish them, designate each piece of the Diabolical cube by the number of unit cubes it contains. There are three ways in which the two largest pieces, 6 and 7, can go:

1. Parallel and side by side. When properly placed, with the 5-piece wrapped around a projecting cube of 6, the 4-piece can go in three places. There are five solutions.

2. Parallel but on opposite sides of the cube. There are two solutions.

3. Perpendicular to each other. Crossing in one way yields four solutions, another way two, or six solutions in all.

Philpott also sent a proof that a pattern of 14 unit holes, each surrounded by eight cells, cannot be achieved with the 12 pentominoes. The proof establishes that at least 59 squares are needed to surround 14 holes. On all such patterns each pentomino will fit except the *P* and *W* pieces. Adding a 60th cell will accommodate only one of the two pieces, proving that the 60 cells of the pentomino set are not enough. Essentially the same proof had earlier been formulated by Joseph Madachy.

Christer Lindstedt of Göteborg, Sweden, had found a 13-hole solution before the problem appeared in September. Other solutions (no two alike) were found by Robert Bart, Bruce Beckwith, Neil E. Beckwith, Greg Buckingham, Andrew L. Clarke, H. I. da Costa, John D. Determan, Victor G. Feser, William J. Flora, William Grolz, Thomas M. Napier, Jack M. Welch, David N. Yetter and Thomas Zaslavsky, and jointly by three readers in Paris, L. J. Francois, J. R. Ponce de Leon Pina and J. F. Vincent.

Only Clarke (of Freshfield in England) found solutions with bilateral symmetry. He sent seven, one of which has two axes of symmetry [*see bottom illustration on page 113*].

Readers interested in Greco-Latin squares may wish to write Joseph Arkin, 197 Old Nyack Turnpike, Spring Valley, N.Y. 10977, for a remarkable construction. A self-taught mathematician working without computer aid, Arkin succeeded last year in constructing an order-10 cube on which three Latin cubes are mutually orthogonal, provided that all three are considered together. The construction leads to an order-10 magic cube on which all orthogonals and the four diagonals are magic.

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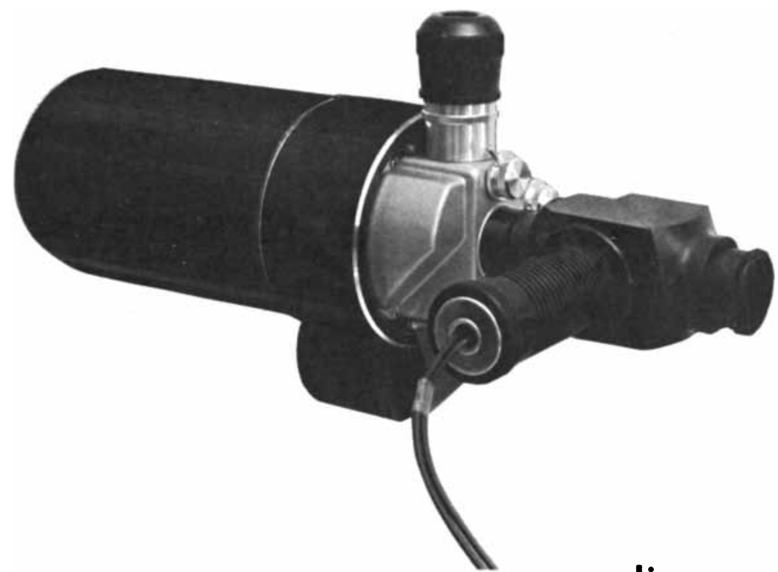

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