STOCHASTIC MODELS FOR THE LEARNING PROCESS

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INTRODUCTION

SINCE 1949 psychologists have increasingly used mathematics in the study of the learning process. Probability theory has been the keynote in these applications, and thus there is considerable unity in the published work. The applications described here are typical of those made by W. K. Estes, C. J. Burke, and their students (2, 3, 11, 12, 13, 14, 15, 16, 17, 18, 19), by G. A. Miller, F. C. Frick, and W. J. McGill (25, 26), by F. Restle (27, 28), and by R. R. Bush and F. Mosteller (1, 4, 5, 6, 7, 8, 9, 10, 23).

The purpose of this paper is to give the flavor of these new developments through a few examples, rather than to survey the literature. Three experiments illustrate the use of the new mathematics to provide: (1) a summary description of the course of learning in an experiment; (2) a qualitative distinction between two theoretical positions; (3) a test of the correspondence between elements in one theory and those in the physical world. We also mention a new mathematical problem, interesting in its own right, that has arisen from these applications.

PROBABILITY AND LEARNING

Let us suppose that to learn a list of words you read through the list and then recite those words that you recall. With successive readings and recitations, the number recalled correctly increases, and, if the list is short, you ultimately learn all the words. Early writers on mathematical methods in the psychology of learning described the course of such learning by finding a mathematical curve whose shape is appropriate to this general improvement in recall. Such curves were invented by many authors. More generally, for a variety of experiments, mathematical functions were sought to describe how some measure of performance of a task increased with some measure of practice.

Early favorites among the functions were the hyperbola, the exponential, and the arc cotangent. The main features of these curves are that they rise monotonically with increased practice and that they tend to an asymptote or ceiling corresponding to the best possible performance. Let us focus attention on the monotonically increasing character of these curves. The course of learning, like that of true love, does not run quite so smoothly. You will find in learning a long list of words that on some trials you do not remember words that you recalled on earlier trials and, worse yet, that on some late trials you do not recall as many words as you did on an earlier trial.

This erratic character of learning in turn suggests that a better description of the process might be achieved by the use of probabilistic, statistical, or random processes rather than by deterministic curves. In present day mathematics, such descriptions are often called *stochastic models*. The word "stochastic" is now almost synonymous with "probabilistic" or "random," but usually implies that a time variable is present. In the learning of lists, the successive readings and recitations are thought of as a sequence in time. The word "model" in applied mathematics has come to mean a mathematical description of certain aspects of a physical process rather than a small physical replica of the real thing. Thus one might speak of the Mendelian model for inheritance, or the binomial model for coin tosses.

Thurstone (31) seems to have developed the first of these stochastic models for learning in 1930, but he used it only as a vehicle to get a deterministic curve. Gulliksen and Wolfle (21) developed modified versions of such learning curves. These curves were designed to describe average performance as it depended upon the number of practice units.

The stochastic models developed since 1949 are designed to describe the responses made by subjects in simple repetitive experiments. The subject of the experiment receives a stimulus, he makes one of a number of responses, and some outcome of this response occurs, perhaps reward or shock. It is assumed that at the start of a trial each possible response has its own probability of occurring. It is assumed further that the event that occurs during the trial changes the probabilities of these responses for the next trial. The mathematical counterpart of the event is a mathematical operator which adjusts the probabilities in a predetermined manner. Usually an event consists of the response of the subject together with the outcome of the trial, such as reward or punishment.

Thus from the point of view of the model, the learning process consists of the changing probabilities of the responses and the rules that change them. These changes are reflected in the data by the changing frequencies of the responses through time. Such a probability process generates mathematically an erratic sequence of "responses" like those that occur in the successive recalls of lists of words.

These models have been used to describe experiments in reward training including partial reinforcement, rote learning, discrimination, spontaneous recovery, avoidance training, time and rate problems, and experimental extinction.

We turn now to some specific experiments.

AN ESCAPE–AVOIDANCE EXPERIMENT

One use of stochastic models is to provide a summary description of the learning process. For example, Solomon and Wynne (30), in an escape-avoidance experiment with dogs as subjects, placed the dog in one side of a symmetrical box divided by a movable barrier. At the start of a trial, the barrier was raised leaving the dog free to jump to the other side across a shoulder-high fence. If the dog did not jump within 10 seconds, he received a shock and thereupon usually jumped to the other side. Such a trial is called an "escape." If the dog jumped before the 10 seconds were up, he is said to have "avoided." Thus the two responses are "escape" and "avoidance." All normal dogs tested learned to avoid almost perfectly.

The question arises, how does the probability of escape change as the trials continue. In this experiment the event changing the probability is assumed to be in perfect correspondence with the response of the animal. Bush and Mosteller (7) assume that the probabilities of the responses, escape and avoidance, are p and $1 - p$, respectively, at some given time in the course of the experiment. They assume further that on the next trial either response will reduce the probability of escape. The reduction takes the form of multiplication by a factor α_1 for escape, α_2 for avoidance. Both α_1 and α_2 are assumed to have values between 0 and 1. These α's measure the slowness of learning, and in this experiment their values were estimated as $\alpha_1 = 0.92$, $\alpha_2 = 0.80$. Thus if the present probability of escape is $p = 0.4$, an escape on the next trial would reduce this probability of escape to $\alpha_1 p = (0.92)(0.4) = 0.368$, whereas an avoidance would reduce it to $\alpha_2 p = (0.80)(0.4) = 0.32$. In other words, an escape reduces the probability of escape by 8 per cent $(100 - 92 = 8)$, and an avoidance reduces the probability of escape by 20 per cent. The values 8 and 20 per cent can be thought of as the speed of learning corresponding to the two events.

The particular form these changes in probability are assumed to take in this experiment are a specialization of the more general operators used by Bush and Mosteller.

Initially, the dogs almost never avoided (one avoidance in 300 pretraining trials), therefore the initial probability of escape could be taken as very close to unity. Using $p = 1$ as the initial probability, the model states that after a avoidances and b escapes the probability of escape is given by $\alpha_1^b \alpha_2^a$. Thus as soon as α_1 and α_2 are measured, one can estimate, for dogs with a given history of escapes and avoidances, the fraction that will escape on the next trial. And, of course, many other predictions can be made.

In this experiment the values of the parameters themselves give special information. Of course, we note that the learning rate is faster for avoidance than for escape. But, more important, we can find out how many escape trials are required to change the probability by the same amount as one avoidance trial. Since $(0.92)^{2.7}$ is approximately 0.80, the answer is that 2.7 escape trials are roughly equivalent to one avoidance trial. It should, of course, be understood that this and other calculations are made within

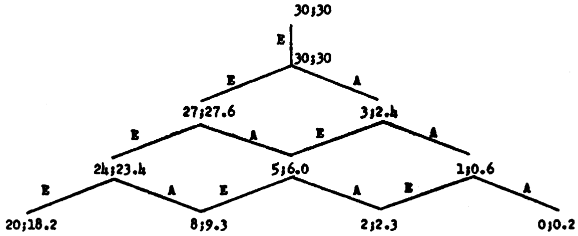

FIG. 1. Branching diagram for the first few trials of Solomon-Wynne escape-avoidance experiment. The first number at each intersection is observed, the second is the corresponding computed number. The letter E stands for escape, A for avoidance.

the framework of this model with its special assumptions. The validity of the interpretation depends on how well the whole model describes the process.

Figure 1 is a branching diagram showing the results for 30 dogs for the first few trials of the Solomon-Wynne experiment. The number before the semicolon at each point of the branching diagram gives the number of dogs that have arrived at that position, the number after the semicolon is computed on the basis of the values of the parameters already given. Thus 30 dogs started the experiment and all escaped (E) on the first trial. Assuming that all dogs had the same $\alpha_1 = 0.92$, approximately, their new probability of E is 0.92. (The experimenters adjusted the shock and the height of the barrier for each dog separately in an effort to equate the situation from dog to dog.) The mathematical expectation of the number of dogs escaping on the next trial is $(0.92)(30) = 27.6$, and 27 dogs actually escaped, the other 3 avoided. After the third trial there are three kinds of dogs—those that escaped thrice, twice, or once. The only new kind of calculation is that for those escaping twice. These 5 dogs are composed of the 3 who avoided for the first time on the third trial and the 2 who escaped after a previous avoidance. The theoretical calculation is $(27.6)(1 - (0.92)^2) + 2.4(0.92)(0.80)$, or about 6.0. (Thus the theoretical calculation was done entirely on the basis of the original parameter values and does not use the actual outcomes for the dogs on intermediate trials.)

It will readily be noted that the observed and computed frequencies in figure 1 are quite close—closer, if anything, than ordinary sampling variation would lead one to expect.

Once the three numbers are given—the initial probability and the two learning parameters—we can compute statistics other than those given in figure 1 to see how well they fit the data. While some of these new statistics involve routine probability calculations, others are quite complicated, and it is convenient to run a number of stat-dogs using random number tables to evaluate them.

Given the values of the parameters and the method of changing the probabilities, we can carry out a mock experiment with the aid of a random number table. Thus our first stat-dog has probability 1 of escaping on the first trial. This escape reduces his probability of escaping to 0.92. We draw a two-digit random number from a table in which the 100 numbers 00, 01, . . . , 99 are equally likely. If the number is less than 92 (i.e., 00, 01, . . . , 91), we say the stat-dog escaped on the second trial, otherwise (92, 93, . . . , 99) that he avoided. Now we adjust his probability, appropriately, and continue the process. Thus the stat-dog is carried through the "experiment," and he produces a succession of E's and A's just as the real dogs do. By carrying many such stat-dogs through the process, we can generate artificial data that indicate the properties of the mathematical model. These artificial data can be compared with the real data. In this manner questions that are too complicated for theoretical calculation can still be answered. The technique is an application of the Monte Carlo method which is widely used by physicists and statisticians. Many such statistics are compared in table 1. Not all of these measures are of psychological interest; rather the table illustrates the point that the three basic parameters supply satisfactory answers to a wealth of questions about the sequences that occur in such an experiment.

The standard deviations provided in table 1 could be used to test the difference of the means obtained in the experiment and the pseudoexperiment, but they are given for a different reason. The mathematical model is supposed to predict not only the mean value of the statistics listed in table 1 but also the distribution of values for many dogs. Rather than provide the whole distribution, we supply the standard deviations so that the variability of the stat-dogs can be compared with that of the real dogs. This supplementary comparison shows that the stat-dogs are slightly less variable, generally, than the real dogs.

Table 1 illustrates an important difference between stochastic models and the earlier deterministic ones. After fitting the parameters, the deterministic models yield a "learning curve," which in this experiment would estimate the fraction of escapes on each trial. The stochastic model can do the same. But the deterministic model had no answer to such questions as, what is the average length of the longest run of escapes, or what fraction of those who have avoided once and escaped twice will avoid next time, whereas the stochastic model does.

Most acquisition curves have much the same shape, and it is a shape that a good many mathematical curves in common use can fit quite well. Consequently, a good fit to the learning curve cannot give much support to the theory leading to it. If, however, one can use the fitted param-

TABLE 1

COMPARISONS OF THE STAT-DOG "DATA" AND THE SOLOMON-WYNNE DATA FOR 30 DOGS, EACH RUN THROUGH 25 TRIALS

	Stat-dogs		Real dogs	
	Mean	S.D.*	Mean	S.D.*
Trials before first avoidance	4.13	2.08	4.50	2.25
Trials before second avoidance	6.20	2.06	6.47	2.62
Total shocks	7.60	2.27	7.80	2.52
Trial of last shock	12.53	4.78	11.33	4.36
Alternations	5.87	2.11	5.47	2.72
Longest run of shocks	4.33	1.89	4.73	2.03
Trials before first run of four avoidances	9.47	3.48	9.70	4.14

* To obtain standard deviation of the mean, divide by $\sqrt{30}$.

eters to forecast accurately a variety of other statistics which have not been fitted directly but are consequences of the mathematical process, the results are rather more satisfying and informative. The statistics given in table 1 are of this type.

This example illustrates some of the kinds of calculations that are involved in stochastic models. It shows how a particular model fits in one experiment, and especially it displays the use of the model to describe the fine-grained structure of the data over and above the mean performance curves provided by earlier models. The close agreement between data and theory indicates that the model describes the data quite well. Thus, the model plus the parameter values summarizes the data rather completely. And, incidentally, the example illustrates the Monte Carlo method.

AN EXPERIMENT WITH PARADISE FISH

In an experiment with paradise fish, by Bush and Wilson (10), the fish had two choices—to swim to the right-hand side or to the left-hand side of the far end of a tank after the starting gate was raised. One of these sides—the favorable side—gave the fish a reward, caviar, 75 per cent of the times he chose it, the other side gave the reward only 25 per cent of the times he chose it. Thus, from the point of view of the model there are four possible events: right-reward, right-nonreward, left-reward, left-nonreward. It would be generally thought that being rewarded on a given side would improve the probability that that side was chosen on the next trial. But about nonrewarded trials, the reasoning is not so clear. An extinction or information theory would suggest a reduction in the probability of going to an unrewarded side on the next trial, but a theory based on habit formation or secondary reinforcement would suggest that merely going to a side would make that side more likely to be chosen on the next trial.

The mathematical operators for the four events in this experiment would differ according to which of these formulations one adopted.[1] Let us assume that the probability of choosing the right-hand side of the tank is p at some stage of the experiment. Then if the fish chooses the right-hand side and is rewarded, his probability of choosing the right-hand side on the following trial is increased. Bush and Mosteller assume that the new probability of choosing the right-hand side has the form $\alpha_1 p + 1 - \alpha_1$. (See table 2.) As before, α_1 is the learning parameter appropriate to this particular outcome, and α_1 is between 0 and 1. If $p = 0.4$ and $\alpha_1 = 0.8$, the new probability is $(0.8)(0.4) + 1 - 0.8 = 0.52$. It is assumed that if the left side is chosen and rewarded, the new probability of turning right is smaller. Furthermore, from the symmetry of the experiment it is assumed that the rate of learning on the left is the same as that on the right. It turns out that $\alpha_1 p$ makes the proper reduction in the probability of turning right. (The algebraic asymmetry between $\alpha_1 p$ and $\alpha_1 p + 1 - \alpha_1$ comes from the fact that we discuss the problem from the point of view of the probability of turning right, rather than from the point of view of the effect on the probability of the side just chosen.)

When we consider non-reinforcement on the right-hand side, the model for a theory of extinction suggests a reduction in the probability of choosing the right-hand side on the next trial ($\alpha_2 p$). A theory of habit formation or secondary reinforcement suggests that an increase, no doubt smaller than that for reward, in the probability will occur ($\alpha_2 p + 1 - \alpha_2$). These possibilities are listed in table 2.

These two models make quite different forecasts about the long-run behavior of the animals. The reinforcement-extinction model im-

[1] They would depend, too, on whose mathematical analysis one adopted. In this paper we present some of the spirit of the applications of probability theory to experiments in learning. A much more technical discussion would be required to compare different mathematical theories for the same experiment. Here we choose one form for the operators and use this same form to describe two different psychological positions. It is not implied that this is the only mathematical form that could be used.

TABLE 2

OPERATORS FOR CHOICE EXPERIMENTS

Operators for reinforcement-extinction model (p = Prob (right)):

	Left	Right
Reinforcement	$\alpha_1 p$	$\alpha_1 p + 1 - \alpha_1$
Non-reinforcement	$\alpha_2 p + 1 - \alpha_2$	$\alpha_2 p$

Operators for habit-formation model:

	Left	Right
Reinforcement	$\alpha_1 p$	$\alpha_1 p + 1 - \alpha_1$
Non-reinforcement	$\alpha_2 p$	$\alpha_2 p + 1 - \alpha_2$

plies that the animals never stabilize on a side. The reason is that if a very high probability of choosing the right-hand side is achieved, non-rewards there will reduce the probability, and ultimately the animal will switch to the left. Nonrewards are sure to occur because the fish is rewarded on only 75 per cent of the choices of the favorable side. A similar argument shows that he cannot stabilize on the unfavorable side according to this model.

On the other hand, the habit-formation model implies that the animal will stabilize on one side or the other, but, surprisingly enough, states that some animals will stabilize on the favorable side and some on the unfavorable side. Again the idea is simple. Whether rewarded or not, going to a side increases its probability. Once a high probability is achieved for a side, the animal is very likely to go there, and going there makes it the more probable that he will go there again. (A somewhat technical argument is required to prove that all organisms are ultimately absorbed by one side or the other.)

In the paradise fish experiment, there were two conditions: (1) with opaque divider between the two goal boxes, (2) with transparent divider. The opaque divider prevented the fish from seeing the goal box he did not choose, but the transparent one permitted the fish to see the food placed in the other goal box if that one was to be rewarded in the given trial. There was behavioral evidence that the fish was usually aware of the food in the other goal box when he could see but not obtain it.

A tabulation was made of the number of trials on which each fish turned to the favorable side in the last 49 of its total of 140 trials. This tabulation is shown in table 3. Clearly, most of the fish go nearly all the time to one side. Ten have almost all their late trials on the favorable side, 4 have almost all on the unfavorable side. And generally, the clustering is at the extremes. This result is in better agreement with the habit-

TABLE 3

PARADISE FISH EXPERIMENT

Cell entry is number of fish with trials to the favorable side indicated in first column (last 49 of 140 trials).

Trials to favorable side	Real fish: transparent divider	Stat-fish	Real fish: opaque divider
0–4	4	4	1
5–9	1	2	0
10–14	2	0	0
15–19	0	0	1
20–24	0	0	2
25–29	0	1	0
30–34	1	0	2
35–39	2	2	3
40–44	2	3	7
45–49	10	10	11
Total fish	22	22	27

formation model than with the reinforcement-extinction model, because the latter predicts that the fish will cluster around the value of 37 trials to the favorable side, contrary to the data which cluster at one end or the other of the distribution.

The stat-fish figures of table 3 are the results of Monte Carlo runs using the habit-formation model. The parameters (α_1, α_2) have been fitted to those of the fish run with transparent divider. No stat-fish were run to compare with the opaque divider group.

The treatment of this experiment illustrates the use of stochastic models to describe different theoretical positions and to make qualitative distinctions between them.

A COMPOUND STIMULUS EXPERIMENT

In deriving a form for the mathematical operators used to change the probabilities, Estes (11) assumes that the environment is composed of elementary elements. Each of these elements is assumed to be conditioned to or associated with one and only one of the responses. The subject is assumed to take a sample of the elements. His probability of giving a particular response is assumed to be identical with the fraction of elements in the sample that are conditioned to that response. Such a theory can be used to derive operators like those used by Bush and Mosteller (7).

Among the experiments performed to provide a test of Estes' theory of conditioned elements is that of Schoeffler (29). A group of 24 lights was randomly divided into three groups of 8 each. Subjects were taught to move a lever to the left when one set of 8 lights was flashed, to move it to the right when another set of 8 flashed (hereafter we call these the left and right sets, respectively). The third set of 8 lights was used only for testing purposes (we call these the neutral set). After subjects learned to respond perfectly to these two sets of lights, a composite set was flashed composed of some lights from two or all three sets—the left, right, and neutral sets.

After each test trial, the subject was re-trained to discriminate between the original left and right sets of lights. The theory would suggest that the fraction of times the subject would respond "left" to the compound stimulus would be

$$\frac{l + \frac{1}{2}n}{r + l + n},$$

where l = number of lights in the test stimulus previously conditioned to left, r = number of lights in the test stimulus previously conditioned to right, n = number of lights in the test stimulus not previously presented in a training series. In constructing this formula, it is a convenience to suppose that a light in the compound stimulus is an elementary element. However, a light might correspond to many elements, so we suppose that the lights each correspond to the same number of elements—that is, each light has the same weight in the formula. There is a tacit assumption that, for lights not presented in the training trials, half the elements are conditioned to left, half to right. There is a further tacit assumption that the lights in the stimulus set contributed all the elements in the environment (in some later experiments, Estes and Burke (18) have questioned this assumption).

The results are shown in table 4. The predicted and observed fractions agree quite well except possibly for the pattern shown in the second line.

TABLE 4

Schoeffler's Compound Stimulus Experiment

Test trial	l = number of left lights	r = number of right lights	n = number of neutral lights	Theoretical fraction "left" responses	Fraction* "left" responses
1	8	8	0	.50	.54
2	8	4	0	.67	.79
3	8	2	0	.80	.81
4	4	2	0	.67	.63
5	8	4	8	.60	.62
6	8	2	8	.67	.67
7	4	2	8	.57	.54
8	8	0	8	.75	.73
9	8	8	8	.50	.54

* Each based on 180 responses, one by each of 180 subjects.

It is not intended here to claim that other theories might not account for these same results, but rather to display an experiment suggested by the theory and to indicate the degree to which the results were in agreement with it.

A MATHEMATICAL PROBLEM

If the habit-formation model is appropriate for the paradise fish experiment, what fraction of the fish will stabilize on the favorable side? This question is typical of a number of mathematical problems that have arisen in the mathematical study of learning.

For a mathematical description of this problem we take a slightly simpler question. Suppose that there are two responses A_1 and A_2, and that the outcome of both is pleasant though not necessarily equally so. Suppose that the initial probabilities of the responses A_1 and A_2 are p_0 and $1 - p_0$, respectively. If p is the probability of A_1 on some trial and A_1 is performed, the new probability of A_1 is $\alpha_1 p + 1 - \alpha_1$, but if A_2 is performed, the new probability of A_1 is $\alpha_2 p$. It can be shown that an organism obeying this description will in the long run stop giving one of the responses and respond only with the other (with probability one). Now, given p_0, α_1, α_2, what is the probability that the organism stops giving A_2's—that is, is absorbed by A_1? Let us call this probability $f(p_0, \alpha_1, \alpha_2)$. After one trial the organism has a new probability $\alpha_1 p_0 + 1 - \alpha_1$ (if A_1 occurs) or $\alpha_2 p_0$ (if A_2 occurs) with probabilities p_0 and $1 - p_0$, respectively. Thus, if his first trial is A_1, his new probability of absorption by A_1 is $f(\alpha_1 p_0 + 1 - \alpha_1, \alpha_1, \alpha_2)$, but if the first trial is A_2, the new probability of absorption by A_1 is $f(\alpha_2 p_0, \alpha_1, \alpha_2)$. Weighting these by their respective probabilities, we find the functional equation

$$f(p_0, \alpha_1, \alpha_2) = p_0 f(\alpha_1 p_0 + 1 - \alpha_1, \alpha_1, \alpha_2) + (1 - p_0) f(\alpha_2 p_0, \alpha_1, \alpha_2).$$

This functional equation and related ones have been studied by Bellman and Shapiro (22) and by Karlin (24). Bellman showed that a limiting continuous solution exists, that it is analytic and unique, and described other properties. Shapiro studied methods of solving the functional equation and gave the rate of convergence of iterative solutions. Karlin developed a different approach to the study of such random walk problems, and studied the case of two reflecting barriers as well as the present one of two absorbing barriers, and other extensions of this problem.

SUMMARY

In summary, the erratic nature of the learning process suggests that a mathematical description might better be probabilistic rather than deterministic. The manner in which stochastic models describe the fine-grained structure of the learning process has been illustrated by the data of the escape-avoidance experiment. The use of these models to describe alternative psychological theories, and thus suggest tests of them, was illustrated by the paradise fish experiment. Similarly, the Schoeffler experiment was a test of Estes' theory of conditioned stimuli. It was also noted that work in learning had created new mathematical problems of interest in their own right.

In closing, I should mention that this is a quite young field and that we have a long way to go before we can explain even a good fraction of the many well-established principles already known to the psychologist. Progress can best be made if we are willing to consider and then destroy a great many of our mathematical efforts.

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