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Strategy-proof school choice mechanisms with minimum quotas and initial endowments ☆

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ABSTRACT

We consider a school choice program where minimum quotas are imposed for each school, i.e., a school must be assigned at least a certain number of students to operate. We require that the obtained matching must respect the initial endowments, i.e., each student must be assigned to a school that is at least as good as her initial endowment school. Although minimum quotas are relevant in school choice programs and strategy-proofness is important to many policymakers, few existing mechanisms simultaneously achieve both. One difficulty is that no strategy-proof mechanism exists that is both efficient and fair under the presence of minimum quotas. Furthermore, existing mechanisms require that all students consider all schools acceptable to obtain a feasible matching that respects minimum quotas. This assumption is unrealistic in a school choice program.

We consider the environment where a student considers her initial endowment school acceptable and the initial endowments satisfy all the minimum quotas. We develop two strategy-proof mechanisms. One mechanism, which we call the Top Trading Cycles among Representatives with Supplementary Seats (TTCR-SS), is based on the Top Trading Cycles (TTC) mechanism and is significantly extended to handle the supplementary seats of schools while respecting minimum quotas. TTCR-SS is Pareto efficient. The other mechanism, which we call Priority List-based Deferred Acceptance with Minimum Quotas (PLDA-MQ), is based on the Deferred Acceptance (DA) mechanism. PLDA-MQ is fair, satisfies a concept called Priority List-based (PL-) stability, and obtains the student-optimal matching within all PL-stable matchings. Our simulation results show that our new mechanisms are significantly better than simple extensions of the existing mechanisms.

☆ This paper is partially based on the authors' conference publication [30], where TTCR-SS was presented. In this paper, we introduce another mechanism (Section 4) and compare two mechanisms by simulations (Section 5). We also add new theoretical results of TTCR-SS in Section 6.

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1. Introduction

Traditionally, a student who plans to attend a public school is assigned to one based on where she lives. School choice programs are implemented to give students and their parents opportunities to choose which public schools to attend. In such programs, students submit their preferences over schools to a centralized matching mechanism, which assigns students to schools. A seminal work by Abdulkadiroğlu and Sönmez [3] introduced the idea of using a mechanism design approach to study this issue by formalizing it as a problem of allocating indivisible objects with multiple supplies (seats in schools) to agents (students). This problem is referred to as the school choice problem.

In this paper, we consider a school choice problem with two requirements. First, we assume that a minimum quota constraint is imposed on each school. A school is required to enroll a certain number of students. This is a reasonable assumption since each school needs a minimum number of students to operate. Second, we assume that each student has a default school that she would have attended without a school choice program, which we refer to as her *initial endowment*. We further assume that initial endowments satisfy all minimum quotas. The objective of this paper is to design school choice mechanisms so that each student who participates in the matching process will be able to attend a school that is at least as good as her initial endowment school. On the other hand, strategy-proofness, i.e., no student ever has any incentive to misreport her preference regardless of other students' reports, is critical to many policymakers. We focus on strategy-proof mechanisms in this paper.

Several desirable properties of a matching mechanism have been proposed in the literature. Two widely discussed properties are Pareto efficiency and stability. Pareto efficiency is a welfare notion that rules out incidents that can improve agents' well-being without making others worse off. Stability rules out *justified envy*, which is an incident that violates priority in a school.[1] However, [3] showed that a matching mechanism cannot be both stable and Pareto efficient in the setting of school choice problem, even when there is no distributional constraint. As a result, a policymaker needs to choose between Pareto efficiency and stability.

In this paper, we develop two strategy-proof mechanisms: the *Top Trading Cycles among Representatives with Supplementary Seats* (TTCR-SS) and the *Priority List-based Deferred Acceptance with Minimum Quotas* (PLDA-MQ). The first achieves Pareto efficiency,[2] and the second achieves PL-stability, which is a version of stability we consider in this paper.

Before we introduce our mechanisms, we first introduce two simple mechanisms that can handle minimum quotas. The first is based on the *Top Trading Cycles* (TTC) mechanism of [44]. Since in our setting, each student is endowed with a seat in a default school, a simple way to improve students' welfare is to allow them to trade their seats in schools. Moreover, since we assume that the initial matching satisfies maximum and minimum quotas, the new matching that resulted from trading also satisfies these distributional constraints. This is because trading happens only when a group of students wants to exchange seats, and therefore, the numbers of students who are matched to a school are the same both in the initial and new matching. We call this simple mechanism the *Top Trading Cycles among Representatives* (TTCR). Note that we design a list over the students, the *Master List* (ML), in TTCR to prioritize their rights to form a trading cycle.[3] Note that a student will exchange her seat in her default school with another student only when she can obtain a seat in a school that she prefers to her default school. Thus, in the new matching, each student is weakly better off.

The second simple mechanism is the *Artificial Cap Deferred Acceptance* (ACDA), which is identical as the *Deferred Acceptance* (DA) mechanism [16] except for two adjustments. First, we created an artificial maximum quota for each school that is equal to the number of students who are matched to this school in the initial matching. The mechanism uses artificial maximum quotas instead of true maximum quotas to create matchings. Second, we adjust the priorities of the students in schools so that a student has higher priority in her default school than a student whose default school is different. These two adjustments guarantee that the new matchings created by DA satisfy the distributional constraints and that every student is weakly better off in the new matching.

ACDA is a popular mechanism to handle minimum quotas in practice,[4] and its properties have been analyzed in several studies, for example, [15] and [23]. The real-world applications of ACDA include the hospital-doctor matching in Japan and the cadet-branch matching in the United States ([15] and [23]).

We find that both TTCR and ACDA have severe shortcomings emanating from the fact that the number of students who are assigned to a school is weakly less than its capacity in both mechanisms. The mechanisms developed in this paper, TTCR-SS and PLDA-MQ, are designed to properly exploit these extra seats. TTCR-SS is designed to achieve efficiency, and PLDA-MQ is designed to improve students' welfare while achieving a certain degree of fairness.

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[1] Individual rationality, fairness, and nonwastefulness constitute stability [5]. In our setting, a matching is fair if any school, which a student prefers to her matched school, is occupied by students with higher priority or who initially endow the school. A matching is nonwasteful if moving a student from her currently assigned school to a more preferred one violates minimum or maximum quotas.

[2] In our setting, a Pareto efficient matching is not Pareto dominated by another feasible matching. A feasible matching satisfies both maximum and minimum quotas.

[3] In real-world applications, their GPAs can be used to create ML.

[4] To the best of our knowledge, in practice, ACDA is used in settings where there are no initial endowments. The second adjustment mentioned above is not used in such a setting. We introduce the second adjustment to guarantee that the matching created by this mechanism makes every student weakly better off in the new matching. In the following discussion in this paper, we continue to use the second adjustment.

In developing both of these mechanisms, the presence of minimum quotas plays a crucial role. If there are no minimum quotas and if only the initial endowments are addressed, we can use a simple extension of the TTC-based mechanism described in [2] to achieve Pareto efficiency. Also, to achieve stability, we can use the standard DA mechanism with the modified priorities so that each school respects its initial endowment students.

The design of TTCR-SS is based on TTC. We introduce *dummy students* who are used to facilitate the formation of trading cycles in the mechanism.[5] We carefully design the dummy students' preferences to satisfy the distributional constraints and show that TTCR-SS is strategy-proof and Pareto efficient. Note that the notion of Pareto efficiency is stronger than nonwastefulness.

The design of PLDA-MQ is based on DA. Although the design objective is to create a mechanism that is fair and nonwasteful, such fair and nonwasteful matching might not exist in our setting. This impossibility result precludes the existence of a fair and nonwasteful mechanism. We find that this impossibility result comes from the standard definitions of fairness and nonwastefulness. In fact, these definitions do not prioritize students' rights to leave their default schools for other schools when leaving might cause the constraints of minimum quotas to fail to be satisfied. To accommodate this situation, we use the notions of *PL-fairness* and *PL-nonwastefulness*.[6] These two notions are based on an ordering over all pairs of schools and students, called the *Priority List* (PL). With PL, if two students want to move to two (possibly different) schools but only one can, PL determines which student can move. We only make two assumptions about PL. First, PL respects students' priorities in each school. Second, it respects their initial endowments. This gives the policymaker great flexibility in designing PL. In particular, it can be designed to prioritize which schools have the right to receive students from other schools if moving all of the students might violate distributional constraints. In the setting of cadet-branch matching, policymakers might prioritize one branch over another for receiving more personnel, while all branches still satisfy the minimum quotas constraints. A matching is *PL-stable* if it is PL-fair and PL-nonwasteful. We show that ACDA does not create PL-stable matching.[7]

We show that PLDA-MQ is strategy-proof and PL-stable. Moreover, it obtains the student-optimal matching within all PL-stable matchings. We use the techniques developed in [27] to develop PLDA-MQ. Although they provide a useful toolkit, developing a concrete mechanism that works for new types of constraints remains challenging; we need to appropriately design a choice function of schools so that their framework is applicable for achieving the required design goals. PLDA-MQ is an instance of the Generalized Deferred Acceptance (GDA) mechanism [20][8] whose skeleton resembles DA. Thus, PLDA-MQ can be easily adopted to policymakers, schools, and students. This is a major advantage compared to applying a completely new or unfamiliar mechanism.

Our mechanisms can be used in a setting where students' preferences are changed after they receive their initial assignments, and there is a minimum quota constraint in each school. In many universities in Japan, undergraduate engineering students must be assigned to a laboratory to conduct projects. However, students sometimes have difficulty choosing appropriate laboratories since their familiarity with them is limited. Our mechanisms can be used to reallocate students after they experience a certain trial period. In the NYC high school choice program, the allocation of students is determined by a centralized matching mechanism in December, and students are allowed to participate in a reapplication process in April [35]. He empirically showed that some students change their preferences after they receive their first assignments. Such reapplication processes are a potential application of our mechanisms.

Very few studies have addressed strategy-proof mechanisms that can handle minimum quotas. One example is [15], who developed mechanisms based on DA. However, there is a severe limitation to apply their mechanisms to our settings. They require that every student consider all schools acceptable. This assumption is not realistic in our setting.

The rest of this paper is organized as follows. We show a more detailed literature review in the rest of this section and introduce a formal model of our problem setting in Section 2. In Section 3, we describe TTCR and show its deficiency and then introduce our first main mechanism, TTCR-SS, and describe its properties. In Section 4 we describe ACDA and show its deficiency and then introduce our second main mechanism, PLDA-MQ, and show its properties. We evaluate our mechanisms by computer simulation in Section 5. In Section 6 we discuss some relevant issues and provide a conclusion in Section 7. All long proofs are found in the Appendix.

1.1. Related literature

In the context of school choice, distributional constraints are often imposed on different *types* of students, e.g., gender and socioeconomic status [7,11,19,26,29,45]. The crucial difference between our setting and those works that consider minimum quotas is that the minimum quotas in our paper are hard constraints that must be satisfied by any matching, while these works treat minimum quotas as "soft" constraints that may or may not be satisfied.

[5] A similar idea is also used in [12] who considered a model where some schools have the right to refuse to join a centralized school choice program. They showed that when all schools participate in a stable centralized school choice program, there is always a school that has an incentive to leave the program and admit students afterwards. One proposed remedy is called the virtual school mechanism. In their mechanism, a virtual school is used if a school leaves. The virtual school mimics the school that leaves the mechanism and acts as if the school were still in the mechanism. They show that there is a Nash equilibrium in which no schools leave the virtual school mechanism.

[6] Both notions are also used in [17].

[7] Recall that ACDA is not a stable mechanism, since no stable mechanism exists.

[8] See also [13].

Table 1
Properties of mechanisms.

	resp. initial endow.	PL-fairness	fairness	PL-NW	NW	PE
PLDA-MQ	yes	yes	yes	yes	no	no
TTCR-SS	yes	no	no	yes	yes	yes
ESDA	no	no	yes	no	no	no
MSDA	no	no	no	yes	yes	no
TTCR	yes	no	no	no	no	no
ACDA	yes	no	yes	no	no	no

[11] showed that if the constraints are interpreted as hard constraints, no mechanism satisfies fairness, constrained nonwastefulness,[9] and strategy-proofness. Due to this impossibility result, [15] developed two strategy-proof mechanisms. However, their mechanisms cannot simultaneously satisfy fairness and nonwastefulness. One is called the *Extended Seat Deferred Acceptance* (ESDA) mechanism, which is fair but wasteful. The other is called the *Multi-Stage Deferred Acceptance* (MSDA) mechanism, which is nonwasteful but not fair. Based on their work, [17] developed a strategy-proof mechanism that can handle hierarchical minimum quotas. We cannot use these mechanisms in our setting since they do not respect initial endowments.

The computer science community has also been studying the problem of matching with minimum quotas [6,14,22]. These works examined the complexity of verifying the existence of types of *stable* matchings, but they did not provide explicit mechanisms or consider incentive issues. Our approach to this problem resembles those used in [15] and [17], and we study strategy-proof mechanisms that achieve desirable outcomes in polynomial time.

There is other closely related literature that starts from [44] who introduced the housing market problem, which addresses the problem of allocating objects to agents when they are initially owned by agents who have strict preferences over them. In their paper, they introduced TTC due to David Gale and showed that the core is nonempty. [42] showed that, when the preferences of agents are strict, the core is a singleton. For the incentive property, [41] showed that TTC is strategy-proof. [31] argued that a trading mechanism is individual rational, Pareto efficient, and strategy-proof if and only if it is TTC. Later, TTC was generalized to the Hierarchical Exchange mechanism [37] and to the Trading Cycles mechanism [40]. [39] generalized [40] to an environment where each object can have more than one copy[10]. [2], who considered a setting that resembles ours, modified TTC to a setting where some houses are initially owned by tenants, while others are not. The differences between [2] and our work are that we consider a setting where multiple copies of objects (school seats) exist and minimum quotas are imposed.

School choice programs have also been identified as an important application domain of TTC [3]. This work formulated a school choice problem, in which school seats are allocated to students. A school can have multiple seats, and each school can have an idiosyncratic priority among students. They also introduced a modified version of the original TTC that is specific to a school choice problem and show that the mechanism is Pareto efficient and strategy-proof. Since then in the setting of school choice problems, TTC has drawn independent research interest and many research directions. One direction is to design the priority structure of students for a given mechanism [18,25]. Another direction is using an axiomatic approach to characterize it [1,10,32,33].

There are other related works on matching theory in AI literature. For example, [28] studied the house allocation problem, and [4,43], and [24] studied the housing market problem. [21] investigated a dynamic matching problem, and [8] and [38] studied matching problems with couples.

We conclude this section by comparing the properties of the mechanisms that respect minimum quotas. Table 1 summarizes the comparison ("NW" stands for nonwastefulness and "PE" stands for Pareto efficiency). Here, for ESDA and MSDA, we assume all students consider all schools acceptable. Without this assumption, they cannot satisfy minimum quotas.

2. Model

A market is a tuple $(S, C, X, q_C, p_C, \omega, \succ_S, \succ_C)$.

- $S = \{s_1, \dots, s_n\}$ is a finite set of students.
- $C = \{c_1, \dots, c_m\}$ is a finite set of schools.
- $X = S \times C$ is a finite set of contracts. Contract $x = (s, c) \in X$ represents that student s is assigned to school c. For any $X' \subseteq X$, let X'_s denote $\{(s, c) \in X' \mid c \in C\}$, i.e., the sets of contracts related to student s who is involved in X', and let X'_c denote $\{(s, c) \in X' \mid s \in S\}$, i.e., the sets of contracts related to school c involved in X'.
- $q_C = (q_c)_{c \in C}$ is a vector of the schools' maximum quotas.
- $p_C = (p_c)_{c \in C}$ is a vector of the schools' minimum quotas.

[9] This was introduced in [11].
[10] An earlier version of [40] was made available before [39].

- ω: $S \to C$ is an initial endowment function. $\omega(s)$ returns $c \in C$, which is s's initial endowment. When $\omega(s) = c$, we say school c is student s's initial endowment school, and student s is school c's initial endowment student. Let X^* denote $\bigcup_{s \in S}\{(s, \omega(s))\}$, i.e., X^* is the set of contracts, where each element is a contract between a student and her initial endowment school. We assume $\sum_{c \in C} p_c \le n \le \sum_{c \in C} q_c$ holds. Also, we assume X^* satisfies minimum and maximum quotas, i.e., for all $c \in C$, $p_c \le |X^*_c| \le q_c$ holds.
- $\succ_S = (\succ_s)_{s \in S}$ is a profile of the students' preferences. For each student s, $\succ_s$ represents the preference of s over X_s. We assume $\succ_s$ is strict for each s. We say (s, c) is *acceptable* for s if $(s, c) \succ_s (s, \omega(s))$ or $c = \omega(s)$ holds. We sometimes use such notation as $c \succ_s c'$ instead of $(s, c) \succ_s (s, c')$ and write $\succ_{-s} = (\succ_{s'})_{s' \in S \setminus \{s\}}$.
- $\succ_C = (\succ_c)_{c \in C}$ is a profile of the schools' priorities. For each school c, $\succ_c$ represents the priorities of c over X_c. We assume $\succ_c$ is strict for each c. We sometimes write $s \succ_c s'$ instead of $(s, c) \succ_c (s', c)$.

With a slight abuse of notation, for two sets of contracts, X' and X'', we denote $X'_s \succ_s X''_s$ if either (i) $X'_s = \{x'\}$, $X''_s = \{x''\}$, and $x' \succ_s x''$ for some $x', x'' \in X_s$ that are acceptable for s, or (ii) $X'_s = \{x'\}$ for some $x' \in X_s$ that is acceptable for s and $X''_s = \emptyset$. We denote $X'_s \succeq_s X''_s$ if either $X'_s \succ_s X''_s$ or $X'_s = X''_s$. Also, for $X'_s \subseteq X_s$, we say X'_s is acceptable for s if $X'_s = \{x\}$ and x is acceptable for s.[11]

Next, we define the concept of a feasible outcome.

Definition 1 *(feasibility).* $X' \subseteq X$ is *student-feasible* if $\forall s \in S$, X'_s is acceptable for s. X' is *school-feasible* if $\forall c \in C$, $p_c \le |X'_c| \le q_c$ holds. X' is *feasible* if it is both student- and school-feasible.

We call a feasible set of contracts a *matching*. Note that by definition, any matching is *individually rational*, i.e., every student is matched with a school that is at least as good as her initial endowment school. Also note that since X^* is school-feasible, it is a matching.

A *mechanism* is function φ that takes a profile of students' preferences $\succ_S$ as input and returns set of contracts $\varphi(\succ_S) \subseteq X$. We write $\varphi_s(\succ_S)$, which is the set of contracts assigned to student s given submitted preference profile $\succ_S$. A mechanism is called feasible if it always returns a matching.

In the following, we describe several properties that are widely discussed in the literature. One important property regards students' incentives when they submit preferences.

Definition 2 *(strategy-proofness).* We say mechanism φ is *strategy-proof* if $\varphi_s(\succ_s, \succ_{-s}) \succeq_s \varphi_s(\succ'_s, \succ_{-s})$ holds $\forall$ s, $\succ_s$, $\succ'_s$, and $\succ_{-s}$.

In words, a mechanism is strategy-proof if no student ever has any incentive to misreport her preference, regardless of the reports of other students. Another important concept concerns students' welfare.

Definition 3 *(Pareto efficiency).* Matching X' *Pareto dominates* another matching X'' if $\forall s \in S$, $X'_s \succeq_s X''_s$ and $\exists s \in S$, $X'_s \succ_s X''_s$ hold, i.e., compared with X'', X' makes all students weakly better off and at least one student strictly better off. A matching is *Pareto efficient* if no other matching Pareto dominates it. A mechanism is Pareto efficient if it always selects a Pareto efficient matching.

In other words, a policymaker cannot make a student better off without making another student worse off in a Pareto efficient matching. Pareto efficiency may not be the only property with which a policymaker is concerned. One might think that the priorities of students should be respected. This concept is formalized by the following definition.

Definition 4 *(fairness).* We say student s has justified envy toward $s' \neq s$ in matching X', where $(s, c) \in X'$, $(s', c') \in X' \setminus X^*$, and $(s, c') \in X \setminus X'$, if $(s, c') \succ_s (s, c)$ and $(s, c') \succ_{c'} (s', c')$ hold. Matching X' is *fair* if no student has justified envy. A mechanism is fair if it always gives a fair matching.

In words, student s, who is allocated to c, has justified envy toward student s' who is allocated to c' if s prefers c' over c, s has a higher priority than s' in c', and c' is not the initial endowment of s'. Finally, we introduce a weaker welfare notion than Pareto efficiency. Basically, it says that empty seats should not be wasted.

Definition 5 *(nonwastefulness).* We say student s claims an empty seat of c' in matching X', where $(s, c) \in X'$ and $(s, c') \in X \setminus X'$, if $(s, c') \succ_s (s, c)$, $|X'_{c'}| < q_{c'}$, and $|X'_c| > p_c$ hold. Matching X' is *nonwasteful* if no student claims an empty seat. A mechanism is nonwasteful if it always gives a nonwasteful matching.

[11] With this definition, student s cannot accept more than one contract, since X'_s must be a singleton to be acceptable.

The above definition says that student s, who is allocated to c, claims an empty seat of c' if s prefers c' over c and the set of contracts obtained by moving s from c to c' is school-feasible. If X' is Pareto efficient, then X' is nonwasteful, but not vice versa.

We say a matching is *stable* if it is fair and nonwasteful. A mechanism is stable if it always gives a stable matching.

3. Top trading cycles among representatives with supplementary seats

In this section, we study TTC-based mechanisms. We begin by describing a straightforward extension of TTC and show its deficiency. Then we introduce our first main mechanism and describe its properties.

3.1. Top trading cycles among representatives

Let us introduce the Top Trading Cycles among Representatives (TTCR). Since a student is indifferent between multiple seats within the same school, we cannot directly apply the standard TTC mechanism. TTCR, which exploits ML to treat such indifferences in a simple manner, is a special case of Algorithm III in [25].

Before we formally introduce TTCR, let us introduce some notions that will be used in its description. *Directed graph* G is a pair (V, E) where V is a set of vertices and $E \subseteq \{(i, j) \mid i, j \in V\}$ is a collection of ordered pairs of vertices in V. Ordered pair (i, j), where $i, j \in V$, is called a *directed edge* from i to j. A sequence of vertices $(i_1, \dots, i_k)$, $k \geq 2$, is a *directed path* from vertex i_1 to vertex i_k if $(i_h, i_{h+1}) \in E$ for $h = 1, \dots, k-1$. If $i_1 = i_k$, then we call this directed path a *cycle*. In particular, (i, i), where $(i, i) \in E$, is called a *self-loop* cycle.

$\succ_{ML}$ ranks all the students in the market. One example is using the scores of nation-wide exams. ML can be exogenously given and therefore it might be completely independent from the schools' priorities. Without loss of generality, in the rest of the paper we assume $\succ_{ML}$ such that $s_1 \succ_{ML} s_2 \succ_{ML} \cdots \succ_{ML} s_n$. Note that, given a market, any ordering on S can be ML. In Subsection 6.2, we discuss how the choice of such a list affects the outcome of the introduced mechanism.

This mechanism repeats several rounds. At Round k, Y^{k-1} represents the set of remaining initial endowment contracts and Z^{k-1} represents the set of contracts that has already been finalized. TTCR is defined in Mechanism 1.

Mechanism 1 Top Trading Cycles among Representatives (TTCR).

Initialize $Y^0 = X^*$, $Z^0 = \emptyset$, $k = 1$
Round k
Step 1 Create directed graph $G^k = (V^k, E^k)$ as follows:

- V^k is a set of contracts, each of which is selected from each school. More specifically, for each school $c \in C$ s.t. $Y_c^{k-1} \neq \emptyset$, select (s, c) where s has the highest priority among students in Y_c^{k-1} according to ML.
- E^k is the set of directed edges between contracts in V^k. There exists a directed edge $((s, c), (s', c')) \in E^k$ if c' is the most preferred school according to $\succ_s$ within the schools in V^k.

Step 2 Let $\mathscr{C}^k$ denote a set of contracts, each of which is included in a cycle within G^k.
Step 3 For each contract $(s, c) \in \mathscr{C}^k$, let $((s, c), (s', c'))$ denote the direct edge from (s, c). Add (s, c') to Z^k. $Y^k \leftarrow Y^{k-1} \setminus \mathscr{C}^k$.
Step 4 If $Y^k = \emptyset$, then return Z^k. Otherwise, $k \leftarrow k + 1$ and go to the next round.

Intuitively, we can assume in TTCR that each school chooses one representative student from its initial endowment students based on ML. Then within these representative students, the standard TTC mechanism is applied. By choosing one representative for each school, we can ignore the fact that a student is indifferent among multiple seats within the same school. Since a student considers her initial endowment school acceptable, at least one cycle always exists. TTCR can be considered one instance of Algorithm III [25]. In Algorithm III, each school has its own priority ordering among students. Student s, who has the highest priority in school c's ordering, obtains all the seats of c. Then the standard TTC mechanism is applied among the students who own seats. When a student is involved in a cycle and obtains her desired seat, she returns the remaining seats to each school. Then the mechanism repeats the same procedure for the remaining students. If we assume the number of seats available for a school equals the number of its initial endowment students, and school c gives the highest priority to student s according to ML within her initial endowment students, Algorithm III becomes identical to TTCR.

The obtained matching of TTCR satisfies all the minimum and maximum quotas, since the number of students matched to a school are identical in the initial matching and in the resulting matching. However, this mechanism is not Pareto efficient, as shown in the following example:

Example 1. Assume $S = \{s_1, s_2, s_3, s_4, s_5, s_6, s_7\}$, $C = \{c_1, c_2, c_3\}$, where $\omega(s_1) = \omega(s_2) = \omega(s_3) = c_1$, $\omega(s_4) = \omega(s_5) = \omega(s_6) = c_2$, and $\omega(s_7) = c_3$. $q_c = 3$ for all $c \in C$. $p_{c_1} = 2$ and $p_{c_2} = p_{c_3} = 0$.

The preferences of students are given as follows:

$$\begin{aligned} \succ_{s_1}:&\quad c_2 \quad c_1,\\ \succ_{s_2},\succ_{s_3}:&\quad c_3 \quad c_1,\\ \succ_{s_4},\succ_{s_5},\succ_{s_6}:&\quad c_3 \quad c_2,\\ \succ_{s_7}:&\quad c_1 \quad c_3. \end{aligned}$$

First, Y^0 is determined: $\{(s_1,c_1),(s_2,c_1),(s_3,c_1),(s_4,c_2),(s_5,c_2),(s_6,c_2),(s_7,c_3)\}$.

At Step 1 of Round 1, since $Y^0_c \neq \emptyset$ for all $c \in C$, the mechanism selects each (s,c) where s has the highest priority according to ML within the students in Y^0_c for all $c \in C$ and adds (s_1,c_1), (s_4,c_2), and (s_7,c_3) to V^1. Then each selected student points to her most preferred school according to $\succ_s$ within the schools in V^1; s_1, s_4, and s_7 point to c_2, c_3, and c_1, respectively. Therefore, G^1 is determined as follows:

$$\begin{aligned} V^1 &= \{(s_1,c_1),(s_4,c_2),(s_7,c_3)\},\\ E^1 &= \{((s_1,c_1),(s_4,c_2)),((s_4,c_2),(s_7,c_3)),((s_7,c_3),(s_1,c_1))\}. \end{aligned}$$

There exists one cycle: $((s_1,c_1),(s_4,c_2),(s_7,c_3),(s_1,c_1))$. At Step 2, $\mathscr{C}^1$ is $\{(s_1,c_1),(s_4,c_2),(s_7,c_3)\}$. At Step 3, (s_1,c_2), (s_4,c_3), and (s_7,c_1) are added to Z^0, and the contracts in $\mathscr{C}^1$ are removed from Y^0. Z^1 and Y^1 are determined as follows:

$$\begin{aligned} Z^1 &= \{(s_1,c_2),(s_4,c_3),(s_7,c_1)\},\\ Y^1 &= \{(s_2,c_1),(s_3,c_1),(s_5,c_2),(s_6,c_2)\}. \end{aligned}$$

At Step 4, go to Round 2 because $Y^1 \neq \emptyset$.

At Step 1 of Round 2, since $Y^1_{c_3} = \emptyset$, there is no representative student from c_3. The mechanism selects (s_2,c_1) and (s_5,c_2) according to ML and adds them to V^2. Then each selected student points to her most preferred school according to $\succ_s$ within the schools in V^2. Therefore, G^2 is determined as follows:

$$\begin{aligned} V^2 &= \{(s_2,c_1),(s_5,c_2)\},\\ E^2 &= \{((s_2,c_1),(s_2,c_1)),((s_5,c_2),(s_5,c_2))\}. \end{aligned}$$

There are two self-loop cycles. At Step 2, $\mathscr{C}^2$ is $\{(s_2,c_1),(s_5,c_2)\}$. Therefore, at Step 3, Z^2 and Y^2 are given as follows:

$$\begin{aligned} Z^2 &= \{(s_1,c_2),(s_4,c_3),(s_7,c_1),(s_2,c_1),(s_5,c_2)\},\\ Y^2 &= \{(s_3,c_1),(s_6,c_2)\}. \end{aligned}$$

At Step 4, go to Round 3 because $Y^2 \neq \emptyset$.

At Step 1 of Round 3, G^3 is determined as follows:

$$\begin{aligned} V^3 &= \{(s_3,c_1),(s_6,c_2)\},\\ E^3 &= \{((s_3,c_1),(s_3,c_1)),((s_6,c_2),(s_6,c_2))\}. \end{aligned}$$

There are two self-loop cycles. At Step 2, $\mathscr{C}^2$ is $\{(s_3,c_1),(s_6,c_2)\}$. Therefore, at Step 3, Z^3 and Y^3 are given as follows:

$$\begin{aligned} Z^3 &= \{(s_1,c_2),(s_4,c_3),(s_7,c_1),(s_2,c_1),(s_5,c_2),(s_3,c_1),(s_6,c_2)\},\\ Y^3 &= \emptyset. \end{aligned}$$

At Step 4, return Z^3 because $Y^3 = \emptyset$.

In the end, obtained matching Z^3 is:

$$\begin{aligned} c_1:&\quad s_2 \quad s_3 \quad s_7,\\ c_2:&\quad s_1 \quad s_5 \quad s_6,\\ c_3:&\quad s_4. \end{aligned}$$

Consider another matching Z':

$$\begin{aligned} c_1:&\quad s_3 \quad s_7,\\ c_2:&\quad s_1 \quad s_6,\\ c_3:&\quad s_2 \quad s_4 \quad s_5. \end{aligned}$$

We find that $Z'_s \succeq_s Z^3_s$ for all $s \in S$ and $Z'_s \succ_s Z^3_s$ for $s \in \{s_2,s_5\}$ hold. Therefore, Z^3 is not Pareto efficient. Also, $|Z^3_c| = |X^*_c|$ for all $c \in C$.

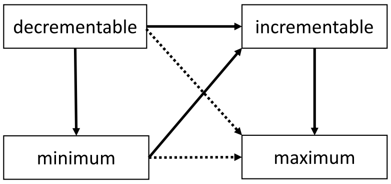


Fig. 1. Transition of school categories.

TTCR's limitation is that it cannot allocate supplementary seats, as shown in Example 1. However, if we allocate supplementary seats too generously, constraints on minimum quotas can be violated. In the following, we develop a new Pareto efficient mechanism, which is called the Top Trading Cycles among Representatives with Supplementary Seats (TTCR-SS), that utilizes the notion of dummy students to control the supplementary seats at each school. In TTCR-SS, if a school has already "consumed" its initial endowment students and has supplementary seats, it selects a dummy student as its representative.

3.2. Top trading cycles among representatives with supplementary seats

TTCR-SS repeats several rounds like TTCR. At Round k, Y^{k-1} represents the set of remaining initial endowment contracts and Z^{k-1} represents the set of contracts that has already been finalized. We divide each school c at Round k into the following four categories:

minimum: $|Y_c^{k-1}| > 0$ and $|Z_c^{k-1}| + |Y_c^{k-1}| = p_c$, i.e., c has the remaining initial endowment contracts and the total number of students in the finalized contracts and the initial endowment contracts equals the minimum quota. Thus, a student in its initial endowment contracts cannot move to another school without violating the constraint on minimum quotas.

decrementable: $|Y_c^{k-1}| > 0$ and $|Z_c^{k-1}| + |Y_c^{k-1}| > p_c$, i.e., c has the remaining initial endowment contracts and a student in its initial endowment contracts can move to another school.

maximum: $|Y_c^{k-1}| = 0$ and $|Z_c^{k-1}| = q_c$, i.e., c has no remaining initial endowment contracts and it has already accepted students up to its maximum quota.

incrementable: $|Y_c^{k-1}| = 0$ and $|Z_c^{k-1}| < q_c$, i.e., c has no remaining initial endowment contract and can accept another student without violating its maximum quota constraint.

Let C^k_{min}, C^k_{dec}, C^k_{max}, and C^k_{inc} represent the sets of schools in each of the above categories at Round k.

TTCR-SS resembles TTCR, but if school c has exhausted its initial endowment students (i.e., $Y_c^{k-1} = \emptyset$ holds) while it has a supplementary seat (i.e., $|Z_c^{k-1}| < q_c$), it is incrementable and can send dummy student s_d as its representative. If a dummy student points to (s, c) and obtains c's seat, in reality, the number of assigned students in c is decremented by one. To ensure that the obtained matching respects the minimum quotas, we carefully design the "preference" of each dummy student. If $|Y_c^{k-1}| + |Z_c^{k-1}| = p_c$ holds for school c, i.e., if c is minimum, then c cannot afford to "accept" a dummy student. Thus each dummy student points to the contract, in which the student has the highest priority among students whose initial endowment schools are decrementable. Note that all dummy students point to the same contract. Thus, at most one cycle exists that includes a dummy student. TTCR-SS is defined in Mechanism 2.

Mechanism 2 Top Trading Cycles among Representatives with Supplementary Seats (TTCR-SS).

Initialize $Y^0 = X^*, Z^0 = \emptyset, k = 1$
Round k
Step 1 Create directed graph $G^k = (V^k, E^k)$ as follows:

- V^k is a set of contracts, each of which is selected from each school. More specifically, for each school $c \in C^k_{\text{min}} \cup C^k_{\text{dec}}$, select (s, c) where s has the highest priority among students in Y_c^{k-1} according to ML. Also, for each school $c \in C^k_{\text{inc}}$, select (s_d, c), where s_d is a dummy student, as long as $C^k_{\text{dec}} \neq \emptyset$.
- E^k is the set of directed edges among contracts. There exists a directed edge $((s, c), (s', c')) \in E^k$ if c' is the most preferred school according to $\succ_s$ within schools in V^k. For each contract related to a dummy student (s_d, c), there exists a directed edge $((s_d, c), (s, c')) \in E^k$, where s has the highest priority according to ML among the students in V^k and $c' \in C^k_{\text{dec}}$.

Step 2 Let $\mathscr{C}^k$ denote a set of contracts, each of which is included in a cycle within G^k.
Step 3 For each contract $(s, c) \in \mathscr{C}^k$, let $((s, c), (s', c'))$ denote the direct edge from (s, c). Add (s, c') to Z^k when s is not a dummy student. $Y^k \leftarrow Y^{k-1} \setminus \mathscr{C}^k$.
Step 4 If $Y^k = \emptyset$, then return Z^k. Otherwise, $k \leftarrow k + 1$ and go to the next round.

Fig. 1 shows the possible transitions of the school categories. Typically, school c is initially decrementable. If $|X_c^*| = p_c$, c is initially minimum. If $|X_c^*| = 0$, c is initially incrementable. As long as all the schools are decrementable or minimum, no dummy student is introduced. Thus, for each contract in a cycle, the related school is either decrementable or minimum. Then at some Round k, Y_c^k eventually becomes $\emptyset$ for some school c. c typically becomes incrementable, and a dummy student is introduced. After a dummy student is introduced, for each contract in a cycle, the related school can be incrementable, decrementable, or minimum, and a student whose initial endowment school is decrementable can obtain a seat of an incrementable school from a dummy student. A decrementable school can become minimum, and an incrementable school can become maximum. As a special case (represented by a dotted line in Fig. 1), if for school c, the number of initial endowment students exactly equals q_c, and no student whose initial endowment is c gives a seat to a dummy student, then c becomes maximum when Y_c^k becomes $\emptyset$. As another special case (represented by a dotted line in Fig. 1), if $p_c = q_c$ holds for school c, then c is initially minimum and directly moves to maximum when Y_c^k becomes $\emptyset$. The assignment of maximum schools becomes fixed. When no decrementable school exists, no dummy student is introduced. Thus, for each contract in a cycle, the related school is minimum. Once this happens, there will be no decrementable school in the later rounds. Thus, no dummy student will be introduced at any later round.

Let us describe how TTCR-SS works.

Example 2. Consider the same instance as in Example 1. Y^0 is the same as Example 1.

The mechanism behaves exactly the same as the previous example until a dummy student is introduced. The following is the result of Round 1:

$$Z^1 = \{(s_1, c_2), (s_4, c_3), (s_7, c_1)\},$$
$$Y^1 = \{(s_2, c_1), (s_3, c_1), (s_5, c_2), (s_6, c_2)\}.$$

At Step 1 of Round 2, c_1 and c_2 are decrementable, and c_3 is incrementable. Schools c_1 and c_2 select their representative students s_2 and s_5, and (s_2, c_1) and (s_5, c_2) are added to V^2. Since there are decrementable schools, c_3 sends a dummy student and (s_d, c_3) is added to V^2. Then each selected student points to her most preferred school according to $\succ_s$ among the schools in V^2; s_2 and s_5 indicate c_3. On the other hand, dummy student s_d points to the school whose initial endowment student has the highest priority according to ML within C^2_{dec}; s_d of c_3 points to c_1. Therefore, G^2 is given as follows:

$$V^2 = \{(s_2, c_1), (s_5, c_2), (s_d, c_3)\},$$
$$E^2 = \{((s_2, c_1), (s_d, c_3)), ((s_5, c_2), (s_d, c_3)), ((s_d, c_3), (s_2, c_1))\}.$$

There is one cycle $((s_2, c_1), (s_d, c_3), (s_2, c_1))$. At Step 2, $\mathscr{C}^2$ is $\{(s_2, c_1), (s_d, c_3)\}$. Therefore, at Step 3, Z^2 and Y^2 are given as follows:

$$Z^2 = \{(s_1, c_2), (s_4, c_3), (s_7, c_1), (s_2, c_3)\},$$
$$Y^2 = \{(s_3, c_1), (s_5, c_2), (s_6, c_2)\}.$$

At Step 1 of Round 3, c_1 is minimum, c_2 is decrementable, and c_3 is incrementable. Thus the mechanism adds (s_3, c_1), (s_5, c_2), and (s_d, c_3) to V^3. Then s_3 and s_5 point to c_3. Here, although s_3 has higher priority than s_5 according to ML, since s_3's initial endowment school c_1 is minimum, s_d points to c_2 instead of c_1. Therefore, G^3 is given as follows:

$$V^3 = \{(s_3, c_1), (s_5, c_2), (s_d, c_3)\},$$
$$E^3 = \{((s_3, c_1), (s_d, c_3)), ((s_5, c_2), (s_d, c_3)), ((s_d, c_3), (s_5, c_2))\}.$$

There is one cycle $((s_5, c_2), (s_d, c_3), (s_5, c_2))$. At Step 2, $\mathscr{C}^3$ is $\{(s_5, c_2), (s_d, c_3)\}$. Z^3 and Y^3 are given as follows:

$$Z^3 = \{(s_1, c_2), (s_4, c_3), (s_7, c_1), (s_2, c_3), (s_5, c_3)\},$$
$$Y^3 = \{(s_3, c_1), (s_6, c_2)\}.$$

At Step 1 of Round 4, c_1 is minimum, c_2 is decrementable, and c_3 is maximum. Then the mechanism adds (s_3, c_1) and (s_6, c_2) to V^4. Since c_3 is maximum, it cannot send a representative. Thus no dummy student is added. Therefore, G^4 is given as follows:

$$V^4 = \{(s_3, c_1), (s_6, c_2)\},$$
$$E^4 = \{((s_3, c_1), (s_3, c_1)), ((s_6, c_2), (s_6, c_2))\}.$$

There are two self-loop cycles. At Step 2, $\mathscr{C}^4$ is $\{(s_3, c_1), (s_6, c_2)\}$. Z^4 and Y^4 are determined as follows:

$$Z^4 = \{(s_1, c_2), (s_4, c_3), (s_7, c_1), (s_2, c_3), (s_5, c_3), (s_3, c_1), (s_6, c_2)\},$$
$$Y^4 = \emptyset.$$

At Step 4, since $Y^4 = \emptyset$, the mechanism returns Z^4.

The obtained matching is identical to Z' in Example 1.

Next, we describe the theoretical properties of TTCR-SS.

Theorem 1. *TTCR-SS is feasible, strategy-proof, and Pareto efficient.*

Proof. See Appendix A.1. □

The fact that TTCR-SS is Pareto efficient, while TTCR is not, does not imply that all students weakly prefer the matching of TTCR-SS over that of TTCR. This is because there can be multiple Pareto efficient matchings. Let us show a simple example. Assume two students, s_1 and s_2, and three schools, c_1, c_2, and c_3. The minimum quota of c_1 is 1, and the minimum quotas of c_2 and c_3 are 0. The maximum quotas of all the schools are 1. The initial endowment schools of s_1 and s_2 are c_1 and c_2. The preference of s_1 is $c_2 \succ_{s_1} c_1$, and the preference of s_2 is $c_3 \succ_{s_2} c_1 \succ_{s_2} c_2$. In TTCR-SS, s_d of c_3 and s_2 swap seats, and s_1 cannot move to c_2. Thus the obtained matching is $\{(s_1, c_1), (s_2, c_3)\}$. On the other hand, in TTCR, no dummy student is introduced and s_1 and s_2 swap seats. The obtained matching is $\{(s_1, c_2), (s_2, c_1)\}$. Here s_1 prefers the TTCR matching. In the next section, we experimentally show that the overwhelming majority of students prefer the matching obtained by TTCR-SS.

Finally, we show that TTCR-SS can be done in polynomial time in $|S|$ and $|C|$.

Theorem 2. *The time complexity of TTCR-SS is $O(|S| \cdot |C|)$.*

Proof. At each round, there exists at least one cycle that contains at least one student $s \in S$, and the assignment of s is fixed. Thus the number of rounds required for TTCR-SS is at most $|S|$. Also, for each round, there are at most $|C|$ contracts, and finding the cycles can be done in $O(|C|)$. Therefore, the time complexity of TTCR-SS is $O(|S| \cdot |C|)$. □

4. Priority list-based deferred acceptance with minimum quotas

In this section, we study DA-based mechanisms. We first describe a simple extension of DA and show its deficiency. Then we introduce our second main mechanism and show its properties.

4.1. Artificial Cap DA

In this subsection, we introduce the Artificial Cap Deferred Acceptance mechanism (ACDA), which is a simple extension of DA that can handle minimum and maximum quotas. The idea of ACDA is used in the Japan Residency Matching Program. It works by reducing the maximum quotas of hospitals in such urban areas as Tokyo so that more doctors are assigned to rural hospitals [23].

For each school c, original maximum quota q_c is decreased to $|X_c^*|$, i.e., the number of its initial endowment students. Also, original school priority $\succ_c$ is modified so that each of its initial endowment contracts has a higher priority than any contract in $X_c \setminus X_c^*$. Then the mechanism performs the standard DA procedure described as follows, which repeats the following rounds.

Mechanism 3 Artificial Cap Deferred Acceptance (ACDA).

Round $k \geq 1$

A student applies to her most preferred school from which she has not been rejected so far. Then each school tentatively accepts students applying to it up to its maximum quotas. The rest of the students are rejected. If no student is rejected, then return the current assignment as a final matching. Otherwise, $k \leftarrow k+1$ and go to the next round.

In the following, we show that ACDA is not a stable mechanism.

4.2. PL-stability

We show that the standard notion of stability is not appropriate in our setting because a stable outcome might not exist. This impossibility result is shown in the following example.[12]

Example 3. Assume $S = \{s_1, s_2\}, C = \{c_1, c_2, c_3\}$, where $\omega(s_1) = \omega(s_2) = c_1$. $q_c = 2$ for all $c \in C$. $p_{c_1} = 1$, $p_{c_2} = p_{c_3} = 0$. The priorities of schools are given as follows:

$$\begin{aligned} \succ_{c_1}&: \quad s_1 \quad s_2, \\ \succ_{c_2}&: \quad s_2 \quad s_1, \\ \succ_{c_3}&: \quad s_1 \quad s_2. \end{aligned}$$

[12] This example is based on one used in the proof of Theorem 1 in [11].

We assume the preferences of students are given as follows:

$\succ_{s_1}$: c_2 c_3 c_1,
$\succ_{s_2}$: c_3 c_2 c_1.

Here c_1 is the least popular school for both s_1 and s_2, but at least one student must be assigned to it since $p_{c_1} = 1$. Assume s_1 is allocated to c_1. Then s_2 must be allocated to her most preferred school c_3, or otherwise s_2 claims an empty seat of c_3. However, then s_1 has justified envy toward s_2 since $s_1 \succ_{c_3} s_2$. Similarly, assume s_2 is allocated to c_1. Then s_1 must be allocated to her most preferred school c_2, or otherwise s_1 claims an empty seat of c_2. However, then s_2 has justified envy toward s_1 since $s_2 \succ_{c_2} s_1$.

This example also shows that there exists no stable mechanism. Therefore ACDA is not a stable mechanism. We use an alternative concept of stability, Priority List-based (PL-) stability. PL-stability consists of PL-fairness and PL-nonwastefulness.[13] PL-stability relies on priority list, $\succ_{PL}$, that ranks all of the contracts. Such a list breaks ties between students or contracts when necessary.

We only make two assumptions about PL. First, we assume $\succ_{PL}$ respects $\succ_C$, i.e., for any $(s, c), (s', c) \in X \setminus X^*$, $(s, c) \succ_{PL} (s', c)$ holds if $(s, c) \succ_c (s', c)$ holds. Second, we assume $\succ_{PL}$ respects the initial endowments,[14] i.e., for each $x \in X^*$ and $x' \in X \setminus X^*$, it holds that $x \succ_{PL} x'$.

The way we construct PL gives much flexibility to policymakers, since we only make two assumptions for it. One feature of PL is that in addition to the fact that it respects students' priorities in schools, it also prioritizes the rights of schools for receiving students. One application of such prioritization is in cadet-branch matching where policymakers want to prioritize the rights of individual branches for accepting personnel.

One simple way of creating $\succ_{PL}$ is based on the ranking of students and the tie-breaking order among schools. Let $rank(c, s)$ denote the ranking of student s for school c based on $\succ_c$, i.e., if s is ranked highest for c among students except c's initial endowment students, $rank(c, s) = 1$, and if she is ranked second, $rank(c, s) = 2$, and so on. If s is c's initial endowment student, we assume $rank(c, s) = 0$. Also, assume the tie-breaking order among schools is defined as $c_1, c_2, \ldots, c_m$. Then $(s, c_i) \succ_{PL} (s', c_j)$ holds when $rank(c_i, s) < rank(c_j, s')$, or $rank(c_i, s) = rank(c_j, s')$ and $i < j$ hold.

In the following we formally define PL-fairness and PL-nonwastefulness.

Definition 6 *(PL-fairness).* We say student s has justified envy toward $s' \neq s$ in matching X' based on PL, where $(s, c), (s', c') \in X'$ and $(s, c'') \in X \setminus X'$, if $(s, c'') \succ_s (s, c)$, $|X'_{c''}| < q_{c''}$, $|X'_{c'}| > p_{c'}$, and $(s, c'') \succ_{PL} (s', c')$ hold. Matching X' is *PL-fair* if no student has justified envy or justified envy based on PL. A mechanism is PL-fair if it always gives a PL-fair matching.

In words, if student s is assigned to c even though she hopes to be assigned to c'', which can accept one more student, while another student s' is assigned to c' even though it has already satisfied its minimum quota, then s has justified envy toward s' if the tie-breaking rule supports this, i.e., $(s, c'') \succ_{PL} (s', c')$ holds. This condition is stronger than what is required by the standard fairness, since a student may justifiably envy another student whose matched school is not exactly where she wants to go.

Intuitively, PL-fairness requires that if we need to reject a contract to satisfy the minimum quota of a school, and when there exist several candidate contracts to reject, the mechanism should fairly reject one based on PL, i.e., the contract with the lowest priority according to PL.

Definition 7 *(PL-nonwastefulness).* Student s claims an empty seat of c' in matching X' based on PL, where $(s, c) \in X'$ and $(s, c') \in X \setminus X'$, if $(s, c') \succ_s (s, c)$, $|X'_{c'}| < q_{c'}$, $|X'_c| > p_c$ and $(s, c') \succ_{PL} (s, c)$ hold. Matching X' is *PL-nonwasteful* if no student claims an empty seat based on PL. A mechanism is PL-nonwasteful if it always gives a PL-nonwasteful matching.

This definition weakens standard nonwastefulness. Here, the claim of student s who is assigned to c to obtain an empty seat of c' is legitimate only if the tie-breaking rule supports this, i.e., $(s, c') \succ_{PL} (s, c)$.

Recall that a matching is PL-stable if it is PL-fair and PL-nonwasteful. We say a mechanism is PL-stable if it always gives a PL-stable matching. Note that a stable matching might not be PL-stable and vice versa.

Consider the situation of Example 3, where all the schools prefer s_2 over s_1, and $\succ_{PL}$ is given as follows (which respects the schools' priorities):

$$\succ_{PL}: \quad (s_1, c_1) \quad (s_2, c_1) \quad (s_2, c_2) \quad (s_2, c_3) \quad (s_1, c_2) \quad (s_1, c_3).$$

Since $p_{c_1} = 1$, at least one student must be assigned to c_1 even though c_1 is the least popular school for both students. A mechanism needs to decide which contract should be rejected, e.g., among $\{(s_1, c_3), (s_2, c_2)\}$. Here since both schools

[13] PL-fairness and PL-nonwastefulness are used in [17] for handling regional quotas.

[14] This assumption becomes redundant if one assumes that each school prioritizes its initial endowment students.

c_2 and c_3 unanimously prefer s_2 over s_1, it is natural to assume that (s_1, c_3) is rejected. Indeed, PL-fairness requires the mechanism to reject (s_1, c_3), which has the lowest priority according to PL (which respects the schools' priorities).

Note that from these definitions, the initial endowment matching X^* is PL-stable. Since a student cannot have justified envy toward another student allocated to her initial endowment, X^* is fair. Also, for any $x \in X^*$ and $x' \in X \setminus X^*$, $x \succ_{PL} x'$ holds. Then a student cannot have justified envy based on PL or claim an empty seat based on PL. Thus X^* is PL-stable. Therefore, there always exists at least one PL-stable matching.

In the next example we show that ACDA does not satisfy PL-stability.

Example 4. Assume $S = \{s_1, s_2, s_3, s_4\}$, $C = \{c_1, c_2, c_3\}$, where $\omega(s_1) = c_1$, $\omega(s_2) = \omega(s_3) = c_2$, $\omega(s_4) = c_3$. $q_c = 4$ for all $c \in C$. $p_{c_1} = p_{c_2} = 0$, $p_{c_3} = 1$. The priorities of the schools and the preferences of students are given as follows:

$$\begin{aligned}
&\succ_{c_1}: \; s_1 \; s_2 \; s_4 \; s_3,\\
&\succ_{c_2}: \; s_2 \; s_3 \; s_4 \; s_1,\\
&\succ_{c_3}: \; s_4 \; s_1.
\end{aligned}$$

$$\begin{aligned}
&\succ_{s_1}: \; c_2 \; c_3 \; c_1,\\
&\succ_{s_2}: \; c_1 \; c_2,\\
&\succ_{s_3}: \; c_1 \; c_2,\\
&\succ_{s_4}: \; c_1 \; c_2 \; c_3.
\end{aligned}$$

We assume $\succ_{PL}$ is given as follows:

$$\begin{aligned}
\succ_{PL}: \; &(s_1, c_1) \; (s_2, c_2) \; (s_3, c_2) \; (s_4, c_3) \; (s_1, c_3)\\
&(s_2, c_1) \; (s_4, c_1) \; (s_4, c_2) \; (s_1, c_2) \; (s_3, c_1).
\end{aligned}$$

Now, assume q_1, q_2, and q_3 are artificially reduced to 1, 2, and 1, respectively.

In Round 1, each student applies to her most preferred school, i.e., s_2, s_3, s_4 apply to c_1, and s_1 applies to c_2. Since three students are applying to c_1, which is larger than the (artificially decreased) maximum quota $q_{c_1} = 1$, c_1 deferred-accepts s_2, who has the highest priority according to $\succ_{c_1}$, and rejects s_3 and s_4. c_2 deferred-accepts s_1.

In Round 2, s_3 and s_4, who were rejected in the previous step, apply to c_2. Thus, three students in total are applying to c_2, which is larger than the (artificially decreased) maximum quota $q_{c_2} = 2$. Then c_2 deferred-accepts s_3 and s_4 and rejects s_1.

In Round 3, s_1 applies to c_3. Then all students are deferred-accepted and the mechanism terminates.

In the end, the obtained matching becomes the following:

$$\begin{aligned}
&c_1: \; s_2,\\
&c_2: \; s_3 \; s_4,\\
&c_3: \; s_1.
\end{aligned}$$

In this matching, s_4 claims an empty seat of c_1 based on PL since moving s_4 from c_2 to c_1 is possible and $(s_4, c_1) \succ_{PL} (s_4, c_2)$ holds. If $\succ_{PL}$ is defined such that $(s_1, c_2) \succ_{PL} (s_2, c_1)$ holds, then s_1 has justified envy toward s_2 based on PL. Thus, ACDA is neither PL-fair nor PL-nonwasteful.

4.3. Priority-list based DA with minimum quotas

In this subsection, we introduce the Priority-List based Deferred Acceptance mechanism with Minimum Quotas (PLDA-MQ) and describe its properties. We utilize a general framework for developing a strategy-proof mechanism with various distributional constraints recently developed by [27]. Their framework exploits *choice functions* for students (Ch_S) and schools (Ch_C), defined as follows:

Definition 8 *(students' choice function).* For each student s, her choice function Ch_s is defined as follows. Given $X' \subseteq X$, let Y_s denote that $\{x \in X'_s \mid x$ is acceptable for $s\}$. $Ch_s(X')$ returns $\{x\}$, s.t. $x \in Y_s$ and x is the most preferred contract in Y_s for s. If $Y_s = \emptyset$, then $Ch_s(X')$ returns $\emptyset$. Then the choice function of all students Ch_S is defined as $Ch_S(X') := \bigcup_{s \in S} Ch_s(X')$.

Definition 9 *(schools' choice function).* Given $X' \subseteq X$, the choice function of the schools returns set of contracts $Ch_C(X')$, defined as follows:

$$Ch_C(X') := \arg\max_{X'' \subseteq X'} f(X'').$$

Here $f : 2^X \to \mathbb{R} \cup \{-\infty\}$ is an evaluation function that aggregates the schools' priorities and distributional constraints. We sometime say f represents the priorities of schools. We assume f is unique-selecting, i.e., for all $X' \subseteq X$, there exists a unique $X'' \subseteq X'$ that maximizes $f(X'')$.

Based on these choice functions, GDA is defined as follows:

Mechanism 4 Generalized Deferred Acceptance (GDA).

Initialize $X' = \emptyset$, $X'' = \emptyset$, $Re = \emptyset$, $k = 1$
Round $k \geq 1$

Step 1 $X' \leftarrow Ch_S(X \setminus Re)$, $X'' \leftarrow Ch_C(X')$.
Step 2 If $X' = X''$, then return X'. Otherwise, $Re \leftarrow Re \cup (X' \setminus X'')$, $k \leftarrow k + 1$, and go to the next round.

Here Re represents a set of contracts proposed by the students and rejected by the schools. Students are not allowed to propose a contract in Re, which is initially empty. They can choose their most preferred contracts and propose them to the schools. This set is X'. Then schools choose the most preferred subset X'' from X'. If no contract is rejected, the mechanism terminates. Otherwise, the rejected contracts are added to Re, and the mechanism repeats the same procedure.

Next we show how to appropriately define f. To exploit previous results by [27], we decompose f into two parts, $\widehat{f}$ and $\widetilde{f}$, such that f is expressed as $f = \widehat{f} + \widetilde{f}$. Roughly speaking, given a market, $\widehat{f}$ is responsible for the distributional constraints and $\widetilde{f}$ takes care of the priorities of the schools.

Definition 10 *(hard constraint).* Given $X' \subseteq X$, $\widehat{f}(X')$ is defined as follows:

$$\widehat{f}(X') := \begin{cases} 0 & \text{if } |X'_c| \leq q_c \forall c \in C \text{ and } \sum_{c \in C} \max(|X'_c|, p_c) \leq n, \\ -\infty & \text{otherwise.} \end{cases}$$

Intuitively, if $\widehat{f}(X') = -\infty$ holds, no $X'' \supseteq X'$ exists such that X'' is feasible. This is because, if count $\sum_{c \in C} \max(|X'_c|, p_c)$ exceeds n, there are not enough remaining students (who are not involved in X' or cannot be accepted in X') to fill the seats of schools whose minimum quotas are not satisfied under X'. Let the effective domain of $\widehat{f}$ (denoted as $\mathrm{dom}\,\widehat{f}$) be defined as $\mathrm{dom}\,\widehat{f} := \{X' \subseteq X \mid \widehat{f}(X') \neq -\infty\}$.

Defining $\widetilde{f}$ is straightforward. Let $v : X \to (0, \infty)$ with $v(\emptyset) = 0$ be a function such that $x \neq x'$ implies $v(x) \neq v(x')$. We can assume that $v(x)$ represents the value of contract x and that v respects PL; we define $v(\cdot)$ based on $\succ_{PL}$ such that $x \succ_{PL} x'$ implies $v(x) > v(x')$. Then given a market, PL, and v that respects PL, $\widetilde{f}$ is defined such that $\widetilde{f}(X') = \sum_{x \in X'} v(x)$ holds for $X' \subseteq X$.

With these definitions of $\widehat{f}$ and $\widetilde{f}$, we call GDA, where $f(X')$ is defined by $\widehat{f}(X') + \widetilde{f}(X')$, the Priority List-based Deferred Acceptance mechanism with Minimum Quotas (PLDA-MQ).

Let us show the execution of PLDA-MQ.

Example 5. We consider the same instance as Example 4. First, each student chooses her most preferred acceptable contract. Thus, $X' = \{(s_1, c_2), (s_2, c_1), (s_3, c_1), (s_4, c_1)\}$. Here $\widehat{f}(X') = -\infty$ since $\sum_{c \in C} \max(|X'_c|, p_c) = 3 + 1 + 1 = 5 > 4$. Then the schools choose $X'' = \{(s_1, c_2), (s_2, c_1), (s_4, c_1)\}$ and $X' \setminus X'' = \{(s_3, c_1)\}$ is rejected, since $v((s_3, c_1))$ has the lowest value, i.e., the lowest priority according to $\succ_{PL}$, within X'.

Next s_3 chooses her second preferred contract (s_3, c_2), while other students choose the same schools as before. $X' = \{(s_1, c_2), (s_2, c_1), (s_3, c_2), (s_4, c_1)\}$. $\widehat{f}(X') = -\infty$ since $\sum_{c \in C} \max(|X'_c|, p_c) = 2 + 2 + 1 = 5 > 4$. Then (s_1, c_2) is rejected since it has the lowest priority according to $\succ_{PL}$ within X'.

Finally, s_1 chooses her second preferred contract (s_1, c_3). Thus $X' = \{(s_1, c_3), (s_2, c_1), (s_3, c_2), (s_4, c_1)\}$. $\widehat{f}(X') = 0$ since $\sum_{c \in C} \max(|X'_c|, p_c) = 2 + 1 + 1 = 4$. Since no contract is rejected, the mechanism terminates.

In the end, the obtained matching becomes the following:

$$\begin{aligned} c_1 &: \; s_2 \quad s_4, \\ c_2 &: \; s_3, \\ c_3 &: \; s_1. \end{aligned}$$

This matching is PL-fair and PL-nonwasteful.

Consider the situation described in Example 3, where $\succ_{PL}$ is given as follows (which respects schools' priorities):

$$\succ_{PL}: \quad (s_1, c_1) \quad (s_2, c_1) \quad (s_2, c_2) \quad (s_1, c_3) \quad (s_1, c_2) \quad (s_2, c_3).$$

The matching obtained by PLDA-MQ is $\{(s_1, c_1), (s_2, c_2)\}$. This is PL-stable; s_2 cannot claim an empty seat of c_3 based on PL since $(s_2, c_2) \succ_{PL} (s_2, c_3)$ holds.

Now we describe the theoretical properties of PLDA-MQ. In general there are multiple PL-stable matchings, including the initial endowment matching X^*. We show that PLDA-MQ selects a particular stable matching. We say a matching is student-optimal if every student weakly prefers her matching to any other matching.

Definition 11 *(student optimality).* For set of matchings $\mathcal{X}$, $X' \in \mathcal{X}$ is *student-optimal* within $\mathcal{X}$ if $X'_s \succeq_s X''_s$ holds $\forall X'' \in \mathcal{X}$, $\forall s \in S$.

It is possible that no student-optimal matching exists in $\mathcal{X}$. If a student-optimal matching does exist in $\mathcal{X}$, it must be unique.[15] Also, if there exists a unique Pareto efficient matching, it is student-optimal within all matchings. The following theorem describes the properties of PLDA-MQ.

Theorem 3. *PLDA-MQ is feasible, strategy-proof, and PL-stable. Moreover, it obtains the student-optimal matching within all the PL-stable matchings.*

Proof. See Appendix A.2. □

Let us briefly discuss the time complexity of PLDA-MQ with the following theorem.

Theorem 4. *The time complexity of PLDA-MQ is $O(|X|^3)$.*

Proof. It is clear that f can be calculated in $O(|X|)$, since both $\widehat{f}$ and $\widetilde{f}$ can be calculated in it. Since the f used in PLDA-MQ is M$^\natural$-concave and Condition 2 of Theorem 5, the time complexity of PLDA-MQ is $O(|X|^3)$. □

5. Evaluation

In this section, we conduct quantitative evaluations by computer simulations. In the previous sections we theoretically showed that TTCR-SS is an efficient mechanism and that PLDA-MQ is a fair mechanism in our setting. What we are mainly concerned with here is TTCR-SS's fairness and PLDA-MQ's efficiency. As benchmarks we use TTCR and ACDA, two simple mechanisms described above. Both are feasible and strategy-proof in our setting. TTCR, which is based on TTC, is a special case of Algorithm III in [25]. ACDA is based on DA, and a modified version of it is used as a benchmark in [15] and [17], for example.

We consider two markets, A and B, which have different sizes. Market A has $n = 720$ students and $m = 36$ schools. We assume the schools and the residences of the students are located in a 6×6 grid space, where the right-end school is connected to the left-end school in each row, and the lower-end school is connected to the upper-end school in each column, i.e., the grid space constitutes a torus.[16] Market B has $n = 1280$ students and $m = 64$ schools, distributed in an 8×8 grid space constituting a torus.

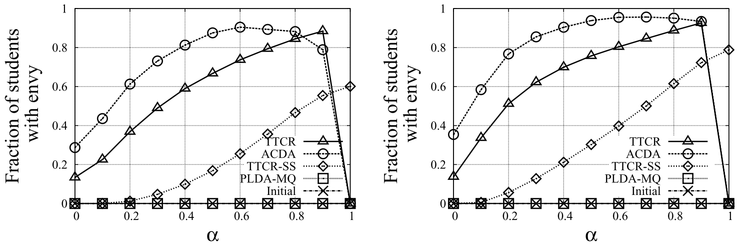
In both markets, we assume that the schools and students are distributed evenly in the grid space and that a student is initially endowed with the school at her location. Therefore, each location has one school and 20 initial endowment students. We set the minimum and maximum quotas of each school to 5 and 60 in both markets. Note that the artificial maximum quota of each school is set to 20, which is the number of initial endowment students of the school.

Regarding student preferences over the schools, we assume a student considers a school unacceptable if it is located too far from her residence. More precisely, a student considers a school unacceptable if the Manhattan distance between her residence and the school exceeds 2 in Market A. In Market B, a student will not consider attending a school whose Manhattan distance exceeds 3 from her location. Each student considers at most 13 schools acceptable in Market A and at most 25 schools acceptable in Market B.

We generate the student preferences over the schools as follows. We first draw one common vector v of the cardinal utilities from set $[0,1]^m$ uniformly at random. We then draw private vector u_s of the cardinal utilities of each student s from the same set, again uniformly at random. Next we construct the cardinal utilities over all m schools for student s as $\alpha v + (1-\alpha)u_s$ for $\alpha \in [0,1]$. Here the i-th element of this vector represents the cardinal utility for school c_i. A student's preference is then determined by converting her cardinal utilities into an ordinal preference relation over the following schools: (i) those located within the above distance from her location, and (ii) those whose cardinal utilities are weakly higher than that of her initial endowment school. Here parameter α reflects the similarity of the student preferences. The higher the value of α, the greater the difference in popularity among schools becomes. In this experiment, we varied α from 0.0 to 1.0 in increments of 0.1.

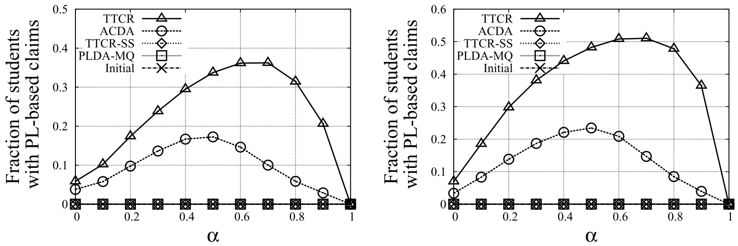
[15] More precisely, assume X' is student-optimal within $\mathcal{X}$. Then for any $X'' \in \mathcal{X} \setminus \{X'\}$ and s, it holds that $X'_s \succeq_s X''_s$. Since X' and X'' are different and students' preferences are strict, there must be at least one student s' such that $X'_{s'} \succ_{s'} X''_{s'}$ holds. Thus, X'' cannot be another student-optimal matching.

[16] We employ this structure to avoid boundary effects. In fact, we also conducted experiments where the grid space was not a torus, and the qualitative trends turned out to be quite similar.



(a) Market A (b) Market B

Fig. 2. Fraction of students with envy.



(a) Market A (b) Market B

Fig. 3. Fraction of students with PL-based claims.

Regarding the schools' priorities over students, we assume here that a school prioritizes students who live closer to it. Among students whose locations are equidistant from a school, we use one common vector u, drawn from set $[0, 1]^n$ uniformly at random, to break ties; if s_i and s_j are equally distant from a school, then the school prioritizes student s_i over s_j if $u_i > u_j$ holds. An ML is generated by converting u into an ordering over students, i.e., for two students, s_i and s_j, $s_i \succ_{ML} s_j$ holds if $u_i > u_j$.

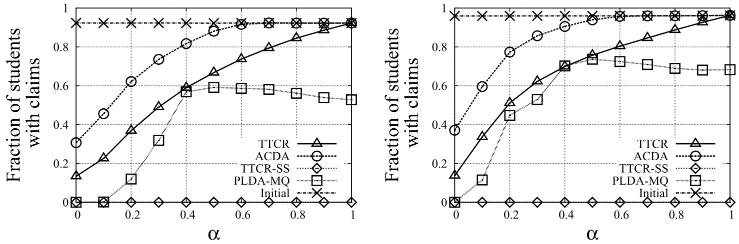
For obtaining a PL, we use the simple method introduced in Subsection 4.2. We first fix a round-robin ordering among schools as $c_1 \to c_2 \to \cdots \to c_m \to c_1 \to c_2 \to \cdots$. Then at each of its turns according to this ordering, school c removes its most prioritized contract from X_c and adds it to the bottom of the current list, starting from empty. Note that the priorities of the schools in our experimental setting guarantee that school c first adds contracts in X_c^* to the list. We obtain a PL by repeating this procedure until all the contracts are added to the list.

We created 100 problem instances for each parameter setting and compared the outcomes of four mechanisms (TTCR, TTCR-SS, ACDA, and PLDA-MQ) and a trivial mechanism that assigns each student to her initial endowment school, i.e., it always returns X^*. We refer to this mechanism as *Initial* in the figures. Figs. 2, 3, and 4 concern the stability of the mechanisms, and Figs. 5 and 6 show the results of the students' welfare.

Fig. 2 shows the average ratios of the students with justified envy or justified envy based on PL. Since PLDA-MQ and the initial endowments are PL-fair, no student has such envy for any α. For the other three mechanisms, the ratio of students with envy increases as α increases, as a result of more conflicts in the student preferences. With TTCR-SS, however, these conflicts are somehow mitigated by exploiting the supplementary seats. We can see that for small α, TTCR-SS efficiently assigns supplementary seats so that no student has envy, and the slope in the figures is much gentler than those of TTCR and ACDA. In Market A, for example, when $\alpha = 0.6$, fewer than 30% of the 720 students have envy in TTCR-SS, while at least 70% are envious under the simple extensions. When $\alpha = 1$, since all the student preferences are basically the same, no trade occurs in TTCR and its outcome is X^*. The ACDA outcome becomes the initial endowments if $\alpha = 1$ because, for any school, there is a round in the mechanism where the school receives applications from all of its initial endowment students. In Market B, the slopes shift upward, meaning conflicts of interest are more difficult to resolve. This is because in Market B, more students consider a school acceptable, and thus a student might have many more students to envy.

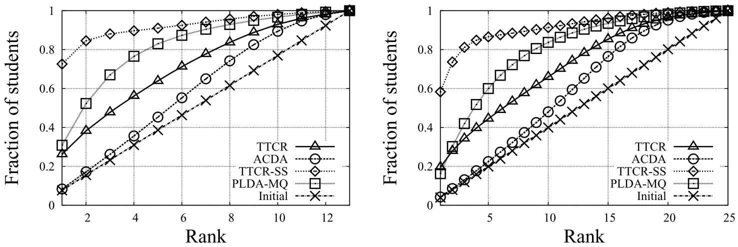
Figs. 3 and 4 show the student claims for empty seats. Fig. 3 shows the ratio of students who claimed empty seats based on PL.

We showed in the previous sections that PLDA-MQ is a PL-stable mechanism and that the TTCR-SS outcome satisfies Pareto efficiency, which is a stronger notion than PL-nonwastefulness. Initial is by definition also PL-nonwasteful. Thus, no student claims an empty seat based on PL under these mechanisms, as shown in the figure. On the other hand, we observe an inverted U-shaped relationship between α and both the TTCR and ACDA ratios. In both mechanisms, if conflicts of interest among the students are very high, then more students are likely to be assigned to their initial endowment schools. Since PL respects the initial endowments, a student's claim is not supported by PL if she is matched to her initial endowment school. As α gets closer to 1, the number of claims based on PL decreases.



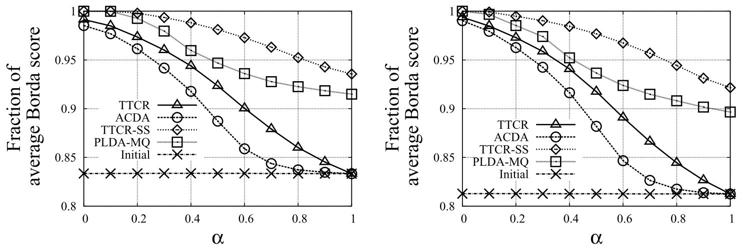
(a) Market A (b) Market B

Fig. 4. Fraction of students with claims.



(a) Market A (b) Market B

Fig. 5. CDFs of students' welfare when $\alpha = 0.6$.



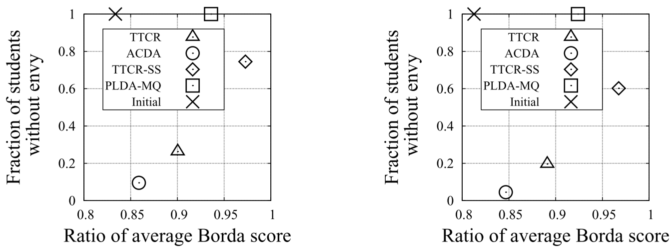
(a) Market A (b) Market B

Fig. 6. Borda scores.

Fig. 4 shows the average ratio of students who claimed empty seats in the sense of Definition 5. The difference between Figs. 3 and 4 reveals the amount of claims supported by PL. Note that Initial gives the highest ratio for any α. Since all the mechanisms are individually rational, the worst outcome in terms of efficiency for students is X^*: the Initial outcome. On the other hand, no student claims an empty seat under TTCR-SS, since it is Pareto efficient. Among the remaining mechanisms, the PLDA-MQ ratio stays the lowest for almost all the cases.[17] Although PLDA-MQ and its properties depend on PL, this figure shows that this mechanism outperforms the simple extensions with respect to the amount of conventional claims. For a large α, the PLDA-MQ ratio begins to decrease. A rationale is that, for a large α, popular schools begin to fill their maximum quotas, and no student can claim their seats any more, implying that the ratio declines.

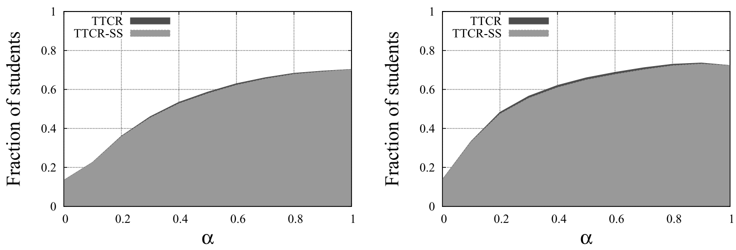
Regarding the efficiency of the mechanisms, Figs. 5 and 6 compare them in terms of students' welfare. Fig. 5 plots the cumulative distribution functions (CDFs) of the average number of students matched with their kth or higher ranked schools under each mechanism when $\alpha = 0.6$. The ordering of the mechanisms in terms of their performance is consistent with the previous figures; TTCR-SS outperforms the other mechanisms in terms of students' welfare, since it is Pareto efficient, and PLDA-MQ, the runner-up, outperforms TTCR and ACDA. Therefore, the developed mechanisms in this paper obtain better matchings than the simple extensions of existing mechanisms in terms of both fairness and efficiency. Fig. 6 compares the outcomes for different α with respect to the Borda scores of the students. If a student is assigned to her kth-choice school, her score is $m - k + 1$; a higher score is more desirable. The figure's y-axis shows the ratio of the average scores to m. This figure supports the result of Fig. 5 and suggests that the ordering of mechanisms in terms of students' welfare is consistent

[17] This ratio is slightly above the TTCR ratio only in the case in Market B with $\alpha = 0.4$.



(a) Market A (b) Market B

Fig. 7. Comparison of efficiency/fairness.



(a) Market A (b) Market B

Fig. 8. Fraction of students who prefer TTCR-SS/TTCR.

over α. By comparing Figs. 6a and 6b, it can also be seen that students' welfare slightly worsens in Market B due to more competition among students.

To illustrate the relationship between efficiency and fairness obtained by these mechanisms, we plot the average points of the obtained matchings in the x–y plane in Fig. 7 (when $\alpha = 0.6$), where the x-axis shows the ratio of the average Borda scores of the students and the y-axis shows the ratio of students without envy or envy based on PL. The point in the upper-right corner is preferable, and the figure shows that the developed mechanisms outperformed the simple extensions of the existing mechanisms in both aspects. This figure displays a trade-off between the mechanisms; TTCR-SS does not maintain fairness but achieves an efficient outcome, while PLDA-MQ maintains fairness to a certain extent by giving up some efficiency. In Market B, the matching outcomes become worse for all the mechanisms due to a more competitive setting in terms of the number of schools a student might consider acceptable.

We conclude this section by comparing the welfare of TTCR-SS and TTCR. Section 3 argues that the outcome of TTCR-SS may not Pareto dominate TTCR, i.e., some students might be worse off by exploiting the supplementary seats. The gray area in Fig. 8 shows the ratio of students who prefer TTCR-SS matching, and the black area shows the ratio of students who prefer TTCR matching. Note that this area is very narrow and resembles a line. The white area shows the ratio of students whose assignments are the same. For example, when $\alpha = 0.6$, only 1% of the students prefer TTCR matching, while more than 60% prefer TTCR-SS matching. These figures clearly show that the fraction of students who are worse off due to TTCR-SS is rather small compared to the overwhelming majority of students who benefit from it.

6. Discussion

In this section we discuss some relevant issues about TTCR-SS and PLDA-MQ. In the first subsection, we discuss some relationships between TTCR-SS and the hierarchical exchange mechanism [37], which generalizes TTC. We show that TTCR-SS, as with the hierarchical exchange mechanism, satisfies the notion of group strategy-proofness. In the second subsection, we discuss the effects of choosing different MLs on the TTCR-SS outcome and different PLs on the PLDA-MQ outcome. We experimentally show that such effects are rather small.

6.1. Relationship between TTCR-SS and hierarchical exchange mechanism

In this subsection, we discuss some relationships between TTCR-SS and the hierarchical exchange mechanism [37], which is a generalization of TTC. As mentioned in the literature review, [37] proposed a class of exchange mechanisms called hierarchical exchange mechanisms, which characterize Pareto efficient, group strategy-proof, and reallocation proof mechanisms, when each object (or school in our context) has only one copy (or seat). Since she does not consider any distributional constraints, her notion of Pareto efficiency and ours are different. In particular, one can construct an example of a Pareto efficient (in the sense of [37]) allocation that does not satisfy minimum quotas and thus is not Pareto efficient according to our notion. On the other hand, one can construct an example of a Pareto efficient allocation that satisfies minimum quota (in the sense of our definition) but is not Pareto efficient among matchings that do not satisfy minimum quotas. Recall from the literature review that [40] generalized the characterization of [37] for environments where each object has one copy, and [39] generalized [40] to environments where each object can have more than one copy. Since the notion of Pareto efficiency in these papers and ours are different, TTCR-SS, which produces a Pareto efficient allocation based on our notion, is not an instance of the mechanisms proposed in [37,40], or [39].

6.1.1. Group strategy-proofness of TTCR-SS

[37] showed that the hierarchical exchange mechanism is group strategy-proof. Given $S' \subseteq S$, let $\succ_{S'} = (\succ_{s'})_{s' \in S'}$ and $\succ_{-S'} = (\succ_{s'})_{s' \in S \setminus S'}$. The notion of group strategy-proofness is described as follows.

Definition 12 *(group strategy-proofness).* We say mechanism φ is *group strategy-proof* if $\forall \succ_S$, there is no $S' \subseteq S$ and $\succ'_{S'}$ such that $\varphi_s(\succ'_{S'}, \succ_{-S'}) \succeq_s \varphi_s(\succ_{S'}, \succ_{-S'})$ $\forall s \in S'$ and for some $s \in S'$, $\varphi_s(\succ'_{S'}, \succ_{-S'}) \succ_s \varphi_s(\succ_{S'}, \succ_{-S'})$.

In other words, a mechanism is group strategy-proof if no set of students ever has any incentive to jointly misreport their preferences, regardless of the reports of the rest of the students.[18]

Note that group strategy-proofness is a stronger notion than strategy-proofness. Although TTCR-SS is different from the hierarchical exchange mechanism, in the following, we show that TTCR-SS is also group strategy-proof by following the previously developed techniques of [37]. First, we introduce the definition of nonbossiness.

Definition 13 *(nonbossiness).* We say mechanism φ is *nonbossy* if $\varphi_s(\succ_s, \succ_{-s}) = \varphi_s(\succ'_s, \succ_{-s})$ implies $\varphi(\succ_s, \succ_{-s}) = \varphi(\succ'_s, \succ_{-s})$, $\forall s$, $\succ_s$, $\succ'_s$, and $\succ_{-s}$.

In words, a mechanism is nonbossy if no student can affect other students' assignments by changing her reporting preference, as long as her own assignment does not change.

Lemma 1. *TTCR-SS is nonbossy.*

Proof. See Appendix A.3. □

From this result and the strategy-proofness of TTCR-SS, it immediately follows (as shown in [37]) that TTCR-SS is group strategy-proof.

Lemma 2 *(Lemma 1 in [37]). Strategy-proofness and nonbossiness imply group strategy-proofness.*

Although [37] assumes that each school has only one seat, one can easily verify from the proof of Lemma 1 in [37] that Lemma 2 holds in our setting, where each object (school) may have more than one copy (seat). Thus we have the following corollary.

Corollary 1. *TTCR-SS is group strategy-proof.*

6.2. Effect of choices of lists

We exploit two kinds of lists for our mechanisms to work: ML for TTCR-SS and PL for PLDA-MQ. While PL is generated to reflect the schools' priorities, the choice of ML can be completely independent from the market. Introducing such a list would be controversial if the obtained matching by a mechanism varied significantly, depending on the chosen list. In this subsection we examine whether our mechanisms are sensitive to the choices of lists. We focus on the number of assigned students to schools, which should reflect their popularity. Thus it is desirable that such numbers do not heavily depend on the choice of a list.

6.2.1. Effect of ML choices

To see the effect of the choice of ML, we fixed one problem instance of Market A in Section 5 and ran TTCR-SS for 100 randomly generated different MLs. Fig. 9 shows the differences of the number of assigned students under the cases of $\alpha = 0$, 0.6, and 1. We show the average, minimum, and maximum numbers of allocated students for each school. The x-axis represents the schools that are sorted in decreasing order of their average number of assigned students. When $\alpha = 0$, the

[18] More precisely, there are two different versions of group strategy-proofness whose definitions differ in terms of whether a student in the group that misreports their preferences is better off. The stronger version allows some students in the group to be indifferent between truth-telling and misreporting, while the weaker version requires that every student in the group strictly prefers the outcome obtained by misreporting. In this paper, we use the stronger version. Regarding the weaker one in the context of the school choice problem, see for example, [9].

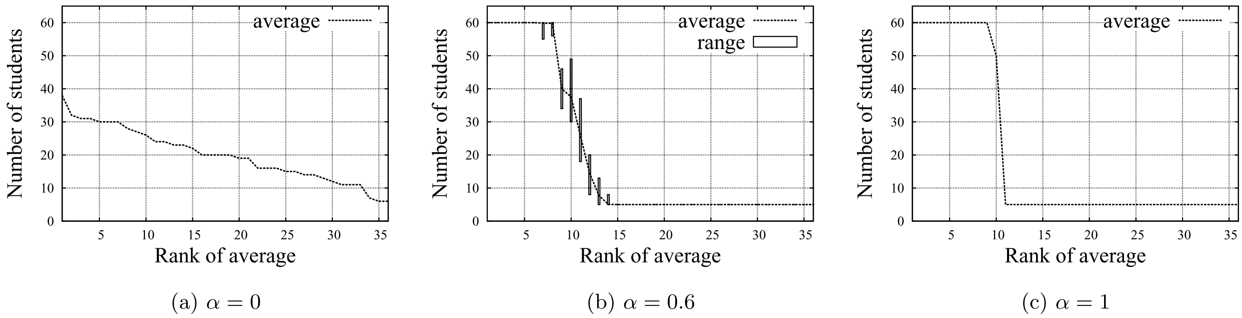


Fig. 9. Differences in number of assigned students for different MLs under TTCR-SS.

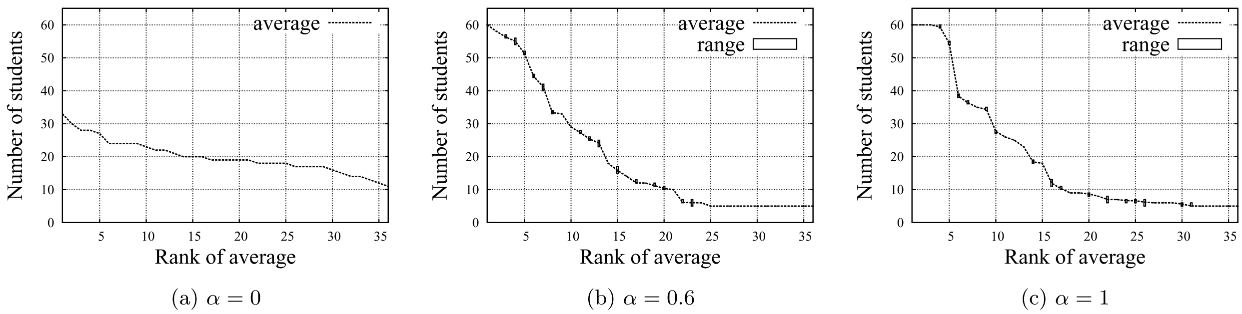


Fig. 10. Differences in number of assigned students for different PLs under PLDA-MQ.

preferences of the students are independent from each other and there is virtually no competition. Then the choice of ML did not affect the outcome very much; the average, minimum, and maximum numbers are almost the same (Fig. 9a). When $\alpha = 1$, the preferences of the students are the same and they are all competing for the seats of the same popular schools. Thus the choice of ML affects who will be assigned to the popular schools, but it does not affect the number of students assigned to them. The average, minimum, and maximum numbers are identical (Fig. 9c). When $\alpha = 0.6$, the number of allocated students can vary by the choice of ML. In this case, however, the numbers of students allocated to either popular or unpopular schools are almost the same under different MLs, and it varies in the schools that are neither popular nor unpopular (Fig. 9b). Thus we conjecture that the choice of ML is not very controversial; the choice only slightly affects the popularity of schools.

6.2.2. Effect of PL choices

We also ran PLDA-MQ under different PLs. We fixed one problem instance of Market A in Section 5 and ran PLDA-MQ for 100 different PLs as follows. First we randomly generated 100 different orderings among schools. For each ordering σ, we fixed the round-robin ordering among the schools as $c_{\sigma(1)} \to c_{\sigma(2)} \to \cdots \to c_{\sigma(m)} \to c_{\sigma(1)} \to c_{\sigma(2)} \to \cdots$ and created a PL based on it, as in Section 5. Fig. 10 shows the differences of the number of assigned students under the cases of $\alpha = 0$, 0.6, and 1. When $\alpha = 0$, each school has enough students who consider it their favorite to satisfy its minimum quota. In this case, schools can just accept their prioritized students without referring to PL. Since there is no correlation in the student preferences, every student is assigned to her favorite school; recall Fig. 6a with $\alpha = 0$. Using a different PL does not even change the outcome. As α increases, the preferences of students are more correlated, resulting in fluctuations for some schools (Figs. 9b and 9c). However, the difference in the numbers in a school under different PLs is at most 2. This shows the robustness of PLDA-MQ over different PLs in this respect, even in comparison with TTCR-SS over MLs. Thus we experimentally show evidence that the choice of PL has only a limited effect on the mechanism's outcome in terms of the number of filled seats of each school.

Note that the choice of PL only affects the PLDA-MQ process when the mechanism needs to break ties among contracts of the same rank, e.g., when either one of the two contracts, (s_1, c_1) and (s_2, c_2) where $rank(s_1, c_1) = rank(s_2, c_2)$, must be rejected. Intuitively, such ties are more likely to occur in the later rounds of the process, and at that time the choice of PL, or rather the choice of σ, begins to matter. On the other hand, ML affects the TTCR-SS process from the beginning to the end, which results in more fluctuations among schools that are neither extremely popular nor unpopular.

7. Conclusion

In this paper, we develop two strategy-proof mechanisms, TTCR-SS and PLDA-MQ, for a school choice program where the obtained matching must respect minimum quotas and initial endowments. TTCR-SS is Pareto efficient, while PLDA-MQ is PL-stable and obtains the student-optimal matching within all PL-stable matchings. Our simulation results show that our new mechanisms significantly outperform simple extensions of existing mechanisms.

Our immediate future work will extend our mechanisms to handle different types of distributional constraints besides minimum quotas [27].

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Appendix A. Proofs

A.1. Proof of Theorem 1

We separately show the feasibility, strategy-proofness, and Pareto efficiency of TTCR-SS.

A.1.1. Proof of feasibility for TTCR-SS

Proof. The outcome is clearly student-feasible since a student never selects a contract that is related to her unacceptable school.

As for the school-feasibility of the outcome, we show by induction that $\{Y^k \cup Z^k\}$ is school-feasible for any k. For $k = 0$, $\{Y^0 \cup Z^0\}$ is clearly school-feasible because $Y^0 = X^*$ is school-feasible and $Z^0 = \emptyset$. Suppose $\{Y^k \cup Z^k\}$ is school-feasible for some k. The induction is completed if we show that for any $c \in C$ it holds that $p_c \le |Y_c^{k+1}| + |Z_c^{k+1}| \le q_c$. If a contract related to c is not included in $\mathscr{C}^{k+1}$, the assignment related to c never changes and from the induction argument $p_c \le |Y_c^{k+1}| + |Z_c^{k+1}| \le q_c$ holds. Next, assume $(s, c) \in \mathscr{C}^{k+1}$ for some s. It is clear that $c \notin C_{\text{max}}^{k+1}$. If $c \in C_{\text{inc}}^{k+1}$ is the case, it holds that $Y_c^k = Y_c^{k+1} = \emptyset$ and $p_c \le |Z_c^k| < q_c$. Furthermore, $s = s_d$. The fact $(s_d, c) \in \mathscr{C}^{k+1}$ implies $|Z_c^{k+1}| = |Z_c^k| + 1$, and therefore $p_c \le |Y_c^{k+1}| + |Z_c^{k+1}| = |Z_c^{k+1}| \le q_c$ holds. If $c \in C_{\text{dec}}^{k+1}$ is the case, it follows that $p_c < |Y_c^k| + |Z_c^k| \le q_c$ with $Y_c^k \ne \emptyset$. The fact $(s, c) \in \mathscr{C}^{k+1}$ implies $|Y_c^{k+1}| = |Y_c^k| - 1$. It holds either $Z_c^{k+1} = Z_c^k$ or $|Z_c^{k+1}| = |Z_c^k| + 1$, depending on who (a dummy or a non-dummy student) obtains the seat of c in $\mathscr{C}^{k+1}$. In either case, $p_c \le |Y_c^{k+1}| + |Z_c^{k+1}| \le q_c$ holds. Finally, if $c \in C_{\text{min}}^{k+1}$ is the case, it holds that $p_c = |Y_c^k| + |Z_c^k| \le q_c$ with $Y_c^k \ne \emptyset$. The fact $(s, c) \in \mathscr{C}^{k+1}$ implies $|Y_c^{k+1}| = |Y_c^k| - 1$. It also implies that $|Z_c^{k+1}| = |Z_c^k| + 1$, since a dummy student never obtains a seat of c in $\mathscr{C}^{k+1}$. Thus $p_c = |Y_c^{k+1}| + |Z_c^k| \le q_c$ holds. □

A.1.2. Proof of strategy-proofness for TTCR-SS

We need several lemmas to prove strategy-proofness for TTCR-SS.

School c is *available* at Round k if either (i) $C_{\text{dec}}^k \ne \emptyset$ and $c \in C \setminus C_{\text{max}}^k$ or (ii) $C_{\text{dec}}^k = \emptyset$ and $c \in C_{\text{min}}^k$ hold. Let C_{ava}^k denote the set of all available schools. At Round k, a contract related to school c is clearly included in V^k if and only if c is available at Round k.

It is obvious that the following lemma holds from the category transition of the schools and the definition of C_{ava}^k.

Lemma 3. *For any two Rounds k and k' with $k < k'$, $C_{dec}^k \supseteq C_{dec}^{k'}$ and $C_{ava}^k \supseteq C_{ava}^{k'}$ hold.*

Intuitively, this lemma means that the possible choices for a student weakly monotonically shrink in the later rounds. As a result, the following lemma argues that a student's choice is the best within all the schools that are available in the later rounds.

Lemma 4. *Suppose TTCR-SS obtains X'. For any k and $c \in C_{ava}^k$, and any student s who is included in $\mathscr{C}^{k'}$, i.e., a cycle at Round $k' \le k$, $X'_s \succeq_s \{(s, c)\}$ holds.*

Proof. From Lemma 3, $C_{\text{ava}}^k \subseteq C_{\text{ava}}^{k'}$ holds. Also, the fact that s is included in $\mathscr{C}^{k'}$ means that $\{(s, c')\} = X'_s$ and c' is the most preferred school for s within $C_{\text{ava}}^{k'}$. Thus $X'_s \succeq_s \{(s, c)\}$ holds. □

The following lemma implies if there exists a directed path toward a contract at some round, it remains in the later rounds unless the contract is removed because it is included in a cycle. As a consequence, if a student is assigned to a school by her manipulative report, then she can be assigned to the school when it becomes her best available school during the mechanism.

Lemma 5. *Suppose there is a directed path from contract (s, c) to (s', c') in G^k, and suppose $(s', c') \in V^{k'}$ for some $k' > k$. Then exactly the same directed path from (s, c) to (s', c') exists in $G^{k'}$.*

Proof. It is sufficient to show that $((s, c), (s', c')) \in E^k$ and $(s', c') \in V^{k+1}$ imply $((s, c), (s', c')) \in E^{k+1}$, since a directed path is a sequence of directed edges. First, suppose $s \neq s_d$. From Lemma 3, $C^{k+1}_{\text{ava}} \subseteq C^k_{\text{ava}}$ holds. Since c' is the most preferred school for s within C^k_{ava}, if $c' \in C^{k+1}_{\text{ava}}$, c' remains the most preferred school for s within C^{k+1}_{ava}. Thus $((s, c), (s', c')) \in E^{k+1}$ holds. Second, suppose $s = s_d$. The fact that $((s_d, c), (s', c')) \in E^k$ implies that c' is decrementable at Round k and that s' has the highest priority in ML among all the remaining students in all the decrementable schools. According to Lemma 3, the set of decrementable schools never expands. As long as (s', c') remains in V^{k+1}, c' remains decrementable and s' still has the highest priority in ML among all the remaining students in all the decrementable schools at Round $k + 1$. Thus $((s, c), (s', c')) \in E^{k+1}$ holds. □

The following lemma means that the declared preference of a student does not affect the outcome of the rounds before she is included in a cycle.

Lemma 6. *Fix the reported preferences of all students except s at $\succ_{-s}$. Suppose that $(s, \omega(s)) \in \mathscr{C}^k$ if she reports $\succ_s$ and $(s, \omega(s)) \in \mathscr{C}^{k'}$ if she reports $\succ'_s$, where $k \leq k'$. Then C^k_* does not change, where "$*$" can be either "max," "min," "inc," "dec," or "ava," regardless of whether student s reports $\succ_s$ or $\succ'_s$.*

Proof. Since $(s, \omega(s)) \notin \mathscr{C}^{\widehat{k}}$ holds for any $\widehat{k} < k$, the same contracts form cycles before Round k whether student s reports $\succ_s$ or $\succ'_s$. Then the remaining contracts in Y^{k-1} are identical in both cases. Also, the same sets of contracts are added to Z in both cases. This implies that C^k_* does not change. □

Now, we are ready to prove that TTCR-SS is strategy-proof.

Proof of strategy-proofness for TTCR-SS

Proof. Fix the reported preferences of all the students except s at $\succ_{-s}$ and denote $\succ = (\succ_s, \succ_{-s})$ and $\succ' = (\succ'_s, \succ_{-s})$, where $\succ_s$ is her true preference and $\succ'_s$ is fake. For Round k, explicitly write $V^k(\succ)$, $G^k(\succ)$, $E^k(\succ)$, and $\mathscr{C}^k(\succ)$ to denote V^k, G^k, E^k, and $\mathscr{C}^k$ when the reported preference profile is $\succ$ and so on. Explicitly write $C^k_*(\succ)$ to denote C^k_* when the reported preference profile is $\succ$ and so on. Suppose that $(s, \omega(s)) \in \mathscr{C}^k(\succ)$, i.e., if s reports her true preference $\succ_s$, she belongs to a cycle at Round k, and $(s, \omega(s)) \in \mathscr{C}^{k'}(\succ')$, i.e., if she reports some other preference $\succ'_s$, she belongs to a cycle at Round k'.

First, assume $k \leq k'$. Since $(s, \omega(s)) \in \mathscr{C}^k(\succ)$, s must be matched with her most preferred school within $C^k_{\text{ava}}(\succ)$. Also, $(s, \omega(s)) \in \mathscr{C}^{k'}(\succ')$ means that s is matched with a school within $C^{k'}_{\text{ava}}(\succ')$. Therefore, it is sufficient to show that $C^{k'}_{\text{ava}}(\succ') \subseteq C^k_{\text{ava}}(\succ)$ holds. Since $k \leq k'$, it follows from Lemma 6 that $C^k_{\text{ava}}(\succ') = C^k_{\text{ava}}(\succ)$ holds. Also, from Lemma 3, it follows that $C^{k'}_{\text{ava}}(\succ') \subseteq C^k_{\text{ava}}(\succ')$. Combining these results, we have $C^{k'}_{\text{ava}}(\succ') \subseteq C^k_{\text{ava}}(\succ)$.

Next assume $k > k'$. Since $(s, \omega(s)) \in \mathscr{C}^{k'}(\succ')$, there exists a directed path from (s', c') to $(s, \omega(s))$ in $G^{k'}(\succ')$, where $((s, \omega(s)), (s', c')) \in E^{k'}(\succ')$. From Lemma 6, $C^{k'}_{\text{ava}}(\succ') = C^{k'}_{\text{ava}}(\succ)$ holds. Thus, there exists the same directed path from (s', c') to $(s, \omega(s))$ in $G^{k'}(\succ)$. The fact that $(s, \omega(s)) \in \mathscr{C}^k(\succ)$ implies $(s, \omega(s)) \in V^k(\succ)$, and thus from Lemma 5 there exists the same directed path from (s', c') to $(s, \omega(s))$ in $G^k(\succ)$. Then s's assignment under $\succ$ is at least as good as c', which is the assignment under $\succ'$. Thus s cannot be better off by reporting $\succ'_s$. □

A.1.3. Proof of Pareto efficiency for TTCR-SS

Proof. We show, by induction, that the following statement is true when we run TTCR-SS: A student, who is matched at Round r, cannot be better off without making a student, who is matched before r, worse off.

When $r = 1$, the statement is trivially true because a matched student at Round 1 is assigned to her top choice.

Assume the supposition is true up to $r = k$. Consider $r = k + 1$. Take student s who is matched to a school c at Round $k + 1$, and assume her assignment is not her top choice. If no such s exists, then every student goes to her top choice, and thus the statement is true.

From Lemma 4, for any school c' such that $c' \succ_s c$, it holds that $c' \notin C^{k+1}_{\text{ava}}$. Then either (i) $c' \in C^{k+1}_{\text{max}}$ or (ii) $c' \in C^{k+1}_{\text{inc}}$ and $C^{k+1}_{\text{dec}} = \emptyset$ holds.

If (i) $c' \in C^{k+1}_{\text{max}}$ is the case, then c' is already matched to students up to its maximum quota before Round $k+1$. Improving s, by assigning her to c', therefore necessarily forces a student who is matched before Round $k+1$ to leave c' to make room for s. However, because of the induction argument, it is not possible to improve the departing student without making someone worse off.

Suppose (ii) $c' \in C^{k+1}_{\text{inc}}$ and $C^{k+1}_{\text{dec}} = \emptyset$ is the case. Then at Round $k + 1$, the number of seats that must be filled to satisfy the minimum quotas equals the number of students who have not been matched yet. Therefore, improving student s, by assigning her a school that is not available at $k + 1$, necessarily forces a student who was matched before $k + 1$ to change her assignment to satisfy the minimum quotas. From the induction argument, however, such a change will make someone worse off. □

A.2. Proof of Theorem 3

We first introduce an equilibrium concept based on choice functions proposed by [20].

Definition 14 *(HM-stability).* $X' \subseteq X$ is *Hatfield–Milgrom (HM-) stable* if $X' = Ch_S(X') = Ch_C(X')$ and there exists no $x \in X \setminus X'$ such that $x \in Ch_S(X' \cup \{x\})$ and $x \in Ch_C(X' \cup \{x\})$ hold.

Note that if Ch_S and Ch_C are the ones for PLDA-MQ defined in Subsection 4.3, then it always holds that $X' = Ch_S(X') = Ch_C(X')$ whenever X' is a matching. Next we introduce some mathematical tools used in the proofs.

Definition 15 *($M^\natural$-concavity).* We say that f is *$M^\natural$-concave* if for all $Y, Z \subseteq X$ and $y \in Y \setminus Z$, there exists $z \in (Z \setminus Y) \cup \{\emptyset\}$ such that $f(Y) + f(Z) \leq f((Y \setminus \{y\}) \cup \{z\}) + f((Z \setminus \{z\}) \cup \{y\})$ holds.

$M^\natural$-concavity, introduced in [34], is a discrete analogue of concavity. We also exploit the notion of matroid, see for example, [36].

Definition 16 *(matroid).* Let X be a finite set and let $\mathcal{F}$ be a family of the subsets of X. Pair $(X, \mathcal{F})$ is a *matroid* if it satisfies the following conditions:

1. $\emptyset \in \mathcal{F}$,
2. If $X' \in \mathcal{F}$ and $X'' \subset X'$, then $X'' \in \mathcal{F}$ holds,
3. If $X', X'' \in \mathcal{F}$ and $|X'| > |X''|$, then there exists $x \in X' \setminus X''$ such that $X'' \cup \{x\} \in \mathcal{F}$.

Next we present the existing results in [27], which connects the aforementioned concepts. Theorem 5, which is identical to Theorem 1 in their paper, clarifies the conditions so that GDA satisfies several desirable properties.

Theorem 5 *(Theorem 1 in [27]). Suppose that the preference of the schools can be represented by $M^\natural$-concave function f. Then*

1. *GDA is strategy-proof and the obtained matching is student-optimal in all of the HM-stable matchings.*
2. *The time complexity of GDA is $O(T(f) \cdot |X|^2)$, assuming f can be calculated in $T(f)$ time.*

Furthermore, they show a sufficient condition where f becomes $M^\natural$-concave.

Theorem 6 *(Theorem 3 in [27]). Suppose $f = \widehat{f} + \widetilde{f}$, where (i) $\widehat{f}$ returns $-\infty$ if X' violates the hard constraint and otherwise it returns 0, and (ii) $\widetilde{f}(X') = \sum_{x \in X'} v(x)$, where $v : X \to (0, \infty)$ is a function such that $x \neq x'$ implies $v(x) \neq v(x')$. Then f is $M^\natural$-concave if $(X, \mathrm{dom}\,\widehat{f})$ is a matroid.*

Note that function f, defined in Subsection 4.3, satisfies conditions (i) and (ii). Let $\widehat{f}$ be the function proposed in Definition 10. The following lemma shows that we can use these results in PLDA-MQ.

Lemma 7. *$(X, \mathrm{dom}\,\widehat{f})$ is a matroid.*

Proof. It is clear that $\widehat{f}(\emptyset) = 0$, and thus $\emptyset \in \mathrm{dom}\,\widehat{f}$ holds. Also, if $\widehat{f}(X') = 0$, then for any $X'' \subset X'$, $\widehat{f}(X'') = 0$ holds since $\max(|X''_c|, p_c) \leq \max(|X'_c|, p_c)$ holds for all $c \in C$ when $X'' \subset X'$ holds.

Finally, we show that for any $X', X'' \in \mathrm{dom}\,\widehat{f}$, where $|X'| > |X''|$, there exists $x \in X' \setminus X''$ such that $X'' \cup \{x\} \in \mathrm{dom}\,\widehat{f}$ holds. Let $\tilde{C}$ denote $\{c \in C \mid |X''_c| < p_c\}$. If there exists $c \in \tilde{C}$ with $|X'_c| > |X''_c|$, we can choose any $(s, c) \in X'_c \setminus X''_c$ such that $X'' \cup \{(s, c)\} \in \mathrm{dom}\,\widehat{f}$ holds, since $|X''_c| < p_c$. Thus assume for all $c \in \tilde{C}$, $|X'_c| \leq |X''_c|$ holds.[19] Since $|X'| > |X''|$ and $|X'_c| \leq |X''_c|$ hold for all $c \in \tilde{C}$, $\sum_{c \in C \setminus \tilde{C}} |X'_c| > \sum_{c \in C \setminus \tilde{C}} |X''_c|$ holds. Also, there exists $c' \in C \setminus \tilde{C}$ such that $|X'_{c'}| > |X''_{c'}|$ holds. Thus we can choose $(s, c') \in X'_{c'} \setminus X''_{c'}$. Let Y denote $X'' \cup \{(s, c')\}$. Note that $\sum_{c \in C} \max(|X''_c|, p_c) = \sum_{c \in C \setminus \tilde{C}} |X''_c| + \sum_{c \in \tilde{C}} p_c < \sum_{c \in C \setminus \tilde{C}} |X'_c| + \sum_{c \in \tilde{C}} p_c \leq n$ holds, and therefore $\sum_{c \in C} \max(|Y_c|, p_c) = \sum_{c \in C} \max(|X''_c|, p_c) + 1 \leq n$ holds. Thus $\widehat{f}(Y) = 0$ and $Y \in \mathrm{dom}\,\widehat{f}$ hold. $\square$

The following lemmas show a relationship between HM-stability and PL-stability.

Lemma 8. *$Ch_C(X')$ is equivalent to the following procedure:*

[19] Note that $\tilde{C}$ can be an empty set.

1. *$Y \leftarrow \emptyset$.*
2. *Remove (s,c) from X' such that $v((s,c))$ is largest in X'. If there exists no such contract, terminate the procedure and return Y.*
3. *If $\widehat{f}(Y \cup \{(s,c)\}) = 0$, then $Y \leftarrow Y \cup \{(s,c)\}$. Go to 2.*

Proof. Since $(X, \mathrm{dom}\ \widehat{f})$ is a matroid, the above greedy procedure, which selects a contract from X' one by one according to v, is guaranteed to obtain $\arg\max_{X'' \subseteq X'} f(X'')$, i.e., the optimal solution [36]. □

Lemma 9. *Student s is never rejected by her initial endowment school $\omega(s)$, i.e., whenever $(s,\omega(s)) \in X'$, $(s,\omega(s)) \in Ch_C(X')$ holds.*

Proof. If $(s,\omega(s)) \in X'$, since $\succ_{PL}$ respects the initial endowments, when $(s,\omega(s))$ is selected in the procedure of Lemma 8, $Y \cup \{(s,\omega(s))\} \subseteq X^*$ holds. Since $\widehat{f}(X^*) = 0$, $\widehat{f}(Y \cup \{(s,\omega(s))\}) = 0$ holds. Thus $(s,\omega(s))$ is included in $Ch_C(X')$. □

Lemma 10. *Matching X' is HM-stable iff it is PL-stable.*

Proof. We first show that HM-stability implies PL-stability. First, assume for contradiction, X' is HM-stable but there exists a student who has justified envy; there exist $(s,c) \in X'$, $(s',c') \in X' \setminus X^*$, and $(s,c') \in X \setminus X'$ such that $(s,c') \succ_s (s,c)$ and $(s,c') \succ_{c'} (s',c')$ hold. Then it is clear that $(s,c') \in Ch_S(X' \cup \{(s,c')\})$ holds. Also, in the procedure of Lemma 8, when calculating $Ch_C(X' \cup \{(s,c')\})$, (s,c') is selected before (s',c') since $\omega(s) \neq c', \omega(s') \neq c'$, and $\succ_{PL}$ respects $\succ_C$. In the calculation of $Ch_C(X')$, when (s',c') is selected, $\widehat{f}(Y \cup \{(s',c')\}) = 0$ holds since $X' = Ch_C(X')$. In the calculation of $Ch_C(X' \cup \{(s,c')\})$, when (s,c') is selected, $\widehat{f}(Y \cup \{(s,c')\}) = 0$ also holds. Then $(s,c') \in Ch_C(X' \cup \{(s,c')\})$ holds; X' is not HM-stable.

Second, assume X' is HM-stable but there exists a student who has justified envy based on PL, i.e., there exist $(s,c),(s',c') \in X'$ and $(s,c'') \in X \setminus X'$ such that $(s,c'') \succ_s (s,c)$, $|X'_{c''}| < q_{c''}$, $|X'_{c'}| > p_{c'}$ and $(s,c'') \succ_{PL} (s',c')$ hold. It is clear that $(s,c'') \in Ch_S(X' \cup \{(s,c'')\})$ holds. In the procedure of Lemma 8, when calculating $Ch_C(X' \cup \{(s,c'')\})$, (s,c'') is selected before (s',c'). In the calculation of $Ch_C(X')$, when (s',c') is selected, $\widehat{f}(Y \cup \{(s',c')\}) = 0$ holds since $X' = Ch_C(X')$. In the calculation of $Ch_C(X' \cup \{(s,c'')\})$, when (s,c'') is selected, $\widehat{f}(Y \cup \{(s,c'')\}) = 0$ also holds since $|X'_{c''}| < q_{c''}$ and $|X'_{c'}| > p_{c'}$ hold. Then $(s,c'') \in Ch_C(X' \cup \{(s,c'')\})$ holds; X' is not HM-stable.

Third, assume X' is HM-stable but not PL-nonwasteful, i.e., there exist $(s,c) \in X'$ and $(s,c') \in X \setminus X'$ such that $(s,c') \succ_s (s,c)$, $|X'_{c'}| < q_{c'}$, $|X'_c| > p_c$, and $(s,c') \succ_{PL} (s,c)$ hold. It is clear that $(s,c') \in Ch_S(X' \cup \{(s,c')\})$ holds. Also, in the procedure of Lemma 8, when calculating $Ch_C(X' \cup \{(s,c')\})$, (s,c') is selected before (s,c). In the calculation of $Ch_C(X')$, when (s,c) is selected, $\widehat{f}(Y \cup \{(s,c)\}) = 0$ holds since $X' = Ch_C(X')$. In the calculation of $Ch_C(X' \cup \{(s,c')\})$, when (s,c') is selected, $\widehat{f}(Y \cup \{(s,c')\}) = 0$ also holds since $|X'_{c'}| < q_{c'}$ and $|X'_c| > p_c$ hold. Then $(s,c') \in Ch_C(X' \cup \{(s,c')\})$ holds; X' is not HM-stable.

Next we show that PL-stability implies HM-stability. Assume X' is not HM-stable, i.e., there exists contract $(s,c') \in X \setminus X'$ such that $(s,c') \in Ch_S(X' \cup \{(s,c')\})$ and $(s,c') \in Ch_C(X' \cup \{(s,c')\})$ hold. It is clear that $(s,c') \succ_s (s,c)$ holds for some $(s,c) \in X'$. Since $|X'| = n$, there exists at least one contract $(s'',c'') \in Ch_C(X')$ and $(s'',c'') \notin Ch_C(X' \cup \{(s,c')\})$. (s'',c'') is rejected as a consequence of accepting (s,c'). Since (s'',c'') is rejected, $\omega(s'') \neq c''$ holds from Lemma 9. Also, it is clear that $(s,c') \succ_{PL} (s'',c'')$ holds. First, assume $s'' \neq s$ and $c'' = c'$ hold. Then $(s,c') \succ_{PL} (s'',c')$ and $(s,c') \succ_{c'} (s'',c')$ hold since $\omega(s) \neq c'$, $\omega(s'') \neq c'$, and $\succ_{PL}$ respects $\succ_C$. Therefore, s has justified envy toward s''. Second, assume $c'' \neq c'$ holds. Then $|X'_{c''}| > p_{c''}$ holds since (s'',c'') is rejected by accepting s to another school c'. It is clear that $|X'_{c'}| < q_{c'}$ holds. If $s'' \neq s$, s has justified envy toward s'' based on PL. If $s'' = s$, s claims an empty seat of c' based on PL since $c'' = c$ holds. In either case, X' is not PL-stable. □

With these results, we obtain the following.

A.2.1. Proof of feasibility for PLDA-MQ

Proof. Assume PLDA-MQ obtains set of contracts X'. If $X'_s = \emptyset$, then $(s,\omega(s)) \in Ch_S(X' \cup \{(s,\omega(s))\})$ holds. Also, from Lemma 9, $(s,\omega(s)) \in Ch_C(X' \cup \{(s,\omega(s))\})$ holds. This violates the fact that GDA obtains an HM-stable matching. If $X'_s = \{(s,c)\}$ holds, for each student s, (s,c) is clearly acceptable for s since $(s,c) \in Ch_S(X')$. Thus X' is student-feasible. Since each student is accepted by some school, $|X'| = n$ holds.

Next we show that X' is school-feasible. It is clear that all contracts in X'_c are acceptable for c. Also, for all $c \in C$, $|X'_c| \leq q_c$ holds, since otherwise, $\widehat{f}(X') = -\infty$ and $Ch_C(X')$ cannot be X'. Assume for contradiction there exists $c \in C$ such that $|X'_c| < p_c$. Then $\sum_{c \in C} \max(|X'_c|, p_c) > |X'| = n$. Thus $\widehat{f}(X') = -\infty$ and $Ch_C(X')$ cannot be X'. □

A.2.2. Proof of strategy-proofness, PL-stability, and obtaining the student-optimal matching within all the PL-stable matchings for PLDA-MQ

Proof. From Lemma 7 and Theorem 6, the f used in PLDA-MQ is M$^\natural$-concave. Thus from Condition 1 of Theorem 5, PLDA-MQ is strategy-proof and obtains the student-optimal matching within all HM-stable matchings. From Lemma 10, HM-stability is equivalent to PL-stability. Thus PLDA-MQ satisfies these properties. □

A.3. Proof of Lemma 1

Proof. Fix $\succ$, s, and $\succ'_s$ such that $\succ' = (\succ'_s, \succ_{-s})$, and suppose TTCR-SS assigns s to the same school under $\succ$ and $\succ'$. Let k and k' be such that $(s, \omega(s)) \in \mathscr{C}^k(\succ)$ and $(s, \omega(s)) \in \mathscr{C}^{k'}(\succ')$. If $k = k'$, then the assumption that s receives the same assignment implies that exactly the same cycles are formed at each round under $\succ$ and $\succ'$. Thus consider when $k \neq k'$ holds, and without loss of generality assume $k < k'$.

Let $k^* \geq k$ be such that TTCR-SS terminates at Round k^* when the input is $\succ$. Given some $z \leq k^*$, take any cycle in $G^z(\succ)$. We show that there exists z' such that the cycle is formed in $G^{z'}(\succ')$, which implies that the outcomes are identical.

First, suppose $1 \leq z < k$. Obviously, this cycle is formed in $G^z(\succ')$. Next consider $k \leq z \leq k^*$. The rest of the proof is done by induction on z.

When $z = k$, which is the base case, then all the cycles formed in $G^k(\succ)$, except the one in which $(s, \omega(s))$ is involved, are formed in $G^k(\succ')$. From Lemma 5 and since the assignments of s are identical in both cases, the same cycle containing $(s, \omega(s))$ in $G^k(\succ)$ is formed in $G^{k'}(\succ')$. Thus it is true for the base case.

Assume the supposition is true up to some z, $k \leq z < k^*$. Take any cycle in $G^{z+1}(\succ)$, and let $((\widehat{s}_1, \widehat{c}_1), \ldots, (\widehat{s}_t, \widehat{c}_t), (\widehat{s}_1, \widehat{c}_1))$ be the cycle. Assume first that it does not contain any contract related to a dummy student. For each ℓ, $1 \leq \ell \leq t$, there must exist Round z'_ℓ such that $(\widehat{s}_\ell, \widehat{c}_\ell) \in \mathscr{C}^{z'_\ell}(\succ')$. We show that $z'_\ell \geq z'_{\ell+1}$ holds for all ℓ, $1 \leq \ell \leq t$ with $t + 1 = 1$, where under $\succ'$, the contract $(\widehat{s}_\ell, \widehat{c}_\ell)$ does not leave the mechanism before $(\widehat{s}_{\ell+1}, \widehat{c}_{\ell+1})$. Given that the supposition is true, $\widehat{s}_\ell$ cannot obtain any of the seats that are gone by the end of Round z under $\succ$, when the input is $\succ'$. Thus $\widehat{c}_{\ell+1}$, which is the school she receives at Round $z + 1$ under $\succ$, must be the best possible assignment for her under the supposition and $\succ'$. On the other hand, $\widehat{s}_{\ell+1}$ has the highest priority in $\succ_{ML}$ to represent $\widehat{c}_{\ell+1}$ among those whose assignments are not fixed by the supposition. Therefore, if $((\widehat{s}_\ell, \widehat{c}_\ell), (\widehat{s}_{\ell+1}, \widehat{c}_{\ell+1})) \notin E^{z'_\ell}(\succ')$ is the case, then $(\widehat{s}_{\ell+1}, \widehat{c}_{\ell+1})$ is already gone before Round z'_ℓ, i.e., $z'_\ell > z'_{\ell+1}$. If $((\widehat{s}_\ell, \widehat{c}_\ell), (\widehat{s}_{\ell+1}, \widehat{c}_{\ell+1})) \in E^{z'_\ell}(\succ')$ holds, then it obviously follows that $z'_\ell = z'_{\ell+1}$. In any case, $z'_\ell \geq z'_{\ell+1}$ holds. This argument implies that if the cycle is a self-loop, then it must be formed under $\succ'$.

Next suppose the cycle contains a contract related to a dummy student. There is at most one such contract in the cycle, because all dummy students point to the same contract in a round. Without loss of generality, assume $\widehat{s}_t = s_d$ and let $((\widehat{s}_1, \widehat{c}_1), \ldots, (\widehat{s}_{t-1}, \widehat{c}_{t-1}), (s_d, \widehat{c}_t), (\widehat{s}_1, \widehat{c}_1))$ be the cycle. There must exist z'_ℓ such that $(\widehat{s}_\ell, \widehat{c}_\ell) \in \mathscr{C}^{z'_\ell}(\succ')$ for each ℓ, $1 \leq \ell \leq t - 1$, and the previous argument leads to $z'_1 \geq \cdots \geq z'_{t-1}$. What follows shows that z'_t exists, i.e., supplementary seat $(s_d, \widehat{c}_t)$ leaves the mechanism under $\succ'$ by forming a cycle. Since $((s_d, \widehat{c}_t), (\widehat{s}_1, \widehat{c}_1)) \in E^{z+1}(\succ)$ is the case, it follows that $\widehat{c}_1 \in C^{z+1}_{dec}(\succ)$. The supposition then implies that $\widehat{c}_1 \in C^{z'_1}_{dec}(\succ')$. Since $z'_1 \geq z'_{t-1}$ holds, it follows from Lemma 3 that $\widehat{c}_1 \in C^{z'_{t-1}}_{dec}(\succ')$. Therefore, if $((\widehat{s}_{t-1}, \widehat{c}_{t-1}), (s_d, \widehat{c}_t)) \notin E^{z'_{t-1}}(\succ')$ occurs, it implies that $\widehat{c}_t \in C^{z'_{t-1}}_{max}(\succ')$, i.e., supplementary seat $(s_d, \widehat{c}_t)$ has been taken by another student at some round before Round z'_{t-1}. If $((\widehat{s}_{t-1}, \widehat{c}_{t-1}), (s_d, \widehat{c}_t)) \in E^{z'_{t-1}}(\succ')$ is the case, then z'_t clearly exists with $z'_{t-1} = z'_t$. In both cases, z'_t exists and $z'_{t-1} \geq z'_t$ holds. Then the previous argument can be applied to pair of contracts $(s_d, \widehat{c}_t)$ and $(\widehat{s}_1, \widehat{c}_1)$, and it must hold that $z'_t \geq z'_1$.

Therefore, $z'_1 \geq \cdots \geq z'_t \geq z'_1$ should be satisfied in all cases. This condition can only be satisfied with equalities. Thus all the contracts in the cycle are in $V^{z'}(\succ')$ for some z', and all leave the mechanism at Round z'. The requirement for equalities directly implies $((\widehat{s}_\ell, \widehat{c}_\ell), (\widehat{s}_{\ell+1}, \widehat{c}_{\ell+1})) \in E^{z'}(\succ')$ for all ℓ, i.e., the cycle is formed in $G^{z'}(\succ')$. $\square$

References

[1] A. Abdulkadiroğlu, Y.K. Che, The Role of Priorities in Assigning Indivisible Objects: A Characterization of Top Trading Cycles, Working Paper, 2010.
[2] A. Abdulkadiroğlu, T. Sönmez, House allocation with existing tenants, J. Econ. Theory 88 (1999) 233–260.
[3] A. Abdulkadiroğlu, T. Sönmez, School choice: a mechanism design approach, Am. Econ. Rev. 93 (2003) 729–747.
[4] H. Aziz, B. de Keijzer, Housing markets with indifferences: a tale of two mechanisms, in: Proceedings of the Twenty-Sixth AAAI Conference on Artificial Intelligence, AAAI-2012, 2012, pp. 1249–1255.
[5] M. Balinski, T. Sönmez, A tale of two mechanisms: student placement, J. Econ. Theory 84 (1999) 73–94.
[6] P. Biró, T. Fleiner, R. Irving, D. Manlove, The college admissions problem with lower and common quotas, Theor. Comput. Sci. 411 (2010) 3136–3153.
[7] S. Braun, N. Dwenger, D. Kübler, A. Westkamp, Implementing quotas in university admissions: an experimental analysis, Games Econ. Behav. 85 (2014) 232–251.
[8] J. Drummond, A. Perrault, F. Bacchus, Sat is an effective and complete method for solving stable matching problems with couples, in: Proceedings of the Twenty-Fourth International Joint Conference on Artificial Intelligence, IJCAI-2015, 2015, pp. 518–525.
[9] L.E. Dubins, D.A. Freedman, Machiavelli and the Gale–Shapley algorithm, Am. Math. Mon. 88 (1981) 485–494.
[10] U.M. Dur, A Characterization of the Top Trading Cycles Mechanism for School Choice Problem, Working Paper, 2013.
[11] L. Ehlers, I.E. Hafalir, M.B. Yenmez, M.A. Yildirim, School choice with controlled choice constraints: hard bounds versus soft bounds, J. Econ. Theory 153 (2014) 648–683.
[12] M. Ekmekci, M.B. Yenmez, Integrating Schools for Centralized Admissions, Working Paper, 2016.
[13] T. Fleiner, A fixed-point approach to stable matchings and some applications, Math. Oper. Res. 28 (2003) 103–126.
[14] T. Fleiner, N. Kamiyama, A matroid approach to stable matchings with lower quotas, Math. Oper. Res. 41 (2016) 734–744.
[15] D. Fragiadakis, A. Iwasaki, P. Troyan, S. Ueda, M. Yokoo, Strategyproof matching with minimum quotas, ACM Trans. Econ. Comput. 4 (2015) 6:1–6:40.
[16] D. Gale, L.S. Shapley, College admissions and the stability of marriage, Am. Math. Mon. 69 (1962) 9–15.
[17] M. Goto, A. Iwasaki, Y. Kawasaki, R. Kurata, Y. Yasuda, M. Yokoo, Strategyproof matching with regional minimum and maximum quotas, Artif. Intell. 235 (2016) 40–73.
[18] G. Haeringer, F. Klijn, Constrained school choice, J. Econ. Theory 144 (2009) 1921–1947.
[19] I.E. Hafalir, M.B. Yenmez, M.A. Yildirim, Effective affirmative action in school choice, Theor. Econ. 8 (2013) 325–363.

[20] J.W. Hatfield, P.R. Milgrom, Matching with contracts, Am. Econ. Rev. 95 (2005) 913–935.
[21] H. Hosseini, K. Larson, R. Cohen, Matching with dynamic ordinal preferences, in: Proceedings of the Twenty-Ninth AAAI Conference on Artificial Intelligence, AAAI-2015, 2015, pp. 936–943.
[22] C.C. Huang, Classified stable matching, in: Proceedings of ACM–SIAM Symposium on Discrete Algorithms, SODA-2010, 2010, pp. 1235–1253.
[23] Y. Kamada, F. Kojima, Efficient matching under distributional constraints: theory and applications, Am. Econ. Rev. 105 (2015) 67–99.
[24] S. Kannan, J. Morgenstern, R. Rogers, A. Roth, Private Pareto optimal exchange, in: Proceedings of the Sixteenth ACM Conference on Economics and Computation, ACM EC-2015, 2015, pp. 261–278.
[25] O. Kesten, On two competing mechanisms for priority-based allocation problems, J. Econ. Theory 127 (2006) 155–171.
[26] F. Kojima, School choice: impossibilities for affirmative action, Games Econ. Behav. 75 (2012) 685–693.
[27] F. Kojima, A. Tamura, M. Yokoo, Designing matching mechanisms under constraints: an approach from discrete convex analysis, in: Proceedings of the Seventh International Symposium on Algorithmic Game Theory, SAGT-2014, 2014. The full version is available at http://mpra.ub.uni-muenchen.de/62226.
[28] P. Krysta, D. Manlove, B. Rastegari, J. Zhang, Size versus truthfulness in the house allocation problem, in: Proceedings of the Fifteenth ACM Conference on Economics and Computation, ACM EC-2014, 2014, pp. 453–470.
[29] R. Kurata, M. Goto, A. Iwasaki, M. Yokoo, Controlled school choice with soft bounds and overlapping types, in: Proceedings of the Twenty-Ninth AAAI Conference on Artificial Intelligence, AAAI-2015, 2015, pp. 951–957.
[30] R. Kurata, N. Hamada, C.L. Hsu, T. Suzuki, S. Ueda, M. Yokoo, Pareto efficient strategy-proof school choice mechanism with minimum quotas and initial endowments, in: Proceedings of the Sixteenth International Conference on Autonomous Agents and Multiagent Systems, AAMAS-2016, 2016, pp. 59–67.
[31] J. Ma, Strategy-proofness and the strict core in a market with indivisibilities, Int. J. Game Theory 23 (1994) 75–83.
[32] T. Morrill, An alternative characterization of top trading cycles, Econ. Theory 54 (2013) 181–197.
[33] T. Morrill, Making just school assignments, Games Econ. Behav. 92 (2015) 18–27.
[34] K. Murota, A. Shioura, M-convex function on generalized polymatroid, Math. Oper. Res. 24 (1999) 95–105.
[35] Y. Narita, Match or Mismatch: Learning and Inertia in School Choice, Working Paper, 2016.
[36] J. Oxley, Matroid Theory, Oxford University Press, 2011.
[37] S. Pápai, Strategyproof assignment by hierarchical exchange, Econometrica 68 (6) (2000) 1403–1433.
[38] A. Perrault, J. Drummond, F. Bacchus, Strategy-proofness in the stable matching problem with couples, in: Proceedings of the Sixteenth International Conference on Autonomous Agents and Multiagent Systems, AAMAS-2016, 2016, pp. 132–140.
[39] M. Pycia, M.U. Ünver, Trading Cycles for School Choice, Working Paper, 2011.
[40] M. Pycia, M.U. Ünver, Incentive compatible allocation and exchange of discrete resources, Theor. Econ. 12 (2017) 287–329.
[41] A.E. Roth, Incentive compatibility in a market with indivisible goods, Econ. Lett. 9 (1982) 127–132.
[42] A.E. Roth, A. Postlewaite, Weak versus strong dominance, J. Math. Econ. 4 (1977) 131–137.
[43] D. Saban, J. Sethuraman, House allocation with indifferences: a generalization and a unified view, in: Proceedings of the Fourteenth ACM Conference on Electronic Commerce, ACM EC-2013, 2013, pp. 803–820.
[44] L. Shapley, H. Scarf, On cores and indivisibility, J. Math. Econ. 1 (1974) 23–37.
[45] A. Westkamp, An analysis of the German university admissions system, Econ. Theory 53 (2013) 561–589.