# **Emphatic TD Bellman Operator is a Contraction**

#### **Assaf Hallak**

Electrical Engineering Dept.
Technion
ifogph@gmail.com

# **Aviv Tamar**

Electrical Engineering Dept.
Technion
avivt@tx.technion.ac.il

#### **Shie Mannor**

Electrical Engineering Dept. Technion shie@ee.technion.ac.il

#### **Abstract**

Recently, Sutton et al. (2015) introduced the emphatic temporal differences (ETD) algorithm for off-policy evaluation in Markov decision processes. In this short note, we show that the projected fixed-point equation that underlies ETD involves a contraction operator, with a  $\sqrt{\gamma}$ -contraction modulus (where  $\gamma$  is the discount factor). This allows us to provide error bounds on the approximation error of ETD. To our knowledge, these are the first error bounds for an off-policy evaluation algorithm under general target and behavior policies.

## 1 Introduction

In Reinforcement Learning (RL; Sutton & Barto 1998), *policy-evaluation* refers to the problem of evaluating the value function – a mapping from states to their long-term discounted return under a given policy, using sampled observations of the system dynamics and reward. Policy-evaluation is important both for assessing the quality of a policy, but also as a sub-procedure for policy optimization (Sutton & Barto, 1998).

For systems with large or continuous state-spaces, an exact computation of the value function is often impossible. Instead, an *approximate* value-function is sought using various function-approximation techniques (Sutton & Barto 1998; a.k.a. approximate dynamic-programming; Bertsekas 2012). In this approach, the parameters of the value-function approximation are tuned using machine-learning inspired methods, often based on the *temporal-difference* idea (TD;Sutton & Barto 1998).

The method generating the sampled data leads to two different types of policy evaluation. In the *on-policy* case, the samples are generated by the *target-policy* – the policy under evaluation, while in the *off-policy* setting, a different *behavior-policy* generates the data. In the on-policy setting, TD methods are well understood, with classic convergence guarantees and approximation-error bounds, based on a contraction property of the projected Bellman operator underlying TD (Bertsekas & Tsitsiklis, 1996). For the off-policy case, however, standard TD methods no longer maintain this contraction property, the error bounds do not hold, and these methods may even diverge (Baird, 1995).

Recently, Sutton et al. (2015) proposed the *emphatic TD* (ETD) algorithm: a modification of the TD idea that can be shown to converge off-policy (Yu, 2015). In this paper, we show that the projected Bellman operator underlying ETD also possesses a contraction property, which allows us to derive approximation-error bounds for ETD.

In recent years, several different off-policy policy-evaluation algorithms have been proposed and analyzed, such as importance-sampling based least-squares TD (Yu, 2012), gradient-based TD (Sutton et al., 2009), and ETD (Sutton et al., 2015). While these algorithms were shown to converge, to our knowledge there are no guarantees on the *error* of the converged solution. The only exception that we are aware of, is a contraction-based argument for importance-sampling based LSTD, under the restrictive assumption that the behavior and target policies are very similar (Bertsekas & Yu,

2009). This paper presents the first approximation-error bounds for off-policy policy evaluation under general target and behavior policies.

## 2 Preliminaries

We consider an MDP  $M=(S,A,P,R,\gamma,\rho)$ , where S is the state space, A is the action space, P is the transition probability matrix, R is the reward function,  $\gamma \in [0,1)$  is the discount factor, and  $\rho$  is the initial state distribution.

Given a target policy  $\pi$ , our goal is to evaluate the *value function*:

$$V^{\pi}(s) \doteq \mathbb{E}^{\pi} \left[ \sum_{t=0}^{\infty} R(s_t, a_t) \middle| s_0 = s \right].$$

Temporal difference methods (Sutton & Barto, 1998), approximate the value function by

$$V^{\pi}(s) \approx \theta^{\top} \phi(s),$$

where  $\phi(s) \in \mathbb{R}^n$  are state features, and  $\theta \in \mathbb{R}^n$  are weights, and use sampling to find a suitable  $\theta$ . Let  $\mu$  denote a behavior policy that generates the samples  $s_0, a_0, s_1, a_1, \ldots$  according to  $a_t \sim \mu(\cdot|s_t)$  and  $s_{t+1} \sim P(\cdot|s_t, a_t)$ . We denote by  $\rho_t$  the ratio  $\pi(a_t|s_t)/\mu(a_t|s_t)$ , and we assume, similarly to Sutton et al. (2015), that  $\mu$  and  $\pi$  are such that  $\rho_t$  is well-defined for all t.

Let  $T^{\pi}$  denote the Bellman operator for policy  $\pi$ , given by

$$T^{\pi}V = R_{\pi} + \gamma P_{\pi}V,$$

where  $R_\pi$  and  $P_\pi$  are the reward vector and transition matrix induced by policy  $\pi$ , and let  $\Phi$  denote a matrix whose columns are the feature vectors for all states. Let  $d_\mu$  and  $d_\pi$  denote the stationary distributions over states induced by the policies  $\mu$  and  $\pi$ , respectively. For some  $d \in \mathbb{R}^{|S|}$  satisfying d>0 element-wise, we denote by  $\Pi_d$  a projection to the subspace spanned by  $\phi(s)$  with respect to the d-weighted Euclidean-norm.

Similarly to Sutton et al. (2015), we divide the analysis to the 'pure bootstrapping' case  $\lambda=0$ , and the more general case with  $\lambda\in[0,1)$ . The ETD(0) algorithm iteratively updates the weight vector  $\theta$  according to:

$$\theta_{t+1} := \theta_t + \alpha F_t \rho_t (R_{t+1} + \gamma \theta_t^{\top} \phi_{t+1} - \theta_t^{\top} \phi_t) \phi_t$$
  
 
$$F_t = \gamma \rho_{t-1} F_{t-1} + 1, \quad F_0 = 1.$$

The emphatic weight vector f is defined by

$$f^{\top} = d_{\mu}^{\top} (I - \gamma P_{\pi})^{-1}. \tag{1}$$

The ETD( $\lambda$ ) algorithm iteratively updates the weight vector  $\theta$  according to

$$\theta_{t+1} := \theta_t + \alpha (R_{t+1} + \gamma \theta_t^{\top} \phi_{t+1} - \theta_t^{\top} \phi_t) e_t$$

$$e_t = \rho_t (\gamma \lambda e_{t-1} + M_t \phi_t), \quad e_{-1} = 0$$

$$M_t = \lambda i (S_t) + (1 - \lambda) F_t$$

$$F_t = \rho_{t-1} \gamma F_{t-1} + i (S_t), \quad F_0 = i (S_0),$$

where  $i:S\to \mathcal{R}^+$  is a known given function signifying the importance of the state. Note that Sutton et al. (2015) consider state-dependent discount factor  $\gamma(s)$  and bootstrapping parameter  $\lambda(s)$ , while in this paper we consider the special case where  $\gamma$  and  $\lambda$  are constant.

The emphatic weight vector m is defined by

$$m^{\top} = \mathbf{i}^{\top} (I - P_{\pi}^{\lambda})^{-1}, \tag{2}$$

where:

$$\mathbf{i}(s) = i(s) \cdot d_{\mu}(s),$$
  

$$P_{\pi}^{\lambda} = I - (I - \gamma \lambda P_{\pi})^{-1} (I - \gamma P_{\pi}).$$

Notice that in the case of general  $\lambda$ , the Bellman operator is:

$$T^{(\lambda)}v = (I - \gamma \lambda P_{\pi})^{-1}r_{\pi} + P_{\pi}^{\lambda}v. \tag{3}$$

Mahmood et al. (2015) show that ETD converges to some  $\theta^*$  that is a solution of the projected fixed-point equation:

$$\theta^{\top} \Phi = \Pi_m T^{(\lambda)} (\theta^{\top} \Phi).$$

In this paper, we establish that the projected Bellman operator  $\Pi_m T^{(\lambda)}$  is a contraction, which allows us to bound the error  $\|\Phi^\top \theta^* - V^\pi\|_m$ .

## 3 Results

We start from ETD(0). It is well known that  $T^{\pi}$  is a  $\gamma$ -contraction with respect to the  $d_{\pi}$ -weighted Euclidean norm (Bertsekas & Tsitsiklis, 1996). However, it is not immediate that the concatenation  $\Pi_f T^{\pi}$  is a contraction in any norm. Indeed, for the TD(0) algorithm Sutton & Barto (1998), a similar representation as a projected Bellman operator holds, but it may be shown that in the off-policy setting the algorithm diverges (Baird, 1995).

The following theorem shows that for ETD(0), the projected Bellman operator  $\Pi_f T^{\pi}$  is indeed a contraction.

**Theorem 1.** Denote by  $\kappa = \min_s \frac{d_{\mu}(s)}{f(s)}$ , then  $\Pi_f T^{\pi}$  is a  $\sqrt{\gamma(1-\kappa)}$ -contraction with respect to the Euclidean f-weighted norm, namely,

$$\|\Pi_f T^{\pi} v_1 - \Pi_f T^{\pi} v_2\|_f \le \sqrt{\gamma (1 - \kappa)} \|v_1 - v_2\|_f, \quad \forall v_1, v_2 \in \mathbb{R}^{|S|}.$$

*Proof.* Let F = diag(f). We have

$$||v||_{f}^{2} - \gamma ||P_{\pi}v||_{f}^{2} = v^{\top} F v - \gamma v^{\top} P_{\pi}^{\top} F P_{\pi} v$$

$$\geq^{a} v^{\top} F v - \gamma v^{\top} diag(f^{\top} P_{\pi}) v$$

$$= v^{\top} [F - \gamma diag(f^{\top} P_{\pi})] v$$

$$= v^{\top} \left[ diag\left(f^{\top} (I - \gamma P_{\pi})\right) \right] v$$

$$=^{b} v^{\top} diag(d_{u}) v = ||v||_{d}^{2},$$

$$(4)$$

where (a) follows from the Jensen inequality:

$$v^{\top} P_{\pi}^{\top} F P_{\pi} v = \sum_{s} f(s) (\sum_{s'} P_{\pi}(s'|s) v(s'))^{2}$$

$$\leq \sum_{s} f(s) \sum_{s'} P_{\pi}(s'|s) v^{2}(s')$$

$$= \sum_{s'} v^{2}(s') \sum_{s} f(s) P_{\pi}(s'|s)$$

$$= v^{\top} diag(f^{\top} P_{\pi}) v,$$
(5)

and (b) is by the definition of f in (1).

Notice that for every v:

$$||v||_{d_{\mu}}^{2} = \sum_{s} d_{\mu}(s)v^{2}(s) \ge \sum_{s} \kappa f(s)v^{2}(s) = \kappa ||v||_{f}^{2}$$
(6)

Therefore:

$$||v||_{f}^{2} \ge \gamma ||P_{\pi}v||_{f}^{2} + ||v||_{d_{\mu}}^{2} \ge \gamma ||P_{\pi}v||_{f}^{2} + \kappa ||v||_{f}^{2},$$

$$\Rightarrow \gamma ||P_{\pi}v||_{f}^{2} \le (1 - \kappa) ||v||_{f}^{2}$$
(7)

and:

$$||T^{\pi}v_{1} - T^{\pi}v_{2}||_{f}^{2} = ||\gamma P_{\pi}(v_{1} - v_{2})||_{f}^{2}$$

$$= \gamma^{2}||P_{\pi}(v_{1} - v_{2})||_{f}^{2}$$

$$\leq \gamma(1 - \kappa)||v_{1} - v_{2}||_{f}^{2}.$$
(8)

Hence, T is a  $\sqrt{\gamma(1-\kappa)}$ -contraction. Since  $\Pi_f$  is a non-expansion in the f-weighted norm (Bertsekas & Tsitsiklis, 1996),  $\Pi_f T$  is a  $\sqrt{\gamma(1-\kappa)}$ -contraction as well.

Notice that  $\kappa$  obtains values ranging from  $\kappa=0$  (when there is a state visited by the target policy, but not the behavior policy), to  $\kappa=1-\gamma$  (when the two policies are identical). In the latter case we obtain the classical bound:  $\sqrt{\gamma(1-\kappa)}=\gamma$ . This result resembles that of Kolter (2011) who used the discrepancy between the behavior and the target policy to bound the TD-error.

An immediate consequence of Theorem 1 is the following error bound, based on Lemma 6.9 of Bertsekas & Tsitsiklis (1996).

Corollary 1. We have

$$\|\Phi^{\top}\theta^* - V^{\pi}\|_f \le \frac{1}{1 - \sqrt{\gamma(1 - \kappa)}} \|\Pi_f V^{\pi} - V^{\pi}\|_f.$$

In a sense, the error  $\|\Pi_f V^{\pi} - V^{\pi}\|_f$  is the best approximation we can hope for, within the capability of our linear approximation architecture. Corollary 1 guarantees that we are not too far away from it

Now we move on to the analysis of  $ETD(\lambda)$ :

**Theorem 2.**  $\Pi_m T^{(\lambda)}$  is a  $\sqrt{\beta}$ -contraction with respect to the Euclidean f-weighted norm, where  $\beta = \frac{\gamma(1-\lambda)}{1-\lambda \alpha}$ . Namely,

$$\|\Pi_m T^{(\lambda)} v_1 - \Pi_m T^{(\lambda)} v_2\|_m \le \sqrt{\beta} \|v_1 - v_2\|_m, \quad \forall v_1, v_2 \in \mathbb{R}^{|S|}.$$

*Proof.* The proof is almost identical to the proof of Theorem 1, only now we cannot apply Jensen's inequality directly, since the rows of  $P_{\pi}^{\lambda}$  do not sum to 1. However:

$$P_{\pi}^{\lambda} \mathbf{1} = \left( I - (I - \gamma \lambda P_{\pi})^{-1} (I - \gamma P_{\pi}) \right) \mathbf{1} = \beta \mathbf{1}, \tag{9}$$

and each entry of  $P_{\pi}^{\lambda}$  is positive. Therefore  $\frac{P_{\pi}^{\lambda}}{\beta}$  will hold for Jensen's inequality. Let M=diag(m), we have

$$||v||_{m}^{2} - \frac{1}{\beta}||P_{\pi}v||_{m}^{2} = v^{\top}Mv - \beta v^{\top} \frac{P_{\pi}^{\lambda}}{\beta}^{\top} M \frac{P_{\pi}^{\lambda}}{\beta} v$$

$$\geq^{a} v^{\top}Mv - \beta v^{\top} diag(m^{\top} \frac{P_{\pi}^{\lambda}}{\beta}) v$$

$$= v^{\top} [M - diag(m^{\top} P_{\pi}^{\lambda})] v$$

$$= v^{\top} \left[ diag\left(m^{\top} (I - P_{\pi}^{\lambda})\right) \right] v$$

$$=^{b} v^{\top} diag(\mathbf{i}) v = ||v||_{\mathbf{i}}^{2},$$

$$(10)$$

where (a) follows from the Jensen inequality and (b) from Equation 2.

Therefore:

$$||v||_{m}^{2} \ge \frac{1}{\beta} ||P_{\pi}^{\lambda}v||_{m}^{2} + ||v||_{\mathbf{i}}^{2} \ge \frac{1}{\beta} ||P_{\pi}^{\lambda}v||_{m}^{2}, \tag{11}$$

and:

$$||T^{(\lambda)}v_1 - T^{(\lambda)}v_2||_m^2 = ||P_{\pi}^{\lambda}(v_1 - v_2)||_m^2 \le \beta ||v_1 - v_2||_m^2.$$
(12)

Hence,  $T^{(\lambda)}$  is a  $\sqrt{\beta}$ -contraction. Since  $\Pi_m$  is a non-expansion in the m-weighted norm (Bertsekas & Tsitsiklis, 1996),  $\Pi_m T^{(\lambda)}$  is a  $\sqrt{\beta}$ -contraction as well.

As before, Theorem 2 leads to the following error bound, based on Theorem 1 of Tsitsiklis & Van Roy (1997).

Corollary 2. We have

$$\|\Phi^{\top}\theta^* - V^{\pi}\|_m \le \frac{1}{1 - \sqrt{\beta}} \|\Pi_m V^{\pi} - V^{\pi}\|_m.$$

We now show in an example that our contraction modulus bounds are tight.

**Example** Consider an MDP with two states: Left and Right. In each state there are two identical actions leading to either Left or Right deterministically. The behavior policy will choose Right with probability  $\epsilon$ , and the target policy will choose Left with probability  $\epsilon$ . Calculating the quantities of interest:

$$P_{\pi} = \begin{pmatrix} \epsilon & 1 - \epsilon \\ \epsilon & 1 - \epsilon \end{pmatrix}, \quad d_{\mu} = (1 - \epsilon, \epsilon)$$
$$f = \frac{1}{1 - \gamma} (1 + 2\epsilon \gamma - \epsilon - \gamma, -2\epsilon \gamma + \epsilon + \gamma)^{\top}.$$

So for  $v = (0, 1)^{\top}$ :

$$||v||_f^2 = \frac{\epsilon + \gamma - 2\epsilon\gamma}{1 - \gamma}, \quad ||P_{\pi}v||_f^2 = \frac{(1 - \epsilon)^2}{1 - \gamma},$$

and for small  $\epsilon$  we obtain that  $\frac{\|\gamma P_\pi v\|^2}{\|v\|_f^2} \approx \gamma.$ 

## 4 Discussion

Interestingly, the ETD error bounds in Corollary 1 and 2 are more conservative by a factor of square root than the error bounds for standard on-policy TD (Bertsekas & Tsitsiklis, 1996; Tsitsiklis & Van Roy, 1997). Thus, it appears that there is a price to pay for off-policy convergence. Future work should address the implications of the different norms in these bounds.

Nevertheless, we believe that the results in this paper motivate ETD (or its least-squares counterpart; Yu 2015) as the method of choice for off-policy policy-evaluation in MDPs.

# References

Baird, L. Residual algorithms: Reinforcement learning with function approximation. In ICML, 1995.

Bertsekas, D. Dynamic Programming and Optimal Control, Vol II. Athena Scientific, 4th edition, 2012.

Bertsekas, D. and Tsitsiklis, J. Neuro-Dynamic Programming. Athena Scientific, 1996.

Bertsekas, D. and Yu, H. Projected equation methods for approximate solution of large linear systems. *Journal of Computational and Applied Mathematics*, 227(1):27–50, 2009.

Kolter, J Zico. The fixed points of off-policy td. In *Advances in Neural Information Processing Systems*, pp. 2169–2177, 2011.

Mahmood, A. R., Yu, H., White, M., and Sutton, R. S. Emphatic Temporal-Difference Learning. *ArXiv e-prints*, 2015.

Sutton, R. S. and Barto, A. Reinforcement learning: An introduction. Cambridge Univ Press, 1998.

Sutton, R. S., Maei, H. R., Precup, D., Bhatnagar, S., Silver, D., Szepesvári, C., and Wiewiora, E. Fast gradient-descent methods for temporal-difference learning with linear function approximation. In *ICML*, 2009.

Sutton, R. S., Mahmood, A. R., and White, M. An emphatic approach to the problem of off-policy temporal-difference learning. *CoRR*, abs/1503.04269, 2015.

Tsitsiklis, John N and Van Roy, Benjamin. An analysis of temporal-difference learning with function approximation. *Automatic Control, IEEE Transactions on*, 42(5):674–690, 1997.

- Yu, H. Least squares temporal difference methods: An analysis under general conditions. *SIAM Journal on Control and Optimization*, 50(6):3310–3343, 2012.
- Yu, H. On convergence of emphatic temporal-difference learning. In *COLT*, 2015.