An Introduction to Loop Quantum Gravity through Cosmology

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This introductory review is addressed to beginning researchers. Some of the distinguishing features of loop quantum gravity are illustrated through loop quantum cosmology of FRW models. In particular, these examples illustrate: i) how 'emergent time' can arise; ii) how the technical issue of solving the Hamiltonian constraint and constructing the *physical* sector of the theory can be handled; iii) how questions central to the Planck scale physics can be answered using such a framework; and, iv) how quantum geometry effects can dramatically change physics near singularities and yet naturally turn themselves off and reproduce classical general relativity when space-time curvature is significantly weaker than the Planck scale.

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I. INTRODUCTION

My lectures at the first Stückleberg symposium began with an introduction to the challenges of quantum gravity, then discussed key features of loop quantum gravity (LQG) and finally summarized applications to cosmology. However, in view of the page limit, in these proceedings I decided to present the material from a different perspective, through the lens of loop quantum cosmology (LQC). Thus, rather than starting with general considerations and then descending to applications, here I will follow an opposite approach, using applications to illustrate some of the key problems of quantum gravity and distinguishing features of LQG.¹ The advantage of this reverse strategy is that students and other beginning researchers can appreciate the key issues, challenges and advances in a rather simple setting, without having to first grasp the technical intricacies of full LQG. The main drawback is that this approach leaves out some of the most interesting developments such as statistical mechanical calculations of black hole entropy, discussions of the issue of information loss, and spin foams, particularly the recent advances in calculating graviton propagators in the non-perturbative setting of LQG [7].

With these caveats in mind, let us begin with a list of some of the long standing conceptual and technical issues of quantum gravity on which loop quantum cosmology has a bearing.

In general relativity, gravity is encoded in the very geometry of space-time. The most

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¹ A brief history of quantum gravity can be found in [1] and a detailed discussion of conceptual problems in [2]. A brief mathematical introduction to loop quantum gravity can be found in [3] and much more detailed reviews in [4, 5, 6].

dramatic features of general relativity can be traced back to this dual role of geometry: the expansion of the universe, the big-bang, formation of black holes and emergence of gravitational waves as ripples of space-time curvature. Already in the classical theory, it took physicists several decades to truly appreciate the dynamical nature of geometry and to get used to the absence of a kinematic background geometry. In quantum gravity, this paradigm shift leads to a new level of conceptual and technical difficulties.²

- The absence of a background geometry implies that classical dynamics is generated by constraint equations. In the quantum theory, physical states are solutions to quantum constraints. All of physics, including the dynamical content of the theory, has to be extracted from these solutions. But there is no external time to phrase questions about evolution. Therefore we are led to ask: Can we extract, from the arguments of the wave function, one variable which can serve as emergent time with respect to which the other arguments 'evolve'? If not, how does one interpret the framework? What are the physical (i.e., Dirac) observables? In a pioneering work, DeWitt proposed that the determinant of the 3-metric can be used as an 'internal' time [8]. Consequently, in much of the literature on the Wheeler-DeWitt (WDW) approach to quantum cosmology, the scale factor is assumed to play the role of time, although often only implicitly. However, in closed models the scale factor fails to be monotonic due to classical recollapse and cannot serve as a global time variable already in the classical theory. Are there better alternatives at least in the simple setting of quantum cosmology? If not, can we still make physical predictions?
- Can one construct a framework that cures the short-distance difficulties faced by the classical theory near singularities, while maintaining an agreement with it at large scales? By their very construction, perturbative and effective descriptions have no problem with the second requirement. However, physically their implications can not be trusted at the Planck scale and mathematically they generally fail to provide a deterministic evolution across the putative singularity. In LQG the situation is just the opposite. Quantum geometry gives rise to new discrete structures at the Planck scale that modify the classical theory in such a way that, at least in simple models, space-like singularities of general relativity are resolved. However, since the emphasis is on background independence and non-perturbative methods, a priori it is not clear whether the theory also has a rich semi-classical sector. Do the novel dynamical corrections unleashed by the underlying quantum geometry naturally fade away at macroscopic distances or do they have unforeseen implications that prevent the theory from reproducing general relativity at large scales? Some of such unforeseen problems are discussed in, e.g., [9, 10].

Next, the dual role of the space-time metric also implies that space-time itself ends when the gravitational field becomes infinite. This is in striking contrast with Minkowskian physics, where singularity of one specific field has no bearing at all on the space-time structure or on the rest of physics. In general relativity singularities of the gravitational field represent an absolute boundary of space-time where all of physics comes to a halt.

² There is a significant body of literature on this issue; see e.g., [2] and references therein. These difficulties are now being discussed also in the string theory literature in the context of the AdS/CFT conjecture.

Now, it is widely believed that the prediction of a singularity —such as the big-bang—in classical general relativity is primarily a signal that the theory has been pushed beyond the domain of its validity and cannot therefore be trusted. One needs a quantum theory of gravity to analyze true physics. This expectation immediately leads to a host of questions which have been with us for several decades now:

- How close to the big-bang does a smooth space-time of GR make sense? Inflationary scenarios, for example, are based on a space-time continuum. Can one show from some first principles that this is a safe approximation already at the onset of inflation?
- Is the Big-Bang singularity naturally resolved by quantum gravity? This possibility led to the development of the field of quantum cosmology in the late 1960s. The basic idea can be illustrated using an analogy to the theory of the hydrogen atom. In classical electrodynamics the ground state energy of this system is unbounded below. Furthermore, because an accelerating electron radiates continuously, it must fall into the proton making the atom extremely unstable. Quantum physics intervenes and, thanks to a non-zero Planck's constant, the ground state energy is lifted to a finite value, $-me^4/2\hbar^2 \approx -13.6$ ev. The hope was that a similar mechanism would resolve the big-bang and big crunch singularities in simple cosmological models. However, as we will see in some detail in section II, the WDW quantization did not realize this hope. Can the quantum nature of geometry underlying LQG make a fundamental difference to this status-quo?
- Is a new principle/ boundary condition at the big bang or the big crunch essential to provide a deterministic evolution? The most well known example of such a boundary condition is the 'no boundary proposal' of Hartle and Hawking [11]. Or, do quantum Einstein equations suffice by themselves even at the classical singularities?
- Do quantum dynamical equations remain well-behaved even at these singularities? If so, do they continue to provide a deterministic evolution? The idea that there was a pre-big-bang branch to our universe has been advocated by several approaches, most notably by the pre-big-bang scenario in string theory [12] and ekpyrotic and cyclic models inspired by brane world ideas [13, 14]. However, in such perturbative treatments there is always a smooth continuum in the background and hence the dynamical equations break down at the singularity. Consequently, the pre-big-bang branch is not joined to the current post-big-bang branch by a deterministic evolution. The hope has been that non-perturbative effects would remedy this situation in the future. LQG, on the other hand, does not have a continuum geometry in the background and the treatment is non-perturbative. So, one is led to ask: Can the LQC quantum Einstein's equation provide a deterministic evolution across the big bang and the big-crunch?
- If there is a deterministic evolution, what is on the 'other side'? Is there just a quantum foam from which the current post-big-bang-branch is born, say a 'Planck time after the putative big-bang'? Or, was there another classical universe as in the pre-big-bang and cyclic scenarios, joined to ours by deterministic equations?

The goal of this article is show that such questions can be answered satisfactorily in the simplest LQC models. The first significant results appeared already five years ago in the

pioneering work of Bojowald [15]. Since then there has been a steady stream of papers by a dozen or so researchers and the field achieved a new degree of maturity over the last year. However, as discussed in section IV this is not the end of the story because in LQC one first symmetry reduces the classical theory and then quantizes it. Nonetheless, since the quantization procedure mimics that of full LQG, the LQC answers provide not only the much needed intuition but also a strategy for answering the larger questions.

The article is organized as follows. Section II introduces the reader to quantum cosmology and reviews the WDW theory. While one can answer some of the questions posed above, the WDW theory fails to resolve the big bang and the big crunch singularities. The situation is drastically different in LQC where not only are the singularities resolved but most of the questions have physically desired answers. The key ideas and the structure of LQC are summarized in section III, which also explains the sense in which it 'descends' form full LQG and emphasizes the main differences between LQC the WDW theory. For definiteness, I will use the k=1 FRW models coupled to a massless scalar field [16, 17] to illustrate in some detail both the difficulties discussed above and the way they can be handled. For results on the k=0 model, see [18, 19], for a first discussion of the k=1 and of the anisotropic models, see [20, 21] and for a detailed review of the developments in LQC till 2005, see [22]. I conclude in section IV with a brief summary of these other developments and a discussion of the many challenges that still remain.

II. QUANTUM COSMOLOGY: THE WDW THEORY

Both on the mathematical and phenomenological fronts, most of the work to date in classical cosmology involves spatially homogeneous models and perturbations thereof. In the late 60's DeWitt [8] and Misner [23] began investigations in quantum cosmology with a similar viewpoint: first reduce general relativity by imposing homogeneity and then quantize the resulting system. Since this system has only a finite number of degrees of freedom, the key field-theoretic difficulties are bypassed. However, since there is no classical space-time in the background, most of the conceptual problems of full quantum gravity still remain. Therefore one can hope that resolution of these problems in this technically simpler context would provide valuable insights for the full theory. In particular, as mentioned above, an initial expectation was that quantum effects —particularly the uncertainty principle— would intervene and resolve the big bang and the big crunch singularities. While subsequent work in the seventies and eighties did shed light on several conceptual issues, this specific expectation was not met. I will begin by briefly explaining the strategy used in these analyses and point out its limitation. This summary will also serve to bring out the new elements of LQC.

In this early work, the configuration space consisted of positive definite 3-metrics. In the simplest, spatially homogeneous, isotropic context, this metric is completely determined by the scale factor a. Since a is restricted to be positive, one generally introduced a new variable $\alpha = \ln a$ and, as in quantum mechanics, considered wave functions $\Psi(\alpha, \phi) \in L^2(\mathbb{R}^2, d\alpha d\phi)$, where ϕ denotes possible matter fields. As in quantum mechanics, the basic operators were defined via $\hat{\alpha} \Psi = \alpha \Psi$, $\hat{p}_{\alpha} \Psi = -i\hbar (\partial \Psi/\partial \alpha)$, $\hat{\phi} \Psi = \phi \Psi$, and $\hat{p}_{\phi} \Psi = -i\hbar (\partial \Psi/\partial \phi)$. Quantization of the classical Hamiltonian constraint,

$$\frac{2\pi G}{3a^3} p_{\alpha}^2 + \frac{3\pi^2}{2G} e^{\alpha} = H_{\phi} \,, \tag{2.1}$$

where H_{ϕ} is the matter Hamiltonian, then led to a differential equation, called the

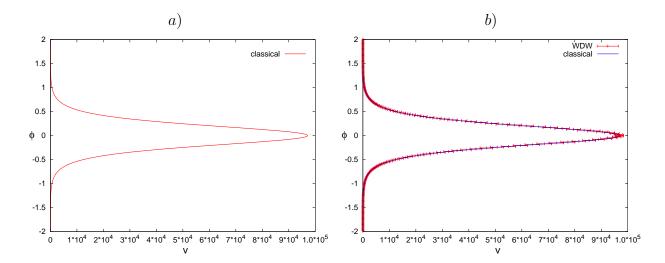


FIG. 1: a) Classical solutions. Since p_{ϕ} is a constant of motion, a classical trajectory can be plotted in the v- ϕ plane, where v is essentially the volume in Planck units (see Eq (3.15)). b) Expectation values (and dispersions) of $|\hat{v}|_{\phi}$ for the WDW wave function and comparison with the classical trajectory. The WDW wave function follows the classical trajectory into the big-bang and big-crunch singularities. (In this simulation, the parameters were: $p_{\phi}^{\star} = 5000$, and $\Delta p_{\phi}/p_{\phi}^{\star} = 0.02$.)

WDW equation. If the matter field is a zero rest mass scalar field —the case I will focus on for simplicity— the WDW equation becomes

$$\frac{4\pi}{3G}\partial_{\alpha}^{2}\Psi - \frac{3\pi^{2}}{G\hbar^{2}}e^{4\alpha}\Psi = \partial_{\phi}^{2}\Psi \tag{2.2}$$

(For details, see, e.g., [24]). Physical quantum states are solutions to this equation. To extract physics, one has to introduce an inner product and suitable observables on the space of these states. The task of finding these observables is not entirely straightforward because they must preserve the space of solutions to (2.2), i.e., they must be Dirac observables. Using these observables, one can then ask if classical singularities are resolved in quantum theory. The older WDW literature does not appear to analyze this issue in any detail since its focus was on the WKB approximation which, unfortunately, becomes unreliable near the singularity. However, they were analyzed in some detail recently (see e.g. [10, 16, 18, 19]). I will now provide a summary of the main results.

Each classical trajectory begins with the big bang, undergoes a subsequent expansion and a recollapse and finally contracts into the big crunch singularity (see Fig. 1 a). Since there is no potential, the matter momentum p_{ϕ} is a constant of motion. Consequently, ϕ is a monotonic function on each classical trajectory. Therefore, it can be thought of as 'internal time' in the classical theory. The form of the WDW equation (2.2) suggests that the same is true in quantum theory. Thus the WDW equation can be regarded as an 'evolution equation' with respect to this emergent time, ϕ . Next, $\hat{p}_{\phi} = -i\hbar\partial_{\phi}$ is a Dirac observable in the quantum theory. Given any value ϕ_{o} of ϕ , one can also define a second Dirac observable, $\hat{V}|_{\phi_{o}}$, corresponding to the volume of the universe at any given 'time' ϕ_{o} [10, 16, 18, 19]. The pair $(\hat{p}_{\phi}, \hat{V}_{\phi_{o}})$ constitutes a complete set of Dirac observables. Finally, the requirement that these observables be self-adjoint fixes the physical inner product (which can also be determined by the more precise group averaging technique [25]). Thus, the first

set of questions raised in section I can be answered satisfactorily in the WDW theory.

Now to answer the remaining questions about singularity, one can proceed as follows. Consider a point (v^*, ϕ^*) on a classical trajectory with $p_\phi = p_\phi^*$, where the universe is large and the curvature is low compared to the Planck scale. At 'time' $\phi = \phi^*$, one can construct an initial quantum state which is peaked at values p_ϕ^* , v^* of the Dirac observables \hat{p}_ϕ and \hat{V}_{ϕ_o} . One can then evolve it using the WDW equation.³ The solutions can be expressed in terms of modified Bessel functions. A resulting plot of the expectation values and dispersions of \hat{V}_ϕ as a function of ϕ is shown in Fig. 1 b. The good news is that, when the universe is large, the quantum state remains sharply peaked at the classical trajectory, showing that the theory has a good semi-classical limit. The bad news is that the quantum state continues to be peaked at this trajectory even as the classical trajectory hits the big bang and the big crunch singularities! Thus, although the WDW theory is background independent and non-perturbative, its results are similar to that of background dependent, perturbative treatments: Good semi-classical limit but no improvement on the classical short distance pathologies.

Remark: In the classical solution, the scalar field ϕ tends to $-\infty$ at the big bang and to ∞ at the big crunch. Therefore, the WDW evolution is unitary on the entire real line of the 'emergent time'. However, as we just saw, even semi-classical states simply track the classical solution and therefore to the extent classical solutions are singular –singularities are reached in finite *proper time*– so are the quantum solutions. Thus, while the emergent time ϕ is useful in enabling us to think of 'evolution', unitary evolution over its entire range does not guarantee non-singular behavior. Additional considerations, such as the relation between the emergent time and proper time in semi-classical settings are essential.

III. LOOP QUANTUM COSMOLOGY: THE GRAVITATIONAL SECTOR

A. Classical Theory

Since readers may not be familiar with the Hamiltonian description used in LQC, I will summarize its key elements before proceeding with quantization.

Let us consider space-time manifolds which have the form $M \times \mathbb{R}$, where M has the topology of a 3-sphere, \mathbb{S}^3 . As Misner showed in the late 60's [23], one can identify M with the symmetry group SU(2) (which ensures spatial homogeneity and isotropy) and endow it with a fixed fiducial basis of 1-forms \mathcal{W}_a^i and vectors \mathcal{C}_i^a . The resulting fiducial metric is

$${}^{o}q_{ab} := {}^{o}\omega_{a}^{i} {}^{o}\omega_{b}^{j} k_{ij}, \qquad k_{ij}$$
: the Cartan-Killing metric on su(2). (3.1)

 ${}^o\!q_{ab}$ turns out to be the metric of the round 3-sphere with radius $a_o=2$ (rather than $a_o=1$). The volume of M w.r.t. this fiducial metric ${}^o\!q_{ab}$ is $V_o=2\pi^2a_o^3=16\pi^2$ and the scalar curvature is ${}^o\!R=6/a_o^2=3/2$. We shall set $\ell_o:=V_o^{1/3}$.

In LQG, the dynamical variables are SU(2) (gravitational) spin-connections A_a^i . Their conjugate momenta E_i^a are the analogs of electric fields in Yang-Mills theory. However,

³ Technically, physical states satisfy the 'positive frequency' square root $i\partial_{\phi}\Psi = -\sqrt{\underline{\Theta}}\Psi$ of (2.2), where the positive definite self-adjoint operator $\underline{\Theta}$ is the negative of the operator on the right side of (2.2). One uses this 'first order in time' equation to 'evolve' the wave function. For details, see [10, 16, 19].

since the 'internal' group SU(2) is now tied with the group SO(3) of rotations in the tangent space at any point of M, E_i^a now acquire a direct geometric meaning: they represent spatial, orthonormal triads (with density weight 1). In the present isotropic, homogeneous setting, these canonically conjugate fields are parameterized by constants c and p respectively:

$$A_a^i = c \ell_o^{-1} {}^o \omega_a^i$$
, and $E_i^a = p \ell_o^{-2} \sqrt{q} {}^o e_i^a$ (3.2)

which are functions of time. The factors of l_o facilitate comparison with the spatially flat, k=0 case [10, 18, 19]. In both cases, c is dimensionless while p has dimensions of area. In terms of geometrodynamical variables, p determines the scale factor a (which determines the spatial metric) and c, the canonically conjugate momentum, or, \dot{a} (which determines the extrinsic curvature). More precisely, at the point (c, p) of the phase space, the physical 3-metric q_{ab} and the extrinsic curvature K_{ab} are given by:

$$q_{ab} = |p| \ell_o^{-2} {}^{o}q_{ab} \quad \text{and} \quad \gamma K_{ab} = (c - \frac{\ell_o}{2}) |p|^{\frac{1}{2}} \ell_o^{-2} {}^{o}q_{ab}$$
 (3.3)

The corresponding physical volume of M is $V=|p|^{\frac{3}{2}}$. The scale factor a associated with a physical metric q_{ab} is generally expressed via $q_{ab}=a^2\,{}^o\!\underline{\mathbf{q}}_{ab}$ where ${}^o\!\underline{\mathbf{q}}_{ab}$ is the unit 3-sphere metric. Then, the scale factor is related to p via $|p|=a^2\ell_o^2/4$. p can take both positive and negative values, the change in sign corresponds to a flip in the orientation of the triad which leaves the physical metric q_{ab} invariant.

Using the fact that A_a^i and E_i^a are canonically conjugate, it is easy to calculate the fundamental Poisson bracket between c and p:

$$\{c, p\} = \frac{8\pi G\gamma}{3} \tag{3.4}$$

where γ is the so called 'Barbero-Immirzi parameter' which labels quantization ambiguity of LQG. (Black hole entropy considerations show that $\gamma \approx 0.24$.) Finally, using the fact that the co-triad \mathcal{W}_a^i satisfies the Cartan identity

$$d^o \omega^k + \frac{1}{2} \epsilon_{ij}{}^{ko} \omega^i \wedge {}^o \omega^j = 0, \qquad (3.5)$$

it is straightforward to calculate the field strength ${\cal F}^k_{ab}$ of the connection A^i_a on M:

$$F_{ab}^{k} = \ell_o^{-2} \left[c^2 - c \, \ell_o \right] \epsilon_{ij}^{k} \, {}^o \! \omega_a^j \, {}^o \! \omega_b^k \,. \tag{3.6}$$

Because of our choice of parametrization of A_a^i and E_i^a , the only non-trivial constraint is the Hamiltonian one. Its gravitational part reduces to [16]:

$$C_{\text{grav}} = -\frac{1}{\gamma^2} \int_M d^3 x \, \epsilon^{ij}_k \, e^{-1} E_i^a E_j^b \left[F_{ab}^k - \left(\frac{1+\gamma^2}{4} \right) {}^o \! \epsilon_{ab}{}^c {}^o \! \omega_c^k \right]$$
(3.7)

$$= -\frac{6\sqrt{p}}{\gamma^2} \left[\left(c - \frac{\ell_o}{2} \right)^2 + \frac{\gamma^2 \ell_o^2}{4} \right] \tag{3.8}$$

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B. Quantum Kinematics

Since the phase space is now parameterized by $(c, p; \phi, p_{\phi})$, as in the WDW theory one's first impulse would be to follow standard quantum mechanics. In the WDW case this strategy was natural because one did not have access to full quantum geometrodynamics to take guidance from. In LQC it is not: a more appropriate strategy would be to follow the procedures used in full LQG. One's first reaction may be: How can this make any difference? After all the system has only a finite number of degrees of freedom and von Neumann's theorem assures us that, under appropriate assumptions, the resulting quantum mechanics is unique. The only remaining freedom appears to be in the factor-ordering of the Hamiltonian constraint and this is generally insufficient to lead to qualitatively different predictions. It turns out however that if one mimics full LQG, one naturally violates a key assumption of von Neumann's uniqueness theorem and, in spite of the presence of only a finite number of degrees of freedom, one is led to new quantum mechanics.

To explain this point, let me begin with a quick summary of the relevant structure of full LQG. Since novel elements of interest to us appear in the gravitational sector, in the remainder of this subsection I will ignore matter. To maintain gauge covariance, configuration variables are taken to be the 'Wilson lines' or the holonomies $h_e := \mathcal{P} \exp - \int_e A$ along arbitrary edges e, determined by the SU(2) connection A_a^i . Geometrically, being 1-forms, connections are naturally 'smeared' along 1-dimensional curves. The conjugate momenta are density-weighted triads E_i^a which (being duals of a triplet of 2-forms) are naturally smeared along 2-surfaces S with test fields f_i : $E(f) = \int_S {}^{\star} E_i f^i$. These are referred to as triad fluxes. Note that these two sets of phase space functions $(h_e(A), E(S, f))$ have been constructed without any reference to a background metric or any other background field. They generate an algebra, rather analogous to that generated by functions $\exp i\lambda q$ and p on the phase space of a particle. Our job is to find a representation of this 'holonomy-flux' algebra which preserves background independence of the theory. The surprising result is that the representation is unique |26|! More precisely, the algebra admits a unique (internal, i.e. SU(2)gauge and) diffeomorphism invariant state and the representation is generated by the action of the algebra on this state. While the uniqueness result is rather recent, the representation itself was constructed over a decade ago and is well understood [4, 5, 6]. For our purposes a key feature of this representation is the following: While there are well-defined operators h_e corresponding to holonomies, there is no operator corresponding to the connection itself.

In quantum cosmology one follows the same overall procedure. In the k=1 model now under consideration, holonomies along integral curves of the left invariant vector fields ${}^o\!e_i^a$ already suffice to separate homogeneous, isotropic connections. Let us denote by $\lambda \ell_o$ the

⁴ In the framework of algebraic quantum field theory, a \star -algebra \mathfrak{a} of operators is first constructed abstractly, without reference to a Hilbert space. A state f is a positive linear functional on the algebra, i.e. a linear functional which satisfies $f(A^*A) \geq 0$ for all $A \in \mathfrak{a}$. Such a state determines, via a standard construction due to Gel'fand, Naimark and Segal, a \star representation of \mathfrak{a} on a Hilbert space, in which the abstract state f is now represented by a vector $|\Psi_f\rangle$ in the Hilbert space and its action by the expectation value functional: $f(A) = \langle \Psi_f | A \Psi_f \rangle$. The vector Ψ_f is 'cyclic' in the sense that a dense subspace of the full Hilbert space is generated by repeated action of operators in \mathfrak{a} on Ψ_f . Finally, note that the uniqueness result refers to the holonomy-flux algebra whose elements are not diffeomorphism invariant. An algebra of diffeomorphism invariant operators would admit many diffeomorphism invariant states.

directed length (w.r.t. the fiducial metric ${}^{o}q_{ab}$) of the curve along ${}^{o}e_{i}^{a}$ (so that λ is positive if the curve is oriented in the direction of ${}^{o}e_{i}^{a}$ and negative if has the opposite orientation). Then the holonomy along the edge of length $\lambda \ell_{o}$ in the kth direction is given by

$$h_{(\lambda)}^{k}(c) = \cos(\lambda c/2) \mathbb{I} + 2\sin(\lambda c/2) \tau^{k}$$
(3.9)

where \mathbb{I} is the identity 2×2 matrix. (The holonomy is of course independent of the background metric ${}^{o}q_{ab}$.) The functions of c which enter as coefficients are 'almost periodic' functions of c, i.e. are of the form $N_{(\lambda)}(c) := \exp i\lambda(c/2)$. In the quantum theory, then, we are led to a representation in which operators $\hat{h}^k_{(\lambda)}$, $\hat{N}_{(\lambda)}$ and \hat{p} are well-defined, but there is no operator corresponding to the connection component c.

At first this seems surprising because our experience with quantum mechanics suggests that one should be able to obtain the operator analog of c by differentiating $\hat{N}_{(\lambda)}$ with respect to the parameter λ . However, in the representation of the basic quantum algebra that descends to LQC from full LQG, although the $\hat{N}_{(\lambda)}$ provide a 1-parameter group of unitary transformations, the group fails to be weakly continuous in λ . Therefore one can not differentiate and obtain the operator analog of c. In quantum mechanics, this would be analogous to having well-defined (Weyl) operators corresponding to the classical functions $\exp i\lambda x$ but no operator \hat{x} corresponding to x itself. This violates one of the assumptions of the von-Neumann uniqueness theorem. New representations then become available which are inequivalent to the standard Schrödinger one. In quantum mechanics, these representations are not of direct physical interest because we need the operator \hat{x} . In LQC, on the other hand, full LQG naturally leads us to a new representation. This theory is inequivalent to the WDW type theory already at a kinematical level.

The structure of this theory can be summarized as follows. Since \hat{p} is a self-adjoint operator, the gravitational part $\mathcal{H}_{\rm kin}^{\rm grav}$ of the kinematic Hilbert space can be described in terms of its eigenbasis. A general state $|\Psi\rangle$ has the form

$$|\Psi\rangle = \sum_{i} \Psi_{i} |p_{i}\rangle \quad \text{with} \quad \sum_{i} |\Psi_{i}|^{2} < \infty ,$$
 (3.10)

where i ranges over a countable set and $|p_i\rangle$ is an orthonormal basis:

$$\langle p_i | p_j \rangle = \delta_{ij} \,. \tag{3.11}$$

While the general form of these equations may seem familiar from quantum mechanics, there are key differences: the state is a *countable* sum of eigenstates of \hat{p} rather than a direct integral and the right side of (3.11) is a Kronecker delta rather than the Dirac delta distribution.⁵ Consequently, the intersection of \mathcal{H}_{kin}^{grav} and $L^2(\mathbb{R}, dp)$ consists only of the zero function. However, the action of the basic operators on wave functions $\Psi(p) := \langle p | \Psi \rangle$ is the expected one:

$$\hat{p}\Psi(p) = p\Psi(p)$$
 and $\hat{N}_{(\lambda)}\Psi(p) = \Psi(p+\lambda)$ (3.12)

⁵ Thus, in contrast to the Schrödinger Hilbert space $L^2(\mathbb{R}, dc)$, the kinematical LQC Hilbert space \mathcal{H}_{kin}^{grav} is non-separable. However, as we will see, quantum dynamics is governed by a discrete evolution equation. This leads to superselection sectors so that the final physical Hilbert spaces are all separable. For details, see [10, 16].

The forms of these operators are the same as in ordinary quantum mechanics. However, in the new Hilbert space $\hat{N}_{(\lambda)} \equiv \exp i\lambda(c/2)$ fails to be (weakly) continuous in λ because $\Psi(p+\lambda)$ is orthogonal to $\psi(p+\lambda')$ for all $\lambda \neq \lambda'$. Consequently, one can not take a derivative of $\hat{N}_{(\lambda)}$ w.r.t. λ and define an operator \hat{c} . Finally, let us note a fact that will be useful in section III C: from the discussion of the classical theory of section III A it follows that the physical volume operator of M is given by: $\hat{V} = |\hat{p}|^{3/2}$.

Remark: Kinematics of LQC provides a useful, broad-brush picture of the kinematics of full LQG (for details, see [4, 5, 6]). We saw that the LQC kinematical Hilbert space is not the space $L^2(\mathbb{R}, dc)$ that one would use in the WDW theory. But it can in fact be written as the space of square integrable functions, not on the classical configuration space \mathbb{R} , but on a certain completion, \mathbb{R}_{Bohr} thereof, called the Bohr compactification of the real line. This is an Abelian group with a natural Haar measure. The kinematical Hilbert space can be written as $\mathcal{H}_{kin}^{grav} = L^2(\bar{\mathbb{R}}_{Bohr}, d\mu_{Haar})$. Thus, one way to understand the difference between the Schrödinger and the new quantum mechanics is to realize that while in the Schrödinger theory the 'quantum configuration space' continues to be the classical configuration space, \mathbb{R} , in the new theory it is replaced by a certain completion \mathbb{R}_{Bohr} thereof. The same phenomenon occurs in full LQG: While the classical configuration space \mathcal{A} is the space of smooth SU(2) connections on a 3-manifold Σ , the quantum configuration space \mathcal{A} is a certain completion thereof, containing also 'generalized connections' which can be arbitrarily discontinuous. The Haar measure on SU(2) induces a natural, faithful, regular Borel measure $d\mu_o$ on \mathcal{A} . The kinematical Hilbert space of LQG is $L^2(\bar{\mathcal{A}}, d\mu_o)$. Thus the completion $\bar{\mathbb{R}}_{Bohr}$ of LQC is replaced by $\bar{\mathcal{A}}$ and the measure μ_{Haar} by the measure μ_o . Technically both completions \mathcal{A} and \mathbb{R}_{Bohr} arise as the 'Gel'fand spectrum' of the holonomy algebras in the two theories.

Next, let us consider the algebraic approach. The \star -algebra generated by N_{λ} , \hat{p} is the LQC analog of the holonomy flux algebra of full LQG. The analog of the unique LQG diffeomorphism state on this \star -algebra is the positive-linear functional f which has the following action on the basic operators: $f(\hat{N}_{\lambda}) = \delta_{\lambda,0}$ and f(p) = 0. As in full LQG, the Gel'fand, Naimark, Segal construction then leads us to the kinematical Hilbert space $\mathcal{H}_{\text{kin}}^{\text{grav}}$. In this space, the abstract state is represented by the concrete vector $|p=0\rangle$, or equivalently $\Psi(c) = 1$, in which the triad vanishes and, heuristically, the connection is completely undetermined. As in LQG, the full Hilbert space is generated by a repeated action of the algebra on this state. A natural basis in the LQG kinematical Hilbert space is given by the so-called spin network functions. These are analogous to the states $N_{\lambda}(c)$ in the connection representation of LQC. Just as $N_{\lambda}(c)$ diagonalize operators \hat{p} , (appropriately chosen) spin networks diagonalize triads and hence geometric operators. This is why a firm grasp of the LQC quantum kinematics can provide good intuition for full LQG.

Finally, a word of caution: the analogy does fail in a few but important respects because of the homogeneity assumption of LQC. So, it should be pursued as a guideline to aid

The space $\mathbb{R}_{\mathrm{Bohr}}$ was introduced by the mathematician Harold Bohr, Niels's brother, in his theory of almost periodic functions. In the connection representation, elements of $\mathcal{H}_{\mathrm{kin}}^{\mathrm{grav}}$ are of the form $\Psi(c) = \sum_{i} \alpha_{i} \exp i \lambda_{i} c$ where α_{i} are complex numbers, λ_{i} real and the sum extends over a countable set. These $\Psi(c)$ are almost periodic functions of c, where the word 'almost' refers to the fact that λ_{i} are arbitrary real numbers. The scalar product given by (3.11) now becomes: $\langle \Psi_{1} | \Psi_{2} \rangle = \lim_{L \to \infty} (1/2L) \int_{-L}^{L} \bar{\Psi}_{1}(c) \Psi_{2}(c) dc$.

intuition rather than a literal dictionary.

C. Quantum Dynamics

To describe quantum dynamics, we have to first introduce a well-defined operator on $\mathcal{H}_{\rm kin}^{\rm grav}$ representing the Hamiltonian constraint $C_{\rm grav}$. Since there is no operator corresponding to c itself, we cannot directly use the final expression (3.8). Rather, we return to the form (3.7) from full general relativity and promote it to an operator. This procedure also serves to bring LQC closer to LQG. The general strategy is the same as in ordinary quantum mechanics: One first expresses various terms in (3.7) in terms of 'elementary variables', namely the holonomies $h_{(\lambda)}^k$ and momenta p which have direct quantum analogs and then replaces them with operators $\hat{h}_{(\lambda)}^k$ and \hat{p} . Quantization of the first term, $\epsilon^{ij}{}_k e^{-1} E_i^a E_j^b$, can then be carried out by directly following a procedure given by Thiemann in the full theory [6, 27, 28].

The second term is the field strength $F_{ab}{}^k$. As in gauge theories, one can obtain $F_{ab}{}^k$ by first calculating the holonomy around a closed loop in the a-b plane, multiplying it by τ^k , dividing by the area and taking the limit as the area shrinks to zero. However, precisely because there is no connection operator \hat{A}^i_a in LQG, the limit can not exist in quantum theory. However, the same feature that prevents \hat{A}^i_a from existing leads to quantum Riemannian geometry in LQG and in particular implies that there is a smallest non-zero eigenvalue, Δ , of the area operator. Therefore the non-existence of the limit is interpreted as telling us that the procedure of shrinking the area of the loop to zero is inappropriate in quantum theory. Rather, we should shrink the loop till its area equals Δ . Now, given a point (c, p) of the classical phase space, a 'square' loop each of whose sides has length $\bar{\lambda}l_o$ with respect to the fiducial metric q_{ab} has area $\bar{\lambda}^2|p|$ with respect to the physical metric $q_{ab} = |p|l_o^{-2} q_{ab}$. Therefore, to define $\hat{F}_{ab}{}^k$, the holonomy is evaluated around a 'square' loop whose edges are formed by the integral curves of right and left invariant vector fields on \mathbb{S}^3 , with $\bar{\lambda}$ given by

$$\bar{\lambda}^2 |p| = \Delta \equiv 2\sqrt{3} \pi \gamma \,\ell_{\rm Pl}^2 \,, \tag{3.13}$$

so that *physical* area of the surface enclosed by the loop is Δ . (Because of homogeneity, the precise location of the loop is irrelevant.) The resulting $\hat{F}_{ab}^{\ k}$, given by

$$\hat{F}_{ab}{}^{k} = \frac{1}{\bar{\lambda}l_o^2} \left(\sin^2 \bar{\lambda} (c - l_o/2) - \sin^2(\bar{\lambda}l_o/2) \right) \epsilon_{ij}{}^{k}{}^o \omega_a^i {}^o \omega_b^j, \tag{3.14}$$

is well-defined but, as in full LQG, has a built in fundamental non-locality.

It was clear from the beginning that something like this must occur because there is no connection operator in LQG. Recall that passage to quantum theory from the classical one *always* requires new physical inputs. Quantization of $F_{ab}^{\ k}$ is well motivated in the representation of the holonomy-flux algebra we are forced to use by the requirement of back

⁷ In full LQG, one often defines a 'Master Hamiltonian constraint' only on diffeomorphism invariant states [6]. On this restricted class of states, the limit of the constraint operator as the area shrinks to zero is well defined. In LQC this avenue is not easily available because the diffeomorphism freedom is gauge fixed in the very beginning while parameterizing the connection and triad by c and p.

ground independence. Furthermore, one can show that $\hat{F}_{ab}{}^k$ does reduce, in a precise sense, to the standard local expression of curvature $F_{ab}{}^k$ in the classical limit. (For details, see [16, 19].) Hence $\hat{F}_{ab}{}^k$ is a well-defined quantization of $F_{ab}{}^k$. However, the non-locality makes a crucial difference in the Planck regime. In particular, as we will now see, it makes the gravitational part \hat{C}_{grav} of the constraint a difference operator, rather than a differential operator as in the Wheeler-DeWitt theory. As one might expect, the 'step size' is governed by the area gap Δ .

Using (3.14) and the the operator corresponding to $\epsilon^{ij}_k e^{-1} E^a_i E^b_j$ given by the Thiemann procedure it is straightforward to construct the gravitational part \hat{C}_{grav} of the constraint operator corresponding to (3.7). It turns out [16, 19] that its expression is simplest if one considers wave functions which are naturally diagonal in the volume operator \hat{V} rather than in \hat{p} . (Mathematically, this is a rather trivial transformation since $\hat{V} = |\hat{p}|^{3/2}$.) It is convenient to parameterize the eigenvalues of the volume operator by v such that:

$$\hat{V}|v\rangle = (\frac{8\pi\gamma}{6})^{\frac{3}{2}} \frac{|v|}{K} \ell_{\text{Pl}}^3 |v\rangle \quad \text{where} \quad K = \frac{2\sqrt{2}}{3\sqrt{3\sqrt{3}}}.$$
 (3.15)

Then, the gravitational part of the constraint is given by:

$$\hat{C}_{\text{grav}} \Psi(v) = e^{if\ell_o} \sin \bar{\lambda} c \, \hat{A} \sin \bar{\lambda} c \, e^{-i\ell_o f} \, \Psi(v)
- \left[\sin^2 \frac{\bar{\lambda}\ell_o}{2} - \frac{\bar{\lambda}^2 \ell_o^2}{4} - \frac{\ell_o^2}{9|K^2 v|^{\frac{2}{3}}} \right] \hat{A} \, \Psi(v)$$
(3.16)

Here, f is a simple function of v, $f(v) = (\operatorname{sgn} v/4) |v/K|^{2/3}$, and the operator \hat{A} , given by

$$\hat{A}\Psi(v) = -\frac{27K}{4}\sqrt{\frac{8\pi}{6}}\frac{\ell_{\text{Pl}}}{\gamma^{3/2}}|v| ||v-1| - |v+1|| \Psi(v), \qquad (3.17)$$

is also diagonal in v.

To summarize, if we mimic full LQG, we are naturally led to a theory that is inequivalent to the WDW theory even at the kinematical level. Indeed the two Hilbert spaces have no non-zero element in common. A key distinguishing feature of this theory is that while the holonomies of the connection are well defined operators, connections themselves are not. Consequently, in the definition of the quantum Hamiltonian constraint (3.7), we are led to define the field strength through holonomies along loops which enclose area Δ , the smallest non-zero eigenvalue of the area operator. The gravitational part of the resulting Hamiltonian constraint is a difference operator, rather than a differential operator that would have resulted had we worked in a Schrödinger type representation used in the WDW theory. However, in a well-defined sense it reduces to the WDW differential operator for large v [16].

D. The Full Hamiltonian Constraint and Main Results

As in the WDW theory, we will now assume that the only matter field is a massless scalar field (although it is straightforward to allow additional matter fields). The form of the Hamiltonian constraint is such that this field can serve as an internal clock also in LQC and we can again use relational dynamics. This simplifies the intermediate technical steps

and makes the physical meaning of results more intuitive.

To write the complete Hamiltonian constraint we also need the matter part. For the massless scalar field, in the classical theory it is given by:

$$C_{\text{matt}} = 8\pi G |p|^{-\frac{3}{2}} p_{\phi}^2 \tag{3.18}$$

The non-trivial part in the passage to quantum theory is the function $|p|^{-3/2}$. However, we can define this operator again by using the method introduced by Thiemann in the full theory [6, 27, 28]. The final result is [19]:

$$\widehat{|p|^{-\frac{3}{2}}}\Psi(v) = \left(\frac{6}{8\pi\gamma\ell_{\rm Pl}^2}\right)^{3/2} B(v) \Psi(v)$$
(3.19)

where

$$B(v) = \left(\frac{3}{2}\right)^3 K |v| \left| |v+1|^{1/3} - |v-1|^{1/3} \right|^3.$$
 (3.20)

This operator is self-adjoint on \mathcal{H}_{kin}^{grav} and diagonal in the eigenstates of the volume operator. We can express the total constraint

$$\hat{C}\Psi(v) = \left(\hat{C}_{\text{grav}} + \hat{C}_{\text{matt}}\right)\Psi(v) = 0, \qquad (3.21)$$

as follows:

$$\partial_{\phi}^{2} \Psi(v, \phi) = -\Theta \Psi(v, \phi)
= -\Theta_{o} \Psi(v, \phi) + \frac{\pi G}{2} [B(v)]^{-1} \left[3K(\sin^{2}(\frac{\bar{\lambda}\ell_{o}}{2}) - \frac{\bar{\lambda}^{2}\ell_{o}^{2}}{4}) |v| \right]
- \frac{1}{3} \ell_{o}^{2} \gamma^{2} \left| \frac{v}{K} \right|^{\frac{1}{3}} \left[\left| |v - 1| - |v + 1| \right| \right] \Psi(v, \phi).$$
(3.22)

Here, Θ_o is a difference operator,

$$\Theta_o \Psi(v,\phi) = -[B(v)]^{-1} \left(C^+(v) \, \Psi(v+4,\phi) + C^o(v) \, \Psi(v,\phi) + C^-(v) \, \Psi(v-4,\phi) \right) , \quad (3.23)$$

where the coefficients $C^{\pm}(v)$ and $C^{o}(V)$ are given by:

$$C^{+}(v) = \frac{3\pi KG}{8} |v+2| ||v+1| - |v+3||$$

$$C^{-}(v) = C^{+}(v-4)$$

$$C^{o}(v) = -C^{+}(v) - C^{-}(v) .$$
(3.24)

For the k=0 (i.e., flat) FRW model the quantum constraint is obtained by simply replacing Θ by Θ_o in (3.22) [19]. Thus, the k=0 quantum constraint has the same form as in the k=1 case. As one would expect from the classical expression (3.21), the difference $\Theta - \Theta_o$ is diagonal in the v-representation and vanishes when we set $\ell_o = 0$.

Let us continue with the k=1 case. The form of the Hamiltonian constraint is similar to that of a massless Klein-Gordon field in a static space-time, with a static potential. ϕ is the analog of the static time coordinate and the difference operator Θ , of the spatial

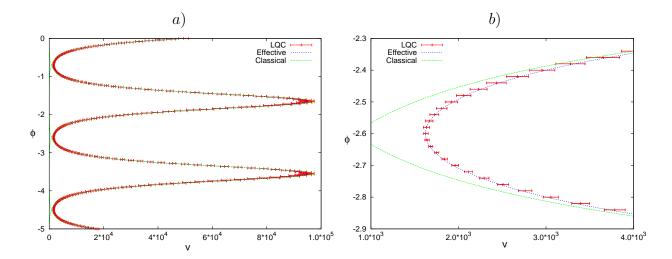


FIG. 2: In the LQC evolution the big bang and big crunch singularities are replaced by quantum bounces. a) Expectation values and dispersion of $|\hat{v}|_{\phi}$ are compared with the classical trajectory and the trajectory from effective Friedmann dynamics (see (3.27)). The classical trajectory deviates significantly from the quantum evolution at Planck scale and evolves into singularities. The effective trajectory provides an excellent approximation to quantum evolution at all scales. b) Zoom on the portion near the bounce point of comparison between the expectation values and dispersion of $\hat{v}|_{\phi}$, the classical trajectory and the solution to effective dynamics. At large values of $|v|_{\phi}$ the classical trajectory approaches the quantum evolution. In this simulation $p_{\phi}^{\star} = 5 \times 10^3$, $\Delta p_{\phi}/p_{\phi}^{\star} = 0.018$, and $v^{\star} = 5 \times 10^4$.

Laplace-type operator plus the static potential. Hence, the scalar field ϕ can again be used as 'emergent time' in the quantum theory. As in the WDW case, \hat{p}_{ϕ} and $\hat{V}|_{\phi_o}$ provide a complete set of Dirac observables. The physical inner product can again be fixed either by requiring that these operators be self-adjoint or by the group averaging method [25]. This provides us with the *physical sector* of the theory. As in the WDW case, one can construct states which are sharply peaked at given values p_{ϕ}^{\star} and v^{\star} of \hat{p}_{ϕ} and $\hat{V}|_{\phi^{\star}}$ at an instant of 'time' ϕ^{\star} and use (3.22) to 'evolve' them. While the general structure of the resulting LQC theory is thus analogous to that of the WDW theory, there is a dramatic difference the final results.

Numerical techniques play a vital role in this analysis. More precisely, numerical evolutions have been performed for several values of parameters and using two different techniques. They attest to the robustness of results. While solutions mimic the WDW behavior when the universe is large, there is a radical departure in the strong curvature region: the big bang and the big crunch singularities are resolved and replaced by big-bounces. The main results can be summarized as follows. (For details, see [16].)

• Consider a classical solution depicted in Fig.1 a which evolves from the big-bang to the big crunch, reaching a large maximum radius a_{max} . Fix a point on this trajectory where the universe has reached macroscopic size and consider a semi-classical state peaked at this point. Such states remain sharply peaked throughout the given 'cycle', i.e., from the quantum bounce near the classical big-bang to the quantum bounce near the classical big-crunch. Note that the notion of semi-classicality used here is rather weak: these results hold even for universes with $a_{\text{max}} \approx 23\ell_{\text{Pl}}$ and the 'sharply peaked' property improves greatly as a_{max}

grows.

- The trajectory defined by the expectation values of the Dirac observable $\hat{V}|_{\phi}$ in the full quantum theory is in good agreement with the trajectory defined by the classical Friedmann dynamics until the 4-d scalar curvature (the only independent curvature invariant in isotropic, homogeneous models) attains the value $\approx 13\pi/\ell_{\rm Pl}^2$, or, equivalently, the energy density of the scalar field becomes comparable to a critical energy density $\rho_{\rm crit} \approx 0.82 \rho_{\rm Pl}$. Then the classical trajectory deviates from the quantum evolution. In the classical solution, scalar curvature and the matter energy density keeps increasing on further evolution, eventually leading to a big bang (respectively, big crunch) singularity in the backward (respectively, forward) evolution, when $v \to 0$. The situation is very different with quantum evolution. Now the universe bounces at $\rho \approx 0.82 \rho_{\rm Pl}$, avoiding the past (or the big bang) and future (or the big crunch) singularities.
- The volume of the universe takes its minimum value V_{\min} at the bounce point. V_{\min} scales linearly with p_{ϕ} : ⁸

$$V_{\min} = \left(\frac{4\pi G \gamma^2 \Delta}{3}\right)^{\frac{1}{2}} p_{\phi} \approx (1.28 \times 10^{-33} \,\text{cm}) \,p_{\phi}$$
 (3.25)

Consequently, $V_{\rm min}$ can be much larger than the Planck size. Consider for example a quantum state describing a universe which attains a maximum radius of a megaparsec. Then the quantum bounce occurs when the volume reaches the value $V_{\rm min} \approx 5.7 \times 10^{16} \, {\rm cm}^3$, some 10^{115} times the Planck volume. Deviations from the classical behavior are triggered when the density or curvature reaches the Planck scale, even when the volume is large. Since $V_{\rm min}$ is large at the bounce point of macroscopic universes, the so-called 'inverse volume effects'—i.e. the fact that $B(v) \neq 1/v$ — are largely insignificant to the quantum dynamics of such universes.

• After the quantum bounce the energy density of the universe decreases and, when $\rho \ll \rho_{\text{max}}$, the quantum evolution is well-approximated by the classical trajectory. On subsequent evolution, the universe recollapses both in classical and quantum theory at the value $V = V_{\text{max}}$ when energy density reaches a minimum value ρ_{min} . V_{max} scales as the 3/2-power of p_{ϕ} :

$$V_{\text{max}} = (16\pi G/3\ell_o^2)^{\frac{3}{4}} p_\phi^{\frac{3}{2}} \approx 0.6 p_\phi^{\frac{3}{2}}$$
(3.26)

Quantum corrections to the classical Friedmann formula $\rho_{\min} = 3/8\pi G a_{\max}^2$ are of the order $O(\ell_{\rm Pl}/a_{\max})^4$. For a universe with $a_{\max} = 23\ell_{\rm Pl}$, the correction is only one part in 10^{-5} . For universes which grow to macroscopic sizes, classical general relativity is essentially exact near the recollapse.

• There is a Hamiltonian analog of the 'effective action' framework [16, 29] which provides a systematic procedure to obtain 'effective equations' from quantum theory. While the classical Friedmann equation is $(\dot{a}/a)^2 = (8\pi G/3)(\rho - 3/8\pi Ga^2)$, the effective Friedmann equation turns out to be

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3} \left(\rho - \rho_1(v)\right) \left[\rho_2(v) - \rho/\rho_{\text{crit}}\right]$$
(3.27)

⁸ Here and in what follows, numerical values are given in the classical units G=c=1. In these units p_{ϕ} has the same physical dimensions as \hbar and the numerical value of \hbar is $2.5 \times 10^{-66} \text{cm}^2$.

where ρ_1 and ρ_2 are specific functions of v. The term in the square bracket is the key quantum correction; away from the Planck regime, $\rho_2 \approx 1$ and $\rho/\rho_{\rm crit} \approx 0$. Bounces occur when \dot{a} vanishes, i.e. at the value of v at which the matter density equals $\rho_1(v)$ or $\rho_2(v)$. $\rho(v) = \rho_1(v)$ at the classical recollapse while $\rho(v) = \rho_2(v)$ at the quantum bounce.

- For quantum states under discussion, the density $\rho_{\rm max}$ is well approximated by $\rho_{\rm crit} \approx 0.82 \rho_{\rm Pl}$ up to terms $O(\ell_{\rm Pl}^2/a_{\rm min}^2)$, independently of the details of the state and values of p_{ϕ} . (For a universe with maximum radius of a megaparsec, $\ell_{\rm Pl}^2/a_{\rm min}^2 \approx 10^{-76}$.) The density $\rho_{\rm min}$ at the recollapse point also agrees with the value $(3/8\pi G a_{\rm max}^2)$ predicted by the classical evolution to terms of the order $O(\ell_{\rm Pl}^4/a_{\rm max}^4)$. Furthermore the scale factor $a_{\rm max}$ at which recollapse occurs in the quantum theory agrees to a very good precision with the one predicted by the classical dynamics.
- The trajectory obtained from effective Friedmann dynamics is in excellent agreement with quantum dynamics throughout the evolution. In particular, the maximum and minimum energy densities predicted by the effective description agree with the corresponding expectation values of the density operator $\hat{\rho} \equiv p_{\phi}^2/|p|^3$ computed numerically.

 The state remains sharply peaked for a very large number of 'cycles'. Consider the
- The state remains sharply peaked for a very large number of 'cycles'. Consider the example of a semi-classical state with an almost equal relative dispersion in p_{ϕ} and $|v|_{\phi}$ and peaked at a large classical universe of the size of a megaparsec. When evolved, it remains sharply peaked with relative dispersion in $|v|_{\phi}$ of the order of 10^{-6} even after 10^{50} cycles of contraction and expansion! Any given quantum state eventually ceases to be sharply peaked in $|v|_{\phi}$ (although it continues to be sharply peaked in p_{ϕ}). Nonetheless, the quantum evolution continues to be deterministic and well-defined for an infinite number of cycles. This is in sharp contrast with the classical theory where the equations break down at singularities and there is no deterministic evolution from one cycle to the next. In this sense, in LQC the k=1 universe is cyclic, devoid of singularities. This non-singular evolution holds for all states, not just the ones which are semi-classical at late times. There is no fine tuning of initial conditions. Also, there is no violation of energy conditions. Indeed, quantum corrections to the matter Hamiltonian do not play any role in the resolution of the singularity. The standard singularity theorems are evaded because the geometrical side of the classical Einstein's equations is modified by the quantum geometry corrections of LQC.

To summarize, the issues raised in section I have all been answered in the FRW models. The main departures from the WDW theory occur due to quantum geometry effects of LQG. These effects are small but dominate the Planck scale physics by creating an effective repulsive force which can overwhelm gravitational attraction. While these effects are small outside the Planck regime, in principle, they could have accumulated and led to departures from general relativity even in weak field regime on the very long time scales that are relevant to cosmology. This does not happen. While they dominate the Planck regime, the quantum geometry effects die extremely quickly outside this regime so that in the weak field regime LQC is in excellent agreement with general relativity even on the very large cosmological time scales.

⁹ For k=0, i.e. open universes, the Friedmann equation $(\dot{a}/a)^2=(8\pi G/3)\,\rho$ is replaced just by $(\dot{a}/a)^2=(8\pi G/3)\,(\rho-\rho_{\rm crit})$. Since there is no classical recollapse, there is a single pre-big-bang contracting branch which is joined deterministically to the post-big-bang expanding branch. For details see [19].

IV. DISCUSSION

In the last two section, I focused on a simple model to illustrate how one might hope to solve the long standing problems of quantum gravity using LQG. As I mentioned in the beginning of section II, several other cosmological models have been analyzed. Roughly, work to date can be divided into three categories.

- In the isotropic models with free scalar fields, with and without cosmological constant, there are detailed analytical as well as numerical frameworks to answer all questions of physical interest. In these models, there is only one non-trivial curvature invariant—the scalar curvature R— and in the classical theory it is related very simply to energy density. Therefore the onset of the quantum epoch which signals departures from classical general relativity can be described by critical values of either the scalar curvature or the density [16, 19].¹⁰
- For the Kasner model with anisotropies as well as models with physically interesting potentials for scalar fields, the physical sector of the theory can be constructed along the same lines. However, the analysis of effective equations is still somewhat incomplete and numerical analysis is still in infancy. In the anisotropic models the key features of the isotropic models —resolution of singularities and emergence of semiclassical pre-big-bang branches— persist. But new phenomena also emerge. With massless scalar fields as sources, the scalar curvature R is again determined by the matter density but there are also other curvature invariants in particular because the Weyl tensor is no longer zero. Now the quantum epoch is reached when any one of these invariants reaches the Planck regime, whence multiple bounces can occur [21]. Density is no longer the governing factor; rather it is space-time curvature. In models with potentials, the scalar field does not always serve as internal time globally. However, it is still possible to construct the physical Hilbert space using group averaging [25] and introduce relational observables. A global emergent time aids our intuition immensely but is not essential in the construction of the physical sector of the theory.
- In more complicated models —black hole interiors [31] and the so called midisuperspaces [32] which are symmetry reduced but have an *infinite* number of degrees of freedom— the Hamiltonian constraint has been written down. It does not break down near the putative singularities signalling that they are resolved. However, the physical sector of the theory is yet to be constructed and numerical analysis has not yet been undertaken. But there do not appear to be any unsurmountable difficulties for these investigations to reach maturity in the near future.

These advances are encouraging because they deal with the long standing questions I discussed in section I. Furthermore, in contrast to string theory, space-like singularities that are resolved are of direct physical interest. However, the major downside is that these advances are based on symmetry reduction and the precise relation between these models and the full theory is still to be spelled out. While significant efforts are being made on this key problem [33], I think we are still at a preliminary stage largely because we do not have a clear candidate for full LQG.

¹⁰ Quantum geometry effects have also been shown to provide a deterministic evolution in certain cyclic models [30].

What lessons do these cosmological investigations have for full LQG?

By now, the kinematic structure of full LQG is well controlled. Key open issues involve dynamical issues. Work in this area has been primarily focused on mathematical problems in full generality. In the leading approaches, one first solves the diffeomorphism constraint and then imposes the Hamiltonian constraint on the resulting diffeomorphism invariant states. The non-trivial achievement is that there are well-defined candidates for constraint operators [6, 27]. Furthermore advances related to Thiemann and Dittrich's 'Master Constraint' program [34] strongly indicate that, for each admissible choice of the master constraint, one would be able to construct the physical sector of the theory. However, there is still a great deal of freedom in the definition of the constraint operators and, more importantly, the issue of existence of a sufficiently rich semi-classical sector has remained largely open. More recently, via their 'algebraic quantum gravity program,' Giesel and Thiemann [35] have introduced new strategies to address both these issues. Here, the diffeomorphism and the Hamiltonian constraints can be treated on an equal footing and imposed simultaneously. This enables one to address the long standing problem of recovering the classical constraint algebra and it is in fact possible to recover it using suitably defined semi-classical states [36]. Thus there is ongoing progress.

Nonetheless, fresh insights are needed to address key physical problems such as the fate of classical singularities in full LQG and the detailed recovery of Einstein dynamics in the classical limit. Since LQC has provided concrete solutions to these problems in simple models, a useful strategy would be to work 'from bottom up' to less and less symmetric models. For example, in the symmetry reduced systems, one inevitably carries out a gauge fixing which provides a good control on individual space-time geometries rather than equivalence classes of them under diffeomorphisms. Similarly, the standard description of the low energy world involves specific space-time metrics. While one can translate both these descriptions in a manifestly diffeomorphism invariant language, the result would be quite cumbersome. The procedures used in models can complement the more general and more systematic programs that are being pursued in full LQG.

The idea would be to work one's way up by incorporating, at each step, lessons learned from the symmetry reduced models. These models suggest that, to address physically important issues, it may be essential to restrict oneself to interesting sectors of the theory —finite, non-linear neighborhoods of the FRW solutions, or of Minkowski space-time, in the phase space of general relativity— and exploit the additional structure such restrictions make available. Secondly, in LQC one could get truly valuable insights by analyzing in detail states which are semi-classical in a suitable sense. By contrast, the discussion of dynamics in the full theory generally focuses on 'elementary states'—the spin networks. These are analogous to the 'n-photon states' of the Maxwell theory, while the semi-classical states are analogous to the coherent states. Both span the full Hilbert space in the Maxwell theory and are convenient in different regimes. Genuinely new insights could be gained in interesting sectors of LQG if one revisits the issue of constructing and solving the Hamiltonian constraint from a new perspective. As in mini and midi superspace analyses, one could exploit an astute gauge-fixing of (a part of) the diffeomorphism constraint, and the extra structure provided by a basis of semi-classical states. Results of LQC appear to provide valuable guidelines not only for constructing the physical sector of the theory along these lines but also for answering some of the most challenging physical questions.

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