# Mechanics of higher-dimensional black holes in asymptotically anti-de Sitter space-times

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# Abstract

We construct a covariant phase space for Einstein gravity in dimensions  $d \geq 4$  with negative cosmological constant, describing black holes in local equilibrium. Thus, space-times under consideration are asymptotically anti-de Sitter and admit an inner boundary representing an isolated horizon. This allows us to derive a first law of black hole mechanics that involves only quantities defined quasi-locally at the horizon, without having to assume that the bulk space-time is stationary. The first law proposed by Gibbons et al. for the Kerr-AdS family follows from a special case of this much more general first law.

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## I. STATEMENT OF THE PROBLEM

Although our universe has only four large space-time dimensions and a positive cosmological constant, higher dimensional, asymptotically anti-de Sitter (AdS) space-times have featured prominently in the recent mathematical physics literature, especially in connection with the AdS/CFT conjecture. A natural conceptual question in this setting is whether the first law of black hole mechanics

$$\delta M = \frac{\kappa}{8\pi G} \delta a + \Omega \delta J + \Phi \delta Q \tag{1.1}$$

continues to hold, where M, J, Q and a are the black hole mass, angular momentum, charge and horizon area, and  $\kappa, \Omega, \Phi$  denote the surface gravity, angular velocity and the electric potential. Since the last two terms have the interpretation of work done on the black hole, similarity with the first law of thermodynamics suggests that we interpret the first term as the analog of  $T \delta S$  where T denotes temperature and S, entropy. This in turn opens door to the fertile and challenging field of black hole thermodynamics. Do these considerations, which are well-established in 4 dimensions, go through also in more general situations?

In 4 dimensions, the Kerr-Newman family of stationary black holes admits a natural generalization to the asymptotically AdS context and one can explicitly verify that the family continues to satisfy (1.1). Recall however that, to obtain the first law of thermodynamics, one needs to assume only that the system under consideration is in equilibrium and makes a transition to a nearby equilibrium state; there may well be dynamical processes away from the system. By analogy, one would physically expect the first law of black hole mechanics to hold even in situations in which the black hole itself is in equilibrium, although there may be time dependence in the universe away from the black hole. This expectation is borne out in the isolated horizon framework [1-4] where only the *intrinsic* geometry and physical fields at the horizon are assumed to be stationary; space-time itself need not admit any Killing field. In the resulting extension, all quantities that enter the expression of the first law are defined quasi-locally, at the horizon. In particular, the mass M and the angular momentum J refer to the horizon itself. In absence of global Killing vectors, they are different from the ADM mass and angular momentum at infinity which receive contributions also from matter and gravitational waves in the region between the horizon and infinity. In the vacuum, globally stationary, axi-symmetric space-times the horizon mass and angular momentum coincides with the ADM quantities and the first law of the isolated horizon framework reduces to that on the AdS (or de Sitter) Kerr-Newman family.

In higher dimensions the situation is more complicated. First, even in the stationary, axisymmetric context, solutions describing charged black holes are not known in dimensions higher than 5. Hence one can not explicitly verify (1.1) even in this limited context. Kerr-AdS solutions, on the other hand, are known in all dimensions [5, 6]. Hence it is natural to ask if (1.1) holds without the last term on the right side. Somewhat surprisingly, there has been considerable confusion on this issue. While there is unanimity about the definition of area, surface gravity and angular velocities, there has been lack of agreement on the expressions of total mass and total angular momentum. In particular, different proposals have been put forward by Hawking, Hunter and Taylor-Robinson [5], Berman and Parikh [7], Hawking and Reall [8], and Awad and Johnson [9]. However, Caldarelli, Cognola and Klemm pointed out in 4 dimensions and, more recently, Gibbons, Perry and Pope, in higher dimensions that *none* of these proposed conserved quantities are compatible with the first law [10].

Now, in 4-dimensions and in absence of a cosmological constant, a satisfactory covariant framework describing the asymptotic fields and conserved quantities at spatial infinity,  $i^{o}$ , has been available since the late seventies [11, 12]. Here, one first uses field equations to establish certain identities involving the asymptotic Weyl tensor and other physical fields and then uses the asymptotic Weyl tensor to associate conserved quantities with asymptotic symmetries. Some time ago, Ashtekar and Das (AD) showed that this framework admits a natural generalization to higher dimensional asymptotically AdS space-times [13]. Thus, there is a systematic procedure to use field equations and asymptotic symmetries  $\xi^{a}$  to define conserved quantities  $\mathcal{Q}_{\mathscr{I}}^{(\xi)}$  also in higher dimensions. More recently, using the standard second-order Einstein-Hilbert action, Hollands, Ishibashi and Marolf [17] constructed a covariant phase space of asymptotically AdS solutions and provided a Hamiltonian basis for the AD quantities  $\mathcal{Q}_{\mathscr{I}}^{(\xi)}$ . Finally, Gibbons, Perry and Pope [10] have shown that if one uses the AD definitions of mass and angular momenta in Kerr-AdS solutions, then the first law does hold for this family. This provides a satisfactory resolution of the question of the first law for the Kerr-AdS family in all higher dimensions.

A natural question is whether this law can be extended to situations without global Killing fields through an isolated horizon framework. The purpose of this paper is to answer this question in the affirmative.

This task will require us to construct a Hamiltonian framework for higher dimensional space-times which are asymptotically AdS and admit an inner boundary which is an isolated horizon (representing a black hole in equilibrium). Thus, we will extend three sets of constructions: i) the 4-dimensional isolated horizon framework of [3, 4] with a cosmological constant; ii) the Hollands, Ishibashi, Marolf Hamiltonian framework of [17] for higher dimensional, asymptotically AdS space-times without internal boundaries; and iii) the higher dimensional isolated horizon framework without a cosmological constant of Korzyński, Lewandowski and Pawlowski [18]. As in [3, 4] and [18], we will begin with a first order Palatini action and construct a covariant phase space based on vielbeins and Lorentz connections. The conserved quantities of interest will again emerge from two surface terms, one at infinity and one at the horizon. The terms at infinity will agree with those obtained by Hollands, Ishibashi and Marolf [17] using a covariant phase space based on metrics, i.e., will reproduce the AD quantities  $\mathcal{Q}_{\mathscr{J}}^{(\xi)}$ . The terms at the horizon will define the horizon angular momenta and mass. In general, the surface integrals at the horizon will differ from those at infinity, the difference accounting for the energy and angular momentum in the matter and gravitational radiation in the region between the horizon and infinity. However, in presence of global Killing fields, the two integrals coincide. In particular, when restricted to the Kerr-AdS family, our first law will reduce to that obtained by Caldarelli, Cognola and Klemm in four dimensions and by Gibbons, Perry and Pope in higher dimensions. Because several of the necessary techniques and intermediate steps have been discussed in [3, 4, 17, 18], we will skip the corresponding details. A more complete discussion can be found in [19].

This paper is structured as follows. In section II we construct the covariant phase space and in section III we show how conserved quantities emerge as Hamiltonian functions associated with symmetries. Detailed calculations of conserved quantities at infinity are presented in section IV where the AD quantities  $\mathcal{Q}_{\mathscr{I}}^{(\xi)}$  are retrieved. In section V we derive the expres-

<sup>&</sup>lt;sup>1</sup> This generalization was based on the 4-dimensional analysis of [14]. In 4 dimensions, there also exist alternative but essentially equivalent frameworks. See, e.g., [15, 16].

sions of energy and angular momenta at the horizon and obtain the first law. Section VI summarizes the simplifications that occur in presence of global Killing vectors and shows that the general first law obtained in section V reduces to the one obtained by Gibbons, Perry and Pope [10] in the Kerr-AdS family.

In any Hamiltonian framework, the theory of interest has to be specified right in the beginning. In 4 dimensions, the isolated horizon framework has been constructed for general relativity (with and without  $\Lambda$ ) coupled to a large class of matter fields, including Maxwell, dilaton, Yang-Mills, and Higgs fields. However, as noted above, relatively little is known even about the charged Kerr solutions in higher dimensions. Therefore, to keep the discussion simple, in most of the paper we will restrict ourselves to the vacuum case. Also, to avoid detours involving technical subtleties, we will only consider non-extremal isolated horizons. The extremal ones can be handled along the lines of [3, 4, 20]. We will find it convenient to define N := d-2 where d is the number of space-time dimensions. Finally, we will set the 'AdS radius'  $L = \sqrt{-N(N+1)/(2\Lambda)}$  equal to 1 in the intermediate steps and restore it only in the final results.

#### II. COVARIANT PHASE SPACE

We wish to construct a covariant phase space  $\Gamma$  for general relativity in d space-time dimensions with a negative cosmological constant.  $\Gamma$  consists of solutions to Einstein's (and matter-field) equations that are asymptotically AdS and admit a weakly isolated horizon as their inner boundary. We will work with the first order, Palatini framework. Thus the basic variables will be co-frames  $e_a^I$  and Lorentz connections  $A_a^{IJ}$  where the lower case indices  $a, b, \ldots$  denote space-time indices and the upper case indices  $I, J, \ldots$ , the internal (or frame) indices. The space-time metric is defined by  $g_{ab} = \eta_{IJ}e_a^Ie_b^J$  where  $\eta_{IJ}$  is a fix Minkowskian metric on the internal space  $\mathbb{R}^d$ . Throughout we will assume that  $g_{ab}$  satisfies the Einstein's equations  $G_{ab} + \Lambda g_{ab} = 8\pi G T_{ab}$ , for a suitable matter stress-energy tensor  $T_{ab}$  (which will be set to zero in later sections).

In the first order framework, one of the equations of motion implies that the Lorentz connection  $A_a^{IJ}$  is compatible with the co-frame  $e_a^I$  via  $A_a^{IJ} = e^{bI} \nabla_a e_b^J$ , where  $\nabla$  is the derivative operator compatible with  $g_{ab}$ . Although the number of basic variables is larger than the metrics  $g_{ab}$  used in the second order Einstein-Hilbert framework, detailed calculations are considerably simpler and generally more transparent because they only involve forms and exterior calculus.

This section is divided into two parts. In the first, we specify boundary conditions both at the outer boundary  $\mathscr{I}$  at infinity and at the inner boundary representing the isolated horizon  $\Delta$ . In the second, we construct the covariant phase space.

#### A. Boundary conditions

Let  $\mathcal{M}$  be an (N+2)-dimensional manifold bounded by three (N+1)-dimensional manifolds,  $M_1$ ,  $M_2$  and  $\Delta$  (see Fig. 1). Dynamical fields  $e_a^I$  will be restricted such that  $M_1$  and  $M_2$  are space-like, while the surface  $\Delta$  connecting them is null, with topology  $\mathbb{S} \times \mathbb{R}$  where  $\mathbb{S}$  is a compact N-dimensional manifold. Fields  $(e_a^I, A_a^{IJ})$  will be subject to boundary conditions both at infinity and at the horizon, specified below.

Conditions at infinity will ensure that space-times  $(\mathcal{M}, g_{ab})$  are asymptotically anti-de

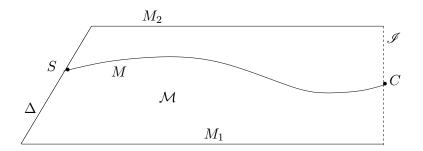


FIG. 1: The N+2 dimensional space-time  $\mathcal{M}$  under consideration has an internal boundary  $\Delta$  and is bounded by two N+1-dimensional space-like surfaces  $M_1$  and  $M_2$ . M is a partial Cauchy surface in  $\mathcal{M}$  which intersects  $\Delta$  in a compact N-dimensional surface S and  $\mathscr{I}$  in a N dimensional sphere C.

Sitter. More precisely, we will restrict ourselves to fields  $(e_a^I, A_a^{IJ})$  such that there exists a manifold  $\tilde{\mathcal{M}}$  with outer boundary  $\mathscr{I}$ , equipped with a metric  $\tilde{g}_{ab}$  and a diffeomorphism from  $\mathcal{M}$  onto  $\tilde{\mathcal{M}} - \mathscr{I}$  such that:

- 1. There exists a smooth function  $\Omega$  on  $\tilde{\mathcal{M}}$  for which  $\tilde{g}_{ab} = \Omega^2 g_{ab}$  on  $\mathcal{M}$ ;
- 2.  $\mathscr{I}$  is topologically  $\mathbb{S}^N \times [0,1]$ , and  $\Omega$  vanishes on  $\mathscr{I}$  but its gradient  $\nabla_a \Omega$  is nowhere vanishing on  $\mathscr{I}$ ;
- 3. On  $\mathcal{M}$ , the stress-energy tensor  $T_{ab}$  is such that  $\Omega^{-N}T_{ab}$  admits a smooth limit to  $\mathscr{I}$ . Further,  $A_a^{IJ}$  is compatible with  $e_I^a$  (i.e.,  $A_a^{IJ}=e^{bI}\nabla_a e_b^J$ ) in some neighborhood of  $\mathscr{I}$ .

In what follows, we will identify  $\mathcal{M}$  with its image in  $\tilde{\mathcal{M}}$ . The fall-off condition on the physical stress-energy  $T_{ab}$  is tailored to make (the energy and) the total angular momentum of matter fields well-defined in the (stationary and) axi-symmetric context. These boundary conditions are the standard ones [13].

We will now recall from [13] and especially [17] some consequences of these boundary conditions which will play an important role in the subsequent sections. The main idea is to use the freedom in the conformal factor and the choice of coordinates together with asymptotic field equations to obtain a convenient Taylor expansion of the rescaled metric  $\tilde{g}_{ab}$  in a neighborhood of  $\mathscr{I}$ . The leading order coefficients in the Taylor expansion then have direct geometrical meaning, being related to the asymptotic Weyl curvature. This idea was first implemented by Beig and Schmidt [12] in the asymptotically flat, 4-dimensional context already in the early eighties.

In the present case, it immediately follows from the field equations (i.e., condition 3 in the definition) that  $\mathscr{I}$  is time-like. Hence, there exists a neighborhood of  $\mathscr{I}$  in which the (N+1)-dimensional surfaces  $\Omega=$  const are all time-like. We will restrict ourselves to this neighborhood. Next, let us set  $\tilde{n}_a=\tilde{\nabla}_a\Omega$  so the conformally rescaled metric has the form  $\tilde{g}_{ab}=\tilde{n}_a\tilde{n}_b/(\tilde{n}\cdot\tilde{n})+\tilde{h}_{ab}$ , where  $\tilde{h}_{ab}$  is the induced metric (of signature (1,N)) on the level surfaces of  $\Omega$ . One can now Taylor-expand  $\tilde{g}_{ab}$  in  $\Omega$  and use the conformal freedom to simplify its form. More precisely, by a judicious modification  $\Omega\mapsto e^\alpha\Omega$  of the conformal factor and use of Einstein's equations, one can set  $\tilde{n}^c\tilde{n}_c=1$  to all orders in  $\Omega$  so the metric takes the form

$$\tilde{g}_{ab} = \tilde{n}_a \tilde{n}_b + \tilde{h}_{ab}. \tag{2.1}$$

Next, let us Taylor expanding  $\tilde{h}_{ab}$  as

$$\tilde{h}_{ab} = \sum_{j=0}^{n} \Omega^{j} \, \tilde{h}_{ab}^{(j)} + \mathcal{R}_{ab} \tag{2.2}$$

for a sufficiently high n, where the remainder  $\mathcal{R}_{ab}$  is of the order  $\mathcal{O}(\Omega^{n+1})$  (i.e., is such that even when multiplied by  $\Omega^{-(n+1)}$ , it admits a smooth limit to  $\mathscr{I}$ ). Using Einstein's equations, one arrives at a recursive expression for the  $\tilde{h}_{ab}^{(j)}$ . The  $\tilde{h}_{ab}^{(j)}$  can be made to agree with those of the pure AdS space-time up to order  $\mathcal{O}(\Omega^N)$ . If the space-time dimension d is different from 5, the first correction, of the order  $\mathcal{O}(\Omega^{N+1})$ , is given by

$$\tilde{h}_{ab}^{(N+1)} = -\frac{2}{N+1}\tilde{E}_{ab},\tag{2.3}$$

where  $\tilde{E}_{ab}$  is the appropriately rescaled, leading-order electric part of the Weyl tensor [13]:

$$\tilde{E}_{ab} := \frac{1}{N-1} \Omega^{1-N} \, \tilde{C}_{acb}{}^d \tilde{n}^c \tilde{n}_d. \tag{2.4}$$

Hence the conformally rescaled metric takes the following form.

$$\tilde{g}_{ab} = \tilde{\nabla}_a \Omega \tilde{\nabla}_b \Omega - \left(1 + \frac{1}{4}\Omega^2\right)^2 \tilde{\nabla}_a t \tilde{\nabla}_b t + \left(1 - \frac{1}{4}\Omega^2\right)^2 S_{ab}$$

$$-\frac{2}{N+1} \Omega^{N+1} \tilde{E}_{ab} + \mathcal{O}(\Omega^{N+2}),$$
(2.5)

where  $S_{ab}$  is the metric on an unit N sphere. The case d = 5 (i.e. N = 3) is exceptional. In this case, the leading correction to the AdS metric turns out to be:

$$\tilde{h}_{ab}^{(N+1)} = -\frac{2}{N+1}\tilde{E}_{ab} - \frac{1}{16}\tilde{h}_{ab}^{0},\tag{2.6}$$

where  $\tilde{h}_{ab}^0$  is the intrinsic metric on  $\mathscr{I}$  (that of the Einstein cylinder). In this case, the asymptotic form of  $\tilde{g}_{ab}$  is given by

$$\tilde{g}_{ab} = \tilde{\nabla}_a \Omega \, \tilde{\nabla}_b \Omega - \left(1 + \frac{1}{2} \Omega^2\right) \, \tilde{\nabla}_a t \, \tilde{\nabla}_b t + \left(1 - \frac{1}{2} \Omega^2\right) \, S_{ab}$$

$$- \frac{2}{N+1} \Omega^{N+1} \tilde{E}_{ab} + \mathcal{O}(\Omega^{N+2}).$$
(2.7)

Without loss of generality one can assume that all the metrics  $\tilde{g}_{ab}$  of interest have the above asymptotic forms for a fixed conformal factor  $\Omega$  in a neighborhood of  $\mathscr{I}$  in  $\tilde{\mathcal{M}}$ . This restriction fixes the conformal freedom in the neighborhood and the diffeomorphism freedom to a certain asymptotic order.<sup>2</sup>

Next, let us consider the inner boundary  $\Delta$ . Now the physical fields  $(e_a^I, A_a^{IJ})$  will be assumed to be restricted to ensure that  $\Delta$  is a weakly isolated horizon. More precisely, we assume that  $\Delta$  is a null (N+1)-dimensional sub-manifold of  $\mathcal{M}$ , equipped with a preferred family  $[\ell]$  of future directed null normals such that:

<sup>&</sup>lt;sup>2</sup> For details, see [17]. Although this reference assumes vacuum equations, the results quoted above hold even if one allows matter fields whose stress energy tensor is such that  $\Omega^{-N}T_{ab}$  has a smooth limit to  $\mathscr{I}$ .

- 1.  $\ell$  and  $\ell'$  are in  $[\ell]$  if and only if there exists a positive constant c such that  $\ell' \cong c\ell$  where  $\cong$  denotes equality at  $\Delta$ ;
- 2. If  $q_{ab}$  denotes the pull-back to  $\Delta$  of the space-time metric  $g_{ab}$  (so it has signature (0, N)), then  $\mathcal{L}_{\ell}q_{ab} = 0$ ; and, if  $t^a$  is any vector field tangential to  $\Delta$  satisfying  $\mathcal{L}_{\ell}t^a = 0$ , then  $\mathcal{L}_{\ell}(t^a\nabla_a\ell^b) = 0$ ;
- 3. On  $\Delta$ , the stress energy tensor satisfies  $T_{ab}\ell^a\ell^b \geq 0$  and  $A_a^{IJ}$  is compatible with the co-frame, (i.e.,  $A_a^{IJ} = e^{bI}\nabla_a e_b^J$ ).

These conditions capture the idea that  $\Delta$  represents a horizon in equilibrium in the following sense. First, the condition  $\mathcal{L}_{\ell}q_{ab} = 0$  implies that the intrinsic horizon metric is time-independent. In particular, it implies that  $\ell^a$  is expansion free, whence all of its cross-sections have the same area. The Raychaudhuri equation and the energy condition then ensure  $T_{ab}\ell^a\ell^b = 0$ , i.e., that there is no flux of matter across  $\Delta$ . Finally, the condition  $\mathcal{L}_{\ell}q_{ab} = 0$  also implies that there is a 1-form  $\omega_a$  on  $\Delta$  such that  $t^a\nabla_a\ell^b = t^a\omega_a\ell^b$  for all vectors  $t^a$  tangential to  $\Delta$ .  $\omega_a$  is called the rotation 1-form; we will see in section V that it captures all the information pertaining to the horizon angular momentum. The second condition requires that the rotation 1-form also be 'time-independent'. (For further details, see [3, 4, 18].)

Just as we had fixed once and for all  $\Omega$  and hence  $\tilde{n}^a$  in a neighborhood of  $\mathscr{I}$ , we will now fix an equivalence class  $[\ell^a]$  on  $\Delta$ , common to all geometries in the phase space. As at  $\mathcal{I}$ , it is convenient to partially fix the gauge on  $\Delta$ . We will restrict our dynamical frame fields such that the (N+1)th frame field  $e^a_{N+1}$  is in the equivalence class of preferred null normals  $[\ell^a]$  and denote it by  $\ell^a$ . We will also denote the pull-back  $e^{N+1}_a$  of the (N+1)th co-frame field  $e^{N+1}_a$  by  $n_a$ , so that  $\ell^a n_a = -1$ . Then it is easy to verify [3, 4] that, because  $A^{IJ}_a$  is required to be compatible with the co-frame  $e^I_a$ , the condition  $\mathcal{L}_\ell q_{ab} = 0$  is equivalent to

$$A_{\underline{a}}^{I N+1} \stackrel{\frown}{=} 0 \tag{2.8}$$

for I = 1, ..., N, where again the 'under-bar' denotes 'pull-back to  $\Delta$ '. Similarly, the condition  $\mathcal{L}_{\ell}\omega_a = 0$  is equivalent to:

$$\mathcal{L}_{\ell}A_a^{N+1} \stackrel{N+2}{=} 0. \tag{2.9}$$

These facts will be used in the subsequent sections.

## B. The Symplectic Structure

The covariant phase space  $\Gamma$  consists of smooth solutions  $(e_a^I, A_a^{IJ})$  to the field equations on  $\mathcal{M}$  which satisfy the boundary conditions specified above.

However, to obtain the expression of the symplectic structure on  $\Gamma$ , one first suspends the field equations, considers all pairs  $(e_a^I, A_a^{IJ})$  on  $\mathcal{M}$  satisfying the boundary conditions and introduces the action principle. The symplectic structure results from the second variation of the action. As in the previous literature on isolated horizons, we will use the first order, Palatini action. For vacuum gravity, it is given by:

$$S_P(e,A) = -\frac{1}{16\pi G} \int_{\mathcal{M}} \left[ F^{IJ} \wedge \Sigma_{IJ} - 2\Lambda \epsilon \right] + \frac{1}{16\pi G} \lim_{\Omega_o \to 0} \int_{\Omega = \Omega_o} A^{IJ} \wedge \Sigma_{IJ}. \tag{2.10}$$

Here  $\epsilon$  is the space-time volume element determined by the frame-field,

$$\epsilon = e^1 \wedge e^2 \wedge \dots \wedge e^{N+2},\tag{2.11}$$

 $F^{IJ}$  is the curvature of  $A_a^{IJ}$ ,

$$F^{IJ} = dA^{IJ} + A^{I}{}_{K} \wedge A^{KJ}, \tag{2.12}$$

and  $\Sigma_{IJ}$  is defined by

$$\Sigma_{IJ} := \frac{1}{N!} \epsilon_{IJK_1...K_N} e^{K_1} \wedge \ldots \wedge e^{K_N}, \qquad (2.13)$$

with  $\epsilon_{I_1...I_{N+2}}$  the internal alternating tensor. The integral in the last term of (2.10) is over the level surfaces  $\Omega = \Omega_o$  of the conformal factor defined in the intersection of a neighborhood of  $\mathcal{I}$  with  $\mathcal{M}$ .

The boundary term in the Palatini action, together with our boundary conditions, ensures that the action is differentiable. The resulting equation of motion for  $A_a^{IJ}$  simply says that the connection is determined by the co-frame  $e_a^I$ :

$$D\Sigma_{IJ} := d\Sigma_{IJ} - A^{K}{}_{I} \wedge \Sigma_{KJ} + A^{K}{}_{J} \wedge \Sigma_{KI} = 0$$
(2.14)

The equation of motion for the frame yields:

$$F^{IJ} \wedge \Sigma_{IJK} - 2\Lambda \Sigma_K = 0, \tag{2.15}$$

where the space-time (N+2-k)-forms  $\Sigma_{I_1...I_k}$  are straightforward generalizations of  $\Sigma_{IJ}$  (defined in (2.13)), given by

$$\Sigma_{I_1...I_k} := \frac{1}{(N-k+2)!} \epsilon_{I_1...I_k K_{k+1}...K_{N+2}} e^{K_{k+1}} \wedge ... \wedge e^{K_{N+2}}.$$
 (2.16)

By using (2.14) to express the curvature  $F_{ab}{}^{IJ}$  in terms of the frame fields  $e_a^I$ , it is easy to show that (2.15) reduces to the familiar field equations:

$$G_{ab} + \Lambda g_{ab} = 0. \tag{2.17}$$

Before embarking on the calculation of the second variation of the action and the symplectic structure, let us note a few facts that will be useful later. First, from the definition of  $\Sigma_{I_1,...I_k}$  it follows that

$$\delta \Sigma_{I_1 \dots I_k} = \delta e^L \wedge \Sigma_{I_1 \dots I_k L} \tag{2.18}$$

and

$$\chi \bot \Sigma_{I_1 \dots I_k} = \chi^L \Sigma_{I_1 \dots I_k L} \tag{2.19}$$

where  $\chi^a$  is any vector and  $\chi^L = e_a^L \chi^a$ . Second, taking the variation of the second Einstein equation (2.15), we obtain a useful identity:

$$(\delta F^{IJ}) \wedge \Sigma_{IJK} = -F^{IJ} \wedge \delta e^L \wedge \Sigma_{IJKL} + 2\Lambda \delta e^L \wedge \Sigma_{KL}. \tag{2.20}$$

We are now ready to derive the symplectic structure by considering second variations of the action.<sup>3</sup> Consider any region  $\mathcal{M}'$  in  $\mathcal{M}$  and restrict the Palatini action to this region. For any two vector fields  $\delta_1$  and  $\delta_2$  tangent to the space of our dynamical fields, one has

$$\delta_1 \delta_2 S_P - \delta_2 \delta_1 S_P - [\delta_1, \delta_2] S_P = 0 \tag{2.21}$$

<sup>&</sup>lt;sup>3</sup> As usual, the procedure followed here actually leads to a pre-symplectic structure: a closed 2-form on phase space which is degenerate. The kernel corresponds to infinitesimal gauge transformations. The physical phase space can then be obtained by taking the quotient of the space of solutions by these gauge transformations.

where  $\delta f$  denotes the Lie derivative of the function f along the vector field  $\delta$  and  $[\delta_1, \delta_2]$  is the Lie bracket of the vector fields  $\delta_1$  and  $\delta_2$ . Now, when Einstein equations (2.14) and (2.15) are satisfied, the bulk term in the left hand side vanishes and we have

$$\delta_1 \delta_2 S_P - \delta_2 \delta_1 S_P - [\delta_1, \delta_2] S_P = \frac{1}{16\pi G} \int_{\partial \mathcal{M}'} [\delta_1 A^{IJ} \wedge \delta_2 \Sigma_{IJ} - \delta_2 A^{IJ} \wedge \delta_1 \Sigma_{IJ}]. \tag{2.22}$$

Therefore, if we restrict ourselves to the the covariant phase space  $\Gamma$ , and restrict the vector fields  $\delta_1, \delta_2$  to be tangential to  $\Gamma$ , the right side of (2.22) vanishes for any  $\mathcal{M}'$ . Hence,

$$j(\delta_1, \delta_2) = \frac{1}{16\pi G} [\delta_1 A^{IJ} \wedge \delta_2 \Sigma_{IJ} - \delta_2 A^{IJ} \wedge \delta_1 \Sigma_{IJ}]$$
 (2.23)

is a closed (N+1)-form on  $\mathcal{M}$ . This is the symplectic current.

Let  $\mathcal{M}'$  now be any sub-region in  $\mathcal{M}$ , bounded by partial Cauchy slices  $M'_1, M'_2$  and a portion  $\Delta'$  of the isolated horizon  $\Delta$ . Since  $j(\delta_1, \delta_2)$  is closed on  $\mathcal{M}'$ , we have

$$\int_{M_1'} j(\delta_1, \delta_2) + \int_{M_2'} j(\delta_1, \delta_2) + \int_{\Delta'} j(\delta_1, \delta_2) + \int_{\mathscr{I}} j(\delta_1, \delta_2) = 0.$$
 (2.24)

The idea is to analyze these three fluxes to arrive at a conserved 2-form on the covariant phase space  $\Gamma$ .

First consider the integral of  $j(\delta_1, \delta_2)$  over  $\Delta'$ . Now we can take over arguments from the isolated horizon analysis of [3, 4, 18] in a straightforward manner because they are insensitive to the precise number of space-time dimensions and the value of the cosmological constant. We therefore simply provide the final result. Denote by  $S'_1$  and  $S'_2$  the cross-sections of  $\Delta$  at which  $M'_1$  and  $M'_2$  intersect  $\Delta$ . Given any point  $(e^I_a, A^{IJ}_a)$  of  $\Gamma$ , let  $\psi$  on  $\Delta$  be the potential for the surface gravity  $\kappa_{(\ell)} := \ell^a \omega_a$  defined via:  $\mathcal{L}_{\ell} \psi = \kappa_{(\ell)}$  and  $\psi \mid_{S_1} = 0$  where  $S_1$  is the past boundary of  $\Delta$ . Then,

$$\int_{\Delta'} j(\delta_1, \delta_2) = \frac{1}{8\pi G} \left( \oint_{S_2'} - \oint_{S_1'} \right) \left[ \delta_1 \bar{\epsilon} \, \delta_2 \psi - \delta_2 \bar{\epsilon} \, \delta_1 \psi \right], \tag{2.25}$$

where  $\bar{\epsilon}$  is the area element of the induced geometry on  $S'_{1,2}$ . All integrals in questions are well-defined because the integrands involve smooth fields and the domains are compact (possibly with boundary).

Next, let us consider the integrals over  $M'_1$  and  $M'_2$ . Since these manifolds are non-compact, we need to first establish the finiteness of the flux of the symplectic current. The idea is to express the integrands in terms of conformally rescaled fields which are well-behaved on  $\tilde{\mathcal{M}}$  and carry out the integrals on  $M'_{1,2} \cup C'_{1,2}$  where  $C'_{1,2}$  are the cross-sections at which  $M'_{1,2}$  intersect  $\mathscr{I}$ . Since these manifolds with boundary are compact, the integrals are guaranteed to be well-defined if the integrands are smooth.

Thus, for an arbitrary vector field  $\delta$  in the tangent space of  $\Gamma$ , we need to express  $\delta \Sigma_{IJ}$  and  $\delta A^{IJ}$  in terms of suitably conformally rescaled fields. To compute these, we use the expressions (2.7) and (2.5) for the conformal metric, together with the fact that since  $\Omega$  is fixed on phase space,  $\delta\Omega = 0$ . For arbitrary  $N \geq 2$  (i.e.,  $d \geq 4$ ) we then obtain

$$\delta g_{ab} = -\frac{2}{N+1} \Omega^{N+1} \delta \tilde{E}_{ab} + \mathcal{O}(\Omega^{N+2}). \tag{2.26}$$

We are now ready to evaluate  $\delta \Sigma_{IJ}$  and  $\delta A^{IJ}$ . Using the definition (2.13), for the former we have

$$\delta\Sigma_{IJ} = \frac{1}{(N-1)!} \epsilon_{IJK_1...K_N} e^{K_1} \wedge \ldots \wedge e^{K_{N-1}} \wedge \delta e^{K_N}. \tag{2.27}$$

Using

$$\delta e_a^I = \frac{1}{2} (\delta g_{ab}) e^{bI} = -\frac{1}{N+1} \Omega^N \delta \tilde{E}_{ab} \tilde{e}^{bI} + \mathcal{O}(\Omega^{N+1}), \tag{2.28}$$

together with  $\tilde{e}_a^I = \Omega e_a^I$  and  $\delta\Omega = 0$ , we arrive at

$$\delta \Sigma_{IJa_1...a_N} = -\frac{N}{N+1} \Omega \epsilon_{IJK_1...K_N} \tilde{e}_{[a_1}^{K_1} \dots \tilde{e}_{a_{N-1}}^{K_{N-1}} \delta \tilde{E}_{a_N]c} \tilde{e}^{cK_N} + \mathcal{O}(\Omega^2).$$
 (2.29)

Thus,  $\delta \Sigma_{IJ}$  is of order  $\mathcal{O}(\Omega)$ . Computing the variation of  $A^{IJ}$  is somewhat more involved; a detailed calculation shows that

$$\delta A_a^{IJ} = \delta(e^{bI}\nabla_a e_b^J) = -2\frac{N}{N+1}\Omega^N \delta \tilde{E}_{ab} \tilde{e}^{b[I} \tilde{e}^{c|J]} \tilde{n}_c + \mathcal{O}(\Omega^{N+1}). \tag{2.30}$$

Thus  $\delta A_a^{IJ}$  is of order  $\mathcal{O}(\Omega^N)$ . Collecting these two results we conclude for any two vector fields  $\delta_1$ ,  $\delta_2$  on  $\Gamma$ ,

$$\delta_1 A^{IJ} \wedge \delta_2 \Sigma_{IJ} = \tilde{f} \, \tilde{\alpha},\tag{2.31}$$

where  $\tilde{f}$  is a function on  $\mathcal{M}$  which asymptotically falls off as  $\mathcal{O}(\Omega^{N+1})$  and  $\tilde{\alpha}$  is a (N+1)-form on  $\mathcal{M}$  which admits a smooth limit to  $\mathscr{I}$ . This observation has two immediate consequences:

- i) The integrals of the symplectic current j on any partial Cauchy surfaces M' is well defined;
- ii) The flux of j across  $\mathscr{I}$  vanishes identically.

These two facts and equations (2.24) and (2.25) now imply that the right side of

$$\mathbf{\Omega}(\delta_1, \delta_2) = -\frac{1}{16\pi G} \int_M [\delta_1 A^{IJ} \wedge \delta_2 \Sigma_{IJ} - \delta_2 A^{IJ} \wedge \delta_1 \Sigma_{IJ}] + \frac{1}{8\pi G} \oint_S [\delta_1 \bar{\epsilon} \, \delta_2 \psi - \delta_2 \bar{\epsilon} \, \delta_1 \psi], \quad (2.32)$$

is well-defined and independent of the choice of the partial Cauchy slice M in  $\mathcal{M}$  (where S is the intersection of M with  $\Delta$ ). Since the right hand side is bilinear and anti-symmetric in the tangent vector fields  $\delta_1$ ,  $\delta_2$  on  $\Gamma$ , it defines a 2-form on  $\Gamma$ . The second variation procedure ensures that it is closed (see e.g. [22]). This is the (pre-)symplectic 2-form on our covariant phase space  $\Gamma$ .

While the general procedure used above is well-known (see, e.g. [22]), it had to be supplemented by two technical steps, both arising from non-trivial boundary conditions. First, even though there are no fluxes of physical quantities such as energy or angular momentum across the inner boundary  $\Delta$ , the flux of the symplectic current across  $\Delta$  does not vanish. This fact has to be folded-in appropriately in the analysis. As in other isolated horizon discussions, this flux gave rise to a surface term in the final expression of the symplectic structure. The second technical point concerns behavior of the integrand of the symplectic current at infinity. We showed that the fall-off is such that the flux across  $\mathscr I$  vanishes so that there is no surface term coming from  $\mathscr I$ . We also showed that the fall-off is such that the flux across partial Cauchy slices M—and hence the expression of the symplectic structure itself— is well-defined. This step is more delicate and less transparent in the second order framework [17].

Finally, for reasons given in Section I, we have restricted ourselves to vacuum general relativity. However, inclusion of matter fields, such as the Maxwell, Yang-Mills, and Higgs

fields will not involve new conceptual elements. To include any of them one would add the corresponding term to the action and carry out the second variation also of this term. In the gravitational sector, our asymptotic analysis will go through provided the stress energy tensor falls off so that  $\Omega^{-N}T_{ab}$  admits a smooth limit to  $\mathscr{I}$ —an assumption that is necessary and sufficient for the matter angular momentum to be finite. Thus, the inclusion of these fields will not affect the gravitational part of the symplectic structure. The total symplectic structure will simply have additional terms corresponding to matter fields.

### III. SYMMETRIES AND HAMILTONIANS

Phase-space frameworks provide a universal procedure to arrive at conserved quantities: they are constructed from the Hamiltonians generating canonical transformations representing appropriate symmetries. In particular, energy is the generator of a time translation and angular momenta, generators of rotations. This prescription is universal in the sense that it applies to all physical systems —particles, fields in Minkowski (or any stationary, axi-symmetric) space-time, and space-time geometry itself.

Consider then smooth vector fields  $\xi^a$  on  $\mathcal{M}$  which preserve the boundary conditions imposed in section II A. This implies in particular that  $\xi^a$  are tangential to  $\Delta$  and admit smooth extensions to  $\widetilde{\mathcal{M}}$  which are tangential to  $\mathscr{I}$ . We will spell out other consequences in the next two sections where we obtain explicit expressions for conserved quantities at  $\mathscr{I}$  and a first law at  $\Delta$ . Here, we confine ourselves to general observations which will be needed in that discussion.

Since we do not have any background fields, if  $(e_a^I, A_a^{IJ})$  satisfies the field equations so does its image under smooth diffeomorphisms of  $\mathcal{M}$ . Since each  $\xi^a$  furthermore respects boundary conditions, the infinitesimal diffeomorphisms it generates define a vector field  $\delta_{\xi}$  on  $\Gamma$ , acting on A and e through Lie derivation along  $\xi$ ,

$$\delta_{\xi}e = \mathcal{L}_{\xi}e; \qquad \delta_{\xi}A = \mathcal{L}_{\xi}A;$$

$$(3.1)$$

 $(\mathcal{L}_{\xi}e, \mathcal{L}_{\xi}A)$  automatically solve the linearized equations. One can now ask whether this vector field  $\delta_{\xi}$  is a phase space symmetry, i.e., if  $\mathcal{L}_{\delta_{\xi}}\Omega$  vanishes identically on  $\Gamma$ . The necessary and sufficient condition for this to be the case is that there exist a function  $H^{(\xi)}$  on  $\Gamma$  such that for any vector field  $\delta$  tangent to  $\Gamma$ ,

$$\delta H^{(\xi)} = \mathbf{\Omega}(\delta, \delta_{\xi}). \tag{3.2}$$

 $H^{(\xi)}$  is the Hamiltonian generating the infinitesimal symmetry (i.e. infinitesimal canonical transformation)  $\delta_{\xi}$ . In this section we will simplify the right side of (3.2) and bring it to a form that can be used directly to obtain conserved quantities at  $\mathscr{I}$  and  $\Delta$ .

Recall that the symplectic structure  $\Omega(\delta_1, \delta_2)$  consists of a bulk term (an integral over a partial Cauchy surface M) and a surface term (an integral over a cross-section of  $\Delta$ ). (See Eq. (2.32).) As shown in [4], if  $\xi$  is a symmetry of  $\Delta$ , the surface term does not contribute to  $\Omega(\delta, \delta_{\xi})$  (although it does contribute if  $\delta_{\xi}$  is replaced by a general vector field  $\delta'$  on  $\Gamma$ ). That argument is insensitive to the presence of a cosmological constant or higher dimensions. Therefore, we have:

$$\Omega(\delta, \delta_{\xi}) = -\frac{1}{16\pi G} \int_{M} [\delta A^{IJ} \wedge \mathcal{L}_{\xi} \Sigma_{IJ} - \mathcal{L}_{\xi} A^{IJ} \wedge \delta \Sigma_{IJ}] 
= \frac{(-1)^{N+1}}{16\pi G} \int_{M} [\delta \Sigma_{IJ} \wedge \mathcal{L}_{\xi} A^{IJ} - \mathcal{L}_{\xi} \Sigma_{IJ} \wedge \delta A^{IJ}],$$
(3.3)

where the rewriting in the second step was done for later convenience. Using the Cartan identity we obtain

$$\Omega(\delta, \delta_{\xi}) = \frac{(-1)^{N+1}}{16\pi G} \int_{M} [\delta \Sigma_{IJ} \wedge d(\xi \Box A^{IJ}) + \delta \Sigma_{IJ} \wedge (\xi \Box dA^{IJ}) 
-d(\xi \Box \Sigma_{IJ}) \wedge \delta A^{IJ} - (\xi \Box d\Sigma_{IJ}) \wedge \delta A^{IJ}].$$
(3.4)

Next, by performing partial integration on the first and third terms, we have:

$$\Omega(\delta, \delta_{\xi}) = \frac{(-1)^{N+1}}{16\pi G} \int_{M} [(-1)^{N+1} \delta d\Sigma_{IJ} \wedge (\xi \Box A^{IJ}) + \delta\Sigma_{IJ} \wedge (\xi \Box dA^{IJ}) 
+ (-1)^{N-1} (\xi \Box\Sigma_{IJ}) \wedge \delta dA^{IJ} - (\xi \Box d\Sigma_{IJ}) \wedge \delta A^{IJ}] 
+ \frac{(-1)^{N+1}}{16\pi G} \int_{\partial M} [(-1)^{N} \delta\Sigma_{IJ} \wedge (\xi \Box A^{IJ}) - (\xi \Box\Sigma_{IJ}) \wedge \delta A^{IJ}].$$
(3.5)

Note that the surface term in this (and subsequent) expression(s) arises from the bulk term in the symplectic structure (2.32) and is unrelated to the surface term in (2.32).

Next, we can use the definition of the curvature  $F^{IJ}$  (2.12) and the field equation (2.14), i.e.,  $D\Sigma_{IJ} = 0$  to simplify the right side:

$$\Omega(\delta, \delta_{\xi}) = \frac{(-1)^{N+1}}{16\pi G} \int_{M} [\delta \Sigma_{IJ} \wedge (\xi \Box F^{IJ}) + (-1)^{N-1} (\xi \Box \Sigma_{IJ}) \wedge \delta F^{IJ}] 
- \frac{1}{16\pi G} \int_{\partial M} [\delta \Sigma_{IJ} \wedge (\xi \Box A^{IJ}) + \delta A^{IJ} \wedge (\xi \Box \Sigma_{IJ})].$$
(3.6)

The final step is to show that the bulk term (i.e., the integral over M) in this expression vanishes. Note first that it can be written as

$$\frac{-1}{16\pi G} \int_{M} \left[ -\delta F^{IJ} \wedge (\xi \bot \Sigma_{IJ}) + (\xi \bot F^{IJ}) \wedge \delta \Sigma_{IJ} \right]. \tag{3.7}$$

Using eq. (2.19) and the linearized Einstein equation (2.20), we have

$$-\delta F^{IJ} \wedge (\xi \bot \Sigma_{IJ}) = -\xi^K \delta F^{IJ} \wedge \Sigma_{IJK}$$
  
=  $\xi^K F^{IJ} \wedge \delta e^L \wedge \Sigma_{IJKL} - 2\Lambda \xi^K \delta e^L \wedge \Sigma_{KL},$  (3.8)

where  $\xi^K = e_a^K \xi^a$ . Hence the integrand of (3.7) can be written as:

$$-\delta F^{IJ} \wedge (\xi \Box \Sigma_{IJ}) + (\xi \Box F^{IJ}) \wedge \delta \Sigma_{IJ}$$

$$= \delta e^{L} \wedge [-F^{IJ} \wedge (\xi \Box \Sigma_{IJL}) + 2\Lambda (\xi \Box \Sigma_{L}) - (\xi \Box F^{IJ}) \wedge \Sigma_{IJL}]$$

$$= -\delta e^{L} \wedge [\xi \Box (\text{Einstein equation})], \tag{3.9}$$

where, by 'Einstein equation' we mean the left hand side of (2.15). Thus, the bulk contribution to  $\Omega(\delta, \delta_{\varepsilon})$  indeed vanishes and the  $\Omega(\delta, \delta_{\varepsilon})$  reduces just to surface terms:

$$\Omega(\delta, \delta_{\xi}) = -\frac{1}{16\pi G} \int_{\partial M} [(\xi \cdot A^{IJ}) \delta \Sigma_{IJ} + \delta A^{IJ} \wedge (\xi \Box \Sigma_{IJ})] 
= -\frac{1}{16\pi G} \left( \oint_{C} - \oint_{S} \right) [(\xi \cdot A^{IJ}) \delta \Sigma_{IJ} + \delta A^{IJ} \wedge (\xi \Box \Sigma_{IJ})],$$
(3.10)

where C and S are compact N-manifolds in which M intersects  $\mathscr{I}$  and  $\Delta$  respectively. This reduction of the expression of  $\Omega(\delta, \delta_{\xi})$  to boundary terms is a universal feature of field theories which are 'generally covariant' i.e. which have no background fields.<sup>4</sup>

With this simplified form of  $\Omega(\delta, \delta_{\xi})$  at hand we can return to the issue discussed in the beginning of this section: Is  $\delta_{\xi}$  a phase space symmetry? From (3.2) we know that it is if and only if there exists a function  $H^{(\xi)}$  on  $\Gamma$  such that  $\Omega(\delta, \delta_{\xi}) = \delta H^{(\xi)}$  for all vector fields  $\delta$  on  $\Gamma$ . This condition will be met if and only if one can 'pull  $\delta$  out' and put it in front of the integrals on the right side of (3.10). If  $\xi^a$  is a vector field for which this can be done, then  $H^{(\xi)}$  would be a difference of two terms, one defined at  $\mathscr I$  and the other at  $\Delta$ :

$$H^{(\xi)} = \mathcal{Q}_{\mathscr{I}}^{(\xi)} - \mathcal{Q}_{\Lambda}^{(\xi)}. \tag{3.11}$$

In the next two sections we will investigate conditions under which  $\delta_{\xi}$  is a phase space symmetry. When these conditions are met,  $\mathcal{Q}_{\mathcal{J}}^{(\xi)}$  and  $\mathcal{Q}_{\Delta}^{(\xi)}$  provide us the conserved quantities at infinity and on the horizon.

In the space-time description, the symmetry vector fields  $\xi^a$  are restricted only near the boundaries where their action must respect the boundary conditions. In the interior, they can be arbitrary smooth fields. In the phase space description this is reflected by the fact that the total Hamiltonian  $H^{(\xi)}$  receives contributions only from surface terms. Therefore, in general, symmetries and conserved quantities at infinity are disconnected from those on the horizon (except in the sector of the phase space admitting global Killing fields). Hence, on the full phase space one can calculate  $\mathcal{Q}^{(\xi)}_{\mathscr{I}}$  and  $\mathcal{Q}^{(\xi)}_{\Delta}$  separately. To compute the  $\mathcal{Q}^{(\xi)}_{\mathscr{I}}$  it will be convenient to use vector fields  $\xi^a$  that generate non-trivial symmetries at  $\mathscr{I}$  but vanish near  $\Delta$ . Similarly, to calculate  $\mathcal{Q}^{(\xi)}_{\Delta}$  we will use vector fields that are non-trivial symmetries at the horizon but vanish outside some neighborhood of  $\Delta$ . The  $\mathcal{Q}^{(\xi)}_{\mathscr{I}}$  will turn out to be the AD [13] quantities, obtained in [17] through a second order Hamiltonian framework. The quantities  $\mathcal{Q}^{(\xi)}_{\Delta}$  will provide us with the desired generalized first law.

quantities  $\mathcal{Q}_{\Delta}^{(\xi)}$  will provide us with the desired generalized first law. In the next section we will compute the  $\mathcal{Q}_{\mathscr{I}}^{(\xi)}$ , while the quantities  $\mathcal{Q}_{\Delta}^{(\xi)}$  will be considered in section V.

# IV. CONSERVED QUANTITIES AT $\mathscr{I}$

Consider a vector field  $\xi^a$  which has support outside of some cylinder  $\Omega = \Omega_o$  and is an asymptotic symmetry. Since the conformal factor  $\Omega$  has been fixed on our entire phase space in a neighborhood of  $\mathscr{I}$ , we must have  $\mathcal{L}_{\xi}\Omega = 0$  in this neighborhood. Thus  $\xi^a$  must be tangential to the level surfaces of  $\Omega$  there. Since  $\xi^a$  must also admit a smooth extension to  $\mathscr{I}$ , it must be tangential to  $\mathscr{I}$ . Finally, since the metric at  $\mathscr{I}$  is also fixed to be the AdS metric (order  $\Omega^N$  in the Taylor expansion),  $\xi^a$  must be the limit to  $\mathscr{I}$  of an AdS Killing field. Thus, the Lie algebra of asymptotic symmetries at  $\mathscr{I}$  is d(d+1)/2 = (N+2)(N+3)/2-dimensional. In this section we will show that for each of these symmetry vector fields  $\xi^a$  on  $\mathscr{M}$ , the vector field  $\delta_{\xi}$  on  $\Gamma$  is a phase space symmetry and calculate the corresponding conserved quantity  $\mathcal{Q}_{\mathscr{I}}^{(\xi)}$ .

<sup>&</sup>lt;sup>4</sup> Thus, if  $\mathcal{M}$  were spatially compact,  $\Omega(\delta, \delta_{\xi})$  would vanish identically for all  $\delta$  whence  $\delta_{\xi}$  would be in the kernel of the symplectic structure for all smooth  $\xi^{a}$ . This is why all diffeomorphisms represent gauge and all the corresponding conserved quantities vanish in the spatially compact case.

Let C be a cross-section of  $\mathscr{I}$  —the intersection of M and  $\mathscr{I}$ . Then, from (3.10) we have

$$\Omega(\delta, \delta_{\xi}) = -\frac{1}{16\pi G} \oint_C [(\xi \cdot A^{IJ}) \delta \Sigma_{IJ} + (-1)^{N-1} (\xi \Box \Sigma_{IJ}) \wedge \delta A^{IJ}]. \tag{4.1}$$

Our goal is to show that the right side is the Lie derivative  $\delta_{\xi} \mathcal{Q}_{\mathscr{I}}^{(\xi)}$  of a function  $\mathcal{Q}_{\mathscr{I}}^{(\xi)}$  on  $\Gamma$ . Let us begin with the first term in the integrand,

$$(\xi \cdot A^{IJ}) \, \delta \Sigma_{IJ}. \tag{4.2}$$

From (2.29) we have

$$\delta \Sigma_{IJc_1...c_N} = -\frac{N}{N+1} \Omega \epsilon_{IJK_1...K_N} \tilde{e}_{[c_1}^{K_1} \dots \tilde{e}_{c_{N-1}}^{K_{N-1}} \delta \tilde{E}_{c_N]d} \tilde{e}^{dK_N} + \mathcal{O}(\Omega^2). \tag{4.3}$$

Next, let us express  $A^{IJ}$  in terms of conformally rescaled fields which are well-behaved at  $\mathscr{I}$ :

$$A_a^{IJ} = e^{bI} \nabla_a e_b^J = \tilde{e}^{bI} \tilde{\nabla}_a \tilde{e}_b^J + \frac{2}{\Omega} \tilde{n}_b \tilde{e}^{b[I} \tilde{e}_a^{J]}. \tag{4.4}$$

Collecting the two terms, we have

$$(\xi \cdot A_{IJ}) \,\delta \Sigma^{IJ}{}_{c_1...c_N} = 2 \frac{N}{N+1} \xi^a \tilde{n}^b \tilde{\epsilon}_{ab[c_1...c_{N-1}|e|} \delta \tilde{E}_{c_N]d} \tilde{g}^{de} + \mathcal{O}(\Omega). \tag{4.5}$$

Next, consider the second term in the integrand of (4.1):

$$(\xi \bot \Sigma_{IJ}) \wedge \delta A^{IJ}. \tag{4.6}$$

From (2.30) we have

$$\delta A_a^{IJ} = -2 \frac{N}{N+1} \Omega^N \delta \tilde{E}_{ab} \, \tilde{e}^{b[I} \tilde{e}^{|d|J]} \, \tilde{n}_d + \mathcal{O}(\Omega^{N+1}). \tag{4.7}$$

Using the definition (2.13) of  $\Sigma_{IJ}$ , we have

$$\xi^{a} \Sigma_{IJac_{1}...c_{N-1}} := \Omega^{-N} \xi^{a} \epsilon_{IJLK_{1}...K_{N-1}} \tilde{e}_{[a}^{L} \tilde{e}_{c_{1}}^{K_{1}} \wedge \dots \wedge e_{c_{N-1}]}^{K_{N-1}}.$$

$$(4.8)$$

Combining these expressions we arrive at

$$[(\xi \bot \Sigma_{IJ}) \land \delta A^{IJ}]_{c_1...c_N} = 2(-1)^{N-1} \frac{N^2}{N+1} \xi^a \tilde{n}^b \tilde{\epsilon}_{ab[c_1...c_{N-1}|e|} \delta \tilde{E}_{c_N]d} \tilde{g}^{de} + \mathcal{O}(\Omega). \tag{4.9}$$

Substituting the expressions (4.5) and (4.9) into (4.1), we find

$$\Omega(\delta, \delta_{\xi}) = -\frac{1}{16\pi G} \oint_{C} 2N \, \xi^{a} \tilde{n}^{b} \, \tilde{\epsilon}_{ab[c_{1}...c_{N-1}|e|} \, \delta \tilde{E}_{c_{N}]d} \, \tilde{g}^{de} 
= -\frac{1}{16\pi G} \oint_{C} \left[ \frac{2}{(N-1)!} \, \xi^{a} \tilde{n}^{b} \, \tilde{\epsilon}_{abc_{1}...c_{N-1}e} \delta \tilde{E}_{c_{N}d} \, \tilde{g}^{de} \, \tilde{\epsilon}^{c_{1}...c_{N}} \right] \, \tilde{\epsilon}_{a_{1}...a_{N}} 
= -\frac{1}{8\pi G} \oint_{C} \delta \tilde{E}_{ab} \, \xi^{a} \tilde{u}^{b} \, \tilde{\epsilon},$$
(4.10)

where  $\tilde{\tilde{\epsilon}}$  is again the area element on C induced by  $\tilde{g}_{ab}$ . Using  $\delta \xi = 0$ , and the fact that  $\delta \tilde{u}^a$  and  $\delta \tilde{\tilde{\epsilon}}$  vanish on  $\mathscr{I}$ , we have

$$\Omega(\delta, \delta_{\xi}) = -\frac{L}{8\pi G} \delta \left[ \oint_C \tilde{E}_{ab} \xi^a \tilde{u}^b \,\tilde{\tilde{\epsilon}} \right]$$
(4.11)

where we have reinstated the AdS radius L. Thus,  $\Omega(\delta, \delta_{\xi})$  is an exact differential whence  $\delta_{\xi}$  is a phase space symmetry,  $\mathcal{L}_{\xi}\Omega = 0$ . Since  $\delta$  is an arbitrary vector field on  $\Gamma$ , (4.11) determines the Hamiltonian  $H^{(\xi)}$  generating this symmetry up to an additive constant (see (3.11)). As in [13], this constant is determined by requiring that (3.11) vanish in the pure AdS space.<sup>5</sup> Then, we have:

$$H^{(\xi)} \equiv \mathcal{Q}_{\mathscr{I}}^{(\xi)} = -\frac{L}{8\pi G} \oint_C \tilde{E}_{ab} \xi^a \tilde{u}^b \,\tilde{\bar{\epsilon}}. \tag{4.12}$$

These are precisely the AD conserved quantities [13].

To summarize, in this section we considered space-time vector fields  $\xi^a$  which have support only in a neighborhood of  $\mathscr{I}$  and are symmetries at  $\mathscr{I}$ , showed that  $\delta_{\xi}$  are phase space symmetries, and obtained the expressions of the Hamiltonians  $H^{(\xi)}$ . This procedure associates with every symmetry field  $\xi^a$  on  $\mathscr{I}$ , i.e., to every Killing field of the asymptotic AdS metric, a conserved quantity  $\mathcal{Q}^{(\xi)}_{\mathscr{I}}$ , thereby providing another justification for the AD conserved quantities, now using a first order, covariant Hamiltonian framework.

For comparison with the situation at the inner boundary  $\Delta$  discussed in the next section, we note that in the above derivation we assumed that the vector field  $\xi^a$  was fixed on  $\mathscr{I}$  once and for all, i.e. did not vary from one phase space point to another. Indeed, the derivation made a crucial use of the fact that  $\delta\xi$  vanishes on  $\mathscr{I}$ . From a space-time perspective, if  $\xi^a$  is an asymptotic symmetry, so is  $k\xi^a$ , where k is a constant on  $\mathscr{I}$ . A priori one can let k depend on the phase space point under consideration and still obtain a symmetry on each individual space-time. However, in this case  $\delta_{\xi}$  would not be a phase space symmetry because we would not be able to express  $\Omega(\delta, \delta_{\xi})$  as  $\delta H^{(\xi)}$ . At the horizon, by contrast, we will find that physically interesting time-translation symmetries  $\xi^a$  must vary from one phase space point to another. This will make the notion of horizon energy/mass more subtle.

# V. CONSERVED QUANTITIES AT $\Delta$ AND THE FIRST LAW

Let us now consider symmetries and conserved quantities at the horizon. Recall from section II that the definition of the weakly isolated horizon features two fields induced directly by the space-time metric  $g_{ab}$ ; a preferred family  $[\ell^a]$  of null normals and the intrinsic (degenerate) metric  $q_{ab}$ . A vector field  $\xi^a$  on  $\mathcal{M}$  is said to be an infinitesimal symmetry of the weakly isolated horizon  $\Delta$  if  $\xi^a$  is tangential to  $\Delta$  and satisfies  $\mathcal{L}_{\xi}\ell^a \cong c\ell^a$ , and  $\mathcal{L}_{\xi}q_{ab} \cong 0$  for all  $\ell^a \in [\ell^a]$  and some positive constant c. (As before  $\cong$  denotes 'equality on  $\Delta$ '.)

Note that there is a key difference from the situation at  $\mathscr{I}$ . All metrics  $g_{ab}$  in our phase space  $\Gamma$  approach a fixed AdS metric at infinity and asymptotic symmetries are Killing fields

<sup>&</sup>lt;sup>5</sup> More precisely, we require that  $Q_{\mathscr{J}}^{(\xi)}$  should vanish in the limit in which the horizon area goes to zero of the Schwarzschild AdS family. The reason for this somewhat indirect condition is that the AdS space-time itself does not belong to our phase space since it does not admit an isolated horizon inner boundary.

of this metric. The horizon  $\Delta$  on the other hand is in a strong field region and the metric  $q_{ab}$  varies from one phase space point to another. Even in 4 dimensions  $q_{ab}$  is spherically symmetric in the Schwarzschild space-time and only axi-symmetric in the Kerr space-time. Therefore even the number of horizon symmetries is not universal on the phase space  $\Gamma$ .

Clearly  $\xi^a = k\ell^a$  is a horizon symmetry for any constant k. However, since  $q_{ab}$  can vary from one space-time to another, in general there are no other symmetries. To introduce a useful definition of angular momentum and obtain a first law analogous to (1.1), we will now restrict the phase space.<sup>6</sup> First, we will assume that the topology of the inner boundary  $\Delta$  is  $\mathbb{S}^N \times \mathbb{R}$  and fix N(N+1)/2 vector fields  $\phi_i^a$  on  $\Delta$ , satisfying the commutation relations of the rotation group SO(N+1) and the condition  $\mathcal{L}_{\ell}\phi_i^a = 0$ . Second, will now restrict the phase space so that each  $q_{ab}$  induced on  $\Delta$  admits at least one  $\phi_i^a$  as its symmetry and every of its symmetries is a linear combination of the type

$$\xi^a = k\ell^a + \sum_i \Omega_i \phi_i^a \tag{5.1}$$

where the coefficients k,  $\Omega_i$  are constants on  $\Delta$  (but can vary from one phase space point to another). The constants  $\Omega_i$  are unrelated to the conformal factor used to attach  $\mathscr I$  and, as we will see, can finally be thought of as 'angular velocities'. In the physically interesting case of higher dimensional Kerr solutions,  $q_{ab}$  admits [(N+1)/2] commuting, rotational Killing fields where [...] stands for 'integral part of'.

With a slight abuse of notation, we will continue to denote the restricted phase space by  $\Gamma$  and its (pulled-back) symplectic structure by  $\Omega$ . Note that geometries in  $\Gamma$  need not admit any Killing vectors even in a neighborhood of  $\Delta$ ; the restriction is only on the metrics  $q_{ab}$  induced on  $\Delta$ .

We will now show that each  $\phi_i^a$  gives rise to a conserved angular momentum  $J_{\Delta}^i$  on the sector of the phase space which admits a horizon symmetry  $\xi_i^a$  with  $\xi_i^a = \phi_i^a$ . Let us furthermore assume for simplicity that  $\xi^a$  vanishes outside some neighborhood of  $\Delta$ . Then Eq. (3.10) yields:

$$\Omega(\delta, \delta_{\xi_i}) = \frac{1}{16\pi G} \oint_S \left[ (\xi_i \cdot A^{IJ}) \delta \Sigma_{IJ} + \delta A^{IJ} \wedge (\xi_i \sqcup \Sigma_{IJ}) \right], \tag{5.2}$$

where S is the intersection of M with the horizon  $\Delta$ . We now use the consequences (2.8) and (2.9) of the horizon boundary conditions to conclude

$$A_{\underline{a}}^{N+1\ N+2} = -\omega_a \tag{5.3}$$

where, as before,  $\omega_a$  the rotation potential 1-form, and

$$\Sigma_{N+1 N+2 \underline{a_1...a_N}} = \bar{\epsilon}_{a_1...a_N}. \tag{5.4}$$

<sup>&</sup>lt;sup>6</sup> One can obtain first laws [3] even in absence of these restrictions, but they are not as closely related to the standard first law.

Expression (5.2) then reduces to

$$\Omega(\delta, \delta_{\xi_{i}}) = -\frac{1}{8\pi G} \oint_{S} [(\xi_{i} \sqcup \omega) \, \delta\bar{\epsilon} + \delta\omega \wedge (\xi_{i} \sqcup \bar{\epsilon})] 
= -\frac{1}{8\pi G} \oint_{S} [(\phi_{i} \sqcup \omega) \, \delta\bar{\epsilon} + (\phi_{i} \sqcup \delta\omega) \, \bar{\epsilon}] 
= -\frac{1}{8\pi G} \delta \oint_{S} [(\phi_{i} \sqcup \omega) \, \bar{\epsilon}].$$
(5.5)

The right side provides us the expression of  $\delta J_{\Delta}^{i}$ . Since  $\delta$  is arbitrary, this determines the angular momenta  $J_{\Delta}^{i}$  up to an additive constant. We will eliminate this freedom by a natural requirement:  $J_{\Delta}^{i}$  should vanish in spherically symmetric space-times. In these space-times the pull-back of the rotational 1-form  $\omega_{a}$  to any spherically symmetric cross-section vanishes on  $\Delta$ , whence the integral on the right side also vanishes. Therefore, the angular momenta  $J_{\Delta}^{i(\xi)}$ —the Hamiltonians generating the infinitesimal symmetry  $\delta_{\xi_{i}}$  on  $\Gamma$ — are given by the obvious expression

$$J_{\Delta}^{i} := -\frac{1}{8\pi G} \oint_{S} (\phi_{i} \sqcup \omega) \,\bar{\epsilon}. \tag{5.6}$$

(5.6) brings out the reason why  $\omega_a$  is referred to as the rotation 1-form.

Remark: For some applications, it is useful to note that the expression (5.6) can be recast in terms of the Weyl tensor at the horizon. Since each  $\phi_i^a$  is a Killing vector on the intrinsic metric on S, it is divergence-free. Hence it admits a (N-2)-form potential  $f_i$ :  $\phi_i = {}^{\star} df_i$  where  ${}^{\star}$  denotes the Hodge-dual on S.<sup>7</sup> Using this expression in (5.6) and integrating by parts, one obtains:

$$J_{\Delta}^{i} = \frac{N+1}{8\pi G} \oint_{S} {}^{\star} f_{i}^{ab} C_{abc}{}^{d} \ell^{c} n_{d} \bar{\epsilon}$$

$$(5.7)$$

This is the generalization of the 4-dimensional expression of the angular momentum in terms of the Weyl tensor component  $\text{Im}\Psi_2$ .

As in 4-dimensions, the definition of the horizon energy is more subtle because we can not fix once and for all a vector field on  $\Delta$  and require that the time translation symmetry coincide with it. For, in a spherically symmetric space-time the appropriate time-translation symmetry would be along the null generators  $\ell^a$  of  $\Delta$ , while for a rotating black hole, it would be a linear combination of  $\ell^a$  and rotational symmetry-fields. Thus, while physically interesting time translations will be horizon symmetries, i.e., will be of the type (5.1), in general one would expect the coefficients k and  $\Omega_i$  to vary from one point on the phase space to another (although they will be constants on the  $\Delta$  of any one space-time).

For later convenience, let us set  $k = \kappa_{(\xi)}/\kappa_{(\ell)}$ , where  $\kappa_{(\ell)}$  is the 'surface gravity' associated with the null normal  $\ell^a$  via  $\ell^a \nabla_a \ell^b \cong \kappa_{(\ell)} \ell^b$  and  $\kappa_{(\xi)}$ , with the component  $c\ell^a$  of  $\xi^a$  along  $\ell^a$ . The symmetry field  $\xi^a$  is then given by

$$\xi^a = \frac{\kappa_{(\xi)}}{\kappa_{(\ell)}} \ell^a - \sum_i \Omega^i_{(\xi)} \phi^a_i, \tag{5.8}$$

<sup>&</sup>lt;sup>7</sup> We follow the standard convention: On an *n*-manifold, the hodge-dual \**f* of an *m*-form *f* is an n-m form defined by:  $(*f)_{a_1,...a_{n-m}} = \frac{1}{(n-m)!} \epsilon_{a_1,...a_{n-m}} b_1...b_m f_{b_1...b_m}$ .

so that  $\kappa_{(\xi)}$  is now a constant on  $\Delta$  in any solution but can vary from one solution to another. Then, Eq. (3.10) yields:

$$\Omega(\delta, \delta_{\xi}) = \frac{1}{16\pi G} \oint_{S} [(\xi \cdot A^{IJ}) \delta \Sigma_{IJ} + \delta A^{IJ} \wedge (\xi \Box \Sigma_{IJ})] 
= -\frac{1}{8\pi G} \oint_{S} [(\xi \Box \omega) \delta \bar{\epsilon} + \delta \omega \wedge (\xi \Box \bar{\epsilon})] 
= -\frac{1}{8\pi G} \oint_{S} [\kappa_{(\xi)} \delta \bar{\epsilon} - \sum_{i} \Omega^{i}_{(\xi)} (\phi_{i} \Box \omega) \delta \bar{\epsilon} - \sum_{i} \Omega^{i}_{(\xi)} (\phi_{i} \Box \delta \omega) \bar{\epsilon}] 
= -\frac{\kappa_{(\xi)}}{8\pi G} \delta a_{\Delta} - \sum_{i} \Omega^{i}_{(\xi)} \delta J^{i}_{\Delta}.$$
(5.9)

Since  $\kappa_{(\xi)}$  and  $\Omega^i_{(\xi)}$  are allowed to be functions on phase space, in general the right side is not an exact differential. That is, in general  $\delta_{\xi}$  is *not* a Hamiltonian vector field on  $\Gamma$ . The necessary and sufficient condition for this is precisely that there should exist a phase space function, call it  $E_{\Delta}^{(\xi)}$ , such that

$$\Omega(\delta, \delta_{\xi}) = \delta E_{\Lambda}^{(\xi)} \tag{5.10}$$

i.e. such that

$$\delta E_{\Delta}^{(\xi)} = \frac{\kappa_{(\xi)}}{8\pi G} \delta a_{\Delta} + \sum_{i} \Omega_{(\xi)}^{i} \delta J_{\Delta}^{i}. \tag{5.11}$$

Note that in any one space-time in our phase space  $\Gamma$ , the acceleration on  $\Delta$  of the 'null part' of  $\xi^a$  is given by  $\kappa_{(\xi)}$  (see (5.8)). Therefore  $\kappa_{(\xi)}$  is the surface gravity associated with the symmetry field  $\xi^a$ . Similarly, integral curves of  $\xi^a$  'move' with angular velocities  $\Omega^i_{(\xi)}$ . These geometrical properties of  $\xi^a$  imply that (5.11) has precisely the same form as the standard first law of black hole mechanics,  $\xi^a$  replacing the globally stationary Killing field which need not exist.

Let us call a vector field  $\xi^a$  permissible if  $\delta_{\xi}$  is a Hamiltonian vector field, i.e., if (5.11) is satisfied. Now, (5.11) can be rewritten using exterior derivatives d and wedge-products  $\wedge$  on the infinite dimensional phase space  $\Gamma$ :

$$0 = dI E_{\Delta}^{(\xi)} = \frac{1}{8\pi G} dI \kappa_{(\xi)} \wedge dI a_{\Delta} + \sum_{i} dI \Omega_{(\xi)}^{i} \wedge dI J_{\Delta}^{i}, \qquad (5.12)$$

This immediately implies that  $\xi^a$  is permissible if and only if its surface gravity  $\kappa_{(\xi)}$  and angular velocities  $\Omega^i_{(\xi)}$ , regarded as functions on the phase space  $\Gamma$ , depend only on the horizon area  $a_{\Delta}$  and angular momenta  $J^i_{\Delta}$ , and furthermore satisfy the integrability conditions:

$$\frac{\partial \kappa_{(\xi)}}{\partial J_{\Delta}^{i}} = 8\pi G \frac{\partial \Omega_{(\xi)}^{i}}{\partial a_{\Delta}}, \quad \text{and} \quad \frac{\partial \Omega_{(\xi)}^{i}}{\partial J_{\Delta}^{k}} = \frac{\partial \Omega_{(\xi)}^{k}}{\partial J_{\Delta}^{i}}.$$
 (5.13)

When these conditions are satisfied,  $\delta E_{\Delta}^{(\xi)}$  also depends only on  $a_{\Delta}$  and  $J_{\Delta}^{i}$ . Finally, since  $\delta$  is arbitrary, (5.12) determines  $E_{\Delta}^{(\xi)}$  up to an additive constant. This freedom is eliminated by requiring that when  $J_{\Delta}^{i}$  all vanish,  $E_{\Delta}^{(\xi)}$  should vanish in the limit  $a_{\Delta}$  tends to zero.

Thus, the first law has been generalized from stationary space-times to isolated horizons. In the standard first law —e.g. the one in the Kerr-AdS family obtained in [10]— one uses the global time-translation Killing field to define surface gravity and angular velocities. In the

present framework, by contrast, a typical solution in  $\Gamma$  does not admit any Killing vector. The role of the Killing field is assumed by a horizon symmetry (5.8). These symmetries exist on the entire, infinite-dimensional phase space, including space-times which admit dynamical processes away from  $\Delta$ . Thus, (5.11) represents a considerable generalization of the standard first law. In addition, the framework brings out a deeper significance of the first law: it is the necessary and sufficient condition for the evolution generated by the symmetry vector field  $\xi^a$  to be Hamiltonian. However, there is now a first laws for each permissible vector field  $\xi^a$  and using (5.13) it is easy to give a step by step procedure to construct an infinite number of permissible vector fields [4]. In the next section we will show that, when restricted to finite dimensional subspaces of  $\Gamma$  consisting of solutions admitting global Killing fields, one can naturally recover the more familiar first law from (5.11).

#### VI. GLOBAL SYMMETRIES

We will begin with angular momentum. Let us restrict ourselves to solutions admitting a global rotational Killing vector  $\varphi^a$ , fixed once and for all on  $\mathcal{M}$ . Denote by  $\underline{\Gamma}$  a (maximal) connected component of  $\Gamma$  consisting of these axi-symmetric solutions which also contains at least one spherically symmetric solution.<sup>8</sup> Then, in particular  $\varphi^a$  is a symmetry both at  $\mathscr{I}$  and at  $\Delta$ . Furthermore, since  $\mathcal{L}_{\varphi}g_{ab}=0$  on  $\mathcal{M}$ , it follows that  $(\mathcal{L}_{\varphi}e_a^I,\mathcal{L}_{\varphi}A_a^{IJ})$  is an (internal) gauge transformation on the frames and Lorentz connections in  $\underline{\Gamma}$ . Since these belong to the kernel of the symplectic structure we have:

$$\underline{\Omega}(\underline{\delta}, \underline{\delta}_{\varphi}) = 0 \tag{6.1}$$

where  $\underline{\Omega}$  is the pull-back of the symplectic structure to  $\underline{\Gamma}$ ,  $\underline{\delta}$  any vector field thereon and  $\underline{\delta}_{\varphi}$  the restriction of  $\delta_{\varphi}$  to  $\underline{\Gamma}$ . Therefore, from (3.11) we must have

$$\underline{\delta}H^{(\varphi)} \equiv \underline{\delta}J_{\mathscr{I}}^{(\varphi)} - \underline{\delta}J_{\Delta}^{(\varphi)} = 0. \tag{6.2}$$

Thus, on  $\underline{\Gamma}$ ,  $J^{(\varphi)}_{\mathscr{I}}$  and  $J^{(\varphi)}_{\Delta}$  differ just by a constant. Let us evaluate the two quantities in any spherically symmetric space-time in  $\underline{\Gamma}$ . Since  $J^{(\varphi)}_{\mathscr{I}}$  can be evaluated on any cross-section C of  $\mathscr{I}$ , let us choose to evaluate it on one to which all rotational Killing fields are tangential. Then by spherical symmetry, the field  $\tilde{E}_{ab}\tilde{u}^b$  in the integrand of the AD conserved quantity (4.12) must be proportional to the normal  $\tilde{u}^a$  to C within  $\mathscr{I}$ , whence  $\tilde{E}_{ab}\varphi^a\tilde{u}^b$  must vanish. This implies that  $J^{(\varphi)}_{\mathscr{I}}$  must vanish on  $\underline{\Gamma}$ . Furthermore, by construction, on a spherical symmetric space-time  $J^{(\varphi)}_{\Delta}$  vanishes. Therefore, the constant relating the two conserved charges is zero and we have:

$$J_{\mathscr{I}}^{(\varphi)} = J_{\Delta}^{(\varphi)}. \tag{6.3}$$

Now, the Killing vector  $\varphi^a$  also gives rise to a Komar integral:

$$J_{\text{Komar}}^{(\varphi)} = \frac{N!}{32\pi G} \oint_{\hat{S}} {}^{\star} d\varphi \tag{6.4}$$

<sup>&</sup>lt;sup>8</sup> Because the black hole uniqueness theorem fails in higher dimensions,  $\Gamma$  may well admit several disconnected sets  $\underline{\Gamma}$  (and  $\tilde{\Gamma}$ ) of axi-symmetric solutions (or stationary solutions). Sectors considered here are the most interesting ones for us because they encompass the Kerr-AdS family.

where the integral is evaluated on any N-sphere  $\hat{S}$  homologous to the cross section C of  $\mathscr{I}$  and S of  $\Delta$  to which  $\varphi^a$  is tangential. A simple calculation shows that if  $\hat{S}_1$  and  $\hat{S}_2$  lie on a N+1 dimensional manifold  $\hat{M}$  to which  $\varphi^a$  is everywhere tangential, Einstein's equations  $G_{ab} + \Lambda g_{ab} = 0$  imply that the Komar integral evaluated on  $\hat{S}_1$  equals that evaluated on  $\hat{S}_2$ . In this sense  $J_{\text{Komar}}^{(\varphi)}$  is a conserved quantity.

Denote the restriction of  $\varphi^a$  to  $\Delta$  by  $\phi$ . Then, a natural question is: What is the relation between  $J_{\text{Komar}}^{(\varphi)}$  and  $J_{\Delta}^{(\varphi)}$ ? We will now show that they are necessarily equal. Consider a cross-section S of  $\Delta$  and let  $n_a$  be the null normal to it satisfying  $\ell^a n_a = -1$ . Extend it to  $\Delta$  by demanding  $\mathcal{L}_{\ell} n_a = 0$ . Then the definition of the rotation 1-form  $\omega_a$  implies:  $\omega_a = -n_b \nabla_{\underline{a}} \ell^b = \ell^b \nabla_b n_{\underline{a}}$ , where a bar under an index denotes the pull-back to  $\Delta$ . Therefore, from (5.6) we have:

$$J_{\Delta}^{(\varphi)} = -\frac{1}{8\pi G} \oint_{S} (\phi \sqcup \omega) \,\bar{\epsilon} = \frac{1}{8\pi G} \oint_{S} (\nabla_{\ell} \phi) \cdot n \,\bar{\epsilon} = \frac{N!}{32\pi G} \oint_{S} {}^{\star} d\phi \tag{6.5}$$

Next, let us explore the relation between the Komar integral  $J_{\text{Komar}}^{(\varphi)}$  and the angular momentum  $J_{\mathscr{I}}^{(\varphi)}$  at  $\mathscr{I}$ . In the expression of the Komar integral, let us express  $d\varphi$  in terms of the conformal factor  $\Omega$  and rescaled fields which are smooth at  $\mathscr{I}$ :  $\nabla_a \varphi_b = \tilde{\nabla}_{[a}(\Omega^{-2}\hat{g}_{b]c}\tilde{\varphi}^c)$ . Then, using the asymptotic forms (2.5) and (2.7) of the metric  $\tilde{g}_{ab}$  and simplifying, one obtains:

$$J_{\text{Komar}}^{\varphi} = -\frac{L}{8\pi G} \oint_{C} \tilde{E}_{ab} \tilde{\varphi}^{a} \tilde{u}^{b} \,\bar{\tilde{\epsilon}} \,. \tag{6.6}$$

Thus, in presence of a global rotational Killing field, the Komar integral  $J_{\rm Komar}^{(\varphi)}$ , the horizon angular momentum  $J_{\Delta}^{(\varphi)}$  and the angular momentum  $J_{\mathscr{I}}^{(\varphi)}$  at  $\mathscr{I}$  all agree, even though each is defined using quite different fields. In particular the higher dimensional Kerr-AdS spacetimes belong to our phase space  $\Gamma$  since they are asymptotically AdS and their event horizons are special cases of weakly isolated horizons. In these space-times, the horizon angular momenta  $J_{\Delta}^{i}$  agree with the Komar integrals.

Let us next consider space-times which admit a stationary Killing field  $\xi^a$ . Since  $\xi^a$  is in particular a horizon symmetry, it has the form (5.8) on  $\Delta$ . Denote by  $\tilde{\Gamma}$  the connected component of  $\Gamma$  consisting of stationary solutions which also contains the Schwarzschild-AdS family. Then arguments completely analogous to those given above for the axi-symmetric Killing field imply  $\mathcal{Q}^{(\xi)}_{\mathscr{I}} = E^{(\xi)}_{\Delta}$ . Thus, in this case, the horizon energy coincides with the AD mass at  $\mathscr{I}$  and the first law (5.11) takes the standard form:

$$\delta M = \frac{\kappa_{(\xi)}}{8\pi G} \delta a_{\Delta} + \sum_{i} \Omega^{i}_{(\xi)} \delta J^{i}_{\Delta}. \tag{6.7}$$

This is in particular true for the Kerr-AdS family.

We will conclude with two remarks. First, note that one can again define a Komar integral  $\mathcal{Q}_{\text{Komar}}^{(\xi)}$  on any N-sphere S and, if  $S_1$  and  $S_2$  are boundaries of a (N+1)-manifold M to which  $\xi^a$  is everywhere tangential, the integrals defined on  $S_1$  and  $S_2$  will be equal. However, since  $\xi^a$  is stationary, manifolds M to which it is tangential can not join an N-sphere in the asymptotic region to that in the interior. Therefore, the conservation law is very restricted and not of great physical interest. This is in striking contrast with situation vis a vis angular momentum where M can join N-spheres which are in the interior to those in the asymptotic

region.<sup>9</sup> Finally, since the angular momentum Komar integral  $J_{\text{Komar}}^{(\varphi)}$  agrees with  $J_{\mathscr{I}}^{(\varphi)}$ , in particular it follows that  $J_{\text{Komar}}^{(\varphi)}$  vanishes in the AdS space time. The corresponding Komar integral  $\mathcal{Q}_{\text{Komar}}^{(\xi)}$  for energy diverges in the limit as the cross-section is taken to  $\mathscr{I}$  even in the AdS space-time. One could try to renormalize the Komar integral by 'subtracting out' this infinity. But straightforward subtractions lead to ambiguous results.

Second, it is instructive to compare the above derivation of (6.7) with that by Gibbons, Perry and Pope [10] for the Kerr-AdS family. In our derivation, the angular momenta  $J_{\Delta}^{i}$  were defined using the rotation 1-form at the horizon (see (5.6)). The mass M arose as the horizon energy  $E_{\Delta}^{(\xi)}$ . The first law (5.11) only guarantees the existence of  $E_{\Delta}^{(\xi)}$ . To obtain its explicit expression we used general facts about symplectic spaces to argue that  $E_{\Delta}^{(\xi)}$  must equal the AD quantity  $\mathcal{Q}_{\mathscr{I}}^{(\xi)}$ . In [10], angular momenta were computed by evaluating the Komar integrals  $J_{\text{Komar}}^{(\varphi_i)}$  using explicit Kerr-AdS metrics. As shown above, these do agree with  $J_{\Delta}^{i}$  in presence of global Killing fields. The mass was determined by integrating the first law (6.7) and eliminating the freedom in the choice of a constant by requiring that the mass should vanish when the parameter m in the explicit expression of the Kerr-AdS metric vanishes. They then showed that the mass so defined agrees with the AD quantity  $\mathcal{Q}_{\mathscr{I}}^{(\xi)}$  associated with the stationary Killing field  $\xi^{a}$ . Thus, in the final picture both procedures give the same results for angular momenta and mass and the same first law in the Kerr-AdS family, although the starting points are very different.

## VII. DISCUSSION

In sections II and III, we constructed a covariant phase space of asymptotically AdS solutions to Einstein equations. While this construction is conceptually similar to that of [17], there are two main differences: i) we allowed an internal boundary  $\Delta$ , a weakly isolated horizon representing a black hole in local equilibrium; and, ii) we used a first order framework based on vielbeins and Lorentz connections, which is especially well suited to handling this internal boundary (and to incorporate fermions). In section IV we focused on  $\mathscr{I}$ . Through Hamiltonian considerations, we associated a conserved quantity  $Q_{\mathscr{I}}^{(\xi)}$  to each symmetry  $\xi^a$  at infinity and, as in [17], showed that these agree with the AD quantities [13] that were previously defined using the standard asymptotic techniques [11]. In section V we focused on the inner boundary  $\Delta$  and first obtained expressions of angular momenta using the so-called 'rotation 1-form'  $\omega_a$  on  $\Delta$ .

We then considered general horizon symmetries  $\xi^a$  representing time-translations (see (5.8)). As in 4-dimensions [3, 4], now there is a qualitatively new element: physically interesting  $\xi^a$  must be allowed to vary from one space-time to another. For example, on spherically symmetric horizons,  $\xi^a$  should point along the null normal  $\ell^a$  to the horizon while in the axi-symmetric case, it should be a linear combination of  $\ell^a$  and rotational symmetries  $\phi_i^a$ . As a result, in general the vector field  $\delta_{\xi}$  on the infinite dimensional phase space  $\Gamma$ , defined by infinitesimal motions along  $\xi^a$  in the space-time manifold  $\mathcal{M}$ , fails to be Hamiltonian, i.e., fails to Lie-drag the symplectic structure. The necessary and sufficient

<sup>&</sup>lt;sup>9</sup> In both cases conservation requires that the Killing vector be tangential to M because, in the presence of a cosmological constant, a Killing vector  $K^a$  satisfies  $d(^*dK) \propto \Lambda K \bot \epsilon$ . When  $\Lambda$  vanishes,  $d(^*dK) = 0$  and conservation holds even when  $K^a$  is not tangential to M.

condition for  $\delta_{\xi}$  to be Hamiltonian is precisely the generalized first law (5.11). This is a significant generalization of the standard first law because it does not require that spacetimes under consideration be globally stationary. It also sheds new light by tying the first law with Hamiltonian evolutions along space-time symmetry vector fields  $\xi^{a}$ .

Finally, in section VI we showed that in presence of global symmetries, the quantities defined at infinity agree with those defined on the horizon. Furthermore, for angular momenta, these quantities also agree with Komar integrals. This tight relation is a non-trivial consequence of field equations since definitions of these three different sets of quantities involve three different sets of geometrical fields. These relations enabled us to show that the first law for the Kerr-AdS family obtained in [10] results as a special case of the much more general first law of the isolated horizon framework. In non-stationary contexts, the framework continue to provide us with conserved quantities associated with symmetries at  $\Delta$  and at  $\mathscr{I}$ . But now there is no simple relation between the two. In particular, while quantities defined at  $\Delta$  refer only to the horizon, i.e., to the black hole in local equilibrium, quantities at infinity receive contributions also from dynamical fields in the region between the horizon and infinity. It is only the horizon quantities that feature in the generalized first law.

To summarize, we have constructed a coherent framework that encompasses and extends three sets of results: the discussion of the first law for the Kerr-AdS family of [10]; the known results on isolated horizons in four dimensions[2–4], and in higher dimensions but without a cosmological constant [18]; and the Hamiltonian framework without internal boundaries of [17]. Our detailed considerations were restricted to the source-free case. However, as indicated in section II, it should be relatively straightforward to extend the phase space to allow matter sources such as Maxwell, Yang-Mills and Higgs fields along the lines of [3].

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