The Issue of the Beginning in Quantum Gravity

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I.   INTRODUCTION

Treatise on Time, the Beginning and the End date back at least twenty five centuries. Does the flow of time have an objective, universal meaning beyond human perception? Or, it is fundamentally only a convenient, and perhaps merely psychological, notion? Are its properties tied to the specifics of observers such as their location and state of motion? Did the physical universe have a finite beginning or has it been evolving eternally? Leading thinkers across cultures meditated on these issues and arrived at definite but strikingly different answers. For example, in the sixth century BCE, Gautama Buddha taught that 'a period of time' is a purely conventional notion, time and space exist only in relation to our experience and the universe is eternal. In the Christian thought, by contrast, the universe had a finite beginning and there was debate whether time represents 'movement' of bodies or if it flows only in the soul. In the fourth century CE, St. Augustine held that time itself started with the world.

Founding fathers of modern Science from Galileo to Newton accepted that God created the universe but Newton posited an absolute time which is to run uniformly from the infinite past to the infinite future. This paradigm became a dogma over centuries. Some philosophers used it to argue that the universe itself *had* to be eternal. For, otherwise one could ask "what was there before?" General relativity toppled this paradigm in one fell swoop. Since the gravitational field is now encoded in space-time geometry, geometry itself now becomes dynamical. The universe could have had a finite beginning —the big-bang— at which not only matter but space-time itself is 'born'. General relativity takes us back to St. Augustine's paradigm but in a detailed, specific and mathematically precise form. In semi-popular articles and radio shows, relativists now like to emphasize that the question "what was there before?" is rendered meaningless because the notions of 'before' and 'after' refer to a space-time geometry. We now have a new paradigm, a new dogma: In the Beginning there was the big bang.

But general relativity is incomplete. For, it ignores quantum effects entirely. Over the last century, we have learned that these effects become important in the physics of the small and should in fact dominate in parts of the universe where matter densities become enormous. So, there is no reason to trust the predictions of general relativity near space-time singularities. The classical physics of general relativity does come to a halt at the big-bang. But applying general relativity near a singularities is an extrapolation which has no justification whatsoever. We need a theory that incorporates not only the dynamical

nature of geometry but also the ramifications of quantum physics. Does the 'correct' or 'true' physics stop at the big-bang also in quantum gravity? Or, is there yet another paradigm shift waiting in the wings?

The goal of this article is to present an up to date summary of the status of these age-old issues within loop quantum gravity (LQG). Detailed calculations in simple cosmological models have shown that the quantum nature of geometry does dominate physics near the big bang, altering dynamics and drastically changing the paradigm provided by general relativity. In particular, the quantum space-time may be much larger than what general relativity has us believe, whence the big bang may not, after all, be the Beginning. Some of the mathematics underlying the main results is subtle. However, I have made an attempt to also include a descriptive summary of the viewpoint, ideas, constructions and physical ramifications of the results.

II. LOOP QUANTUM GRAVITY

A. Conceptual Issues

Remarkably, the necessity of a quantum theory of gravity was pointed out by Einstein already in 1916. In a paper in the Preussische Akademie Sitzungsberichte he wrote:

> *Nevertheless, due to the inneratomic movement of electrons, atoms would have to radiate not only electromagnetic but also gravitational energy, if only in tiny amounts. As this is hardly true in Nature, it appears that quantum theory would have to modify not only Maxwellian electrodynamics but also the new theory of gravitation.*

Ninety years later, our understanding of the physical world is vastly richer but a fully satisfactory unification of general relativity with quantum physics still eludes us. Indeed, the problem has now moved to the center-stage of fundamental physics. (For a brief historical account of the evolution of ideas see, e.g., [1].)

A key reason why the issue is still open is the lack of experimental data with direct bearing on quantum gravity. As a result, research is necessarily driven by theoretical insights on what the key issues are and what will 'take care of itself' once this core is understood. As a consequence, there are distinct starting points which seem natural. Such diversity is not unique to this problem. However, for other fundamental forces we have had clear-cut experiments to weed-out ideas which, in spite of their theoretical appeal, fail to be realized in Nature. We do not have this luxury in quantum gravity. But then, in absence of strong experimental constraints, one would expect a rich variety of internally consistent theories. Why is it then that we do not have a single one? The reason, I believe, lies the deep conceptual difference between the description of gravity in general relativity and that of non-gravitational forces in other fundamental theories. In those theories, space-time is given a priori, serving as an inert background, a stage on which the drama of evolution unfolds. General relativity, on the other hand, is not only a theory of gravity, it is also a theory of space-time structure. Indeed, as I remarked in section I, in general relativity gravity is

encoded in the very geometry of space-time. Therefore, a quantum theory of gravity has to simultaneously bring together *gravity, geometry and the quantum.* This is a band new adventure and our past experience with other forces can not serve as a reliable guide.

LQG is an approach that attempts to face this challenge squarely (for details, see, e.g., [2, 3, 4]). Recall that Riemannian geometry provides the appropriate mathematical language to formulate the physical, kinematical notions as well as the final dynamical equations of any classical theory of relativistic gravity. This role is now assumed by *quantum* Riemannian geometry. Thus, in LQG both matter and geometry are quantum mechanical 'from birth'.

In the classical domain, general relativity stands out as the best available theory of gravity. Therefore, it is natural to ask: *Does quantum general relativity, coupled to suitable matter* (or supergravity, its supersymmetric generalization) *exist as consistent theories non-perturbatively?* In particle physics circles the answer is often assumed to be in the negative, not because there is concrete evidence which rules out this possibility, but because of the analogy to the theory of weak interactions. There, one first had a 4-point interaction model due to Fermi which works quite well at low energies but which fails to be renormalizable. Progress occurred not by looking for non-perturbative formulations of the Fermi model but by replacing the model by the Glashow-Salam-Weinberg renormalizable theory of electro-weak interactions, in which the 4-point interaction is replaced by $W^{\pm}$ and Z propagators. It is often assumed that perturbative non-renormalizability of quantum general relativity points in a similar direction. However this argument overlooks a crucial and qualitatively new element of general relativity. Perturbative treatments pre-suppose that space-time is a smooth continuum *at all scales* of interest to physics under consideration. This assumption is safe for weak interactions. In the gravitational case, on the other hand, the scale of interest is *the Planck length* and there is no physical basis to pre-suppose that the continuum approximation should be valid down to that scale. The failure of the standard perturbative treatments may largely be due to this grossly incorrect assumption and a non-perturbative treatment which correctly incorporates the physical micro-structure of geometry may well be free of these inconsistencies.

Are there any situations, outside LQG, where such physical expectations are borne out by detailed mathematics? The answer is in the affirmative. There exist quantum field theories (such as the Gross-Neveu model in three dimensions) in which the standard perturbation expansion is not renormalizable although the theory is *exactly soluble*! Failure of the standard perturbation expansion can occur because one insists on perturbing around the trivial, Gaussian point rather than the more physical non-trivial fixed point of the renormalization group flow. Interestingly, thanks to the recent work by Lauscher, Reuter, Percacci, Perini and others, there is now growing evidence that situation may be similar with general relativity (see [5] and references therein). Impressive calculations have shown that pure Einstein theory may also admit a non-trivial fixed point. Furthermore, the requirement that the fixed point should continue to exist in presence of matter constrains the couplings in physically interesting ways [6].

Let me conclude this discussion with an important caveat. Suppose one manages to establish that non-perturbative quantum general relativity (or, supergravity) does exist as a mathematically consistent theory. Still, there is no a priori reason to assume that the result would be the 'final' theory of all known physics. In particular, as is the case with classical

general relativity, while requirements of background independence and general covariance do restrict the form of interactions between gravity and matter fields and among matter fields themselves, the theory would not have a built-in principle which *determines* these interactions. Put differently, such a theory would not be a satisfactory candidate for unification of all known forces. However, just as general relativity has had powerful implications in spite of this limitation in the classical domain, quantum general relativity should have qualitatively new predictions, pushing further the existing frontiers of physics. In section III we will see an illustration of this possibility.

B.   Salient features

Detailed as well as short and semi-qualitative reviews of LQG have recently appeared in the literature (see, e.g., [2] and [1] respectively). Therefore, here I will only summarize the key features of the theory that are used in section III.

The starting point of LQG is a Hamiltonian formulation of general relativity based on spin connections [7]. Since all other basic forces of nature are also described by theories of connections, this formulation naturally leads to an unification of all four fundamental forces at a *kinematical* level. Specifically, the phase space of general relativity is the same as that of a Yang-Mills theory. The difference lies in dynamics: whereas in the standard Yang-Mills theory the Minkowski metric features prominently in the definition of the Hamiltonian, there are no background fields whatsoever once gravity is switched on.

Let us focus on the gravitational sector of the theory. Then, the phase space $\mathbf{\Gamma}_{\text{grav}}$ consists of canonically conjugate pairs (A_a^i, P_a^i), where A_a^i is a connection on a 3-manifold M and P_a^i a vector density of weight one, both of which take values in the Lie-algebra su(2). The connection A enables one to parallel transport chiral spinors (such as the left handed fermions of the standard electro-weak model) along curves in M. Its curvature is directly related to the electric and magnetic parts of the space-time *Riemann tensor*. P_i^a plays a double role. Being the momentum canonically conjugate to A, it is analogous to the Yang-Mills electric field. In addition, $E_i^a := 8\pi G \gamma P_i^a$, has the interpretation of a frame or an orthonormal triad (with density weight 1) on M, where γ is the 'Barbero-Immirzi parameter' representing a quantization ambiguity. Each triad E_i^a determines a positive definite 'spatial' 3-metric q_{ab}, and hence the Riemannian geometry of M. This dual role of P is a reflection of the fact that now SU(2) is the (double cover of the) group of rotations of the orthonormal spatial triads on M itself rather than of rotations in an 'internal' space associated with M.

To pass to quantum theory, one first constructs an algebra of 'elementary' functions on $\mathbf{\Gamma}_{\text{grav}}$ (analogous to the phase space functions x and p in the case of a particle) which are to have unambiguous operator analogs. In LQG, the configuration variables are the holonomies h_e built from A_a^i which enable us to parallel transport chiral spinors along edges e and fluxes $E_{S,f}$ of 'electric fields' or 'triads' (smeared with test fields f) across 2-surfaces S. These functions generate a certain algebra $\mathfrak{a}$ (analogous to the algebra generated by operators $\widehat{\exp i\lambda x}$ and $\hat{p}$ in quantum mechanics). The first principal task is to find representations of this algebra. In that representation, *quantum* Riemannian geometry can be probed through the traid operators $\hat{E}_{S,f}$, which stem from classical orthonormal triads. Quite surprisingly the

requirement of diffeomorphism covariance on M suffices to single out a *unique* representation of $\mathfrak{a}$ [8, 9]! This recent result is the quantum geometry analog to the seminal results by Segal and others that characterized the Fock vacuum in Minkowskian field theories. However, while that result assumes not only Poincaré invariance of the vacuum but also specific (namely free) dynamics, it is striking that the present uniqueness theorems make no such restriction on dynamics. The requirement that there be a diffeomorphism invariant state is surprisingly strong and makes the 'background independent' quantum geometry framework surprisingly tight.

This unique representation was in fact introduced already in the mid-nineties [10, 11, 12, 13, 14] and has been extensively used in LQG since then. The underlying Hilbert space is given by $\mathcal{H} = L^2(\bar{\mathcal{A}}, \mathrm{d}\mu_o)$ where $\bar{\mathcal{A}}$ is a certain completion of the classical configuration space $\mathcal{A}$ consisting of smooth connections on M and μ_o is a diffeomorphism invariant, faithful, regular Borel measure on $\bar{\mathcal{A}}$. The holonomy (or configuration) operators $\hat{h}_e$ act just by multiplication. The momentum operators $\hat{P}_{S,f}$ act as Lie-derivatives. In the classical theory, by taking suitable derivatives in M of holonomies h_e along arbitrary edges e, one can recover the connection from which the holonomy is built. However, in the quantum theory, the operators $\hat{h}_e$ are discontinuous and there is no operator $\hat{A}$ corresponding to the connection itself.

Key features of this representation which distinguish it from, say, the standard Fock representation of the quantum Maxwell field are the following. While the Fock representation of photons makes a crucial use of the background Minkowski metric, the above construction is manifestly 'background independent'. Second, as remarked above, the connection itself is not represented as an operator (valued distribution). Holonomy operators, on the other hand, are well-defined. Third, the 'triads' or 'electric field' operators now have purely discrete eigenvalues. Given a surface S and a region R one can express the area A_S and volume V_R using the triads. Although they are non-polynomial functions of triads, the operators $\hat{A}_S$ and $\hat{V}_R$ are well-defined and also have discrete eigenvalues. By contrast, such functions of electric fields can not be promoted to operators on the Fock space. Finally, and most importantly, the Hilbert space $\mathcal{H}$ and the associated holonomy and (smeared) triad operators only provide a *kinematical* framework —the quantum analog of the full phase space. Thus, while elements of the Fock space represent physical states of photons, elements of $\mathcal{H}$ are *not* the physical states of LQG. Rather, like the classical phase space, the kinematic setup provides a home for *formulating* quantum dynamics.

In the Hamiltonian framework, the dynamical content of any background independent theory is contained in its constraints. In quantum theory, the Hilbert space $\mathcal{H}$ and the holonomy and (smeared) triad operators thereon provide the necessary tools to write down quantum constraint operators. Physical states are solutions to these quantum constraints. Thus, to complete the program, one has to: i) obtain the expressions of the quantum constraints; ii) solve the constraint equations; iii) construct the physical Hilbert space from the solutions (e.g. by the group averaging procedure); and iv) extract physics from this physical sector (e.g., by analyzing the expectation values, fluctuations of and correlations between Dirac observables). While strategies have been developed —particularly through Thiemann's 'Master constraint program' [15]— to complete these steps, important open issues remain in the full theory. However, as section III illustrates, the program has been

completed in mini and midi superspace models, leading to surprising insights and answers to some long-standing questions.

III.   APPLICATION: HOMOGENEOUS ISOTROPIC COSMOLOGY

There is long list of questions about the quantum nature of the big bang. For example:

- How close to the Big Bang does a smooth space-time of general relativity make sense? In particular, can one show from first principles that this approximation is valid at the onset of inflation?
- Is the Big-Bang singularity naturally resolved by quantum gravity? Or, is some external input such as a new principle or a boundary condition at the Big Bang essential?
- Is the quantum evolution across the 'singularity' deterministic? Since one needs a fully non-perturbative framework to answer this question in the affirmative, in the Pre-Big-Bang [16] and Ekpyrotic/Cyclic [17, 18] scenarios, for example, so far the answer is in the negative.
- If the singularity is resolved, what is on the 'other side'? Is there just a 'quantum foam', far removed from any classical space-time, or, is there another large, classical universe?

For many years, these and related issues had been generally relegated to the 'wish list' of what one would like the future, satisfactory quantum gravity theory to eventually address. It seems likely that these issues can be met head-on only in a background independent, non-perturbative approach. One such candidate is LQG. Indeed, starting with the seminal work of Bojowald some five years ago [19], notable progress has been made in the context of symmetry reduced, minisuperspaces. Earlier papers focussed only on singularity resolution. However, to describe physics in detail, it is essential to construct the physical Hilbert space and introduce interesting observables and semi-classical states by completing the program outlined at the end of the last section. These steps have been completed recently. In this section, I will summarize the state of the art, emphasizing these recent developments. (For a comprehensive review of the older work see, e.g., [20].)

Consider the spatially homogeneous, isotropic, $k{=}0$ cosmologies with a massless scalar field. It is instructive to focus on this model because *every* of its classical solutions has a singularity. There are two possibilities: In one the universe starts out at the big bang and expands, and in the other it contracts into a big crunch. The question is if this unavoidable classical singularity is naturally tamed by quantum effects. This issue can be analyzed in the geometrodynamical framework used in older quantum cosmology. Unfortunately, the answer turns out to be in the negative. For example, if one begins with a semi-classical state representing an expanding classical universe at late times and evolves it back via the Wheeler DeWitt equation, one finds that it just follows the classical trajectory into the big bang singularity [22, 23].

In loop quantum cosmology (LQC), the situation is very different [21, 22, 23]. This may seem surprising at first. For, the system has only a finite number of degrees of freedom

and von Neumann's theorem assures us that, under appropriate assumptions, the resulting quantum mechanics is unique. The only remaining freedom is factor-ordering and this is generally insufficient to lead to qualitatively different predictions. However, for reasons I will now explain, LQC does turn out to be qualitatively different from the Wheeler-DeWitt theory [24].

Because of spatial homogeneity and isotropy, one can fix a fiducial (flat) triad $\overset{o}{e}{}^a_i$ and its dual co-triad $\overset{o}{\omega}{}^i_a$. The SU(2) gravitational spin connection A^i_a used in LQG has only one component c which furthermore depends only on time; $A^i_a = c\, \overset{o}{\omega}{}^i_a$. Similarly, the triad E^a_i (of density weight 1) has a single component p; $E^a_i = p\,(\det \overset{o}{\omega})\, \overset{o}{e}{}^a_i$. p is related to the scale factor a via $a^2 = |p|$. However, p is not restricted to be positive; under $p \to -p$ the metric remains unchanged but the spatial triad flips the orientation. The pair (c, p) is 'canonically conjugate' in the sense that the only non-zero Poisson bracket is given by:

$$\{c,\, p\} = \frac{8\pi G\gamma}{3}\,, \tag{3.1}$$

where as before γ is the Barbero-Immirzi parameter.

Since a precise quantum mechanical framework was not available for full geometrodynamics, in the Wheeler-DeWitt quantum cosmology one focused just on the reduced model, without the benefit of guidance from the full theory. A major difference in Loop quantum cosmology (LQC) is that although the symmetry reduced theory has only a finite number of degrees of freedom, quantization is carried out by closely mimicking the procedure used in *full* LQG, outlined in section II. Key differences between LQC and the older Wheeler-DeWitt theory can be traced back to this fact.

Recall that in full LQG diffeomorphism invariance leads one to a specific kinematical framework in which there are operators $\hat{h}_e$ representing holonomies and $\hat{P}_{S,f}$ representing (smeared) momenta but there is no operator(-valued distribution) representing the connection A itself [8, 9]. In the cosmological model now under consideration, it is sufficient to evaluate holonomies along segments $\mu\, \overset{o}{e}{}^a_i$ of straight lines determined by the fiducial triad $\overset{o}{e}{}^a_i$. These holonomies turn out almost periodic functions of c, i.e. are of the form $N_{(\mu)}(c) := \exp i\mu(c/2)$, where the word 'almost' refers to the fact that μ can be any real number. These functions were studied exhaustively by the mathematician Harold Bohr, Niels' brother. In quantum geometry, the $N_{(\mu)}$ are the LQC analogs of the spin-network functions of full LQG.

In quantum theory, then, we are led to a representation in which operators $\hat{N}_{(\mu)}$ and $\hat{p}$ are well-defined, but there is *no* operator corresponding to the connection component c. This seems surprising because our experience with quantum mechanics suggests that one should be able to obtain the operator analog of c by differentiating $\hat{N}_{(\mu)}$ with respect to the parameter μ. However, in the representation of the basic quantum algebra that descends to LQC from full LQG, although the $\hat{N}_{(\mu)}$ provide a 1-parameter group of unitary transformations, it fails to be weakly continuous in μ. Therefore one can not differentiate and obtain the operator analog of c. In quantum mechanics, this would be analogous to having well-defined (Weyl) operators corresponding to the classical functions $\exp i\mu x$ but no operator $\hat{x}$ corresponding to x itself. This violates one of the assumptions of the von-Neumann uniqueness theorem. New representations then become available which are

inequivalent to the standard Schrödinger one. In quantum mechanics, these representations are not of direct physical interest because we need the operator $\hat{x}$. In LQC, on the other hand, full LQG naturally leads us to a new representation, i.e., to *new quantum mechanics.* This theory is inequivalent to the Wheeler-DeWitt type theory already at a kinematical level. In the Wheeler-Dewitt theory, the gravitational Hilbert space would be $L^2(\mathbb{R}, \mathrm{d}c)$, operators $\hat{c}$ would act by multiplication and $\hat{p}$ would be represented by $-i\hbar \mathrm{d}/\mathrm{d}c$. In LQC the 'quantum configuration space' is different from the classical configuration space: Just as we had to complete the space $\mathcal{A}$ of smooth connections to the space $\bar{\mathcal{A}}$ of generalized connections in LQG, we are now led to consider a completion —called the Bohr compactification $\bar{\mathbb{R}}_{\mathrm{Bohr}}$— of the '$c$-axis'. The gravitational Hilbert space is now $L^2(\bar{\mathbb{R}}_{\mathrm{Bohr}}, \mathrm{d}\mu_{\mathrm{Bohr}})$ [24] where $\mathrm{d}\mu_{\mathrm{Bohr}}$ is the LQC analog of the measure $\mathrm{d}\mu_o$ selected by the uniqueness results [8, 9] in full LQG. The operators $\hat{N}_{(\mu)}$ act by multiplication and $\hat{p}$ by differentiation. However, there is no operator $\hat{c}$. In spite of these differences, in the semi-classical regime LQC is well approximated by the Wheeler-DeWitt theory. However, important differences manifest themselves at the Planck scale. These are the hallmarks of quantum geometry [2, 20].

The new representation also leads to a qualitative difference in the structure of the Hamiltonian constraint operator: the gravitational part of the constraint is a *difference* operator, rather than a differential operator as in the Wheeler-DeWitt theory. The derivation [22, 23, 24] can be summarized briefly as follows. In the classical theory, the gravitational part of the constraint is given by $\int d^3x\, \epsilon^{ijk} e^{-1} E_i^a E_j^b F_{ab\,k}$ where $e = |\det E|^{1/2}$ and F_{ab}^k the curvature of the connection A_a^i. The part $\epsilon^{ijk} e^{-1} E_i^a E_j^b$ of this operator involving triads can be quantized [19, 24] using a standard procedure introduced by Thiemann in the full theory [4]. However, since there is no operator corresponding to the connection itself, one has to express F_{ab}^k as a limit of the holonomy around a loop divided by the area enclosed by the loop, as the area shrinks to zero. Now, quantum geometry tells us that the area operator has a minimum non-zero eigenvalue, Δ, and in the quantum theory it is natural to shrink the loop only till it attains this minimum. There are two ways to implement this idea in detail (see [22, 23, 24]). In both cases, it is the existence of the 'area gap' Δ that leads one to a difference equation. So far, most of the LQC literature has used the first method [22, 24]. In the resulting theory, the classical big-bang is replaced with a quantum bounce with a number of desirable features. However, it also has one serious drawback: at the bounce, matter density can be low even for physically reasonable choices of quantum states. Thus, that theory predicts certain departures from classical general relativity even in the low curvature regime (for details, see [22, 23]). The second and more recently discovered method [23] cures this problem while retaining the physically appealing features of the first and, furthermore, has a more direct motivation. For brevity, therefore, I will confine myself only to the second method.

Let us represent states as functions $\Psi(v, \phi)$, where ϕ is the scalar field and the dimensionless real number v represents geometry. Specifically, $|v|$ is the eigenvalue of the operator

$\hat{V}$ representing volume[1] (essentially the cube of the scale factor):

$$\hat{V}|v\rangle = K \, (\frac{8\pi\gamma}{6})^{\frac{3}{2}} \, |v| \, \ell_{\mathrm{Pl}}^3 \, |v\rangle \quad \text{where} \quad K = \frac{3\sqrt{3\sqrt{3}}}{2\sqrt{2}} \, . \tag{3.2}$$

Then, the LQC Hamiltonian constraint assumes the form:

$$\begin{aligned} \partial_\phi^2 \Psi(v,\phi) &= [B(v)]^{-1} \left(C^+(v) \, \Psi(v+4,\phi) + C^o(v) \, \Psi(v,\phi) + C^-(v) \, \Psi(v-4,\phi) \right) \\ &=: -\Theta \, \Psi(v,\phi) \end{aligned} \tag{3.3}$$

where the coefficients $C^\pm(v)$, $C^o(V)$ and $B(v)$ are given by:

$$\begin{aligned} C^+(v) &= \frac{3\pi KG}{8} \, |v+2| \, \big||v+1| - |v+3|\big| \\ C^-(v) &= C^+(v-4) \quad \text{and} \quad C^o(v) = -C^+(v) - C^-(v) \\ B(v) &= \left(\frac{3}{2}\right)^3 K \, |v| \, \left||v+1|^{1/3} - |v-1|^{1/3}\right|^3 \, . \end{aligned} \tag{3.4}$$

Now, in each classical solution, ϕ is a globally monotonic function of time and can therefore be taken as the dynamical variable representing an *internal* clock. In quantum theory there is no space-time metric, even on-shell. But since the quantum constraint (3.3) dictates how $\Psi(v,\phi)$ 'evolves' as ϕ changes, it is convenient to regard the argument ϕ in $\Psi(v,\phi)$ as *emergent time* and v as the physical degree of freedom. A complete set of Dirac observables is then provided by the constant of motion $\hat{p}_\phi$ and operators $\hat{v}|_{\phi_o}$ determining the value of v at the 'instant' $\phi = \phi_o$.

Physical states are the (suitably regular) solutions to Eq (3.3). The map $\hat{\Pi}$ defined by $\hat{\Pi} \, \Psi(v,\phi) = \Psi(-v,\phi)$ corresponds just to the flip of orientation of the spatial triad (under which geometry remains unchanged); $\hat{\Pi}$ is thus a large gauge transformation on the space of solutions to Eq. (3.3). One is therefore led to divide physical states into sectors, each providing an irreducible, unitary representation of this gauge symmetry. Physical considerations [22, 23] imply that we should consider the symmetric sector, with eigenvalue $+1$ of $\hat{\Pi}$.

To endow this space with the structure of a Hilbert space, one can proceed along one of two paths. In the first, one defines the action of the Dirac observables on the space of suitably regular solutions to the constraints and selects the inner product by demanding that these operators be self-adjoint [25]. A more systematic procedure is the 'group averaging method' [26]. The technical implementation [22, 23] of both these procedures is greatly simplified by the fact that the difference operator Θ on the right side of (3.3) is independent of ϕ and can be shown to be self-adjoint and positive definite on the Hilbert space $L^2(\bar{\mathbb{R}}_{\mathrm{Bohr}}, B(v)\mathrm{d}\mu_{\mathrm{Bohr}})$.

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[1] In non-compact spatially homogeneous models, integrals of physical interest over the full spatial manifold diverge. Therefore, to obtain a consistent Hamiltonian description, one has to introduce an elementary cell $\mathcal{V}$ and restrict all integrals to $\mathcal{V}$ already in the classical theory. This is necessary also in geometrodynamics. $\hat{V}$ is the volume operator associated with $\mathcal{V}$.

The final result can be summarized as follows. Since Θ is a difference operator, the physical Hilbert space $\mathcal{H}_{\text{phy}}$ has sectors $\mathcal{H}_\epsilon$ which are superselected; $\mathcal{H}_{\text{phy}} = \oplus_\epsilon \mathcal{H}_\epsilon$ with $\epsilon \in (0, 2)$. The overall predictions are insensitive to the choice of a specific sector (for details, see [22, 23]). States $\Psi(v, \phi)$ in $\mathcal{H}_\epsilon$ are symmetric under the orientation inversion $\hat{\Pi}$ and have support on points $v = |\epsilon| + 4n$ where n is an integer. Wave functions $\Psi(v, \phi)$ in a generic sector solve (3.3) and are of positive frequency with respect to the 'internal time' ϕ: they satisfy the 'positive frequency' square root

$$-i\partial_\phi \Psi = \sqrt{\Theta}\, \Psi\,. \tag{3.5}$$

of Eq (3.3). (The square-root is a well-defined (positive self-adjoint) operator because Θ is positive and self-adjoint.) The physical inner product is given by:

$$\langle \Psi_1 \,|\, \Psi_2 \rangle \;=\; \sum_{v \in \{|\epsilon|+4n\}} B(v)\, \bar{\Psi}_1(v, \phi_o) \Psi_2(v, \phi_o) \tag{3.6}$$

and is 'conserved', i.e., is independent of the 'instant' ϕ_o chosen in its evaluation. On these states, the Dirac observables act in the expected fashion:

$$\hat{p}_\phi \Psi \;=\; -i\hbar \partial_\phi \Psi$$
$$\hat{v}|_{\phi_o} \Psi(v, \phi) \;=\; e^{i\sqrt{\Theta}(\phi - \phi_o)}\, v\, \Psi(v, \phi_o) \tag{3.7}$$

What is the relation of this LQC description with the Wheeler-DeWitt theory? It is straightforward to show [23] that, for $v \gg 1$, there is a precise sense in which the difference operator Θ approaches the Wheeler DeWitt differential operator $\underline{\Theta}$, given by

$$\underline{\Theta}\Psi(v, \phi) = 12\pi G\, v\partial_v\big(v\partial_v \Psi(v, \phi)\big) \tag{3.8}$$

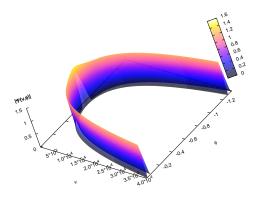Thus, if one ignores the quantum geometry effects, Eq (3.3) reduces to the Wheeler-DeWitt equation

$$\partial_\phi^2 \Psi = -\underline{\Theta}\,\Psi. \tag{3.9}$$

Note that the operator $\underline{\Theta}$ is positive definite and self-adjoint on the Hilbert space $L_s^2(\mathbb{R}, \underline{B}(v)\mathrm{d}v)$ where the subscript s denotes the restriction to the symmetric eigenspace of Π and $\underline{B}(v) := Kv^{-1}$ is the limiting form of $B(v)$ for large v. Its eigenfunctions $\underline{e}_k$ with eigenvalue $\omega^2 (\geq 0)$ are 2-fold degenerate on this Hilbert space. Therefore, they can be labelled by a real number k:

$$\underline{e}_k(v) := \frac{1}{\sqrt{2\pi}}\, e^{ik \ln |v|} \tag{3.10}$$

where k is related to ω via $\omega = \sqrt{12\pi G}|k|$. They form an orthonormal basis on $L_s^2(\mathbb{R}, \underline{B}(v)\mathrm{d}v)$. A 'general' positive frequency solution to (3.9) can be written as

$$\Psi(v, \phi) = \int_{-\infty}^{\infty} \mathrm{d}k\, \tilde{\Psi}(k)\, \underline{e}_k(v) e^{i\omega\phi} \tag{3.11}$$
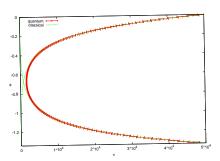
FIG. 1: The figure on left shows the absolute value of the wave function Ψ as a function of ϕ and v. Being a physical state, Ψ is symmetric under $v \to -v$. The figure on the right shows the expectation values of Dirac observables $\hat{v}|_\phi$ and their dispersions. They exhibit a quantum bounce which joins the contracting and expanding classical trajectories marked by fainter lines. In this simulation, the parameters in the initial data are: $v^\star = 5 \times 10^4$, $p_\phi^\star = 5 \times 10^3 \sqrt{G}\hbar$ and $\Delta p_\phi / p_\phi = 0.0025$.

for suitably regular $\tilde{\Psi}(k)$. This expression will enable us to show explicitly that the singularity is not resolved in the Wheeler-DeWitt theory.

With the physical Hilbert space and a complete set of Dirac observables at hand, we can now construct states which are semi-classical at late times —e.g., now— and evolve them 'backward in time' numerically. There are three natural constructions to implement this idea in detail, reflecting the freedom in the notion of semi-classical states. In all cases, the main results are the same [22, 23]. Here I will report on the results obtained using the strategy that brings out the contrast with the Wheeler DeWitt theory most sharply.

As noted before, p_ϕ is a constant of motion. For the semi-classical analysis, we are led to choose a large value $p_\phi^\star$ ($\gg \sqrt{G}\hbar$). In the closed model, for example, this condition is necessary to ensure that the universe can expand out to a macroscopic size. Fix a point $(v^\star, \phi_o)$ on the corresponding classical trajectory which starts out at the big bang and then expands, choosing $v^\star \gg 1$. We want to construct a state which is peaked at $(v^\star, p_\phi^\star)$ at a 'late initial time' $\phi = \phi_o$ and follow its 'evolution' backward. At 'time' $\phi = \phi_o$, consider then the function

$$\Psi(v, \phi_o) = \int_{-\infty}^{\infty} dk\, \tilde{\Psi}(k)\, \underline{e}_k(v)\, e^{i\omega(\phi_o - \phi^\star)}, \quad \text{where} \quad \tilde{\Psi}(k) = e^{-\frac{(k - k^\star)^2}{2\sigma^2}} \tag{3.12}$$

where $k^\star = -p_\phi^\star / \sqrt{12\pi G\hbar^2}$ and $\phi^\star = -\sqrt{1/12\pi G}\, \ln(v^\star) + \phi_o$. One can easily evaluate the integral in the approximation $|k^*| \gg 1$ and calculate mean values of the Dirac observables

and their fluctuations. One finds that, as required, the state is sharply peaked at values $v^\star, p_\phi^\star$. The above construction is closely related to that of coherent states in non-relativistic quantum mechanics. The main difference is that the observables of interest are not v and its conjugate momentum but rather v and p_ϕ —the momentum conjugate to 'time', i.e., the analog of the Hamiltonian in non-relativistic quantum mechanics. Now, one can evolve this state backwards using the Wheeler-DeWitt equation (3.9). It follows immediately from the form (3.11) of the general solution to (3.9) and the fact that p_ϕ is large that this state would remain sharply peaked at the chosen classical trajectory and simply follow it into the big-bang singularity.

In LQC, we can use the restriction of (3.12) to points $v = |\epsilon| + 4n$ as the initial data and evolve it backwards numerically. Now the evolution is qualitatively different (see Fig.1). The state does remains sharply peaked at the classical trajectory till the matter density reaches a critical value:

$$\rho_{\mathrm{crit}} = \frac{\sqrt{3}}{16\pi^2\gamma^3G^2\hbar}, \tag{3.13}$$

which is about 0.82 times the Planck density. However, *then it bounces*. Rather than following the classical trajectory into the singularity as in the Wheeler-DeWitt theory, the state 'turns around'. What is perhaps most surprising is that it again becomes semi-classical and follows the 'past' portion of a classical trajectory, again with $p_\phi = p_\phi^\star$, which was headed towards the big crunch. Let us we summarize the forward evolution of the full quantum state. In the distant past, the state is peaked on a classical, contracting pre-big-bang branch which closely follows the evolution dictated by Friedmann equations. But when the matter density reaches the Planck regime, quantum geometry effects become significant. Interestingly, they make gravity *repulsive*, not only halting the collapse but turning it around; the quantum state is again peaked on the classical solution now representing the post-big-bang, expanding universe. Since this behavior is so surprising, a very large number of numerical simulations were performed to ensure that the results are robust and not an artifact of the special choices of initial data or of the numerical methods used to obtain the solution [22, 23].

For states which are semi-classical at late times, the numerical evolution in exact LQC can be well-modelled by an effective, modified Friedman equation :

$$\frac{\dot{a}^2}{a^2} = \frac{8\pi G}{3}\rho\left[1 - \frac{\rho}{\rho_{\mathrm{crit}}}\right] \tag{3.14}$$

where, as usual, a is the scale factor. In the limit $\hbar \to 0$, ρ_{crit} diverges and we recover the standard Friedmann equation. Thus the second term is a genuine quantum correction. Eq. (3.14) can also be obtained analytically from (3.3) by a systematic procedure [27]. But the approximations involved are valid only well outside the Planck domain. It is therefore surprising that the bounce predicted by the exact quantum equation (3.3) is well approximated by a naive extrapolation of (3.14) across the Planck domain. While there is some understanding of this seemingly 'unreasonable success' of the effective equation (3.14), further work is needed to fully understand this issue.

Finally let us return to the questions posed in the beginning of this section. In the model, LQC has been able to answer all of them. One can deduce from first principles that classical

general relativity is an excellent approximation till very early times, including the onset of inflation in standard scenarios. Yet quantum geometry effects have a profound, global effect on evolution. In particular, the singularity is naturally resolved without any external input and there is a classical space-time also in the pre-big-bang branch. LQC provides a deterministic evolution which joins the two branches.

IV. DISCUSSION

Even though there are several open issues in the formulation of full quantum dynamics in LQG, detailed calculations in simple models have provided hints about the general structure. It appears that the most important non-perturbative effects arise from the replacement of the local curvature term F_{ab}^i by non-local holonomies. This non-locality is likely to be a central feature of the full LQG dynamics. In the cosmological model considered in section III, it is this replacement of curvature by holonomies that is responsible for the subtle but crucial differences between LQC and the Wheeler-DeWitt theory.[2]

By now a number of mini-superspaces and a few midi-superspaces have been studied in varying degrees of detail. In all cases, the classical, space-like singularities are resolved by quantum geometry *provided one treats the problem non-perturbatively.* For example, in anisotropic mini-superspaces, there is a qualitative difference between perturbative and non-perturbative treatments: if anisotropies are treated as perturbations of a background isotropic model, the big-bang singularity is not resolved while if one treats the whole problem non-perturbatively, it is [28].

A qualitative picture that emerges is that the non-perturbative quantum geometry corrections are *'repulsive'*. While they are negligible under normal conditions, they dominate when curvature approaches the Planck scale and halt the collapse that would classically have lead to a singularity. In this respect, there is a curious similarity with the situation in the stellar collapse where a new repulsive force comes into play when the core approaches a critical density, halting further collapse and leading to stable white dwarfs and neutron stars. This force, with its origin in the Fermi-Dirac statistics, is *associated with the quantum nature of matter.* However, if the total mass of the star is larger than, say, 5 solar masses, classical gravity overwhelms this force. The suggestion from LQC is that, a new repulsive force *associated with the quantum nature of geometry* may come into play near Planck density, strong enough to prevent the formation of singularities irrespective of how large the mass is. Since this force is negligible until one enters the Planck regime, predictions of classical relativity on the formation of trapped surfaces, dynamical and isolated horizons would still hold. But assumptions of the standard singularity theorems would be

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[2] Because early presentations emphasized the difference between $B(v)$ of LQC and $\underline{B}(v) = Kv^{-1}$ of the Wheeler-DeWitt theory, there is a misconception in some circles that the difference in quantum dynamics is primarily due to the non-trivial 'inverse volume' operator of LQC. This is not correct. In deed, in the model considered here, qualitative features of quantum dynamics, including the bounce, remain unaffected if one replaces by hand $B(v)$ with $\underline{B}(v)$ in the LQC evolution equation (3.3).

violated. There would be no singularities, no abrupt end to space-time where physics stops. Non-perturbative, background independent quantum physics would continue.

Returning to the issue of the Beginning, the big-bang in particular appears to be an artifact of the assumption that the continuum, classical space-time of general relativity should hold at all scales. LQC strongly suggests that this approximation breaks down when the matter reaches Planck density. One might have at first thought that, since this is a tiny portion of space-time, whatever quantum effects there may be, they would have negligible effect on global properties of space-time and hence almost no bearing on the issue of The Beginning. However, detailed LQC calculations have shown that this intuition may be too naive. The 'tiny portion' may actually be a bridge to another large universe. The physical, quantum space-time of could be significantly larger than what general relativity had us believe. The outstanding open issue is whether this scenario persists when inhomogeneities are adequately incorporated in the analysis.

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