How Black Holes Grow*

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Abstract

A summary of how black holes grow in full, non-linear general relativity is presented. Specifically, a notion of *dynamical horizons* is introduced and expressions of fluxes of energy and angular momentum carried by gravitational waves across these horizons are obtained. Fluxes are local and the energy flux is positive. Change in the horizon area is related to these fluxes. The flux formulae also give rise to balance laws analogous to the ones obtained by Bondi and Sachs at null infinity and provide generalizations of the first and second laws of black hole mechanics.

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I. INTRODUCTION

Black holes are perhaps the most fascinating manifestations of the curvature of space and time predicted by general relativity. Properties of isolated black holes in equilibrium have been well-understood for quite some time. However, in Nature, black holes are rarely in equilibrium. They grow by swallowing stars and galactic debris as well as electromagnetic and gravitational radiation. For such dynamical black holes, the only known major result in exact general relativity has been a celebrated area theorem, proved by Stephen Hawking in the early seventies: if matter satisfies the dominant energy condition, the area of the black hole event horizon can never decrease. This theorem has been extremely influential because of its similarity with the second law of thermodynamics. However, it is a 'qualitative' result; it does not provide an explicit formula for the amount by which the area increases in any given physical situation. One might hope that the change in area is related, in a direct manner, to the flux of matter fields and gravitational radiation falling into the black hole. Is this in fact the case? If so, the formula describing this dynamical evolution of the black hole would give us a 'finite' generalization of the first law of black hole mechanics: The standard first law, $\delta E = (\kappa/8\pi G)\delta a + \Omega \delta J$, relates the infinitesimal change δa in the black hole area due to the infinitesimal influx δE of energy and angular momentum δJ as the black hole makes a transition from one equilibrium state to a nearby one, while the exact evolution law would provide its 'integral version', relating equilibrium configurations which are far removed from one another.

From a general, physical viewpoint, these expectations seem quite reasonable. Why, then, had this question remained unresolved for three decades? The reason is that when one starts thinking of possible strategies to carry out these generalizations, one immediately encounters severe difficulties. To begin with, to carry out this program one would need a precise notion of the gravitational energy flux falling into the black hole. Now, as is well known, in full general relativity, there is no gauge invariant, quasi-local notion of gravitation radiation. The standard notion refers to null infinity, where one can exploit the weakness of curvature to introduce the notion of asymptotic translations, define Bondi 4-momentum as the generator of these translations, calculate energy fluxes, and prove the 'balance law' relating the change in the Bondi 4-momentum with the momentum flux across portions of null infinity. Even this structure at null infinity is highly non-trivial. Indeed there was considerable confusion about physical reality of gravitational waves well into the late fifties because it was difficult to disentangle coordinate effects from physical ones. Even Arthur Eddington who, according to a popular legend, was 'one of the only three wise men' to understand general relativity from the very early days, is said not to have believed in the reality of gravitational waves. Apparently, he referred to them as pure coordinate effects which 'travelled at the speed of thought'! Therefore, the introduction of the Bondi framework in the sixties was hailed as a major breakthrough: it established, once and for all, the physical reality of gravitational waves. Bondi famously said: They are real; they carry energy; you can heat water with them!

However, to address the issues discussed above pertaining to black holes, one does not have the luxury of working in the asymptotic region; one must face the highly curved geometry near black holes. Since this is the strong field, highly non-linear regime of general relativity, it seems hopeless to single out a translation group unambiguously. What would the notion of energy even mean without the usual simplifications at infinity? There do exist formulas for the amount of gravitational energy contained in a given region. But typi-

cally they use pseudo-tensors and are coordinate and/or frame dependent. Consequently, in strong curvature regions, these expressions fail to be gauge invariant whence, from a physical perspective, they are simply not meaningful. Thus, even a broad conceptual framework or paradigm was not available within which one could hope to establish a formula relating the change in area to the flux of energy and angular momentum falling into the black hole. The issues had remained unresolved because it appeared that one has to simultaneously develop the conceptual framework which is to provide a natural home for the required notions in the strong field regime of a black hole, and manipulate field equations with their full non-linearity to get explicit expressions for energy and angular momentum fluxes.

At first these challenges seem formidable. However, it turns out that an appropriate paradigm is in fact suggested by the strategy used in numerical simulations of black hole formation and merger. Using it, a program was initiated in collaboration with Badri Krishnan [1, 2] a few months before the Plebanskifest and has been further developed by the two of us as well as by Ivan Booth and Stephen Fairhurst since then. The final results are surprising because one can introduce the necessary notions of energy and angular momentum fluxes in the strong field, fully non-linear regime. But the methods used are all well-established and quite conservative; there is nothing here that was not available in the seventies.

The purpose of this contribution is to provide a bird's eye view of the present situation and of prospects for near future. A detailed and more comprehensive treatment will appear elsewhere.

II. CONCEPTUAL PARADIGM

A. The idea

Our first task is to give a precise definition of the black hole surface whose area is to increase during evolution.

Heuristically, one thinks of black holes as regions of space-time from which no signal can escape to infinity. In mathematical general relativity, this idea is captured in the notion of event horizons. Standard space-times —such as the Kerr family—that we physically think of as containing black holes, all have event horizons. However, the notion of event horizons is extremely non-local and teleological: it is the future boundary of the past of future null infinity. Consequently, one can use event horizons to detect the presence of a black hole only if one has access to the full space-time metric, all the way to the infinite future. This extreme non-locality is dramatically illustrated by the fact that there may well be an event horizon developing in your room as you read this page because a million years from now, there may be a gravitational collapse in a nearby region in our galaxy!

Because of these features, generally, the notion of an event horizon is not very useful in practice. A striking example is provided by numerical simulations of a gravitational collapse leading to the formation of a black hole (or, of binary black holes which merge). Here, one is interested in evolving suitable initial data sets and one needs to know, at each time step, if a black hole has formed and, if so, where it is. The teleological nature of the event horizon makes it totally unsuitable to detect black holes during these evolutions. One needs a notion that can sense the presence of the black hole and narrow down its approximate location quasi-locally, e.g., using only the initial data that can be accessed at each instant of time. Marginally trapped surfaces and apparent horizons provide such notions. A marginally

trapped surface S is a 2-dimensional sub-manifold of space-time \mathcal{M} , topologically S^2 , such that the expansion of one null normal to it, say ℓ^a , is everywhere zero and that of the other null normal, n^a , is negative. The second condition merely says that n^a is the inward pointing null normal. The non-trivial feature of S is that the expansion of the other null normal, ℓ^a , is zero rather than positive. Thus, none of the light rays emerging from any point on S are directed towards the 'asymptotic' or 'outside' region. Apparent horizons are associated with (partial) Cauchy surfaces M. Given M, the apparent horizon S is the outermost marginally trapped surface lying in M. In numerical simulations then, one keeps track of black holes by monitoring the behavior of apparent horizons that emerge during the evolution. Assuming cosmic censorship, once an apparent horizon develops, there is an event horizon which lies outside it.

Thus, in numerical relativity, one typically has a foliation of the given region of the space-time \mathcal{M} by partial Cauchy surfaces M, each equipped with an apparent horizon. The apparent horizons can 'jump' discontinuously. For example, in the black hole coalescence problem, there are two distinct apparent horizons up to a certain time step and then there is a sudden jump to a single connected apparent horizon. However, in practice, these discrete jumps happen only at a few places during the course of numerical simulation. Here, we will be concerned with those evolutionary epochs during which the 'stack of apparent horizons' formed by evolution span a continuous world tube. Along these apparent horizon world-tubes, the area increases and our aim is to present formulas relating this increase with the influx of energy and angular momentum.

Before proceeding to give precise definitions, however, let me emphasize that the above considerations suggested by numerical relativity serve only as motivation. In particular, we will not need a foliation of space-time \mathcal{M} by partial Cauchy surfaces M. The object of direct physical interest —a $dynamical\ horizon$ — can be located using only the space-time geometry, although in applications to numerical relativity, it will typically arise as the world tube of apparent horizons.

B. Definition and methodology

Let us then begin with the precise definition.

Definition: A smooth, three-dimensional, space-like sub-manifold H in a space-time \mathcal{M} is said to be a *dynamical horizon* if it can be foliated by a family of two-spheres such that, on each leaf S, the expansion $\theta_{(\ell)}$ of a null normal ℓ^a vanishes and the expansion $\theta_{(n)}$ of the other null normal n^a is strictly negative.

Thus, a dynamical horizon H is a 3-manifold which is foliated by marginally trapped 2-spheres. Note first that, in contrast to event horizons, dynamical horizons can be located quasi-locally; knowledge of the full space-time is not required. Thus, for example, you can rest assured that no dynamical horizon has developed in the room you are sitting in ever since it was built! On the other hand, while in asymptotically flat space-times black holes are characterized by event horizons, there is no one-to-one correspondence between black holes and dynamical horizons. It follows from properties of trapped surfaces that, assuming cosmic censorship, dynamical horizons must lie inside an event horizon. However, in the interior of an expanding event horizon, there may be many dynamical horizons. 1 Nonetheless, the

¹ At this still preliminary stage, there is essentially no control on how many dynamical horizons there

framework is likely to have powerful applications to black hole physics because its results apply to each and every one of these dynamical horizons.

Apart from the requirement that H be foliated by marginally trapped surfaces, the definition contains only two conditions. The first asks that H be space-like. This property is implied by a stronger but physically reasonable restriction that the 'inward' derivative $\mathcal{L}_n \theta_{(\ell)}$ of $\theta_{(\ell)}$ be negative and the flux of energy across H be non-zero. The second condition is that the leaves be topologically S^2 . This can be replaced by the weaker condition that they be compact. One can then show that the topology of S is necessarily S^2 if the flux of matter or gravitational energy across H is non-zero. Thus, the conditions imposed in the definition appear to be minimal. If these fluxes were to vanish identically, H would become isolated and replaced by a null, non-expanding horizon [5].

Dynamical horizons are closely related to Hayward's [3] future, outward trapping horizons H'. These H' are 3-manifolds, foliated by compact 2-manifolds S with $\theta_{(\ell)} = 0$, $\theta_{(n)} < 0$, and $\mathcal{L}_n \theta_{(\ell)} < 0$. Assuming that the null energy condition holds, one can show that H is either space-like or null. If it is space-like, it is a dynamical horizon, while if it is null it is a non-expanding horizon, studied extensively in [5]. This notion is especially useful in the study of 'weakly-dynamical' horizons [4] which can be viewed as perturbations of isolated horizons [5–8]. The main difference from the notion of dynamical horizons used in this report is that while the definition of trapping horizons imposes a condition on the derivative of $\theta_{(\ell)}$ off H, dynamical horizons refer only to geometric quantities which are intrinsically defined on H. But this difference is not physically significant because, in cases of interest, the additional condition would probably be satisfied and dynamical horizons will be future, outer trapping horizons. Rather, the significant difference with respect to Hayward's work lies in the way we analyze consequences of these conditions and in the results we obtain. While Hayward's framework is based on a 2+2 decomposition, ours will be based on the ADM 3+1 decomposition. We use different parts of Einstein's equations, our discussion includes angular momentum, our flux formulae are new and our generalization of black hole mechanics is different.

Let us begin by fixing notation. Let $\hat{\tau}^a$ be the unit time-like normal to H and denote by ∇ the space-time derivative operator. The metric and extrinsic curvature of H are denoted by q_{ab} and $K_{ab} := q_a{}^c q_b{}^d \nabla_c \hat{\tau}_d$ respectively; D is the derivative operator on H compatible with q_{ab} and \mathcal{R}_{ab} its Ricci tensor. If H admits more than one foliation by marginally trapped surfaces, we will simply fix any one of these and work with it. Thus, our results will apply to all such foliations. Leaves of this foliation are called *cross-sections* of H. The unit space-like

can be inside an *evolving* event horizon, but the expectation is that there will be several distinct ones. *Stationary* event horizons, by contrast, do not admit any dynamical horizons because, for simplicity, we have tailored our definition to cases where the area increases monotonically. In the closely related notion of *outer trapping horizons*, introduced by Hayward and discussed below, the area either increases or remains constant. So, the event horizon of a stationary space-time is an example of a trapping horizon.

² This may seem surprising at first. As Carlo Rovelli asked after my talk, in the problem of black hole merger, initially one has two distinct horizons and finally there is only one. So, the topology changes. How can one reconcile this with the universality of topology of H? Note first that to address this issue we should examine apparent —not event— horizons. Secondly, and more importantly, by definition H can be realized only by *continuous* segments of the world tubes of apparent horizons. Before the merger, each continuous segment does have topology $S^2 \times R$ and after the merger the final world tube has the same topology. In between the apparent horizon simply jumps and there is no dynamical horizon.

vector orthogonal to S and tangent to H is denoted by \widehat{r}^a . Quantities intrinsic to S will be generally written with a tilde. Thus, the two-metric on S is \widetilde{q}_{ab} , the extrinsic curvature of $S \subset H$ is $\widetilde{K}_{ab} := \widetilde{q}_a{}^c \widetilde{q}_b{}^d D_c \widehat{r}_d$, the derivative operator on (S, \widetilde{q}_{ab}) is \widetilde{D} and its Ricci tensor is $\widetilde{\mathcal{R}}_{ab}$. Finally, we will fix the rescaling freedom in the choice of null normals via $\ell^a := \widehat{\tau}^a + \widehat{r}^a$ and $n^a := \widehat{\tau}^a - \widehat{r}^a$.

We first note an immediate consequence of the definition. Since $\theta_{(\ell)} = 0$ and $\theta_{(n)} < 0$, it follows that

$$\widetilde{K} = \widetilde{q}^{ab} D_a \widehat{r}_b = \frac{1}{2} \widetilde{q}^{ab} \nabla_a (\ell_b - n_b) > 0.$$

Hence the area a_S of S increases monotonically along \hat{r}^a . Thus the second law of black hole mechanics holds on H. Our first task is to obtain an explicit expression for the change of area.

Our main analysis is based on the fact that, since H is a space-like surface, the Cauchy data (q_{ab}, K_{ab}) on H must satisfy the usual scalar and vector constraints

$$H_S := \mathcal{R} + K^2 - K^{ab}K_{ab} = 16\pi G T_{ab} \widehat{\tau}^a \widehat{\tau}^b$$
(2.1)

$$H_V^a := D_b \left(K^{ab} - K q^{ab} \right) = 8\pi G T^{bc} \hat{\tau}_c q_b^a.$$
 (2.2)

We will often fix two cross-sections S_1 and S_2 and focus our attention on a portion $\Delta H \subset H$ which is bounded by them.

III. ENERGY FLUXES AND AREA BALANCE

Let us now turn to the task of relating the change in area to the flux of energy across H. As is usual in general relativity, the notion of energy is tied to a choice of a vector field. The definition of a dynamical horizon provides a preferred direction field; that along ℓ^a . To extract a vector field, we need to fix the proportionality factor, or the 'lapse' N, let us first introduce the area radius R, a function which is constant on each S and satisfies $a_S = 4\pi R^2$. Since we already know that area is monotonically increasing, R is a good coordinate on H. Now, the 3-volume d^3V on H can be decomposed as $d^3V = |\partial R|^{-1}d^2VdR$. Therefore, as we will see, our calculations will simplify if we choose $N_R = |\partial R|$. Let us begin with this simple choice, obtain an expression for the change in area and then generalize the result to include a more general family of lapses.

Fix two cross sections S_1 and S_2 of H and denote by ΔH the portion of H they bound. We are interested in calculating the flux of energy associated with $\xi^a_{(R)} = N_R \ell^a$ across ΔH . Denote the flux of matter energy across ΔH by $\mathcal{F}_m^{(R)}$:

$$\mathcal{F}_m^{(R)} := \int_{\Delta H} T_{ab} \widehat{\tau}^a \xi_{(R)}^b d^3 V. \tag{3.1}$$

By taking the appropriate combination of (2.1) and (2.2) we obtain

$$\mathcal{F}_{m}^{(R)} = \frac{1}{16\pi G} \int_{\Lambda H} N_R \{ H_S + 2\hat{r}_a H_V^a \} d^3 V.$$
 (3.2)

Since H is foliated by two-spheres, we can perform a 2+1 split of the various quantities on H. Using the Gauss Codazzi relation we rewrite \mathcal{R} in terms of quantities on S:

$$\mathcal{R} = \widetilde{\mathcal{R}} + \widetilde{K}^2 - \widetilde{K}_{ab}\widetilde{K}^{ab} + 2D_a\alpha^a \tag{3.3}$$

where $\alpha^a = \hat{r}^b D_b \hat{r}^a - \hat{r}^a D_b \hat{r}^b$. Next, the fact that the expansion $\theta_{(\ell)}$ of ℓ^a vanishes leads to the relation

$$K + \widetilde{K} = K_{ab}\widehat{r}^a\widehat{r}^b. \tag{3.4}$$

Using (3.3) and (3.4) in (3.2) and simplifying, we obtain the result

$$\int_{\Delta H} N_R \widetilde{\mathcal{R}} \, d^3 V = 16\pi G \int_{\Delta H} T_{ab} \widehat{\tau}^a \xi_{(R)}^b \, d^3 V + \int_{\Delta H} N_R \left\{ |\sigma|^2 + 2|\zeta|^2 \right\} \, d^3 V \tag{3.5}$$

where $|\sigma|^2 = \sigma_{ab}\sigma^{ab}$ with σ_{ab} being the shear of ℓ^a , and $|\zeta|^2 = \zeta^a\zeta_a$ with $\zeta^a := \tilde{q}^{ab}\hat{r}^c\nabla_c\ell_b$; both σ_{ab} and ζ^a are tensors intrinsic to S. To simplify the left side of this equation, recall that the volume element d^3V on H can be written as $d^3V = N_R^{-1}dR\,d^2V$ where d^2V is the area element on S. Using the Gauss-Bonnet theorem, the integral of $N_R\widetilde{\mathcal{R}}$ can then be written as

$$\int_{\Delta H} N_R \widetilde{\mathcal{R}} \, d^3 V = \int_{R_1}^{R_2} dR \left(\oint_S \widetilde{\mathcal{R}} \, d^2 V \right) = 8\pi (R_2 - R_1) \,. \tag{3.6}$$

(It is this manipulation that dictated our choice of N_R .) Substituting this result in (3.5) we finally obtain

$$\left(\frac{R_2}{2G} - \frac{R_1}{2G}\right) = \int_{\Delta H} T_{ab} \hat{\tau}^a \xi_{(R)}^b d^3 V + \frac{1}{16\pi G} \int_{\Delta H} N_R \left\{ |\sigma|^2 + 2|\zeta|^2 \right\} d^3 V.$$
(3.7)

This is the first key result we were looking for. Let us now interpret the various terms appearing in this equation. The left side gives us the change in the horizon 'radius' caused by the dynamical process under consideration. The first integral on the right side of this equation is the flux $\mathcal{F}_m^{(R)}$ of matter energy associated with the vector field $\xi_{(R)}^a$. Since $\xi_{(R)}^a$ is null and $\hat{\tau}$ time-like, if T_{ab} satisfies, say, the dominant energy condition, this quantity is guaranteed to be non-negative. Since the second term is purely geometrical and emerged as the 'companion' of the matter term, it is tempting to interpret it as the flux $\mathcal{F}_g^{(R)}$ of $\xi_{(R)}^a$ -energy in the gravitational radiation:³

$$\mathcal{F}_g^{(R)} := \frac{1}{16\pi G} \int_{\Delta H} N_R \left\{ |\sigma|^2 + 2|\zeta|^2 \right\} d^3 V. \tag{3.8}$$

Is this proposal physically viable? We will now argue that the answer is in the affirmative in the sense that it passes all the 'text-book' tests one uses to demonstrate the viability of the Bondi flux formula at null infinity.

First, since we did not have to introduce any structure, such as coordinates or tetrads, which is auxiliary to the problem, the expression is obviously 'gauge invariant'. Second, the energy flux is manifestly non-negative. Third, all fields used in it are local; we did not have to perform, e.g., a radial integration to define any of them. Fourth, the expression vanishes in the spherically symmetric case: Since the only spherically symmetric vector field and trace-free, second rank tensor field on a 2-sphere are the zero fields, if the Cauchy data

³ While the presence of the shear term $|\sigma|^2$ in the flux formula (3.8) is natural from one's expectations based on the weak field limit, the term $|\zeta|^2$ is surprising. Booth and Fairhurst have shown that this term is of two orders higher than the shear term in the weak field expansion. Thus, it captures some genuinely non-linear, strong field physics which is yet to be understood fully.

 (q_{ab}, K_{ab}) and the foliation on H are spherically symmetric, $\sigma_{ab} = 0$ and $\zeta^a = 0$. Next, one might be concerned that the flux may not vanish in stationary space-times. Even in the Schwarzschild space-time, could one not construct a clever, non-spherical dynamical horizon H? If one could, the area law (3.8) would hold and then we would be led to an absurd conclusion that there is flux of gravitational energy across this H! Even if this is not possible in the Schwarzschild space-time, could it not happen in a more general stationary space-time? If it can, the 'gravitational energy-flux' interpretation would not be viable. Now, since H is foliated by marginally trapped surfaces, it follows from general results that it must lie inside the event horizon. Using the fact that there is a Killing field in that region, one can show that there are no dynamical horizons in this interior region, whence the concern is unfounded. Thus, the expression on the right side of (3.8) shares with the Bondi-Sachs energy flux at null infinity all its key properties. We will therefore interpret it as the $\xi_{(R)}$ -energy flux of carried by gravitational waves. Recently, Booth and Fairhurst [4] have verified that on 'weakly-dynamical' horizons, the expression reduces to the familiar one from perturbation theory. They have also shown that this formula can be derived from a Hamiltonian framework where H is treated as the inner boundary of the space-time region of interest. These results provide considerable further support for our interpretation. Nonetheless, it is important to continue to think of new criteria and make sure that (3.8) passes these tests.

Remark: The emergence of a precise formula for the flux of energy across ΔH is very surprising. What would happen if we repeat the above procedure for a general space-like surface \tilde{H} ? The analog of the flux term would be much more complicated and fail to be positive definite. This happens even if we assume that \tilde{H} is foliated by strictly —rather than marginally— trapped surfaces \tilde{S} , i.e. if we replaced the condition $\sigma_{(\ell)} = 0$ by $\sigma_{(\ell)} < 0$. Thus, there is no satisfactory candidate for the flux formula across \tilde{H} . To summarize, although the calculation is straightforward, it crucially depends on subtle cancellations which occur precisely because H is a dynamical horizon.

To conclude this section, let us discuss the possibility of choosing more general lapse functions. In the above calculation, we needed a specific form of the lapse to cast $N_R d^3 V$ into the form $d^2 V dR$. This suggests that we use more general functions r which are constant on each leaf S of the foliation and set $N_r = |\partial r|$. If we use a different radial coordinate r', then the lapse is rescaled according to the relation

$$N_{r'} = \frac{dr'}{dr} N_r \,. \tag{3.9}$$

Thus, although the lapse itself will in general be a function of all three coordinates on H, the relative factor between any two permissible lapses can be a function only of r. Recall that, on an isolated horizon, physical fields are time independent and null normals —which play the role of $N_r\ell^a$ there— can be rescaled by a positive constant [5, 7]. In the present case, the horizon fields are 'dynamical', i.e., r-dependent, and the rescaling freedom is by a positive function of r. Thus, the freedom in the choice of lapse is just what one would expect.

Given a lapse N_r , following the terminology used in the isolated horizon framework, the resulting vector fields $\xi_{(r)}^a := N_r \ell^a$ will be called *permissible*. By repeating the above calculation, it is easy to arrive at a generalization of (3.8) for any permissible vector field:

$$\left(\frac{r_2}{2G} - \frac{r_1}{2G}\right) = \int_{\Delta H} T_{ab} \hat{\tau}^a \xi_{(r)}^b d^3 V + \frac{1}{16\pi G} \int_{\Delta H} N_r \left\{ |\sigma|^2 + 2|\zeta|^2 \right\} d^3 V, \qquad (3.10)$$

where the constants r_1 and r_2 are values the function r assumes on the fixed cross-sections S_1 and S_2 . This generalization of (3.7) will be useful in section V.

IV. ANGULAR MOMENTUM

To obtain the integral version of the full first law, we need the notion of angular momentum and angular momentum flux. It turns out that the angular momentum analysis is rather straightforward and is, in fact, applicable to an arbitrary space-like hypersurface. Fix any vector field φ^a on H which is tangential to the cross-sections of H. Contract φ^a with both sides of (2.2). Integrate the resulting equation over the region $\Delta H \subset H$, perform an integration by parts and use the identity $\mathcal{L}_{\varphi}q_{ab} = 2D_{(a}\varphi_{b)}$ to obtain

$$\frac{1}{8\pi G} \oint_{S_2} K_{ab} \varphi^a \widehat{r}^b d^2 V - \frac{1}{8\pi G} \oint_{S_1} K_{ab} \varphi^a \widehat{r}^b d^2 V$$

$$= \int_{\Delta H} \left(T_{ab} \widehat{\tau}^a \varphi^b + \frac{1}{16\pi G} P^{ab} \mathcal{L}_{\varphi} q_{ab} \right) d^3 V \tag{4.1}$$

where $P^{ab} := K^{ab} - Kq^{ab}$. It is natural to identify the surface integrals with the generalized angular momentum $J^{(\varphi)}$ associated with those surfaces and set:

$$J_S^{(\varphi)} = -\frac{1}{8\pi G} \oint_S K_{ab} \varphi^a \hat{r}^b d^2 V \tag{4.2}$$

where we have chosen the overall sign to ensure compatibility with conventions normally used in the asymptotically flat context. The term 'generalized' emphasizes the fact that the vector field φ^a need not be an axial Killing field even on S; it only has to be tangential to our cross-sections.

The fluxes of this angular momentum due to matter fields and gravitational waves are, respectively,

$$\mathcal{J}_{\mathbf{m}}^{(\varphi)} = -\int_{\Lambda H} T_{ab} \widehat{\tau}^a \varphi^b \, d^3 V \,, \tag{4.3}$$

$$\mathcal{J}_{g}^{(\varphi)} = -\frac{1}{16\pi G} \int_{\Delta H} P^{ab} \mathcal{L}_{\varphi} q_{ab} d^{3} V, \qquad (4.4)$$

and we get the balance equation

$$J_2^{(\varphi)} - J_1^{(\varphi)} = \mathcal{J}_m^{(\varphi)} + \mathcal{J}_{\sigma}^{(\varphi)}.$$
 (4.5)

As expected, if φ^a is a Killing vector of the three-metric q_{ab} , then the gravitational flux vanishes: $\mathcal{J}_{g}^{(\varphi)} = 0$. For the discussion of the integral version of the first law, it is convenient to introduce the angular momentum current

$$j^{\varphi} := -K_{ab}\varphi^a \hat{r}^b \tag{4.6}$$

so that the angular momentum formula becomes

$$J_S^{(\varphi)} = \frac{1}{8\pi G} \oint_S j^{\varphi} d^2 V.$$
 (4.7)

V. FINITE VERSION OF THE FIRST LAW

Let us now combine the results of sections III and IV to obtain the physical process version of the first law for H and a mass formula for an arbitrary cross-section of H.

To begin with, let us ignore angular momentum and consider the vector field $\xi_{(R)}$ of section III. Denote by $E^{\xi_{(R)}}$ the $\xi_{(R)}$ -energy of cross-sections S of H. While we do not yet have the explicit expression for it, it is natural to assume that, because of the influx of matter and gravitational energy, $E^{\xi_{(R)}}$ will change by an amount $\Delta E^{\xi_{(R)}} = \mathcal{F}_m^{(R)} + \mathcal{F}_g^{(R)}$ as we move from one cross-section to another. Then, the infinitesimal form of (3.7), $dR/2G = dE^{\xi_{(R)}}$, suggests that we define effective surface gravity k_R associated with $\xi_{(R)}^a$ as $k_R := 1/2R$ so that the infinitesimal expression is recast into the familiar form $(k_R/8\pi G)da = dE^{\xi_{(R)}}$ where a is the area of a generic cross-section. For a general choice of the radial coordinate r, (3.10) yields a generalized first law:

$$\frac{k_r}{8\pi G} da = dE^{\xi_{(r)}} \tag{5.1}$$

provided we define the effective surface gravity k_r of $\xi^a_{(R)}$ by

$$k_r = \frac{dr}{dR} k_R$$
 where $\xi_{(r)}^a = \frac{dr}{dR} \xi_{(R)}^a$. (5.2)

Note that this rescaling freedom in surface gravity is completely analogous to the rescaling freedom which exists for Killing horizons, or, more generally, isolated horizons [5, 7]. The new feature in the present case is that we have the freedom to rescale the ℓ^a and the surface gravity by a function of the radius R rather than by a constant. This is just what one would expect in a dynamical situation since R plays the role of 'time' along H. Finally, note that the differentials appearing in (5.1) are actual, physical variations along the dynamical horizon due to an infinitesimal change in r and are not variations in phase space as in the formulations [5, 7, 10] of the first law based on Killing or isolated horizons. Thus, (5.1) is a physical version of the first law, whence (3.10) is the finite version of the first law in the absence of rotation.

Next, let us include rotation. Pick a vector field φ^a on H such that φ^a is tangent to the cross-sections of H, has closed orbits and has affine length 2π . (At this point, φ^a need not be a Killing vector of q_{ab} .) Consider time evolution vector fields t^a which are of the form $t^a = N_r \ell^a - \Omega \varphi^a$ where N_r is a permissible lapse associated with a radial function r and Ω an arbitrary function of r.⁴ (On an isolated horizon, the analogs of these two fields are constant.) Evaluate the quantity $\int_{\Delta H} T_{ab} \hat{\tau}^a t^b d^3 V$ using (4.1) and (3.7):

$$\frac{r_2 - r_1}{2G} + \frac{1}{8\pi G} \left\{ \oint_{S_2} \Omega j^{\varphi} d^2 V - \oint_{S_1} \Omega j^{\varphi} d^2 V - \int_{\Omega_1}^{\Omega_2} d\Omega \oint_{S} j^{\varphi} d^2 V \right\} = \int_{\Delta H} T_{ab} \widehat{\tau}^a t^b d^3 V + \frac{1}{16\pi G} \int_{\Delta H} N_r \left(|\sigma|^2 + 2|\zeta|^2 \right) d^3 V - \frac{1}{16\pi G} \int_{\Delta H} \Omega P^{ab} \mathcal{L}_{\varphi} q_{ab} d^3 V (5.3)$$

⁴ Constancy of Ω on cross-sections implies rigid rotation, although the frequency of rotation is allowed to change in 'time'. After my talk, Alberto Garcia raised the interesting issue of allowing differential rotations. This can be done by letting Ω be a general function on H. In this case, one can still obtain the integral version of the first law but, as one would expect, if Ω has angular dependence, one can not recover the familiar infinitesimal form of the first law.

This is our finite version of the familiar first laws of the isolated horizon framework [5, 7]. For, if we now restrict ourselves to infinitesimal ΔH , the three terms in the curly brackets combine to give $d(\Omega J) - Jd\Omega$ and we obtain

$$\frac{dr}{2G} + \Omega dJ = \frac{k_r}{8\pi G} da + \Omega dJ = dE^t. \tag{5.4}$$

This equation is just the familiar first law but now in the setting of dynamical horizons. Since the differentials in this equation are variations along H, this can be viewed as a physical process version of the first law. Note that for each allowed choice of lapse N_r , angular velocity $\Omega(r)$ and vector field φ^a on H, we obtain a 'permissible' time evolution vector field $t^a = N_r \ell^a - \Omega \varphi^a$ and a corresponding first law. This situation is very similar to what happens in the isolated horizon framework [5, 7] where we obtain a first law for each permissible time translation on the horizon. Again, the generalization from that time independent situation consists of allowing the lapse and the angular velocity to become r-dependent, i.e., 'dynamical'.

For every allowed choice of $(N_r, \Omega(r), \varphi^a)$, we can integrate (5.4) on H to obtain a formula for E^t on any cross section but, in general, the result may not be expressible just in terms of geometric quantities defined locally on that cross-section. However, in some physically interesting cases, the expression is local. For example, in the case of spherical symmetry, it is natural to choose $\Omega = 0$ and R as the radial coordinate in which case we obtain $E^t = R/2G$. This is just the irreducible (or Hawking) mass of the cross-section. Even in this simple case, (5.3) provides a useful balance law, with clear-cut interpretation. Physically, perhaps the most interesting case is the one in which q_{ab} is only axi-symmetric with φ^a as its axial Killing vector. In this case we can naturally apply, at each cross-section S of H, the strategy used in the isolated horizon framework to select a preferred t^a : Calculate the angular momentum J defined by the axial Killing field φ , choose the radial coordinate r (or equivalently, the lapse N_r) such that

$$k_r = k_o(R) := \frac{R^4 - 4G^2J^2}{2R^3\sqrt{R^4 + 4G^2J^2}}$$
(5.5)

and choose Ω such that

$$\Omega = \Omega_o(R) := \frac{2GJ}{R\sqrt{R^4 + 4G^2J^2}}.$$
(5.6)

This functional dependence of k_r on R and J is exactly that of the Kerr family. With this choice of N_r and Ω , the energy E_S^t is given by the well known Smarr formula

$$E^{t_o} = 2\left(\frac{k_o a}{8\pi G} + \Omega_o J\right) = \frac{\sqrt{R^4 + 4G^2 J^2}}{2GR}.$$
 (5.7)

Thus, as a function of its angular momentum and area, each cross-section is assigned simply that mass which it would have in the Kerr family. This may seem like a reasonable but rather trivial strategy. The non-triviality lies in two facts. First, with this choice, there is still a balance equation in which the flux of gravitational energy $\mathcal{F}_g^{(t_o)}$ is local and positive definite (see (5.3)). (The gravitational angular momentum flux which, in general, has indeterminate sign vanishes due to axi-symmetry.) Second, as mentioned in section III, Booth and Fairhurst have recently shown that this expression of the dynamical horizon energy emerges from a systematic Hamiltonian framework on \mathcal{M} where H is treated as an inner boundary.

Motivated by the isolated horizon framework, we will refer to this canonical E^{t_o} as the mass associated with cross-sections S of H and denote it simply by M. Thus, among the infinitely many first laws (5.4), there is a canonical one:

$$dM = \frac{k_o}{8\pi G} da + \Omega_o dJ. (5.8)$$

Note that the mass and angular momentum depend only on geometrical fields on each cross section and, furthermore, the dependence is local. Yet, thanks to the constraint part of Einstein's equations, changes in mass over *finite* regions ΔH of H can be related to the expected matter fluxes and to the flux of gravitational radiation which is local and positive.

I will conclude this section with a conceptual subtlety, emphasized by Stephen Fairhurst in recent conferences. The first law (5.3) discussed here is really a conservation law and as such it is a finite or integral version of the first law of black hole mechanics. In the discussion of the first law in its standard, infinitesimal form, one explicitly or implicitly considers transitions between an equilibrium state and a nearby equilibrium state. Conceptually, this is the same setting as in laws of equilibrium thermodynamics. In particular, in infinitesimal processes involving black holes, the change in surface gravity can be ignored just as the change in the temperature is ignored in the first law dE = TdS + W of thermodynamics. During fully non-equilibrium thermodynamical processes, by contrast, the system does not have time to come to equilibrium and there is no canonical notion of its temperature. Similarly, in the case of dynamical horizons, we only have a notion of 'effective' or 'average' surface gravity; in striking contrast to what happens on isolated horizons which describe equilibrium configurations, κ_r does not have the geometrical interpretation of surface gravity. However, if one considers 'weakly-dynamical' horizons and regards them as perturbations of isolated horizons, there is a geometrical notion of surface gravity (of which κ_r is a 2-sphere average). In this situation, the geometrical surface gravity appears to be a good analog of the temperature, the idea being that system is evolving slowly so that it can reach approximate equilibrium in spite of time dependence. In these situations, the dynamical first law (5.3) can be simplified by keeping terms only up to second order in perturbation [4] and that expression can be regarded as the integral version of the first law of black hole thermodynamics.

VI. DISCUSSION

I will conclude the report by pointing out some important open problems whose solutions will add very significantly to our knowledge of dynamical horizons and suggest some applications of this framework.

i) An important open problem is to obtain a complete characterization of the 'initial data' on dynamical horizons. Can one characterize the solutions to the constraint equations such that (H, q_{ab}, K_{ab}) is a dynamical horizon? Note that this would, in particular, provide a complete control on the geometry of the world tube of apparent horizons that will emerge in all possible numerical simulations! One can further ask: Can one isolate the freely specifiable data in a useful way? Are these naturally related to the freely specifiable data on isolated horizons [8]? In the spherically symmetric case, these issues are straightforward to address and an essentially complete solution is known. It would be very interesting to answer these questions in the axi-symmetric case.

- ii) In the analysis of section V, let us drop the restriction to t_o^a and consider general permissible vector fields t^a . Unlike the vector fields $\xi_{(r)}^a = N_r \ell^a$, the vector field t^a is not necessarily causal. Therefore the matter flux $\int_{\Delta H} T_{ab} t^a \hat{\tau}^b d^3V$ need not be positive. Similarly, if φ^a is not a Killing field of q_{ab} , the gravitational flux need not be positive. Therefore, although the area a always increases with R, E^t can decrease as R increases. This is the analog of the Penrose process in which 'rotational energy' is extracted from the dynamical horizon. Do Einstein's equations with physically reasonable matter allow one to extract all the rotational energy? Once the first question in i) above is answered, one would have an essentially complete control on such issues in fully dynamical processes.
- iii) In a gravitational collapse or a black hole merger, one expects the dynamical horizon in the distant future to asymptotically approach an isolated horizon. Is this expectation correct? If so, what can one say about the rate of approach? There exists a useful characterization of the Kerr isolated horizon [14]. Under what conditions is one guaranteed that the asymptotic isolated horizon is Kerr? On an isolated horizon one can define multipoles invariantly [15] and the definition can be carried over to each cross-section of the dynamical horizon. Can one physically justify this generalization? If so, what can one say about the rate of change of these multipoles? Can one, for example, gain insight into the maximum amount of energy that can be emitted in gravitational radiation, from the knowledge of the horizon quadrupole and its relation to the Kerr quadrupole? Is the quasi-normal ringing of the final black hole coded in the rate of change of the multipoles, as was suggested by heuristic considerations using early numerical simulations?
- iv) As the vast mathematical literature on black holes shows, the infinitesimal version (5.4) of the first law is conceptually very interesting. The finite balance equation (5.3) is likely to be even more directly useful in the analysis of astrophysical situations. In particular, there exist an infinite number of balance equations. Can they provide useful checks on numerical simulations in the strong field regime? Similarly, the Hamiltonian framework of Booth and Fairhurst could be used as a point of departure for quantum mechanical treatments beyond equilibrium situations. Can one extend the non-perturbative quantization of [15, 16] to incorporate these dynamical situations? To naturally incorporate back reaction in the Hawking process?
- v) There has been considerable interest in the geometric analysis community in the inequalities conjectured by Penrose, which say that the total (ADM) mass of space-time must be greater than the area of the apparent horizon on any Cauchy slice. In the time symmetric case (i.e., when the extrinsic curvature on the Cauchy slice vanishes) this conjecture was proved last year by Huisken and Ilmamen [18] and Bray [17]. The area law of dynamical horizons provides a nice setting to extend this analysis not only beyond the time symmetric case but to establish a stronger version relating the area of apparent horizons to the future limit of the Bondi energy at null infinity.

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