Action and Hamiltonians in higher dimensional general relativity: First order framework

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We consider $d > 4$-dimensional space-times which are asymptotically flat at spatial infinity and show that, in the first order framework, the action principle is well-defined *without the need of infinite counter terms.* It naturally leads to a covariant phase space in which the Hamiltonians generating asymptotic symmetries provide the total energy-momentum and angular momentum of the isolated system. This work runs parallel to our previous analysis in four dimensions [1]. The higher dimensional analysis is in fact simpler because of absence of logarithmic and super translation ambiguities.

I. INTRODUCTION

The motivation and underlying ideas of this paper are the same as that of our earlier analysis in 4 dimensions [1]. However, for completeness, we will summarize the main points.

In most field theories the action depends only on fundamental fields and their first derivatives. By contrast, the Einstein-Hilbert action of general relativity depends also on the second derivatives of the fundamental field, the space-time metric g. As a consequence, stationary points of this action do not yield Einstein's equations unless both the metric and its first derivatives are kept fixed at the boundary; strictly we do not have a well-defined variational principle. To remedy this situation, Gibbons and Hawking [2, 3] proposed that we add a surface term to the Einstein Hilbert action. We are then led to

$$S_{\text{EH+GH}}(g) = \frac{1}{2\kappa} \left(\int_{\mathcal{M}} R \, d^d V + 2 \int_{\partial \mathcal{M}} K \, d^{d-1} V + C \right). \qquad (1.1)$$

Here $\kappa = 8\pi G$, $\mathcal{M}$ is a d-manifold representing an appropriate portion of space-time, $\partial \mathcal{M}$ its boundary, R the Ricci scalar of the metric g, K the trace of the extrinsic curvature of $\partial \mathcal{M}$, and C is an arbitrary function of the metric h induced on $\partial \mathcal{M}$ by g.

Let us restrict ourselves to cases where g has signature $(-,+,...,+)$, is smooth and globally hyperbolic. We will let $\mathcal{M}$ be the space-time region bounded between two Cauchy surfaces. If $\mathcal{M}$ is spatially compact, by setting $C = 0$, we obtain a well-defined variational principle. However, in the asymptotically flat case, it is well-known that this strategy has some important limitations (see e.g., [4]). In particular, the action is typically infinite even 'on-shell', and indeed even when g is the Minkowski metric. To remedy this problem, Gibbons and Hawking [2, 3] proposed an infinite subtraction: Carry out an isometric embedding

of $(\partial\mathcal{M}, h)$ in Minkowski space, calculate the trace K_o of the extrinsic curvature of $\partial\mathcal{M}$ defined by the Minkowski metric, and set $C = -K_o$. However, the required embedding does not always exist. For, in a d dimensional space-time the metric h on the boundary $\partial\mathcal{M}$ has $[d(d-1)/2] - [d-1]$ degrees of freedom (after removing the diffeomorphism gauge) while the choice of embedding provides a freedom worth only one function on $\partial\mathcal{M}$. Thus, even at this heuristic level, if $d \geq 4$ the freedom is not sufficient whence this infinite subtraction procedure will not work for generic metrics g.

Over the last few years, a new set of proposals for infinite counter terms C have appeared in the literature. In particular, Kraus, Larsen, Siebelink [5] have constructed a counter-term using a (non-polynomial) function of the Ricci curvature of the boundary. Mann and Marolf [4] have introduced a counter-term which is closer to the spirit of the Gibbons-Hawking proposal. They replace K_o with the trace of a tensor field $\hat{K}_{ab}$ which generalizes the extrinsic curvature K^o_{ab} of $\partial\mathcal{M}$ with respect to the Minkowski metric, used by Gibbons and Hawking, to situations in which the boundary can not be isometrically embedded in Minkowski space. Not only do these improved actions S_{imp} lead to well-defined action principles, but they also overcome another limitation of the original proposal: Now $\delta S_{\text{imp}} = 0$ at asymptotically flat solutions for *all* permissible variations δ.

Since we are dealing just with *classical* field theories with *smooth* fields, one might wonder if there is a way to avoid infinite subtractions altogether and construct an action principle which is manifestly finite from the beginning. The first goal of this paper is to show in some detail that this is indeed possible if one uses a first order framework based on orthonormal frames and Lorentz connections. The second goal of the paper is to use this action to construct a covariant Hamiltonian framework by keeping careful track of boundary conditions.

If $d \geq 5$, the super translations and logarithmic translations encountered in the $d = 4$ [1] case are absent: the asymptotic symmetry group is the Poincaré group even with the 'obvious' choice of boundary conditions. Heuristically, the difference can be understood as follows. Since the solutions to the Poisson equation in d-1 spatial dimensions fall off at spatial infinity as $1/r^{d-3}$, to obtain non-zero mass at spatial infinity, we have to allow metrics which approach a flat metric only as $1/r^{d-3}$. If $d=4$, the resulting $1/r$ fall-off makes the asymptotic symmetry group infinite dimensional because of possible super translations [6–9] and logarithmic translations [10]. The boundary conditions needed to remove these potential asymptotic symmetries are subtle (see [1] for a concise summary). In $d \geq 5$, the $1/r^{d-3}$ deviation from flat space do not allow these extra symmetries and the group of asymptotic symmetries is naturally the Poincaré group. The only subtlety is that for the Hamiltonians generating boosts to be well-defined, one has to impose a 'reflection symmetry' condition on the leading, $1/r^{d-3}$ part of permissible metrics (more precisely, of ortho-normal frames). Thus, overall, the higher dimensional analysis is considerably simpler than that in four space-time dimensions.

The paper is organized as follows. In section II we introduce the Lagrangian and Hamiltonian framework using the first order framework. In section III we calculate the Hamiltonians generating these Poincaré symmetries. Our approach has several similarities to that used by Mann, Marolf, McNees and Virmani [4, 11, 12]. However, there are also two significant differences. First, we use a first order —rather than a second order— framework, thereby avoiding infinite counter-terms altogether. Second, our boundary conditions are weaker in the sense that we do not require that the space-time metric have an asymptotic 'Beig-Schmidt expansion'. On the other hand we require that the coefficient of the leading, $1/r^{d-3}$ part of the frame field be reflection symmetric. We are not aware of a complete Hamiltonian

treatment in absence of this (or analogous) condition.

II.   ACTION AND THE COVARIANT PHASE SPACE

Our basic gravitational variables will be co-frames e_a^I and Lorentz connections A_a^{IJ} on space-time $\mathcal{M}$. Co-frames e are 'square-roots' of metrics and the transition from metrics to frame fields is motivated by the fact that these frames are essential if one is to introduce spinorial matter. e_a^I is an isomorphism between the tangent space $T_p(\mathcal{M})$ at any point p and a fixed internal vector space V equipped with a metric η_{IJ} with Lorentzian signature $(- + .. +)$. The internal indices can be freely lowered and raised using this fiducial η_{IJ} and its inverse η^{IJ}. Each co-frame defines a space-time metric by $g_{ab} := e_a^I e_b^J \eta_{IJ}$ which also has signature $(- + .. +)$. Then the co-frame e is automatically orthonormal with respect to g. Since the connection 1-forms A take values in the Lorentz Lie algebra, $A_a^{IJ} = -A_a^{JI}$. The connection acts only on internal indices and defines a derivative operator

$$D_a k_I := \partial_a k_I + A_{aI}{}^J k_J \,,$$

where ∂ is a fiducial derivative operator which, as usual, will be chosen to be torsion-free and compatible with η_{IJ}. As fundamental fields, e and A are independent. However, the equation of motion of A implies that A is compatible with e, i.e., is fully determined by e. Therefore, boundary conditions on A will be motivated by those on e.

In the Lagrangian and Hamiltonian frameworks we have to first introduce the precise space of dynamical fields of interest. Let us fix, once and for all, a co-frame ${}^o\!e_a^I$ such that $g_{ab}^o = \eta_{IJ} \, {}^o\!e_a^I \, {}^o\!e_b^J$ is flat and $\partial_{[a} \, {}^o\!e_{b]}^I = 0$. The cartesian coordinates x^a of g_{ab}^o and the associated radial-hyperboloid coordinates (ρ, Φ^i) will be used in asymptotic expansions near spatial infinity.

Detailed analysis shows that to define the Lorentz angular momentum e_a^I has to admit an expansion to order $d-2$ (see Sec. III B). Therefore, we will assume that e_a^I can be expanded as follows: Setting $n = d - 3$,

$$e = {}^o\!e(\Phi) + \frac{{}^n\!e(\Phi)}{\rho^n} + \frac{{}^{n+1}\!e(\Phi)}{\rho^{n+1}} + o(\frac{1}{\rho^{n+1}}) \tag{2.1}$$

where the leading non-trivial term, ${}^n\!e(\Phi)$, is assumed to be reflection symmetric and where the remainder $o(\rho^{-m})$ has the property that $\lim_{\rho \to \infty} \rho^m \, o(\rho^{-m}) = 0$.

Since the equation of motion for A implies that A is compatible with e, without any loss of generality we can require that, in the expansion near spatial infinity, A_a^{IJ} is completely determined by e_a^I to appropriate leading orders. This leads us to require that A_a^{IJ} have the following asymptotic behavior:

$$A = {}^o\!A(\Phi) + \frac{{}^1\!A(\Phi)}{\rho} + ... + \frac{{}^{n+2}\!A(\Phi)}{\rho^{n+2}} + o(\rho^{n+2}) \tag{2.2}$$

where ${}^0\!A = ... = {}^n\!A = 0$ and ${}^{n+1}\!A$ is given by

$${}^{n+1}\!A_a^{IJ}(\Phi) = 2\rho^{n+1} \, \partial^{[J} \left((\rho^{-n}) \, {}^n\!e_a^{I]} \right) \tag{2.3}$$

(In spite of the explicit factors of ρ the right side is in fact independent of ρ because $\partial_a \, {}^n\!e \sim \rho^{-1} \times$ (angular derivatives of ${}^n\!e$).) We will not need the corresponding expression of ${}^{n+2}\!A$ in

terms of e and therefore demand compatibility between A and e only via (2.3). Appendix A 1 shows that the boundary conditions are readily satisfied by the d-dimensional Schwarzschild solution.

A.   Action Principle

Consider as before the d-manifold $\mathcal{M}$ bounded by space-like surfaces M_1 and M_2. In this section we will restrict ourselves to the case $d > 4$; for the case $d = 4$ see [1]. We will consider smooth histories (e, A) on M such that (e, A) are asymptotically flat in the sense specified above, and are such that M_1, M_2 are Cauchy surfaces with respect to the space-time metrics g defined by e, and the pull-back of A to M_1, M_2 is determined by the pull-back of e. The last condition is motivated by the fact that, since the compatibility between e and A is an equation of motion, boundary values where this compatibility is violated are not of interest to the variational principle. Finally it is convenient to partially fix the internal gauge on the boundaries. We will fix a constant, time-like internal vector n^I so that $\partial_a n^I = 0$ and require that the histories be such that $n^a := n^I e_I^a$ is the unit normal to M_1 and M_2.

The first order gravitational action on these histories is given by (see e.g. [13–15])

$$S(e, A) = -\frac{1}{2\kappa} \int_{\mathcal{M}} \Sigma^{IJ} \wedge F_{IJ} + \frac{1}{2\kappa} \int_{\partial \mathcal{M}} \Sigma^{IJ} \wedge A_{IJ}, \qquad (2.4)$$

where the d-2-forms Σ^{IJ} are constructed from the co-frames and F is the curvature A:

$$\Sigma_{IJ} := \tfrac{1}{(d-2)!} \, \epsilon_{IJK...L} e^K \wedge ... \wedge e^L \qquad \text{and} \qquad F_I{}^J = \mathrm{d}A_I{}^J + A_I{}^K \wedge A_K{}^J \qquad .$$

As in more familiar field theories, the action now depends only on the fundamental fields and their first derivatives. Although the connection A itself appears in the surface term at infinity, action is in fact gauge invariant. Indeed, it is not difficult to show that the compatibility between the pull-backs to M_1 and M_2 of e and A and the property $\partial_a n^I = 0$ implies that, on boundaries M_1 and M_2, $\Sigma^{IJ} \wedge A_{IJ} = 2K \, {}^{d-1}\epsilon$ where K is the trace of the extrinsic curvature of M_1 or M_2 and ${}^{d-1}\epsilon$ is the volume element thereon (see e.g., section 2.3.1 of [16]). Thus, on M_1 and M_2, the surface term in (2.4) is *precisely the Gibbons-Hawking surface term* with $C = 0$ in (1.1). Therefore, these surface contributions are clearly gauge invariant. This leaves us with just the surface term at the time-like cylinder τ_∞ at infinity. However, since e has to tend to the fixed co-frame ${}^o e$ at infinity, permissible gauge transformations must tend to identity on τ_∞. Since the surface integral on τ_∞ involves only the pull-back of A to τ_∞, it follows immediately that this surface integral is also gauge invariant. Note also that on τ_∞ this term is *not* equal to the Gibbons-Hawking surface term (because $\partial_a \partial_b \rho$ falls off only as $1/\rho$). Therefore, even if we were to assume compatibility between e and A everywhere and pass to a second order action, (2.4) would not reduce to the Gibbons-Hawking action with $C = 0$. It is also inequivalent to the Gibbons-Hawking prescription of setting $C = K_o$ because, as we now show, (2.4) is well-defined although it does not make any reference to an embedding in flat space.

Our boundary conditions allow us to rewrite this action as

$$S(e, A) = \frac{1}{2\kappa} \int_M \mathrm{d}\Sigma \wedge A - \Sigma \wedge A \wedge A \qquad (2.5)$$

Boundary conditions also imply that the integrand falls off as ρ^{4-2d}. Since the volume element on any Cauchy slice goes as $\rho^{d-2} \, \mathrm{d}\rho \, \mathrm{d}^{d-2}\Phi$, the action is manifestly finite *even off shell* if the two Cauchy surfaces M_1, M_2 are asymptotically time-translated or even boosted with respect to each other. Such space-times $\mathcal{M}$ are referred to as *cylindrical slabs* and *boosted slabs*, respectively.[1]

It is easy to check that the functional derivatives of the action are well defined with respect to both e and A on our class of histories. Variation with respect to the connection yields $D\Sigma = 0$. This condition implies that the connection D defined by A acts on internal indices in the same way as the unique torsion-free connection ∇ compatible with the co-frame e (i.e., defined by $\nabla_a \, e_b^I = 0$). When this equation of motion is satisfied, the curvature F is related to the Riemann curvature R of ∇ by

$$F_{ab}{}^{IJ} = R_{ab}{}^{cd} e_c^I e_d^J \, .$$

Varying the action with respect to e_a^I and taking into account the above relation between curvatures, one obtains Einstein's equations $G_{ab} = 0$. Inclusion of matter is straightforward because the standard matter actions contain only first derivatives of fundamental fields without any surface terms and the standard fall-off conditions on matter fields imply that the matter action is finite on cylindrical or boosted slabs even off shell.

B. Covariant Phase Space

We will now let $\mathcal{M}$ be $\mathbb{R}^d$. The covariant phase space Γ will consist of smooth, asymptotically flat *solutions (e, A) to field equations* on $\mathcal{M}$. (In contrast to section II A, the pull-backs of (e, A) are no longer fixed on any Cauchy surfaces.) Our task is to use the action (2.4) to define the symplectic structure Ω on this Γ.

Following the standard procedure (see, e.g. [17]), let us perform second variations of the action to associate with each phase space point $\gamma \equiv (e, A)$ and tangent vectors $\delta_1 \equiv (\delta_1 e, \delta_1 A)$ and $\delta_2 \equiv (\delta_2 e, \delta_2 A)$ at that point, a 3-form J on $\mathcal{M}$, called the symplectic current:

$$J(\gamma; \delta_1, \delta_2) = -\frac{1}{2\kappa} \left[\delta_1 \Sigma^{IJ} \wedge \delta_2 A_{IJ} - \delta_2 \Sigma^{IJ} \wedge \delta_1 A_{IJ} \right]. \tag{2.6}$$

Using the fact that the fields (e, A) satisfy the field equations and the tangent vectors δ_1, δ_2 satisfy the linearized equations off (e, A), one can directly verify that $J(\gamma; \delta_1, \delta_2)$ is closed as guaranteed by the general procedure involving second variations. Let us now consider a portion $\mathcal{M}'$ of $\mathcal{M}$ bounded by two Cauchy surfaces M_1, M_2. These are allowed to be general Cauchy surfaces so $\mathcal{M}'$ may in particular be a cylindrical or a boosted slab in the sense of section III A. Consider now a region $\mathcal{R}'$ within $\mathcal{M}'$, bounded by finite portions M_1', M_2' of M_1 and M_2 and a time-like cylinder τ joining $\partial M_1'$ and $\partial M_2'$. Since $dJ = 0$, integrating it over $\mathcal{R}'$ one obtains

$$\int_{M_1'} J + \int_{M_2'} J + \int_\tau J = 0 \tag{2.7}$$

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[1] In specifying our boundary conditions on e, A, we set $n = d - 3$. If we restrict ourselves only to the issue of finiteness of the action, we can weaken these conditions. The action is finite if $n > \frac{d-2}{2}$ in the case of boosted slabs, and $n > \frac{d-3}{2}$ for cylindrical slabs. We introduced stronger requirements to ensure that the Hamiltonian framework and conserved charges are well-defined.

The idea is to take the limit as τ expands to the cylinder τ_∞ at infinity. Suppose the first two integrals continue to exist in this limit *and* the third integral goes to zero. Then, in the limit the sum of the first two terms would vanish and, taking into account orientation signs, we would conclude that $\int_M J$ is a 2-form on Γ which is *independent* of the choice of the Cauchy surface M. This would be the desired pre-symplectic structure. However, the issue of whether the boundary conditions ensure that the integrals over Cauchy surfaces converge and the flux across τ_∞ vanishes is somewhat delicate and often overlooked in the literature.[2] If either of these properties failed, we would not obtain a well-defined symplectic structure on Γ.

Let us first consider the integral over the time-like boundary τ. As τ tends to τ_∞, the integrand $J_{a..bc}\epsilon^{a..bc}$ tends to

$$\lim_{\tau \to \tau_\infty} \left(\delta_{[1}\frac{{}^n\Sigma_{a..b}}{\rho^n}\right)\left(\delta_{2]}\frac{{}^{n+1}A_c}{\rho^{n+1}}\right)\epsilon^{a...bc} = \lim_{\tau \to \tau_\infty} \epsilon_{IJK...L}\, {}^oe_a^K\, ...(\delta_{[1}{}^n e_b^L)\,(\delta_{2]}{}^{n+1}A_c^{IJ})\,\rho^{-1-2n}\,\epsilon^{a...bc} \tag{2.8}$$

where $\epsilon^{a...bc}$ is the metric compatible $(d\text{-}1)$-form on τ. Since the volume element on τ goes as $\rho^{d-1} = \rho^{n+2}$, the integral of the symplectic flux over τ_∞ is *zero*.

The next question is whether the integral over $\tilde{M}_1$ (and $\tilde{M}_2$) continues to be well-defined in the limit as we approach M_1 (resp. M_2). The leading term is again given by the integral of (2.8) over M_1, the only difference being that $\epsilon^{ab...c}$ is now the metric compatible $(d\text{-}1)$-form on M_1. Since the volume element on M_1 goes as $\rho^{d-2}\, \mathrm{d}\rho\, \mathrm{d}^{d-2}\Phi$, a power counting argument shows that the integrand of this leading term falls off as ρ^{3-d}. Thus, because of our boundary conditions, we are led to a well-defined pre-symplectic structure, i.e., a closed 2-form, on Γ

$$\Omega(\delta_1, \delta_2) = \frac{1}{2\kappa}\int_M \mathrm{Tr}\,[\delta_1\Sigma \wedge \delta_2 A - \delta_2\Sigma \wedge \delta_1 A]\,, \tag{2.9}$$

where M is any Cauchy surface in $\mathcal{M}$ and trace is taken over the internal indices. Ω is not a symplectic structure because it is degenerate. The vectors in its kernel represent infinitesimal 'gauge transformations'. The physical phase space is obtained by quotienting Γ by gauge transformations and inherits a true symplectic structure from Ω. We will not carry out the quotient however because the calculation of Hamiltonians can be carried out directly on (Γ, Ω).

III.   GENERATORS OF ASYMPTOTIC POINCARE SYMMETRIES

Let ${}^oe$ and ${}^oe'$ be any two flat co-frames in the phase space Γ and denote the corresponding space-time metrics by g^o and $g^{o\prime}$. Let V be a Killing vector field of g^o. Then, it is easy to check that $g^{o\prime}$ admits a Killing vector V' such that $\lim \rho \to \infty\,(V' - V) = 0$. Killing vectors of any of these flat metrics will be referred to as asymptotic symmetries.

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[2] Furthermore, even when such issues are discussed, one often considers only the restricted action $\Omega(\delta, \delta_V)$ of the pre-symplectic structure Ω, where one of the tangent vectors, δ_V, is associated with an asymptotic symmetry V^a on $\mathcal{M}$ because, as we will see, it is this restricted action that directly enters the discussion of conserved quantities. Typically the 3-form integrands of $\Omega(\delta_1, \delta_V)$ on $\mathcal{M}$ have a better asymptotic behavior than those of generic $\Omega(\delta_1, \delta_2)$. However, unless $\Omega(\delta_1, \delta_2)$ is well-defined for all δ_1, δ_2, one does not have a coherent Hamiltonian framework and cannot start constructing conserved quantities.

Let V^a be an asymptotic symmetry. Then, at any point (e, A) of Γ, the pair $(\mathcal{L}_V e, \mathcal{L}_V A)$ of fields satisfies the linearized field equations, whence $\delta_V := (\mathcal{L}_V e, \mathcal{L}_V A)$ is a vector field on Γ. (In the definition of the Lie-derivative, internal indices are treated as scalars; thus $\mathcal{L}_V e_a^I = V^b \partial_b e_a^I + e_b^I \partial_a V^b$.) The question is whether δ_V is a phase space symmetry, i.e., whether it satisfies $\mathcal{L}_{\delta_V} \Omega = 0$.

Consider the 1-form X_V on Γ defined by

$$X_V(\delta) = \Omega(\delta, \delta_V). \tag{3.1}$$

$\mathcal{L}_{\delta_V} \Omega = 0$ on Γ if and only if X_V is closed, i.e.,

$$\mathbb{d} X_V = 0$$

where $\mathbb{d}$ denotes the exterior derivative on (the infinite dimensional) phase space Γ. If this is the case then, up to an additive constant, the Hamiltonian is given by

$$\mathbb{d} H_V = X_V.$$

The constant is determined by requiring that all Hamiltonians generating asymptotic symmetries at the phase space point $({}^{o}e, A = 0)$ corresponding to Minkowski space-time must vanish. To calculate the right side of (3.1), it is useful to note the Cartan identities

$$\mathcal{L}_V A = V \cdot F + D(V \cdot A) \qquad \text{and} \qquad \mathcal{L}_V \Sigma = V \cdot D\Sigma + D(V \cdot \Sigma) - [(V \cdot A), \Sigma] \tag{3.2}$$

Using these, the field equations satisfied by (e, A) and the linearized field equations for δ, one obtains the required expression of X_V (see e.g. [13, 15]):

$$X_V(\delta) := \Omega(\delta, \delta_V) = \frac{1}{2\kappa} \oint_{S_\infty} \mathrm{Tr} \left[(V \cdot A)\delta\Sigma + \delta A \wedge (V \cdot \Sigma) \right] . \tag{3.3}$$

Note that the expression involves integrals *only* over the d-2-sphere boundary S_∞ of the Cauchy surface M (i.e., the intersection of M with the hyperboloid $\mathcal{H}$ at infinity); there is no volume term. This is a reflection of the fact that, because there are no background fields, all diffeomorphisms which are asymptotically identity represent gauge transformations.

A.   Energy-Momentum

Let us begin by setting $V^a = T^a$, an infinitesimal asymptotic translation. Since $\delta\Sigma \sim 1/\rho^{d-3}$, $A \sim 1/\rho^{d-2}$ and since the area element of the d-2-sphere grows as ρ^{d-2}, the first term on the right side of (3.3) vanishes in the limit and we are left with

$$X_T(\delta) := \Omega(\delta, \delta_T) = \frac{1}{2\kappa} \oint_{S_\infty} \mathrm{Tr} \left[\delta A \wedge (T \cdot {}^{o}\Sigma) \right] \tag{3.4}$$

which is manifestly well-defined. Furthermore, since ${}^{o}\Sigma$ and T are constant on the phase space Γ, we can pull the variation δ out of the integral. The resulting Hamiltonian H_T generating an asymptotic translation T^a is then given by:

$$H_T = \frac{1}{2\kappa} \oint_{S_\infty} \mathrm{Tr} \left[A \wedge (T \cdot {}^{o}\Sigma) \right] \tag{3.5}$$

Had we selected a translational Killing field $\bar{T}^a$ of another flat metric $\bar{\eta}_{ab}$ in our phase space Γ, we would have obtained the same answer because $\lim_{\rho \to \infty}(\bar{T}^a - T^a) = 0$.

Substituting for the leading order term ^{n+1}A in the asymptotic expansion (2.3) of A, H_T can be expressed in terms of the leading order co-frame field $^n e_{ab} = {}^n e_a{}^J \, {}^o e_{bJ}$:

$$
\begin{aligned}
H_T = \lim_{\rho \to \infty} \frac{1}{\kappa} \oint_{S_\rho} \Big[& (\rho \cdot T) \left(\rho \, n^b \, \partial^a \, (^n e_{ab}) - n\rho^a n^b \, {}^n e_{ab} - \rho n^a \partial_a \, (^n e_{ab}) \right) \\
& + (n \cdot T) \left(n\rho^a \rho^b \, {}^n e_{ab} - n \, {}^n e_a{}^a - \rho\rho^b \, \partial^a \, (^n e_{ab}) \right) \\
& + nT^a n^b \, (^n e_{ab}) + \rho T^a \rho^b n^c \, \partial_c \, (^n e_{ab}) \Big] \, \mathrm{d}^{d-2} S_o
\end{aligned}
\tag{3.6}
$$

where S_ρ is a d-2 sphere cross-section of the hyperboloid $\rho = $ const, n^a is the unit normal to S_ρ within the hyperboloid and $\mathrm{d}^n S_o$ is the area element of the *unit* d-2-sphere. The terms with an explicit multiplicative factor of ρ have well-defined limits because $\partial_a {}^n e_{ab}$ falls off as $1/\rho$. Thus, energy-momentum is determined directly by the leading correction $^n e_{ab}$ to the Minkowskian co-frame $^o e_{ab}$ in the asymptotic expansion. Our boundary conditions required that $^n e_{ab}$ be even under reflection. However, this condition is not needed to arrive at the expression (3.6). Indeed, an examination of the integrand shows that only the even part of $^n e_{ab}$ contributes to this energy-momentum. This expression is 'universal' in the sense that it holds in all higher dimensions. In Appendix A 2 we show that it reduces to the more special expression derived in [1] in 4 dimensions using the Beig-Schmidt form of the metric.

If T^a is a unit time-translation, we can choose a hyperplane M which is orthogonal to it (with respect to η_{ab}) and let S_ρ be the intersection of the hyperboloids $\rho = $ const with M. Then, $\rho \cdot T$ vanishes and the remaining terms can be easily shown to equal the familiar expression of the ADM energy:

$$
H_T = \frac{1}{2\kappa} \oint_{S_\infty} \left(\underline{\partial}_a q_{bc} - \underline{\partial}_c q_{ab} \right) q^{ab} \, \mathrm{d}^{d-2} S^c
\tag{3.7}
$$

where q_{ab} is the intrinsic physical metric on the Cauchy surface M and $\underline{\partial}$ is the derivative operator compatible with a flat metric induced on M by η_{ab}.

If T^a is a space-translation, we can choose a hyperplane M (w.r.t. η_{ab}) to which T^a is tangential and let S_ρ be the intersection of the hyperboloids $\rho = $ const with M. Now, the pull-back $A_{\underline{a}}{}^{IJ}$ to any Cauchy surface M of the Lorentz connection $A_a{}^{IJ}$ satisfies $A_{\underline{a}}{}^{IJ} n_J = K_a{}^I$ where $n^a := e_a^J n_J$ is the unit normal to M and $K_{ab} := K_a{}^J e_{bJ}$ is the extrinsic curvature of M. (See, e.g., Section 2.3.1 of [16].) Therefore, the expression of the Hamiltonian $H_{\vec{T}}$ generating a spatial translation simplifies to the more familiar form:

$$
H_{\vec{T}} = \frac{1}{\kappa} \oint_{S_\infty} \left(K_{ab} - K q_{ab} \right) T^a \, \mathrm{d}^{d-2} S^b
\tag{3.8}
$$

B. Relativistic Angular Momentum

Let us now set $V^a = L^a$, an infinitesimal asymptotic Lorentz symmetry. For definiteness, we will assume that it is a Lorentz Killing field of $\eta_{ab} := \eta_{IJ} \, {}^o e_a^I \, {}^o e_b^J$ so that it is tangential to the $\rho = $ const hyperboloids $\mathcal{H}$.

The question is whether the vector field δ_L on Γ is Hamiltonian. Let us begin by examining the 1-form X_L on Γ. Using (3.3), we have:

$$X_L(\delta) := \Omega(\delta, \delta_L) = \frac{1}{2\kappa} \lim_{\rho \to \infty} \oint_{S_\rho} \mathrm{Tr}\left[(L \cdot A)\,\delta\Sigma + \delta A \wedge (L \cdot \Sigma)\right] \tag{3.9}$$

where S_ρ is a d-2-sphere, the intersection of the $\rho = $ constant hyperboloid $\mathcal{H}_\rho$ with the Cauchy surface M used to evaluate the symplectic structure. Now, as ρ tends to infinity, $A \sim \rho^{d-2}$, $\delta A \sim \rho^{d-2}$, $\Sigma \to {}^o\Sigma$, $\delta\Sigma \sim \rho^{d-3}$ and $L \sim \rho$. Therefore for $d > 4$, the first term vanishes in the limit. However, the second term in (3.9) is potentially divergent. It is here we use the parity condition on ${}^n e$: Using the fact that this field is even under reflections, one can show that the potentially divergent term in fact vanishes. Furthermore, since the integral of $A \wedge \mathcal{L} \cdot \delta\Sigma$ vanishes in the limit, we can take the variation δ out of the integral and obtain the Hamiltonian H_L representing the component of the relativistic angular momentum along L:

$$H_L = \frac{1}{2\kappa} \lim_{\rho \to \infty} \oint_{S_\rho} \mathrm{Tr}\left(A \wedge L \cdot \Sigma\right), \tag{3.10}$$

To simplify further, one can use the form (2.3) of the leading order piece ${}^{n+1}A$ of the connection and using reflection symmetry of ${}^n e$ show that its contribution to (3.10) vanishes. Therefore, we have:

$$H_L = \frac{1}{2\kappa} \oint_{S_\infty} \mathrm{Tr}\left({}^{n+2}A \wedge \hat{L} \cdot {}^o\Sigma\right), \tag{3.11}$$

where $\hat{L}^a = L^a/\rho$ is the Lorentz Killing field on the *unit* hyperboloid $(\mathcal{H}, h^o_{ab})$. As in 4 dimensions [1], in contrast to the energy momentum the angular momentum is not determined by the leading order deviation of (e, A) from the ground state $({}^o e, A = 0)$, but by sub-leading terms.

Finally, if L^a is a spatial rotation ϕ^a, we can recast (3.10) in a more familiar form. In this case, only the part $A_{\underline{a}}^{IJ} n_J$ of the connection contributes to the integral, where $n^a = n^J e^a_J$ is the normal to the Cauchy surface M, chosen such that ϕ^a is tangential to it, and the underbar below a denotes that this index is pulled back to M. As in the case of spatial momentum, one notes the relation between the Lorentz connection $A_a{}^{IJ}$ and the extrinsic curvature K_{ab} on M to rewrite (3.10) as:

$$H_\phi = \frac{1}{\kappa} \lim_{\rho \to \infty} \oint_{S_\rho} (K_{ab} - K q_{ab})\,\phi^a\,\mathrm{d}^{d-2}S^b \tag{3.12}$$

IV. DISCUSSION

In this paper we have shown that in the first order formalism based on co-frames and Lorentz connections, the Lagrangian and Hamiltonian frameworks can be constructed without having to introduce an infinite counter term subtraction in the action. The analysis was considerably simpler than the four dimensional case [1] because of absence of logarithmic translations and super translations in higher dimensions.

For simplicity, in this paper we focused on vacuum Einstein's equations. However, inclusion of standard matter —in particular, scalar, Maxwell and Yang-Mills fields— with standard boundary conditions used in Minkowski space is straightforward. There are no surface terms in the action associated with matter. Similarly, the expressions of the Hamiltonians generating asymptotic Poincaré transformations are the same as the ones we found in section III. In particular, the Hamiltonians consist entirely of d-2-sphere surface integrals

at spatial infinity and their integrands do not receive any explicit contributions from matter. Matter makes its presence felt through constraint equations which, in response to matter, modify the asymptotic gravitational fields.

Acknowledgment:

APPENDIX A: EXAMPLES

1. The Schwarzschild Metric

In this Appendix we show that d-dimensional Schwarzschild space-times satisfy our boundary conditions.

Recall that the schwarzschild line element can be expressed as

$$\mathrm{d}s^2 = -(1 - \frac{2M}{r^{d-3}})\mathrm{d}t^2 + (1 - \frac{2M}{r^{d-3}})^{-1}\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2 \tag{A1}$$

Where Ω is the unit solid angle. We can recast this expression in terms of our variables as:

$$\mathrm{d}s^2 = {}^o\mathrm{d}s^2 + \frac{{}^n\mathrm{d}s^2}{\rho^n} + O(\rho^-2) \tag{A2}$$

where ${}^o\mathrm{d}s^2$ is the flat (Minkowski) metric adapted to the (t, r, Φ^i) coordinates and

$$^n\mathrm{d}s^2 = \frac{2M}{\cosh(\chi)}\left[(\sinh^2\chi + \cosh^2\chi)\mathrm{d}\rho^2 + 4\rho\cosh(\chi)\sinh(\chi)\mathrm{d}\rho\mathrm{d}\chi + \rho^2(\sinh^2\chi + \cosh^2\chi)\mathrm{d}\chi^2\right] \tag{A3}$$

Note that the leading-order deviation from the flat metric contains is an off-diagonal, $\mathrm{d}\rho\mathrm{d}\chi$ term. It is often assumed (see, e.g. [4, 11]) that this off-diagonal term is absent because we know from general considerations that it is possible to eliminate it by a suitable redefinition of coordinates [8]. However, it is not trivial to carry out this step *explicitly* even for the Schwarzschild metric. But it is easy to construct a co-frame compatible with this metric satisfying the boundary conditions of section II:

$$^ne_a^I = \frac{M}{\cosh(\chi)}\left[(\sinh^2\chi + \cosh^2\chi)\rho_a\rho^I + \frac{4}{\rho}\sinh(\chi)\cosh(\chi)\rho_a\chi^I + \frac{1}{\rho^2}(\sinh^2\chi + \cosh^2\chi)\chi_a\chi^I\right] \tag{A4}$$

The boundary condition on the leading-order, non-trivial contribution, $^{n+1}A_a^{IJ}$ of the connection A_a^{IJ} is readily satisfied because the connection is *everywhere* compatible with the co-frame.

Inserting the expression (A4) of the co-frame into our expression (3.6) for energy-momentum and (3.11) of angular momentum we find that we recover the expected result: $E = M$, $\vec{P} \cdot \vec{T} = 0$, $H_L = 0$.

2. 4-dimensions: Beig-Schmidt form

In 4 dimensions, one makes an extensive use of a the 'Beig-Schmidt form' of the metric:

$$\mathrm{d}s^2 = (1 + \frac{2\sigma}{\rho})\mathrm{d}\rho^2 + (1 - \frac{2\sigma}{\rho})\,\rho^2\,h_{ab}\mathrm{d}x^a\mathrm{d}x^b + o(\rho^{-1}) \tag{A5}$$

where h_{ab} is the metric on the unit hyperboloid. It is easy to construct a co-frame compatible with this metric by setting:

$$^1e_a^I = \sigma(2\rho_a\rho^I - {}^oe_a^I) \tag{A6}$$

In [1], this asymptotic form of the co-frame was used to obtain an expression for energy-momentum in terms of σ and angular momentum in terms of the sub-leading term ^3A in the expansion of the Lorentz connection. In this paper, on the other hand, we have obtained more general forms of these conserved quantities without assuming the Beig-Schmidt form of the metric. Do these more general forms directly reduce to those obtained in [1] in 4 dimensions once the co-frames are assumed to admit the Beig-Schmidt form? The answer is in the affirmative: our expression (3.6) of energy-momentum and (3.11) of angular momentum of section III directly simplify to yield

$$E = \frac{2}{\kappa}\oint_{S_\infty}\sigma\mathrm{d}S_o \tag{A7}$$

$$\vec{P}\cdot\vec{T} = \frac{2}{\kappa}\oint_{S_\infty}(\frac{\rho\cdot T}{\rho})\,\frac{\partial\sigma}{\partial\chi}\,\mathrm{d}S_o \tag{A8}$$

$$J_L = \frac{1}{2\kappa}\oint_{S_\infty}(L\cdot{}^0\Sigma_{IJ})\wedge{}^3A^{IJ}\,. \tag{A9}$$

These are exactly the same expressions that we found in [1]. Note, however, that in 4-dimensions the Beig-Schmidt form was essential to eliminate ambiguities arising from the logarithmic translations and super translations and to ensure that the symplectic structure is well-defined. It is just that the final expressions of conserved quantities have the same 'universal' form that we found in this paper for higher dimensions.

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