

Certified Defense to Image Transformations via Randomized Smoothing

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Abstract

We extend randomized smoothing to cover parameterized transformations (e.g., rotations, translations) and certify robustness in the parameter space (e.g., rotation angle). This is particularly challenging as interpolation and rounding effects mean that image transformations do not compose, in turn preventing direct certification of the perturbed image (unlike certification with ℓ^p norms). We address this challenge by introducing three different defenses, each with a different guarantee (heuristic, distributional and individual) stemming from the method used to bound the interpolation error. Importantly, in the individual case, we show how to efficiently compute the inverse of an image transformation, enabling us to provide individual guarantees in the online setting. We provide an implementation of all methods at <https://github.com/eth-sri/transformation-smoothing>.

1 Introduction

Deep neural networks are vulnerable to adversarial examples [1] – small changes that preserve semantics (e.g., ℓ^p -noise or geometric transformations such as rotations) [2], but can affect the output of a network in undesirable ways. As a result, there has been substantial recent interest in methods which aim to ensure the network is certifiably robust to adversarial examples [3–13].

Certification guarantees There are two principal robustness guarantees a certified defense can provide at inference time: (i) the (standard) distributional guarantee, where a robustness score is computed offline on the test set to be interpreted in expectation for images drawn from the data distribution, and (ii) an individual guarantee, where a certificate is computed online for the (possibly perturbed) input. The choice of guarantee depends on the application and regulatory constraints.

Guarantees with ℓ^p norms When considering ℓ^p norms, existing certification methods can be directly used to obtain either of the above two guarantees: for an image \mathbf{x} and adversarial noise δ , $\|\delta\|_p < r$, proving that a classifier f is r -robust around $\mathbf{x}' := \mathbf{x} + \delta$ is enough to guarantee $f(\mathbf{x}) = f(\mathbf{x}')$. That is, it suffices to prove robustness of a perturbed input in order to certify that the perturbation did not change the classification, as the r -ball around \mathbf{x}' includes \mathbf{x} .

Key challenge: guarantees for geometric perturbations Perhaps not intuitively however, for more complex perturbations such as geometric transformations, proving robustness around an image \mathbf{x}' via existing methods (e.g., [9–12]) does not imply that $f(\mathbf{x}) = f(\mathbf{x}')$ for the original image \mathbf{x} . To illustrate this issue, consider the rotation R_γ , with angle γ of an image \mathbf{x} , followed by an interpolation I . Certifying that the classification of the rotated image $\mathbf{x}' := I \circ R_\gamma(\mathbf{x})$ for $\|\gamma\| < r$ is robust under further rotations $I \circ R_\beta$ for $\|\beta\| < r$ is not sufficient to imply that \mathbf{x} and \mathbf{x}' classify the same, as rotating \mathbf{x}' back by $\beta = -\gamma$ does not return the original image \mathbf{x} due to interpolation. A central challenge then is to develop techniques that are able to handle more involved perturbations.

This work: certification beyond ℓ^p norms In this work we address this challenge and introduce the first certification methods for geometric transformations based on smoothing which provide the above two guarantees. Concretely, we extend randomized smoothing [7] to handle parameterized transformations (SPT) by adding (Gaussian) noise to the parameters of a transformation, enabling us to handle large models and datasets (e.g., ImageNet). We present three methods, BASESPT, DISTSPT and INDIVSPT yielding different guarantees.

BASESPT The guarantees provided by SPT hold for composable parameterized perturbations ψ_γ , that is $\psi_{\beta+\gamma} = \psi_\gamma \circ \psi_\beta$. SPT can be applied directly to obtain both a distributional guarantee and an individual guarantee. However, if used with non-composable transformations (e.g., rotations with interpolation), BASESPT will yield a heuristic guarantee.

DISTSPT By estimating a probabilistic upper bound for $\epsilon = \|\psi_\beta \circ \psi_\gamma(\mathbf{x}) - \psi_{\beta+\gamma}(\mathbf{x})\|_2$ offline on the training dataset, SPT can be combined with randomized smoothing (yielding DISTSPT) to provide the standard distributional guarantee for non-composable transformations.

INDIVSPT To obtain an individual guarantee for non-composable transformations, we calculate at inference time for each input \mathbf{x}' an individual upper bound of the expression ϵ *without access* to \mathbf{x} , which is combined with SPT and smoothing to yield INDIVSPT. A key step here is computing the inverse $\psi_\gamma^{-1}(\mathbf{x}')$ of a transformed image \mathbf{x}' , for which we introduce an efficient technique.

We remark that all three methods are suitable for online use as a defense and the choice of method depends on the particular trade-offs. To summarize, our core contributions are:

- A generalization of randomized smoothing to parameterized transformations.
- A certification method which provides a distributional guarantee by calculating an upper bound on the interpolation error on the training dataset.
- A certification method which provides an individual guarantee based on an algorithm that efficiently calculates the inverse of an image.
- A thorough evaluation of all methods on common image datasets, achieving provable distributional robust accuracy of 73% for rotations with up to $\pm 30^\circ$ on Restricted ImageNet.

2 Related Work

We now survey the most closely related work in neural network certification and defenses.

ℓ^p norm based certification and defenses The discovery of adversarial examples [1, 14] triggered interest in training and certifying robust neural networks. An attempt to improve model robustness are empirical defenses [15, 16], strategies which harden a model against an adversary. While this may improve robustness to current adversaries, typically robustness cannot be formally verified with current certification methods. This is because complete methods [17–19] do not scale and incomplete methods relying on over approximation lose too much precision [3, 20, 21, 6, 10, 22], even for networks trained to be amenable to certification. Recently, randomized smoothing was introduced, which could for the first time, certify a (smoothed) classifier against norm bound ℓ^2 noise on ImageNet [23, 24, 7, 8, 25], by relaxing exact certificates to high confidence probabilistic ones. Smoothing scales to large models, however, it is currently limited to norm-based perturbations.

Semantic perturbations Transformations such as translations and rotation can produce adversarial examples [2, 26]. An enumerative approach certifying against semantic perturbations was presented in [9]. There, the search space is reduced by only considering next neighbor interpolation. Unfortunately, for more elaborate interpolations, like bilinear interpolation, this approach becomes infeasible. The first certification against rotations with bilinear interpolations was carried out in [10], which was later significantly improved on by [11]. Both methods generate linear relaxations and propagate them through the network. However, these methods do not yet scale to large networks (i.e., ResNet-50) or complex data sets (i.e., ImageNet). The approaches of [12] is similar to the one in [10] for rotation. The method in [13] is a combination of enumeration and smoothing. The methods for translation in [12] and [13] are in fact equivalent to Pei et al. [9]. Further, [13] can only invoke their method for unperturbed images, making it inapplicable as a defense.

3 Generalization of Smoothing

A smoothed classifier $g: \mathbb{R}^m \mapsto \mathcal{Y}$ can be constructed out of an ordinary classifier $f: \mathbb{R}^m \mapsto \mathcal{Y}$, by calculating the most probable result of $f(\mathbf{x} + \epsilon)$ where $\epsilon \sim \mathcal{N}(0, \sigma^2 \mathbf{1})$:

$$g(\mathbf{x}) := \arg \max_c \mathbb{P}_{\epsilon \sim \mathcal{N}(0, \sigma^2 \mathbf{1})}(f(\mathbf{x} + \epsilon) = c).$$

One then obtains the following robustness guarantee:

Theorem 3.1 (From [7]). *Suppose $c_A \in \mathcal{Y}$, $\underline{p}_A, \overline{p}_B \in [0, 1]$. If*

$$\mathbb{P}_\epsilon(f(\mathbf{x} + \epsilon) = c_A) \geq \underline{p}_A \geq \overline{p}_B \geq \max_{c \neq c_A} \mathbb{P}_\epsilon(f(\mathbf{x} + \epsilon) = c),$$

then $g(\mathbf{x} + \delta) = c_A$ for all δ satisfying $\|\delta\|_2 \leq \frac{\sigma}{2}(\Phi^{-1}(\underline{p}_A) - \Phi^{-1}(\overline{p}_B)) =: r_\delta$.

We now generalize this theorem to parameterized transformations. Consider the composable transformations $\psi_\beta: \mathbb{R}^m \rightarrow \mathbb{R}^m$, satisfying $\psi_\beta \circ \psi_\gamma = \psi_{\beta+\gamma}$ for all $\beta, \gamma \in \mathbb{R}^d$. Then we can define a smoothed classifier $g: \mathbb{R}^m \rightarrow \mathcal{Y}$ analogously for a parametric transformation ψ_β by

$$g(\mathbf{x}) = \arg \max_c \mathbb{P}_{\beta \sim \mathcal{N}(0, \sigma^2 \mathbf{1})}(f \circ \psi_\beta(\mathbf{x}) = c). \quad (1)$$

With that, we obtain the following robustness guarantee:

Theorem 3.2. *Let $\mathbf{x} \in \mathbb{R}^m$, $f: \mathbb{R}^m \rightarrow \mathcal{Y}$ be a classifier and $\psi_\beta: \mathbb{R}^m \rightarrow \mathbb{R}^m$ be a composable transformation as above. If*

$$\mathbb{P}_\beta(f \circ \psi_\beta(\mathbf{x}) = c_A) \geq \underline{p}_A \geq \overline{p}_B \geq \max_{c_B \neq c_A} \mathbb{P}_\beta(f \circ \psi_\beta(\mathbf{x}) = c_B),$$

then $g \circ \psi_\gamma(\mathbf{x}) = c_A$ for all γ satisfying $\|\gamma\|_2 \leq \frac{\sigma}{2}(\Phi^{-1}(\underline{p}_A) - \Phi^{-1}(\overline{p}_B)) =: r_\gamma$. Further, if g is evaluated on a proxy classifier f' that behaves like f with probability $1 - \rho$ and else returns an arbitrary answer, then $r_\gamma := \frac{\sigma}{2}(\Phi^{-1}(\underline{p}_A - \rho) - \Phi^{-1}(\overline{p}_B + \rho))$.

The proof is similar to the one presented in Cohen et al. [7] and is given in App. A. The key difference is that we allow parameterized transformations ψ , while Cohen et al. [7] only allows additive noise.

4 Certification with interpolation and rounding errors

We now instantiate Theorem 3.2 for parameterized geometric image transformations T_β , $\beta \in \mathbb{R}^d$, followed by interpolation I , denoted as T_β^I . A geometric transformation T_β is followed by an interpolation I in order to express the result on the pixel grid. In general, even if T_β composes, T_β^I does not (see Fig. 1 in the case where T_β is a rotation R_β by an angle β). This prevents us from directly instantiating Theorem 3.2 with $\psi_\beta := T_\beta^I$.

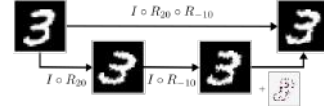


Figure 1: Rotations with interpolation do not compose.

To address this issue, we now show how to construct a classifier g_E with the desired guarantees, namely that $g_E \circ T_\gamma^I(\mathbf{x}) = g_E(\mathbf{x})$ for γ with $\|\gamma\|_2 \leq r_\gamma$, thus enabling certification of image transformations (which may not compose). Our proposed construction consist of two steps.

First, for a fixed but arbitrary \mathbf{x} , let h_E be a classifier satisfying interpolation invariance:

$$h_E \circ T_\beta^I \circ T_\gamma^I(\mathbf{x}) = h_E \circ T_{\beta+\gamma}^I(\mathbf{x}) \quad \forall \beta, \gamma \in \mathbb{R}^d. \quad (2)$$

We now instantiate Theorem 3.2 with $f := h_E \circ I$ and $\psi_\beta := T_\beta$, obtaining a smoothed classifier $g_E(\mathbf{x}) := \arg \max_c \mathbb{P}_{\beta \sim \mathcal{N}(0, \sigma^2 \mathbf{1})}(h_E \circ I \circ T_\beta(\mathbf{x}) = c)$, such that $g_E \circ T_\gamma(\mathbf{x}) = c_A = g_E(\mathbf{x})$ for γ with $\|\gamma\|_2 \leq r_\gamma$ by Theorem 3.2. Further, since

$$\begin{aligned} g_E \circ T_\gamma(\mathbf{x}) &= \arg \max_c \mathbb{P}_{\beta \sim \mathcal{N}(0, \sigma^2 \mathbf{1})}(h_E \circ I \circ T_\beta \circ T_\gamma(\mathbf{x}) = c) \\ &= \arg \max_c \mathbb{P}_{\beta \sim \mathcal{N}(0, \sigma^2 \mathbf{1})}(h_E \circ T_\beta^I \circ T_\gamma^I(\mathbf{x}) = c) \\ &= g_E \circ T_\gamma^I(\mathbf{x}), \end{aligned}$$

where the first and last equities hold by the definition of g_E and the second one due to Eq. (2). Thus, we obtain a classifier g_E with the desired property.

Second, we discuss the construction of the desired h_E (from step 1). Consider the interpolation error

$$\epsilon(\beta, \gamma, \mathbf{x}) := T_\beta^I \circ T_\gamma^I(\mathbf{x}) - T_{\beta+\gamma}^I(\mathbf{x}), \quad (3)$$

$$\text{bounded by } E \in \mathbb{R}^{\geq 0} \text{ s.t. } \forall \beta, \gamma \in \mathbb{R}^d. \|\epsilon(\beta, \gamma, \mathbf{x})\|_2 \leq E \quad (4)$$

for a given but arbitrary \mathbf{x} . Thus if h_E is ℓ^2 -robust with radius E around $T_{\beta+\gamma}^I(\mathbf{x})$, interpolation invariance holds. While many choices for such h_E are possible in the rest of the paper we instantiate h_E by applying Theorem 3.1 to a base classifier b .

Obtaining probabilistic guarantees from Theorem 3.2 So far we assumed that \mathbf{x} is arbitrary but fixed and constructed E and h_E for this \mathbf{x} specifically. In general, finding a tight deterministic bound E that holds $\forall \beta, \gamma$ is computationally challenging. Thus, we relax this deterministic guarantee into a probabilistic one:

$$\mathbb{P}_{\beta \sim \mathcal{N}(0, \sigma^2 \mathbf{1})} (\|\epsilon(\beta, \gamma, \mathbf{x})\|_2 \leq E) \geq 1 - \alpha_E \quad \forall \gamma \in \mathbb{R}^d. \quad (5)$$

meaning Eq. (4) holds with probability at least $1 - \alpha_E$, in turn implying that Eq. (2) also holds at least with probability $1 - \alpha_E$. This can also be formulated as having a proxy classifier h'_E which behaves like h_E with probability at least $1 - \alpha_E$ on the inputs specified by Eq. (2). In practice, we construct h'_E which behaves like h_E with probability at least $1 - \alpha_E$ on all inputs, implying this behavior on the inputs from Eq. (2). From h'_E , we then obtain $f' := h'_E \circ I$ which behaves like f with probability at least $1 - \alpha_E$ on all inputs. Then, we can apply Theorem 3.2 by setting ρ to α_E and obtain the desired guarantee. In Section 5, we show how to obtain E for DISTSPT and INDIVSPT.

5 Calculation of error bounds

In Section 5.1 we derive a distributional error bound over a dataset and in Section 5.2 a per-image bound. Throughout this section, we assume the attacker model $\gamma \in \Gamma \subseteq \mathbb{R}^d$. As we compute E with this assumption, our obtained certificate proves robustness of g_E to T_γ^I for $\gamma \in \Gamma$ with $\|\gamma\|_2 \leq r_\gamma$.

5.1 Distributional bounds for DISTSPT

For a fixed $E \in \mathbb{R}^{\geq 0}$, $\alpha_E \in [0, 1]$, the probability that ϵ is bounded by E for $\mathbf{x} \sim \mathcal{D}$ is

$$q_E := \mathbb{P}_{\mathbf{x} \sim \mathcal{D}} (\mathbb{P}_{\beta \sim \mathcal{N}(0, \sigma^2 \mathbf{1})} (\max_{\gamma \in \Gamma} \|\epsilon(\beta, \gamma, \mathbf{x})\|_2 \leq E) \geq 1 - \alpha_E). \quad (6)$$

In practice, we evaluate q_E by sampling \mathbf{x} and counting how often the inner property holds. We compute the inner probability by: (i) sampling multiple realizations of β , (ii) computing their corresponding error ϵ and checking how many are successfully bounded by E , and (iii) bounding the inner probability using Clopper-Pearson. If this lower bound is larger than $1 - \alpha_E$ we count this as a positive sample, else a negative. Once these counts are obtained for a number of sampled points \mathbf{x} , we can apply Clopper-Pearson and obtain a lower bound \underline{q}_E with the desired confidence.

To compute the maximization over γ we employ standard interval analysis, which allows us to efficiently propagate lower and upper bounds [27]. By propagating the hyperrectangle containing Γ along with the sampled β and \mathbf{x} , we eventually obtain a lower and upper bound for the norm calculation of which we take the maximum:

$$\max_{\gamma \in \Gamma} \|\epsilon(\beta, \gamma, \mathbf{x})\|_2 \leq \max \|T_\beta^I \circ T_\Gamma^I(\mathbf{x}) - T_{\beta+\Gamma}^I(\mathbf{x})\|_2. \quad (7)$$

The result can be refined by splitting the hyperrectangle Γ into smaller hyperrectangles Γ_k for $k \in \{0, \dots, N\}$. The refined bound is

$$\max_{\gamma \in \Gamma} \|\epsilon(\beta, \gamma, \mathbf{x})\|_2 \leq \max_{k \in \{0, \dots, N\}} \max \|T_\beta^I \circ T_{\Gamma_k}^I(\mathbf{x}) - T_{\beta+\Gamma_k}^I(\mathbf{x})\|_2. \quad (8)$$

To obtain E in the first place, we perform the same sampling operations as above (sample \mathbf{x} and β) but do not compute any probabilities, that is, for each sample (\mathbf{x}, β) , we simply keep the values attained by Eq. (8). We then pick an E that bounds many of these values, choosing α_E to be small. Once E is obtained, we compute q_E as described above. Instantiating the construction of Section 4 with this E yields the guarantee that for a random image $\mathbf{x} \sim \mathcal{D}$ the guarantees provided by Theorem 3.2 hold with probability q_E .

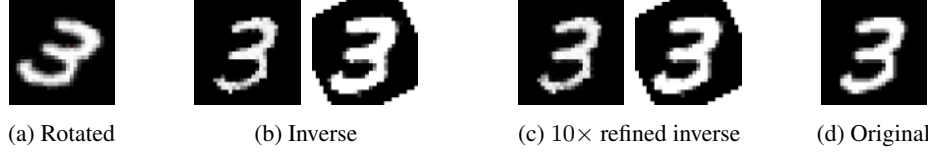


Figure 2: Over approximation of the inverse image. The image pairs (b) and (c) depict the lower (left) and upper (right) interval pixel bounds for the inverse image and the $10\times$ refined image respectively.

5.2 Individual bounds for INDIVSPT

At inference time, we are given $\mathbf{x}' := T_\gamma^I(\mathbf{x})$ but neither the original \mathbf{x} nor the parameter $\gamma \in \Gamma$, and we would like to certify that $g_E(\mathbf{x}') = g_E(\mathbf{x})$. When ψ_β composes as required in Section 3, this can be certified by showing g is robust with a sufficient radius r_γ . However, when ψ_β does not compose, this can be accomplished by applying Theorem 3.2 to show $g_E(\mathbf{x})$ is robust with radius r_γ that includes Γ . In turn, this requires a bound E (see Eq. (5)) for \mathbf{x} (rather than \mathbf{x}'):

$$\mathbb{P}_{\beta \sim \mathcal{N}(0, \sigma^2 \mathbf{1})} \left(\max_{\gamma \in \Gamma} \|\epsilon(\beta, \gamma, \mathbf{x})\|_2 \leq E \right) \geq 1 - \alpha_E. \quad (9)$$

Now, we would like to compute an upper bound on the max term *without* having access to \mathbf{x} . This is accomplished as follows. First, in the above equation we replace ϵ by its definition (Eq. (3)) and $T_\gamma^I(\mathbf{x})$ by \mathbf{x}' . We then replace \mathbf{x} with a symbolic set of possible inputs that could have generated \mathbf{x}' , denoted as $(T_\gamma^I)^{-1}(\mathbf{x}') := \{\mathbf{x} \in \mathbb{R}^m \mid T_\gamma^I(\mathbf{x}) = \mathbf{x}', \gamma \in \Gamma\}$ which we can use instead of \mathbf{x} due to the maximization over γ . As in Section 5.1, we obtain the resulting bound via interval analysis:

$$\max_{\gamma \in \Gamma} \|\epsilon(\beta, \gamma, \mathbf{x})\|_2 \leq \max \|T_\beta^I(\mathbf{x}') - T_{\beta+\Gamma}^I \circ (T_\Gamma^I)^{-1}(\mathbf{x}')\|_2. \quad (10)$$

The computation of the inverse $(T_\Gamma^I)^{-1}(\mathbf{x}')$ is explained in Section 6. By substituting Eq. (10) in Eq. (9) we can obtain and verify E as in Section 5.1 (except we do not need to sample \mathbf{x} 's). As before, we can refine the upper bound of Eq. (10) by splitting Γ into Γ_k . We note as the inverse does not depend on β , given \mathbf{x}' , it only needs to be computed once and can be reused whenever we evaluate Section 5.1 for a given sample β .

6 Inverse Computation

We now discuss how to obtain a set containing all possible inverse images. That is, given $\mathbf{x}' := T_\gamma^I(\mathbf{x})$ and $\gamma \in \Gamma$, we compute the set $(T_\Gamma^I)^{-1}(\mathbf{x}')$ which contains all possible \mathbf{x} . First we cover the necessary background. To ease presentation, we assume even image height and width. We embed the images in \mathbb{R}^2 by placing the pixels on an odd-integer grid $G := (2\mathbb{Z} + 1) \times (2\mathbb{Z} + 1)$ and centered at 0. We denote the value of a pixel at $(i, j) \in G$ by $p_{i,j} \in [0, 1]$.

Transformations The pixel values $p'_{i',j'}$ for $(i', j') \in G$ of an image, produced by a transformation $T_\gamma: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ with parameter $\gamma \in \mathbb{R}^d$, is calculated by interpolating at the inversely transformed coordinate $T_\gamma^{-1}(i', j')$, followed by the interpolation I resulting in $p'_{i',j'} = I \circ T_\gamma^{-1}(i', j')$.

Bilinear interpolation A prominent interpolation is *bilinear interpolation*, given by

$$I(x, y) = p_{v,w} \frac{2+v-x}{2} \frac{2+w-y}{2} + p_{v,w+2} \frac{2+v-x}{2} \frac{y-w}{2} + p_{v+2,w} \frac{x-v}{2} \frac{2+w-y}{2} + p_{v+2,w+2} \frac{x-v}{2} \frac{y-w}{2}, \quad (11)$$

where $(v, w) \in G$ is the coordinate such that (x, y) lies in the (v, w) -interpolation region, that is $(x, y) \in [v, v+2] \times [w, w+2]$. We use v and w as grid indices in the context of the interpolation I . If $p_{v,w}$ has no defined value because (v, w) is out of range for the image, we set $p_{v,w}$ to 0.

We start by giving a procedure to calculate constraints of a single pixel (i, j) for a single color channel, after which we present an iterative procedure to refine that constraint. The inverse image is then obtained by following this procedure for every pixel in every color channel. We illustrate the steps in Section 6.1 using the example of a rotated image \mathbf{x}' (Fig. 2a).

The attacker transformed the original image \mathbf{x} (Fig. 2d) using T_γ^I for $\gamma \in \Gamma$ and therefore obtained the pixel values $p'_{i',j'}$ of the transformed image \mathbf{x}' by evaluating $p'_{i',j'} = I \circ T_\gamma^{-1}(i', j')$. The interpolation I uses the pixel values $p_{i,j}$ of \mathbf{x} . The following steps invert this relation for every coordinate (i, j) :

Step 1 For every $(i', j') \in G$, we over-approximate the region the pixel value $p'_{i', j'}$ could have been interpolated from, which is $c_{i', j'} := T_\Gamma^{-1}(i', j')$, $C := \{c_{i', j'} := T_\Gamma^{-1}(i', j') \mid (i', j') \in G\}$. In practice, only a finite subset of C is used. In App. B, we shows how to calculate this subset efficiently.

Step 2 The interpolation I is defined piecewise per (v, w) -interpolation region $[v, v+2] \times [w, w+2]$, so the algebraic form of I , Eq. (11) holds for each interpolation region separately. For every interpolation region cornering (i, j) that $c_{i', j'}$ intersects with, the pixel value $p'_{i', j'}$ yields constraints for value $p_{i, j}$. Here, we describe just the constraint $q_{i, j}^*$ associated with the (i, j) -interpolation region; others $((i-2, j-2), (i-2, j), (i, j-2))$ work analogously. First, for every $c_{i', j'} \in C$ we calculate its intersection with the (i, j) -interpolation region, yielding

$$[x_l, x_u] \times [y_l, y_u] := c_{i', j'} \cap [i, i+2] \times [j, j+2].$$

We can plug this into the interpolation I , where we instantiate $(v, w) \leftarrow (i, j)$, resulting into

$$p'_{i', j'} \in I([x_l, x_u], [y_l, y_u]) = p_{i, j} \frac{2+i-x_l}{2} \frac{2+j-y_l}{2} + p_{i+2, j} \frac{2+i-x_l}{2} \frac{[y_l, y_u]-j}{2} + p_{i+2, j+2} \frac{[x_l, x_u]-i}{2} \frac{2+j-y_l}{2} + p_{i, j+2} \frac{[x_l, x_u]-i}{2} \frac{[y_l, y_u]-j}{2}. \quad (12)$$

Next, we solve for the pixel value of interest $p_{i, j}$. Then, we replace all other three pixel values $p_{i+2, j}$, $p_{i+2, j+2}$, and $p_{i, j+2}$ with the (trivial) $[0, 1]$ constraint, covering all possible pixel values. While this results into sound constraints for $p_{i, j}$, instantiating $[x_l, x_u]$ and $[y_l, y_u]$ with its corner (x, y) furthest from (i, j) , yields still a sound but more precise constraint $q_{i, j}^*$ for $p_{i, j}$. Here, this amounts to $x \leftarrow x_u$ and $y \leftarrow y_u$. App. B presents a detailed explanation of the derivation. The result is

$$q_{i, j}^* = [p'_{i', j'} - (\frac{2+i-x_u}{2} \frac{y_u-j}{2} + \frac{x_u-i}{2} \frac{2+j-y_u}{2} + \frac{x_u-i}{2} \frac{y_u-j}{2}), p'_{i', j'}] (\frac{2+i-x_u}{2} \frac{2+j-y_u}{2})^{-1}.$$

Step 3 In order to be sound, we need to take the union over $q_{i-2, j-2}^*$, $q_{i-2, j}^*$, $q_{i, j-2}^*$, $q_{i, j}^*$ for each $c_{i', j'}$. To gain precision, we can intersect all of those unions and finally, we can intersect this constraint with the trivial one, $[0, 1]$, resulting in the final pixel constraint for pixel $p_{i, j}$:

$$p_{i, j} \in [0, 1] \cap \left(\bigcap_{c_{i', j'} \in C} p_{i-2, j-2}^*(c_{i', j'}) \sqcup q_{i-2, j}^*(c_{i', j'}) \sqcup q_{i, j-2}^*(c_{i', j'}) \sqcup q_{i, j}^*(c_{i', j'}) \right), \quad (13)$$

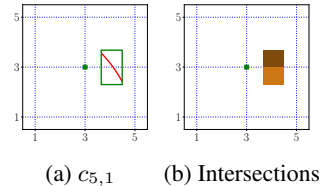
where \sqcup denotes the *join* operation, that is $[a, b] \sqcup [c, d] := [\min(a, c), \max(b, d)]$. If the intersection of $c_{i', j'}$ with the respective (v, w) -interpolation region is empty, we omit $q_{v, w}^*$ in Eq. (13).

In Section 5.2, we splitted Γ into Γ_k . It often happens that one of the resulting intervals is empty. Then we know for sure that γ lies in a different Γ_k , speeding up the process substantially.

Refined Inverse The constraints can be refined by following the same steps as for calculating the inverse, but instead of replacing the (unknown) pixel values in Eq. (12) with $[0, 1]$, we replace them with the intervals calculated previously. However, replacing $[x_l, x_u] \times [y_l, y_u]$ with the corner furthest away from (i, j) would be unsound. To be sound, one needs to consider all 4 corners of every non-empty intersection $[x_l, x_u] \times [y_l, y_u]$ and join all interval constraints. Similarly, we use the previously calculated constraint for $p_{i, j}$ instead of $[0, 1]$ in Eq. (13). This procedure can be repeated to further increase precision. The final result after applying the refinement 10 times is shown in Fig. 2c representing the lower (left) and upper (right) interval bound for all pixels.

6.1 Example

We calculate the constraint for pixel $(3, 3)$ of the original image (Fig. 2d), depicted as the green dot in Fig. 3 under the assumption $\gamma \in [23^\circ, 26^\circ]$. We elaborate the constraints that pixel $(5, 1)$ of the rotated image (Fig. 2a) yields for pixel $(3, 3)$ of the original image.



Step 1 We illustrate the calculation of the set C for $c_{5,1} := R_{[23^\circ, 26^\circ]}^{-1} \begin{pmatrix} 5 \\ 1 \end{pmatrix} = \begin{pmatrix} [4.06, 4.21] \\ [2.85, 3.11] \end{pmatrix}$. The result is depicted as the green box in Fig. 3a enclosing the red arc. The red shows the precise set of coordinates where the pixel value $p'_{5,1}$ could have been interpolated from the original image x .

Figure 3: To improve presentation, the red arc is $3 \times$ longer.

Step 2 The only non-empty intersections of $c_{5,1}$ with interpolation regions (blue squares in Fig. 3), cornering (3, 3) are the (3, 1) and the (3, 3)-interpolation regions, hence we omit $q_{1,1}^*$ and $q_{1,3}^*$. The intersection with the (3, 3)-interpolation region yields $[x_l, x_u] = [4.06, 4.21]$ and $[y_l, y_u] = [3, 3.11]$ (dark brown rectangle in Fig. 3b), hence at the furthest corner $(x, y) = (4.21, 3.11)$, we get

$$q_{3,3}^* = [0.73, 2.48] = [p'_{5,1} - (\frac{5-x}{2} \frac{y-3}{2} + \frac{x-3}{2} \frac{5-y}{2} + \frac{x-3}{2} \frac{y-3}{2}), p'_{5,1}] (\frac{5-x}{2} \frac{5-y}{2})^{-1},$$

and the intersection with the (3, 1)-interpolation region yields $[x_l, x_u] = [4.06, 4.21]$ and $[y_l, y_u] = [2.85, 3]$ (light brown rectangle in Fig. 3b), hence at the furthest corner $(x, y) = (4.21, 2.85)$, we get

$$q_{3,1}^* = [0.72, 2.48] = [p'_{5,1} - (\frac{5-x}{2} \frac{3-y}{2} + \frac{x-3}{2} \frac{3-y}{2} + \frac{x-3}{2} \frac{y-1}{2}), p'_{5,1}] (\frac{5-x}{2} \frac{y-1}{2})^{-1}.$$

Step 3 The join $q_{3,1}^* \sqcup q_{3,3}^*$ yields $[0.72, 2.48]$. After intersecting this with $[0, 1]$ and the constraints from the other $c_{i',j'} \in C$ (as in Eq. (13)), we are left with the final result $p_{3,3} \in [0.73, 1]$.

The final result of the inverse calculation for all pixels is shown in Fig. 2b representing the lower (left) and upper (right) interval bounds for all pixels. The iterative refinement is shown in Fig. 2c.

7 Experimental Evaluation

We now present our extensive evaluation of the different defenses discussed so far.

7.1 Instantiation in Practice

In Section 4 we showed how to certify robustness of g_E to T_γ^I , obtained from Eq. (1) with $f := h_E \circ I$ and $\psi_\beta := T_\beta$. Because in practice $h_E \circ I$ and T_β cannot be evaluated as I and T_β are not available independently, in order to evaluate g_E in practice, we need to re-write it as follows:

$$\begin{aligned} g_E(\mathbf{x}) &= \arg \max_c \mathbb{P}_{\beta \sim \mathcal{N}(0, \sigma^2 \mathbb{1})} ((h_E \circ I) \circ T_\beta(\mathbf{x}) = c) \\ &= \arg \max_c \mathbb{P}_{\beta \sim \mathcal{N}(0, \sigma^2 \mathbb{1})} (h_E \circ (I \circ T_\beta)(\mathbf{x}) = c) =: g(\mathbf{x}), \end{aligned}$$

which is an instantiation of Eq. (1) with $f := h_E$ and $\psi_\beta := T_\beta^I$, both of which can be evaluated.

Further, as the probability in Eq. (1) cannot be computed exactly, in practice we use the approximation introduced in Cohen et al. [7]: by taking n samples around a given \mathbf{x} with standard deviation σ , we can obtain $g(\mathbf{x})$ and the corresponding robustness radius r with confidence $1 - \alpha$. Here, n can be too small to make a statement with confidence $1 - \alpha$, in which case the classifier abstains. Further, we let $\sigma_\gamma, \alpha_\gamma, n_\gamma, r_\gamma$ and $\sigma_\delta, \alpha_\delta, n_\delta, r_\delta$ denote the parameters and radius required to use Theorem 3.2 and Theorem 3.1 in practice, respectively. As we require h_E to be robust with radius at least E , we treat abstentions and certification with $r_\delta < E$ as incorrect classifications. As the certificate of h_E can be inaccurate with probability α_δ , we need to further increase $\rho = \alpha_E$ (from Section 4) to $\rho = \alpha_\delta + \alpha_E$.

7.2 Setup

All experiments were performed on a machine with 2 GeForce RTX 2080 Tis and an Intel(R) Core(TM) i9-9900K CPU. As base classifiers b we utilize neural networks in PyTorch [28], using robustness [29] for training. Further, we implemented the interval analysis (cf. §5 and §6) of the interpolation error and inverse computation in C++/CUDA.

In our evaluation we consider rotations R_γ^I by γ degrees and translations Δ_γ^I by $\gamma \in \mathbb{R}^2$ with bilinear interpolation I . Here, we allow the adversary to choose $\gamma \in \Gamma$. For a scalar $\Gamma_\pm \in \mathbb{R}^{\geq 0}$, we permit $\Gamma := [-\Gamma_\pm, \Gamma_\pm]$ for rotations and $\Gamma := [-\Gamma_\pm, \Gamma_\pm]^2$ for translations. All estimates of E include interpolation errors as well as 8-bit representation (“rounding”) errors.

We evaluate on ImageNet [30], Restricted ImageNet (RImageNet)[31], a subset of ImageNet with 10 classes, CIFAR-10 (CIFAR) [32], and MNIST [33]. For the base classifier, in Section 7.3 we use standard models without any additional training, while in the other sections we use models trained with data augmentation (transformations, ℓ^2 -noise) using [8].

In §7.4 and §7.5, we apply a circular or rectangular vignette for rotation and translation respectively, to reduce error estimates in areas of the image where information is lost. We also apply a Gaussian blur prior to classification to further reduce the high-frequency components of the interpolation error. App. D contains further details on preprocessing, model training and parameters. Note that pre-processing does not impact the theoretical guarantees as long as it is consistently applied. We provide an ablation study regarding vignetting and Gaussian blur in App. F. Additional experiments, including other interpolation methods or audio classification are provided in App. E, highlighting the generality of our methods. Throughout the section we use $\alpha_\delta = 0.002$ and $\alpha_\gamma = 0.01$ for confidences.

Table 1: Evaluation of BASESPT. We obtain Acc for b on the test set and evaluate adv. Acc. on 3000 images obtained by the worst-of-100 attack. t denotes the average run time of g .

Dataset	T^I	Γ_\pm	Acc.		adv. Acc.		t [s]
			b	b	g		
MNIST	R^I	30°	0.99	0.73	0.99	0.97	
CIFAR	R^I	30°	0.91	0.26	0.85	0.95	
ImageNet	R^I	30°	0.76	0.56	0.76	5.43	
MNIST	Δ^I	4	0.99	0.03	0.53	0.86	
CIFAR	Δ^I	4	0.91	0.44	0.79	0.95	
ImageNet	Δ^I	20	0.76	0.65	0.75	6.70	

7.3 BASESPT

We can quickly obtain a well-motivated but empirical defense by instantiating Theorem 3.2 with $\psi_\beta := T_\beta^I$ and ignoring both the interpolation error Eq. (3) and the construction in Section 4. Table 1 shows results on an undefended classifier b and the BASESPT smoothed version g . Here Acc. is obtained over the whole dataset. To evaluate adv. Acc. we use the worst-of- k proposed by Engstrom et al. [2], which returns the γ yielding the highest cross-entropy loss out of k randomly sampled $\gamma \sim \mathcal{U}(\Gamma)$. We apply worst-of- k to 1000 images and produce 3 attacked images each, resulting 3000 samples on which we then evaluate b and g . For g , the average inference time per image t is generally fast, where most time is spent on sampling transformations. The actual inference, invoking b on the samples, is not slowed down as all samples fit into a single batch. In this section we use $n_\gamma = 1000$, $\sigma_\gamma = \Gamma_\pm$. We do not obtain certificates here as the assumptions of Theorem 3.2 are violated. However, we investigate in App. E if the certification radius holds practically.

7.4 DISTSPT

In Table 2, we show our results for DISTSPT with rotations. We restrict the attacker model to $\Gamma_\pm = 30^\circ$. To obtain E , we used 1000 images from the training set for MNIST and CIFAR and 700000 for ImageNet to compute the bounded interpolation error, Eq. (7). The largest of these is shown as ϵ_{\max} . We choose E roughly 1.5 times larger. For small images this bound can be computed quickly. However for large images (ImageNet), the optimization over γ for many images is computationally expensive. Thus, for ImageNet we replace the max in Eq. (6) with the maximum over 10 samples $\gamma \in \mathcal{U}(\Gamma)$. This formally restricts the certificate to only hold against random attacks. However, if sufficient computational resources are available, the max method can still be applied (we empirically find the method to obtain similar values). On (R)ImageNet (variable image size) we resize all images so that the short side is 512 pixel prior to applying transformations. As RImageNet is a subset of ImageNet we use the same bound. Here we use $\alpha_E = 0.001$ and expect q_E to be close to 1 for all datasets. We show $q_E \geq 0.99$ with confidence 0.999 by using 1000 samples for x and 8000 for β (and correction for possible test errors over β). Subsequently, we evaluate the accuracy of b and g . For b we use the whole test set, while for g we use 1000 samples. We clip obtained robustness radius r_γ to Γ (indicated by †) in order to provide a sound guarantee. Additionally we evaluate rotations on MNIST with $\Gamma_\pm = 180^\circ$, where we also obtain $E = 0.55$ (thus not further restricting the 30° results) and show robustness for 0.84 of images with an median r_γ of 180^\dagger . Finally, we evaluate translations on MNIST ($E = 0.72$, $\Gamma_\pm = 2$) and achieve 0.95 certified accuracy with r_γ of 2.41 and cover all of Γ for the 25th and 50th percentile respectively. The results on RImageNet indicate that the limiting factor for our method is the robustness and quality of the base classifier, not the size of the image. $\sigma_\delta = 0.3$ for MNIST, $\sigma_\delta = 0.25$ for CIFAR and $\sigma_\delta = 0.5$ for (R)ImageNet.

Comparison to other work Balunovic et al. [11] certifies model accuracy on the test set and thus provides a distributional bound. On MNIST they report 87.01% of certified accuracy for rotations with $\pm 30^\circ$ (35s per image), which with further refinement (at cost of run time) can be increased to 97%, and for translations with ± 2 pixels 76.30% (263 s per image). We significantly improve upon

Table 2: Evaluation of DISTSPT for $T^I := R^I$. ϵ_{\max} is computed on the training set. We show the test set accuracy of b , certified accuracy of g and distribution of the obtained certification radius r_γ , along with the average run time t and the number of used samples n_γ, n_δ .

Dataset	ϵ_{\max}	E	Acc.		r_γ percentile			t [s]	n_γ	n_δ
			b	g	25 th	50 th	75 th			
MNIST	0.36	0.55	0.99	0.97	38.24	44.90	54.07	2.55	200	200
CIFAR	0.51	0.77	0.72	0.54	1.87	18.37	30.00 [†]	23.93	50	10000
CIFAR	0.51	0.77	0.72	0.56	6.79	25.44	30.00 [†]	96.45	200	10000
RImageNet	0.91	1.20	0.84	0.73	12.15	30.00 [†]	30.00 [†]	85.58	50	2000
ImageNet	0.91	1.20	0.32	0.23	1.86	16.17	26.31	85.58	50	2000

this. On CIFAR they certify rotation up to 10° for 62.51%, but unlike our work does not scale to larger image sizes and models, such as a ResNet-50 on ImageNet. We provide further comparison with Balunovic et al. [11] in App. F. Results, similar to ours, are obtained in [13]. However, they require access to the unperturbed image to evaluate their classifier, which is practically not feasible. Pei et al. [9] certify $\pm 2^\circ$ in 714s per image on ImageNet. However, they report the failure rate per image and use nearest-neighbor interpolation and are thus incomparable.

7.5 INDIVSPT

Finally, we evaluate our certified online defense: where we compute E on the given input. The bound computed by interval analysis is always sound, but may be quite large due to the loss of precision inherent in interval analysis. We show results for MNIST and discuss challenges on larger datasets in App. C. To this end, we attack images as in Section 7.3, and subsequently apply INDIVSPT. We use the worst-of-100 attack on a base classifier b to obtain a set of attacked images. To these images we then apply INDIVSPT. For rotations ($\Gamma_\pm = 10, \sigma_\gamma = 30, n_\gamma = 2000$, 3 attacks per image, 1000 images) we fix $E = 0.7$ and use 100 samples of β to obtain the correct α_E (Eq. (9)). g was correct on 98% of attacked images. For 76% of these we could certify that the attacked image that classifies the same as the original. The analysis of E took on average 0.95s and the randomized smoothing 44.81s. For translation ($\Gamma_\pm = 1, \sigma_\gamma = 1.5, \sigma_\delta = 0.3, n_\gamma = 200$, 3 attacks per image, 100 images) we started with $E = 0.35$. g classified 71% of attacked images correctly and could certify 88% of these while on average taking 58.40s for analysis and 100.47s for smoothing per image. The reason for the higher run time is that compared to rotation less possible inverses can be discarded. We use 10 refinement steps and $n_\delta = 200$ for both rotations and translations. In both settings the certification rate can be increased further, while also increasing evaluation time, by increasing $n_\gamma, \sigma_\gamma, n_\delta, \sigma_\delta$.

Limitations & Generalization While we showcased translation and rotation, our approach is not limited to these transformations or to specific interpolation methods. BASESPT and DISTSPT can be directly adapted to other transformations, interpolation schemes or domains such as audio (see App. E). INDIVSPT can also be adapted but requires additional care. Generally, Theorem 3.2 can be applied to all parameterized data transformations that are additive in the parameter space. If this holds up to a small error, as discussed here, DISTSPT and INDIVSPT can be applied. While many data transformations, e.g., image scaling are additive in their parameter space, their compositions are often not (e.g., rotation and translation). As we are most limited by the ℓ^p -robustness of b , any gains in ℓ^p certification will directly improve our method. Further, INDIVSPT can incur a large loss of precision in the inverse computation. Improving this directly increases the applicability of the method.

8 Conclusion

We presented a generalization of randomized smoothing to image transformations, a challenging task as image transformations do not compose. Based on this generalization, we presented two certified defenses with distributional and individual guarantees (which relies on efficient inverse computation). Finally, we showed both defenses are applicable in an online setting and realistic datasets.

9 Broader Impact

In general, methods from artificial intelligence can be applied in beneficial and malicious ways. While this poses a threat in itself, verification techniques provide formal guarantees for the robustness of the model, independently of the intended use case. Certification techniques could therefore distinguish a potentially unstable model from a stable one in safety critical settings, e.g., autonomous driving. However, especially for regulators, it is of utter importance to understand the certified properties of different certification methods precisely, so to avoid legal model deployment in safety critical applications based on misconceptions.

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Supplementary Material for Certified Defense to Image Transformations via Randomized Smoothing

A Proof of Theorem 3.2

We now proceed to proof Theorem 3.2. We achieve this by first proofing an auxiliary Theorem and Lemma, and then instantiating as a special case Theorem 3.2 of these slightly more general results.

Theorem A.1. *Let $\mathbf{x} \in \mathbb{R}^n$, $f : \mathbb{R}^m \rightarrow \mathcal{Y}$ be a classifier, $\psi_\beta : \mathbb{R}^n \rightarrow \mathbb{R}^m$ a composable transformation for $\beta \sim \mathcal{N}(0, \Sigma)$ with a symmetric, positive-definite covariance matrix $\Sigma \in \mathbb{R}^{m \times m}$. If*

$$\mathbb{P}_\beta(f \circ \psi_\beta(\mathbf{x}) = c_A) = p_A \geq \underline{p}_A \geq \overline{p}_B \geq p_B = \max_{c_B \neq c_A} \mathbb{P}_\beta(f \circ \psi_\beta(\mathbf{x}) = c_B),$$

then $g \circ \psi_\gamma(\mathbf{x}) = c_A$ for all γ satisfying

$$\sqrt{\gamma^T \Sigma^{-1} \gamma} < \frac{1}{2}(\Phi^{-1}(\underline{p}_A) - \Phi^{-1}(\overline{p}_B)) =: r_\gamma.$$

Proof. The assumption is

$$\mathbb{P}((f \circ \psi_\beta)(\mathbf{x}) = c_A) = p_A \geq \underline{p}_A \geq \overline{p}_B \geq p_B = \mathbb{P}((f \circ \psi_\beta)(\mathbf{x}) = c_B).$$

By the definition of g we need to show that

$$\mathbb{P}((f \circ \psi_{\beta+\gamma})(\mathbf{x}) = c_A) \geq \mathbb{P}((f \circ \psi_{\beta+\gamma})(\mathbf{x}) = c_B).$$

We define the set $A := \{\mathbf{z} \mid \gamma^T \Sigma^{-1} \mathbf{z} \leq \sqrt{\gamma^T \Sigma^{-1} \gamma} \Phi(\underline{p}_A)\}$. We claim that for $\beta \sim \mathcal{N}(0, \Sigma)$, we have

$$\mathbb{P}(\beta \in A) = \underline{p}_A \tag{14}$$

$$\mathbb{P}(f \circ \psi_{\beta+\gamma}(\mathbf{x}) = c_A) \geq \mathbb{P}(\beta + \gamma \in A). \tag{15}$$

First, we show that Eq. (14) holds.

$$\begin{aligned} \mathbb{P}(\beta \in A) &= \mathbb{P}(\gamma^T \Sigma^{-1} \beta \leq \sqrt{\gamma^T \Sigma^{-1} \gamma} \Phi(\underline{p}_A)) \\ &= \mathbb{P}(\gamma^T \Sigma^{-1} \mathcal{N}(0, \Sigma) \leq \sqrt{\gamma^T \Sigma^{-1} \gamma} \Phi(\underline{p}_A)) \\ &= \mathbb{P}(\gamma^T \sqrt{\Sigma^{-1}} \mathcal{N}(0, \mathbb{1}) \leq \sqrt{\gamma^T \Sigma^{-1} \gamma} \Phi(\underline{p}_A)) \\ &= \mathbb{P}(\mathcal{N}(0, \gamma^T \Sigma^{-1} \gamma) \leq \sqrt{\gamma^T \Sigma^{-1} \gamma} \Phi(\underline{p}_A)) \\ &= \mathbb{P}(\sqrt{\gamma^T \Sigma^{-1} \gamma} \mathcal{N}(0, 1) \leq \sqrt{\gamma^T \Sigma^{-1} \gamma} \Phi(\underline{p}_A)) \\ &= \mathbb{P}(\mathcal{N}(0, 1) \leq \Phi(\underline{p}_A)) \\ &= \Phi(\Phi^{-1}(\underline{p}_A)) \\ &= \underline{p}_A \end{aligned}$$

Thus Eq. (14) holds. Next we show that Eq. (15) holds. For a random variable $v \sim \mathcal{N}(\mu_v, \Sigma_v)$ we write $p_v(z)$ for the evaluation of the Gaussian cdf at point z .

$$\begin{aligned}
& \mathbb{P}(f \circ \psi_{\beta+\gamma}(x) = c_A) - \mathbb{P}(\beta + \gamma \in A) \\
&= \int_{\mathbb{R}^d} [f \circ \psi_{\mathbf{z}} = c_A] p_{\beta+\gamma}(z) dz - \int_A p_{\beta+\gamma}(z) dz \\
&= \int_{\mathbb{R}^d \setminus A} [f \circ \psi_{\mathbf{z}}(x) = c_A] p_{\beta+\gamma}(z) dz + \int_A [f \circ \psi_{\mathbf{z}}(x) = c_A] p_{\beta+\gamma}(z) dz - \int_A p_{\beta+\gamma}(z) dz \\
&= \int_{\mathbb{R}^d \setminus A} [f \circ \psi_{\mathbf{z}}(x) = c_A] p_{\beta+\gamma}(z) dz + \int_A [f \circ \psi_{\mathbf{z}}(x) = c_A] p_{\beta+\gamma}(z) dz \\
&\quad - \left(\int_A [f \circ \psi_{\mathbf{z}}(x) = c_A] p_{\beta+\gamma}(z) dz + \int_A [f \circ \psi_{\mathbf{z}}(x) \neq c_A] p_{\beta+\gamma}(z) dz \right) \\
&= \int_{\mathbb{R}^d \setminus A} [f \circ \psi_{\mathbf{z}}(x) = c_A] p_{\beta+\gamma}(z) dz - \int_A [f \circ \psi_{\mathbf{z}}(x) \neq c_A] p_{\beta+\gamma}(z) dz \\
&\stackrel{\text{Lemma 1}}{\geq} t \left(\int_{\mathbb{R}^d \setminus A} [f \circ \psi_{\mathbf{z}}(x) = c_A] p_{\beta}(z) dz - \int_A [f \circ \psi_{\mathbf{z}}(x) \neq c_A] p_{\beta}(z) dz \right) \\
&= t \left(\int_{\mathbb{R}^d} [f \circ \psi_{\mathbf{z}}(x) = c_A] p_{\beta}(z) dz - \int_A p_{\beta}(z) dz \right) \\
&\stackrel{\text{Eq. (14)}}{\geq} 0.
\end{aligned}$$

Thus also Eq. (15) holds.

Next, we claim that for $B := \{z \mid \gamma^T \Sigma^{-1} \mathbf{z} \geq \sqrt{\gamma^T \Sigma^{-1} \gamma} \Phi^{-1}(1 - \overline{p_B})\}$ holds that

$$\mathbb{P}(f \circ \psi_{\beta}(x) = c_B) \leq \mathbb{P}(\beta \in B) \quad (16)$$

$$\mathbb{P}(f \circ \psi_{\beta+\gamma}(x) = c_B) \leq \mathbb{P}(\beta + \gamma \in B) \quad (17)$$

The proof for Eq. (16) and Eq. (17) are analogous to the proofs for Eq. (14) and Eq. (15).

Now we derive the conditions that lead to $\mathbb{P}(\beta + \gamma \in A) > \mathbb{P}(\beta + \gamma \in B)$:

$$\begin{aligned}
\mathbb{P}(\beta + \gamma \in A) &= \mathbb{P}\left(\gamma^T \Sigma^{-1}(\beta + \gamma) \leq \sqrt{\gamma^T \Sigma^{-1} \gamma} \Phi^{-1}(\underline{p_A})\right) \\
&= \mathbb{P}\left(\gamma^T \Sigma^{-1}(\Sigma^{\frac{1}{2}} \mathcal{N}(0, \mathbb{1}) + \gamma) \leq \sqrt{\gamma^T \Sigma^{-1} \gamma} \Phi^{-1}(\underline{p_A})\right) \\
&= \mathbb{P}\left(\gamma^T \sqrt{\Sigma^{-1}} \mathcal{N}(0, \mathbb{1}) + \gamma^T \Sigma^{-1} \gamma \leq \sqrt{\gamma^T \Sigma^{-1} \gamma} \Phi^{-1}(\underline{p_A})\right) \\
&= \mathbb{P}\left(\sqrt{\gamma^T \Sigma^{-1} \gamma} \mathcal{N}(0, \mathbb{1}) + \gamma^T \Sigma^{-1} \gamma \leq \sqrt{\gamma^T \Sigma^{-1} \gamma} \Phi^{-1}(\underline{p_A})\right) \\
&= \mathbb{P}\left(\mathcal{N}(0, \mathbb{1}) + \sqrt{\gamma^T \Sigma^{-1} \gamma} \leq \Phi^{-1}(\underline{p_A})\right) \\
&= \mathbb{P}\left(\mathcal{N}(0, \mathbb{1}) \leq \Phi^{-1}(\underline{p_A}) - \sqrt{\gamma^T \Sigma^{-1} \gamma}\right) \\
&= \Phi(\Phi^{-1}(\underline{p_A}) - \sqrt{\gamma^T \Sigma^{-1} \gamma})
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
\mathbb{P}(\beta + \gamma \in B) &= \mathbb{P}\left(\mathcal{N}(0, \mathbb{1}) \geq \Phi^{-1}(1 - \overline{p_B}) - \sqrt{\gamma^T \Sigma^{-1} \gamma}\right) \\
&= \Phi(\sqrt{\gamma^T \Sigma^{-1} \gamma} - \Phi^{-1}(1 - \overline{p_B}))
\end{aligned}$$

Thus, we get

$$\begin{aligned}
& \mathbb{P}(\beta + \gamma \in A) > \mathbb{P}(\beta + \gamma \in B) \\
\Leftrightarrow & \Phi(\Phi^{-1}(\underline{p_A}) - \sqrt{\gamma^T \Sigma^{-1} \gamma}) > \Phi(\sqrt{\gamma^T \Sigma^{-1} \gamma} - \Phi^{-1}(1 - \overline{p_B})) \\
\Leftrightarrow & \Phi^{-1}(\underline{p_A}) - \sqrt{\gamma^T \Sigma^{-1} \gamma} > \sqrt{\gamma^T \Sigma^{-1} \gamma} - \Phi^{-1}(1 - \overline{p_B}) \\
\Leftrightarrow & \Phi^{-1}(\underline{p_A}) + \Phi^{-1}(1 - \overline{p_B}) > 2\sqrt{\gamma^T \Sigma^{-1} \gamma} \\
\Leftrightarrow & \frac{1}{2}(\Phi^{-1}(\underline{p_A}) - \Phi^{-1}(\overline{p_B})) > \sqrt{\gamma^T \Sigma^{-1} \gamma}.
\end{aligned}$$

□

Next we show the lemma used in the proof.

Lemma 1. *There exists $t > 0$ such that $p_{\beta+\gamma}(z) \leq p_{\beta}(z) \cdot t$ for all $z \in A$. And further $p_{\beta+\gamma}(z) > p_{\beta}(z) \cdot t$ for all $z \in \mathbb{R}^d \setminus A$.*

Proof.

$$\begin{aligned} \frac{p_{\beta+\gamma}(z)}{p_{\beta}(z)} &= \exp\left(-\frac{1}{2}(\mathbf{z} - \gamma)^T \Sigma^{-1}(\mathbf{z} - \gamma) + \frac{1}{2}\mathbf{z}^T \Sigma^{-1}\mathbf{z}\right) \\ &= \exp\left(-\frac{1}{2}\mathbf{z}^T \Sigma^{-1}\mathbf{z} + \mathbf{z}^T \Sigma^{-1}\gamma - \frac{1}{2}\gamma^T \Sigma^{-1}\gamma + \frac{1}{2}\mathbf{z}^T \Sigma^{-1}\mathbf{z}\right) \\ &= \exp\left(\mathbf{z}^T \Sigma^{-1}\gamma - \frac{1}{2}\gamma^T \Sigma^{-1}\gamma\right) \end{aligned}$$

What is the lowest t if it exists such that $\frac{p_{\beta+\gamma}(z)}{p_{\beta}(z)} \leq t$?

$$\begin{aligned} \frac{p_{\beta+\gamma}(z)}{p_{\beta}(z)} &\leq t \\ \Leftrightarrow \exp\left(\mathbf{z}^T \Sigma^{-1}\gamma - \frac{1}{2}\gamma^T \Sigma^{-1}\gamma\right) &\leq t \\ \Leftrightarrow \mathbf{z}^T \Sigma^{-1}\gamma - \frac{1}{2}\gamma^T \Sigma^{-1}\gamma &\leq \log t \\ \Leftrightarrow \mathbf{z}^T \Sigma^{-1}\gamma &\leq \log t + \frac{1}{2}\gamma^T \Sigma^{-1}\gamma \end{aligned}$$

Because $z \in A$, we know that

$$\mathbf{z}^T \Sigma^{-1}\gamma \leq \sqrt{\gamma^T \Sigma^{-1}\gamma} \Phi^{-1}(\underline{p}_A).$$

Does there exist a t such that both upper bound coincide? Yes, namely

$$t = \exp\left(\sqrt{\gamma^T \Sigma^{-1}\gamma} \Phi^{-1}(\underline{p}_A) - \frac{1}{2}\gamma^T \Sigma^{-1}\gamma\right).$$

□

Lemma 2. *If we evaluate on a proxy classifier f' instead of f , behaving with probability $(1 - \rho)$ the same as f and with probability ρ differently than f and if*

$$\mathbb{P}_{\beta, f'}(f' \circ \psi_{\beta}(x) = c_A) \geq \underline{p}'_A \geq \overline{p}'_B \geq \max_{c_B \neq c_A} \mathbb{P}_{\beta, f'}(f' \circ \psi_{\beta}(x) = c_B),$$

then $g \circ \psi_{\gamma}(\mathbf{x}) = c_A$ for all γ satisfying

$$\|\gamma\|_2 < \frac{\sigma}{2}(\Phi^{-1}(\underline{p}'_A - \rho) - \Phi^{-1}(\overline{p}'_B + \rho)).$$

Proof. By applying the union bound we can relate the output probability p of f for a class c with the output probability of f' and p' :

$$\begin{aligned} p' &:= \mathbb{P}_{\beta, f'}(f' \circ \psi_{\beta}(x) = c) \\ &= \mathbb{P}_{\beta, f'}((f \circ \psi_{\beta}(x) = c) \vee (f' \text{ error})) \\ &\leq \mathbb{P}_{\beta}(f \circ \psi_{\beta}(x) = c) + \mathbb{P}_{f'}(f' \text{ error}) \\ &= p + \rho \end{aligned}$$

Thus we can obtain new bounds $\underline{p}_A \geq \underline{p}'_A - \rho$ and $\overline{p}_B \leq \overline{p}'_B + \rho$ from \underline{p}'_A and \overline{p}'_B measured on f' . Plugging these bounds in Theorem 3.2 yields the result. □

We now show Theorem 3.2 (restarted below): Setting $\Sigma = \sigma^2 \mathbb{1}$ in Theorem A.1 directly recovers Theorem 3.2 up to the last sentence, which in turn is a direct consequence of Lemma 2.

Theorem (Theorem 3.2 restated). *Let $\mathbf{x} \in \mathbb{R}^m$, $f : \mathbb{R}^m \rightarrow \mathcal{Y}$ be a classifier and $\psi_{\beta} : \mathbb{R}^m \rightarrow \mathbb{R}^m$ be a composable transformation as above. If*

$$\mathbb{P}_{\beta}(f \circ \psi_{\beta}(\mathbf{x}) = c_A) \geq \underline{p}_A \geq \overline{p}_B \geq \max_{c_B \neq c_A} \mathbb{P}_{\beta}(f \circ \psi_{\beta}(\mathbf{x}) = c_B),$$

then $g \circ \psi_{\gamma}(\mathbf{x}) = c_A$ for all γ satisfying $\|\gamma\|_2 \leq \frac{\sigma}{2}(\Phi^{-1}(\underline{p}_A) - \Phi^{-1}(\overline{p}_B)) =: r_{\gamma}$. Further, if g is evaluated on a proxy classifier f' that behaves like f with probability $1 - \rho$ and else returns an arbitrary answer, then $r_{\gamma} := \frac{\sigma}{2}(\Phi^{-1}(\underline{p}_A - \rho) - \Phi^{-1}(\overline{p}_B + \rho))$.

B Inverse and Refinement

B.1 Details for Step 2

In this section, we elaborate on the details of Step 2 in Section 6. We consider the intersection of $c_{i',j'}$ with the (i, j) -interpolation region, $[x_l, x_u] \times [y_l, y_u] := c_{i',j'} \cap [i, i+2] \times [j, j+2]$. This yields,

$$p'_{i',j'} \in I([x_l, x_u], [y_l, y_u]) = p_{i,j} \frac{2+i-[x_l, x_u]}{2} \frac{2+j-[y_l, y_u]}{2} + p_{i,j+2} \frac{2+i-[x_l, x_u]}{2} \frac{[y_l, y_u]-j}{2} \\ + p_{i+2,j} \frac{[x_l, x_u]-i}{2} \frac{2+j-[y_l, y_u]}{2} + p_{i+2,j+2} \frac{[x_l, x_u]-i}{2} \frac{[y_l, y_u]-j}{2}.$$

Next, we solve for the pixel value $p_{i,j}$ to get the constraint $q_{i,j}^*$:

$$q_{i,j}^* = \left(p'_{i',j'} - p_{i,j+2} \frac{2+i-[x_l, x_u]}{2} \frac{[y_l, y_u]-j}{2} - p_{i+2,j} \frac{[x_l, x_u]-i}{2} \frac{2+j-[y_l, y_u]}{2} \right. \\ \left. - p_{i+2,j+2} \frac{[x_l, x_u]-i}{2} \frac{[y_l, y_u]-j}{2} \right) \left(\frac{2+i-[x_l, x_u]}{2} \frac{2+j-[y_l, y_u]}{2} \right)^{-1}$$

Because we don't have any constraints for the pixel values $p_{i+2,j}$, $p_{i,j+2}$ and $p_{i+2,j+2}$, we replace their values by the $[0, 1]$ constraint and obtain:

$$q_{i,j}^* = \left(p'_{i',j'} - \left(\frac{2+i-[x_l, x_u]}{2} \frac{[y_l, y_u]-j}{2} - \frac{[x_l, x_u]-i}{2} \frac{2+j-[y_l, y_u]}{2} \right. \right. \\ \left. \left. - \frac{[x_l, x_u]-i}{2} \frac{[y_l, y_u]-j}{2} \right) [0, 1] \right) \left(\frac{2+i-[x_l, x_u]}{2} \frac{2+j-[y_l, y_u]}{2} \right)^{-1}$$

Instead of using standard interval analysis to compute the constraints for $p_{i,j}$, we use the following more efficient transformer: We replace $[x_l, x_u]$ and $[y_l, y_u]$ with the coordinate $(x, y) \in [x_l, x_u] \times [y_l, y_u]$ furthest away from (i, j) , which is in our case (x_u, y_u) to obtain

$$q_{i,j}^* = \left(p'_{i',j'} - \left(\frac{2+i-x_u}{2} \frac{y_u-j}{2} + \frac{x_u-i}{2} \frac{2+j-y_u}{2} + \frac{x_u-i}{2} \frac{y_u-j}{2} \right) [0, 1] \right) \left(\frac{2+i-x_u}{2} \frac{2+j-y_u}{2} \right)^{-1} \\ = \left[p'_{i',j'} - \left(\frac{2+i-x_u}{2} \frac{y_u-j}{2} + \frac{x_u-i}{2} \frac{2+j-y_u}{2} + \frac{x_u-i}{2} \frac{y_u-j}{2} \right), p'_{i',j'} \right] \left(\frac{2+i-x_u}{2} \frac{2+j-y_u}{2} \right)^{-1}.$$

B.2 Algorithm

Here, we present the algorithm used to compute the inverse of a transformation. For the construction of the set C , we iterate only over the index set P . The set P is constructed to include all points in G that could yield non empty intersections $c_{i',j'}$, thus this is just to speed up the evaluation and equivalent otherwise to the algorithm described in the main part.

Data: Image $x' \in \mathbb{R}^{m \times m}$, transform T , parameter range B , coordinates i, j

Result: Range for the pixel value $p_{i,j}$.

- 1 $N \leftarrow \begin{pmatrix} [i-2, i+2] \\ [j-2, j+2] \end{pmatrix}$
- 2 $\begin{pmatrix} [i', i'_u] \\ [j', j'_u] \end{pmatrix} \leftarrow T_B(N)$
- 3 $P \leftarrow \left\{ \begin{pmatrix} i' \\ j' \end{pmatrix} \mid i' \in \text{range}([i'_l], \dots, [i'_u], 2) \right. \\ \left. j' \in \text{range}([j'_l], \dots, [j'_u], 2) \right\}$
- 4 $C \leftarrow \left\{ c_{i',j'} := T_B^{-1} \left(\begin{pmatrix} i' \\ j' \end{pmatrix} \right) \cap N \mid c_{i',j'} \neq \emptyset, (i', j') \in P \right\}$
- 5 $p_{i,j} \leftarrow [0, 1] \cap \left(\bigcap_{c_{i',j'} \in C} q_{i-2,j-2}^*(c_{i',j'}) \cup q_{i,j-2}^*(c_{i',j'}) \cup q_{i-2,j}^*(c_{i',j'}) \cup q_{i,j}^*(c_{i',j'}) \right)$

Algorithm 1: Procedure to calculate the range for the pixel values of the inverse image.

B.3 Experimental Evaluation

To investigate the impact of refinement on the downstream error estimate we used 20 MNIST images, rotated each with 3 random angles and then proceeded to calculate the inverse. In the calculation we considered the range $\Gamma_{\pm} = 10$. We see that a low number of refinements have a large impact on the error but the returns become quickly diminishing. The impact on the run time of a single additional refinement step is negligible.

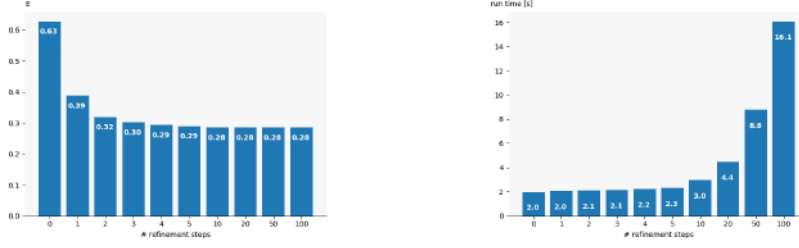


Figure 4: Interpolation and rounding error E as well as run time for different numbers of refinement steps.

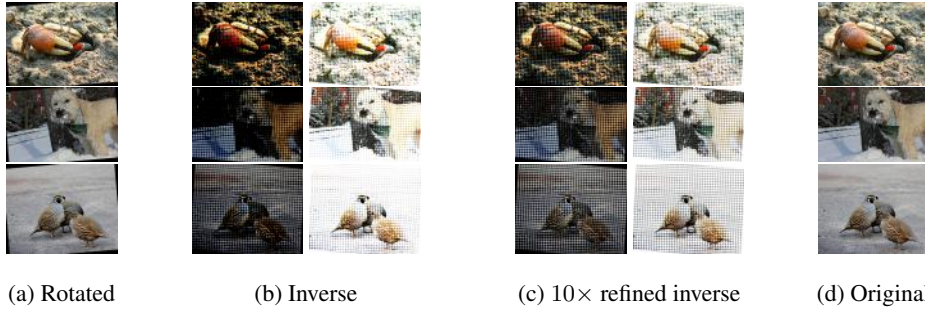


Figure 5: Computation of the inverse, analogous to Fig. 2, for images from ImageNet [30].

C Inverse for Rich Images

INDIVSPT performs poorly on large images, such as those from ImageNet as the inverse computation outlined in Section 6 produces a too large over-approximation of \mathbf{x} leading to E estimates of around 40, while manageable value would be ≤ 2 .

Fig. 5 shows the computed inverse for such images. We observe a pattern of artefacts in the inverse, where the pixel value can not be narrowed down sufficiently resulting in the large estimate of E . The result of the refined inverse is perfectly recognizable to a human observer (or a neural network), highlighting the promise of the algorithm for future applications.

D Experiment Details

D.1 Details for Section 7.3

To evaluate BASESPT we use the following classifiers. Note that Table 3 in App. E.1 contains results for further datasets:

MNIST [33] We trained a convolutional network consisting of $\text{CONV2D}(k, n)$, with $k \times k$ filter size, n filter channels and stride 1, batch norm BN [34], maximum pooling $\text{MAXPOOL}(k)$ on $k \times k$ grid, $\text{DROPOUT}(p)$ [35] with probability p and linear layers $\text{LIN}(a, b)$ from \mathbb{R}^a to \mathbb{R}^b .

CONV2D(5, 32), RELU, BN
 CONV2D(5, 32), RELU, MAXPOOL(2), DROPOUT(0.2)
 CONV2D(3, 64), RELU, BN
 CONV2D(3, 64), RELU, BN, MAXPOOL(2), DROPOUT(0.2)
 CONV2D(3, 128), RELU, BN
 CONV2D(1, 128), RELU, BN, FLATTEN
 LIN(128, 100), RELU
 LIN(100, 10)

We used data normalization for MNIST and trained for 180 epochs with SGD, starting from learning rate 0.01, decreasing it by a factor of 10 every 60 epochs. No other pre-processing was used.

FashionMNIST [36] We trained a ResNet-18 with data normalization. We trained for 180 epochs with SGD with an initial learning rate of 0.01, lowering it by a factor of 10 every 60 epochs.

CIFAR [32] & GTSRB [37] We trained a ResNet-18 with data normalization. We trained for 90 epochs with SGD with an initial learning rate of 0.1, lowering it by a factor of 10 every 30 epochs. We resized GTSRB images to $32 \times 32 \times 3$.

ImageNet [30] We used the pre-trained ResNet50 from torchvision: <https://pytorch.org/docs/stable/torchvision/models.html>.

D.2 Details for Section 7.4

In Section 7.4 we use the same architectures (with the exception of MNIST, for which we use a ResNet-18) as discussed in App. D.1, however we trained them to be robust to image transformations (rotation, translation) as well as ℓ^2 noise.

To train networks that are when applying randomized smoothing we utilize the training procedure SMOOTHADV_{PGD} as outlined in Salman et al. [8]. For each batch of samples we apply a randomized data augmentation, vignetting for rotation, and Gaussian blur. After this preprocessing we then apply SMOOTHADV_{PGD} with one noise sample using σ and 1 PGD pass (with step size 1; as in Salman et al. [8]) and then evaluate or train on the batch.

The intuition behind the Gaussian blur is that many artefacts, such as the interpolation error are have high frequencies. The blur acts as a low-pass filter and discards high frequency noise. This does not strongly impact the classification accuracy, but drastically reduces the error estimate and therefore the amount of noise that needs to be added for robust classification. The filter is parameterized by σ_b and the filter size s_b . Formally the filter is a convolution with a filter matrix $F \in \mathbb{R}^{s_b \times s_b}$. Each entry in F is filled with values of a two dimensional Gaussian distribution centered at the center of the matrix and evaluated at the center of the entry. Afterwards the matrix is normalized such that $\sum_{i,j} F_{i,j} = 1$.

In the error estimation and inference we use the same preprocessing as during training.

MNIST For MNIST we use a ResNet-18 (that takes a single color channel in the input layer), which we trained with $\sigma = 0.3$, PGD step size 0.2, batch size 1024, and initial learning rate 0.01 over 180 epochs, lowering the learning rate every 60 epochs. For data augmentation we used rotations in $[-30, 30]$, $[-180, 180]$ degrees and translations of $\pm 50\%$ for each model respectively. For the Gaussian blur we use $\sigma_b = 2.0$ with filter size $s_b = 5$.

CIFAR-10 & German Traffic Sign Recognition Benchmark (GTSRB) For both datasets we use a ResNet-18 trained on 32×32 images. We use the same pipeline as for ImageNet, but with $\sigma = 0.12$, PGD step size 0.25, batch size 128, and lowered the learning rate every 70 epochs over 500 total epochs. For both datasets we used data augmentation with ± 30 degree rotations. For the Gaussian blur we use $\sigma_b = 1.0$ and $s_b = 5$.

(Restricted) ImageNet We trained with a batch size of 400 for 90 epochs using stochastic gradient decent with a learning rate starting at 0.1, which is decreased by a factor 10 every 30 epochs.

We trained either with random rotation (uniformly in $[-30, 30]$ degrees; with bilinear interpolation). Further, we used $\sigma = 0.5$ and PGD step size 1.0, as well as $\sigma_b = 2.0$ and $s_b = 5$.

All training on ImageNet with 4 GeForce RTX 2080 Tis and a 16-core node of aw Intel(R) Xeon(R) Gold 6242 CPU @ 2.80GHz takes roughly 1 hour per epoch and Restricted ImageNet 10 minutes per epoch.

D.3 Details for Section 7.5

We used the same networks as outlined in App. D.2.

E Additional Experiments

E.1 Additional Results for Section 7.3

Table 3: Extended version of Table 1. Evaluation of BASESPT on 1000 images. The attacker used worst-of-100. We use $n_\gamma = 1000$, $\sigma_\gamma = \Gamma_\pm$.

Dataset	T^I	Γ_\pm	Acc.	adv. Acc.		t [s]
			b	b	g	
MNIST	R^I	30°	0.99	0.73	0.99	0.97
FMNIST	R^I	30°	0.91	0.13	0.87	7.98
CIFAR	R^I	30°	0.91	0.26	0.85	0.95
GTSRB	R^I	30°	0.91	0.30	0.88	8.00
ImageNet	R^I	30°	0.76	0.56	0.76	5.43
MNIST	Δ^I	4	0.99	0.03	0.53	0.86
FMNIST	Δ^I	4	0.91	0.10	0.50	6.12
CIFAR	Δ^I	4	0.91	0.44	0.79	0.95
GTSRB	Δ^I	4	0.91	0.30	0.63	5.17
ImageNet	Δ^I	20	0.76	0.65	0.75	6.70

Table 3 is an extended version of Table 1 and provides results for additional datasets.

Table 4: We first use BASESPT to obtain the certification radius r_γ on 30 images and subsequently sample from the parameter space indicated by $\Gamma_\pm = r_\gamma$ and checked whether the certificate holds for them. We use 30 samples and $n_\gamma = 2000$ samples for the smoothed classifier. The last column shows the number of images for which we found violations.

Dataset	T^I	Γ_\pm	median r_γ	r_γ violated
MNIST	R^I	30°	28.34	0
FMNIST	R^I	30°	13.45	1
CIFAR	R^I	30°	19.16	14
GTSRB	R^I	30°	20.93	0
ImageNet	R^I	10°	27.13	1
MNIST	Δ^I	4	1.12	0
FMNIST	Δ^I	4	1.78	1
CIFAR	Δ^I	4	4.76	14
GTSRB	Δ^I	4	2.58	0
ImageNet	Δ^I	20	16.43	0

Table 5: Same setup as in Table 4, but with circular vignetting.

Dataset	T^I	Γ_{\pm}	median r_{γ}	r_{γ} violated	r_{γ} violated, no interpolation
MNIST	R^I	30°	28.34	0	0
FMNIST	R^I	30°	17.07	0	0
CIFAR	R^I	30°	11.49	10	0
GTSRB	R^I	30°	25.28	0	0

E.2 “Certification Radius” of BASESPT

As BASESPT uses Theorem 3.2 to justify the heuristic, this also makes it tempting to use the bound r_{γ} provided by it. However, as the assumptions of Theorem 3.2 are violated it does not formally present a certification radius. Here we investigate if and how much it holds nevertheless. To do this we construct a smoothed classifier g from an undefended base classifier b and calculated the certification radius r_{γ} . Subsequently we sampled 100 new rotated images in the parameter space induced by $\Gamma_{\pm} = r_{\gamma}$ and evaluated on them. The results are shown in Table 4. While generally robust, the radius does not constitute a certificate, as we can clearly find violations.

In the context of rotation R^I we add circular vignetting (as we do for DISTSPT and INDIVSPT) to make the behavior closer to a composing transformation. For this experiment we retrained the same networks, but applied the vignette during training. Results are shown in Table 5 where we can see that this already decreases the number of violations for CIFAR and FMNIST. In a final step we assume knowledge of the attacker parameter γ and replace $R_{\beta}^I \circ R_{\gamma}^I$ (for the same images) with $R_{\beta+\gamma}^I$ in the evaluation of the classifier, in which case Theorem 3.2 should hold and indeed we don’t observe anymore violations.

E.3 Additional Results for Section 7.4

Beyond Bilinear Interpolation BASESPT and DISTSPT can directly be applied to image transformations using other interpolation schemes without any adaption. INDIVSPT, however, requires the adaption of the inverse algorithm. While this is generally possible, we consider it beyond the scope of this work.

Table 6 shows the estimated ϵ (via sampling for $N = 1000$). These values are generally higher than the ones observed for bilinear interpolation, making classification more challenging. For E 1.5 times larger than ϵ_{\max} we initiated classifier (with 100 images each) and evaluate them in App. E.3. These results indicate that we can obtain similar guarantees as for bilinear although might need to apply higher computation resources (n_{δ}). For (R)ImageNet taking E to 1.5 the maximal error this error becomes too large to handle. Realistically one could use an E between 1.84 and 2.76 depending on the desired q_E .

Table 6: The observed ϵ values. Showing the maximum ϵ_{\max} and the 99-th percentile ϵ_{99} .

Dataset	ϵ_{\max}	ϵ_{99}
MNIST	0.44	0.29
CIFAR	0.88	0.74
ImageNet	1.84	1.52

Table 7: Results for DISTSPT on bicubic interpolation. $\sigma_{\gamma} = \Gamma$.

Dataset	T^I	E	Γ_{\pm}	Acc.		r_{γ} percentile			T [s]	n_{γ}	n_{δ}
				b	g	25 th	50 th	75 th			
MNIST	R^I	0.66	30°	0.99	0.99	30.00 [†]	30.00 [†]	30.00 [†]	8.30	200	2000
CIFAR	R^I	1.11	30°	0.72	0.54	1.87	18.37	30.00 [†]	24.67	50	10000
RImageNet	R^I	1.84	30°	0.72	0.68	14.34	30.00 [†]	30.00 [†]	169.25	50	4000
RImageNet	R^I	2.76	30°	0.72	0.00	-	-	-	151.88	50	4000

E.4 Audio Volume Change

To show that our method can be used beyond image transformation we showcase the adaption to audio volume changes. The volume of an audio signal can be changed by multiplying the signal with a constant. In order to change the signal \mathbf{x} by β (measured in decibel $[\beta] = \text{dB}$) we multiply \mathbf{x} by $10^{\beta/20}$. Thus the transformation is $\psi_\beta(\mathbf{x}) := 10^{\beta/20} \cdot \mathbf{x}$, which composes:

$$\psi_\beta \circ \psi_\gamma(\mathbf{x}) = 10^{(\beta+\gamma)/20} \cdot \mathbf{x} = \psi_{\beta+\gamma}(\mathbf{x}).$$

In practice such signals are stored in final precision, e.g. 16-bit, thus potentially introducing rounding errors, with an ℓ^2 -norm bound by E . If this is ignored BASESPT can be applied to obtain guarantees. Otherwise, DISTSPT and INDIVSPT can be used to obtain sound bounds.

To evaluate this we use the speech commands dataset [38], consisting of 30 different commands, spoken by people, which are to be classified. The length of the recordings are one second each. We use a classification pipeline that converts audio wave forms into MFCC spectra [39] and then treats these as images and applies normal image classification. We use a ResNet-50, that was trained with Gaussian noise, but not SMOOTHADV_{PGD}. We apply the noise before the waveform is converted to the MFCC spectrum.

For DISTSPT we estimate E to be 0.005 with the parameters $\alpha_E = 0.05, \sigma_\gamma = 3$ and $\Gamma = 3$. On 100 samples, the base classifier f was correct 94 times, and the smoothed classifier g 51 times for r_γ of 0.75, 1.96 and 3.12 for the 25th, 50th, 75th percentile respectively, corresponding to $\pm 1.09, \pm 1.25$ and ± 1.43 dB. At $n_\gamma = 100$ and $n_\epsilon = 400$ the average certification time was 138.06s.

To investigate INDIVSPT we use $\sigma_\gamma = 0.85, \Gamma = 1.05$. For 92 out of 100 perturbed audio signals to compute ϵ . We obtained $\epsilon_{\max} \leq 0.0055$ and for 68 an $\epsilon \leq 0.005$, which together with our results for DISTSPT suggests the applicability of the method. For each signal we used 100 samples for β . For cases with $\epsilon_{\max} > 0.0055$ we in fact observed $\epsilon_{\max} \gg 0.0055$, as here many parts of the signal were amplified beyond the precision of the 16-bit representation and clipped to ± 1 . This makes the information unrecoverable and sound error bound estimates large.

Table 8: Maximum observed errors and without gaussian blur (G) and without vignetting (V).

Dataset	Both	-V	-G	-V-G
MNIST	0.36	0.36	2.47	2.51
CIFAR10	0.51	6.08	2.66	18.17
ImageNet	0.91	70.66	9.25	75.69

Table 9: Correct classifications and by the model and verifications by DeepG [11], with and without vignetting (V), out of 100 images.

Model	Correct	[11]	[11]+V
MNIST	98	86	87
CIFAR10	74	65	32
CIFAR10+V	78	63	23

F Further Comparison and Ablation

To show that the vignette and Gaussian blur are essential to our algorithm we perform a small ablation study. Table 8 shows the maximal error observed when sampling as in DISTSPT. We use the same setup as in Section 7.4, but with 10000 samples for ImageNet.

Both, vignetting and Gaussian blur reduce the error bound significantly for DISTSPT and INDIVSPT. On CIFAR10 and ImageNet vignetting is very impactful because the corners of images are rarely black in contrast to MNIST. Li et al. [13] uses vignetting for the same reason. Without either of the methods bounding the error would not be feasible.

For INDIVSPT vignetting is crucial, even for MNIST, as we can make no assumptions for parts that are rotated into the image. Thus we need to set these pixels to the full $[0, 1]$ interval (see Fig. 2). Without Gaussian blur the certification rate drops to 0.11.

Further we extend this comparison to related work: We extended Balunovic et al. [11] (Table 1 in their paper) to include vignetting. The results are shown in Table 9. We also retrained their CIFAR10 model with vignetting (CIFAR10+V) for completeness. While vignetting on MNIST slightly helps

(+1 image verified) on CIFAR10 it leads to a significant drop. Including Gaussian blur into [11] would require non-trivial adaption of the method. However, we implemented this for interval analysis (on which their method is build) and found no impact on results.