Alan Turing and Number Theory

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Recognition of Merits

Andrew R. Booker.
Artin's Conjecture, Turing's Method, and the Riemann Hypothesis.
Experiment. Math., Volume 15, Issue 4 (2006), 385-408.

"Reading Turing's paper on the subject, which was one of his last, one marvels at what he accomplished with the limited computational resources of the day. His method was truly ahead of its time."

Distribution of Prime Numbers

$$\pi(x) = \text{the number of primes not exceeding } x$$

Prime Number Theorem (Hadamard [1896] and de la Vallée Poussin[1896]).

$$\pi(x) \approx \frac{x}{\ln(x)} \qquad \frac{\pi(x)}{\frac{x}{\ln(x)}} \to_{x\to\infty} 1 \qquad \frac{\pi(x)}{\mathrm{Li}(x)} \to_{x\to\infty} 1$$

$$\pi(x) - \frac{x}{\ln(x)} \qquad \pi(x) - \mathrm{Li}(x)$$

$$\mathrm{Li}(x) = \int_2^x \frac{1}{\ln(t)} \mathrm{d}t \approx \frac{x}{\ln(x)} \approx \pi(x)$$

Numerical Values

x	$\pi(x) - x/\log(x)$	$\pi(x) - \mathrm{Li}(x)$
10^1	0	-1
10^2	4	-4
10^3	24	-8
10^4	144	-16
10^5	907	-36
10^6	6116	-128
10^7	44159	-338
10^8	332774	-753
10^9	2592592	-1699
10^{10}	20758030	-3102
10^{11}	169923160	-11586
10^{12}	1416705193	-38261
10^{13}	11992858452	-108970
10^{14}	102838308636	-314888

Riemann's formula for $\pi(x)$

$$\pi(x) = \mathrm{Li}(x) - \frac{1}{2}\mathrm{Li}(x^{\frac{1}{2}}) + \sum_{\zeta(\rho)=0} \mathrm{Li}(x^{\rho}) + \text{smaller terms}$$

Theorem (Littlewood [1914]). *For infinitely many x*

$$\pi(x) > \mathrm{Li}(x). \qquad (*)$$

Theorem (Skewes [1933]). *There exists an x satisfying* $(*)$ *such that*

$$x < 10^{10^{10^{34}}}.$$

Hardy: "the largest number that have served any definite purpose in mathematics"

Ingham's Book

A. E. Ingham.
The Distribution of Prime Numbers.
Cambridge University Press, 1932

Reissued in 1990

Riemann's zeta function

$$\pi(x) = \mathrm{Li}(x) - \frac{1}{2}\mathrm{Li}(x^{\frac{1}{2}}) + \sum_{\zeta(\rho)=0} \mathrm{Li}(x^{\rho}) + \text{smaller terms}$$

Dirichlet series:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

$$s = \sigma + \mathrm{i}t$$

The series converges in the semi-plane $\Re(s) > 1$ and defines a function that can be analytically extended to the entire complex plane except for the point $s = 1$, its only (and simple) pole.

The Riemann Hypothesis

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \qquad s = \sigma + \mathrm{i}t$$

Theorem (Riemann [1859]). *All non-real zeros of $\zeta(s)$ lie in the critical strip $0 \leq \Re(s) \leq 1$.*

Riemann Hypothesis (RH). *All non-real zeros of $\zeta(s)$ lie on the critical line $\Re(s) = \frac{1}{2}$.*

Equivalent formulation of RH.
$\pi(x) - \mathrm{Li}(x) = O(x^{\frac{1}{2}} \log(x))$.

Theorem (Miller [1975]). *If the Generalized Riemann Hypothesis is true, then* **Primes**$\in$**P**.

Numerical checking of RH

Year	Number of zeroes	Author
1903	15	J.P.Gram
1914	79	R. J. Backlund
1925	138	J. I. Hutchinson
1936	1041	E. C. Titchmarsh

Riemann's formula for $\pi(x)$

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Theorem (Littlewood [1914]). For infinitely many x

$$\pi(x) > \mathrm{Li}(x) \qquad (*)$$

Theorem (Skewes [1933]). If the Riemann Hypothesis is true, then there exists an x satisfying $(*)$ such that

$$x < 10^{10^{10^{34}}}.$$

Theorem (Skewes [1955]). If the Riemann Hypothesis is false, then there exists an x satisfying $(*)$ such that

$$x < 10^{10^{10^{10^{3}}}}.$$

Application for a grant of Royal Society

3. A. M. Turing.....£40.

King's College,
Cambridge.
24 *March* 1939.

"1. It is proposed to make calculations of the Riemann zeta-function on the critical line for $1{,}450 < t < 6{,}000$ with a view to discovering whether all the zeros of the function in this range of t lie on the critical line. An investigation for $0 < t < 1{,}464$ has already been made by Titchmarsh. The most laborious part of such calculations consists in the evaluation of certain trigonometrical sums

$$\sum_{r=1}^{m} \frac{1}{\sqrt{r}} \cos(t \log r - \vartheta) \qquad m = \left[\sqrt{\frac{t}{2\pi}}\right]$$

"Turing machine"

$$\sum_{r=1}^{m} \frac{1}{\sqrt{r}} \cos(t \log r - \vartheta) \qquad m = \left[\sqrt{\frac{t}{2\pi}}\right]$$

In the present calculation it is intended to evaluate these sums approximately in most cases by the use of apparatus somewhat similar to what is used for tide prediction. When this method does not give sufficient accuracy it will be necessary to revert to the straightforward calculation of the trigonometric sums, but this should be only very rarely necessary. I am hoping that the use of the tide-predicting machine will reduce the amount of such calculation necessary in a ratio of 50 : 1 or better. It will not be feasible to use already existing tide predictors because the frequencies occurring in the tide problem are entirely different from those occurring in the zeta-function problem. I shall be

The cost

from those occurring in the zeta-function problem. I shall be working in collaboration with D. C. MacPhail, a research student who is an engineer. We propose to do most of the machine-shop work ourselves, and are therefore applying only for the cost of materials, and some preliminary computation.

"2. Cost of materials for making tide predictor, estimated at £25, and not exceeding £35. Cost of preliminary computation, estimated at £3 10s., and not exceeding £5. Some further computation may be necessary after the work with the tide predictor, but the amount of this cannot be accurately estimated at this stage, and might be negligible. No application is being made on this account at present.

Tide-predicting Machine

$$\sum_{r=1}^{m} a_r \cos(\omega_r t - \theta)$$

Sir William Thomson's (Lord Kelvin) design
for his third tide-predicting machine, 1879-81.
http://en.wikipedia.org/wiki/Tide-predicting_machine

Tide-predicting Machine

http://tidesandcurrents.noaa.gov/predma3.html

Tide-predicting Machines

March, 1939

A. M. Turing.
A method for the calculation of the zeta-function.
Proceedings of the London Mathematical Society
ser. 2, vol. 48, 1943, pp. 180–197.
The paper was submitted on March 7, 1939

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Methods for the calculation of the zeta-function

Euler–Maclaurin summation

Riemann–Siegel formula

A. M. Turing. A method for the calculation of the zeta-function. *Proceedings of the London Mathematical Society* ser. 2, vol. 48, 1943, pp. 180–197

A. M. Odlyzko, A. Schönhage. Fast algorithms for multiple evaluations of the Riemann zeta function.

"Only one other method, besides the Euler–Maclaurin and Riemann–Siegel ones, seems to have been proposed for computing $\zeta(s)$ to moderate accuracy at large heights, namely the one due to Turing. It was designed to provide higher accuracy than was guaranteed by the crude bounds on the remainder term in the Riemann–Siegel formula that were available at that time, and at the same time be more efficient than the Euler–Maclaurin formula. However, very good estimates for remainder terms in the Riemann–Siegel formula are now available, which seem to make Turing's method unnecessary."

Proceedings of the London Mathematical Society
ser. 3, vol. 3, 1953, pp. 99–117

SOME CALCULATIONS OF THE RIEMANN ZETA-FUNCTION

By A. M. TURING

[Received 29 February 1952.—Read 20 March 1952]

Introduction

In June 1950 the Manchester University Mark 1 Electronic Computer was used to do some calculations concerned with the distribution of the zeros of the Riemann zeta-function. It was intended in fact to determine whether there are any zeros not on the critical line in certain particular intervals.

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The calculations were done in an optimistic hope that a zero would be found off the critical line, and the calculations were directed more towards finding such zeros than proving that none existed. The procedure was such that if it had been accurately followed, and if the machine made no errors in the period, then one could be sure that there were no zeros off the critical line in the interval in question. In practice only a few of the results were checked by repeating the calculation, so that the machine might well have made an error.

Proceedings of the London Mathematical Society
ser. 3, vol. 3, 1953, pp. 99–117

If more time had been available it was intended to do some more calculations in an altogether different spirit. There is no reason in principle why computation should not be carried through with the rigour usual in mathematical analysis. If definite rules are laid down as to how the computation is to be done one can predict bounds for the errors throughout. When the computations are done by hand there are serious practical difficulties about this. The computer will probably have his own ideas as to how certain steps should be done. When certain steps may be omitted without serious loss of accuracy he will wish to do so. Furthermore he will probably not see the point of the 'rigorous' computation and will probably say 'If you want more certainty about the accuracy why not just take more figures?' an argument difficult to counter. However, if the calculations are being done by an automatic computer one can feel sure that this kind of indiscipline does not occur. Even with the automatic computer this rigour can be rather tiresome to achieve, but in connexion with such a subject as the analytical theory of numbers, where rigour is the essence, it seems worth while.

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Cauchy integral

Let $N(T)$ denote the number of zeros of $zeta(s)$ inside the rectangular region

$$0 \leq \Re(s) \leq 1, \; 1 \leq \Im(s) \leq T$$

$$N(T) = \frac{1}{2\pi \mathrm{i}} \oint \frac{\zeta'(s)}{\zeta(s)} \, \mathrm{d}s$$

Alongside the Critical Line

$$Z(t) = e^{i\theta(t)}\zeta(\frac{1}{2} + it)$$

$$\theta(t) = \Im \ln\left(\Gamma(\frac{it}{2} + \frac{1}{4})\right) - \frac{t}{2}\ln(\pi)$$

Classical Method for Checking Riemann's Hypothesis

- Calculate $N(T) = \frac{1}{2\pi \mathrm{i}} \oint \frac{\zeta'(s)}{\zeta(s)} \, \mathrm{d}s$, the number of zeros of $\zeta(s)$ inside the rectangular region

$$0 \leq \Re(s) \leq 1, \; 1 \leq \Im(s) \leq T.$$

- Select $N(T) + 1$ numbers ("green points") $f_0, \ldots, f_{N(T)}$ such that

$$1 \leq f_0 < \cdots < f_k < f_{k+1} < \cdots < f_{N(T)} \leq T$$

  and verify that

$$Z(f_{k-1})Z(f_k) < 0, \qquad k = 1, \ldots, N(T).$$

In case of success all zeros of $\zeta(s)$ with imaginary part between 1 and T do lie *exactly* on the critical line $\Re(s) = \frac{1}{2}$.

Gram Points

$$Z(t)=e^{\mathrm{i}\theta(t)}\zeta\left(\tfrac{1}{2}+\mathrm{i}t\right) \qquad \theta(t)=\Im\ln\left(\Gamma\left(\tfrac{\mathrm{i}t}{2}+\tfrac{1}{4}\right)\right)-\tfrac{t}{2}\ln(\pi)$$

$$\theta(t)=\tfrac{1}{2}\left(\ln\left(\tfrac{t}{2\pi}\right)-1\right)t-\tfrac{\pi}{8}+o(t^{-1})$$

Gram Points

$$Z(t) = e^{i\theta(t)}\zeta\left(\tfrac{1}{2} + it\right) \qquad \theta(t) = \Im \ln\left(\Gamma\left(\tfrac{it}{2} + \tfrac{1}{4}\right)\right) - \tfrac{t}{2}\ln(\pi)$$

$$e^{i\theta(t)} = \cos(\theta(t)) + i\sin(\theta(t)) \qquad \sin(\theta(g_m)) = 0 \qquad \theta(g_m) = \pi m$$

Gram's "law"

$Z(t) = e^{i\theta(t)}\zeta(\frac{1}{2} + it)$ $\qquad$ $\sin(\theta(t)) = \sin\left(\Im \ln\left(\Gamma(\frac{it}{2} + \frac{1}{4})\right) - \frac{t}{2}\ln(\pi)\right)$

Gram's "Law". $(-1)^m Z(g_m) > 0$

Violation of Gram's "Law"

$Z(t) = e^{i\theta(t)}\zeta(\frac{1}{2} + it)$ $\qquad$ $\sin(\theta(t)) = \sin\left(\Im \ln\left(\Gamma(\frac{it}{2} + \frac{1}{4})\right) - \frac{t}{2}\ln(\pi)\right)$

Gram's "Law". $(-1)^m Z(g_m) > 0$

$Z(g_{133}) = -0.7763\ldots$ $\quad$ $Z(g_{134}) = -0.0169\ldots$ $\quad$ $Z(g_{135}) = -3.4698\ldots$

Good Gram point: $(-1)^m Z(g_m) > 0$

Bad Gram point: $(-1)^m Z(g_m) < 0$

Violation of Gram's "Law"

$Z(t) = e^{i\theta(t)}\zeta(\frac{1}{2} + it)$ $\qquad$ $\sin(\theta(t)) = \sin\left(\Im \ln\left(\Gamma(\frac{it}{2} + \frac{1}{4})\right) - \frac{t}{2}\ln(\pi)\right)$

Gram's "Law". $(-1)^m Z(g_m) > 0$

$$Z(g_{133}) = -0.7763\ldots \qquad Z(g_{134}) = -0.0169\ldots \qquad Z(g_{135}) = -3.4698\ldots$$

$$Z(g_{134} + h_{134}) = +0.1958\ldots$$

$$g_{133} < g_{134} + h_{134} < g_{135} \qquad h_{134} = -0.2$$

Classical Method for Checking Riemann's Hypothesis

- Calculate $N(T) = \frac{1}{2\pi \mathrm{i}} \oint \frac{\zeta'(s)}{\zeta(s)} \, \mathrm{d}s$, the number of zeros of $\zeta(s)$ inside the rectangular region

$$0 \leq \Re(s) \leq 1, \; 1 \leq \Im(s) \leq T.$$

- Find $N(T) + 1$ "small" numbers $h_{-1}, \ldots, h_{N(T)-1}$ such that

$$1 \leq g_{-1} + h_{-1} < \cdots < g_k + h_k < g_{k+1} + h_{k+1} < \cdots < g_{N(T)-1} + h_{N(T)-1} \leq T$$

  and verify that

$$Z(g_{k-1} + h_{k-1}) Z(g_k + h_k) < 0, \qquad k = 0, \ldots, N(T) - 1.$$

In case of success all zeros of $\zeta(s)$ with imaginary part between 1 and T do lie *exactly* on the critical line $\Re(s) = \frac{1}{2}$.

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To summarize. The method recommended is first to find the total number of zeros in the rectangle by methods to be described later. Then to calculate the function at sufficient points to account for all the zeros, either by changes of sign or as complex zeros determined by the use of Rouché's theorem. We know no way of dealing with multiple zeros, and simply hope that none are present.

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This paper is divided into two parts. The first part is devoted to the analysis connected with the problem. All the results obtained in this part are likely to be applicable to any further calculations to the same end, whether carried out on the Manchester Computer or by any other means. The second part is concerned with the means by which the results were achieved on the Manchester Computer.

Function $S(t)$

$N(T)$, the number of zeros of $zeta(s)$ inside the rectangular region $0 \leq \Re(s) \leq 1,\ 1 \leq \Im(s) \leq T$ should be close to the number of Gram's points below T.

$$\begin{aligned} g_m < T \quad &\Leftrightarrow \quad \theta(g_m) < \theta(T) \\ &\Leftrightarrow \quad \pi m < \theta(T) \\ &\Leftrightarrow \quad m < \frac{\theta(T)}{\pi} \end{aligned}$$

$$N(T) = \frac{\theta(T)}{\pi} + 1 + S(T)$$

Theorem (Littlewood [1924]).

$$\int_0^T S(t)\,\mathrm{d}t = O(\ln(T))$$

Function $S(t)$

$N(T)$, the number of zeros of $zeta(s)$ inside the rectangular region $0 \leq \Re(s) \leq 1,\ 1 \leq \Im(s) \leq T$ should be close to the number of Gram's points below T.

$$N(T) = \frac{\theta(T)}{\pi} + 1 + S(T)$$

Theorem (Littlewood [1924]).

$$\int_0^T S(t)\,\mathrm{d}t = O(\ln(T))$$

Theorem (Turing [1953]). *If* $168\pi < t_1 < t_2$ *then*

$$\left| \int_{t_1}^{t_2} S(t)\,\mathrm{d}t \right| \leq 2.30 + 0.128 \ln\left(\frac{t_2}{2\pi}\right).$$

modulo 2 in this interval, the additive even integer being the same throughout. Hence $S_1(t_0+L)-S_1(t_0)$ will be known modulo $2L$, and if L is sufficiently large this will determine throughout the interval. In orde it is necessary to replace the O resu to try and arrange to manage wit of the zeros.

THEOREM 4. *If* $t_2 > t_1 > 168\pi$

$$S_1(t_2)-S_1(t_1) =$$

The proof of this follows Theo step is

LEMMA 1. *If* $t_2 > t_1 > 0$, *then*

$$\pi\{S_1(t_2)-S_1(t_1)\} = \mathscr{R}\int_{\frac{1}{2}+i}^{\infty+}$$

We apply Cauchy's theorem t $\frac{1}{2}+it_1$, $\frac{1}{2}+it_2$, $R+it_2$, $R+it_1$ anc lines from zeros within the recta

$$-\mathscr{R}\int_{\frac{1}{2}+it_2}^{R+it_2}\log\zeta(s)\,ds+\int_{\frac{1}{2}+it_1}^{\frac{1}{2}+it_2}\text{ar}$$

$$+\mathscr{R}$$

no contribution arising from the to 0 as $R\to\infty$ and the second is π

LEMMA 2. *If* $\tau \geqslant 64$, *we have*

$$|\zeta(\tfrac{1}{2}+$$

Since $|h(\zeta)| < 0{\cdot}95$ we have, b

$$|\tau(\tfrac{1}{2}+2\pi i\tau)| = |Z(\tau)|$$

and $$\sum_{1\leqslant r\leqslant\tau^{\frac{1}{2}}} r^{-\frac{1}{2}} < 1+$$

LEMMA 3.

$$|\zeta(1{\cdot}25+it)|$$

$$\left|\int_{1\cdot25+it}^{\infty}\log\zeta(s)\,ds\right|$$

$$\left|\frac{\zeta'}{\zeta}(1{\cdot}5+it)\right|$$

$$\left|\int_{1\cdot5+it}^{2\cdot5+it}\log\zeta(s)\,ds\right|$$

$$\left|\int_{1\cdot5+it}^{\infty}\log\zeta(s)\,ds\right|$$

$$\tfrac{1}{2}\log\pi$$

These results are all based on tl error of two units in the last plac not know how these tables were c the 'rigorous computation'.

LEMMA 4. *If* $\frac{1}{2} < \sigma < \frac{5}{4}$ *and* t

$$|\zeta(s)$$

Consider $f(s)$:

$$f(s) = \zeta(s)\left(\frac{s-\frac{1}{2}}{i}\right)^{-}$$

Now $$\left|\exp\left[\frac{-4\pi i}{s-\frac{1}{2}-127{\cdot}5\pi i}\right]\right|$$

Hence, by Lemma 3, $|f(s)| < 4$ o $t > 128\pi$, we have

$$\log\left|\left(\frac{s-\frac{1}{2}}{i}\right)^{-3/8+\frac{1}{4}s}\right| = \tfrac{1}{2}(-\tfrac{3}{8}$$

$$\leqslant -\tfrac{1}{32}\text{l}$$

Hence on the line $\sigma = \frac{5}{4}$, $t \geqslant 128$

$$|f(s)| < \zeta(\tfrac{5}{4})$$

Finally on the line $t = 128\pi$, $\frac{1}{2} \leqslant$

$$|f(s)| < |\zeta(s)|\exp\left[\frac{-}{\frac{1}{4}\pi}\right.$$

by equation (8) on p. 27 of Ing boundary of the strip $t \geqslant 128\pi$, $\frac{1}{2}$

we have $|f(s)| < 4$ throughout the strip by the Phragmén–Lindelöf theorem. From this it follows that

$$|\zeta(s)| < 4e^{0\cdot1}\left|\right.$$

$$< 4{\cdot}5t^{\frac{3}{8}-\frac{1}{4}}$$

The purpose of the factor exp

do without accurate knowledge

LEMMA 5. *If* $t > 168\pi$, *then*

$$\mathscr{R}\int_{\frac{1}{2}+it}^{\infty+it}\log\zeta(s)$$

For $$\mathscr{R}\int_{\frac{1}{2}+it}^{\infty+it}\log\zeta(s)\,ds \leqslant 1\cdot$$

$$\leqslant 1\cdot$$

$$= 1\cdot$$

$$< 2\cdot$$

It is certainly possible to improv expense of the constant. The c rate to 0·052 using results stated

LEMMA 6.

$$\frac{\zeta(s)\zeta(s+2)}{\{\zeta(s+1)\}^2} = \frac{s^2}{s^2-1}\frac{\{1}{\Gamma(\frac{1}{2}s}$$

where the product is over the non-

This is an immediate consequ zeta-function.

LEMMA 7. *If* $k = 1{\cdot}49$, $\mathscr{R}a \geqslant$

$$\mathscr{R}\left(\psi(a)+\frac{k}{a}\right) = \mathscr{R}$$

It is easily seen that if $\mathscr{R}a =$ $\mathscr{R}(k/a) \geqslant 0$ for $\mathscr{R}a \geqslant 0$ and that as $a\to\infty$. Hence applying the m minimum real part principle) to

$$\mathscr{R}a \geqslant 0,$$

we see by allowing $\epsilon\to 0$, $R\to\infty$ that the minimum real part must be achieved either on the boundary be a singularity). It only rema for the real axis. At any station

$$0 = \mathscr{R}\left(\psi'(a)-\frac{k}{a^2}\right)$$

This equation only has solutions spond to minima of $\psi(a)+(k/a)$. them, but there is a singularity minima, and knowledge of an up function in intervals enclosing tl minima are positive. The value 0·0087. Hence $\psi(a)+(k/a) > 0$ c

LEMMA 8. *If* $\mathscr{R}z > 0$, *then*

$$\frac{\Gamma'(z)}{\Gamma(z)} = \log z-$$

We use the formula

$$\frac{\Gamma'(z)}{\Gamma(z)} = \log z-$$

and take the line of integration

$$\left|\frac{u}{e^{2\pi u}-1}\right| < \frac{e^{-\pi v}}{\pi\sqrt{2}},$$

No poles are encountered in the c

LEMMA 9. *If* $t > 50$, *then*

$$-\mathscr{R}\int_{\frac{1}{2}+it}^{\infty}\log\zeta(s)$$

We have

$$\mathscr{R}\int_{\frac{1}{2}+it}^{\infty}\log\zeta(s)\,ds = \mathscr{R}\int_{\frac{1}{2}+it}^{\infty}\text{l}$$

$$\geqslant \mathscr{R}\int_{\frac{1}{2}+it}^{\infty}\log$$

by Lemma 3.

Also, by Lemma 6,

$$\int_{\frac{1}{2}+it}^{\infty}\log\frac{\zeta(s)\zeta(s+2)}{\{\zeta(s+1)\}^2}\,ds = \sum_{\rho}\int_{\frac{1}{2}+it}^{\frac{3}{2}+it}\log\frac{s-\rho}{s-\rho+1}\,ds-$$

$$-\int_{\frac{1}{2}+it}^{\frac{3}{2}+it}\log\frac{\Gamma(\frac{1}{2}s+1)}{\Gamma(\frac{1}{2}s+\frac{3}{2})}\,ds+\int_{\frac{1}{2}+it}^{\frac{3}{2}+it}\log\frac{s}{s-1}\,ds.$$

Now if $\frac{1}{2} < \sigma < \frac{3}{2}$, then $|s-1| < |s|$, and therefore

$$\mathscr{R}\int_{\frac{1}{2}+it}^{\frac{3}{2}+it}\log\frac{s}{s-1}\,ds \geqslant 0.$$

Also $$\mathscr{R}\int_{\frac{1}{2}+it}^{\frac{3}{2}+it}\log\frac{\Gamma(\frac{1}{2}s+1)}{\Gamma(\frac{1}{2}s+\frac{3}{2})}\,ds = -\tfrac{1}{2}\mathscr{R}\frac{\Gamma'}{\Gamma}(\tfrac{1}{2}it+\sigma)$$

for some σ, $\frac{5}{4} < \sigma < \frac{9}{4}$, by the mean value theorems,

$$\leqslant -\tfrac{1}{4}\log(\tfrac{1}{4}t^2+\tfrac{25}{16})-\frac{\frac{5}{2}}{t^2+\frac{25}{4}}+\frac{2}{\pi^2(\frac{1}{4}t^2-\frac{81}{16})}$$

by Lemma 8,

$$< -\tfrac{1}{2}\log\tfrac{1}{2}t \quad (\text{since } t > 50)$$

$$< -\tfrac{1}{2}\log\frac{t}{2\pi}-0{\cdot}572, \text{ by Lemma 3.}$$

Finally

$$\mathscr{R}\sum_{\rho}\int_{\frac{1}{2}+it}^{\frac{3}{2}+it}\log\frac{s-\rho}{s-\rho+1}\,ds \geqslant -1{\cdot}49\mathscr{R}\sum_{\rho}\frac{1}{it-\rho+\frac{3}{2}},$$

by Lemma 7,

$$= -1{\cdot}49\mathscr{R}\left[\frac{\zeta'}{\zeta}(it+\tfrac{3}{2})-\tfrac{1}{2}\log\pi+\tfrac{1}{2}\frac{\Gamma'}{\Gamma}(\tfrac{1}{2}it+\tfrac{7}{4})\right],$$

by the Mittag–Leffler series for $\frac{\zeta'}{\zeta}(s)$,

$$\geqslant -1{\cdot}49\left[\mathscr{R}\frac{\zeta'}{\zeta}(it+\tfrac{3}{2})-\tfrac{1}{2}\log\pi+\tfrac{1}{4}\log(\tfrac{1}{4}t^2+\tfrac{49}{16})-\frac{7}{4t^2+49}\right],$$

by Lemma 8,

$$\geqslant -1{\cdot}49\left[\tfrac{1}{2}\log\frac{t}{2\pi}+2{\cdot}63\right],$$

using Lemma 3 and $t > 50$.

Combining these results gives the asserted inequality.

Turing's Method

$N(T)$, the number of zeros of $zeta(s)$ inside the rectangular region $0 \leq \Re(s) \leq 1,\ 1 \leq \Im(s) \leq T$ should be close to the number of Gram's points below T.

$$N(T) = \frac{\theta(T)}{\pi} + 1 + S(T)$$

Let us try to find T_0 such that $T < T_0$ and

$$\begin{aligned} S(T_0) = 0 \quad &\Rightarrow \quad \frac{\theta(T_0)}{\pi} \text{ is an integer} \\ &\Rightarrow \quad \sin(\theta(T_0)) = 0 \\ &\Rightarrow \quad T_0 = g_m \end{aligned}$$

Turing's Method

$$T_0 = g_m \qquad N(T_0) = N(g_m) = \frac{\theta(g_m)}{\pi} + 1 + S(g_m)$$

$$\begin{aligned} \sin(\theta(g_m)) = 0 \quad &\Rightarrow \quad \theta(g_m) \text{ is a multiple of } \pi \\ &\Rightarrow \quad \frac{\theta(g_m)}{\pi} \text{ is an integer} \\ &\Rightarrow \quad S(g_m) \text{ is an integer} \end{aligned}$$

$$(-1)^m Z(g_m) > 0 \quad \Rightarrow \quad S(g_m) \text{ is an even integer}$$

$$|S(g_m)| < 2 \Rightarrow S(g_m) = 0$$

Turing's Method

$$
\begin{aligned}
(-1)^m Z(g_m) &> 0 \\
(-1)^{m+1} Z(g_{m+1} + h_{m+1}) &> 0 \\
&\vdots \\
(-1)^{m+k-1} Z(g_{m+k-1} + h_{m+k-1}) &> 0 \\
(-1)^{m+k} Z(g_{m+k}) &> 0
\end{aligned}
$$

Theorem (Turing [1953]).

$$S(g_m) \leq 1 + \frac{2.30 + 0.128 \ln\left(\frac{g_{m+k}}{2\pi}\right) + \sum_{j=1}^{k-1} h_{m+j}}{g_{m+k} - g_m}$$

$$S(g_m) \geq -1 - \frac{2.30 + 0.128 \ln\left(\frac{g_m}{2\pi}\right) - \sum_{j=1}^{k-1} h_{m-j}}{g_m - g_{m-k}}$$

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Part II. The Computations

Essentials of the Manchester Computer

It is not intended to give any detailed account of the Manchester Computer here, but a few facts must be mentioned if the strategy of the computation is to be understood. The storage of the machine is of two kinds, known as 'electronic' and 'magnetic' storage. The electronic storage consisted of four 'pages' each of thirty-two lines of forty binary digits. The magnetic storage consisted of a certain number of tracks each of two pages of similar capacity. Only about eight of these tracks were available for the zeta-function calculations. It was possible at any time to transfer one or both pages of a track to the electronic storage by an appropriate instruction. This operation takes about 60 ms. (milliseconds). Transfers to the magnetic store from the electronic were also possible, but were in fact only used for preparatory loading of the magnetic store. The course of the calculations is controlled by instructions each of twenty binary digits. These are normally magnetically stored, but must be transferred to the electronic

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The storage available was distributed as follows:

Magnetic store

Logarithms routine (for κ)	1 page
Table of logarithms and reciprocal square roots	4 pages
Routine for calculating the terms $n^{-\frac{1}{2}}\cos 2\pi(\tau \log n - \kappa)$ and table of cosines	2 pages
Remainder of routine for calculating the function $Z(\tau)$	2 pages
Input routine	2 pages
Output routine	2 pages

Electronic store, as occupied during the greater part of the time

Instructions and cosines	2 pages
Logarithms and reciprocal square roots	1 page
Miscellaneous data and working space	1 page

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The results of the calculations are punched out on teleprint tape. This is a slow process in comparison with the calculations, taking about 150 ms. per character. The content of a tape may afterwards automatically be printed out with a typewriter if desired. The significance of what is printed out is determined by the 'programmer'. In the present case the output consisted mainly of numbers in the scale of 32 using the code

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
/	E	@	A	:	S	I	U	¼	D	R	J	N	F	C	K	T	Z	L	W

20	21	22	23	24	25	26	27	28	29	30	31
H	Y	P	Q	O	B	G	‖	M	X	V	£

and writing the most significant digit on the right. More conventionally the scale of 10 can be used, but this would require the storage of a conversion routine, and the writer was entirely content to see the results in the scale of 32, with which he is sufficiently familiar.

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Unfortunately, although the details were all worked out, practically nothing was done on these lines. The interval $1414 < t < 1608$ was investigated and checked, but unfortunately at this point the machine broke down and no further work was done. Furthermore this interval was subsequently found to have been run with a wrong error value, and the most that can consequently be asserted with certainty is that the zeros lie on the critical line up to $t = 1540$, Titchmarsh having investigated as far as 1468 (Titchmarsh (**5**)).

Unfortunately $0{\cdot}31E$ was given the inappropriate value of $\frac{1}{128}$ and consequently we are only able to assert the validity of the Riemann hypothesis as far as $t = 1540$, a negligible advance.

Numerical checking of RH (continued)

Year	Number of zeroes	Author
1903	15	J.P.Gram
1914	79	R. J. Backlund
1925	138	J. I. Hutchinson
1936	1041	E. C. Titchmarsh
1953	1104	A. M. Turing
1956	25000	D. H. Lehmer
⋮	⋮	⋮
1968	3500000	J. B. Rosser, J. M. Yohe, and L. Schoenfeld
⋮	⋮	⋮
2004	100000000000000	X. Gourdon

Turing's papers on Riemann's Hypothesis

A. M Turing.
Systems of logic based on ordinals.
Proceedings of the London Mathematical Society
ser.2, volume 45, 1939, pp. 161–228;
Reproduced in *The undecidable. Basic papers on undecidable propositions, unsolvable problems and computable functions*
Edited by M. Davis, Raven Press, Hewlett, N.Y. 1965.

A. M. Turing.
A method for the calculation of the zeta-function.
Proceedings of the London Mathematical Society
ser. 2, vol. 48, 1943, pp. 180–197.

A. M. Turing
Some calculations of the Riemann zeta-function
Proceedings of the London Mathematical Society
ser. 3, vol. 3, 1953, pp. 99–117.

Systems of logic based on ordinals.

Proc. London Math. Soc., ser.2, vol. 45, 1939, pp. 161–228

3. Number-theoretic theorems

By a number-theoretic theorem[5] we shall mean a theorem of the form "$\theta(x)$ vanishes for infinitely many natural numbers x", where $\theta(x)$ is a primitive recursive function.

⋮

An alternative form for number-theoretic theorems is "for each natural number x there exists a natural number y such that $f(x, y)$ vanishes", where $f(x, y)$ is primitive recursive.

[5] I believe that there is no generally accepted meaning for this term, but it should be noticed that we are using it in a rather restricted sense.

RH in the Arithmetical Hierarchy

$$\Pi_0^0 = \Sigma_0^0 = \{\phi | \text{ all quantifiers in } \phi \text{ are bounded}\}$$

$$\Pi_1^0 = \{\forall x_1 \ldots x_m \phi | \phi \in \Sigma_0^0\} \qquad \Sigma_1^0 = \{\exists x_1 \ldots x_m \phi | \phi \in \Pi_0^0\}$$

$$\Pi_2^0 = \{\forall x_1 \ldots x_m \phi | \phi \in \Sigma_1^0\} \qquad \Sigma_2^0 = \{\exists x_1 \ldots x_m \phi | \phi \in \Pi_1^0\}$$

$$\vdots \qquad \vdots$$

$$\Pi_{n+1}^0 = \{\forall x_1 \ldots x_m \phi | \phi \in \Sigma_n^0\} \qquad \Sigma_{n+1}^0 = \{\exists x_1 \ldots x_m \phi | \phi \in \Pi_n^0\}$$

$$\vdots \qquad \vdots$$

Where does RH lie in this hierarchy? $\quad$ RH$\in \Pi_0^0 = \Sigma_0^0$

Either $\quad$ RH $\Leftrightarrow 0 = 0 \quad$ or $\quad$ RH $\Leftrightarrow 0 = 1$

RH in the Arithmetical Hierarchy

$$\Pi_0^0 = \Sigma_0^0 = \{\phi | \text{ all quantifiers in } \phi \text{ are bounded}\}$$

$$\Pi_1^0 = \{\forall x_1 \dots x_m \phi | \phi \in \Sigma_0^0\} \qquad \Sigma_1^0 = \{\exists x_1 \dots x_m \phi | \phi \in \Pi_0^0\}$$

$$\Pi_2^0 = \{\forall x_1 \dots x_m \phi | \phi \in \Sigma_1^0\} \qquad \Sigma_2^0 = \{\exists x_1 \dots x_m \phi | \phi \in \Pi_1^0\}$$

$$\vdots \qquad \vdots$$

$$\Pi_{n+1}^0 = \{\forall x_1 \dots x_m \phi | \phi \in \Sigma_n^0\} \qquad \Sigma_{n+1}^0 = \{\exists x_1 \dots x_m \phi | \phi \in \Pi_n^0\}$$

$$\vdots \qquad \vdots$$

Given what we know today, where in this hierarchy can we find a formula equivalent to RH?

Theorem (Turing [1939]). RH$\in \Pi_2^0 = \{\forall x_1 \dots x_m \exists y_1 \dots y_n \phi | \phi \in \Sigma_1^0\}$.

Theorem (Kreisel [1958]). RH$\in \Pi_1^0 = \{\forall x_1 \dots x_m \phi | \phi \in \Sigma_0^0\}$.

Corollary of DPRM-Theorem [1970]. RH$\Leftrightarrow \forall x_1 \dots x_m P(x_1, \dots, x_m) \neq 0$.