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FINITE APPROXIMATIONS TO LIE GROUPS

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A certain sense in which a finite group may be said to approximate the structure of a metrical group will be discussed. On account of Jordan's theorem on finite groups of linear transformations[1] it is clear that we cannot hope to approximate a general Lie group with finite subgroups. I shall show that we cannot approximate even with groups which are 'approximately subgroups': in fact the only approximable Lie groups are the compact Abelian groups. The key to the situation is again afforded by Jordan's theorem, but it is not immediately applicable. It is necessary to find representations of the approximating groups whose degree depends only on the group approximated.

Approximability of metrical groups. Suppose G is a group with a metric D invariant under left transformations i.e. $D(ax, ay) = D(x, y)$ for all x, y, a of G. Let H_ϵ be a finite subset of G in which is defined a second product with respect to which it forms a group (if a and b are in H_ϵ their product as elements of G will be written ab; the product of them as elements of H_ϵ will be written a_0b, the inverse of a as an element of H_ϵ is written $[a]^{-1}$ and the identities of G and H_ϵ are written e, e_ϵ), and suppose each element x of G is within distance ϵ of an element $r(x)$ of H_ϵ, and for each a, b of H_ϵ $D(a_0b, ab) > \epsilon$. Then H_ϵ will be said to be an ϵ-approximation to G.

A group is said to be approximable if it has an ϵ-approximation for each $\epsilon > 0$.

Immediately from the definition we see that an approximable group is totally bounded i.e. conditionally compact. It is therefore possible to find a metric which is both left and right invariant and equivalent to the given metric, in the sense that the class of open sets is the same for either metric. In future therefore we shall suppose that our metric is both ways invariant, and we shall denote the distance between x and y by $D(x, y)$.

It has been shown by J. v. Neumann[2] that with a conditionally compact

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[1] This theorem states that a finite group of linear transformations has an Abelian self conjugate subgroup whose index does not exceed a certain bound depending only on the degree.

[2] J. v. Neumann, *Zum Haarschen Mass in topologischen Gruppen*, Compositio Mathematica, vol. 1 (1934) pp. 106–114; or alternatively, J. v. Neumann, *Almost periodic functions in a group*, Transactions of the American Mathematical Society, vol. 36 (1934) pp. 445-492, (remember that every continuous function in a conditionally compact group is a.p.). If the reader prefers to restrict the group in some way and use some other mean he has only to verify the inequality (1).

group we can define a mean for each continuous (complex-valued) function in the group, in such a way that (denoting the mean of $f(x)$ by $\int_G f(x)\,dx$)

$$\int_G (f(x) + g(x))\,dx = \int_G f(x)\,dx + \int_G g(x)\,dx$$

$$\int_G f(ax)\,dx = \int_G f(xa)\,dx = \int_G f(x)\,dx$$

and so that if $\epsilon > 0$ (and $f(x)$ is continuous), then there is a finite set of elements $a_1, a_2, \cdots, a_N$ of G such that

$$\left| \frac{1}{N} \sum_{i=1}^{N} f(xa_i) - \int_G f(x)\,dx \right| < \epsilon. \tag{1}$$

Before proceeding to the proofs of our main theorems we shall establish some elementary inequalities following immediately from our definition. Suppose the function $r(x)$ belongs to an ϵ-approximation H_ϵ to G, then

$$\begin{aligned} D(r(x)_0 r(y), xy) &\leqq D(r(x)_0 r(y), r(x)r(y)) + D(r(x)r(y), xr(y)) \\ &\qquad + D(xr(y), xy) < 3\epsilon \end{aligned} \tag{2}$$

and for any a, c of H_ϵ

$$D(c_0 a_0 [c]^{-1}, cac^{-1}) < 4\epsilon \tag{3}$$

for

$$\begin{aligned} D(c_0 a_0[c]^{-1}, cac^{-1}) &< D(ca[c]^{-1}, cac^{-1}) + 2\epsilon \\ &= D([c]^{-1}c, e) + 2\epsilon \\ &\leqq D([c]^{-1}c, [c]_0^{-1}c) + D(e_\epsilon, e) + 2\epsilon \\ &\leqq D(e_\epsilon, e) + 3\epsilon \\ &= D(e_\epsilon^2, e_\epsilon) + 3\epsilon \\ &\leqq D(e_{\epsilon 0} e_\epsilon, e_\epsilon) + 4\epsilon = 4\epsilon. \end{aligned}$$

THEOREM 1. *Let G be an approximable group with a true continuous representation by matrices of degree n. Then it may be approximated by finite groups with true representations of the same degree n.*

LEMMA. *If H_η is an η-approximation (of order h_η) to the group G and if $f(x)$ is a continuous function in G such that*

$$| f(x) - f(x') | < \Delta \quad \text{when} \quad D(x, x') < \eta$$

then

$$\left| \frac{1}{h_\eta} \sum_{a \in H_\eta} f(a) - \int_G f(x)\,dx \right| \leqq 2\Delta. \tag{4}$$

We put

$$\int_G f(x)\,dx = A \qquad \frac{1}{h_\eta} \sum_{a \in H_\eta} f(a) = B$$

then given $\epsilon > 0$ there are $a_1, a_2, \cdots, a_N$ such that

$$\left| \frac{1}{N} \sum_{i=1}^{N} f(xa_i) - A \right| < \epsilon \tag{5}$$

for each element x of G. If in (5) we successively put x equal to each member of H_η and combine the resulting inequalities we obtain

$$\left| \frac{1}{Nh_\eta} \sum_{i=1}^{N} \sum_{c \in H_\eta} f(ca_i) - A \right| < \epsilon,$$

but $D(ca_i, c_0 r(a_i)) < 2\eta$, so that $| f(ca_i) - f(c_0 r(a_i)) | < 2\Delta$ and therefore

$$\left| \frac{1}{Nh_\eta} \sum_{i=1}^{N} \sum_{c \in H_\eta} f(c_0 r(a_i)) - A \right| < \epsilon + 2\Delta. \tag{6}$$

However

$$\frac{1}{h_\eta} \sum_{c \in H_\eta} f(c_0 r(a_i)) = B$$

so that (6) yields (4) since ϵ was arbitrary.

Proof of the theorem. Without loss of generality we may suppose that the given representation of G does not contain any irreducible component more than once. Let $\chi(x)$ be the character of the representation. This function will satisfy

$$\chi(x) = \int_G \chi(xy)\overline{\chi(y)}\,dy \tag{7}$$

$$\chi(x) = \chi(cxc^{-1}) \tag{8}$$

$$| \chi(x) | \leqq n \tag{9}$$

and since it is the character of a true representation

$$\chi(x) \neq \chi(e) = n \quad \text{if} \quad x \neq e.$$

Let $\epsilon > 0$. Then for some α, $1 > \alpha > 0$, $| \chi(x) - n | > \alpha$ when $D(x, e) \geqq \frac{1}{4}\epsilon$. Now let η be so chosen that $\epsilon/16 > \eta > 0$ and

$$| \chi(x) - \chi(x') | < \alpha/(50n^2) \quad \text{when} \quad D(x, x') < 4\eta \tag{10}$$

$$| \chi(ay)\overline{\chi(y)} - \chi(ay')\overline{\chi(y')} | < \alpha/(50n) \text{ all } a, \text{ when } D(y, y') < 2\eta \tag{11}$$

and take a corresponding η-approximation H_η. If we put

$$\varphi(a) = \frac{1}{h_\eta} \sum_{c \in H_\eta} \chi(c_0 a_0 [c]^{-1}) \tag{12}$$

then

$$|\varphi(a) - \chi(a)| \leqq \frac{1}{h_\eta} \sum_{c \in H_\eta} |\chi(c_0 a_0 [c]^{-1}) - \chi(cac^{-1})| < \frac{\alpha}{50n^2} \tag{13}$$

for $D(c_0 a_0 [c]^{-1}, cac^{-1}) < 4\eta$ by (3) and therefore each summand is less than $\alpha/(50n^2)$. We have

$$\begin{aligned} \left| \frac{1}{h_\eta} \sum_{b \in H_\eta} \varphi(a_0 b)\overline{\varphi(b)} - \chi(a) \right| &\leqq \frac{1}{h_\eta} \left| \sum_{b \in H_\eta} (\varphi(a_0 b)\overline{\varphi(b)} - \chi(a_0 b)\overline{\chi(b)}) \right| \\ &+ \frac{1}{h_\eta} \left| \sum_{b \in H_\eta} (\chi(a_0 b) - \chi(ab))\overline{\chi(b)} \right| + \left| \frac{1}{h_\eta} \sum_{b \in H_\eta} \chi(ab)\overline{\chi(b)} - \int_G \chi(ay)\overline{\chi(y)}\, dy \right|. \end{aligned} \tag{14}$$

Applying the lemma to $\chi(ay)\overline{\chi(y)}$ and making use of (11) we have

$$\left| \frac{1}{h_\eta} \sum_{b \in H_\eta} \chi(ab)\overline{\chi(b)} - \int_G \chi(ay)\overline{\chi(y)}\, dy \right| < \frac{2\alpha}{50n} \tag{15}$$

and from (9), (10) we obtain

$$\frac{1}{h_\eta} \left| \sum_{b \in H_\eta} (\chi(a_0 b) - \chi(ab))\overline{\chi(b)} \right| < \frac{\alpha}{50n}. \tag{16}$$

Finally

$$\begin{aligned} &\frac{1}{h_\eta} \left| \sum_{b \in H_\eta} (\varphi(a_0 b)\overline{\varphi(b)} - \chi(a_0 b)\overline{\chi(b)}) \right| \\ &\leqq \frac{1}{h_\eta} \sum_{b \in H_\eta} |(\varphi(a_0 b) - \chi(a_0 b))\overline{\varphi(b)}| + \frac{1}{h_\eta} \sum_{b \in H_\eta} |(\overline{\varphi(b)} - \overline{\chi(b)})\chi(a_0 b)| \\ &< \frac{2\alpha}{50n} \end{aligned} \tag{17}$$

by (9) and (13). Combining (14), (15), (16), (17),

$$\left| \frac{1}{h_\eta} \sum \varphi(a_0 b)\overline{\varphi(b)} - \chi(a) \right| < \frac{\alpha}{10n} \tag{18}$$

$$\left| \frac{1}{h_\eta} \sum \varphi(a_0 b)\overline{\varphi(b)} - \varphi(a) \right| < \frac{\alpha}{8n}. \tag{19}$$

Now $\varphi(a) = \varphi(c_0 a_0 [c]^{-1})$ for each a, c of H_η. This function is therefore expressible as a sum of characters

$$\varphi(a) = \sum_{\lambda=1}^{M} \alpha_\lambda \chi^{(\lambda)}(a),$$

$\chi^{(1)}(a), \cdots, \chi^{(M)}(a)$ being the characters of the different irreducible representations of H_η. From the general theory of representations

$$\frac{1}{h_\eta} \sum_{b \in H_\eta} \chi^{(\lambda)}(a_0 b)\overline{\chi^{(\mu)}(b)} = \delta_{\lambda\mu} \chi^{(\lambda)}(a)$$

(19) therefore becomes

$$\left| \sum_{\lambda=1}^{M} \alpha_\lambda(\bar{\alpha}_\lambda - 1)\chi^{(\lambda)}(a) \right| < \frac{\alpha}{8n}.$$

Squaring each side of this inequality and summing over H_η,

$$\frac{1}{h_\eta} \sum_{\lambda=1}^{M} \sum_{a \in H_\eta} | \alpha_\lambda |^2 | 1 - \alpha_\lambda |^2 | \chi^{(\lambda)}(a) |^2 = \sum_{\lambda=1}^{M} | \alpha_\lambda |^2 | 1 - \alpha_\lambda |^2 < \frac{\alpha^2}{64n^2}.$$

If we define $\xi(a)$ by

$$\xi(a) = \sum_{|1-\alpha_\lambda| > |\alpha_\lambda|} \chi^{(\lambda)}(a)$$

it will satisfy

$$\frac{1}{h_\eta} \sum_{a \in H_\eta} \xi(a_0 b)\overline{\xi(b)} = \xi(a) \tag{20}$$

and

$$\begin{aligned} \frac{1}{h_\eta} \sum_{a \in H_\eta} | \xi(a) - \varphi(a) |^2 &= \sum_{\lambda=1}^{M} \operatorname{Min}\,(| \alpha_\lambda |^2, | 1 - \alpha_\lambda |^2) \\ &\leqq 4 \sum_{\lambda=1}^{M} | \alpha_\lambda |^2 | 1 - \alpha_\lambda |^2 < \frac{\alpha^2}{16n^2}. \end{aligned} \tag{21}$$

We now wish to infer from the inequality (21) that the functions $\varphi(a)$ and $\xi(a)$ differ only slightly at each point of H_η. This is possible on account of the relations (19), (20).

$$\begin{aligned} &\left| \frac{1}{h_\eta} \sum_{b \in H_\eta} (\xi(a_0 b)\overline{\xi(b)} - \varphi(a_0 b)\overline{\varphi(b)}) \right| \\ &\leqq \sum_{b \in H_\eta} \frac{1}{h_\eta} \left| (\xi(a_0 b) - \varphi(a_0 b))\overline{\xi(b)} \right| + \sum_{b \in H_\eta} \frac{1}{h_\eta} \left| (\overline{\xi(b)} - \overline{\varphi(b)})\varphi(a_0 b) \right| \\ &\leqq \left\{ \frac{1}{h_\eta} \sum_{b \in H_\eta} | \xi(b) - \varphi(b) |^2 \right\}^{\frac{1}{2}} \left(\left\{ \frac{1}{h_\eta} \sum_{b \in H_\eta} | \xi(b) |^2 \right\}^{\frac{1}{2}} + \left\{ \frac{1}{h_\eta} \sum_{b \in H_\eta} | \varphi(b) |^2 \right\}^{\frac{1}{2}} \right) \\ &< \frac{\alpha}{4n}(n + n) = \frac{1}{2}\alpha, \end{aligned} \tag{22}$$

since $| \xi(b) | \leqq n$ and $| \varphi(b) | \leqq n$ for each b of H_η. Now combine (18), (20), (22), and we have

$$| \xi(a) - \chi(a) | < \frac{1}{2}\alpha + \frac{\alpha}{10n} < \alpha.$$

This implies that $\xi(e_\eta) = \chi(e) = n$ and that if $D(a, e) \geqq \frac{1}{4}\epsilon$ then $\xi(a) \neq \chi(e) = \xi(e_\eta)$. $\xi(a) = \xi(e_\eta)$ only for elements of a certain self-conjugate subgroup N entirely contained within distance $\frac{1}{4}\epsilon$ of the identity of G. The factor group has a true representation of degree n, and I shall show that it can be taken as a

ϵ-approximation to G. We choose an element in each coset of N as a representative of that coset and define the function $v(a)$ (a in H_η) to be the representative of the coset in which a lies. The totality of elements $v(a)$ we call K. Putting $v(a) \otimes v(b) = v(a_0 b)$, K forms a group with respect to the product $\otimes$. For each a of H_η there is an element m of N for which $v(a) = a_0 m$ and therefore

$$D(a, v(a)) \leqq D(a, am) + D(am, a_0 m) < \tfrac{1}{4}\epsilon + \eta.$$

Consequently if we put $R(x) = v(r(x))$ we have

$$D(R(x), x) \leqq D(v(r(x)), r(x)) + D(r(x), x) < (\tfrac{1}{4}\epsilon + \eta) + \eta < \epsilon$$

and

$$D(v(a) \otimes v(b), v(a)v(b)) \leqq D(v(a_0 b), a_0 b) + D(a_0 b, ab) + D(ab, v(a)v(b)) < 3(\tfrac{1}{4}\epsilon + \eta) + \eta < \epsilon,$$

which shows that K is an ϵ-approximation to G.

THEOREM 2. *An approximable Lie group is compact and Abelian.*

LEMMA. *A closed subgroup of a connected group cannot have a finite index greater than* 1.

Suppose H is a closed subgroup of G and has index i, $1 < i < \infty$. Then $G - H$ is not void and is closed, being the sum of a finite number of closed sets, the cosets of H. G is the sum of two closed disjoint sets neither of which is void, and therefore is not connected.

If G is a compact Lie group it cannot have a closed subgroup of positive measure different from the whole group.

PROOF OF THE THEOREM. An approximable Lie group is complete and conditionally compact, i.e. it is compact, and is therefore a group of linear transformations,[3] of degree n say. By theorem 1 we can approximate it by finite groups H_ϵ of linear transformations of degree n. But by Jordan's theorem[4] each finite group of linear transformations has an Abelian subgroup whose index does not exceed a certain bound $Z(n)$ depending only on the degree. Let A_ϵ be this Abelian subgroup in H_ϵ. Then there is a finite number $c_1, c_2, \cdots c_N$ ($N \leqq Z(n)$) of elements of H_ϵ such that every element of H_ϵ is of the form $c_{i0} a$ where a is in A_ϵ. For any x of G we have

$$D(x, r(x)) < \epsilon$$

$$r(x) = c_{i0} a, \; a \in A_\epsilon, \; i \leqq N$$

$$D(c_{i0} a, c_i a) < \epsilon.$$

[3] J. v. Neumann, *Die Einführung analytischer Parameter in topologischen Gruppen*, Annals of Mathematics, vol. 34 (1933), pp. 170-190.

[4] A. Speiser, *Theorie der Gruppen von endlicher Ordnung*, (Berlin 1927) 2nd ed., p. 215.

Hence every element of G is of the form $c_i ad$ where d is within distance 2ϵ of the identity of G and $i \leqq N$. The points ad must therefore form a set E_ϵ of measure $1/Z(n)$ at least. Now put $x = ad\ y = a'd'$:

$$
\begin{aligned}
D(xy, yx) &= D(ada'd', a'd'ad) \\
&\leqq 2D(d, d') + D(a_0 a', a_0' a) + D(aa', a_0 a') + D(a_0' a, a'a) \qquad (23)\\
&< 6\epsilon.
\end{aligned}
$$

In the product group $G \times G$ we have therefore a set $E_\epsilon \times E_\epsilon$ of pairs (x, y) of measure $1/(Z(n))^2$ at least, in which $D(xy, yx) < 6\epsilon$. Now take a sequence ϵ_i tending to 0, and put $F_i = \sum_{j \geqq i} E_{\epsilon_j} \times E_{\epsilon_j}$, $E = \prod F_i$. For each $i \leqq N$,

$$mF_i \geqq m(E_{\epsilon_i} \times E_{\epsilon_i}) \geqq \frac{1}{(Z(n))^2}.$$

Then $mE \geqq 1/(Z(n))^2$ since the F_i are a decreasing sequence. If $(x, y) \in E$ then for each i, $(x, y) \in F_i$, i.e. $x \in E_{\epsilon_j}$, $y \in E_{\epsilon_j}$ for some $j \geqq i$. Then by (23), $D(xy, yx) < 6\epsilon_j \leqq 6\epsilon_i$: but i was arbitrary so that $D(xy, yx) = 0$, $xy = yx$.

Now let N_x be the set of those y for which $xy = yx$, i.e. the normaliser of x. Then

$$\int_G mN_x \, dx \geqq mE \geqq \frac{1}{(Z(n))^2}.$$

Consequently $mN_x > 0$ in an x-set of positive measure. But if $mN_x > 0$ we have $N_x = G$ by the lemma, for N_x is certainly closed. This shows that the centre of G is of positive measure, and again applying the lemma we see that G is Abelian.

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