# EQUIVALENCE OF LEFT AND RIGHT ALMOST PERIODICITY

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In his paper "Almost periodic functions in a group", J. v. Neumann† has used independently the ideas of left and right periodicity. I shall show that these are equivalent.

$f(x)$ is a complex-valued function of a variable $x$ which runs through an arbitrary group $\mathfrak{G}$. $f(x)$ is said to be right almost periodic (r.a.p.) if for each $\epsilon > 0$ we can find a finite set $b_1, \ldots, b_m$ of elements of $\mathfrak{G}$ such that to each $t$ of $\mathfrak{G}$ there corresponds a $\mu = \mu(t)$ satisfying

$$|f(xt) - f(xb_\mu)| < \epsilon \quad \text{for all } x \epsilon \mathfrak{G}. \tag{D}$$

The definition of left almost periodicity is obtained from this by replacing the inequality (D) by

$$|f(tx) - f(b_\mu x)| < \epsilon.$$

Suppose now that $f(x)$ is r.a.p., then to prove $f(x)$ l.a.p. it is sufficient to find, for each $\epsilon > 0$, a finite number of elements $c_1, \ldots, c_n$ of $\mathfrak{G}$ such that to each $s$ of $\mathfrak{G}$ there corresponds a $\nu = \nu(s)$ satisfying

$$|f(sb_\pi) - f(c_\nu b_\pi)| < \epsilon \quad \text{for each } \pi; \tag{K}$$

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† J. v. Neumann, *Trans. American Math. Soc.*, 36 (1934), 445–492.

for then, by the r.a.p. property of $f(x)$,

$$|f(sb_\mu)-f(st)|<\epsilon,$$

$$|f(c_\nu b_\mu)-f(c_\nu t)|<\epsilon,$$

where $\mu=\mu(t)$.

Putting $\pi=\mu(t)$ in the inequality (K), we have

$$|f(st)-f(c_\nu t)|<3\epsilon \quad \text{for each } t,$$

*i.e.* $f(x)$ is l.a.p.

To prove the existence of the elements $c_1, \ldots, c_n$ let us introduce a space $R$ of $m$ complex dimensions. Consider the set $S$ of points $P_y$ of $R$ whose coordinates are $[f(yb_1), \ldots, f(yb_m)]$ ($y$ runs through $\mathfrak{G}$). $f(x)$ being r.a.p. is bounded*; $S$ is therefore bounded and can be covered with a finite number of spheres of diameter $\epsilon$ each containing some point of $S$. Let the finite set of elements of $S$ obtained in this way be $P_{c_1}, \ldots, P_{c_n}$; then for each $s$ of $\mathfrak{G}$ there is a $\nu=\nu(s)$ with $P_s$ distant less than $\epsilon$ from $P_{c_\nu}$; hence, for each $\mu$,

$$|f(sb_\mu)-f(c_\nu b_\mu)|<\epsilon,$$

*i.e.* $c_1, \ldots, c_n$ have the required property.

Thus $f(x)$ is r.a.p. implies that $f(x)$ is l.a.p. and the converse follows similarly or by the use of the inverse group. v. Neumann's theory can now be used to show that each l.a.p. function has a unique left mean. Previously it was necessary to suppose $f(x)$ to be both l.a.p. and r.a.p. The theory of a.p. functions in a group can now be taken over to sets of objects which admit transitive transformations by the group. Let $\mathfrak{A}$ be a set of objects admitting (left-) transformations by the group $\mathfrak{G}$. Represent the elements of $\mathfrak{A}$ by small Gothic letters. Then to a function $f(\mathfrak{y})$ in $\mathfrak{A}$ corresponds a function $f(y)$ in $\mathfrak{G}$ defined by $f(y)=f(\mathfrak{y})$, whenever $y\mathfrak{t}=\mathfrak{y}$, $\mathfrak{t}$ being some fixed element of $\mathfrak{A}$. $f(\mathfrak{y})$ may be said to be a.p. if $f(y)$ is l.a.p., and will then have a unique left mean.

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* Putting $x=e$ in (D), we have

$$|f(t)-f(b_\pi^*)|<\epsilon.$$

Then $$|f(t)|<\epsilon+\max\{|f(b_1)|, \ldots, |f(b_m)|\}.$$

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