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Turing on "Common Sense": Cambridge Resonances

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Abstract Turing was a philosopher of logic and mathematics, as well as a mathematician. His work throughout his life owed much to the Cambridge *milieu* in which he was educated and to which he returned throughout his life. A rich and distinctive tradition discussing how the notion of "common sense" relates to the foundations of logic was being developed during Turing's undergraduate days, most intensively by Wittgenstein, whose exchanges with Russell, Ramsey, Sraffa, Hardy, Littlewood and others formed part of the backdrop which shaped Turing's work. Beginning with a Moral Sciences Club talk in 1933, Turing developed an "anthropological" approach to the foundations of logic, influenced by Wittgenstein, in which "common sense" plays a foundational role. This may be seen not only in "On Computable Numbers" (1936/7) and Turing's dissertation (written 1938, see (1939)), but in his exchanges with Wittgenstein in 1939 and in two later papers, "The Reform of Mathematical Phraseology and Notation" (1944/5) and "Solvable and Unsolvable Problems" (1954).

4.1. Introduction

Turing's philosophical attitude has often been distorted by controversies in recent philosophy of mind—above all, those that have associated his thought with computationalist and behaviorist reductionisms. Of course, Turing pioneered philosophical discussion of computational explanation and modelling in such far-flung fields as cognitive science, artificial intelligence, neurological connectionism, and morphogenesis. He also formulated the famed "Turing Test" in an explicitly philosophical essay (1950). His model of computability is regarded by philosophers of mind and language as a standard model of cognition ---although its ability adequately to explain or reduce general semantic and representational elements of intelligence and thought, as well as physical and evolutionary processes, is increasingly questioned.[1] In all this fray, appreciation of Turing's own *philosophical* sophistication, especially about uncritical notions of meaning and mentality, has tended to be obscured and underplayed.

[1] Cf. Putnam (1988), Duwell Chap. 8, Rescorla Chap. 11.

I will argue that the standard views here---that Turing was an arch-mechanist and reductive computationalist or functionalist, if not a behaviorist and "strong" proponent of AI---are just wrong. Rather than offering general theories of mind, matter, emergence and/or meaning in the traditional sense of a foundation for knowledge, reference, truth and/or ontology, Turing artfully dodged head-on confrontation with such controversies, shifting their flavor. *Un*decidability and *ir*reducibility were central issues for him: how to think about the *limits* of computational, mathematical and philosophical explanations, as well as the consequent need for what logicians and philosophers, as well as computer scientists, call *types*. These are the delimited, surveyable ordering and organizing of objects, concepts, terms, logical particles, definitions, proofs, procedures and algorithms into surveyable wholes. What he provided for future philosophy and science was a way of thinking, a logical framework in which the cost of the absoluteness and ubiquity of computability, as a notion, is flexibility at the basis and on the margins. His universal machine entails a fluidity between hardware, software and data in the realm of computability (cf. Davis Chap. 5). Philosophically Turing also emphasized, however, what we can *do* by way of articulation in ordinary language (cf. Parikh and Renero Chap. 3). We can only grasp his *philosophical* sophistication if we keep what he called "common sense" firmly in mind.

Turing took "common sense" to be a communicable, meaningful level of human discussion awaiting study of its "phraseology". These notions of "common sense" and "phraseology" he developed in conversations with Wittgenstein, as he explicitly acknowledged (See Sect. 4.7 below). Wittgenstein in turn developed these notions in response to an ongoing Cambridge philosophical tradition in logic. Wittgenstein's later philosophy was partly shaped in response to Turing, as some of Turing's work was shaped by him. Here we focus on the Wittgenstein-to-Turing direction. Our point is that Turing's engagement with "common sense", throughout his life, bespeaks his having embedded himself, quite self-consciously, in Cambridge's extended philosophical circles of debate and pedagogy. We shall argue that it influenced several of his most important papers, and ongoing work throughout his life.

For Turing, as for Wittgenstein, "common sense" and "in-the-street" "phraseology" are something received and in turn continually being altered through human discussion, invention, characterization, technology and discovery. With our limited individual cognitive architecture, we humans have long offloaded tasks to calculative routine. Nevertheless, a residue of what is not offloaded has always remained. This end-user conversation is where the *significance* of results, social processes, and logical standards may be contested, redesigned, and re-interpreted. Turing took this residue of "common sense" to be evolving under the pressure of human culture, intellectual development, integration of technology and biology. But its *logical* role he saw as an irreducible part of the foundations. Already when he was thirteen, he had written, "I always seem to want to make things from the thing that is commonest in nature and with the least waste in energy".[2] "Common sense" and ordinary "phraseology" became a logico-philosophical realization of this idea.

Turing was primarily a mathematician, but one of a very special and reflective kind, and the range and breadth of his contributions were due in part, as he realized, to their deep resonances with philosophical controversies of his day. He himself attempted to articulate and explore this in academic and popular philosophical settings. By tracing the course of his logical and foundational work with an eye on what he was doing *vis-à-vis* his predecessors and his contemporaries at Cambridge, we shall argue that his distinctive, sophisticated attitude---one whose philosophical importance it is easy to overlook or underestimate---comes into view.

Turing's interests in physics, and the extent and limitations of mechanistic explanations of nature, will thus not be our focus. Instead, we shall focus on his foundational interests as they were directed toward the nature of mathematics and logic. Turing's main philosophical contribution here was to recast and clarify the primordial *notion* of logic---by which we shall mean here nothing less than all the different routines, subroutines, patterns and regularities that humans design and follow in their procedures of describing, informing, proving, inferring, arguing about and coming to terms with one another and nature. Turing shifted attention away from the aim of "grounding" logic, mathematical truth, or meaning, and toward an emphasis on *us*---though not primarily, and certainly not only, *via* a theory of mind.

Logic, the technology of deduction, is entangled with what we can write down and take in in the way of patterns and procedures---but also what we can *say* about and do *with* them. In a computational world, applications *via* all kinds of architecture and modelling in physics and mathematics are of great interest, as are cognitive, biological and neurological implementation of routines (whether general purpose or highly specific and modular in character). But these did not frame the fundamental points of departure, philosophically speaking, for Turing. Instead, for him it was always the user perspective, the *interface* that mattered to logic. Here he focused, not only on issues of artifice and accessibility, but also the *reality* of ordinary "phraseology". This seemingly ephemeral and amorphous point is one that he rigorously pursued into the very foundations of logic and mathematics.

The period leading up to "On Computable Numbers" has been a bit of a mystery to those analyzing Turing's intellectual development, insofar there is little direct evidence for how his activities as an undergraduate (1931-34) pointed toward this great work. We do know from Hodges that Newman's course on foundations of

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[2] Hodges (1983/2012), p. 19.

mathematics, in the spring of Turing's first post-graduate Fellowship year (1935) had an immediate impact, introducing him to mathematical logic and setting him onto the *Entscheidungsproblem*; yet we cannot be sure of exactly which lecture courses he attended during his undergraduate years.[3] Nevertheless, there are some important philosophical features of his Cambridge *milieu* that may be traced, and plausibility arguments may be given about these. Our thesis—a development of some of what Turing's biographer Hodges has suggested—is that philosophical ideas around him at Cambridge influenced Turing significantly during his undergraduate years, and especially his discussions with Wittgenstein, who in turn was responding to a longstanding Cambridge tradition of "common sense".[4]

Gödel, struggling in the 1930s with the notion of "general recursive" alongside Church and Kleene, remained focused on devising axiomatic and equational formulations necessary for a logically and mathematically rigorous analysis of "effective calculability", but here "effective" was a broadly "heuristic" notion, deriving from examples intuitively presented, not something philosophically analyzed.[5] By contrast, the philosophical backdrop at Cambridge, in which the *nature* of logic, meaning and mathematics were under discussion, brought to the fore intuitive, human aspects of logic as evolving, purposive technology. Logic was approached, not first and foremost axiomatically, but practically and in thought experiments. These features of the backdrop mattered centrally to Turing. If inspiration is to be found for the distinctive characteristics of his analysis of computability in his undergraduate years 1931-34, it is here, in the philosophical foundations of logic.

We shall reconstruct in what follows how Turing's notion of "common sense" works its way into his argumentation in several works: a talk on "Mathematics and Logic" to the Moral Sciences Club, December 1933; "On Computable Numbers" (1936/7); Turing's exchanges with Wittgenstein in Wittgenstein's 1939 Cambridge lectures (1976/1989, hereafter "LFM"); a later, unpublished paper

[3] Hodges (1983/2012) pp. 63-64 and (1999) pp. 5-7 focus on the importance of Turing's essay "The Nature of Spirit" (1932), where Turing discusses McTaggart, and his reading of Eddington's (1929) before coming to Cambridge. Hodges also argues (1983/2012) that Eddington's autumn 1933 course on the methodology of science also had a serious impact on Turing (p. 87), not least setting him on a course of thinking about the Central Limit Theorem, proved in his Fellowship essay. We shall contest none of this.

[4] In (1983/2012) Hodges weaves general aspects of Wittgenstein's thought into his discussion of Turing's work leading up to "On Computable Numbers". In (1999), however, he stresses the absence of any direct positive evidence for Wittgenstein's impact on Turing, arguing that Wittgenstein's 1939 lectures (1976/1989) (LFM) "shed no light on Turing's view of mind and machine" (pp. 22-24). In what follows we agree, but offer a different, though admittedly circumstantial, reconstruction of Turing's development spelling out Hodges' prior hints. As we argue, the nature of mind was not the driving issue behind Turing's initial work in logic, whereas the nature of logic was.

[5] Cf. Mundici and Sieg Chap. 1, Kennedy Chap. 2, section 2 above.

"The Reform of Mathematical Notation" (1944/5); and his last paper, "Solvable and Unsolvable Problems" (1954). We first shall move thematically (Sect. 4.2-4.3) and then chronologically (Sect. 4.4-4.9), using the arc of Turing's engagement with his Cambridge philosophical *habitus* as the basic line through.

My overall suggestion is that we take philosophical reflection to have been an essential component of Turing's development and contributions, rather than a spectator-sport or a sideline distraction vaguely lying in the penumbra of his thought. New ways of critically thinking about meaning and logic served in this case as a genuine intellectual resource—one all too easy to underestimate as a force.

4.2 The Human Interface

"Common sense", I am arguing, mattered centrally to Turing's logico-mathematical work throughout his life. This is especially evident in his resolution of the *Entscheidungsproblem* in "On Computable Numbers" (1936/7), his most important and groundbreaking paper. In this paper what Turing *did* was to inject common sense into the heart of the foundations of logic and mathematics. We shall make this out in a more detailed discussion of the paper below, when we revisit the heart of his *specific* argumentation (Sect. 4.5). Here we address the human interface and the importance of "common sense" to the foundations of logic as these figured in the general logico-philosophical context at the time.

In the Hilbert school, the notion of a "general method" for deciding problems---the key notion at issue in the *Entscheidungsproblem*---was framed in terms of the idea of a "finitistic", step-by-step calculation, "derivation" or proof in a formal system of logic. In attacking the intuitive idea of "effective calculability", logicians of the early 1930s generated a more and more crowded and refined set of different logical systems and approaches, finding themselves awash in ideological, philosophical and heuristic discussions. In describing the situation, Post aptly referred to "the forbidding, diverse and alien formalisms" in which the work of logic had been embedded.[6] Mathematical clarification, the theoretical assimilation of this work, would have to take place in *informal* discussion, where the significance and conceptual articulation of the situation could be settled, even if all the reasoning used therein could be, with labor, coded back into the symbolic setting.

By 1935 a desideratum, given the differing precisifications, was an analysis of the general notion of *formal system* itself. In order to gain this, it would not have been enough to write down another, slightly different formalism: one had instead

[6] Post (1944), p. 281 is looking backward here, summarizing the developments historically.

to place the very idea of what such writing down *is* in a new light, one that did not turn on choice of a particular notation, a set of principles, or a specific "metalogic". What was needed was a philosophical clarification of the *first* step, a novel way of thinking. This is what Turing provided: he made the notion of a "formal system" *plain*. Turing's analysis of computability is, as Hodges remarks, "profoundly ordinary".[7]

The vernacular vividness of Turing's analysis in "On Computable Numbers" is widely appreciated. Gödel offered enthusiastic praise when he wrote that Turing offered "the precise and unquestionably adequate definition of the general concept of formal system", one which finally allowed the 1931 incompleteness theorems to be "proved rigorously for *every* consistent formal system containing a certain amount of finitary number theory".[8] Gödel had in mind here the mathematically distinctive features of Turing's contribution, first and foremost that the analysis does not depend upon the formalism chosen. This is crucial, not only for demonstrating the precise scope of Gödel's incompleteness results, but also for showing that the notion of "computable" is *absolute*, i.e., it does not change depending upon the particular formal system or set of axioms used. As Gödel had earlier remarked (1946),

> In all other cases treated previously, such as demonstrability or definability, one has been able to define [the fundamental notions] only relative to the given language, and for each individual language it is clear that the one thus obtained is not the one looked for. For the concept of computability, however, although it is merely a special kind of demonstrability or decidability, the situation is different. By a kind of miracle it is not necessary to distinguish orders, and the diagonal procedure does not lead outside the defined notion.[9]

We shall scrutinize the Wittgensteinian air that directly resonates with the heart of Turing's diagonal argumentation in "On Computable Numbers" below (Sect. 4.5). For now, we need to draw out the general role that the themes of "common sense" and "phraseology"[10] *had* to play at the time, anticipating Turing's use of the latter phrases in two of his later works ((1944/5), (1954); see Sects. 4.7-8 below).

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[7] Hodges (1983/2012), p. 96.

[8] Note added 1964 to Gödel (1934), in Gödel (1986), p. 369; cf. Davis (1982).

[9] Gödel (1946) in his (1990), pp. 150-153. Compare Sieg (2006), especially pp. 472ff and Kennedy (2013, 2014) and Chap. 2.

[10] The idea of "phraseology" occurs explicitly in *The Blue Book* (1965), p. 69; it appears in Wittgenstein's manuscripts and lectures around differing conceptions of numbers and mathematics, cf. e.g. (1999) MS 121, p. 76 (1939), MS 126, p 131 (1942-43); MS 127, p. 194 (1943) and (1976), pp. 18,91,98. Floyd and Mühlhölzer (unpublished) discuss this notion at length in the context of an analysis of Wittgenstein's annotations to a 1941 edition Hardy's *Course of Pure Mathematics*.

Each and every individual Turing Machine may be regarded as a precise mathematical object—in fact, a formal system or system of equations, or a collection of quintuples. But, as Gödel noted, the distinctively wide applicability and robustness of Turing's *analysis* of formal logic comes from its other face: the fact that it bypasses entanglement with this or that particular formal language or this or that choice of logic (within a range of recursively axiomatizable, finitary languages of the usual kind). It reduces our general notion of a formal system to an ordinary, informal *comparison* between the step-by-step routine of a human calculator and a machine---in other words, to something belonging to "common sense". In this way Turing avoided entangling himself with dogmatism or ideology, formalism vs. Platonism, intuitionism vs. classical logic, and so on. He showed that it is no part of our general notion of a human being following a (formal) rule that the person must be a classical, or intuitionistic, or any other particular kind of theorist of logic and/or mathematics.[11] As Gandy wrote of Turing's cutting through the knot of formalisms in 1936, "let us praise the uncluttered mind".[12] Turing's comparison allows us to recognize his novel construal of the *Entscheidungsproblem* as a remarkable piece of "applied philosophy"[13], a "paradigm of philosophical analysis".[14]

Turing *unvarnished* logic. How did he do this? His basic move was to utilize what Wittgenstein would have been calling, during Turing's undergraduate days---and in dictations widely circulated among mathematics students at Cambridge---the method of a "language game" or an "object of comparison".[15] Later, perhaps reacting to "On Computable Numbers", Wittgenstein himself explicitly contrasted the use of an informal, everyday, comparative understanding of the foundations of logic with an axiomatic approach, stressing that the former offers a more realistic conception of foundations of logic, whereas the latter invites needless dogmatism and dispute.[16]

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[11] Cf. Floyd (2012b).

[12] Gandy (1988), p. 85.

[13] Davis (1982), p. 14.

[14] Gandy (1988), p. 86.

[15] The method of "comparisons" is evinced in *The Blue Book* (1965) (BB), pp. 5,70 and especially in *The Brown Book* BB §§13,10 (p. 140), §13 (p. 153), §15 (p. 158), §16 (pp. 162, 164), §22, p. 179).

[16] Wittgenstein's use of "*Vergleichsobjekte*" in this particular context of the foundations of logic first appears in (1999) MS 157b, pp. 33-4, drafted in February 1937, just after he would have likely received "On Computable Numbers" from Turing. It reads:

> Our exact language games are not preparations for a future regimentation of our everyday language, like first approximations without friction and air resistance. This idea leads to *Un*games (*Unspiele)* (Nicod & Russell [JF: who were pursuing the axiomatic approach to logic].) Much more, the language games stand as *objects of comparison*, which, through similarities and differences should throw light on the relationships of our language.

Turing's fundamental step in "On Computable Numbers" rests on his drawing a "comparison" between a human computor[17] and a machine (1936/7 §1). This method is applied at the first step, in Turing's appeal to an ordinary snapshot of a limited portion of human behavior, a calculation made using pen and paper. He was making mathematics out of a "language game", a simplified snapshot of a portion of human language use designed to elicit from us insights into the workings of logic. This move is, philosophically speaking, *fundamental* to the power of his analysis.

In order to appreciate its philosophical significance, we must understand—contrary to what many have suggested elsewhere—that Turing's analysis did *not* rest on any general theory of language, logic, or mind, even if he utilizes the notion of "state of mind" explicitly in "On Computable Numbers". Just here the Cambridge resonance comes into play. For Wittgenstein's lectures at Cambridge during Turing's years there had stressed, not only the need to extrude issues about inner, mental processes from the foundations of logic; it also brought to the fore an "anthropological" perspective on logic, one that stressed everyday human interfaces with logic in informal language as the place where a common basis or "foundation" would have to appear.[18]

Before we delve into this Cambridge backdrop, it is important to qualify our general remarks. As Mundici and Sieg show (Chap. 1), Turing's analysis is subject to perfectly precise axiomatic presentation within the theory of computability. So in saying that Turing connected his analysis with an *informal* model of what we do—thereby doing something of great philosophical interest—we are not saying that he defeated, delimited or circumscribed in any way the mathematician's autonomy to mathematically clarify mathematics: far from it. Turing—and independently Post, with his "workers"[19]—showed how the finistic, constructivistic elements of the proof theory of the Hilbert program, motivated by an idea of rigor that aimed to make mathematical arguments humanly controllable, *had* to be construed in terms of the model of a human computor with limited powers of taking in symbolic configurations *in order to be made mathematically precise*. It had to be done, this self-reflection, this beginning and ending with a snapshot of what we do in doing mathematics.

Turing's specific Cambridge perspective included in this, however, something else: the *general* importance and interest of how we ordinarily phrase ourselves,

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For the clarity after which we strive is in any case a *complete* one. But this only means that the philosophical problems should *completely* disappear. (cf. (2009) (PI) §130).

[17] Until the late 1940's "computer" referred to a person, often a woman, who carried out calculations and computations in the setting of an office or research facility. Nowadays "computor" is used to make the human user explicit.

[18] Englemann (2013b).

[19] Cf. Post (1936).

i.e., "common sense", not only for mathematics, but also for what he would later call its "merging" with logic (1944/5, cf. Sect. 4.7 below). The mathematical precisification had to be done with minimal fuss and maximal range of applicability. Unlike Post and Gödel, Turing did not believe that his proof rested upon, or even necessarily applied to, limits of the human mind *per se*.[20] Such issues he dodged entirely. Taking up an "anthropological", rather than an ideological perspective on logic, he was able to bypass or transform the issue of psychological fidelity at the foundations, *de*-psychologizing logic, thereby leaving his interest in the mind and its capacities to later works.

Turing's model connected the *Entscheidungsproblem* both to mechanical procedures and to an analysis of the primordial mathematical notion of an algorithm, *via* a kind of snapshot modeling of human mathematical practice. But it therefore also allowed for a generalization across all of logic itself, every recipe of a sort that we may devise and apply, including the recursive aspects of human symbolism and narrative (cf. Winston Chap. 10). It was Turing's insight into the *human purposiveness* of the notion of computability specifically—and logic more generally—that enabled him to boil the relevant notions down to their most vivid, ordinary, essential and simple elements. This turns, however, on Turing's having *not* offered a general definition of logicality, the nature of thought, or the nature of meaning or mind.

This was a bold philosophical innovation in the mid-1930s. The point may be made out by contrasting Turing's approach to philosophical problems with that of Carnap, whose "logical syntax" program also aimed to achieve an appropriate philosophical clarification of formal logic within a largely empiricist framework (one that rejected "self-evidence" of logical principles as an *a priori* feature of logic). Carnap took axioms to constitute the meanings of terms in a language, including the logical terms. Rejecting a crude version of verificationism about meaning, he adopted the "principle of tolerance"---that "in logic there are no morals"---and reduced philosophy to the comparative and pragmatic study of the (formal, axiomatized) logical syntax of the language of science.[21] He soon integrated Tarskian "semantics" and the definability of truth in metalanguages into this ambitious program, thereby making the development of scientifically

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[20] Compare Post's remarks about "psychological fidelity" (1936), p. 105, and Gödel (1972a) in (1990), at p. 306, including the introductory note Webb (1990). The reader may contrast Hodges (1999), p. 22 and see Floyd (2012b) for further argumentation. On Gödel vs. Turing, see Copeland and Shagrir (2013), Sieg (2013a).

[21] Carnap (1934); cf. Carnap, Maund and Reeves (1934) and Carnap (1935).

respectable criteria of logicality and analyticity of truth for the languages and metalanguages of science central to the philosophical enterprise.[22]

Turing cut through the whirl very differently from the way that Carnap did, and concertedly so. He used the notion of a "computable number" as a stand-in for the general notion of an effective, indefinitely applicable, mechanical, but humanly operated rule. The comprehensiveness of this treatment---its lack of "morals", if you will---is rooted in his fundamental analytical device, a snapshot of what a human being *does* in computing: scanning a finite sequence of symbols in a step-by-step manner, taking in sequences of them at a glance, moving to a locally differentiated, shareable state, and then operating step-by-step in accordance with finitely articulated, definite "orders" or shareable commands. Turing analyzed what a step in a formal system *is* by thinking through what it is *for*, i.e., what is *done* with it.

Unlike Carnap, Turing was not invested in any attempt to generally characterize the notion of purely logical, meaning-theoretic, or "analytic" truth in terms of formal systems. As we shall soon see, he explicitly and repeatedly bemoaned the idea of forcing the logician or mathematician into a "cast iron system" in which all language would be paraphrased (see Sect. 4.7 below). As opposed to Carnap's proposal that logic (even when relativized to a particular framework) should be able to plan and determine in advance all the purely conceptual, logico-analytic truths about meanings and structures, Turing focused on what it is that we *do* in mathematics or logic when we succeed in setting up a systematic search or effective calculational routine. He was happy to let contexts vary, as we clip one routine out of another and amalgamate it with other routines. Unlike Carnap, Turing valorized "common sense", rather than bemoaning its unclarities: mathematical notation and ordinary language are the result of long evolution of a cooperative enterprise taking place over centuries, and encode "common sense".

These very different philosophical attitudes toward mathematics and logic mattered. The circles and steps among the notions of proof, argument, and step-by-step calculability in a logic, easily regarded as prosaic, are rigorous and mathematically tight, demonstrably as well as intuitively so.[23] For throughout Turing's reasoning in "On Computable Numbers", formal derivations, routines, computable processes, alphabets and functions may be regarded as mathematical or logical objects receiving precise characterization. Nevertheless, at the heart of this mathematical analysis remains the fact that the very idea of dynamic, step-by-step motions, actions, processes and states of a Turing machine are given *to* it by our common sense ways of regarding and using it. In fact, implicit throughout this

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[22] Carnap's earlier program of "general axiomatics", broached during Gödel's student days with him in the late 1920s, had also been intended to make the proper analysis of symbolic logic (Carnap 2000).

[23] See Kripke (2013).

whole range of thinking is something obvious, yet terribly easy to underestimate: *proofs and algorithms are used by humans*.

It is this that Wittgenstein had in mind when, reminiscing ten years after his 1937 conversations with Turing about "On Computable Numbers", he wrote:

> Turing's 'Machines': these machines are *humans* who calculate. And one might express what he says also in the form of *games* ((1980b) (RPP I) §1095).[24]

What we want to do is to explore how and why Turing embedded himself in a Cambridge context in which this brief philosophical remark could resonate so well with what he had done.

4.3 Turing's Path to "On Computable Numbers", 1931-1935

A distinguished historian of logic, Grattan-Guinness, recently argued that Newman's course in mathematical logic, which Turing followed during Lent Term 1935---in the first year of his graduate fellowship at King's College (1934-37)---was the sole significant stimulus to Turing's great work, "On Computable Numbers". On this account, without Newman's tutelage, Turing would never have become interested in the foundations of logic at all. Grattan-Guinness writes:

> A possible alternative source of logic for Newman and Turing were a few Cambridge philosophers; but they were largely concerned with revising logicism (especially Frank Ramsey, who went to see Wittgenstein in Austria in 1923 and 1924, and died in 1930). Further, Wittgenstein, back in Cambridge from 1929, was a monist, and so distinguished between what can be said and what can only be shown. So while these philosophers did indeed engage with Turing [Hodges 1983, 152–154], they would not have directed [Turing] toward recursive functions or decision problems. There, by contrast, the Hilbertians stressed the *hierarchy* of "mathematics" and "metamathematics", upon which the actual Turing seized. In the same style Carnap coined "metalogic" in 1931 from seeing it playing a central role in Gödel's [1931] paper, and Tarski was starting to speak of "metalanguage." ((2013), p. 61).

Grattan-Guinness was right to argue that Newman's course was a crucial stimulus for Turing's quick advances in logic, and in two other papers he has established the importance of philosophy in general, and Russell in particular, to Newman's own development.[25] Nevertheless, his claims above misdescribe Turing's relation to his *milieu* in Cambridge, especially during his undergraduate

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[24] See Floyd (2012b) for detailed discussion of this remark, and compare Sieg (2009) whose more general analysis quotes the remark with approval.

[25] Grattan-Guinness (2012a), (2012b), (2013); Newman (1955), p. 254 states explicitly that "it was in 1935 that [Turing] first began to work in mathematical logic".

years (1931-1934). They certainly err in characterizing Wittgenstein's views.[26] Most importantly, they miscast the importance of Turing's own philosophical stance toward logic, especially toward Carnap's and Tarski's "metalogic", both before and after he wrote "On Computable Numbers". Grattan-Guinness has missed Turing's *philosophy*, and thereby failed to appreciate the depth of his contributions.

Turing did not merely "seize upon" metalogic in the style of Gödel, Tarski and/or Carnap in 1935. Instead he *analyzed* it, and in a manner resonant with certain themes *critical* of "metalogic" stressed frequently among his teachers at Cambridge, especially Wittgenstein. What is most striking about "On Computable Numbers" is what Turing does *not* do. He does *not* ascend to an infinite hierarchy of metalanguages. He does *not* embroil himself in disputes over the law of excluded middle, infinitary objects, or the nature of logic. He does *not* spin off from Gödel's theorem into speculations about the human mind. He does *not* begin by coding up or revising a particular formal or axiomatic system already in use. Issues of consistency and contradiction, even of negation, are *dodged*. So are philosophical disputes. Instead, Turing simply shows what he shows, and directly, in terms of the *human* interface.

In fact, *nothing else would have done*. Turing's analogy, his perspective on "common sense" is a logical *must*. In his remarkable article it is the simplification and the argumentation that matter, and here is where Turing broke through to a novel way of thinking, one it is difficult to imagine Gödel or anyone else having originated.

How did he do it? And why?

4.3.1 Turing's Way In to Logic

In March of 1933, in his second year as an undergraduate, Turing acquired a copy of Russell's *Introduction to Mathematical Philosophy* (1919)—a book that was also a gateway for Gödel and Kleene on their ways in to logic, philosophy and computability.[27] Here Russell engagingly set out, without fuss about symbolism, his treatment of descriptions, the Peano Axioms for arithmetic, the basic logical constructions of number, the fundamentals of set theory and the theories of order

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[26] *Pace* Grattan-Guinness, by 1931, when Turing arrived at Cambridge, Wittgenstein was an *anti*-"monist", stressing, against his earlier *Tractatus* view, the importance of plurality and variety in systems of logic and grammar. Wittgenstein was hardly engaged in "revising" logicism in 1931-5, having never embraced it in the higher-order style of Whitehead and Russell: for him logic consisted of tautologies, but mathematics did not; see Floyd (2001a), Weiss (forthcoming).

[27] Russell's (1919) had stimulated Kurt Gödel to turn toward logic from physics at the age of 19 (in a seminar of Moritz Schlick's on the book in Vienna 1925-6), and had been read by Stephen Kleene before he attended Church's seminar on logic at Princeton in the fall of 1931-2. Cf. Floyd and Kanamori (forthcoming); Kleene (1981).

and types, including an account of the paradoxes. It excited Turing; he discussed it for hours with a friend.[28]

Significantly, the final chapter of the book, "Mathematics and Logic", was purely philosophical. Here Russell alluded to Wittgenstein's (then) novel conception of logic as "tautologous". Logicist that he was, Russell equated "mathematics" to "logic" without argument, an equation Wittgenstein never accepted.[29] Offering no clear explanation or defense of "tautology[30]", Russell wrote:

> It is clear that the definition of "logic" or "mathematics" must be sought by trying to give a new definition of the old notion of "analytic" propositions. Although we can no longer be satisfied to define logical propositions as those that follow from the law of contradiction, we can and must still admit that they are a wholly different class of propositions from those that we come to know empirically. They all have the characteristic which, a moment ago, we agreed to call "tautology".
>
> …For the moment I do not know how to define "tautology". [Note: The importance of "tautology" for a definition of mathematics was pointed out to me by my former pupil Ludwig Wittgenstein, who was working on the problem. I do not know whether he has solved it, or even whether he is alive or dead.] (1919, p. 206)

This dramatic ending, calling for a reworking of the traditional distinction between "analytic" and "synthetic" truth (i.e., truth-in-virtue-of-concepts-or-meaning vs. truth-in-virtue-of-fact), would not have escaped Turing's notice. Nor, fatefully enough, would the notion of "tautology". For by 1935 or 36, having taken Newman's course, Turing would import the use of a tautology-like construction into the heart of his argumentation in "On Computable Numbers" (Sect. 4.5 below). Turing would thus vindicate Russell's suggestion, drawn from Wittgenstein in 1918, that appeal to the laws of contradiction, excluded middle and bivalence are no longer sufficient as a basis for an analysis of logic, whereas the idea of an empty, senseless, repetitive remark "saying the same thing over again" (*tauto- logoi*) *is*.

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[28] Hodges (1983/2012), p. 85; (1999), p. 6.

[29] This identification of mathematical sentences with tautologies occurred as a logicist appropriation of Wittgenstein's thought in the Vienna Circle, but also in Cambridge. On January 24, 1941, G. H. Hardy gave a talk to the Moral Sciences Club on "Mathematical Reality," §§20-22 of his book *A Mathematician's Apology* (1940), a book Wittgenstein would call "miserable", probably because it took so little account of his philosophical criticisms (1999, 124, p. 35, from 1941). Mays (1967, p. 82 recalls: "Hardy mentioned that he did not accept Wittgenstein's view that mathematics consisted of tautologies. Wittgenstein denied that he had ever said this, and pointed to himself saying in an incredulous tone of voice, 'Who, I?"' (cf. Klagge and Nordmann eds. (2003), p. 336.) For discussion of Wittgenstein's resulting annotations to Hardy's *Course of Pure Mathematics* (1941), see Floyd and Mühlhölzer (unpublished).

[30] Dreben and Floyd (1991).

But why would Turing have been reading Russell at all? Two possible answers present themselves.

The first may be gleaned from Turing's annotated copy of Littlewood's *Elements of the Theory of Real Functions* (1926). In his preface Littlewood explained that he had "aimed at excluding as far as possible anything that could be called philosophy" from his presentation, hoping to "inculcate the proper attitude of enlightened simple-mendedness" in his students "by concentrating attention on matters which are abstract but not complicated". But, conceding some readers might be interested in the foundations of mathematics, he recommended that the reader consult Russell's *Introduction to Mathematical Philosophy*, pointing out that its content would be necessary for understanding his presentation of the Peano axioms and the theory of types. Turing's annotations to the inside cover of his copy of Littlewood use the idea of a "propositional function" as well as logical notation.[31] So it is plausible to suppose that Turing turn toward logic in the spring of 1933, during his second undergraduate year, and that this was his introduction to, or motivation for, his study of the theory of types.

There is something else even more interesting in the preface and first sections Littlewood's lecture notes. Lamenting the tendency of modern mathematics papers to eliminate, in their terseness, any discussion of the "*point*" of the subject matter, Littlewood suggests that a lecture, or lecture notes, are perhaps just the sort of place for such "provisional nonsense". He picks the example of his presentation of the Burali-Forti paradox in terms of types, and states that he thinks it possible to

> …make both the underlying ideas seem intuitive and the official proof natural. The infinitely greater flexibility of speech enables me here to do without a blush what I shrink from doing in print.

Moreover Littlewood emphasizes that with respect to certain undefined or "logical" terms ("proposition" for example),

> …we hold ourselves free to choose that mode of expression in ordinary language which is most familiar, vivid, or convenient (1926, p. 2).

Littlewood thereby explicitly refuses to concede "to the cantankerous [logicist or formalist] reader" (p. 2) explicit definitions at all points, insisting that

> No line can strictly be drawn between mathematics and logic, and we are merely using a more or less popular distinction to indicate our starting point (1926, p. 1).

Here, we may say, is a hint at the direction in which Turing's own sensibility would go. The "point" of a mathematical conceptualization needs to be discussed. If "nonsense" emerges in the course of communication, it might only reflect a

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[31] Turing Digital Archive, AMT/B/46, http://www.turingarchive.org/browse.php/B/46.

"provisional" situation, further articulable in terms of natural, familiar, ordinary language.

There is a second possible answer as to why Turing would have been reading Russell in the spring of 1933. Wittgenstein's course "Philosophy for Mathematicians" was given 1932-1933 (Turing's second undergraduate year), and in the beginning of fall 1933 (in the first term of Turing's third and final undergraduate year). It may have influenced Turing (directly or indirectly), drawing him toward logic and foundations of mathematics. Wittgenstein had studied with Littlewood in 1908-9 and been given his own Cambridge fellowship (1930-36) largely on the basis of Littlewood's positive assessment of his philosophical work, made after several discussions; they shared some students and Wittgenstein invited them to his lectures.[32] In May 1932, in his "Philosophy" course, Wittgenstein had turned toward applications of his views to the foundations of logic and mathematics, formulating the need for an explicit uniqueness rule in (quantifier-free) equational specifications of recursive definitions as a replacement for an explicit principle of mathematical induction.[33] Seeking a mathematical audience---and throughout the semester when Turing was reading Russell---Wittgenstein began teaching a new course, "Philosophy for Mathematicians", in the fall of 1932, running it through the spring of 1933. It is possible that Turing attended, although we cannot be sure.

What is even more likely is that, as a rising third-year undergraduate, Turing was aware of controversy about Wittgenstein's views by the fall of 1933. In May 1933 Braithwaite published a survey of Cambridge philosophy since the War, specifically designed to help undergraduates in their choice of a course of study. In this essay he held that "common sense" and "science" are opposed, attributing to Wittgenstein a strong form of verificationism, based on what he knew of his lectures (1933). In a letter to *Mind* Wittgenstein objected heatedly to Braithwaite's characterization of his philosophy, and Braithwaite replied, somewhat testily, that the world awaited Wittgenstein's publication of his views.[34] Perhaps as a result of this, so many students showed up for "Philosophy for Mathematicians" in the fall of 1933 that useful discussion became impossible.

Wittgenstein surprised everyone by cancelling the class and dictating notes to a small group of mathematics students as a substitute for the lectures. Mimeographed and bound, *The Blue Book* (1933-4) and *The Brown Book* (1934-

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[32] McGuinness (2008), pp. 182-87, 207, 256. Cf. Monk (1990), p. 30; McGuinness (1988), pp. 62, 96, 155.

[33] The Lecture was given on May 20, 1932, cf. Stern, Rogers, and Citron eds. (forthcoming); cf. Goodstein (1945), p. 407n, von Plato (2014), Marion and Okada (unpublished).

[34] The letters appeared in *Mind* vol. 42, no. 167 July 1933, pp. 415-16; they are reprinted with editorial commentary in Wittgenstein, Klagge and Nordmann (eds.) (1993), pp. 156-9 and in McGuinness (2008), p. 210.

5) (1965 (BB))---named for the colors of their covers----were widely distributed, and presumably fairly widely discussed, among Cambridge mathematics students.[35] The so-called *Yellow Book* (a compilation of notes of the lectures given before the cancellation of the course and other dictations 1933-4) was also dictated, and though perhaps less widely circulated; it gives us a first-hand account of the beginning of the originally envisioned course.[36]

These lectures and dictations mark the first public emergence of Wittgenstein's mature philosophy; the only mimeographed materials that would appear and be widely read in his lifetime. In them we see the issues of "phraseology" and "nonsense" explicitly raised and explored---both notions we shall encounter again in Wittgenstein's and Turing's later writings and discussions. We also see an "anthropological" approach to logic and meaning---due in part to Wittgenstein's discussions with Sraffa[37]---in which various "tribes" are investigated in thought-experiments. Calculations and rules are construed in terms of tables with command lines directing humans to use signs in a step-by-step manner. The problem of how to regard the amalgamazation of such "logic-free" routines is in full view,[38] as is the repeated imaginative circumscribing of human actions, step-by-step, with directive rules, and the difficulty of accurately describing *what* following a rule or set of rules *is* in ordinary English. Perhaps most significantly, Wittgenstein used a series of "language games" to extrude psychological states from his characterization of rule-following and meaning. What it *is* to be "guided by" a calculative rule is neither a particular mental state nor a specific conscious process. It is something emerging against the backdrop of a series of mergings of routines, and in a shared context of "common sense"---i.e., *sense* held in *common* among us, as we speak, argue, and converse with one another.

4.3.2 "Common Sense" at Cambridge

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[35] Wittgenstein dictated *The Blue Book* to Turing's fellow mathematics students H.S.M. Coxeter, R.L. Goodstein, and Francis Skinner, along with Alice Ambrose (writing with Wittgenstein and Newman), as well as Margaret Masterman (later Braithwaite); cf. BB, Preface.

[36] Cf. BB and also the Francis Skinner archives, now at the Wren Library of Trinity College, Cambridge. These were donated by his close friend and colleague (and fellow mathematics student of Turing) R.L. Goodstein to the Mathematical Association of Great Britain, who held them until his death; Wittgenstein had given him the materials soon after Skinner's own death. These contain more extensive and precise dictated material from the period 1933-35, including a different version of *The Yellow Book* and some hitherto unpublished and unknown lecture notes on self-evidence in logic (cf. Gibson (2010)).

[37] Sen (2003), Engelmann (2013a).

[38] BB §41, p. 98. Wittgenstein remained unsatisfied with *The Brown Book*; after an attempted revision in the fall of 1936, it was abandoned. The beginning of MS 152, a manuscript begun presumably in early 1937, revisits rule-following, but begins concretely, with a series of calculations from the theory of continued fractions.

The points above about Russell, Wittgenstein and Littlewood are indicative. At the center of the tradition Turing entered in 1931, when he first came up to Cambridge (the very year of Gödel's (1931) paper), was the question, "What is the *nature* of the logical?" This was in part a purely philosophical, and in part a mathematical question. First, it raised the issue of how and whether to draw a line between mathematics and logic. Should we take mathematics to be nothing but a branch of logic conceived as a theory of concepts, properties, propositional functions, as Russell claimed? Or is it different in character, though subject to logic, whose essential feature is the symbolic representation of well-founded, step-by-step, iterable formal procedures, some ending in tautologies, as Wittgenstein held? Finally, is mathematics autonomous from logic, rooted ultimately in the "prejudice" and experience of "the mathematician-in-the-street", as Hardy called it (1929 p. 4)? Most importantly, how was one to argue such questions? *Who* was prejudiced, the logician or the mathematician-in-the-street?

Second, this tradition of asking about the *nature* of logic drove logicians themselves back to the foundations, as we have seen above (Sect. 4.2). These, as Wittgenstein would argue, lie in our everyday judgments about what follows from what, what makes sense and what does not, what is convincing and what is not: the human interface with logic.

Littlewood's and Wittgenstein's remarks resonate directly with a longstanding Cambridge tradition of philosophical argument over the notion of "common sense". It was spearheaded by Moore, Keynes and Russell, and developed by Hardy, Ramsey, Sraffa, Wittgenstein, Broad and Braithwaite during Turing's undergraduate years. It concerned philosophy's and logic's relation to what ordinary, mortal, untutored, unspecialized humans might be said to think and mean. May "common sense" be dislodged or changed by a logician or philosopher, or is it instead something to be protected from metaphysical---especially Idealist---onslaught?

For Moore, "common sense" and "plain meanings" were not to be overturned by metaphysics---though Moore himself invoked quite abstract metaphysical ideas in the course of his efforts to refute Idealism and to secure a subject matter for psychology. For Russell, by contrast, untutored "common sense" amounted to nothing more than the primitive, pre-scientific "metaphysics of the Stone Age", and it should be revised, if not overturned, by philosophers responsive to the course of scientific progress.[39] For Ramsey, the important point in discussions of logic

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[39] Russell's remark is quoted without attribution in J.L. Austin's "Plea for Excuses" (from 1956, in Austin (1979), p. 185; it was apparently widely known. Wittgenstein sent Russell *The Blue Book*, hoping for a response, in 1935 (McGuinness (2008), p. 250). As Cavell has suggested (2012, pp. 30ff), the opening of Wittgenstein's *Blue Book*, with its truncated, cave-man-like language game of builders, may be regarded as either a stimulus or a response to Russell's remark about "the metaphysics of the Stone Age".

had been to find a "common basis" from which to argue with an opponent.[40] For Wittgenstein, the basis of logic was to be achieved only through logic itself; but the key to his post-1930 method was exploration, in informal language, of specific analogies and comparisons used in philosophical discussion of formal and mathematical proof.

The point is that at Cambridge during Turing's undergraduate years the general issue of "common sense" was deeply entrenched, not only in mathematics common room discussions, but in the teaching of logic, philosophy and mathematics. As for the issue of decidability, it was Wittgenstein who had first posed the *Entscheidungsproblem* for general logic (in a letter to Russell of 1913).[41] Ramsey, aiming to develop a foundation for classical mathematics from Wittgenstein's *Tractatus*, contributed a brilliant partial resolution in 1928 (1930). Arriving at King's College just one year after Ramsey's death, Turing would have been aware of this tradition.

Such considerations also apply to G.H. Hardy, the great number theorist and colleague of Littlewood's who returned to Cambridge from Oxford in 1931, in Turing's first year (Turing had begun reading Hardy's *Course of Pure Mathematics* the semester before he arrived at Cambridge).[42] One of Hardy's main aims in his widely-known paper "Mathematical Proof" (1929) was to urge the study of Hilbertian metamathematics upon Cambridge students of mathematics. He had a sense that, due to *philosophical* objections, the contributions of Hilbert were being overlooked. This required him to criticize the philosophical, though not the mathematical elements of Hilbert's work, the "formalism" that it seemed to contain.

Hardy's whole essay was intended to locate a place where "the-mathematician-in-the-street" could find a haven, safe from both formalist and logicist reductions. By insisting that there is "a [purely mathematical] content given independently of all logic", perhaps "some sort of concrete perceptible basis" of signs, Hilbert and Weyl *precluded*, Hardy argued, any real treatment of mathematical concepts. Hardy (rightly) took Hilbert to believe that there are such concepts, and he also (rightly) took Hilbert's metamathematical investigations of consistency to be mathematically and conceptually important (p. 11). By contrast, Russell, Ramsey and Wittgenstein, the "logisticians" had developed a theory of concepts directly, from logic. This approach Hardy preferred; but at the same time he felt the

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[40] See Ramsey, (1927), p. 161 and (1929). For an argument that Ramsey's pragmatism influenced Wittgenstein in 1929-30, see Misak (2016).

[41] McGuinness (2008), pp. 56-69.

[42] Hodges 1983/2012, p. 58; cf. the early editions (in 1933, 1938) of Hardy (1941); presumably Wittgenstein read out passages from the 1933 edition in his "Philosophy for Mathematicians" of 1932-33 and fall 1933. Later Wittgenstein would annotate his copy of Hardy (1941) and copy these annotations into his manuscripts; cf. Floyd and Mühlhölzer (unpublished).

"logisticians" had wandered too far from the common sense, the "prejudices", of "the-mathematician-in-the-street"---a figure with whom Hardy identified (p. 4).

Hardy made what he took to be a "fatal" objection to the idea of purely mathematical experiences of "strings" and "marks":

> If Hilbert has made the Hilbert mathematics with a particular series of marks on a particular sheet of paper, and I copy them on another sheet, have I made a *new* mathematics? Surely it is *the same* mathematics, and that even if he writes in pencil and I in ink, and his marks are black while mine are red. Surely the Hilbert mathematics must be in some sense something which is common to all such sets of marks. I make this point here, because there are two questions which suggest themselves at once about Hilbert's marks. The first is whether we are studying the physical signs themselves or general formal relations in which they stand, and the second is whether these signs or relations have 'meaning' in the sense in which the symbols of mathematics are usually supposed to have meaning. It seems to me that the two questions are quite distinct (1929, p. 11).

Hardy then invoked Wittgenstein's *Tractatus* theory of "general form". Yet his argument that the ability to copy *the same* sign down is fundamental to the symbolism of mathematics is, after all nothing but "common sense". The idea was granted, developed and discussed later on by Wittgenstein and Turing;[43] it was a problem known and addressed by other logician-mathematicians, such as Post.[44]

Hardy next criticized the "logisticians" for doing away with the idea of a pure "proposition" as a conceptual complex carrying within itself the nature of being true or being false. He argued that *this* idea (Moore's and Russell's, 1899-1910) belonged to "the-mathematician-in-the-street". When he offered a proof in mathematics, Hardy insisted, he ultimately *pointed* at an abstract proposition's intrinsic truth (or falsity), as if pointing at a distant mountaintop, aiming with chains of reasoning to get his pupil also to *see* (p. 18). Hardy conceded that in the end, this analogy was partial, and risked a vicious regress into ineffable pointing, making of proof something merely "gas", an ethereal atmosphere of chatter inessential to the mathematical content at issue. But he "detested" Russell's, Wittgenstein's and Ramsey's rejection of the view, their "multiple relation theory of judgment", which made judging *itself* into a complex ordered fact, and truth into a matter of projection, a greater-than-2-arity structure given by the relations, orderings and objects involved in what is judged.

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[43] After discussing these issues with Turing, both in 1937 and in his 1939 seminar, Wittgenstein wrote a long series of manuscript pages exploring the idea that a proof must be "surveyable" (*Übersichtlich*), i.e., "can be copied, in the manner of a picture" by a human being, and so "taken in", communicated, archived, recognized, and acted upon; cf. Mühlhölzer (2006), Floyd (2015).

[44] Cf. Post (1941/1994), at n. 8 (p. 377); Post is referring to his initial work on "operational logic" for sequences done in 1924.

As Hardy explained, he and Littlewood called "gas" all those dispensable heuristics, "rhetorical flourishes designed to affect psychology, pictures on the board in the lecture, devices to stimulate the imagination of pupils" (1929 p. 18). Thus were philosophy and ordinary language reduced by Hardy---though not by Littlewood---to mere fluff, a trading of opinions and feelings, not something belonging to the *content* of mathematics. Hardy alluded to the *Entscheidungsproblem*:

> The image [of "gas", that which manages to "point" at mathematical propositions as if they are distant peaks] gives us a genuine approximation to the processes of mathematical pedagogy on the one hand, and of mathematical discovery on the other; it is only the very unsophisticated outsider who imagines that mathematicians make discoveries by turning the handle of some miraculous machine. (1929, p. 18).

In this way, using what was ultimately an overstatement, Hardy rejected the idea of reducing all of mathematics to step-by-step calculative routine.

The primary aim of Wittgenstein's 1932-3 course "Philosophy for Mathematicians" was to offer a more articulate conception of the *positive* role to be played both by calculation and by everyday mathematical language in pedagogy and in mathematical discovery. He was working through Hardy's idea of a "pure" proposition by getting to the "gas".[45] The issue of how we can make sense in ordinary language---how *we* can lay specific conditions down on reality, truly or falsely, how *we* reason at all---had to be accommodated within the foundations of mathematics itself, and not dismissed. For this Wittgenstein explored the ideas of "mechanism" and "calculation" *in* ordinary mathematics and in ordinary language. The idea was to get rid of Hardy's idea of a "miraculous machine" by bringing the figure down to earth.[46] Given Hardy's argumentation, an alternative analysis of metamathematics would be required, as well as an alternative analysis of the very notions of "proposition", "ground" and conviction. And there would be no way to draw in and work through the "gas" other than to plough through everyday language, analogies, and "phraseology", mathematical and otherwise.

Wittgenstein's 1932-33 "Philosophy for Mathematicians" opened with consideration of two major questions, and an answer:

> 1. Is there a substratum on which mathematics rests? Is logic the foundation of mathematics? In my view mathematical logic is simply part of mathematics. Russell's calculus [I.e., *Principia Mathematica*] is not fundamental; it is just another calculus. There is nothing wrong with a science before the foundations are laid. ((1979 (AWL), p. 205).

In answer to Russell and Hardy—as he made explicit in the course of these lectures, by reading out passages from Hardy's *Course of Pure Mathematics* and

[45] On "pure propositions" as gaseous, see AWL, p. 55.

[46] Discussions of a variety of conceptions of "machines" and "mechanism" appear in Wittgenstein's "Philosophy" lectures 1934-5 and *The Yellow Book*; cf. AWL, pp. 52-3, 72, 80.

working through many examples—Wittgenstein was denying that mathematics rested upon any "substratum", logical or otherwise. *Its* foundations were all right. Moreover, *Principia Mathematica*, far from "reducing" mathematics to logic, should be regarded as "just another calculus", simply a *part* of mathematics, and no such "substratum". Wittgenstein would not offer any independent "logical" foundation of mathematics, he was turning instead toward a philosophical "merging" of logic and mathematics, as Turing would later call it (see Sect. 4.7 below)---something different from what either Hilbert or Carnap had envisaged.

He next turned to "a=b", and "a=a", the fundamental steps to be taken in in the grasp of recursive or inductive specifications involving a uniqueness rule. At issue here is the very basis of substitution as a logical step, and the basis of generalizations based on our grasp of step-by-step routines. Where are we to take ourselves to *begin* in grasping the number series? This problem isolated and made vivid a more general and basic difficulty Wittgenstein faced with logical analysis itself, given his own emphasis on well-founded, calculative procedures as the heart of logic and mathematics: What is the starting point of a well-founded procedure, if there are no absolute starting points?[47] How and where does analysis bottom out?[48] This entangled him in the question of what standpoint to use to make sense of the whole idea of *sameness* or *difference* of symbolic steps and substitutions, the "recurrence" of terms in the carrying out of operations.

In his final lecture Wittgenstein sketched some implications of his perspective. First, he positioned himself orthogonally to the reigning "schools" of logicism, formalism and intuitionism. There was some truth in formalism, he held, in that "what counts in mathematics is what is written down", and "if a mathematician exhibits a piece of reasoning one does not inquire about a psychological process" (AWL, p. 225). Thus an intuitionist "should be asked to show how meaning operates", for it is not enough to insist that the object of mathematical thinking is not signs, but objects. However, Russell was wrong to think that logic is a kind of "zoology" of the conceptual world. And Hardy was wrong to think of mathematical content as lodged in pure conceptual structures at which we can only point, with "gas". So far as his own investigations were concerned,

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[47] In the *Tractatus* (1922) there had been posited an ultimate starting point, the "simple objects" of the final analysis. Wittgenstein had rejected these by 1929.

[48] The problem is evinced by his use of images of a spiral to gauge infinitary, rule or law-governed series at Wittgenstein AWL, p. 206. The image had been tethered to a particular origin in *Philosophical Remarks*, Wittgenstein's fellowship submission (1980a) (PR) §§158, 189, 192, 197. But in the "Big Typescript" of 1932-3 (2005), p. 379, as well as "Philosophy for Mathematicians" 1932-3, AWL, p. 206 the *issue* of a point of origin and how to assign it in a single space has come to the fore. After this point, the image of the spiral is not used by Wittgenstein, but is replaced with that of a more localized, free-standing and "portable" table, or finite set, of commands.

> There is no retreat in mathematics except in the gaseous part. (You may find that some of mathematics is uninteresting--that Cantor's paradise is not a paradise.) (AWL p. 225)

The point was to revise, not the actual mathematics the students were learning, but rather its mode of presentation, by exploring and comparing the images and heuristic ways of thinking characteristic of the presentation of modern mathematics. "Gas" could and should---indeed *must*---be worked on and discussed. Wittgenstein would admit no sharp break between "gas" or ordinary heuristic discussion and mathematics itself. He thus echoed Littlewood's earlier plea for allowing, in the face of "cantankerous" requests for explicitness, an appeal to plain and ordinary language. He also refused, like Littlewood, to draw a sharp line between mathematics and logic.

Unlike Hardy and Littlewood, Wittgenstein drew a philosophical consequence: by focussing on such fundamental questions as what we are to make of the *first* step in the specification of a class by way of a recursive definition, his students might end up feeling that abstract modern mathematics (e.g. Cantor) is less interesting to them than such foundational questions about the fundamentals. Thus back and forth about how to present mathematical issues, the philosophical "gas" *could* play a serious role in mathematics.

He ended his course this way:

> The talk of mathematicians becomes absurd when they leave mathematics, for example, Hardy's description of mathematics as not being a creation of our minds. He conceived philosophy as a decoration, an atmosphere, around the hard realities of mathematics and science. These disciplines, on the one hand, and philosophy on the other, are thought of as being like the necessities and decoration of a room. Hardy is thinking of philosophical opinions. I conceive of philosophy as an activity of clearing up thought (AWL, p. 225).

Wittgenstein's challenge to Hardy was explicit: philosophy is not a *mere* "decoration", not a *mere* swapping of common room opinions or feelings. The "gaseous" "decorations" in which results are dressed up in what Littlewood called "provisional nonsense" are not only a necessity, they are a serious matter, as much a working part of the communication of mathematics as the "hard" results. So Wittgenstein would again insist in 1939, when he and Turing revisited the discussion of "gas" and the metaphor of "decorations" drawn from Hardy in the very first lecture of his Cambridge course (cf. Sect. 4.6 below).

All this shows that by 1933 at Cambridge a number of mathematics and philosophy students had been exposed to fundamental logical questions about recursive processes, representations of calculational activities, forms of generality, and philosophical questions about the foundations of logic and mathematics. They worked closely with Wittgenstein, Hardy, and Littlewood, each of whom, like Newman, had published on philosophical issues connected with *Principia*. Each of these figures were to be found in discussion with one another, sometimes in public venues (the Moral Sciences Club, the Trinity Mathematical Society, Wittgenstein's courses), sometimes in the advising of graduate students, sometimes in the common rooms and dining halls of the colleges.

Before we turn to review the theme of "common sense" in Turing's writings, we need to make a few final remarks about Turing and Wittgenstein.

4.3.3 Turing and Wittgenstein

We do not know exactly when Turing first heard Wittgenstein. There is no direct evidence about this in the period of Turing's undergraduate (1931-34) and early King's College fellowship years (1934-1936). Hodges tells us, however, that they would have known one another during Turing's undergraduate years at the Moral Sciences Club.[49] There was also the Trinity Math Society, where Wittgenstein, Littlewood and Hardy spoke[50] and where Wittgenstein's intimate friend and amenuensis, Francis Skinner---undergraduate 1930-33, mathematics Fellowship student at the same time as Turing 1933-6, dictatee of *The Yellow Book*, *The Blue Book* and *The Brown Book*, and other Wittgenstein's lectures---became Secretary in 1931.[51] In addition, of course, there was the mathematics common room, and, around the colleges---and at King's specifically---there were many go-betweens with Wittgenstein among the mathematics students who had fellowships during Turing's years.[52]

What we have so far argued, as a partial explanation for, or result of, his turn to Russell in March 1933, is that Turing may well have attended, heard about or read dictations of Wittgenstein's lectures during his second or third year as an undergraduate 1932-33. We shall further argue that a lecture Turing gave to the Moral Sciences Club in December 1933 lecture is fully consistent with this hypothesis (Sec. 4.4 below). Finally we shall argue that Wittgenstein's idea of "tautology" resonates strikingly with Turing's *specific* method of argumentation in "On Computable Numbers" (see Sect. 4.5), written 1935-6.

Turing and Wittgenstein certainly held one-on-one discussions about logic well before the 1939 lecture course of Wittgenstein's (on which see Sect. 4.6 below). Hodges tells us that they were introduced in 1937 by Wittgenstein's close student, mathematician and philosopher Alister Watson, a King's College undergraduate

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[49] Though it is unclear exactly when, because after 1931 there was a hiatus in Wittgensteinstein's attendance. Cf. Klagge and Nordmann (eds.) (2003), p. 377.

[50] See the archive of talks at https://www.srcf.ucam.org/tms/talks-archive/#earlier.

[51] Klagge and Nordman (eds.) (2003), p. 362.

[52] McGuinness (2008), p. 207 mentions Mary Cartwright, George Temple, L.C. Young, and H.D. Ursell; there were also Goodstein and Skinner, on which see Gibson (2010).

with Turing during Turing's second undergraduate year[53] and, like Turing, afterwards a Fellow of King's College (1933-1939), beginning two years ahead of Turing.[54] Watson attended many of Wittgenstein's lectures 1930-1938 (including the 1939 ones) and was close to Wittgenstein.[55] As Hodges also tells us, during the academic year of 1935-6, just after his election to a King's Fellowship, Turing was observed discussing the nature of Cantor's Diagonal argument with Watson and asking a few questions of Braithwaite.[56]

To Newman Turing apparently did not talk at all about his machines before handing him the manuscript of "On Computable Numbers" in April 1936.[57] Given the unfamiliar character of Turing's argumentation, Newman at first did not know what to make of it, thinking it must be wrong.[58] As we have said, Turing's particular way of resolving the *Entscheidungsproblem* was not the application of a preexisting blueprint of ideas and methods in the metamathematics literature, it was instead a philosophically reflective analytic exercise.

It was thus with philosophers and peers, and not with Newman, that Turing was chatting during his crucial year of conceiving "On Computable Numbers".[59] Moreover, Turing continued philosophical discussions immediately after his great paper was published. In February 1937 he sent off the first round of offprints of "On Computable Numbers".[60] He reported to his mother that in addition to King's College colleagues he had sent them to---in order---Littlewood, Wittgenstein, Newman and two others. He asked his mother to send one to Russell (warning her

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[53] Watson (1908-1982) entered King's College in 1926, receiving firsts in both parts of the Mathematical Tripos; he was awarded Studentship prizes in 1929, 1930 and 1932. Though he failed in his first bid for a King's Fellowship, "Chance and Determinism in Relation to Modern Physics" (1932) he succeeded in 1933, with a new thesis on "The Logic of the Quantum Theory". He helped proofread Braithwaite's edition of Ramsey's papers (1931). A man of the left and a friend of Anthony Blunt, his alleged entanglement with the Cambridge Spy Ring (he never confessed) is discussed in Wright (1987).

[54] Hodges 1983/2012.

[55] Cf. McGuinness (2008), pp. 253, 280.

[56] Hodges (1983/2012), p. 109.

[57] Hodges (1983/2012), p. 109.

[58] Hodges (1983/2012), p. 112.

[59] Hodges (1983/2012), p. 109 tells us that Turing did discussed his ideas about machines with David Champernowne, who, along with Alister Watson, would later be relied on by Sraffa to check the mathematics of Sraffa (1960/1975). See Kurz and Salvadori (2001).

[60] Turing was disappointed at receiving only two requests for offprints, one from Braithwaite and one from Heinrich Scholz, who gave a seminar on it at Münster (Hodges (1983/2012), pp. 123-4.

not to address him as "Lord").[61] Moreover, when six months later Turing returned to Cambridge for an August visit from his sojourn at Princeton, he held discussions with Watson and Wittgenstein about the implications of his work on Gödelian incompleteness and the *Entscheidungsproblem* for the foundations of mathematics, meeting with them sometimes in the Botanical Gardens.[62]

This discussion occurred at a significant time. Turing was apparently highly optimistic about rewriting the philosophical foundations of analysis.[63] He was deeply involved in writing his Princeton dissertation on ordinal logics (1939), where his (once again) anthropological comparison of higher-order recursion to an "oracle" would be used to explore a procedural analysis of Gödelian incompleteness in terms of higher types (cf. Sect. 4.6 below). Wittgenstein was in the midst of drafting his second book, *Philosophical Investigations*. He lacked the the second half that he had always sought, the application of his mature perspective to logic and the foundations of mathematics. But immediately after talking with Watson and Turing he travelled to Norway and completed the first full draft of his book.[64] His manuscripts commence with a summary of his reactions to the discussions, including remarks on Gödelian incompleteness, and one in which a machine is imagined to be capable of "symbolizing its own action", this clearly indebted to "On Computable Numbers".[65]

Over the next year Wittgenstein would write at length about two alternative ways of looking at Cantor's diagonal argumentation: one bottom-up, in terms of calculative procedures, the other top-down, in terms of theories of cardinality.[66] It is plausible to think that he was directly inspired by Turing's great paper, for in his later 1947 remarks about Turing's "Machines" he not only explicitly reminisced about his 1937 discussions with Turing and Watson, he wrote down a reconstruction of Turing's variation on Cantor's diagonal argument in "On

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[61] Turing to his mother February 11, 1937 AMT/K/1/54 in the Turing Digital Archive http://www.turingarchive.org/browse.php/K/1/54.

[62] Hodges (1983/2012), p. 136.

[63] Hodges tells us that Turing was "overoptimistic" at this time "in thinking he could re-write the foundations of analysis" ((1999), p. 19). It was just this question that interested Watson in his (1938) paper that they discussed in the summer of 1937 with Wittgenstein; our best guess is that the three urged one another on in conversation to ponder the question.

[64] Wittgenstein (2001), "Frühversion"; in (1999) TS 225, 220, 221.

[65] (1999), MS 119, pp. 28ff; cf. PI §190. The first passages Wittgenstein wrote up on Gödelian incompleteness are also then, in Sept 1937 ((1999), 118, p. 106ff; cf. 117, pp. 147, 152; 121, pp. 75v-78v, 81v-84r), and 122, p. 28v; cf. Floyd (2001b) and Floyd and Putnam (2000) for discussion).

[66] Cf. Wittgenstein (1978) (RFM), Part II, written in 1938, analyzed in Floyd and Mühlhölzer (unpublished).

Computable Numbers" (cf. Sect. 4.5 below).[67] For his part Watson published a paper in *Mind* (1938) in which he thanked Turing and Wittgenstein for his understanding of Gödel's theorem.[68]

The fact is that philosophers were among those to whom Turing turned both before and after publishing his great paper.[69] And again: in 1939, back teaching mathematical logic at Cambridge, Turing attended Wittgenstein's lectures on the foundations of mathematics (cf. Sect. 4.6 below).

Let us next turn, one by one, to consideration of the theme "common sense" in a series of Turing's lectures, remarks, and writings.

4.4 Turing's Moral Sciences Club Lecture, December 1933

In December 1933—nine months after reading Russell, at the end of the semester when Wittgenstein was generating *The Yellow Book* and the beginning of *The Blue Book*---the third-year undergraduate Turing read a paper to the Moral Sciences Club on the topic of "Mathematics and Logic". Although all we have as a record is Braithwaite's minutes of the meeting, given what we have just said, it is striking to see what he reports:

> On 1 December 1933, the minutes in the Moral Science Club records: "A.M. Turing read a paper on 'Mathematics and Logic'. He suggested that a purely logistic view of mathematics was inadequate; and that mathematical propositions possessed a variety of interpretations, of which the logistic was merely one." Signed, R.B. Braithwaite (Hodges 1999, p. 6).

These remarks---made to a philosophical audience---raise crucial issues about what may have been Turing's and others' thinking about logic at this time. We shall argue that they may be taken to anticipate the development of a distinctive attitude that Turing would put to practical use later on. If our suggestions are at all plausible, this would show that he had already begun to assemble a distinctive way of thinking about logic while still an undergraduate.

What was meant by "a purely logistic view of mathematics"? And what could have been meant by it an "inadequate" "interpretation" of mathematics?

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[67] Wittgenstein (1980b) (RPP I) §1096ff written 1947 and discussed in Floyd (2012b).

[68] Cf. Floyd (2001b).

[69] Of course, conversations with many others went on, especially mathematicians. Newman himself went to Princeton in 1937-38, joining Turing there, cf. Newman (2010).

It is first of all striking that the term "logistic" was used at all.[70] We cannot know whether Turing or only Braithwaite used it. But we do know that Russell, Wittgenstein and Ramsey, the logicists, wouldn't and didn't use it.[71] Hilbert and Ackermann's *Grundzüge der Logik* (1928) used the different term "*Logikkalkül*". Hardy had called Russell, Ramsey and Wittgenstein the "logisticians" while surveying their differences with Hilbert's "formalism" and the "intuitionism" of Brouwer and Weyl (1929, pp. 5-6); on this use "logistic" simply meant "symbolic logic". What was meant by Turing's point about "a purely logistic view" would, however, have probably been something else. For by 1933 "logistic" was embroiled, in philosophical discussion, in debates over the *nature* of logic and the *nature* of mathematics.

Turing (or Braithwaite himself) could have picked up the term "logistic" from Lewis and Langford's textbook *Symbolic Logic* (1932), which Braithwaite was just then reviewing (1934).[72] There was interest in the book at Cambridge, as Wisdom, a close student of Wittgenstein's who arrived in 1934, reviewed it in *Mind* (1934). Lewis's earlier *Survey of Symbolic Logic* (1918) may also have been known. Or Cambridge could already have gotten wind of Carnap's *Abriss der Logistic* (1929) or related essays (1931) and his syntax program (1934). Whether or not Braithwaite or Turing knew these or related works and ideas, it is useful to contrast what Turing is reported to have said both with Lewis's and with Carnap's perspectives, reprising our earlier differences with Grattan-Guinness's account.

For Lewis and Langford, "logistic" did *not* mean a particular "view" or "interpretation", but rather a technique or method of approach, roughly coextensive with "symbolic logic". A particular example or "system" of "logistic" would have been something itself already "interpreted" or "applied", the term

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[70] A brief history of the term is given in Church (1956), pp. 56-7 n. 125; Church traces the concerted use of "logistic" for mathematical logic back to the 1904 World Congress of philosophy. He notes that "sometimes 'logistic' has been used with special reference to the school of Russell or to the Frege-Russell doctrine that mathematics is a branch of logic", but the "more common usage … attaches no such special meaning to this word". For an overview of *logos, logic* vs. *logistiké* in connection with incompleteness and modern mathematics see Stein (1988).

[71] Except for one loose sheet Wittgenstein inserted into the very end of the manuscript of *Philosophical Investigations* (PI Part II/PPF xiv §372).

[72] Braithwaite (1934) regards Lewis and Langford (1932) as "eminently successful", predicting that "it will probably not be superseded for some time as the standard work" on the subject of symbolic logic, though he objects to its treatment of the theory of types and finds it wanting as a text in the foundations of mathematics. He notes that Langford's use of postulates is analogous to Hilbert's *Entscheidungsproblem*. Langford, a student of H.M. Sheffer, spent 1925-6 at Cambridge visiting from Harvard. He met Ramsey and indeed proved some of the earliest significant results on the completeness and decideability of first-order theories, pioneering the use of quantifier elimination (Urquhardt (2012))---another point of contact between Cambridge philosophy and the *Enscheidungsproblem* in the 1920s.

"logistic" connoting, in general, no specific step of "interpretation" at all, but instead a set of "more or less mechanical operations upon the symbols" (1932, p. 118).[73] In terms of the history of philosophy, since Berkeley "logistic" involved a general algebraic technique of using symbols to fix ideas, reason and calculate---thereby *extruding* vagaries of more ephemeral "ideas" from the mind by means of a kind of substitutive move.[74] Lewis's earlier (1918) *Survey of Symbolic Logic* had actually labelled "heterodox" a conception of logistic and mathematics in which every reference to meaning is eliminated from one's conception of the workings of the formal symbolic system, reducing this to the totality of kinds of symbolic operative order on "strings" of signs. His formulation influenced Post (1936), in his work that would be bound, conceptually, so closely with Turing's, though concocted independently of it.[75]

In Carnap's *Abriss der Logistik* (1929) and his remarks at Königsberg (1931) the term "logistic" had been explicitly ushered toward a full-throated philosophical perspective, or "view of mathematics". Carnap explicitly construed the term in a narrower, philosophically radicalized, sense. He held that Lewis's conception of "logistic" was oriented toward *extra*-logical applications of symbolic logic (§1) whereas he used it, by contrast, more "purely" (the adjective Braithwaite reported Turing having used), as a philosophical point of view *on* logic and mathematics. This "purely logistic view" was a radical proposal for developing formal systems to analyze philosophical positions. In lectures in London (1934) Carnap would share with an English audience his vision of philosophy as "the logical syntax of the language of science". On this view, the distinction between "analytic" and experiential truths was relativized to particular formal languages, and "every significant question asked by philosophers" was taken to concern, either the logical syntax of a specific formal system or an issue proper to a particular science, thus making the fundamental object of study in logic "direct consequence" and the fundamental problem of philosophy the construction of formal language-systems.[76]

I have no proof that Turing knew in 1933 of any of Carnap's philosophical works, though he did know German, and by 1933 the 1931 Königsberg Conference, where Carnap spoke and Gödel had first announced his incompleteness results, would surely have been heard of and discussed in England.[77] The important point for our purposes, whatever the case, is this.

[73] Wisdom (1934), p. 101 rejects this on the ground that these rules "are principles of logic". If Turing had read this, he might have demurred, taking the mathematical notion of "calculation" as basic, and, having analyzed *it*, applying it to the whole idea of "logic".

[74] See Detlefsen (2005).

[75] Cf. Mundici and Sieg, Chap. 1 and Davis and Sieg (2015).

[76] Carnap, Maund and Reeves (1934), p. 47.

[77] Carnap (1931). Carnap himself never once refers to Turing; see Floyd (2012a).

Although in his Moral Sciences Club lecture Turing is said to have invoked the notion of "interpretation", his use of it would have been loose and intuitive, not mathematical. He meant to say that a "purely logistic view of mathematics" is just one among other "interpretations" that purport to tell us what mathematics *really* is, and it is "inadequate" on its own to deliver such a story. By "mathematics" he would have meant mathematics as practiced in ordinary, informal language.

If this is right, and if the gist of Braithwaite's report is accurate, then, like Russell and Wittgenstein and unlike Carnap, in 1933 the undergraduate Turing was surely not ceding the notion of "interpretation" to metamathematics and model theory. After Carnap's assimilation of Tarski's work, "interpretation" would come to be routinely applied by logician-philosophers to the notion of a *structure* or *model* of a formal system, a notion developed by Tarski to clarify the notion of a "definable set". With this notion, collections of sentences may be shown, through an extensionally adequate definition, to be "satisfied ('true') in a model" in a metalanguage, allowing one rigorously to prove the *un*definability of a truth predicate for any sufficiently powerful formalized language within that language and to nevertheless be able to analyze the notion of logical consequence in the system.[78] It is important to stress, however, that notions of meaning, reference and "interpretation" in the intuitive sense—the one we imagine Turing would have used in his Moral Science Club lecture of 1933—are *replaced* by Tarski in the ascent to a metalanguage, and not analyzed, so that no particular philosophical analysis of meaning or truth follows from Tarski's work.[79] The fact is, we see in Turing's subsequent writings no general philosophical interest in truth *per se*.[80]

Instead, in 1933, in this particular Cambridge *milieu*, I suggest that Turing was positioning himself. He was engaged in a program of putting Russell's new logic into place, critiquing, not only its position or office, but perspectives on mathematics being forwarded by the "purely logistic" views. Turing was insisting that mathematical discourse stand on its own autonomous feet. And he was rejecting the idea that *Principia*—or any other construal of mathematics as a whole in terms of symbolic logic—offers mathematics a "foundation" of any ultimate or privileged sort, or reduces it to logic. A "purely logistic view" is but one (philosophically tendentious) "interpretation" among others, and is *therefore* "inadequate". This is thoroughly consonant with the opening of Wittgenstein's 1932 Cambridge lectures "Philosophy for Mathematicians" we have quoted above:

[78] Tarski (1933). Gödel, having studied the Introduction to *Principia Mathematica* and the *Tractatus* carefully, apparently came to the undefinability of truth already in 1930 studying *Principia* directly; Wang (1996), p. 82 and Floyd and Kanamori (forthcoming).

[79] Cf. Putnam (2015).

[80] Turing of course duly refers to Tarski in his dissertation on ordinal logics ((1939), p. 197) but here he is interested in *operative* accounts of definability. For discussion of his notion of an "oracle" in (1939), cf. 4.6 below.

Principia is simply part of mathematics, and there is nothing wrong with mathematics before its foundations are laid.

It is true that Turing's words, as reported by Braithwaite, may be taken to express nothing more than a "typical" mathematician's attitude. On this mild view of the remarks, the mathematician should simply regard logic as but one tool in the arsenal, and mathematics as wholly autonomous and self-authenticating. This certainly is consistent with what Turing is reported to have said, and it would echo what Hilbert himself really thought.[81] What is interesting, however, is how directly what Turing is said to have argued resonates with Cambridge philosophical themes central to discussion of his time.

4.5 Turing's "On Computable Numbers" (1936/7): the "Do-What-You-Do Machine"

Let us turn to Turing' specific argumentation in "On Computable Numbers" (1936/7), with an eye toward Cambridge resonances. As is well-known, Turing shows that, with their partiality, it is not possible to diagonalize out of the class of computable functions. As Cantor had demonstrated much earlier, one may diagonalize out of any purported enumeration of the real numbers (or the infinite sequences) by constructing a sequence not on the list: just go down the diagonal and change one digit of each expansion.[82] Turing's (1936/7) applies diagonal argumentation, but differently, as he says explicitly (§8). His particular manner of argumentation adapts Wittgenstein's notion of a tautology, central to the philosophy of logic at Cambridge since the *Tractatus*, as Wittgenstein later explicitly recognized in his 1947 recapitulation of Turing's diagonal argument in terms of "games".[83] Let us summarize the argumentation. We shall dub it Turing's "Do-What-You-Do Machine" argument, for reasons that will become clear.

A Turing Machine is a rigorous mathematical structure, but it is also, from another point of view, a crucial heuristic, or blueprint. It is also, as we have already insisted, an *everyday* picture, a simplified snapshot of human calculation as it is done, boiled down to its simplest and most vivid elements. This second point of view would have counted for Hardy as "gas", a merely rhetorical flourish that somehow nevertheless nonsensically points at a mathematical object. In contrast, for Turing this second aspect of his intuitive model was much more than that. It

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[81] Sieg (2013b).
[82] Cf. Kanamori (2012a), (2012b).
[83] Wittgenstein RPP I §§1096ff, cf. Floyd (2012).

was an essential part of his making a *philosophically* satisfying analysis of the notion of formal system.

After setting out his analogy with a human computer, Turing gives his definitions. A ***circle-free machine*** is one that, placed in a particular initial configuration, prints an infinite sequence of 0's and 1's. A ***circular machine*** fails to do this, never writing down more than a finite number of 0s and 1s. For Turing the *satisfactory* machines print out infinite sequences of 0's and 1's, whereas the *unsatisfactory* ones "get stuck".[84] A ***computable sequence*** of 0's and 1's is one that can be represented by (is the output of) a circle-free machine. A ***computable number*** is a real number differing by an integer from a number computed by a circle-free machine (i.e., its decimal (binary) expansion will, in the non-integer part, coincide with an infinite series of 0's and 1's printed by some circle-free machine); this is a real number whose decimal (binary) expression is said to be ***calculable by finite means***. Here Turing is applying an intuitive idea, and one incidentally discussed *ad nauseum* by Wittgenstein in his Cambridge lectures (and many other writings), and one also procedurally quite familiar to mathematicians: the notion of a "computable number", conceived as a *rule* for decimal expansion, differs from that of a "real number" conceived *extensionally*, regarded as a finished member of a finished totality.

After presenting examples of machines and variables ranging over machines in the form of "skeleton tables", Turing argues that because there is an enumeration of all the machines (since each can be associated with a "description number"), there is in addition—as a result—a single Universal Turing Machine U that can simulate any Turing Machine.

Next comes Turing's crucial, distinctive argument, his "Application of the Diagonal Process" to show that there is no circle-free machine that enumerates all and only the computable sequences by finite means (§8). His argument turns on a particular and *directly* constructed limit point at which the method itself must end, resolving the *Entscheidungsproblem* in the negative. *Here* is his ultimate reduction of the problem to "common sense".

First Turing defines a hypothetical "decision machine" D, which takes the standard description number k of an arbitrary Turing machine M, and tests to see whether k is the number of a circle-free machine or not, outputting "s" ("satisfactory") if it is, and "u" ("unsatisfactory") if not. He supposes that D computes its enumeration of the description numbers of all and only the circle-free machines by drawing from the enumeration of all machines. Let α_n be the nth

[84] Watson uses the metaphor that the machine "gets stuck" ((1938), p. 445), but I have not found that metaphor either in Wittgenstein or Turing. In LFM the metaphor is criticized (LFM, pp. 178-9), as well as the idea that we have to fear contradictions more than empty commands.

computable sequence in this supposed enumeration, and let $\phi_n(m)$, computable under the hypothesis, be the *m*th figure in α_n. By combining D with the universal machine U, Turing next constructs H, a machine that draws from along the diagonal sequence $\phi_n(n)$ to enumerate β, the sequence whose *n*th figure is the output of the *n*th circle-free machine on input n. Since by hypothesis D is circle-free, so is H.

Turing now argues that there can be no such H, and hence, no such D: these machines may be defined, but they cannot compute computable sequences. He does this by showing that H is infected with a defective command that cannot be followed, viz., "Do-What-You-Do".

The "Do-What-You-Do Machine", H, would by its design enumerate β as follows. Its action is divided into step-by-step sequential sections. In the first N-1 sections the integers 1, 2, … N-1 have been tested by D, and a certain number of these, say $R(N\text{-}1)$, have been marked "*s*", i.e., are description numbers of circle-free machines. In the *N*th section the machine D tests the number N. If N is satisfactory, then $R(N) = 1+ R(N\text{-}1)$ and the first $R(N)$ figures of the sequence whose description number is N are calculated. H writes down the $R(N)$th figure of this sequence. This figure will be a figure of β, for it is the output on n of the *n*th circle-free machine in the enumeration of α_n by finite means that D has been assumed to provide. Otherwise, if N is not satisfactory, then $R(N) = R(N\text{-}1)$ and H goes on to the $(N+1)$th section of its action.

Consider now K, the description number of H itself. We may ask, "What does H do on input K?" Since K is the description number of H, and H is circle-free, the verdict delivered by D on K cannot be "*u*". But the verdict also cannot be "*s*". For if it were, H would write down as the *K*th digit of β the *K*th digit of the sequence computed by the *K*th circle-free machine in α_n, namely by H itself. But the *instruction* for H on input K would be: "calculate the first $R(K\text{-}1)$ figures computed by the machine with description number K (that is, H) and write down the $R(K)$th". The computation of the first $R(K\text{-}1)$ figures would be carried out without trouble. But the instruction for calculating the $R(K)$th figure would amount to "calculate the first $R(K)$ figures computed by H and write down the $R(K)$th". This digit "would never be found", as Turing says. For at the *K*th step, it would be "circular", contrary to the verdict "*s*" and the original assumption that D exists (1936/7, p. 247). Its instructions at the *K*th step amount to the "circular" order "Do What You Do".

The ending of Turing's argument is analogous to drawing a card in a game that says "Do what this card tells you to do". The difficulty is not one of "nonsense" in the sense of gibberish, or a paradox, or a contradiction forming a kind of barrier to pursuing the hypothesis. The difficulty is that a rule has been formulated, using bits of language we well understand and perfectly acceptable rules of grammar, but which cannot be *followed* without a clear understanding of being in a "certain position" in a game. With a context in which actions are purposeful and directed against a backdrop of well-understood possibilities, "Do what you do" makes perfect sense. Without such a context, it does not. Such is common sense.

The last point matters. Turing's "Do-What-You-Do Machine" argument involves no special appeal to any specific principles of logic, any logical constants, or any interpretation of logic. It does not contain any negation or logical complexity, as most of the other classical uses of diagonal argumentation---including Gödel's---do. It is not a *reduction ad absurdam* argument. Instead, it is comprehensible directly, with---and *only* with---"common sense". Rather than a contradiction, it is the production of this *empty* command—one that cannot be *followed*—that defeats the idea of a decision procedure for logic. The reduction to "common sense" is what makes Turing's resolution so general.

In the usual modern presentation of Turing's proof, one uses the different "Halting argument", due to Martin Davis.[85] In this argument one defines a "contrary" machine *C* that changes 0 to 1 and vice versa along $\phi_n(n)$ to enumerate a sequence whose *n*th figure is the "contrary" output of the *n*th circle-free machine on input *n*. When *C* comes to apply itself to its own description number, it faces a *contradictory* command: "If your output is 0 on this input then you are to output 1, and if your output is 1, then you are to output 0". But no output can both be 0 and 1.

The difference between Turing's and the Halting argument is significant, for Turing's reaches into the "profoundly ordinary". The Halting argument, by contrast, works by *reductio ad absurdum*, applying a contradiction and utilizing the law of excluded middle, just as Gödel's argument for incompleteness did in his famous (1931). Turing, by contrast, does not oblige himself to apply any logical law---of non-contradiction, bi-valence, or the excluded middle---in his proof. Nor is his argument a regress argument, though it can be reconstrued in that way.[86]

Turing's "On Computable Numbers" provides, one might say, a kind of *ex post facto* justification for Wittgenstein's post-1933 "language game" approach to definability and to logic: a piecemeal, step-by-step exploration of possible meanings, processes, rules, using simplified snapshots of an evolving series of possible routines and modes of argumentation, portrayed as embedded and embodied in an evolving environment and culture. "Logic" is on this view fundamentally erected in the course of the investigation, carried forward in pieces of technology embedded in everyday language, and not by devising a particular formal system or finding any particular bottom level of analysis or uncontroversial starting point.

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[85] Martin Davis first gave this argument in 1952; see *http://en.wikipedia.org/wiki/Halting_problem#History_of_the_halting_problem* and Copeland (2004), p. 40 n 61.

[86] Copeland (2004), p. 38 nicely adapts *H* to a regress argument.

The notion of *tautology* was Wittgenstein's central contribution to logic and philosophy, as Turing knew. The whole argument of "On Computable Numbers" is a practical realization---and profound generalization---of an idea Wittgenstein had injected into the heart of Cambridge discussion. The fact that in our language we can put together declarative sayings that obey all the ordinary rules of grammar and yet "cancel out" their saying anything *shows* us something important *in* being empty, something important about logic and its limits. This perspective---as Russell noted at the end of *Introduction to Mathematical Philosophy*---transforms the idea of analytic or self-evident "truth", sinking it into logical activity itself, now regarded as a manifestation of our ability to appreciate limits in the totality of what we can sensibly say, represent, or derive. The role of such limiting cases of expression may be clarified by reflecting on tautologies in a suitable symbolism, just as Wittgenstein had done in the *Tractatus*, and as Turing also did, far more generally and rigorously, in "On Computable Numbers".

4.6 Turing 1939: Wittgenstein's Cambridge Lectures

Space prevents us from offering a detailed interpretation of the most well-known record of Turing's discussions with Wittgenstein (LFM). We shall content ourselves with a brief account of the importance to these exchanges of the theme of "common sense".

The usual focus of interpreters of these lectures has been Wittgenstein's disputes with Turing over the role of contradictions and paradoxes in logic. Wittgenstein says that such contradictions (such as the Liar paradox) are "useless", and he questions why anyone would ever has worried about them (LFM, pp. 207ff). Turing puts up a spirited defense, insisting that contradictions can have real-world consequences. (As a matter of fact, he was already working at Bletchley Park, where he would help design machines to decode cyphers from the German Enigma machine, and part of the implementation of logic used was to knock out possible interpretive hypotheses by finding contradictions.)

Commentators have tended to extract two points from these exchanges, each of which we shall dispute. First, there is the claim that Wittgenstein is attacking mathematical logic, pitting it against philosophy, so that the whole discussion is a pro vs. con debate, Turing pro mathematical logic and Wittgenstein con.[87] Second, there is the idea that Turing is some kind of Platonist, and Wittgenstein some kind of constructivist, perhaps even rejecting use of negation in infinite contexts.[88]

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[87] Monk (1990), pp. 419-20.

[88] Turing himself worries about constructivism creeping in LFM, pp. 31, 67,105.

The fact is that both thinkers were primarily interested, instead, in the workings of "common sense" and ordinary "phraseology". From the beginning of the very first lecture, they were picking up where they were before in discussion. Wittgenstein knew that "On Computable Numbers" had made a profound contribution to our understanding of logic; he was interested in developing his own philosophical thoughts to see how far he could better defend them before a sophisticated audience. Sometime in 1938 he had given a lecture to his circle of students (including Watson) in which he attempted, somewhat lamely, to transmute the "Do-What-You-Do-Machine" argument of Turing into the language of Gödel's 1931 paper.[89] This brainstorming gave way, in the spring 1939 lectures, to a businesslike reworking of his own philosophical ideas about common sense with Turing in the audience. Gödel's incompleteness theorem is only briefly mentioned (LFM, pp. 56, 188-89), presumably because they had already discussed it before; however certain Gödelian themes, refracted through Turing's (1939) work, shine through in their exchanges, as we shall explain.[90]

The fundamental themes under discussion concerned "gas" and the role of calculations, rules, and concept-formation in mathematics---ultimately, as we have seen from Hardy onward, the issue of how to regard "ground" and "conviction" in logic (cf. 1929, p. 17). Turing and Wittgenstein agreed that the "ground" would not be an object language leading to "an infinite hierarchy", as Wittgenstein put it (LFM, p. 14): neither advocated Hardy's idea of a purely abstract proposition. The "ground", instead, would be methods of procedure *as they are used*, of which, as Turing had himself shown in "On Computable Numbers" there are a variety. Wittgenstein's main aim was to stress how subtle or dramatic conceptual shifts may happen, as reflected in phraseology, when one kind of method gives way to a wholly different one, as opposed to being smoothly amalgamated into a prior whole. Turing repeatedly questioned, not Wittgenstein's views, but his applications of arguments to specific cases. Their most heated exchanges took place around the idea of "common sense".

The trouble with Hardy's idea of "gas" is that it made of such conversation as theirs something merely "aesthetic". This for Wittgenstein was a wholly wrong point of view of the relation of logic and philosophy to mathematics. ("You smoke cigarettes every now and then and work. But if you said your work was smoking cigarettes, the whole picture would be different" (LFM, p. 16)). Wittgenstein's basic move was to insist that there is no *general* dichotomy between form and content, between aesthetic or heuristic and result, between formal procedure and meaning, between philosophical discussion of mathematics and mathematics, and

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[89] This, at least, is one guess as to the contents of the lecture, which exists only in notes taken down by Smythies. See Munz and Ritter (eds.) (forthcoming).

[90] For more on Wittgenstein's own remarks on Gödel and "phraseology", see Floyd (2001b).

between previously given understandings and their revision---though of course there are *local* distinctions that may be drawn in the face of particular cases, and which particular "phraseology" we choose, and how we respond and go on from it, often very much matters. Wittgenstein wanted to argue that this followed from Turing's own analysis of the very idea of a formal system, in the form of a problem about what it *is* to follow a rule, correctly or incorrectly.

The trouble with the notion of "common sense" as an unexamined idea is philosophical. How are we to distinguish erroneous dogma, or mere stipulation, from common understanding that provides us with sufficient backdrop for the notions of correctness and incorrectness to have a grip? How are we to understand the notion of the "reasonable", as opposed to the formally possible? How are we to correct—or even just expose and probe---what is taken to *be* "common sense" by a mathematician or philosopher? How are we to make heuristic discussion and the metaphors used something other than "decoration", "a lot of jaw", "like squiggles on the wall of a room", as Hardy had said (1929, pp. 13-14).

We have already seen that this had long been the main divide between the Wittgenstein-Turing Cambridge approach, on the one hand, and the Hardy approach on the other. In *The Blue Book* and *The Brown* Book, as well as in his 1932-33 lectures and dictations for mathematicians, Wittgenstein had developed a method of thinking through a series of small variations and comparisons in the backdrop of imagined, unformalized human uses of logic ("language games") to draw out the critical importance, not only of "common sense", but of the kind of conversation and discussion that a reasonable notion of "common sense" requires. Here, in 1939, he applies the method to well-known classical impossibility and other proofs in mathematics.

Wittgenstein's overall suggestion to Turing was to express a concern. Without the right orientation toward his machines, Turing risked being seen, or seeing himself, as nothing but a reducer of mathematics to logic, and logic to "squiggles" (LFM, p. 14). The first step here---just where Wittgenstein began Lecture I---was to insist that a key constraint on his remarks was "not to interfere with the mathematicians" (LFM, p. 13). That is, to say, he wanted to make it clear that he had no interest in revising logic, or Turing's mathematical work. In fact Turing's remarks were of special importance to Wittgenstein, and he even chastised himself later on in the lectures when he appeared to swerve dangerously close to interfering with ongoing mathematics in discussing the notion of contradiction (LFM, p. 223). He wanted to keep his conversation with Turing going, pressing the importance of not covering up the variety of methods at work in logic and mathematics. And Turing represented, within the context of the course, "common sense" as it was understood by mathematicians.

The main question under discussion was how to accept what Turing had done in analyzing the notion of a formal system, and yet draw out its philosophical significance. Lecture I begins with an insider's joke directed at Turing: a kind of tribute. First, Wittgenstein reverted to his extrusion of mental states from the foundations of logic, pointing out that since an expression has different kinds of use, it makes no sense to think that one can "have the use before one's mind" in a single instant. What one has is a *modus operandi*, and not an image-in-itself, or a state of mind as a property or ontological particular. This he argued for by reverting to the older argument---initiated by Hardy in response to Hilbert, as we have seen---about *sameness* of signs (LFM, p. 20):

> What is a 'representative piece of the application'? Take the following example. Suppose I say to Turing, "This is the Greek letter sigma", pointing to the sign σ. Then when I say, "Show me a Greek sigma in this book", he cuts out the sign I showed him and puts it in the book.

This is obviously a parody of the idea of a sign as used by a Turing Machine. Wittgenstein continues:

> --Actually these things don't happen. These misunderstandings only immensely rarely arise---although my words might have been taken either way. This is because we have all been trained from childhood to use such phrases as "This is the letter so-and-so" in one way rather than another.
>
> When I said to Turing, "This is the Greek sigma", did he get the wrong picture? No, he got the right picture. But he didn't understand the application (LFM, p. 20).

The point to Turing was the importance of "taking a wider look 'round", at the general *cultural* and "application" setting within which a machine or routine is set to work. This might be said to be a "common sense" remark *about* "common sense", as well as about computations themselves. But Wittgenstein's point was that the philosopher is duty-bound to insist on the importance and complexity of this notion, for only in this way may meaning, and actual logic itself, get a grip. Attending to all the panoply of apparent contingencies and "phraseology" surrounding our uses of logic, and discussing them one-by-one in turn, learning to contrast and compare cases, is the only way to rightly---or responsibly---see what it *is* to be "responsible to a mathematical reality" (LFM, p. 240).[91]

Further on in the lectures (LFM, p. 35), the issue of how time may or may not enter into an analysis of proof is discussed, a significant feature of Turing's model, for it has a double face: the first that of a static mathematical object or formalism, the other that of a dynamic machine whose movements are traceable by mechanisms and human step-by-step computations as we regard them ordinarily. Wittgenstein explores with Turing the question of enormously long computations and proofs, whose outcomes constantly change (LFM, pp. 37, 105-6), how to think about the relation between formalized and unformalized proofs and the notion of rigor (LFM, pp. 127-133, 261ff), how to demarcate the realm of calculation from

[91] Cf. Diamond (1996).

that of experiment (LFM, pp. 96ff), how to discuss certainty vs. skepticism about calculated or proved results (LFM, pp. 101-3). All this talk of "faces" and length-of-proof would have been counted by Hardy as "gas", but to Wittgenstein and Turing it forms a crucial matter for discussion, a way of thinking about what it is to "build new roads" in mathematics, as opposed to ineffably pointing at distant mountaintops (LFM, p. 139).

This whole issue of "phraseology" also formed the key to their debates over contradictions. Turing took Wittgenstein at times to be insisting that when a conceptual shift occurs, when the character or aspect of use shifts, there is a "change in meaning", and he repeatedly worried that Wittgenstein was attacking the use of negation or indirect argument in general, as an intuitionist or finitist would---something Wittgenstein explicitly and repeatedly denied (LFM, pp. 31, 67). Turing also worried that Wittgenstein was veering too close to saying that mathematicians merely "invent" procedures, rather than "discovering" them---again something Wittgenstein denied (LFM, p. 68). Instead, Wittgenstein wanted to discuss what discoveries in mathematics *are*. He reminded the students, after an exchange with Turing, of his "slogan":

> Don't treat your common sense like an umbrella. When you come into a room to philosophize, don't leave it outside but bring it in with you (LFM, p. 68).

Turing challenged this response, especially applied to the arising of possible contradictions: "You seem", he later remarked, "to be saying that if one uses a little common sense, one will not get into trouble" (LFM, p. 219). Wittgenstein responded vehemently,

> No, that is *NOT* what I mean at all. --- the trouble described is something you get into if you apply the calculation in a way that leads to something breaking. This you can do with *any* calculation, contradiction or no contradiction (LFM, p. 219).

The issue reaches back to Russell's *Introduction to Mathematical Philosophy*, and the question whether appealing to the laws of non-contradiction, bivalence, and excluded middle suffice for the foundations of logic. For Wittgenstein, as for Russell, they did not. Wittgenstein had long argued that instead, we need to attend to *tautologies*, emptinesses of words and rule-commands, in order to probe the limits of the logical, and see how it is grounded, not in facts or self-evident principles, but in our own manner of representing the world in language by doing things *with* language. This, as we have seen, was confirmed by the "Do-What-You-Do" machine Turing had constructed in "On Computable Numbers"; even there the argument was explicitly portrayed as a response to the kind of worries about negation Wittgenstein was still expressing in 1939.[92]

[92] In (1936/7) §8 Turing writes, of the more "direct proof" using a *reductio*, that "although [it is] perfectly sound, [it] has the disadvantage that it may leave the reader with a feeling that

Turing and Wittgenstein agreed that the use of mathematical symbols in extra-mathematical contexts is crucial to their meaning; such would anyone having learned from logicism.[93] Wittgenstein, however, worried that this could leave parts of mathematics out of account. For some of these (e.g. set theory) he regarded as taking up, as their very *raison d'être*, a way of regarding concepts that abstracts from all the procedures and methods with which we are familiar from those parts of mathematics that are applied (LFM, p. 29, 102-3). Moreover, Wittgenstein rejected Turing's suggestion that the relation between everyday language and mathematical symbolism is merely one of "abbreviation" or "definition" as Turing at one point suggested (cf. LFM, p. 42). *Conceptual*, not merely verbal work is required to make unsurveyable routines surveyable, and this is a matter of proof in mathematics, not merely experiment or psychology (LFM, p. 226). The relation between logic and mathematics was to be found in the concepts, but this was not properly clarified by Frege and by Russell, with their "theory of types".

Wittgenstein eventually acknowledged that he wanted to say "something rather similar" about "common sense" to what Turing had suggested (LFM, p. 223). "Common sense" had "some truth" in it (LFM, p. 229). His idea was that the basis and the necessities of logic lie in what we *do*, in what we *recognize* in practice, rather than in our being convinced of a particular truth (LFM, p. 230). Our *uses* of the law of contradiction in particular venues and contexts shows us what adherence to it *is*, and this is constituted, not only by the myriad ways we have been trained and train others in a variety of methods, but also by what we are inclined to say: how we are inclined to express ourselves, draw analogies, pictures, and so on, in conversations *about* particular cases---the "gas".

On this view, the law of contradiction is not to be understood as a stopping point of argument, a potential flaw in engineering routines, or a piece of psychological reality. Rather, we must see the relation of logic to mathematics in terms of what we *grant*, normatively, what we *recognize as*: reasoning, calculating, inferring, and so on. Here Wittgenstein did not simply mean, as Hardy had, our ability to recognize one sign as the same as another. He meant the more general, older notion of recognition, the sense in which a king *recognizes* or *acknowledges* the

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'there must be something wrong'". His "Dow-What-You-Do" argument offers a response to such a reader. In his corrections (1937), stimulated by Bernays, Turing develops this point with respect to intuitionism explicitly. See Floyd (2012b) for a discussion.

[93] Russell (1919), p. 6:

> This point, that "o" and number and "successor" cannot be defined by means of Peano's five axioms, but must be independently understood, is important. We want our numbers not merely to verify mathematical formula, but to apply in the right way to common objects. We want to have ten fingers and two eyes and one nose. A system in which "1" meant 100, and "2" meant 102, and so on, might be all right for pure mathematics, but would not suit daily life.

dominion of a lord over a land. Such recognition, or granting of authority is bound up in serious ways, not simply with the law of contradiction as applied in physics and mathematics, but with how we *phrase* ourselves about hard and transition cases, with what we are able actually to share, with "common sense" in the sense of a working harmony and agreement. This was the domain, at least for Wittgenstein, of philosophy. It is far more than "smoke" or "gas".

Of course, Turing was a mathematician, not a philosopher. His job was to do something mathematical. In the winter of 1937, back at Princeton after his summer discussion group with Wittgenstein and Watson, he had written his dissertation, devoted to the question of how, logically, one might get around Gödelian incompleteness operationally, in theory (1939). He framed a notion of *relative* computability or solubility in a higher-order logic, invoking the picturesque idea of an "oracle": the instantaneous, black-box delivery of a solution otherwise unavailable, getting one to a next system. This anthropological image---one that eliminates the internal psychology of the individual human mind entirely from the mathematical step---was, in the words of Solovay, to "change the face of recursion theory".[94]

It was this very image of an "oracle" to which Wittgenstein---probably aware of Turing's (1939)---explored in his 1939 lectures:

> [*Wittgenstein*]: So what about our case: this new calculation and these people disagree [half go one way, half go the other[95]]. What are we to say? --Shall we say, "Why aren't our minds stronger?" or "Where is an oracle?" But is there anything for it to know? Aren't you right---or wrong---as you please?
>
> *Turing*: We'd better make up our minds what we want to do.
>
> *Wittgenstein*: Then it isn't a message from God or an intuition, which you pray for, but it is a *decision* you want. But doesn't that contradict [your] idea of an experiment? Where is the experiment now?
>
> *Turing*: I should probably only speak of an experiment where there is agreement.
>
> *Wittgenstein*: Don't you mean that in that case the experiment will show what the rule is? The fact is that we all multiply in the same way---that actually there are no difficulties about multiplication. If I ask Wisdom to write out a multiplication and get the result, and he tells me, then I am perfectly certain that that's the right thing, the adopted thing (LFM, p. 109).

Wittgenstein is here arguing *against* the idea, at first casually suggested by Turing, that different understandings of particular phraseologies represent nothing

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[94] Solovay (1988), p. 126; cf. discussion of Turing (1939) in Copeland (2004) and Hodges (1999).

[95] Our best guess is that Wittgenstein is alluding to the situation after Gödel 1931, in which both the Gödel sentence P and its negation, not-P, may each be added to the original system of arithmetic consistently, presenting a kind of branch of possible paths. Cf. Wittgenstein's 1938 lecture on Gödel in Munz and Ritter (eds). (forthcoming).

but different "decisions", "intuitions", or "experiments" (LFM, pp. 109ff, cf. 31, 147).[96] "I should say," Wittgenstein replied, "that if it was a mathematical proof, God didn't know more than any one of us what the result of the calculation was" (LFM, p. 103). His point would have been that Turing's notion of computation relative to an oracle is *not* the idea of a computation less certain than any other mathematical step, and it need not be regarded as something analogous to a proposition of physics, or an approximation.[97] And so it is with a *relative* computation in Turing's (1939) sense. If we take up the result of a higher-order computation, then we take *it* as it is. But it is *we* who acknowledge the result of the "oracle", and we proceed conditionally, as logic and mathematics allow us to do.[98]

What Turing went on to do in the 1940s with Wittgenstein's notion of "phraseology" is what we shall discuss next.

4.7 Turing's "The Reform of Mathematical Notation and Phraseology" (1944/5)

As we have seen, to Turing the ordinary talk or "common sense" of what Hardy (1929) called the "mathematician-in-the-street" is a richly rewarding subject of scrutiny, as well as a repository of longstanding mathematical experience and culture. It was not *just* "gas". As he saw it, it should be respected, exploited, improved, and given its due. On the other hand, as he lamented in an unpublished paper on "The Reform of Mathematical Notation and Phraseology" (1944/5), the variegation of this talk remained "exceedingly unsystematic", constituting "a definite handicap both to the would-be-learner and to the writer who is unable to express ideas because the necessary notation for expressing them is not widely known" (1944/5, p. 215). He was calling for a "reform" of mathematical notation and "phraseology". As he acknowledged here, he drew his ideas from Wittgenstein:

> We are taught that the theory of types is necessary for the avoidance of paradoxes, but we are not usually taught how to work the theory of types into our day-to-day mathematics: rather we are encouraged to think that it is of no practical importance for

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[96] Turing works through the notion of "intuition" himself in (1939), in (2001) §11, pp 214-216. Cf. Solovay (1988) and Copeland (2004) for discussion of the 1938 thesis, the basis for (1939).

[97] It is just this idea to which Wittgenstein reverts when, probably reminiscing about these exchanges, he revisits the notion of an "oracle" much later. See RPP I §817 and (1974) (OC) §609, written just six days before his death.

[98] In (1944) Post criticized Turing for his "picturesque" use of the idea of an "oracle", writing "the 'if' of mathematics is … more conductive to the development of a theory", p. 311 n.23.

> anything but symbolic logic. This has a most unfortunate psychological effect. We tend to suspect the soundness of our arguments all the time because we do not know whether we are respecting the theory of types or not. Actually it is not difficult to put the theory of types into a form in which it can be used by the mathematician-in-the-street without having to study symbolic logic, much less use it. The statement of the type principle given below was suggested by lectures of Wittgenstein, but its shortcomings should not be laid at his door (1944/5, p. 217)[99]

In the essay Turing appeals to argumentation he and Wittgenstein had discussed in Wittgenstein's seminar:

> The type principle is effectively taken care of in ordinary language by the fact that there are nouns as well as adjectives. We can make the statement 'All horses are four-legged', which can be verified by examination of every horse, at any rate if there only a finite number of them. If however we try to use words like 'thing' or 'thing whatever' trouble begins. Suppose we understand 'thing' to include everything whatever, books, cats, men, women, thoughts, functions of men with cats as values, numbers, matrices, classes of classes, procedures, propositions,. . . Under these circumstances what can we make of the statement 'All things are not prime multiples of 6'? We are of course inclined to maintain that it is true, but that is merely a form of prejudice. What do we mean by it? Under no circumstances is the number of things to be examined finite. It may be that some meaning can be given to statements of this kind, but for the present we do not know of any. In effect then the theory of types requires us to refrain from the use of such nouns as 'thing', 'object' etc., which are intended to convey the idea 'anything whatever' (1944/5, p. 218).

Turing's view, as expressed in this paper, was that "it has long been recognized that mathematics and logic are virtually the same and that they may be expected to merge imperceptibly into one another", although "this merging process has not gone at all far, and mathematics has profited very little [so far] from researches in symbolic logic" (1944/5, p. 245). Though Turing was nodding to what he called the "Russellian *Weltanschauung*" of *Principia Mathematica*---whose theory of types he took to be held by "the majority of mathematicians-in-the-street"---he was not arguing for logicism as a conceptual or metaphysical doctrine in the foundations of mathematics. He was making an observation. Turing's aim was to revise both symbolic logic and mathematics to effect a better "merging", where this would be a matter of degree rather than a matter of either a sharply "perceptible" line or a reduction of one subject to the other.

His diagnosis of the difficulty hindering productive "merging" was one of culture and missed opportunities for cooperative communication, rather than any in-principle philosophical mistake:

> The chief reasons for this [merging not going at all far] seem to be a lack of liaison between the logician and the mathematician-in-the-street. Symbolic logic is a very alarming mouthful for most mathematicians, and the logicians are not very much interested in making it more palatable. It seems however that symbolic logic has a

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[99] Cf. Floyd (2013).

> number of small lessons for the mathematician which may be taught without it being necessary for him to learn very much of symbolic logic.
>
> In particular it seems that symbolic logic will help the mathematicians to improve their notation and phraseology…By notation I do not of course refer to such trivial questions as whether pressure should be denoted by *p* or P, but deeper ones such as whether we should say 'the function *f*(*z*) of *z*' or 'the function *f*' (1944/5, p. 215)

Whether to refer to letters of the alphabet, parameters such as "*z*", casually, mixing what Carnap called "formal" and the "material" modes of speech, or whether to objectualize functions; whether rigorously to obey use vs. mention and/or to avail oneself of Russell's theory of incomplete symbols and types, whether to explicitly bind all variables or leave them free, but typed: these issues were on Turing's mind at the time he wrote this essay. Such issues lay at the heart---the philosophical heart---of "symbolic logic". Just here Turing took a particular stand:

> It would not be advisable to let the reform take the form of a cast-iron logical system into which all the mathematics of the future are to be expressed. No democratic mathematical community would stand for such an idea, nor would it be desirable. Instead one must put forward a number of definite small suggestions for improvement, each backed up by good argument and examples. It should be possible for each suggestion to be adopted singly. Under these circumstances one may hope that some of the suggestions will be adopted in one quarter or another, and that the use of all will spread (1944/5, p. 215).

On Turing's view, formalisms should never be "cast-iron" straightjackets, but opportunistic, targeted aids to the avoidance of ambiguity and unclarity, bridging the gap between the formal and the informal in order to lessen the need for detailed knowledge of formalized structures. Rather than machines for calculating necessary and sufficient conditions, they would be operational, useful interventions. They would respond to the language of the "mathematician-in-the-street", clarifying and aiding it. Turing advised that we make a study of "current mathematical and physical and engineering books and papers with a view to listing all commonly used forms of notation", and then examine the notations

> …to discover what they really mean. This will usually involve statements of various implicit understandings as between writer and reader, it may also include the equivalent of the notation in question in a standard notation…[Then proceed in] laying down a code of minimum requirements for desirable notations. These requirements should be exceedingly mild (1944/5 p. 215).

The meanings involved in mathematics and mathematical talk are various, but for Turing they are dynamic, evolving, and purposive, subject to notational clarification and systematization in the context of ongoing cultural and intellectual developments. His envisioned "code of minimum requirements" was to serve communication and clarity, not machine implementation alone. It was certainly not intended to fix a "cast-iron" semantics for our language---one provably unavailable anyway, on his view, due to incompleteness and undecidability.

Turing's paper was written after he had accomplished much work at Bletchley Park. It is forward-looking in its call for the development of what computer scientists nowadays call "types" in higher-level programming languages: categories and ontologies that humans can use, develop, visualize, organize and communicate with, among one another and utilizing machines. Even today the "semantic web" involves a quest for massive archival sorting, exploitation, higher-type organization.[100] Turing never lost sight of the need for articulation in higher-level languages and meanings in his implementation of the very low-level, step-by-step computational and representational modularity of the Turing Machine. He saw the difference in levels and types as a complex series of systematizations sensitive to everyday "phraseology" and common sense, not a divide of principle. This was because he always saw "types" or "levels" as lying on an evolving continuum, shaped by practical aspects, the user end, *and* mathematics. *This* is the aspect of his thought that drew from the Cambridge tradition of "common sense".

4.8 Turing's "Solveable and Unsolveable Problems" (1954)

This brings us to the final published words of Turing, the closing paragraph of his last paper, a popular presentation of logical theory in terms of word problems and puzzles. Here Turing adapts some of Post's work on word problems to this accessible setting, revisiting issues of undecideability and incompleteness.[101] He closes the essay this way:

> These [limitative] results, and some other results of mathematical logic may be regarded as going some way towards a demonstration, within mathematics itself, of the inadequacy of 'reason' unsupported by common sense (1954, p. 23).

Turing's idea of "common sense" is the idea of something *not* given in an algorithm, but generally shared among language users, and ineradicable in light of incompleteness. His stance implies that the reality of what we do and say with our words is part of reality itself, not a mandatory "add on" at "the meta-level" to a purely formalized or idealized computational "object language". The mathematician-in-the-street should not be regarded by the logician as someone with metaphysical prejudices to be argued with, for or against (as Hardy thought), but rather as someone in need of practical help. Help would come from inside mathematics itself, but only with appropriate philosophical understanding.

As Turing's student Gandy put it,

> Turing was first and foremost a mathematician. He believed that the chief purpose of mathematical logic and the study of the foundations of mathematics was to help

[100] Cf. Wolfram (2013), a commentary on Turing's (1944/5).

[101] See Davis and Sieg (2015).

> mathematicians to understand what they were doing, and could do. In pursuit of this goal, mathematical logicians must perforce construct and manipulate complex formal systems. But they have a duty to explain to mathematicians, in as non-technical way as possible, what they have accomplished. (A good example is Turing's account of the purpose of his ordinal logics - see his paper [(1939] in Part I). Turing disliked those high priests of logic who sought (like Quine in his 'Mathematical Logic' [1940, 1st. ed.]) to blind the mathematician-in-the-street with arcane formalisms (Turing (2001), p. 213).

Quine's emphasis on syntactic *finesse* and the use of formalized notation to enunciate a view of the world as a whole stemmed from the Russellian and Carnapian traditions. Turing set his face against these. These philosophies of logic were oriented *cosmically*, toward an explicit enunciation or renunciation of an ontology of the world as a whole, and the articulation of meaning through logical consequence, implemented, especially in the hands of Quine and Carnap, through emphasis on syntax. Turing's work stemmed from a different quarter. He was an artful and practical dodger in matters of ontology and meaning, oriented toward the values of use and simplicity all the way down.

4.9 Concluding Remark

This essay is intended a contribution to our understanding of how philosophical progress takes place. Genuine philosophical contributions do not always come labelled with the term "philosophy", and progress in philosophy does not always come labelled with the term "philosophical progress". Sometimes the most crucial moves in philosophy are not labelled as such *per se*, being pitch and not wind-up. In such cases, when what is to count as "common sense" is at stake, what is required is for participants to articulate rather than merely expound, to inspire and adapt ways of thinking, to pursue and cultivate open-minded reflection, argumentation, a sense of internal criticism and careful development of particular, differing points of view. From this may emerge the true value of intelligibility, directness and simplicity—especially at the initial steps.

Such are the earmarks of Turing's philosophical and scientific sensibility. They are to be celebrated and prized.

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Abbreviations of Wittgenstein's Works

AWL	Wittgenstein (1979)
BB	Wittgenstein (1965)
LFM	Wittgenstein (1976/1989)
PR	Wittgenstein (1980a)
OC	Wittgenstein (1974)
PI	Wittgenstein (2009)
RFM	Wittgenstein (1978)
RPP I	Wittgenstein (1980b)

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