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J. v. Neumann, On an algebraic generalization of the quantum mechanical formalism (Part I), *Rec. Math. [Mat. Sbornik] N.S.*, 1936, Volume 1(43), Number 4, 415–484

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On an algebraic generalization of the quantum mechanical formalism (Part I)

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Introduction

1. The present work is a continuation and extension of a paper by P. Jordan, E. Wigner and the author, which appeared — under the same title — in the „Annals of Mathematics", **35**, (1934), 29—64. This paper will be referred to as „loc. cit.", but a detailed knowledge of its contents will not be necessary for the understanding of the present note.

Both, loc. cit. and this note, are based on an idea of P. Jordan [„Zeitschr. für Physik", **80**, (1933), 285; „Göttinger Nachr.", 1932, p. 569 and 1933, p. 209] which may be described briefly as follows:

Quantum mechanics deal with a physical system S and the class of its „observables" $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \ldots$ for which certain algebraic operations have a meaning, while others do not — or at least do not always — have one.

In particular:

(i) If $\mathfrak{a}$ is an „observable" and

$$p(x) = a_0 + a_1 x + \ldots + a_n x^n$$

a polynomial ($n = 0, 1, 2, \ldots, a_0, a_1, \ldots, a_n$ — real numbers), then $p(\mathfrak{a})$ is an „observable" too. This statement is justified by the observation that a method to measure $p(\mathfrak{a})$ can be defined explicitly: $p(\mathfrak{a})$ is measured by measuring first $\mathfrak{a}$, and then forming $p(x_0)$ for the result x_0.

(ii) If $\mathfrak{a}, \mathfrak{b}$ are two „observables" and $p(x, y)$ is a two-variable polynomial, then $p(\mathfrak{a}, \mathfrak{b})$ cannot be formed in general along the lines of (i). In fact this would necessitate a simultaneous measurement of $\mathfrak{a}$ and $\mathfrak{b}$, and it is well known that this is not always possible for quantum-mechanical „observables". In particular for

$$p(x, y) = x \cdot y$$

it is not possible to attach any immediate meaning to $\mathfrak{a} \cdot \mathfrak{b}$.

(iii) In spite of (ii),

$$p(x, y) = x + y$$

can always be formed, that is has always a meaning. Not that we cannot explicitly construct a method to measure $\mathfrak{a} + \mathfrak{b}$, but we postulate the existence of an „observable" $\mathfrak{c}$ with the following property:

In every collective of specimens $S_1, S_2, \ldots$ of the system S we have

$$\text{mean value of } \mathfrak{c} = \text{mean value of } \mathfrak{a} + \text{mean value of } \mathfrak{b}. \qquad (*)$$

If such a $\mathfrak{c}$ exists (which we postulated) it is unique: two such $\mathfrak{c}_1$, $\mathfrak{c}_2$ would have the same mean value in every collective, and therefore be identical.

(For the detailed discussion of (i)—(iii) cf. the author's book „Mathematische Begründung der Quantenmechanik", J. Springer, Berlin, 1932, part IV, 1, as well as a note in „Göttinger Nachr.", 1927, p. 1—55. The essential points to be observed are:

First, that even if $\mathfrak{a}$, $\mathfrak{b}$ are not simultaneously observable, their mean values in a collective are simultaneously observable — one can use two different parts of the collective, and use Bernoulli's „law of great numbers".

Second, that the privileged character of $\mathfrak{a}+\mathfrak{b}$ is due to the fact that (*) is true in statistics irrespective of whether $\mathfrak{a}$ and $\mathfrak{b}$ are „correlated" or not — $\mathfrak{a}\cdot\mathfrak{b}$ is not such, its mean value depending essentially on the „correlation" of $\mathfrak{a}$ and $\mathfrak{b}$.)

Thus a philosophically satisfactory system of quantum theory should use the operations (i), (iii) only, that is the operations:

(α) $\rho\mathfrak{a}$ (ρ a real coefficient),

(β) $\mathfrak{a}^n$ ($n=0, 1, 2, \ldots$),

(γ) $\mathfrak{a}+\mathfrak{b}$.

Now in actual quantum theory

$$\mathfrak{A}=(\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \ldots)$$

is represented by the system $\mathcal{A}$ of all Hermitean matrices of Hilbert space:

$$\mathcal{A}=(A, B, C, \ldots).$$

But $\mathcal{A}$ must be considered as a subsystem of the system $\mathcal{B}$ of all matrices of Hilbert space:

$$\mathcal{B}=(X, Y, Z, \ldots)$$

and the fundamental operations of $\mathcal{B}$ are these:

(α') ρX (ρ — a complex coefficient),

(β') XY,

(γ') $X+Y$,

(δ') X^* (the „adjoint").

It is highly unsatisfactory that (α')—(δ') are not operations within the physically significant part $\mathcal{A}$ of $\mathcal{B}$: (α'), (β') form quantities not necessarily in $\mathcal{A}$, even if X, Y are in $\mathcal{A}$; and (δ') is void in $\mathcal{A}$. ($X^*=X$ for X in $\mathcal{A}$.)

Therefore P. Jordan proposed to study the algebraic properties of $\mathcal{B}$ only, using the operations (α)—(γ) instead of (α')—(δ'). His program consists of two steps:

First, to formulate those formal properties of (α)—(γ) in $\mathcal{A}$ which seem to be essential and physically significant.

Second, to replace $\mathcal{A}$ by any abstract system $\mathfrak{A}$ with operations (α)—(γ), introduce the above-mentioned formal properties of $\mathcal{A}$ as axioms for $\mathfrak{A}$, and to treat $\mathcal{A}$ axiomatically: that is, to determine which other systems $\mathfrak{A}$ (besides the above-described part $\mathcal{A}$ of $\mathcal{B}$) fulfil those axioms.

2. The decisive axioms are these:

(I) Addition ($\mathfrak{a}+\mathfrak{b}$) is commutative and associative, and with respect to $\rho\mathfrak{a}$ (ρ is a real number) also distributive.

(II) For polynomials of one variable the usual rules of computation hold, that is: if $p(x)$, $q(x)$, $r(x)$ are polynomials of one (numerical) variable and if

$$p(q(x)) \equiv r(x)$$

then

$$p(q(\mathfrak{a})) = r(\mathfrak{a})$$

(cf. (i) in § 1).

Jordan pointed out that a „quasi"-multiplication $\mathfrak{a} \circ \mathfrak{b}$ can be defined on the basis of (α)—(γ):

$$\mathfrak{a} \circ \mathfrak{b} = \frac{1}{2}(\mathfrak{a} + \mathfrak{b})^2 + \left(-\frac{1}{2}\right)(\mathfrak{a}^2 + \mathfrak{b}^2).$$

In the case of $\mathcal{A}$ and $\mathcal{B}$, that is of matrices, this is clearly the „symmetrized" product:

$$X \circ Y = \frac{1}{2}(X + Y)^2 + \left(-\frac{1}{2}\right)(X^2 + Y^2) = \frac{1}{2}(XY + YX).$$

If X, Y are Hermitean matrices, then $X \circ Y$ is one too. $\mathfrak{a} \circ \mathfrak{b}$ is obviously commutative, but not necessarily associative.

In fact, in the case of $\mathcal{A}$ and $\mathcal{B}$, the non-associativity of $X \circ Y$ is closely connected with the non-commutativity of the ordinary product XY: using the abbreviations

$$[X, Y, Z] = (X \circ Y) \circ Z - X \circ (Y \circ Z), \quad [X, Y] = XY - YX,$$

we have:

$$[X, Y, Z] = \frac{1}{4}\{(XY + YX)Z + Z(XY + YX) - X(YZ + ZY) -$$
$$-(YZ + ZY)X\} = \frac{1}{4}\{YXZ + ZXY - XZY - YZX\} = \frac{1}{4}[Y, [X, Z]].$$

The meaning of this relation was discussed (loc. cit., p. 45), and will be treated more fully in this note, in part of § 5, and in part H. The importance of this non-associativity, as an equivalent on a broader basis of the non-commutativity of the ordinary theory, was emphasized by Jordan in his papers quoted at the beginning of § 1.

On the other hand, an algebraic discussion will be scarcely possible, if the distributive law does not hold for $\mathfrak{a} \circ \mathfrak{b}$:

$$(\mathfrak{a} + \mathfrak{b}) \circ \mathfrak{c} = \mathfrak{a} \circ \mathfrak{c} + \mathfrak{b} \circ \mathfrak{c}.$$

Thus it holds for $X \circ Y$ in $\mathcal{A}$ and $\mathcal{B}$. The distributive law happens to be equivalent to

(III)

$$(\mathfrak{a} + \mathfrak{b})^2 + (\mathfrak{a} - \mathfrak{b})^2 = 2\mathfrak{a}^2 + 2\mathfrak{b}^2.$$

(Cf. loc. cit., p. 32, and Remark 2 following the group II of Axioms in this note.) Thus we must require axiom (III) too.

It is appropriate to observe that while there are obvious physical (phenomenological) reasons to require (I) and (II), no such reasons are known in favor of (III). We require (III) merely on the basis of its truth in the present system of quantum mechanics, and its algebraic rôle in connection with the distributive law. It seems to be one of the essential features of quantum theory, although its true (phenomenological) meaning is obscure.

(I), (III) permit replacement of (II) by the weaker requirement

(II′)

$$(\mathfrak{a}^m + \mathfrak{a}^n)^2 = (\mathfrak{a}^{2m} + \mathfrak{a}^{2n}) + 2\mathfrak{a}^{m+n}.$$

(That is: $$p(x) = x^2, \quad q(x) = x^m + x^n, \quad r(x) = x^{2m} + x^{2n} + 2x^{m+n}.$$
Cf. loc. cit., p. 32, and Remark 3 following the group II of Axioms in this note.) Besides (I), (II′), (III) an axiom on definite, that is on essentially non-negative quantities, is needed. These are best characterized as being squares $\mathfrak{a}^2$. The physically obvious axiom then is:

(IV) $$\mathfrak{a}^2 + \mathfrak{b}^2 + \mathfrak{c}^2 + \ldots = 0 \quad \text{implies} \quad \mathfrak{a} = \mathfrak{b} = \mathfrak{c} = \ldots = 0.$$

(This is the „reality"-condition of E. Artin and J. Schreier.)

Loc. cit., all systems $\mathfrak{A}$ which satisfy (I), (II′), (III), (IV) [cf. loc. cit., p. 32, our (II′), (III), (IV) correspond to (II′), (*), (I′) there] and possess a finite linear basis were discussed.

It was found that only these solutions exist:

(1) $\mathfrak{A}$ consists of all Hermitean matrices of some fixed degree $N = 1, 2, \ldots,$ the elements of which all belong to $\mathfrak{Z}$, $\mathfrak{Z}$ being either the set of all real numbers, or the set of all complex numbers, or the set of all common (real) quaternions. $\mathfrak{a} + \mathfrak{b}$ is the common matrix-addition, $\mathfrak{a} \circ \mathfrak{b}$ is $\frac{1}{2}(\mathfrak{a} \cdot \mathfrak{b} + \mathfrak{b} \cdot \mathfrak{a})$, where $\cdot$ expresses the common matrix-multiplication.

(2) Some further solutions exist, but they do not seem to be fit for quantum mechanical purposes.

(Cf. loc. cit., p. 42, and in Part II, the continuation of this note.)

The scope of this paper is to free ourselves of the requirement of a finite linear basis for $\mathfrak{A}$. This restriction is a very suspicious one from the point of view of quantum mechanics. Thus Heisenberg's „commutation relation"

$$X \cdot Y - Y \cdot X = i\,1$$

($i = \sqrt{-1}$, 1 — the unit-matrix, we omit the numerical factor $\frac{h}{2\pi}$ which is unessential in this connection) is known to be impossible in systems of a finite linear basis. Therefore the hope, necessarily underlying loc. cit., that all $\mathfrak{A}$ fulfilling our axioms may be limiting cases of ones with finite linear basis, is a very uncertain one.

On the other hand it is obvious that if we drop the requirement of the existence of a finite linear basis for $\mathfrak{A}$, the axioms (I)—(IV) cannot be sufficient. At least some topological axioms must be introduced.

3. Thus we must add to the „undefined operations" (α)—(γ) in § 1 of our abstract system $\mathfrak{A}$ an „undefined" topology $\mathcal{T}$ of $\mathfrak{A}$. As we shall always remember the situation for matrices, and particularly that one where $\mathfrak{A} = \mathcal{A}$ consists of all bounded Hermitean operators of Hilbert space, we declare: it is our intention that $\mathfrak{a} + \mathfrak{b}$ be the analogue of matrix-addition $X + Y$, $\mathfrak{a}^n$ of matrix-nth-power X^n [so $\mathfrak{a} \circ \mathfrak{b} = \frac{1}{2}(\mathfrak{a} + \mathfrak{b})^2 + \left(-\frac{1}{2}\right)(\mathfrak{a}^2 + \mathfrak{b}^2)$, the analogue of $\frac{1}{2}(X \cdot Y + Y \cdot X)$, $\cdot$ being matrix-multiplication], and $\mathcal{T}$ the analogue of the weak topology of operators. (Cf. for the latter the author's discussion in „Math. Annalen", **102**, (1929), 370—427, particularly p. 382—384. As to topologies in general, cf. F. Hausdorff, Mengenlehre, Berlin & Leipzig, 1927, p. 226—232. We shall always use conditions (1)—(3), (5) eod.)

Furthermore it seems to be appropriate to introduce two further notions as „undefined" ones, although they could be described in terms of the above ones. These are two fixed subsets $\mathfrak{D}$ and $\mathfrak{U}$ of $\mathfrak{A}$. We shall characterize them so as to make them the

analogues of the set of all (non-negative semi-) definite matrices respectively the set of all matrices of „absolute value" $\leqslant 1$. (That is: matrices $\{a_{ij}\}$ with

$$\sum_{i,j} a_{ij} x_i \overline{x_j} \geqslant 0$$

respectively

$$\left| \sum_{i,j} a_{ij} x_i \overline{x_j} \right| \leqslant \sum_i |x_i|^2$$

for all choices of the $x_1, x_2, \ldots$) Physically speaking $\mathfrak{D}$ and $\mathfrak{U}$ represent the sets of all those „observables" which are essentially $\geqslant 0$, respectively, essentially of absolute value $\leqslant 1$.

We introduce $\mathfrak{D}$ and $\mathfrak{U}$ as independent entities, because we can describe them by axioms which are more natural than those which it would otherwise be necessary to impose on $\mathfrak{A}$.

Finally it is preferable not to introduce $\rho\mathfrak{a}$ [ρ — a real number, cf. (α) in § 1] as an „undefined relation", nor $\mathfrak{a}^n$ for $n=0$ [cf. (β) in § 1]. These notions can be defined with the aid of the others. And the fact that we can prove their existence, chiefly that one of the „unity" $\mathfrak{a}^0=1$, has certain technical advantages. (Cf. Explanation III following the group III of Axioms, and D, 5.)

After these preliminaries we are able to formulate our axioms.

The Axioms

4. It is more convenient to give the list of axioms which follows, with some interruptions: we will divide our axioms into several (four) groups. Each group will be followed by an „Explanation" developing its meaning in the system $\mathcal{A}$ of all bounded Hermitean matrices of Hilbert space $\mathfrak{H}$, and by one or more „Remarks" in which some immediate consequences of those axioms are derived. A fifth group of axioms will be introduced later (in Part II). It will not be used in the preceding paragraphs, that is in Part I.

We axiomatize a system $\mathfrak{A}$ of elements, and with it

1) *an operation* $\mathfrak{a}+\mathfrak{b}$: $\mathfrak{a}, \mathfrak{b} \in \mathfrak{A}$, $\mathfrak{a}+\mathfrak{b} \in \mathfrak{A}$,

2) *an operation* $\mathfrak{a}^n$: $\mathfrak{a} \in \mathfrak{A}$, $n=1, 2, \ldots$, $\mathfrak{a}^n \in \mathfrak{A}$ *(here* $\mathfrak{a}^1=\mathfrak{a}$*)*,

3) *a certain subset* $\mathfrak{D}$ *of* $\mathfrak{A}$,

4) *a certain subset* $\mathfrak{U}$ *of* $\mathfrak{A}$,

5) *a certain topology* $\mathcal{T}$ *of* $\mathfrak{A}$. (In the sense of Fréchet-Hausdorff, cf. the quotations in § 3.)

We assume that $\mathfrak{A}$ has at least two different elements.

The axioms are these:

Group I:

I_1) $\mathfrak{a}+\mathfrak{b}=\mathfrak{b}+\mathfrak{a}$.

I_2) $(\mathfrak{a}+\mathfrak{b})+\mathfrak{c}=\mathfrak{a}+(\mathfrak{b}+\mathfrak{c})$.

I_3) $\mathfrak{b}+\mathfrak{x}=\mathfrak{a}$ *($\mathfrak{a}$, $\mathfrak{b}$ given) has a unique solution* $\mathfrak{x}$.

I_4) $\mathfrak{x}+\mathfrak{x}=\mathfrak{a}$ *($\mathfrak{a}$ given) has a unique solution* $\mathfrak{x}$.

Explanation I. I_1) — I_3) are the axioms of „vector-addition". I_4) will permit, together with the axioms of Group IV, definition of multiplication of „vectors" $\mathfrak{a} \in \mathfrak{A}$ by real numbers.

Remark 1. Denote the solution of I_3) by

$$\mathfrak{x} = \mathfrak{a} - \mathfrak{b}.$$

As

$$\mathfrak{a} + (\mathfrak{a} - \mathfrak{a}) = \mathfrak{a}, \quad (\mathfrak{a} + \mathfrak{y}) + (\mathfrak{a} - \mathfrak{a}) = \mathfrak{a} + \mathfrak{y},$$

so if we choose $\mathfrak{y}$ with $\mathfrak{a} + \mathfrak{y} = \mathfrak{b}$, then

$$\mathfrak{b} + (\mathfrak{a} - \mathfrak{a}) = \mathfrak{b}, \quad \mathfrak{a} - \mathfrak{a} = \mathfrak{b} - \mathfrak{b}.$$

Thus $\mathfrak{a} - \mathfrak{a}$ is independent of $\mathfrak{a}$; denote it by 0. Denote $0 - \mathfrak{a}$ by $-\mathfrak{a}$. So we have:

$$\mathfrak{a} + 0 = \mathfrak{a}, \quad \mathfrak{a} + (-\mathfrak{a}) = 0, \quad -(-\mathfrak{a}) = \mathfrak{a}, \tag{1}$$

and as

$$(\mathfrak{a} + \mathfrak{b}) + ((-\mathfrak{a}) + (-\mathfrak{b})) = 0,$$

so

$$-(\mathfrak{a} + \mathfrak{b}) = (-\mathfrak{a}) + (-\mathfrak{b}). \tag{2}$$

Besides

$$\mathfrak{b} + (\mathfrak{a} + (-\mathfrak{b})) = \mathfrak{a} + (\mathfrak{b} + (-\mathfrak{b})) = \mathfrak{a} + 0 = \mathfrak{a},$$

so

$$\mathfrak{a} - \mathfrak{b} = \mathfrak{a} + (-\mathfrak{b}). \tag{3}$$

Denote the solution of I_4) by

$$\mathfrak{x} = \frac{1}{2}\mathfrak{a}.$$

For any dyadically rational $\rho = \frac{n}{2^m}$ $(n = 0, \pm 1, \pm 2, \ldots, \quad m = 1, 2, \ldots)$ define:

$$\left.\begin{array}{ll} = \frac{1}{2}\left(\frac{1}{2}\left(\ldots\left(\frac{1}{2}\mathfrak{a}\right)\ldots\right)\right) & \text{if } \rho = \frac{1}{2^m} \ (m \text{ factors } \tfrac{1}{2}), \\ \rho\mathfrak{a} = \frac{1}{2^m}\mathfrak{a} + \ldots + \frac{1}{2^m}\mathfrak{a} & \text{,, } \rho = \frac{n}{2^m}, \ n = 1, 2, \ldots \ (n \text{ addends}), \\ = 0 & \text{,, } \rho = 0, \\ = -(-\rho)\,\mathfrak{a} & \text{,, } \rho = \frac{n}{2^m}, \ n = -1, -2, \ldots \end{array}\right\} \tag{4}$$

One verifies easily that the two representations $\rho = \frac{n}{2^m}$ and $\rho = \frac{2n}{2^{m+1}}$ of ρ give the same $\rho\mathfrak{a}$, thus $\rho\mathfrak{a}$ depends really on ρ and $\mathfrak{a}$ only.

We have

$$(\rho + \sigma)\,\mathfrak{a} = \rho\mathfrak{a} + \sigma\mathfrak{a}, \tag{5}$$

$$\rho\,(\mathfrak{a} + \mathfrak{b}) = \rho\mathfrak{a} + \rho\mathfrak{b}, \tag{6}$$

$$(\rho\sigma)\,\mathfrak{a} = \rho\,(\sigma\mathfrak{a}). \tag{7}$$

(5) verifies immediately. (6), (7) hold for $\rho = \frac{1}{2}$, because

$$\left(\frac{1}{2}\mathfrak{a} + \frac{1}{2}\mathfrak{b}\right) + \left(\frac{1}{2}\mathfrak{a} + \frac{1}{2}\mathfrak{b}\right) = \mathfrak{a} + \mathfrak{b}, \quad \left(\frac{1}{2}\sigma\right)\mathfrak{a} + \left(\frac{1}{2}\sigma\right)\mathfrak{a} = \sigma\mathfrak{a}$$

[by (5)]. By iteration they extend to all $\rho = \frac{1}{2^m}$, and then by (5) to all our ρ. One verifies easily the customary rules of vector algebra for $-$ and 0.

Group II:

II_1) $(\mathfrak{a}^m + \mathfrak{a}^n)^2 = \mathfrak{a}^{2m} + \mathfrak{a}^{2n} + 2\mathfrak{a}^{m+n}$.

II_2) $(\mathfrak{a} + \mathfrak{b})^2 + (\mathfrak{a} - \mathfrak{b})^2 = 2\mathfrak{a}^2 + 2\mathfrak{b}^2$.

II_3) $\mathfrak{a}^2 = 0$ *implies* $\mathfrak{a} = 0$.

Explanation II. II_1) is a weakened form of the obvious matrix rule (II) in § 2; it expresses (II′) eod. II_2) is (III) in § 2, and its character was discussed there. II_3) is obvious for matrices.

Remark 2. Define an operation $\mathfrak{ab}$, $\mathfrak{a}, \mathfrak{b} \in \mathfrak{A}$, $\mathfrak{ab} \in \mathfrak{A}$, in the following way:

$$\mathfrak{ab} = \frac{1}{4}((\mathfrak{a}+\mathfrak{b})^2 - (\mathfrak{a}-\mathfrak{b})^2). \tag{8}$$

We now derive some properties of $\mathfrak{a}^2$ and $\mathfrak{ab}$:

(i) Put $\mathfrak{a} = \mathfrak{b} = 0$ in II_2), then $2 \cdot 0^2 = 2 \cdot 0^2 + 2 \cdot 0^2$ results, so $2 \cdot 0^2 = 0$, $0^2 = \frac{1}{2} \cdot (2 \cdot 0^2) = \frac{1}{2} \cdot 0 = 0$. So

$$0^2 = 0.$$

(ii) Put $\mathfrak{a} = 0$ in II_2), then $\mathfrak{b}^2 + (-\mathfrak{b})^2 = 2 \cdot \mathfrak{b}^2 + 2 \cdot 0^2 = 2 \cdot \mathfrak{b}^2 = \mathfrak{b}^2 + \mathfrak{b}^2$, $(-\mathfrak{b})^2 = \mathfrak{b}^2$ results. So

$$\mathfrak{b}^2 = (-\mathfrak{b})^2.$$

(iii) Interchange $\mathfrak{a}$, $\mathfrak{b}$ in (8), as $\mathfrak{b} - \mathfrak{a} = -(\mathfrak{a} - \mathfrak{b})$, the right side is unchanged. So

$$\mathfrak{ab} = \mathfrak{ba}. \tag{9}$$

(iv) Put $\mathfrak{b} = 0$ in (8); this gives

$$\mathfrak{a}0 = 0. \tag{10}$$

(v) Combining (8) with II_2) gives

$$\mathfrak{ab} = \frac{1}{2}((\mathfrak{a}+\mathfrak{b})^2 - \mathfrak{a}^2 - \mathfrak{b}^2) = \frac{1}{2}(\mathfrak{a}^2 + \mathfrak{b}^2 - (\mathfrak{a}-\mathfrak{b})^2). \tag{11}$$

(vi) Put $m = n = 1$ in II_1), then the first relation in (11) gives

$$\mathfrak{aa} = \mathfrak{a}^2.$$

(vii) Now II_1) may be written $(\mathfrak{a}^m + \mathfrak{a}^n)^2 = (\mathfrak{a}^m)^2 + (\mathfrak{a}^n)^2 + 2\mathfrak{a}^{m+n}$, and therefore the first relation of (11) gives

$$\mathfrak{a}^m \mathfrak{a}^n = \mathfrak{a}^{m+n}. \tag{12}$$

In particular

$$\mathfrak{a}^1 = \mathfrak{a}, \quad \mathfrak{a}^{n+1} = \mathfrak{a}^n \mathfrak{a}, \tag{13}$$

which is the inductive characterization of $\mathfrak{a}^n$ in terms of $\mathfrak{ab}$.

(viii) Replace $\mathfrak{a}$ by $\mathfrak{a} + \mathfrak{c}$ or $\mathfrak{a} - \mathfrak{c}$ respectively in II_2), and subtract. Then

$$(\mathfrak{a}+\mathfrak{c}+\mathfrak{b})^2 - (\mathfrak{a}-\mathfrak{c}+\mathfrak{b})^2 + (\mathfrak{a}+\mathfrak{c}-\mathfrak{b})^2 - (\mathfrak{a}-\mathfrak{c}-\mathfrak{b})^2 = \\ = 2(\mathfrak{a}+\mathfrak{c})^2 - 2(\mathfrak{a}-\mathfrak{c})^2,$$

and so by (8)

$$4 \cdot (\mathfrak{a}+\mathfrak{b})\mathfrak{c} + 4 \cdot (\mathfrak{a}-\mathfrak{b})\mathfrak{c} = 8 \cdot \mathfrak{ac}.$$

Multiply by $\frac{1}{4}$, and replace $\mathfrak{a}$, $\mathfrak{b}$ by $\frac{1}{2}(\mathfrak{a}+\mathfrak{b})$, $\frac{1}{2}(\mathfrak{a}-\mathfrak{b})$, then

$$\mathfrak{ac} + \mathfrak{bc} = 2 \cdot \left(\frac{1}{2}(\mathfrak{a}+\mathfrak{b})\right)\mathfrak{c}$$

results. Now $\mathfrak{b} = 0$ gives $\mathfrak{ac} = 2\left(\frac{1}{2}\mathfrak{a}\right)\mathfrak{c}$, and so $2 \cdot \left(\frac{1}{2}(\mathfrak{a}+\mathfrak{b})\right)\mathfrak{c} = (\mathfrak{a}+\mathfrak{b})\mathfrak{c}$. Thus

$$\mathfrak{ac} + \mathfrak{bc} = (\mathfrak{a}+\mathfrak{b})\mathfrak{c}.$$

Using (9) we have:

$$\mathfrak{a}\mathfrak{c}+\mathfrak{b}\mathfrak{c}=(\mathfrak{a}+\mathfrak{b})\,\mathfrak{c}, \quad \mathfrak{a}\mathfrak{c}+\mathfrak{a}\mathfrak{d}=\mathfrak{a}\,(\mathfrak{c}+\mathfrak{d}). \tag{14}$$

(ix) (14) gives immediately:

$$(n\mathfrak{a})\,\mathfrak{b}=n\cdot\mathfrak{a}\mathfrak{b}, \text{ for } n=1,\ 2,\ \ldots,\ 0\mathfrak{b}=0$$

[cf. (10)],

$$(-\mathfrak{a})\,\mathfrak{b}=-\mathfrak{a}\mathfrak{b}, \quad \left(\frac{1}{2}\,\mathfrak{a}\right)\mathfrak{b}=\frac{1}{2}\cdot\mathfrak{a}\mathfrak{b}.$$

Combining these,

$$(\rho\mathfrak{a})\,\mathfrak{b}=\rho\cdot\mathfrak{a}\mathfrak{b}$$

obtains, for all dyadically rational ρ. This gives, together with (9), that

$$(\rho\mathfrak{a})\,\mathfrak{b}=\mathfrak{a}\,(\rho\mathfrak{b})=\rho\cdot\mathfrak{a}\mathfrak{b}. \tag{15}$$

Remark 3. (i)—(ix) in Remark 2 prove that $\mathfrak{a}\pm\mathfrak{b}$, 0, $\rho\mathfrak{a}$ (ρ dyadically rational) and $\mathfrak{a}\mathfrak{b}$ fulfil all rules of (commutative ring) algebra, except the associative law of multiplication. Besides $\mathfrak{a}^n$ ($n=1,\ 2,\ \ldots$) is derivable from $\mathfrak{a}\mathfrak{b}$ in the usual way [by (13)]. Now for the $\mathfrak{a}^n$ (of one $\mathfrak{a}$) even the associative law holds: (12) gives

$$(\mathfrak{a}^m\mathfrak{a}^n)\,\mathfrak{a}^p=\mathfrak{a}^{m+n}\,\mathfrak{a}^p=\mathfrak{a}^{m+n+p},$$
$$\mathfrak{a}^m\,(\mathfrak{a}^n\mathfrak{a}^p)=\mathfrak{a}^m\,\mathfrak{a}^{n+p}=\mathfrak{a}^{m+n+p}.$$

This motivates the following definition:

Let $\mathcal{P}'$ be the set of all polynomials $p(x)$ of the form

$$p(x)\equiv\alpha_1 x+\ldots+\alpha_n x^n \tag{16}$$

where $n=1,\ 2,\ \ldots$ and the $\alpha_1,\ \ldots,\ \alpha_n$ are dyadically rational numbers. Then define for any $\mathfrak{a}\,\varepsilon\,\mathfrak{A}$

$$p(\mathfrak{a})=\alpha_1\mathfrak{a}+\ldots+\alpha_n\mathfrak{a}^n. \tag{17}$$

Observe that $p(x)$ could be written redundantly, by adding a term $0\cdot x^{n+1}$ to it. But this would alter $p(\mathfrak{a})$ merely by a term $0\cdot\mathfrak{a}^{n+1}=0$, that is, not alter it at all. Thus any way of writing $p(x)$ gives the same value of $p(\mathfrak{a})$, that is: $p(\mathfrak{a})$ depends really on the function $p(x)$ and on $\mathfrak{a}$ only.

We now derive some properties of $p(\mathfrak{a})$:

(i) If $p(x)+q(x)\equiv r(x)$, then $p(\mathfrak{a})+q(\mathfrak{a})=r(\mathfrak{a})$. This is obvious.

(ii) If $p(x)\,q(x)\equiv r(x)$, then $p(\mathfrak{a})\,q(\mathfrak{a})=r(\mathfrak{a})$. For $p(x)\equiv x^m$, $q(x)\equiv x^n$ this follows from (12); by (15) it generalizes to $p(x)\equiv\alpha x^m$, $q(x)\equiv\beta x^n$; and by (14) to all $p(x)$, $q(x)$.

(iii) If $p(q(x))\equiv r(x)$, then $p(q(\mathfrak{a}))=r(\mathfrak{a})$. (ii) and an induction with the help of (13) establish this for $p(x)\equiv x^m$; by (15) it generalizes to $p(x)\equiv\alpha x^m$; and by (14) to all $p(x)$.

(iv) Thus the rules of computation for polynomials $p(\mathfrak{a})$ ($p(x)\,\varepsilon\,\mathcal{P}'$) of a given $\mathfrak{a}\,\varepsilon\,\mathfrak{A}$ coincide with those which hold for the polynomials $p(x)$ themselves. In particular the associative law holds [this is, of course, a direct consequence of (ii)]:

$$(p(\mathfrak{a})\,q(\mathfrak{a}))\,r(\mathfrak{a})=p(\mathfrak{a})\,(q(\mathfrak{a})\,r(\mathfrak{a})). \tag{18}$$

Group III:

III$_1$) *For each* $\mathfrak{a}\,\varepsilon\,\mathfrak{A}$ *we have* $\frac{1}{2}\left(\frac{1}{2}\left(\ldots\left(\frac{1}{2}\,\mathfrak{a}\right)\ldots\right)\right)\varepsilon\,\mathfrak{U}$ *for a convenient choice of the number of factors* $\frac{1}{2}$.

III_2) $\mathfrak{a}, \mathfrak{b} \in \mathfrak{D}$ *implies* $\mathfrak{a}+\mathfrak{b} \in \mathfrak{D}$.

III_3) $\mathfrak{a}, \mathfrak{b} \in \mathfrak{D}$, $\mathfrak{a}+\mathfrak{b} \in \mathfrak{U}$ *implies* $\mathfrak{a} \in \mathfrak{U}$.

III_4) $\mathfrak{a} \in \mathfrak{D}$ *implies* $\frac{1}{2}\mathfrak{a} \in \mathfrak{D}$.

III_5) $\mathfrak{a} \in \mathfrak{U}$ *is equivalent to* $(p(\mathfrak{a}))^2 \pm \mathfrak{a}(p(\mathfrak{a}))^2 \in \mathfrak{D}$ *and* $(p(\mathfrak{a}))^2 - \mathfrak{a}^2(p(\mathfrak{a}))^2 \in \mathfrak{D}$ *for all polynomials* $p(x) \in \mathcal{P}'$.

Explanation III. As mentioned in § 3, in the case of Hermitean matrices $\mathfrak{D}$ stands for the set of all (non-negative semi-) definite matrices, and $\mathfrak{U}$ for the set of all matrices of „absolute value" $\leqslant 1$. The formulation of III_5) is somewhat complicated by our not having introduced any equivalent of the unit matrix. The reasons for this omission are purely technical, namely: we will prove in Theorem VI (D, 4) the existence of a unit 1 of $\mathfrak{A}$. As certain subsets $\mathfrak{M}$ of $\mathfrak{A}$ („rings", cf. A, 1) fulfil our axioms too, this applies immediately the existence of a unit $1(\mathfrak{M})$ for each such $\mathfrak{M}$ (cf. the Corollary to Theorem VI). This has important consequences for the theory of idempotents, as we will see in D, 6. Had we postulated the existence of $1 = 1(\mathfrak{A})$ it would have still been necessary to prove the existence of all $1(\mathfrak{M})$.

We must now verify III_1) — III_5) for the Hermitean matrices (= Hermitean operators) of finite dimensional Euclidean spaces and of Hilbert space. As we pointed out in § 2, our product corresponds to $X \circ Y = \frac{1}{2}(XY + YX)$, XY being common matrix multiplication. For commutative quantities X, Y this gives

$$X \circ Y = XY.$$

Thus for all polynomials $p(X)$ of one given Hermitean matrix X our multiplication coincides with common matrix multiplication.

III_1) is obvious for a finite dimensional Euclidean space, and for Hilbert space it expresses the „boundedness" of the matrices X. III_2) and III_4) are obvious. In order to verify III_3), and III_5), we will use the notations of the author's paper, „Math. Annalen", **102**, (1929), 49—131, particularly p. 63—74, or M. H. Stone, Linear transformations in Hilbert space, New York, (1932), particularly p. 2—16, 53—65.

Ad III_3): $X, Y \in \mathfrak{D}$, $X+Y \in \mathfrak{U}$ mean:

$$(Xf, f) \geqslant 0, \quad (Yf, f) \geqslant 0, \quad \|(X+Y)f\| \leqslant \|f\|$$

for all f. Thus

$$0 \leqslant (Xf, f) \leqslant ((X+Y)f, f) \leqslant \|(X+Y)f\| \cdot \|f\| \leqslant \|f\|^2.$$

Now (by Schwarz's inequality for definite operators)

$$(Xf, g) \leqslant \sqrt{(Xf, f)(Xg, g)} \leqslant \|f\| \cdot \|g\|.$$

Putting $g = Xf$:

$$\|Xf\|^2 \leqslant \|f\| \cdot \|Xf\|,$$

and so

$$\|Xf\| \leqslant \|f\|.$$

That is

$$X \in \mathfrak{U}.$$

Ad III_5): $X \in \mathfrak{U}$ means

$$\|Xf\| \leqslant \|f\|,$$

so

$$|(Xf, f)| \leqslant \|Xf\| \cdot \|f\| \leqslant \|f\|^2, \quad (X^2 f, f) = (Xf, Xf) = \|Xf\|^2 \leqslant \|f\|^2.$$

Thus

$$((1 \pm X)f, f) \quad \text{and} \quad ((1 - X^2)f, f) \geqslant 0.$$

Now if Y commutes with X, then

$$((1 \pm X)\, Y^2 f, f) = (Y(1 \pm X)\, Yf, f) = ((1 \pm X)\, Yf,\ Yf) \geqslant 0,\ ((1 - X^2)\, Y^2 f, f) = \\ = (Y(1 - X^2)\, Yf, f) = ((1 - X^2)\, Yf,\ Yf) \geqslant 0.$$

So $Y^2 \pm XY^2$ and $Y^2 - X^2Y^2 \in \mathfrak{D}$. Thus $Y = p(X)$ shows that the condition of III_5) is necessary for $X \in \mathfrak{U}$.

But even III_5) 's second form with $p(x) \equiv x$, that is $X^2 - X^4 \in \mathfrak{D}$, is sufficient for $X \in \mathfrak{U}$: then

$$((X^2 - X^4)\, f, f) \geqslant 0,$$

and as

$$((X^2 - X^4)f, f) = (X^2 f, f) - (X^4 f, f) = (Xf, Xf) - (X^2 f, X^2 f) = \| Xf \|^2 - \| X^2 f \|^2,$$

so

$$\| X^2 f \| \leqslant \| Xf \|.$$

Now

$$\| Xf \|^2 = (Xf, Xf) = (X^2 f, f) \leqslant \| X^2 f \| \cdot \| f \| \leqslant \| Xf \| \cdot \| f \|,$$

and therefore

$$\| Xf \| \leqslant \| f \|.$$

Thus $X \in \mathfrak{U}$.

The phenomenological meaning of III_1)—III_5) is obvious, and they are clearly justified from this point of view. $X \in \mathfrak{D}$ should mean: „the observable X can assume values $\geqslant 0$ only", and $X \in \mathfrak{U}$: „the observable X can assume values of absolute value $\leqslant 1$ only".

Group IV:

IV_1) $\mathfrak{a} + \mathfrak{b}$ *is a* $\mathcal{T}$*-continuous two-variable function of the variables* $\mathfrak{a}$, $\mathfrak{b}$.

IV_2) $-\mathfrak{a}$, $\frac{1}{2}\mathfrak{a}$ *are* $\mathcal{T}$*-continuous functions of* $\mathfrak{a}$.

IV_3) $\mathfrak{a}\mathfrak{b}$ *is a* $\mathcal{T}$*-continuous function of* $\mathfrak{a}$ *alone (for any fixed* $\mathfrak{b}$*).*

IV_4) *If* $\mathfrak{a}_n$, $\mathfrak{b}_n \in \mathfrak{D}$, *then* $\underset{n \to \infty}{\mathcal{T}\text{-lim}}\, (\mathfrak{a}_n + \mathfrak{b}_n) = 0$ *implies* $\underset{n \to \infty}{\mathcal{T}\text{-lim}}\, \mathfrak{a}_n = 0$.

IV_5) *The set* $\mathfrak{D}$ *is* $\mathcal{T}$*-closed.*

IV_6) *The set* $\mathfrak{U}$ *is* $\mathcal{T}$*-separable and* $\mathcal{T}$*-compact.*

Explanation IV. All these axioms are easily verified in the case of matrices in Hilbert space, if $\mathcal{T}$ is weak topology. Observe that IV_6) holds for weak, but not for strong topology, and that $\mathcal{T}$ is not separable in the entire $\mathfrak{A}$. [Cf. the author's discussion in «Math. Annalen», **102**, (1929), 370—427, particulary p. 385—386.] Using weak topology $\mathfrak{a}^2$ is not continuous, and therefore the two-variable function $\mathfrak{a}\mathfrak{b}$ cannot be continuous either, this motivates the peculiar form of IV_3).

The topology $\mathcal{T}$ could be replaced in all our applications of it in this note by the notion of convergence which it includes: continuity of functions, closure of sets, etc., will always be used in such a way that only the notion of convergence (and not the general one of condensation points) is really essential. But we do not propose to discuss this matter exhaustively.

For matrices in a finite-dimensional Euclidean space all these distinctions become of course unimportant.

This completes provisionally our list of axioms.

Analysis of contents

5. The present paper contains only „Part I" of the program described in the introduction; „Part II" will follow soon.

The content of this „Part I" is indicated by its title, and more details are given in the detailed „Table of contents". We wish, however, to make some remarks about the general character of this part.

The main contents of this part are: the classification of all $\mathfrak{a} \varepsilon \mathfrak{A}$ with respect to one or more idempotents (D, G respectively), the spectral theory (E and F), the theory of commutativity (H). The first topic is taken over unchanged from loc. cit., except for the discussion of „units" in D, 4, and that one of the „lattice" character of the set of all idempotents in D, 5, which are new. The treatment of the two other topics is new and characteristic for the more general basis of the present work. The following technical circumstance seems to be worth pointing out: the treatment of the spectral theory contains (among others) elements of F. Riesz's method to establish the spectral form of bounded Hermitean operators. On the other hand the theory of commutativity is based (in the proof of the decisive Theorem XIX in H, 1, of the Lemmas H, 1, 3 — H, 1, 5 respectively, which lead to it) on a method of T. Carleman, used by this author to obtain the spectral form of Hermitean operators. Thus two different methods, which fulfil the same purpose in common spectral theory, appear here in two essentially different rôles, and it seems as if neither could replace the other in its own field (in this work).

Our theory of commutativity has a certain amount of interest even in the special case where $\mathfrak{A}$ consists of bounded Hermitean operators in Hilbert space (cf. Introduction, § 3) with the multiplication rule $X \circ Y = \frac{1}{2}(XY + YX)$. In this case our notion of commutativity coincides with the usual one: $XY = YX$. (This is so for all X, Y if it is so for idempotents: use Theorem XIX twice. For idempotents it holds by Theorem XXI, because $\leqslant$ and orthogonality for idempotents mean in both cases the same thing, cf. the first definitions given in A, 3.) On the other hand it means (use formula (77) in H, 1 and the formula $[X, Y, Z] = \frac{1}{4}[[X, Y], Z]$ of Introduction, § 2), that the „commutator" $[X, Y]$ commutes (in the ordinary sense) with every Z. But as we could choose $\mathfrak{A}$ as the minimum operator-ring containing X, Y it would suffice to state that $XY - YX$ commutes with X and Y. Thus this implies $XY - YX = 0$, which is a non-trivial property of bounded Hermitean operators. (For unbounded ones it does not hold, as Heisenberg's „commutation-relation" shows. Thus it is a „non-formal" theorem.)

The procedure by which functions $\varphi(\mathfrak{a})$ of an $\mathfrak{a} \varepsilon \mathfrak{A}$ can be formed, and which is closely connected with spectral theory, is only expounded for continuous (numerical) functions $\varphi(x)$. It was found convenient to do this in three successive steps (Theorems II, III and VIII), the ultimate connection with spectral theory being established in Lemma F, 5, 1.

The operation $\rho\mathfrak{a}$, for an arbitrary real number ρ, is defined as $\varphi(\mathfrak{a})$ for $\varphi(x) \equiv \rho x$; the axioms secure directly merely its existence for $\rho = -1, \frac{1}{2}$ [axioms I_3) and I_4)].

Discontinuous functions $\varphi(x)$ could be treated easily by the routine methods, an example of this being the argument in C, 3 which leads to Theorem I.

We have carried the discussion on several points farther than absolutely necessary for the mathematical structure of this paper. This was done whenever the notions in question seemed to be important for a general understanding of the theory or for its physical interpretation.

In particular, the following notions are to be interpreted in the usual quantum mechanical way: idempotents (= properties), elements of $\mathfrak{A}$ (= observables), spectral interval (= smallest interval containing all possible values of the observable), commutativity (= simultaneous observability), and the spectral form of $\mathfrak{a}$, and functions $\varphi(\mathfrak{a})$.

The statistical statements, which are based on the notions of „dimensionality" and of „trace", will be dealt with in the second part. There the physical interpretation will be discussed systematically too.

In „Part II" of this paper we will discuss the properties of irreducible $\mathfrak{A}$'s, subdividing them into „discrete" and „continuous" types, according to the existence or non-existence of „minimum idempotents". (An idempotent $\mathfrak{e} \neq 0$ is „minimum", if each idempotent $\mathfrak{f} \leqslant \mathfrak{e}$ is $= 0$ or $\mathfrak{e}$.) We will see, that only the latter type leads to e s s e n t i a l l y new physical possibilities.

These questions are very closely connected with some recent work of F. J. Murray and the author on rings of operators [«Annals of Math.», **37**, (1936), 116—229, particularly the notions of „relative dimensionality" and „relative trace" on p. 165—168, 212—221 eod. The property $\mathfrak{M} \cdot \mathfrak{M}' =$ Set of all $\rho 1$, discussed in the Corollary of our Theorem XXII in H, 3, coincides with the definition of those rings which form the subject of the work quoted above].

Other connections exist with the theory of „lattices". These are hinted in D, 5, and will come out fully in Part II. This aspect of the problem permits an independent axiomatic treatment too, and will be dealt with in a forthcoming paper of G. Birkhoff and the author.

The subdivision of this paper can be seen in the detailed Table of Contents. The main results are formulated as Theorems I—XXII, the auxiliary ones as formulae (1)—(83) and Lemmas B, 2, 1—H, 3, 1.

PART I. ELEMENTARY AND SPECTRAL THEORY

A. Fundamental definitions

A, 1. A subset $\mathfrak{M}$ of $\mathfrak{A}$ is a m o d u l e if it has the following properties:

1) $\mathfrak{a}, \mathfrak{b} \varepsilon \mathfrak{M}$ imply $\mathfrak{a} + \mathfrak{b}, -\mathfrak{a}, \frac{1}{2}\mathfrak{a} \varepsilon \mathfrak{M}$,

2) $\mathfrak{M}$ is $\mathcal{T}$-closed.

Obviously 1) has this consequence:

1′) $\mathfrak{a} \varepsilon \mathfrak{M}$, ρ dyadically rational imply $\rho\mathfrak{a} \varepsilon \mathfrak{M}$.

$\mathfrak{M}$ is a r i n g if it is a module and if it has the further property:

1°) $\mathfrak{a}, \mathfrak{b} \varepsilon \mathfrak{M}$ imply $\mathfrak{a}\mathfrak{b} \varepsilon \mathfrak{M}$.

A special case of 1°) (for $\mathfrak{a} = \mathfrak{b}$) is

1°°) $\mathfrak{a} \varepsilon \mathfrak{M}$ implies $\mathfrak{a}^2 \varepsilon \mathfrak{M}$.

But as

$$\mathfrak{a}\mathfrak{b} = \frac{1}{4}((\mathfrak{a} + \mathfrak{b})^2 - (\mathfrak{a} - \mathfrak{b})^2),$$

so $1^{\circ\circ}$) implies 1°). So 1°) is equivalent to $1^{\circ\circ}$). Again 1°) implies [by (13) and induction]

$1^{\circ\circ\circ}$) $\mathfrak{a} \varepsilon \mathfrak{M}$, $n = 1, 2, \ldots$, imply $\mathfrak{a}^n \varepsilon \mathfrak{M}$.

As $1^{\circ\circ\circ}$) implies $1^{\circ\circ}$), it is equivalent to 1°).

$\mathfrak{M}$ is an ideal if it is a module and if it has the further property

1^*) $\mathfrak{a} \varepsilon \mathfrak{M}$, $\mathfrak{b} \varepsilon \mathfrak{A}$ imply $\mathfrak{a}\mathfrak{b} \varepsilon \mathfrak{M}$.

Clearly every ideal is at the same time a ring also.

As the intersection of any number of modules, rings and ideals respectively is one again, and as $\mathfrak{A}$ itself belongs to all of these three categories, therefore for every set $\mathfrak{S} \subset \mathfrak{A}$ a minimum module, ring, or ideal respectively, which is $\supset \mathfrak{S}$, exists. Denote it by $\mathfrak{M}(\mathfrak{S})$, $\mathfrak{P}(\mathfrak{S})$, $\mathfrak{J}(\mathfrak{S})$, respectively. In what follows we will use only $\mathfrak{P}(\mathfrak{S})$.

If a module $\mathfrak{M}$ is not empty it must contain 0. So we see: if $\mathfrak{M}$ is not the empty set Θ or the set with the unique element 0: (0), then it contains 0 and elements $\neq 0$. This holds a fortiori for rings.

We see immediately: every ring $\mathfrak{M} \neq \Theta$, (0) (that is: which has at least two different elements) fulfils all our axioms I—IV, along with $\mathfrak{A}$, with the same definitions of $\mathfrak{a} + \mathfrak{b}$, $\mathfrak{a}^n$ and consequently of $\rho\mathfrak{a}$, $\mathfrak{a}\mathfrak{b}$, and with $\mathfrak{D} \cdot \mathfrak{M}$, $\mathfrak{U} \cdot \mathfrak{M}$ in place of $\mathfrak{D}$, $\mathfrak{U}$ and the „relative topology" $\mathcal{T}$ in $\mathfrak{M}$ in place of $\mathcal{T}$.

A, 2. An element $\mathfrak{e}$ is idempotent if

$$\mathfrak{e}^2 = \mathfrak{e}.$$

Clearly 0 is idempotent. Denote the set of all idempotents in $\mathfrak{A}$ by $\mathfrak{E}$.

An element $\mathfrak{u}$ is a unit (of $\mathfrak{A}$) if

$$\mathfrak{u}\mathfrak{a} = \mathfrak{a}$$

for all $\mathfrak{a} \varepsilon \mathfrak{A}$. There exists exactly one unit (of $\mathfrak{A}$) or none at all: because if $\mathfrak{u}_1$, $\mathfrak{u}_2$ are units, then

$$\mathfrak{u}_1 = \mathfrak{u}_2\mathfrak{u}_1 = \mathfrak{u}_1\mathfrak{u}_2 = \mathfrak{u}_2.$$

Observe that if $\mathfrak{M}$ is a ring, the notion of an idempotent is the same in $\mathfrak{M}$ as in $\mathfrak{A}$ for an $\mathfrak{e} \varepsilon \mathfrak{M}$; but a unit $\mathfrak{u}$ of $\mathfrak{M}$ need not be one of $\mathfrak{A}$; it will however necessarily be idempotent.

If $\mathfrak{M}$ has the unit 0, then $\mathfrak{a} = 0\mathfrak{a} = 0$ for all $\mathfrak{a} \varepsilon \mathfrak{M}$, so $\mathfrak{M} = (0)$. Conversely, $\mathfrak{M} = (0)$ has the unit 0, thus it is characterized by this fact.

A, 3. Let $\mathfrak{e}$, $\mathfrak{f}$ be two idempotents. We determine when $\mathfrak{e} \pm \mathfrak{f}$ respectively are idempotents also. As

$$(\mathfrak{e} \pm \mathfrak{f})^2 = \mathfrak{e}^2 + \mathfrak{f}^2 \pm 2\mathfrak{e}\mathfrak{f} = \mathfrak{e} + \mathfrak{f} \pm 2\mathfrak{e}\mathfrak{f},$$

therefore $\mathfrak{e} + \mathfrak{f}$ is idempotent if and only if $2\mathfrak{e}\mathfrak{f} = 0$, $\mathfrak{e}\mathfrak{f} = 0$, and $\mathfrak{e} - \mathfrak{f}$ is idempotent if and only if $2\mathfrak{e}\mathfrak{f} = 2\mathfrak{f}$, $\mathfrak{e}\mathfrak{f} = \mathfrak{f}$. We will say that $\mathfrak{e}$, $\mathfrak{f}$ are orthogonal, respectively that $\mathfrak{f}$ is part of $\mathfrak{e}$. In the latter case we use the symbols $\mathfrak{e} \geqslant \mathfrak{f}$ or $\mathfrak{f} \leqslant \mathfrak{e}$.

As $\mathfrak{e}\mathfrak{e} = \mathfrak{e}^2 = \mathfrak{e}$ we have $\mathfrak{e} \leqslant \mathfrak{e}$. If we have $\mathfrak{e} \leqslant \mathfrak{f}$ and $\mathfrak{f} \leqslant \mathfrak{e}$, then $\mathfrak{e} = \mathfrak{f}\mathfrak{e} = \mathfrak{e}\mathfrak{f} = \mathfrak{f}$, hat is $\mathfrak{e} = \mathfrak{f}$. As $\mathfrak{e}0 = 0$, so $0 \leqslant \mathfrak{e}$. If $\mathfrak{u}$ is a unit (of $\mathfrak{A}$), then $\mathfrak{u}\mathfrak{e} = \mathfrak{e}$, so $\mathfrak{e} \leqslant \mathfrak{u}$.

B. Elementary properties of $\mathfrak{D}$, $\mathfrak{U}$, $\mathcal{T}$

B, 1. Combination of III_2) and III_4) gives:

$$\mathfrak{a} \varepsilon \mathfrak{D} \text{ and } \rho > 0 \ (\rho \text{ dyadically rational}) \text{ imply } \rho\mathfrak{a} \varepsilon \mathfrak{D}. \tag{19}$$

[This holds even for $\rho = 0$, cf. after (20).]

If $\mathfrak{a} \varepsilon \mathfrak{U}$, then III_5) (with $p(x) \equiv x$) gives $\mathfrak{a}^2 \pm \mathfrak{a}^4 \varepsilon \mathfrak{D}$, and so by III_2) and III_4) $\mathfrak{a}^2 \varepsilon \mathfrak{D}$. If $\mathfrak{a}$ is arbitrary, then some $\frac{1}{2^n}\mathfrak{a} \varepsilon \mathfrak{U}$ by III_1), and so $\left(\frac{1}{2^n}\mathfrak{a}\right)^2 \varepsilon \mathfrak{D}$. Now (19) gives

$$2^{2n}\left(\frac{1}{2^n}\mathfrak{a}\right)^2 \varepsilon \mathfrak{D},$$

but

$$2^{2n}\left(\frac{1}{2^n}\mathfrak{a}\right)^2 = \left(2^n \frac{1}{2^n}\mathfrak{a}\right)^2 = \mathfrak{a}^2$$

[use (7), (15)]. So we have:

$$\mathfrak{a}^2 \varepsilon \mathfrak{D}. \tag{20}$$

Thus in particular $0 = 0^2 \varepsilon \mathfrak{D}$, securing (19) for $\rho = 0$ too.

Another useful relation is this: assume $\mathfrak{a} \varepsilon \mathfrak{U}$. Then III_2) [with $p(x) \equiv x^n$] gives

$$\mathfrak{a}^{2n} - \mathfrak{a}^{2(n+1)} \varepsilon \mathfrak{D}.$$

If $m < l$, then put $n = m, \ldots, l-1$, and add [by III_2)],

$$\mathfrak{a}^{2m} - \mathfrak{a}^{2l} \varepsilon \mathfrak{D}$$

results. This is clearly the case for $m = l$ too:

$$\mathfrak{a}^{2m} - \mathfrak{a}^{2m} = 0 \varepsilon \mathfrak{D}.$$

Now assume $k \geqslant 2m$, and form $(\mathfrak{a}^m \pm \mathfrak{a}^{k-m})^2$. It is $= \mathfrak{a}^{2m} + \mathfrak{a}^{2(k-m)} \pm 2\mathfrak{a}^k$ and $\varepsilon \mathfrak{D}$ by (20). As $2m \leqslant 2(k-m)$, we have $\mathfrak{a}^{2m} - \mathfrak{a}^{2(k-m)} \varepsilon \mathfrak{D}$ too. Adding gives $2\mathfrak{a}^{2m} \pm \pm 2\mathfrak{a}^k \varepsilon \mathfrak{D}$ by III_2), and then III_4) gives $\mathfrak{a}^{2m} \pm \mathfrak{a}^k \varepsilon \mathfrak{D}$. So we have:

$$\mathfrak{a} \varepsilon \mathfrak{U} \text{ and } k \geqslant 2m \text{ imply } \mathfrak{a}^{2m} \pm \mathfrak{a}^k \varepsilon \mathfrak{D}. \tag{21}$$

B, 2. We prove some auxiliary facts about $\mathcal{T}$-convergence.

Lemma B, 2, 1. *If a sequence* $\mathfrak{a}_1, \mathfrak{a}_2, \ldots$ *with* $\mathfrak{a}_i \varepsilon \mathfrak{D}$, $\mathfrak{a}_1 + \ldots + \mathfrak{a}_i \varepsilon \mathfrak{U}$ *for all* $i = 1, 2, \ldots$ *is given, then*

$$\mathcal{T}\text{-}\lim_{i \to \infty} (\mathfrak{a}_1 + \ldots + \mathfrak{a}_i)$$

exists.

Proof. The set of all $\mathfrak{a}_1 + \ldots + \mathfrak{a}_i$, $i = 1, 2, \ldots$, has a condensation point $\mathfrak{a}^*$ by IV_6), and again by IV_6) a sequence $n_1 < n_2 < \ldots$ with

$$\mathcal{T}\text{-}\lim_{i \to \infty} (\mathfrak{a}_1 + \ldots + \mathfrak{a}_{n_i}) = \mathfrak{a}^*$$

exists. If $n_i > j$, then

$$(\mathfrak{a}_1 + \ldots + \mathfrak{a}_{n_i}) - (\mathfrak{a}_1 + \ldots + \mathfrak{a}_j) = \mathfrak{a}_{j+1} + \ldots + \mathfrak{a}_{n_i} \varepsilon \mathfrak{D}$$

by III_2). So $i \to \infty$ gives by IV_1), IV_2) and IV_5)

$$\mathfrak{a}^* - (\mathfrak{a}_1 + \ldots + \mathfrak{a}_j) \varepsilon \mathfrak{D}.$$

For each $j > n_1$ let $i = i(j)$ be the greatest i with $j > n_i$. Then clearly $\lim_{j \to \infty} i(j) = \infty$ and so

$$\mathcal{T}\text{-}\lim_{j \to \infty} (\mathfrak{a}_1 + \ldots + \mathfrak{a}_{n_{i(j)}}) = \mathfrak{a}^*, \quad \mathcal{T}\text{-}\lim_{j \to \infty} (\mathfrak{a}^* - (\mathfrak{a}_1 + \ldots + \mathfrak{a}_{n_{i(j)}})) = 0$$

[by IV_1), IV_2)]. Now

$$\mathfrak{a}^* - (\mathfrak{a}_1 + \ldots + \mathfrak{a}_{n_{i(j)}}) = [\mathfrak{a}^* - (\mathfrak{a}_1 + \ldots + \mathfrak{a}_j)] + [\mathfrak{a}_{n_{i(j)}+1} + \ldots + \mathfrak{a}_j]$$

and

$$\mathfrak{a}^* - (\mathfrak{a}_1 + \ldots + \mathfrak{a}_j) \varepsilon \mathfrak{D}, \quad \mathfrak{a}_{n_{i(j)}+1} + \ldots + \mathfrak{a}_j \varepsilon \mathfrak{D}.$$

So IV_4) gives

$$\mathcal{T}\text{-}\lim_{j\to\infty}(\mathfrak{a}^* - (\mathfrak{a}_1 + \ldots + \mathfrak{a}_j)) = 0$$

(the $j \leqslant n_1$ are unessential), and IV_1), IV_2) give

$$\mathcal{T}\text{-}\lim_{j\to\infty}(\mathfrak{a}_1 + \ldots + \mathfrak{a}_j) = \mathfrak{a}^*.$$

Lemma B, 2, 2. *If* $\mathfrak{a} \varepsilon \mathfrak{D}$ *and* $n\mathfrak{a} \varepsilon \mathfrak{U}$ *for all* $n = 1, 2, \ldots,$ *then* $\mathfrak{a} = 0$.

Proof. Apply Lemma B, 2, 1 to $\mathfrak{a}_1 = \mathfrak{a}_2 = \ldots = \mathfrak{a}$, thus

$$\mathcal{T}\text{-}\lim_{n\to\infty} n\mathfrak{a} = \mathfrak{a}^*$$

exists. Thus $\mathcal{T}\text{-}\lim_{n\to\infty}(n+1)\mathfrak{a} = \mathfrak{a}^*$ too, and so by IV_1), IV_2)

$$\mathcal{T}\text{-}\lim_{n\to\infty}\mathfrak{a} = \mathcal{T}\text{-}\lim_{n\to\infty}(n+1)\mathfrak{a} - \mathcal{T}\text{-}\lim_{n\to\infty} n\mathfrak{a} = \mathfrak{a}^* - \mathfrak{a}^* = 0.$$

Now $\mathcal{T}\text{-}\lim_{n\to\infty}\mathfrak{a} = \mathfrak{a}$, so $\mathfrak{a} = 0$.

Lemma B, 2, 3. *If a sequence* $\mathfrak{a}_1, \mathfrak{a}_2, \ldots$ *and a numerical sequence* $n_1, n_2, \ldots$ *with*

$$\mathfrak{a}_i \varepsilon \mathfrak{D}, \quad n_i\mathfrak{a}_i \varepsilon \mathfrak{U}, \quad \lim_{i\to\infty} n_i = \infty$$

is given, then

$$\mathcal{T}\text{-}\lim_{i\to\infty}\mathfrak{a}_i = 0.$$

Proof. If this were not so, a neighbourhood $\mathcal{N}$ of 0 would exist so that infinitely many $\mathfrak{a}_i \not\varepsilon \mathcal{N}$. As we could replace $\mathfrak{a}_1, \mathfrak{a}_2, \ldots$ by a subsequence, we may assume $\mathfrak{a}_i \not\varepsilon \mathcal{N}$ for all $i = 1, 2, \ldots$ By IV_6) these $\mathfrak{a}_i$, $i = 1, 2, \ldots,$ have at least one condensation point $\mathfrak{a}^*$, which must be $\mathcal{T}$-lim of some subsequence. Replacing the sequence by this subsequence we obtain:

$$\mathcal{T}\text{-}\lim \mathfrak{a}_i = \mathfrak{a}^*.$$

As all $\mathfrak{a}_i \not\varepsilon \mathcal{N}$, so $\mathfrak{a}^* \not\varepsilon \mathcal{N}$, and so $\mathfrak{a}^* \neq 0$.

Consider an $n = 1, 2, \ldots$ Then IV_1) gives

$$\mathcal{T}\text{-}\lim_{i\to\infty} n\mathfrak{a}_i = n\mathfrak{a}^*.$$

If i is sufficiently great, then $n_i > n$, so $n\mathfrak{a}_i$ and $(n_i - n)\mathfrak{a}_i \varepsilon \mathfrak{D}$ by III_2). As $n_i\mathfrak{a}_i \varepsilon \mathfrak{U}$, so III_3) gives $n\mathfrak{a}_i \varepsilon \mathfrak{U}$. As $\mathfrak{U}$ is closed by IV_6), so $n\mathfrak{a}^* \varepsilon \mathfrak{U}$ too.

As all $\mathfrak{a}_i \varepsilon \mathfrak{D}$, so IV_5) gives $\mathfrak{a}^* \varepsilon \mathfrak{D}$. Now Lemma B, 2, 2 gives $\mathfrak{a}^* = 0$, which we saw to be impossible.

B, 3. Consider a polynomial $p(x) \varepsilon \mathcal{P}'$, with $p(x) \geqslant 0$ for $|x| \leqslant 1$. As $p(x)$ is real, has no constant term, and is $\geqslant 0$ for all $|x| \leqslant 1$, therefore the fundamental theorem of algebra permits the conclusion: 0 is a root of $p(x)$ and has even multiplicity, say $2r$ $(r = 1, 2, \ldots)$; each root $\xi_1, \ldots, \xi_s > -1, < 1, \neq 0$ of $p(x)$ $(s = 0, 1, 2, \ldots)$ has even multiplicity, say $2r_1, \ldots, 2r_s$ $(r_1, \ldots, r_s = 1, 2, \ldots)$; the complex roots of $p(x)$ occur in conjugate pairs $\xi_1' \pm i\xi_1'', \ldots, \xi_t' \pm i\xi_t''$ $(t = 0, 1, 2, \ldots)$; and the real roots $\geqslant 1$ or $\leqslant -1$ respectively can be written in the form $1 + \eta_1', \ldots, 1 + \eta_u'$ and

$-1-\eta_1'', \ldots, -1-\eta_v''$ $(u, v=0, 1, 2, \ldots)$ respectively. So we have:

$$p(x) \equiv \alpha x^{2r} \prod_{i=1}^{s} (x-\xi_i)^{2r_i} \prod_{i=1}^{t} ((x-\xi_i')^2+\xi_i''^2) \cdot$$

$$\cdot \prod_{i=1}^{u} ((1+x)+\eta_i') \prod_{i=}^{v} ((1-x)+\eta_i'').$$

This makes it obvious that $\alpha \geqslant 0$ too. By carrying out all multiplications indicated we see that $p(x)$ is a sum of a finite number of terms:

$$\gamma (q(x))^2 (1+x)^j (1-x)^k,$$

$\gamma \geqslant 0$, $q(x)$ a real polynomial, divisible by x^r, $j, k=0, 1, 2, \ldots$ If j', k' are the integer parts of $\frac{1}{2}j$, $\frac{1}{2}k$ respectively, then replacement of $q(x)$ by $\sqrt{\gamma}\, q(x)(1+x)^{j'}(1-x)^{j'}$ gives terms of this form:

$$(r(x))^2 (1+x)^l (1-x)^m,$$

(x) a real polynomial, divisible by x^r, $l, m=0, 1$. So we have:

$$p(x) \equiv \sum_{\nu=1}^{N} (r_\nu(x))^2 (1+x)^{l_\nu} (1-x)^{m_\nu},$$

$r(x), l_\nu, m_\nu$ as above. Let M be the highest degree of any $r_\nu(x)$, $\nu=1, \ldots, N$, then $p(x)$'s is $\leqslant 2M$.

Assume $\rho > 0$, dyadically rational. We can replace each $r_\nu(x)$ by an $r_\nu^0(x)$ with dyadically rational coefficients, divisible by x^r, and of degree $\leqslant M$, so that all coefficients of

$$S^0(x) \equiv p(x) - \sum_{\nu=1}^{N} (r_\nu^0(x))^2 (1+x)^{l_\nu} (1-x)^{m_\nu}$$

have absolute values $\leqslant \frac{1}{2M}\rho$. Thus

$$S^0(x) \equiv \sum_{i=2r}^{2M} \beta_i x^i,$$

all β_i dyadically rational, $|\beta_i| \leqslant \frac{1}{2M}\rho$. So we have:

$$p(x)+\rho x^{2r} \equiv \sum_{\nu=1}^{N} (r_\nu^0(x))^2 (1+x)^{l_\nu} (1-x)^{m_\nu} + \sum_{i=2r}^{2M} |\beta_i| (x^{2r} \pm x^i) +$$

$$+ \Big(\rho - \sum_{i=2r}^{2M} |\beta_i|\Big) x^{2r}. \tag{22}$$

Assume now that an $\mathfrak{a} \varepsilon \mathfrak{U}$ is given. (22) shows that $p(\mathfrak{a})+\rho\mathfrak{a}^{2r} \varepsilon \mathfrak{D}$ holds by III_2), if all terms on the right side, with $\mathfrak{a}$ for x are $\varepsilon \mathfrak{D}$. Now the second and third terms are easily taken care of: $\mathfrak{a}^{2r} \pm \mathfrak{a}^i \varepsilon \mathfrak{D}$ by (21), $\mathfrak{a}^{2r} \varepsilon \mathfrak{D}$ by (20), besides

$$|\beta_i| \geqslant 0 \text{ and } \rho - \sum_{i=2r}^{2M} |\beta_i| \geqslant \rho - 2M \cdot \frac{1}{2M}\rho = 0,$$

and thus (19) settles this. The first term remains, that is we have to prove $t(\mathfrak{a}) \varepsilon \mathfrak{D}$ for $t(x) \equiv (r(x))^2 (1+x)^l (1-x)^m$, $l, m = 0, 1$. For $l = m = 0$, $t(x) \equiv (r(x))^2$ this follows from (20); for $l = 1$, $m = 0$ or $l = 0$, $m = 1$, $t(x) \equiv (r(x))^2 (1 \pm x)$ this follows from III_5)'s first part; and for $l = m = 1$, $t(x) \equiv (r(x))^2 (1 - x^2)$ this follows from III_5)'s second part.

So we have proved:

Lemma B, 3, 1. *Consider a polynomial $p(x) \varepsilon \mathcal{P}'$ with $p(x) \geqslant 0$ for $|x| \leqslant 1$, a dyadically rational $\rho > 0$, and an $\mathfrak{a} \varepsilon \mathfrak{U}$. Then 0 is a root of $p(x)$ of a multiplicity $2r$, $r = 1, 2, \ldots$, and $p(\mathfrak{a}) + \rho \mathfrak{a}^{2r} \varepsilon \mathfrak{D}$.*

We next prove:

Lemma B, 3, 2. *Consider a polynomial $p(x) \varepsilon \mathcal{P}'$ with $p(x) \geqslant 0$ for $|x| \leqslant 1$ and an $\mathfrak{a} \varepsilon \mathfrak{U}$. Then $p(\mathfrak{a}) \varepsilon \mathfrak{D}$.*

Proof. Let r again be the multiplicity of $p(x)$'s root 0. Choose m with $\frac{1}{2m} \mathfrak{a}^{2r} \varepsilon \mathfrak{U}$ by III_1). Now put

$$\mathfrak{a}_i = \frac{1}{2m+i} \mathfrak{a}^{2r}, \quad n_i = 2^i.$$

Then $\mathfrak{a}_i = \frac{1}{2m+i} \mathfrak{a}^{2r} \varepsilon \mathfrak{D}$ by (19), (20), and $n_i \mathfrak{a}_i \varepsilon \mathfrak{U}$, besides $\lim\limits_{i \to \infty} n_i = \infty$. So Lemma B, 2, 3 applies:

$$\mathcal{T}\text{-}\lim_{i \to \infty} \frac{1}{2m+i} \mathfrak{a}^{2r} = \mathcal{T}\text{-}\lim_{i \to \infty} \mathfrak{a}_i = 0,$$

and by IV_1) $\mathcal{T}\text{-}\lim\limits_{i \to \infty} \left(p(\mathfrak{a}) + \frac{1}{2m+i} \mathfrak{a}^{2r} \right) = p(\mathfrak{a})$.

But Lemma B, 3, 1 gives $p(\mathfrak{a}) + \frac{1}{2m+i} \mathfrak{a}^{2r} \varepsilon \mathfrak{D}$ and so by IV_5) $p(\mathfrak{a}) \varepsilon \mathfrak{D}$ too.

Lemma B, 3, 3. *Consider a polynomial $p(x) \varepsilon \mathcal{P}'$ with $|p(x)| \leqslant 1$ for $|x| \leqslant 1$ and an $\mathfrak{a} \varepsilon \mathfrak{U}$. Then $p(\mathfrak{a}) \varepsilon \mathfrak{U}$.*

Proof. Apply III_5) to $p(\mathfrak{a})$ (in place of $\mathfrak{a}$). Consider any $q(x) \varepsilon \mathcal{P}'$, and put

$$r(x) \equiv (q(p(x)))^2 + p(x)(q(p(x)))^2$$

or

$$\equiv (q(p(x)))^2 - (p(x))^2 (q(p(x)))^2$$

respectively, then III_5) establishes $p(\mathfrak{a}) \varepsilon \mathfrak{U}$ if we can prove $r(\mathfrak{a}) \varepsilon \mathfrak{D}$ for all these $r(x)$'s. Now $|p(x)| \leqslant 1$ holds for all $|x| \leqslant 1$, and clearly implies $r(x) \geqslant 0$. So Lemma B, 3, 2 proves that $r(\mathfrak{a}) \varepsilon \mathfrak{D}$.

B, 4. We now derive convergence criteria for $p(\mathfrak{a})$'s:

Lemma B, 4, 1. *Consider a sequence $p_1(x), p_2(x), \ldots$ of polynomials, all $p_i(x) \varepsilon \mathcal{P}'$, with $p_i(x) \geqslant 0$ for $|x| \leqslant 1$ and $\lim\limits_{i \to \infty} p_i(x) = 0$ uniformly for $|x| \leqslant 1$. Assume $\mathfrak{a} \varepsilon \mathfrak{U}$. Then $\mathcal{T}\text{-}\lim\limits_{i \to \infty} p_i(\mathfrak{a}) = 0$.*

Proof. Put $M_i = \max\limits_{|x| \leqslant 1} p_i(x)$, our convergence assumption means $\lim\limits_{i \to \infty} M_i = 0$. So for $i \geqslant i_0$ $M_i \leqslant 1$; choose $n = n_i$ as the greatest $n = 1, 2, \ldots$ with $nM_i \leqslant 1$. Clearly $\lim\limits_{i \to \infty} n_i = \infty$.

Thus $|n_i p_i(x)| \leqslant 1$ for $|x| \leqslant 1$, and so $n_i p_i(\mathfrak{a}) \varepsilon \mathfrak{U}$ by Lemma B, 3, 3. Besides $p_i(\mathfrak{a}) \varepsilon \mathfrak{D}$ by Lemma B, 3, 2. So Lemma B, 2, 3 applies [with $\mathfrak{a}_i = p_i(\mathfrak{a})$], giving $\mathcal{T}\text{-}\lim\limits_{i \to \infty} p_i(\mathfrak{a}) = 0$.

Let now $\mathcal{P}''$ be the set of all polynomials $p(x)$ of the form

$$p(x) \equiv \alpha_2 x^2 + \ldots + \alpha_n x^n, \tag{23}$$

where $n = 2, 3, \ldots$, and the $\alpha_2, \ldots, \alpha_n$ are dyadically rational numbers. So $p(x) \varepsilon \mathcal{P}''$ means $p(x) \varepsilon \mathcal{P}'$ and $p(x)$ divisible by x^2.

Lemma B, 4, 2. *Consider a sequence* $p_1(x), p_2(x), \ldots$ *of polynomials, all* $p_i(x) \varepsilon \mathcal{P}''$, *with* $\lim\limits_{i \to \infty} p_i(x) = 0$ *uniformly for* $|x| \leqslant 1$. *Assume* $\mathfrak{a} \varepsilon \mathfrak{U}$. *Then* $\mathcal{T}\text{-}\lim\limits_{i \to \infty} p_i(\mathfrak{a}) = 0$.

Proof. Under these assumptions $\frac{p_i(x)}{x^2}$ is everywhere continuous in $|x| \leqslant 1$, and with it $\max\left(0, \frac{p_i(x)}{x^2}\right) + \frac{1}{i}$. Thus a polynomial $q_i(x)$ with

$$\left|\left[\max\left(0, \frac{p_i(x)}{x^2}\right) + \frac{1}{i}\right] - q_i(x)\right| \leqslant \frac{1}{i} \quad \text{for} \quad |x| \leqslant 1$$

exists (by the Weierstrass approximation-theorem). Clearly we may even choose $q_i(x)$ with dyadically rational coefficients.

The above inequality implies (for $|x| \leqslant 1$)

$$\max(0, p_i(x)) \leqslant x^2 q_i(x) \leqslant \max(0, p_i(x)) + \frac{2}{i} x^2.$$

Thus we have for $r_i'(x) \equiv x^2 q_i(x)$ and for $r_i''(x) \equiv x^2 p_i(x) - q_i(x)$: $r_i'(x), r_i''(x) \geqslant 0$ for $|x| \leqslant 1$, $r_i'(x), r_i''(x) \varepsilon \mathcal{P}'' \subset \mathcal{P}'$. Finally

$$|r_i'(x)| = |x^2 q_i(x)| \leqslant |p_i(x)| + \frac{2}{i} x^2 \leqslant M_i + \frac{2}{i}$$

and so $\lim\limits_{i \to \infty} r_i'(x) = 0$ uniformly in $|x| \leqslant 1$. As $r_i''(x) = r_i'(x) - p_i(x)$, so $\lim\limits_{i \to \infty} r_i''(x) = 0$ uniformly in $|x| \leqslant 1$ too.

So Lemma B, 4, 1 applies to $r_i'(\mathfrak{a})$, $r_i''(\mathfrak{a})$:

$$\mathcal{T}\text{-}\lim_{i \to \infty} r_i'(\mathfrak{a}) = \mathcal{T}\text{-}\lim_{i \to \infty} r_i''(\mathfrak{a}) = 0.$$

As $p_i(x) \equiv r_i'(x) - r_i''(x)$, $p_i(\mathfrak{a}) = r_i'(\mathfrak{a}) - r_i''(\mathfrak{a})$, so III_1) and III_2) give finally, that

$$\mathcal{T}\text{-}\lim_{i \to \infty} p_i(\mathfrak{a}) = 0.$$

Lemma B, 4, 3. *Consider a sequence* $p_1(\mathfrak{a}), p_2(\mathfrak{a}), \ldots$ *of polynomials, all* $p_i(x) \varepsilon \mathcal{P}''$, *with* $\lim\limits_{i \to \infty} p_i(x)$ *existing uniformly for* $|x| \leqslant 1$. *Assume* $\mathfrak{a} \varepsilon \mathfrak{U}$. *Then* $\mathcal{T}\text{-}\lim\limits_{i \to \infty} p_i(\mathfrak{a})$ *exists.*

Proof. As $\lim\limits_{i \to \infty} p_i(x)$ exists uniformly for $|x| \leqslant 1$, so the $p_i(x)$ are uniformly bounded for $i = 1, 2, \ldots$, $|x| \leqslant 1$. So we may choose an m with $|p_i(x)| \leqslant 2^m$.

Thus $\left|\frac{1}{2^m} p_i(x)\right| \leqslant 1$ for $|x| \leqslant 1$, therefore by Lemma B, 3, 3, $\frac{1}{2^m} p_i(\mathfrak{a}) \varepsilon \mathfrak{U}$. Thus by IV_6) a condensation point $\mathfrak{a}^*$ of the $\frac{1}{2^m} p_i(\mathfrak{a})$, $i = 1, 2, \ldots$, exists, and it is the limit of a subsequence $n_1 < n_2 < \ldots$:

$$\mathcal{T}\text{-}\lim_{\to \infty} \frac{1}{2^m} p_{n_j}(\mathfrak{a}) = \mathfrak{a}^*.$$

Now as $\lim_{j\to\infty} n_j = \infty$, therefore the uniform existence of $\lim_{i\to\infty} p_i(x)$ for $|x| \leqslant 1$ involves

$$\lim_{j\to\infty} \left(\frac{1}{2^m} p_j(x) - \frac{1}{2^m} p_{n_j}(x)\right) = 0$$

uniformly for $|x| \leqslant 1$.

Thus Lemma B, 4, 2 gives

$$\mathcal{T}\text{-}\lim_{j\to\infty} \left(\frac{1}{2^m} p_j(\mathfrak{a}) - \frac{1}{2^m} p_{n_j}(\mathfrak{a})\right) = 0.$$

Addition of our two $\mathcal{T}\text{-}\lim_{j\to\infty}$-relations gives

$$\mathcal{T}\text{-}\lim_{j\to\infty} \frac{1}{2^m} p_j(\mathfrak{a}) = \mathfrak{a}^*,$$

and adding this 2^m-times [use IV_1)]

$$\mathcal{T}\text{-}\lim_{j\to\infty} p_j(\mathfrak{a}) = 2^m \mathfrak{a}^*.$$

Thus $\mathcal{T}\text{-}\lim_{j\to\infty} p_j(\mathfrak{a})$ exists.

C. The idempotent of $\mathfrak{a}$. Continuous functions of $\mathfrak{a}$

C, 1. Denote the set of all $p(\mathfrak{a})$, $p(x) \varepsilon \mathcal{P}'$, by $\mathfrak{P}^0(\mathfrak{a})$, and the $\mathcal{T}$-closure of $\mathfrak{P}^0(\mathfrak{a})$ by $\mathfrak{P}(\mathfrak{a})$.

$$\mathfrak{b}, \mathfrak{c} \varepsilon \mathfrak{P}^0(\mathfrak{a}) \text{ imply } -\mathfrak{b}, \frac{1}{2}\mathfrak{b}, \mathfrak{b}+\mathfrak{c}, \mathfrak{b}\mathfrak{c} \varepsilon \mathfrak{P}^0(\mathfrak{a}),$$

and a fortiori $\varepsilon \mathfrak{P}(\mathfrak{a})$. Now IV_1), IV_2) give that even $\mathfrak{b}, \mathfrak{c} \varepsilon \mathfrak{P}(\mathfrak{a})$ imply $-\mathfrak{b}, \frac{1}{2}\mathfrak{b}, \mathfrak{b}+\mathfrak{c} \varepsilon \mathfrak{P}(\mathfrak{a})$. If $\mathfrak{c} \varepsilon \mathfrak{P}^0(\mathfrak{a})$ is fixed, IV_3) gives $\mathfrak{b}\mathfrak{c} \varepsilon \mathfrak{P}(\mathfrak{a})$ for all $\mathfrak{b} \varepsilon \mathfrak{P}(\mathfrak{a})$; and if now $\mathfrak{b} \varepsilon \mathfrak{P}(\mathfrak{a})$ is fixed, IV_3) and (9) give $\mathfrak{b}\mathfrak{c} \varepsilon \mathfrak{P}(\mathfrak{a})$ for all $\mathfrak{c} \varepsilon \mathfrak{P}(\mathfrak{a})$. So $\mathfrak{b}, \mathfrak{c} \varepsilon \mathfrak{P}(\mathfrak{a})$ imply $\mathfrak{b}\mathfrak{c} \varepsilon \mathfrak{P}(\mathfrak{a})$ too. Besides $\mathfrak{P}(\mathfrak{a})$ is clearly closed. Thus A, 1 shows that $\mathfrak{P}(\mathfrak{a})$ is a ring containing $\mathfrak{a}$.

Clearly every ring containing $\mathfrak{a}$ must be $\supset \mathfrak{P}(\mathfrak{a})$. So we see:

$$\mathfrak{P}(\mathfrak{a}) \text{ is the minimum ring containing } \mathfrak{a}. \tag{24}$$

The relation $(\mathfrak{b}\mathfrak{c})\mathfrak{d} = \mathfrak{b}(\mathfrak{c}\mathfrak{d})$ holds by (18) whenever $\mathfrak{b}, \mathfrak{c}, \mathfrak{d} \varepsilon \mathfrak{P}^0(\mathfrak{a})$. If $\mathfrak{c}, \mathfrak{d} \varepsilon \mathfrak{P}^0(\mathfrak{a})$ are fixed, IV_3) gives this relation for $\mathfrak{b} \varepsilon \mathfrak{P}(\mathfrak{a})$; if $\mathfrak{b} \varepsilon \mathfrak{P}(\mathfrak{a})$, $\mathfrak{d} \varepsilon \mathfrak{P}^0(\mathfrak{a})$ are fixed, IV_3) and (9) extend it to $\mathfrak{c} \varepsilon \mathfrak{P}(\mathfrak{a})$; and finally if $\mathfrak{b}, \mathfrak{c} \varepsilon \mathfrak{P}(\mathfrak{a})$ are fixed, IV_3) and (9) extend it to $\mathfrak{d} \varepsilon \mathfrak{P}(\mathfrak{a})$. So we have:

$$\mathfrak{b}, \mathfrak{c}, \mathfrak{d} \varepsilon \mathfrak{P}(\mathfrak{a}) \text{ imply } (\mathfrak{b}\mathfrak{c})\mathfrak{d} = \mathfrak{b}(\mathfrak{c}\mathfrak{d}). \tag{25}$$

C, 2. If $\mathfrak{e}$ is an idempotent, then $\mathfrak{e}^n = \mathfrak{e}$ for all $n = 1, 2, \ldots$ For $n = 1$ this is obvious, and if it holds for an $n = 1, 2, \ldots$ then it holds for $n+1$ too:

$$\mathfrak{e}^{n+1} = \mathfrak{e}^n \mathfrak{e} = \mathfrak{e}\mathfrak{e} = \mathfrak{e}^2 = \mathfrak{e}.$$

Therefore

$$\alpha_1 \mathfrak{e} + \ldots + \alpha_n \mathfrak{e}^n = (\alpha_1 + \ldots + \alpha_n)\mathfrak{e},$$

that is:

For every $p(x) \varepsilon \mathcal{P}'$ and every idempotent $\mathfrak{e}$

$$p(\mathfrak{e}) = p(1) \cdot \mathfrak{e}. \tag{26}$$

Clearly $\mathfrak{e} = \mathfrak{e}^2 \varepsilon \mathfrak{D}$ [by (20)]. If $r(x) \equiv (p(x))^2 \pm x(p(x))^2$ or $\equiv (p(x))^2 - x^2(p(x))^2$, then $r(1) \geqslant 0$, so by (26) $r(\mathfrak{e}) = r(1) \cdot \mathfrak{e} = (\sqrt{r(1)} \cdot \mathfrak{e})^2 \varepsilon \mathfrak{D}$. Thus III_5) gives $\mathfrak{e} \varepsilon \mathfrak{U}$. So we have proved:

For every idempotent $\mathfrak{e} \varepsilon \mathfrak{D} \cdot \mathfrak{U}$. (27)

C, 3. Assume $\mathfrak{a} \varepsilon \mathfrak{U}$. Define

$$p_i(x) \equiv 1 - (1 - x^2)^i, \quad i = 1, 2, \ldots$$

Clearly $p_i(x) \varepsilon \mathcal{P}''$. Besides $0 \leqslant p_1(x) \leqslant p_2(x) \leqslant \ldots \leqslant 1$ for $|x| \leqslant 1$. Thus if $q_i(x) = p_i(x) - p_{i-1}(x)$ (put $p_0(x) \equiv 0$), then

$$q_i(x) \varepsilon \mathcal{P}'', \quad q_i(x) \geqslant 0, \quad |q_1(x) + \ldots + q_i(x)| = |p_i(x)| \leqslant 1$$

for $|x| \leqslant 1$. Thus Lemmas B, 3, 2 and B, 3, 3 give for $\mathfrak{a}_i = q_i(\mathfrak{a})$, $i = 1, 2, \ldots$: $\mathfrak{a}_i \varepsilon \mathfrak{D}$, $\mathfrak{a}_1 + \ldots + \mathfrak{a}_i \varepsilon \mathfrak{U}$. Now Lemma B, 2, 1 guarantees the existence of

$$\mathcal{T}\text{-}\lim_{i \to \infty} (\mathfrak{a}_1 + \ldots + \mathfrak{a}_i) = \mathfrak{a}^0.$$

As

$$\mathfrak{a}_1 + \ldots + \mathfrak{a}_i = q_1(\mathfrak{a}) + \ldots + q_i(\mathfrak{a}) = p_i(\mathfrak{a}),$$

therefore we can write:

$$\mathcal{T}\text{-}\lim_{i \to \infty} p_i(\mathfrak{a}) = \mathfrak{a}^0.$$

Clearly $p_i(\mathfrak{a}) \varepsilon \mathfrak{P}^0(\mathfrak{a})$, and so $\mathfrak{a}^0 \varepsilon \mathfrak{P}(\mathfrak{a})$.

As

$$(1 - x^2)(1 - p_i(x)) \equiv 1 - p_{i+1}(x), \quad p_{i+1}(x) - p_i(x) \equiv x^2 - x^2 p_i(x),$$
$$p_{i+1}(\mathfrak{a}) - p_i(\mathfrak{a}) = \mathfrak{a}^2 - \mathfrak{a}^2 p_i(\mathfrak{a}),$$

therefore we obtain, by applying $\mathcal{T}\text{-}\lim_{i \to \infty}$ to both sides [use IV_1) — IV_3)]:

$$0 = \mathfrak{a}^0 - \mathfrak{a}^0 = \mathfrak{a}^2 - \mathfrak{a}^2 \mathfrak{a}^0,$$

that is:

$$\mathfrak{a}^2 \mathfrak{a}^0 = \mathfrak{a}^2.$$

As $\mathfrak{a}$, $\mathfrak{a}^0 \varepsilon \mathfrak{P}(\mathfrak{a})$ (25) may be used unrestrictedly. So we obtain:

$$(\mathfrak{a}\mathfrak{a}^0 - \mathfrak{a})^2 = \mathfrak{a}^2 (\mathfrak{a}^0)^2 - 2\mathfrak{a}^2\mathfrak{a}^0 + \mathfrak{a}^2 = (\mathfrak{a}^2\mathfrak{a}^0)\mathfrak{a}^0 - 2\mathfrak{a}^2\mathfrak{a}^0 + \mathfrak{a}^2 = \mathfrak{a}^2 - 2\mathfrak{a}^2 + \mathfrak{a}^2 = 0,$$

and by II_3)

$$\mathfrak{a}\mathfrak{a}^0 - \mathfrak{a} = 0, \quad \mathfrak{a}\mathfrak{a}^0 = \mathfrak{a}.$$

If $n = 2, 3, \ldots$ then

$$\mathfrak{a}^n \mathfrak{a}^0 = \mathfrak{a}^{n-1} \cdot \mathfrak{a}\mathfrak{a}^0 = \mathfrak{a}^{n-1} \cdot \mathfrak{a} = \mathfrak{a}^n$$

results. Thus $\mathfrak{b}\mathfrak{a}^0 = \mathfrak{b}$ holds for all $\mathfrak{b} = \mathfrak{a}^n$, $n = 1, 2, \ldots$ (for $n = 1$ we had it originally). Therefore it holds for all $\mathfrak{b} \varepsilon \mathfrak{P}^0(\mathfrak{a})$, and by IV_3) for all $\mathfrak{b} \varepsilon \mathfrak{P}(\mathfrak{u})$. In other words: $\mathfrak{a}^0$ is the unit of the ring $\mathfrak{P}(\mathfrak{a})$.

In all this we have assumed $\mathfrak{a} \varepsilon \mathfrak{U}$. If $\mathfrak{a}$ is arbitrary, then choose m by III_1) with $\frac{1}{2m} \mathfrak{a} \varepsilon \mathfrak{U}$. Clearly $\mathfrak{P}^0(\mathfrak{a}) = \mathfrak{P}^0\left(\frac{1}{2m}\mathfrak{a}\right)$, so $\mathfrak{P}(\mathfrak{a}) = \mathfrak{P}\left(\frac{1}{2m}\mathfrak{a}\right)$. Thus $\mathfrak{P}(\mathfrak{a})$ again possesses a unit.

Summing up:

Theorem I. *The minimum ring containing* $\mathfrak{a}$, $\mathfrak{P}(\mathfrak{a})$, *possesses a unit* $\mathfrak{a}^0$.

From this we infer immediately:

Corollary. Every ring $\mathfrak{M} \neq \Theta, (0)$ contains an idempotent $\mathfrak{e} \neq 0$.

Proof. $\mathfrak{M}$ possesses an element $\mathfrak{a} \neq 0$, as $\mathfrak{a} \varepsilon \mathfrak{M}$, so $\mathfrak{P}(\mathfrak{a}) \subset \mathfrak{M}$. Thus $\mathfrak{a}^0 \varepsilon \mathfrak{M}$. Now as $\mathfrak{a}\mathfrak{a}^0 = \mathfrak{a} \neq 0$ so $\mathfrak{a}^0 \neq 0$; and being unit of $\mathfrak{M}$, $\mathfrak{a}^0$ is idempotent (by A, 2). Thus $\mathfrak{e} = \mathfrak{a}^0$ meets our requirements.

C, 4. We prove next

Lemma C, 4, 1. *Let $\varphi(x)$ be a real function, defined and everywhere continuous for $|x| \leqslant 1$; $\varphi(0) = 0$. Assume $\mathfrak{a} \varepsilon \mathfrak{U}$.*

Then we have:

(i) *Sequences $p_1(x), p_2(x), \ldots$ of polynomials $p_i(x) \varepsilon \mathcal{P}''$, with*

$$\lim_{i \to \infty} p_i(x) = \varphi(x)$$

uniformly for $|x| \leqslant 1$, do exist.

(ii) *For each such sequence $\mathcal{T}\text{-}\lim_{i \to \infty} p_i(\mathfrak{a})$ exists.*

(iii) *This $\mathcal{T}\text{-}\lim_{i \to \infty} p_i(\mathfrak{a})$ depends on $\varphi(x)$ and $\mathfrak{a}$ only, and not on the*

$$p_1(x), \; p_2(x), \ldots$$

Proof. Ad (i): choose $\varepsilon_i > 0$, < 1 so that $|x| \leqslant \varepsilon_i$ implies

$$|\varphi(x)| \leqslant \frac{1}{i}, \quad i = 1, 2, \ldots$$

Now define

$$\psi_i(x) \begin{cases} = 0 & \text{for} \quad |x| \leqslant \varepsilon_i, \\ = \varphi(x) - \varphi(-\varepsilon_i) & \text{"} \quad -1 \leqslant x \leqslant -\varepsilon_i, \\ = \varphi(x) - \varphi(\varepsilon_i) & \text{"} \quad \varepsilon_i \leqslant x \leqslant 1. \end{cases}$$

$\psi_i(x)$ is everywhere continuous for $|x| \leqslant 1$; $|\varphi(x) - \psi_i(x)| \leqslant \frac{1}{i}$ for $|x| \leqslant 1$; and $\psi_i(x) = 0$ for $|x| \leqslant \varepsilon_i$. Thus $\frac{\psi_i(x)}{x^2}$ is everywhere continuous for $|x| \leqslant 1$.

Now choose (by the Weierstrass approximation-theorem) a polynomial $q_i(x)$ with $|\psi_i(x) - q_i(x)| \leqslant \frac{1}{i}$ for $|x| \leqslant 1$. Clearly we may even choose $q_i(x)$ with dyadically rational coefficients. Thus

$$p_i(x) = x^2 q_i(x) \varepsilon \mathcal{P}'',$$

and

$$|\varphi(x) - p_i(x)| \leqslant |\varphi(x) - \psi_i(x)| + |\psi_i(x) - p_i(x)| \leqslant \frac{1}{i} + \frac{1}{i} x^2 \leqslant \frac{2}{i}$$

for $|x| \leqslant 1$. Thus

$$\lim_{i \to \infty} p_i(x) = \varphi(x)$$

uniformly for $|x| \leqslant 1$.

Ad (ii): follows immediately from Lemma B, 4, 3.

Ad (iii): if $p_1'(x), p_2'(x), \ldots$ and $p_1''(x), p_2''(x), \ldots$ are sequences in the sense of (i), then the sequence $p_1'(x) - p_1''(x), p_2'(x) - p_2''(x), \ldots$ fulfils the requirements of Lemma B, 4, 2. So

$$\mathcal{T}\text{-}\lim_{i \to \infty} (p_i'(\mathfrak{a}) - p_i''(\mathfrak{a})) = 0,$$

that is

$$\mathcal{T}\text{-}\lim_{i \to \infty} p_i'(\mathfrak{a}) = \mathcal{T}\text{-}\lim_{i \to \infty} p_i''(\mathfrak{a}).$$

On the basis of these results we will denote $\mathcal{T}\text{-}\lim_{i \to \infty} p_i(\mathfrak{a})$ by $\varphi((\mathfrak{a}))$. The double bracket is necessary in order to avoid ambiguity when $\varphi(x) \equiv p(x) \varepsilon \mathcal{P}'$. But we will see soon (in Lemma C, 4, 3) that this precaution is unnecessary, as $p(\mathfrak{a}) = p((\mathfrak{a}))$ for all $p(x) \varepsilon \mathcal{P}'$. As all $p_i(\mathfrak{a}) \varepsilon \mathfrak{P}^0(\mathfrak{a})$, therefore $\varphi((\mathfrak{a})) \varepsilon \mathfrak{P}(\mathfrak{a})$.

Lemma C, 4, 2. $\varphi((\mathfrak{a})), \mathfrak{a} \varepsilon \mathfrak{U}$, *has the following properties:*

(i) $\chi(x) \equiv \rho\varphi(x)$ *(ρ dyadically rational) implies* $\chi((\mathfrak{a})) = \rho\,\varphi((\mathfrak{a}))$.

(ii) $\chi(x) \equiv \varphi(x) + \psi(x)$ *implies* $\chi((\mathfrak{a})) = \varphi((\mathfrak{a})) + \psi((\mathfrak{a}))$.

(iii) $\chi(x) \equiv \varphi(x)\,\psi(x)$ *implies* $\chi((\mathfrak{a})) = \varphi((\mathfrak{a}))\,\psi((\mathfrak{a}))$.

(iv) $\varphi(x) \geqq 0$ *for* $|x| \leqq 1$ *implies* $\varphi((\mathfrak{a})) \varepsilon \mathfrak{D}$.

(v) $|\varphi(x)| \leqq 1$ *for* $|x| \leqq 1$ *implies* $\varphi((\mathfrak{a})) \varepsilon \mathfrak{U}$.

Proof. Observe first that if $p(x) \varepsilon \mathcal{P}''$, then $p(\mathfrak{a}) = p((\mathfrak{a}))$, as we might choose $p_1(x) = p_2(x) = \ldots = p(x)$ in Lemma C, 4, 1, (i).

Ad (i), (ii): obvious if $\varphi(x) \equiv p(x) \varepsilon \mathcal{P}''$, $\psi(x) \equiv q(x) \varepsilon \mathcal{P}''$. By IV_1), IV_2) it carries over to all $\varphi(x)$, $\psi(x)$.

Ad (iii): obvious if $\varphi(x) \equiv p(x) \varepsilon \mathcal{P}''$, $\psi(x) \equiv q(x) \varepsilon \mathcal{P}''$. For a fixed $\psi(x) \equiv q(x) \varepsilon \mathcal{P}''$ it extends by IV_3) to all $\varphi(x)$, and then for a fixed $\varphi(x)$ by IV_3) to all $\psi(x)$.

Ad (iv): put $\psi(x) \equiv \sqrt{\varphi(x)}$ for $|x| \leqq 1$. Then $\psi(x)\,\psi(x) = \varphi(x)$ for $|x| \leqq 1$, so by (iii) $\psi(\mathfrak{a})\,\psi(\mathfrak{a}) = \varphi(\mathfrak{a})$. Thus

$$\varphi((\mathfrak{a})) = (\psi((\mathfrak{a})))^2 \varepsilon \mathfrak{D}$$

by (20).

Ad (v): (i)—(iii) give $\psi((\mathfrak{a})) = p(\varphi((\mathfrak{a})))$ if $\psi(x) \equiv p(\varphi(x))$, $p(x) \varepsilon \mathcal{P}'$. Put

$$\omega(x) \equiv q(\varphi(x))^2 \pm \varphi(x)\,q(\varphi(x))^2 \text{ and } \equiv q(\varphi(x))^2 - \varphi(x)^2\,q(\varphi(x))^2$$

respectively, for any $q(x) \varepsilon \mathcal{P}'$. Then $|\varphi(x)| \leqq 1$ for $|x| \leqq 1$ implies $\omega(x) \geqq 0$ for $|x| \leqq 1$, so $\omega((\mathfrak{a})) \varepsilon \mathfrak{D}$ by (iv). Now the above remark shows that this means

$$(q(\varphi((\mathfrak{a}))))^2 \pm \varphi((\mathfrak{a}))\,(q(\varphi((\mathfrak{a}))))^2, \quad (q(\varphi((\mathfrak{a}))))^2 - (\varphi((\mathfrak{a})))^2\,(q(\varphi((\mathfrak{a}))))^2 \varepsilon \mathfrak{D}$$

and so by III_5)

$$\varphi((\mathfrak{a})) \varepsilon \mathfrak{U}.$$

Lemma C, 4, 3. *If* $\varphi(x) \equiv p(x) \varepsilon \mathcal{P}'$ *then* $\varphi(\mathfrak{a}) = \varphi((\mathfrak{a}))$.

Proof. $p(x) \varepsilon \mathcal{P}'$ means

$$p(x) \equiv \alpha x + q(x), \quad q(x) \varepsilon \mathcal{P}''.$$

$q(\mathfrak{a}) = q((\mathfrak{a}))$ was shown at the beginning of the proof of Lemma C, 4, 2, and so we need to consider $p(x) \equiv x$ only, owing to Lemma C, 4, 2, (i), (ii). Put

$$\overline{p_i}(x) = 1 - (1 - x^2)^i.$$

Then $x\,\overline{p_i}(x) \varepsilon \mathcal{P}''$, and

$$\lim_{i \to \infty} x\,\overline{p_i}(x) = x$$

uniformly for $|x| \leqq 1$. So Lemma C, 4, 1 gives

$$p((\mathfrak{a})) = \mathcal{T}\text{-}\lim_{i \to \infty} \mathfrak{a}\,\overline{p_i}(\mathfrak{a})$$

[put $p_i(x) \equiv x\,\overline{p_i}(x)$]. But we know from Theorem I and the considerations which preceeded it in C, 3 that

$$\mathcal{T}\text{-}\lim_{i \to \infty} \overline{p_i}(\mathfrak{a}) = \mathfrak{a}^0,$$

and that $\mathfrak{a}\mathfrak{a}^0 = \mathfrak{a}$. So $p((\mathfrak{a})) = \mathfrak{a}\mathfrak{a}^0 = \mathfrak{a}$. On the other hand, clearly $p(\mathfrak{a}) = \mathfrak{a}$, so $p(\mathfrak{a}) = p((\mathfrak{a}))$.

Lemma C, 4, 4. *Given a sequence* $\varphi_1(x), \varphi_2(x), \ldots$ *with* $\lim_{i \to \infty} \varphi_i(x) = \varphi(x)$ *uniformly for* $|x| \leqq 1$, *and assuming* $\mathfrak{a} \varepsilon \mathfrak{U}$, *we have*

$$\mathcal{T}\text{-}\lim_{i \to \infty} \varphi_i((\mathfrak{a})) = \varphi((\mathfrak{a})).$$

Proof. Assume first $\varphi_i(x) \geqslant 0$ for all $i = 1, 2, \ldots$, $|x| \leqslant 1$, and $\varphi(x) \equiv 0$. Put $M_i = \max_{|x| \leqslant 1} \varphi_i(x)$, our convergence assumption means $\lim_{i \to \infty} M_i = 0$. So for $i \geqslant i_0$ $M_i \leqslant 1$, choose $n = n_i$ as the greatest $n = 1, 2, \ldots$ with $nM_i \leqslant 1$. Clearly $\lim_{i \to \infty} n_i = \infty$.

Thus $|n_i \varphi_i(x)| \leqslant 1$ for $|x| \leqslant 1$ and so by Lemma C, 4, 2, (i), (v) $n_i \varphi_i((\mathfrak{a})) \varepsilon \mathfrak{U}$. Besides $\varphi_i(x) \geqslant 0$ gives by Lemma C, 4, 2, (iv) $\varphi_i((\mathfrak{a})) \varepsilon \mathfrak{D}$. Now Lemma B, 2, 3 applies [with $\mathfrak{a}_i = \varphi_i((\mathfrak{a}))$] giving $\mathcal{T}\text{-}\lim_{i \to \infty} \varphi_i((\mathfrak{a})) = 0$.

Let now $\varphi_i(x) \gtreqless 0$, but still $\varphi(x) \equiv 0$. Put $\varphi'_i(x) = \frac{1}{2}(|\varphi_i(x)| + \varphi_i(x))$, $\varphi''_i(x) = \frac{1}{2}(|\varphi_i(x)| - \varphi_i(x))$. Then $\varphi'_i(x) \geqslant 0$, $\varphi''_i(x) \geqslant 0$ for $|x| \leqslant 1$, and $\lim_{i \to \infty} \varphi'_i(x) = \lim_{i \to \infty} \varphi''_i(x) = 0$ uniformly for $|x| \leqslant 1$. Thus

$$\mathcal{T}\text{-}\lim_{i \to \infty} \varphi'_i((\mathfrak{a})) = \mathcal{T}\text{-}\lim_{i \to \infty} \varphi''_i((\mathfrak{a})) = 0.$$

But $\varphi_i(x) = \varphi'_i(x) - \varphi''_i(x)$, so $\varphi_i((\mathfrak{a})) = \varphi'_i((\mathfrak{a})) - \varphi''_i((\mathfrak{a}))$ [Lemma C, 4, 2, (ii)], and thus

$$\mathcal{T}\text{-}\lim_{i \to \infty} \varphi_i((\mathfrak{a})) = 0.$$

Finally, let $\varphi(x)$ too be arbitrary. As $\lim_{i \to \infty} (\varphi_i(x) - \varphi(x)) = 0$ uniformly for $|x| \leqslant 1$ the above result and Lemma C, 4, 2, (ii) give

$$\mathcal{T}\text{-}\lim_{i \to \infty} (\varphi_i((\mathfrak{a})) - \varphi((\mathfrak{a}))) = 0,$$

that is,

$$\mathcal{T}\text{-}\lim_{i \to \infty} \varphi_i((\mathfrak{a})) = \varphi((\mathfrak{a})).$$

Lemma C, 4, 5. *If* $|\varphi(x)| \leqslant 1$ *for* $|x| \leqslant 1$, *and* $\mathfrak{a} \varepsilon \mathfrak{U}$, *then* $\chi(x) \equiv \psi(\varphi(x))$ *gives*

$$\chi((\mathfrak{a})) = \psi((\varphi((\mathfrak{a})))).$$

[The right side is defined owing to Lemma C, 4, 2, (v).]

Proof. For $\psi(x) \equiv p(x) \varepsilon \mathcal{P}''$ this follows from Lemma C, 4, 2, (i)—(iii). Then Lemmas C, 4, 1 and C, 4, 4 extend it to all $\psi(x)$.

Summing up:

Theorem II. *If* $\mathfrak{a} \varepsilon \mathfrak{U}$, *then for each real function* $\varphi(x)$, *defined and everywhere continuous for* $|x| \leqslant 1$, *and with* $\varphi(0) = 0$ *a* $\varphi((\mathfrak{a})) \varepsilon \mathfrak{P}(\mathfrak{a})$ *is defined by Lemma C, 4, 1. These* $\varphi((\mathfrak{a}))$ *are isomorphic to the* $\varphi(x)$ *themselves* [the algebraic side of this is described in Lemma C, 4, 2, (i)—(iii) and Lemma C, 4, 5; the topological side in Lemma C, 4, 4]; *and if* $\varphi(x) \equiv p(x) \varepsilon \mathcal{P}'$, *then* $p((\mathfrak{a})) = p(\mathfrak{a})$. *Furthermore* $\varphi(x) \geqslant 0$ *and* $|\varphi(x)| \leqslant 1$ *respectively for* $|x| \leqslant 1$ *imply* $\varphi((\mathfrak{a})) \varepsilon \mathfrak{D}$ *and* $\varphi((\mathfrak{a})) \varepsilon \mathfrak{U}$ *respectively.*

C, 5. Consider now an arbitrary $\mathfrak{a} \varepsilon \mathfrak{A}$. There exists an $m = 0, 1, 2, \ldots$ [by III_1)] with $\frac{1}{2^m} \mathfrak{a} \varepsilon \mathfrak{U}$. If $\varphi(x)$ is a real function, defined and everywhere continuous for $|x| \leqslant 2^m$ and with $\varphi(0) = 0$, then put $\varphi_m(x) \equiv \varphi(2^m x)$. Then Theorem II applies to $\varphi_m(x)$ and to $\frac{1}{2^m} \mathfrak{a} \varepsilon \mathfrak{A}$, so that we may form $\varphi_m\left(\frac{1}{2^m} \mathfrak{a}\right)$.

$\varphi_m\left(\frac{1}{2^m}\mathfrak{a}\right)$ does not depend on the choice of m. Let m, n be such that $\frac{1}{2^m}\mathfrak{a}$, $\frac{1}{2^n}\mathfrak{a}\,\varepsilon\,\mathfrak{U}$, $m \neq n$. We may assume that $m < n$. Then for $\chi(x) \equiv \frac{1}{2^{n-m}}x$

$$|\chi(x)| \leqslant 1 \text{ for } |x| \leqslant 1,$$

and

$$\varphi_m(x) \equiv \varphi_n(\chi(x)).$$

So Theorem II gives

$$\varphi_m\left(\left(\frac{1}{2^m}\mathfrak{a}\right)\right) = \varphi_n\left(\left(\chi\left(\left(\frac{1}{2^m}\mathfrak{a}\right)\right)\right)\right) = \varphi_n\left(\left(\frac{1}{2^{n-m}}\frac{1}{2^m}\mathfrak{a}\right)\right) = \varphi_n\left(\left(\frac{1}{2^n}\mathfrak{a}\right)\right).$$

We will denote this common value of all $\varphi_m\left(\left(\frac{1}{2^m}\mathfrak{a}\right)\right)$ for all $m = 0, 1, 2, \ldots$ with $\frac{1}{2^m}\mathfrak{a}\,\varepsilon\,\mathfrak{U}$ by $\varphi(((\mathfrak{a})))$. If in particular $\mathfrak{a}\,\varepsilon\,\mathfrak{U}$ then we may choose $m = 0$ and as $\varphi_0(x) \equiv \equiv \varphi(x)$, $\frac{1}{2^0}\mathfrak{a} = \mathfrak{a}$, we then have: $\varphi(((\mathfrak{a}))) = \varphi((\mathfrak{a}))$. Thus $\varphi(((\mathfrak{a})))$ is an extension of the $\varphi((\mathfrak{a}))$ of Theorem II.

One verifies immediately:

Theorem III. *If* $\mathfrak{a}\,\varepsilon\,\mathfrak{A}$, *then for each real function* $\varphi(x)$, *defined and everywhere continuous for* $|x| \leqslant 2^m$ [$m = 0, 1, 2, \ldots$ being such that $\frac{1}{2^m}\mathfrak{a}\,\varepsilon\,\mathfrak{U}$, cf. III_1)], *and with* $\varphi(0) = 0$, *a* $\varphi(((\mathfrak{a})))\,\varepsilon\,\mathfrak{P}(\mathfrak{a})$ *is defined as above. These* $\varphi(((\mathfrak{a})))$ *are isomorphic to the* $\varphi(x)$ *themselves* [the algebraic side of this is expressed by the equivalents of Lemma C, 4, 2, (i)—(iii) (replace there $|x| \leqslant 1$ by $|x| \leqslant 2^m$) and Lemma C, 4, 5 (replace there $|\varphi(x)| \leqslant 1$ for $|x| \leqslant 1$ by $|\varphi(x)| \leqslant 2^n$ for $|x| \leqslant 2^m$, the $n = 0, 1, 2, \ldots$ being such that $\psi(x)$ is defined and everywhere continuous in $|x| \leqslant 2^n$; the topological side by the equivalent of Lemma C, 4, 4 (replace there $|x| \leqslant 1$ by $|x| \leqslant 2^m$)], *and if* $\varphi(x) \equiv p(x)\,\varepsilon\,\mathcal{P}'$ *then* $p(((\mathfrak{a}))) = p(\mathfrak{a})$. *If* $\mathfrak{a}\,\varepsilon\,\mathfrak{U}$ *then* $\varphi(((\mathfrak{a}))) = = \varphi((\mathfrak{a}))$. *Finally* $\varphi(x) \geqslant 0$ *and* $|\varphi(x)| \leqslant 1$ *respectively for* $|x| \leqslant 2^m$ *imply* $\varphi(((\mathfrak{a})))\,\varepsilon\,\mathfrak{D}$ *and* $\varphi(((\mathfrak{a})))\,\varepsilon\,\mathfrak{U}$ *respectively.*

C, 6. An immediate application of Theorem III is this:

For any real number ρ put $\varphi(x) \equiv \rho x$, and define $\rho\mathfrak{a}$ by $\varphi(((\mathfrak{a}))) = \rho\mathfrak{a}$. If ρ is dyadically rational, then $\varphi(x) \equiv p(x)\,\varepsilon\,\mathcal{P}'$, and so our $\rho\mathfrak{a}$ agrees then with the original one of Remark 1. Besides, Theorem III secures the validity of (5), (7) for general real ρ, σ:

$$(\rho + \sigma)\mathfrak{a} = \rho\mathfrak{a} + \sigma\mathfrak{a}, \tag{28}$$

$$(\rho\sigma)\mathfrak{a} = \rho(\sigma\mathfrak{a}). \tag{29}$$

For the same reason we have:

$\rho\mathfrak{a}$ is a $\mathcal{T}$-continuous function of ρ alone (for any fixed $\mathfrak{a}$). (30)

Now (30) extends (6) from the dyadically rational ρ to all real ρ

$$\rho(\mathfrak{a} + \mathfrak{b}) = \rho\mathfrak{a} + \rho\mathfrak{b}. \tag{31}$$

Thus $\rho\mathfrak{a}$ is defined for all real ρ and $\mathfrak{a}\,\varepsilon\,\mathfrak{A}$, and has the properties which one is led to expect.

D. Theory of idempotents

D,1. We first observe

Lemma D, 1, 1. *Assume* $\mathfrak{a}_{n_1 \ldots n_i} \varepsilon \mathfrak{A}$ *and fixed,* $\rho_1, \ldots, \rho_i$ *real numerical variables (dyadically rational would do too). If*

$$\sum_{n_1=0}^{N_1} \cdots \sum_{n_i=0}^{N_i} \rho_1^{n_1} \cdots \rho_i^{n_i} \mathfrak{a}_{n_1 \ldots n_i} = 0$$

for all values of $\rho_1, \ldots, \rho_i$, *then all* $\mathfrak{a}_{n_1 \ldots n_i} = 0$.

Proof. Assume that the statement holds generally for a certain $i = 1, 2, \ldots$ In this case it holds for $i = 1$ too. Consider now $i + 1$, that is: assume that

$$\sum_{n_1=1}^{N_1} \cdots \sum_{n_{i+1}=}^{N_{i+1}} \rho_1^{n_1} \cdots \rho_{i+1}^{n_{i+1}} \mathfrak{a}_{n_1 \ldots n_{i+1}} = 0$$

or all $\rho_1, \ldots, \rho_{i+1}$. Hold $\rho_1, \ldots, \rho_i$ fixed, put

$$\sum_{n_1=0}^{N_1} \cdots \sum_{n_i=0}^{N_i} \rho_1^{n_1} \cdots \rho_i^{n_i} \mathfrak{a}_{n_1 \ldots n_i n} = \mathfrak{b}_n$$

and let $\rho_{i+1} = \rho$ vary. Then $\sum_{n=0}^{N_{i+1}} \mathfrak{b}_n \rho^n = 0$ for all ρ, so by assumption all $\mathfrak{b}_n = 0$, thatis

$$\sum_{n_1=0}^{N_1} \cdots \sum_{n_i=0}^{N_i} \rho_1^{n_1} \cdots \rho_i^{n_i} \mathfrak{a}_{n_1 \ldots n_i n} = 0.$$

As this holds for all $\rho_1, \ldots, \rho_i$ it gives by assumption that all $\mathfrak{a}_{n_1 \ldots n_i n} = 0$.

Thus it suffices to consider the case $i = 1$. If $\sum_{n=}^{N} \rho^n \mathfrak{a}_n = 0$ for all ρ, then all $\mathfrak{a}_n = 0$. For $N = 0$ this is obvious, therefore it suffices to prove it for $N + 1$ assuming its validity for $N = 0, 1, 2, \ldots$

Put $\rho = 0$, this gives $\mathfrak{a}_0 = 0$. So $\sum_{n=1}^{N+1} \rho_n \mathfrak{a}_n = 0$. Assume $\rho \neq 0$, multiply by $\frac{1}{\rho}$:

$$\sum_{n=1}^{N+1} \rho^{n-1} \mathfrak{a}_n = 0, \quad \sum_{n=0}^{N} \rho^n \mathfrak{a}_{n+1} = 0$$

results. By continuity [cf. (30)] the latter formula extends to $\rho = 0$ too. Now our assumption gives: all $\mathfrak{a}_{n+1} = 0$, that is $\mathfrak{a}_n = 0$ if $n \neq 0$. But we have proved already $\mathfrak{a}_0 = 0$, so all $\mathfrak{a}_n = 0$.

Consider now the equation

$$\mathfrak{a}^2 \mathfrak{a}^2 = (\mathfrak{a}^2 \mathfrak{a})\, \mathfrak{a}$$

[by (18) or (25)]. Replace $\mathfrak{a}$ by $\rho_1 \mathfrak{a} + \rho_2 \mathfrak{b}$ and compare the coefficients of $\rho_1^3 \rho_2$ and $\rho_1^2 \rho_2^2$. Then Lemma D, 1, 1 gives:

$$4\,(\mathfrak{a}\mathfrak{b})\,\mathfrak{a}^2 = \mathfrak{a}^3 \mathfrak{b} + \mathfrak{a}\,(\mathfrak{a}^2 \mathfrak{b}) + 2\mathfrak{a}\,(\mathfrak{a}\,(\mathfrak{a}\mathfrak{b})), \tag{32}$$

$$4\,(\mathfrak{a}\mathfrak{b})^2 + 2\mathfrak{a}^2 \mathfrak{b}^2 = \mathfrak{a}\,(\mathfrak{a}\mathfrak{b}^2) + \mathfrak{b}\,(\mathfrak{b}\mathfrak{a}^2) + 2\mathfrak{a}\,(\mathfrak{b}\,(\mathfrak{a}\mathfrak{b})) + 2\mathfrak{b}\,(\mathfrak{a}\,(\mathfrak{a}\mathfrak{b})) \tag{33}$$

Put in (32) $\mathfrak{a}=\mathfrak{e}$, $\mathfrak{b}=\mathfrak{x}$, where $\mathfrak{e}$ is an idempotent. Then

$$2\mathfrak{e}(\mathfrak{e}(\mathfrak{e}\mathfrak{x}))-3\mathfrak{e}(\mathfrak{e}\mathfrak{x})+\mathfrak{e}\mathfrak{x}=0 \tag{34}$$

results. The same substitution in (33) gives

$$4(\mathfrak{e}\mathfrak{x})^2+2\mathfrak{e}\mathfrak{x}^2-\mathfrak{e}(\mathfrak{e}\mathfrak{x}^2)-(\mathfrak{e}\mathfrak{x})\mathfrak{x}-2\mathfrak{e}((\mathfrak{e}\mathfrak{x})\mathfrak{x})-2(\mathfrak{e}(\mathfrak{e}\mathfrak{x}))\mathfrak{x}=0.$$

Replace $\mathfrak{x}$ by $\rho_1\mathfrak{x}+\rho_2\mathfrak{y}$ and compare the coefficients of $\rho_1\rho_2$. Then Lemma D, 1, 1 gives:

$$8(\mathfrak{e}\mathfrak{x})(\mathfrak{e}\mathfrak{y})+4\mathfrak{e}(\mathfrak{x}\mathfrak{y})-2\mathfrak{e}(\mathfrak{e}(\mathfrak{x}\mathfrak{y}))-(\mathfrak{e}\mathfrak{x})\mathfrak{y}-(\mathfrak{e}\mathfrak{y})\mathfrak{x}-2\mathfrak{e}((\mathfrak{e}\mathfrak{x})\mathfrak{y})-2\mathfrak{e}((\mathfrak{e}\mathfrak{y})\mathfrak{x})-2(\mathfrak{e}(\mathfrak{e}\mathfrak{x}))\mathfrak{y}-$$
$$-2(\mathfrak{e}(\mathfrak{e}\mathfrak{y}))\mathfrak{x}=0. \tag{35}$$

These identities (32)—(35) will be used in what follows.

D, 2. If $\mathfrak{e}$ is an idempotent, and ρ any real number, then let $N_\rho(\mathfrak{e})$ be the set of all $\mathfrak{x}\varepsilon\mathfrak{A}$ with $\mathfrak{e}\mathfrak{x}=\rho\mathfrak{x}$. We prove:

Lemma D, 2, 1. *$N_\rho(\mathfrak{e})$ is a module for every ρ. However $N_\rho(\mathfrak{e})=(0)$, except when $\rho=0,\frac{1}{2},1$. $\mathfrak{A}$ is the direct sum of the three $N_\rho(\mathfrak{e})$, $\rho=0,\frac{1}{2},1$, that is: every $\mathfrak{x}\varepsilon\mathfrak{A}$ permits a unique decomposition:*

$$\mathfrak{x}=\mathfrak{x}_0+\mathfrak{x}_{\frac{1}{2}}+\mathfrak{x}_1,\quad \mathfrak{x}_\rho\varepsilon N_\rho(\mathfrak{e}). \tag{36}$$

Herein

$$\left.\begin{aligned}\mathfrak{x}_0&=2\mathfrak{e}(\mathfrak{e}\mathfrak{x})-3\mathfrak{e}\mathfrak{x}+\mathfrak{x},\\ \mathfrak{x}_{\frac{1}{2}}&=-4\mathfrak{e}(\mathfrak{e}\mathfrak{x})+4\mathfrak{e}\mathfrak{x},\\ \mathfrak{x}_1&=2\mathfrak{e}(\mathfrak{e}\mathfrak{x})-\mathfrak{e}\mathfrak{x}.\end{aligned}\right\} \tag{37}$$

Proof. That $N_\rho(\mathfrak{e})$ is a module, is clear. If $\mathfrak{x}\varepsilon N_\rho(\mathfrak{e})$ then (34) becomes

$$(2\rho^3-3\rho^2+\rho)\mathfrak{x}=0.$$

If $2\rho^3-3\rho^2+\rho\neq 0$, then multiplication by $\frac{1}{2\rho^3-3\rho^2+\rho}$ gives $\mathfrak{x}=0$. Thus $N_\rho(\mathfrak{e})\neq(0)$ necessitates $2\rho^3-3\rho^2+\rho=0$, that is $\rho=0,\frac{1}{2},1$.

(36) and $\mathfrak{e}\mathfrak{x}_\rho=\rho\mathfrak{x}_\rho$ give immediately (37), so we must only show that the $\mathfrak{x}_\rho$ from (37) do actually solve (36). $\mathfrak{x}=\mathfrak{x}_0+\mathfrak{x}_{\frac{1}{2}}+\mathfrak{x}_1$ is obvious, so we must only prove $\mathfrak{x}_\rho\varepsilon N_\rho(\mathfrak{e})$, that is $\mathfrak{e}\mathfrak{x}_\rho=\rho\mathfrak{x}_\rho$. But this coincides with (34) for all three $\rho=0,\frac{1}{2},1$.

Lemma D, 2, 2:

$$\begin{array}{lll}\mathfrak{a},\mathfrak{b}\varepsilon N_0(\mathfrak{e}) & \textit{imply} & \mathfrak{a}\mathfrak{b}\varepsilon N_0(\mathfrak{e}),\\ \mathfrak{a},\mathfrak{b}\varepsilon N_1(\mathfrak{e}) & \text{»} & \mathfrak{a}\mathfrak{b}\varepsilon N_1(\mathfrak{e}),\\ \mathfrak{a}\varepsilon N_0(\mathfrak{e}),\mathfrak{b}\varepsilon N_1(\mathfrak{e}) & \text{»} & \mathfrak{a}\mathfrak{b}=0,\\ \mathfrak{a},\mathfrak{b}\varepsilon N_{\frac{1}{2}}(\mathfrak{e}) & \text{»} & \mathfrak{a}\mathfrak{b}\varepsilon N_0(\mathfrak{e})+N_1(\mathfrak{e}),\\ \mathfrak{a}\varepsilon N_0(\mathfrak{e}),\mathfrak{b}\varepsilon N_{\frac{1}{2}}(\mathfrak{e}) & \text{»} & \mathfrak{a}\mathfrak{b}\varepsilon N_{\frac{1}{2}}(\mathfrak{e})+N_1(\mathfrak{e}),\\ \mathfrak{a}\varepsilon N_1(\mathfrak{e}),\mathfrak{b}\varepsilon N_{\frac{1}{2}}(\mathfrak{e}) & \text{»} & \mathfrak{a}\mathfrak{b}\varepsilon N_{\frac{1}{2}}(\mathfrak{e})+N_0(\mathfrak{e}).\end{array}$$

(The $+$ signs indicate direct sums.)

Proof. We have $\mathfrak{a} \varepsilon N_\rho(\mathfrak{e})$, $\mathfrak{b} \varepsilon N_\sigma(\mathfrak{e})$, $\rho, \sigma = 0, \frac{1}{2}, 1$, that is $\mathfrak{e}\mathfrak{a} = \rho\mathfrak{a}$, $\mathfrak{e}\mathfrak{b} = \sigma\mathfrak{b}$. Put [by (36)]

$$\mathfrak{a}\mathfrak{b} = (\mathfrak{a}\mathfrak{b})_0 + (\mathfrak{a}\mathfrak{b})_{\frac{1}{2}} + (\mathfrak{a}\mathfrak{b})_1, \quad (\mathfrak{a}\mathfrak{b})_\rho \varepsilon N_\rho(\mathfrak{e}).$$

Now (35) becomes, with $\mathfrak{x} = \mathfrak{a}$, $\mathfrak{y} = \mathfrak{b}$:

$$-2\mathfrak{e}(\mathfrak{e}(\mathfrak{a}\mathfrak{b})) + (4 - 2\rho - 2\sigma)\,\mathfrak{e}(\mathfrak{a}\mathfrak{b}) + (8\rho\sigma - 2\rho^2 - 2\sigma^2 - \rho - \sigma)\,\mathfrak{a}\mathfrak{b} = 0,$$

and if we use the decomposition of $\mathfrak{a}\mathfrak{b}$:

$$(8\rho\sigma - 2\rho^2 - 2\sigma^2 - \rho - \sigma)(\mathfrak{a}\mathfrak{b})_0 + \left(8\rho\sigma - 2\rho^2 - 2\sigma^2 - 2\rho - 2\sigma + \frac{3}{2}\right)(\mathfrak{a}\mathfrak{b})_{\frac{1}{2}} +$$
$$+ (8\rho\sigma - 2\rho^2 - 2\sigma^2 - 3\rho - 3\sigma + 2)(\mathfrak{a}\mathfrak{b})_1 = 0.$$

As these three addends belong to $N_0(\mathfrak{e})$, $N_{\frac{1}{2}}(\mathfrak{e})$, $N_1(\mathfrak{e})$ respectively, they must vanish separately (use Lemma D, 2, 1 for $\mathfrak{x} = 0$). Therefore

$$8\rho\sigma - 2\rho^2 - 2\sigma^2 - \rho - \sigma = 0$$

or

$$8\rho\sigma - 2\rho^2 - 2\sigma^2 - 2\rho - 2\sigma + \frac{3}{2} = 0$$

or

$$8\rho\sigma - 2\rho^2 - 2\sigma^2 - 3\rho - 3\sigma + 2 = 0$$

imply $(\mathfrak{a}\mathfrak{b})_0 = 0$ respectively $(\mathfrak{a}\mathfrak{b})_{\frac{1}{2}} = 0$ respectively $(\mathfrak{a}\mathfrak{b})_1 = 0$. Substituting $\rho, \sigma = 0, \frac{1}{2}, 1$ leads to the desired results.

The two last statements of this Lemma can even be strengthened. In those cases even $\mathfrak{a}\mathfrak{b} \varepsilon N_{\frac{1}{2}}(\mathfrak{e})$ holds, as we will prove in Lemma G, 1, 1.

Theorem IV. *$N_0(\mathfrak{e})$ and $N_1(\mathfrak{e})$ are rings, and they are orthogonal, that is $\mathfrak{a} \varepsilon N_0(\mathfrak{e})$, $\mathfrak{b} \varepsilon N_1(\mathfrak{e})$ imply $\mathfrak{a}\mathfrak{b} = 0$.*

Proof. As $N_0(\mathfrak{e})$, $N_1(\mathfrak{e})$ are moduli, these statements coincide with the three first statements of Lemma D, 2, 2.

D, 3. Assume $\mathfrak{e}, \mathfrak{f}, \mathfrak{g}$ idempotents, $\mathfrak{e} \leqslant \mathfrak{f}$, $\mathfrak{f} \leqslant \mathfrak{g}$. (Cf., for this and what follows, A, 3.) Then $\mathfrak{f} - \mathfrak{e}$ and $\mathfrak{g} - \mathfrak{f}$ are idempotents too. As $\mathfrak{f} - (\mathfrak{f} - \mathfrak{e}) = \mathfrak{e}$ is idempotent, we have $\mathfrak{f} - \mathfrak{e} \leqslant \mathfrak{f}$, $\mathfrak{f}(\mathfrak{f} - \mathfrak{e}) = \mathfrak{f} - \mathfrak{e}$, $\mathfrak{f} - \mathfrak{e} \varepsilon N_1(\mathfrak{f})$. As $(\mathfrak{g} - \mathfrak{f}) + \mathfrak{f} = \mathfrak{g}$ is idempotent, we have the orthogonality of $\mathfrak{f}$, $\mathfrak{g} - \mathfrak{f}$, $\mathfrak{f}(\mathfrak{g} - \mathfrak{f}) = 0$, $\mathfrak{g} - \mathfrak{f} \varepsilon N_0(\mathfrak{f})$. So Theorem IV gives $(\mathfrak{f} - \mathfrak{e})(\mathfrak{g} - \mathfrak{f}) = 0$, $\mathfrak{f} - \mathfrak{e}$, $\mathfrak{g} - \mathfrak{f}$ are orthogonal. Therefore $(\mathfrak{f} - \mathfrak{e}) + (\mathfrak{g} - \mathfrak{f}) = \mathfrak{g} - \mathfrak{e}$ is idempotent, $\mathfrak{e} \leqslant \mathfrak{g}$.

We have proved:

$$\mathfrak{e} \leqslant \mathfrak{f}, \quad \mathfrak{f} \leqslant \mathfrak{g} \text{ imply } \mathfrak{e} \leqslant \mathfrak{g}. \tag{38}$$

A, 3 and (38) give together:

Theorem V. *Writing $\mathfrak{e} < \mathfrak{f}$ for $\mathfrak{e} \leqslant \mathfrak{f}$, $\mathfrak{e} \neq \mathfrak{f}$ ($\mathfrak{e}$, $\mathfrak{f}$ idempotents), and $\mathfrak{e} > \mathfrak{f}$ for $\mathfrak{f} < \mathfrak{e}$ we have in $\mathfrak{e} \gtreqless \mathfrak{f}$ a partial, non-reflexive, transitive ordering of all idempotents in $\mathfrak{A}$.*

D, 4. Lemma D, 4, 1. *$\mathfrak{a} \varepsilon N_{\frac{1}{2}}(\mathfrak{e})$ implies $\mathfrak{a}^2 \varepsilon N_0(\mathfrak{e}) + N_1(\mathfrak{e})$* (by Lemma D, 2, 2), *but if $\mathfrak{a}^2 \varepsilon N_0(\mathfrak{e})$ or $\varepsilon N_1(\mathfrak{e})$, then $\mathfrak{a} = 0$.*

Proof. Assume $\mathfrak{a} \varepsilon N_{\frac{1}{2}}(\mathfrak{e})$, and $\mathfrak{a}^2 \varepsilon N_\rho(\mathfrak{e})$, $\rho = 0, 1$. Put $\mathfrak{b} = \mathfrak{e}$ in (32); then

$$2\mathfrak{a}^3 = \mathfrak{e}\mathfrak{a}^3 + \rho\mathfrak{a}^3 + \mathfrak{a}^3, \quad \mathfrak{e}\mathfrak{a}^3 = (1-\rho)\,\mathfrak{a}^3, \quad \mathfrak{a}^3 \varepsilon N_{1-\rho}(\mathfrak{e})$$

obtains. Thus Theorem IV gives $\mathfrak{a}^5 = \mathfrak{a}^2 \cdot \mathfrak{a}^3 = 0$, and therefore $((\mathfrak{a}^2)^2)^2 = \mathfrak{a}^8 = \mathfrak{a}^3 \cdot \mathfrak{a}^5 = 0$. Now three applications of II_3) give $\mathfrak{a} = 0$.

Lemma D, 4, 2. *If the idempotent* $\mathfrak{e}$ *is not the unit of* $\mathfrak{A}$, *then an idempotent* $\mathfrak{f} > \mathfrak{e}$ *exists.*

Proof. If $N_0(\mathfrak{e}) \neq (0)$ [it is $\neq \Theta$ as $0 \varepsilon N_0(\mathfrak{e})$], then the Corollary to Theorem I secures the existence of an idempotent $\mathfrak{e}_1 \varepsilon N_0(\mathfrak{e})$, $\mathfrak{e}_1 \neq 0$, because $N_0(\mathfrak{e})$ is a ring by Theorem IV. Now $\mathfrak{e}\mathfrak{e}_1 = 0$, $\mathfrak{e}$, $\mathfrak{e}_1$ are orthogonal, and so $\mathfrak{f} = \mathfrak{e} + \mathfrak{e}_1$ is idempotent. As $\mathfrak{f} - \mathfrak{e} = \mathfrak{e}_1$ is idempotent, $\mathfrak{f} \geqslant \mathfrak{e}$. As $\mathfrak{f} - \mathfrak{e} = \mathfrak{e} \neq 0$, so $\mathfrak{f} \neq \mathfrak{e}$. Thus $\mathfrak{f} > \mathfrak{e}$.

Assume now that no idempotent $\mathfrak{f} > \mathfrak{e}$ exists. Then our above considerations prove $N_0(\mathfrak{e}) = (0)$. If $\mathfrak{a} \varepsilon N_{\frac{1}{2}}(\mathfrak{e})$ then

$$\mathfrak{a}^2 \varepsilon N_0(\mathfrak{e}) + N_1(\mathfrak{e}) = (0) + N_1(\mathfrak{e}) = N_1(\mathfrak{e}),$$

and so by Lemma D, 4, 1, $\mathfrak{a} = 0$. Thus $N_{\frac{1}{2}}(\mathfrak{e}) = (0)$ too. Now Lemma D, 2, 1 gives

$$\mathfrak{A} = N_0(\mathfrak{e}) + N_{\frac{1}{2}}(\mathfrak{e}) + N_1(\mathfrak{e}) = (0) + (0) + N_1(\mathfrak{e}) = N_1(\mathfrak{e}).$$

That is: for every $\mathfrak{a} \varepsilon \mathfrak{A}$ $\mathfrak{e}\mathfrak{a} = \mathfrak{a}$. Thus $\mathfrak{e}$ is the unit of $\mathfrak{A}$.

Lemma D, 4, 3. *Let* $\mathfrak{e}_1, \mathfrak{e}_2, \ldots$ *be a sequence of idempotents with* $\mathfrak{e}_1 \leqslant \mathfrak{e}_2 \leqslant \ldots$ *Then*

$$\mathcal{T}\text{-}\lim_{i \to \infty} \mathfrak{e}_i = \mathfrak{e}^*$$

exists. $\mathfrak{e}^*$ *is idempotent, and all* $\mathfrak{e}_i \leqslant \mathfrak{e}^*$.

Proof. Put $\mathfrak{a}_i = \mathfrak{e}_i - \mathfrak{e}_{i-1}$ (with $\mathfrak{e}_0 = 0$) for $i = 1, 2, \ldots$ As $\mathfrak{e}_{i-1} \leqslant \mathfrak{e}_i$, so $\mathfrak{a}_i$ is idempotent, and $\mathfrak{a}_1 + \ldots + \mathfrak{a}_i = \mathfrak{e}_i$ is idempotent too. Thus (27) (in C, 2) gives $\mathfrak{a}_i \varepsilon \mathfrak{D}$, $\mathfrak{a}_1 + \ldots + \mathfrak{a}_i \varepsilon \mathfrak{U}$, and so by Lemma B, 2, 1,

$$\mathcal{T}\text{-}\lim_{i \to \infty} (\mathfrak{a}_1 + \ldots + \mathfrak{a}_i) = \mathcal{T}\text{-}\lim_{i \to \infty} \mathfrak{e}_i = \mathfrak{e}^*$$

exists.

As $\mathfrak{e}_i \leqslant \mathfrak{e}_{i+1}$ so Theorem V gives $\mathfrak{e}_i \leqslant \mathfrak{e}_j$ for all $j > i$. For $j = i$ this holds too, so $i \leqslant j$ suffices. In other words: $\mathfrak{e}_i \mathfrak{e}_j = \mathfrak{e}_i$ if $i \leqslant j$. Forming $\mathcal{T}\text{-}\lim\limits_{j \to \infty} \mathfrak{e}_i \mathfrak{e}^* = \mathfrak{e}_i$ obtains [use IV_3)]. Now form $\mathcal{T}\text{-}\lim\limits_{i \to \infty}$, then $(\mathfrak{e}^*)^2 = \mathfrak{e}^*$ obtains [use IV_3)]. Thus $\mathfrak{e}^*$ is idempotent. And now we see that $\mathfrak{e}_i \mathfrak{e}^* = \mathfrak{e}_i$ means $\mathfrak{e}_i \leqslant \mathfrak{e}^*$.

Lemma D, 4, 4. *A set* $\mathfrak{S}$ *of mutually orthogonal idempotents is necessarily finite or enumerably infinite.*

Proof. By (27) $\mathfrak{S} \subset \mathfrak{U}$, so $\mathfrak{S}$ is separable along with $\mathfrak{U}$, by IV_6). Let $\mathfrak{e}_1, \mathfrak{e}_2, \ldots$ be a subset of $\mathfrak{S}$ which is dense in $\mathfrak{S}$.

Consider an $\mathfrak{e} \varepsilon \mathfrak{S}$, $\mathfrak{e} \neq \mathfrak{e}_1, \mathfrak{e}_2, \ldots$ Then

$$\mathfrak{e} = \mathcal{T}\text{-}\lim_{i \to \infty} \mathfrak{e}_{n_i}$$

for a suitable subsequence $\mathfrak{e}_{n_1}, \mathfrak{e}_{n_2}, \ldots$ As $\mathfrak{e}, \mathfrak{e}_{n_i}$ are two different elements of $\mathfrak{S}$, we have $\mathfrak{e}\mathfrak{e}_{n_i}=0$, application of $\underset{i\to\infty}{\mathcal{T}\text{-lim}}$ gives $\mathfrak{e}^2=0$ [use IV_3)]. Thus $\mathfrak{e}=\mathfrak{e}^2=0$. So $\mathfrak{S}$ is contained in the set $(0, \mathfrak{e}_1, \mathfrak{e}_2, \ldots)$. Therefore it must be finite or enumerably infinite.

We are now in the position to prove:

Theorem VI. *The unit of $\mathfrak{A}$ exists.*

Proof. Assume the contrary. Let Ω be the first non-enumerable (infinite) ordinal of Cantor. For each ordinal $\alpha<\Omega$ define an idempotent $\mathfrak{e}_\alpha$ so that $\alpha<\beta<\Omega$ implies $\mathfrak{e}_\alpha<\mathfrak{e}_\beta$ in the following manner:

$\alpha=0$, define

$$\mathfrak{e}_0=0.$$

$\alpha=\beta+1$, $\mathfrak{e}_\beta$ being already defined: as $\mathfrak{e}_\beta$ cannot be the unit of $\mathfrak{A}$, so an idempotent $\mathfrak{f}>\mathfrak{e}_\beta$ must exist by Lemma D, 4, 2. Choose such an $\mathfrak{f}$, and put

$$\mathfrak{e}_\alpha=\mathfrak{f}.$$

Then $\mathfrak{e}_\alpha=\mathfrak{e}_{\beta+1}>\mathfrak{e}_\beta$.

α a limit number, $\mathfrak{e}_\beta$ being already defined for all $\gamma<\alpha$, and so that $\gamma<\delta<\alpha$ implies $\mathfrak{e}_\gamma<\mathfrak{e}_\delta$: as α is a limit number $<\Omega$, an enumerably infinite sequence of ordinals $\alpha_1<\alpha_2<\ldots<\alpha$ with $\lim\limits_{i\to\infty}\alpha_i=\alpha$ exists. Then $\mathfrak{e}_{\alpha_1}<\mathfrak{e}_{\alpha_2}<\ldots$, so Lemma D, 4, 3 applies: $\underset{i\to\infty}{\mathcal{T}\text{-lim}}\,\mathfrak{e}_{\alpha_i}$ exists. Put

$$\mathfrak{e}_\alpha=\underset{i\to\infty}{\mathcal{T}\text{-lim}}\,\mathfrak{e}_{\alpha_i}.$$

By Lemma D, 4, 3 this $\mathfrak{e}_\alpha$ is idempotent, and all $\mathfrak{e}_{\alpha_i}\leqslant\mathfrak{e}_\alpha$. So if $\gamma<\alpha$, then as an α_i with $\alpha_i>\gamma$ can be found, we have $\mathfrak{e}_\gamma<\mathfrak{e}_{\alpha_i}\leqslant\mathfrak{e}_\alpha$, $\mathfrak{e}_\gamma<\mathfrak{e}_\alpha$.

Thus all $\mathfrak{e}_\alpha$, $\alpha<\Omega$, are defined, and $\alpha<\beta<\Omega$ implies $\mathfrak{e}_\alpha<\mathfrak{e}_\beta$.

Put $\mathfrak{f}_\alpha=\mathfrak{e}_{+1}-\mathfrak{e}_\alpha$, as $\mathfrak{e}_\alpha<\mathfrak{e}_{\alpha+1}$ so $\mathfrak{f}_\alpha$ is an idempotent and $\neq 0$. Assume $\alpha<\beta<\Omega$. $\mathfrak{f}_\beta+\mathfrak{e}_\beta=\mathfrak{e}_{\beta+1}$ being idempotent, $\mathfrak{f}_\beta$, $\mathfrak{e}_\beta$ are orthogonal, that is $\mathfrak{f}_\beta\,\varepsilon\,N_0(\mathfrak{e}_\beta)$. Now $\mathfrak{e}_{\alpha+1}-\mathfrak{f}_\alpha=\mathfrak{e}_\alpha$ being idempotent, $\mathfrak{f}_\alpha\leqslant\mathfrak{e}_{\alpha+1}$. As $\alpha+1\leqslant\beta$ so $\mathfrak{e}_{\alpha+1}\leqslant\mathfrak{e}_\beta$, thus (by Theorem V) $\mathfrak{f}_\alpha\leqslant\mathfrak{e}_\beta$, $\mathfrak{f}_\alpha\,\varepsilon\,N_1(\mathfrak{e}_\beta)$. So Theorem IV gives $\mathfrak{f}_\alpha\mathfrak{f}_\beta=0$, $\mathfrak{f}_\alpha$, $\mathfrak{f}_\beta$ are orthogonal. By symmetry this is true for $\beta<\alpha<\Omega$ too, that is it holds for all $\alpha\neq\beta$.

$\alpha\neq\beta$ and $\mathfrak{f}_\alpha=\mathfrak{f}_\beta$ would give $\mathfrak{f}_\alpha=(\mathfrak{f}_\alpha)^2=\mathfrak{f}_\alpha\mathfrak{f}_\beta=0$, which is not the case. Thus the $\mathfrak{f}_\alpha$, $\alpha<\Omega$, are mutually different, and so their set $\mathfrak{S}$ is unenumerably infinite. But as $\mathfrak{S}$ is a set of mutually orthogonal idempotents, this contradicts Lemma D, 4, 4.

Thus the original assumption must have been wrong, and therefore $\mathfrak{A}$ must possess a unit.

Corollary. Every ring $\mathfrak{M}\neq\Theta$ possesses a unit.

Proof. $\mathfrak{M}=(0)$ possesses a unit: 0. If $\mathfrak{M}\neq\Theta$, (0) then $\mathfrak{M}$ satisfies our axioms by A, 1, and so Theorem VI applies to $\mathfrak{M}$ in place of $\mathfrak{A}$.

This Corollary is clearly a strengthened form of the Corollary to Theorem I. In fact it contains Theorem I as the special case $\mathfrak{M}=\mathfrak{P}(\mathfrak{a})$.

Denote the unit of $\mathfrak{M}$ by $1(\mathfrak{M})$. For the unit of $\mathfrak{A}$ write $1=1(\mathfrak{A})$.

D, 5. Theorem VI and its Corollary permit a number of applications to the theory of idempotents.

First some properties of 0, 1:

Lemma D, 5, 1. *Let* $\mathfrak{e}$, $\mathfrak{f}$ *be idempotents.*

(i) *Always* $0 \leqslant \mathfrak{e} \leqslant 1$.

(ii) $1-\mathfrak{e}$ *is idempotent along with* $\mathfrak{e}$.

(iii) $\mathfrak{e} \gtreqless \mathfrak{f}$ *is equivalent to* $1-\mathfrak{e} \lesseqgtr 1-\mathfrak{f}$.

(iv) $\mathfrak{e}$, $\mathfrak{f}$ *are orthogonal if and only if* $\mathfrak{e} \leqslant 1-\mathfrak{f}$.

Proof. We prove (i)—(iv) in a changed order:

Ad (ii): $(1-\mathfrak{e})^2 = 1 - 2\mathfrak{e} + \mathfrak{e}^2 = 1 - 2\mathfrak{e} + \mathfrak{e} = 1 - \mathfrak{e}$.

Ad (i): $\mathfrak{e} - 0 = \mathfrak{e}$ is idempotent, and so is $1-\mathfrak{e}$ [by (ii)].

Ad (iii): $\mathfrak{e} \leqslant \mathfrak{f}$ implies $1-\mathfrak{e} \geqslant 1-\mathfrak{f}$, as $(1-\mathfrak{e})-(1-\mathfrak{f}) = \mathfrak{f}-\mathfrak{e}$; $\mathfrak{e} \neq \mathfrak{f}$ clearly implies $1-\mathfrak{e} \neq 1-\mathfrak{f}$. So $\mathfrak{e} < \mathfrak{f}$ implies $1-\mathfrak{e} > 1-\mathfrak{f}$. Interchanging $\mathfrak{e}$, $\mathfrak{f}$ shows that $\mathfrak{e} > \mathfrak{f}$ implies $1-\mathfrak{e} < 1-\mathfrak{f}$. $\mathfrak{e} = \mathfrak{f}$ clearly implies $1-\mathfrak{e} = 1-\mathfrak{f}$. So $\mathfrak{e} \gtreqless \mathfrak{f}$ imply $1-\mathfrak{e} \lesseqgtr 1-\mathfrak{f}$ respectively. The inverse follows by replacing $\mathfrak{e}$, $\mathfrak{f}$ by $1-\mathfrak{e}$, $1-\mathfrak{f}$.

Ad (iv): $\mathfrak{e} \leqslant 1-\mathfrak{f}$ means $\mathfrak{e}(1-\mathfrak{f}) = \mathfrak{e}$, that is $\mathfrak{e} - \mathfrak{e}\mathfrak{f} = \mathfrak{e}$, $\mathfrak{e}\mathfrak{f} = 0$, that is, that $\mathfrak{e}$, $\mathfrak{f}$ are orthogonal.

Lemma D, 5, 2. $N_\rho(\mathfrak{e}) = N_{1-\rho}(1-\mathfrak{e})$.

Proof. Obvious.

Lemma D, 5, 3. (i) $\mathfrak{e} \leqslant \mathfrak{f}$ *is equivalent to* $N_1(\mathfrak{e}) \subset N_1(\mathfrak{f})$, *or just as well to* $N_0(\mathfrak{e}) \supset N_0(\mathfrak{f})$.

(ii) $\mathfrak{e}$, $\mathfrak{f}$ *orthogonal is equivalent to* $N_1(\mathfrak{e}) \subset N_0(\mathfrak{f})$, *or just as well to* $N_1(\mathfrak{f}) \subset N_0(\mathfrak{e})$.

Proof. Ad (i): replacing $\mathfrak{e}$, $\mathfrak{f}$ by $1-\mathfrak{e}$, $1-\mathfrak{f}$ and using Lemma D, 5, 1, (iii), and Lemma D, 5, 2, shows that the first statement implies the second one. So it suffices to consider the first one.

$N_1(\mathfrak{e}) \subset N_1(\mathfrak{f})$ implies, as $\mathfrak{e}\mathfrak{e} = \mathfrak{e}$, $\mathfrak{e} \varepsilon N_1(\mathfrak{e})$ that $\mathfrak{e} \varepsilon N_1(\mathfrak{f})$, $\mathfrak{f}\mathfrak{e} = \mathfrak{e}$, that is $\mathfrak{e} \leqslant \mathfrak{f}$.

Conversely $\mathfrak{e} \leqslant \mathfrak{f}$ implies this: $\mathfrak{e}$, $1-\mathfrak{f}$ are orthogonal by Lemma D, 5, 1, (iv), so $1-\mathfrak{f} \varepsilon N_0(\mathfrak{e})$. Thus $\mathfrak{x} \varepsilon N_1(\mathfrak{e})$ implies $(1-\mathfrak{f})\mathfrak{x} = 0$ (by Theorem IV), $\mathfrak{f}\mathfrak{x} = \mathfrak{x}$, $\mathfrak{x} \varepsilon N_1(\mathfrak{f})$. In other words: $N_1(\mathfrak{e}) \subset N_1(\mathfrak{f})$.

Ad (ii): owing to the symmetry in $\mathfrak{e}$, $\mathfrak{f}$ it suffices to prove the first statement. $\mathfrak{e}$, $\mathfrak{f}$ orthogonal means $\mathfrak{e} \leqslant 1-\mathfrak{f}$ by Lemma D, 5, 1, (iv), and so by (i) and Lemma D, 5, 2

$$N_1(\mathfrak{e}) \subset N_1(1-\mathfrak{f}) = N_0(\mathfrak{f}).$$

If $\mathfrak{S}$ is a set of idempotents, then an idempotent $\mathfrak{e}_0$ is a least upper bound, abbreviated: l. u. b., of $\mathfrak{S}$ if $\mathfrak{e} \geqslant \mathfrak{e}_0$ is equivalent to $\mathfrak{e} \geqslant \mathfrak{e}'$ for all $\mathfrak{e}' \varepsilon \mathfrak{S}$; and it is a greatest lower bound, abbreviated: gr. l. b., of $\mathfrak{S}$ if $\mathfrak{e} \leqslant \mathfrak{e}_0$ is equivalent to $\mathfrak{e} \leqslant \mathfrak{e}'$ for all $\mathfrak{e}' \varepsilon \mathfrak{S}$. It is clear that $\mathfrak{S}$ possesses no l. u. b., or exactly one, and similarly for the gr. l. b. We now prove:

Theorem VI'. *Every set* $\mathfrak{S}$ *of idempotents possesses a l. u. b. and a gr. l. b.*

Proof. Use of the mapping $\mathfrak{e} \longrightarrow 1-\mathfrak{e}$ shows, by Lemma D, 5, 1, (iii), that the first statement implies the second one. Thus it suffices to consider the first.

$\mathfrak{S} = \Theta$ has the l. u. b. $\mathfrak{e}_0 = 0$, so we may assume $\mathfrak{S} \neq \Theta$.

Let $\mathfrak{M} = \mathfrak{P}(\mathfrak{S})$ be the minimum ring containing $\mathfrak{S}$. (Cf. A, 1.) Put $\mathfrak{e}_0 = 1(\mathfrak{M})$.

Being the unit of $\mathfrak{M}$, $\mathfrak{e}_0$ is idempotent. Each $\mathfrak{e}' \varepsilon \mathfrak{S}$ is $\varepsilon \mathfrak{M}$, so $\mathfrak{e}_0\mathfrak{e}' = \mathfrak{e}'$, $\mathfrak{e}' \leqslant \mathfrak{e}_0$. Thus $\mathfrak{e} \geqslant \mathfrak{e}_0$ implies $\mathfrak{e} \geqslant \mathfrak{e}'$ for all $\mathfrak{e}' \varepsilon \mathfrak{S}$.

Assume now conversely $\mathfrak{e} \geqslant \mathfrak{e}'$ for all $\mathfrak{e}' \varepsilon \mathfrak{S}$. Then for each $\mathfrak{e}' \varepsilon \mathfrak{S}$ $\mathfrak{e}' \leqslant \mathfrak{e}$, $\mathfrak{e}\mathfrak{e}' = \mathfrak{e}'$, $\mathfrak{e}' \varepsilon N_1(\mathfrak{e})$. Thus $N_1(\mathfrak{e})$ is a ring (use Theorem IV) and $\supset \mathfrak{S}$, therefore $N_1(\mathfrak{e}) \supset \mathfrak{M}$. Now $\mathfrak{e}_0 = 1(\mathfrak{M}) \varepsilon \mathfrak{M} \subset N_1(\mathfrak{e})$, $\mathfrak{e}\mathfrak{e}_0 = \mathfrak{e}_0$, $\mathfrak{e}_0 \leqslant \mathfrak{e}$.

We know that these l. u. b. and gr. l. b. of $\mathfrak{S}$ are unique, and we will denote them by $\Sigma(\mathfrak{S})$ and $\Pi(\mathfrak{S})$ respectively. For two-element sets $\mathfrak{S}=(\mathfrak{e}, \mathfrak{f})$ we use the notation

$$\Sigma((\mathfrak{e}, \mathfrak{f}))=\mathfrak{e} \smile \mathfrak{f}, \quad \Pi((\mathfrak{e}, \mathfrak{f}))=\mathfrak{e} \frown \mathfrak{f}.$$

One verifies immediately:

$$\mathfrak{e} \smile \mathfrak{f}=\mathfrak{f} \smile \mathfrak{e}, \tag{A}$$

$$(\mathfrak{e} \smile \mathfrak{f}) \smile \mathfrak{g}=\mathfrak{e} \smile (\mathfrak{f} \smile \mathfrak{g}) \tag{B}$$

[both sides of (B) are $=\Sigma((\mathfrak{e}, \mathfrak{f}, \mathfrak{g}))$] and

$$\mathfrak{e} \frown \mathfrak{f}=\mathfrak{f} \frown \mathfrak{e}, \tag{C}$$

$$(\mathfrak{e} \frown \mathfrak{f}) \frown \mathfrak{g}=\mathfrak{e} \frown (\mathfrak{f} \frown \mathfrak{g}) \tag{D}$$

[both sides of (D) are $=\Pi((\mathfrak{e}, \mathfrak{f}, \mathfrak{g}))$] as well as

$$\mathfrak{e} \smile \mathfrak{f}=\mathfrak{f} \quad \text{and} \quad \mathfrak{e} \frown \mathfrak{f}=\mathfrak{e} \tag{E}$$

are equivalent to each other and to $\mathfrak{e} \leqslant \mathfrak{f}$. Thus the set $\mathfrak{E}$ of all idempotents of $\mathfrak{A}$ is a lattice with the „join" $\mathfrak{e} \smile \mathfrak{f}$, the „meet" $\mathfrak{e} \frown \mathfrak{f}$, and the „partial ordering" $\mathfrak{e} \leqslant \mathfrak{f}$. [Cf. F. Klein, «Math. Annalen», **105**, (1931), 308—323; G. Birkhoff, «Proc. Cambridge Phil. Soc.», **29**, (1933), 441—464; O. Ore, «Annals of Mathematics», **36**, (1935), 406—437. We follow the terminology of G. Birkhoff.] Theorem VI′ expresses that this lattice is „completely additive and multiplicative". (Cf. also H. M. MacNeille, „Proc. Nat. Acad.", January 1936, p. 45—50.)

If $\mathfrak{e}$, $\mathfrak{f}$ are orthogonal, then $\mathfrak{e}+\mathfrak{f}$ is idempotent. As $\mathfrak{e}+\mathfrak{f}-\mathfrak{e}=\mathfrak{f}$ is idempotent, so $\mathfrak{e} \leqslant \mathfrak{e}+\mathfrak{f}$, similarly $\mathfrak{f} \leqslant \mathfrak{e}+\mathfrak{f}$. So $\mathfrak{e}' \geqslant \mathfrak{e}+\mathfrak{f}$ implies $\mathfrak{e}' \geqslant \mathfrak{e}, \mathfrak{f}$. On the other hand $\mathfrak{e}' \geqslant \mathfrak{e}, \mathfrak{f}$ gives $\mathfrak{e}'\mathfrak{e}=\mathfrak{e}$, $\mathfrak{e}'\mathfrak{f}=\mathfrak{f}$, so $\mathfrak{e}'(\mathfrak{e}+\mathfrak{f})=\mathfrak{e}+\mathfrak{f}$, $\mathfrak{e}' \geqslant \mathfrak{e}+\mathfrak{f}$. Thus in this case $\mathfrak{e}+\mathfrak{f}$ is the l. u. b. of $(\mathfrak{e}, \mathfrak{f})$, $\mathfrak{e}+\mathfrak{f}=\mathfrak{e} \smile \mathfrak{f}$. Conversely: if this equation holds, then $\mathfrak{e}+\mathfrak{f}$ is idempotent, so $\mathfrak{e}$, $\mathfrak{f}$ are orthogonal. So we have:

$$\mathfrak{e} \smile \mathfrak{f}=\mathfrak{e}+\mathfrak{f} \text{ if and only if } \mathfrak{e}, \mathfrak{f} \text{ are orthogonal.} \tag{39}$$

D, 6. We are able to determine all ideals in $\mathfrak{A}$ (cf. A, 1):

Lemma D, 6, 1. *The ideals* $\neq \Theta$ *coincide with the sets* $N_1(\mathfrak{e})$ *for idempotents* $\mathfrak{e}$ *with* $N_{\frac{1}{2}}(\mathfrak{e})=(0)$.

Proof. Sufficiency: $N_1(\mathfrak{e})$ is a ring by Theorem IV, and $\neq \Theta$ as $0 \varepsilon N_1(\mathfrak{e})$. Thus we must only prove that $\mathfrak{a} \varepsilon N_1(\mathfrak{e})$, $\mathfrak{b} \varepsilon \mathfrak{A}$ imply $\mathfrak{a}\mathfrak{b} \varepsilon N_1(\mathfrak{e})$. Now by Lemma D, 2, 1, $\mathfrak{b} \varepsilon \mathfrak{A}=N_0(\mathfrak{e})+N_{\frac{1}{2}}(\mathfrak{e})+N_1(\mathfrak{e})$, and as $N_{\frac{1}{2}}(\mathfrak{e})=(0)$, so $\mathfrak{b} \varepsilon N_0(\mathfrak{e})+N_1(\mathfrak{e})$. Thus Theorem IV gives $\mathfrak{a}\mathfrak{b} \varepsilon N_1(\mathfrak{e})$.

Necessity: assume that $\mathfrak{M} \neq \Theta$ is an ideal. As $\mathfrak{M}$ is a ring, it possesses a unit by the Corollary to Theorem VI: $1(\mathfrak{M})=\mathfrak{e}$. Thus $\mathfrak{a} \varepsilon \mathfrak{M}$ implies $\mathfrak{e}\mathfrak{a}=\mathfrak{a}$, $\mathfrak{a} \varepsilon N_1(\mathfrak{e})$ and so $\mathfrak{M} \subset N_1(\mathfrak{e})$. On the other hand $\mathfrak{a} \varepsilon N_\rho(\mathfrak{a})$, $\rho=\frac{1}{2}$, 1, implies, as $\mathfrak{e} \varepsilon \mathfrak{M}$ and as $\mathfrak{M}$ is an ideal, $\rho\mathfrak{a}=\mathfrak{e}\mathfrak{a} \varepsilon \mathfrak{M}$ and thus $\mathfrak{a}=\frac{1}{\rho}\rho\mathfrak{a} \varepsilon \mathfrak{M}$. So $N_{\frac{1}{2}}(\mathfrak{e})+N_1(\mathfrak{e}) \subset \mathfrak{M}$.

Thus $N_{\frac{1}{2}}(\mathfrak{e}) + N_1(\mathfrak{e}) \subset \mathfrak{M} \subset N_1(\mathfrak{e})$, that is $\mathfrak{M} = N_1(\mathfrak{e}) \supset N_{\frac{1}{2}}(\mathfrak{e})$. Now $\mathfrak{a} \varepsilon N_{\frac{1}{2}}(\mathfrak{e})$ has the consequence $\mathfrak{a} \varepsilon N_1(\mathfrak{e})$ and so Lemma D, 2, 1 [applied to the decomposition $0 = 0 + \mathfrak{a} + (-\mathfrak{a})$] gives $\mathfrak{a} = 0$. As $0 \varepsilon N_{\frac{1}{2}}(\mathfrak{e})$, this means $N_{\frac{1}{2}}(\mathfrak{e}) = (0)$. $\mathfrak{M} = N_1(\mathfrak{e})$ we have already proved.

The following result expresses an essential algebraic aspect of $\mathfrak{A}$:

Theorem VII. *$\mathfrak{A}$ is* completely reducible, *that is: to every ideal $\mathfrak{M}$ in $\mathfrak{A}$ one and only one ideal $\mathfrak{N}$ can be found, such that:*

(i) *$\mathfrak{A}$ is the direct sum of $\mathfrak{M}$, $\mathfrak{N}$:*

$$\mathfrak{A} = \mathfrak{M} + \mathfrak{N}.$$

(ii) *$\mathfrak{M}$, $\mathfrak{N}$ are orthogonal: $\mathfrak{a} \varepsilon \mathfrak{M}$, $\mathfrak{b} \varepsilon \mathfrak{N}$ imply $\mathfrak{a}\mathfrak{b} = 0$.*

Proof. Existence of $\mathfrak{N}$: by Lemma D, 6, 1, $\mathfrak{M} = N_1(\mathfrak{e})$, with $N_{\frac{1}{2}}(\mathfrak{e}) = (0)$.

Now

$$\mathfrak{N} = N_0(\mathfrak{e})$$

is an ideal too, as

$$\mathfrak{N} = N_0(\mathfrak{e}) = N_1(1 - \mathfrak{e}), \quad N_{\frac{1}{2}}(1 - \mathfrak{e}) = N_{\frac{1}{2}}(\mathfrak{e}) = (0)$$

(use Lemma D, 5, 2). (i) holds by Lemma D, 2, 1:

$$\mathfrak{A} = N_0(\mathfrak{e}) + N_{\frac{1}{2}}(\mathfrak{e}) + N_1(\mathfrak{e}) = \mathfrak{N} + (0) + \mathfrak{M} = \mathfrak{M} + \mathfrak{N}.$$

(ii) holds by Theorem IV: $\mathfrak{M} = N_1(\mathfrak{e})$ and $\mathfrak{N} = N_0(\mathfrak{e})$ are orthogonal.

Unicity of $\mathfrak{N}$: by (ii) $\mathfrak{b} \varepsilon \mathfrak{N}$ implies $\mathfrak{e}\mathfrak{b} = 0$ (as $\mathfrak{e} \varepsilon N_1(\mathfrak{e}) = \mathfrak{M}$), $\mathfrak{b} \varepsilon N_0(\mathfrak{e})$, so $\mathfrak{N} \subset N_0(\mathfrak{e})$. Now we saw above that

$$\mathfrak{A} = N_0(\mathfrak{e}) + N_{\frac{1}{2}}(\mathfrak{e}) + N_1(\mathfrak{e}) = N_0(\mathfrak{e}) + (0) + \mathfrak{M} = \mathfrak{M} + N_0(\mathfrak{e}).$$

So (i) gives: $\mathfrak{M} + \mathfrak{N} = \mathfrak{A} = \mathfrak{M} + N_0(\mathfrak{e})$, both sums being direct. Thus $\mathfrak{N} \subset N_0(\mathfrak{e})$ necessitates

$$\mathfrak{N} = N_0(\mathfrak{e}).$$

This theorem will play an essential rôle at the formulation of the axiom-group V in Part II.

E. The spectral interval and the absolute value

E, 1. Consider an $\mathfrak{a} \varepsilon \mathfrak{A}$ and choose an m by III_1) with $\frac{1}{2^m} \mathfrak{a} \varepsilon \mathfrak{U}$. Let $\varphi(x)$ be a real function, defined and everywhere continuous for $|x| \leqq 2^m$. Then

$$\overset{\circ}{\varphi}(x) \equiv \varphi(x) - \varphi(0)$$

is such a function too, and besides $\overset{\circ}{\varphi}(0) = 0$. So we can form $\overset{\circ}{\varphi}(((\mathfrak{a})))$ in the sense of Theorem III. Now define $\varphi((((\mathfrak{a))))$ by

$$\varphi((((\mathfrak{a)))) = \varphi(0) \cdot 1 + \overset{\circ}{\varphi}(((\mathfrak{a))). \tag{40}$$

We now check the properties of $\varphi((\mathfrak{a}))$ described in Lemmas C, 4, 3—C, 4, 5, which hold for $\varphi(((\mathfrak{a)))$ too by Theorem III, for $\varphi((((\mathfrak{a))))$.

Ad Lemma C, 4, 2: (i)—(iii) carry over immediately from $\varphi(((\mathfrak{a})))$ to $\varphi((((\mathfrak{a}))))$. (i) holds for any real (not necessarily dyadically rational) ρ considering the definition of $\rho\mathfrak{a}$ in C, 6. (iii) implies (iv) and (iv) implies (v) in literally the same way as there.

Ad Lemma C, 4, 3: more than this is true: whenever $\varphi(((\mathfrak{a})))$ is defined, that is for $\varphi(0)=0$, $\varphi((((\mathfrak{a))))) = \varphi(((\mathfrak{a})))$. This is obvious by (40).

Ad Lemma C, 4, 4: this holds by Theorem III for $\varphi(((\mathfrak{a})))$ and (40) extends it immediately to $\varphi((((\mathfrak{a}))))$.

Ad Lemma C, 4, 5: this is obvious for $\psi(x)\equiv 1$ and $\psi(x)\equiv x$ and so by Lemma C, 4, 2, (i)—(iii) for all polynomials $\psi(x)$. Now by use of the Weierstrass approximation theorem and Lemma C, 4, 4 it extends to all $\psi(x)$ (cf. the restrictions in Theorem III).

Summing up:

Theorem VIII. *If* $\mathfrak{a}\varepsilon\mathfrak{A}$, *then for each real function* $\varphi(x)$, *defined and everywhere continuous for* $|x|\leqslant 2^m$ [$m=0, 1, 2, \ldots$ *being such that* $\frac{1}{2^m}\mathfrak{a}\varepsilon\mathfrak{U}$, *cf.* III_1)], *a* $\varphi((((\mathfrak{a))))}$ *is defined as above.*

These $\varphi((((\mathfrak{a))))}$ *are isomorphic to the* $\varphi(x)$ *themselves: the algebraic side of this is expressed by the equivalents of Lemma C, 4, 2,* (i)—(iii) (replace there $|x|\leqslant 1$ by $|x|\leqslant 2^m$, the ρ in (i) may be any real number) *and Lemma C, 4, 5* (replace there $|\varphi(x)|\leqslant 1$ for $|x|\leqslant 1$ by $|\varphi(x)|\leqslant 2^n$ for $|x|\leqslant 2^m$, the $n=0, 1, 2, \ldots$ being such that $\psi(x)$ is defined and everywhere continuous in $|x|\leqslant 2^n$); *the topological side by the equivalent of Lemma C, 4, 4* (replace there $|x|\leqslant 1$ by $|x|\leqslant 2^m$); *and if* $\varphi(x)\equiv p(x)\,\varepsilon\,\mathcal{P}'$ *then* $p((((\mathfrak{a))))) = p(\mathfrak{a})$. *If* $\varphi(0)=0$, *then* $\varphi((((\mathfrak{a))))) = \varphi(((\mathfrak{a})))$. *Finally* $\varphi(x)\geqslant 0$ *and* $|\varphi(x)|\leqslant 1$ *respectively for* $|x|\leqslant 2^m$ *imply* $\varphi((((\mathfrak{a))))) \varepsilon\mathfrak{D}$ *and* $\varphi((((\mathfrak{a))))) \varepsilon\mathfrak{U}$ *respectively.*

These facts, combined with $\varphi((((\mathfrak{a))))) = 1$ and $\mathfrak{a}$ respectively for $\varphi(x)\equiv 1$ and x respectively, give:

If

$$\varphi(x) = \alpha_0 + \alpha_1 x + \ldots + \alpha_n x^n, \tag{41}$$

the $\alpha_0, \alpha_1, \ldots, \alpha_n$ being arbitrary real numbers, then

$$\varphi((((\mathfrak{a))))) = \alpha_0 1 + \alpha_1 \mathfrak{a} + \ldots + \alpha_n \mathfrak{a}^n.$$

Another relation of some interest is this:

$$\varphi((((\alpha 1)))) = \varphi(\alpha)\cdot 1. \tag{42}$$

This follows from (41) for $\varphi(x)\equiv$ polynomial, and carries over by $\mathcal{T}$-continuity to all $\varphi(x)$.

E, 2. The definition of $\varphi((\mathfrak{a}))$ ($\mathfrak{a}\varepsilon\mathfrak{U}$) in C, 4 makes it obvious that $\varphi((\mathfrak{a}))$ depends only on the function $\varphi(x)$ in $|x|\leqslant 1$. Therefore $\varphi(((\mathfrak{a})))$ ($\frac{1}{2^m}\mathfrak{a}\varepsilon\mathfrak{U}$, cf. C, 5) depends only on the function $\varphi(x)$ in $|x|\leqslant 2^m$, and the same is true for $\varphi((((\mathfrak{a))))) $ ($\frac{1}{2^m}\mathfrak{a}\varepsilon\mathfrak{U}$, cf. E, 1).

Now $|1-x|=1-x$ for $|x|\leqslant 1$, therefore if we put $\varphi(x)=|1-x|$, then for $\mathfrak{a}\varepsilon\mathfrak{U}$ $\varphi((((\mathfrak{a}))))=1-\mathfrak{a}$. Now put $\psi(x)=+\sqrt{|1-x|}$, as $(\psi(x))^2=\varphi(x)$, so $(\psi((((\mathfrak{a}))))\,)^2=\varphi((((\mathfrak{a}))))=1-\mathfrak{a}$ (by Theorem VIII). Therefore we have:

$$\text{If } \mathfrak{a}\varepsilon\mathfrak{U}, \text{ then a } \mathfrak{b} \text{ with } \mathfrak{b}^2=1-\mathfrak{a} \text{ exists.} \tag{43}$$

We now prove:

Lemma E, 2, 1. $\mathfrak{a}\varepsilon\mathfrak{D}$ *if and only if a* $\mathfrak{b}$ *with* $\mathfrak{b}^2=\mathfrak{a}$ *exists.*

Proof. The condition is sufficient by (20), so we must only consider its necessity.

Assume first $\mathfrak{a}\varepsilon\mathfrak{D}\cdot\mathfrak{U}$. Then (43) gives $1-\mathfrak{a}=\mathfrak{b}_1^2$, so $1-\mathfrak{a}\varepsilon\mathfrak{D}$ by (20). So $\mathfrak{a}$, $1-\mathfrak{a}\varepsilon\mathfrak{D}$, and $(1-\mathfrak{a})+\mathfrak{a}=1\varepsilon\mathfrak{U}$ because 1 is idempotent and by (27) in C, 2. So III_3) gives $1-\mathfrak{a}\varepsilon\mathfrak{U}$, and now by (43) $\mathfrak{a}=1-(1-\mathfrak{a})=\mathfrak{b}_2^2$.

Assume now $\mathfrak{a}\varepsilon\mathfrak{D}$ only. Choose $m=0, 1, 2, \ldots$ with $\frac{1}{2^m}\mathfrak{a}\varepsilon\mathfrak{U}$ [by III_1)]. As $\frac{1}{2^m}\mathfrak{a}\varepsilon\mathfrak{D}$ too, so $\frac{1}{2^m}\mathfrak{a}\varepsilon\mathfrak{D}\cdot\mathfrak{U}$. Thus a $\mathfrak{b}_3$ with $\mathfrak{b}_3^2=\frac{1}{2^m}\mathfrak{a}$ exists. Then $(\sqrt{2^m}\mathfrak{b}_3)^2=2^m\mathfrak{b}_3^2=$ $=2^m\cdot\frac{1}{2^m}\mathfrak{a}=\mathfrak{a}$. So $\mathfrak{b}=\sqrt{2^m}\mathfrak{b}_3$ fulfils $\mathfrak{b}^2=\mathfrak{a}$.

Lemma E, 2, 2. *If* $\mathfrak{a}\varepsilon\mathfrak{D}$, *then* $\varphi((((\mathfrak{a))))}$ *depends only on* ($\mathfrak{a}$ *and*) *the function* $\varphi(x)$ *in* $x\geqslant 0$.

Proof. Put $\mathfrak{a}=\mathfrak{b}^2$ by Lemma E, 2, 1. Put $\psi(x)\equiv\varphi(x^2)$. Then $\psi(\mathfrak{b})=\varphi(\mathfrak{b}^2)=$ $=\varphi(\mathfrak{a})$ (by Theorem VIII). Thus $\varphi((((\mathfrak{a))))}$ depends only on $\mathfrak{b}$, that is on $\mathfrak{a}$, and the function $\psi(x)\equiv\varphi(x^2)$, that is $\varphi(x)$ in $x\geqslant 0$.

Lemma E, 2, 3. $\mathfrak{a}\rightarrow\mathfrak{a}^2$ *is a one-to-one mapping of* $\mathfrak{D}$ *on itself. Its inverse is* $\mathfrak{a}\rightarrow\psi((((\mathfrak{a))))}$, *where* $\psi(x)\equiv+\sqrt{|x|}$.

Proof. $\mathfrak{a}\rightarrow\mathfrak{a}^2$ clearly maps $\mathfrak{D}$ on part of itself. If $\mathfrak{a}\varepsilon\mathfrak{D}$, then as $|x|\equiv x$ for $x\geqslant 0$, so Lemma E, 2, 2 gives for $\varphi(x)\equiv|x|$

$$\varphi((((\mathfrak{a))))}=\mathfrak{a}.$$

Now put $\psi(x)\equiv+\sqrt{|x|}$. As $\psi(x)\geqslant 0$ everywhere, so $\psi(\mathfrak{a})\varepsilon\mathfrak{D}$, and as $(\psi(x))^2\equiv\varphi(x)$, so

$$(\psi((((\mathfrak{a)))))^2=\varphi((((\mathfrak{a))))}=\mathfrak{a}.$$

(Use Theorem VIII.) So $\mathfrak{b}=\psi((((\mathfrak{a))))}$ fulfils $\mathfrak{b}\varepsilon\mathfrak{D}$, $\mathfrak{b}^2=\mathfrak{a}$. Thus $\mathfrak{a}\rightarrow\mathfrak{a}^2$ maps $\mathfrak{D}$ exactly on itself, and

$$\mathfrak{b}^2=\mathfrak{a}$$

has the solution

$$\mathfrak{b}=\psi((((\mathfrak{a))))}.$$

It remains to prove that $\mathfrak{a}\rightarrow\mathfrak{a}^2$ is one-to-one in $\mathfrak{D}$, that is, that $\mathfrak{a}$, $\mathfrak{b}\varepsilon\mathfrak{D}$, $\mathfrak{a}^2=\mathfrak{b}^2$ imply $\mathfrak{a}=\mathfrak{b}$. Now $\psi(x^2)\equiv\varphi(x)$, so $\psi((((\mathfrak{a}^2))))=\varphi((((\mathfrak{a))))}$ (by Theorem VIII), and for $\mathfrak{a}\varepsilon\mathfrak{D}$ this is $=\mathfrak{a}$. Thus an $\mathfrak{a}\varepsilon\mathfrak{D}$ is uniquely determined by its $\mathfrak{a}^2$:

$$\mathfrak{a}=\psi((((\mathfrak{a}^2)))).$$

Lemma E, 2, 4. $\mathfrak{a}\varepsilon\mathfrak{U}$ *is equivalent to* $1-\mathfrak{a}^2\varepsilon\mathfrak{D}$, *as well as to the two statements* $1\pm\mathfrak{a}\varepsilon\mathfrak{D}$.

Proof. $\mathfrak{a}\varepsilon\mathfrak{U}$ implies $1-\mathfrak{a}^2\varepsilon\mathfrak{D}$:

$$\varphi(x)\equiv 1-x^2\geqslant 0$$

for $|x|\leqslant 1$, so $\mathfrak{a}\varepsilon\mathfrak{U}$ implies by Theorem VII $\varphi((((\mathfrak{a}))))=1-\mathfrak{a}^2\varepsilon\mathfrak{U}$.

$1-\mathfrak{a}^2\varepsilon\mathfrak{D}$ implies $1\pm\mathfrak{a}\varepsilon\mathfrak{D}$: as $(1\pm\mathfrak{a})^2\varepsilon\mathfrak{D}$, so III_1) and III_2) give

$$\frac{1}{2}((1-\mathfrak{a}^2)+(1\pm\mathfrak{a})^2)=1\pm\mathfrak{a}\varepsilon\mathfrak{D}.$$

$1\pm\mathfrak{a}\varepsilon\mathfrak{D}$ implies $\mathfrak{a}\varepsilon\mathfrak{U}$: by Lemma E, 2, 3 then

$$1\pm\mathfrak{a}=(\psi((((1\pm\mathfrak{a))))))^2.$$

So

$$1-\mathfrak{a}^2=(1+\mathfrak{a})(1-\mathfrak{a})=(\psi((((1+\mathfrak{a))))))^2(\psi((((1-\mathfrak{a))))))^2.$$

Now

$$(p(\mathfrak{a}))^2\pm\mathfrak{a}(p(\mathfrak{a}))^2=(1\pm\mathfrak{a})(p(\mathfrak{a}))^2=(\psi((((1\pm\mathfrak{a))))))^2(p((((\mathfrak{a))))))^2=$$

$$=(r((((\mathfrak{a))))))^2\varepsilon\mathfrak{D}$$

with

$$r(x)=\psi(1\pm x)\,p(x),$$

and

$$(p(\mathfrak{a}))^2-\mathfrak{a}^2(p(\mathfrak{a}))^2=(1-\mathfrak{a}^2)(p(\mathfrak{a}))^2=$$
$$=(\psi((((1+\mathfrak{a))))))^2(\psi((((1-\mathfrak{a))))))^2(p((((\mathfrak{a))))))^2=(s((((\mathfrak{a))))))^2\varepsilon\mathfrak{D}$$

with $s(x)\equiv\psi(1+x)\,\psi(1-x)\,p(x)$. (Use Theorem VIII.) So III_5) gives $\mathfrak{a}\varepsilon\mathfrak{U}$.

We repeat some of these results:

Theorem IX. *$\mathfrak{a}\varepsilon\mathfrak{D}$ if and only if a $\mathfrak{b}$ with $\mathfrak{b}^2=\mathfrak{a}$ exists. $\mathfrak{a}\varepsilon\mathfrak{U}$ if and only if $1-\mathfrak{a}^2\varepsilon\mathfrak{D}$, or just as well, if and only if $1\pm\mathfrak{a}\varepsilon\mathfrak{D}$.*

E, 3. We can now start determining the „spectral interval" of any $\mathfrak{a}\varepsilon\mathfrak{A}$.

Lemma E, 3, 1. *$\alpha 1\varepsilon\mathfrak{D}$ (α a real number) if and only if $\alpha\geqslant 0$.*

Proof. If $\alpha\geqslant 0$, then $\sqrt{\alpha}$ can be formed, and

$$(\sqrt{\alpha}\,1)^2=(\sqrt{\alpha})^2\,1^2=\alpha\,1,$$

so $\alpha 1\varepsilon\mathfrak{D}$. If $\alpha<0$, then $\alpha 1\varepsilon\mathfrak{D}$ would imply this: $-\alpha>0$, so $(-\alpha)\,1=-\alpha 1\varepsilon\mathfrak{D}$ too, now apply IV_4) with

$$\mathfrak{a}_1=\mathfrak{a}_2=\ldots=\alpha 1,\quad \mathfrak{b}_1=\mathfrak{b}_2=\ldots=-\alpha 1\quad (\mathfrak{a}_n+\mathfrak{b}_n=\alpha 1-\alpha 1=0),$$

then

$$\mathcal{T}\text{-}\lim_{n\to\infty}\mathfrak{a}_n=0,\quad \alpha 1=0.$$

Thus $1=\frac{1}{\alpha}\cdot\alpha 1=0$, contradicting $1\neq 0$ which follows from A, 1. (1 is the unit of $\mathfrak{A}\neq\Theta$, (0).) Thus $\alpha<0$ excludes $\alpha 1\varepsilon\mathfrak{D}$.

Lemma E, 3, 2. *For each $\mathfrak{a}\varepsilon\mathfrak{D}$ there exists a unique real number α_1 such that $\mathfrak{a}-\rho 1\varepsilon\mathfrak{D}$ is equivalent to $\rho\leqslant\alpha$; and a unique real number β, such that $-(\mathfrak{a}-\rho 1)\varepsilon\mathfrak{D}$ is equivalent to $\rho\geqslant\beta$.*

Proof. The first statement implies the second one by replacement of $\mathfrak{a}$, ρ by $-\mathfrak{a}$, $-\rho$. So it suffices to consider the first one.

Consider the set P of all ρ with $\mathfrak{a}-\rho 1\varepsilon\mathfrak{D}$. $\rho\varepsilon\mathrm{P}$, $\sigma\leqslant\rho$ give:

$$\mathfrak{a}-\rho 1\varepsilon\mathfrak{D},\quad (\rho-\sigma)\,1\varepsilon\mathfrak{D}$$

(by Lemma E, 3, 1), so

$$\mathfrak{a} - \sigma 1 = (\mathfrak{a} - \rho 1) + (\rho - \sigma) 1 \varepsilon \mathfrak{D}$$

[by III_2)], $\sigma \varepsilon \mathrm{P}$. Considering (30) in C, 6 and IV_5), P is a closed set.

Choose m by III_1) with $\frac{1}{2^m} \mathfrak{a} \varepsilon \mathfrak{U}$. Then Theorem VIII gives $1 \pm \frac{1}{2^m} \mathfrak{a} \varepsilon \mathfrak{D}$, and so

$$\pm \mathfrak{a} + 2^m \cdot 1 = 2^m \left(1 \pm \frac{1}{2^m} \mathfrak{a} \right) \varepsilon \mathfrak{D}.$$

$\mathfrak{a} + 2^m \cdot 1 \varepsilon \mathfrak{D}$ means that $-2^m \varepsilon \mathrm{P}$. $-\mathfrak{a} + 2^m \cdot 1 \varepsilon \mathfrak{D}$ excludes $2^m + 1 \varepsilon \mathrm{P}$: if this were the case we would have $\mathfrak{a} - (2^m + 1) \cdot 1 \varepsilon \mathfrak{D}$, $-\mathfrak{a} + 2^m \cdot 1 \varepsilon \mathfrak{D}$, so $-1 \varepsilon \mathfrak{D}$ [by III_1)], thus contradicting Lemma E, 3, 1.

So P is not empty, nor is its complement empty; $\rho \varepsilon \mathrm{P}$, $\sigma \leqslant \rho$ imply $\sigma \varepsilon \mathrm{P}$; P is a closed set. Therefore P must be a set of this form: $\rho \varepsilon \mathrm{P}$ is equivalent to $\rho \leqslant \alpha$, for a suitable α. That is: $\mathfrak{a} - \rho 1 \varepsilon \mathrm{P}$ is equivalent to $\rho \leqslant \alpha$.

We will denote the numbers α and β obtained in Lemma E, 3, 2 by $\|\mathfrak{a}>$ and $<\mathfrak{a}\|$. We define the absolute value of $\mathfrak{a}$, $\|\mathfrak{a}\|$, by

$$\|\mathfrak{a}\| = \max(-\|\mathfrak{a}>, <\mathfrak{a}\|). \tag{44}$$

Finally the set of real numbers

$$\mathcal{J}(\mathfrak{a}): \qquad \|\mathfrak{a}> \leqslant x \leqslant <\mathfrak{a}\|, \tag{45}$$

is the spectral interval of $\mathfrak{a}$.

Lemma E, 3, 3:

(i)

$$\|1> = <1\| = 1, \quad \|0> = <0\| = 0.$$

(ii)

$$\|\mathfrak{a}> = -<-\mathfrak{a}\|.$$

(iii)

$$\|\mathfrak{a}> + \|\mathfrak{b}> \leqslant \|\mathfrak{a}+\mathfrak{b}> \leqslant \begin{Bmatrix} \|\mathfrak{a}> + <\mathfrak{b}\| \\ <\mathfrak{a}\| + \|\mathfrak{b}> \end{Bmatrix} \leqslant <\mathfrak{a}+\mathfrak{b}\| \leqslant <\mathfrak{a}\| + <\mathfrak{b}\|.$$

(iv)

$$\|\rho\mathfrak{a}> = \rho\|\mathfrak{a}>, \quad <\rho\mathfrak{a}\| = \rho<\mathfrak{a}\| \quad \text{for } \rho \geqslant 0.$$

(v)

$$\|\mathfrak{a} + \rho 1> = \|\mathfrak{a}> + \rho, \quad <\mathfrak{a} + \rho 1\| = <\mathfrak{a}\| + \rho.$$

(vi)

$$\|\mathfrak{a}> \leqslant <\mathfrak{a}\|.$$

Proof. Ad (i): obvious by Lemma E, 2, 1.

Ad (ii): obvious as $-(\mathfrak{a} - \rho 1) = (-\mathfrak{a}) - (-\rho) 1$.

Ad (iii): the two first $\leqslant$ imply the two last ones by replacing $\mathfrak{a}$, $\mathfrak{b}$ by $-\mathfrak{a}$, $-\mathfrak{b}$ using (ii). So it suffices to consider the former ones. Furthermore the upper second implies the lower one by interchangeing $\mathfrak{a}$ and $\mathfrak{b}$. So we may omit the latter one. Then we need to prove

$$\|\mathfrak{a}> + \|\mathfrak{b}> \leqslant \|\mathfrak{a}+\mathfrak{b}> \leqslant \|\mathfrak{a}> + <\mathfrak{b}\|$$

only.

Put $\|\mathfrak{a}> = \alpha$, $\|\mathfrak{b}> = \beta$, then $\mathfrak{a} - \alpha 1$, $\mathfrak{b} - \beta 1 \varepsilon \mathfrak{D}$, so by III_2)

$$(\mathfrak{a} + \mathfrak{b}) - (\alpha + \beta) 1 = (\mathfrak{a} - \alpha 1) + (\mathfrak{b} - \beta 1) \varepsilon \mathfrak{D}, \quad \alpha + \beta \leqslant \|\mathfrak{a}+\mathfrak{b}>,$$

proving the first relation. The second relation may be written

$$\|\mathfrak{a}+\mathfrak{b}> - <\mathfrak{b}\| \leqslant \|\mathfrak{a}>$$

or by (ii)

$$\|\mathfrak{a}+\mathfrak{b}> + \|-\mathfrak{b}> \leqslant \|\mathfrak{a}>.$$

Thus it results from the first one by replacing $\mathfrak{a}$, $\mathfrak{b}$ by $\mathfrak{a}+\mathfrak{b}$, $-\mathfrak{b}$.

Ad (iv): the second statement follows from the first one by replacing $\mathfrak{a}$ by $-\mathfrak{a}$, using (ii). So it suffices to consider the later one. For $\rho=0$ it follows from (i), so we may assume $\rho>0$.

Put $\|\mathfrak{a}>=\alpha$ then

$$\mathfrak{a}-\alpha 1 \varepsilon \mathfrak{D}, \quad \rho\mathfrak{a}-\rho\alpha 1=\rho(\mathfrak{a}-\alpha 1)\varepsilon\mathfrak{D}, \quad \rho\alpha \leqslant \|\rho\mathfrak{a}>.$$

So

$$\rho\|\mathfrak{a}> \leqslant \|\rho\mathfrak{a}>.$$

Replacing ρ, $\mathfrak{a}$ by $\frac{1}{\rho}$, $\rho\mathfrak{a}$ gives $\rho\|\mathfrak{a}> \geqslant \|\rho\mathfrak{a}>$, thus

$$\rho\|\mathfrak{a}>=\|\rho\mathfrak{a}>.$$

Ad (v): obvious.

Ad (vi): apply (iii) to $\mathfrak{a}$, $-\mathfrak{a}$ and use (i), (ii):

$$<\mathfrak{a}\|-\|\mathfrak{a}>=<\mathfrak{a}\|+<-\mathfrak{a}\| \geqslant <\mathfrak{a}+(-\mathfrak{a})\|=<0\|=0,$$

$$\|\mathfrak{a}> \leqslant <\mathfrak{a}\|.$$

Theorem X. *$\mathfrak{a}\varepsilon\mathfrak{D}$ is equivalent to $\|\mathfrak{a}> \geqslant 0$, that is to $\mathfrak{J}(\mathfrak{a})\subset D$, D being the set of all $x \geqslant 0$. $\mathfrak{a}\varepsilon\mathfrak{U}$ is equivalent to $-1 \leqslant \|\mathfrak{a}> \leqslant <\mathfrak{a}\| \leqslant 1$, that is to $\|\mathfrak{a}\| \leqslant 1$, that is to $\mathfrak{J}(\mathfrak{a})\subset U$, U being the set of all $-1 \leqslant x \leqslant 1$.*

Proof. That the two formulations given as equivalents of $\mathfrak{a}\varepsilon\mathfrak{D}$ are equivalent to each other, and that the same is true for the three given for $\mathfrak{a}\varepsilon\mathfrak{U}$ is obvious. Thus we consider the first formulation in each case.

$\mathfrak{a}\varepsilon\mathfrak{D}$ means $\mathfrak{a}-0\cdot 1\varepsilon\mathfrak{D}$, that is $\|\mathfrak{a}> \geqslant 0$. $\mathfrak{a}\varepsilon\mathfrak{U}$ means $1\pm\mathfrak{a}\varepsilon\mathfrak{D}$ by Theorem IX, and so $\|1\pm\mathfrak{a}> \geqslant 0$ by our above result. Now Lemma E, 3, 3, (iii), (v) give

$$\|1+\mathfrak{a}>=1+\|\mathfrak{a}>, \quad \|1-\mathfrak{a}>=1-<\mathfrak{a}\|.$$

So we obtain $-1 \leqslant \|\mathfrak{a}>$, $<\mathfrak{a}\| \leqslant 1$, and by Lemma E, 3, 3, (vi) this means

$$-1 \leqslant \|\mathfrak{a}> \leqslant <\mathfrak{a}\| \leqslant 1.$$

Corollary. If $\mathfrak{b}\varepsilon\mathfrak{D}$, then

$$\|\mathfrak{a}+\mathfrak{b}> \geqslant \|\mathfrak{a}>, \quad <\mathfrak{a}+\mathfrak{b}\| \geqslant <\mathfrak{a}\|.$$

Proof. This follows from Lemma E, 3, 3, (iii), because we have $\|\mathfrak{b}> \geqslant 0$ by Theorem X.

Theorem XI. *For all real ρ, σ $\mathfrak{J}(\rho\mathfrak{a}+\sigma 1)$ is the $x \longrightarrow \rho x+\sigma$-image of the set $\mathfrak{J}(\mathfrak{a})$.*

Proof. For $x \longrightarrow x+\sigma$ $(\rho=1)$ this follows from Lemma E, 3, 3, (v), so it suffices to consider $x \longrightarrow \rho x$ $(\sigma=0)$. For $x \longrightarrow -x$ $(\rho=-1, \sigma=0)$ it follows from (ii) eod., so we may assume $\rho \geqslant 0$ $(\sigma=0)$. This case is settled by (iv) eod.

E, 4. The „absolute value" $\|\mathfrak{a}\|$ will be studied next, and the „uniform topology" $\mathfrak{U}$ which we will define with its help.

Lemma E, 4, 1:

(i) $\|1\|=1$, $\|0\|=0$.

(ii) $\|\mathfrak{a}\|0$ *if* $\mathfrak{a}\neq 0$.

(iii) $\|\mathfrak{a}+\mathfrak{b}\|\leqslant\|\mathfrak{a}\|+\|\mathfrak{b}\|$.

(iv) $\|\rho\cdot\mathfrak{a}\|=|\rho|\cdot\|\mathfrak{a}\|$.

(v) $\lim\limits_{i\to\infty}\|\mathfrak{a}_i\|=0$ *implies* $\mathcal{T}\text{-}\lim\limits_{i\to\infty}\mathfrak{a}_i=0$.

Proof. Ad (i): obvious by Lemma E, 3, 3, (i).

Ad (ii): if $\|\mathfrak{a}\|=\max(-\|\mathfrak{a}>, <\mathfrak{a}\|)$ were $\leqslant 0$ this would necessitate $\|\mathfrak{a}>\geqslant 0$, $<\mathfrak{a}\|\leqslant 0$. Now by Lemma E, 3, 3, (vi), $\|\mathfrak{a}>\leqslant<\mathfrak{a}\|$, so we have $\|\mathfrak{a}>=<\mathfrak{a}\|=0$. This gives $\mathfrak{a}$, $-\mathfrak{a}\varepsilon\mathfrak{D}$. Now apply IV$_4$) to $\mathfrak{a}_1=\mathfrak{a}_2=\ldots=\mathfrak{a}$, $\mathfrak{b}_1=\mathfrak{b}_2=\ldots=-\mathfrak{a}$ (so $\mathfrak{a}_n+\mathfrak{b}_n=0$) then $\mathcal{T}\text{-}\lim\limits_{n\to\infty}\mathfrak{a}_n=0$, $\mathfrak{a}=0$. So $\mathfrak{a}\neq 0$ implies $\|\mathfrak{a}\|>0$.

Ad (iii): immediate by Lemma E, 3, 3, (iii).

Ad (iv): Lemma E, 3, 3, (ii) gives this for $\rho=-1$, so it suffices to consider the $\rho\geqslant 0$. But for these it follows immediately from (iv) eod.

Ad (v): put $\|\mathfrak{a}_i\|=\alpha_i$, then $\lim\limits_{i\to\infty}\alpha_i=0$ and so

$$\mathcal{T}\text{-}\lim_{i\to\infty}(2\alpha_i 1)=0.$$

Now by definition

$$-\alpha_i\leqslant\|\mathfrak{a}_i>\leqslant<\mathfrak{a}_i\|\leqslant\alpha_i,$$

so

$$\alpha_i 1\pm\mathfrak{a}\varepsilon\mathfrak{D}.$$

As

$$(\alpha_i 1+\mathfrak{a})+(\alpha_i 1-\mathfrak{a})=2\alpha_i 1,$$

so IV$_4$) applies (replace its $\mathfrak{a}_i$, $\mathfrak{b}_i$ by $\alpha_i 1+\mathfrak{a}$, $\alpha_i 1-\mathfrak{a}$ respectively, $i=1, 2, \ldots$) giving

$$\mathcal{T}\text{-}\lim_{i\to\infty}(\alpha_i 1+\mathfrak{a}_i)=0.$$

As $\mathcal{T}\text{-}\lim\limits_{i\to\infty}\alpha_i 1=0$ so we have

$$\mathcal{T}\text{-}\lim_{i\to\infty}\mathfrak{a}_i=\mathcal{T}\text{-}\lim_{i\to\infty}(\alpha_i 1+\mathfrak{a}_i)-\mathcal{T}\text{-}\lim_{i\to\infty}(\alpha_i 1)=0.$$

Theorem XII. $\|\mathfrak{a}\|$ *is a „linear distance" in* $\mathfrak{A}$, *if* $\mathfrak{A}$ *is looked at as a „linear space"*. (Cf. for instance F. Hausdorff, Mengenlehre, W. de Gruyter & Co., Berlin, 1935, p. 94—97.) *The metric topology* $\mathcal{U}$ *which originates from this distance* $\|\mathfrak{a}\|$ *(the distance of* $\mathfrak{a}$, $\mathfrak{b}$ *being by definition* $\|\mathfrak{a}-\mathfrak{b}\|$*) is „stronger" than the topology* $\mathcal{T}$*; that is: if* $\mathcal{U}\text{-}\lim\limits_{i\to\infty}\mathfrak{a}_i$ *exists, then* $\mathcal{T}\text{-}\lim\limits_{i\to\infty}\mathfrak{a}_i$ *exists too, and coincides with it.*

Proof. $\|\mathfrak{a}\|$ is a „linear distance" in $\mathfrak{A}$ by Lemma E, 4, 1, (i)—(iv), and its topology $\mathcal{U}$ is „stronger" than $\mathcal{T}$ by (v) eod.

Some further facts about $\|\mathfrak{a}\|$, etc.:

Lemma E, 4, 2. $\|\mathfrak{a}\mathfrak{b}\|\leqslant\|\mathfrak{a}\|\cdot\|\mathfrak{b}\|$.

Proof. $\|\mathfrak{a}\|\leqslant 1$ means $\mathfrak{a}\varepsilon\mathfrak{U}$ (by Theorem IX). Thus $1-\mathfrak{a}^2\varepsilon\mathfrak{D}$, and so $<\mathfrak{a}^2\|\leqslant 1$ (by definition). On the other hand $\mathfrak{a}^2\varepsilon\mathfrak{D}$, $\|\mathfrak{a}^2>\geqslant 0$ (by Theorem X), so

$$0\leqslant\|\mathfrak{a}>\leqslant<\mathfrak{a}\|\leqslant 1.$$

Thus

$$\mathfrak{a}^2|\leqslant 1.$$

If $\mathfrak{a}$ is arbitrary and $\rho > 0$, then application of the above result to $\frac{1}{\rho}\mathfrak{a}$ shows: $\|\mathfrak{a}\| \leqslant \rho$ implies

$$\left\|\frac{1}{\rho}\mathfrak{a}\right\| = \frac{1}{\rho}\|\mathfrak{a}\| \leqslant 1, \quad \left\|\left(\frac{1}{\rho}\mathfrak{a}\right)^2\right\| \leqslant 1,$$

but

$$\left\|\left(\frac{1}{\rho}\mathfrak{a}\right)^2\right\| = \left\|\frac{1}{\rho^2}\mathfrak{a}^2\right\| = \frac{1}{\rho^2}\|\mathfrak{a}^2\|,$$

so

$$\|\mathfrak{a}^2\| \leqslant \rho^2.$$

Thus no $\rho > 0$ with $\|\mathfrak{a}\| \leqslant \rho < +\sqrt{\|\mathfrak{a}^2\|}$ can exist, therefore

$$\|\mathfrak{a}\| \geqslant +\sqrt{\|\mathfrak{a}^2\|}, \quad \|\mathfrak{a}^2\| \leqslant \|\mathfrak{a}\|^2.$$

Consider now two arbitrary $\mathfrak{a}$, $\mathfrak{b}$. As

$$\left(\frac{1}{2}\mathfrak{a}^2 + \frac{1}{2}\mathfrak{b}^2\right) - \mathfrak{a}\mathfrak{b} = \frac{1}{2}(\mathfrak{a} - \mathfrak{b})^2 \in \mathfrak{D},$$

so the Corollary to Theorem X gives

$$<\mathfrak{a}\mathfrak{b}\| \leqslant <\frac{1}{2}\mathfrak{a}^2 + \frac{1}{2}\mathfrak{b}^2\| \leqslant \|\frac{1}{2}\mathfrak{a}^2 + \frac{1}{2}\mathfrak{b}^2\| \leqslant \frac{1}{2}\|\mathfrak{a}^2\| +$$
$$+ \frac{1}{2}\|\mathfrak{b}^2\| \leqslant \frac{1}{2}\|\mathfrak{a}\|^2 + \frac{1}{2}\|\mathfrak{b}\|^2.$$

Thus

$$<\mathfrak{a}\mathfrak{b}\| \leqslant \frac{1}{2}\|\mathfrak{a}\|^2 + \frac{1}{2}\|\mathfrak{b}\|^2.$$

Replacing $\mathfrak{a}$, $\mathfrak{b}$ by $-\mathfrak{a}$, $\mathfrak{b}$ gives

$$-\|\mathfrak{a}\mathfrak{b}> = <-\mathfrak{a}\mathfrak{b}\| \leqslant \frac{1}{2}\|\mathfrak{a}\|^2 + \frac{1}{2}\|\mathfrak{b}\|^2.$$

Thus

$$\|\mathfrak{a}\mathfrak{b}\| \leqslant \frac{1}{2}\|\mathfrak{a}\|^2 + \frac{1}{2}\|\mathfrak{b}\|^2.$$

Replace now $\mathfrak{a}$, $\mathfrak{b}$ by $\rho\mathfrak{a}$, $\frac{1}{\rho}\mathfrak{b}$ for some $\rho > 0$. As

$$(\rho\mathfrak{a}) \cdot \left(\frac{1}{\rho}\mathfrak{b}\right) = \left(\rho\frac{1}{\rho}\right) \cdot \mathfrak{a}\mathfrak{b} = 1 \cdot \mathfrak{a}\mathfrak{b} = \mathfrak{a}\mathfrak{b},$$

so

$$\|\mathfrak{a}\mathfrak{b}\| \leqslant \frac{1}{2}\rho^2\|\mathfrak{a}\|^2 + \frac{1}{2\rho^2}\|\mathfrak{b}\|^2.$$

If $\mathfrak{a}, \mathfrak{b} \neq 0$ then $\|\mathfrak{a}\|, \|\mathfrak{b}\| > 0$, and so $+\sqrt{\frac{\|\mathfrak{b}\|}{\|\mathfrak{a}\|}}$ is finite and > 0, put $\rho = +\sqrt{\frac{\|\mathfrak{b}\|}{\|\mathfrak{a}\|}}$. Thus

$$\|\mathfrak{a}\mathfrak{b}\| \leqslant \|\mathfrak{a}\| \cdot \|\mathfrak{b}\|$$

tains. This is a fortiori true if $\mathfrak{a} = 0$ or $\mathfrak{b} = 0$, because then both sides vanish.

Lemma E, 4, 3. $\mathfrak{a} + \mathfrak{b}$, $\rho\mathfrak{a}$, $\mathfrak{a}\mathfrak{b}$, $\|\mathfrak{a}\|$, $\|\mathfrak{a}>$, $<\mathfrak{a}\|$ *are $\mathfrak{U}$-continuous functions of all their variables.*

Proof. Ad $\mathfrak{a} + \mathfrak{b}$:

$$\|(\mathfrak{a}' + \mathfrak{b}') - (\mathfrak{a} + \mathfrak{b})\| = \|(\mathfrak{a}' - \mathfrak{a}) + (\mathfrak{b}' - \mathfrak{b})\| \leqslant \|\mathfrak{a}' - \mathfrak{a}\| + \|\mathfrak{b}' - \mathfrak{b}\|$$

proves this.

Ad $\rho\mathfrak{a}$: as $\mathfrak{a}+\mathfrak{b}$ is continuous, we may restrict ourselves to neighbourhoods of $\rho=0$, $\mathfrak{a}$ arbitrary, or ρ arbitrary, $\mathfrak{a}=0$. Thus at any rate $\rho\mathfrak{a}=0$. Now

$$\|\rho'\mathfrak{a}'\|=|\rho'|\cdot\|\mathfrak{a}'\|\begin{cases}\leqslant|\rho'|\cdot(\|\mathfrak{a}\|+\|\mathfrak{a}'-\mathfrak{a}\|),\\ \leqslant(|\rho|+|\rho'-\rho|)\cdot\|\mathfrak{a}'\|,\end{cases}$$

proves the statement.

Ad $\mathfrak{a}\mathfrak{b}$: as $\mathfrak{a}+\mathfrak{b}$ is continuous, we may restrict ourselves to the neighbourhoods of $\mathfrak{a}=0$, $\mathfrak{b}$ arbitrary or $\mathfrak{a}$ arbitrary, $\mathfrak{b}=0$. By symmetry it suffices to consider the first case. Now $\mathfrak{a}\mathfrak{b}=0$, and so

$$\|\mathfrak{a}'\mathfrak{b}'\|\leqslant\|\mathfrak{a}'\|\cdot\|\mathfrak{b}'\|\leqslant\|\mathfrak{a}'\|\cdot(\|\mathfrak{b}\|+\|\mathfrak{b}'-\mathfrak{b}\|)$$

proves the statement.

Ad $\|\mathfrak{a}>$: $\|\mathfrak{a}>-\|\mathfrak{b}>=\|\mathfrak{a}>+<-\mathfrak{b}\|\leqslant<\mathfrak{a}-\mathfrak{b}\|\leqslant\|\mathfrak{a}-\mathfrak{b}\|$, similarly

$$\|\mathfrak{b}>-\|\mathfrak{a}>\leqslant\|\mathfrak{b}-\mathfrak{a}\|=\|\mathfrak{a}-\mathfrak{b}\|,$$

so

$$|\|\mathfrak{a}>-\|\mathfrak{b}>|\leqslant\|\mathfrak{a}-\mathfrak{b}\|,$$

proving this statement.

Ad $<\mathfrak{a}\|$, $\|\mathfrak{a}\|$: $<\mathfrak{a}\|=-\|-\mathfrak{a}>$, $\|\mathfrak{a}\|=\max\,(-\|\mathfrak{a}>,\ <\mathfrak{a}\|)$.

Lemma E, 4, 4:

$$\|\varphi((((\mathfrak{a}))))>\geqslant\min_{x\varepsilon\mathfrak{J}(\mathfrak{a})}\varphi(x),$$

$$<\varphi((((\mathfrak{a))))\|\leqslant\max_{x\varepsilon\mathfrak{J}(\mathfrak{a})}\varphi(x),$$

$$\|\varphi((((\mathfrak{a))))\|\leqslant\max_{x\varepsilon\mathfrak{J}(\mathfrak{a})}|\varphi(x)|.$$

Proof. The third statement results by combining the two first ones; the second follows from the first by replacing $\varphi(x)$ by $-\varphi(x)$ [use Lemma E, 3, 3, (ii)]; so we need to consider the first one only. This could be formulated as follows:

$$\|\varphi((((\mathfrak{a))))>\geqslant\rho,\ \text{if}\ \varphi(x)\geqslant\rho\ \text{for}\ x\varepsilon\mathfrak{J}(\mathfrak{a}).$$

Replacing $\varphi(x)$ by $\varphi(x)-\rho$ [use Lemma E, 3, 3, (v)] we see that we may assume $\rho=0$. Then the statement becomes (use Theorem X):

$$\varphi((((\mathfrak{a))))\varepsilon\mathfrak{D},\ \text{if}\ \varphi(x)\geqslant 0\ \text{for}\ x\varepsilon\mathfrak{J}(\mathfrak{a}).$$

We can replace $\varphi(x)$, $\mathfrak{a}$ by $\varphi\left(\frac{1}{\rho}x-\frac{\sigma}{\rho}\right)$, $\rho\mathfrak{a}+\sigma\mathfrak{1}$ for any ρ, σ with $\rho\neq 0$. (Use Theorem VIII.) Then $\mathfrak{J}(\mathfrak{a})$ is transformed into its $x\to\rho x+\sigma$-image (by Theorem XI). Thus if $\mathfrak{J}(\mathfrak{a})$ is not a single point, we can make it coincide with the interval $-1\leqslant x\leqslant 1$ and if it is a single point, with the one-point-set $x=0$. In other words: $\mathfrak{J}(\mathfrak{a})$ may be assumed to be one of these two sets.

If $\mathfrak{J}(\mathfrak{a})$ is $-1\leqslant x\leqslant 1$, then $\mathfrak{a}\varepsilon\mathfrak{U}$ (by Theorem X) and $\varphi(x)\geqslant 0$ for $|x|\leqslant 1$, so the next to the last statement of Theorem X gives

$$\varphi((((\mathfrak{a))))\varepsilon\mathfrak{D}.$$

If $\mathfrak{J}(\mathfrak{a})$ is $x=0$, then $\|\mathfrak{a}>=<\mathfrak{a}\|=0$, so

$$\mathfrak{a}\|=0,\quad\mathfrak{a}=0,$$

and

$$\varphi(0)\geqslant 0.$$

Now

$$\varphi((((\mathfrak{a})))) = \varphi((((0)))) = \varphi(0)\cdot 1$$

[by (42) in E, 1], and this is $\varepsilon\mathfrak{D}$ (by Lemma E, 3, 1).

Thus we have $\varphi((((\mathfrak{a}))))\,\varepsilon\,\mathfrak{D}$ in both cases.

Lemma E, 4, 5. $\lim\limits_{i\to\infty} \varphi_i(x) = \varphi(x)$ *uniformly in* $\mathfrak{J}(\mathfrak{a})$ *implies*

$$\mathfrak{U}\text{-}\lim_{i\to\infty} \varphi_i((((\mathfrak{a})))) = \varphi((((\mathfrak{a})))).$$

(Therefore by Theorem XII a fortiori

$$\mathfrak{T}\text{-}\lim_{i\to\infty} \varphi_i((((\mathfrak{a})))) = \varphi(((((\mathfrak{a}))))).$$

Proof. Put $\max\limits_{x\varepsilon\mathfrak{J}(\mathfrak{a})} |\varphi_i(x) - \varphi(x)| = \alpha_i$, our assumption means

$$\lim_{i\to\infty} \alpha_i = 0.$$

By Lemma E, 4, 4 we have

$$\|\varphi_i((((\mathfrak{a})))) - \varphi((((\mathfrak{a})))) \| \leqslant \alpha_i,$$

thus

$$\lim_{i\to\infty} \|\varphi_i((((\mathfrak{a})))) - \varphi((((\mathfrak{a})))) \| = 0, \qquad \mathfrak{U}\text{-}\lim_{i\to\infty} \varphi_i((((\mathfrak{a})))) = \varphi((((\mathfrak{a})))).$$

Observe that Lemma E, 4, 5 is the strongest convergence theorem we have proved so far. Furthermore we obtain from it, by putting $\varphi_1(x) \equiv \varphi_2(x) \equiv \ldots \equiv \psi(x)$:

If $\varphi(x) = \psi(x)$ for $x\varepsilon\mathfrak{J}(\mathfrak{a})$, then $\varphi((((\mathfrak{a})))) = \psi((((\mathfrak{a})))).$ In other words:

$$\varphi((((\mathfrak{a})))) \text{ depends only on the function } \varphi(x) \text{ in } \mathfrak{J}(\mathfrak{a}). \tag{46}$$

E, 5. We prove for later applications:

Lemma E, 5, 1. *For an idempotent* $\mathfrak{e} \neq 0,\ 1$

$$\|\mathfrak{e}> = 0, \quad <\mathfrak{e}\| = 1.$$

For $\mathfrak{e} = 0$ *both are* $= 0$, *for* $\mathfrak{e} = 1$ *both are* $= 1$.

Proof. The statements for $\mathfrak{e} = 0,\ 1$ are only repetitions of Lemma E, 3, 3, (i). Consider now $\mathfrak{e} \neq 0,\ 1$.

(27) in C, 2 gives $\mathfrak{e}\varepsilon\mathfrak{D}\cdot\mathfrak{U}$, so by Theorem X $\|\mathfrak{e}> \geqslant 0$ and $\|\mathfrak{e}\| \leqslant 1$, $<\mathfrak{e}\| \leqslant 1$. Thus

$$0 \leqslant \|\mathfrak{e}> \leqslant <\mathfrak{e}\| \leqslant 1.$$

Now $<\mathfrak{e}\| \neq 1$ would clearly imply $\|\mathfrak{e}\| < 1$. By Lemma E, 4, 2 $\|\mathfrak{e}\| = \|\mathfrak{e}^2\| \leqslant \|\mathfrak{e}\|^2$ so in this case $\|\mathfrak{e}\| = 0$, $\mathfrak{e} = 0$, contradicting our assumption. Thus $<\mathfrak{e}\| = 1$. Now $1 - \mathfrak{e}$ too is idempotent and $\neq 0,\ 1$, so $<1 - \mathfrak{e}\| = 1$. But $<1 - \mathfrak{e}\| = 1 - \|\mathfrak{e}>$ [use Lemma E, 3, 3, (ii), (v)], and so $\|\mathfrak{e}> = 0$.

And:

Lemma E, 5, 2. $\mathfrak{A}$ *is a „complete" space in the (metric)* $\mathfrak{U}$*-topology.*

Proof. Assume $\lim\limits_{i,j\to\infty} \|\mathfrak{a}_i - \mathfrak{a}_j\| = 0$, then we have to prove the existence of $\mathfrak{U}\text{-}\lim\limits_{i\to\infty} \mathfrak{a}_i$ that is: the existence of an $\mathfrak{a}^*$ with $\lim\limits_{i\to\infty} \|\mathfrak{a}_i - \mathfrak{a}^*\| = 0$.

Our assumption means that for every $\varepsilon > 0$ there exists an $n(\varepsilon)$, such that $i, j \geqslant n(\varepsilon)$ imply

$$\mathfrak{a}_i - \mathfrak{a}_j\| \leqslant \varepsilon.$$

Thus for $i \geqslant n(1)$

$$\|\mathfrak{a}_i - \mathfrak{a}_{n(1)}\| \leqslant 1, \quad \mathfrak{a}_i - \mathfrak{a}_{n(1)} \varepsilon \mathfrak{U}$$

(by Theorem X). Now apply IV_6): the $\mathfrak{a}_i - \mathfrak{a}_{n(1)}$ $[i \geqslant n(1)]$ possess a $\mathcal{T}$-condensation point $\mathfrak{a}^{**}$ and a sequence $n_1 < n_2 < \ldots$ with

$$\mathcal{T}\text{-}\lim_{j \to \infty} (\mathfrak{a}_{n_j} - \mathfrak{a}_{n(1)}) = \mathfrak{a}^{**}$$

must exist. Then

$$\mathcal{T}\text{-}\lim_{j \to \infty} \mathfrak{a}_{n_j} = \mathfrak{a}^{**} + \mathfrak{a}_{n(1)} = \mathfrak{a}^{*}.$$

If $i \geqslant n(\varepsilon)$, then if j is sufficiently great [for $n_j \geqslant n(\varepsilon)$]

$$\|\mathfrak{a}_i - \mathfrak{a}_{n_j}\| \leqslant \varepsilon,$$

that is

$$\langle \pm (\mathfrak{a}_i - \mathfrak{a}_{n_j}) \| \leqslant \varepsilon,$$

and so

$$\| \varepsilon 1 \mp (\mathfrak{a}_i - \mathfrak{a}_{n_j}) \rangle \geqslant 0$$

[use Lemma E, 3, 3, (ii), (vi)],

$$\varepsilon 1 \mp (\mathfrak{a}_i - \mathfrak{a}_{n_j}) \varepsilon \mathfrak{D}$$

use Theorem X). Now $j \to \infty$ gives [by IV_5)]

$$\varepsilon 1 \mp (\mathfrak{a}_i - \mathfrak{a}^*) \varepsilon \mathfrak{D},$$

and thus (using the same facts in the inverse sense)

$$\| \varepsilon 1 \mp (\mathfrak{a}_i - \mathfrak{a}^*) \rangle \geqslant 0, \quad \langle \pm (\mathfrak{a}_i - \mathfrak{a}^*) \| \leqslant \varepsilon, \quad \| \mathfrak{a}_i - \mathfrak{a}^* \| \leqslant \varepsilon.$$

In other words (as $\varepsilon > 0$ was arbitrary):

$$\lim_{i \to \infty} \| \mathfrak{a}_i - \mathfrak{a}^* \| = 0.$$

Theorem XII and Lemma E, 5, 2 express together that $\mathfrak{A}$ is a (not necessarily separable) „Hahn-Banach space".

F. The spectral form

F, 1. Consider an $\mathfrak{a} \varepsilon \mathfrak{A}$. Form the functions

$$B(x) = \max(0, x), \quad C(x) = \max(0, -x), \tag{47}$$

and then form $B((((\mathfrak{a}))))$, $C((((\mathfrak{a}))))$. We have $B(0) = 0$, $C(0) = 0$, so $B((((\mathfrak{a})))) = B(((\mathfrak{a})))$, $C((((\mathfrak{a})))) = C(((\mathfrak{a})))$, and thus

$$B((((\mathfrak{a})))), \quad C((((\mathfrak{a})))) \varepsilon \mathfrak{P}(\mathfrak{a}); \tag{48}$$

furthermore $B(x)$, $C(x) \geqslant 0$ for all x, so

$$B((((\mathfrak{a})))), \quad C((((\mathfrak{a})))) \varepsilon \mathfrak{D}, \tag{49}$$

and finally $B(x) - C(x) \equiv x$, $B(x)\, C(x) \equiv 0$, so

$$B((((\mathfrak{a})))) - C((((\mathfrak{a})))) = \mathfrak{a}, \quad B((((\mathfrak{a})))) \, C((((\mathfrak{a})))) = 0. \tag{50}$$

Now form $(B((((\mathfrak{a}))))))^\circ$, $(C((((\mathfrak{a}))))))^\circ$ (by Theorem I), and denote them by $\beta(\mathfrak{a})$, $\gamma(\mathfrak{a})$. We have:

$$\beta(\mathfrak{a}) \varepsilon \mathfrak{P}(B((((\mathfrak{a})))))) \subset \mathfrak{P}(\mathfrak{a}), \quad \gamma(\mathfrak{a}) \varepsilon \mathfrak{P}(C((((\mathfrak{a})))))) \subset \mathfrak{P}(\mathfrak{a}),$$

so

$$\beta(\mathfrak{a}), \quad \gamma(\mathfrak{a}) \varepsilon \mathfrak{P}(\mathfrak{a}), \tag{51}$$

and

$$\beta(\mathfrak{a}), \quad \gamma(\mathfrak{a}) \text{ are idempotents.} \tag{52}$$

Consider the set $\mathfrak{M}$ of all $\mathfrak{x} \varepsilon \mathfrak{P}(\mathfrak{a})$ with $\mathfrak{x} C((((\mathfrak{a})))) = 0$. If $\mathfrak{x}$, $\mathfrak{y} \varepsilon \mathfrak{M}$ then $\mathfrak{x} + \mathfrak{y}$ and $\mathfrak{x}\mathfrak{y} \varepsilon \mathfrak{M}$ [use (25) in C, 1], and $\mathfrak{M}$ is $\mathcal{T}$-closed [by IV_3)]. So $\mathfrak{M}$ is a ring. Now $B((((\mathfrak{a})))) \varepsilon \mathfrak{M}$ by (50), so $\mathfrak{P}(B((((\mathfrak{a})))))) \subset \mathfrak{M}$. Therefore $\beta(\mathfrak{a}) \varepsilon \mathfrak{M}$, $\beta(\mathfrak{a}) C((((\mathfrak{a})))) = 0$. Similarly $\gamma(\mathfrak{a}) B((((\mathfrak{a})))) = 0$. So:

$$\beta(\mathfrak{a}) C((((\mathfrak{a})))) = \gamma(\mathfrak{a}) B((((\mathfrak{a})))) = 0. \tag{53}$$

(50) and (53) give, as $B((((\mathfrak{a})))$ belongs to $\mathfrak{P}(B((((\mathfrak{a})))))$ with the unit $\beta(\mathfrak{a})$, and $C((((\mathfrak{a})))$ belongs to $\mathfrak{P}(C((((\mathfrak{a})))))$ with the unit $\gamma(\mathfrak{a})$, that

$$\beta(\mathfrak{a})\mathfrak{a} = B((((\mathfrak{a})))) \varepsilon \mathfrak{D}, \quad -\gamma(\mathfrak{a})\mathfrak{a} = C((((\mathfrak{a})))) \varepsilon \mathfrak{D}. \tag{54}$$

(50) and the second equation of (54) give

$$(1 - \gamma(\mathfrak{a}))\mathfrak{a} = B((((\mathfrak{a})))) \varepsilon \mathfrak{D}. \tag{55}$$

Another consequence of (53) is

$$C((((\mathfrak{a})))) \varepsilon N_0(\beta(\mathfrak{a})),$$

and as $N_0(\beta(\mathfrak{a}))$ is a ring, $\mathfrak{P}(C((((\mathfrak{a})))))) \subset N_0(\beta(\mathfrak{a}))$. So $\gamma(\mathfrak{a}) \varepsilon N_0(\beta(\mathfrak{a}))$, that is

$$\beta(\mathfrak{a})\gamma(\mathfrak{a}) = 0. \tag{56}$$

Now Lemma D, 5, 3, (ii) gives

$$N_1(\beta(\mathfrak{a})) \subset N_0(\gamma(\mathfrak{a})).$$

Thus we have

$$B((((\mathfrak{a})))) \varepsilon N_1(\beta(\mathfrak{a})) \subset N_0(\gamma(\mathfrak{a})), \quad C((((\mathfrak{a})))) \varepsilon N_1(\gamma(\mathfrak{a})),$$

and so (50) gives:

The decomposition $\mathfrak{a} = \mathfrak{a}_0 + \mathfrak{a}_{\frac{1}{2}} + \mathfrak{a}_1$ of $\mathfrak{a}$ with respect to the idempotent $\gamma(\mathfrak{a})$ in the sense of Lemma D, 2, 1 is

$$\mathfrak{a} = B((((\mathfrak{a})))) + 0 + (-C((((\mathfrak{a})))))). \tag{57}$$

Apply now the above construction to $\mathfrak{a} - \rho 1$, and write

$$\beta(\mathfrak{a} - \rho 1) = \beta_\rho(\mathfrak{a}), \quad \gamma(\mathfrak{a} - \rho 1) = \gamma_\rho(\mathfrak{a}).$$

Put

$$B_\rho(x) \equiv \max(0, x - \rho), \quad C_\rho(x) = \max(0, \rho - x), \tag{58}$$

then Theorem VIII gives

$$B((((\mathfrak{a} - \rho 1)))) = B_\rho((((\mathfrak{a})))), \quad C((((\mathfrak{a} - \rho 1)))) = C_\rho((((\mathfrak{a})))). \tag{59}$$

Now $\rho \geqslant 0$ implies $B_\rho(0) = 0$, so

$$B_\rho((((\mathfrak{a})))) = B_\rho(((\mathfrak{a}))) \varepsilon \mathfrak{P}(\mathfrak{a}), \quad \beta_\rho(\mathfrak{a}) \varepsilon \mathfrak{P}(B_\rho((((\mathfrak{a})))))) \subset \mathfrak{P}(\mathfrak{a});$$

and $\rho \leqslant 0$ implies $C_\rho(0) = 0$, so

$$C_\rho((((\mathfrak{a})))) = C_\rho(((\mathfrak{a}))) \varepsilon \mathfrak{P}(\mathfrak{a}), \quad \gamma_\rho(\mathfrak{a}) \varepsilon \mathfrak{P}(C_\rho((((\mathfrak{a})))))) \subset \mathfrak{P}(\mathfrak{a}).$$

Assume $\rho \leqslant 0$. Then one verifies easily

$$C_\rho(x) = \max(0, \ \rho - x) \equiv \max(0, \ \max(0, \ -x) + \rho) = B_{-\rho}(C(x)),$$

so

$$C_\rho((((\mathfrak{a})))) = B_{-\rho}((((C((((\mathfrak{a})))))))).$$

As $-\rho \geqslant 0$, so the right side is $\varepsilon \mathfrak{P}(C((((\mathfrak{a))))))$. This is a ring, so

$$\mathfrak{P}(C_\rho((((\mathfrak{a)))))) \subset \mathfrak{P}(C((((\mathfrak{a)))))).$$

Thus $\gamma_\rho(\mathfrak{a}) \varepsilon \mathfrak{P}(C((((\mathfrak{a)))))) \subset N_1(\gamma(\mathfrak{a}))$, $\gamma(\mathfrak{a})\gamma_\rho(\mathfrak{a}) = \gamma_\rho(\mathfrak{a})$, that is

$$\gamma_\rho(\mathfrak{a}) \leqslant \gamma(\mathfrak{a}).$$

Replace $\mathfrak{a}$ by $\mathfrak{a} - \sigma 1$, then $B((((\mathfrak{a)))))$, ..., $B_\sigma((((\mathfrak{a)))))$, ... become $B_\sigma((((\mathfrak{a)))))$, ..., $B_{\rho+\sigma}((((\mathfrak{a)))))$, ... Now write $\rho - \sigma$, σ for ρ, σ, then $B_\sigma((((\mathfrak{a)))))$, ..., $B_\rho((((\mathfrak{a)))))$, ... obtain, and our results become:

$$\left.\begin{array}{ll} B_\rho((((\mathfrak{a)))))\text{,} & \beta_\rho(\mathfrak{a}) \varepsilon \mathfrak{P}(\mathfrak{a} - \sigma 1) \text{ if } \rho \geqslant \sigma, \\ C_\rho((((\mathfrak{a)))))\text{,} & \gamma_\rho(\mathfrak{a}) \varepsilon \mathfrak{P}(\mathfrak{a} - \sigma 1) \quad \text{„} \ \rho \leqslant \sigma, \end{array}\right\} \tag{60}$$

$$\gamma_\rho(\mathfrak{a}) \leqslant \gamma_\sigma(\mathfrak{a}) \text{ if } \rho \leqslant \sigma. \tag{61}$$

F, 2. We establish a continuity property of the $\gamma_\rho(\mathfrak{a})$.

Lemma F, 2, 1. *For an idempotent* $\mathfrak{e}$ *and an arbitrary* $\mathfrak{a}$ *the following statements are equivalent:*

(i) $\mathfrak{a}\mathfrak{e} = 0$,

(ii) $\mathfrak{a}^0\mathfrak{e} = 0$ *($\mathfrak{a}^0$ from Theorem I)*,

(iii) $\mathfrak{x}\mathfrak{e} = 0$ *for all* $\mathfrak{x} \varepsilon \mathfrak{P}(\mathfrak{a})$.

Proof. As $\mathfrak{a}$, $\mathfrak{a}^0 \varepsilon \mathfrak{P}(\mathfrak{a})$ so (iii) implies (i), (ii).

(i) $\longrightarrow$ (iii): if (i) holds, then $\mathfrak{a} \varepsilon N_0(\mathfrak{e})$ and as $N_0(\mathfrak{a})$ is a ring, $\mathfrak{P}(\mathfrak{a}) \subset N_0(\mathfrak{e})$, implying (iii).

(ii) $\longrightarrow$ (iii): if (ii) holds, then $[\mathfrak{a}^0$ is idempotent, being the unit of $\mathfrak{P}(\mathfrak{a})]\,\mathfrak{e} \varepsilon N_0(\mathfrak{a}^0)$. For any $\mathfrak{x} \varepsilon \mathfrak{P}(\mathfrak{a})$

$$\mathfrak{a}^0\mathfrak{x} = \mathfrak{x}, \quad \mathfrak{x} \varepsilon N_1(\mathfrak{a}^0),$$

so (by Theorem IV) $\mathfrak{x}\mathfrak{e} = 0$, proving (iii).

Lemma F, 2, 2. *For the* $\gamma_\rho(\mathfrak{a})$ *of F, 1 and every sequence* $\rho_1, \rho_2, \ldots$ *with*

$$\rho_1 \leqslant \rho_2 \leqslant \ldots \leqslant \rho, \ \lim_{i \to \infty} \rho_i = \rho$$

we have

$$\mathcal{T}\text{-}\lim_{i \to \infty} \gamma_{\rho_i}(\mathfrak{a}) = \gamma_\rho(\mathfrak{a}).$$

Proof. By (61) $\gamma_{\rho_1}(\mathfrak{a}) \leqslant \gamma_{\rho_2}(\mathfrak{a}) \leqslant \ldots \leqslant \gamma_\rho(\mathfrak{a})$. So by Lemma D, 4, 3,

$$\mathcal{T}\text{-}\lim_{i \to \infty} \gamma_{\rho_i}(\mathfrak{a}) = \mathfrak{e}^*$$

exists, and all $\gamma_{\rho_i}(\mathfrak{a}) \leqslant \mathfrak{e}^*$. As all

$$\gamma_{\rho_i}(\mathfrak{a}) \leqslant \gamma_\rho(\mathfrak{a}), \quad \gamma_\rho(\mathfrak{a})\gamma_{\rho_i}(\mathfrak{a}) = \gamma_{\rho_i}(\mathfrak{a}),$$

so application of $\mathcal{T}\text{-}\lim_{i \to \infty}$ gives $\gamma_\rho(\mathfrak{a})\mathfrak{e}^* = \mathfrak{e}^*$, $\mathfrak{e}^* \leqslant \gamma_\rho(\mathfrak{a})$.

On the other hand $\gamma_{\rho i}(\mathfrak{a}) \leqslant \mathfrak{e}^*$ gives [by Lemma D, 5, 1, (iv)] the orthogonality of $\gamma_{\rho i}(\mathfrak{a})$ and $1-\mathfrak{e}^*$, so

$$\gamma_{\rho i}(\mathfrak{a})(1-\mathfrak{e}^*)=0.$$

But $\gamma_{\rho i}(\mathfrak{a})=(C_{\rho i}((((\mathfrak{a}))))\,)^\circ$ (cf. F, 1), so Lemma F, 2, 1 applies (replace its $\mathfrak{a}$, $\mathfrak{e}$ by $C_{\rho i}((((\mathfrak{a}))))$, $1-\mathfrak{e}^*$):

$$C_{\rho i}((((\mathfrak{a})))(1-\mathfrak{e}^*)=0.$$

But

$$\mathcal{T}\text{-}\lim_{i\to\infty} C_{\rho i}((((\mathfrak{a})))) = C_\rho((((\mathfrak{a}))))$$

(because $\lim\limits_{i\to\infty} C_{\rho i}(x)\equiv C_\rho(x)$, uniformly for all x, use (59) and Lemma E, 4, 5), and so

$$C_\rho((((\mathfrak{a})))(1-\mathfrak{e}^*)=0$$

obtains. Applying Lemma E, 2, 1 again (replacing $\mathfrak{a}$, $\mathfrak{e}$ by $C_\rho((((\mathfrak{a}))))$, $1-\mathfrak{e}^*$):

$$\gamma_\rho(\mathfrak{a})(1-\mathfrak{e}^*)=0$$

results, that is [by Lemma D, 5, 1, (iv)]

$$\gamma_\rho(\mathfrak{a})\leqslant\mathfrak{e}^*.$$

So we have proved

$$\mathcal{T}\text{-}\lim_{i\to\infty}\gamma_{\rho i}(\mathfrak{a})=\mathfrak{e}^*=\gamma_\rho(\mathfrak{a}).$$

F, 3. Let us now return to the connection between $\mathfrak{a}$ and the $\gamma_\rho(\mathfrak{a})$. We need the following preparatory Lemma:

Lemma F, 3, 1. *If* $\mathfrak{b}$, $\mathfrak{c}\,\varepsilon\,\mathfrak{P}(\mathfrak{a})$, *then* $\mathfrak{b}$, $\mathfrak{c}\,\varepsilon\,\mathfrak{D}$ *imply* $\mathfrak{b}\mathfrak{c}\,\varepsilon\,\mathfrak{D}$.

Proof. If $\mathfrak{a}=0$, then $\mathfrak{b}=\mathfrak{c}=0$, so we may assume $\mathfrak{a}\neq 0$. Then Theorem IX may be applied to $\mathfrak{P}(\mathfrak{a})$ (instead of $\mathfrak{A}$), and so $\mathfrak{b}$, $\mathfrak{c}\,\varepsilon\,\mathfrak{D}$. $\mathfrak{P}(\mathfrak{a})$ imply the existence of two $\mathfrak{b}_1$, $\mathfrak{c}_1\,\varepsilon\,\mathfrak{P}(\mathfrak{a})$ with

$$\mathfrak{b}=\mathfrak{b}_1^2,\quad \mathfrak{c}=\mathfrak{c}_1^2.$$

Now (25) in C, 1 gives:

$$\mathfrak{b}\mathfrak{c}=\mathfrak{b}_1^2\mathfrak{c}_1^2=(\mathfrak{b}_1\mathfrak{c}_1)^2\,\varepsilon\,\mathfrak{D}.$$

Lemma F, 3, 2. *If* $\sigma\,\varepsilon\,\mathcal{J}(\mathfrak{a})$ (cf. E, 3), *then* $\mathfrak{P}(\mathfrak{a}-\sigma 1)$ *is the minimum ring containing* $\mathfrak{a}$ *and* 1: $\mathfrak{P}(\mathfrak{a},\ 1)$.

Proof. The statement is unaffected if we replace $\mathfrak{a}$, σ by $\mathfrak{a}-\sigma 1$, 0 (use Theorem XI), so we may assume $\sigma=0$. Thus $0\,\varepsilon\,\mathcal{J}(\mathfrak{a})$, and so an $\varepsilon>0$ exists, such that $|x|\leqslant\varepsilon$ excludes $x\,\varepsilon\,\mathcal{J}(\mathfrak{a})$.

Put $\varphi(x)\equiv 1$, $\psi(x)\equiv\min\left(1,\ \frac{|x|}{\varepsilon}\right)$. Then $\varphi(x)=\psi(x)$ for all $x\,\varepsilon\,\mathcal{J}(\mathfrak{a})$ (as $|x|\geqslant\varepsilon$), and so $\varphi((((\mathfrak{a))))=\psi((((\mathfrak{a))))$, [by (46) in E, 4]. Now $\varphi((((\mathfrak{a))))=1$, and as $\psi(0)=0$, so

$$\psi((((\mathfrak{a))))=\psi(((\mathfrak{a)))\,\varepsilon\,\mathfrak{P}(\mathfrak{a})$$

(by Theorem III). Thus $1\,\varepsilon\,\mathfrak{P}(\mathfrak{a})$.

So $\mathfrak{P}(\mathfrak{a})$ contains 1, and of course $\mathfrak{a}$, and it is a ring. But every ring containing $\mathfrak{a}$ and 1 is $\supset\mathfrak{P}(\mathfrak{a})$ by definition. Thus $\mathfrak{P}(\mathfrak{a})$ (remember that $\sigma=0$) is the minimum ring containing $\mathfrak{a}$ and 1.

Corollary. For $\mathfrak{b}$, $\mathfrak{c}$, $\mathfrak{d}\,\varepsilon\,\mathfrak{P}(\mathfrak{a}1)$:

(i) $(\mathfrak{b}\mathfrak{c})\mathfrak{d} = \mathfrak{b}(\mathfrak{c}\mathfrak{d})$.

(ii) $\mathfrak{c}, \mathfrak{d} \varepsilon \mathfrak{D}$ imply $\mathfrak{c}\mathfrak{d} \varepsilon \mathfrak{D}$.

Proof. (i) results from (25) in C, 1, and (ii) from Lemma F, 3, 1, by replacing in both cases $\mathfrak{a}$ by an $\mathfrak{a} - \sigma 1$, $\sigma \not\varepsilon \mathfrak{J}(\mathfrak{a})$.

Observe that always $\mathfrak{P}(\mathfrak{a} - \tau 1) \subset \mathfrak{P}(\mathfrak{a}, 1)$, and so (60) gives:

$$B_\rho((((\mathfrak{a})))), \quad C_\rho((((\mathfrak{a})))), \quad \beta_\rho(\mathfrak{a}), \ \gamma_\rho(\mathfrak{a}) \varepsilon \mathfrak{P}(\mathfrak{a}, 1). \tag{62}$$

Now assume $\rho \leqslant \sigma$. By (54), (55) we have

$$-\gamma_\sigma(\mathfrak{a})(\mathfrak{a} - \sigma 1) = \gamma_\sigma(\mathfrak{a})(-\mathfrak{a} + \sigma 1) \varepsilon \mathfrak{D}$$

and $(1 - \gamma_\rho(\mathfrak{a}))(\mathfrak{a} - \rho 1) \varepsilon \mathfrak{D}$, and the idempotents $1 - \gamma_\rho(\mathfrak{a})$ and $\gamma_\sigma(\mathfrak{a})$ are obviously $\varepsilon \mathfrak{D}$ too. Therefore Lemma E, 3, 2, Corollary, (ii), gives

$$(1 - \gamma_\rho(\mathfrak{a})) \cdot \gamma_\sigma(\mathfrak{a})(-\mathfrak{a} + \sigma 1) \varepsilon \mathfrak{D}$$

and

$$\gamma_\sigma(\mathfrak{a}) \cdot (1 - \gamma_\rho(\mathfrak{a}))(\mathfrak{a} - \rho 1) \varepsilon \mathfrak{D}.$$

But (i) eod. gives

$$\begin{aligned} &(1 - \gamma_\rho(\mathfrak{a})) \cdot \gamma_\sigma(\mathfrak{a})(-\mathfrak{a} + \sigma 1) = (1 - \gamma_\rho(\mathfrak{a}))\gamma_\sigma(\mathfrak{a}) \cdot (-\mathfrak{a} + \sigma 1) = \\ &= (\gamma_\sigma(\mathfrak{a}) - \gamma_\rho(\mathfrak{a})\gamma_\sigma(\mathfrak{a}))(-\mathfrak{a} + \sigma 1) = (\gamma_\sigma(\mathfrak{a}) - \gamma_\rho(\mathfrak{a}))(-\mathfrak{a} + \sigma 1), \end{aligned}$$

$$\begin{aligned} &\gamma_\sigma(\mathfrak{a}) \cdot (1 - \gamma_\rho(\mathfrak{a}))(\mathfrak{a} - \rho 1) = \gamma_\sigma(\mathfrak{a})(1 - \gamma_\rho(\mathfrak{a})) \cdot (\mathfrak{a} - \rho 1) = \\ &= (\gamma_\sigma(\mathfrak{a}) - \gamma_\sigma(\mathfrak{a})\gamma_\rho(\mathfrak{a}))(\mathfrak{a} - \rho 1) = (\gamma_\sigma(\mathfrak{a}) - \gamma_\rho(\mathfrak{a}))(\mathfrak{a} - \rho 1). \end{aligned}$$

Combining these facts:

$$(\gamma_\sigma(\mathfrak{a}) - \gamma_\rho(\mathfrak{a}))(-\mathfrak{a} + \sigma 1) \text{ and } (\gamma_\sigma(\mathfrak{a}) - \gamma_\rho(\mathfrak{a}))(\mathfrak{a} - \rho 1) \varepsilon \mathfrak{D} \tag{63}$$

if $\rho \leqslant \sigma$.

Assume $\rho > < \mathfrak{a}\|$. Then

$$C_\rho(x) = C(x - \rho) \geqslant \rho - < \mathfrak{a}\| = \eta > 0$$

for all $x \leqslant < \mathfrak{a}\|$, that is in all $\mathfrak{J}(\mathfrak{a})$. So $C_\rho(x) - \eta \geqslant 0$ for $x \varepsilon \mathfrak{J}(\mathfrak{a})$, and so by (46) in E, 4

$$C_\rho((((\mathfrak{a})))) - \eta 1 \varepsilon \mathfrak{D}.$$

Thus

$$\| C_\rho((((\mathfrak{a})))) > \geqslant \eta > 0, \quad 0 \not\varepsilon \mathfrak{J}(C_\rho((((\mathfrak{a}))))).$$

Therefore Lemma F, 3, 2 permits inferring $1 \varepsilon \mathfrak{P}(C_\rho((((\mathfrak{a})))))$, and so the unit of $\mathfrak{P}(C_\rho((((\mathfrak{a})))))$, $\gamma_\rho(\mathfrak{a}) = (C_\rho((((\mathfrak{a})))))^\circ$, is necessarily 1. So we have

$$\gamma_\rho(\mathfrak{a}) = 1.$$

If $\rho < \|\mathfrak{a} >$, then replacement of $\mathfrak{a}$, ρ by $-\mathfrak{a}$, $-\rho$ gives $\beta_\rho(\mathfrak{a}) = 1$, and so (56) implies

$$\gamma_\rho(\mathfrak{a}) = 0.$$

Owing to Lemma F, 2, 2 this extends even to $\rho = \|\mathfrak{a} >$. So we have proved:

$$\left.\begin{array}{l} \rho > < \mathfrak{a}\| \text{ and } \rho \leqslant \|\mathfrak{a} > \text{ respectively implies } \gamma_\rho(\mathfrak{a}) = 1 \\ \text{and } \gamma_\rho(\mathfrak{a}) = 0 \text{ respectively.} \end{array}\right\} \tag{64}$$

F, 4. We call a family of idempotents $\mathfrak{e}_\rho$, $-\infty < \rho < +\infty$, a bounded spectral family if it possesses the following properties:

(i) $\rho \leqslant \sigma$ implies $\mathfrak{e}_\rho \leqslant \mathfrak{e}_\sigma$.

(ii) $\rho_1 \leqslant \rho_2 \leqslant \ldots \leqslant \rho$, $\lim\limits_{i\to\infty} \rho_i = \rho$ imply $\mathcal{T}\text{-}\lim\limits_{i\to\infty} \mathfrak{e}_{\rho_i} = \mathfrak{e}_\rho$.

(iii) Two α, β exist, for which $\rho > \beta$ and $\rho \leqslant \alpha$ respectively imply $\mathfrak{e}_\rho = 1$ and $\mathfrak{e}_\rho = 0$ respectively.

Clearly (iii) necessitates $\alpha \leqslant \beta$; we say that the interval $\alpha \leqslant x \leqslant \beta$ covers the family $\mathfrak{e}_\rho$.

Owing to (61), Lemma F, 2, 2, and (64) the $\gamma_\rho(\mathfrak{a})$, $-\infty < \rho < +\infty$, as defined in F, 1, form a bounded spectral family, and are covered by the interval $\mathcal{J}(\mathfrak{a})$.

A finite sequence of numbers $\lambda_0, \xi_1, \lambda_1, \xi_2, \lambda_2, \ldots, \xi_n, \lambda_n$ is a grating if

(i) $\lambda_0 \leqslant \xi_1 \leqslant \lambda_1 \leqslant \xi_2 \leqslant \lambda_2 \leqslant \ldots \leqslant \xi_n \leqslant \lambda_n$,

(ii) $\lambda_0 \leqslant \alpha$, $\lambda_n > \beta$

and it is of order ε for a given $\varepsilon > 0$, if

(iii) $\lambda_i - \lambda_{i-1} \leqslant \varepsilon$ for all $i = 1, \ldots, n$.

An $\mathfrak{a} \varepsilon \mathfrak{A}$ belongs to a given bounded spectral family $\mathfrak{e}_\rho$, if we have for every grating $\lambda_0, \xi_1, \lambda_1, \xi_2, \lambda_2, \ldots, \xi_n, \lambda_n$ of order ε (for any $\varepsilon > 0$)

$$\left\| \mathfrak{a} - \sum_{i=1}^{n} \xi_i (\mathfrak{e}_{\lambda_i} - \mathfrak{e}_{\lambda_{i-1}}) \right\| \leqslant \varepsilon. \tag{*}$$

We prove now:

Lemma F, 4, 1. *Let $\mathfrak{e}_\rho$ be a bounded spectral family, and $\lambda_0, \xi_1, \lambda_1, \xi_2, \lambda_2, \ldots, \xi_n, \lambda_n$ and $\mu_0, \eta_1, \mu_1, \eta_2, \mu_2, \ldots, \eta_m, \mu_m$ two gratings of order ε and δ respectively. Then*

$$\left\| \sum_{i=1}^{n} \xi_i (\mathfrak{e}_{\lambda_i} - \mathfrak{e}_{\lambda_{i-1}}) - \sum_{j=1}^{m} \eta_j (\mathfrak{e}_{\mu_j} - \mathfrak{e}_{\mu_{j-1}}) \right\| \leqslant \varepsilon + \delta.$$

Proof. Consider the grating $\lambda_0, \xi_1, \lambda_1, \xi_2, \lambda_2, \ldots, \xi_n, \lambda_n$ and another grating $\varkappa_0, \zeta_1, \varkappa_1, \zeta_2, \varkappa_2, \ldots, \zeta_p, \varkappa_p$ which contains it, that is for which $\lambda_i = \varkappa_{s_i}$ for $i = 0, 1, \ldots, n$ for some $0 \leqslant s_0 < s_1 < s_2 < \ldots < s_n \leqslant p$. Now compute:

$$\begin{aligned}
&\sum_{i=1}^{n} \xi_i (\mathfrak{e}_{\lambda_i} - \mathfrak{e}_{\lambda_{i-1}}) - \sum_{l=1}^{p} \zeta_l (\mathfrak{e}_{\varkappa_l} - \mathfrak{e}_{\varkappa_{l-1}}) = \\
&= -\sum_{l=1}^{s_0} \zeta_l (\mathfrak{e}_{\varkappa_l} - \mathfrak{e}_{\varkappa_{l-1}}) - \sum_{l=s_n+1}^{p} \zeta_l (\mathfrak{e}_{\varkappa_l} - \mathfrak{e}_{\varkappa_{l-1}}) + \\
&+ \sum_{i=1}^{n} \Big(\xi_i (\mathfrak{e}_{\lambda_i} - \mathfrak{e}_{\lambda_{i-1}}) - \sum_{l=s_{i-1}+1}^{s_i} \zeta_l (\mathfrak{e}_{\varkappa_l} - \mathfrak{e}_{\varkappa_{l-1}}) \Big) = \\
&= -\sum_{l=1}^{s_0} \zeta_l (0 - 0) - \sum_{l=s_n+1}^{p} \zeta_l (1 - 1) + \\
&+ \sum_{i=1}^{n} \Big(\xi_i (\mathfrak{e}_{\varkappa_{s_i}} - \mathfrak{e}_{\varkappa_{s_{i-1}}}) - \sum_{=s_{i-1}+1}^{s_i} \zeta_l (\mathfrak{e}_{\varkappa_l} - \mathfrak{e}_{\varkappa_{l-1}}) \Big) = \\
&= \sum_{i=1}^{n} \sum_{l=s_{i-1}+1}^{s_i} (\xi_i - \zeta_l)(\mathfrak{e}_{\varkappa_l} - \mathfrak{e}_{\varkappa_{l-1}}) = \sum_{l=s_0+1}^{s_n} \upsilon_l (\mathfrak{e}_{\varkappa_l} - \mathfrak{e}_{\varkappa_{l-1}})
\end{aligned}$$

where υ_l is defined for $l = s_0 + 1, \ldots, s_n$, namely $\upsilon_l = \xi_i - \zeta_l$ for $s_{l-1} + 1 \leqslant l \leqslant s_l$. Considering $\lambda_{l-1} \leqslant \xi_i \leqslant \lambda_i$, $\lambda_{l-1} = \varkappa_{s_{i-1}} \leqslant \zeta_l \leqslant \varkappa_{s_i} = \lambda_i$, we have

$$|\upsilon_l| = |\xi_i - \zeta_l| \leqslant \lambda_i - \lambda_{l-1} \leqslant \varepsilon.$$

So we see:

$$\varepsilon 1 \pm \left\{ \sum_{i=1}^{n} \xi_i (\mathfrak{e}_{\lambda_i} - \mathfrak{e}_{\lambda_{i-1}}) - \sum_{l=1}^{p} \zeta_l (\mathfrak{e}_{\varkappa_l} - \mathfrak{e}_{\varkappa_{l-1}}) \right\} =$$

$$= \sum_{l=s_0+1}^{s_n} (\varepsilon \pm \upsilon_l)(\mathfrak{e}_{\varkappa_l} - \mathfrak{e}_{\varkappa_{l-1}}).$$

Now $\mathfrak{e}_{\varkappa_{l-1}} \leqslant \mathfrak{e}_{\varkappa_l}$, so $\mathfrak{e}_{\varkappa_l} - \mathfrak{e}_{\varkappa_{l-1}}$ is a projection, and therefore $\varepsilon \mathfrak{D}$, besides all $\varepsilon \pm \upsilon_l \geqslant 0$, therefore

$$\varepsilon 1 \pm \left\{ \sum_{i=1}^{n} \xi_i (\mathfrak{e}_{\lambda_i} - \mathfrak{e}_{\lambda_{i-1}}) - \sum_{l=1}^{p} \zeta_l (\mathfrak{e}_{\varkappa_l} - \mathfrak{e}_{\varkappa_{l-1}}) \right\} \varepsilon \mathfrak{D}.$$

From this

$$\left\| \sum_{i=1}^{n} \xi_i (\mathfrak{e}_{\lambda_i} - \mathfrak{e}_{\lambda_{i-1}}) - \sum_{l=1}^{p} \zeta_l (\mathfrak{e}_{\varkappa_l} - \mathfrak{e}_{\varkappa_{l-1}}) \right\| \leqslant \varepsilon$$

follows by definition.

Similarly, if $\varkappa_0, \zeta_1, \varkappa_1, \zeta_2, \varkappa_2, \ldots, \zeta_p, \varkappa_p$ contains $\mu_0, \eta_1, \mu_1, \eta_2, \mu_2, \ldots, \eta_m, \mu_m$, that is if $\mu_j = \varkappa_{t_j}$ for $j = 0, 1, \ldots, m$, for some $0 \leqslant t_0 < t_1 < t_2 < \ldots < t_n \leqslant p$, then

$$\left\| \sum_{j=1}^{m} \eta_j (\mathfrak{e}_{\mu_j} - \mathfrak{e}_{\mu_{j-1}}) - \sum_{l=1}^{p} \zeta_l (\mathfrak{e}_{\varkappa_l} - \mathfrak{e}_{\varkappa_{l-1}}) \right\| \leqslant \delta.$$

Combining these two inequalities gives

$$\left\| \sum_{i=1}^{n} \xi_i (\mathfrak{e}_{\lambda_i} - \mathfrak{e}_{\lambda_{i-1}}) - \sum_{j=1}^{m} \eta_j (\mathfrak{e}_{\mu_j} - \mathfrak{e}_{\mu_{j-1}}) \right\| \leqslant \varepsilon + \delta.$$

Lemma F, 4, 2. *Let $\mathfrak{e}_\rho$ be a bounded spectral family. Then there exists exactly one $\mathfrak{a} \varepsilon \mathfrak{A}$ which belongs to it.*

Proof. Unicity: if $\mathfrak{a}, \mathfrak{b}$ belong both to the same $\mathfrak{e}_\rho$, then (*) implies $\|\mathfrak{a} - \mathfrak{b}\| \leqslant 2\varepsilon$ for every $\varepsilon > 0$. So $\|\mathfrak{a} - \mathfrak{b}\| = 0$, $\mathfrak{a} = \mathfrak{b}$.

Existence: consider a sequence of gratings

$$\lambda_0^{(\nu)}, \quad \xi_1^{(\nu)}, \quad \lambda_1^{(\nu)}, \quad \xi_2^{(\nu)}, \quad \lambda_2^{(\nu)}, \ldots, \xi_{n^{(\nu)}}^{(\nu)}, \quad \lambda_{n^{(\nu)}}^{(\nu)} \text{ for } \nu = 1, 2, \ldots,$$

of the respective orders $\varepsilon^{(\nu)}$, and with $\lim\limits_{\nu \to \infty} \varepsilon^{(\nu)} = 0$. Then

$$\left\| \sum_{i=1}^{n^{(\nu)}} \xi_i^{(\nu)} (\mathfrak{e}_{\lambda_i^{(\nu)}} - \mathfrak{e}_{\lambda_{i-1}^{(\nu)}}) - \sum_{i=1}^{n^{(\mu)}} \xi_i^{(\mu)} (\mathfrak{e}_{\lambda_i^{(\mu)}} - \mathfrak{e}_{\lambda_{i-1}^{(\mu)}}) \right\| \leqslant \varepsilon^{(\nu)} + \varepsilon^{(\mu)}$$

by Lemma F, 4, 1, and thus

$$\lim_{\nu, \mu \to \infty} \left\| \sum_{i=1}^{n^{(\nu)}} \xi_i^{(\nu)} (\mathfrak{e}_{\lambda_i^{(\nu)}} - \mathfrak{e}_{\lambda_{i-1}^{(\nu)}}) - \sum_{i=1}^{n^{(\mu)}} \xi_i^{(\mu)} (\mathfrak{e}_{\lambda_i^{(\mu)}} - \mathfrak{e}_{\lambda_{i-1}^{(\mu)}}) \right\| = 0.$$

So Lemma E, 5, 2 applies to the sequence

$$\sum_{i=1}^{n^{(\nu)}} \xi_i^{(\nu)} (\mathfrak{e}_{\lambda_i^{(\nu)}} - \mathfrak{e}_{\lambda_{i-1}^{(\nu)}}), \quad \nu = 1, 2, \ldots,$$

securing the existence of

$$\mathcal{U}\text{-}\lim_{\nu\to\infty} \sum_{i=1}^{n^{(\nu)}} \xi_i^{(\nu)} (\mathfrak{e}_{\lambda_i^{(\nu)}} - \mathfrak{e}_{\lambda_{i-1}^{(\nu)}}) = \mathfrak{a}.$$

Now let $\lambda_0, \xi_1, \lambda_1, \xi_2, \lambda_2, \ldots, \xi_n, \lambda_n$ be any other grating of order ε. Then Lemma F, 4, 1 applies again, and gives

$$\left\| \sum_{i=1}^{n^{(\nu)}} \xi_i^{(\nu)} (\mathfrak{e}_{\lambda_i^{(\nu)}} - \mathfrak{e}_{\lambda_{i-1}^{(\nu)}}) - \sum_{i=1}^{n} \xi_i (\mathfrak{e}_{\lambda_i} - \mathfrak{e}_{\lambda_{i-1}}) \right\| \leqslant \varepsilon^{(\nu)} + \varepsilon.$$

Applying $\mathcal{U}\text{-}\lim\limits_{\nu\to\infty}$ to this (that is: to the first term inside the $\| \ldots \|$) we obtain:

$$\left\| - \sum_{i=1}^{n} \xi_i (\mathfrak{e}_{\lambda_i} - \mathfrak{e}_{\lambda_{i-1}}) \right\| \leqslant \varepsilon.$$

Thus (*) is verified, and $\mathfrak{a}$ belongs to $\mathfrak{e}_\rho$.

Lemma F, 4, 3. *To the bounded spectral family* $\mathfrak{e}_\rho = \gamma_{\rho\mathfrak{a}}^{\mathfrak{r}}(\mathfrak{a})$ (cf. F, 1) *this* $\mathfrak{a}$ *itself belongs.*

Proof. Let $\lambda_0, \xi_1, \lambda_1, \xi_2, \lambda_2, \ldots, \xi_n, \lambda_n$ be a grating of order ε. We have

$$\sum_{i=1}^{n} (\gamma_{\lambda_i}(\mathfrak{a}) - \gamma_{\lambda_{i-1}}(\mathfrak{a}))\, \mathfrak{a} = (\gamma_{\lambda_n}(\mathfrak{a}) - \gamma_{\lambda_0}(\mathfrak{a}))\, \mathfrak{a} = (1 - 0)\, \mathfrak{a} = \mathfrak{a},$$

and so by (63) both

$$\sum_{i=1}^{n} (\gamma_{\lambda_i}(\mathfrak{a}) - \gamma_{\lambda_{i-1}}(\mathfrak{a}))(-\mathfrak{a} + \lambda_i 1) = -\mathfrak{a} + \sum_{i=1}^{n} \lambda_i (\gamma_{\lambda_i}(\mathfrak{a}) - \gamma_{\lambda_{i-1}}(\mathfrak{a}))$$

and

$$\sum_{i=1}^{n} (\gamma_{\lambda_i}(\mathfrak{a}) - \gamma_{\lambda_{i-1}}(\mathfrak{a}))(\mathfrak{a} - \lambda_{i-1} 1) = \mathfrak{a} - \sum_{i=1}^{n} \lambda_{i-1} (\gamma_{\lambda_i}(\mathfrak{a}) - \gamma_{\lambda_{i-1}}(\mathfrak{a}))$$

are $\varepsilon\mathfrak{D}$. Now as $\xi_i + \varepsilon - \lambda_i \geqslant 0$, $\lambda_{i-1} + \varepsilon - \xi_i \geqslant 0$ and $\gamma_{\lambda_i}(\mathfrak{a}) - \gamma_{\lambda_{i-1}}(\mathfrak{a})$ is idempotent, and so $\varepsilon\mathfrak{D}$, therefore

$$\sum_{i=1}^{n} (\xi_i + \varepsilon - \lambda_i)(\gamma_{\lambda_i}(\mathfrak{a}) - \gamma_{\lambda_{i-1}}(\mathfrak{a}))$$

and

$$\sum_{i=1}^{n} (\lambda_{i-1} + \varepsilon - \xi_i)(\gamma_{\lambda_i}(\mathfrak{a}) - \gamma_{\lambda_{i-1}}(\mathfrak{a}))$$

$\varepsilon\mathfrak{D}$. Adding, we see that

$$-\mathfrak{a} + \sum_{i=1}^{n} (\xi_i + \varepsilon)(\gamma_{\lambda_i}(\mathfrak{a}) - \gamma_{\lambda_{i-1}}(\mathfrak{a})) = -\mathfrak{a} + \sum_{i=1}^{n} \xi_i (\gamma_{\lambda_i}(\mathfrak{a}) - \gamma_{\lambda_{i-1}}(\mathfrak{a})) + \varepsilon 1 =$$

$$= \varepsilon 1 - \Big(\mathfrak{a} - \sum_{i=1}^{n} \xi_i (\gamma_{\lambda_i}(\mathfrak{a}) - \gamma_{\lambda_{i-1}}(\mathfrak{a}))\Big)$$

and

$$\mathfrak{a}-\sum_{i=1}^{n}(\xi_i-\varepsilon)(\gamma_{\lambda_i}(\mathfrak{a})-\lambda_{\lambda_{i-1}}(\mathfrak{a}))=\mathfrak{a}-\sum_{i=1}^{n}\xi_i(\gamma_{\lambda_i}(\mathfrak{a})-\gamma_{\lambda_{i-1}}(\mathfrak{a}))+\varepsilon 1=$$

$$=\varepsilon 1+\Big(\mathfrak{a}-\sum_{i=1}^{n}\xi_i(\gamma_{\lambda_i}(\mathfrak{a})-\gamma_{\lambda_{i-1}}(\mathfrak{a}))\Big)$$

are $\varepsilon\mathfrak{D}$. Therefore

$$\Big\|\mathfrak{a}-\sum_{i=1}^{n}\xi_i(\gamma_{\lambda_i}(\mathfrak{a})-\gamma_{\lambda_{i-1}}(\mathfrak{a}))\Big\|\leqslant\varepsilon.$$

Thus (*) is fulfilled, and $\mathfrak{a}$ belongs to $\mathfrak{e}_\rho=\gamma_\rho(\mathfrak{a})$.

F, 5. Consider a finite sequence of idempotents $\mathfrak{f}_0\leqslant\mathfrak{f}_1\leqslant\ldots\leqslant\mathfrak{f}_n$. Then $i<j$ (both $=1,\ldots,n$) gives $i\leqslant j-1$, $\mathfrak{f}_i\leqslant\mathfrak{f}_{j-1}$. But $\mathfrak{f}_i-\mathfrak{f}_{i-1}\leqslant\mathfrak{f}_i$, and as $(1-(\mathfrak{f}_j-\mathfrak{f}_{j-1}))-\mathfrak{f}_{j-1}=1-\mathfrak{f}_j$ is idempotent, so $\mathfrak{f}_{j-1}\leqslant 1-(\mathfrak{f}_j-\mathfrak{f}_{j-1})$. So

$$\mathfrak{f}_i-\mathfrak{f}_{i-1}\leqslant 1-(\mathfrak{f}_j-\mathfrak{f}_{j-1})$$

that is $\mathfrak{f}_i-\mathfrak{f}_{i-1}$ and $\mathfrak{f}_j-\mathfrak{f}_{j-1}$ are orthogonal. (Use the various parts of Lemma D, 5, 1.) As this statement is symmetric in i, j so it holds whenever $i\neq j$. So we see:

$$(\mathfrak{f}_i-\mathfrak{f}_{i-1})(\mathfrak{f}_j-\mathfrak{f}_{j-1})\begin{cases}=\mathfrak{f}_i-\mathfrak{f}_{i-1} & \text{for } i=j,\\ =0 & \text{»}\ i\neq j.\end{cases}$$

Therefore (ξ_i, η_i are arbitrary real numbers):

$$\sum_{i=1}^{n}\xi_i(\mathfrak{f}_i-\mathfrak{f}_{i-1})\cdot\sum_{i=1}^{n}\eta_i(\mathfrak{f}_i-\mathfrak{f}_{i-1})=\sum_{i=1}^{n}\xi_i\eta_i(\mathfrak{f}_i-\mathfrak{f}_{i-1}).\tag{65}$$

We now prove:

Lemma F, 5, 1. *Let $\mathfrak{a}$ belong to the bounded spectral family $\mathfrak{e}_\rho$, $-\infty<\rho<+\infty$, let $\varphi(x)$ be an everywhere defined and continuous function, and let $\lambda_0^{(\nu)}$, $\xi_1^{(\nu)}$, $\lambda_1^{(\nu)}$, $\xi_2^{(\nu)}$, $\lambda_2^{(\nu)}$, $\ldots$, $\xi_{n^{(\nu)}}^{(\nu)}$, $\lambda_{n^{(\nu)}}^{(\nu)}$ for $\nu=1,2,\ldots$ be a sequence of gratings, of the respective orders $\varepsilon^{(\nu)}$ with $\lim\limits_{\nu\to\infty}\varepsilon^{(\nu)}=0$.*

Then

$$\mathfrak{U}\text{-}\lim_{\nu\to\infty}\sum_{i=1}^{n^{(\nu)}}\varphi(\xi_i^{(\nu)})(\mathfrak{e}_{\lambda_i^{(\nu)}}-\mathfrak{e}_{\lambda_{i-1}^{(\nu)}})=\varphi((((\mathfrak{a})))).$$

Proof. If this holds for $\varphi(x)$, $\psi(x)$ then it is necessarily true for $\rho\varphi(x)$, $\varphi(x)+\psi(x)$ and by (65) for $\varphi(x)\psi(x)$ too (use Lemma E, 4, 3). Besides it is evidently true for $\varphi(x)\equiv 1$ and it holds for $\varphi(x)\equiv x$ owing to the definitory relation (*) in F, 4. Thus it holds whenever $\varphi(x)$ is a polynomial.

Let now $\varphi(x)$ be arbitrary. Choose α, β so that $\alpha\leqslant x\leqslant\beta$ covers $\mathfrak{e}_\rho$, $-\infty<\rho<+\infty$ (cf. F, 4). Choose an $\varepsilon>0$ and a polynomial $p(x)$ with $|p(x)-\varphi(x)|\leqslant\varepsilon$ for all $x\,\varepsilon\,\mathfrak{J}(\mathfrak{a})$ and all $x\geqslant\alpha-\varepsilon$, $\leqslant\beta+\varepsilon$.

Thus by Lemma E, 4, 4,

$$\|p((((\mathfrak{a))))-\varphi((((\mathfrak{a))))\|\leqslant\varepsilon.$$

If ν is sufficiently great, say $\nu \geqslant \nu_0(\varepsilon)$, then

$$\left\| p((((\mathfrak{a})))) - \sum_{i=1}^{n^{(\nu)}} p(\xi_i^{(\nu)})(\mathfrak{e}_{\lambda_i^{(\nu)}} - \mathfrak{e}_{\lambda_{i-1}^{(\nu)}}) \right\| \leqslant \varepsilon.$$

Furthermore

$$\varepsilon 1 \pm \left(\sum_{i=1}^{n^{(\nu)}} p(\xi_i^{(\nu)})(\mathfrak{e}_{\lambda_i^{(\nu)}} - \mathfrak{e}_{\lambda_{i-1}^{(\nu)}}) - \sum_{i=1}^{n^{(\nu)}} \varphi(\xi_i^{(\nu)})(\mathfrak{e}_{\lambda_i^{(\nu)}} - \mathfrak{e}_{\lambda_{i-1}^{(\nu)}}) \right) =$$

$$= \sum_{i=1}^{n^{(\nu)}} \left(\varepsilon \pm (p(\xi_i^{(\nu)}) - \varphi(\xi_i^{(\nu)})) \right) (\mathfrak{e}_{\lambda_i^{(\nu)}} - \mathfrak{e}_{\lambda_{i-1}^{(\nu)}}).$$

If $\alpha - \varepsilon \leqslant \xi_{i-1}^{(\nu)} \leqslant \xi_i^{(\nu)} \leqslant \beta + \varepsilon$ the numerical coefficient of $\mathfrak{e}_{\lambda_i^{(\nu)}} - \mathfrak{e}_{\lambda_{i-1}^{(\nu)}}$ is $\geqslant 0$. If this is not the case, then either $\xi_i^{(\nu)} > \beta + \varepsilon$, so

$$\lambda_i^{(\nu)} \geqslant \lambda_{i-1}^{(\nu)} \geqslant \xi_i^{(\nu)} - \varepsilon > \beta, \quad \mathfrak{e}_{\lambda_i^{(\nu)}} = \mathfrak{e}_{\lambda_{i-1}^{(\nu)}} = 1,$$

or $\xi_i^{(\nu)} < \alpha - \varepsilon$, so

$$\lambda_{i-1}^{(\nu)} \leqslant \lambda_i^{(\nu)} \leqslant \xi_i^{(\nu)} + \varepsilon < \alpha, \quad \mathfrak{e}_{\lambda_i^{(\nu)}} = \mathfrak{e}_{\lambda_{i-1}^{(\nu)}} = 0,$$

so at any rate

$$\mathfrak{e}_{\lambda_i^{(\nu)}} - \mathfrak{e}_{\lambda_{i-1}^{(\nu)}} = 0.$$

These facts imply, as all $\mathfrak{e}_{\lambda_i^{(\nu)}} - \mathfrak{e}_{\lambda_{i-1}^{(\nu)}}$ are idempotents, and so $\varepsilon \mathfrak{D}$, that

$$\varepsilon 1 \pm \left(\sum_{i=1}^{n^{(\nu)}} p(\xi_i^{(\nu)})(\mathfrak{e}_{\lambda_i^{(\nu)}} - \mathfrak{e}_{\lambda_{i-1}^{(\nu)}}) - \sum_{i=1}^{n^{(\nu)}} \varphi(\xi_i^{(\nu)})(\mathfrak{e}_{\lambda_i^{(\nu)}} - \mathfrak{e}_{\lambda_{i-1}^{(\nu)}}) \right) \varepsilon \mathfrak{D}.$$

Thus

$$\left\| \sum_{i=1}^{n^{(\nu)}} p(\xi_i^{(\nu)})(\mathfrak{e}_{\lambda_i^{(\nu)}} - \mathfrak{e}_{\lambda_{i-1}^{(\nu)}}) - \sum_{i=1}^{n^{(\nu)}} \varphi(\xi_i^{(\nu)})(\mathfrak{e}_{\lambda_i^{(\nu)}} - \mathfrak{e}_{\lambda_{i-1}^{(\nu)}}) \right\| \leqslant \varepsilon.$$

Our three inequalities give together:

$$\left\| \varphi((((\mathfrak{a})))) - \sum_{i=}^{n^{(\nu)}} \varphi(\xi_i^{(\nu)})(\mathfrak{e}_{\lambda_i^{(\nu)}} - \mathfrak{e}_{\lambda_{i-1}^{(\nu)}}) \right\| \leqslant 3\varepsilon$$

if $\nu \geqslant \nu(\varepsilon)$. As this holds for every $\varepsilon > 0$ we have

$$\mathcal{U}\text{-}\lim_{\nu \to \infty} \sum_{i=1}^{n^{(\nu)}} \varphi(\xi_i^{(\nu)})(\mathfrak{e}_{\lambda_i^{(\nu)}} - \mathfrak{e}_{\lambda_{i-1}^{(\nu)}}) = \varphi((((\mathfrak{a})))).$$

Lemma F, 5, 2. *$\mathfrak{a}$ cannot belong to two different bounded spectral families.*

Proof. Assume that $\mathfrak{a}$ belongs to both bounded spectral families $\mathfrak{e}_\rho$ and $\mathfrak{f}_\rho$. Choose $\lambda_0^{(\nu)}, \xi_1^{(\nu)}, \lambda_1^{(\nu)}, \xi_2^{(\nu)}, \lambda_2^{(\nu)}, \ldots, \xi_{n^{(\nu)}}^{(\nu)}, \lambda_{n^{(\nu)}}^{(\nu)}$ as in Lemma F, 5, 1, and apply this Lemma to both $\mathfrak{a}, \mathfrak{e}_\rho$ and $\mathfrak{a}, \mathfrak{f}_\rho$. Then

$$\mathcal{U}\text{-}\lim_{\nu \to \infty} \left(\sum_{i=1}^{n^{(\nu)}} \varphi(\xi_i^{(\nu)})(\mathfrak{e}_{\lambda_i^{(\nu)}} - \mathfrak{e}_{\lambda_{i-1}^{(\nu)}}) - \sum_{i=1}^{n^{(\nu)}} \varphi(\xi_i^{(\nu)})(\mathfrak{f}_{\lambda_i^{(\nu)}} - \mathfrak{f}_{\lambda_{i-1}^{(\nu)}}) \right) = 0$$

results, for each continuous $\varphi(x)$.

Assume $\rho < \sigma$. Choose the $\xi_i^{(\nu)}$, $\lambda_i^{(\nu)}$ so that both ρ and σ occur as $\lambda_i^{(\nu)}$'s:

$$\lambda_{r_\nu}^{(\nu)} = \rho, \quad \lambda_{s_\nu}^{(\nu)} = \sigma, \quad 1 \leqslant r_\nu < s_\nu \leqslant n^{(\nu)}.$$

Now put

$$\varphi(x) \begin{cases} = 1 & \text{for } x \leqslant \rho, \\ = \dfrac{x-\rho}{\sigma-\rho} & \text{»} \quad \rho \leqslant x \leqslant \sigma, \\ = 0 & \text{»} \quad x \geqslant \sigma. \end{cases}$$

Then

$$-\sum_{i=1}^{n^{(\nu)}} \varphi(\xi_i^{(\nu)})(\mathfrak{e}_{\lambda_i^{(\nu)}} - \mathfrak{e}_{\lambda_{i-1}^{(\nu)}}) + \mathfrak{e}_\sigma =$$

$$= -\sum_{i=1}^{n^{(\nu)}} \varphi(\xi_i^{(\nu)})(\mathfrak{e}_{\lambda_i^{(\nu)}} - \mathfrak{e}_{\lambda_{i-1}^{(\nu)}}) + \sum_{i=1}^{s_\nu} (\mathfrak{e}_{\lambda_i^{(\nu)}} - \mathfrak{e}_{\lambda_{i-1}^{(\nu)}}) =$$

(as $\varphi(\xi_i^{(\nu)}) = 0$ for $i \geqslant s_\nu + 1$, $\xi_i^{(\nu)} \geqslant \xi_{s_\nu+1}^{(\nu)} \geqslant \lambda_{s_\nu}^{(\nu)} = \sigma$)

$$= \sum_{i=1}^{s_\nu} (1 - \varphi(\xi_i^{(\nu)}))(\mathfrak{e}_{\lambda_i^{(\nu)}} - \mathfrak{e}_{\lambda_{i-1}^{(\nu)}}) \,\varepsilon\, \mathfrak{D}$$

(as $1 - \varphi(\xi_i^{(\nu)}) \geqslant 0$), and

$$\sum_{i=1}^{n^{(\nu)}} \varphi(\xi_i^{(\nu)})(\mathfrak{f}_{\lambda_i^{(\nu)}} - \mathfrak{f}_{\lambda_{i-1}^{(\nu)}}) - \mathfrak{f}_\rho =$$

$$= \sum_{i=1}^{n^{(\nu)}} \varphi(\xi_i^{(\nu)})(\mathfrak{f}_{\lambda_i^{(\nu)}} - \mathfrak{f}_{\lambda_{i-1}^{(\nu)}}) - \sum_{i=1}^{r_\nu} (\mathfrak{f}_{\lambda_i^{(\nu)}} - \mathfrak{f}_{\lambda_{i-1}^{(\nu)}}) =$$

(as $\varphi(\xi_i^{(\nu)}) = 1$ for $i \leqslant r_\nu$, $\xi_i^{(\nu)} \leqslant \xi_{r_\nu}^{(\nu)} \leqslant \lambda_{r_\nu}^{(\nu)} = \rho$)

$$= \sum_{i=r_\nu+1}^{n^{(\nu)}} \varphi(\xi_i^{(\nu)})(\mathfrak{f}_{\lambda_i^{(\nu)}} - \mathfrak{f}_{\lambda_{i-1}^{(\nu)}}) \,\varepsilon\, \mathfrak{D}$$

(as $\varphi(\xi_i^{(\nu)}) \geqslant 0$).

Furthermore, if ε is sufficiently great, then

$$\left\| \sum_{i=1}^{n^{(\nu)}} \varphi(\xi_i^{(\nu)})(\mathfrak{e}_{\lambda_i^{(\nu)}} - \mathfrak{e}_{\lambda_{i-1}^{(\nu)}}) - \sum_{i=1}^{n^{(\nu)}} \varphi(\xi_i^{(\nu)})(\mathfrak{f}_{\lambda_i^{(\nu)}} - \mathfrak{f}_{\lambda_{i-1}^{(\nu)}}) \right\| \leqslant \varepsilon,$$

$$\varepsilon 1 + \sum_{i=1}^{n^{(\nu)}} \varphi(\xi_i^{(\nu)})(\mathfrak{e}_{\lambda_i^{(\nu)}} - \mathfrak{e}_{\lambda_{i-1}^{(\nu)}}) - \sum_{i=1}^{n^{(\nu)}} \varphi(\xi_i^{(\nu)})(\mathfrak{f}_{\lambda_i^{(\nu)}} - \mathfrak{f}_{\lambda_{i-1}^{(\nu)}}) \,\varepsilon\, \mathfrak{D}.$$

Adding these three quantities $\varepsilon\,\mathfrak{D}$, we obtain

$$\varepsilon 1 + \mathfrak{e}_\sigma - \mathfrak{f}_\rho \,\varepsilon\, \mathfrak{D}.$$

In this final formula the $\xi_i^{(\nu)}$, $\lambda_i^{(\nu)}$, $\varepsilon^{(\nu)}$ do no more appear. So we may let $\varepsilon \to 0$ then $\mathfrak{e}_\sigma - \mathfrak{f}_\rho \,\varepsilon\, \mathfrak{D}$ results [use IV_5)]. This holds whenever $\rho < \sigma$.

Now choose $\rho_1, \rho_2, \ldots$ with $\rho_1 < \rho_2 < \ldots < \sigma$, $\lim\limits_{i\to\infty} \rho_i = \sigma$. Then

$$\mathcal{T}\text{-}\lim_{i\to\infty} \mathfrak{f}_{\rho_i} = \mathfrak{f}_\sigma,$$

so the relations $\mathfrak{e}_\sigma - \mathfrak{f}_{\rho i} \varepsilon \mathfrak{D}$ give [by IV_5)]

$$\mathfrak{e}_\sigma - \mathfrak{f}_\sigma \varepsilon \mathfrak{D}.$$

The rôle of the $\mathfrak{e}_\rho$ and of the $\mathfrak{f}_\rho$ being perfectly symmetric, we have similarly

$$\mathfrak{f}_\sigma - \mathfrak{e}_\sigma \varepsilon \mathfrak{D}.$$

Now IV_4) (with $\mathfrak{a}_1 = \mathfrak{a}_2 = \ldots = \mathfrak{e}_\sigma - \mathfrak{f}_\sigma$, $\mathfrak{b}_1 = \mathfrak{b}_2 = \ldots = \mathfrak{f}_\sigma - \mathfrak{e}_\sigma$) gives

$$\mathfrak{e}_\sigma - \mathfrak{f}_\sigma = 0, \quad \mathfrak{e}_\sigma = \mathfrak{f}_\sigma$$

for all σ.

Lemmas F, 4, 2, F, 4, 3, and F, 5, 2 permit us to state:

Theorem XIII. *The relation „$\mathfrak{a}$ belongs to the family $\mathfrak{e}_\rho$, $-\infty < \rho < +\infty$", as defined in F, 4, establishes a one-to-one relation between all $\mathfrak{a} \varepsilon \mathfrak{A}$ and all bounded spectral families $\mathfrak{e}_\rho$, $-\infty < \rho < +\infty$. If the $\mathfrak{e}_\rho$ are given, then $\mathfrak{a}$ is characterized by (*) in F, 4; if $\mathfrak{a}$ is given, the $\mathfrak{e}_\rho$ are described by $\mathfrak{e}_\rho = \gamma_\rho(\mathfrak{a})$* (cf. F, 1).

Relation (*) in F, 4 suggests strongly an analogy with Stieltjes-integrals. This is even more marked, considering the more general final formula of Lemma F, 5, 1. We will therefore write symbolically:

$$\varphi((((\mathfrak{a})))) = \int_\alpha^\beta \varphi(\rho)\, d\mathfrak{e}_\rho, \tag{66}$$

where $\mathfrak{a}$ belongs to the spectral family $\mathfrak{e}_\rho$ which is covered by the interval $\alpha \leqslant x \leqslant \beta$ (cf. F, 4), and $\varphi(x)$ is everywhere defined and continuous. A special case of (66) is

$$\mathfrak{a} = \int_\alpha^\beta \rho\, d\mathfrak{e}_\rho. \tag{67}$$

As we observed in F, 4, we can choose $\alpha = \|\mathfrak{a}>$, $\beta = <\mathfrak{a}\|$.

F, 6. Theorem XIV. *The (finite) linear aggregates of idempotents are $\mathcal{U}$-everywhere dense (and so, by Theorem XII, a fortiori $\mathcal{T}$-everywhere dense) in $\mathfrak{A}$.*

Proof. Obvious by Theorem XIII, considering the definitory relations (*) in F, 4.

Corollary. For every ring $\mathfrak{M}$ (in $\mathfrak{A}$) the (finite) linear aggregates of idempotents from $\mathfrak{M}$ are $\mathcal{U}$-everywhere dense (and so, by Theorem XIII, a fortiori $\mathcal{T}$-everywhere dense) in $\mathfrak{M}$.

Proof. This is obvious for $\mathfrak{M} = \Theta$ or (0). If $\mathfrak{M} \neq \Theta$, (0), then $\mathfrak{M}$ satisfies our axioms by A, 1, and so we can apply Theorem XIII to $\mathfrak{M}$ in place of $\mathfrak{A}$.

G. Theory of idempotents (Continuation)

G, 1. We prove first a certain refinement of one of the statements of Lemma D, 2, 2.

Lemma G, 1, 1. *$\mathfrak{a} \varepsilon N_{\frac{1}{2}}(\mathfrak{e})$ and $\mathfrak{b} \varepsilon N_0(\mathfrak{e})$ or $N_1(\mathfrak{e})$ imply $\mathfrak{a}\mathfrak{b} \varepsilon N_{\frac{1}{2}}(\mathfrak{e})$.*

Proof. Since replacement of $\mathfrak{e}$ by $1 - \mathfrak{e}$ would replace $N_0(\mathfrak{e})$, $N_{\frac{1}{2}}(\mathfrak{e})$, $N_1(\mathfrak{e})$ by $N_1(\mathfrak{e})$, $N_{\frac{1}{2}}(\mathfrak{e})$, $N_0(\mathfrak{e})$ (by Lemma D, 5, 2), it suffices to consider the case where $\mathfrak{b} \varepsilon N_0(\mathfrak{e})$. Since $N_0(\mathfrak{e})$ is a ring, we may apply to it Theorem XIV, Corollary. Since $N_{\frac{1}{2}}(\mathfrak{e})$ is a module, this has the consequence that we may restrict ourselves to the idempotents $\mathfrak{b} \varepsilon N_0(\mathfrak{e})$.

So $\mathfrak{b} = \mathfrak{e}' \varepsilon N_0(\mathfrak{e})$ is idempotent, $\mathfrak{e}\mathfrak{e}' = 0$, $\mathfrak{e}$, $\mathfrak{e}'$ orthogonal. Thus $\mathfrak{e} + \mathfrak{e}'$ is idempotent too. Resolve now $\mathfrak{a}$ with respect to $\mathfrak{e} + \mathfrak{e}'$, in the sense of Lemma D, 2, 1:

$$\mathfrak{a} = \mathfrak{a}_0^* + \mathfrak{a}_{\frac{1}{2}}^* + \mathfrak{a}_1^*, \quad \mathfrak{a}_\rho^* \varepsilon N_\rho(\mathfrak{e} + \mathfrak{e}') \quad \text{for } \rho = 0, \frac{1}{2}, 1.$$

Since $\mathfrak{a} \varepsilon N_{\frac{1}{2}}(\mathfrak{e})$ so $\mathfrak{e}\mathfrak{a} = \frac{1}{2}\mathfrak{a}$, therefore

$$\mathfrak{e}\mathfrak{a}_0^* + \mathfrak{e}\mathfrak{a}_{\frac{1}{2}}^* + \mathfrak{e}\mathfrak{a}_1^* = \frac{1}{2}\mathfrak{a}_0^* + \frac{1}{2}\mathfrak{a}_{\frac{1}{2}}^* + \frac{1}{2}\mathfrak{a}_1^*.$$

As $(\mathfrak{e} + \mathfrak{e}')\mathfrak{e} = \mathfrak{e}$, $\mathfrak{e} \varepsilon N_1(\mathfrak{e} + \mathfrak{e}')$, therefore Lemma D, 2, 2 gives (when applied to $\mathfrak{e} + \mathfrak{e}'$):

$$\mathfrak{a}_0^* \mathfrak{e} = 0, \quad \mathfrak{a}_{\frac{1}{2}}^* \mathfrak{e} \varepsilon N_0(\mathfrak{e} + \mathfrak{e}') + N_{\frac{1}{2}}(\mathfrak{e} + \mathfrak{e}'),$$

$$\mathfrak{a}_1^* \mathfrak{e} \varepsilon N_1(\mathfrak{e} + \mathfrak{e}').$$

Thus by Lemma D, 2, 1 (as the previous formula give two decompositions of $\frac{1}{2}\mathfrak{a}$ with respect to $\mathfrak{e} + \mathfrak{e}'$):

$$\mathfrak{a}_{\frac{1}{2}}^* \mathfrak{e} = \frac{1}{2}\mathfrak{a}_0^* + \frac{1}{2}\mathfrak{a}_{\frac{1}{2}}^*, \quad \mathfrak{a}_1^* \mathfrak{e} = \frac{1}{2}\mathfrak{a}_1^*.$$

By definition

$$\mathfrak{a}_0^*(\mathfrak{e} + \mathfrak{e}') = 0, \quad \mathfrak{a}_{\frac{1}{2}}^*(\mathfrak{e} + \mathfrak{e}') = \frac{1}{2}\mathfrak{a}_{\frac{1}{2}}^*, \quad \mathfrak{a}_1^*(\mathfrak{e} + \mathfrak{e}') = \mathfrak{a}_1^*.$$

This gives, using the preceding equations as well as $\mathfrak{a}_0^* \mathfrak{e} = 0$, that

$$\mathfrak{a}_0^* \mathfrak{e}' = 0, \quad \mathfrak{a}_{\frac{1}{2}}^* \mathfrak{e}' = -\frac{1}{2}\mathfrak{a}_0^*, \quad \mathfrak{a}_1^* \mathfrak{e}' = \frac{1}{2}\mathfrak{a}_1^*.$$

Hence $(\mathfrak{a}_{\frac{1}{2}}^* \mathfrak{e}')\mathfrak{e}' = 0$, implying $((\mathfrak{a}_{\frac{1}{2}}^* \mathfrak{e}')\mathfrak{e}')\mathfrak{e}' = 0$, and so by (34) in D, 1

$$\mathfrak{a}_{\frac{1}{2}}^* \mathfrak{e}' = 0, \quad \mathfrak{a}_{\frac{1}{2}}^* \varepsilon N_0(\mathfrak{e}').$$

Now $\mathfrak{a}_0^* = -2\mathfrak{a}_{\frac{1}{2}}^* \mathfrak{e}' = 0$. Thus

$$\mathfrak{a}\mathfrak{e}' = \mathfrak{a}_0^* \mathfrak{e}' + \mathfrak{a}_{\frac{1}{2}}^* \mathfrak{e}' + \mathfrak{a}_1^* \mathfrak{e}' = 0 + 0 + \frac{1}{2}\mathfrak{a}_1^* = \frac{1}{2}\mathfrak{a}_1^*.$$

Since $\mathfrak{a}_1^* \mathfrak{e} = \frac{1}{2}\mathfrak{a}_1^*$, $\mathfrak{a}_1^* \varepsilon N_{\frac{1}{2}}(\mathfrak{e})$, so we have

$$\mathfrak{a}\mathfrak{b} = \mathfrak{a}\mathfrak{e}' = \frac{1}{2}\mathfrak{a}_1^* \varepsilon N_{\frac{1}{2}}(\mathfrak{e}').$$

Lemma G, 1, 2. *If* $\mathfrak{e}$, $\mathfrak{f}$ *are orthogonal idempotents, then for every* $\mathfrak{x}$

$$\mathfrak{e}(\mathfrak{f}\mathfrak{x}) = \mathfrak{f}(\mathfrak{e}\mathfrak{x}).$$

Proof. By Lemma D, 2, 1 it suffices to prove this for the $\mathfrak{x} \varepsilon N_\rho(\mathfrak{e})$ only, for $\rho = 0, \frac{1}{2}, 1$. Besides $\mathfrak{e}\mathfrak{f} = 0$, $\mathfrak{f} \varepsilon N_0(\mathfrak{e})$. Now Lemmas D, 2, 2 and G, 1, 1 prove our

statement: they show that for $\rho = 0, \frac{1}{2}, 1$ both sides are equal to $0, \frac{1}{2}\mathfrak{f}\mathfrak{x}, 0$ respectively.

We are now in the position to prove:

Theorem XV. *If* $\mathfrak{b}, \mathfrak{c} \varepsilon \mathfrak{P}(\mathfrak{a}, 1)$ *then for every* $\mathfrak{x}$

$$\mathfrak{b}(\mathfrak{c}\mathfrak{x}) = \mathfrak{c}(\mathfrak{b}\mathfrak{x}).$$

Proof. Replacing $\mathfrak{a}$ by $\mathfrak{a} - \sigma 1$, $\sigma \not\varepsilon \mathcal{J}(\mathfrak{a})$ shows that we need to consider $\mathfrak{b}, \mathfrak{c} \varepsilon \mathfrak{P}(\mathfrak{a})$ only (use Lemma F, 3, 2). It suffices even to prove it for $\mathfrak{b}, \mathfrak{c} \varepsilon \mathfrak{P}^0(\mathfrak{a})$ only (cf. C, 1): by IV_3) it extends first to all $\mathfrak{b} \varepsilon \mathfrak{P}^0(\mathfrak{a})$, $\mathfrak{c} \varepsilon \mathfrak{P}(\mathfrak{a})$, and then to all $\mathfrak{b}, \mathfrak{c} \varepsilon \mathfrak{P}(\mathfrak{a})$. So we may assume $\mathfrak{b} = p((((\mathfrak{a}))))$, $\mathfrak{c} = q((((\mathfrak{a}))))$ where $p(x)$, $q(x)$ are polynomials with $p(0) = q(0) = 0$. But then we may restrict ourselves to the polynomials $p(x) \equiv x^n$, $q(x) \equiv x^m$, $m, n = 1, 2, \ldots$ So we must prove

$$\mathfrak{a}^n(\mathfrak{a}^m\mathfrak{x}) = \mathfrak{a}^m(\mathfrak{a}^n\mathfrak{x})$$

only.

By Lemma E, 4, 3 we can restrict ourselves to any $\mathcal{U}$-everywhere dense set of $\mathfrak{a}$. By Theorem XIII the $\sum_{i=1}^{l} \xi_i(\mathfrak{e}_{\lambda_i} - \mathfrak{e}_{\lambda_{i-1}})$ will do, where $\lambda_0 \leqslant \xi_1 \leqslant \lambda_1 \leqslant \xi_2 \leqslant \lambda_2 \leqslant \ldots$ $\ldots \leqslant \xi_n \leqslant \lambda_n$ and $\mathfrak{e}_\rho$ is any bounded spectral family. A fortiori the $\mathfrak{a} = \sum_{=1}^{l} \xi_i(\mathfrak{f}_i - \mathfrak{f}_{i-1})$ with $\mathfrak{f}_0 \leqslant \mathfrak{f}_1 \leqslant . \; . \leqslant {}_n$ will do. Now our formula becomes [by (65) in F, 5]

$$\sum_{i,j=1}^{l} \xi_i^n \xi_j^m (\mathfrak{f}_i - \mathfrak{f}_{i-1})((\mathfrak{f}_j - \mathfrak{f}_{j-1})\mathfrak{x}) =$$
$$= \sum_{i,j=1}^{l} \xi_i^n \xi_j^m (\mathfrak{f}_j - \mathfrak{f}_{j-1})((\mathfrak{f}_i - \mathfrak{f}_{i-1})\mathfrak{x}).$$

So we must prove

$$(\mathfrak{f}_i - \mathfrak{f}_{i-1})((\mathfrak{f}_j - \mathfrak{f}_{j-1})\mathfrak{x}) = (\mathfrak{f}_j - \mathfrak{f}_{j-1})((\mathfrak{f}_i - \mathfrak{f}_{i-1})\mathfrak{x}).$$

For $i = j$ this is obvious. For $i \neq j$ $\mathfrak{f}_i - \mathfrak{f}_{i-1}$ and $\mathfrak{f}_j - \mathfrak{f}_{j-1}$ are orthogonal [cf. the derivation of (65) in F, 5], and so Lemma G, 1, 2 gives the desired result (replace in it $\mathfrak{e}, \mathfrak{f}, \mathfrak{x}$ by $\mathfrak{f}_i - \mathfrak{f}_{i-1}$, $\mathfrak{f}_j - \mathfrak{f}_{j-1}$, $\mathfrak{x}$).

We introduce the abbreviation $[\mathfrak{a}, \mathfrak{b}, \mathfrak{c}]$:

$$[\mathfrak{a}, \mathfrak{b}, \mathfrak{c}] = (\mathfrak{a}\mathfrak{b})\mathfrak{c} - \mathfrak{a}(\mathfrak{b}\mathfrak{c}). \tag{68}$$

One verifies easily (owing to the commutative law)

$$\left.\begin{aligned} &[\mathfrak{a}, \mathfrak{b}, \mathfrak{c}] + [\mathfrak{c}, \mathfrak{b}, \mathfrak{a}] = 0, \\ &[\mathfrak{a}, \mathfrak{b}, \mathfrak{c}] + [\mathfrak{b}, \mathfrak{c}, \mathfrak{a}] + [\mathfrak{c}, \mathfrak{a}, \mathfrak{b}] = 0. \end{aligned}\right\} \tag{69}$$

Lemma G, 1, 2 becomes:

$$[\mathfrak{e}, \mathfrak{x}, \mathfrak{f}] = 0 \quad \text{if } \mathfrak{e}, \mathfrak{f} \text{ are orthogonal idempotents.} \tag{70}$$

And Lemma G, 1, 3:

$$[\mathfrak{b}, \mathfrak{x}, \mathfrak{c}] = 0 \quad \text{if } \mathfrak{b}, \mathfrak{c} \varepsilon \mathfrak{P}(\mathfrak{a}, 1). \tag{71}$$

We finally prove:

Theorem XVI:

$$[\mathfrak{a}\mathfrak{b}, \mathfrak{x}, \mathfrak{c}] + [\mathfrak{b}\mathfrak{c}, \mathfrak{x}, \mathfrak{a}] + [\mathfrak{c}\mathfrak{a}, \mathfrak{x}, \mathfrak{b}] = 0.$$

Proof. (71) gives $[\mathfrak{a}^2, \mathfrak{x}, \mathfrak{a}] = 0$. Now replace herein $\mathfrak{a}$ by $\rho_1\mathfrak{a} + \rho_2\mathfrak{b} + \rho_3\mathfrak{c}$, and apply Lemma D, 1, 1 to the coefficient of $\rho_1\rho_2\rho_3$. This coefficient is $[\mathfrak{a}\mathfrak{b}, \mathfrak{x}, \mathfrak{c}] + [\mathfrak{b}\mathfrak{c}, \mathfrak{x}, \mathfrak{a}] + [\mathfrak{c}\mathfrak{a}, \mathfrak{x}, \mathfrak{b}]$, and so the desired formula results.

Corollary. $[\mathfrak{a}^2, \mathfrak{x}, \mathfrak{b}] = -2[\mathfrak{a}\mathfrak{b}, \mathfrak{x}, \mathfrak{a}]$.

Proof. Put $\mathfrak{a} = \mathfrak{c}$ in Theorem XVI.

G, 2. Lemma D, 2, 1 gave a decomposition of an arbitrary $\mathfrak{x} \varepsilon \mathfrak{A}$ with respect to an idempotent $\mathfrak{e}$. We will now give a more refined simultaneous decomposition with respect to a system of mutually orthogonal idempotents $\mathfrak{e}_1, \ldots, \mathfrak{e}_n$.

Theorem XVII. *Let $\mathfrak{e}_1, \ldots, \mathfrak{e}_n$ be mutually orthogonal idempotents with $\mathfrak{e}_1 + \ldots + \mathfrak{e}_n = 1$. Define $N_{ij} = N_{ij}(\mathfrak{e}_1, \ldots, \mathfrak{e}_n)$ as the set of all $\mathfrak{x}$ with*

$$\mathfrak{x}\mathfrak{e}_k = \frac{1}{2}(\delta_{ik} + \delta_{jk})\mathfrak{x} \quad \textit{for all } k = 1, \ldots, n \tag{72}$$

(δ_{hl} is the Kronecker-Weierstrass symbol:

$$\delta_{hl} \begin{cases} = 1 & \textit{for } h = l, \\ = 0 & \text{»} \quad h \neq l.) \end{cases}$$

Clearly $N_{ij} = N_{ji}$. Each N_{ij} is a module, the N_{ii}'s are even rings.

$\mathfrak{A}$ is the direct sum of all N_{ij}, $i \leqslant j$, $i, j = 1, \ldots, n$, that is: every $\mathfrak{x} \varepsilon \mathfrak{A}$ permits a unique decomposition

$$\mathfrak{x} = \sum_{\substack{i,j=1 \\ (i \leqslant j)}}^{n} \mathfrak{x}_{ij}, \quad \mathfrak{x}_{ij} \varepsilon N_{ij}. \tag{73}$$

Herein

$$\left.\begin{aligned} \mathfrak{x}_{ii} &= 2\mathfrak{e}_i(\mathfrak{e}_i\mathfrak{x}) - (\mathfrak{e}_i\mathfrak{x}), \\ \mathfrak{x}_{ij} &= 4\mathfrak{e}_i(\mathfrak{e}_j\mathfrak{x}) = 4\mathfrak{e}_j(\mathfrak{e}_i\mathfrak{x}) \quad \textit{for} \quad i \neq j. \end{aligned}\right\} \tag{74}$$

Proof. That each N_{ij} is a module, is clear by (72), and so is $N_{ij} = N_{ji}$. For $i = j$ (72) states that $\mathfrak{x}\mathfrak{e}_i = \mathfrak{x}$, and $\mathfrak{x}\mathfrak{e}_j = 0$ for $j \neq i$. But as $\mathfrak{e}_i\mathfrak{e}_j = 0$, $\mathfrak{e}_j \varepsilon N_0(\mathfrak{e}_i)$ the irst relation that is $\mathfrak{x} \varepsilon N_1(\mathfrak{e}_i)$, implies the others: $\mathfrak{x}\mathfrak{e}_j = 0$. So $N_{ii} = N_1(\mathfrak{e}_i)$. Thus N_{ii} is a ring (use Theorem IV).

The self-consistency of the second set of definitions in (74) (for $i \neq j$) follows from Lemma G, 1, 2.

If (73) holds, one verifies (74) with the help of (72) (which holds for $\mathfrak{x}_{ij}$). So we must only prove that (74) does really solve (73).

The $\mathfrak{x}_{ii}$ of (74) belong to $N_1(\mathfrak{e}_i)$ by Lemma D, 2, 1, (37), and thus to N_{ii}.

Consider next (74) for $i \neq j$. Use Theorem XVI, Corollary, with $\mathfrak{e}_i$, $\mathfrak{x}$, $\mathfrak{e}_j$ instead of $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{x}$. Then

$$[\mathfrak{e}_i, \mathfrak{e}_j, \mathfrak{x}] = -2[\mathfrak{e}_i\mathfrak{x}, \mathfrak{e}_j, \mathfrak{e}_i]$$

results. This means

$$-\mathfrak{e}_i(\mathfrak{e}_j\mathfrak{x}) = -2((\mathfrak{e}_i\mathfrak{x})\mathfrak{e}_j)\mathfrak{e}_i, \quad \mathfrak{e}_i(\mathfrak{e}_j(\mathfrak{e}_i\mathfrak{x})) = \frac{1}{2}\mathfrak{e}_i(\mathfrak{e}_j\mathfrak{x}),$$

$$\mathfrak{e}_i\mathfrak{x}_{ij} = \frac{1}{2}\mathfrak{x}_{ij}.$$

By symmetry

$$\mathfrak{e}_j\mathfrak{x}_{ij} = \frac{1}{2}\mathfrak{x}_{ij}$$

holds too. Thus (72) holds for $k = i$ or j. If $k \neq i, j$, then

$$(\mathfrak{e}_i + \mathfrak{e}_j)\mathfrak{e}_k = 0, \quad \mathfrak{e}_k \varepsilon N_0(\mathfrak{e}_i + \mathfrak{e}_j),$$

while
$$(\mathfrak{e}_i+\mathfrak{e}_j)\,\mathfrak{x}_{ij}=\frac{1}{2}\mathfrak{x}_{ij}+\frac{1}{2}\mathfrak{x}_{ij}=\mathfrak{x}_{ij}\,,\quad \mathfrak{x}_{ij}\in N_1(\mathfrak{e}_i+\mathfrak{e}_j),$$
so that $\mathfrak{e}_k\mathfrak{x}_{ij}=0$ (by Theorem IV). Thus (72) holds for $k\neq i,\ j$ too. So $\mathfrak{x}_{ij}\in N_{ij}$.

Finally
$$\sum_{\substack{(i,j=1\\(i\leq j)}}^{n}\mathfrak{x}_{ij}=\sum_{i=1}^{n}(2\mathfrak{e}_i(\mathfrak{e}_i\mathfrak{x})-\mathfrak{e}_i\mathfrak{x})+2\sum_{\substack{i,j=1\\(i<j)}}^{n}(\mathfrak{e}_i(\mathfrak{e}_j\mathfrak{x})+\mathfrak{e}_j(\mathfrak{e}_i\mathfrak{x}))=$$
$$=2\sum_{i,j=1}^{n}\mathfrak{e}_i(\mathfrak{e}_j\mathfrak{x})-\sum_{i=1}^{n}\mathfrak{e}_i\mathfrak{x}=$$
$$=2\Big(\sum_{i=1}^{n}\mathfrak{e}_i\Big)\Big(\sum_{i=1}^{n}\mathfrak{e}_i\Big)\mathfrak{x}-\Big(\sum_{i=1}^{n}\mathfrak{e}_i\Big)\mathfrak{x}=$$
$$=2\cdot1\cdot1\cdot\mathfrak{x}-1\cdot\mathfrak{x}=2\mathfrak{x}-\mathfrak{x}=\mathfrak{x}.$$

Thus all parts of (73) are fulfilled by the $\mathfrak{x}_{ij}$ from (74).

Theorem XVIII. *Suppose* $\mathfrak{x}\in N_{ij}$, $\mathfrak{y}\in N_{kl}$. *Then*:

(i) *If both i, j differ from k, l, then* $\mathfrak{x}\mathfrak{y}=0$.

(ii) *If i, j and k, l have one common index (order does not matter*, cf. Theorem XVII*), say $j=k$, but $i\neq l$, then* $\mathfrak{x}\mathfrak{y}\in N_{il}$.

(iii) *If i, j and k, l are the same pair (order does not matter,* cf. above*), say $i=k$, $j=l$, then* $\mathfrak{x}\mathfrak{y}\in N_{ii}+N_{jj}$ *if* $i\neq j$, *and* $\mathfrak{x}\mathfrak{y}\in N_{ii}$ *if* $i=j$.

Proof. Put
$$\mathfrak{e}\begin{cases}=\mathfrak{e}_i+\mathfrak{e}_j & \text{for } i\neq j,\\ =\mathfrak{e}_i & \text{«}\quad i=j,\end{cases}$$
$$\mathfrak{f}\begin{cases}=\mathfrak{e}_k+\mathfrak{e}_l & \text{for } k\neq l,\\ =\mathfrak{e}_k & \text{«}\quad k=l.\end{cases}$$
Then (72) gives
$$\mathfrak{e}\mathfrak{x}=\mathfrak{x},\quad \mathfrak{f}\mathfrak{y}=\mathfrak{y},$$
so
$$\mathfrak{x}\in N_1(\mathfrak{e}),\quad \mathfrak{y}\in N_1(\mathfrak{f}).$$

Ad (i): here $\mathfrak{e}\mathfrak{f}=0$, so $\mathfrak{f}\in N_0(\mathfrak{e})$ (use Theorem IV in what follows), thus $\mathfrak{f}\mathfrak{x}=0$, $\mathfrak{x}\in N_0(\mathfrak{f})$, and so
$$\mathfrak{x}\mathfrak{y}=0.$$

Ad (ii): assume first $j=k\neq i,\ l,\ \ i\neq l$.
$$\mathfrak{M}=N_1(\mathfrak{e}_i+\mathfrak{e}_j+\mathfrak{e}_l)$$
is an algebra. It is defined by
$$(\mathfrak{e}_i+\mathfrak{e}_j+\mathfrak{e}_l)\,\mathfrak{z}=\mathfrak{z}.$$
Resolving $\mathfrak{z}$ by Theorem XVII, (73), we see that this means $\mathfrak{z}_{pq}=0$, except when $p,\ q=i,\ j,\ l$. Hence
$$\mathfrak{M}=\sum_{p,\,q=i,\,j,\,l}N_{pq}\,.$$
In a similar way it appears that in the ring $\mathfrak{M}$ (not in $\mathfrak{A}$!) we have the relations
$$N_1^{(\mathfrak{M})}(\mathfrak{e}_i+\mathfrak{e}_j)=\sum_{p,\,q=i,\,j}N_{pq}\,,\quad N_{\frac{1}{2}}^{(\mathfrak{M})}(\mathfrak{e}_i+\mathfrak{e}_j)=\sum_{p=i,\,j}N_{pl}\,,\quad N_0^{(\mathfrak{M})}(\mathfrak{e}_l+\mathfrak{e}_j)=N_{ll}\,.$$

So

$$\mathfrak{x} \varepsilon N_1^{(\mathfrak{M})}(\mathfrak{e}_i + \mathfrak{e}_j), \quad \mathfrak{y} \varepsilon N_{\frac{1}{2}}^{(\mathfrak{M})}(\mathfrak{e}_l + \mathfrak{e}_j),$$

and thus by Lemma G, 1, 1 (replacing $\mathfrak{e}$, $\mathfrak{A}$ by $\mathfrak{e}_i + \mathfrak{e}_j$, $\mathfrak{M}$)

$$\mathfrak{x}\mathfrak{y} \varepsilon N_{\frac{1}{2}}^{(\mathfrak{M})}(\mathfrak{e}_i + \mathfrak{e}_j) = N_{il} + N_{jl}.$$

By symmetry (replace i, $j = k$, l and $\mathfrak{x}$, $\mathfrak{y}$ by l, $k = j$, i and $\mathfrak{y}$, $\mathfrak{x}$)

$$\mathfrak{x}\mathfrak{y} \varepsilon N_{il} + N_{ij}$$

too. Because of the uniqueness of the resolution of Theorem XVII (73) (applied to $\mathfrak{x}\mathfrak{y}$) $\mathfrak{x}\mathfrak{y} \varepsilon N_{il}$ results.

Assume next $j = k = i \neq l$. Then put $\mathfrak{M} = N(\mathfrak{e}_i + \mathfrak{e}_l)$, and find in the same way as above:

$$\mathfrak{M} = \sum_{p,\, q = i,\, l} N_{pq},$$

and

$$N_1^{(\mathfrak{M})}(\mathfrak{e}_i) = N_{ii}, \quad N_{\frac{1}{2}}^{(\mathfrak{M})}(\mathfrak{e}_i) = N_{il}, \quad N_0^{(\mathfrak{M})}(\mathfrak{e}_i) = N_{ll}.$$

Then

$$\mathfrak{x} \varepsilon N_1^{(\mathfrak{M})}(\mathfrak{e}_i), \quad \mathfrak{y} \varepsilon N_{\frac{1}{2}}^{(\mathfrak{M})}(\mathfrak{e}_i),$$

so by Lemma G, 1, 1 (replacing $\mathfrak{e}$, $\mathfrak{A}$ by $\mathfrak{e}_i$, $\mathfrak{M}$)

$$\mathfrak{x}\mathfrak{y} \varepsilon N_{\frac{1}{2}}^{(\mathfrak{M})}(\mathfrak{e}_i) = N_{il}.$$

By symmetry $j = k = l \neq i$ (replace $i = j = k$, l and $\mathfrak{x}$, $\mathfrak{y}$ by $l = k = j$, i and $\mathfrak{y}$, $\mathfrak{x}$) too implies $\mathfrak{x}\mathfrak{y} \varepsilon N_{ll}$.

Thus all possibilities arising in (ii) are exhausted.

Ad (iii): here $\mathfrak{e} = \mathfrak{f}$, so $\mathfrak{x}$, $\mathfrak{y} \varepsilon N_1(\mathfrak{e})$ implying

$$\mathfrak{x}\mathfrak{y} \varepsilon N_1(\mathfrak{e}).$$

If $i = j$ then

$$N_1(\mathfrak{e}) = N_1(\mathfrak{e}_i) = N_{ii}.$$

Thus $\mathfrak{x}\mathfrak{y} \varepsilon N_{ii}$.

If $i \neq j$, then

$$N_1(\mathfrak{e}) = N_1(\mathfrak{e}_i + \mathfrak{e}_j) = \sum_{p,\, q = i,\, j} N_{pq}$$

(verify this as above). Now in $\mathfrak{M} = N_1(\mathfrak{e}) = \sum_{p,\, q = i,\, j} N_{pq}$ we have

$$N_1^{(\mathfrak{M})}(\mathfrak{e}_i) = N_{ii}, \quad N_{\frac{1}{2}}^{(\mathfrak{M})}(\mathfrak{e}_i) = N_{ij}, \quad N_0^{(\mathfrak{M})}(\mathfrak{e}_i) = N_{jj}.$$

So $\mathfrak{x}$, $\mathfrak{y} \varepsilon N_{\frac{1}{2}}^{(\mathfrak{M})}$ and thus Theorem IV gives (replacing $\mathfrak{e}$, $\mathfrak{A}$ by $\mathfrak{e}_i$, $\mathfrak{M}$)

$$\mathfrak{x}\mathfrak{y} \varepsilon N_0^{(\mathfrak{M})} + N_1^{(\mathfrak{M})} = N_{ii} + N_{jj}.$$

Thus all possibilities arising in (iii) are exhausted.

It is obvious that Theorems XVII and XVIII indicate a decomposition of along the well-known lines of the matrix scheme (corresponding to Wedderburn's theorem on

finite-basis algebras). The $i, j = 1, \ldots, n$ correspond to the indices, and the N_{ij} have a certain analogy to the „field" of a matrix-algebra. While this analogy will be very important in later parts of our analysis, we do not discuss it any further now.

H. Commutativity

H, 1. We wish to define the commutativity of two elements $\mathfrak{a}$, $\mathfrak{b}$ of $\mathfrak{A}$. Since $\mathfrak{a}\mathfrak{b} = \mathfrak{b}\mathfrak{a}$ is always true, so this cannot be done in the customary way. The natural way to do it is therefore as follows:

For each $\mathfrak{a} \varepsilon \mathfrak{A}$ denote by $\mathcal{V}_{\mathfrak{a}}$ the operation:

$$\mathcal{V}_{\mathfrak{a}}\, \mathfrak{x} \equiv \mathfrak{a}\mathfrak{x}, \quad \mathfrak{x} \varepsilon \mathfrak{A}. \tag{75}$$

Two $\mathfrak{a}$, $\mathfrak{b}$ commute if the operations $\mathcal{V}_{\mathfrak{a}}$, $\mathcal{V}_{\mathfrak{b}}$ commute:

$$\mathcal{V}_{\mathfrak{a}}\, \mathcal{V}_{\mathfrak{b}} = \mathcal{V}_{\mathfrak{b}}\, \mathcal{V}_{\mathfrak{a}},$$

that is

$$\mathcal{V}_{\mathfrak{a}}\, \mathcal{V}_{\mathfrak{b}}\, \mathfrak{x} \equiv \mathcal{V}_{\mathfrak{b}}\, \mathcal{V}_{\mathfrak{a}}\, \mathfrak{x} \quad \text{for all} \quad \mathfrak{x} \varepsilon \mathfrak{A}. \tag{76}$$

Observe that commutativity of $\mathfrak{a}$, $\mathfrak{b}$ thus means

$$\mathfrak{a}(\mathfrak{b}\mathfrak{x}) = \mathfrak{b}(\mathfrak{a}\mathfrak{x}),$$

that is

$$\mathfrak{a}(\mathfrak{x}\mathfrak{b}) = (\mathfrak{a}\mathfrak{x})\,\mathfrak{b},$$

that is

$$[\mathfrak{a}, \mathfrak{x}, \mathfrak{b}] = 0 \quad \text{for all} \quad \mathfrak{x} \varepsilon \mathfrak{A}. \tag{77}$$

Observe that we do not have in general $\mathcal{V}_{\mathfrak{a}}\, \mathcal{V}_{\mathfrak{b}} = \mathcal{V}_{\mathfrak{a}\mathfrak{b}}$, this would mean

$$\mathfrak{a}(\mathfrak{b}\mathfrak{x}) = (\mathfrak{a}\mathfrak{b})\,\mathfrak{x},$$

that is

$$[\mathfrak{a}, \mathfrak{b}, \mathfrak{x}] = 0 \quad \text{for all} \quad \mathfrak{x} \varepsilon \mathfrak{A}. \tag{78}$$

We repeat: commutativity of $\mathfrak{a}$, $\mathfrak{b}$ is defined by (77).

A set $\mathfrak{S} \subset \mathfrak{A}$ is Abelian if any two $\mathfrak{a}, \mathfrak{b} \varepsilon \mathfrak{S}$ commute. For any set $\mathfrak{S} \subset \mathfrak{A}$ we denote by $\mathfrak{S}'$ the set of all those $\mathfrak{a} \varepsilon \mathfrak{A}$, which commute with every $\mathfrak{b} \varepsilon \mathfrak{S}$. Thus a set $\mathfrak{S}$ is Abelian if and only if $\mathfrak{S} \subset \mathfrak{S}'$.

We can show at once:

Lemma H, 1, 1. (i) *Any two orthogonal idempotents* $\mathfrak{e}$, $\mathfrak{f}$ *commute.*

(ii) *Any two idempotents* $\mathfrak{e}$, $\mathfrak{f}$ *with* $\mathfrak{e} \gtreqless \mathfrak{f}$ *commute.*

(iii) *The ring* $\mathfrak{P}(\mathfrak{a}, 1)$ *is always Abelian.*

Proof. Ad (i): coincides with Lemma G, 1, 2.

Ad (ii): by symmetry we may assume that $\mathfrak{e} \leqslant \mathfrak{f}$. Then $\mathfrak{e}$, $\mathfrak{f} - \mathfrak{e}$ are orthogonal idempotents, so they commute by (i). Thus $\mathfrak{e}$ and $(\mathfrak{f} - \mathfrak{e}) + \mathfrak{e} = \mathfrak{f}$ commute too.

Ad (iii): coincides with Theorem XV.

We now wish to prove that every $\mathfrak{S}'$ is a ring. While it is easy to see that it is a module, it is not so with $\mathfrak{a}, \mathfrak{b} \varepsilon \mathfrak{S}'$ implying $\mathfrak{a}\mathfrak{b} \varepsilon \mathfrak{S}'$. In other words: the essential difficulty consists in showing that if $\mathfrak{a}$, $\mathfrak{b}$ both commute with $\mathfrak{c}$, then $\mathfrak{a}\mathfrak{b}$ does too. This is a very characteristic difference between the behavior of „commutativity" in $\mathfrak{A}$, and in the usual type of non-commutative but associative algebras.

We will reach our goal in several steps, with an essential use of spectral theory, and particularly of a certain trick due to T. Carleman (cf. Lemma H, 1, 3).

Lemma H, 1, 2. *For an idempotent* $\mathfrak{e}$

$$(\mathfrak{e})' = N_0(\mathfrak{e}) + N_1(\mathfrak{e}).$$

Proof. Sufficiency: in order to show that all $\mathfrak{x} \varepsilon N_0(\mathfrak{e}) + N_1(\mathfrak{e})$ commute with $\mathfrak{e}$, it suffices to prove this for the $\mathfrak{x} \varepsilon N_0(\mathfrak{e})$ and the $\mathfrak{x} \varepsilon N_1(\mathfrak{e})$. Now each $N_\rho(\mathfrak{e})$, $\rho = 0, 1$, is a ring, and so Theorem XIV, Corollary, applies to it. Thus we need to consider only idempotents $\mathfrak{x} \varepsilon N_\rho(\mathfrak{e})$, $\rho = 0, 1$. That is: $\mathfrak{x}$ is either an idempotent which is orthogonal to $\mathfrak{e}$, or an idempotent $\leqslant \mathfrak{e}$. Therefore it commutes with $\mathfrak{e}$ by Lemma H, 1, 1, (i), (ii), respectively.

Necessity: assume that $\mathfrak{x}$, $\mathfrak{e}$ commute. Apply the decomposition (36) of Lemma D, 2, 1. Then (37) eod. gives:

$$\mathfrak{x}_{\frac{1}{2}} = -4\mathfrak{e}(\mathfrak{e}\mathfrak{x}) + 4(\mathfrak{e}\mathfrak{x}) = 0,$$

as $\mathfrak{e}(\mathfrak{e}\mathfrak{x}) = (\mathfrak{e}\mathfrak{e})\mathfrak{x} = \mathfrak{e}\mathfrak{x}$. So

$$\mathfrak{x} = \mathfrak{x}_0 + \mathfrak{x}_1 \varepsilon N_0(\mathfrak{e}) + N_1(\mathfrak{e}).$$

We prove now a series of Lemmas establishing a connection between commutativity and the spectral form (67). In what follows $\mathfrak{e}_\rho$, $-\infty < \rho < +\infty$, will always be the bounded spectral family to which $\mathfrak{a}$ belongs. (Cf. the beginning of F,4 and Theorem XIII.)

Lemma H, 1, 3. *Let* $\mathfrak{a}$, $\mathfrak{b}$ *commute. If* $\rho_1 \leqslant \rho_2 < \sigma_1 \leqslant \sigma_2$ *are four numbers with* $(\rho_2 - \rho_1) + (\sigma_2 - \sigma_1) < \sigma_1 - \rho_2$, *then*

$$(\mathfrak{e}_{\rho_2} - \mathfrak{e}_{\rho_1})((\mathfrak{e}_{\sigma_2} - \mathfrak{e}_{\sigma_1})\mathfrak{b}) = 0.$$

Proof. Put $\mathfrak{e}_1 = \mathfrak{e}_{\rho_2} - \mathfrak{e}_{\rho_1}$, $\mathfrak{e}_2 = \mathfrak{e}_{\sigma_2} - \mathfrak{e}_{\sigma_1}$. As $\mathfrak{e}_{\rho_1} \leqslant \mathfrak{e}_{\rho_2} \leqslant \mathfrak{e}_{\sigma_1} \leqslant \mathfrak{e}_{\sigma_2}$, so $\mathfrak{e}_1$, $\mathfrak{e}_2$ are orthogonal [cf. the derivation of (65) in F, 5]. Thus $\mathfrak{e}_1 + \mathfrak{e}_2$ is idempotent too, and so is $\mathfrak{e}_3 = 1 - \mathfrak{e}_1 - \mathfrak{e}_2$. As $\mathfrak{e}_1, \mathfrak{e}_2 \leqslant \mathfrak{e}_1 + \mathfrak{e}_2 = 1 - \mathfrak{e}_3$, they are orthogonal to $\mathfrak{e}_3$. So we see: $\mathfrak{e}_1$, $\mathfrak{e}_2$, $\mathfrak{e}_3$ are mutually orthogonal idempotents with $\mathfrak{e}_1 + \mathfrak{e}_2 + \mathfrak{e}_3 = 1$.

Apply the decomposition (73) of Theorem XVII to $\mathfrak{a}$, $\mathfrak{b}$ with respect to these $\mathfrak{e}_1$, $\mathfrak{e}_2$, $\mathfrak{e}_3$:

$$\mathfrak{a} = \sum_{\substack{i,\, j=1 \\ (i \leqslant j)}}^{3} \mathfrak{a}_{ij}, \qquad \mathfrak{b} = \sum_{\substack{i,\, j=1 \\ (i \leqslant j)}}^{3} \mathfrak{b}_{ij}.$$

By (62) in F, 3, $\mathfrak{e}_1$, $\mathfrak{e}_2$, $\mathfrak{e}_3$, $\mathfrak{a} \varepsilon \mathfrak{P}(\mathfrak{a}, 1)$, and so Lemma F, 3, 2, Corollary, (i), gives [use (74) in Theorem XVII]:

$$\mathfrak{a}_{ii} = 2\mathfrak{e}_i(\mathfrak{e}_i\mathfrak{a}) - \mathfrak{e}_i\mathfrak{a} = 2(\mathfrak{e}_i\mathfrak{e}_i)\mathfrak{a} - \mathfrak{e}_i\mathfrak{a} = 2\mathfrak{e}_i\mathfrak{a} - \mathfrak{e}_i\mathfrak{a} = \mathfrak{e}_i\mathfrak{a}, \quad \mathfrak{a}_{ij} = 4\mathfrak{e}_i(\mathfrak{e}_j\mathfrak{a}) = 4(\mathfrak{e}_i\mathfrak{e}_j)\mathfrak{a} =$$
$$= 4 \cdot 0 \cdot \mathfrak{a} = 0, \text{ for } i \neq j.$$

Apply now (77), which characterizes the commutativity of $\mathfrak{a}$, $\mathfrak{b}$ with $\mathfrak{x} = \mathfrak{e}_1$: $\mathfrak{a}(\mathfrak{e}_1\mathfrak{b}) = (\mathfrak{a}\mathfrak{e}_1)\mathfrak{b}$, that is:

$$\sum_{i=1}^{3} \mathfrak{e}_i\mathfrak{a} \cdot \Big(\mathfrak{e}_1 \Big(\sum_{\substack{j,\, k=1 \\ (j \leqslant k)}}^{3} \mathfrak{b}_{jk}\Big)\Big) = \Big(\Big(\sum_{i=1}^{3} \mathfrak{e}_i\mathfrak{a}\Big)\mathfrak{e}_1\Big) \cdot \sum_{\substack{j,\, k=1 \\ (j \leqslant k)}}^{3} \mathfrak{b}_{jk}.$$

Herein

$$\mathfrak{e}_i\mathfrak{a} = \mathfrak{a}_{ii} \varepsilon N_{ii}, \quad \mathfrak{e}_1 \varepsilon N_{11}, \quad \mathfrak{b}_{jk} \varepsilon N_{jk}.$$

Apply the decomposition (73) of Theorem XVII (with respect to $\mathfrak{e}_1$, $\mathfrak{e}_2$, $\mathfrak{e}_3$) to both sides of this equation. Compare the N_{12}-terms on both sides: they must be equal. Now the multiplication-rules of Theorem XVIII show that the N_{12}-term of the left side is

$$(\mathfrak{e}_1\mathfrak{a})(\mathfrak{e}_1\mathfrak{b}_{12}) + (\mathfrak{e}_2\mathfrak{a})(\mathfrak{e}_1\mathfrak{b}_{12})$$

while that of the right side is

$$((\mathfrak{e}_1\mathfrak{a})\,\mathfrak{e}_1)\,\mathfrak{b}_{12}\,.$$

Owing to (72) in Theorem XVII the former expression is

$$\frac{1}{2}(\mathfrak{e}_1\mathfrak{a})\,\mathfrak{b}_{12} + \frac{1}{2}(\mathfrak{e}_2\mathfrak{a})\,\mathfrak{b}_{12}\,,$$

and as we saw above that $\mathfrak{e}_1(\mathfrak{e}_1\mathfrak{a}) = (\mathfrak{e}_1\mathfrak{e}_1)\,\mathfrak{a} = \mathfrak{e}_1\mathfrak{a}$, so the latter is

$$(\mathfrak{e}_1\mathfrak{a})\,\mathfrak{b}_{12}\,.$$

These two expressions must be equal: $\frac{1}{2}(\mathfrak{e}_1\mathfrak{a})\,\mathfrak{b}_{12} + \frac{1}{2}(\mathfrak{e}_2\mathfrak{a})\,\mathfrak{b}_{12} = (\mathfrak{e}\mathfrak{a})\,\mathfrak{b}_{12}$, that is:

$$(\mathfrak{e}_1\mathfrak{a})\,\mathfrak{b}_{12} = (\mathfrak{e}_2\mathfrak{a})\,\mathfrak{b}_{12}\,.$$

Now (72) of Theorem XVII gives:

$$\frac{1}{2}(\sigma_2 - \rho_1)\,\mathfrak{b}_{12} = (\sigma_2\mathfrak{e}_2 - \rho_1\mathfrak{e}_1)\,\mathfrak{b}_{12} = (-\mathfrak{e}_2\mathfrak{a} + \sigma_2\mathfrak{e}_2)\,\mathfrak{b}_{12} + \\ + (\mathfrak{e}_1\mathfrak{a} - \rho_1\mathfrak{e}_1)\,\mathfrak{b}_{12}\,.$$

Taking $\|\ldots\|$ on both sides, and using Lemma E, 4, 1 and E, 4, 2, we obtain:

$$\frac{1}{2}|\sigma_2 - \rho_1|\cdot\|\mathfrak{b}_{12}\| \leqslant \|-\mathfrak{e}_2\mathfrak{a} + \sigma_2\mathfrak{e}_2\|\cdot\|\mathfrak{b}_{12}\| + \|\mathfrak{e}_1\mathfrak{a} - \rho_1\mathfrak{e}_1\|\cdot\|\mathfrak{b}_{12}\|,$$

$$(|\sigma_2 - \rho_1| - 2\|-\mathfrak{e}_2\mathfrak{a} + \sigma_2\mathfrak{e}_2\| - 2\|\mathfrak{e}_1\mathfrak{a} - \rho\mathfrak{e}_1\|)\cdot\|\mathfrak{b}_{12}\| \leqslant 0. \qquad (\sharp)$$

We now evaluate this bracket $(\ldots)$.

First $\rho_1 < \sigma_2$, so $|\sigma_2 - \rho_1| = \sigma_2 - \rho_1$.

Second (63) in F, 3 becomes for $\rho = \rho_1$, $\sigma = \rho_2$:

$$-\mathfrak{e}_1\mathfrak{a} + \rho_2\mathfrak{e}_1 \text{ and } \mathfrak{e}_1\mathfrak{a} - \rho_1\mathfrak{e}_1 \varepsilon \mathfrak{D}.$$

The second relation means

$$\|\mathfrak{e}_1\mathfrak{a} - \rho_2\mathfrak{e}_1 > \geqslant 0;$$

the first one implies, as $\rho_2 - \rho_1 \geqslant 0$ and the idempotent $1 - \mathfrak{e}_1 \varepsilon \mathfrak{D}$, that

$$(\rho_2 - \rho_1)\,1 - (\mathfrak{e}_1\mathfrak{a} - \rho_1\mathfrak{e}) = (-\mathfrak{e}_1\mathfrak{a} + \rho_2\mathfrak{e}) + (\rho_2 - \rho_1)(1 - \mathfrak{e}_1)\varepsilon\mathfrak{D},$$

so

$$< \mathfrak{e}_1\mathfrak{a} - \rho_1\mathfrak{e}_1\| \leqslant \rho_2 - \rho_1\,.$$

Thus

$$\|\mathfrak{e}_1\mathfrak{a}_1 - \rho_1\mathfrak{e}_1\| \leqslant \rho_2 - \rho_1\,.$$

Similarly by (63) eod. for $\rho = \sigma_1$, $\sigma = \sigma_2$:

$$-\mathfrak{e}_2\mathfrak{a} + \sigma_2\mathfrak{e}_2 \text{ and } \mathfrak{e}_2\mathfrak{a} - \sigma_1\mathfrak{e}_2 \varepsilon \mathfrak{D}.$$

The first relation means

$$\|-\mathfrak{e}_2\mathfrak{a} + \sigma_2\mathfrak{e}_2 > \geqslant 0,$$

the second one implies, as $\sigma_2 - \sigma_1 \geqslant 0$ and the idempotent $1 - \mathfrak{e}_2 \varepsilon \mathfrak{D}$, that

$$(\sigma_2 - \sigma_1)\,1 - (-\mathfrak{e}_2\mathfrak{a} + \sigma_2\mathfrak{e}_2) = (\mathfrak{e}_2\mathfrak{a} - \sigma_1\mathfrak{e}_2) + (\sigma_2 - \sigma_1)(1 - \mathfrak{e}_2)\varepsilon\mathfrak{D},$$

so

$$< -\mathfrak{e}_2\mathfrak{a} + \sigma_2\mathfrak{e}_2 \| \leqslant \sigma_2 - \sigma_1 .$$

Thus

$$\| -\mathfrak{e}_2\mathfrak{a} + \sigma_2\mathfrak{e}_2 \| \leqslant \sigma_2 - \sigma_1 .$$

So our bracket $(\ldots)$ is

$$\geqslant (\sigma_2 - \rho_1) - 2(\rho_2 - \rho_1) - 2(\sigma_2 - \sigma_1) \geqslant (\sigma_1 - \rho_2) - (\rho_2 - \rho_1) - (\sigma_2 - \sigma_1) > 0.$$

Therefore the inequality (♯) becomes $\|\mathfrak{b}_{12}\| = 0$, and so $\mathfrak{b}_{12} = 0$.

Now (74) in Theorem XVII shows that $\mathfrak{b}_{12} = 0$ means

$$\mathfrak{e}_1(\mathfrak{e}_2\mathfrak{b}) = 0,$$

which coincides with the desired relation

$$(\mathfrak{e}_{\rho_2} - \mathfrak{e}_{\rho_1})((\mathfrak{e}_{\sigma_2} - \mathfrak{e}_{\sigma_1})\mathfrak{b}) = 0.$$

Lemma H, 1, 4. *Let* $\mathfrak{a}$, $\mathfrak{b}$ *commute. Then for any four numbers* $\rho_1 \leqslant \rho_2 < \sigma_1 \leqslant \sigma_2$

$$(\mathfrak{e}_{\rho_2} - \mathfrak{e}_{\rho_1})((\mathfrak{e}_{\sigma_2} - \mathfrak{e}_{\sigma_1})\mathfrak{b}) = 0.$$

Proof. Choose an $n = 1, 2, \ldots$ with

$$\frac{\rho_2 - \rho_1}{n} + \frac{\sigma_2 - \sigma_1}{n} < \sigma_1 - \rho_2 .$$

Define

$$\tau_\nu = \frac{(n-\nu)\rho_1 + \nu\rho_2}{n}, \quad \upsilon_\nu = \frac{(n-\nu)\sigma_1 + \nu\sigma_2}{n}$$

for $\nu = 0, 1, \ldots, n$. Then

$$\rho_1 = \tau_0 \leqslant \tau_1 \leqslant \ldots \leqslant \tau_n = \rho_2 < \sigma_1 = \upsilon_0 \leqslant \upsilon_1 \leqslant \ldots \leqslant \upsilon_n = \sigma_2 ,$$

and

$$\tau_\nu - \tau_{\nu-1} = \frac{\rho_2 - \rho_1}{n}, \quad \upsilon_\nu - \upsilon_{\nu-1} = \frac{\sigma_2 - \sigma_1}{n},$$

for $\nu = 1, \ldots, n$. Besides clearly $\upsilon_{\mu-1} - \tau_\nu \geqslant \sigma_1 - \rho_2$ for $\mu, \nu = 1, \ldots, n$. So we have

$$(\tau_\nu - \tau_{\nu-1}) + (\upsilon_\mu - \upsilon_{\mu-1}) < \upsilon_{\mu-1} - \tau_\nu ,$$

and therefore Lemma H, 1, 3 applies to $\tau_{\nu-1}, \tau_\nu, \upsilon_{\mu-1}, \upsilon_\mu$ (in place of $\rho_1, \rho_2, \sigma_1, \sigma_2$). So

$$(\mathfrak{e}_{\tau_\nu} - \mathfrak{e}_{\tau_{\nu-1}})((\mathfrak{e}_{\sigma_\mu} - \mathfrak{e}_{\sigma_{\mu-1}})\mathfrak{b}) = 0,$$

and summing over all $\mu, \nu = 1, \ldots, n$ gives

$$(\mathfrak{e}_{\tau_n} - \mathfrak{e}_{\tau_0})((\mathfrak{e}_{\sigma_n} - \mathfrak{e}_{\sigma_0})\mathfrak{b}) = 0$$

that is the desired relation

$$(\mathfrak{e}_{\rho_2} - \mathfrak{e}_{\rho_1})((\mathfrak{e}_{\sigma_2} - \mathfrak{e}_{\sigma_1})\mathfrak{b}) = 0.$$

Lemma H, 1, 5. *Let* $\mathfrak{a}$, $\mathfrak{b}$ *commute. Then*

$$\mathfrak{e}_\rho((1 - \mathfrak{e}_\rho)\mathfrak{b}) = 0.$$

Proof. Choose a sequence $\tau_1 < \tau_2 < \ldots < \rho$ with $\lim_{i\to\infty} \tau_i = \rho$. Then Lemma H, 1, 4 with $\rho_1 \leqslant \|\mathfrak{a}\rangle$, $\rho_2 = \tau_i$, $\sigma_1 = \rho_1\sigma_2 \rangle\langle \mathfrak{a}\|$ gives $[\mathfrak{e}_{\rho_1} = 0$, $\mathfrak{e}_{\sigma_2} = 1$ by (iii) in F, 4

$$\mathfrak{e}_{\tau_i}((1 - \mathfrak{e}_\rho)\mathfrak{b}) = 0.$$

Now $i \to \infty$ gives $[\mathcal{T}\text{-}\lim_{i\to\infty} \mathfrak{e}_{\tau_i} = \mathfrak{e}_\rho$ by (ii) in F, 4]

$$\mathfrak{e}_\rho((1-\mathfrak{e}_\rho)\mathfrak{b}) = 0.$$

We are now in the position to prove:

Theorem XIX. *$\mathfrak{b}$ commutes with $\mathfrak{a}$ if and only if it commutes with every element of the spectral family $\mathfrak{e}_\rho$, $-\infty < \rho < +\infty$, to which $\mathfrak{a}$ belongs.*

Proof. Sufficiency: if $\mathfrak{b}$ commutes with $\mathfrak{e}_\rho$, then it commutes with the

$$\sum_{i=1}^{n} \xi_i (\mathfrak{e}_{\lambda_i} - \mathfrak{e}_{\lambda_{i-1}})$$

of (*) in F, 4 as well. So it commutes with $\mathfrak{a}$ too (use Lemma E, 4, 3).

Necessity: assume that $\mathfrak{b}$ commutes with $\mathfrak{a}$. Decompose $\mathfrak{b}$ by (36) in Lemma D, 2, 1 for $\mathfrak{e} = \mathfrak{e}_\rho$. Then (37) eod. gives, together with the Lemma H, 1, 5, that

$$\mathfrak{b}_{\frac{1}{2}} = -4\mathfrak{e}_\rho(\mathfrak{e}_\rho\mathfrak{b}) + 4\mathfrak{e}_\rho\mathfrak{b} = 4\mathfrak{e}_\rho((1-\mathfrak{e}_\rho)\mathfrak{b}) = 0.$$

So

$$\mathfrak{b} = \mathfrak{b}_0 + \mathfrak{b}_1 \varepsilon N_0(\mathfrak{e}_\rho) + N_1(\mathfrak{e}_\rho).$$

By Lemma H, 1, 2 this means $\mathfrak{b} \varepsilon (\mathfrak{e}_\rho)'$ and so $\mathfrak{b}$ commutes with $\mathfrak{e}_\rho$.

Theorem XIX is the key to the behavior of commutativity. The further deductions are easy.

Theorem XX. *$\mathfrak{S}'$ is always a ring.*

Proof. $\mathfrak{b} \varepsilon \mathfrak{S}'$ means that $\mathfrak{b}$ commutes with $\mathfrak{a} \varepsilon \mathfrak{S}$, and so by Theorem IX, that it commutes with all $\mathfrak{e}_\rho$, $-\infty < \rho < +\infty$, where $\mathfrak{e}_\rho$ is any bounded spectral family to which $\mathfrak{a} \varepsilon \mathfrak{S}$ belongs. So $\mathfrak{S}'$ is the intersection of all sets $(\mathfrak{e})'$, where $\mathfrak{e}$ runs over all $\mathfrak{e}_\rho$'s we described above. If all these $(\mathfrak{e})'$ are rings, then $\mathfrak{S}'$ being their intersection will be a ring too. In other words: it suffices to prove that $(\mathfrak{e})'$ is a ring if $\mathfrak{e}$ is idempotent.

Now $(\mathfrak{e})'$ is clearly a module, so we must only prove that $\mathfrak{a}, \mathfrak{b} \varepsilon (\mathfrak{e})'$ imply $\mathfrak{a}\mathfrak{b} \varepsilon (\mathfrak{e})'$ But by Lemma H, 1, 2

$$(\mathfrak{e})' = N_0(\mathfrak{e}) + N_1(\mathfrak{e}),$$

and $N_0(\mathfrak{e}) + N_1(\mathfrak{e})$ possesses this property by Theorem IV.

Corollary:

(i) $\mathfrak{S} \subset \mathfrak{T}$ implies $\mathfrak{S}' \supset \mathfrak{T}'$.

(ii) $\mathfrak{S} \subset \mathfrak{T}'$ and $\mathfrak{T} \subset \mathfrak{S}'$ are equivalent.

(iii) $\mathfrak{S}' = (\mathfrak{P}(\mathfrak{S}))'$.

(iv) $\mathfrak{S} \subset \mathfrak{P}(\mathfrak{S}) \subset \mathfrak{S}''$.

(v) If any of the sets in (iv) is Abelian, all are.

(vi) $\mathfrak{S}' = \mathfrak{S}''' = \mathfrak{S}^{V} = \ldots$, $\mathfrak{S}'' = \mathfrak{S}^{IV} = \mathfrak{S}^{VI} = \ldots$

Proof. We proceed in a somewhat changed order.

Ad (i): obvious.

Ad (ii): both statements mean: whenever $\mathfrak{a} \varepsilon \mathfrak{S}$, $\mathfrak{b} \varepsilon \mathfrak{T}$, then $\mathfrak{a}$, $\mathfrak{b}$ commute.

Ad (iv): $\mathfrak{S}' \subset \mathfrak{S}'$ gives $\mathfrak{S} \subset \mathfrak{S}''$ by (ii) with $\mathfrak{T} = \mathfrak{S}'$. As $\mathfrak{S}''$ is a ring, this necessitates $\mathfrak{P}(\mathfrak{S}) \subset \mathfrak{S}''$. $\mathfrak{S} \subset \mathfrak{P}(\mathfrak{S})$ is obvious.

Ad (vi): by (iv) $\mathfrak{S} \subset \mathfrak{S}''$, application of $'$ gives [use (i)] $\mathfrak{S}' \supset \mathfrak{S}'''$, but substitution of $\mathfrak{S}'$ for $\mathfrak{S}$ gives $\mathfrak{S}' \subset \mathfrak{S}'''$, so $\mathfrak{S}' = \mathfrak{S}'''$. Apply $'$, $''$, $^{\mathrm{III}}, \ldots$, then $\mathfrak{S}' = \mathfrak{S}''' = \mathfrak{S}^{\mathrm{V}} = \ldots$, $\mathfrak{S}'' = \mathfrak{S}^{\mathrm{IV}} = \mathfrak{S}^{\mathrm{VI}} = \ldots$ result.

Ad (iii): apply $'$ to (iv), and use (i), (vi):

$$\mathfrak{S}' \supset (\mathfrak{P}(\mathfrak{S}))' \supset \mathfrak{S}''' = \mathfrak{S}'$$

results, and thus

$$\mathfrak{S}' = (\mathfrak{P}(\mathfrak{S}))'.$$

Ad (v): if $\mathfrak{S}''$ is Abelian, then its subset $\mathfrak{P}(\mathfrak{S})$ is too; similarly if $\mathfrak{P}(\mathfrak{S})$ is Abelian, so is its subset $\mathfrak{S}$. [Use (iv).] If $\mathfrak{S}$ is Abelian, then $\mathfrak{S} \subset \mathfrak{S}'$; apply now $''$, then (i) gives $\mathfrak{S}'' \subset \mathfrak{S}'''$, and thus $\mathfrak{S}''$ is Abelian. Thus all the equivalences are established.

Theorems XIX and XX, and the Corollary to the latter, establish a great analogy between our notion of commutativity and our operation $\mathfrak{S}'$, and the corresponding notions for bounded linear operators in Hilbert space. [Cf. the author's paper, «Math. Annalen», **102**, (1929), 370—427.] Only two essential facts which were proved there have not been derived here, namely:

(A) Every Abelian ring has the form $\mathfrak{P}(\mathfrak{a})$ for a suitable $\mathfrak{a}$. [$\mathfrak{P}(\mathfrak{a})$ is always Abelian by Lemma G, 3, 1, (iii).]

(B) For every ring $\mathfrak{M}$ which contains 1,

$$\mathfrak{M} = \mathfrak{M}''.$$

(Cf. loc. cit. above.)

(A) could be proved with the same methods as loc. cit. above, all necessary technicalities being already established. (B) however, is not true for all systems $\mathfrak{A}$ fulfilling our axioms, as will be seen in Part II.

H, 2. Assume that the idempotents $\mathfrak{e}$, $\mathfrak{f}$ commute. Then $\mathfrak{e}\mathfrak{f}$ and $\mathfrak{f}$ commute too [as $(\mathfrak{f})'$ is a ring], and so we have

$$(\mathfrak{e}\mathfrak{f})(\mathfrak{e}\mathfrak{f}) = ((\mathfrak{e}\mathfrak{f})\mathfrak{e})\mathfrak{f}.$$

The right side is $= ((\mathfrak{f}\mathfrak{e})\mathfrak{e})\mathfrak{f}$, and as $\mathfrak{f}$, $\mathfrak{e}$ commute

$$(\mathfrak{f}\mathfrak{e})\mathfrak{e} = \mathfrak{f}(\mathfrak{e}\mathfrak{e}) = \mathfrak{f}\mathfrak{e}.$$

So the above expression is

$$= (\mathfrak{f}\mathfrak{e})\mathfrak{f} = (\mathfrak{e}\mathfrak{f})\mathfrak{f},$$

and using the commutativity of $\mathfrak{e}$, $\mathfrak{f}$ again

$$= \mathfrak{e}(\mathfrak{f}\mathfrak{f}) = \mathfrak{e}\mathfrak{f}.$$

Thus $(\mathfrak{e}\mathfrak{f})^2 = \mathfrak{e}\mathfrak{f}$, $\mathfrak{e}\mathfrak{f}$ idempotent. (Observe that the same rule holds in non-commutative but associative algebras, only the rôles of the commutative and associative laws of multiplication respectively are interchanged.)

We saw that $(\mathfrak{e}\mathfrak{f})\mathfrak{f} = \mathfrak{e}\mathfrak{f}$, so $\mathfrak{e}\mathfrak{f} \leqslant \mathfrak{f}$. Similarly $\mathfrak{f} \leqslant \mathfrak{e}\mathfrak{f}$. So if an idempotent $\mathfrak{g} \leqslant \mathfrak{f}\mathfrak{e}$, then $\mathfrak{g} \leqslant \mathfrak{e}$ and $\mathfrak{f}$ too. Conversely: if for an idempotent $\mathfrak{g} \leqslant \mathfrak{e}$ and $\mathfrak{f}$, then we have: $\mathfrak{e}\mathfrak{g} = \mathfrak{f}\mathfrak{g} = \mathfrak{g}$ (by definition), $\mathfrak{e}$, $\mathfrak{g}$ commute [by Lemma H, 1, 1, (ii)], so $(\mathfrak{e}\mathfrak{f})\mathfrak{g} = \mathfrak{e}(\mathfrak{f}\mathfrak{g}) = \mathfrak{e}\mathfrak{g} = \mathfrak{g}$. Thus $\mathfrak{e}\mathfrak{f}$ coincides with the idempotent $\Pi((\mathfrak{e}, \mathfrak{f})) = \mathfrak{e} \frown \mathfrak{f}$ (cf. D, 5).

So we have:

$$\text{If } \mathfrak{e}, \mathfrak{f} \text{ commute, then } \mathfrak{e} \frown \mathfrak{f} = \mathfrak{e}\mathfrak{f}. \tag{79}$$

If $\mathfrak{e}$, $\mathfrak{f}$ commute, then $\mathfrak{e}$, $1-\mathfrak{f}$ commute too, and so $1-\mathfrak{e}$, $1-\mathfrak{f}$ too. Applying (79) to $1-\mathfrak{e}$, $1-\mathfrak{f}$ gives [use Lemma D, 5, 1, (iii), which gives $(1-\mathfrak{e})\frown(1-\mathfrak{f})=$ $=1-(\mathfrak{e}\smile\mathfrak{f})$]:

$$\text{If } \mathfrak{e},\ \mathfrak{f} \text{ commute, then } \mathfrak{e}\smile\mathfrak{f}=\mathfrak{e}+\mathfrak{f}-\mathfrak{e}\mathfrak{f}. \tag{80}$$

(39) in D, 5 is a special case of this.

We prove finally:

Theorem XXI. *An idempotent* $\mathfrak{f}$ *commutes with an idempotent* $\mathfrak{e}$ *if and only if* $\mathfrak{f}=\mathfrak{f}_1+\mathfrak{f}_2$, *where* $\mathfrak{f}_1$, $\mathfrak{f}_2$ *are idempotents,* $\mathfrak{f}_1\leqslant\mathfrak{e}$, $\mathfrak{f}_2$ *orthogonal to* $\mathfrak{e}$.

Proof. Sufficiency: obvious by Lemma H, 1, 1, (i) and (ii). Necessity: assume that $\mathfrak{e}$, $\mathfrak{f}$ commute. Then $\mathfrak{f}=\mathfrak{f}_1+\mathfrak{f}_2$ with $\mathfrak{f}_1=\mathfrak{e}\mathfrak{f}$, $\mathfrak{f}_2=(1-\mathfrak{e})\mathfrak{f}$. These are idempotents by (79), and by the same formula

$$\mathfrak{f}_1=\mathfrak{e}\frown\mathfrak{f}\leqslant\mathfrak{e},\quad \mathfrak{f}_2=(1-\mathfrak{e})\frown\mathfrak{f}\leqslant 1-\mathfrak{e},$$

so $\mathfrak{e}$, $\mathfrak{f}_2$ are orthogonal.

H, 3. $\mathfrak{C}=\mathfrak{A}'$ is the center of $\mathfrak{A}$.

Applying the formula (77) of H, 1, we see that $\mathfrak{a}\varepsilon\mathfrak{C}$ means

$$[\mathfrak{a},\ \mathfrak{x},\ \mathfrak{y}]=0 \quad \text{for all } \mathfrak{x},\ \mathfrak{y}\varepsilon\mathfrak{A}. \tag{81}$$

Now the first equation of (69) in G, 1 gives $[\mathfrak{y},\ \mathfrak{x},\ \mathfrak{a}]=0$, and so $[\mathfrak{x},\ \mathfrak{y},\ \mathfrak{a}]=0$. And this, together with (81) gives, owing to the second equation of (69), that $[\mathfrak{y},\ \mathfrak{a},\ \mathfrak{x}]=0$. In other words:

$$[\mathfrak{a},\ \mathfrak{b},\ \mathfrak{c}]=0 \quad \text{if at least one of } \mathfrak{a},\ \mathfrak{b},\ \mathfrak{c} \text{ is in } \mathfrak{C}. \tag{82}$$

$\mathfrak{C}$ is a ring by Theorem XX. This can be seen directly too, owing to M. Zorn's (directly verifiable) identity

$$[\mathfrak{a}\mathfrak{b},\ \mathfrak{x},\ \mathfrak{y}]=[\mathfrak{a},\ \mathfrak{b}\mathfrak{x},\ \mathfrak{y}]-[\mathfrak{a},\ \mathfrak{b},\ \mathfrak{x}\mathfrak{y}]+\mathfrak{a}[\mathfrak{b},\ \mathfrak{x},\ \mathfrak{y}]+[\mathfrak{a},\ \mathfrak{b},\ \mathfrak{x}]\mathfrak{y} \tag{83}$$

and (81).

Clearly $1\varepsilon\mathfrak{C}$, therefore all $\rho 1\varepsilon\mathfrak{C}$ (ρ any real number).

We now prove:

Lemma H, 3, 1. (i) $\mathfrak{a}\varepsilon\mathfrak{C}$ *means* $(\mathfrak{a})'=\mathfrak{A}$.

(ii) *The ideals* $\neq\Theta$ *coincide with the sets* $N_1(\mathfrak{e})$ *for idempotents* $\mathfrak{e}\varepsilon\mathfrak{C}$.

Proof. Ad (i): $\mathfrak{a}\varepsilon\mathfrak{C}$ means that every $\mathfrak{b}\varepsilon\mathfrak{A}$ commutes with $\mathfrak{a}$, that is $(\mathfrak{a})'=\mathfrak{A}$.

Ad (ii): by Lemma D, 6, 1 an ideal $\neq\Theta$ is a set $N_1(\mathfrak{e})$ with $N_{\frac{1}{2}}(\mathfrak{e})=(0)$. Owing to Lemma D, 2, 1 this means $\mathfrak{A}=N_0(\mathfrak{e})+N_1(\mathfrak{e})$, owing to Lemma H, 1, 2 this is $\mathfrak{A}=(\mathfrak{e})'$, and by (i) this is equivalent to $\mathfrak{e}\varepsilon\mathfrak{C}$.

Theorem XXII. $\mathfrak{A}$ *is called* irreducible, *if for two ideals* $\mathfrak{M}$, $\mathfrak{N}$ *which fulfil* (i), (ii) *in Theorem VII, necessarily* $\mathfrak{M}=(0)$, $\mathfrak{N}=\mathfrak{A}$ *or* $\mathfrak{M}=\mathfrak{A}$, $\mathfrak{N}=(0)$. $\mathfrak{A}$ *is irreducible if and only if its center* $\mathfrak{C}$ *contains the* $\rho 1$ *(*ρ *any real number) only.*

Proof. $\mathfrak{M}=(0)$ implies $\mathfrak{N}=\mathfrak{A}$ [use Theorem VII, (i)], $\mathfrak{M}=\mathfrak{A}$ implies $\mathfrak{N}=(0)$ [use (ii) eod.], so the irreducibility of $\mathfrak{A}$ can just as well be characterized by $\mathfrak{M}=(0)$ or $\mathfrak{A}$. Owing to Theorem VII this means: every ideal $\mathfrak{M}\neq\Theta$ is $=(0)$ or $\mathfrak{A}$. And by Lemma H, 3, 1: for every idempotent $\mathfrak{e}\varepsilon\mathfrak{C}$ $N_1(\mathfrak{e})=(0)$ or $\mathfrak{A}$. But the former means clearly $\mathfrak{e}=0$, and the latter $\mathfrak{e}=1$. So we see: $\mathfrak{A}$ is irreducible if and only if 0, 1 are the only idempotents in $\mathfrak{C}$.

If $\mathfrak{C}$ consists of $\rho 1$'s only, then the idempotents in $\mathfrak{C}$ have this form. $(\rho 1)^2 = \rho 1$ implies $\rho^2 = \rho$, $\rho = 0$, 1, so 0, 1 are the only idempotents in $\mathfrak{C}$.

Assume conversely that 0, 1 are the only idempotents in $\mathfrak{C}$. Then $\mathfrak{C}$ is by Theorem XIV, Corollary (put $\mathfrak{M} = \mathfrak{C}$) the $\mathcal{U}$-closure of their finite linear aggregates, that is of the $\rho 1$. So we must only show that this set is $\mathcal{U}$-closed. $\mathfrak{a} = \rho 1$ is equivalent to $\|\mathfrak{a} - \rho 1\| = 0$, so to $\|\mathfrak{a} - \rho 1 > = < \mathfrak{a} - \rho 1\| = 0$, that is to $\|\mathfrak{a} > = < \mathfrak{a}\| = \rho$. Thus the above set is characterized by $\|\mathfrak{a} > = < \mathfrak{a}\|$, and as these are $\mathcal{U}$-continuous functions of $\mathfrak{a}$, it is $\mathcal{U}$-closed. (Use Lemmas E, 4, 1 and E, 4, 3.)

Corollary. A ring $\mathfrak{M} \neq \Theta$ in $\mathfrak{A}$ is irreducible if and only if $\mathfrak{M} \cdot \mathfrak{M}'$ contains the $\rho 1\,(\mathfrak{M})$ only (ρ being any real number, and $1\,(\mathfrak{M})$ the unit of $\mathfrak{M}$).

Proof. The case $\mathfrak{M} = (0)$ is obvious, because then $\mathfrak{M} \cdot \mathfrak{M}' = (0)$, $1\,(\mathfrak{M}) = 0$. So we may assume $\mathfrak{M} \neq \Theta$, (0). Then we may apply Theorem XXII to $\mathfrak{M}$ instead of $\mathfrak{A}$, cf. A, 1. As the center of $\mathfrak{M}$ is, using notations with respect to $\mathfrak{A}$, the set $\mathfrak{M} \cdot \mathfrak{M}'$ the desired statement results.

(Поступило в редакцию 2/IV 1936 г.)

Обобщение математического аппарата квантовой механики методами абстрактной алгебры. (Часть I)

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(Резюме)

Эта статья является продолжением и распространением работы P. Jordan'a, E. Wigner'a и автора [1].

Обе статьи — цитированная и настоящая — основаны на идее P. Jordan'a, имеющей целью избежать введения в квантовую теорию понятий, не допускающих непосредственного физического истолкования.

Наблюдаемые величины квантовой механики соответствуют (в обычной теории) эрмитовым матрицам в гильбертовом пространстве, — а потому желательно иметь аксиоматическое построение систем гиперкомплексных чисел, аналогичных эрмитовым матрицам, а не всем матрицам пространства Гильберта.

Основные операции для эрмитовых матриц будут: aA (a — действительное число), $A+B$, A^n ($n=1, 2, \ldots$), но не aA (a — комплексное число), AB (что приводит к неэрмитовым матрицам), A^* (сопряженная, которая равна A, если A — эрмитова матрица).

Проведена аксиоматизация первых трех упомянутых операций. В то время как P. Jordan, E. Wigner и автор рассматривали только случай систем гиперкомплексных чисел с конечным базисом, теперь такого предположения не делается, а это требует введения некоторых топологических аксиом.

При помощи A^n определяется особого рода произведение

$$A \circ B = \frac{1}{4}\left((A+B)^2 - (A-B)^2\right).$$

[Для матриц, если AB — обыкновенное произведение матриц, это $A \circ B$ будет известное „симметризированное произведение“:

$$A \circ B = \frac{1}{2}(AB + BA).]$$

$A \circ B$ коммутативно и $A^{m+n} = A^m \circ A^n$, но, вообще, оно не ассоциативно.

В настоящей I части излагаются:

1) теория идемпотентных величин (§ A,2; C,1 — C,3; D,1 — D,6; G,1 — G,2),

2) спектральная теория (§ E,1 — E,3; F,1 — F,4),

3) теория непрерывных функций, в применении к элементам системы (§ C,4 — C,6; E,1 — E,3; F,5),

4) теория „коммутативности“ (§ H,1 — H,2),

5) теория приводимости системы (§ D,6; H,3).

„Коммутативность“ A, B не определяется через $A \circ B = B \circ A$ (что является тождеством), а через

$$(A \circ X) \circ B = A \circ (X \circ B)$$

для всех X.

[1] Статья того же названия, „Annals of Mathematics“, **35**, (1934), 29—64.

Математическое и квантовомеханическое обоснования этого построения даются в § H,1.

Во всех этих исследованиях видна строгая аналогия со свойствами эрмитовых матриц в пространстве Гильберта.

Во II части работы будет исследована подробнее структура идемпотентных величин, установлена глубокая связь с работой Murray'я и автора [1], и показано, как далеко это построение позволяет ввести обобщения в квантовую механику. В частности, будет показано, что все наши системы (за одним исключением, приведенным в предыдущей статье, которое, вероятно, не имеет физического смысла) могут быть включены в некоммутативные, но ассоциативные системы, т. е. в системы с операциями aA, $A+B$, AB, A^* с обычными свойствами. В них они характеризуются „эрмитовским" условием $A=A^*$.

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[1] „On rings of operators", „Annals of Mathematics", 37, (1936), 116—229.