GRAVITATION

Introduction

So far we have discussed various forces : pushes and pulls, elastic forces, friction, and other forces that act when one body is in contact with another. In this chapter we study the properties of one particularly important noncontact force, gravitation, which is one of the fundamental and universal forces of nature.

Origin of the Law of Gravitation

From at least the time of the ancient Greeks, two problems were puzzling : (1) the falling of objects released near the Earth's surface, and (2) the motions of the planets. Although there was no reason at that time to connect these two problems, today we recognize that they result from the effect of the same force – gravitation. In fact, this force also determines the motion of the Sun in our Milky Way galaxy, as well as the motion of the galaxy in our Local Cluster of galaxies, the motion of the galaxy in our Local Cluster of galaxies, the motion of the Local Cluster in the Local Supercluster, and so on through the universe. In short, the gravitational force, and the law that describes that force, controls the structure, the developement, and the eventual fate of the universe.

The earliest serious attempt to explain the motions of the planets was due to Claudius Ptolemy (A.D. 2nd century), who developed a model of the solar system in which the planets, including the Sun and Moon, revolved about the Earth. Unfortunately, to explain the complicated orbits of the planets in this geocentric frame of reference, Ptolemy was forced to introduce epicycles, in which a planet moves around a small circle whose center moves around another larger circle centered on the Earth. Of course, today we would reject such a model because it violates the law that every accelerated motion must be accounted for by a force due to a body in its environment - there is no boy at the center of the small circles that would supply the force necessary for the centripetal acceleration.

Famous Indian astronomer and mathematician, Aryabhat, studied motion of earth in great detail, most likely in the 5th century A.D., and wrote his conclusions in his book Aryabhatiy. He established that the earth revolves about its own axis and moves in a circular orbit about the sun, and that the moon moves in a circular orbit about the earth. But these ideas could not be communicated to the world.

It was not until the 16th century that Nicolaus Copernicus (1473-1543) proposed a heliocentric (Sun-centered scheme, in which the Earth and the other planets move about the Sun. Like Ptolemy's model, Copernicus' solar system was still based only on geometry because the notion of a force had not yet between introduced.

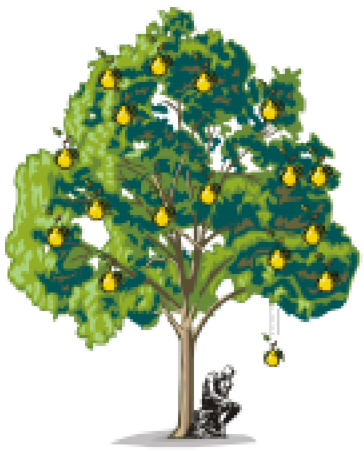
Based on careful analysis of observational data of histeacher tyco brahe (1546-1601) on planetary motions, Johannes Kepler (1571-1630) proposed three laws that describe those motions. However, Kepler's laws were only empirical-they simply described the motions of the planets without any basis in terms of forces. It was a great triumph for the newly developed field of mechanics later in the 17th century when Isaac Newton was able to derive Kepler's laws from his laws of mechanics and his proposed law of gravitation. With this tunning development, Newton was able to use the same concept to account for the motion of the planets and of bodies falling near the Earth's surface.

The year 1665 was very fruitful for Isaac Newton aged 23. He was forced to take rest at his home in Lincolnshire after his college at Cambridge was closed for an indefinite period due to plague. In this year, he performed brilliant theoretical and experimental tasks mainly in the field of mechanics and optics. In this same year he focussed his attention on the moiton of the moon about the earth.

The moon makes a revolution about the earth in T = 27.3 days. The distance of the moon from the earth T = 27.3 days. The distance of the moon from the earth is R = 3.85×10^5 km. The acceleration of the moon is, therefore,

$$a = \omega^2 R = \frac{4\pi^2 \times (3.85 \times 10^5\,\text{km})}{(27.3\ \text{days})^2} = 0.0027\ \text{m s}^{-2}.$$

The first question before Newton was that what is the force that produces this acceleration. The acceleration is towards the center of the orbit, that is towards the centre of the earth. Hence the force must act towards the centre of the earth. A natural guess was that the earth is attracting the moon. The saying goes that Newton was sitting under an apple tree when an apple fell down from the tree on the earth. This sparked the idea that the earth attracts all bodies towards its centre. The next question was what is the law governing this force.



Newton had to make several daring assumptions which proved to be turning points in science and philosophy. He declared that the law of nature are the same for earthly and celestial bodies. The force operating between the earth and an apple and that operating between the earth and the moon, must be governed by the same laws. This statement may look vary obvious today but in the era before Newton, there was a general belief in the western countries that the earthly bodies are governed by certain rules and the heavenly bodies are governed by different rules. In particular, this heavenly structure was supposed to be so perfect that there could not be any change in the sky. This distincition was so sharp that when Tycho Brahe saw a new star in the sky, he did not believe his eyes as there could be no change in the sky. So the Newton's declaration was indeed revolutionary.

The acceleration of a body falling near the earth's surface is about 9.8 ms^{-2}. Thus,

$$\frac{a_{\text{apple}}}{a_{\text{moon}}} = \frac{9.8\,\text{ms}^{-2}}{0.0027\ \text{ms}^{-2}} = 3600.$$

Also,

$$\frac{\text{distance of the moon from the earth}}{\text{distance of the apple from the earth}}$$

$$= \frac{d_{\text{moon}}}{d_{\text{apple}}} = \frac{3.85 \times 10^5\ \text{km}}{6400\,\text{km}} = 60$$

Thus, $$\frac{a_{\text{apple}}}{a_{\text{moon}}} = \left(\frac{d_{\text{moon}}}{d_{\text{apple}}}\right)^2.$$

Newton guessed that the acceleration of a body towards the earth is inversely proportional to the square of the distance of the body from the centre of the earth.

Thus, $a \propto \frac{1}{r^2}$

Also, the force is mass times acceleration and so it is proportional to the mass of the body. Hence,

$$F \propto \frac{m}{r^2}$$

By the third law of motion, the force on a body due to the earth must be equal to the force on the earth due to the body. Therefore, this force should also be proportional to the mass of the earth. Thus, the force between the earth and a body is

$$F \propto \frac{Mm}{r^2} \text{ or } F = \frac{GMm}{r^2}$$

Newton further generalised the law by saying that not only the earth but all material bodies in the universe attract each other.

Here G, called the gravitational constant, has the experimentally determined value

$G = 6.67 \times 10^{-11}\ N.m^2/kg^2$

G is a universal constant, with the same value for any pair of particles at any location in the universe.

Note :

1. Gravitation, the force that acts between bodies due only to their masses, is one of the four basic forces of physics. It acts throughout the universe : between bodies on Earth, where it is weak and difficult to measure ; between the Earth and bodies in its vicinity, where it is the controlling feature of our lives ; and among the stars and galaxies, where it controls their evolution and structure.

2. Normally, however, it is only when the mass of at least one of the interacting bodies is large (planet-sized) that the effects of the gravitational force become significant.

3. In this argument, the distance of the apple from the earth is taken to be equal to the radiusof the earth. This means we have assumed that earth can be treated as a single particle placed at its centre. This is of course not obvious. Newton had spent several years to prove that indeed this can be done. A spherically symmetric body can be replaced by a point particle of equal mass placed at its centre for the purpose of calculating gravitational force.

Characteristics of The Gravitational force :

(a) Gravitational force is always attractive and directed along the line joining the particles.

(b) It is independent of the nature of the medium surrounding the particles.

(c) It holds good for long distances like inter-planetary distances and also short distances like inter-atomic distances.

(d) Interaction means that, both the particles experience force of equal magnitude in opposite directions. If $\vec{F}_1$, $\vec{F}_2$ are the forces acting on particle 1 by particle 2 and particle 2 by particle 1 respectively, then $\vec{F}_1 = -\vec{F}_2$. Since the forces $\vec{F}_1$ and $\vec{F}_2$ are exerted on different bodies, they are known as action-rection pair.

(e) It is a conservative force. Therefore the work done by the gravitational force on a particle is independent of the path described by the particle. It depends upon the initial and final position of the particle. Therefore no work is done by the gravity if a particle moves in a closed path.

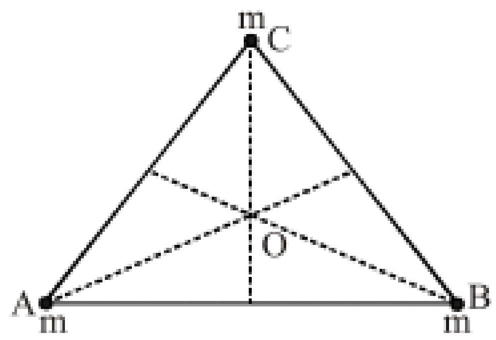
(f) If a particle is acted by n particles, say, the net force $\vec{F}$ exerted on it must be equal the vector sum of the forces due to surrounding particles.

$$\Rightarrow \quad \vec{F} = \sum_{i=1}^{i=n} \vec{F}_i$$

where $\vec{F}_i$ = force acted on the particle , by the i^{th} particle.

Illustration :

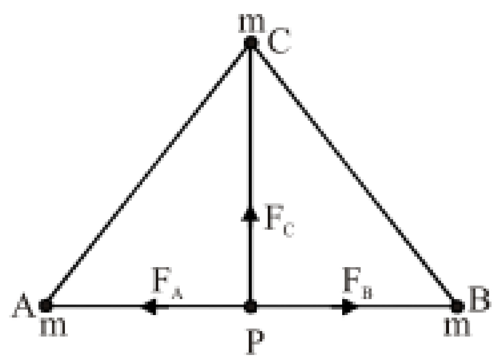
Three identical particles each of mass m are placed at the vertices of an equilateral triangle of side a. Find the force exerted by this system on a particle P of mass m placed at the



(a) the mid point of a side *(b) centre of the triangle*

Sol. *Using the superposition principle, the net gravitational force on P is* $\vec{F} = \vec{F}_A + \vec{F}_B + \vec{F}_C$

(a) *As shown in the figure, when P is at the mid point of a side,* $\vec{F}_A$ *and* $\vec{F}_B$ *will be equal in magnitude but opposite in direction. Hence they will cancel each other. So the net force on the particle P will be the force due to the particle placed at C only.*



$$\Rightarrow \quad F = F_C = G\frac{m\,.\,m}{(CP)^2} = G\frac{m^2}{(a\sin 60)^2} = \frac{4Gm^2}{3a^2} \text{ along PC.}$$

(b) *At the centre of triangle O, the forces* $\vec{F}_A$, $\vec{F}_B$ *and* $\vec{F}_C$ *will be equal in magnitude and will subtend 120° with each other. Hence the resultant force on P at O is* $\vec{F} = \vec{F}_A + \vec{F}_B + \vec{F}_C = 0$.

Illustration :

Two balls of mass m each are hung side by side by side by two long threads of equal length l. If the distance between upper ends is r, show that the distance r' between the centres of the ball is given by $g\,r^2\,(r - r') = 2\,l\,G\,m$

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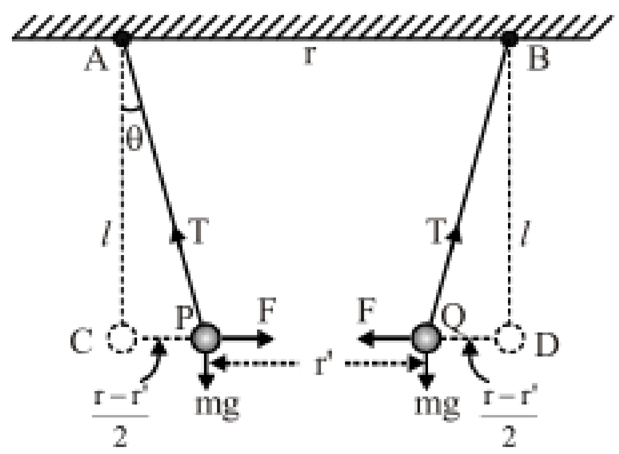
Sol. *The situation is shown in figure*



Following force act on each ball

(i) Weight of the ball m g in downward direction

(ii) Tension in thread T along string

(iii) Force of gravitation attraction towards each other

$$F = G\frac{m\,m}{r'^2}$$

Here for equilibrium of balls we have

$$T\sin\theta = \frac{Gm^2}{r'^2} \qquad ...(i)$$

$$T\cos\theta = mg \qquad ...(ii)$$

Dividing equation (i) and (ii), we get

or $$\tan\theta = \frac{Gm^2}{mgr'^2} \qquad ...(iii)$$

In ΔACP $$\tan\theta = \frac{r-r'}{2l} \qquad ...(iv)$$

From equation (iii) and (iv)

$$\frac{r-r'}{2l} = \frac{Gm^2}{mgr^2}$$

$$g\,r^2\,(r-r') = 2\,l\,G\,m$$

Practice Exercise

Q.1 Four particles of equal masses M move along a circle of radius R under the action of their mutual gravitational attraction. Find the speed of each particle.

Q.2 In a double star, two stars (one of mass m and the other of 2 m) distance d apart rotate about their common centre of mass. Deduce an expression for the period of revolution. Show that the ratio of their angular momenta about the centre of mass in the same as the ratio of their kinetic energies.

Answers

Q.1 $\sqrt{\frac{GM}{R}\left(\frac{2\sqrt{2}+1}{4}\right)}$

Gravitational Field or gravitational field strength:

All the bodies on or above earth's surface experience gravitational force known as the weight of the bodies. Therefore the space surrounding the earth, where the gravitational force (weight) is experienced is known as the gravitational field of the earth. Similarly the space surrounding each and every material particle is known as gravitational field of that particle.

Gravitational field strength at any point is defined as gravitational force exerted on a unit point mass. It is equal to acceleration due to gravity.

If we ask yourself, what is your strength ? Definitely you will think of your muscular power. A boxer is stronger than an ordinary man. That means he can exert a larger force. This reveals that strength is related to force.

Now if we want to measure the strength of the gravitational field at any point we will have to calculate the force acting on a point mass placed at that point. We see that , different masses experience different forces. The larger the mass, the larger the force it will experience. When we take the ratio of the gravitational force $\vec{F}_g$ and the point mass m we obtain a constant value for that point. This constant is known as the strength of the gravitational field. If the field exerts a large force on the point mass, we say that the strength of the gravitational field is stronger at that point and vice-versa.

$\Rightarrow$ The strength of the gravitational field $\vec{g} = (\vec{F}_g / m)$

$\Rightarrow$ Gravitational field strength is defined as gravitational force per unit mass.

In earth's gravitational field $\vec{g} = \frac{\text{weight of the particle}}{\text{mass of the particle}} = \frac{\vec{W}}{m}$

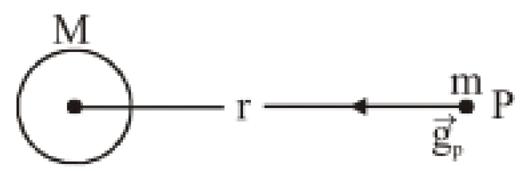
The above expression is equal to the acceleration due to gravity '$\vec{g}$'

Gravitational field unit is N/kg and dimensions LT^{-2}.

Gravitational field due to point mass at a distance r :

We want to find g due to M at point P as per the procedure, place a point mass m at P. Measure the force imparted by M on the test mass m. That is equal to $F_g = \dfrac{GMm}{r^2}$

$$\Rightarrow g_p = \frac{F_g}{m} = \frac{1}{m}\left(\frac{GMm}{r^2}\right)$$



$$\Rightarrow g_p = \frac{GM}{r^2}$$ and it is directed towards the mass M.

Hence, $\vec{g}_p = \dfrac{GM}{r^2}\hat{a}_r$, where $\hat{a}_r = (\vec{r}/r)$

Acceleration due to gravity on surface on earth

$$g = \frac{GM}{r^2}$$

Where $G = 6.67 \times 10^{-11} \dfrac{N-m^2}{Kg^2}$ (universal gravitational constant)

$M = 5.983 \times 10^{24}$ Kg (mass of earth)
$R = 6.378 \times 10^6$ m (equation radius of earth)
r = distance between the particle and centre of earth

If the particle is very close to the earth's surface) then $= R + h \simeq R$.
Putting all the values we obtain

$$g = 9.8 \text{ m/sec}^2$$

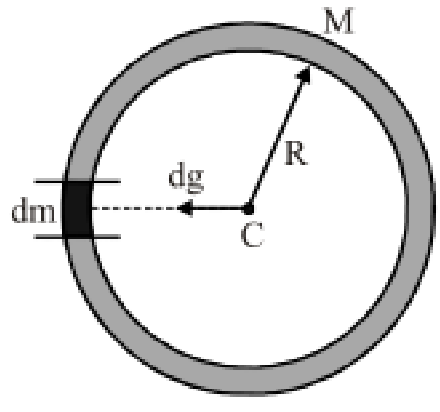
Note : (i) All objects on or above the earth's surface (at low altitudes) experience an acceleration of 9.8 m/sec^2 (approximately). The motion of particles under gravity is known as free fall. For example (a) releasing any object in earth's gravity (b) falling of fruits from the trees (c) falling of meteorites (d) motion of the satellites and (e) projectile motion.

(ii) Newton's second law of motion for any particle falling freely under gravity can be written as $\vec{F} = m\vec{a} = m\vec{g}$. The acceleration due to gravity is independent of mass of the particle. That means all the particles move with same acceleration $\vec{g}$ at a particular point.

Gravitational Field Strength due to a Ring

Case-I: At the centre of ring

To find gravitational field strength at the centre of a ring of mass M and radius R, we consider an elemental mass dm on it as shown in figure.

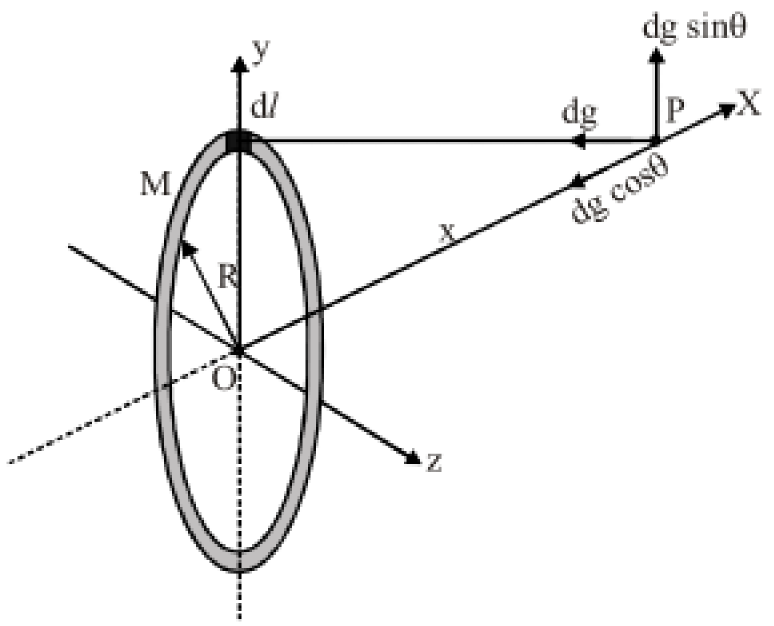


Here we can simply state that another element of same exactly opposite to dm on other half of ring will produce an equal gravitational field at C in opposite direction. Thus due to all the elements on ring, the net gravitational field at centre C will be vectorially nullified and hence net gravitational field strength at C will be 0.

Case-II: At a point on the axis of ring

To find this we consider an element dl on ring as shown figure. The mass dm of this element can be given as

$$dm = \frac{M}{2\pi R} dl$$



Let the gravitational field strength at point P due to the element dm is dg then it is given as

$$dg = \frac{Gdm}{(x^2 + R^2)}$$

Thus here net gravitational field strength at P is given as

$$g = \int dg \cos\theta = \int_0^{2\pi R} \frac{GM\ dl}{2\pi R(x^2 + R^2)} \times \frac{x}{\sqrt{x^2 + R^2}}$$

$$= \frac{GM\ x}{2\pi R(x^2+R^2)^{3/2}}[2\pi R]$$

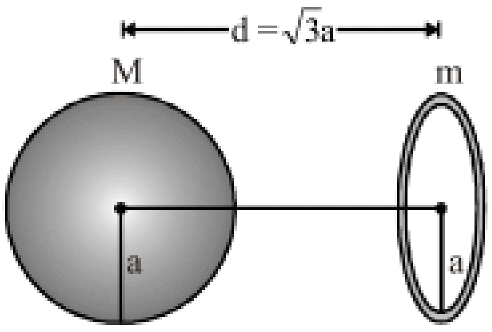
$$= \frac{GMx}{(x^2+R^2)^{3/2}}$$

Illustration :

A uniform ring of mass m and radius a is placed directly above a uniform sphere of mass M and of equal radius. The centre of the ring is at a distance $\sqrt{3}$ a from the centre of the sphere. Find the gravitational force exerted by the sphere on the ring.

Sol. *The gravitational field at any point on the ring due to the sphere is equal to the field due to single particle of mass M Placed at the centre of the sphere. Thus, the force on the ring due to the sphere is also equal to the force on it by particle of mass M placed at this point. By Newton's third law it is equal to the force on the particle by the ring. Now the gravitational field due to the ring at a distance d = $\sqrt{3}$ a on its axis is given as*

$$g = \frac{Gmd}{(a^2+d^2)^{3/2}} = \frac{\sqrt{3}Gm}{8a^2}$$



The force on sphere of mass M placed here is

$$F = Mg$$

$$= \frac{\sqrt{3}GMm}{8a^2}$$

Illustration :

If the radius of the earth were to shrink by one percent, its mass remaining the same. What would happen to the acceleration due to gravity on the earth's surface.

Sol. *Consider the case of a body of mass m placed on the earth's surface (mass of the earth M and radius R).*

$$g_s = \frac{GM_e}{R_e^2} \qquad ...(i)$$

Now, when the radius reduced by 1%, i.e. become 0.99 R, let acceleration due to gravity be g', then

$$g' = \frac{GM}{(0.99)^2} \qquad ...(ii)$$

From equation (i) and (ii), we get

$$\frac{g'}{g} = \frac{R^2}{(0.99)R^2} = \frac{1}{(0.99)^2}$$

or $$g' = g \times \left(\frac{1}{(0.99)}\right)^2$$

or $$g' = 1.02\, g$$

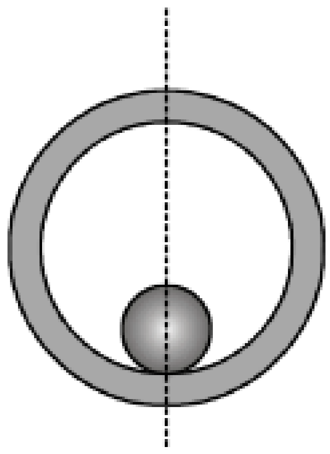
Thus the value of g is increased by 2%

Practice Exercise

Q.1 Two concentric spherical shells have masses M_1 , M_2 and radii R_1, R_2 ($R_1 < R_2$). What is the force exerted by this system on a particle of mass m_1 if it is placed at a distance $(R_1 + R_2)/2$ from the centre ?

Q.2 A particle would take a time t_1 to move down a straight tunnel from the surface of earth to its centre. If g is assumed to be constant, time would be t_2. Find t_1 / t_2 .

Q.3 A solid sphere of mass m and radius r is placed inside a hollow thin spherical shell of mass M and radius R as shown in figure. A particle of mass m' is placed on the line joining the two centre at a distance x from the point of contact of the sphere and the shell. Find the magnitude of the resultant gravitational force on this particle due to the sphere and the shell if (a) $r < x < 2r$, (b) $x < 2R$ and (c) $x > 2R$.



Answers

Q.1 $\dfrac{2GM_1m}{(R_1+R_2)}$ Q.2 $\dfrac{\pi}{2\sqrt{2}}$ Q.3 (a) $\dfrac{Gmm'(x-r)}{r^3}$ (b) $\dfrac{Gmm'}{(x-r)^2}$ (c) $\dfrac{GMm'}{(x-R)^2} + \dfrac{Gmm'}{(x-r)^2}$

Variation in the value of g

The acceleration due to gravity is given by

$$g = \frac{F}{m} = \frac{GM}{R^2}$$

where F is the force exerted by the earth on an object of mass m. This force is affected by a number of factors and hence g also depends on these factors.

(a) height from the surface of the Earth

If the object is placed at a distance h above the surface of the earth, the force of gravitation on it due to the earth is

$$F = \frac{GMm}{(R+h)^2}$$

where M is the mass of the earth and R is its radius.

Thus, $$g = \frac{F}{m} = \frac{GM}{(R+h)^2}$$

We see that the value of g decreases as one goes up. We can write,

$$g = \frac{GM}{R^2\left(1+\frac{h}{R}\right)^2} = \frac{g_0}{\left(1+\frac{h}{R}\right)^2}$$

where $g_0 = \frac{GM}{R^2}$ is the value of g at the surface of the earth. If h < < R,

$$g_0 = g_0\left(1+\frac{h}{R}\right)^{-2} \approx \left(1-\frac{2h}{R}\right).$$

If one goes a distance h inside the earth such as in mines, the value of g again decreases. The force by the earth is, by equation

$$F = \frac{GMm}{R^3}(R-h)$$

or, $$g = \frac{F}{m} = \frac{GM}{R^2}\left(\frac{R-h}{R}\right)$$

$$g = g_0\left(1-\frac{h}{g}\right)$$

The value of g is maximum at the surface of the earth and decreases with the increase in height as well as with depth.

Illustration :

Calculate the value of acceleration due to gravity at a point (a) 5.0 km above the earth's surface and (b) 5.0 km below the earth's surface. Radius of earth = 6400 km and the valeu of g at the surface of the earth is 9.80 m s^{-2}.

Sol. *(a) The value of g at a height h is (for h < < R)*

$$g_0 = g_0\left(1-\frac{2h}{R}\right)$$

$$= (9.80\ m\ s^{-2})\left(1-\frac{2\times 5.0\,km}{6400\,km}\right)$$

$$= 9.78\ ms^{-2}$$

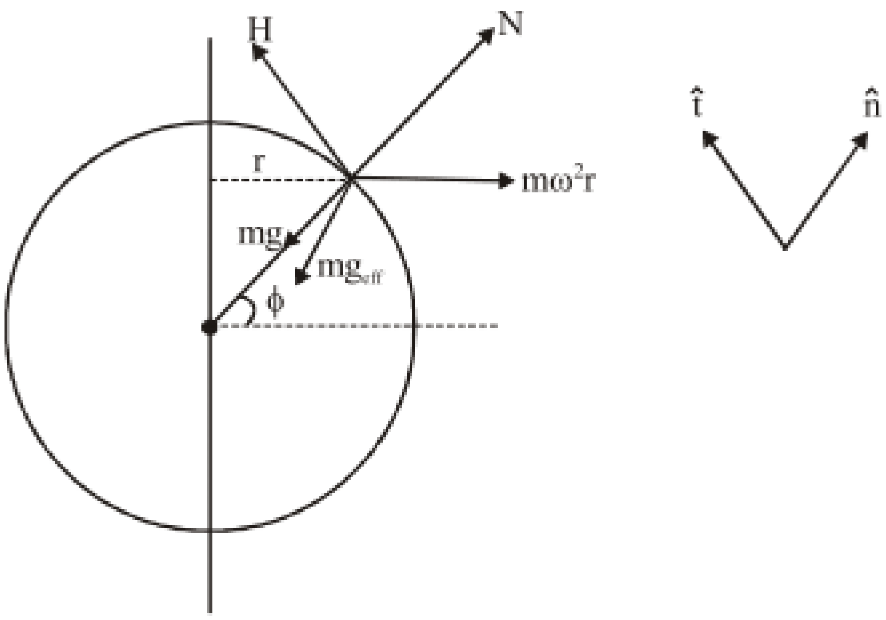
(b) The value at a depth h is

$$g = g_0\left(1 - \frac{h}{R}\right)$$

$$= (9.8\ m\ s^{-2})\ \left(1 - \frac{5.0\,km}{6400\,km}\right)$$

$$= 9.79\ m\ s^{-2}$$

Variation with latitude (Due to rotation of earth)



In figure m is a mass placed on a weighing machine situated at a latitude of ϕ the real forces acting on it are :

(i) the gravitational force mg towards the center of earth.

(ii) the normal reaction N of the weighing machine directed away from the center of earth,

(iii) The horizontal reaction H of the weighing machine directed along the tangent as shown.

In the reference frame fixed to the earth's surface the body would also be acted upon by the pseudo force (centrifugal force) mw^2r directed as shown.

where r is the distance of P from earth's axis, $r = R_e\cos\phi$, where R_e is radius of earth.

Now in this reference frame m is at rest. Resolving forces along the radial and normal direction we get,

$$mg = m\omega^2 r\cos\phi + N$$

$$= m\omega^2 R_e\cos^2\phi + N \Rightarrow N = mg - m\omega^2 R_e\cos^2\phi$$

and $\quad H = m\omega^2 R_e\cos\phi\sin\phi$

Now the effective weight of the body is the net force experienced by the weighing machine which is equal and opposite to the force exerted by the weighing machine on the body.

$$\therefore\ m\vec{g}_{eff} = -\left(\vec{N} + \vec{H}\right) = -\left(N\hat{n} + H\hat{t}\right)$$

$$\Rightarrow \vec{g}_{eff} = -g\left[\left(\frac{1 - \omega^2 R_e\cos^2\phi}{g}\right)\hat{n} - \frac{\omega^2 R_e\cos\phi\sin\phi\,\hat{t}}{g}\right]$$

Now $\dfrac{\omega^2 R_e}{g} \simeq 0.0337$, Hence its square may be neglected and thus $\boxed{g_{eff} \simeq g - \omega^2 R\cos^2\phi}$

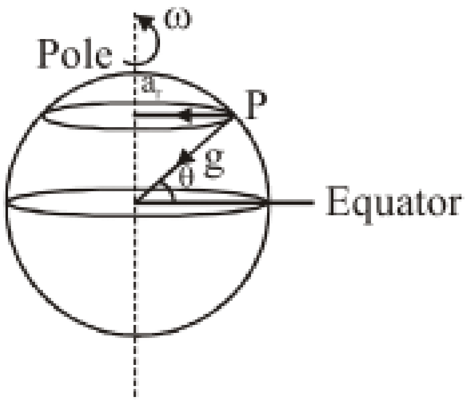
g_{eff} = apparant according due to gravity at a latitude of ϕ

* At poles, $\phi = \pi/2$ and hence $g_{eff} = g$
* At the equator, $\phi = 0$ and hence $g_{eff} = g - \omega^2 R$

Illustration :

If earth stops spinning about its own axis, what will be the change in acceleration due to gravity on its equation ? The radius of earth is 6.4×10^6 m and its angular speed is 7.27×10^{-5} rad/s.

Sol. *Effective acceleration due to gravity is given by*



$g' = g - R\omega^2 \cos^2\theta.$

Hence change in acceleration due to gravity is given as

$\Delta g = g - g' = R\omega^2 \cos^2\theta,$

at equator $\theta = 0$

$= 6.4 \times 10^6 \times (7.27 \times 10^{-5})^2$

$= 0.0338\ m/s^2$

(c) Nonsphericity of the Earth

All formulae and equations have been derived by assuming that the earth is a uniform solid sphere. the shape of the earth slightly deviates from the perfect sphere. The radius in the equatorial plane is about 21 km larger than the radius along the poles. Due to this the force of gravity is more at the poles and less at the equator. The value of g is accordingly larger at the poles and less at the equator. Note that due to rotation of earth also, the value of g is smaller at the equator than that at the poles.

(d) Nonuniformity of the Earth

The earth is not a uniformly dense object. There are a veriety of minerals, metals, water, oil, etc., inside the earth. Then at the surface there are mountains, seas, etc. Due to these nonuniformities in the mass distribution, the value of g is locally affected.

"Weighing" the Earth

The exerted force by the earth on a body is called the weight of the body. In this sence "weight of the earth" is a meaningless concept. However, the mass of the earth can be determined by noting the acceleration due to gravity near the surface of the earth. We have,

$$g = \frac{GM}{R^2}$$

or, $M = gR^2/G$

Putting $g = 9.8\ m\ s^{-2}$, $R = 6400$ km

and $G = 6.67 \times 10^{-11} \dfrac{N-m^2}{kg^2}$

the mass of the earth comes out to be 5.98×10^{-24} kg.

Illustration :

At what rate should the earth rotate so that the apparent g at the equator becomes zero ? What will be the length of the day in this situation ?

Sol. *At earth's equator effective value of gravity is*

$$g_{eq} = g_s - \omega^2 R_e$$

If g_{eff} at equator to be zero, we have

$$g_s - \omega^2 R_e = 0$$

or
$$\omega = \sqrt{\frac{g_e}{R_e}}$$

Thus length of the day will be

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{R_e}{g_s}}$$

$$= 2 \times 3.14\sqrt{\frac{6.4\times 10^6}{9.8}} = 5074.77s$$

$$\simeq 84.57\ min.$$

Practice Exercise

Q.1 Earth's mass is 80 times that of the moon and their diameters are 12800 and 3200 kms respectively. What is the value of g at the moon ? g on earth = 980 cm/s^2.

Q.2 The diameter of a planet is four times that of the earth. Find the time period of pendulum on the planet, if it is a second pendulum on the earth. Take the mean density of the planet equal to that of the earth,

Q.3 A tunnel is dug along a chord of the earth at perpendicular distance R/2 from the earth's centre. The wall of the tunnel may be assumed to be frictionless. Find the force exerted by the wall on particle of mass m when it is at a distance x from the centre of the tunnel.

Q.4 Find the height over earth's surface at which the weight of a body becomes half of its value at the surface.

Answers

Q.1 196 cm/s^2 Q.2 1 s Q.3 $\dfrac{GM_e m}{2R^2}$ Q.4 $(\sqrt{2}-1)$

Gravitational potential energy

In analyzing the motion of planets and satellites, it is often easier and more informative to use energy rather than force. In this section we shall evaluate the potential energy of a system consisting of two bodies that interact through the gravitational force. we obtained the potential energy change due to gravity for a body that moves through a height y near the Earth's surface : $\Delta U = mgy$. However, this applies only near the Earth's surface, where (for changes in height that are small compared with the distance from the center of the Earth) we can regard the gravitational force as approximately constant. Our goal here is to find a general expression that applies at all location, such as at the altitude of an orbiting satellite.

The potential energy difference can be found from equation. $\Delta U = U_b - U_a = -W_{ab}$, where W_{ab} is the work done to configuration b. However, this equation applies only if the force is conservative.

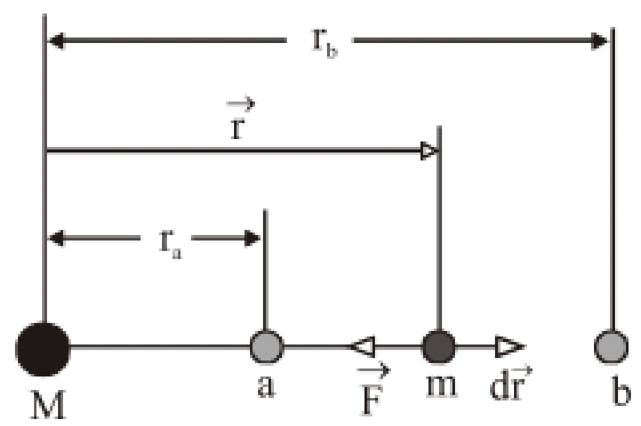
Calculating the Potential Energy

The gravitational force is conservative so we can calculate the potential energy. Figure shows a particle of mass m moving from a to b along a radial path. A particle of mass M, which we assume to be at rest at the origin, exerts a gravitational force on m. The vector $\vec{r}$ locates the position of m relative to M by the gravitational force is

$$W_{ab} = -\int_a^b F dr$$

$$= -\int_{r_a}^{r_b} \frac{GMm}{r^2} dr = -GMm \int_{r_a}^{r_b} \frac{dr}{r^2}$$

$$= -GMm \left(-\frac{1}{r} \right) \Bigg|_{r_a}^{r_b} = GMm \left(\frac{1}{r_b} - \frac{1}{r_a} \right)$$



The negative sign in the first line of this equation arises because the (attractive) force $\vec{F}$ and the infinitesimal radial vector $d\vec{r}$ point in opposite directions. Equation shows that, when $r_b > r_a$ (as in figure), the work W_{ab} is negative, as we expect.

Applying equation ($\Delta U = -W_{ab}$), we can find the change in the potential energy of the system as m oves between points a and b

$$\Delta U = U_b = -W_{ab} = GMm \left(\frac{1}{r_a} - \frac{1}{r_b} \right)$$

A particle of mass M exerts a gravitational force $\vec{F}$ on a particle of mass m that moves from a to b. If m moves outward from a to b, the change in potential energy is positive ($U_b > U_a$). That is, if the particle passes through point a with a certain kinetic energy K_a, as it travels to b its gravitational potential energy increases as its kinetic energy decreases ($K_b < K_a$). Conversely, if the particle is moving inward, its potential energy decreases as its kinetic energy increases.

Instead of differences in potential energy, we can consider the value of the potential energy at a single point if wedefine a reference point. We choose our reference configuration to be an infinite separation of the particles, and the we define the potential energy to be zero in that configuration. Let us evaluate equation for $r_b = \infty$ and $U_b = 0$. If a represents any arbitrary point, where the separation between the particles is r, then equation becomes

$$U(\infty) - U(r) = GMm\left(\frac{1}{r} - 0\right)$$

or $$U(r) = -\frac{GMm}{r}$$

Note :

1. We can reverse the previous calculation and derive the gravitational force from the potential energy. For spherically symmetric potential energy functions, the relation $F = -dU/dr$ gives the radial component of the force; see equation. With the potential energy of equation, we obtain

$$F = -\frac{dU}{dr} = -\frac{d}{dr}\left(-\frac{GMm}{r}\right) = -\frac{GMm}{r^2}$$

The minus sign here shows that the force is attractive, directed inward along a radius.

2. We can show that the potential energy defined according to equation leads to the familiar mgy for a small difference in elevation y near the surface of the Earth. Let us evluate equation for the difference in potential energy between the location at a height y above the surface (that is, $r_b = R_E + y$, where R_E is the radius of the Earth) and the surface $(r_a = R_E)$

$$\Delta U = U(R_E + y) - U(R_E) = GM_E m\left(\frac{1}{R_E} - \frac{1}{R_E + y}\right)$$

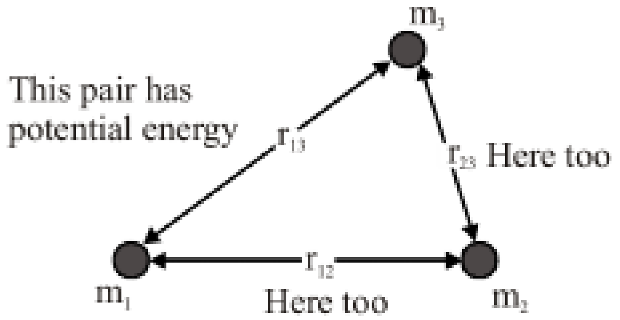
$$= \frac{GM_E m}{R_E}\left(1 - \frac{1}{1 + y/R_E}\right)$$

When $y << R_E$, which would be the case for small displacements of bodies near the Earth's surface, we can use the binomial expansion to approximate the last term as $(1 + x)^{-1} = 1 - x + ... \approx 1 - x$, which gives

$$\Delta U \approx \frac{GM_E m}{R_E}\left[1 - \left(1 - \frac{y}{R_E}\right)\right] = \frac{GM_E my}{R_E^2} = mgy$$

using equation to replace GM_E/R_E^2 with g.

3. If our system contains more than two particles, we consider each pair of particles in turn, calculate the gravitational potential energy of that pair with Equation as if the other particles were not there, and then algebraically sum the results. Each of the three pairs of Figure, for example, gives the potential energy of the system as



If our system contains more than two particles, we consider each pair of particles in turn, calculate the gravitational potential energy of that pair with Eq. below as if the other particles were not there, and then algebraically sum the results. the potential energy of the system as

$$U = -\left(\frac{Gm_1m_2}{r_{12}} + \frac{Gm_1m_3}{r_{13}} + \frac{Gm_2m_3}{r_{23}}\right)$$

Gravitational Potential

The gravitational potential at a point in gravitational field is the gravitational potential energy per unit mass placed at the point in gravitational field. Thus at a certain point in gravitational field, a mass m_0 has a potential energy U the gravitational potential at that point is given as

$$V = \frac{U}{m_0}$$

or if at a point in gravitational field gravitational potential V is known then the interaction potential energy of a point mass m_0 at the point in the field is given as

$$U = m_0 V$$

Interaction energy of a point mass m_0 in a field is defined as work done in bringing that mass from infinity to the point. In the same fashion we can define gravitational potential at a point in field, alternatively as "Work done in bringing a unit mass from infinity to that point against gravitational force."

When a unit mass is brought to a point in a gravitational field, force on the unit mass is $\vec{g}$ at a point in the field. Thus the work done in bringing this unit mass from infinity to a point P in gravitational field or gravitational potential at point P is given as

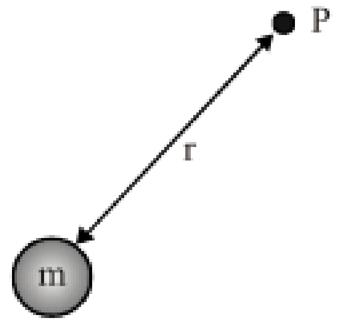
$$V_P = -\int_{\infty}^{P} \vec{g} \cdot dx$$

Here negative sing shown that V_P is the negative of work done by gravitation field or it is the external required work for the purpose against gravitational force.

Gravitational Potential due to a Point Mass

We place a test mass m_0 at P and we find the interaction energy of m_0 with the field of m, which is given as

$$U = -\frac{Gmm_0}{x}$$



Now the gravitational potential at P due to m can be written as

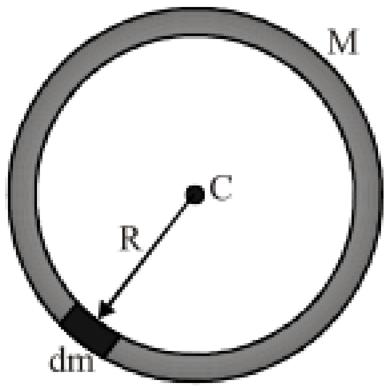
$$V = \frac{U}{m_0} = -\frac{Gm}{r}$$

Gravitational potential due to a ring

At the centre of ring

In gravitational potential the situation is not like this as it is a scaler quantity and here the distane of centre from each element dm on ring circumference is equal to R, thus every element dm produces an equal gravitational potential at C, given as

$$dV = -\frac{Gdm}{R}$$



Now due to the whole ring the gravitational potential at its centre C is given as

$$V_C = \int dV = -\int \frac{Gdm}{R} = -\frac{Gm}{R}$$

Note : Now we have seen that most of the expressions of $\vec{g}$ and V are same as that we have calculated in the topic electrostatics. So we will not prove them again rather we can replace some sysbols as :

<u>**Electrostatics**</u>	<u>**Gravitation**</u>
Q	M
K	G
$\in_0$	$\frac{1}{4\pi G}$

For Formula conversion :

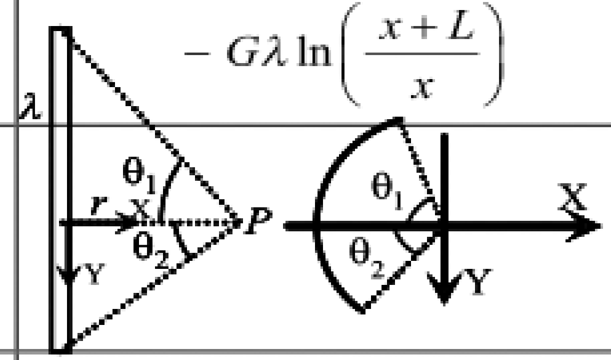
Body	Gravitational Field	Gravitational potential
Point mass	$-\frac{Gm}{r^2}$	$-\frac{Gm}{r}$
Ring	$\frac{Gmx}{\left(R^2+x^2\right)^{3/2}}$; $E_{\max}\big\vert_{x=R/\sqrt{2}}$	$-\frac{Gm}{\sqrt{x^2+R^2}}$
Disc	$2\pi G\sigma\left(1-\frac{x}{\sqrt{x^2+R^2}}\right)$	$-2\pi G\sigma\left[\sqrt{x^2+R^2}-x\right]$
infinite wire	$E=2G\lambda/r$	$\Delta V=2G\lambda \ln\frac{r_1}{r_2}$
wire	$\frac{G\lambda L}{x(x+L)}$	$-G\lambda \ln\left(\frac{x+L}{x}\right)$
circular arc & rod	$-\frac{G\lambda}{r}\begin{bmatrix}(\sin\theta_1+\sin\theta_2)\hat{i}\\-(\cos\theta_1-\cos\theta_2)\hat{j}\end{bmatrix}$	
Apex of cone	∞	$-2\pi G\sigma R$
infinite plate	$E=2\pi G\sigma$	$\Delta V=2\pi G\sigma d$
Hollow Sphere	$\vec{E}=\begin{cases}0 & 0\le r<R\\-\frac{GM}{r^2}\hat{r} & r\ge R\end{cases}$	$V=\begin{cases}-\frac{GM}{R} & 0\le r<R\\-\frac{GM}{r} & r\ge R\end{cases}$
Solid sphere	$\vec{E}=\begin{cases}-\frac{GM\vec{r}}{R^3}=-\frac{4\pi G\rho}{3}\vec{r} & 0\le r<R\\-\frac{Kq}{r^2}\hat{r} & r\ge R\end{cases}$	$V=\begin{cases}-\frac{GM}{2R}\left(3-\frac{r^2}{R^2}\right) & 0\le r<R\\-\frac{GM}{r} & r\ge R\end{cases}$

Illustration :

The magnitude of gravitational field intensities at distance r_1 and r_2 from the centre of a uniform solid sphere of radius R and mass M are I_1 and I_2 respectively. Find the ratio of I_1/I_2 if (a) $r_1 > R$ and $r_2 > R$ and (b) $r_1 < R$ and $r_2 < R$ (c) $r_1 > R$ and $r_2 < R$.

Sol. *Gravitational field intensity for a uniform spherical distribution of mass is given by:*

$$I = \frac{GM}{r^2} \qquad \text{for } r > R \text{ and}$$

$$= \frac{GM}{R^3} \qquad \text{for } r < R.$$

(a) $r_1 > R$ and $r_2 > R$

$$\frac{I_1}{I_2} = \frac{(GM/r_1^2)}{(GM/r_2^2)} = \frac{r_2^2}{r_1^2}$$

(b) $r_1 < R$ and $r_2 < R$

$$\frac{I_1}{I_2} = \frac{(GM/R^3)\,r_1}{(GM/R^3)\,r_2} = \frac{r_1}{r_2}$$

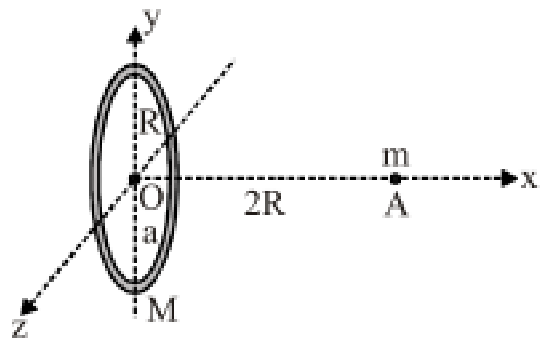
(c) $r_1 > R$ and $r_2 < R$

$$\frac{I_1}{I_2} = \frac{(GM/r_1^2)}{(GM/R^3)\,r_2} = \frac{R^3}{r_1^2 r_2}$$

Illustration :

A circular ring of mass M and radius R is placed in YZ plane with centre at origin. A particle of mass m is released from rest at a point x = 2R. Find the speed with which it will pass the centre of ring

Sol.



As shown in figure, first we find potential at A due to the ring, it is given as

$$V_A = -\frac{GM}{\sqrt{R^2 + (2R)^2}} = -\frac{GM}{\sqrt{5}R}$$

Now potential at origin O due to ring is

$$V_0 = -\frac{GM}{R}$$

When m moves from A to O, work done on it due to gravitational force is

$$W = m\,(V_A - V_0) = m\left(-\frac{GM}{\sqrt{5}R} + \frac{GM}{R}\right)$$

$$= \frac{GMm}{R}\left(\frac{\sqrt{5}-1}{\sqrt{5}}\right)$$

This work done by gravitational force on m must be equal to the increase in kinetic energy of the mass m, thus we have

$$\frac{1}{2}mv^2 = \left(\frac{\sqrt{5}-1}{\sqrt{5}}\right)\frac{GMm}{R}$$

or $$v = \left[\frac{2(\sqrt{5}-1)GM}{\sqrt{5}R}\right]^{1/2}$$

This problem can also be solved simply by using energy conservation. These initially when m was at rest at point A. The total energy of system is only gravitational potential energy given as

$$E_i = m \,.\, V_A = -\frac{GMm}{\sqrt{5}R}$$

Finally when m passes through O, the total energy of system is

$$E_f = \frac{1}{2}mv^2 + mV_0$$

$$= \frac{1}{2}mv^2 - \frac{GMm}{R}$$

As no external work is done on the system in this case, the total energy of system must be conserved, thus according to energy conservation we have

$$E_i - E_f$$

$$-\frac{GMm}{R} = \frac{1}{2}mv^2 - \frac{GMm}{R}$$

$$v = \left[\frac{2(\sqrt{5}-1)GM}{\sqrt{5}R}\right]^{1/2}$$

Illustration :

A small mass m is transferred from the centre of a hollow sphere of mass M to infinity. Find work done in the process. Compare this with the situation if instead of a hollow sphere, a solid sphere of same mass were there.

Sol. *We know at infinity, gravitational potential is taken zero. Thus if V_C be the gravitational potential at centre of hollow sphere then external work required in the process is*

$$W = m\,(0 - V_C)$$

or $$= m\left(0-\left(\frac{GM}{R}\right)\right) = \frac{GMm}{R}$$

Here $V_C = -\frac{GM}{R}$, the potential at the centre of a hollow sphere of mass M and radius R. If a solid sphere we here, we have at its centre

$$V_C = -\frac{3}{2}\frac{GM}{R}$$

Thus work required will be

$$W = m\left[0-\left(-\frac{3}{2}\frac{GM}{R}\right)\right] = \frac{3}{2}\frac{GMm}{R}$$

We can see in second case more work is required for the process.

Practice Exercise

Q.1 Find the kinetic energy needed to project a body of mass m from the centre of a ring of mass M and radius R so that it will never come back.

Q.2 How much work is done in circulating a small object of mass m around a sphere of mass m in a circle of radius R.

Q.3 Find the gravitational potential due to a hemispherical cup of mass M and radius R, at its centre of curvature.

Q.4 Find the gravitational potential energy of system consisting of uniform rod AB of mass M, length l and a point mass m as shown in figure.

Answers

Q.1 $\frac{GMm}{R}$ Q.2 0 Q.3 $-\frac{GM}{R}$ Q.4 $-\frac{GMm}{l}\ln\left(1+\frac{l}{r}\right)$

The motion of planets and satellites

(a) Planets

Planets move round the sun due to the gravitational attraction of the sun. The path of these planets are elliptical with the sun at a focus.

(b) Satellite

Satellites are launched from the earth so as to move round it. A number of rockets are fired from the satellite at proper time to establish the satellite in the desired orbit. Once the satellite is placed in the desired orbit with the correct speed for that orbit, it will continue to move in that orbit under gravitational attraction of the earth.

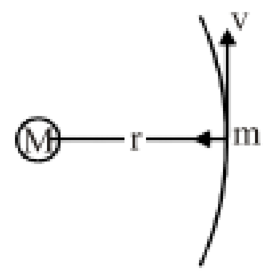
We make two assumptions that simplify the analysis:

(1) We consider the gravitational force only between the orbiting body (the Earthy, for instance and the central body (the Sun), ignoring the perturbing effect of the gravitational force of other bodies (such as other planets)

(2) We assume that the central body is so much more massive than the orbiting body that we can ignore its motion under their mutual interaction.

Let the speed of an artificial earth satellite in its orbits of radius r be v_0. The satellite is accelerating towards the centre of earth by earth's gravitational pull $\frac{GMm}{r^2}$

$$\Rightarrow \quad F_{CP} = \frac{GMm}{r^2}$$



$$\frac{mv_0^2}{r} = \frac{GMm}{r^2}$$

$$\Rightarrow \quad v_0 = \sqrt{\frac{GM}{r}} \qquad \text{...(i)}$$

Putting $\frac{GM}{r^2} = g$ (acceleration due to gravity at the orbit), we obtain,

$$\Rightarrow \quad v_0 = \sqrt{gr} \qquad \text{...(ii)}$$

When it orbits at an altitude h, putting r = (R+h)

$$\Rightarrow \quad v_0 = \sqrt{\frac{GM}{R+h}} = \sqrt{\frac{GM}{R(1+h/R)}} = \sqrt{\frac{gR}{(1+h/R)}}$$

Angular speed

The angluar speed

$$\omega = \frac{v_0}{r}$$

Putting $v_0 = \sqrt{\frac{GM}{r}}$, we obtain

$$\omega = \sqrt{\frac{GM}{r^3}} = \sqrt{\frac{GM}{(R+h)^3}}$$

Angular momentum

The angular momentum of an earth satellite or a planet is given as

L = mvr

$$= m\sqrt{\frac{GM}{r}} \times r$$

$$= m\sqrt{GMr}$$

Time period of Revolution

The period of revolution

$$T = \frac{2\pi}{\omega}$$

Puting $\omega = \sqrt{\frac{GM}{r^3}}$

$$T = 2\pi\sqrt{\frac{r^3}{GM}}$$

Energy consideration in planetary and satellite motion

$$U(r) = -\frac{GMm}{r}$$

Where r is the radius of the circular orbit.
The kinetic energy of the system is

$$K = \frac{1}{2}mv^2 = \frac{1}{2}m\omega^2 r^2$$

the Sun being at rest

$$\omega^2 r^2 = \frac{GM}{r}$$

so that (with $v = \omega r$)

$$K = \frac{GMm}{2r}$$

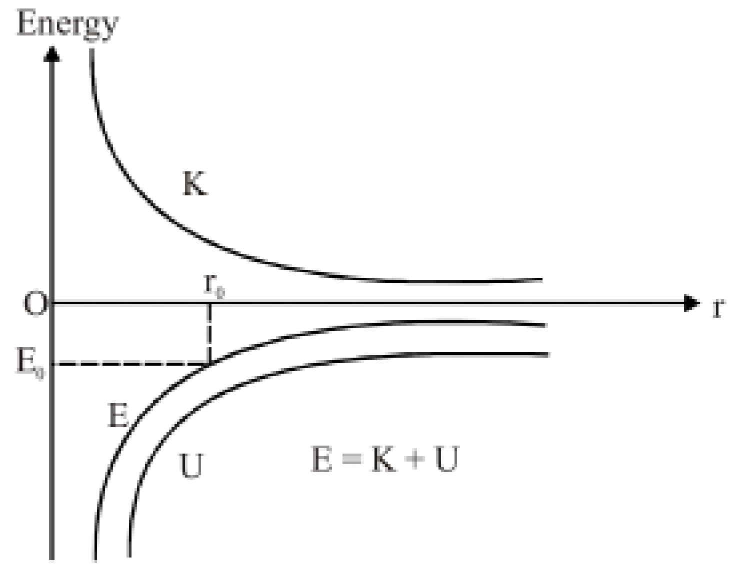
The total mechanical energy is

$$E = K + U = \frac{GMm}{2r} - \frac{GMm}{r} = \frac{GMm}{2r}$$

This energy is constant and negative. The kinetic energy can never be negative, but from equation we see that it

Note :

(1) Graph of kinetic energy K, potential energy U, and total energy $E = K + U$ of a body in circular planetary motion.



A planet with total energy $E_0 < 0$ will remain in a orbit. The greater the distance from the Sun, the greater (that is, less negative) its total energy E.

(2) Gravitational binding energy

We have seen that if a particle of mass m placed on the earth is given an energy $\frac{1}{2}mu^2 = \frac{GMm}{R}$ or more, it finally escapes from earth. The minimum energy needed to take the particle infinitely away from the earth is called the binding energy of the earth-particle system. Thus, the binding energy of the earth-particle system is $\frac{GMm}{R}$

Geostationary satellite :

A satellite which appears to be stationary when seen from earth is called a Geostationary satellite.
For a satellite to be geostationary.

(i) Its orbit must be circular

(ii) It must rotate about the same axis as earth, i.e. it must move in the equatorial* plane.

(iii) It must revolve form west to east.

(iv) Its time period must be 24 hours.

$$\Rightarrow \quad m\omega^2(R+h) = \frac{GMm}{(R+h)^2}$$

$$\left(\frac{2\pi}{T}\right)^2 = \frac{GM}{(R+h)^3}$$

$$(R+h) = \left\{GM\left(\frac{T}{2\pi}\right)^2\right\}^{1/3} \quad \text{where } T = 24 \times 3600 \text{ sec.}$$

Solving $h = 36000$ km (approx)

* Any satellite must rotate about the center of earth.

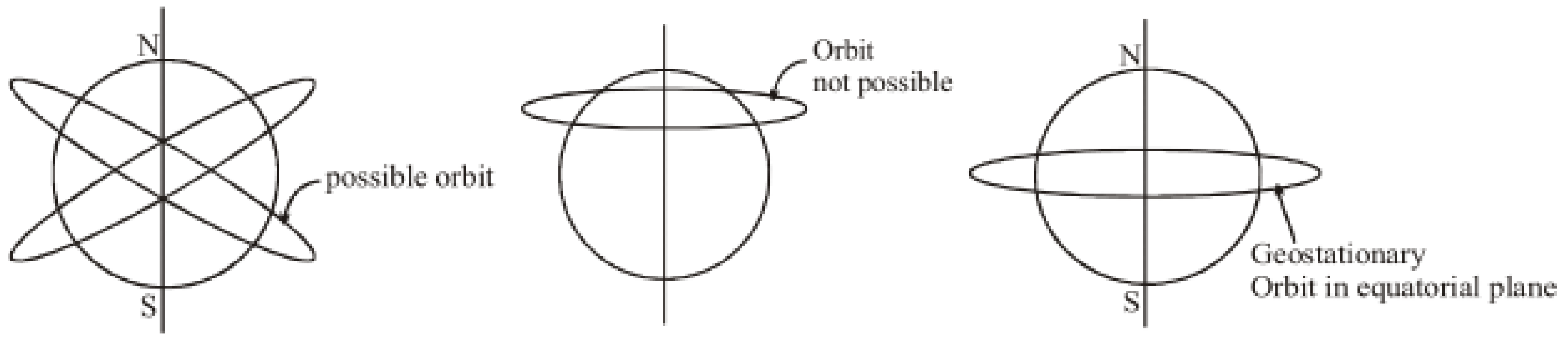


Illustration :

Estimate the mass of the sun, assuming the orbit of the earth round the sun to be a circle. The distance between the sun and earth is 1.49×10^{11} m and $G = 6.66 \times 10^{-11}$ Nm^2/kg^2.

Sol. *Here the revolving speed of earth can be given as*

$$v = \sqrt{\frac{GM}{r}} \qquad \text{[Orbital speed]}$$

Where M is the mass of sun and r is the orbit radius of earth.
We know time period of earth around sun is T = 365 days, thus we have

$$T = \frac{2\pi r}{v}$$

or $$T = 2\pi r\sqrt{\frac{r^3}{GM}}$$

or $$M = \frac{4\pi^2 r^3}{GT^2}$$

$$= \frac{4 \times (3.14)^2 \times (1.49 \times 1011)^3}{(365 \times 24 \times 3600)^2 \times (6.66 \times 10^{-11})} = 1.972 \times 10^{22} \text{ kg}$$

Illustration :

If the earth be one-half of its present distance from the sun, how many days will be in one year ?

Sol. *If orbit of earth's radius is R, in previous example we've discussed that time period is given as*

$$T = 2\pi r\sqrt{\frac{r}{GM}} = \frac{2\pi}{\sqrt{GM}}r^{3/2} \qquad ...(i)$$

If radius changes to $r' = \frac{r}{2}$*, new time period become*

$$T' = \frac{2\pi}{\sqrt{GM}}r^{3/2} \qquad ...(ii)$$

From equation (i) and (ii) we have

$$\frac{T}{T'} = \left(\frac{r}{r'}\right)^{3/2}$$

or

$$T' = T\left(\frac{r'}{r}\right)^{3/2}$$

$$= 365\left(\frac{1}{2}\right)^{3/2} = \left(\frac{365}{2\sqrt{2}}\right) \text{ days}$$

Illustration :

An artificial satellite is describing an equatorial orbit at 1600 km above the surface of the earth. Calculate its orbital speed and the period of revolution. If the satellite is travelling in the same direction as the rotation of the earth (i.e., from west to east), calculate the interval between two successive times at which it will appear vertically overhead to an observer at a fixed point on the equator. Radius of earth = 6400 km.

Sol. *We know that the period of the satellite is*

$$T = \frac{2\pi}{\sqrt{GM}}r^{3/2} = \frac{2\pi}{\sqrt{gR^2}}r^{3/2}$$

Where $r = 6400 + 1600 = 8000 \times 10^3$ m,

$g = 9.8$ m/sec^2 and $R = 6400 \times 10^3$ m

Substituting values we get

$$T = 2 \times 3.14\left[\frac{(8000\times 10^3)^3}{9.8\times(6400\times 10^3)^2}\right]^{1/2}$$

$$= 7096\ x$$

Futher, orbital speed,

$$v = \sqrt{\frac{GM}{r}} = \sqrt{\frac{gR^2}{r}}$$

or

$$v = \sqrt{\left(\frac{9.8}{8000\times 10^3}\right)}\times(6400\times 10^3)$$

$$= 7083.5 \text{ m/s}$$

Let t be the time interval between two successive moments at which the satellite is overhead to an observer at fixed position on the equator. As both satellite and earth are moving in same direction with angular speed ω_S and ω_E respectively, we can write the time of separation as

$$t = \frac{2\pi}{\omega_S - \omega_E}$$

Here $\omega_S = \frac{2\pi}{7096}$ and $\omega_E = \frac{2\pi}{86400}$

Thus we have $t = \frac{86400 \times 7096}{86400 - 7096}$

$= 7731\ s$

Escape Velocity

Definition: If a particle of mass m, kept in an attractive gravitational field is given sufficient kinetic energy, it may escape the gravitational pull due to the field. The particle will escape to infinity depending on whether its path allows it to do so. For example, a particle of mass m kept on the surface of the earth requires a minimum velocity v_{esc} so that it moves to infinity.

$$\frac{1}{2}mv_{esc}^2 - \frac{GMm}{R} = 0$$

Where M is the mass of the earth and R is its radius.

$$v_{esc} = \sqrt{\frac{2GM}{r}}$$

Note :

The escape speed does not depend on the direction in which the projectile is fired. The Earth's rotation which we have not considered in this calculation does play a role, however. Firing eastward has an advantage in that the Earth's tangential surface speed, which is 0.46 km/s at Cape Canaveral, provides part of the kinetic energy needed for escape, and thus less thrust from the rocket engines would be required to escape the Earth's gravity.

Illustration :

The minimum velocity of projection of a body to send it to infinity from the surface of a planet is $\frac{1}{\sqrt{6}}$ times that is required from the surface of the earth. The radius of the planet is $\frac{1}{36}$ times the radius of the earth. The planet is surrounded by an atmosphere which contains monoatomic innert gas ($\gamma = 5/3$) of constant density up to a height h(h << radius of the planet). Find the velocity of sound on the surface of the planet.

Sol. *Escape velocity from the surface of the planet*

$v_p = \sqrt{2g_pR_p}$

Given $v_p = \frac{v_e}{\sqrt{6}} = \sqrt{\frac{2g_eR_e}{6}}$

$$\sqrt{\frac{g_e R_e}{3}} = \sqrt{2g_p R_e/36} \Rightarrow \qquad g_p = 6g_e$$

Pressure exerted by the atmospheric column of height h on the surface of the planet $P = \rho g_p h$

Using equation of state $P = \dfrac{\rho RT}{M}$

Hence speed of the sound $v = \sqrt{\dfrac{\gamma RT}{M}} = \sqrt{\gamma g_p h} = \sqrt{6\gamma\, g_e h} = \sqrt{10\, g_e h}$

Practice Exercise

Q.1 Two satellites A and B of the same mass are orbiting the earth at altitudes R and 3R respectively, where R is the radius of the earth. Taking their orbits to be circular, obtain the ratios of their kinetic and potential energies.

Q.2 A satellite is to revolve round the earth in a circle of radius 8000 km. With what speed should this satellite be projected into orbit ? What will be the time period ? Take g at the surface = 9.8 m./s^2 and radius of the earth = 6400 km.

Q.3 A satellite of mass 2×10^3 kg has to be shifted from an orbit of radius 2R to another of radius 3R, where R is the radius of the earth. Calculate the minimum energy required.

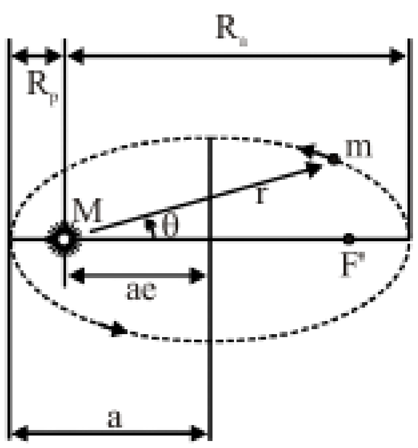
Answers

Q.1 2 : 1 Q.2 7.08 km/s. 118 minutes Q.3 1.04×10^{10} J

Kepler Laws :

The empirical basis for understanding the motions of the planets is there laws deduced by Kepler (1571 - 1630, well before Newton) from studies of the motion of the planet Mars.

1. **The law of orbits :** *All planets move in elliptical orbits having the Sun at one focus.* Newton was the first realize that there is a direct mathematical relationship between inverse-square ($1/r^2$) force and elliptical orbits. Figure shows a typical elliptical orbit.



A planet of mass m moving in a elliptical orbit around the Sun. The Sun, of mass M, is at one focus of the elipse. F' marks the other or "empty" focus. The semimajor axis a of the ellipse, the perihelion distance R_p, and the aphelion distance R_e are also shown. The distance *ae* locates the focal points, e being the eccentricity of the orbit.

other planets in the solar system, the eccentricities are small and the orbits are nearly circular.

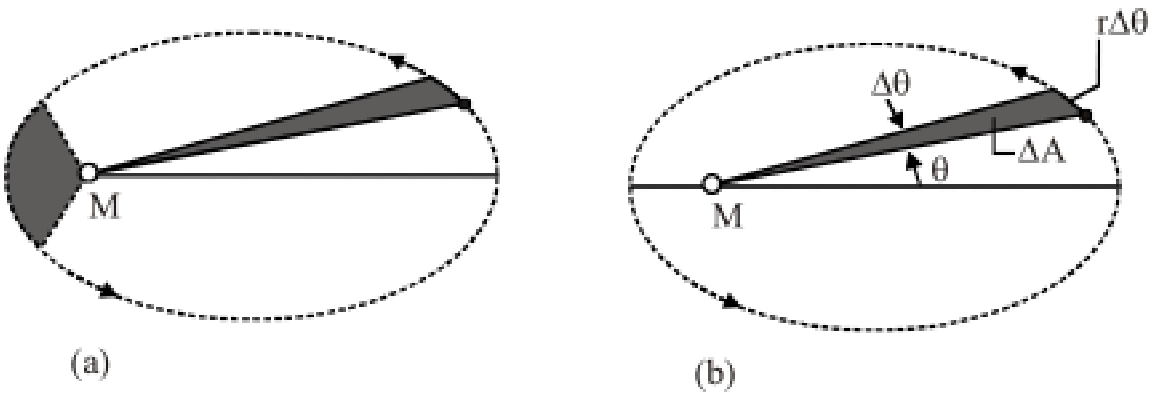
The maximum distance R_a of the orbiting body from the central body is indicated by the prefix apo - (or sometimes ap-), as in aphelion (the maximum distance from the Sun) or apogee (the maximum distance from Earth). Similarly, the cloest distance R_p is indicated by the prefix peri-, as in perihelion or perigee. As you can see from figure $R_a = a\,(1 + e)$ and $R_p = a\,(1 - e)$. For circular orbits. $R_a = R_p = a$

2. **The Law of Areas :** *A line joining any planet to the Sun sweeps out equal area in equal times.* Figure illustrates this law : in effect it says that the orbiting body moves more rapidly when it is close to the central body than it does when it is far aways. We now show that the law of areas is identical with the law of conservation of angular momentum.

Consider the small area increament ΔA covered in a time interval Δt, as shown in figure. The area of this approximately triangular wedge in one-half its base, $r\,\Delta\theta$, times its height r. The rate at which this area is swept out is $\Delta A / \Delta t = \frac{1}{2}\,(r\Delta\theta)\,(r) / \Delta t$. In the instantaneous limit this becomes

$$\frac{dA}{dt} = \lim_{\Delta \to 0} \frac{\Delta A}{\Delta t} = \lim_{\Delta \to 0} \frac{1}{2} r^2 \frac{\Delta\theta}{\Delta t} = \frac{1}{2} r^2 \omega$$

Assuming we can regard the more massive body M as at rest, the angular momentum of the orbiting body m relative



(a) The equal shaded areas are covered in equal times by a line connecting the planet to the Sun, demonstrating the law of areas.

(b) The area ΔA is covered in a time Δt, during which the line sweeps through an angle $\Delta\theta$.

to the origin at the central body is , according to equation $L_z = I\omega = mr^2\omega$ (choosing the z axis perpendicular to the plane of the orbit). Thus

$$\frac{dA}{dt} = \frac{L_z}{2m}$$

If the system of M and m is isolated, meaing that there is no net external torque on the system, the L_z is a constant; therefore dA/dt is also constant. That is in every interval dt in the orbit, the line connecting m and M sweeps out equal areas dA, which verifies kepler's second law. The speeding up of a comet as it passes close to the Sun is an example of this effect and is thus a direct consequence of the law of conservation of angular momentum.

3. **The law of Periods :** *The square of the period of any planet about the Sun is proportional to the cube of seminajor axis of the* . Let us prove this result for circular orbits. The gravitational force provides the necessary centripetal acceleration for circular motion.

If 'T' is the period of revolution and 'a' be the semi-major axis of the path of planet then according to kepler's III Law, we have

$$T^2 \propto a^3$$

For circular orbits, it is a special case of ellipse when its major and minor axis are equal. If a planet is in a circular orbit of radius R around the sun then its revolution speed must be given as

$$v = \sqrt{\frac{GM_s}{r}}$$

Where M_S is the mass of sun. There you can recall that this speed is independent from the mass of planet. Here the time period of revolution can be given as

$$T = \frac{2\pi r}{v}$$

or $$T = \frac{2\pi r}{\sqrt{\frac{GM_s}{r}}} \qquad ...(A)$$

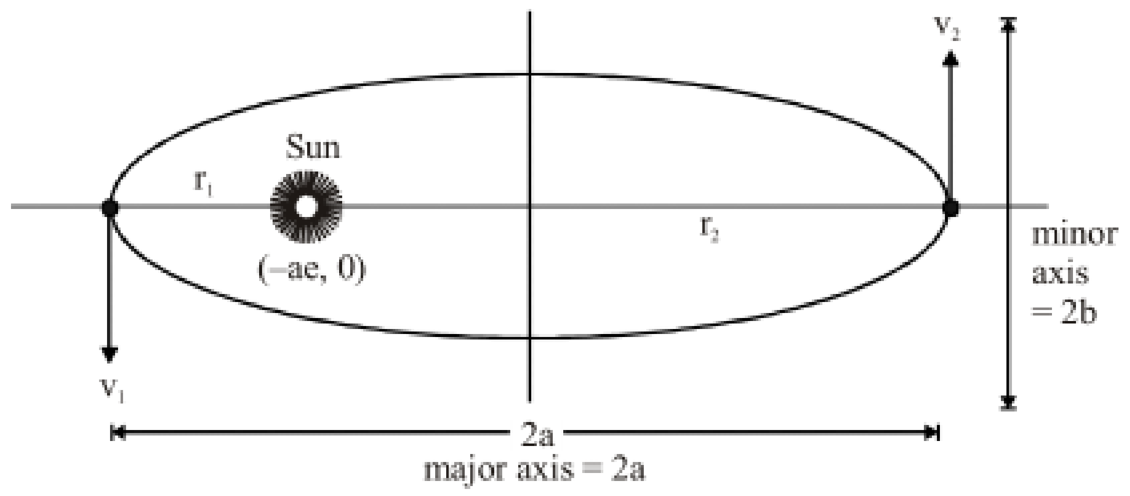
Squaring equation no. (A) we get

$$T^2 = \frac{4\pi^2}{GM_S} r^3 \qquad ...(B)$$

Equation (B) verifies the statement of Kepler's third law for circular orbits. Similarly we can also verify it from elliptical orbits. For this we start from the relation we've derived earlier for rate sweeping area by the position vector of planet with respect to sun which is given as

$$\frac{dA}{dt} = \frac{L}{2m}$$

Where L is the total angular momentum of planet during its motion consider the path of planet shown in figure is an elliptical path with sun on focus (–ae, 0).



Here r_1 and r_2 are the shortest and farthest distance of planet from sun during its motion, which are given as

$$r_1 = a\,(1 - e)$$

and $$r_2 = a\,(1 + e)$$

Where e is the centricity. From geometry we know that the relation in semi major axis and semiminor axis be is given as

$$b = a\sqrt{1-e^2}$$

If v_1 and v_2 are the planet speeds at perihelion and aphelion points then from conservation of momentum we have

$$L = mv_1 r_1 = mv_2 r_2$$

From energy conservation we have

$$\frac{1}{2}m_1 v_1^2 - \frac{GM_S m}{r_1} = \frac{1}{2}mv_2^2 - \frac{GM_s m}{r_2}$$

or $$v_1^2 - v_2^2 = 2GM_S\left[\frac{1}{r_1} - \frac{1}{r_2}\right]$$

$$v_1^2\left[1 - \frac{r_1^2}{r_2^2}\right] = 2GM_s\left[\frac{r_2 - r_1}{r_1 r_2}\right]$$

or $$v_1 = \sqrt{\frac{2GM_S r_2}{(r_1 + r_2)r_1}}$$

From equation (vi) and (vii) we have

$$v_1 = \sqrt{\frac{GM_S}{a}\left(\frac{1+e}{1-e}\right)}$$

Now from equation (v) we have the total area of ellipse traced by the planet is given as

$$A = \frac{L}{2m}T$$

or $$T = \frac{2m}{L}A = \frac{2m\pi ab}{L} = \frac{2m\pi ab}{mv_1 r_1}$$

or $$T = \frac{2m\pi a\left[a\sqrt{1-e^2}\right]}{m\left[\sqrt{\frac{GM_S}{a}\left(\frac{1+e}{1-e}\right)}[a(1-e)]\right]}$$

or $$T^2 = \frac{4\pi^2}{GM_s}a^3$$

Illustration :

A satellite is launched tangentially from a height h above earth's surface as shown.

I. Find v_{min} so that it just touches the earth's surface

II. If h = R and satellite is launched tangentially with speed = $\sqrt{\frac{3GM}{5R}}$ find the mximum distance of satellite from earth's center

Sol. *(I) Satellite just Grazes from surface of earth*

when $2a = 2R + h$

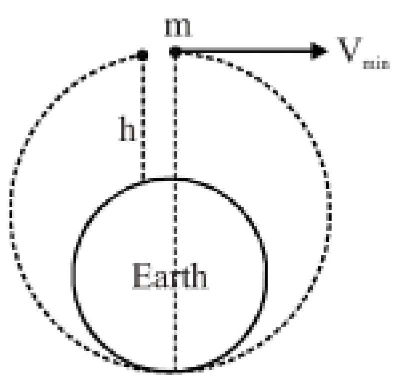
$$a = \left(R + \frac{h}{2}\right)$$

Total energy (E) $= -\dfrac{GMm}{2a} = -\dfrac{GMm}{(2R+h)}$

$$\frac{1}{2}mV_{0\ min}^2 - \frac{GMm}{R+h} = -\frac{GMm}{2R+h}$$

$$V_{0\ min}^2 = 2GMm\left[\frac{1}{R+h} - \frac{1}{2R+h}\right]$$

$$= \frac{2GMR}{(R+h)(2R+h)} = \frac{2GMR}{r(R+r)}$$

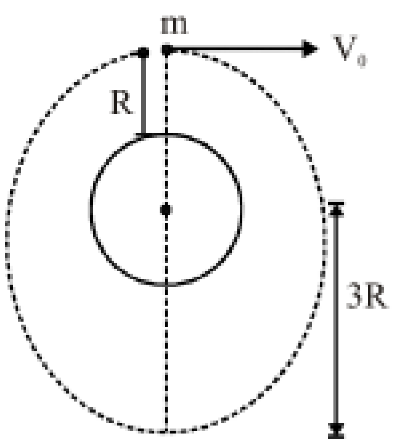


$$V_{0\ min} = \sqrt{\frac{2GMR}{r(R+r)}} \qquad \text{where } r = h + R$$

(II) To find maximum distance from center.

$$E = -\frac{GMm}{2a} = \frac{1}{2}mv_0^2 - \frac{GMm}{2R}$$

$$= \frac{1}{2}m\cdot\frac{3GM}{5R} - \frac{GMm}{2R}$$



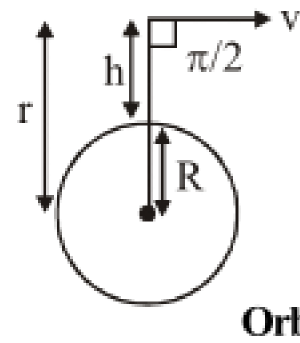
$$-\frac{1}{2a} = \frac{3}{10R} - \frac{1}{2R} = -\frac{2}{10R}$$

$2a = 5R =$ *Major axis*

$= 3R$

Conditions for different trajectroy :

For a body being projected tangentially from above earth's surface, say at a distance r from earth's center, the trajectory would depend on the velocity of projection v.



	Velocity	**Orbit**
1.	velocity, $v < \sqrt{\frac{GM}{r}\left(\frac{2R}{r+R}\right)}$	Body returns to earth following ellipitical Path.
2.	$\sqrt{\frac{GM}{r}} > v > \sqrt{\frac{GM}{r}\left(\frac{2R}{r+R}\right)}$	Body acquires an elliptical orbit with earth as the far-focus w.r.t. the point of projection.
3.	Velocity is equal to the critical velocity of the orbit, $v = \sqrt{\frac{GM_e}{r}}$	Circular orbit with radius r
4.	Velocity is between the critical and escape velocity of the orbit $\sqrt{\frac{2GM_e}{r}} > v > \sqrt{\frac{GM_e}{r}}$	Body acquires an elliptical orbit with earth as the near focus w.r.t. the point of projection.
5.	$v = v_{esc} = \sqrt{\frac{2GM_e}{r}}$	Body just escapes earth's gravity, along a parabolic path.
6.	$v > v_{esc} = \sqrt{\frac{2GM_e}{r}}$	Body escape earth's gravity along a hyperbolic path.

Practice Exercise

Q.1 A planet of mass M moves around the sun along an ellipse so that its minimum distance from the sun is equal to r and the maximum distance to R. Making use of Kepler's law, find the its period of revolution around the sun.

Q.2 What should be the orbit radius of a communication satellite so that it can cover 75% of the surface area of earth during its revolution.

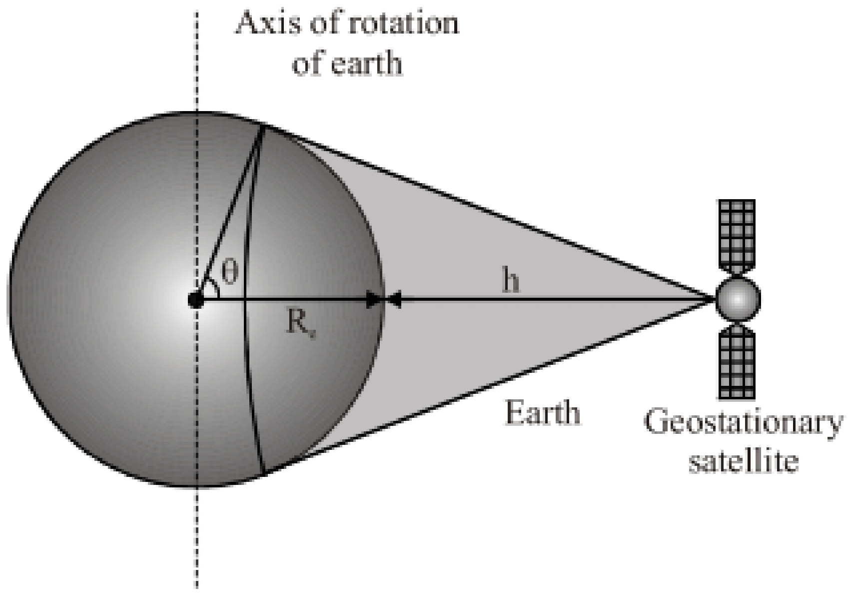
Practice Exercise

Q.1 $\sqrt{\frac{(r+R)^3}{2GM_s}}$ Q.2 $1.15\,R_e$

Broadcasting Region of a Satellite :

Now as we know the height of a geostationary satellite we can easily find the area of earth exposed on the satellite or area of the region in which the communication can be mode using this satelline.
Figure shows earth and its exposed area to a geostationary satellite. Here the angle θ can be given as

$$\theta = \cos^{-1}\left(\frac{R_e}{R_e + h}\right)$$



Now we can find the solid angle Ω which the exposed area subtend on earth's centre as

$$\Omega = 2\pi(1 - \cos\theta)$$

$$= 2\pi\left(1 - \frac{R_e}{R_e + h}\right) = \frac{2\pi h}{R_e + h}$$

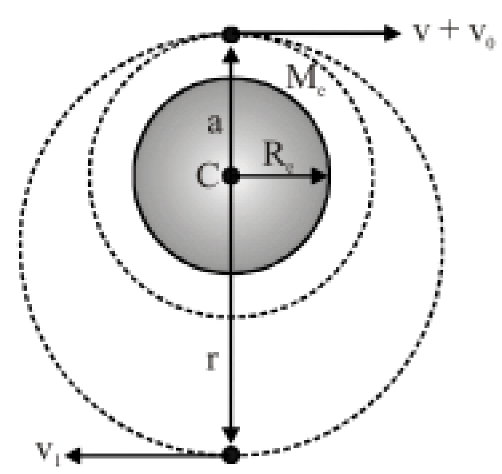
Thus the area of earth's surface to geostationary satellite is

$$S = \Omega R_e^{\,2} = \frac{2\pi h R_e^2}{R_e + h}$$

Solved Example

Q.1 A satellite is revoling round the earth in a circular orbit a rasius a with velocity v_0. A particle is projected from the satellite in forward direction with relative velocity $v = \left(\sqrt{5/4} - 1\right)v_0$. Calculate, during subsequent motion of the particle its minimum and maximum distance from earth's centre.

Sol. The corresponding situation is shown in figure



Initial velocity of statellite $v_0 = \sqrt{\left(\frac{GM}{a}\right)}$

When particle is thrown with the velocity v relative to satellite, the resultant velocity of particle will become

$$v_R = v_0 + v$$

$$= \sqrt{\left(\frac{5}{4}\right)}\, v_0 = \sqrt{\left(\frac{5}{4}\frac{GM}{a}\right)}$$

As the particle velocity is greater than the velocity requried for ciruclar orbit, hence the particle path deviates from circular path to elliptical path. At positions of minimum and maximum distance velocity vector are perpendicular to instantaneous radius vector. In this elliptical path the minimum distance of particle from earth's centre is a and maximum speed in the path is v_R and let the maximum distance and minimum speed in the path is r and v_1 respectively.

Now as angular momentum and total energy remain conserved. Applying the law of conservation of angular momentum, we have

$$mv_1 r = m(v_0 + v)a \qquad [m = \text{mass of particle}]$$

or

$$v_1 = \frac{(v_0 + v)a}{r}$$

$$= \frac{a}{r}\left[\sqrt{\left(\frac{5}{4}\frac{GM}{a}\right)}\right]$$

$$= \frac{1}{r}\left[\sqrt{\left(\frac{5}{4} \times GMa\right)}\right]$$

Applying the law of conservation of energy

$$\frac{1}{2}mv_1^2 - \frac{GMm}{r} = \frac{1}{2}m(v_0 + v)^2 - \frac{GMm}{a}$$

or $$\frac{1}{2}m\left(\frac{5}{4}\frac{GMa}{r^2}\right)-\frac{GMm}{r}=\frac{1}{2}m\left(\frac{5}{4}\frac{GM}{a}\right)-\frac{GMm}{a}$$

$$\frac{5}{8}\times\frac{a}{r^2}-\frac{1}{r}=\frac{5}{8}\times\frac{1}{a}-\frac{1}{a}=\frac{3}{8a}$$

or $$3r^2-8a\,r+5\,a^2=0$$

or $$r=a \text{ or } \frac{5a}{3}$$

Thus minimum disatnce of the particle = a

And maximum distance of the particle = $\frac{5a}{3}$

Q.2 A satellite is revolving around a planet of mass M in an elliptic orbit of semimajor axis a. Show that the orbital speed the satellite when it is at a distance r from the focus will be given by :

$$v^2=GM\left[\frac{2}{r}-\frac{1}{a}\right]$$

Sol. As in case of elliptic orbit with semi major axis a, of a satellite total mechanical energy remains constant, at any position of satellite in the orbit, given as

$$E=-\frac{GMm}{2a}$$

or $$KE+PE=-\frac{GMm}{2a} \qquad ...(i)$$

Now, if at postion r, v is the orbital speed of satellite, we have

$$KE=\frac{1}{2}mv^2 \text{ and } PE=-\frac{GMm}{r} \qquad ...(ii)$$

so from equation (i) and (ii), we have

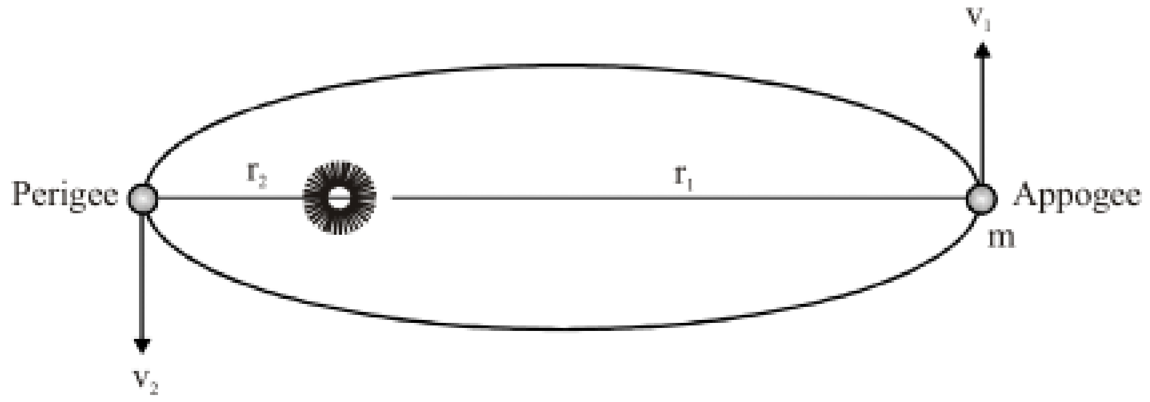
$$\frac{1}{2}mv^2-\frac{GMm}{r}=-\frac{GMm}{2a}, \text{ i.e. } \qquad v^2=GM\left[\frac{2}{r}-\frac{1}{a}\right]$$

Q.3 A planet of mass m moves along an ellipse around the sun so that its maximum and minimum distance from the sun are equal to r_1 and r_2 respectively. Find the angular momentum of this plane relative to the centre of the sun.

Sol. If v_1 and v_2 are the velocity or planet at its apogee and perigee respectively then according to conservation of angular momentum, we have

$$m\,v_1\,r_1=mv_2\,r_2$$

or $$v_1\,r_1=v_2\,r_2$$



As the total energy of the planet is also constant, we have

$$-\frac{GMm}{r_1}+\frac{1}{2}mv_1^2=-\frac{GMm}{r_2}+\frac{1}{2}mv_2^2$$

Where M is the mass of the sun.

or $$GM\left[\frac{1}{r_1}-\frac{1}{r_2}\right]=\frac{v_2^2}{2}-\frac{v_1^2}{2}$$

or $$GM\left(\frac{r_1-r_2}{r_1r_2}\right)=\frac{v_1^2r_1}{2r_2^2}-\frac{v_1^2}{2}$$

or $$GM\left(\frac{r_1-r_2}{r_1r_2}\right)=\frac{v_1^2}{2}\left(\frac{r_1^2}{r_2^2}-1\right)$$

$$=\frac{v_1^2}{2}\left(\frac{v_1^2-v_2^2}{v_2^2}\right)$$

or $$=v_1^2=\frac{2GM(r_1-r_2)r_2^2}{r_1r_2(r_1^2-r_2^2)}=\frac{2Gr_2}{r_1(r_1+r_2)}$$

or $$v_1=\sqrt{\left[\frac{2GMr_2}{r_1(r_1+r_2)}\right]}$$

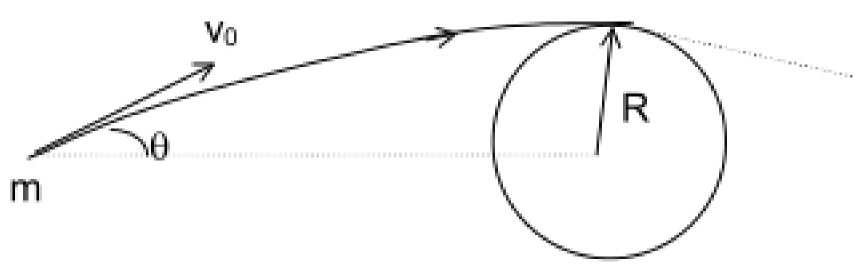
Now Angular momentum of planet is given as

$$L=mv_1r_1$$

$$=m\sqrt{\left[\frac{2GMr_1r_2}{(r_1+r_2)}\right]}$$

Q.4 A spaceship is sent to investigate a planet of mass M and radius R. While hanging motionless in space at a distance 5R from the center of the planet, the spaceship fires an instrument package with speed v_0 as shown in the figure. The packages has mass m, which is much smaller than the mass of the spaceship. For what angle θ will the package just graze the surface of the planet?

Sol. Let the speed of the instrument package is v when it grazes the surface of the planet.



Conserving angular momentum of the package about the centre of the planet $mv_0\times 5R\sin(\pi-\theta)=mvR\sin 90^\circ$

Conserving mechanical energy

$$-\frac{GMm}{5R}+\frac{1}{2}mv_0^2=-\frac{GMm}{R}+\frac{1}{2}mv^2$$

$$\Rightarrow \frac{1}{2}m(v^2-v_0^2)=\frac{4GMm}{5R}$$

$$v^2-v_0^2=\frac{8GM}{5R}$$

substituting the value of v from equation (I) in equation (ii)

$$25\ v_0^2\sin^2\theta-v_0^2=\frac{8GM}{5R}\Rightarrow \sin\theta\ \frac{1}{5}\sqrt{1+\frac{8GM}{5v_0^2R}}$$

$$\text{or } \theta=\sin^{-1}\left[\frac{1}{5}\sqrt{\left(1+\frac{8GM}{5v_0^2R}\right)}\right]$$

Q.5 A missile is launched at an angle of 60° to the vertical with a velocity $\sqrt{0.75gR}$ from the surface of the earth (R is the radius of the earth). Find its maximum height from the surface of earth. (Neglect air resistance and rotation of earth.)

From conservation of mechanical energy

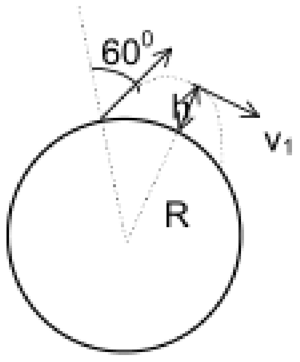
$$\frac{1}{2}mv_0^2-\frac{GMm}{R}=\frac{1}{2}mv_1^2-\frac{GMm}{R+h} \quad ...(i)$$

Also from conservation of angular momentum

$$mv_0R\sin 60^\circ=mv_1(R+h) \quad ...(ii)$$

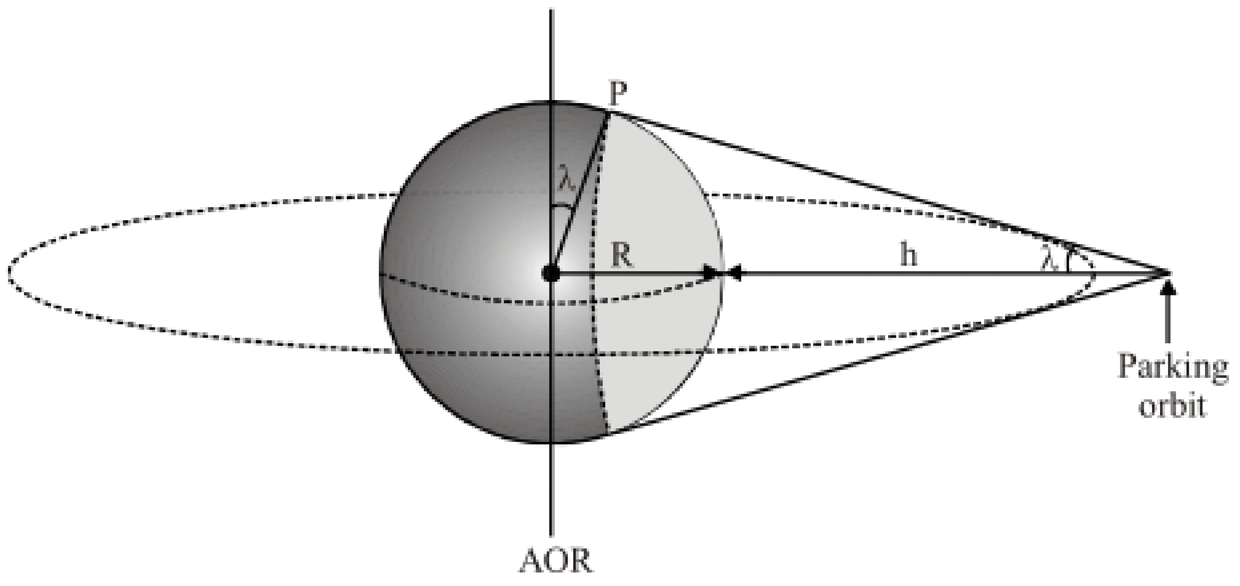
Solving (i) and (ii) and putting $v_0=\sqrt{\frac{3GM}{4R}}$, we get

$h\approx 0.25$ R.



Q.6 Find the minimum colatitude which can directly receive a signed from a geostationary satellite.

Sol. The farthest point on earth, which can receive signals from the parking orbit is the point where a length is drawn on earth surface from satellite as shown in figure



The colatitude λ of point P can be obtained from figure as

$$\sin\lambda = \frac{R_e}{R_e + h} \simeq \frac{1}{7}$$

We know for a partking orbit $h \simeq 6\,R_e$

Thus we have $$l = \sin^{-1}\left(\frac{1}{7}\right)$$

Q.7 A satellite of mass m is orbiting the earth in a circular orbit of radius r. It starts losing energy slowly at a constant rate C due to friction. If M_e and R_e denote the mass and radius of the earth respectively, show that the satellite falls on the earth in a limit time t given by

$$t = \frac{G\,m\,M_e}{2C}\left(\frac{1}{R_e} - \frac{1}{r}\right)$$

Sol. Let velocity of satellite in its orbit of radius r be v then we have

$$v = \sqrt{\frac{G\,M_e}{r}}$$

When satellite approaches earth's surface, if its velocity becomes v' then it is given as

$$v' = \sqrt{\frac{G\,M_e}{R_e}}$$

The total initial energy of satellite at a distance r is

$$E_{T_i} = K_i + U_i$$

$$= \frac{1}{2}mv^2 - \frac{GM_e m}{r}$$

$$= -\frac{1}{2}\frac{GM_e m}{r}$$

The total final energy of satellite at a distance R_e is

$$EE_{T_f} = K_f + U_f$$

$$= \frac{1}{2}mv'^2 - \frac{GM_e m}{R_e}$$

$$= \frac{1}{2}\frac{GM_e m}{R_e}$$

As satellite is loosing energy at a rate C, if it take a time t in reaching earth, we have

$$Ct = E_{T_f} - E_{T_f}$$

$$= \frac{1}{2} GM_e m \left[\frac{1}{R_e} - \frac{1}{r}\right]$$

or
$$t = \frac{GM_e m}{2C}\left[\frac{1}{R_e} - \frac{1}{r}\right]$$

Q.8 Two Earth's satellites move in a common plane along circular orbits. The orbital radius of one satellite r = 7000 km while that of the other satellite is $\Delta r = 70$ km less. What time interval separates the periodic approaches of the satellites to each other over the minimum distance ?

Sol. Now for first satellite which is revolving about the earth (mass M and radius r) the orbital speed is

$$v = \sqrt{\frac{GM}{r}}$$

Let T_1 and T_2 be the time period for first and second satellites respectively. Then we know that

$$T_1 = \frac{2\pi r}{v} = \frac{2\pi r}{\sqrt{GM}} r^{3/2}$$

and
$$T_2 = \frac{2\pi}{\sqrt{GM}} (r - \Delta r)^{3/2}$$

As second satellite is revolving in a radius $(r - \Delta r)$. Know the period interval $(T_1 - T_2)$ is given by

$$T_1 - T_2 = \frac{2\pi}{\sqrt{GM}}\left[r^{3/2} - r^{3/2}\left(1 - \frac{\Delta r}{r}\right)^{3/2}\right]$$

$$= \frac{2\pi}{\sqrt{GM}} r^{3/2}\left[1 - \left(1 - \frac{3}{2}\frac{\Delta r}{r}\right)\right]$$

$$= \frac{2\pi}{\sqrt{GM}} r^{3/2}\left(\frac{3}{2}\frac{\Delta r}{r}\right).$$