UNITS AND DIMENSIONS

PHYSICAL QUANTITIES

All quantities that can be measured are called physical quantities. eg. time, length, mass, force, work done, etc. In physics we study about physical quantities and their inter relationships.

MEASUREMENT

Measurement is the comparison of a quantity with a standard of the same physical quantity.

UNITS

All physical quantities are measured w.r.t. standard magnitude of the same physical quantity and these standards are called UNITS. eg. second, meter, kilogram, etc.

So the four basic properties of units are:—

1. They must be well defined.
2. They should be easily available and reproducible.
3. They should be invariable e.g. step as a unit of length is not invariable.
4. They should be accepted to all.

SET OF FUNDAMENTAL QUANTITIES

A set of physical quanties which are completely independent of each other and all other physical quantities can be expressed in terms of these physical quantities is called Set of Fundamental Quantities.

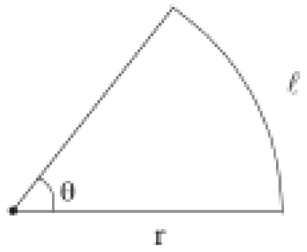
Physical Quantity	Units(SI)	Units(CGS)	Notations
Mass	kg (kilogram)	g	M
Length	m (meter)	cm	L
Time	s (second)	s	T
Temperature	K (kelvin)	°C	θ
Current	A (ampere)	A	I or A
Luminous intensity	cd (candela)	—	cd
Amount of substance	mol	—	mol

Physical Quantity (SI Unit)	Definition
Length (m)	The distance travelled by light in vacuum in $\frac{1}{299,792,458}$ second is called 1 metre.
Mass (kg)	The mass of a cylinder made of platinum-iridium alloy kept at International Bureau of Weights and Measures is defined as 1 kilogram.
Time (s)	The second is the duration of 9,192,631,770 periods of

	the radiation corresponding to the transition between the two hyperfine levels of the ground state of the cesium-133 atom.
Electric Current (A)	If equal currents are maintained in the two parallel infinitely long wires of negligible cross-section, so that the force between them is 2×10^{-7} newton per metre of the wires, the current in any of the wires is called 1 Ampere.
Thermodynamic Temperature (K)	The fraction $\frac{1}{273.16}$ of the thermodynamic temperature of triple point of water is called 1 Kelvin
Luminous Intensity (cd)	1 candela is the luminous intensity of a blackbody of surface area $\frac{1}{600,000}\,m^2$ placed at the temperature of freezing platinum and at a pressure of 101,325 N/m^2, in the direction perpendicular to its surface.
Amount of substance (mole)	The mole is the amount of a substance that contains as many elementary entities as there are number of atoms in 0.012 kg of carbon-12.

There are two supplementary units too:

1. Plane angle (radian) — angle = arc / radius
   $\theta = \ell / r$



2. Solid Angle (steradian)

DERIVED PHYSICAL QUANTITIES

The physical quantities those can be expressed in terms of fundamental physical quantities are called derived physical quantities.eg. speed = distance/time.

DIMENSIONS AND DIMENSIONAL FORMULA

All the physical quantities of interest can be derived from the base quantities.

DIMENSION

The power (exponent) of base quantity that enters into the expression of a physical quantity, is called the dimension of the quantity in that base.

To make it clear, consider the physical quantity "force".

Force = mass × acceleration

$$= \text{mass} \times \frac{\text{length / time}}{\text{time}}$$

$$= \text{mass} \times \text{length} \times (\text{time})^{-2}$$

So the dimensions of force are 1 in mass, 1 in length and –2 in time. Thus

$$[\text{Force}] = MLT^{-2}$$

Similarly energy has dimensional formula given by

$$[\text{Energy}] = ML^2T^{-2}$$

i.e. energy has dimensions, 1 in mass, 2 in length and -2 in time.

Such an expression for a physical quantity in terms of base quantities is called dimensional formula.

DIMENSIONAL EQUATION

Whenever the dimension of a physical quantity is equated with its dimensional formula, we get a dimensional equation.

PRINCIPLE OF HOMOGENEITY

According to this principle, we can multiply physical quantities with same or different dimensional formulae at our convenience, however no such rule applies to addition and substraction, where only like physical quantites can only be added or substracted. e.g. If P + Q $\Rightarrow$ P & Q both represent same physical quantity.

Illustration :

Calculate the dimensional formula of energy from the equation $E = \frac{1}{2}mv^2$.

Sol. *Dimensionally,* $E = \text{mass} \times (\text{velocity})^2$.

Since $\frac{1}{2}$ *is a number and has no dimension.*

or, $[E] = M \times \left(\frac{L}{T}\right)^2 = ML^2T^{-2}$.

Illustration :

Kinetic energy of a particle moving along elliptical trajectory is given by $K = \alpha s^2$ *where s is the distance travelled by the particle. Determine dimensions of* α.

Sol. $K = \alpha s^2$

$$[\alpha] = \frac{(ML^2T^{-2})}{(L^2)}$$

$$[\alpha] = M^1 L^0 T^{-2}$$

$$[\alpha] = (M T^{-2})$$

Illustration :

The position of a particle at time t, is given by the equation, $x(t) = \frac{v_0}{\alpha}(1 - e^{-\alpha t})$*, where* v_0 *is a constant and* $\alpha > 0$*. The dimensions of* v_0 *&* α *are respectively.*

(A) $M^0 L^1 T^0$ & T^{-1}
(B) $M^0 L^1 T^{-1}$ & T
(C*) $M^0 L^1 T^{-1}$ & T^{-1}
(D) $M^1 L^1 T^{-1}$ & LT^{-2}

Sol. $[V_0] = [x]\ [\alpha] = M^0L^1T^{-1}$ & $[\alpha]\ [t] = M^0L^0T^0$, $[\alpha] = M^0L^0T^{-1}$

Illustration :

The distance covered by a particle in time t is going by $x = a + bt + ct^2 + dt^3$ *; find the dimensions of a, b, c and d.*

Sol. *The equation contains five terms. All of them should have the same dimensions. Since [x] = length, each of the remaining four must have the dimension of length.*

Thus, $[a] = \text{length} = L$

$[bt] = L,$ *or* $[b] = LT^{-1}$

$[ct^2] = L,$ *or* $[c] = LT^{-2}$

and $[dt^3] = L$ *or* $[d] = LT^{-3}$

USES OF DIMENSIONAL ANALYSIS

(I) TO CONVERT UNITS OF A PHYSICAL QUANTITY FROM ONE SYSTEM OF UNITS TO ANOTHER :

It is based on the fact that,

Numerical value × unit = constant

So on changing unit, numerical value will also gets changed. If n_1 and n_2 are the numerical values of a given physical quantity and u_1 and u_2 be the units respectively in two different systems of units, then

$$n_1u_1 = n_2u_2$$

$$n_2 = n_1\left[\frac{M_1}{M_2}\right]^a\left[\frac{L_1}{L_2}\right]^b\left[\frac{T_1}{T_2}\right]^c$$

Illustration

Young's modulus of steel is 19×10^{10} N/m². Express it in dyne/cm². Here dyne isthe CGS unit of force.

Sol. *The unit of Young's modulus is N/m².*

This suggest that it has dimensions of $\frac{\text{Force}}{(\text{dis tan ce})^2}$.

Thus, $[Y] = \frac{[F]}{L^2} = \frac{MLT^{-2}}{L^2} = ML^{-1}T^{-2}.$

N/m² is in SI units,

So, $1\ N/m^2 = (1\ kg)(1m)^{-1}\ (1s)^{-2}$

and $1\ dyne/cm^2 = (1g)(1cm)^{-1}\ (1s)^{-2}$

so, $$\frac{1N/m^2}{1dyne/cm^2} = \left(\frac{1kg}{1g}\right)\left(\frac{1m}{1cm}\right)^{-1}\left(\frac{1s}{1s}\right)^{-2} = 1000 \times \frac{1}{100} \times 1 = 10$$

or, $1\ N/m^2 = 10\ dyne/cm^2$

or, $19 \times 10^{10}\ N/m^2 = 19 \times 10^{11}\ dyne/m^2.$

Illustration :

The dimensional formula for viscosity of fluids is,

$\eta = M^1L^{-1}T^{-1}$

Find how many poise (CGS unit of viscosity) is equal to 1 poiseuille (SI unit of viscosity).

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Sol. $\eta = M^1 L^{-1} T^{-1}$

$1\ CGS\ units = g\ cm^{-1} s^{-1}$

$1\ SI\ units = kg\ m^{-1} s^{-1}$

$= 1000\ g\ (100\ cm)^{-1} s^{-1}$

$= 10\ g\ cm^{-1} s^{-1}$

Thus, 1 Poiseuilli = 10 poise

Illustration :

A calorie is a unit of heat or energy and it equals about 4.2 J, where 1 J = 1 kg m²/s². Suppose we employ a system of units in which the unit of mass equals α kg, the unit of length equals β metre, the unit of time is γ second. Show that a calorie has a magnitude $4.2\ \alpha^{-1}\beta^{-2}\gamma^{2}$ *in terms of the new units.*

Sol. $1\ cal = 4.2\ kg\ m^2 s^{-2}$

SI	New system
$n_1 = 4.2$	$n_2 = ?$
$M_1 = 1\ kg$	$M_2 = \alpha\ kg$
$L_1 = 1\ m$	$L_2 = \beta$ metre
$T_1 = 1\ s$	$T_2 = \gamma$ second

Dimensional formula of energy is $[ML^2T^{-2}]$

Comparing with $[M^aL^bT^c]$, *we find that* $a = 1,\ b = 2,\ c = -2$

Now, $$n_2 = n_1\left[\frac{M_1}{M_2}\right]^a\left[\frac{L_1}{L_2}\right]^b\left[\frac{T_1}{T_2}\right]^c$$

$$= 4.2\left[\frac{1\ kg}{\alpha\ kg}\right]^1\left[\frac{1\ m}{\beta\ m}\right]^2\left[\frac{1\ s}{\gamma\ s}\right]^{-2} = 4.2\ \alpha^{-1}\beta^{-2}\gamma^{2}$$

(II) TO CHECK THE DIMENSIONAL CORRECTNESS OF A GIVEN PHYSICAL RELATION:

It is based on principle of homogeneity, which states that a given physical relation is dimensionally correct if the dimensions of the various terms on either side of the relation are the same.

(i) Powers are dimensionless

(ii) $\sin\theta$, e^{θ},, $\cos\theta$, $\log\theta$ gives dimensionless value and in above expression θ is dimensionless

(iii) We can add or subtract quantity having same dimensions.

Illustration :

Let us check the dimensional correctness of the relation $v = u + at$.

Here 'u' represents the initial velocity, 'v' represents the final velocity, 'a' the uniform acceleration and 't' the time.

Dimensional formula of 'u' is $[M^0LT^{-1}]$

Dimensional formula of 'v' is $[M^0LT^{-1}]$

Dimensional formula of 'at' is $[M^0LT^{-2}][T] = [M^0LT^{-1}]$

Here dimensions of every term in the given physical relation are the same, hence the given physical relation is dimensionally correct.

Illustration :

Let us check the dimensional correctness of the relation

$$x = ut + \frac{1}{2}at^2$$

Here 'u' represents the initial velocity, 'a' the uniform acceleration, 'x' the displacement and 't' the time.

Sol. $[x] = L$

$$[ut] = velocity \times time = \frac{length}{time} \times time = L$$

$$\left[\frac{1}{2}at^2\right] = [at^2] = accelecration \times (time)^2$$

$(\because \frac{1}{2}$ *is a number hence dimentionless)*

$$= \frac{velocity}{time} \times (time)^2 = \frac{length/tim\ e}{time} \times (time)^2 = L$$

Thus, the equation is correct as far as the dimensions are concerned.

(III) TO ESTABLISH A RELATION BETWEEN DIFFERENT PHYSICAL QUANTITIES :

If we know the various factors on which a physical quantity depends, then we can find a relation among different factors by using principle of homogeneity.

Illustration :

Let us find an expression for the time period t of a simple pendulum. The time period t may depend upon (i) mass m of the bob of the pendulum, (ii) length ℓ of pendulum, (iii) acceleration due to gravity g at the place where the pendulum is suspended.

Sol. *Let (i)* $t \propto m^a$ *(ii)* $t \propto \ell^b$ *(iii)* $t \propto g^c$

Combining all the three factors, we get

$$t \propto m^a \ell^b g^c \qquad or \qquad t = Km^a \ell^b g^c$$

where K is a dimensionless constant of proportionality.

Writing down the dimensions on either side of equation (i), we get

$$[T] = [M^a][L^b][LT^{-2}]^c = [M^a L^{b+c} T^{-2c}]$$

Comparing dimensions, $a = 0,\ b + c = 0,\ -2c = 1$

$\therefore \quad a = 0,\ c = -1/2,\ b = 1/2$

From equation (i) $t = Km^0 \ell^{1/2} g^{-1/2}$ *or* $t = K\left(\frac{\ell}{g}\right)^{1/2} = K\sqrt{\frac{\ell}{g}}$

Illustration :

When a solid sphere moves through a liquid, the liquid opposes the motion with a force F. The magnitude of F depends on the coefficient of viscosity η of the liquid, the radius r of the sphere and the speed v of the sphere. Assuming that F is proportional to different powers of these quantities, guess a formula for F using the method of dimensions.

Sol. *Suppose the formula is* $F = k\,\eta^a r^b v^c$

Then, $MLT^{-2} = [ML^{-1}T^{-1}]^a L^b \left(\frac{L}{T}\right)^c$

$= M^a L^{-a+b+c} T^{-a-c}$

Equating the exponents of M, L and T from both sides,

$a = 1$

$-a + b + c = 1$

$-a - c = -2$

Solving these, $a = 1$, $b = 1$ *and* $c = 1$

Thus, the formula for F is $F = k\eta rv$.

Illustration :

If P is the pressure of a gas and ρ *is its density, then find the dimension of velocity in terms of P and* ρ.

(A) $P^{1/2}\rho^{-1/2}$ (B) $P^{1/2}\rho^{1/2}$ (C) $P^{-1/2}\rho^{1/2}$ (D) $P^{-1/2}\rho^{-1/2}$

[Sol. $v \propto P^a \rho^b$

$v = kP^a \rho^b$

$[LT^{-1}] = [ML^{-1}T^{-2}]^a\,[ML^{-3}]^b$ *(Comparing dimensions)*

$a = \frac{1}{2},\quad b = -\frac{1}{2} \qquad \Rightarrow \qquad [V] = [P^{1/2}\rho^{-1/2}]$

UNITS AND DIMENSIONS OF SOME PHYSICAL QUANTITIES

Quantity	SI Unit	Dimensional Formula
Density	kg/m^3	M/L^3
Force	Newton (N)	ML/T^2
Work	Joule (J)(=N-m)	ML^2/T^2
Energy	Joule(J)	ML^2/T^2
Power	Watt (W) (=J/s)	ML^2/T^3
Momentum	kg-m/s	ML/T
Gravitational constant	$N\text{-}m^2/kg^2$	L^3/MT^2
Angular velocity	radian/s	T^{-1}
Angular acceleration	$radian/s^2$	T^{-2}
Angular momentum	$kg\text{-}m^2/s$	ML^2/T
Moment of inertia	$kg\text{-}m^2$	ML^2
Torque	N-m	ML^2/T^2
Angular frequency	radian/s	T^{-1}
Frequency	Hertz (Hz)	T^{-1}
Period	s	T
Surface Tension	N/m	M/T^2
Coefficient of viscosity	$N\text{-}s/m^2$	M/LT
Wavelength	m	L
Intensity of wave	W/m^2	M/T^3

Temperature	kelvin (K)	K
Specific heat capacity	J/(kg-K)	L^2/T^2K
Stefan's constant	$W/(m^2–K^4)$	M/T^3K^4
Heat	J	ML^2/T^2
Thermal conductivity	W/(m-K)	ML/T^3K
Current density	A/m^2	I/L^2
Electrical conductivity	1/Ω-m(=mho/m)	I^2T^3/ML^3
Electric dipole moment	C-m	LIT
Electric field	V/m (=N/C)	ML/IT^3
Potential (voltage)	volt (V) (=J/C)	ML^2/IT^3
Electric flux	V-m	ML^3/IT^3
Capacitance	farad (F)	I^2T^4/ML^2
Electromotive force	volt (V)	ML^2/IT^3
Resistance	ohm (Ω)	ML^2/I^2T^3
Permittivity of space	$C^2/N\text{-}m^2$ (=F/m)	I^2T^4/ML^3
Permeability of space	N/A^2	ML/I^2T^2
Magnetic field	Tesla (T) (= Wb/m^2)	M/IT^2
Magnetic flux	Weber (Wb)	ML^2/IT^2
Magnetic dipole moment	N-m/T	IL^2
Inductance	Henry (H)	ML^2/I^2T^2

LIMITATIONS OF DIMENSIONAL ANALYSIS

(i) Dimension does not depend on the magnitude. Due to this reason the equation $x = ut + at^2$ is also dimensionally correct. Thus, a dimensionally correct equation need not be actually correct.
(ii) The numerical constants having no dimensions connot be deduced by the method of dimensions.
(iii) This method is applicable only if relation is of product type. It fails in the case of exponential and trigonometric relations.

SI Prefixes : The magnitudes of physical quantites vary over a wide range. The mass of an electron is 9.1×10^{-31} kg and that of our earth is about 6×10^{24} kg. Standard prefixes for certain power of 10. Table shows these prefixes :

Power of 10	Prefix	Symbol
12	tera	T
9	giga	G
6	mega	M
3	kilo	k
2	hecto	h
1	deka	da
–1	deci	d

–2	centi	c
–3	milli	m
–6	micro	μ
–9	nano	n
–12	pico	p
–15	femto	f

ORDER-OF MAGNITUDE CALCULATIONS

If value of phycal quantity P satisfy

$0.5\times10^{x} < P \le 5 \times 10^{x}$

x is an integer

x is called order of magnitude

Illustration :

The diameter of the sun is expressed as 13.9×10^{9} m. Find the order of magnitude of the diameter ?

Sol. *Diameter = 13.9×10^{9} m*

Diameter = 1.39×10^{10} m

order of magnitude is 10.

SYMBOLS AND THERE USUAL MEANINGS

The scientific group in Greece used following symbols.

θ	Theta
α	Alpha
β	Beta
γ	Gamma
δ	Delta
Δ	Delta
μ	Mu
λ	Lambda
ω, Ω	Omega
π	Pi
ϕ, φ	Phi
ε	epsilon

ψ	Psi
ρ	Roh
ν	Nu
η	Eta
σ	Sigma
τ	Tau
κ	Kappa
χ	chi
≅	Approximately equal to

Solved Examples

Q.1 Find the dimensional formulae of follwoing quantities :

(a) The surface tension S,

(b) The thermal conductivity k and

(c) The coefficient of vescosity η.

Some equation involving these quntities are

$$S = \frac{\rho g r h}{2} \qquad Q = k\frac{A(\theta_2 - \theta_1)t}{d} \qquad \text{and} \qquad F = -\eta A \frac{v_2 - v_1}{x_2 - x_1};$$

where the symbols have their usual meanings. (ρ - density, g - acceleration due to gravity, r - radius, h - height, A - area, θ_1 & θ_2 - temperatures, t - time, d - thickness, v_1 & v_2 - velocities, x_1 & x_2 - positions.)

Sol. (a) $S = \frac{\rho g r h}{2}$

or $[S] = [\rho]\,[g]L^2 = \frac{M}{L^3}\cdot\frac{L}{T^2}\cdot L^2 = MT^{-2}$.

(b) $Q = k\,\frac{A(\theta_2 - \theta_1)t}{d}$

or $k = \frac{Qd}{A(\theta_2 - \theta_1)t}$.

Here, Q is the heat energy having dimension ML^2T^{-2}, $\theta_2 - \theta_1$ is temperature, A is area, d is thickness and t is time. Thus,

$$[k] = \frac{ML^2T^{-2}}{L^2KT} = MLT^{-3}K^{-1}.$$

(d) $F = -hA\frac{v_2 - v_1}{x_2 - x_1}$

or $MLT^{-2} = [\eta]L^2\,\frac{L/T}{L} = [\eta]\,\frac{L^2}{T}$

or, $[\eta] = ML^{-1}T^{-1}$.

Q.2 Suppose $A = B^nC^m$, where A has dimensions LT, B has dimensions L^2T^{-1}, and C has dimensions LT^2. Then the exponents n and m have the values:

(A) 2/3; 1/3 (B) 2; 3 (C) 4/5; -1/5 (D*) 1/5; 3/5

(E) 1/2; 1/2

Sol. $LT = [L^2T^{-1}]^n\,[LT^2]^m$

$LT = L^{2n+m}T^{2m-n}$

$2n + m = 1$ ....(i)

$-n + 2m = 1$ ....(ii)

On sovling $n = \frac{1}{5}, m = \frac{3}{5}$

Q.3 If energy (E), velocity (V) and time (T) are chosen as the fundamental quantities, then the dimensions of surface tension will be. (Surface tension = force / length)

(A) $E V^{-2}T^{-1}$ (B) $E V^{-1}T^{-2}$ (C) $E^{-2} V^{-1}T^{-3}$ (D*) $E V^{-2}T^{-2}$

Sol. [surface tension] = [force/length] = $M^1L^0T^{-2}$

suppose [surface tension] = $E^a V^b T^c$

$\therefore M^1L^0T^{-2} = [M^1L^2T^{-2}]^a [L^1T^{-1}]^b [T]^c$

Matching dimensions of M $\Rightarrow a = 1$

Matching dimensions of L $\Rightarrow 2a + b = 0 \Rightarrow b = -2$

Matching dimensions of T $\Rightarrow -2a - b + c = -2 \Rightarrow c = -2$

$\therefore$ [surface tension] = $EV^{-2} T^{-2}$

Q.4 Given that ln $(\alpha/p\beta) = \alpha z/K_B\theta$ where p is pressure, z is distance, K_B is Boltzmann constant and θ is temperature, the dimensions of β are (useful formula Energy = $K_B \times$ temperature)

(A) $L^0M^0 T^0$ (B) $L^1 M^{-1} T^2$ (C*) $L^2 M^0 T^0$ (D) $L^{-1} M^1 T^{-2}$

Sol. $\ell n\left(\frac{\alpha}{p\beta}\right) = \frac{\alpha z}{k_B\theta}$

$[\alpha z] = [k_\beta\theta]$ Also $[\alpha] = [p\beta]$

$[p\, \beta\, z] = [k_\beta\theta]$

$$[\beta] = \frac{(k_\beta\theta)}{(p\ z)} = \frac{ML^2T^{-2}K^{-1}K}{ML^{-1}T^{-2}L} = L^2$$

Q.5 The SI and CGS units of energy are joule and erg respectively. How many ergs are equal to one joule ?

Sol. Dimensionally, Energy = mass $\times$ (velocity)^2

$$= \text{mass} \times \left(\frac{length}{time}\right)^2 = ML^2T^{-2}$$

Thus, 1 joule = (1 kg) (1 m)^2 (1 s)$^{-2}$

and 1 erg = (1 g) (1 cm)^2 (1 s)$^{-2}$

$$\frac{1\,\text{joule}}{1\,\text{erg}} = \left(\frac{1\,kg}{1\,g}\right)\left(\frac{1\,m}{1\,cm}\right)^2\left(\frac{1\,s}{1\,s}\right)^{-2}$$

$$= \left(\frac{1000g}{1g}\right)\left(\frac{1000\,cm}{1\,cm}\right)^2 = 1000 \times 10000 = 10^7.$$

So 1 joule = 10^7 erg.

Q.6 Young's modulus of steel is 19×10^{10} N/m^2. Express it in dyne/cm^2. Here dyne is the CGS unit of force.

Sol. The unit of Young's modulus is N/m^2.

This suggests that it has dimensions of $\frac{Force}{(area)}$.

Thus, $[Y] = \left[\frac{F}{L^2}\right] = \frac{MLT^{-2}}{L^2} = ML^{-1}T^{-2}$.

N/s^2 is in SI units.

So, 1 N/m^2 = (1kg) (1m)$^{-1}$ (1s)$^{-2}$

and 1 dyne/cm^2 = (1g) (1cm)$^{-1}$ (1s)$^{-2}$

$$\text{So, } \frac{1N/m^2}{1dyne/cm^2} = \left(\frac{1\,kg}{1\,g}\right) = \left(\frac{1\,m}{1\,cm}\right)^{-1}\left(\frac{1s}{1s}\right)^{-2}$$

$$= 1000 \times \frac{1}{100} \times 1 = 10$$

or, 1 N/s^2 = 10 dyne/cm^2

or, 19×10^{10} N/m^2 = 19×10^{11} dyne/cm^2

Q.7 If velcoity, time and force were chosen as basic quantities, find the dimensions of mass.

Sol. Dimensionally, Force = mass × acceleration

$$\text{Force} = \text{mass} \times \frac{velocity}{time}$$

or, $$\text{mass} = \frac{Force \times time}{velocity}$$

or, [mass] = FTV^{-1}

Q.8 The dimension of $\frac{a}{b}$ in the equation $P = \frac{a-t^2}{bx}$ where P is pressure, x is distance and t is time are ________?

Sol. $P = \frac{a-t^2}{bx}$

$\Rightarrow$ $Pbx = a - t^2$

$\Rightarrow$ $[Pbx] = [a] = [T^2]$

or $$[b] = \frac{[T^2]}{[P]\,[x]} = \frac{[T^2]}{[ML^{-1}T^{-2}]\,[L]} = [M^{-1}T^4]$$

$\therefore$ $$\left[\frac{a}{b}\right] = \frac{[T^2]}{[M^{-1}T^4]} = [MT^{-2}]$$

VECTOR

Introduction

History :

There were many scientists involved in developing vectors over several years. Some of them were Caspar Wessel (1745-1818), Jean Robert Argand (1768- 1822), Carl Friedrich Gauss (1777-1855),William Rowan Hamilton. Finally it was Hermann Grassmann (1809-1877) who developed and expanded on the concept of vectors.

Because of this development, physical quantities were divided into two categories

(a) vectors (b) scalars.

Scalars:

Physical quantities are completely specified by a number, which can be positive or negative, and a unit of measure.

An example is temperature, a quantity that is specified completely by a number and a unit (as in 21 °C or 70 F).

Examples of scalar quantities are : mass, distance, average speed, instantaneous speed, energy, pressure, temperature, density, charge etc.

Scalars follow the ordinary arithmetical laws of addition, subtraction, multiplication, and division.

Vectors :

Any physical quantity which in order to be completely specified, requires not only a number and a unit but also a direction in space and also follows laws of vector algebra are known as vectors.

The laws of additioin and substraction of vectors require geometrical techniques.

Examples of vectors are: displacement, velocity, acceleration, force, momentum, impulse, angular momentum, torque, angular impulse, gravitational field, electric field, magnetic field. If we are talking about directions, we should know how to write direction in English. In English sentences, we write directions as 30^o east of north, 60^o south of east, north-east, south-west etc.

Representation of Vector

The representation of vector will be complete if it gives us direction and magnitude.

Symbolic form:

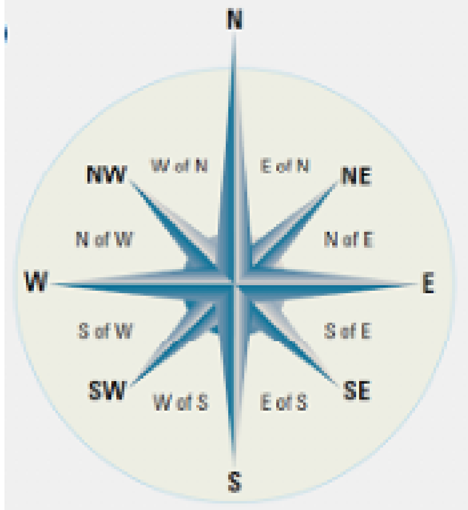
$\vec{v}, \vec{a}, \vec{F}, \vec{s}$ used to separate a vector quantity from scalar quantities (u, i, m). Some books also represent vector by bold letter. Such as **s, v, a**.

Graphical form:

A vector is represented by a directed straight line, having the magnitude and direction of the quantity represented by it.

e.g. if we want to represent a force of 5 N acting 45° N of E

Step:



(i) We choose suitable co-ordinates system.

(ii) We choose a convenient scale like $1\text{cm} \equiv 1\text{ N}$.

(iii) for the direction 45° N of E, the reference is East then turning 45° towards North.

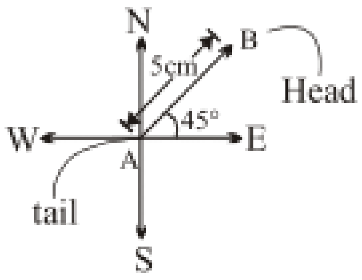
(iv) We draw a line of length equal in magnitude and in the direction of vector to the chosen quantity.

(v) We put arrow in the direction of vector.

$$\overrightarrow{AB}$$

Magnitude of vector:

$$\left|\overrightarrow{AB}\right| = 5\text{N}$$

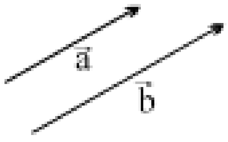


$1\text{ cm} \equiv 1\text{ N}$

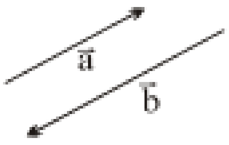
By definition magnitude of a vector quantity is scalar and is always positive.

Terminology of Vectors

Parallel vector: If two vectors have same direction, they are parallel to each other. They may be located anywhere in the space.

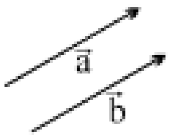


Antiparallel vectors: When two vectors are in opposite direction they are said to be antiparallel vectors.



Equality of vectors: When two vectors have equal magnitude and are in same direction and represent the same quantity, they are equal.

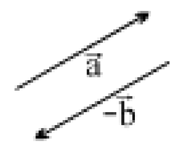
i.e. $\vec{a} = \vec{b}$



Thus when two parallel vectors have same magnitude they are equal. (Their initial point & terminal point may not be same)

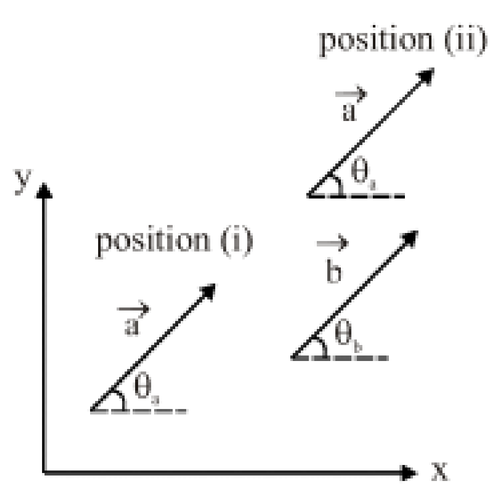
Negative of a vector: When a vector have equal magnitude and is in opposite direction, it is said to be negative vector of the former.

i.e. $\quad \vec{a} = -\vec{b} \qquad$ or $\qquad \vec{b} = -\vec{a}$



Thus when two antiparallel vectors have same magnitude they are negative of each other.
Vector shifting is allowed without change in their direction.

Let two vectors $\vec{a}$ and $\vec{b}$ are represented as –



Vector shifting is allowed without changing their direction as vector $\vec{a}$ shifted in figure, from position (i) to position (ii).

If $|\vec{a}| = |\vec{b}|$ and $\theta_a = \theta_b$ then both vector are equal vectors.

If $\theta_a = \theta_b$ then both vector are parallel vectors.

Illustration :

The vector $-\vec{A}$ is:

(A) greater than $\vec{A}$ in magnitude

(B) less than $\vec{A}$ in magnitude

(C) in the same direction as $\vec{A}$

(D) in the direction opposite to $\vec{A}$*

(E) perpendicular to $\vec{A}$

Sol. *(D)*

ANGLE BETWEEN VECTORS

Angle between two vectors $\vec{a}$ and $\vec{b}$ is defined as the smaller angle between the tail's or the head's of the two vectors.

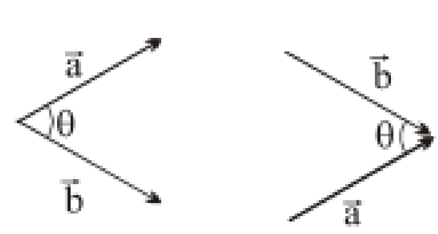
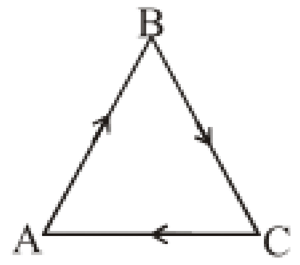


Illustration :

What is the angle between?

(i) $\overline{\text{AB}}$ *and* $\overline{\text{AC}}$ ***Ans.*** *60°*

(ii) $\overline{\text{AB}}$ *and* $\overline{\text{BC}}$ ***Ans.*** *120°*



Equilateral triangle

Laws of addition and subtraction of vectors:

Triangle rule of addition:

Steps for adding two vectors representing same physical quantity by triangle law :

(i) Keep vectors such that tail of one vector coincides with head of other.

(ii) Join tail of first to head of the other by a line with arrow at head of the second.

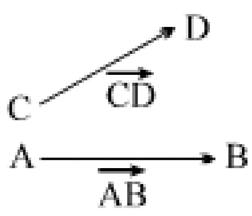
(iii) This new vector is the sum of two vectors. (also called resultant)

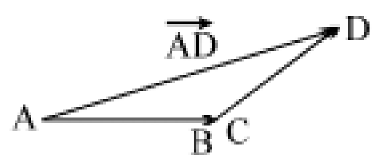
For Example :

To add $\overrightarrow{\text{CD}}$ to $\overrightarrow{\text{AB}}$, place tail of $\overrightarrow{\text{CD}}$ at the head of $\overrightarrow{\text{AB}}$. The sum is the vector from the tail of $\overrightarrow{\text{AB}}$ to head of $\overrightarrow{\text{CD}}$ i.e. $\overrightarrow{\text{AD}}$.

$$\overrightarrow{\text{AB}} + \overrightarrow{\text{CD}} = \overrightarrow{\text{AD}}$$

Graphical Representation :

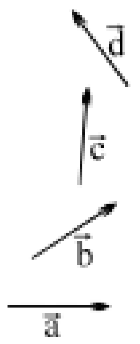
(i) 

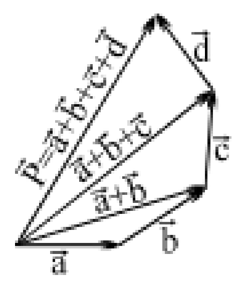
(ii) 

Polygon Law of addition:

This law is used for adding more than two vectors. This is extension of triangle law of addition. We keep on arranging vectors s.t. tail of next vector lies on head of former.

When we connect the tail of first vector to head of last we get resultant of all the vectors.





$\vec{P} = \left(\left((\vec{a}+\vec{b})+\vec{c}\right)+\vec{d}\right) = \left((\vec{c}+\vec{a})+\vec{b}\right)+\vec{d}$ [**Associative Law**]

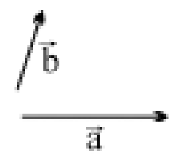
Parallelogram law of addition:

Steps:

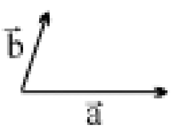
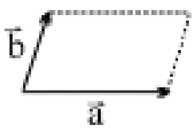
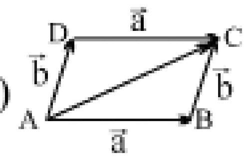
(i) Keep two vectors such that their tails coincide.

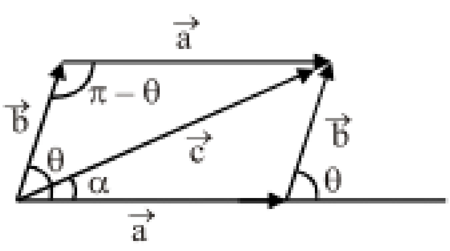
(ii) Draw parallel vectors to both of them considering both of them as sides of a parallelogram.

(iii) Then the diagonal drawn from the point where tails coincide represents the sum of two vectors, with its tail at point of coincidence of the two vectors.

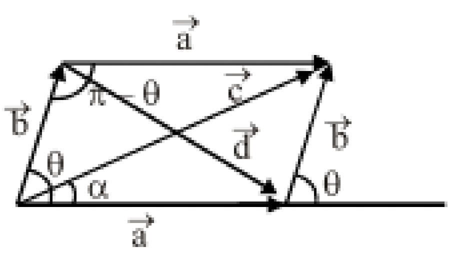


Steps :

(i)  (ii)  (iii)  $\overrightarrow{AC} = \vec{a} + \vec{b}$



$\overrightarrow{AC} = \vec{a} + \vec{b}$ and $\overrightarrow{AC} = \vec{b} + \vec{a}$ thus $\vec{c} = \vec{a} + \vec{b} = \vec{b} + \vec{a}$ [**Commutative Law**]

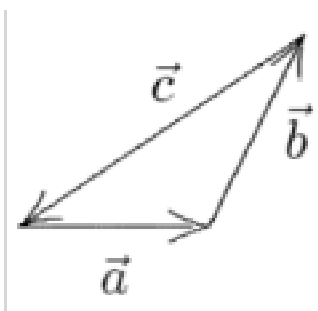
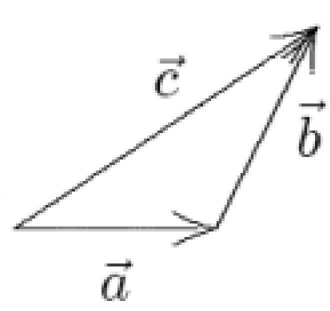


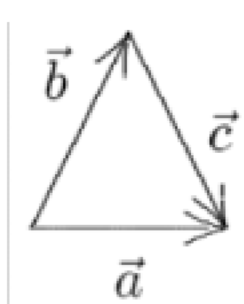
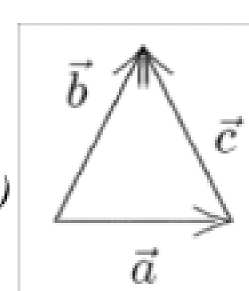
So, the full parallelogram would look like this

$\vec{c} = \vec{a} + \vec{b}$; $\vec{d} = \vec{a} - \vec{b}$

Illustration :

The vectors $\vec{a}, \vec{b}$ *and* $\vec{c}$ *are related by* $\vec{c} = \vec{a} + \vec{b}$. *Which diagram below illustrates this relationship?*

(A)  *(B*)* 

(C)  *(D)*  *(E) None of these*

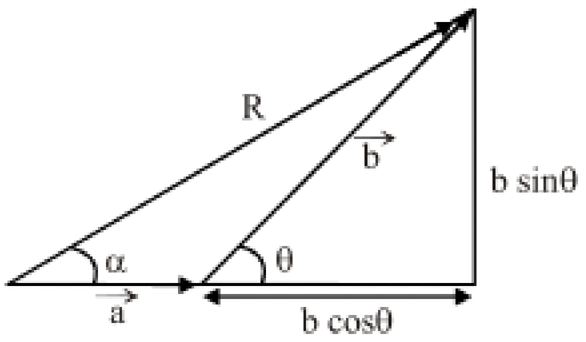
Sol. *(B)* $\vec{a} + \vec{b} = \vec{c}$

Derivation of general formula :

$\vec{R} = \vec{a} + \vec{b}$, $|\vec{R}| = R$, $|\vec{a}| = a$, $|\vec{b}| = b$

$R = \sqrt{(a + b\cos\theta)^2 + (b\sin\theta)^2}$

$= \sqrt{a^2 + b^2 + 2ab\cos\theta}$



as in figure angle between $\vec{a}$ and $\vec{R}$ is α.

$$\tan\alpha = \frac{b\sin\theta}{a + b\cos\theta}$$

Illustration :

Two forces P and Q are in ratio P : Q = 1 : 2. If their resultant is at an angle $\tan^{-1}\left(\frac{\sqrt{3}}{2}\right)$ to vector P, then angle between P and Q is :

(A) $\tan^{-1}\left(\frac{1}{2}\right)$ (B) 45° (C) 30° (D*) 60°

Sol. $\tan\alpha = \frac{Q\sin\theta}{P + Q\cos\theta}$

$$\frac{\sqrt{3}}{2} = \frac{\sin\theta}{\frac{P}{Q} + \cos\theta} \Rightarrow \frac{\sqrt{3}}{2} = \frac{\sin\theta}{(1/2) + \cos\theta} \Rightarrow \frac{3}{4} = \left(\frac{2\sin\theta}{1 + 2\cos\theta}\right)^2$$

$\Rightarrow \quad 3(1 + 2\cos\theta)^2 = 16\sin^2\theta$

$\Rightarrow \quad 3(1 + 4\cos^2\theta + 4\cos\theta) = 16(1 - \cos^2\theta)$

$\Rightarrow \quad 3 + 12\cos^2\theta + 12\cos\theta = 16 - 16\cos^2\theta$

$\Rightarrow \quad 28\cos^2\theta + 12\cos\theta - 13 = 0 \Rightarrow \quad \cos\theta = 1/2, -0.92$

Important Results:

The angle between $\vec{a}$ and $\vec{b}$ is θ and resultant $\vec{R} = \vec{a} + \vec{b}$.

(i) If $\theta = 0°$ $\Rightarrow \vec{a} \parallel \vec{b}$

then, $|\vec{R}| = |\vec{a}| + |\vec{b}|$

& $|\vec{R}|$ is maximum

(ii) If $\theta = \pi$ $\Rightarrow \vec{a}$ anti $\parallel \vec{b}$

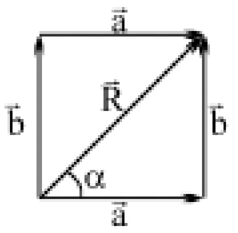
then, $|\vec{R}| = \left||\vec{a}| - |\vec{b}|\right|$

& $|\vec{R}|$ is minimum

(iii) If $\theta = \pi/2 \quad \Rightarrow \vec{a} \perp \vec{b}$

$R = \sqrt{a^2 + b^2}$

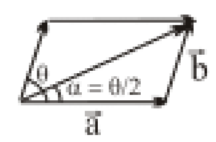
& $\tan\alpha = b/a$ (α is angle with $\vec{a}$)



(iv) If $|\vec{a}| = |\vec{b}| = a$

$|\vec{R}| = 2a\cos\theta/2$

& $\alpha = \theta/2$



(v) If $|\vec{a}| = |\vec{b}| = a$ & $\theta = 2\pi/3$

then $|\vec{R}| = a$

Illustration :

Forces of magnitudes 6N and 4N are acting on the body. Which of the following can be the resultant of the two?

(a) 11 N (b) 2 N (c) 10 N (d) 1 N (e) 8 N (f) 0 N (g) 7 N

Sol. F_{res} *lies between* F_{res} *(max.) and* F_{res} *(min.).*

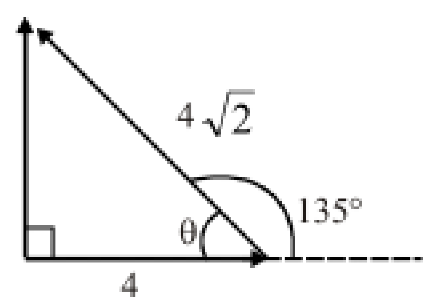
F_{res} *(max.) = 10 N and* F_{res} *(min.) = 2 N*

Thus, options b, c, e, and g are correct.

Illustration :

The resultant of two forces of magnitudes 4 N and $4\sqrt{2}$ *N makes 90° with the smaller force. Then angle between those two forces is ?*

Sol.



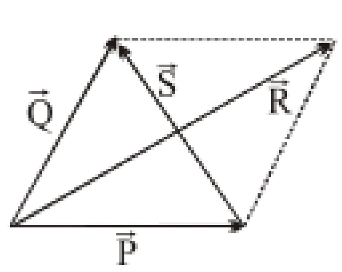
$$\cos\theta = \frac{4}{4\sqrt{2}} = \frac{1}{\sqrt{2}}$$

$\theta = 45°$

angle between forces 4 N and $4\sqrt{2}$ *N is 135°.*

Practice Exercise

Q.1 Two vectors $\vec{P}$ & $\vec{Q}$ are arranged in such a way that they form adjacent sides of a parallelogram as shown in figure



Which of the following options have correct relationship

(A) $\vec{Q} = \vec{R} + \vec{S}$ (B) $\vec{R} = \vec{P} + \vec{Q}$ (C) $\vec{R} = \vec{P} + \vec{S}$ (D) $\vec{S} = \vec{Q} - \vec{P}$

Q.2 Two vectors of 10 units & 5 units make an angle of 120° with each other. Find the magnitude & angle of resultant with vector of 10 unit magnitude.

Answers

Q.1 (B), (D) Q.2 $5\sqrt{3}$, 30°

Multiplication of a vector by a scalar :

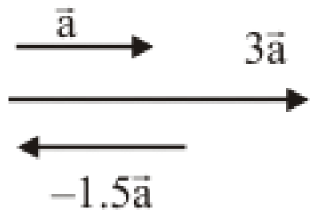
Lets say we have a vector $\vec{a}$ and k is a Scalar. Vector $\vec{b} = k\vec{a}$ is defined as a vector of magnitude |ka|.

If k is a positive then direction of $\vec{b}$ is along $\vec{a}$.

If k is negative then direction of $\vec{b}$ is opposite to $\vec{a}$.

If $|k| > 1 \Rightarrow |\vec{b}| > |\vec{a}|$

$|k| < 1 \Rightarrow |\vec{b}| < |\vec{a}|$



A vector may be multiplied by a pure number or by a scalar.

When a vector is multiplied by a scalar, the new vector may become a different physical quantity for example, when velocity (a vector) is multiplied by time (a scalar) we obtain a displacement (a vector).

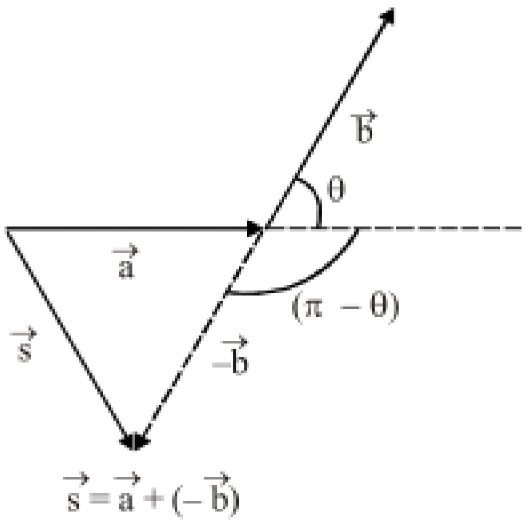
Subtraction of vectors :

To subtract two vectors, reverse the direction of the vector being subtracted and add the inverted vector to the vector from which you are subtracting.

Let say we want to obtain $\vec{a} - \vec{b}$

$$\vec{s} = \vec{a} - \vec{b} = \vec{a} + (-\vec{b})$$

i.e. $\vec{a} - \vec{b}$ can be understood as summation of $\vec{a} + (-\vec{b})$



For $\vec{s} = \vec{a} - \vec{b}$

Steps :

(i) Put tail of $\vec{b}$ at head of $\vec{a}$

(ii) Take $-\vec{b}$

(iii) Resultaul vector $\vec{s}$ from tail of $\vec{a}$ to head of $-\vec{b}$.

(iv) Angle between $\vec{a}$ and $\vec{b}$ is θ, then angle between $\vec{a}$ *and* $-\vec{b}$ or between $-\vec{a}$ *and* $\vec{b}$ is $(180° - \theta)$.

the angle between $\vec{a}$ and $-\vec{b}$ becomes $\pi - \theta$

so we get $s = \sqrt{(a + b\cos(\pi-\theta))^2 + (b\sin(\pi-\theta))^2}$

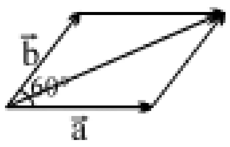
$$= \sqrt{a^2 + b^2 - 2ab\cos\theta}$$

and $\tan\alpha = \dfrac{b\sin(\pi-\theta)}{a + b\cos(\pi-\theta)} = \dfrac{b\sin\theta}{a - b\cos\theta}$

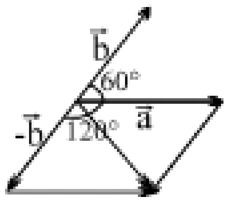
Illustration :

Two vectors of equal magnitude 2 are at an angle of 60° to each other find magnitude of their sum & difference.

Sol. $|\vec{a}+\vec{b}| = \sqrt{2^2 + 2^2 + 2\times2\times2\cos60^\circ} = \sqrt{4+4+4} = 2\sqrt{3}$



$|\vec{a}-\vec{b}| = \sqrt{2^2 + 2^2 + 2\times2\times2\cos120^\circ} = \sqrt{4+4-4} = 2$



Zero vector $\rightarrow$ *When* $\vec{a} = \vec{b}$ *& if want to find* $\vec{a} - \vec{b}$ = *zero vector. It is a vector with zero magnitude & undefined direction.*

Illustration :

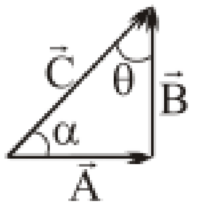
It is given that $\vec{A} + \vec{B} = \vec{C}$ *with* $\vec{A}\perp^{ar}\vec{B}$ *and*

$|\vec{A}| = 10$, $|\vec{C}| = 20$.

Find $|\vec{B}|$ *and angle of* $\vec{C}$ *with* $\vec{A}$

Sol. $\cos\alpha = \dfrac{|\vec{A}|}{|\vec{C}|}$

$\alpha = 60°$ *and* $\theta = 30°$



$\vec{A} + \vec{B} = \vec{C}$

$|\vec{C}|^2 = |\vec{A}|^2 + |\vec{B}|^2$

$|\vec{B}| = 10\sqrt{3}$

Illustration :

Two vectors $\vec{A}$ *and* $\vec{B}$ *are drawn from a common point and* $\vec{C} = \vec{A} + \vec{B}$

(A) If* $C^2 = A^2 + B^2$, *the angle between vectors* $\vec{A}$ *and* $\vec{B}$ *is* $90°$

(B) If* $C^2 < A^2 + B^2$, *the angle between* $\vec{A}$ *and* $\vec{B}$ *is greater than* $90°$

(C) If* $C^2 > A^2 + B^2$ *then angle between the vectors* $\vec{A}$ *and* $\vec{B}$ *is between* $0°$ *and* $90°$.

(D) If* $C = A - B$, *angle between* $\vec{A}$ *and* $\vec{B}$ *is* $180°$.

Sol. $C^2 = A^2 + B^2 + 2AB\cos\theta$

If $\theta = 90°$

then $C^2 = A^2 + B^2$

if $\theta > 90°$

then $C^2 = A^2 + B^2 + 2AB\cos\theta < A^2 + B^2$

$\therefore$ $\cos\theta$ *will be negative*

If $\theta < 90°$

then $C^2 = A^2 + B^2 + 2AB\cos\theta > A^2 + B^2$

$\therefore$ *If* $C = A - B \Rightarrow \theta = 180°$

Unit Vector :

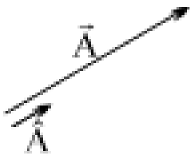
(i) A unit vector is vector of unit length used to specify direction.

(ii) A unit vector is a dimensionless vector with a magnitude of exactly 1.

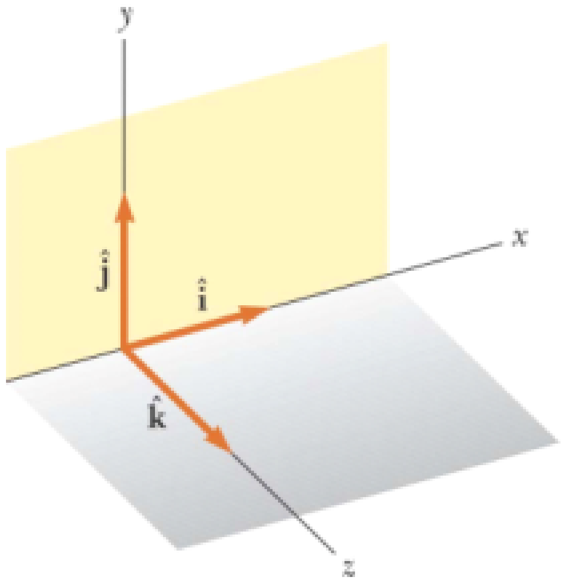
(iii) Unit vectors are used to specify a direction and have no other physical significance

A unit vector in direction of vector $\vec{A}$ is represented as $\hat{A}$

& $\hat{A} = \dfrac{\vec{A}}{|\vec{A}|}$



or $\vec{A}$ can be expressed in terms of a unit vector in its direction i.e. $\vec{A} = |\vec{A}|\,\hat{A}$



Unit Vectors along three coordinates axes:

unit vector along x-axis is $\hat{i}$

unit vector along y-axis is $\hat{j}$

unit vector along z-axis is $\hat{k}$

Illustration :

The value of a unit vector in the direction of vector $\vec{A} = 5\hat{i} - 12\hat{j}$ *is* _______?

Sol. $|\vec{A}| = \sqrt{5^2 + 12^2} = 13$

$$\hat{A} = \frac{\vec{A}}{|\vec{A}|} = \frac{5\hat{i} - 12\hat{j}}{13}$$

Co-ordinate Systems:

Co-ordinate systems are used to describe the position of a point in space.
Coordinate system consists of :
(i) A fixed reference point called the origin
(ii) Specific axes with scales and labels
(iii) Instructions on how to label a point relative to the origin and the axes

Cartesian Coordinate System :

Also called rectangular coordinate system
x- and y- axes intersect at the origin

Points are labeled (x,y)

In figure, points P and Q are shown as (5, 3) and (–3, 4) respectively.

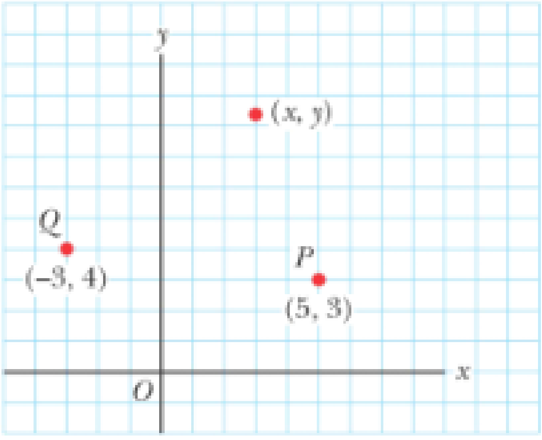


Illustration :

Express the vector in unit - vector notation.

Sol. $2\hat{i}$

Illustration :

Acceleration due to gravity is always downwards. How will you write it vectorially if +Y is (a) downwards (b) upwards.

Sol. *(a) downwards taken as postive Y direcation.*

$$\vec{a} = g\,\hat{j}$$

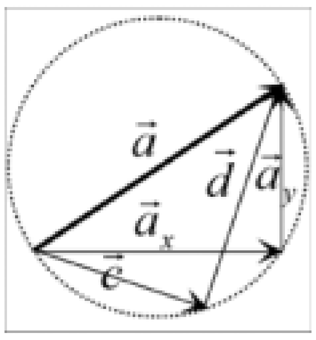
(b) upwards taken as postive Y direcation.

$$\vec{a} = -g\,\hat{j}$$

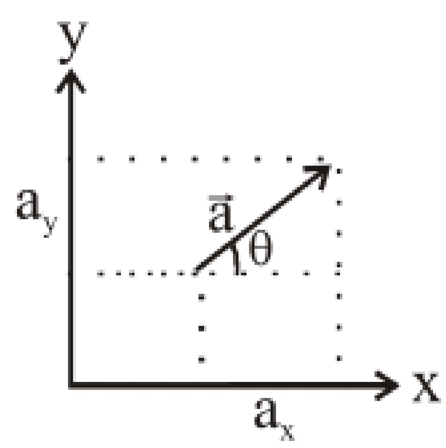
Rectangular Component (resolution) of vectors:

(components means parts)

We can move from tail of $\vec{a}$ to its head via various paths. But, if we move with $\vec{a}$ as the diameter of circle as shown, then the two vectors ($\vec{c}$ & $\vec{d}$ and $\vec{a}_x$ & $\vec{a}_y$) would be perpendicular to each other. Such perpendicular vectors are called rectangular components of $\vec{a}$. But, if



we choose $\vec{a}_x$ & $\vec{a}_y$. Then we can write them in terms of standard unit vectors $\hat{i}$ & $\hat{j}$ respectively. Then we would say that a_x is the component of vector along the x-axis & a_y is the component of vector along y-axis. Now according to triangle law of addition:



$\vec{a} = \vec{a}_x + \vec{a}_y$

So, we write $\vec{a} = a_x(\hat{i}) + a_y(\hat{j}) = a\cos\theta(\hat{i}) + a\sin\theta(\hat{j})$

This is a very convenient form of representing vectors.

$a\cos\theta$ is known as component of $\vec{a}$ along x-axis (a_x)

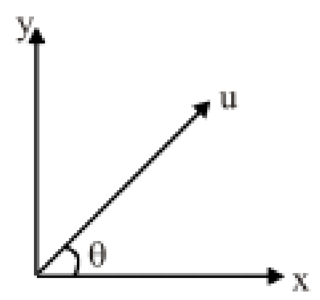
$a\sin\theta$ is known as component of $\vec{a}$ along y-axis (a_y)

This is best form of representing vectors because we can do exact addition and subtraction using simple laws of algebra without needing to draw vectors

Illustration :

A body is thrown from ground making an angle θ with speed u from horizontal. Write its initial velocity in unit vector notation.

Sol.



$$\vec{u} = u\cos\theta\,\hat{i} + u\sin\theta\,\hat{j}$$

Results :

1. Unit vector along $\vec{A}$ is $\hat{A}$.

   Since, $\vec{A} = |\vec{A}|\hat{A}$

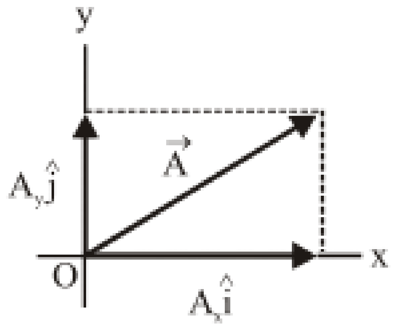
   & $\vec{A} = |\vec{A}|(\cos\theta\hat{i} + \sin\theta\hat{j})$

   $\hat{A} = \hat{i}\cos\theta + \hat{j}\sin\theta$

2. If components of a vector along x & y-axis are known, then that vector can be completely represented as

   $\vec{A} = A_x\hat{i} + A_y\hat{j}$

3. $|\vec{A}| = \sqrt{A_x^2 + A_y^2}$



4. $\tan\theta = \left(\frac{A_y}{A_x}\right)$

   θ is angle with x-axis

5. The components can be positive or negative and will have the same units as the original vector

6. The signs of the components will depend on the angle

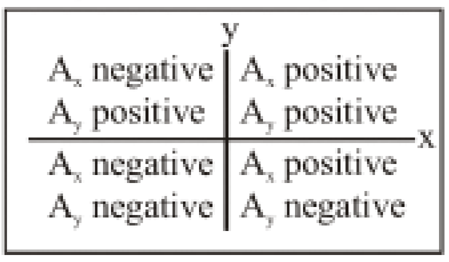
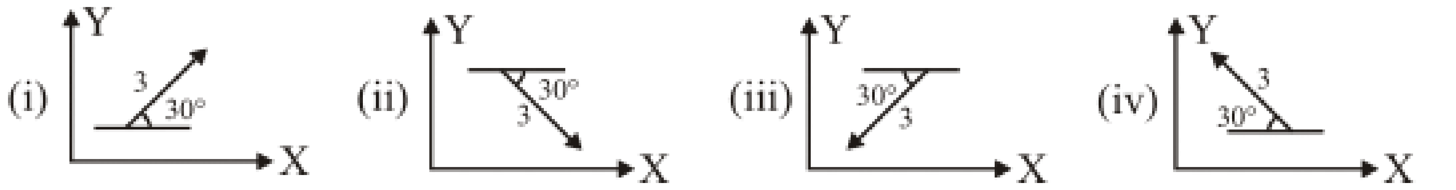


Illustration :

Express the given vector $\vec{A}$ (shown graphically) in unit vector notation.



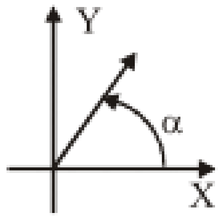
Sol. *(i)* $\vec{A} = 3(\cos 30\hat{i} + \sin 30\hat{j})$ *(ii)* $\vec{A} = 3(\cos 30\hat{i} - \sin 30\hat{j})$ *(iii)* $\vec{A} = 3(-\cos 30\hat{i} - \sin 30\hat{j})$

(iv) $\vec{A} = 3(-\cos 30\hat{i} + \sin 30\hat{j})$

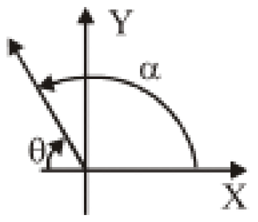
Illustration :

Angle made by the vector with positive direction of X-axis

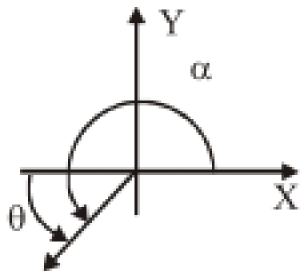
Find angle (α) made by the vectors with the positive direction of X-axis.

$\vec{a} = \hat{i} + \sqrt{3}\hat{j}$ 

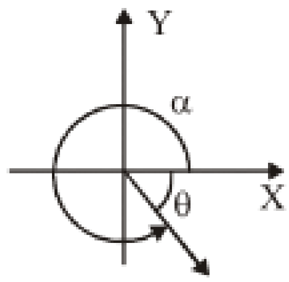
$tan\alpha = \left|\dfrac{a_y}{a_x}\right| = \left|\dfrac{\sqrt{3}}{1}\right|, \therefore \alpha = 60°$ *Ans.* $\alpha = 60°$

$\vec{a} = -\hat{i} + \sqrt{3}\hat{j}$ 

$tan\theta = \left|\dfrac{a_y}{a_x}\right| = \left|\dfrac{\sqrt{3}}{1}\right|, \therefore \theta = 60°, \alpha = 180° - \theta$ *Ans.* $\alpha = 120°$

$\vec{a} = -\hat{i} - \sqrt{3}\hat{j}$ 

$tan\theta = \left|\dfrac{a_y}{a_x}\right| = \left|\dfrac{\sqrt{3}}{1}\right|, \therefore \theta = 60°, \alpha = 180° + \theta$ *Ans.* $\alpha = 240°$

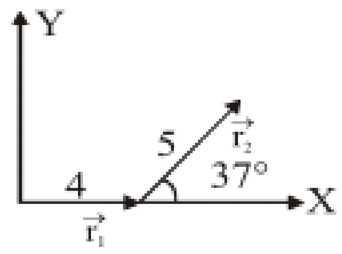
$\vec{a} = \hat{i} - \sqrt{3}\hat{j}$ 

$tan\theta = \left|\dfrac{a_y}{a_x}\right| = \left|\dfrac{\sqrt{3}}{1}\right|, \therefore \theta = 60°, \alpha = 360° - \theta$ *or* $\alpha = -\theta$

Ans. $\alpha = 300°, -60°$

Illustration :

A man moves in the following manner in X-Y plane. Find the magnitude of displacement of man from origin as shown in figure.



Sol. $\vec{r}_1 = 4\hat{i}$

$\vec{r}_2 = 5\cos 37°\hat{i} + 5\sin 37°\hat{j}$

$\vec{r}_2 = 4\hat{i} + 3\hat{j}$

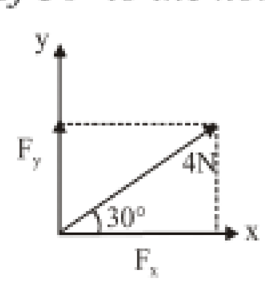
Resultant vector

$\vec{r} = \vec{r}_1 + \vec{r}_2$

$= 8\hat{i} + 3\hat{j}$

Illustration :

A force of 4 N is acting at an angle of 30^o to the horizontal. Find its component along axeses.



Sol $F_y = 4\sin 30° = 2\ N$

$F_x = 4\cos 30° = 2\sqrt{3}\ N$

Illustration :

Find a vector $\vec{F}$ of magnitude 50N parallel to $-4\hat{i}+3\hat{j}$.

Sol. $\vec{F} = 50 \times \frac{(-4\hat{i}+3\hat{j})}{5} = -40\hat{i} + 30\hat{j}$

Illustration :

A particle is moving with speed 6 m/s along the direction of $2\hat{i}+2\hat{j}-\hat{k}$ find the velocity vector of a particle?

Sol. $\vec{v} = |\vec{v}|\,\hat{v} = 6\frac{(2\hat{i}+2\hat{j}-\hat{k})}{3} = 2(2\hat{i}+2\hat{j}-\hat{k})$

Illustration :

Find a vector of magnitude twice of $12\hat{i}-5\hat{j}$ and anti-parallel to $3\hat{i}-4\hat{j}$

Sol. Suppose $\vec{a} = 12\hat{i}-5\hat{j}$, $\vec{b} = 3\hat{i}-4\hat{j}$ required vector $\vec{r}$.

$\vec{r} = 2|\vec{a}|(-\hat{b})$

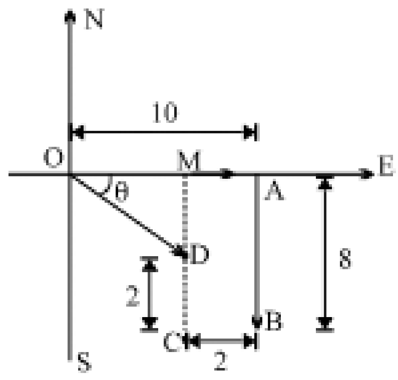
$= 2 \times (13)\left(\frac{-3\hat{i}+4\hat{j}}{5}\right)$

$$= \frac{26}{5}\left(-3\hat{i}+4\hat{j}\right)$$

Illustration :

An insect crawls 10 m towards east, turns to its right, crawls 8 m, and again turns to its right, Now crawling a distance of 2 m it turns to its right and stop after moving 2 m more. Find its net displacement.

Sol. Net displacement is $\overrightarrow{OD}$



$$\overrightarrow{OD} = \overrightarrow{OM} + \overrightarrow{MD}$$

$$OD = \sqrt{(OM)^2 + (MD)^2}$$

$$= \sqrt{(OA - BC)^2 + (AB - CD)^2}$$

$$= \sqrt{8^2 + 6^2}$$

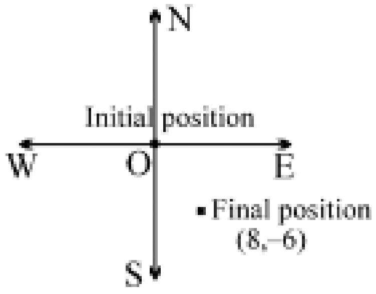
$$OD = 10 \text{ m}$$

In ΔOMD

$$\tan\theta = \frac{MD}{OM} = \frac{6}{8} = \frac{3}{4}$$

Displacement is 10 m at $\theta = \tan^{-1}\left(\frac{3}{4}\right) \approx 37° \text{ S of E}$

Alternate



$$\vec{r}_i = o\hat{i} + o\hat{j}\ ,\ \vec{r}_f = 8\hat{i} - 6\hat{j}$$

$$\vec{d} = \vec{r}_f - \vec{r}_i = (8\hat{i} - 6\hat{j}) - (0\hat{i} + 0\hat{j})$$

$$\vec{d} = 8\hat{i} - 6\hat{j}$$

Concept of Equilibrium :

Equilibrium means net force acting on body is zero.

i.e. $\Sigma\vec{F} = \vec{0}$ (for translational equilibrium)

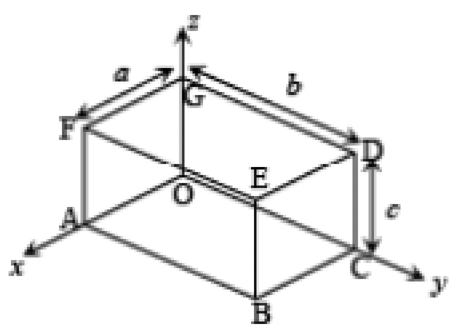
State of rest or moving with constant velocity are the condition of translational equilibrium.

Illustration :

Three forces ($\vec{F}_1$, $\vec{F}_2$, $\vec{F}_3$) are acting on a particle moving vertically up with constant speed. Two forces $\vec{F}_1 = -10\hat{j}$ N, and $\vec{F}_2 = -6\hat{i} + 8\hat{j}$ N are acting on particle respectively find $\vec{F}_3$.

Sol. $\Sigma\vec{F} = \vec{0}$ i.e

$$\vec{F}_1 + \vec{F}_2 + \vec{F}_3 = \vec{0}$$



To find $\overrightarrow{OE}$, You can move in the path $OA \rightarrow AB \rightarrow BE$

Find the vector $\overrightarrow{OA}$, $\overrightarrow{OB}$, $\overrightarrow{OC}$,, $\overrightarrow{OD}$, $\overrightarrow{OE}$, $\overrightarrow{OF}$, $\overrightarrow{OG}$

$\therefore \overrightarrow{OE} = a\hat{i} + b\hat{j} + c\hat{k}$ similarly, $\overrightarrow{OA} = a\hat{i}$ $\overrightarrow{OB} = a\hat{i} + b\hat{j}$

$$\overrightarrow{OC} = b\hat{j} \quad \overrightarrow{OD} = b\hat{j} + c\hat{k}$$

$$\overrightarrow{OG} = c\hat{k} \quad \overrightarrow{OF} = a\hat{i} + c\hat{k}$$

All these vectors are called position vectors as it defines the position of a particle in space with respect to origin.

Illustration :

Can you tell the co-ordinates of A, B, C, D, E, F & G.

Sol. *(A) (a, 0, 0) B(a, b, 0) C(0, b, 0) (D)(0, b, c) E(a, b, c) F(a,, 0, c) G(0, 0, c)*

Illustration :

What is the magnitude of the diagonal $\overrightarrow{OE}$ *?*

Sol. $\overrightarrow{OE} = a\hat{i} + b\hat{j} + c\hat{k}$

$$\left|\overrightarrow{OE}\right| = \sqrt{a^2 + b^2 + c^2}$$

Illustration :

What is the magnitude of the $\overrightarrow{GB}$ *?*

Sol. $\overrightarrow{GB} = \overrightarrow{OB} - \overrightarrow{OG}$

$= a\hat{i} + b\hat{j} - c\hat{k}$

Thus, from now on we shall understand that $|\overrightarrow{GB}| = \sqrt{a^2 + b^2 + c^2}$

Displacement Vector:

Suppose that a particle displaces from position $\vec{r}_1$ to $\vec{r}_2$, then the particle's displacement is given by:

$$\vec{F}_3 = -\left(\vec{F}_1 + \vec{F}_2\right)$$
$$= -\left(-6\hat{i} - 2\hat{j}\right)$$
$$\vec{F}_3 = 6\hat{i} + 2\hat{j}\ N$$

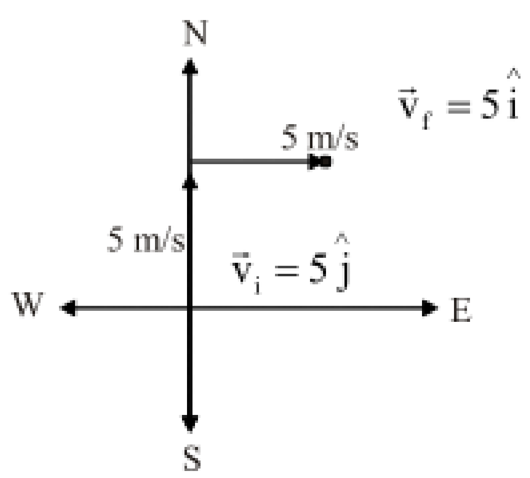
Subtraction of vectors (applications):

To find change in velocity.

Illustration :

A car is moving northwards at a constant speed of 5 m/s. it makes a right turn and continues to move with a constant speed of 5 m/s. Find the magnitude of change in velocity.

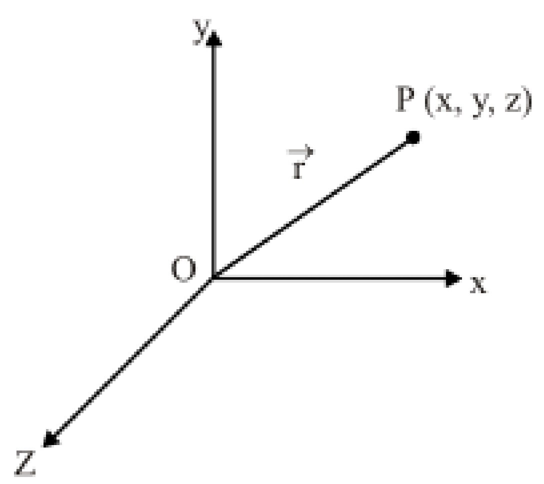
Sol.



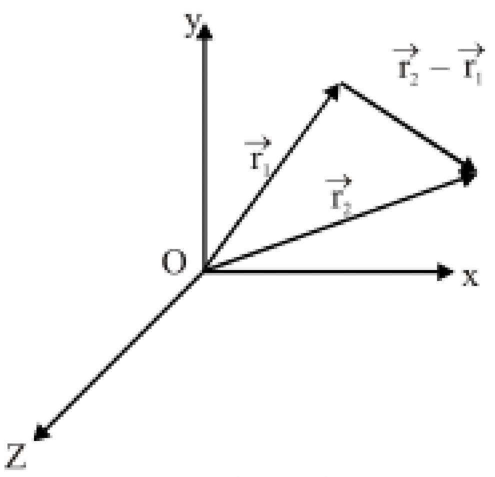
$$\Delta\vec{v} = \vec{v}_f - \vec{v}_i$$
$$\Delta\vec{v} = 5\hat{i} - 5\hat{j}$$
$$|\Delta\vec{v}| = \sqrt{5^2 + 5^2}$$
$$|\Delta\vec{v}| = 5\sqrt{2}\ m/s$$

Position Vector:

A general way of locating a particle is with a position vector $\vec{r}$, which is a vector that extends from a reference point to the particle. This reference point is usually the origin.



In unit vector notation: $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$



$\vec{s}$ = p.v. of $\vec{r}_2$ − p.v. of $\vec{r}_1$ $= \left(x_2\,\hat{i} + y_2\,\hat{j} + z_2\,\hat{k}\right) - \left(x_1\,\hat{i} + y_1\,\hat{j} + z_1\,\hat{k}\right)$

$\therefore\ \vec{s} = \left(\Delta x\,\hat{i} + \Delta y\,\hat{j} + \Delta z\,\hat{k}\right)$

Illustration :

Find the displacement of tip of hour hand of the clock between 1 pm to 5 pm where the length of hour hand is 6 cm.

Sol. $2 \times 6\sin 60°$

$= 2 \times \left(6 \times \frac{\sqrt{3}}{2}\right) = 6\sqrt{3}\ cm$

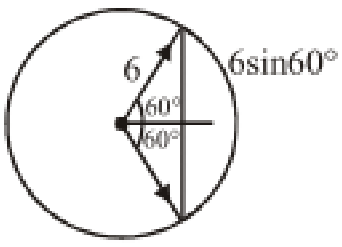


Illustration :

The position vectors of two balls are given by $\vec{r}_1 = 2\hat{i} + 7\hat{j}$, $\vec{r}_2 = -2\hat{i} + 4\hat{j}$ What will be the distance between the two balls?

Sol. $\vec{r} = \vec{r}_2 - \vec{r}_1$

$\vec{r} = -4\hat{i} - 3\hat{j}$

Distance $= \sqrt{4^2 + 3^2} = 5$

***Now**, so far we have learnt about different kinds of vectors. They were velocity vector, displacement vector, position vector etc. Similarly, we can have acceleration vector as well. One would have studied earlier about basic laws of kinematics.*

$v = u + at$

$S = ut + \frac{1}{2}at^2$

In the above equation, other than time, rests physical quantities are vectors. So, a better representation would be:

$\vec{v} = \vec{u} + \vec{a}t$

$\vec{S} = \vec{u}t + \frac{1}{2}\vec{a}t^2$

And we should remember that the above equations are valid when the acceleration is constant. If *the acceleration is zero, then equation will be:*

$\vec{S} = \vec{u}t$ or $\vec{r}_2 - \vec{r}_1 = \vec{u}t$

Illustration :

A particle has initial velocity of $2\hat{i}$ *and has constant acceleration of* $3\hat{j}$. *Find its displacement and velocity after 3s. If initially, the particle is located at* $3\hat{i} + 4\hat{j}$, *find its final location.*

Sol. $\vec{v} = \vec{u} + \vec{a}t = 2\hat{i} + (3\hat{j})3 = (2\hat{i} + 9\hat{j})$

$$\vec{s}_2 - \vec{s}_1 = \vec{u}t + \frac{1}{2}\vec{a}t^2 = (2\hat{i})(3) + \frac{1}{2}(3\hat{j})(9)$$

$$\vec{s}_2 = 6\hat{i} + \frac{27}{2}\hat{j} + 3\hat{i} + 4\hat{j}$$

$$= \left(9\hat{i} + \frac{35}{2}\hat{j}\right)$$

Illustration :

A particle who has a constant speed of 50 m/s. it moves along a straight line from A(2,1) to B(9,25). Find its velocity vector. If at initial instant of time it is located at (2, 0); find its final location after 3 s.

Sol. $\overrightarrow{AB} = \vec{B} - \vec{A} = 7\hat{i} + 24\hat{j}$

$$\vec{v} = (\hat{AB})|\vec{v}| = \frac{(7\hat{i} + 24\hat{j})}{25} \times 50 = 14\hat{i} + 48\hat{j}$$

$\vec{s}_2 - \vec{s}_1 = \vec{v}(3)$, and $\vec{s}_1 = 2\hat{i}$

$\vec{s}_2 = (42\hat{i} + 144\hat{j}) + 2\hat{i} = (44\hat{i} + 144\hat{j})$

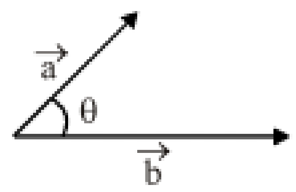
Product of Vectors

There are two ways in which vectors can be multiplied :

(1) Scalar product or dot product.

(2) Vector product or cross product

(1) Scalar product



$\vec{a}.\vec{b} = |\vec{a}|.|\vec{b}|\cos\theta$, where θ is angle between $\vec{a}$ and $\vec{b}$

It is called a scalar product because its product is a scalar quantity.

(i) $\hat{i}\cdot\hat{i} = |\hat{i}|.|\hat{i}|\cos 0° = 1 = \hat{j}.\hat{j} = \hat{k}.\hat{k}$

$\hat{i}\cdot\hat{j} = |\hat{i}|.|\hat{j}|\cos 90° = 0 = \hat{j}.\hat{k} = \hat{k}.\hat{i}$

(ii) $\vec{a}\cdot\vec{b} = 0 \Rightarrow \vec{a} \perp \vec{b}$: used to test orthogonality. (means two vectors are mutually perpendicular to each other).

(iii) If $\vec{a} = a_x\hat{i} + a_y\hat{j} + a_z\hat{k}$ $\quad \vec{b} = b_x\hat{i} + b_y\hat{j} + b_z\hat{k}$

$$\vec{a}.\vec{b} = |\vec{a}||\vec{b}|\cos\theta$$

$$a_x b_x + a_y b_y + a_z b_z = \sqrt{a_x^2 + a_y^2 + a_z^2}\sqrt{b_x^2 + b_y^2 + b_z^2}\cos\theta$$

$\cos\theta = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}$ (this is used to find the angle between two vectors).

(iv) Work done by vector $\vec{F}$ is defined as $w = \vec{F}\cdot\vec{d}$

Where $\vec{d}$ displacement vector.

Illustration :

Find the dot product of vectors $\vec{a} = 2\hat{i} - 3\hat{j} + \hat{k},\ \vec{b} = -\hat{i} + 3\hat{j} + \hat{k}$

Sol. $\vec{a}.\vec{b} = -2 - 9 + 1 = -10$

Illustration :

If $\hat{i} + 2\hat{j} + n\hat{k}$ *is perpendicular to* $4\hat{i} + 2\hat{j} + 2\hat{k}$ *then find the value of n ?*

Sol. $(1\hat{i} + 2\hat{j} + n\hat{k}).(4\hat{i} + 2\hat{j} + 2\hat{k}) = 0$

$4 + 4 + n(2) = 0 \Rightarrow n = -4$

Illustration :

A force of $(-3\hat{i} - \hat{j} + 2\hat{k})$ *N displaced the body from a point (4, - 3, - 5) m to a point (- 1, 4, 3) m in a straight line. Find the work done by the force.*

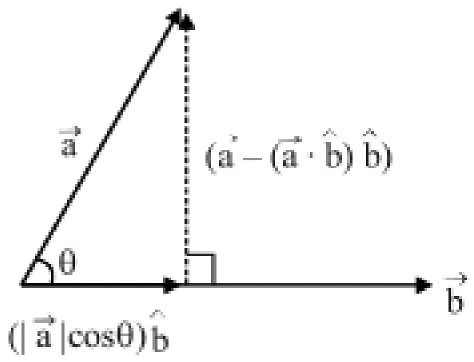
Sol. $\vec{A} = (4\hat{i} - 3\hat{j} - 5\hat{k}),\ \vec{B} = -\hat{i} + 4\hat{j} + 3\hat{k}$

$\overrightarrow{AB} = \vec{B} - \vec{A} = (-5\hat{i} + 7\hat{j} + 8\hat{k})$

$W = \vec{F}\cdot\overrightarrow{AB} = (-3\hat{i} - \hat{j} + 2\hat{k})\cdot(-5\hat{i} + 7\hat{j} + 8\hat{k}) = 24 joule.$

Projection (Component) of Vector : $\vec{a}$ on $\vec{b}$

Find the vector component of (i) $\vec{a}$ along $\vec{b}$ (ii) $\vec{a} \perp$ to $\vec{b}$



(i) *Magnitude of Projection of* $\vec{a}$ on $\vec{b} = |\vec{a}|\cos\theta = \dfrac{\vec{a}.\vec{b}}{|\vec{b}|} = \vec{a}.\hat{b}$

Thus, vector Component of vector $\vec{a}$ in the direction of $\vec{b}$ is $(\vec{a}.\hat{b})\hat{b}$

(ii) *Vector component of vector $\vec{a}$ in the perpendicular direction of $\vec{b}$ is $(\vec{a} - (\vec{a}.\hat{b})\hat{b})$.*

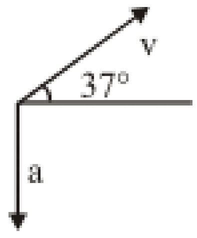
Illustration :

For vector $\vec{a} = 3\hat{i} + 3\hat{j} - 2\hat{k}$, what are its component along X- axis, Y-axis & Z-axis.

Sol. *3 along X-axis, 3 along Y-axis, - 2 along Z-axis.*

Illustration :

Just after firing, a bullet is found to move at an angle of 37º to the horizontal. Its acceleration is 10 m/s² downwards. Find the component of its acceleration in the direction of its velocity. ***Ans: - 6 m/s²***



Ans. $\hat{v} = \frac{1}{5}(4\hat{i} + 3\hat{j})$

$\vec{a} = -g\hat{j}$

vector Component of vector $\vec{a}$ in the direction of $\vec{v}$ is $(\vec{a}.\hat{v})\hat{v}$. Thus – 6 m/s².

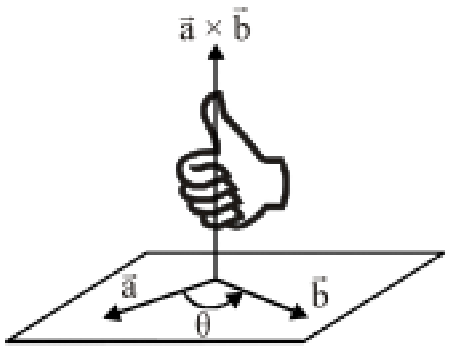
(2) Cross product or vector product of two vectors :

The vector or cross product of two vectors $\vec{a}$ and $\vec{b}$ is defined as

$$\vec{a} \times \vec{b} = ab\ \sin\theta\ \hat{n}$$

The product is defined to be a vector of magnitude ab sin θ that points in the direction of the unit vector $\hat{n}$ normal (perpendicular) to the plane of $\vec{a}$ and $\vec{b}$. The angle θ is the smaller angle between the vector.

The direction of $\hat{n}$ is still ambiguous. This ambiguity is removed by using a convention called the right-hand rule. Curl the fingers of your right hand and stick out your thumb as if you were hitch-hiking as in figure. The sense of rotation of the fingers should be from the first vector $\vec{a}$ to the second vector $\vec{b}$ through the smaller angle between them. The thumb indicates the direction $\hat{n}$.



Because of the right-hand rule, the order of the vectors in the cross product is important. The vector product is **noncommutative** :

$$\vec{b} \times \vec{a} = -\vec{a} \times \vec{b}$$

Properties of vector (or Cross Product)

(i) Cross product non-commutative :

$\vec{a} \times \vec{b} \neq \vec{b} \times \vec{a}$ i.e. $\vec{b} \times \vec{a} = -\vec{a} \times \vec{b}$

(ii) Follows distributive law :

$\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$

(iii) If a and b are any vectors, and m is any real number (positive or negative) then

$(m\,\vec{a}) \times \vec{b} = m(\vec{a} \times \vec{b}) = \vec{a} \times (m\,\vec{b})$

(iv) Does not follow associative law :

$\vec{a} \times (\vec{b} \times \vec{c}) \neq (\vec{a} \times \vec{b}) \times \vec{c}$

(v) $\hat{i} \times \hat{j} = \hat{k}, \hat{j} \times \hat{k} = \hat{i}, \hat{k} \times \hat{i} = \hat{j}$

and $\hat{i} \times \hat{i} = 0, \hat{j} \times \hat{j} = 0, \hat{k} \times \hat{k} = 0$

(vi) $\vec{a} = a_x\hat{i} + a_y\hat{j} + a_z\hat{k}$

$\vec{b} = b_x\hat{i} + b_y\hat{j} + b_z\hat{k}$

$\vec{a} \times \vec{b} = (a_x\hat{i} + a_y\hat{j} + a_z\hat{k}) \times (b_x\hat{i} + b_y\hat{j} + b_z\hat{k})$

$= (a_y b_z - a_z b_y)\,\hat{i} + (a_z b_x - a_x b_z)\,\hat{j} + (a_x b_y - a_y b_x)\,\hat{k}$

(vii) Angle between two vectors

$|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}| \sin\theta$

$\sin\theta = |\vec{a} \times \vec{b}| / |\vec{a}||\vec{b}|$

Let

$\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$

$\vec{a} \times \vec{b} = (a_2b_3 - a_3b_2)\hat{i} - (a_1b_3 - a_3b_1)\hat{j} + (a_1b_2 - a_2b_1)\hat{k}$

$\Rightarrow |\vec{a} \times \vec{b}| = \sqrt{(a_2b_3 - a_3b_2)^2 + (a_1b_3 - a_3b_1)^2 + (a_1b_2 - a_2b_1)^2}$

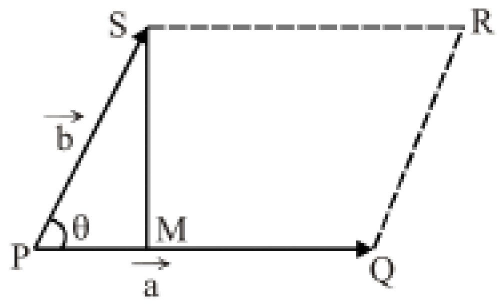
$|\vec{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}, |\vec{b}| = \sqrt{b_1^2 + b_2^2 + b_3^2}$

$$\sin\theta = \frac{\sqrt{(a_2b_3 - a_3b_2)^2 + (a_3b_1 - a_1b_3)^2 + (a_1b_2 - a_2b_1)^2}}{\sqrt{a_1^2 + a_2^2 + a_3^2}\sqrt{b_1^2 + b_2^2 + b_3^2}}$$

(ix) The cross product of a vector with itself is a NULL vector *i.e.,*

$\vec{a} \times \vec{a} = |\vec{a}||\vec{a}| \sin 0^\circ\, \hat{n} = \vec{0}$

(x) The cross product of two vectors represents the area of the parallelogram formed by them, (Figure., shows a parallelogram *PQRS* whose adjacent sides $\overrightarrow{PQ}$ and $\overrightarrow{PS}$ are represented by vectors $\vec{a}$ and $\vec{b}$ respectively.



Now, area of parallelogram = $|\overrightarrow{PQ}| \cdot |\overrightarrow{SM}| = |\overrightarrow{PQ}| \cdot |\overrightarrow{PS}| \sin\theta = |\vec{a}| \cdot |\vec{b}| \sin\theta = |\vec{a} \times \vec{b}|$ hence cross product of two vectors represents the area of parallelogram formed by it. It is worth noting that area vector $\vec{a} \times \vec{b}$ acts along the perpendicular to the plane of two vectors $\vec{a}$ and $\vec{b}$.

Unit Vector Perpendicular to two given vectors

Let be a unit vector perpendicular to two (non-zero) vectors a, b and positive for right handed rotation from a to b and θ be the angle between the vectors a, b then

$$\vec{a} \times \vec{b} = |\vec{a}|\,|\vec{b}| \sin\theta\,\hat{n}$$

$$|\vec{a} \times \vec{b}| = |\vec{a}|\,|\vec{b}| \sin\theta$$

Thus we get $= \vec{a} \times \vec{b} \,/\, |\vec{a} \times \vec{b}| = \hat{n}$.

Illustration :

Prove that

$$\vec{a} \times (\vec{b} + \vec{c}) + \vec{b} \times (\vec{c} + \vec{a}) + \vec{c} \times (\vec{a} + \vec{b}) = 0$$

Sol. $\vec{a} \times (\vec{b} + \vec{c}) + \vec{b} \times (\vec{c} + \vec{a}) + \vec{c} \times (\vec{a} + \vec{b})$

$= \vec{a} \times \vec{b} + \vec{a} \times \vec{c} + \vec{b} \times \vec{c} + \vec{b} \times \vec{a} + \vec{c} \times \vec{a} + \vec{c} \times \vec{b}$

$= \vec{a} \times \vec{b} - \vec{c} \times \vec{a} + \vec{b} \times \vec{c} - \vec{a} \times \vec{b} + \vec{c} \times \vec{a} - \vec{b} \times \vec{c} = 0$

Illustration :

Find $\vec{a} \times \vec{b}$ *and* $\vec{b} \times \vec{a}$ *if*

(i) $\vec{a} = 3\hat{k} + 4\hat{j},\ \vec{b} = \hat{i} + \hat{j} - \hat{k}$

(ii) $\vec{a} = (2, -1, 1)\ ;\ \vec{b} = (3, 4, -1)$

Sol. (i) $$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 4 & 3 \\ 1 & 1 & -1 \end{vmatrix} = -7\hat{i} + 3\hat{j} - 4\hat{k}$$

$$\vec{b} \times \vec{a} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & -1 \\ 0 & 4 & 3 \end{vmatrix} = 7\hat{i} - 3\hat{j} + 4\hat{k}$$

Thus $\vec{a} \times \vec{b} \neq \vec{b} \times \vec{a}$

(ii) $$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -1 & 1 \\ 3 & 4 & -1 \end{vmatrix} = -3\hat{i} + 5\hat{j} + 11\hat{k}$$

$$\vec{b}\times\vec{a}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 3 & 4 & -1\\ 2 & -1 & 1\end{vmatrix}=3\hat{i}-5\hat{j}-11\hat{k}$$

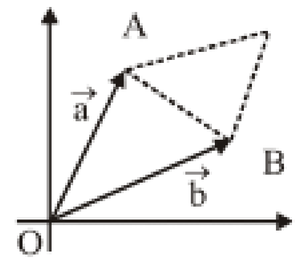
Illustration :

If $\vec{a}=3\hat{i}+\hat{j}+2\hat{k}$ *and* $\vec{b}=2\hat{i}-2\hat{j}+4\hat{k}$

(i) find the magnitude of $\vec{a}\times\vec{b}$

(ii) find a unit vector perpendicular to both a and b.

(iii) find the cosine and sine of the angle between the vectors a and b



Sol. *(i)* $$\vec{a}\times\vec{b}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 3 & 1 & 2\\ 2 & -2 & 4\end{vmatrix}=8\hat{i}-8\hat{j}-8\hat{k}$$

$\therefore$ *Magnitude of* $\vec{a}\times\vec{b}=|\vec{a}\times\vec{b}|$

$$\sqrt{(8)^2+(-8)^2+(-8)^2}=8\sqrt{3}$$

(ii) $$\hat{n}=\frac{\vec{a}\times\vec{b}}{|\vec{a}\times\vec{b}|}=\frac{8\hat{i}-8\hat{j}-8\hat{k}}{8\sqrt{3}}=\frac{1}{\sqrt{3}}(\hat{i}-\hat{j}-\hat{k})$$

There are two unit vectors perpendicular to both $\vec{a}$ *and* $\vec{b}$ *they are*

$$\pm\hat{n}=\pm\frac{1}{\sqrt{3}}(\hat{i}-\hat{j}-\hat{k})$$

(iii) To find cos θ

$$\vec{a}\cdot\vec{b}=|\vec{a}||\vec{b}|\cos\theta=(3\hat{i}+\hat{j}+2\hat{k}).(2\hat{i}-2\hat{j}+4\hat{k})$$

$$=(3)(2)+(1)(-2)+2(4)=12$$

$$|\vec{a}|=\sqrt{3^2+1^2+2^2}=\sqrt{14}$$

$$|\vec{b}|=\sqrt{(2)^2+(-2)^2+(4)^2}=\sqrt{24}$$

$$\cos\theta=\vec{a}\cdot\vec{b}/|\vec{a}||\vec{b}|=\frac{12}{\sqrt{14}\sqrt{24}}=\frac{12}{\sqrt{2}\sqrt{7}.2\sqrt{2}\sqrt{3}}=\sqrt{\frac{3}{7}}$$

$$\text{Also, }\sin\theta=|\vec{a}\times\vec{b}|/|\vec{a}||\vec{b}|=\frac{8\sqrt{3}}{\sqrt{14}\sqrt{24}}=\frac{2}{\sqrt{7}}$$

$$\text{Also, }\sin\theta=\sqrt{1-\cos^2\theta}=\sqrt{1-\frac{3}{7}}=\sqrt{\frac{4}{7}}=\frac{2}{\sqrt{7}}$$

Illustration :

The vectors from origin to the points A and B are $\vec{a}=3\hat{i}-6\hat{j}+2\hat{k}$ *and* $\vec{b}=2\hat{i}+\hat{j}-2\hat{k}$ *respectively. Find the area of :*

(i) the triangle OAB

(ii) the parallelogram formed by $\overrightarrow{OA}$ *and* $\overrightarrow{OB}$ *as adjacent sides.*

Sol. Given $\overrightarrow{OA} = \vec{a} = 3\hat{i} - 6\hat{j} + 2\hat{k}$ and $\overrightarrow{OB} = \vec{b} = 2\hat{i} + \hat{j} - 2\hat{k}$

$$\therefore \quad (\vec{a} \times \vec{b}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & -6 & 2 \\ 2 & 1 & -2 \end{vmatrix}$$

$= (12 - 2)\hat{i} - (-6 - 4)\hat{j} + (3 + 12)\hat{k}$

$= 10\hat{i} + 10\hat{j} + 15\hat{k}$

$\Rightarrow |\vec{a} \times \vec{b}| = \sqrt{10^2 + 10^2 + 15^2} = \sqrt{425} = 5\sqrt{17}$

(i) Area of $OAB = \frac{1}{2}|\vec{a} \times \vec{b}| = \frac{1}{2} \cdot 5\sqrt{17}$ sq. units

$= \frac{5}{2}\sqrt{17}$ sq. units

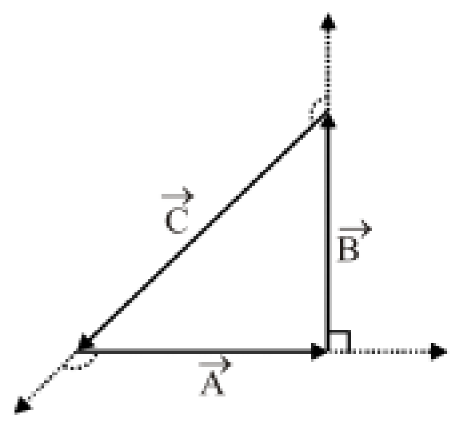
(ii) Area of parallelogram formed by $\overrightarrow{OA}$ and $\overrightarrow{OB}$ as adjacent sides $= |\vec{a} \times \vec{b}| = 5\sqrt{17}$ sq. units.

Solved Example

Q.1 Given that $\vec{A}+\vec{B}+\vec{C}=\vec{0}$, but of three two are equal in magnitude and the magnitude of third vector is $\sqrt{2}$ times that of either of the vectors two having equal magnitude. Then the angles between vectors are given by -

(A) 30°, 60°, 90° (B) 45°, 45°, 90° (C) 45°, 60°, 90° (D) 90°, 135°, 135°

Sol. (D) From polygon law, there vectors having summation zero, should from a closed polygon (triangle). Since the two vectors are having same magnitude and the third vector is $\sqrt{2}$ times that of either of two having equal magnitude. i.e. the triangle should be right angled triangle.



Angle between A and B is 90°

Angle between B and C is 135°

Angle between A and C is 135°

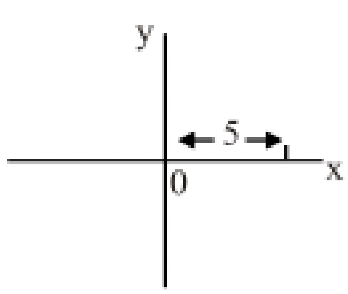
Q.2 If a particle moves 5m in + x-direction. Show the displacement of the particle-

(A) $5\hat{j}$ (B) $5\hat{i}$ (C) $-5\hat{j}$ (D) $5\hat{k}$

Sol. Magnitude of vector = 5

Unit vector in + x direction is $\hat{i}$

displacement = $5\hat{i}$



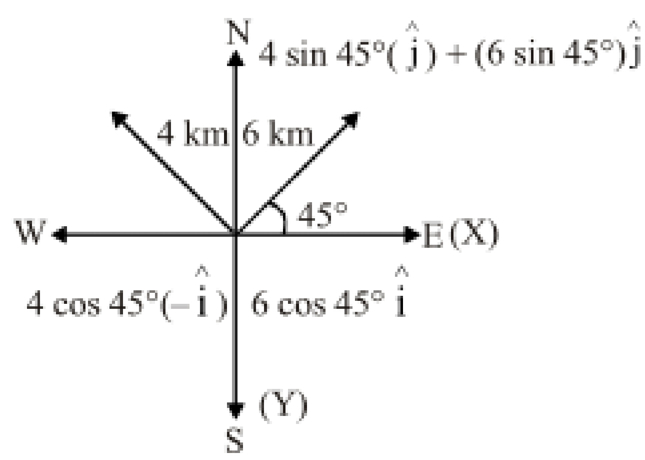
Hence correct answer is (B).

Q.3 A car travels 6 km towards north at an angle of 45° to the east then travels distnace of 4 km towards north at an angle of 135° to the east. How far is its final position due east and due north? How far is the point from the strating point ? What angle does the straight line joining its initial and final position makes with the east ? What is the total distnace travelled by the car ?

Sol. Net movement along X - direction

$$= (6-4)\cos 45°\ \hat{i}$$

$$= 2 \times \frac{1}{\sqrt{2}} = \sqrt{2}\ \text{km}$$



Net movement along Y – direction

$$= (6+4)\sin 45°\ \hat{j}$$

$$= 10 \times \frac{1}{\sqrt{2}} = 5\sqrt{2}\ \text{km}$$

Net movement form starting point (Total distance travelled)

$$= 6 + 4 = 10\ \text{km}$$

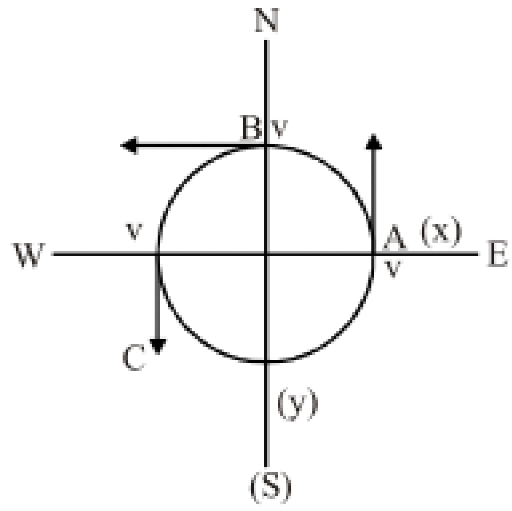
Angle which makes with the east direction

$$\tan\theta = \frac{Y - \text{component}}{X - \text{component}}$$

$$= \frac{5\sqrt{2}}{\sqrt{2}}$$

$$\theta = \tan^{-1}(5)$$

Q.4 A body is moving with uniform speed v on a horizontal circle in anticlockwise direction from A as shown in figure. What is the change in velocity in (a) half revolution (b) first quarter revolution.



Sol. Change in velocity in half revolution

$$\Delta\vec{v} = \vec{v}_C - \vec{v}_A$$

$$= v(-\hat{j}) - v(\hat{j})$$

$$\Delta \vec{v} = -2v\hat{j}$$

$|\Delta \vec{v}| = 2v$ directed towards negative y-axis

change in velocity in first quarter revolution

$$\Delta \vec{v} = \vec{v}_B - \vec{v}_A$$

$$= v(-\hat{i}) - v(\hat{j})$$

$$= -v(\hat{i} + \hat{j})$$

$|\Delta \vec{v}| = \sqrt{2}\ v$ and directed towards south-west.

Q.5 The sum of the magnitudes of two forces acting at a point is 18 and the magnitude of their Resultant is 12. if the resultant is at 90° with the force of smaller magnitude, what are the magnitudes of forces?

(A) 12, 5 (B) 14, 4 (C) 5, 13 (D) 10, 8

Sol. Let P be the smaller force then it is given that

$P + Q = 18$ ....(1)

$R = \sqrt{P^2 + Q^2 + 2PQ\cos\theta} = 12$ ....(2)

$Q \sin \theta / P + Q \cos \theta = \tan \phi = \tan 90° = \infty$

$P + Q \cos \theta = 0$ ....(3)

Substituting the value of P

$Q(1 - \cos \theta) = 18$ ....(4)

and subtracting square of equation (2) from (1)

$2PQ[1 - \cos \theta] = 18^2 - 12^2 = 180$ ....(5)

Dividing equation (5) by (4)

$2P = 10$ i.e. $P = 5$, so $Q = 13$

So the magnitude of forces are (5 and 13)

Hence correct answer is (C)

Q.6 Let $\vec{a} = 2\hat{i} + 3\hat{j} - \hat{k}$; $\vec{b} = -\hat{i} + 3\hat{j} + 4\hat{k}$. Evaluate

(i) $|\vec{a}|$; $|\vec{b}|$

(ii) $\vec{a} \cdot \vec{b}$

(iii) the angle between the vectors $\vec{a}$ and $\vec{b}$

(iv) the projection of $\vec{a}$ on $\vec{b}$

(v) the projection of $\vec{b}$ on $\vec{a}$

(vi) area of the ΔAOB where O is origin

Sol. Give $\vec{a} = 2\hat{i} + 3\hat{j} - \hat{k}$, $\vec{b} = -\hat{i} + 3\hat{j} + 4\hat{k}$

(i) $|\vec{a}| = \sqrt{2^2 + 3^2 + (-1)^2} = \sqrt{4+9+1} = \sqrt{14}$

$|\vec{b}| = \sqrt{(-1)^2 + 3^2 + 4^2} = \sqrt{1+9+16} = \sqrt{26}$

(ii) $\vec{a} \cdot \vec{b} = 2(-1) + 3 \times 3 + (-1)(4) = 3$

(iii) The angle θ between the vectors $\vec{a}$ and $\vec{b}$ is given by

$$\cos\theta = \frac{\vec{a}.\vec{b}}{|\vec{a}||\vec{b}|} = \frac{3}{\sqrt{14}\sqrt{26}} = \frac{3}{2\sqrt{91}}$$

(iv) the projection of $\vec{a}$ on $\vec{b} = |\vec{a}|\cos\theta$

$$\sqrt{14} \times \frac{3}{\sqrt{14}\sqrt{26}} = \frac{3}{\sqrt{26}}$$

(v) The projection of $\vec{b}$ on $\vec{a} = |\vec{b}|\cos\theta$

$$\sqrt{26} \times \frac{3}{\sqrt{14}\sqrt{26}} = \frac{3}{\sqrt{14}}$$

(vi) Area of $\Delta AOB = \frac{1}{2}|\vec{a}||\vec{b}|\sin\theta$

Now $\sin^2\theta = 1 - \cos^2\theta = 1 - \left(\frac{3}{2\sqrt{91}}\right)^2$

$$= 1 - \frac{9}{364} = \frac{355}{364}$$

Area of $\Delta AOB = \frac{1}{2}\sqrt{14}\,\sqrt{26} \cdot \sqrt{\frac{355}{364}}$

$$= \frac{\sqrt{355}}{2} = 9.42 \text{ sq. unit approx.}$$

Q.7 The torque of force $\vec{F} = -3\hat{i} + \hat{j} + 5\hat{k}$ acting at the point $\vec{r} = 7\hat{i} + 3\hat{j} + \hat{k}$ is ______?

(A) $14\hat{i} - 38\hat{j} + 16\hat{k}$ (B) $4\hat{i} + 4\hat{j} + 6\hat{k}$ (C) $-21\hat{i} + 4\hat{j} + 4\hat{k}$ (D) $-14\hat{i} + 34\hat{j} - 16\hat{k}$

Sol. (A) The torque is defined as $\vec{\tau} = \vec{r} \times \vec{F}$

$$\vec{r} \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 7 & 3 & 1 \\ -3 & 1 & 5 \end{vmatrix}$$

$$= \hat{i}\begin{vmatrix} 3 & 1 \\ 1 & 5 \end{vmatrix} + \hat{j}\begin{vmatrix} 1 & 7 \\ 5 & -3 \end{vmatrix} + \hat{k}\begin{vmatrix} 7 & 3 \\ -3 & 1 \end{vmatrix}$$

$$= \hat{i}(15-1) + \hat{j}(-3-35) + \hat{k}(7-(-9))$$

$$= 14\hat{i} - 38\hat{j} + 16\hat{k}$$

Thus the answer is (A)

Q.8 The vectors from origin to the points A and B are $\vec{a} = 3\hat{i} - 6\hat{j} + 2\hat{k}$ and $\vec{b} = 2\hat{i} + \hat{j} - 2\hat{k}$ respectively. Find the area of :

(i) the triangle OAB

(ii) the parallelogram formed by OA and OB as adjacent sides.

Sol. Given $\overrightarrow{OA} = \vec{a} = 3\hat{i} - 6\hat{j} + 2\hat{k}$

and $\overrightarrow{OB} = \vec{b} = 2\hat{i} + \hat{j} - 2\hat{k}$

$$\therefore \quad (\vec{a} \times \vec{b}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & -6 & 2 \\ 2 & 1 & -2 \end{vmatrix}$$

$$= (12-2)\hat{i} - (-6-4)\hat{j} - (3+12)\hat{k}$$

$$= 10\hat{i} + 10\hat{j} + 15\hat{k}$$

$$\Rightarrow |\vec{a} \times \vec{b}| = \sqrt{10^2 + 10^2 + 15^2} = \sqrt{425} = 5\sqrt{17}$$

(i) Area of $\Delta OAB = \frac{1}{2}|\vec{a} \times \vec{b}| = \frac{1}{2}.5\sqrt{17}$ sq. units

$= \frac{5}{2}\sqrt{17}$ sq. units

(ii) Area of parallelogram fromed by OA and OB as adjacent sides $= |\vec{a} \times \vec{b}| = 5\sqrt{17}$ sq. units.

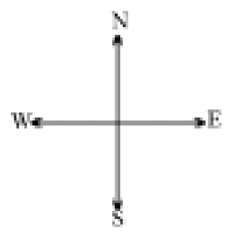
Q.9 A buoy is attached to three tugboats by three ropes. The tugboats are engaged in a tug–of–war. One tugboat pulls west on the buoy with a force $\vec{F}_1$ of magnitude 1000 N. The second tugboat pulls south on the buoy with a force $\vec{F}_2$ of magnitude 2000 N. The third tugboat pulls northeast (that is, half way between north and east), with a force $\vec{F}_3$ of magnitude 2000 N.

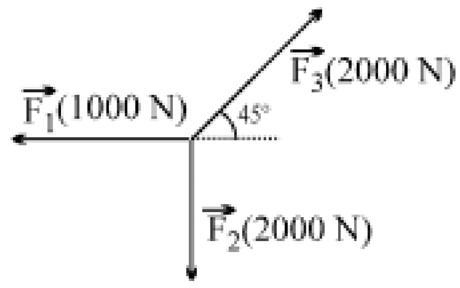
(a) Draw a diagram of forces acting on the buoy to represent this situation.

(b) Express each force in unit vector form ($\hat{i}$, $\hat{j}$).

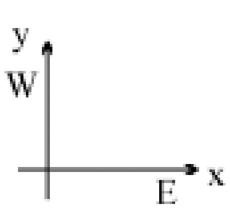
(c) Calculate the magnitude of the resultant force.

Sol. (a)





(b)



$$\vec{F}_1 = -1000\,\hat{i}\ (\text{N}),$$

$$\vec{F}_2 = -2000\ \hat{j}\,(\text{N}),$$

$$\vec{F}_3 = 2000(\cos45°\ \hat{i} + \sin45°\,\hat{j})\,(\text{N}) = 1000\sqrt{2}\ \hat{i} + 1000\sqrt{2}\ \hat{j}$$

(c) $$\vec{F}_{\text{resultant}} = \vec{F}_1 + \vec{F}_2 + \vec{F}_3 = 1000(\sqrt{2}-1)\hat{i} - 1000\ (2-\sqrt{2})\hat{j}\ \ \text{N}$$

$$F_x = 1000\ (\sqrt{2} - 1)\ \text{N}$$

$$F_y = -1000(2 - \sqrt{2}\,)\ \text{N}$$

$$|\vec{F}| = \sqrt{F_x{}^2 + F_y{}^2} = 1000(\sqrt{2} - 1) \times \sqrt{3} = 1000\ (\sqrt{2} - 1)\ \sqrt{3}\ \ \text{N}$$

Trigonometry

Angle

The angle is defined as the amount of revolution that the revolving line makes with its initial position. From θ is positive if it is traced by revolving line in anticlockwise direction and is negative if it is covered in clockwise direction.

$1° = 60'$ (minute), $1'$ (min) $= 60''$ (sec)

1 right angle = 90° (degrees) also 1 right angle = $\frac{\pi}{2}$ rad (radian)

One radian is the angle subtended at the centre of a circle, whose length is equal to the radius of the circle.

$$1 \text{ rad} = \frac{180°}{\pi} \approx 57°\ 17'\ 45'' \approx 57.3°$$

$$\pi = \left(\frac{22}{7}\right)$$

Trigonometric ratios (or T ratios)

Let two fixed lines XOX' and YOY' intersecting at right angles to each other at point O. Then

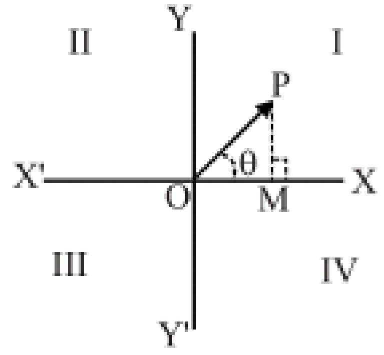
(i) Point O is called origin.

(ii) XOX' known as X-axis and YOY' are Y-axis.

(iii) Portions XOY, YOX', Y'OX' and XOY' are called I, II, III and IV quadrant respectively.

Consider that the revolving line OP has traced out angle θ.

(in I quadrant) in anticlockwise direction. From P, perpendicular PM on OX. Then, side OP (in front of right angle) is called hypotenuse, side MP (in front of angle θ) is called opposite side or perpendicular and side OM (making angle θ with hypotenuse) is called adjacent side or base.



The three sides of a right angled triangle are connected to each other through six different ratios, called trigonometric ratios or simply T-ratios.

$$\sin\theta = \frac{\text{perpendicular}}{\text{hypotenuse}} = \frac{MP}{OP} \qquad \cos\theta = \frac{\text{base}}{\text{hypotenuse}} = \frac{OM}{OP}$$

$$\tan\theta = \frac{\text{perpendicular}}{\text{base}} = \frac{MP}{OM} \qquad \cot\theta = \frac{\text{base}}{\text{perpendicular}} = \frac{OM}{MP}$$

$$\sec\theta = \frac{\text{hypotenuse}}{\text{base}} = \frac{OP}{OM} \qquad \operatorname{cosec}\theta = \frac{\text{hypotenuse}}{\text{perpendicular}} = \frac{OP}{MP}$$

It can be easily proved that

$$\text{cosec}\theta = \frac{1}{\sin\theta} \qquad \sec\theta = \frac{1}{\cos\theta} \qquad \cot\theta = \frac{1}{\tan\theta}$$

$$\sin^2\theta + \cos^2\theta = 1 \qquad 1 + \tan^2\theta + \sec^2\theta \qquad 1 + \cot^2\theta = \text{cosec}^2\theta$$

Illustration :

Given $\sin\theta = \frac{3}{5}$. Find all the other T-ratios, if θ lies in the first quadrant.

Sol. *In ΔOMP, $\sin\theta = \frac{3}{5}$* *So* *$MP = 3$ and $OP = 5$*

$$\because OM = \sqrt{(5)^2 - (3)^2} = \sqrt{25-9} = \sqrt{16} = 4$$

Now, $\cos\theta = \frac{OM}{OP} = \frac{4}{5}$ $\quad \tan\theta = \frac{MP}{OM} = \frac{3}{4}$ $\quad \cot\theta = \frac{OM}{MP} = \frac{4}{3}$

$\sec\theta = \frac{OP}{OM} = \frac{5}{4}$ $\quad \text{cosec}\,\theta = \frac{OP}{MP} = \frac{5}{3}$

The T-ratios of a few standard angles ranging from 0° to 180°

Angle (θ)	0°	30°	45°	60°	90°	120°	135°	150°	180°
$\sin\theta$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{1}{\sqrt{2}}$	$-\frac{\sqrt{3}}{2}$	–1
$\tan\theta$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	∞	$-\sqrt{3}$	–1	$-\frac{1}{\sqrt{3}}$	0

$$\boxed{\begin{aligned}\sin(90°-\theta) &= \cos\theta\\ \cos(90°-\theta) &= \sin\theta\\ \tan(90°-\theta) &= \cot\theta\end{aligned}} \quad \boxed{\begin{aligned}\sin(90°+\theta) &= \cos\theta\\ \cos(90°+\theta) &= -\sin\theta\\ \tan(90°+\theta) &= -\cot\theta\end{aligned}} \quad \boxed{\begin{aligned}\sin(-\theta) &= -\sin\theta\\ \cos(-\theta) &= \cos\theta\\ \tan(-\theta) &= -\tan\theta\end{aligned}} \quad \boxed{\begin{aligned}\sin(180°-\theta) &= \sin\theta\\ \cos(180°-\theta) &= \cos\theta\\ \tan(180°-\theta) &= \tan\theta\end{aligned}}$$

$$\boxed{\begin{aligned}\sin(180°+\theta) &= -\sin\theta\\ \cos(180°+\theta) &= -\cos\theta\\ \tan(180°+\theta) &= \tan\theta\end{aligned}} \quad \boxed{\begin{aligned}\sin(270°-\theta) &= -\cos\theta\\ \cos(270°-\theta) &= -\sin\theta\\ \tan(270°-\theta) &= \cot\theta\end{aligned}} \quad \boxed{\begin{aligned}\sin(270°+\theta) &= -\cos\theta\\ \cos(270°+\theta) &= \sin\theta\\ \tan(270°+\theta) &= -\cot\theta\end{aligned}} \quad \boxed{\begin{aligned}\sin(360°-\theta) &= -\sin\theta\\ \cos(360°-\theta) &= \cos\theta\\ \tan(360°-\theta) &= -\tan\theta\end{aligned}}$$

Illustration :

Find the value of

(i) $\cos(-60°)$ *(ii)* $\tan 210°$ *(iii)* $\sin 300°$ *(iv)* $\cos 120°$

Sol. *(i)* $\cos(-60°) = \cos 60° = \frac{1}{2}$ *(ii)* $\tan 210° = (\tan 180° + 30°) = \tan 30° = \frac{1}{\sqrt{3}}$

(iii) $\sin 300° = \sin(270° + 30°) = -\cos 30° = -\frac{\sqrt{3}}{2}$

(iv) $\cos 120° = \sin(180° - 60°) = -\cos 60° = -\frac{1}{2}$

A few important trigonometric formulae

$\sin(A+B) = \sin A\cos B + \cos A\sin B$ $\sin(A-B) = \sin A\cos B - \cos A\sin B$	$\cos(A+B) = \cos A\cos B - \sin A\sin B$ $\cos(A-B) = \cos A\cos B + \sin A\sin B$
$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A\tan B}$	$\tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A\tan B}$
$\sin 2A = 2\sin A\cos A$	$\cos 2A = \cos^2 A - \sin^2 A$
$\tan 2A = \frac{2\tan A}{1 - \tan^2 A}$	$\cos 2A = 2\cos^2 A - 1 = 1 - 2\sin^2 A$

Small angle approximation :

θ is very small and it must be in radian when you are taking approximation.

$\sin\theta \simeq \theta$, $\tan\theta \simeq \theta$

$\sin\theta \simeq \tan\theta$.

$\cos\theta \simeq 1$

Illustration :

Evaluate $\sin 2°$

Sol. $2° = 2 \times \frac{\pi}{180}\text{ rad} = \frac{\pi}{90}\text{ rad}$

Now

$$\sin 2° = \sin\left(\frac{\pi}{90}\text{rad}\right) \simeq \left(\frac{\pi}{90}\right)$$

Illustration :

Evaluate $\sin 2^\circ (1 - \cos 2^\circ)$

Sol. $\sin 2^\circ (1 - 1 + 2\sin^2 1^\circ)$

$$2\sin 2^\circ \sin^2 1^\circ \simeq 2\left(2\times\frac{\pi}{180^\circ}\right)\left(\frac{\pi}{180^\circ}\right)^2$$

$$= 4\left(\frac{\pi}{180^\circ}\right)^3$$

Practice Exercise

Q.1 Find the value of

(i) $\cos(-30^\circ)$ (ii) $\sin 120^\circ$ (iii) $\sin 135^\circ$ (iv) $\cos 120^\circ$ (v) $\sin 270^\circ$ (vi) $\cos 270^\circ$

Ans. (i) $\frac{1}{\sqrt{3}}$ (ii) $\frac{\sqrt{3}}{2}$ (iii) $\frac{1}{\sqrt{2}}$ (iv) $-\frac{\sqrt{3}}{2}$ (v) -1 (vi) 0

CALCULUS

Introduction:

Suppose on your 8th birthday at midnight you measured your height and found it to be 120 cm. One year later at midnight on your Eight birthday you again measure your height & find it to be 132 cm. Now if we say rate of increase of your height is 12 cm/year it does not mean that at the exact 11:59:59 PM you were 120 cm and as soon as clock moved to 12:00 midnight you streched and became 132 cm.
Now if we express your rate of growth as 1 cm/month we are not implying that every month at midnight of 1st you suddenly strech by 1 cm/month is just a unit and the growth is a continuous process. If we assume you grow uniformly at all times the rate simply implies that if you grow Δh in time Δt at some point of time, the ratio $\left(\frac{\Delta h}{\Delta t}\right)$ will be 12 cm/year or 1 cm/month or whatever unit you may choose.

So we can calculate the rate of growth of height in a finite time internal. If we calculate rate of growth $\left(\frac{\Delta h}{\Delta t}\right)$ in different time interval like

(i) Rate of growth per day $\left(\frac{\Delta h}{\Delta t}\right) = \frac{1}{30}\,\text{cm / day}$

(ii) Rate of growth per hour $\left(\frac{\Delta h}{\Delta t}\right) = \frac{1}{30 \times 24}\,\text{cm / hour}$

(iii) Rate of growth per min $\left(\frac{\Delta h}{\Delta t}\right) = \frac{1}{30 \times 24 \times 60}\,\text{cm / min}$

Similarly if we want to calculate the rate of growth exactly at 12 O'clock mid night then we take time interval Δt is zero. So in that time Δh is also zero becasue nothing can change in zero time. So, rate of growth becomes $\frac{0}{0}$, which is meaningless.

Now the question is how two find the rate of growth exactly at 12 O'clock or at any instant of time.

Differential calculus play an important role to find the rate of growth at any instant. The rate of growth at any instant is known as instantaneous rate. In many situations in physics, it is sometimes necessary to use the concept of instantaneous rate. The basic tool for this is calculus, invented by **Newton and Liebnitz** independently. The use of calculus is fundamental in the treatment of various problems in physics.

Numerical interpretation:

Illustration :

A car moving on a horizontal road whose position changes with time t as $x = 3t^2 + 1$ compute its average speed (average rate of change in position) $\left(v_{avg} = \frac{\text{change in position}}{\text{time interval}} = \frac{\Delta x}{\Delta t}\right)$ *between*

(i) From 2 sec to 3 sec *(ii) From 2 sec to 2.1 sec* *(iii) From 2 sec to 2.001 sec*
(iv) From 2 sec to 2.0001 sec *(v) at 2 sec (instantaneous rate of change)*

Sol. For $t = 2$ sec to 3 sec, we have $\Delta t = 1sec$

$x(\text{at } t = 3 \text{ sec.}) = 3(3)^2 + 1 = 28\ m$

$x(\text{at } t = 2 \text{ sec.}) = 3\ (2)^2 + 1 = 13\ m$

$\Delta x = 28\ m - 13\ m = 15m$

Thus $v_{average} = \dfrac{\Delta x}{\Delta t} = 15\ m/1sec = 15\ ms^{-1}$

(ii) 2 sec and 2.1 sec

Sol. For $t = 2.1$ sec, we have $\Delta t = 0.1$ sec

$x = 3\ (2.\ 1)^2 + 1 = 14.23\ m$ and $\Delta x = 1.23\ m$

Thus $v_{average} = \dfrac{\Delta x}{\Delta t} = 1.23\ \ m/0.1\ sec = 12.3\ ms^{-1}$

(iii) 2 sec and 2.001 sec,

Sol. for $t = 2.001$ sec, we have $\Delta t = 0.001$ sec

$x = 3(2.001)^2 + 1 = 13.012003\ m$ and $\Delta x = 0.012003\ m$

Thus $v_{average} = \dfrac{\Delta x}{\Delta t} = 0.012003\ m/0.001\ sec = 12.003\ ms^{-1}$

(iv) 2 sec and 2.00001 sec.

Sol. The student may verify that for $t = 2.00001$ sec

$v_{avg} = 12.00003\ ms^{-1}$

(v) Also find instantaneous rate of change in position (instantaneous speed) at 2 sec.

$$v_{ins} = \frac{x_2 - x_1}{t_2 - t_1}$$

here $t_1 = 2$ sec. and also $t_2 = 2$ sec.

$$v_{ins} = \frac{3(2)^2 + 1 - 3(2)^2 - 1}{2-2} = \frac{0}{0},\ \text{It is undefined.}$$

So Liebnitz suggest try it like that

$t_1 = 2$ sec., $t_2 = 2 + \Delta t$

$$v_{avg} = \frac{3(2+\Delta t)^2 + 1 - 3(2)^2 - 1}{2 + \Delta t - 2} = \frac{\Delta t^2 + 12\Delta 2}{\Delta t} = \Delta t + 12$$

Then he said let's make $\Delta t \to 0$, means Δt approaches zero not equal to zero. So, t_2 almost becomes 2 sec. and the v_{avg} becomes v_{ins}.

Liebnitz conclude that althrough the method is approximate but result is exact.

In calculus notation

$$v_{ins} = \lim_{\Delta t \to 0} v_{avg}$$

$$v_{ins} = \lim_{\Delta t \to 0} (\Delta t + 12) = 12m/s$$

$$\because \quad v_{avg} = \frac{\Delta x}{\Delta t}$$

$$v_{inst} = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} = \frac{dx}{dt}\ \text{(it is knows as derivative of x with respect to t)}$$

Illustration :

Suppose displacement y (t) (that is, y as a function of t) is given by

$$y(t) = t^3$$

Sol. *Find velocity (dy/dt) at t = 3 sec.*

Displacement at $t + \Delta t$ *is*

$$y(t + \Delta t) = (t + \Delta t)^3$$

$$= (t^3 + 3t^2\,\Delta t + 3t\,\Delta t^2 + \Delta t^3)$$

hence displacement from t *to* $t + \Delta t$ *is* Δy

$$\Delta y = y(t + \Delta t) - y(t) = (3t^2\,\Delta t + 3t\,\Delta t^2 + \Delta t^3)$$

Substituting this into Equation (i) gives

$$\frac{dy}{dt} = \lim_{\Delta t \to 0} \frac{\Delta y}{\Delta t} = \lim_{\Delta t \to 0} [3t^2 + 3t\,\Delta t + \Delta t^2]$$

Here we can see that if we take very small value of Δt *then value of* dy/dt *will approach* $3t^2$ *as all other terms will become negligible and impossible to measure by any instrument available in this world.*

hence, $\dfrac{dy}{dt} = 3t^2 \quad \Rightarrow \quad \dfrac{dy}{dt} = v\ (at\ t = 3\ sec) = 3(3)^2 = 27\ m/s$

Leibneiz derived some general formula for differentiation which is written below.

Differentiation Formulae

1. **Power rule :** $\dfrac{d}{dt}(t^n) = nt^{n-1}$
2. $\dfrac{d}{dt}(t) = 1$
3. $\dfrac{d}{dt}(\sin t) = \cos t$
4. $\dfrac{d}{dt}(\cos t) = -\sin t$
5. $\dfrac{d}{dt}(e^t) = e^t$
6. $\dfrac{d}{dt}(\log_e t) = \dfrac{1}{t}$; $[\log_e t \equiv \ln t]$
7. $\dfrac{d}{dt}(\tan t) = \sec^2 t$
8. $\dfrac{d}{dt}(\sec t) = \sec t \tan t$

Differentiation Rules : u and v are functions of t i.e. $u = f(t)$ and $v = g(t)$

1. $\dfrac{d}{dt}(c) = 0$ ***where c : constant***
2. ***Constant multiple rule:*** $\dfrac{d}{dt}(cu) = c\dfrac{du}{dt}$
3. ***Sum rule:*** $\dfrac{d}{dt}(u + v) = \dfrac{du}{dt} + \dfrac{dv}{dt}$
4. ***Difference rule:*** $\dfrac{d}{dt}(u - v) = \dfrac{du}{dt} - \dfrac{dv}{dt}$
5. ***Product rule :*** $\dfrac{d}{dt}(uv) = u\dfrac{dv}{dt} + v\dfrac{du}{dt}$
6. ***Quotient rule:*** $\dfrac{d}{dt}\left(\dfrac{u}{v}\right) = \dfrac{v\dfrac{du}{dt} - u\dfrac{dv}{dt}}{v^2}$

Example : Differentiate w.r.t. time.

(i) $y = t^2$ (ii) $x = t^{3/2}$ (iii) $y = \dfrac{1}{\sqrt{t}}$

(iv) $x = 4t^3$ (v) $y = 2\sqrt{t}$ (vi) $y = 2t^2 + t - 1$

(vii) $y = 3\sqrt{t} + \dfrac{2}{\sqrt{t}}$ (viii) $y = t^3 \sin t$ (ix) $x = te^t$

(x) $x = \sqrt{t}\,(1 - t)$

Sol. (i) $\dfrac{dy}{dt} = 2t$ (ii) $\dfrac{dx}{dt} = \dfrac{3}{2}t^{1/2}$ (iii) $\dfrac{dy}{dt} = -\dfrac{1}{2}t^{-3/2}$

(iv) $\dfrac{dx}{dt} = 12t^2$ (v) $\dfrac{dy}{dt} = 2\left(\dfrac{1}{2}t^{-1/2}\right) = t^{-1/2}$

(vi) $\dfrac{dy}{dt} = 4t + 1 + 0 = 4t + 1$ (vii) $\dfrac{dy}{dt} = 3t^{-1/2} + 2\left(-\dfrac{1}{2}t^{-3/2}\right)$

$$= \frac{3}{\sqrt{t}} - t^{-3/2}$$

(viii) $\dfrac{dy}{dt} = 3t^2 \sin t + t^3 \cos t$ (ix) $\dfrac{dx}{dt} = e^t + te^t = e^t(1 + t)$

(x) $\dfrac{dx}{dt} = \dfrac{1}{2}t^{-1/2} - \dfrac{3}{2}t^{1/2} = \dfrac{1}{2\sqrt{t}} - \dfrac{3}{2}\sqrt{t}$

Example : Differentiate with respect to x

(i) $y = x\,\ell n\,x$ (ii) $y = x^2 e^x$ (iii) $y = \dfrac{\sin x}{x}$ (iv) $y = \dfrac{3x^2 + 2\sqrt{x}}{x}$

Sol. (i) $\dfrac{dy}{dx} = x\left(\dfrac{1}{x}\right) + \ell n\,x = 1 + \ell n\,x$ (ii) $\dfrac{dy}{dx} = 2xe^x + x^2e^x = xe^x(2 + x)$

(iii) $y = x^{-1} \sin x$

$$\frac{dy}{dx} = \left(-x^{-2}\right)\sin x + x^{-1}\cos x$$

(iv) $y = 3x + 2x^{-1/2}$

$$\frac{dy}{dx} = 3 + 2\left(-\tfrac{1}{2}\,x^{-3/2}\right) = 3 - x^{-3/2}$$

The Chain Rule :

Suppose you are asked to differentiate the function

$$y = \sqrt{x^2 + 1}$$

The differentiation formula you learned in the previous section of this chapter do not enable you to calculate $\dfrac{dy}{dx}$

In fact, if we let $y = \sqrt{u}$ and let $u = x^2 + 1$,

Then we can evaluate $\frac{dy}{du}, \frac{du}{dx}$ easily using the formula that we have learned in previous section.

$$\frac{dy}{du} = \frac{1}{2\sqrt{u}} \qquad \frac{du}{dx} = 2x$$

but our aim is to calculate $\frac{dy}{dx}$ so we can write.

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

and its value will become

$$\frac{dy}{dx} = \frac{1}{2\sqrt{u}} \cdot 2x$$

Substituting value of 'u' we will get

$$\frac{dy}{dx} = \frac{1}{2\sqrt{x^2+1}} \cdot 2x = \frac{x}{\sqrt{x^2+1}}$$

Ex.1 $x = (at+b)^n$ where a, b and n are a real number then $\frac{dx}{dt} = an(at+b)^{n-1}$.

Sol. $u = at + b$ and $x = u^n$

$$\frac{dx}{du} = n\,u^{n-1}$$

$$\frac{du}{dt} = a$$

$$\frac{dx}{dt} = \frac{dx}{du} \times \frac{du}{dt} = n\,(u)^{n-1} \times a = an\,(at+b)^{n-1}$$

Ex.3 It $x = \sin^2\theta$, then find $\frac{dx}{dt}$ where $\frac{d\theta}{dt} = \omega$

Sol. $\frac{dx}{d\theta} = 2\,(\sin\theta)\,(\cos\theta) = \sin 2\theta$

$$\frac{dx}{dt} = \frac{dx}{d\theta} \cdot \frac{d\theta}{dt}$$

$$\frac{dx}{dt} = \sin 2\theta \cdot \omega$$

Application in physics :

velocity $(v) = \frac{dx}{dt}$, acceleration $(a) = \frac{dv}{dt}$; $\left(a = \frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = \frac{dv}{dx}v = v\frac{dv}{dx}\right)$

Force (F) = rate of change of momentum $= \frac{dp}{dt}$;

Current $(i) = \frac{dq}{dt}$; angular speed $(\omega) = \frac{d\theta}{dt}$;

angular acceleration $(\alpha) = \frac{d\omega}{dt}$; $\left(\alpha = \frac{d\omega}{dt} = \frac{d\omega}{d\theta}\frac{d\theta}{dt} = \omega\frac{d\omega}{d\theta}\right)$

Ex. The radius of a circle is increasing at a rate $\frac{dr}{dt} = \alpha$. Find the rate at which its area is increasing when radius is equal to 3 m.

Sol area of cricle (A) = πr^2

$$\frac{dA}{dt} = \pi\,(2r)\,\frac{dr}{dt} = 2\,\pi\,r\left(\frac{dr}{dt}\right) = 2\,\pi\,r\,(\alpha)$$

$$\left(\frac{dA}{dt}\right)_{\text{at } r=3m} = 6\,\pi\,\alpha\ \ m^2/\text{sec.}$$

DERIVATIVE OF A VECTOR :

$$\vec{v} = \frac{d\vec{r}}{dt}, \vec{a} = \frac{d\vec{v}}{dt}\ ;\ \vec{F} = \frac{d\vec{p}}{dt}\ ;\ \vec{\alpha} = \frac{d\vec{\omega}}{dt}$$

Ex. $\vec{r} = 2t\hat{i} + 3t^2\hat{j}$. Find $\vec{v}$ and $\vec{a}$ where $\vec{v} = \frac{d\vec{r}}{dt}$, $\vec{a} = \frac{d\vec{v}}{dt}$

Sol. $\vec{v} = \frac{d\vec{r}}{dt} = 2\hat{i} + 6t\hat{j}$ m/s

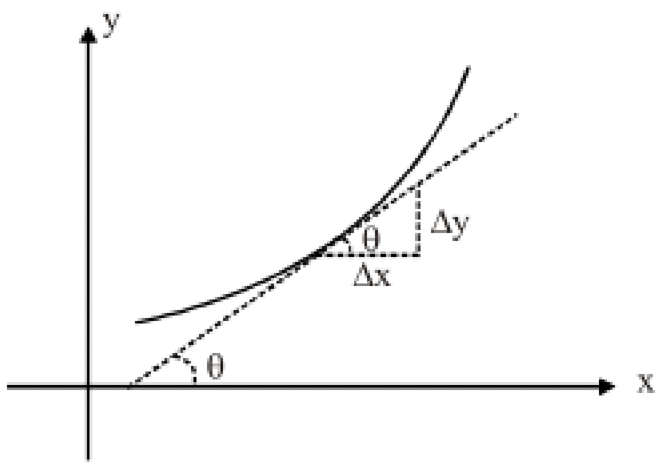
$\vec{a} = \frac{d\vec{v}}{dt} = 0 + 6\hat{j}$ m/s^2

Physical meaning of $\frac{dy}{dx}$

1. The ratio of small change in the function y and the varibalbe x is called the average rate of change of y w.r.t.x. For example, if a body cover a small distance Δs in small time Δ t, then average velocity of the body, $v_{av} = \frac{\Delta s}{\Delta t}$ Also, if the velocity of a body changes by a small amount Δv in small time Δt, then average acceleration of the body, $a_{av} = \frac{\Delta v}{\Delta t}$

2. When $\Delta x \to 0$ The limiting value of $\frac{dy}{dx}$ is $\lim_{\Delta x \to 0}\frac{\Delta y}{\Delta x} = \frac{dy}{dx} = \tan\theta$

= slope of the tangent



It is called the instantaneous rate change of y w.r.t.x.
The differentiation of a function w.r.t. a veriable imples the instantaneous rate change of the funcaiton w.r.t. that variable.

Like wise, instantaneous velocity of the body, $(v) = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} = \frac{ds}{dt}$

and instantaneous acceleration of the body $(a) = \lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t} = \frac{dv}{dt}$

Maxima & Minima :

Maxima & minima of a function $y = f(x)$

for maximum value $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} = \text{negative}$

for minimum value $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} = \text{positive}$

Ex. Find minimum value of $y = 25x^2 - 10x + 5$.

Sol. For maximum / minimum value $\frac{dy}{dx} = 0 \Rightarrow 50x - 10 \Rightarrow x = \frac{1}{5}$

Now at $x = \frac{1}{5}$, $\frac{d^2y}{dx^2} = 50$, which is positive

So $y_{min} = 25\left(\frac{1}{5}\right)^2 - 10\left(\frac{1}{5}\right) + 5 = 1 - 2 + 5 = 4$

Ex. A body is moving vertically upwards under gravity such that its position from ground is given as $y = ut - \frac{1}{2}gt^2$. Find the max height reached by body.

Sol. $\frac{dy}{dt} = u - gt = 0$, $t = \frac{u}{g}$

$\frac{d^2y}{dx^2} = -g < 0$ (which is negative). so we get maximum height y_{max} at $t = \frac{u}{g}$ sec.

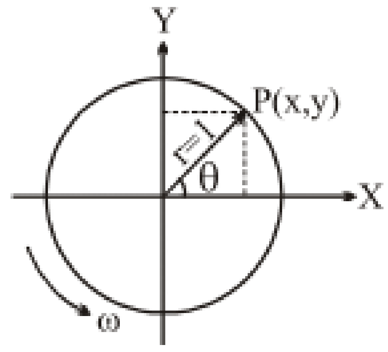
$$y_{max} = u\left(\frac{u}{g}\right) - \frac{1}{2}g\left(\frac{u}{g}\right)^2$$

$$= \frac{u^2}{2g}$$

Equation of Trajectory :

It is path traversed by a particle, independent of time parameter.

Ex. Find the equation of trajectory for the particle moving i n circular path as shown.



$x = r\cos\theta \qquad y = r\sin\theta \qquad x^2 + y^2 = r^2 \quad \Rightarrow \quad x^2 + y^2 = 1$

Integration or Antiderivative :

So far we have studied as how to find velocity of a particle when its position is given. Now if we know the velocity of a particle and we might wish to know its position at any given time or an engineer who can measure the variable rate at which water is leaking from a tank wants to know the amount leaked over a certain time period. This reverse process is called antiderivative of integration.

There are two types of integration :

(a) Indefinite integration **(b)** Definite integration.

(a) Indefinite integration

Since we know that , $\Delta t = t_2 - t_1$

Now, when time t_2 approaches t_1 ; $\Delta t \to dt$

To remind you, $v = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} = \frac{dx}{dt}$

So, If we integrate dt, we should get $t_2 - t_1$; i.e. $\int dt = t_2 - t_1$

Similarly, $\int dx = x_2 - x_1$

but since the differentiation of a constant becomes zero, so i.e. why we can not retrive the value of that constant.

$\therefore$ We should write $\int dx = x + c$.

Where 'c' is integration constant.

Integration Formulae :

1. $\int x^n dx = \frac{x^{n+1}}{n+1} + c \ [n \neq -1]$
2. $\int (ax+b)^n dx = \frac{1}{a}\frac{(ax+b)^{n+1}}{n+1} + c$
3. $\int \cos x \, dx = \sin x + c$
4. $\int \sin x \, dx = -\cos x + c$

5. $\int e^x dx = e^x + c$

6. $\int \frac{1}{x} dx = \ln|x| + c$, $[x \neq 0]$

7. $\int \cos(ax+b)dx = \frac{1}{a}\sin(ax+b) + c$

8. $\int \sin(ax+b)dx = -\frac{1}{a}\cos(ax+b) + c$

9. $\int e^{(ax+b)}dx = \frac{1}{a}e^{(ax+b)} + c$

10. $\int \frac{1}{(ax+b)} dx = \frac{1}{a} ln|(ax+b)| + c$

Rules of Integration :

1. $\int k\, u\, dx = k \int u\, dx$

   where is k constant.

2. $\int (u+v)dx = \int u dx + \int v dx$

Ex.1 Evaluate indefinite integration.

(i) $x = \int dt$ (ii) $x = \int t\, dt$ (iii) $x = \int (2t)\, dt$ (iv) $x = \int (t^2)\, dt$

(v) $x = \int \left(-\frac{2}{t^3}\right) dt$

Sol. (i) $x = t + c$ (ii) $x = \frac{t^2}{2} + c$ (iii) $x = \frac{2t^2}{2} + c = t^2 + c$

(iv) $x = \frac{t^3}{3} + c$ (v) $x = -2\int t^{-3}dt \Rightarrow x = -2\left(\frac{t^{-2}}{-2}\right) + c = t^{-2} + c$

Where 'c' is integration constant.

(b) Definite integration.

When a function is integrated between a lower limit and an upper limit, it is called a definite integral.

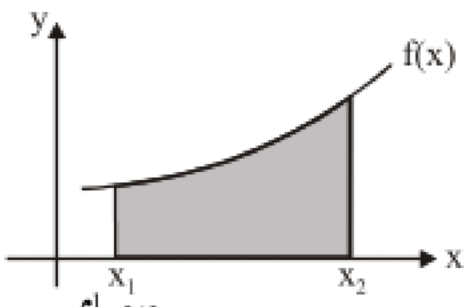
If $\frac{d}{dx}(f(x)) = g(x)$

then $\int g(x)dx$ is called indefinite integral and $\int_a^b g(x)dx = [f(b) - f(a)]$ is called definite integral.

Here, a and b are called lower and upper limits of the variable x.

After carrying out integration, the result is evaluated between upper and lower limits as explained below :

$\int_{x_1}^{x_2} g(x)dx = f(x_2) - f(x_1) =$ Area under the curve.



Ex. Evaluate the integral :

(i) $\int_1^5 x^2 dx$ (ii) $\int_0^1 t^2 dt$ (iii) $\int_3^5 t\, dt$ (iv) $\int_0^1 t^{3/2}\, dx$

Sol. (i) $\int_1^5 x^2 dx = \left[\frac{x^3}{3}\right]_1^5 = \frac{1}{3}\left[x^3\right]_1^5 = \frac{1}{3}\left((5)^3 - (1)^3\right) = \frac{1}{3}\ (125 - 1) = \frac{124}{3}$

(ii) $\int_0^1 t^2 dt = \left[\frac{t^3}{3}\right]_0^1 = \frac{1}{3}$ **(iii)** $\int_3^5 t\, dt = \left[\frac{t^2}{2}\right]_3^5 = \frac{5^2 - 3^2}{2} = 8$

(iv) $\int_0^1 t^{3/2} dx = \left[\frac{t^{5/2}}{5/2}\right]_0^1 = \frac{2}{5}$

Practise Exercise :

(i) $\int_R^{\infty} \frac{GMm}{x^2} dx$ (ii) $\int_u^v Mv dv$ (iii) $\int_0^{\pi/2} \cos x\, dx$

Ans. (i) $\frac{GMm}{R}$ (ii) $\frac{1}{2}M(v^2 - u^2)$ (iii) 1

Ex.3 Find displacement of a particle in 1-D if its velocity is $v = (2t - 5)$ m/s, from $t = 0$ to $t = 4$ sec.

Sol. $\frac{dx}{dt} = 2t - 5$

$\int_{x_1}^{x_2} dx = \int_0^4\ (2t - 5)\ dt$

$x_2 - x_1 = \left(t^2 - 5t\right)_0^4$

displacement $= 16 - 20 = -4$ m

Graphical Interpretation :

Since $\frac{dx}{dt} = v \ \Rightarrow cdx = v\ dt \quad \int dx = \int v\ dt$

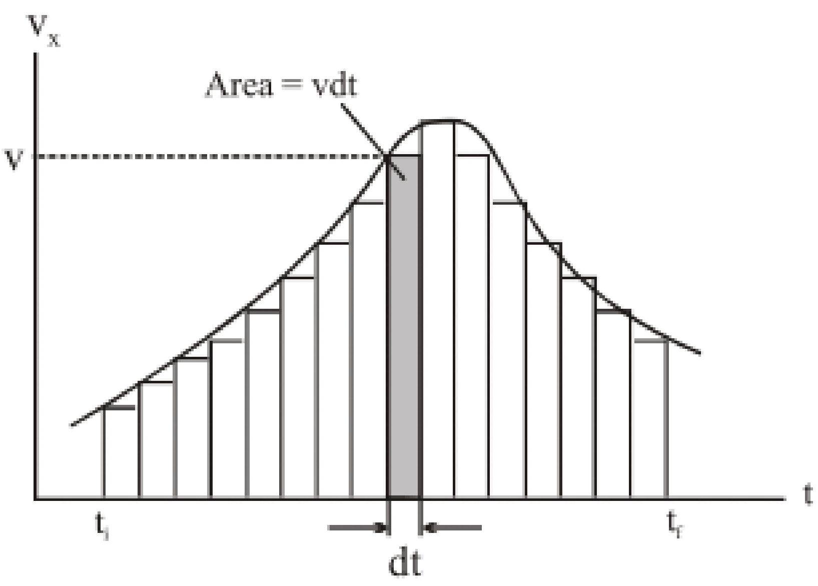
$\therefore\ x_2 - x_1 = \int v\ dt$ ..... (1)

Now, we have to understand the meaning of $\int v\ dt$

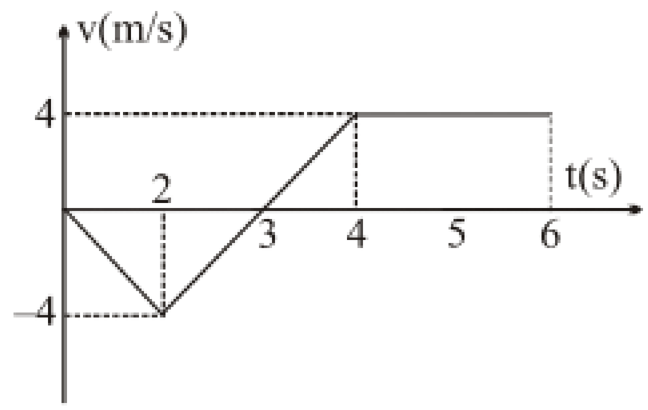
Graphically, it is equivalent to finding the area under a curve. Suppose the $v - t$ graph for a particle is as shown in Fig. given below. We want to find the displacement for time interval $t_i - t_f$. Let us divide the time interval $t_i - t_f$ into many small intervals, each of duration Δt and if $\Delta t \to dt$. We can find the displacement 'dx' of the particle during any small interval, such as the one shaded in figure given below, by $dx = v\,dt$, where 'v' is the velocity in that interval. Therefore, the displacement during this small interval is simply the area of the shaded rectangle. The total displacement for the interval t_i - t_f is the sum of the areas of all the rectangles from t_i to t_f. But by adding up it means integrating.

Thus, Area under curve $= \int v\ dt$ ... (2)

Hence from (1) & (2), Area under curve $= x_2 - x_1 =$ displacement.



Ex.1 From the v versus t graph of figure (a) the time(s) at which the particle is at rest (b) at what time, if any does the particle reverse the direction of its motion? (c) The distance and displacement of the particle from t = 0 s to t = 6s.



Sol. (a) t = 0, 3 sec (b) t = 3 sec

(c) displacement $= -\frac{1}{2}(3\times 4) + \frac{1}{2}(3+2)(4)$

$= -6 + 10 = 4$ m

distance = $\left|-\frac{1}{2}(3\times 4)\right| + \left|\frac{1}{2}(3+2)(4)\right|$

$= 6 + 10 = 16$ m

Finding position and trajectory if velocity is known:

Lets take velocity vector $\vec{v} = v_x\hat{i} + v_y\hat{j} + v_z\hat{k}$

$$\Rightarrow v_x = \frac{dx}{dt};\ v_y = \frac{dy}{dt};\ y_z = \frac{dz}{dt}$$

$$\therefore \quad x = \int v_x dt\ ;\ y = \int v_y dt \qquad z = \int v_z dt$$

Ex.1 Find the equation of trajectory of a particle whose velocity components are
$v_x = 2x + 1$, $v_y = 2y + 3$
Given that particle starts from rest from origin.

Sol. $v_x = 2x + 1,$

$$\frac{dx}{dt} = 2x+1$$

$$\int_0^x \frac{dx}{2x+1} = \int_0^t dt$$

$$\frac{1}{2}\ln(2x+1) = t \qquad \text{.... (1)}$$

$$v_y = 2y + 3$$

$$\int_0^y \frac{dy}{2y+3} = \int_0^t dt$$

$$\frac{1}{2}\ln\frac{(2y+3)}{3} = t \qquad \text{.... (2)}$$

from (1) and (2)

$$\frac{1}{2}\ln(2x+1) = \frac{1}{2}\ln\frac{(2y+3)}{3}$$

$$2x + 1 = \frac{(2y+3)}{3}$$

$$y = 3x$$

Binomial Approximation :

For $x << 1$, the following approximation can be used :

$$(1+x)^n \approx 1 + nx$$

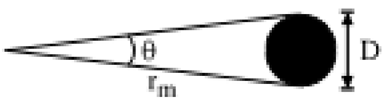
Ex Find $(1.03)^{1/3}$

Sol. $(1 + 0.03)^{1/3} \quad \left(x = 0.03 \,\&\, n = \frac{1}{3}\right)$

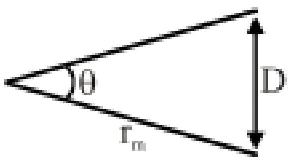
$$\simeq 1 + \frac{0.03}{3} = 1.01$$

Solved Example

Q.1 The angle subtended by the moon's diameter at a point on the earth is about 0.50°. Use this and the fact that the moon is about 384000 km away to find the approximate diameter of the moon.



Sol. $\tan\theta \simeq \theta = \frac{D}{r_m}$ θ is radians



$180° = \pi$-radians

$\Rightarrow \quad 0.5° = \frac{\pi}{180} \times \frac{1}{2}\text{rad} = \frac{\pi}{360}\text{rad}$

$\Rightarrow \quad D = \theta \,.\, r_m$

$= \frac{\pi}{360} \times 384000 \text{ km} = 3350 \text{ km}$

Q.2 Find $\frac{dx}{dt}$ (derivation of x with respect to t)

(i) $x = (t^2+1)^3$ (ii) $x = \sin 2t$

Sol. (i) $\frac{dx}{dt} = 3\,(t^2+1)^2\,(2t)$

$= 6t\,(t^2+1)^2$

(ii) $\frac{dy}{dt} = 2\cos 2t$

Q.3 Find the derivative of $y(x) = x^3/(x+1)^2$ with respect to x.

Sol. We can rewrite this function as $y(x) = x^3\,(x+1)^{-2}$ and apply Equation ( )

$$\frac{dy}{dx} = (x+1)^{-2}\,\frac{d}{dx}(x^3) + x^3\,\frac{d}{dx}(x+1)^{-2}$$

$$= (x+1)^{-2}\,3x^2 + x^3\,(-2)\,(x+1)^{-3}$$

$$\frac{dy}{dx} = \frac{3x^2}{(x+1)^2} - \frac{2x^3}{(x+1)^3}$$

Q.4 The velocity of particle is given by $v = \sqrt{gx}$. Find its acceleration.

Sol. $\frac{dv}{dt} = \frac{1}{2}(gx)^{-1/2}\left(g\frac{dx}{dt}\right)$

$= \frac{1}{2}\,(gx)^{-1/2}\,gv$

$= \frac{1}{2}\,g$

Q.5 If $\vec{r} = \left[u\cos\theta(\hat{i}) + u\sin\theta(\hat{j})\right]t + \frac{1}{2}g(-\hat{j})t^2$ then calculate equation of trajectory.

Sol. $x = u\cos\theta\, t$ & $y = u\sin\theta\, t - \frac{1}{2}\, g\, t^2$

$$y = u\left(\frac{x}{u\cos\theta}\right) - \frac{1}{2}\, g\left(\frac{x}{u\cos\theta}\right)^2$$

$$y = x\tan\theta - \frac{gx^2}{2u^2\cos^2\theta}$$

Q.6 Find the value of definite integral : $\int_0^{\pi}\left(\frac{\pi t}{2} - \frac{t^2}{2}\right)dt$

Sol. $\int_0^{\pi}\left(\frac{\pi t}{2} - \frac{t^2}{2}\right)dt$

$$= \left(\frac{\pi}{2}\left(\frac{t^2}{2}\right) - \frac{t^3}{6}\right)_0^{\pi}$$

$$= \frac{\pi^3}{4} - \frac{\pi^3}{6} = \frac{\pi^3}{12}$$

Q.7 The velocity of a body moving in a straight line is given by $v = (3x^2 + x)$ m/s. Find acceleration at $x = 2$m.

Sol. $v = (3x^2 + x)$

$$\frac{dv}{dx} = 6x + 1$$

$$a = v\frac{dv}{dx} = (3x^2 + x)(6x + 1)$$

at $x = 2$ m

$a = (3 \times 2^2 + 2)(6 \times 2 + 1) = 182$ m/s^2

Q.8 Find $(104)^{1/2}$

Sol. $(100 + 4)^{1/2}$

$= 10\,[1 + 0.04]^{1/2}$

$\simeq 10[1 + 0.02] = 10.2$