Rotational Mechanics

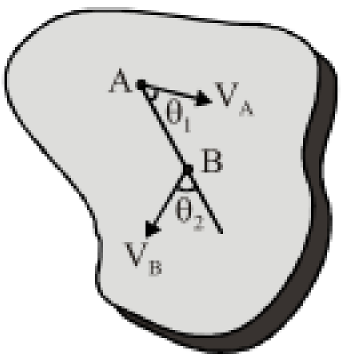
Introduction

The previous chapter we are studied only the translational motion of objects. The most general motion of a rigid body includes rotational as well as translational motions. Thus to study this general motion appropriate kinematics as well as dynamics must be studied in detail. Any body in general interacts with its surroundings through four basic interactions strong, electromagnetic, weak and gravitational. Not all forces produce rotational motion in the body. It is only certain conditions on force and its position and orientation that can produce rotational effects. The rotational variables have a definite relation with corresponding linear variables. This is studied under rotational kinematics. The causes of rotational motion and the factors governing changes in rotational state of motion are subject matter of rotational dynamics. The subject matter of this chapter includes kinematics and dynamics of rotational motion.
There can be pure rotation of a body or can be rotation with translation. We begin with pure rotational cases, go on developing the basics of rotational motion and subsequently deal with rolling motion which is a combination of rotational as well as translational motion.

Rotational kinematics

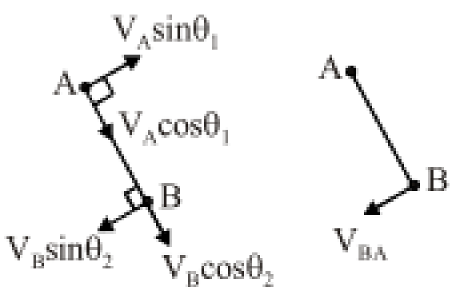
Rigid body :

Rigid body is defined as system of particles in which distance between each pair of partices remain constant (with respect to time) that means the shape and size do not change, during the motion Eg : Fan, Pen, Table, stone and so on.



Our body is not a rigid body, two blocks with a spring attached between them is also not a rigid body.

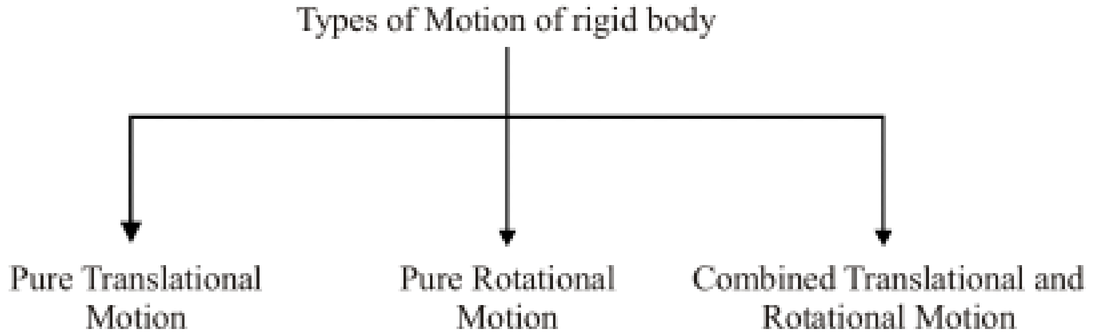
For every pair of particles in a rigid body, there is no velocity of seperation or approach between the particles. In the figure shown velocities of A and B with respect ground re V_A and V_B respectively.



If the above body is rigid

$$V_A \cos\theta_1 = V_B \cos\theta_2$$

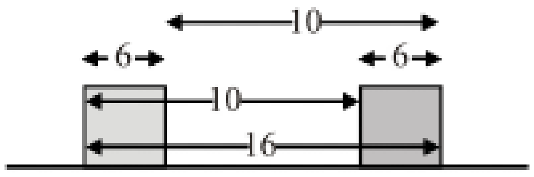
V_{BA} = relative velcoity of B with respect to A.



Pure Translation Motion :

A body is said to be in pure traslational motion if the displacment of each particle is same during any time interval howsoever small or large. In this motion all the particle have same $\vec{S}$, $\vec{V}$ & $\vec{a}$ at an instant.

Ex : A box is being pushed on a horizontal surface.



$$\vec{V}_{CM} = \vec{V} \text{ of any particle}$$

$$\vec{a}_{CM} = \vec{a} \text{ of any particle}$$

$$\Delta\vec{S}_{CM} = \Delta\vec{S} \text{ of any particle}$$

Pure Rotational Motion :

A body is said to be in pure rotational motion if the perpendicular distance of each particle remains constat from a fixed line or point and do not move parallel to the line, and that line is known as axis of rotation. In this motion all the particles have same $\vec{\theta}, \vec{\omega}$ & $\vec{a}$ at an instant. Eg : - a rotating ceiling fan, arms of a clock.

For pure rotation motion -

$\theta = \dfrac{s}{r}$ Where θ = angle rotated by the particle

s = length of arc traced by the particle.

r = distance of particle from the axis of rotation

$\omega = \dfrac{d\theta}{dt}$ Where ω = angular speed of the body

$\alpha = \dfrac{d\omega}{dt}$ Where α = angular acceleration of the body.

All the paramenters θ, ω and α are same for all the particles. Axis of rotation is perpendicular to the plane of rotation of particles.

Specila case : If α = constant,

$\omega = \omega_0 + \alpha t$ Where ω_0 = initial angular speed

$\theta = \omega_0 t + \dfrac{1}{2}\alpha t^2$ t = time interval

$\omega^2 = \omega_0^2 + 2\alpha\theta$

Practice Exercise

Q.1 The motor of an engine is rotation about its axis with an angular velocity of 100 rev/minute. It comes to rest in 15s, after being switched off. Assuming cosntant angular decelaration, calculate the number of revolutions made by it before coming to rest.

Q.2 Starting from rest, a fan takes five seconds to attain the maximum speed of 400 rpm (revolutions per minute). Assuming constant acceleration, find the time taken by the fan in attaining half the maximum speed.

Answers

Q.1 12.5 rev Q.2 2.5 sec.

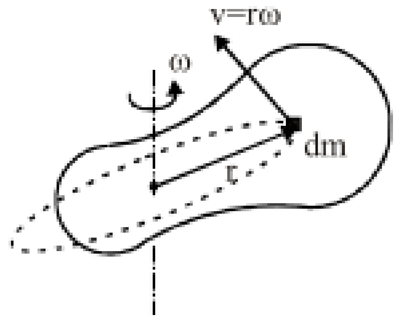
Kinetic Energy of Rotating Body

The rotating blade of a fan has some kinetic energy due to rotational motion which can not be expressed directly as $K.E = \frac{1}{2}mv^2$ since all the points do not have same speed.

To find rotational K.E. we take the fan's blade as a collection of different very small particles called elements. One such element has mass dm and is at distance r from the rotational axis as shown. Its kinetic energy can be given as

$$d(K.E.) = \frac{1}{2}(dm)v^2 = \frac{1}{2}(dm)(r\omega)^2$$

$$= \frac{1}{2}(dm)r^2\omega^2$$



The rotational kinetic energy of the body is given by summing i.e. intergrating the kinetic energy of all the elements of the body.

$$K.E. = \int d(K.E.) = \int \frac{1}{2}(dm)r^2\omega^2$$

Since ω is same for every element of a rigid body so we take ω^2 outside the integral.

$$\therefore \quad K.E. = \frac{1}{2}\omega^2 \int r^2 dm$$

we may write $\mathbf{K.E.} = \frac{1}{2}\mathbf{I}\omega^2$

where $I = \int r^2 dm$ called Moment of Inertia.

The above equation is analogous to the $K.E. = \frac{1}{2}mv^2$ i.e. Kinetic energy of a body having translational motion. Here ω is analogous to v. Also I is analogous to mass m i.e. I plays the same role in rotational motion as that of mass in translational motion. In other words as the inertia to the translational motion is due to the mass, inertia to the rotational motion is due to the quantity **Moment of Inertia**.

Moment of Inertia

Definition

It is the property of a rigid body by virtue of which it opposes change in its rotational motion.

* This is always taken w.r.t. a axis of rotation.

* This plays same role in rotational motion as mass plays in translational motion

* Difference between mass & M.I. (Moment of Inertia) is that mass is property of body & is independent of any reference axis choosen but MI depends on the mass as well as its distribution about the given axis of rotation. In other words it depends on

(i) axis of rotation

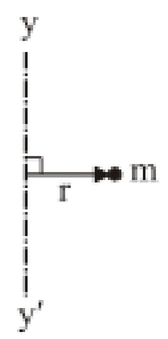
(ii) shape of the body

(iii) size of the body

(iv) density of the material of the body] mass depends only on these two things.

MI of a point mass :

$I_{yy'} = mr^2$

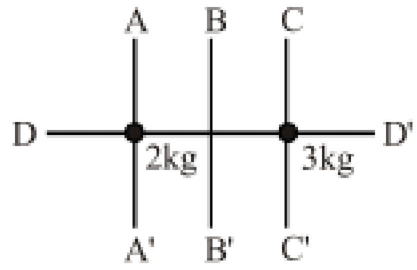


r is perpendicular distance from mass to axis of rotation yy′

Illustration :

Two particles of masses 2 kg and 3 kg are separated by 4 m as shown. Find M.I. of the system of particle about axis.

(A) AA' *(B) BB' (C) CC'* *(D) DD'*



Sol. (A) $I_{AA'} = 2(0)^2 + 3(4)^2 = 48\ kgm^2$ *(2 kg lies on the axis only)*

(B) $I_{BB'} = 2(2)^2 + 3(2)^2 = 20\ kgm^2$

(C) $I_{CC'} = 2(3)^2 + 3(0)^2 = 18\ kgm^2$

(D) $I_{DD'} = 2\,(0)^2 + 3(0)^2 = 0$ *(as both lies on the axis)*

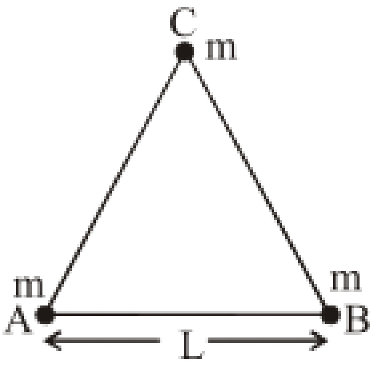
Illustration:

Find M.I. of a system of three particles lying on an equilateral triangle as shown about the axis

(A) Which is ⊥ to AB and passes through its centre.

(B) Passing through the side AC

(C) Passing through centroid and ⊥ to the plane of ABC.

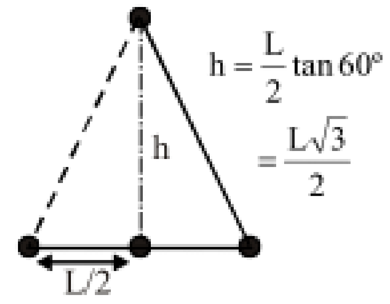


Sol. (A) $I = I_{!} + I_B + I_C$

$= m\,(L/2)^2 + m\,(L/2)^2 + m \times 0$

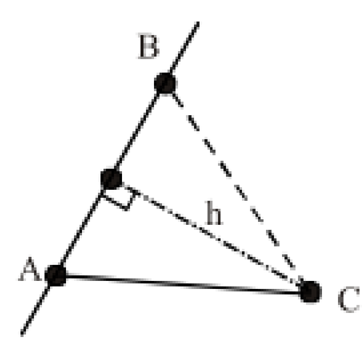
$= mL^2$

(B) $I_{BB'}.\ 0+0+mh^2$ $\qquad [\because I_A = I_B = 0]$

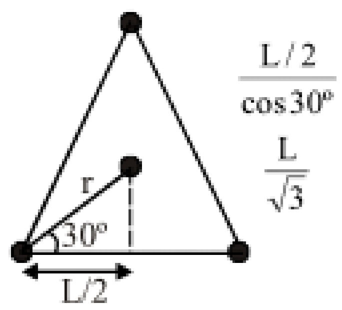


$= \dfrac{3mL^2}{4}$

(C) $I = 3 \times mr^2$



$= 3\,m \left(\dfrac{L}{\sqrt{3}}\right)^2$



$= mL^2$

Illustration:

Find the moment of inertia about C.O.M. system of two particles of mass m & M separated by distance l.

Sol. *Position C.O.M. from m is*

$$r_1 = \frac{m(0) + M(l)}{m+M} = \frac{Ml}{m+M}$$

from M is, $r_2 = l - r_1 = \dfrac{ml}{m+M}$

$$\therefore \qquad I = mr_1^2 + Mr_2^2 = m\left(\frac{Ml}{m+M}\right)^2 + M\left(\frac{ml}{M+m}\right)^2$$

$$= \left(\frac{mM}{M+m}\right)l^2$$

M.I of continuous rigid body

As we have discussed in previous topic for a continuous rigid body, $I = \int r^2 dm$

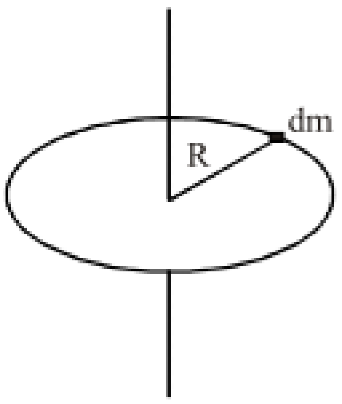
To solve for above equation, we take dm in terms of variable r/dr or both r and dm in terms of another variable.

To substitute dm, we take linear, areal and volume mass densities as we had taken in previous chapter.

M.I. of a ring

Let mass of the ring is M and radius is R.

$\therefore \qquad$ M.I. of small element of mass dm is $dI = (dm)R^2$



$$\therefore \qquad I = \int R^2 (dm)$$

but here distance of each element from the axis is same (= R) and so can be taken out of integral

$$\therefore \qquad I = R^2 \int dm$$

$$\therefore \qquad I = MR^2$$

M.I. of uniform rod

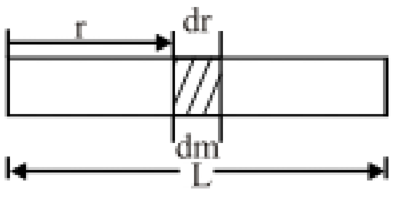
(i) About one of its end

M.I. of the element which is at distance r from the end is

$$I = \int dI = \int r^2 (dm)$$

Where, mass of the dement, dm = (mass per unit length)×(length of the element)

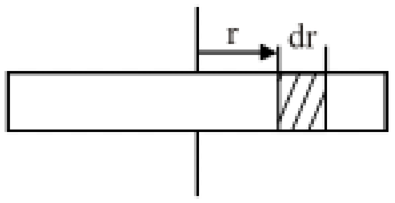
$$dm = \left(\frac{M}{L}\right) dr$$



$$\therefore \qquad I = \int_0^L r^2 \left(\frac{M}{L}\right) dm = \frac{M}{L}\int_0^L r^2 dr = \frac{M}{L}\left[\frac{r^3}{3}\right]_0^L$$

$$\therefore \qquad I = \frac{ML^2}{3}$$

(ii) About its C.O.M



$$dI = (r^2)(dm) = \frac{M}{L} r^2 dr$$

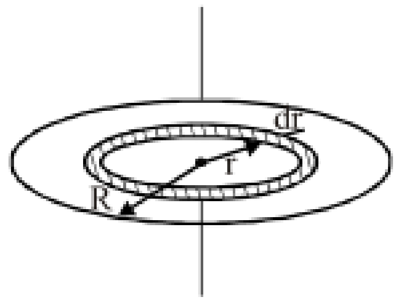
here M.I. can be fround out by integrating the above from left end to rigth end

$$\therefore \qquad I = \frac{M}{L}\int_{-L/2}^{L/2} r^2 dr = \left[\frac{r^2}{3}\right]_{-L/2}^{L/2} = \frac{M}{L}\times\frac{1}{3}\left[\left(\frac{L^3}{8}\right) = \left(\frac{-L^3}{8}\right)\right]$$

$$\therefore \qquad I = \frac{ML^2}{12}$$

Note : Here we can observe M.I. about the axis passing through C.O.M is less then that about end. In fact for various axis which are parallel to each other M.I. is minimum about the axis which passes through C.O.M. This is also proved later on in this chapter only.

M.I. of a uniform disc



Mass – M, Radius R. Lets taken an elementary ring of radius r and thickness dr.

$\therefore$ $\qquad dI = (dm)r^2$ (for ring)

Also $\qquad dm =$ (mass per unit area) $\times$ (area of alimentary ring)

Where $\qquad$ area of elementary ring, $dA = \pi\,[(r + dr)^2 - r^2]$

$$= \pi\,[2\,r\,dr + (dr)^2]$$

$\therefore$ $\qquad dA \simeq 2\,\pi\,r\,dr$ $\qquad$ [neglecting $(dr)^2$ being a ring small quantity]

$$\therefore \qquad dm = \frac{M}{\pi R^2}\times 2\,\pi\,r\,dr = \frac{2M}{R^2}\,r\,dr$$

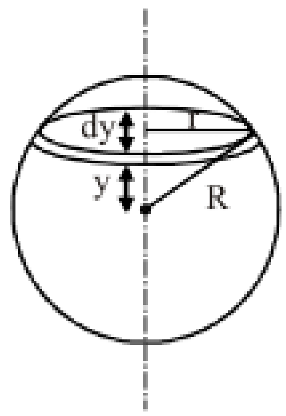
$$\therefore \qquad dI = (dm)\,r^2 = \frac{2M}{R^2}\,r^3\,dr$$

$$\therefore \qquad I = \frac{2M}{R^2}\int_0^R r^3\,dr$$

$$\therefore \qquad I = \frac{MR^2}{2}$$

M.I. of solid sphere (Mass-M, Radius - R)

Let taken n elemntary disc of radius r and at distance from the centre. Let its thikness is dy.



$$dm = \frac{M}{\frac{4}{3}\pi R^3} \times \pi r^2 dy$$

$$= \frac{3M}{4R^3} r^2 dy$$

$\therefore$ M.I. of elementary disc is $dI = \frac{1}{2}(dm)r^2 = \frac{3M}{8R^2} r^4 dy$

Also $r^2 = R^2 - y^2 \Rightarrow r^4 = (R^2 - y^2)^2 = R^4 + y^4 - 2R^2 y^2$

$\therefore$ $dI = \frac{3M}{8R^3}(R^4 + y^4 - 2R^2 y^2)\ dy$

$\therefore$ M.I. of the sphere is

$$I = \int dI = \frac{3M}{8R^3}\left[\int_{-R/2}^{R/2} dy + \int_{-R/2}^{R/2} y^4.dy - 2R^2\int_{-R/2}^{R/2} y^2.dy\right]$$

$$\therefore \quad I = \frac{3M}{8R^3}\left[R^4[y]_{-R/2}^{R/2} + \left[\frac{y^5}{5}\right]_{-R/2}^{R/2} - 2R^2\left[\frac{y^3}{3}\right]_{-R/2}^{R/2}\right]$$

$$\therefore \quad I = \frac{2}{5}MR^2$$

M.I. of thin hollow sphere (mass - M; Radius - R)

Here we take an elementary ring of radius r and thickness = $Rd\theta$

$\therefore$ Mass of elementary ring, $dm = \sigma\, dA$

$$\therefore \quad dm = \frac{M}{4\pi R^2} \times (2\,\pi\, r)\, R\, d\theta$$

$$= \frac{M}{2R}\, r\, d\theta$$

$$\therefore \quad dI = (dm)r^2 = \frac{M}{2R}\, r^3\, d\theta$$

Aslo $r = \sin\theta \Rightarrow dI = \dfrac{M}{2R}\sin^3\theta\, d\theta$

or $dI = \dfrac{M}{2R}\,(1 - \cos^2\theta)\sin\theta\, d\theta$ [Taking $\sin^2\theta = 1 - \cos^2\theta$]

Let $\cos\theta = x \Rightarrow \dfrac{dx}{d\theta} = -\sin\theta \Rightarrow \sin\theta\, d\theta = -\,dx$

$$\therefore \quad dI = \frac{M}{2R}\,(1 - x^2)\,(-\,dx) = \frac{M}{2R}\,(x^2 - 1)\,dx$$

Also When θ varies from 0 to π ; x i.e. $\cos\theta$ varies from 1 to – 1

$$\therefore \quad I = \frac{M}{2R}\int_1^{-1}(x^2 - 1)dx = \frac{M}{2R}\left[\frac{x^3}{3} - x\right]_1^{-1}$$

$$\therefore \quad I = \frac{2}{3}MR^2$$

M.I. of some common bodies

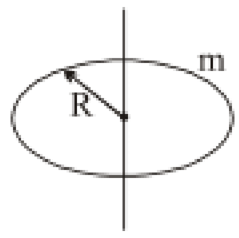
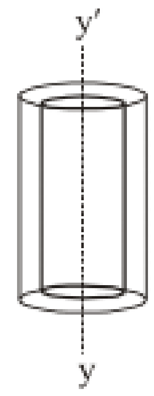
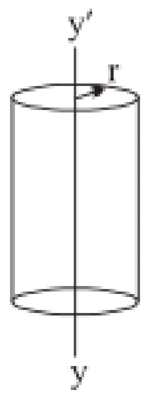
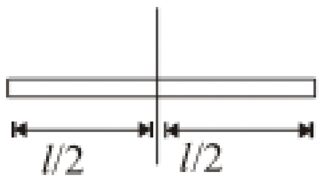
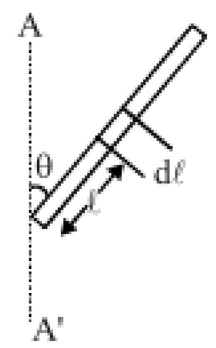
Uniform ring (about an axis passing through centre and perpendicular to the plane.)	$I = Mr^2$
Disc (about axis passing through centre and perpendicular to plane of disc). 	$= \dfrac{MR^2}{2}$
Hollow cylinder (about yy′ axis) Note : Independent of length of cylinder of same mass. 	$= Mr^2$
 **Solid cylinder** (about yy′ axis)	$= \dfrac{MR^2}{2}$
Hollow sphere (about a diameter)	$= \dfrac{2}{3}\ MR^2$
Solid sphere (about a diameter)	$= \dfrac{2M}{5}\ R^2$
Uniform rod mass M length (about axis passing through one end & perpendicular).	$= \dfrac{ml^2}{3}$
uniform rod (about an axis passing at $\dfrac{L}{4}$ from one end and perpendicular)	
	$= \dfrac{ml^2}{12}$
 uniform rod (about axis passing through rod)	$I = 0$

Illustration :

Find M.I. of a unifrom rod of mass M and langth L about an axis which is at angle θ to it as shown .

Sol. *M.I. of the element*



$$dI = (dm)\, r^2 \left(\frac{M}{L} dl\right) r^2$$

Also, $r = l \sin\theta$

$$\therefore \quad dI = \left(\frac{M}{L}\sin^2\theta\right) l^2\, dl$$

$$\therefore \quad I = \frac{M}{L}\sin^2\theta \int_0^L l^2\, dl$$ *[since θ is common to all points, $\sin\theta$ comes out of the integral]*

$$\therefore \quad I = \frac{M}{L}\sin^2\theta\left[\frac{L^3}{3}\right] = \frac{ML^2}{3}\sin^2\theta$$

Illustration :

A rod AB of length L has linear mass density (λ) varying with distance (x) from the end A as $l = \alpha x^2$ where α is a constant. Find M.I. of the rod about the end A.

Sol. $dm = l\, dx$

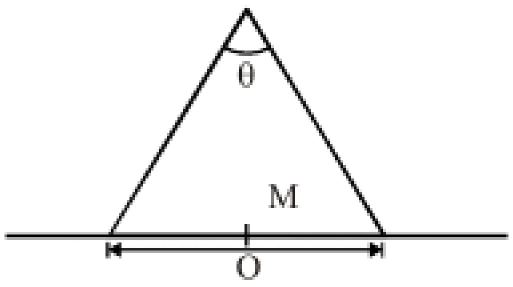
$= \alpha\, x^2\, dx$

$\therefore \quad dI = dm x^2 = \alpha\, x^4\, dx.$

$$\therefore \quad I = \int dI = a\int_0^L x^4 dx \quad \Rightarrow \quad I = \frac{\alpha L^2}{5}$$

Illustration :

Find M.I. of a uniform traingular plate of mass M about its base as shown.



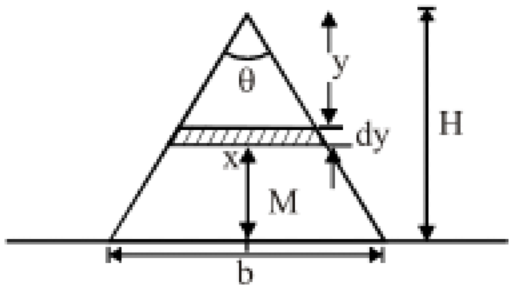
Sol. σ *(mass per unit Area) × (Area of the element)*

$= (\sigma)\,(x\, dy)$

Since each point of the dementary strip is at distance

$(H - y)$ from the axis

$\therefore \quad dI = (dm)\,(H - y)^2 = (\sigma\ xdy)\,(H^2 + y^2 - 2Hy)$



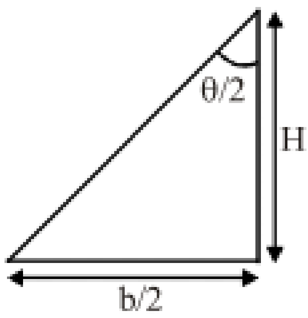
Also $\quad \frac{x}{b} = \frac{y}{H} \Rightarrow x = \left(\frac{b}{H}\right) y$

$$\therefore \quad dI = \sigma \frac{b}{H}\left(H^2 y + y^3 - 2Hy^2\right) dy$$

$$\therefore \quad I = \sigma \frac{b}{H}\left[H^2 \int_0^H y.dy + \int_0^H y^3.dy - 2H \int_0^H y^2.dy\right]$$

$$= \sigma \frac{b}{H}\left[\frac{H^4}{2} + \frac{H^4}{4} - \frac{2H^4}{3}\right]$$

$$= \sigma \frac{bH^3}{12}$$



H = b/2 cot (θ/2)

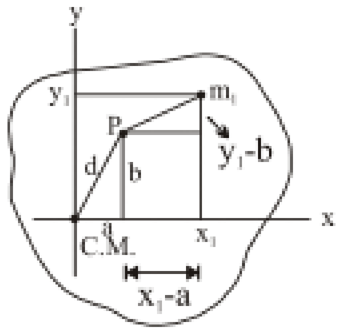
Also $$\sigma = \frac{Total\ mass}{Total\ area} = \frac{M}{\left(\frac{1}{2}bH\right)} = \frac{2M}{bH}$$

$$\therefore \quad I = \frac{MH^2}{6} = \frac{M}{6}\left[\frac{b}{2}cot\left(\frac{\theta}{2}\right)\right]^2$$

$$= \frac{Mb^2}{24} cot\left(\frac{\theta}{2}\right)$$

Parallel axis Theorem

Used to find moment of inertia about an axis which parallel to the axis passing through C.M.



C.M. is at origin

$I_{CM} \rightarrow$ MI of the rigid body about an axis through CM

$I_P \rightarrow$ MI of the rigid body about an axis which is parallel to the above axis through CM & is at distance d from the axis through CM

$$I_{CM} = \sum_i m_i \left(x_i^2 + y_i^2\right)$$

$$I_P = \sum_i m_i \left[(x_i - a)^2 + (y_i - b)^2\right]$$

z coordinates are not involved so m_i can be replaced by sum of mass of all particles placed on z-axis with co-ordinate (x_i, y_i)

$$= \Sigma m_i \left(x_i^2 + y_i^2\right) + \Sigma m_i (a^2 + b^2) - \Sigma\, 2am_i x_i - \Sigma 2bm_i y_i$$

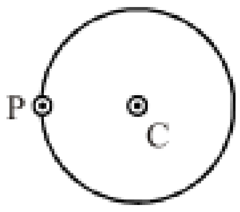
$$= I_{CM} + (a^2 + b^2)\, \Sigma m_i - 2a\, \Sigma m_i x_i - 2b \Sigma m_i y_i$$

$$I_P = I_{CM} + Mh^2$$

If M.I. of a body of mass M about an axis passing through its C.O.M. is I_C them M.I. of another axis which is parallel to the above central axis and is parallel to it is given by $I_{AA'} = I_C + Mh^2$

Illustration :

Find M.I. of a uniform body of mass M and radius R about an axis passing through a point p on its periphery and is perpendicular to its plane. Take the body as

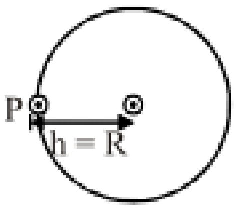


(a) Ring *(b) Disc*

Sol. $I_P = I_C + M(R)^2$

(a) for ring, $I_P = MR^2 + M(R)^2$

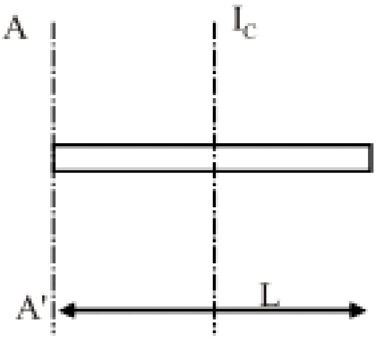
$= 2\ MR^2$



(b) for disc, $I_P = \dfrac{MR^2}{2} + M(R)^2 = \dfrac{3}{2}MR^2$

Illustration :

Find I_C *of the uniform rod of mass M shown, knowing that M.I. about an end is* $\dfrac{ML^2}{3}$



Sol. $I_{AA'} = I_C + Mh^2$

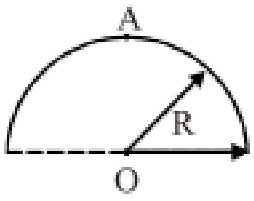
$\therefore$ $I_C = I_{AA'} - Mh^2$

where $h = \dfrac{1}{2}$ and $I_{AA'} = \dfrac{ML^2}{12}$

$\therefore$ $I_C = \dfrac{ML^2}{3} - M\left(\dfrac{L}{2}\right)^2 = \dfrac{ML^2}{12}$

Illustration :

Find M.I. of thin semicircular wire of mass m about axis passing through point A and is perpendicular to its plane as shown.

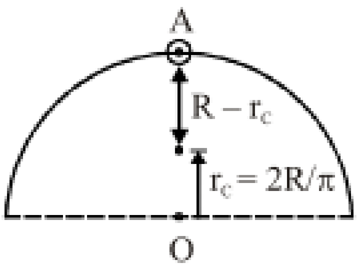


Sol. $I_0 = mR^2$

By parallel axis theorem

$I_0 = I_C + mr^2_C$ $\Rightarrow$ $I_C = I_0 - mr^2_C$

Where r_C *is distance of C.O.M. of the helf ring from its base ;*



$r_C = \dfrac{2R}{\pi}$

$$\therefore \quad I_C = I_0 - m\left(\frac{2R}{\pi}\right)^2$$

Again applying parallel axis theorem, now for axes through A and C.O.M.

$$\therefore \quad I_e = I_C + m\,(R - r_C)^2$$

$$= \left[mR^2 + m\left(\frac{2R}{\pi}\right)^2\right] + m\left(R - \frac{2R}{\pi}\right)^2$$

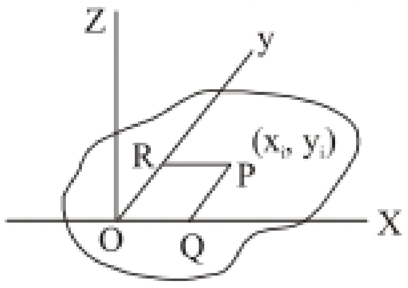
$$\therefore \quad I_A = 2mR^2\left(1 - \frac{2}{\pi}\right)$$

Perpendicular axis theorem

This theorem is applicable only for the laminar bodies (i.e. plane bodies). (e.g. ring, disc, not sphere)

I_x & I_y are MI of body about a common pt. O in two mutually $\perp$ directions in the plane of body

I_z is MI of body about an axis $\perp$ to X & Y axis & passing through pt. O



$I_x = \Sigma m_i\, y_i^2$

$I_y = \Sigma m_i x_i^2$

$I_z = \Sigma m_i\,(x_i^2 + y_i^2)$ $\qquad\qquad$ $I_z = I_x + I_y$

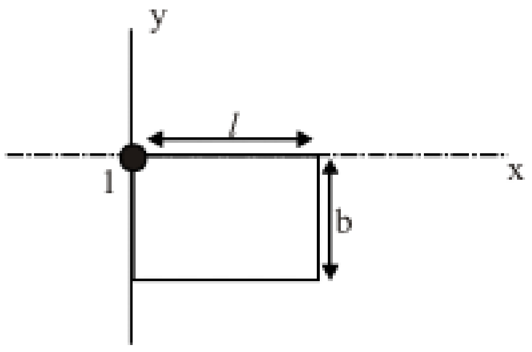
The point of intersection of the three axis need not be centre of mass, it can be any point in the plane of body which lie on the body or even outside it. For relation from perpendicular axis theorem, $I_3 = I_1 + I_2$ axis (3) must be perpendicular the plane of the body and axis (1) and axis (2) must be in the plane of the body.

Illustration :

Find M.I. of a uniform rectangular plate of sides l and b shown, obout the axes passing through

(i) Point 1 i.e. corener

(ii) Point 2 i.e. centre



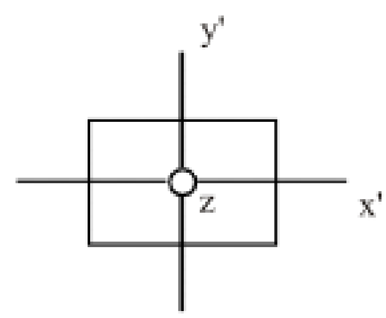
Sol. *(i)* $\quad I_X = \frac{Mb^2}{3}\ ;\ I_y = \frac{Ml^2}{3}$

By perpendicular axis theorem

$$I_z = I_x + I_y$$

$$= \frac{M}{3}\,[l^2 + b^2]$$

(ii) $I_{x'} = \dfrac{Mb^2}{12}, I_y = \dfrac{Ml^2}{12}$



$\therefore \quad I_Z = I_{x'} + I_y = \dfrac{M}{12}(l^2 + b^2)$

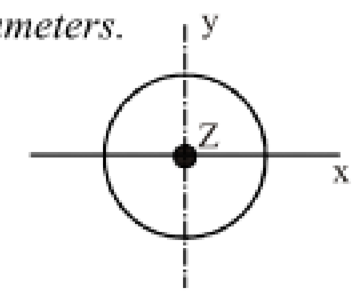
In this question, for squese plate of side b = l

$$I_1 = \frac{M}{3}\ (l^2 + l^2) = \frac{2Ml^2}{3}$$

& $\quad I_Z = \dfrac{M}{12}(l^2 + l^2) = \dfrac{Ml^2}{6}$

Illustation :

Find M.I. of the ring and the disc about the axis passing through their diameters.



Sol. *By perpendicular axis theorem*

$$I_X + I_Y + I_Z$$

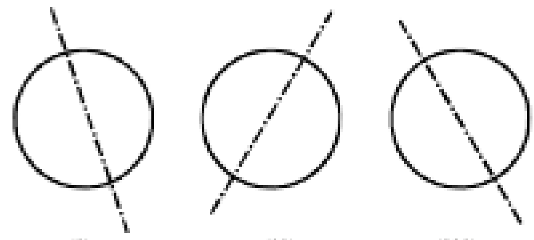
Also, since mass distrioution of the body about x and y axis are similar

$\therefore \quad I_x = I_y \quad \Rightarrow \quad I_x = I_y = I_z/2$

(i) For ring, $I_Z = MR^2 \Rightarrow \quad I_X = I_y = \dfrac{MR^2}{2}$

(ii) For disc, $I_Z = \dfrac{MR^2}{2} \Rightarrow \quad I_X + I_y = \dfrac{MR^2}{4}$

Note :

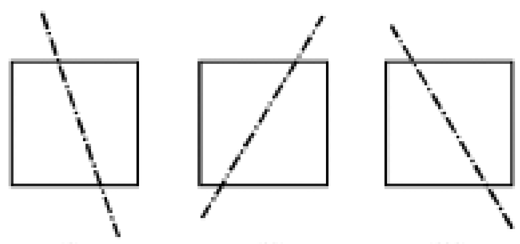


(i) (ii) (iii)

We can observe mass distrbution about the diamenteral axis shown in (i), (ii) and (iii) are similar

$\therefore \quad I_{(i)} = I_{(ii)} = I_{(iii)}$

Similarly for square plate



(i) (ii) (iii)

$$I_{(i)} = I_{(ii)} = I_{(iii)}\ \frac{Ml^2}{6}$$

Illustration :

A hole of radius $\frac{R}{2}$ is made in a uniform ciruclar plate of radius R. The mass of remaining portion shaded is m. Find M-I. of this body about point O.

Sol. *To solve this, we use the concept. M.I. of the remaining postion*

= M.I. of the complete body before removed – M.I. of the removed part.

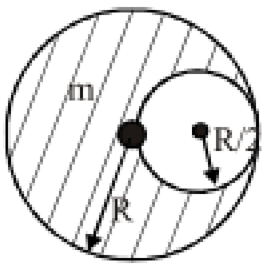
(A) Let 'm' be the mass of reamining disc.

Area of original disc $= \pi r^2$,

Area of removed part $= \pi(r/2)^2$,

Therefore area of the remaining disc $= \pi r^2 - \pi(r/2)^2 = (3/4)\pi r^2$

Mass per unit area $= \sigma = \dfrac{m}{(3/4)\pi 3^2} = \dfrac{4}{3}.\dfrac{m}{\pi r^2}$,



Mass of the removed part $= \dfrac{1}{4}\pi r^2 \sigma = \dfrac{1}{3}m$,

Moment of inertia of the complete disc $(I_{com}) = \dfrac{1}{2}\times\dfrac{4}{3}mr^2 = \dfrac{2}{3}mr^2$

Moment of inertia of the removed part $(I_{removed}) = \dfrac{1}{2}\times\dfrac{m}{3}\left(\dfrac{r}{2}\right)^2 + \dfrac{m}{3}\left(\dfrac{r}{2}\right)^2$

$$= \frac{1}{8}mr^2$$

Therefore the moment of inertia of the remaining disc

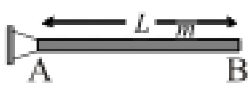
$$I_{remaining} = \left(\frac{2}{3}\right)mr^2 - \left(\frac{1}{8}\right)mr^2$$

$$I_{remaining} = \left(\frac{13}{24}\right)mr^2$$

Illustration :

A non-uniform bar AB of mass m has linear mass density $\lambda = \lambda_0 \dfrac{x}{L}$ *(x is calculated from one end).*

Find the (a) Mass of the rod (b) Moment of inertia of the rod aobut the end A.



(c) Momemt of inertia about centre of mass

Sol. (a) $dm = \lambda\, dx = \lambda_0 \dfrac{x}{L} dx$

$$\therefore \quad M = \int dm = \frac{\lambda_0}{L}\int_0^L x\, dx = \frac{\lambda_0 L}{2}$$

(b) About A, $dI = (dm)x^2 = \dfrac{\lambda_0 x^3}{L} dx$

$$\therefore \quad I_A = \int dI = \frac{\lambda_0}{L}\int_0^L x^3 dx = \frac{\lambda_0 L^3}{4}$$

(c) Position of C.O.M from end A is

$$x_C = \frac{\int (dm)x}{\int dm} = \frac{\frac{\lambda_0}{L}\int_0^L x^2 dx}{\left(\frac{\lambda_0 L}{2}\right)}$$

$$\Rightarrow \quad x_C = \frac{2}{L^2}\left[\frac{L^3}{3}\right] = \frac{2L}{3}$$

By paraell axis theorem, $I_A = I_C + M x_C^2$

$$\therefore \quad I_C = I_A - Mx_c^2 = \frac{\lambda_0 L^3}{4} - \left(\frac{\lambda_0 L}{2}\right)\left(\frac{2L}{3}\right)^2$$

$$= \frac{\lambda_0 L^3}{36}$$

Radius of gyration (K)

If M.I. aobut an axis of a system of mass M is I, then we may write, $I = MK^2$ where k is called radius of gyration of the body about that axis = $\sqrt{\dfrac{I}{M}}$

In other words M.I. aobut an axis of system is same as M.I. of a particle of same mass placed t the distance equal to K from the axis of a uniform disc of radius R, radius of gyration about an axis passing through its centre & is to its plane is

$K = \sqrt{\dfrac{\left(\dfrac{MR^2}{2}\right)}{M}} = \dfrac{R}{\sqrt{2}}$ i.e. the disc will same rotational inertia as that of a particle of same mass placed at distance $\dfrac{R}{\sqrt{2}}$ from the axis

Illustration :

A hole of radius $\frac{R}{2}$ *is made in a uniform ciruclar plate of radius R. The mass of remaining portion shaded is m. Find M-I. of this body about point O.*

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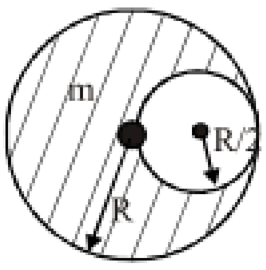

Mass of the removed part $= \frac{1}{4}\pi r^2 \sigma = \frac{1}{3}m$,

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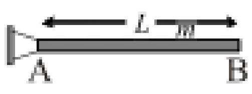

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In other words M.I. aobut an axis of system is same as M.I. of a particle of same mass placed t the distance equal to K from the axis of a uniform disc of radius R, radius of gyration about an axis passing through its centre & is to its plane is

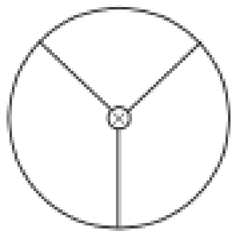
$K = \sqrt{\frac{\left(\frac{MR^2}{2}\right)}{M}} = \frac{R}{\sqrt{2}}$ i.e. the disc will same rotational inertia as that of a particle of same mass placed at distance $\frac{R}{\sqrt{2}}$ from the axis

Note : In case of system of combitionation of various bodies $k = \sqrt{\frac{I_{total}}{M_{total}}}$

Illustration

Each wheels has an outer ring having radius R and mass m. Other than outer ring the wheels comprise of some uniform rods (each of mass m and length R).

Calculate radius of gyration about centre and perpendicular to plane.



Sol. $I_{total} = m(R^2) + 3 \times \left(\frac{mR^2}{3}\right) = 2mR^2$

Also $M_{total} = 4\ m$

$\therefore \quad K = \sqrt{\frac{2mR^2}{4m}} = \frac{R}{\sqrt{2}}$

Illustration :

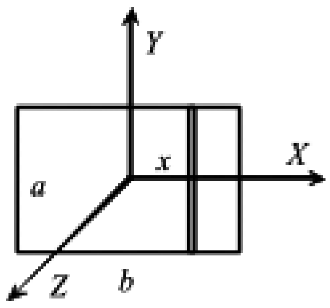
Find the MI of a ring about the chord which is parallel to the diameter of the ring at a distance R/2 from the diameter

MI of Plate:

$I_x = \int \frac{1}{12} dm\, a^2 = \frac{1}{12} m a^2$

$\therefore I_y = \frac{1}{12} m b^2$

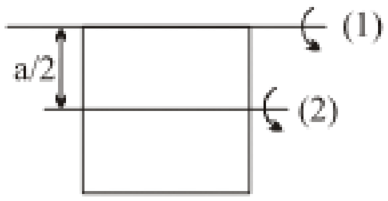
$\therefore I_z = I_x + I_y = \frac{1}{12} m \left(a^2 + b^2\right)$



1. What would have happened if it was a square plate?
2. Prove that MI of square plate of mass m and side a about any axis passing through COM along the surface of plate is $ma^2/12$

Illustration:

Find the moment of inertia of a Square plate about an edge in the plane



$I_{CM} = I_2 = \frac{ma^2}{12} \qquad \therefore I_1 = \frac{ma^2}{12} + m\left(\frac{a}{2}\right)^2 = \frac{ma^2}{12} + \frac{ma^2}{4} = \frac{ma^2}{3}$

Practice Exercise

Q.1. Find the moment of inertia of a pair of spheres, each having a mass m and radius r, kept in contact about the tangent passing through the point of contact.

Q.2 The moment of inertia of a uniform rod of mass $m = 0.50$ kg and length $l = 1$ m is $I = 0.10$ kg $-m^2$ about a line perpendicular to the rod. Find the distance of this line from the middle point of the rod.

Q.3 Find the moment of inertia of a uniform square plate of mass m and edge a about one of its diagonals.

Q.4 The radius of gyration of a uniform disc about a line perpendicular to the disc equals its radius. Find the distance of the line form the centre.

Q.5 Calculate the moment of inertia of a uniform rod of mass m & length l about an axis passing through one end & making angle $\theta = 45^{\circ}$ with its length.

Q.6 The surface density (mass/area) of a circular disc of radius a depends on the distance from the centre of $\rho(r) = A + Br$. Find its moment of inertia about the line perpendicular to the plane of the disc through its centre.

Answers

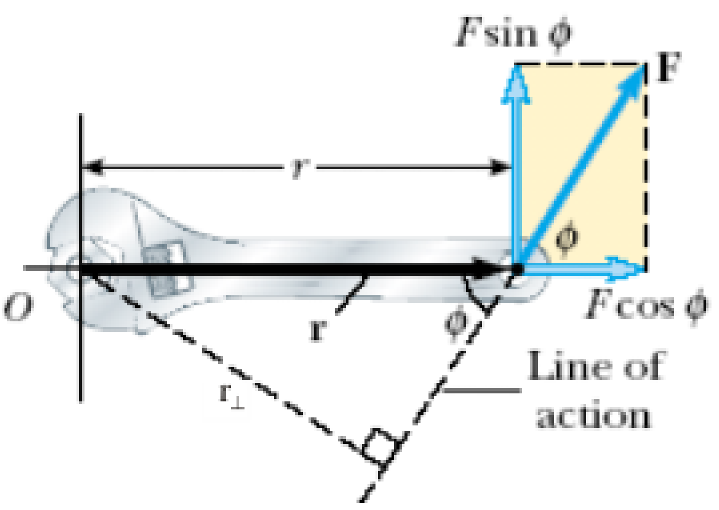
Q.1. $14\ mr^2$ Q.2 $\sqrt{\frac{1}{2}\left(1 - \frac{ml^2}{12}\right)} = \sqrt{\frac{7}{60}} = 0.34m$ Q.3 $ma^2/12$ Q.4 $r/\sqrt{2}$

Q.5 $\frac{ml^2}{6}$ Q.6 $2\pi\left(\frac{Aa^4}{4} + \frac{Ba^5}{5}\right)$

4. Torque

The quantitative measure of the tendency of a force to cause or change the rotational motion of a body is called torque. Consider an example to understand this.

In the figure below, the wrench is trying to open the nut. Now the ability of wrernch to open the nut will depend not only on the applied force, but the distance at which force is applied. This gives birth to **new physical quantity** called torque.



If only radial force F_r were present, the nut could not be turned. Thus the force causing the rotation is tangential force F_T only. The magnitude of the torque about an axis due to a force is given by

τ = (Force causing the rotation)×(distance of point of application of force from the axis)

i.e., $= (F \sin \phi)\, r$

we may also write $\tau = F\,(r \sin\phi) = F\,(r_\perp)$

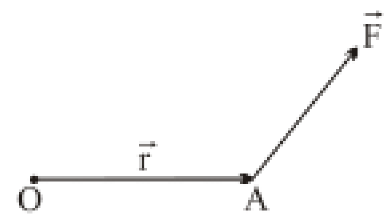
$$\therefore \tau = r F \sin\phi = (r \sin\phi) F$$
$$= r F_\tau = r_\perp F$$

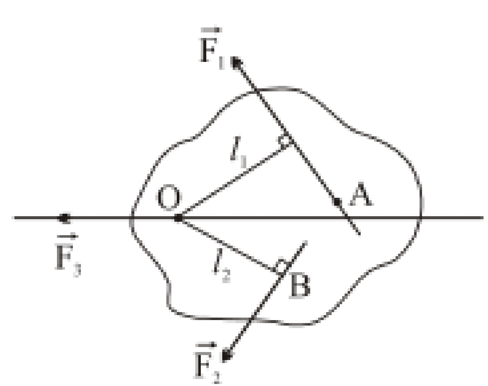
$r_\perp(d)$: moment arm, lever arm

$\vec{\tau} = \vec{r} \, x \, \vec{F}$ Direction of torque is found by sliding the force vector at the axis of rotation and using right hand thumb rule.

Torque also follows superposition principle. $\vec{\tau} = \vec{\tau}_1 + \vec{\tau}_2 + ... + \vec{\tau}_n$

$\overrightarrow{OA} = \vec{r}$ = P.V. of pt. of application of force wrt fixed axis.(centre of rotation)





Note :

1. Torque & force are entirely diff. quantities. As torque is always defined with reference to point about which body is rotating, while force does not depend on it. Like torque of F_3 about O is zero, while about A or B is not zero.
2. When equal & opposite force acts on a body having different line of action is called **couple**.

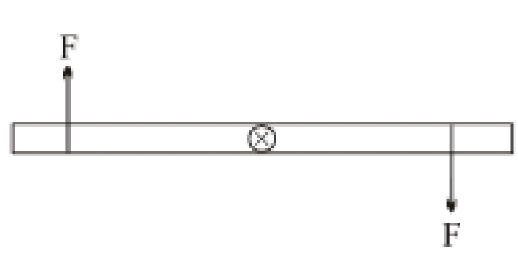
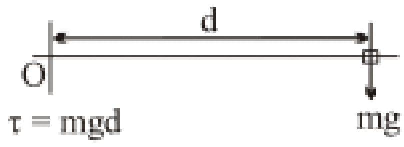


Illustration :

A particle is falling freely along line y = d. Find torque on this particle due to gravity, about origin when it

(i) Crosses x axis *(ii) is at y = d*

Sol. *(i)*



(ii)

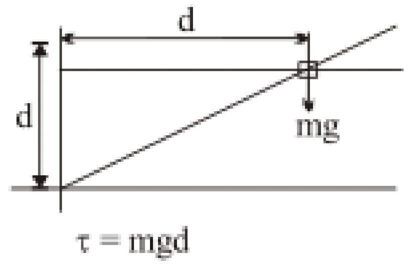


Illustration :

Find torque due to gravity at any time t about pt. of projection, if a body is projected with velocity u at an angle θ.

Sol.

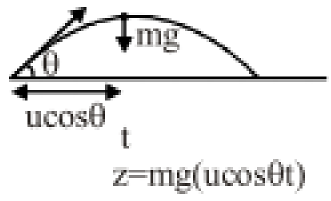
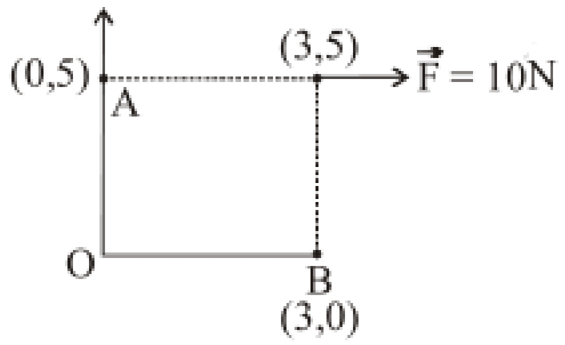
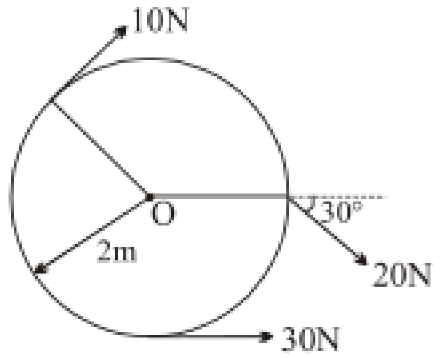


Illustration :

Find out the torque about point A, O and B for fig (i) & about 'O' for fig. (ii).



(i)



(ii)

Sol. $\tau_A = 0$

$\tau_B = 10\times5 = 50$ *N-m*

$\tau_0 = 10\times5 = 50$*N-m*

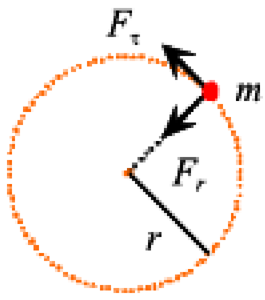
Torque about O

$\tau = -10\times2 - 20 \sin 30^o \times 2 + 30\times2 = 20$ *N-m*

Relationship between torque and angular acceleration

Rotational analog of Newton's second law

Consider a particle of mass m rotating in a circle of radius r under the influence of a tangential force F_τ and a radial force F_r, as shown in figure.



$F_t = m a_t$

Magnitude of torque about the center of circle is:

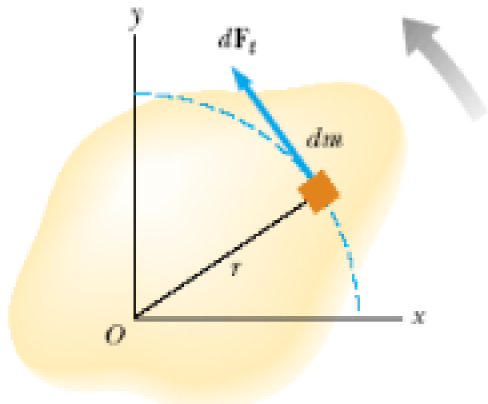
$\tau = F_t\, r = m a_t\, r = m(\alpha\, r) r = \alpha\,(m r^2) = I\alpha$

$\therefore \tau = I\alpha$

That is, the torque acting on the particle is proportional to its angular acceleration.

We can also understand that since torque is written about O, we should write 'I' also about O.

Torque on rigid body:



Proof: Consider a rigid body shown.

Torque acting on *i* th particle will be:

$$\vec{\tau}_i = \left(m_i r_i^2\right)\vec{\alpha} \quad \Rightarrow \vec{\tau}_{net} = \sum \vec{\tau}_i = \sum \left(m_i r_i^2\right)\vec{\alpha}$$

$$\therefore \vec{\tau}_{net} = I\,\vec{\alpha}$$

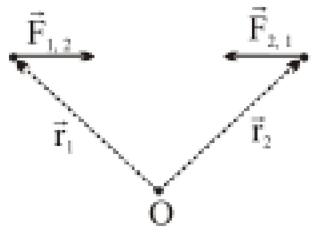
$\vec{\tau} = I\,\vec{\alpha} = \vec{r} \times \vec{F}$: This is the general relation that we are going to use in this chapter.

Note:

a) Torque due to internal forces is alwys zero.

b) Torque due to gravity is found by showing the gravitational force at the COM of the rigid body.

Proof (a): Consider two particles as shown in figure. From Newton's third law, the forces exerted by these particles are equal and opposite.



$$\vec{F}_{2,1} = \vec{F}_{1,2}$$

The sum of torques of these forces about origin O is

$$\begin{aligned}\vec{\tau}_1 + \vec{\tau}_2 &= \vec{r}_1 \times \vec{F}_{2,1} + \vec{r}_2 \times \vec{F}_{1,2} \\ &= \vec{r}_1 \times \vec{F}_{2,1} + \vec{r}_2 \times (-\vec{F}_{2,1}) \\ &= (\vec{r}_1 - \vec{r}_2) \times \vec{F}_{2,1}\end{aligned}$$

The vector $\vec{r}_1 - \vec{r}_2$ is along the line joining the two particles, so $\vec{F}_{2,1}$ is either parallel or antiparallel to $(\vec{r}_1 - \vec{r}_2)$ thus

$$(\vec{r}_1 - \vec{r}_2) \times \vec{F}_{2,1} = 0$$

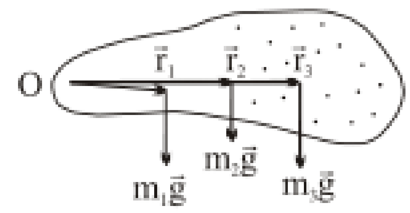
So the internal forces (torques) cancel in pairs.

Proof (b):

τ = torque of gravity about O

$$\vec{\tau} = \vec{r}_1 \times m_1\vec{g} + \vec{r}_2 \times m_2\vec{g} + ...\vec{r}_n \times m_n\vec{g} = (m_1\vec{r}_1 + m_2\vec{r}_2 + m_3\vec{r}_3 + ...+ m_n\vec{r}_n) \times \vec{g}$$

$$\vec{\tau} = \frac{(m_1\vec{r}_1 + m_2\vec{r}_2 + m_3\vec{r}_3 + ...+ m_n\vec{r}_n)}{(m_1 + m_2 + m_3 + ...+ m_n)} \times (M\vec{g})$$



$$\vec{\tau} = \vec{r}_{cm} \times (M\vec{g})$$

Where r_{cm} is PV of C.M. wrt. point O.

Practice Exercise

Q.1 A force $F = A\hat{i} + B\hat{j}$ is applied to a point whose radius vector relative to the origin of coordinates O is equal to $r = a\hat{i} + b\hat{j}$, where a, b & A, B are constants, and $\hat{i}$, $\hat{j}$ are the unit vectors of the x and y axes. Find the Torque due to force.

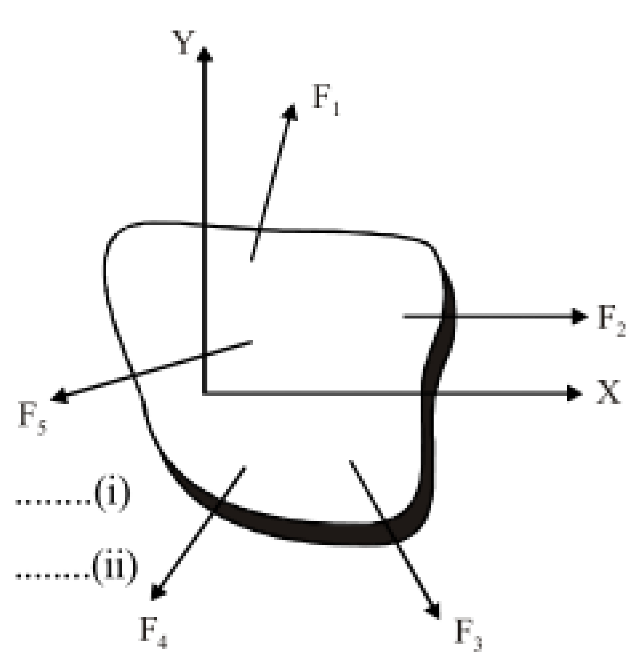
Answers

Q.1 $Z = (aB - bA)\ \hat{k}$

Rotational Equilibrium

If net external torque acting on the body is zero, then the body is said to be in rotational equilibrium.

The centre of mass of a body remains in equilibrium if the total external force acting on the body is zero. Similarly, a body remains in rotational equilibrium if the total external torque acting on the body is zero.



For translational equilibrium.

$$\Sigma F_x = 0 \qquad ....(i)$$

and $$\Sigma F_y = 0 \qquad ....(ii)$$

$$\Sigma F_z = 0$$

The condition of rotational equilibrium is

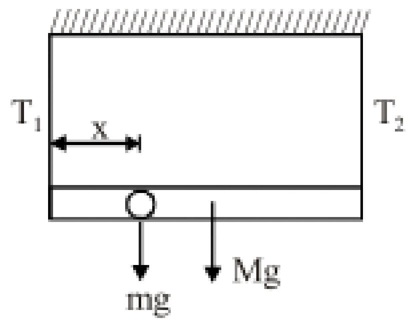
$$\Sigma Z_{ext} = 0$$

The equilibrium of a body is called stable if the body tries to regain its equilibrium position after being slightly displaced and released. It is called unstable if it gets further displaced after being slightly displaced and released. If it can say in equilibrium even after being slightly displaced and released, it is said to be is neutral equilibrium.

Illustration :

A uniform stick of mass M & length L is suspended from the ceiling through two vertical strings of equal lengths fixed at the ends. A small object of mass m is placed on the stick at a distance of x from the left end. Find the tensions in the two strings.

Sol.



By torque balancing about end's

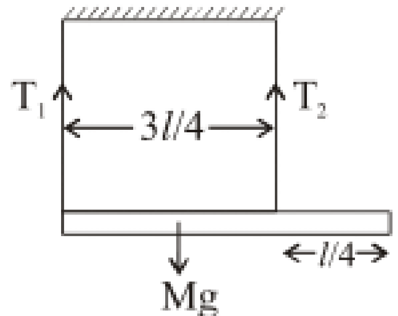
$$T_1 x = mg(L-x) + Mg \times \frac{L}{2}$$

$$T_2 = mgx + Mg\,\frac{L}{2} \qquad \text{....(i)}$$

$$= \left(\frac{M}{L} + \frac{x}{L}m\right)g, \left(\frac{M}{2} + \frac{(L-x)}{L}m\right)g$$

Illustration :

A uniform rod of length l and mass m is hung from two strings of equal length from a ceiling as shown, Determine the tension in the string.



Sol. *Balancing Torque about its left end*

$$T_2 \times \frac{3L}{4} = mg \times \frac{L}{2}$$

$$T_2 = 2\,mg/3$$

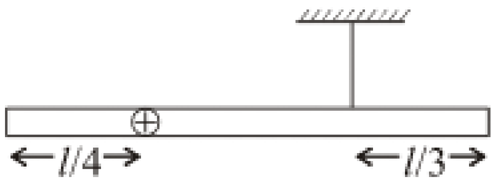
$$T_1 + T_2 = 2mg/3$$

$$T_1 = mg/3$$

$$= \left(\frac{mg}{3}, \frac{2mg}{3}\right)$$

Illustration :

Calculate hinge force in the following diagram.

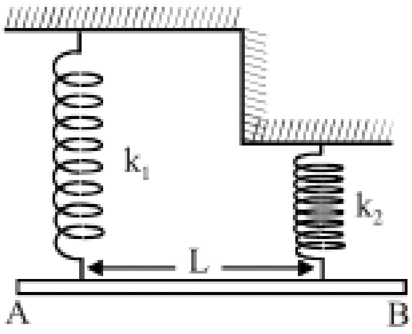


Sol. *By balancing torque about point where string is connected*

$$N \times \frac{5L}{12} = mg \times \frac{L}{6}$$

$$N = \frac{2}{5} mg$$

Q.8 *When a mass M hangs from a spring of length l, it stretches the spring by a distance x . Now the spring is cut in tow parts of lengths l/3 and 2l/3, and the two springs thus formed are connected to a straight rod of mass M which is horizontal in the configuration shown in figure. Find the stretch in each of the spring.*



Sol. *As it is given that the mass M stretches the original spring by a distance x, we have*

$$kx = Mg$$

$$x = \frac{Mg}{k}$$

The new force constants of the two springs can be given by using equation

$$k_1 = 3k \qquad \text{and} \qquad k_2 = \frac{3k}{2}$$

Let we take the stretch in the two springs be x_1 and x_2, we have for the equilibrium of the rod

$$k_1x_1 + k_2x_2 = Mg$$

$$3kx_1 + \frac{3k}{2}x_2 = Mg$$

From equation, we have

$$x_1 + \frac{x_2}{2} = \frac{x}{3}$$

As the rod is horizontal and in static equilibrium, we have net torque acting on the rod about any point on it must be zero. Thus we have torque on it about end A are

$$k_2x_2 L = Mg\frac{L}{2}$$

$$x_2 = \frac{Mg}{2k_2} = \frac{Mg}{3k} = \frac{x}{3}$$

Using this value of x_2 in equation, we have

$$x_1 = \frac{x}{6}$$

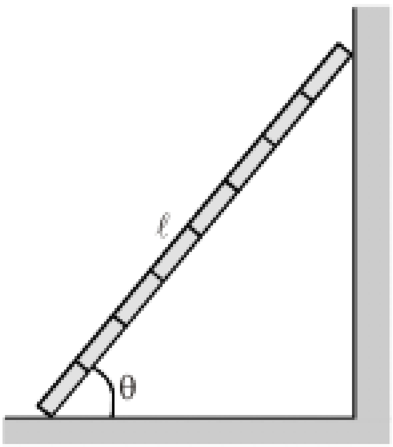
This can also be directly obtained by using torque zero about point B on the rod as

$$k_1x_1 L = Mg\frac{L}{2}$$

$$x_1 = \frac{Mg}{2k_1} = \frac{Mg}{6k} = \frac{x}{6}$$

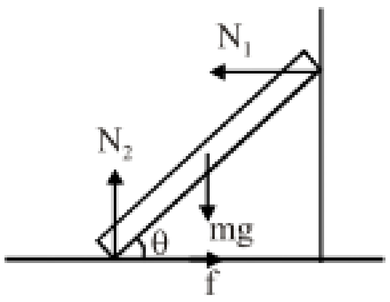
Illustration :

A uniform ladder of length ℓ rests against a smooth, vertical wall (figure). If the mass of the ladder is m and the coefficient of static friction between the ladder and the ground is $\mu_s = 0.375$ *and the minimum angle θ_{min} at which the ladder does not slip.*



A uniform ladder at rest, leaning against a smooth wall. The ground is rough.

Sol.



$$N_2 = mg$$
$$N_1 = f = \mu N_2 = \mu mg$$

By rotational equilibrium

$$mg\ \frac{\ell}{5}\cos\theta = N_1\ \ell \sin\theta$$

$$\mu = \frac{\cot\theta}{2} = \cot\theta = 0.75$$

Illustration :

Two small kids weighting 10 kg and 15 kg are trying to balance a see saw of total length 5.0m with the fulcrum at the centre. If one of the kids is sitting at an end, where should the other sit?

Sol. *By rotational equilibrium*

$$15x = 10 \times \frac{5}{2}$$

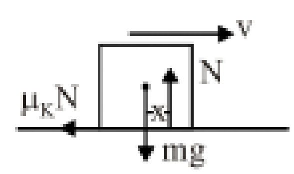
$$x = \frac{5}{3} = 1.7m$$

Illustration :

A block of height h is projected along a rough surface of coefficient of friction μ. Find the point of application of the normal force on the block for $\mu_k = 0.5$.

Sol.

$$\mu_K N \frac{h}{2} = Nx$$

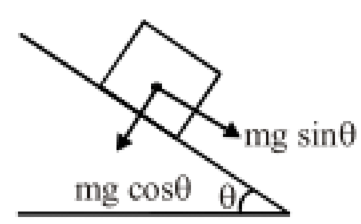


$$x = \frac{\mu_K h}{2}$$

$$x = \frac{h}{4}$$

Illustration :

A cubical block of mass m and edge a slides down a rough inclined plane of inclination with a uniform speed. Find the torque of the normal force acting on the block about its centre.



Sol. $f = mg \sin\theta$

Torque due to friction $= mg \sin\theta\, a/2$

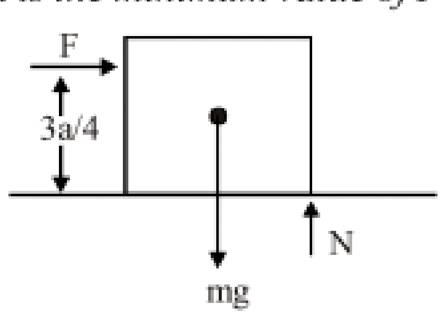
$= 0.5\ mg\ a \sin\theta$

Toppling

Illustration :

A uniform cube of side 'a' and mass m rests on a rough horizontal table. A horizontal force F is applied normal to one of the faces at a point directly above the centre of the face, at a height 3a/4 above the base. What is the minimum value of F for which the cube begins to tip about an edge?

Sol.

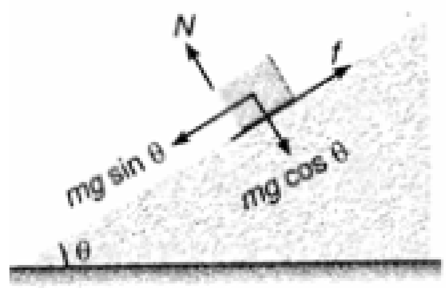


Normal shift upto extreme Right then balancing toqure about that point

$N \times 3a/4 = mg \times a/2$ *then* $N = \dfrac{2mg}{3}$

Illustration :

A uniform cylinder of height h and radius r is placed with its circular face on a rough inclined plane and the inclination of the plane to the horizontal is gradually increased. If μ is the coefficient of friction, then under what conditions the cylinder will slide before toppling.

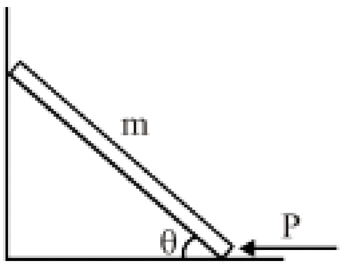


Sol. $f\frac{h}{2} < Nr \quad \Rightarrow \quad \mu N \frac{h}{2} < Nr$

$$\mu < \frac{2r}{h}$$

Practice Exercise

Q.1 Assuming frictionless contacts, determine the magnitude of external horizontal force P applied at the lower end of equilibrium of the rod. The rod is uniform and its mass 'm'



Q.2 A uniform metre stick of mass 200 g is suspended from the ceiling through two vertical strings of equal lengths fixed at the ends. A small object of mass 20 g is placed on the stick at a distance of 70 cm from the left end. Find the tensions in the two strings.

Answers

Q.1 $P = \frac{W}{2}\cot\theta$ or $P = \frac{mg}{2}\cot\theta$

Q.2 1.04 N in the left string and 1.12 N in the right.

Rotation about fixed axis

Since torque is a rotational analong of force, therefore, Newton's second law for rotational motion is given by

$$\tau_{net} = I\alpha \qquad \text{.... (i)}$$

Note that the above equation (i) is not a vector equation.

It is valid in two situation :

(i) The axis is fixed in position and direction.

(ii) The axis passes through the center of mass and is fixed in direction only the equation

$$\tau_{cm} = I_{cm}\alpha_{cm} \qquad \text{.... (ii)}$$

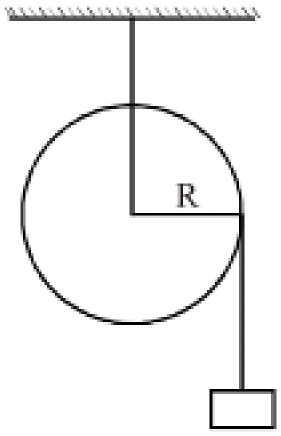
is valid even if the center of mass is accelerating.

Illustration :

A disc - shaped pulley has mass M = 4 kg and radius R = 0.5m. It rotates freely on a horizontal axis, as in figure. A block of mass m = 2 kg hangs by a string that is tightly wrapped around the pulley.

(a) What is the angular velocity of the pully 3 s after the block is released ?

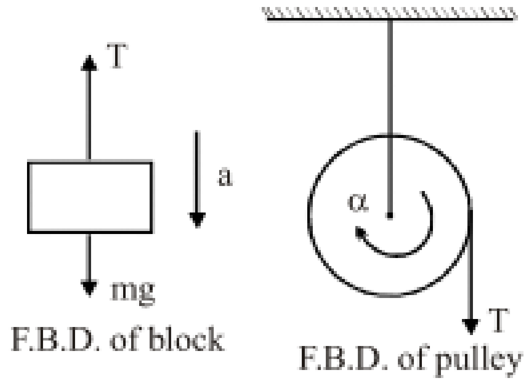
(b) Find the speed of the block after it has fallen 1.6m. Assume the system starts at rest.



Sol. *Since the string is tangential to the pulley, the torque on it due to-the tension is $\tau = TR$. The two forms of Newton's second law for the block and the pulley yield*

Block ($F = ma$) $mg - T = ma$

Pulley ($\tau = I\alpha$) $TR = \left(\frac{1}{2}MR^2\right)\alpha$



Applying Newton's second Law

For the disc, $I = \frac{MR^2}{2}$

$$\therefore \quad TR = \left(\frac{MR^2}{2}\right)\alpha \quad or \quad T = \frac{MR\alpha}{2} \quad ...(i)$$

Applying Newton's second Law on the block

$$F_{net} = ma$$

$$\therefore \quad mg - T = ma \quad ...(ii)$$

Since the block and the rim of the pulley have the same speed (the string does not slip), we have $v = \omega R$. *Thus, form equation (i) we find*

$$T = \frac{1}{2}Ma \quad ....(iii)$$

adding (ii) and (iii) leads to

$$a = \frac{mg}{m + \frac{M}{2}} \quad ....(iv)$$

Putting $m = 2$ *kg ;* $M = 4$ *kg ;* $R = 0\ 0.5$ *m;*

we get $a = 5\ m/s^2$

(a) To find ω *after 3 s, we use equation*

$$\omega = \omega_0 + at = 0 + \left(\frac{a}{R}\right)t = 30\ rad/s$$

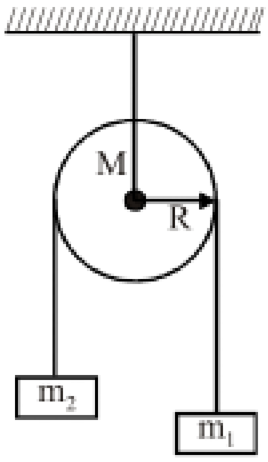
(b) T find the speed of the block we use

$$v_2 = v_0^2 + 2ay = 0 + 2\ (5\ m/s^2)\ (1.6m)$$

Thus $v = 4\ m/s$

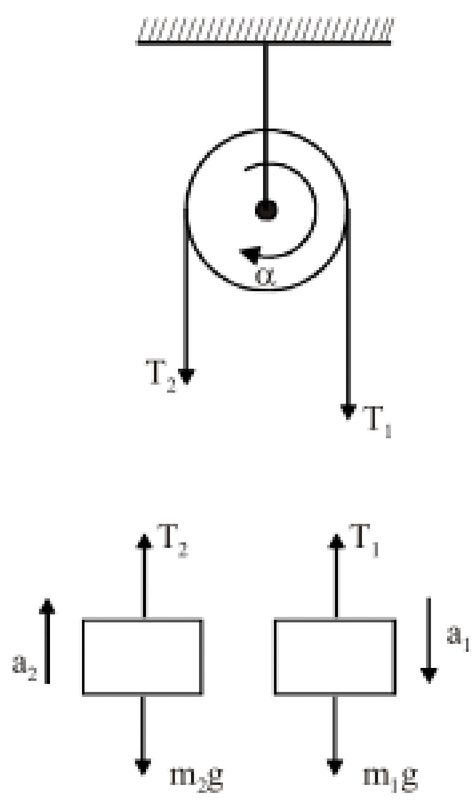
Illustration :

For the arrangement shown in the figure, the string is slightly wrapped over the pulley. Find the acceleration of each when released from rest. The string is not slipping over the pulley.



Sol. *The free body diagrams of the pulley and the blocks shown in the figure.*

Note that tension on two sides of the pulley are different. Why ?



Applying Newton's second law on the pulley, we ge

$$\tau = T_1 R - T_2 R = (T_1 - T_2)\, R$$

Since $$\tau = I\alpha = \left(\frac{MR}{2}\right)\alpha$$

Therefore, $$(T_1 - T_2)\, R = \left(\frac{MR^2}{2}\right)\alpha$$

or $$T_1 - T_2 = \left(\frac{MR}{2}\right)\alpha \qquad .....(i)$$

Applying Newton's Law on the blocks, we get

$$T_2\ mg = m_2 a_2 \qquad ......(ii)$$

$$m_1 g - T_1 = m_1 a_1 \qquad ......(iii)$$

Since the string is tightly wrapped over the pulley, therefore,

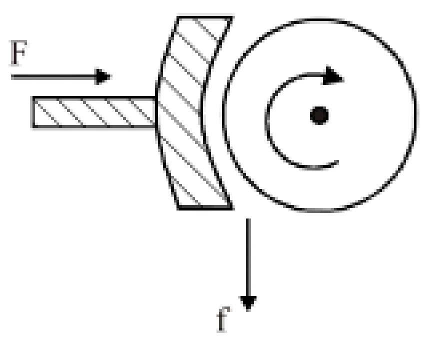
$$a_1 = a_2 = aR = a \qquad ......(iv)$$

Solving equation (i), (ii), (iii) and (iv), we obtain

$$a = \left[\frac{m_1 - m_2}{m_1 + m_2 + \frac{M}{2}}\right] g$$

Illustration :

A fly wheel of mass M = 2 kg and radius R = 40 cm rotates freely at 600 rpm. Its moment of inertia is $\frac{1}{2}MR^2$. A brake applies a force F = 10 N radially inward at the edge as shown in the figure. If the coefficient of frictions $\mu_k = 0.5$, how many revolutions does the wheel make before coming to rest ?



A wheel is slowed down by the application of force F. With the chosen positive sense, the frictional torque is negative.

Sol. *We choose the initial sense of the angular velocity as positive. The force of friction is $f = \mu_k F$ and its (counterclockwise) torque is $\tau = -fR$.*

Using $\tau = I\alpha$, we have

$$-(\mu_k F)R = \left(\frac{1}{2}MR^2\right)\alpha$$

or $$\alpha = -\frac{2\mu_k F}{MR} = 12.5 \text{ rad/s}^2$$

The angular rotation θ of the wheel is given by

$$\omega^2 = \omega_0^2 + 2\alpha\theta$$

Here $$\omega_0 = \frac{2\pi N}{60} = \frac{2\pi(600)}{60} = 20\pi \text{ rad/s}$$

$\therefore$ $0 = (20\,\pi\, rad/s)^2 + 2\,(-12.5\, rad/s^2)\theta$

Thus $\theta = 16\pi^2$ *rad.*

The number of revolutions ($16\pi^2$ rad) (1 rev/2π rad) = 8π revolutions.

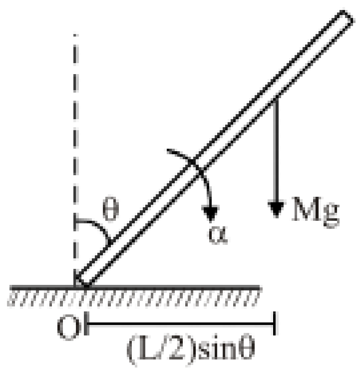
Illustration :

A uniform rod of length L and mass M is pivoted freely at one end as shown in the figure.

(a) what is the angular acceleration of the rod when it is at angle θ to the vertical ?

(b) What is the tangential linear acceleration of the free end when the rod is horizontal ?

The moment of inertia of the rod about one end is $\frac{1}{3}ML^2$.



The angular acceleration of the rod is produced by the torque due to its weight.

Sol. *Figure shown the rod at an angle θ to the vertical.*
Net torque about the point O is

$$\tau_0 = Mg\,\frac{L}{2}\sin\theta$$

Using II law of motion

$$\tau_0 = I_0\alpha$$

$$\frac{MgL}{2}\sin\theta = \frac{ML^2}{3}\alpha$$

Thus, $$\alpha = \frac{3g\sin\theta}{2L}$$

(b) When the rod is horizontal $\theta = \frac{\pi}{2}$ and $\alpha = \frac{3g}{2L}$. From equation the tangential linear acceleration is

$$a_t = \alpha L = \frac{3g}{2}$$

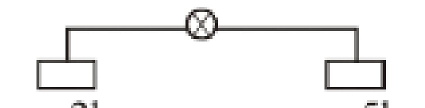
This is greater than the acceleration of an object in free-fall !

Practice Exercise

Q.1 A light rod of length 1 m is pivoted at its centre and two masses of 5 kg and 2 kg are hung from the ends as shown in figure

(a) Find the initial angular acceleration of the rod assuming that it was horizontal in the beginning.

(b) If the rod has a mass of 1 kg distributed uniformly over its length.



(i) Find the initial angular acceleration of the rod

(ii) Find the tension in the supports to the blocks of mass 2 kg and 5 kg.

Q.2 A meter stick is held vertically with one end on a rough horizontal floor. It is gently allowed to fall on the floor . Assuming that the end the floor does not slip, find the angular speed of the rod when it hits the floor.

Answers

Q.1 (a) $\frac{2g(m_1-m_2)}{\ell(m_1+m_2)}=\frac{60}{7}=8.4\,\text{rad/s}^2$

(b) (i) $\frac{2g(m_1-m_2)}{\ell(m_1+m_2+m_3/3)}=\frac{90}{22}=8.4\,\text{rad/s}^2$, (ii) $(m_1g-m_1\alpha\frac{\ell}{2})=29$ N; $(m_2g+m_2\alpha\frac{\ell}{2})=27.6$ N

Q.2 $\sqrt{\frac{3g}{\ell}}=5.4\ \text{rad/s}$

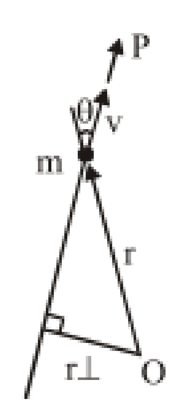
Angular Monentum

The orbital angular momentum : Irrespective of the path or trajectory of the particle, be it a straight line, curved path or a closed orbital path, the orbital angular momentum $\vec{L}$ of the particle at any position w.r.t. a reference point is

$$\vec{L}=\vec{r}\times\vec{P}$$

$$|\vec{L}|=rp\sin\phi$$

$$=r_{\perp}\times mv$$

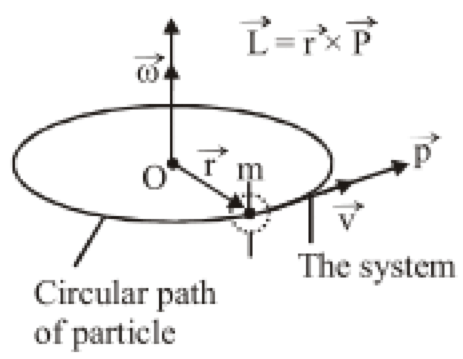


The r sin f is know as the moment arm, or lever arm designated as $r_{\perp}$.

The orbital angular momentum of particle in circular motion is expressed as

$$\vec{L}=mr^2\vec{\omega}$$

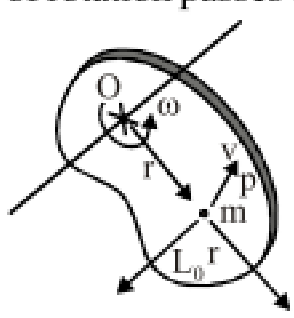
Note that direction of angular momentum vector $\vec{L}$ is parallel to angular velocity $\vec{\omega}$.Figure shows the right hand thumb rule for determining direction of angular momentum. Curl your finger in rotrational sense from $\vec{r}$ vector to $\vec{p}$ vector, then the thumb points in the direction of angular momentum.



Spin angular momentum of a rigid body

We consider two cases:

(i) Axis of rotation passes through centre of mass of the body, referred to as centroidal roation.



(according to right hand thumb rule)

(ii) Axis of rotation is shifted from centre of mass, but passes through the body, referred to as non-centroidal rotation.

For non-centroidal rotation. $\vec{L} = I_0\vec{\omega}$

For non-centroidal rotation. $\vec{L} = I\vec{\omega}$

Where I_0 is moment of inertia about centre of mass and I is moment of inertia about rotational axis, to be calculated with the help of parallel axis theorem.

Simultaneous spin and orbital motion

The total angular momentum is the vector sum of the spin and orbital angular momentum

$$\vec{L}_{total} = \vec{L}_{spin} + \vec{L}_{orbit}$$

$$= I_{CM}\,\vec{\omega}_{spin} + mr_{\perp}^2\,\vec{\omega}_{orbit}$$

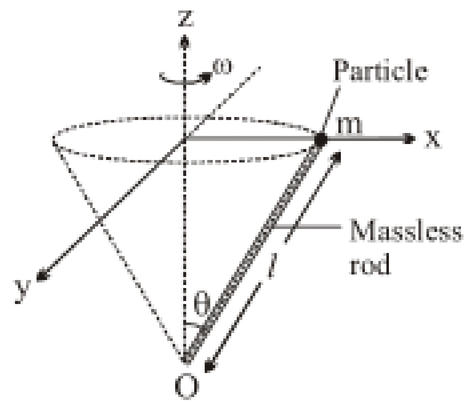
Angular momentum of an inverted conical pendulum

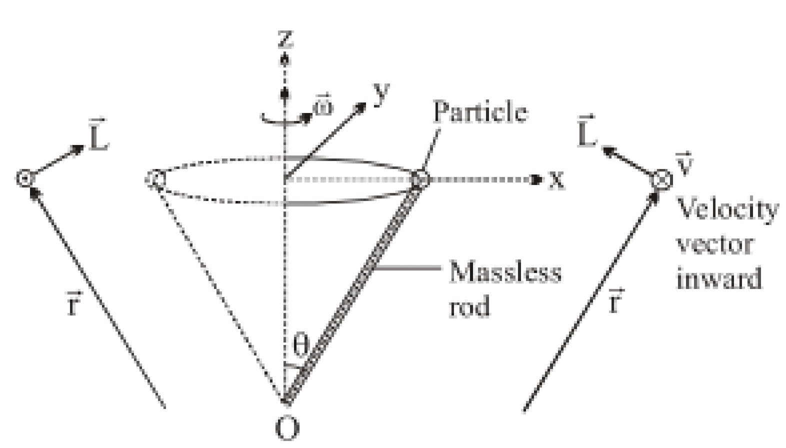
Angular momentum about O,

$\vec{r} = l\sin\theta\hat{i} + l\cos\theta\hat{k}$

$\vec{u} = (l\sin\theta)\omega\hat{j}$

$\vec{L} = \vec{r}\times m\vec{v} = ml^2\sin^2\theta\,\omega\hat{k} - ml\omega^2\sin\theta\cos\theta\,\hat{j}$





Concept : Angular momentum vector $\vec{L}$ is perpendicular to position vector $\vec{r}$ as well as momentum vector $\vec{p}$. The magnitude of $\vec{L}$ is constant but its direction is continuously varying. As the particle swings, $\vec{L}$ vector sweeps out a cone. The z-component of $\vec{L}$ is constant but the horizontal component travels around the circle with the particle.

Torque and Angular Momentum

When a number of force act on a particle, the net torque about origin 0 is sum of the torques due to each force.

$$\tau_{net} = \vec{r}\times\vec{F}_1 + \vec{r}\times\vec{F}_2 + ... = \vec{r}\times\sum_i \vec{F}_1 = \vec{r}\times\vec{F}_{net}$$

From Newton's second Law the net force is equal to rate of change of linear momentum. So we have

$$\tau_{net} = \vec{r}\times\vec{F}_{net} = \vec{r}\times\frac{d\vec{p}}{dt} \qquad ....(i)$$

As rate of change of angular momentum

$$\frac{d\vec{L}}{dt} = \frac{d}{dt}(\vec{r}\times\vec{p}) = \frac{d\vec{r}}{dt}\times\vec{p} + \vec{r}\times\frac{d\vec{p}}{dt} \qquad ....(ii)$$

As $$\frac{d\vec{r}}{dt}\times\vec{p} = \vec{\upsilon}\times m\vec{\upsilon} = 0$$

Thus eqn. (ii) becomes

$$\frac{d\vec{L}}{dt} = \vec{r}\times\frac{d\vec{p}}{dt} \qquad .....(iii)$$

On comparing eqns. (i) and (iii), we get

$$\vec{\tau}_{net} = \frac{d\vec{L}}{dt} \qquad ......(iv)$$

Illustration :

What is the angular momentum of a particle of mass $m = 2$ kg that is located 15 m from the origin in the direction 30° south of west and has a velocity $v = 10$ m/s in the direction 30° east of north ?

Sol. *In the figure, the x-axis points east. We know $r = 15$ m; $p = mv = 20$kg m/s. The angle between r and p is*

$(180^o - 30^o) = 150^o$

Thus, $L = rp\sin\theta = (15)(20)\sin 150^o = 150\text{kg m}^2/\text{s}$

We could also have used the moment arm $r\perp = 15\sin 30^o = 7.5$ m

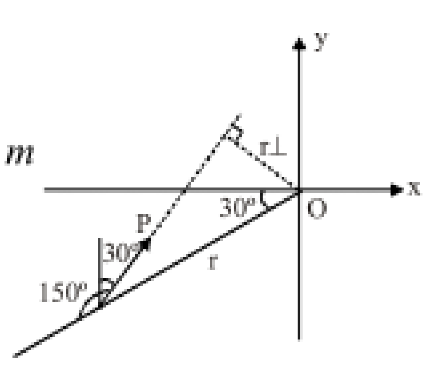
$L = r\perp p = (7.5)\times(20) = 150$ kg m/s

In unit vector notation,

$\vec{r} = 15\cos 30^o\ i - 15\sin 30^o\ j\text{m}$

$\vec{p} = 20\sin 30^o\ i + 20\cos 30^o\ j\ \ \text{kg m/s}$

Therefore,



The angular momentum of each particle may be found by using unit vector notation or by finding the magnitude from r,p and the direction of the right - hand rule.

$$L = \left(-\frac{15\sqrt{3}}{2}i - \frac{15}{2}j\right)\times(10i + 10\sqrt{3}j) = -150\ \text{kg m}^2/\text{s}$$

Illustration :

A disc of mass M and radius R rotating at an angular velocity ω about an axis perpendicular to its plane at a distance R/2 from the center, as shown in the figure. What is its angular momentum ?

The moment of inertia of a disc about the central axis is $\frac{1}{2}MR^2$.

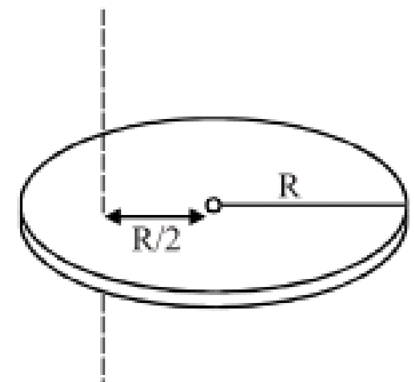
Sol. *The moment of inertia of the disc about the given axis may be found from the parallel axes theorem, equation* $I = I_{cm} + Mh^2$, *where h is the distance between the given axis and a parallel axis through the center of mass.*

Here $h = \frac{R}{2}$, *therefore,*

$$I = \frac{1}{2}MR^2 + M\left(\frac{R}{2}\right)^2 = \frac{3}{4}MR^2$$

The angular momentum is

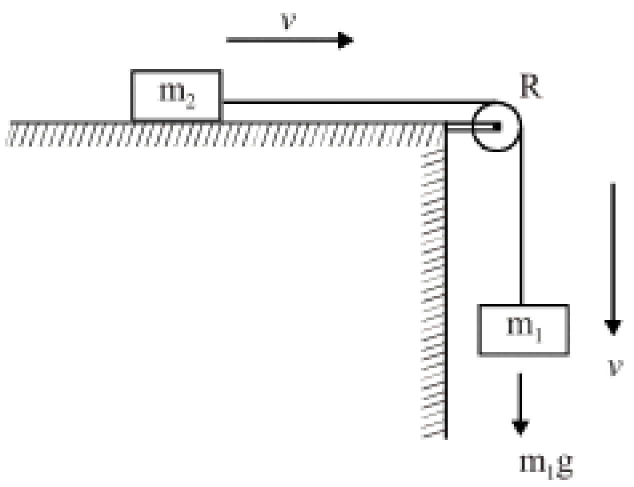
$$L = I\omega = \frac{3}{4}MR^2\,\omega$$



The axis of rotation at a distance R/2 from the center of the disc

Illustration :

Two blocks with masses $m_1 = 3$ kg and $m_2 = 1$ kg are connected by a rope that passes over a pulley of radius R = 0.2 m and mass M = 4 kg. The moment of inertia of the pulley about its center is $I = \frac{1}{2}MR^2$.
Use the concept of angular momentum to find the linear acceleration of the blocks. There is no friction. Assume that the c.m. of the block of mass m_2 is at a distance R above the center of the pulley.



The torque due to the weight of m_1 produces the change in angular momentum of the system

Sol. *If we take the origin at the center of the pulley, the angular momenta of the block are m_1vR and m_2vR and that of the pulley is $I\omega$. Therefore, the angular momentum is*

$$L = m_1vR + m_2vR + I\omega \qquad \text{.....(i)}$$

If the rope does not slip, then $v = \omega R$.

$$\therefore \quad L = (m_1 + m_2)vR + \frac{MR}{2}v$$

The net external torque about the center of the pulley is due to the weight of m_1

$$t_{ext} = r \perp F = R\,(m_1g) \qquad \text{.......(ii)}$$

Applying equation, $\tau_{ext} = \frac{d\vec{L}}{dt}$

We obtain

$$Rm_1g = (m_1 + m_2)\, Ra + \frac{MR}{2} a$$

or $$a = \frac{m_1 g}{m_1 + m_2 + M/2}$$

putting $m_1 = 3\ kg\ ;\ m_2 = 1\ kg;\ M = 4\ kg\ ;\ R = 0.2\ m$

$$a = \frac{(3)(10)}{3 + 1 + 4/2} = 5\ m/s^2$$

Conservation of Angular Momentum

If the next externa torque on a systme is zero, the total angular momentum is constant in magnitude and direction.

That is , if $\tau_{ext} = 0$ $\frac{dL}{dt} = 0$

Thus, L = constant

For rigid body rotating about a fixed axis.

$$L_f = L_i$$

or $$L_f\,\omega_f = I_i\,\omega_i$$

Angular Impulse

In complete analogy with the linear momentum, angular impulse is defined as

$$\tau = \int \tau_{ext}\, dt$$

Using Newton's second law for rotation motion,

$$t_{ext} = \frac{dL}{dt}$$

$$\therefore \quad \vec{\tau}\, t = \Delta\vec{L}_f - \vec{L}_i$$

The net angular impulse acting on a rigid body is equal to the change in angular momentum of the body. This is called the impulse – momentum theorem for rotational dynamics.

Illustration :

A disc of moment of inertia 4 kg m^2 is spinning freely at 3 rad/s. A second disc of moment of inertia 2 kg m^2 slides down the spindle and they together.

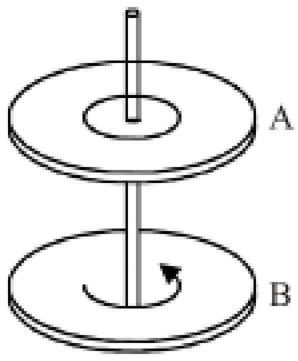
(a) What is the angular velocity of the combination ?

(b) What is the change in kinetic energy of the system ?

Sol. *Since there are no external torques acting, we may apply the conservation of angular momentum.*

$$I_f\omega_f = I_i\omega_i$$
$$6\omega_f = 4\times3$$
$$\omega_f = 2 \text{ rad/s}$$

(b) The kinetic energies before and after the collision are



Disc A, initially not rotating, slip down a spindle into disc B that initially rotating freely

$$K_i = \frac{1}{2}I_i\,\omega_i^2 = 18\ J;$$

$$K_f = \frac{1}{2}I_f\,\omega_f^2 = 12\ J$$

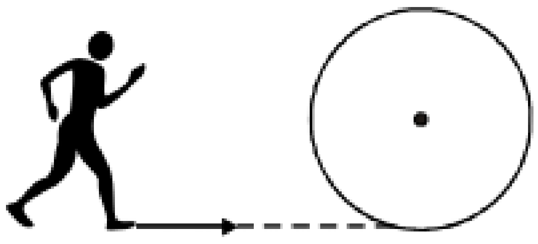
The change is

$$\Delta K = K_f - K_i = -6\ j.$$

In order for the two discs to spin together at the same rate, there had to be friction between them, The lost kinetic energy is converted with thermal energy.

Illustration :

A man of mass m = 80 kg runs at a speed u = 4 m/s along the tangent to disc-shaped horizontal platform of mass M = 160 kg and radius R = 2m. The platform is initially at rest but can rotate freely about an axis through its center.



Take $I = \frac{1}{2}MR^2$.

(a) Find the angular velocity of the platform after the man jumps on it.

(b) He then walk to the center. Find the new angular velocity. Treat the man as a point particle.

Sol. *Can we apply the conservation of linear momentum ?*

No, it con not be applied because the axle exerts an external force on the system man + platform. Can we apply the conservation of angular momentum ! Yes, since the axle dose not exert any torque, we may use the conservation of angular momentum.

Can we apply kinetic energy for the collision between that man and the platform ? Why ?

(a) We choose the origin at the center of platform as shown in figure. When the man runs in a straight line, his initial angular momentum about this origin is $L = r\perp p$,

where in this case $r\perp = R$

so $L_i = muR$

After he jump on, one must take into account his contribution mR^2 to the moment of inertia. The final angular momentum, $L = I\omega$, is

$$L_f = \left(\frac{1}{2}MR^2 + mR^2\right)\omega$$

When we use set $L_f = L_i$, we find

$$\omega = \frac{mu}{(M/2+m)R}$$

Putting $m = 80\ kg$; $M = 160\ kg$; $u = 4\ m/s$; $R = 2\ m$

$$\text{We get } \omega = \frac{(80)(4)}{\left(\frac{160}{2}+80\right)2} = 1\ rad/s$$

(b) When the man reaches the center, his contribution to the moment of inertia is zero. The final angular momentum of part (a) is the initial value for (b);

$$L_i = \left(\frac{1}{2}MR^2 + mR^2\right)\omega_1 = 640\ kg\ m^2/s$$

$$L_f = \left(\frac{MR^2}{2}\right)\omega_2 = 320\,\omega_2$$

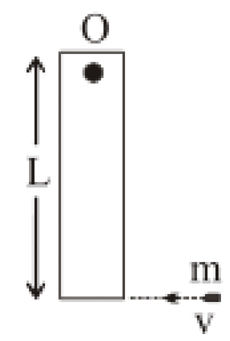
we get $\omega_2 = 2\ rad/s$

Table : Analogy between Rotational Dynamics and Linear Dynamics.

	Quantity	Linear	Rotational
1.	Inertia	m	$S\ m_i r_i^2$ or $\int r^2 dm$
2.	Newton's Second Law	$F_{ext} = ma$	$\tau_{ext} = I\alpha$
		$\vec{F}_{ext} = \frac{d\vec{p}}{dt}$	$\vec{\tau}_{ext} = \frac{d\vec{L}}{dt}$
3.	Work	$W_{lin} = \int \vec{F}d.\vec{s}$	$W_{rot} = \int \vec{\tau}\ .d\theta$
4.	Kinetic Energy	$K_{lin} = \frac{1}{2}mv^2$	$K_{rot} = \frac{1}{2}I\omega^2$
5.	Work Energy Theorem	$W_{lin} = \Delta K_{line}$	$W_{rot} = \Delta K_{rot}$
6.	Impulse	$I = \int F_{ext}.dt$	$J = \int \tau_{ext}.dt$
7.	Momentum	$p = mv$	$L = I\omega$
8.	Impulse momentum Theorem	$\vec{I} = \Delta\vec{p}$	$\vec{J} = \Delta\vec{L}$
9.	Power	$P = \vec{F}.\vec{v}$	$P = \vec{\tau}.\vec{\omega}$

Practice Exercise

Q.1 A wooden long of mass M and length L is hinged by a frictionless nail at O. A bullet of mass m strikes with velocity v and sticks to it. Find angular velocity of the system, immediately after the collision, about O.



Q.2 A disc of mass M and radius r is rotating about its axis with angular velocity ω. Now if a mass m falls vertically on its rim and sticks to it. Then find final angular velocity of the combined system.

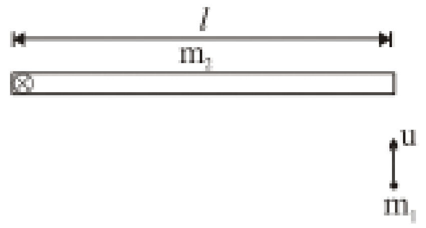
Q.3 A man of mass 100 kg stands at the rim of a turn-table of radius 2m, moment of inertia 4000 kg-m^2 mounted on a vertical frictionless shaft at its centre. The whole system is initially at rest. The man walks along the outer edge of the turn-table with a velocity of 1m/s relative to the earth.

(a) With what angular velocity and in what direction does the turn-table rotate?

(b) Through what angle will it have rotated when the man reaches his initial position on the turn-table?

(c) Through what angle will it have rotated when the man reaches his initial position relative to earth?

Q.4 In horizontal smooth plane. If particle sticks after collision to the rod, find



(a) Final angular velocity

(b) The impulse on particle

Q.5 A particle having mass 2 kg is moving along straight line 3x + 4y = 5 with speed 8 m/s. Find angular momentum of the particle about origin. x and y are in meters.

Q.6 A particle having mass 2 kg is moving with velocity $(2\hat{i} + 3\hat{j})$ m/s. Find angular momentum of the particle about origin when it is at (1, 1, 0).

Q.7 A wheel of moment of inertia 0.500 kg-m^2 and radius 20.0 cm is rotating about its axis at an angular speed of 20.0 rad/s. It picks up a stationary particle of mass 200 g at its edge. Find the new angular speed of the wheel.

Answers

Q.1 $\omega = \dfrac{3mv}{L(M+3m)}$ Q.2 $\omega' = \left(\dfrac{M}{M+2m}\right)\omega$

Q.3 (a) $-\dfrac{1}{20}$ rad/s (b) $-\dfrac{2\pi}{11}$ radian (c) $\theta_t = -\dfrac{\pi}{5}$ radian Q.4 (a) $\omega = \dfrac{3m_1u}{(m_2+3m_1)}$ (b) $\dfrac{m_1m_2u}{m_2+3m_1}$

Q.5 16 kg m^2/s Q.6 $2\hat{k}$ kg m^2/s Q.7 19.7 rad/s

Instantaneous Axis of rotation (IAOR or ICR)

It is the axis about which the motion of a rigid body undergoing plane motion is assumed to be pure rotational motion. It is always perpendicular to the plane of motion of rigid body and instantaneously remains at rest. The point of intersection of instantaneous axis of rotation with the plane of motion of the remains at rest. The point of intersection of instantaneous axis of rotation with the plane of motion of the rigid body is called instantaneous centre of rotation (ICR) about which all points of the rigid body are assumed to be going in circles of different radii equal to their respective distances from ICR with the same ω and α as that about CM of the rigid body at that instant.

$\therefore$ Velocity of IAR = 0 i.e. $v_A = 0$

By finding the position of IAR, we can easily find the velocity of any point of rigid body at that instant.

$\vec{v}_P = \vec{\omega} \times \vec{r}_{P,A}$

We can also find the acceleration of any point P of rigid body at that instant provided the acceleration of IAR should be know.

$$\vec{a}_P = \vec{a}_{P,A} + \vec{a}_A$$

(i) Kinetic energy of the rigid body is $K = \frac{1}{2} I_A \omega^2$

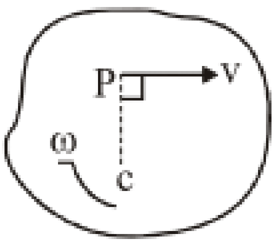
(ii) Angular momentum of rigid body about IAR is $L_A = I_A\omega$

(iii) $\tau_A = I_A\alpha$, where τ_A includes the torque of pseudo force acting on the CM about IAR also.

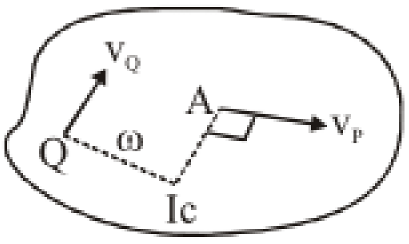
How to find the position of ICR

The position of ICR can be fund out in the following cases.

Case-I :



If the velocity of a point of the body and angular velocity are given.



Draw a line perpendicular to $\vec{v}$, the instantaneous centre must be lying on this line a distance 'r' given by $r = v/\omega$

Case II :

If the lines of action of two non-parallel velocities of two points of the rigid boy are given.

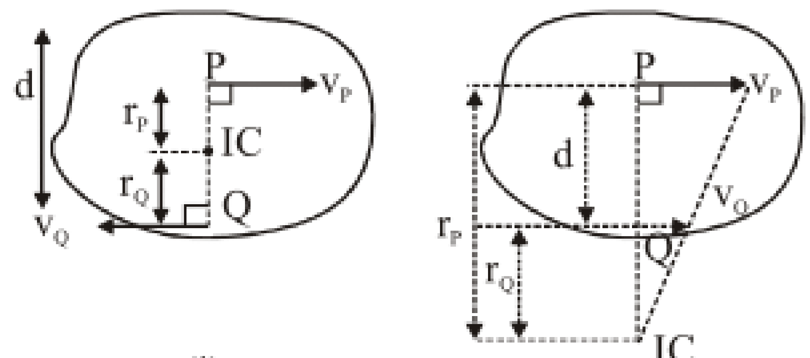
Draw the normals on the two non-parallel velocities $\vec{v}_P$ and $\vec{v}_Q$ at points P and Q, respectively. The point of intersection of these normals is the instantaneous centre at that instant.

Case-III :

If magnitudes and direction of two parallel velocities are given.

In figure (i), if the two velocities $\vec{v}_P$ and $\vec{v}_Q$ are in the opposite direction, then IC must be lying in between P and Q.

$$\frac{v_P}{v_Q} = \frac{r_P}{r_Q} \qquad \text{and } r_P + r_Q = d$$



(i) (ii)

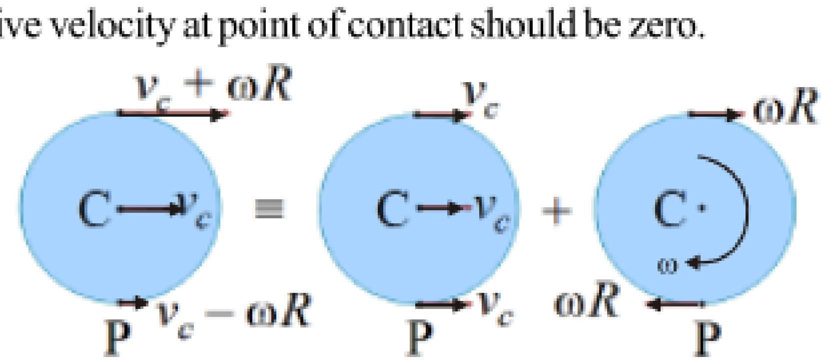
In figure (ii), if the two velocities $\vec{v}_P$ and $\vec{v}_Q$ are in the same direction, the IC must be lying outside PQ (near P if $v_P < v_Q$ and near Q if $v_p > v_Q$)

$$\frac{v_P}{r_Q} = \frac{r_P}{r_Q} \text{ and } r_P - r_Q = d$$

Rolling Motion

Pure rolling means no sliding. Now, the motion of any body can be divided into pure translation & pure rotation. And we can see rotation about any axis. So; for a wheel rolling on a horizontal surface, I m taking its COM as reference point to study its motion. If C is the reference point, then the wheel can be considered rotating about C. And the point C would be translating with velocity vc.

Sliding refers to the condition under which two bodies in contact have relative velocity. And under pure rolling, the relative velocity at point of contact should be zero.



Now for the wheel shown, point P is in contact with the ground. The point P on the ground has zero velocity. Thus, the point P on the wheel should also have zero velocity.

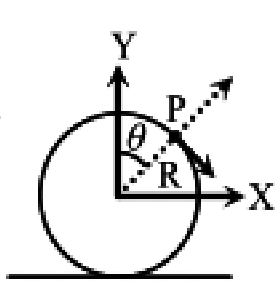
Trajectory of a point on a periphery of the wheel is a cycloid

$$\vec{v} = \frac{d\vec{r}}{dt} = v\left[\left(1+\cos\omega t\right)\hat{i} + \sin\omega t\left(-\hat{j}\right)\right]$$

$$\vec{r} = \int d\vec{r} = v\int\left[\left(1+\cos\omega t\right)\hat{i} + \sin\omega t\left(-\hat{j}\right)\right]dt$$

$$= \omega R\left(t + \frac{1}{\omega}\sin\omega t\right)\hat{i} + \frac{1}{\omega}\cos\omega t\left(\hat{j}\right)$$

$$= R\left[\left(\omega t + \sin\omega t\right)\hat{i} + \cos\omega t\left(\hat{j}\right)\right]$$



$$\frac{d\vec{v}}{dt} = \frac{dv}{dt}\left[(1+\cos\omega t)\hat{i} + \sin\omega t\left(-\hat{j}\right)\right] + \omega v\left[-\sin\omega t\,(\hat{i}) - \cos\omega t\,(\hat{j})\right]$$

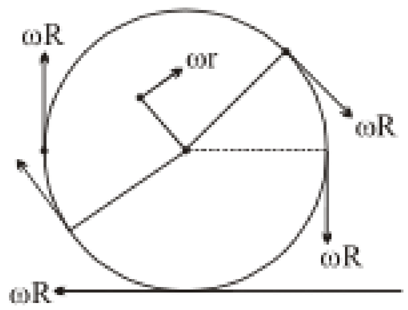
$$= \frac{dv}{dt}(\hat{i}) + \frac{dv}{dt}\left[(\cos\omega t)\hat{i} + \sin\omega t\left(-\hat{j}\right)\right] + \frac{v^2}{R}\left[-\hat{r}\right]$$

$$= a_c\left(\hat{i}\right) + \alpha R\,(\hat{\tau}) + \frac{v^2}{R}\,(-\hat{r})$$

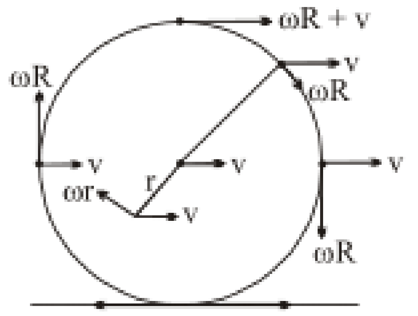
Kinematics of Boby in pure Rolling

1. **Velocity**

wrt centre

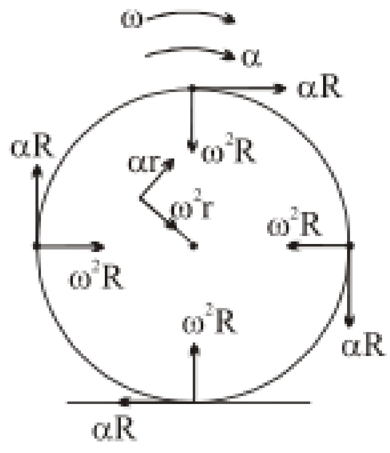


wrt ground



2. **Acceleration**

wrt centre



wrt ground

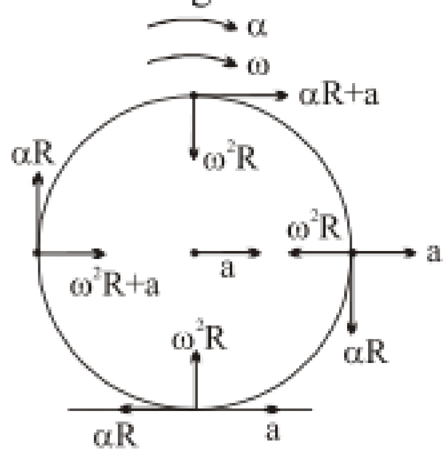
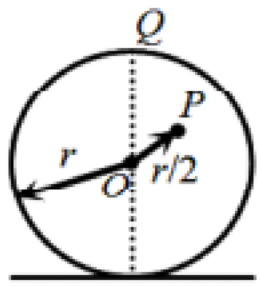


Illustration:

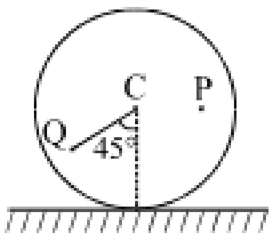
A disc of radius r rolls without slipping on a rough horizontal floor. If velocity of its center of mass is vo, then find the velocity of point P, (*OP* = *r*/2 *and* $\angle QOP = 60^o$).



Ans: 7vo/2

Illustration:

A disc is rolling without slipping with angular velocity ω. P and Q are two points equidistant from the centre C. Find the order of magnitude of velocity ?



Ans. $v_P > v_C > v_Q$

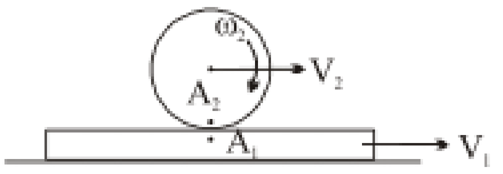
Constraint Equation:

Written at contact points where no slipping takes place.

$$\begin{bmatrix} \text{velocity of a contact} \\ \text{point on } 1^{st} \text{ rigid body} \end{bmatrix} = \begin{bmatrix} \text{velocity of same contact} \\ \text{point on } 2^{nd} \text{ rigid body} \end{bmatrix}$$

Write contraint equation for following exmaples:

Illustration :



Given velocity of C.M. w.r.t. ground is V_2 and V_1 is velocity of platform (w.r.t. ground). Find ω_2 is angular velocity of body about C.M.

Sol. $V_2 - \omega_2 R = V_1$

Friction and rolling

To get the direction of friction in pure rolling, we set two criteria's:

(a) Acceleration : Its direction should be such that vector sum all forces comply with it.

(b) Angular acceleration: Its direction should be such that vector sum all torques comply with it.

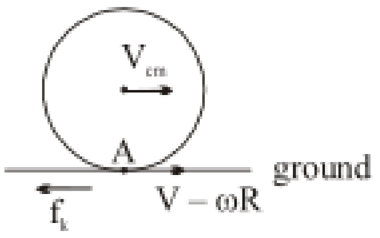
So, from above point we understand that if we show wrong direction of friction in our free body diagram, we will get a negative answer. So, direction of friction force in pure rolling should not be cause of concern.

Here it is also to be noted that in case of sliding, (kinetic friction) we can not take any arbitrary direction of friction and solve that question. In case of rolling static friction is unknown so after sloving, its value comes out negative if taken wrongly, but in case of kinetic friction its value is known beforehand (= μN), so there is no case of value of a known quantity coming negative after solving.

Here are few examples wherein, one might try to guess the direction of friction force.

Rolling with sliping

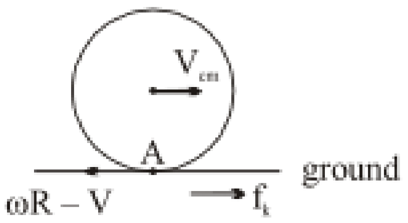
Case-I $V_{cm} > \omega R$



Note : The friction will be kinetic in nature & its magnitude can be determined using $f_k = \mu_k N$. Direction will be opposite to V_{cm} (because pt. of contact A is moving forward w.r.t. ground)

(This is how barkes stop a car)

Case -2 $V_{cm} < \omega R$



(This is how a car accelerate because of friction)

Case -3 $V_{cm} = \omega r$

Known as perfect rolling

Maximum problems we come across this situation

Note : 1. The friction will be be static in nature. Its magnitude be determined, it will very from O to $\mu_s N$.

2. Even its direction can not be perdicted

3. The total work done by static friction is zero. Thus mechanical cnergy of the system will remain conserved.

Application of Newton's Second Law in Rolling Motion

1. Write $F_{net} = M\, a_{cm}$ for the object as if it were a point-mass, that is, ignoring ratation.

2. Write $\tau = I_{cm}\, \alpha$ as if the object were only rotating about the centre of mass, that is ignoring translation.

3. Use of no-slip condition

4. Solve the resulting equations simultaneously for any unknown.

Caution :

- In general, it is not the case that $f = \mu N$
- Be certain that the sign convention of forces and torques are consistent.

Illustration :

Figure shown a shpere of mass M and radius R that rolls without slipping down an incline. Its moment of inertia about a central axis is $\frac{2}{5}MR^2$.

(a) Find the linear acceleration of the center of mass.

(b) What is the minimum coefficient of friction required for the sphere to roll without slipping ?

Sol. *Since the sphere is not driven by a chain or an axle the force of friction must be directed backward, up the slope.*

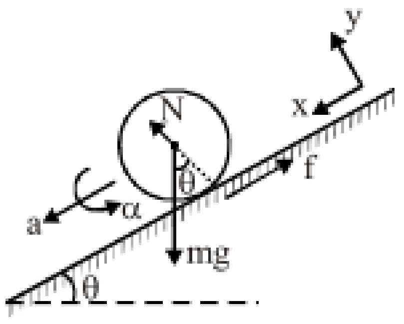
If there is no slipping, the point of contact is simultaneously at rest and so the friction is static.

The linear acceleration a of the c.m. and the angular acceleration α *are assumed as shown in the figure.*

Applying Newon's Second Law

$$\Sigma F_x = ma \qquad Mg\sin\theta - f = Ma \qquad ......(i)$$

$$\Sigma\tau = I\alpha \qquad fR = I\alpha = \frac{2}{5}MR^2\,\alpha \qquad ......(ii)$$



A shpere rolls down an incline. Since the sphere is not driven, the force of friction is direction up the incline.

Since the sphere rolls without slipping, the speed of the centre is $v = \omega R$.

By differentiating it with respect to time, we get

$$a = \alpha R. \qquad ......(iii)$$

Solving equation (i), (ii) & (iii)

we get $$f = \frac{2}{5}\,Ma \qquad ......(iv)$$

and $$a = \frac{5}{7}g\sin\theta \qquad ......(v)$$

(b) Substituting (v) into (iv) yields

$$f = \frac{2}{7}Mg\sin\theta \qquad ......(vi)$$

We use equation (vi) to find the minimum coefficient of friction required for the sphere to roll without slipping.

By definition, $f = \mu N$ *where* $N = Mg\cos\theta$. *Combining this with equation (vi) we have*

$$\mu = \frac{2}{7}\tan\theta$$

If the coefficient of static friction is less than this value, the sphere will slip as it rolls down the incline.

Illustration:

A sphere of mass m and radius R is placed at rest on a plank of mass M which is placed on a smooth horizontal surface as shown in the figure. The coefficient of friction between the sphere and the plank is μ. At t = 0, a horizontal velocity v_0 is given to the plank. Find the time after which the sphere starts rolling.

Sol. *Sphere :*

$$a_c = \frac{f}{m} = \mu g$$

$$\alpha = \frac{\tau_C}{I_C} = \frac{f\ R}{\frac{2}{5}mR^2} = \frac{5}{2}\frac{\mu g}{R}$$

After time t

$$v_c = a_c t = \mu g t$$

$$\omega = at = \frac{5}{2}\frac{\mu g}{R}t$$

The velocity of the point of contact is

$$v = v_c + \omega R = vgt + \frac{5}{2}\mu g t = \frac{7}{2}\mu g t$$

Planck :

$$\text{Retardation } a = \frac{f}{M} = \frac{\mu m g}{M}$$

$$\text{Instantaneous velocity } v = v_0 - \frac{\mu m g}{M}t$$

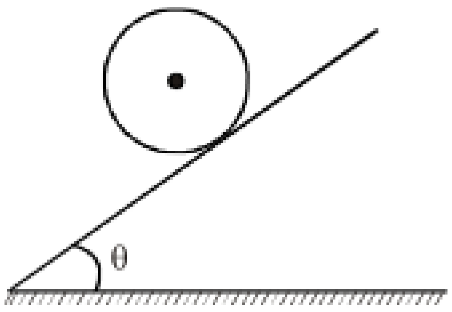
Condition of pure rolling

$$v = \frac{7}{2}\mu g t = v_0 - \frac{\mu m g}{M}t$$

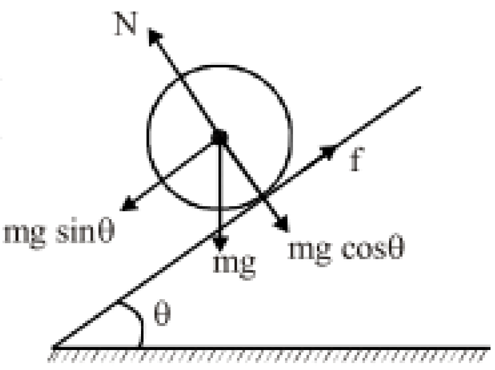
$$\text{or} \qquad t = \frac{v_0}{\left[\frac{7}{5} + \frac{m}{M}\right]\mu g}$$

Illustration :

A uniform disc of mass m and radius R is rolling without slipping up a rough incline plane which makes an angle 30° with the horizontal. If the coefficient of static and kinetic friction are each equal to μ and the only force acting on the disc are gravitational and frictional, then find direction and magnitude of the frictional force acting on it.



Sol. *Since disc does not slip hence frictional force is static and static friction can have any value between 0 and μN. Component of mg parallel to the plane is mg sinθ which is opposite to the direction of motion of the centre of the disc, and hence speed of the centre of mass decreases. For pure rolling the relation $v_{c.m.} = \omega R$ must be obeyed. Therefore ω must decrease. Only frictional force can provide a torque about the centre.*



Torque due to friction must be opposite to the $\vec{\omega}$. There frictional force will act up the plane Now, for translational motion

$$mg\sin\theta - f = ma_{c.m.} \quad \text{....(i)}$$

For rotational motion

$$fR = I\alpha, \text{ where } I = \text{M.I. of the disc about centre.}$$

$$= I\frac{a}{R}, \text{ as } a = \alpha R$$

$$\Rightarrow \quad a_{c.m.} = \frac{fR^2}{I} \quad \text{....(ii)}$$

For (i) and (ii) we get,

$$f = \frac{mg\sin\theta}{1 + \frac{mR^2}{I}}$$

Putting the value of θ and I we get

$$f = mg/6$$

Kinetic Energy of a Rolling Body

Since the rolling motion is a combination of linear velocity of the center and rotational motion about the center. Therefore, the total kinetic energy of a rolling body is given by

$$K = \frac{1}{2}mv_c^2 + \frac{1}{2}I_c\omega^2 \quad \text{(i)}$$

Where $\frac{1}{2}mv_c^2$ is the translational kinetic energy and

$\frac{1}{2} I_c\omega^2$ is the rotational kinetic energy about the center of mass

In pure rolling motion, $v_c = \omega R$

$$\therefore \quad K = \frac{1}{2} m(\omega R)^2 + \frac{1}{2} I_c\omega^2$$

$$\text{or} \quad K = (I_c + mR^2)\,\omega^2$$

Using parallel axes therem, the term $I_c + mR^2$ given the moment of inertia about the point of contact, therefore,

$$I_0 = I_c + mR^2$$

$$\text{and} \quad K = \frac{1}{2} I_0\omega^2 \qquad \text{(ii)}$$

Note that equation (ii) give the rotational kinetic energy of the wheel aobut the point of contact.

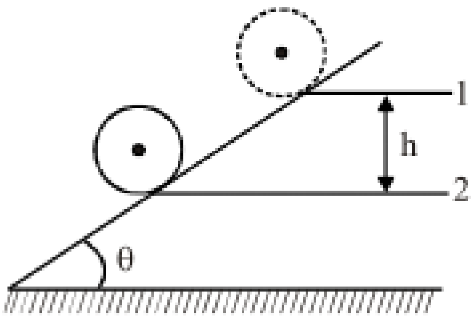
Illustration

A solid cylinder of mass m and radius r starts rolling down an inclined plane f inclination θ. friction is enough to prevent slipping. Find the speed of tis centre of mass when, its centre of mass has fallen a height h.

Sol. *Consider the two shown positions of the cylinder. As it does not slip, total mechanical energy will be conserved.*

Energy at position 1 is $E_1 = mgh$

Energy at position 2 is $E_2 = \frac{1}{2} mv^2_{c.m.} + \frac{1}{2} I_{c.m.}\,\omega^2$



$$\because \quad \frac{V_{c.m.}}{r} = \omega, \text{ and } I_{c.m.} = \frac{mr^2}{2}$$

$$\Rightarrow \quad E_2 = \frac{3}{2} mv^2_{c.m.}$$

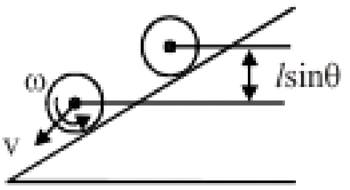
From COE, $E_1 = E_2$

$$\Rightarrow \quad V_{c.m.} = \sqrt{\frac{3}{2} gh}$$

Illustration :

A sphere of radius r starts rolling down an incline of inclination. Find the speed of its CM when it has covered a distance l.

Sol. $mg\, l\sin\theta = \frac{1}{2} I\omega^2 + \frac{1}{2} mv^2$



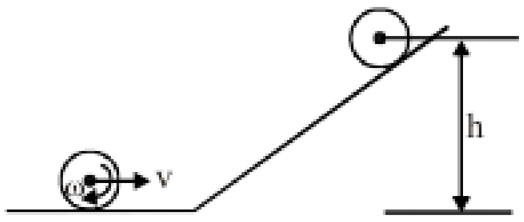
$v = \omega r$

By sloving we get

$v = \sqrt{10g\,l\sin\theta/7}$

Illustration :

A ball of radius R and mass m is rolling without slipping on a horizontal surface with velocity of its centre of mass v. It then rolls without sloping up a hill to a height h before momentarily coming to rets. Find h.



Sol. $\frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 = mgh$

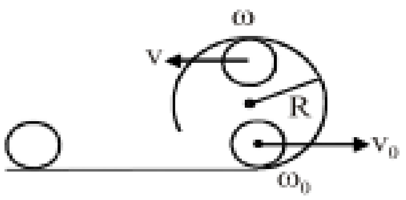
$v = \omega R$, $I = \frac{2}{7}mR^2$

$h = \frac{7v_{CM}^2}{10g}$

Illustration:

Figure shows a rough track, a portion of which is in the form of a cylinder of radius R. With what minimum linear speed should a sphere of radius r be set rolling on the horizontal part so that it completely goes round the circle on the cylindrical part?

Sol.

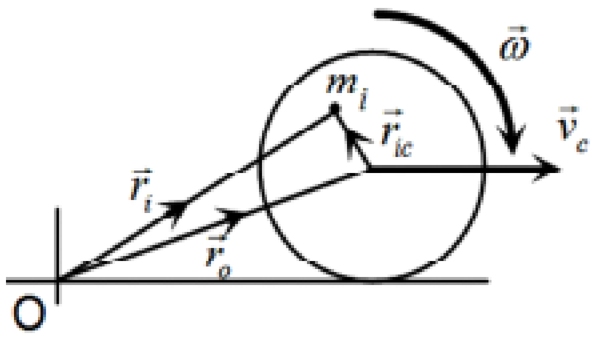


$mg = \frac{mv_C^2}{R-r}$

$\frac{1}{2}mv_0^2 + \frac{1}{2}I\omega_0^2 = mg2(R-r) + \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2$

By solving we get $\sqrt{27g(R-r)/7}$

Angular momentum of a rigid body in planar motion about O



$$\vec{l}_i = \sum_{i=1}^{n} m_i (\vec{r}_i \times \vec{v}_i)$$

$$= \sum_{i=1}^{n} m_i \left[(\vec{r}_o + \vec{r}_{ic}) \times (\vec{v}_c + \vec{\omega} \times \vec{r}_{ic})\right]$$

$$= \sum_{i=1}^{n} m_i (\vec{r}_o \times \vec{v}_c) + \vec{r}_o \times \sum_{i=1}^{n} m_i (\vec{\omega} \times \vec{r}_{ic}) + \sum_{i=1}^{n} m_i (\vec{r}_{ic} \times \vec{v}_c) + \sum_{i=1}^{n} m_i \left[\vec{r}_{ic} \times (\vec{\omega} \times \vec{r}_{ic})\right]$$

$$= \left[\sum_{i=1}^{n} m_i\right](\vec{r}_o \times \vec{v}_c) + \vec{r}_o \times \sum_{i=1}^{n} m_i \vec{v}_{ic} + \left[\sum_{i=1}^{n} m_i \vec{r}_{ic}\right] \times \vec{v}_c + \sum_{i=1}^{n} m_i \left[\vec{r}_{ic}^{\,2} \vec{\omega} - (\vec{r}_{ic} \cdot \vec{\omega}) \vec{r}_{ic}\right]$$

$$= M(\vec{r}_o \times \vec{v}_c) + \vec{\omega} \sum_{i=1}^{n} m_i \vec{r}_{ic}^{\,2} = M(\vec{r}_o \times \vec{v}_c) + I_{COM}\, \vec{\omega}$$

Angular momentum of a rigid body in planar motion about O: $\vec{l} = M(\vec{r}_o \times \vec{v}_c) + I_{COM}\, \vec{\omega}$

Spin and Orbital Angular Momentum

For a rigid body undergoing linear and rotational motion, the total angular momentum may be split into two pats – the orbital angular momentum and the spin angular momentum. The orbital angular momentum L_0 is the angular momentum relative to the center of mass.

The orbital term treats the system as a point particle at the center of mass, where as the spin term is the sum of the angular momenta of the particles relative to the center of mass. The total angular momentum relative to the origin O in an inertial frame is the sum:

$$L = L_0 + L_{cm}$$

Illustration :

A solid sphere of mass M and radius R rolls without slipping on a horizontal surface as shown in the figure. Find the total angular momentum of the sphere with respect to the origin O fixed on the ground.

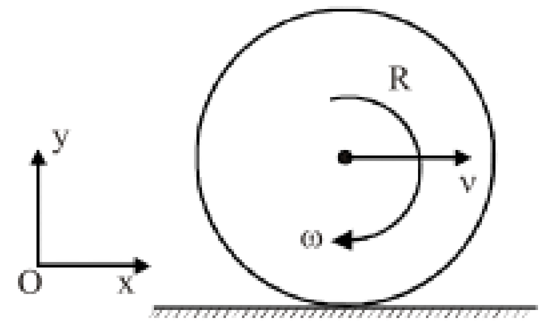
Sol. *Let us assume the clockwise sense of rotation positive.*

Orbital angular momentum about O is

$$L_0 = MvR$$

Spin angular momentum about c.m. is

$$L_{c.m.} = I\omega = \frac{2}{5} MR^2\omega$$



The total angular momentum is

$$L = L_0 + L_{cm} = MvR + \frac{2}{5}MR^2\omega$$

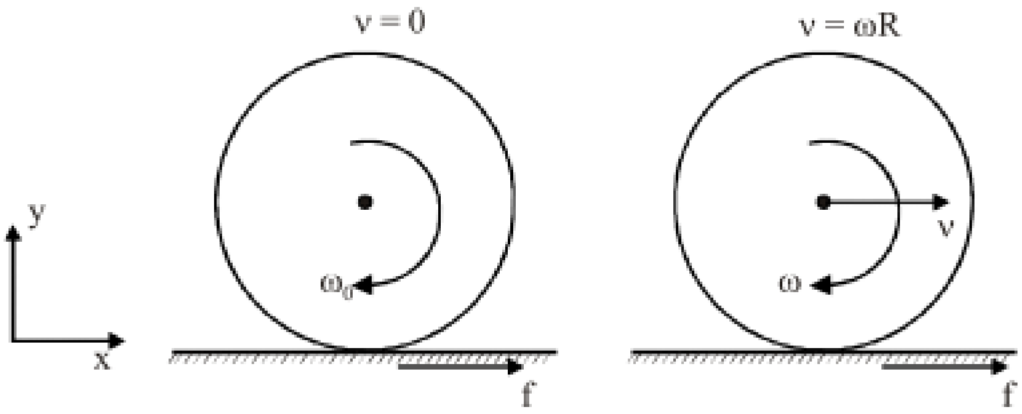
For pure rolling, $v = \omega R$*, therefore,* $L = \frac{7}{5}MvR$

Illustration :

A wheel is held by a handle on its axle and given initial angular velocity ω_0*. The wheel is then placed in contact with the ground. At first the wheel remains stationary, spraining in place. After a short time it begins to move forward and eventually reaches the point where it rolls without slipping. Find the final velocity of the wheel in terms of the initial angular velocity* w_0*.*

Take $I = \frac{MR^2}{2}$.

Sol. *We assume that wheel is initially rotating clockwise. Let the wheel starts rolling after a time t. Then, Using Impulse – Momentum Theorem For translation*



(a) A spinning wheel is placed on a horizontal surface with zero initial velocity. The friction force acts forward.
(b) After a time t the wheel starts rolling.

$$Impulse = \Delta p = p_f - p_i$$

or $\quad f\,t = Mv - 0 \qquad (i)$

For rotation

Angular Impulse $= \Delta L = L_f - L_i$

or $\quad -f\,Rt = I\omega - I\omega_0$

Note that clockwise angular momentum is considered as positive.

Since $I = \frac{MR^2}{2}$*, therefore,* $-fRt = \frac{MR^2}{2}(\omega - \omega_0) \qquad (ii)$

condition of pure rolling

$$v = \omega R \qquad (iii)$$

Using equations (i), (ii) and (iii), we get

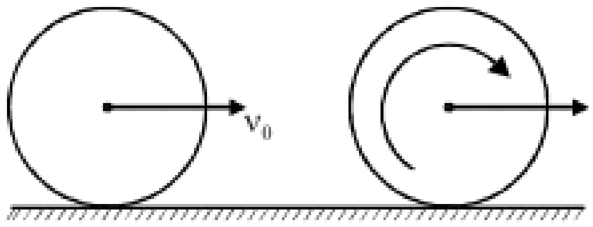
$$-MwR^2 = \frac{MR^2}{2}(\omega - \omega_0)$$

or $\quad \omega = \frac{\omega_0}{3}$

The linear momentum of the wheel is $v = \omega R = \frac{\omega_0 R}{3}$

Illustration :

A unifrom disc of mass m and radius r is projected horizontally with velocity v_0 on a routh horizotnal floor so that it starts off with a purely sliding motion at $t = 0$. At $t = t_0$ seconds it acquries a purely rolling motion.



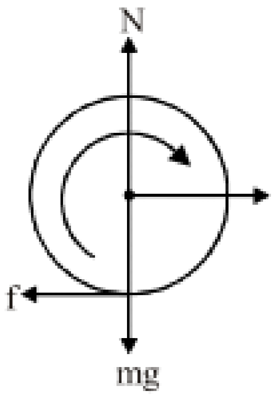
(a) Calculate the velocity of the centre of mass of the disc at $t = t_0$.

(b) Assuming coefficient of friction to be μ calculate t_0.

(c) The work done by frictional force as a function of time

(d) Total work done by the frictional over a time t much longer than t_0.

Sol. *F.B.D. of the disc.*



When the disc is projected it starts sliding and hence there is relative motion between the points of contact. Therefore frictional force acts on the disc in the direction opposite to the motion.

(a) *Now for translational motion*

$$a_{c.m.} = \frac{f}{m}$$

$$f = \mu N \text{ (as it slides)}$$

$$= \mu mg$$

$\Rightarrow$ $a_{c.m.} = -\mu g$, *negative sign indicates that* $a_{c.m.}$ *is opposite* $v_{c.m.}$

$$\Rightarrow \quad v_{c.m.(t)} = v_0 - \mu g t_0$$

$$\Rightarrow \quad t_0 = \frac{(v_0 - v)}{\mu g}, \text{ where } v_{c.m.(t_0)} = v \qquad (i)$$

For rotational motion about centre

$$\tau_f + \tau_{mg} = I_{c.m.}\,\alpha \qquad \Rightarrow \qquad \mu mgr = \frac{mr^2}{2}\alpha$$

$$\Rightarrow \quad a = \frac{2mg}{r} \qquad (ii)$$

$$\text{Therefore } \omega_{(t_0)} = 0 + \frac{2mg}{r} t_0, \text{ using } \omega_t = \omega_0 + at$$

$$\Rightarrow \quad \omega = \frac{2(v_0 - v)}{r} \qquad \text{(iii) using (i)}$$

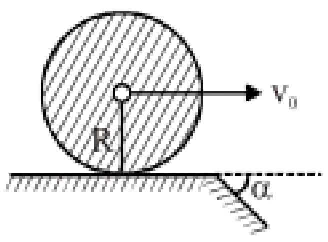
$$v_{c.m.} = \omega r$$

$$\Rightarrow \quad v = 2(v_0 - v) \text{using (iii)}$$

$$\Rightarrow \quad v = \frac{2}{3} v_0$$

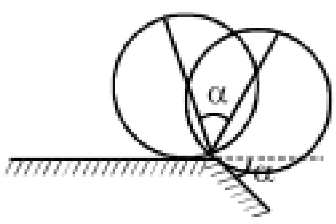
Illustration :

A uniform solid cylinder of radius R =15 cm rolls over a horizontal plane passing into an inclined plane forming an angle $\alpha = 30^o$ with the horizontal. Find the maximum value of the velocity v_0 which still permits the cylinder to roll onto the inclined plane section without a jump. The sliding is assumed to be absent.



Sol. Initial energy $E_i = \frac{1}{2} m v_0^2 + \frac{1}{2} I_{c.m.} \omega^2 + mgR$

For rolling $\frac{v_0}{R} = \omega$



$$\Rightarrow \quad E_I = \frac{1}{2} m v_0^2 + \frac{1}{2} \cdot \frac{1}{2} mR^2 \frac{v_0^2}{R^2} + mgR$$

$$= \frac{3}{4} m v_0^2 + \frac{1}{2} I_{c.m.} w^2 + mgR \cos\alpha$$

From COE (conservation of energy)

$$\Rightarrow \quad mv^2 = mv_0^2 + \frac{4}{3} mgR (1 - \cos\alpha)$$

F.B.D. of the cylinder when it is at the edge.

Centre of mass of the cylinder describes circular motion about P.

Hence $mg \cos\alpha - N = mv^2/R$

$$\Rightarrow \quad N = mg \cos a - mv^2/R$$

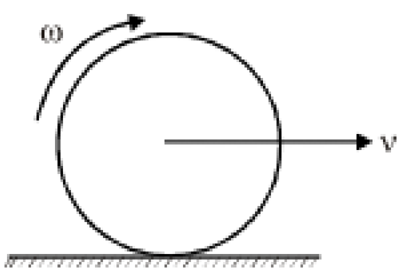
$$= mg \cos\alpha - \frac{mv_0^2}{R} - \frac{4}{3} mg + \frac{4}{3} mg \cos\alpha$$

For no jumping, $N \geq 0$

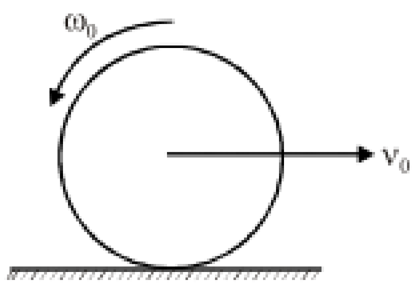
$$\Rightarrow \quad \frac{7}{3} mg \cos a - \frac{4}{3} mg - \frac{mv_0^2}{R} \geq 0 \qquad \Rightarrow \quad v_0 \leq \sqrt{\frac{7gR}{3} \cos\alpha - \frac{4}{3} g}$$

Practice Exercise

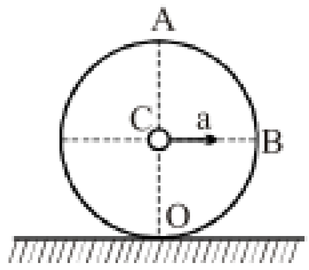
Q.1 A disc of mass and radius R is rolling with slipping on a rough horizontal surface which has coefficient of friction μ. At some instant the velocity of its center of mass is v and angular speed about center of mass is ω. The torque of the frictional force about the point which is instantaneously at rest at this moment is ?



Q.2 A uniform circular disc of radius r placed on a rough horizontal surface has initially a velocity v_0 and angular velocity ω_0 as shown in the figure. The disc comes to rest (neither translates nor rotates) after moving some distance. Then $\dfrac{v_0}{r\omega_0}$ is ?

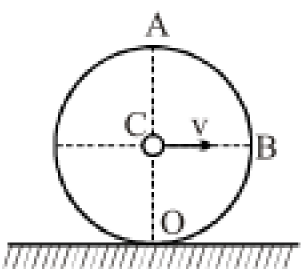


Q.3 A ball of radius R = 10.0 cm rolls without slipping down an inclined plane so that its centre moves with constat acceleration a = 2.50 cm/s^2 ; t = 2.00 s after the beginning of motion its position corresponds to that shown in figure. Find :

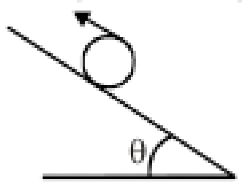


(A) the velocities of the points A, B and O; (B) the acceleration of these points

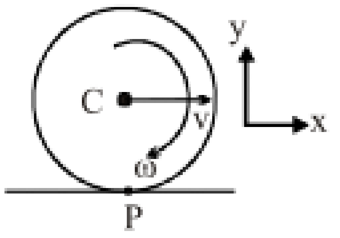
Q.4 A cylinder rolls without slipping over a horizotal plane. The radius of the cylinder is equal to r. Find the curvature radii of trajectories traced out by the points A and B in figure.



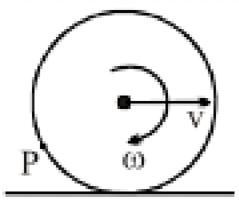
Q.5 A sphere kept on a rough inclined plane is in equilibrium by a string wrapped over it. If the angle of inclination is θ, the tension in the string will be equal to



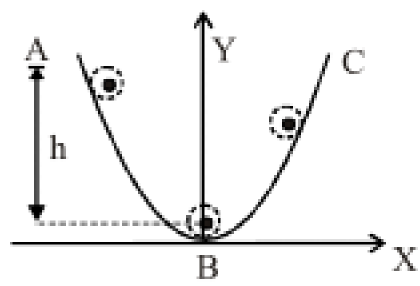
Q.6 A wheel of radius R is rolling without slipping on a stationary horizontal surface. Find the acceleration of point of contact P at the instant shown.



Q.7 A uniform circular disc of radius R rolls without slipping with a constant velocity on a stationary horizontal surface. Find the distance covered by a point P on its circumference in one full rotation.

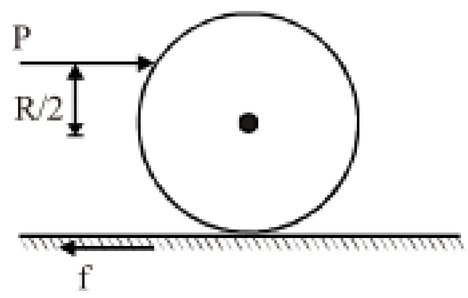


Q.8 A uniform cylinder rolls from rest from A down the side of a trough whose vertical dimension y si given by the equation $y = kx^2$. The cylinder does not slip form A to B but the surface of trough is frictionless from B to C. Then height of ascent of cylinder towards C is



Q.9 A sphere of mass M and radius r slips on a rough horiozntal plane. At some instant it has translational velocity v_0 and rotational velocity about the center $v_0/2r$. The rotational velocity when the sphere starts pure rolling is

Q.10 A solid cylinder of mass M and radius R lying on a rough horizontal surface for which coefficient of friction is μ is pushed by applying a horizontal force P at a distance R/2 from the center as shown in figure. The frictional force acting at the contact is



Answers

Q.1 $\mu mg\left(R - \frac{v}{\omega}\right)$ Q.2 $\frac{1}{2}$

Q.3 (A) $v_A = 10$ cm/s, $v_B = 7.1$cm/s $v_0 = 0$ (B) $a_A = 5.6$ cm/s^2, $a_B = 2.5$ cm/s^2, $a_0 = 2.5$ cm/s^2

Q.4 $R_A = 4r$, $R_B = 2\sqrt{2}r$ Q.5 $\frac{mg\sin\theta}{2}$ Q.6 $\vec{a}_p = \omega^2 R\,\hat{j}$ Q.7 8R

Q.8 $\frac{2h}{3}$ Q.9 $\frac{6}{7}v_0$ Q.10 zero