Circular Motion

Introduction

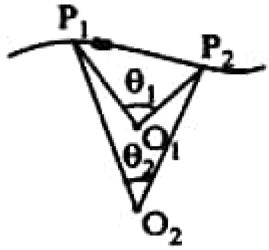
A car rounds a curve. A staellite circles Earth. Electrons revolve around the nucleus. Since they are not in straight lines, their velocities are changing either with direction or magnitude or with both i.e they are accelerated. Newton's Laws tell us that force acts on each. Which is this force and how it does so, will be discussed in this chapter.

Angular Variables

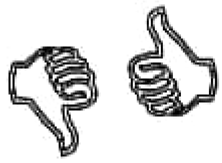
Angular displacement

Angle substended by a moving particle on a fixed point is called angular displacement about the fixed point. Thus in above discussion angular displacement about O_1 is θ_1 & about O_2 is θ_2. It is dimensionless quantity and its unit is radian. It should not be used in degree.

Angular displacement depends on reference frame , but angular displacement is different for different observers in the same frame). (The linear displacement is same for two observers at different positions in same frame) e.g. O_1 & O_2 will observe same linear displacement but different angular displacement although both points are in the same ground frame.



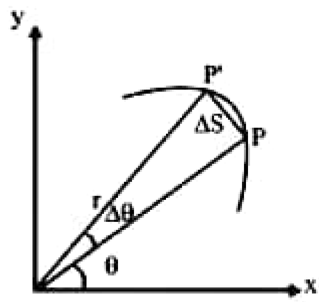
It is a scalar quantity. For small angles it canbe treated as a vector.
Although angular displacement is a scalar quantity, but if a body is rotating in a plane, then we treat it as a vector. Generally, anticlockwise is taken as positive and clockwise is taken as negative
Direction of angular displacement vector is decided by right hand rule.i.e. move your right hand fingers in sense of motion and direction of your thumb will be the direction of angular displacement.



Angular velocity

Rate of change of angular displacement is called angular velocity. Its unit is rad / sec.

Suppose a particle moving in circular path of radius "r" moves from P to P' so that its angular position changes from θ to (θ + Δθ) as shown.



$$\Rightarrow \qquad \omega_{av} = \frac{\Delta\theta}{\Delta T}$$

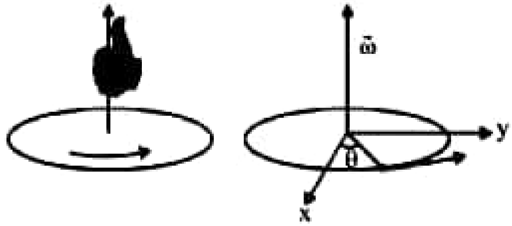
$$\Rightarrow \qquad \omega = \lim_{\Delta t \to 0}\left(\frac{\Delta\theta}{\Delta t}\right) = \frac{d\theta}{dt}$$

It is the measure of how fast a line joining origin and particle is rotating. For example faster rotating fan means it has greater angular velocity.

Angular velocity vector

Angular speed ω is the magnitude of vector called the angular velocity ω of the particle. Direction of $\vec{\omega}$

can be determined from circular motion right hand rule. Curl your fingers of right hand in the sense of revolution of particle,then the extended thumb points in the dirction of $\vec{\omega}$.



Here the angle θ is measured from the x-axis.

Suppose a particle is completing " f" revolutions per second (called frequency). In each revolution, 2π radians are covered. So number of radians covered per second, $\omega = 2\pi f$ rad/s

Time period i.e. time taken to complete one revolution is $T = \frac{1}{f}$

$\Rightarrow \qquad \omega = \frac{2\pi}{T}$

If particle revolutes with n r.p.m., $f = \frac{n}{60}$ cycles/sec $\qquad \Rightarrow \omega = \frac{2\pi n}{60}$ rad/s

Relation between linear velocity (v) and angular velocity (ω)

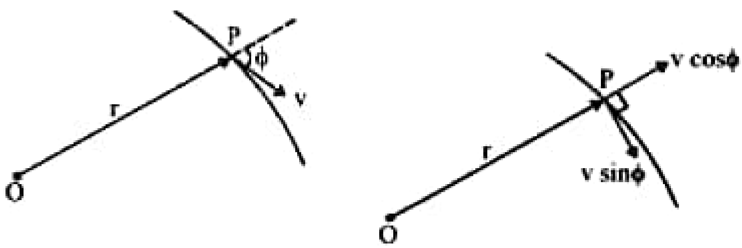
For a particle undergoing circular motion,

$$v = \lim_{\Delta t \to 0}\left(\frac{\Delta S}{\Delta t}\right) = r \lim_{\Delta t \to 0}\left(\frac{\Delta \theta}{\Delta t}\right)$$

$\Rightarrow \qquad v = r\,\omega$

Thus different points of second's hand of a clock are rotating with same angular velocity but with different speeds and its tip has greatest speed.

For any curvilinear motion (like the motion of particle P as shown below)



If the particle has only velocity componant v cosφ (along $\vec{r}$) If the particle has only velocity componant v cosφ (along $\vec{r}$), the observer O need not turn his head to always look at the particle i.e. this componant does not contribute in angular motion. Thus only the componant v sinφ is responsible for changing the angular displacment.

$\therefore \qquad \omega = \frac{v \sin \phi}{r}$

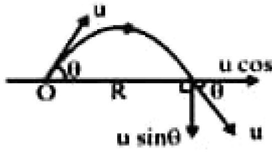
In general, $\omega = \dfrac{\text{velocity componant perpendicular to the line joning the particle and the observer}}{\text{distance between the particle and observer}}$

$$= \frac{v_\perp}{r}$$

Illustration :

A particle is launched from horizontal plane with speed u and angle of projection θ. Find angnlar velocity as observed from the point of projection of the particle at the time of landing.

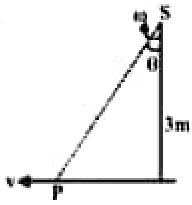
Sol. w.r.t. O, $\omega = \dfrac{u\sin\theta}{R}$



$$\omega = \frac{u\sin\theta}{\left(\dfrac{u^2\sin 2\theta}{g}\right)} = \frac{g}{2u\cos\theta}$$

Illustration :

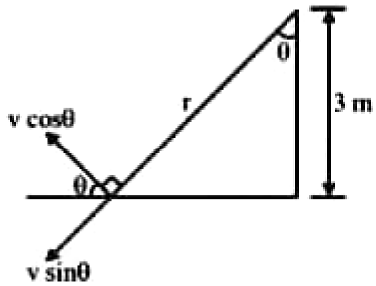
A spotlight S rotates in a horizontal plane with a constant angular velocity of 0.1 rad/sec. The spot of light P moves along the floor at a distance of 3m. Find the velocity of the spot P when θ = 45°



Sol. $\omega = \dfrac{v\cos\theta}{r}$

$$v = \frac{r\omega}{\cos\theta}$$

where $r = \dfrac{3}{\cos\theta}$



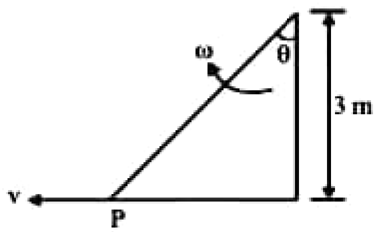
$\therefore \quad v = \dfrac{3\omega}{\cos^2\theta}$

At the instant shown, θ = 45°

$\therefore \quad v = 0.6$ m/s

Alternatively

$x = 3\tan\theta$



Also $v_P = \dfrac{dx}{dt}$

$\therefore \quad v = 3\sec^2\theta \cdot \dfrac{d\theta}{dt}$

$= 3\,\omega\sec^2\theta$

$\therefore$ at the instant shown

$v = 3 \times 0.1 \times \left(\sqrt{2}\right)^2 = 0.6$ m/s

Practice Exercise

Q.1 Find the linear speed of tip of second's hand of length 10 cm of a clock

Q.2 A particle is moving along a stright line y = d with speed u. Find its angular speed w.r.t. origin at the instant it makes angle ϕ with y-axis as shown.

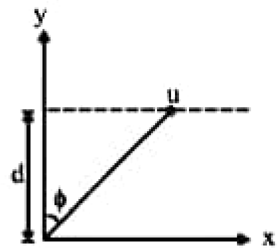


Illustration :

A particle is revoluting in a circular path of radius 0.5 m complating 1200 r.p.m (a) Find its linear speed (b) It now retards at the contant rate of 5π rad/s^2. Find the number of revolutions completed by it from the moment retardation begins till it stops.

Sol. (a) $\omega_0 = \dfrac{2\pi}{60} \times 1200 = 40\,\pi = 20\,\pi$ rad/s

$v = r\,\omega_0 = 0.5 \times 40\pi = 20$ m/s

(b) when it stops $\omega = 0$

$\therefore \quad \omega^2 = \omega_0^2 + 2\,\alpha\,(\Delta\theta) = 0$

$\therefore \quad (20\pi)^2 + 2\,(-5\pi)\Delta\theta = 0$

$\Rightarrow \quad \Delta\theta = 40\pi$ radians

Also number of revolutions is N then

$\therefore \quad N = \dfrac{\Delta\theta}{2\pi} = 20$

Unit vectors along Radial direction ($\hat{r}$) and tangential direction ($\hat{v}$)

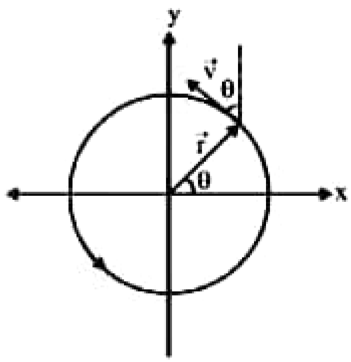
Suppose a particle moving in a cirular path of radius r in x y plane with origin as centre and makes angle θ with x-axis as shown, at any instant. Its position vector at this instant can be given by

$\vec{r} = r\cos\theta\,\hat{i} + r\sin\theta\,\hat{j}$

$\Rightarrow \quad \hat{r} = \dfrac{\vec{r}}{r} = \cos\theta\,\hat{i} + \sin\theta\,\hat{j}$

Also $\quad \vec{v} = -v\sin\theta\,\hat{i} + v\cos\theta\,\hat{j}$

$\therefore \quad \hat{v} = \dfrac{\vec{v}}{v} = -\sin\theta\,\hat{i} + \cos\theta\,\hat{j}$



Uniform circular motion

When a particle is moving with constant speed in a circular path, its motion is called unifrom circular motion. Although magnitude of velocity is constant but its direction changes continusly. It means it is continuonsly having some acceleration. This acceleration is always directing towards the centre. This is called radial acceleration (a_r) or centripetal acceleration (a_c).

As we have discussed in the previous topic

$\vec{r} = r(\cos\theta\,\hat{i} + \sin\theta\,\hat{j})$

and $\vec{v} = -v\sin\theta\,\hat{i} + v\cos\,\hat{j} = v(-\sin\theta\,\hat{i} + \cos\theta\,\hat{j})$

$$\vec{a} = \frac{d\vec{v}}{dt} = v\left(-\cos\theta\,\hat{i} - \sin\theta\,\hat{j}\right)\left(\frac{d\theta}{dt}\right) + v(\sin\theta\,\hat{i} + \cos\theta\,\hat{j})\frac{dv}{dt}$$

$|\vec{v}|$ is constant $\Rightarrow \quad \frac{d|\vec{v}|}{dt} = 0 \qquad$ i.e. $\frac{dv}{dt} = 0$

Also $\frac{d\theta}{dt} = \omega$

$\therefore \qquad \vec{a} = -v\,\omega\,(\cos\theta\,\hat{i} + \sin\theta\,\hat{j})$

$\vec{a} = v\,\omega\,\hat{a}$

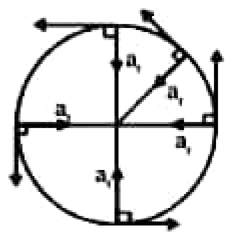
here $\hat{a} = -(\cos\theta\,\hat{i} + \sin\theta\,\hat{j}) = -\hat{r}$

Thus direction of this acceleration is opposite to $\hat{r}$ i.e. radially inwards.

Also $|\vec{a}| = v\,\omega$

$\Rightarrow \qquad a_r = v\,\omega$ or $\frac{v^2}{r}$ or $\omega^2 r \qquad [\because\ v = r\,\omega]$

Magnitude of this acceleration is constant but direction changes continuounsly, always being normal to velcoity as shown in the figure above.



Non-uniform circular motion

For a particle moving in non-uniform cicular motion both direction as well as magnitude of velocity change.

Now, $\vec{v} = v\left(-\sin\theta\,\hat{i} + \cos\theta\,\hat{j}\right)$

$\therefore \qquad \vec{a} = \frac{d\vec{v}}{dt} = -v(\sin\theta\,\hat{i} + \cos\,\hat{j})\cdot\frac{d\theta}{dt} + (-\sin\theta\,\hat{i} + \cos\,\hat{j})\cdot\frac{dv}{dt}$

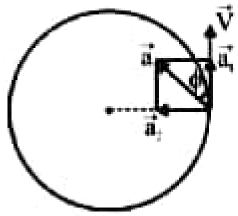
$$= -v\omega\hat{r} + \frac{dv}{dt}\hat{v}$$

we can write $\vec{a} = \vec{a}_r + \vec{a}_t$

where $\vec{a}_r = v\omega(-\hat{r})$ i.e. radial acceleration.

and $\vec{a}_t = \left(\frac{dv}{dt}\right)\hat{v}$ having unit vector equal to $\hat{v}$ i.e. it is also in tangential direction and is called tangential acceleration.

Thus in this case acceleration has two componants one along velocity i.e. tangential acceleration and another normal to velocity i.e. radial acceleration as shown.



If net acceleration ($\vec{a}$) makes angle ϕ with tangential direction, we may write

$$\tan\phi = \frac{a_r}{a_t}$$

also $\quad |\vec{a}| = \sqrt{a_r^2 + a_t^2}$

Some points to remember

- $$a_t = \frac{dv}{dt} = r\frac{d\omega}{dt} = r\alpha$$

$\therefore \quad |\vec{a}| = \sqrt{(\omega^2 r)^2 + (r\alpha)^2}$

- Students need not to be confused with $\left|\frac{d\vec{v}}{dt}\right|$ and $\frac{d}{dt}|\vec{v}|$. $\vec{v}$ contains both magnitude and driection. Thus $\frac{d\vec{v}}{dt}$ means $\vec{a}_{net}$ i.e. $\left|\frac{d\vec{v}}{dt}\right| = |\vec{a}_{net}|$.

Also $|\vec{v}|$ means magnitude of velocity only which changes because of tangential acceleration.

$$\therefore \quad \frac{d|\vec{v}|}{dt} = a_t$$

Illustration :

A particle is revoluting a circular path of radius 0.2 m with angular velocity $\omega = 20\,t^2$ rad/s, where t is in seconds. Find its acceleration at t = 0.5 sec.

Sol. $\quad a_r = \omega^2 r = (20t^2)^2\,(0.2) = 80\,t^4\ m/s^2$

$\therefore$ at $\quad t = 0.5$ sec; $\quad a_r = 80\,(0.5)^4 = 5\ m/s^2$

Also $\quad \alpha = \frac{d\omega}{dt} = 40\,t\ rad/s^2$

$\Rightarrow \quad a_t = r\alpha = 8t\ m/s^2$

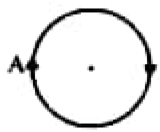
at $\quad t = 0.5$ sec; $\quad a_t = 8\,(0.5) = 4\ m/s^2$

$\therefore \quad |\vec{a}| = \sqrt{a_t^2 + a_r^2} = \sqrt{(4)^2 + (5)^2} = \sqrt{41}\ m/s^2$

$\therefore \quad a = 6.4\ m/s^2$

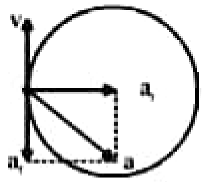
Illustration :

A particle is moving in circular path clockwise as shown with decreating speed. When it is at point A, the direction its acceleration may be given as



(A) $\longrightarrow$ *(B)* $\uparrow$ *(C)* $\nearrow$ *(D)* $\searrow$

Sol. *Since its speed is decreasing so its tangental acceleation is opposite to v. When it is is at A*



so (D) is correct

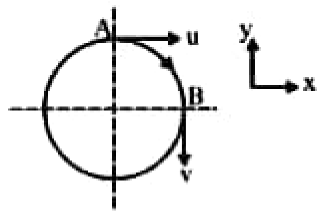
Illustration :

A particle is moving in a circular path of radius R with speed u, when it begins to speed up at a constant rate. After that when it completes one fourth revolution, change in its velocity vector has magnitude 2u. At that moment, find

(i) its radial acceleration

(ii) angle it makes with its velocity vector.

Sol. *(i) Suppose initially it was at point A with speed u and now it is at point B as shown with speed v.*



Now $\Delta\vec{v} = \vec{v}_f - \vec{v}_{in}$

$= (-v\,\hat{j}) - (u\,\hat{i})$

$\therefore \quad |\Delta\vec{v}| = \sqrt{v^2 + u^2}$

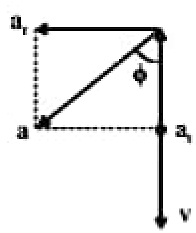
$\Rightarrow \quad \sqrt{v^2 + u^2} = 2u$

$\Rightarrow \quad v^2 = 3u^2$

$a_r = \dfrac{v^2}{R} \quad \Rightarrow \quad a_r = \dfrac{3u^2}{R}$

(ii) *Also* $\omega^2 = \omega_0^{\,2} + 2\,\alpha\,(\Delta\theta)$

$\Rightarrow \quad \left(\dfrac{v}{R}\right)^2 = \left(\dfrac{u}{R}\right)^2 + 2\,\alpha\left(\dfrac{\pi}{2}\right)$



$\Rightarrow \quad \alpha = \dfrac{v^2 - u^2}{\pi R^2} = \dfrac{3u^2 - u^2}{\pi R^2} = \dfrac{2u^2}{\pi R^2}$

$\Rightarrow \quad a_t = R\alpha = \dfrac{2u^2}{\pi R}$

Now $\tan\phi = \dfrac{a_r}{a_t} \quad \Rightarrow \quad \phi = \tan^{-1}\left(\dfrac{3\pi}{2}\right)$

Radius of curvature

If a body is moving in any curvilinear path, then at different locations, the curvature would be different, thus the radius would be different.

For general curvilinear motion, when the particle crosses a point A, it is satisfying condition of moving on an imaginary circle. At this instant, if $a_{\perp} = \frac{V^2}{R_c}$ (where R_c is radius of curvature at this instant.)

$$R_c = \frac{V^2}{a_{\perp}}$$

$$R_c = \frac{(speed)^2}{comp.\ of\ acceleration\ perpendicular\ to\ velocity}$$

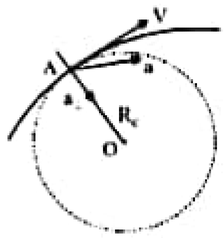
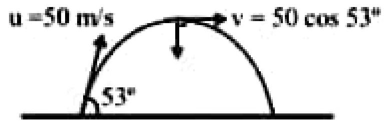


Illustration :

An object is projected with speed 50 m/s at an angle 53° with the horizontal from ground. Find radius of its trajectory (i) at the instant it is at heighest point (ii) at t = 1 sec. after projection.

Sol. *(i) At any instant acceleration of the projectile is 'g' downward. At the heighest point velcoity has magnitude = 50 cos 53° = 30 m/s and is in horizontal direction. Thus acceleration perpendicular to velocity is 'g' itself.*

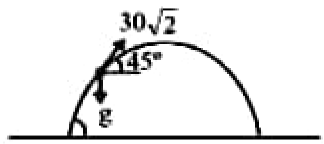


$$\therefore \quad a_r = \frac{v^2}{R} = g$$

$$\Rightarrow \quad R = \frac{v^2}{a_r} = \frac{(30)^2}{10} = 90m$$

(ii) $\quad$ *at t = 1sec* $\quad V_x = 50 \cos 53° = 30\ m/s$

and $\quad V_y = 50 \sin 53° - g\,(1) = 30\ m/s$



$$\therefore \quad V = \sqrt{V_x^2 + V_y^2} = 30\sqrt{2}\ m/s$$

i.e $\vec{v}$ *is at angle 45° with the horizontal*

$$\therefore \quad a_{\perp} = \text{componant of g perpendicular to velocity} = \frac{g}{\sqrt{2}}$$

$$\therefore \quad \frac{(30\sqrt{2})^2}{R} = \frac{10}{\sqrt{2}}$$

$$\Rightarrow \quad R = 180\sqrt{2}\ m$$

Dynamics of circular motion

Here we deal with the forces which are responsible for keeping on object in circular path.

Centripetal force

Force or combination of forces which provide contrepetal acceleration necessary for the revolution of a particle is called centripetal force or radial force. For example

(a) For object tied to a string and is revolving on a smooth horizontal surface, tension is centripetal force.

(b) To revolve satellite around the Earth, gravitational force provides centripetal acceleration, so gravitational force is centripetal force.

(c) For motion of electron around nucleus, the electrostatic force on electron is centripetal force on it.

(d) For an object placed on a rough rotating table, friction on the object due to table is centripetal force.

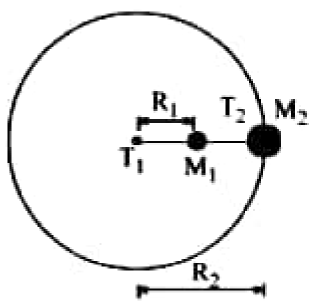
Problem solving strategy

1. Identify the plane of circular motion.
2. Locate the centre of ratation and calculate the radius.
3. Make F.B.D.
4. Resolve force along the radial direction and along the direction perpendicular to it.
5. The net force along radial direction is mass times the radial acceleration i.e. $m\left(\frac{V^2}{R}\right)$ or $m(\omega^2 R)$

Centripetal force $\left(\frac{mV^2}{R}\right)$ is no saperate force like Tension, Weight, Spring force, Normal reaction, Friction ect. In fact anyone of these or their combination may play a role of centripetal force.

Illustration :

Two different masses are connected to two ***light and inextensible*** *strings as shown in the figure. Both masses revolve about a central fixed point with constant angular speed of 10 rad s^{-1} on a smooth horizontal plane. Find the ratio of tensions $\frac{T_1}{T_2}$ in the strings.*



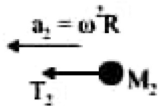
$M_1 = 0.25$ kg
$M_2 = 1.0$ kg
$R_1 = 5$cm
$R_2 = 10$cm

Sol. *Both the masses are moving in horizontal plane with same angular speed 10 rad/s. Here forces in radial directin can be tensions only.*

For M$_2$

$$F_{net} = T_2 = M_2 a_2$$

F.B.D. of M_2

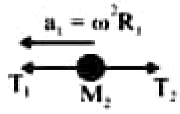


$$\Rightarrow \quad T_2 = M_2 \omega^2 R_2 \qquad ....(i)$$

For M_1

$$T_1 - T_2 = m_1 a_1$$

F.B.D. of M_1



$$T_1 = M_1 a_1 + T_2$$

$$T_1 = M_1 \omega^2 R_1 + M_2 \omega^2 R_2 \qquad \text{..........(ii)}$$

Dividing equation (ii) from (i), we get

$$\therefore \quad \frac{T_1}{T_2} = \frac{M_1R_1 + M_2R_2}{M_2R_2} = \frac{M_1}{M_2} \times \frac{R_1}{R_2} + 1$$

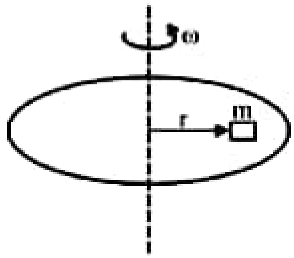
$$= \frac{0.25}{1} \times \frac{5}{10} + 1$$

$$\Rightarrow \quad \frac{T_1}{T_2} = \frac{9}{8}$$

Here centripetal force on M_2 is "T_2" and on M_1 is "$T_1 - T_2$".

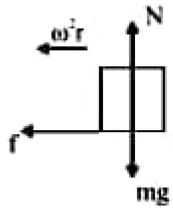
Illustration :

A rough horizontal table can rotate about its axes as shown. A small block is placed at r = 20 cm from its axis. The coefficient of friction between them is 0.5. Find the maximum angular speed that can be given to the block table system so that the block does not slip on the table.



Sol. *The force acting on the block are as shown below.*

N and mg are in vertical direction, so the only force that can provide necessary centrepetal acceleration in horizontal plane is friction. It makes the block to revolve with the table without slipping.



$$(F_{net})_y = 0 \Rightarrow N = mg$$

& $\quad (F_{net})_x = ma.$

$\Rightarrow \quad f = m\omega^2 r$

but $\quad f \le \mu N$

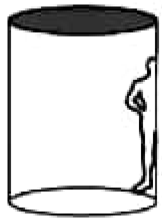
$$\Rightarrow \quad m\omega^2 r \le \mu mg \qquad \Rightarrow \qquad \omega \le \sqrt{\frac{\mu g}{r}}$$

$$\therefore \quad \omega_{max} = \sqrt{\frac{\mu g}{r}} \qquad = \sqrt{\frac{0.5 \times 10}{0.2}}$$

$$\Rightarrow \quad \omega_{max} = 5 \text{ rad/s}^2$$

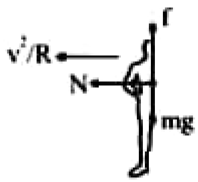
Illustration :

In a rotor, a hollow vertical cylindrical structure rotates about its axis and a person rests against the inner wall. At a particular speed of the rotor, the floor below the person is removed and the person hangs resting against the wall without any floor. If the radius of the rotor is 2m and the coefficient of static friction between the wall and the person is 0.2, find the minimum speed at which the floor may be removed. Take $g = 10\ m/s^2$.



Sol. *Here friction (f) is upward and opposes tendency to move down. Also normal force is radially towards the centre to provide centripetal acceleration.*

$$(F_{net})_x = \frac{mV^2}{R}$$



$$\Rightarrow \quad N = \frac{mV^2}{R}$$

Also as man doesnot fall down,

$$f = mg \le \mu N$$

$$\Rightarrow \quad mg \le \mu \frac{mV^2}{R}$$

$$\Rightarrow \quad \sqrt{\frac{gR}{\mu}} \le V$$

$$\Rightarrow \quad V_{min} = \sqrt{\frac{gR}{\mu}} = \sqrt{\frac{10\times 2}{0.2}} = 10 m/s$$

Illustration :

A block of mass 25 kg rests on a horizontal floor ($\mu = 0.2$). It is attached by a 5m long horizontal rope to a peg fixed on floor. The block is pushed along the ground with an initial velocity of 10 m/s so that it moves in a circle around the peg. Find

(a) *Tangential acceleration of the block*

(b) *Speed of the block at time t.*

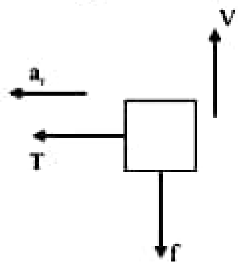
(c) *Time when tension in rope becomes zero.*

Sol. *The block is pushed on a horizontal stationary rough surface, the friction here is kinetic and is always opposite to velocity i.e. it is tangential force and is in horizontal plane.*

Also Normal force is in vertical direction to balance mg

i.e. $N = mg = 250\ N$

F.B.D. of M_2
(Top view)



$$\Rightarrow \quad f = \mu N = 50\ N$$

(a) $f = m a_t$

$$\Rightarrow \quad a_t = \frac{f}{m} = -\frac{50}{25} = -2\ m/s^2$$

(Here –ve sign is taken to indicate that it is always opposite to velocity)

(b) $a_t = \dfrac{dV}{dt} = -2 \Rightarrow \displaystyle\int_{10}^{V} dV = -2\int_{0}^{t} dt$

$$\Rightarrow \quad V - 10 = -2t$$

$$\Rightarrow \quad V = 10 - 2t$$

(c) Tension $T = \dfrac{mV^2}{R}$

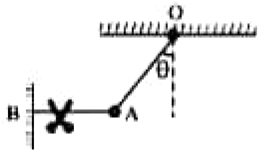
$\therefore$ $T = 0$ when $V = 0$

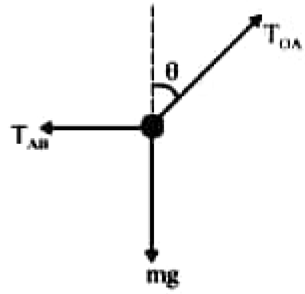
i.e. $10 - 2t = 0$

$\Rightarrow$ $t = 5$ sec

Illustration :

Find tension in OA before and just after AB is cut. The mass of the particle is m.





Sol. **Before AB is cut**, the particule is in static equilibrium i.e. $a = 0$.

Resolving in vertical and horizontal direction,

$$T_{OA} \cos\theta = mg$$

$$\Rightarrow \quad T_{OA} = \frac{mg}{\cos\theta}$$

After AB is cut, the particles moves in a circular path around O, with radius equal to the length of the string OA.

Now the acceleration cannot be directly taken zero, rather we have to consider acceleration in radial and tangential direction.

Resolving the force in radial and tangential direction.

$\therefore$ $T - mg\cos\theta = m\,a_r$

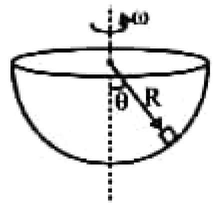
but just after the string AB is cut, speed of the particle is zero $\Rightarrow a_r = \dfrac{V^2}{\ell} = 0$

$\therefore$ $T - mg\cos\theta = 0$

$\Rightarrow$ $T = mg\cos\theta$

Illustration :

A hallow hemispherical bowl having radius of inner smooth surface $R = 80$ cm is rotated with angular velocity $\omega = 5$ rad/s. A small object is placed at rest w.r.t. the bowl at position as shown. Find angle θ.

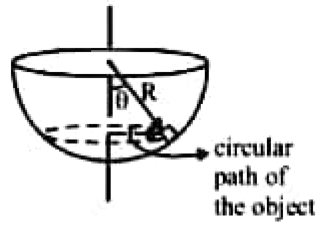


Sol. While the bowl is rotating, the plane of circular path of the particle is horizontal and its radius is r as shown.

$$N\cos\theta = mg.$$

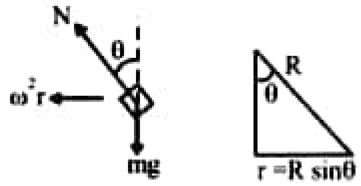
$\Rightarrow \quad N = \dfrac{mg}{\cos\theta}$

Also $\quad N \sin\theta = m\,\omega^2 r$



$\Rightarrow \quad \left(\dfrac{mg}{\cos\theta}\right)\sin\theta = m\omega^2 (R \sin\theta)$

$\Rightarrow \quad \dfrac{g}{\cos\theta} = \omega^2 R$



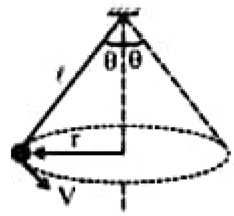
$\Rightarrow \quad \cos\theta = \dfrac{g}{\omega^2 R} = \dfrac{10}{(5)^2(0.8)}$

$\Rightarrow \quad \cos\theta = \dfrac{1}{2}$

$\Rightarrow \quad \theta = 60^\circ$

Conical Pendulum

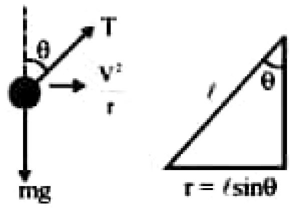
It consists of a small sphere (bob) connected to a light string. The bob is given some speed such that it revolve in a horizontal circular path while the string makes constant angle (say θ) with the vertical as shown.



Resolving in horizontal (radial) direction and vertical direction,

$(F_{net})_y = 0 \Rightarrow \quad T\cos\theta = mg$

$\Rightarrow \quad T = \dfrac{mg}{\cos\theta}$



Also $(F_{net})_x = ma \quad \Rightarrow \quad T\sin\theta = \dfrac{mV^2}{r}$

$\Rightarrow \quad \left(\dfrac{mg}{\cos\theta}\right)\sin\theta = \dfrac{mV^2}{\ell \sin\theta}$

$\Rightarrow \quad V = \sin\theta\sqrt{\dfrac{g\ell}{\cos\theta}}$

Time period of revolution $(t) = \dfrac{2\pi r}{V} = \dfrac{2\pi\ell\sin\theta}{\sin\theta\sqrt{\dfrac{g\ell}{\cos\theta}}} = 2\pi\sqrt{\dfrac{\ell\cos\theta}{g}}$

Results obtained

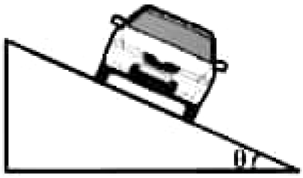
Tension $(T) = \dfrac{mg}{\cos\theta}$

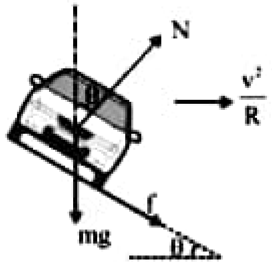
Speed $(V) = \sin\theta\sqrt{\dfrac{g\ell}{\cos\theta}}$

Time period $(t) = 2\pi\sqrt{\dfrac{\ell\cos\theta}{g}}$

Banking of road

When a vehicle is taking on horizontal circular turn, friction on tyres due to the road provides centripetal acceleration. Many times the friction is not sufficient enough, for example when speed is too large or the turn is too narrow. In such cases, instant of keeping the road horizontal, it is made tilted with the outer-side as higher side. This is called banking of road and the angle with which the road is kept inclined is called angle of banking. This enables the normal force to provide a horizontal component to help friction to provide centripetal acceleration. This can be explained below mathematically.





$(F_y) = 0 \Rightarrow N\cos\theta + f\sin\theta = mg$

Under limithing condition, i.e. when the car is about to skid, $f = \mu N$

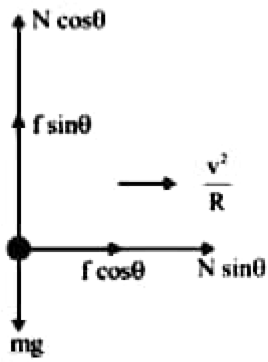
$\Rightarrow \quad N\cos\theta + \mu N\sin\theta = mg$

$$N = \frac{mg}{\cos\theta + \mu\sin\theta} \quad \text{..........(i)}$$

Also $\quad (F_x)_{net} = m\,a_r$

$$\Rightarrow \quad N\sin\theta + f\cos\theta = m\left(\frac{V^2}{R}\right)$$

$$\Rightarrow \quad N(\sin\theta + \mu\cos\theta) = \frac{mV^2}{R}$$



substituting value of N found in equation (i), we get

$$\frac{mg(\sin\theta + \mu\cos\theta)}{\cos\theta + \mu\sin\theta} = \frac{mV^2}{R}$$

$$\Rightarrow \quad V = \sqrt{\frac{gR(\sin\theta + \mu\cos\theta)}{(\cos\theta + \mu\sin\theta)}}$$

This is the speed at which friction achieves its maximum value i.e. it is the maximum speed with which the vehicle can take turn safely on banked road.

Special cases :

I   If friction is neglected i.e. μ is taken zero.

$$V = \sqrt{gR\tan\theta}$$

II   If banking is not provided i.e. $\theta = 0^\circ$

$$V = \sqrt{\mu gR}$$

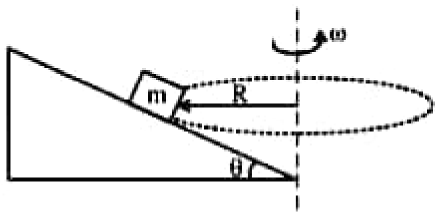
Centrifugal force

This the pseudo force which has to be taken under account when the frame of reference is rotating object itself, since the revolving object is accelerated i.e. non-inertial frame of refernce. The magnitude of this force is $\frac{mV^2}{r}$ or $m\omega^2 r$ (where r - radius of circular path of the object) and its direction is always radially outwards.

Solving problems with the concept of centrifugal force

Illustration :

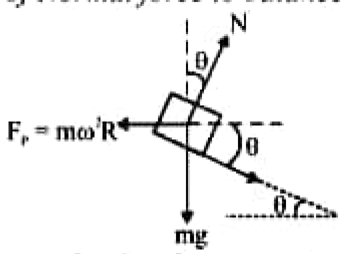
A small block is placed on a rough traingular shaped wedge which is revolving as shown such that the block undergoes circular path of radius R. The coefficient of friction between the block and the wedge is μ. Find the range of angular speed ω so that the block does not slip with respect to the wedge.



Sol. *If we select the frame of reference as the point at which the small block is placed. We take centrifugal force $m\omega^2 R$ outward.*

(i) ***Maximum angular speed (ω_{max})***

More is ω, more is centrifugal force to balance which more horizontal force is required radially inwards. The friction force acts down the incline as shown so that its horizontal componant helps horizontal componant of Normal force to balance centrifugal force.



$(F_{net})_y = 0 \Rightarrow N\cos\theta - f\sin\theta - mg = 0$

Under limiting condition i.e. when the block is about to slip, $f = \mu N$

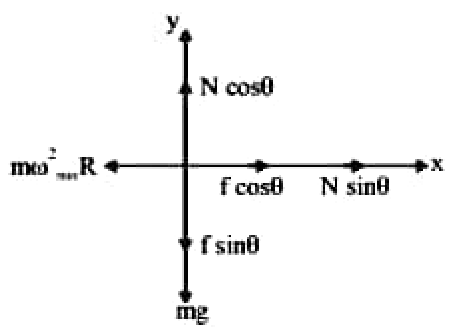
$\therefore \quad N(\cos\theta - \mu\sin\theta) = mg$

$$\Rightarrow \quad N = \frac{mg}{\cos\theta - \mu\sin\theta}$$

$(F_{net})_x = 0 \quad \Rightarrow N\sin\theta + (\mu N)\cos\theta - m\omega_{max}^2 R = 0$

$\Rightarrow N(\sin\theta + \mu\cos\theta) = m\omega_{max}^2 R$

$$\Rightarrow mg\left(\frac{\sin\theta + \mu\cos\theta}{\cos\theta - \mu\sin\theta}\right) = m\omega_{max}^2 R$$



$$\therefore \quad \omega_{max} = \sqrt{\frac{g}{R}\left(\frac{\sin\theta + \mu\cos\theta}{\cos\theta - \mu\sin\theta}\right)}$$

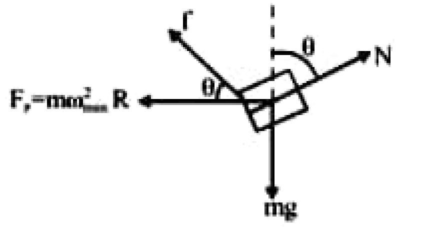
(ii) ***For minimum angular speed (ω_{min})***

Lesser is ω, lesser is centrifugal force thus friction now acts up the incline so as to provide horizontal componant along centifugal force to balance horizontal component of Normal force.

$$(F_{net})_y = 0 \Rightarrow N\cos\theta + f\sin\theta - mg = 0$$

$$\Rightarrow \quad N(\cos\theta + \mu\sin\theta) = mg$$

$$N = \frac{mg}{\cos\theta + \mu\sin\theta}$$

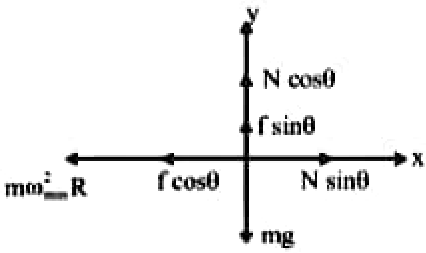


$$(F_{net})_x = 0 \qquad \Rightarrow N\sin\theta - f\cos\theta - m\omega_{min}^2 R = 0$$

$$\Rightarrow N(\sin\theta - \mu\cos\theta) = m\omega_{min}^2 R$$

$$\Rightarrow \frac{mg(\sin\theta - \mu\cos\theta)}{(\cos\theta + \mu\sin\theta)} = m\omega_{min}^2 R$$

$$\Rightarrow \qquad \omega_{min} = \sqrt{\frac{g(\sin\theta - \mu\cos\theta)}{R(\cos\theta + \mu\sin\theta)}}$$

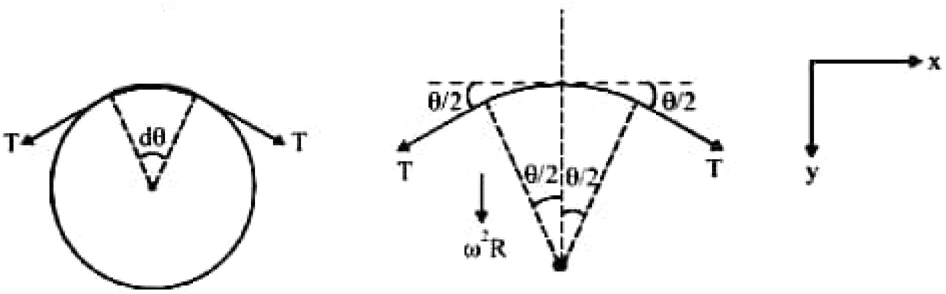


$$\therefore \qquad \sqrt{\frac{g(\sin\theta - \mu\cos\theta)}{R(\cos\theta + \mu\sin\theta)}} \le \omega \le \sqrt{\frac{g(\sin\theta + \mu\cos\theta)}{R(\cos\theta - \mu\sin\theta)}}$$

Tension in rotating ring

Diagram shown top view of a uniforming of mass m roating in horizontal plane.

Here tension inside the ring is an internal force for the ring. So to find it, we take a very small part of the ring which subtends angle dθ as shown.



we take y- axis towards the centre and x-axis in tangential direction and origin as the mid-point of the element

$$\therefore \qquad (F_{net})_y = (dm)\,\omega^2 R$$

$\therefore \quad 2T \sin\left(\frac{d\theta}{2}\right) = (dm)\,\omega^2 R \qquad \text{........(i)}$

but $\quad \sin\left(\frac{d\theta}{2}\right) \simeq \frac{d\theta}{2} \qquad (\because d\theta \text{ is very small})$

Also dm = (Mass per unit length of ring)× length of element

$\therefore \quad dm = \left(\frac{M}{2\pi R}\right)(Rd\theta) = \left(\frac{M}{2\pi}\right)d\theta$

Putting these values in equation (i), we get

$$2T\left(\frac{d\theta}{2}\right) = \left(\frac{M}{2\pi}\right)d\theta \times \omega^2 R$$

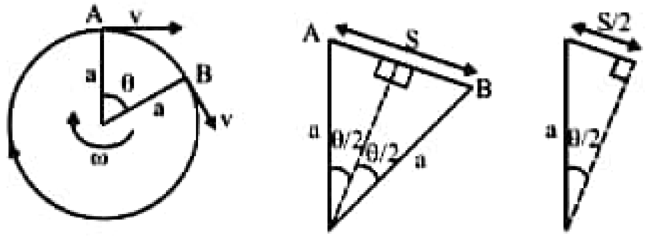
$\therefore \quad T = \frac{M\omega^2 R}{2\pi}$

Solved Example

Q.1 The magnitude of displacement of a particle moving in a circle of radius a with constant angular speed ω varies with time 't' as.

(A) $2\,a \sin\omega t$ (B) $2a \sin \frac{\omega t}{2}$ (C) $2a \cos \omega t$ (D) $2a \cos \frac{\omega t}{2}$

Sol. In time t, let the particle moves from A to B and rotates by angle $\theta = \omega t$



Magnitude of displacement

$$S = |\vec{S}| = A\,B$$

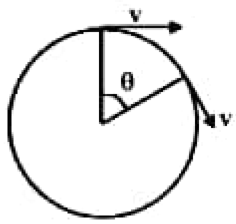
$\Rightarrow$ $S/2 = a \sin(\theta/2)$

$\therefore$ $S = 2a \sin(\theta/2) = 2\,a \sin\left(\frac{\omega t}{2}\right)$

Q.2 The speed of an object undergoing uniform circular motion is 4 m/s. Find the minimum possible centripetal acceleration (in m/s^2) of the object. [Take $\pi = 25/8$]

Sol. Let the particle rotates by angle θ (= ωt) in the given time interval

$\therefore$ $|\Delta\vec{V}| = |\vec{V}_f - \vec{V}_{in}| = \sqrt{|\vec{V}_f|^2 + |\vec{V}_{in}|^2 - 2|\vec{V}_f| \times |\vec{V}_{in}| \times \cos\theta}$



but $|\vec{V}_f| = |\vec{V}_{in}| = V$ and also $|\Delta\vec{V}| = V$

$\therefore$ $V = \sqrt{V^2 + V^2 - 2V^2 \cos\theta}$

$\Rightarrow$ $\cos\theta = \frac{1}{2}$

$\therefore$ θ i.e. $\omega t = \frac{\pi}{3}, \frac{5\pi}{3}$ etc.

$\Rightarrow$ $\omega_{min} t = \frac{\pi}{3}$

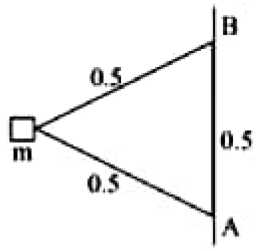
$\therefore \quad \omega_{max} = \frac{\pi}{3t} = \frac{\pi}{3(0.5)} = \frac{2\pi}{3}$

Also $\quad a_r = \omega V$

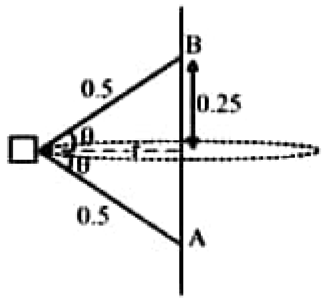
$\Rightarrow \quad a_r = \frac{2\pi}{3} \times 4 = \frac{8\pi}{3}$ m/s^2

$\therefore \quad a_r = \frac{25}{3}$ m/s^2 = 8.33 m/s^2

Q.3 Two strings of length l = 0.5 m each are connected to a block of mass m = 2 kg at one end and their ends are attached to the point A and B 0.5 m apart on a vertical pole which rotates with a constant angular velocity ω = 7 rad/sec. Find the ratio $\frac{T_1}{T_2}$ of tension in the upper string (T_1) and the lower string (T_2). [Use g = 9.8 m/s^2]



Sol. $\sin\theta = \frac{0.25}{0.50} = \frac{1}{2} \quad \Rightarrow \quad \theta = \frac{\pi}{6}$



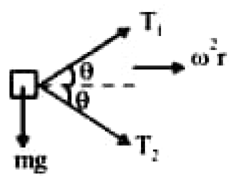
$r = 0.5 \cos\theta = 0.5\left(\frac{\sqrt{3}}{2}\right) = \frac{\sqrt{3}}{4}$

$(F_{net})_y = 0 \Rightarrow T_1 \sin\theta - T_2 \sin\theta - mg = 0$

$\therefore \quad (T_1 - T_2)\sin\theta = mg \qquad .....(i)$

$\quad (F_{net})_x = m\,\omega^2 r$

$\therefore \quad (T_1 + T_2)\cos\theta = m\,\omega^2 r \qquad ......(ii)$



Dividing equation (ii) by (i)

$$\left(\frac{T_1 + T_1}{T_1 - T_2}\right)\cot\theta = \frac{\omega^2 r}{g}$$

$$\therefore \quad \frac{T_1 + T_1}{T_1 - T_2} = \frac{\omega^2 r \tan\theta}{g} = \frac{(7)^2 \times (\sqrt{3}/4) \times 1/\sqrt{3}}{9.8}$$

$$\therefore \quad \frac{T_1 + T_1}{T_1 - T_2} = \frac{5}{4}$$

$$\therefore \quad \frac{T_1}{T_2} = \frac{5+4}{5-4} = 9$$

Q.4 A long horizontal rod has a bead which can slide along its length, and initially placed at a distance L from one end of A of the rod. The rod is set in angular motion about A with constant angular acceleration α. If the coefficient of friction between the rod and the bead is μ and gravity is neglected, then the time after which the bead starts slipping is

(A) $\sqrt{\frac{\mu}{\alpha}}$ (B) $\frac{\mu}{\sqrt{\alpha}}$ (C) $\frac{1}{\sqrt{\mu\alpha}}$ (D) infinitesimal

Sol. $N = ma_t = m\alpha L$

and $f = ma_r = m\omega^2 L$

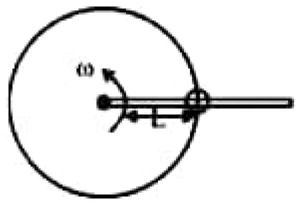
but $\omega = 0 + \alpha t = \alpha t$

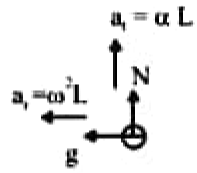
$\therefore$ $f = \mu\alpha^2 t^2 L$

Also at the instant of slipping, $f = \mu N$

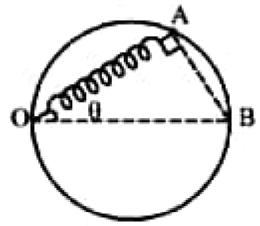
$\therefore$ $m\alpha^2 t^2 L = \mu (m\alpha L)$

$\therefore$ $t = \sqrt{\frac{\mu}{\alpha}}$





Q.5 A bead of mass m = 300gm moves in gravity free region along a smooth fixed ring of radius R = 2m. The bead is attached to a spring having natural length R and spring constant k = 10 N/m. The other end of spring is connected to a fixed point O on the ring.

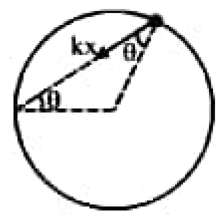


$AB = \frac{6R}{5}$. Line OB is diameter of ring.

Find (a) Speed of bead at A if normal reaction on bead due to ring at A is zero.

(b) The rate of change in speed at this instant.

Sol. The length of the spring, when bead is at A, is



$$OA = \sqrt{(OB)^2 - (AB)^2} = \sqrt{(2R)^2 - \left(\frac{6R}{5}\right)^2} = \frac{8R}{5}$$

Elongation in the spring = OA – natural length

$$= \frac{8R}{5} - R = \frac{3R}{5}$$

Also $\cos\theta = \frac{(8R/5)}{2R} = \frac{4}{5}$ and $\sin\theta = \left(\frac{6R/5}{2R}\right) = \frac{3}{5}$

Radial component of spring force $= kx\cos\theta = \frac{mv^2}{R}$

$$\Rightarrow \quad k\left(\frac{3R}{5}\right)\left(\frac{4}{5}\right)=\frac{mv^2}{R}$$

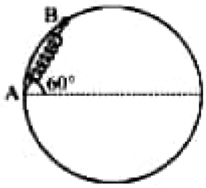
$$\Rightarrow \quad v=\sqrt{\left(\frac{12kR^2}{25m}\right)}=8 \text{ m/s}$$

Tangential component of force $= k\,x\,\sin\theta = k\left(\frac{3R}{5}\right)\left(\frac{3}{5}\right)=\frac{9kR}{25}$

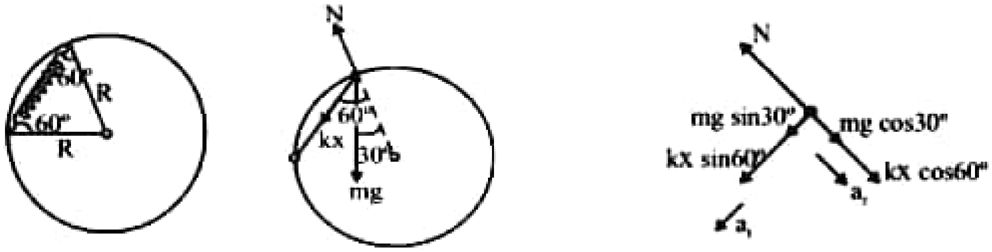
$$\Rightarrow \quad m\frac{dv}{dt}=\frac{9kR}{25}$$

Rate of change in speed $\quad \frac{dv}{dt}=\frac{9kR}{25m} \quad = 24 \text{ m/s}^2$

Q.6 A bead of mass m is attached to one end of a spring of natural length $\sqrt{3}$ R and spring constant $k=\frac{(\sqrt{3}+1)\,mg}{R}$. The other end of the spring is fixed at point A on a smooth fixed vertical ring of radius R as shown in the figure. What is the normal reaction at B just after the bead is released?



Sol.



At the instant of release, compression in the spring is

$$x=\sqrt{3}R-(\sqrt{3}-1)R$$

$\therefore \quad$ spring force, $kx=\frac{(\sqrt{3}+1)mg}{R}\times(\sqrt{3}-1)R=2\,mg$

Also at the instant of release, speed = 0

$\therefore \quad$ radial acceleration, $a_r=\frac{v^2}{R}=0$

$\therefore \quad$ Along radial direction, $F_{net}=0$

$\therefore \quad N-mg\cos 30°-kx\cos 60°=0$

$$N=mg\,\frac{\sqrt{3}}{2}+(2\,mg)\times\frac{1}{2}=\frac{mg(\sqrt{3}+2)}{2}$$

Work Energy and Power

Introduction

Figure (a) shows a skier starting from rest at the top of a uniform slope. What's the skier's speed at the bottom ? You can solve this problem by applying Newton's second law to find the skier's constant acceleration and then the speed. But what about the skier in figure (b) ? Here the slope is continuously changing and so is the acceleration. Constant-acceleration equations are not applicable here, so solving for the details of the skier's motion is difficult.

Figure (a)

Figure (b)

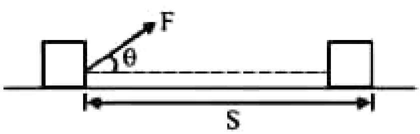
There are many cases where motion involves changing forces and accelerations. In this chapter, we introduce the important physical concepts of work and energy. These powerful concepts enable us to "shortcut" the detailed application of Newton's law to analyze these more complex situations. We begin with the concept of work.

Work

In day-today life, me oftenely use the word "Work". In physics, we define the work in quite different manner than we usually use the word "Work" in daily life. Work done by a constant force $\vec{F}$ on an object when displaced by $\vec{S}$ is given by

$$W = F' S$$

Where F' is the component of force which is in the direction of displacement



In above figure, $F' = F \cos\theta$

$$\therefore \quad W = (F\cos\theta)\,S = F\,S\cos\theta$$

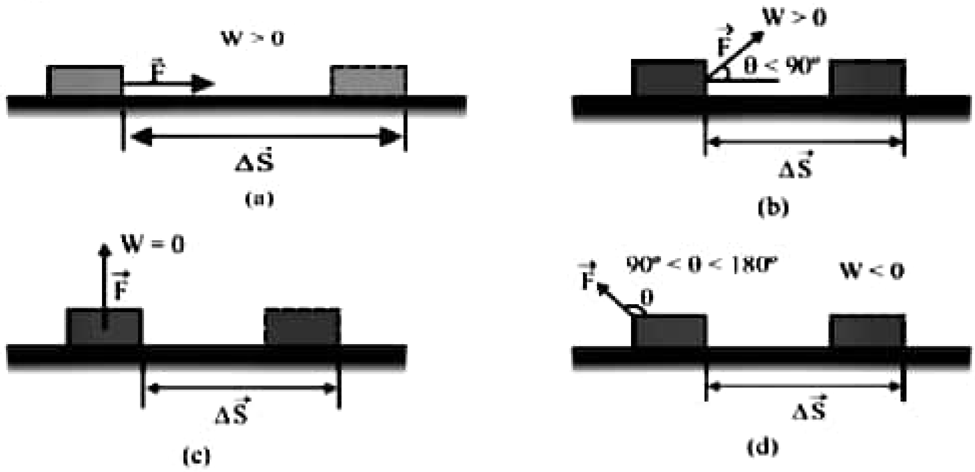
$$\therefore \quad \mathbf{W} = \vec{\mathbf{F}}\cdot\vec{\mathbf{S}}$$

So work may also be given as dot product of force and displacement.

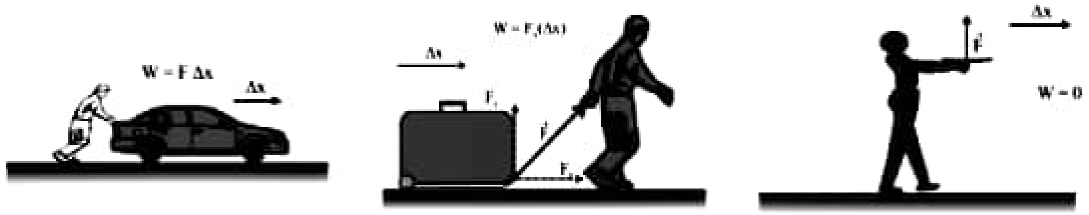
- Its unit is Newton meter (N.m) or Joule (J)
- If $\vec{F} = F_x\,\hat{i} + F_y\,\hat{j} + F_z\,\hat{k}$ and $\vec{S} = \Delta x\,\hat{i} + \Delta y\,\hat{j} + \Delta z\,\hat{k}$, then $W = \vec{F}\cdot\vec{S} = F_x(\Delta x) + F_y(\Delta y) + F_z(\Delta z)$

- Work done force is frame dependent as displacement is frame dependent

- Work can be positive or negative or zero. When a force speed up the praticle, it does positive work. A force acting at 90° to the motion does no work. And when a force slow down the motion, it does negative work.



e.g. According to equation the person pushing the car in figure (a) does work equal to the force he applies times the distance the car moves. But person pulling the suitcase in figure (b) does work equal to only the horizontal component of the force times the distance the suitcase moves. Furthermore, by our definition, the waiter of figure (c) does no work on the tray. Why not ? Because the force on the tray is vertical while the tray's displacement is horizontal; there's no component of force in the direction of the tray's motion.



(a) (b) (c)

Illustration :

The airline passenger in above figure (b) exerts a 80 N force on his suitcase pulling at an angle 60° with the horizontal. What work he does on suitcase while pulling it 50 m on the floor ?

Sol. $W = (80\ N)(50\ m)\cos 60°$

$= 2000\ J = 2\ KJ$

Illustration :

A particle is moving along a straight line from point A to point B. The position vectors for points A and B are $(2\hat{i}+7\hat{j}-3\hat{k})m$ and $(5\hat{i}-3\hat{j}-6\hat{k})m$ respectively. One of the force acting on the particle is $\vec{F}=20\hat{i}-30\hat{j}+15\hat{k}$ N. Find the work done by this force.

Sol. $\vec{S} = (5\hat{i} - 3\hat{j} - 6\hat{k}) - (2\hat{i} + 7\hat{j} - 3\hat{k})$

$= 3\hat{i} - 10\hat{j} + 3\hat{k}$ m

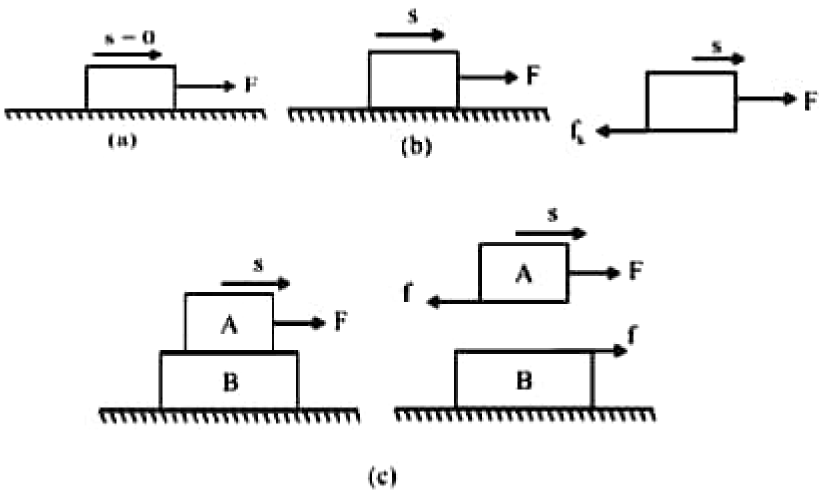
$\vec{F} = 20\hat{i} - 30\hat{j} + 15\hat{k}$

Now $W = \vec{F} \cdot \vec{S}$

$= 60 + 300 - 45 = 315\ J$

Work Done by friction

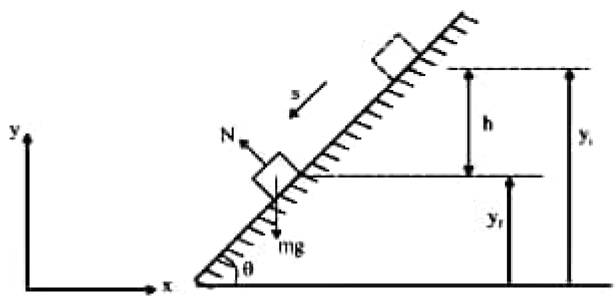
There is a misconception that the force of friction always does negative work. In reality, the work done by fricition may be zero, postive or negative depending upon the situation as shown in the figure.



(a) When a block is pulled by a force F and the block does not move, the work done by friction is zero.
(b) When a block is pulled by a force F on a stationary surface, the work done by the kinetic friction is negative.
(c) Block A is placed on the block B. When the block A is pulled with force F, the friction force does negative work on block A and positive work on block B, which is being accelerated by a force F. the displacement of A relative to the table is in the forward direction. The work done by kinetic friction on block B is positive.

Work Done by Gravity

Consider a block of mass m which slides down a smooth inclined plane of angle θ as shown in figure.



Let us assume the coordinate axes as shown in the figure to specify the components of the two vector although the value of work will not depend onthe orientation of the axes.

Now, the force of gravity, $\vec{F}_g = -mg\,\hat{j}$

and the displacement is given by

$$\vec{s} = \Delta x\,\hat{i} + \Delta y\,\hat{j} + \Delta z\,\hat{k}$$

The work done by gravity is

$$W_g = \vec{F}_g.\vec{s} = -mg\,\hat{j}.(\Delta x\,\hat{i} + \Delta y\,\hat{j} + \Delta z\,\hat{k})$$

or $\quad W_g = -mg(\Delta y)$

Since $\quad \Delta y = y_f - y_i = -h$

$\therefore \quad W_g = +mgh$

If the block moves in the upward direction, then the work done by gravity is negative and is given by

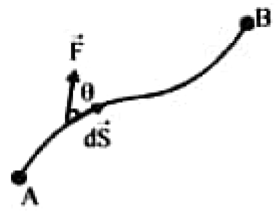
$$W_g = -mgh$$

Important

1. The work done by the force of gravity depends only on the initial and final vertical coordinates, not on the path taken.
2. The work done by gravity is zero for path that returns to its initial point.

Work done by a variable force

Often the force applied to an object varies with position. Important examples include electric and gravitational force, which vary with the distance between interacting objects. The force of a spring that we encountered in previous chapter provides another example: as the spring stretches, the force increases.



In this case we have difficulty to apply $W = \vec{F}.\vec{S}$, since $\vec{F}$ is not same for complete $\vec{S}$.

Thus, we take a very small part $d\vec{S}$ of its path. This displacement $d\vec{S}$ is so small that in variation force may be neglected during it. So we may write, for the work done during this displacement as

$$dW = \vec{F}.d\vec{S}$$
$$= F\,dS\cos\theta$$

The total work done in going from A to B as shown may be calculated by summing up i.e. integrating the work done during its small fractions.

i.e. $$W_{A\to B} = \int_A^B \vec{F}.d\vec{S} = \int_A^B (F\cos\theta)dS$$

In terms of rectangular components,

$$\vec{F} = F_x\hat{i} + F_y\hat{j} + F_z\hat{k}$$

and $\quad d\vec{S} = dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}$

therefore,

$$W_{A\to B} = \int_{x_A}^{x_B} F_x dx + \int_{y_A}^{y_B} F_y dy + \int_{z_A}^{z_B} F_z dz$$

Illustration :

A force $\vec{F} = x\hat{i} + y^2\hat{j}$ N acts on a particle and the particle moves from (1,2) m to (–3,4) m. Find work done by the force $\vec{F}$.

Sol. $dW = \vec{F}\cdot d\vec{S}$

where $d\vec{S} = dx\hat{i} + dy\hat{j}$

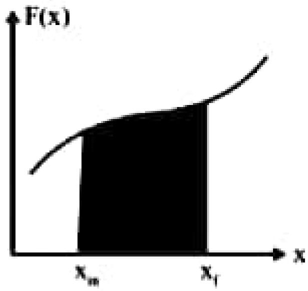
$\therefore \quad dW = xdx + y^2dy$

and $W = \int dW = \int_1^{-3} xdx + \int_2^4 y^2dy = \left.\frac{x^2}{2}\right|_1^{-3} + \left.\frac{y^3}{3}\right|_2^4 = \frac{68}{3} J$

Work done as Area under the force displacement graph

Suppose of particle moving along a straight line and a force acting on it varies with its displacement x as shown.

$$W = \int_{x_m}^{x_f} F.dx$$



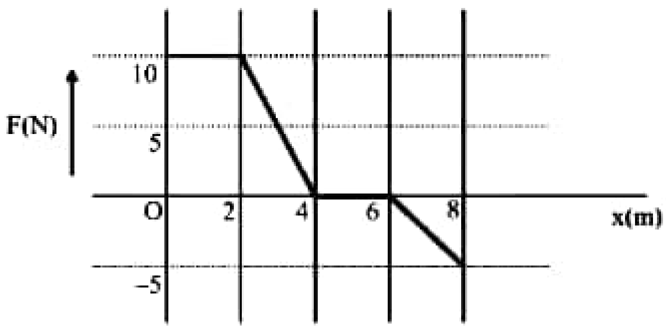
= Area under F vs x graph from $x = x_m$ to x_f

In general, the work done by a point x_m to final point x_f is given by the area under the force - displacement curve as shown in the figure.

Area (work) above the x-axis is taken as positive, and below x-axis as negative.

Illustration :

A 5 kg block moves in a straight line on a horizontal frictionless surface under the influence of a force that varies with position as shown in the figure. Find the work done by this force as the block moves from the origin to x = 8m.

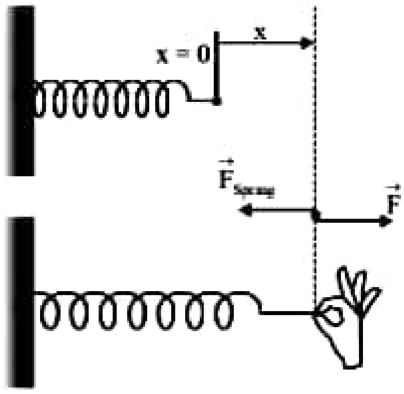


Sol. *The work from x = 0 to x = 8 m is the area under the curve.*

$$W = 10\times 2 + \frac{1}{2}(10)(4-2) + 0 + \frac{1}{2}(-5)(8-6) = 25 J$$

Work done by spring force

A spring provides an important example of a force that varies with position. We've seen that an ideal spring exerts a force proportional to its displacement from equilibrium : $F = -kx$, where k is the spring constant and the minus sign shows that the spring force is opposite the direction of the displacement.



$\therefore$ Work done (W_s) by spring force when its deformation changes from x_m to x_f is

$$W_s = -k\int_{x_{in}}^{x_f} x.dx$$

$$\Rightarrow \quad W_s = -\frac{1}{2}k\left(x_f^2 - x_{in}^2\right)$$

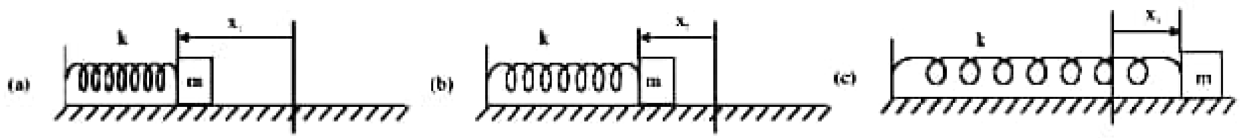
Note : Work done by a spring force to stretch it from its undeformed length to deform it upto x is

$$W = -\frac{1}{2}k(x^2 - 0) = -\frac{1}{2}kx^2$$

- Work done by a spring force may be negative ($x_f > x_{in}$) or may be positve (if $x_f < x_{in}$) or may be zero (if $x_f = x_{in}$).
- Work done by the spring force depends only on initial and final deformation.
- In the equation, $W = -\frac{1}{2}k(x_f^2 - x_{in}^2)$

  x_m & x_f are magnitudes of deformations no matter if these are compressions or extensions.

e.g.



Work done by spring force from position (a) to (b) and that from (a) to (c) are same and equal to $-\frac{1}{2}k\left(x_2^2 - x_1^2\right)$

Work Energy Theorem

Closely related to work is energy – One of the most important concepts in all of physics. Here we introduce the energy associated with motion i.e. kinetic energy. Our goal is to relate kinetic energy and work.

The net work is done by all the forces acting on an object, so we use the net force in our expression for work. We'll consider one-dimensional motion with force and displacement along the same line. In that case, the Equation below gives the net work :

$$W_{net} = \int F_{net}\,dx$$

But net force can also be written as

$$F_{net} = ma = mv\frac{dv}{dx}$$

$$\Rightarrow \quad F_{net}\,dx = mvdv$$

$$W_{net} = \int_{v_{in}}^{v_f} mvdv$$

$$\therefore \quad W_{net} = \frac{1}{2}m\left[v_f^2 - v_{in}^2\right]$$

$$\therefore \quad W_{net} = \Delta\ K.E.$$

Thus change in an object's kinetic energy is equal to the net work done on the object. This called Work Energy Theorm.

Important

(i) The kinetic energy of an object is a measure of the amount of work needed to increase it speed from zero to a given value.

(ii) The kinetic energy of a particle is the work it can do on its surroundings in coming to rest.

(iii) Since the velocity and displacement of a particle depend on the frome of reference, the numerical values of the work and the kinetic energy also depend on the frame.

(iv) If work done by net force is positive, kinetic energy of the system increases. If net work done is negative K.E. decreases and if net work is zero, K.E. remains contact

Illustration :

A 60 gm tennis ball thrown vertically up at 24 m/s rises to a maximum height of 26 m. What was the work done by resistive forces?

Sol. *By Work - Energy theorem,* $W_{net} = \Delta\ K.E$

$$W_g + W_{res} = (0 - \frac{1}{2}mu^2)$$

$$-mgh + W_{res} = -\frac{1}{2}mu^2$$

$$W_{res} = 0.06 \times 10 \times 26 - \frac{1}{2} \times 0.06 \times (24)^2$$

$$= -1.68\ J$$

Illustration :

A force of $(3\hat{i} - 1.5\hat{j})$ N acts on a 5 kg body. The body is at a position of $(2\hat{i} - 3\hat{j})$ m and is travelling at 4 ms^{-1}. The force acts on the body until it is at the position $(\hat{i} + 5\hat{j})$ m. Assuming no other force does work on the body, the final speed of the body.

Sol. Given; Mass of the body = 5 kg

Force $\vec{F} = 3\hat{i} - 1.5\hat{j}$

Displacement $\overline{\Delta s} = \{(\hat{i} + 5\hat{j}) - (2\hat{i} - 3\hat{j})\}\, m = (-\hat{i} + 8\hat{j})\, m$

From Work Energy theorem

$$W = \vec{F}\cdot\Delta\vec{s} = \frac{1}{2}m(v^2 - u^2)$$

$$-3 - 12 = \frac{1}{2}\times 5\left[v^2 - (4)^2\right]$$

$\Rightarrow$ $v = \sqrt{10}$ m/s

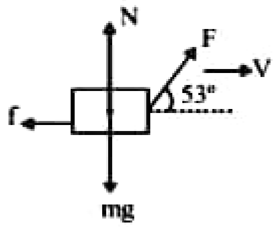
Illustration :

A block of mass m = 4 kg is dragged 2 m along a horizontal surface by a force F = 30 N acting at 53° to the horizontal. The initial speed is 3 m/s and $\mu_k = 1/8$.

(a) Find the change in kinetic energy of the block

(b) Find its final speed

Sol.



(a) The forces acting on the block are shown in the figure.

Clearly, $W_N = 0$ and $W_g = 0$, whereas $W_F = FS\cos\theta$

$W_f = -fs = -(\mu_k N)S$, where $N = mg - F\sin\theta$

The work-energy theorem,

$$\Delta K = W_{net} = W_F + W_f$$

therefore, $\Delta K = F\,S\cos\theta - \mu_k\,(mg - F\sin\theta)\cdot S$

$$= (30)(2)(0.6) - \frac{1}{8}(40 - 24)(2) = 32\ J$$

(b) Now $\Delta K\cdot E\cdot = \frac{1}{2}m\left[v_f^2 - v_i^2\right] = 32\ J$

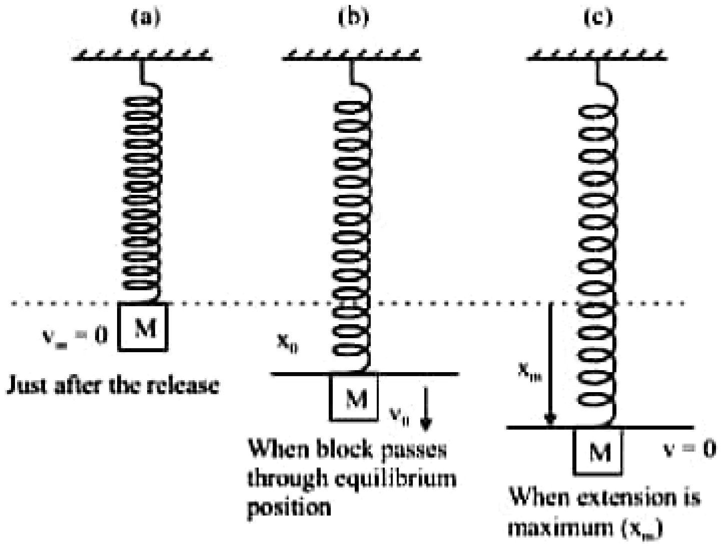
$\Rightarrow$ $\frac{1}{2}\times 4\left[v_f^2 - (3)^2\right] = 32$

$\Rightarrow$ $v_f = 5$ m/s

Illustration :

A sping of spring constant k is attached to the ceiling. A block of mass M is attached to its lower end and is released suddenly. Find (i) maximum extension in the spring (ii) Find its speed at the instant it passes throught the equilibrium position.

Sol.

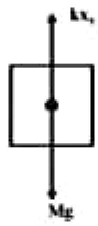


Applying work energy theorem for motion from (a) to (c)

$$W_g + W_{spring} = \Delta KE$$

$$Mg\,x_m - \frac{1}{2}kx_m^2 = \frac{1}{2}M(0-0) = 0$$

$\therefore$ *maximum extension,* $x_m = \dfrac{2Mg}{k}$



(ii) When the block is equilibrium (i.e. $F_{net} = 0$*)*

$$kx_o = Mg$$

$\Rightarrow \quad x_0 = \dfrac{Mg}{k}$

$\therefore$ *Applying work energy therom for motion from (a) to (b)*

$$Mg\,x_0 - \frac{1}{2}kx_0^2 = \frac{1}{2}M(V_0^2 - 0)$$

$$\Rightarrow \quad Mg\left(\frac{Mg}{k}\right) - \frac{1}{2}k\left(\frac{Mg}{k}\right)^2 = \frac{1}{2}MV_0^2$$

$$\Rightarrow \quad v_0 = g\sqrt{\frac{M}{k}}$$

Illustration :

A spring is fixed at the bottom end of an incline of inclination 37°. A small block is released from rest on an incline from a point 4.8 m away from the spring. The block compresses the spring by 20 cm, stops momentarily and then rebounds through a distance of 1 m up the incline. Find (a) the friction coefficient between the plane and the block and (b) the spring constant of the spring. Take g = 10 m/s².

Sol.

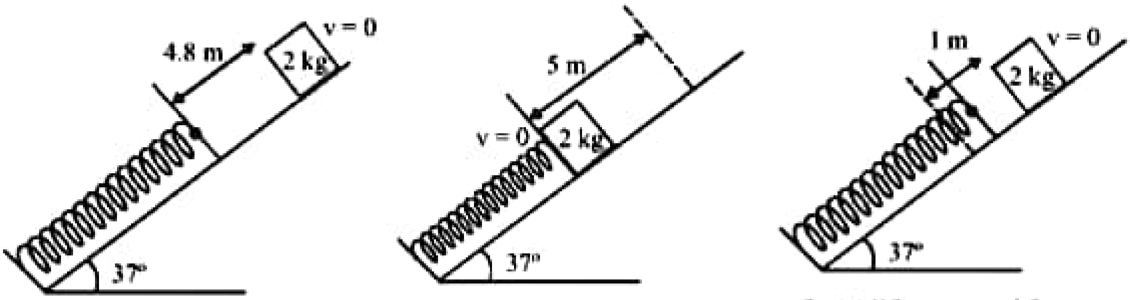


fig (a) Just after release
fig (b) When stopped for the first time
fig (c) When stopped for the second time

Applying work energy theorem for motion from (a) to (b)

$$W_{gravity} + W_{friction} + W_{spring} = \Delta K{\cdot}E = \frac{1}{2}m\,(0-0) = 0$$

$$\therefore \quad 20 \times 5 \sin 37^\circ - \mu\,(20 \cos 37^\circ)5 - \frac{1}{2}\,k\,[(0.2)^2 - 0] = 0 \qquad \text{..........(i)}$$

Applying work energy for motion form (b) to (c)

$$-20 \times 1 \times \sin 37^\circ - \mu\,(20 \cos 37^\circ) \times 1 - \frac{1}{2}k\left[0-(0.2)^2\right] = 0 \qquad \text{..........(ii)}$$

Adding equation (i) and (ii)

$$\therefore \quad -20\,(5-1) \times \frac{3}{5} - \mu\,(20 \times \frac{4}{5})\,(5+1) = 0$$

$$\Rightarrow \quad \mu = 0.5$$

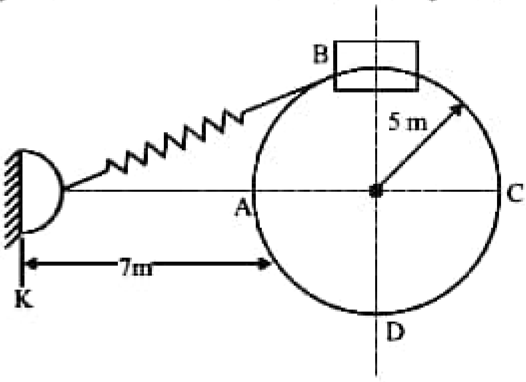
Putting this value in equation (i), we get

$$k = 1000 \text{ N/m}$$

Here velocity is maximum at equilibrium since before this, spring force was less than the weight of the block and the block was accelerating and after this , the spring force is greater than the weight thus retarding the block to zero velocity upto the lowest position.

Illustration :

A collar B of mass 2 kg is constrained to move along a horizontal smooth and fixed circular track of radius 5 m as shown in figure. The spring lying in the plane of the circular track and having spring constant 200 N/m is undeformed when the collar is at A. If the collar starts from rest from B, then find the normal reaction exerted by the track on the collar when it passes through A.



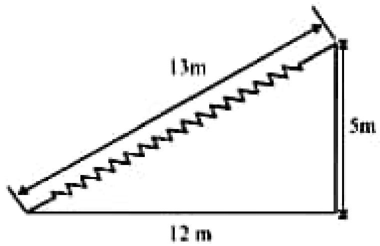
Sol. *Initially ,*

Length of spring = 13 m

undeformed lingth = 7 m

Initial extention (x_{in}) = 13 – 7 = 6

final extension (x_f) = 0

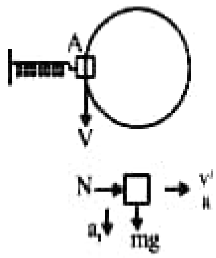


Applying work energy theorem for motion from B to A

$$\frac{1}{2}k\left(x_{in}^2 - x_f^2\right) = \frac{1}{2}mv^2$$

$mv^2 = 200\ (y^2 - 0^2) = 200 \times 49$

$mv^2 = 9800$



At point A, along radial direction, $F_{net} = N = m\dfrac{v^2}{R}$

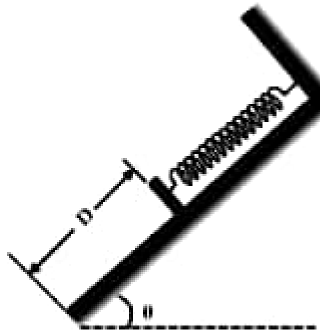
$$\therefore \quad N = \frac{mv^2}{R} = \frac{9800}{5}$$

$$= \frac{200}{5}\ [0 - (6)^2]$$

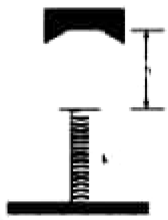
= 1960 Newton

Q.4 In figure a spring with k = 168 N/m is in its relaxed length and is at the top of a frictionless incline of angle $\theta = 37^\circ$. The lower end of the incline is at distance D = 1 m from the end of the spring. A small blcok of mass 2 kg is pushed against the spring until the spring is compressed by 0.2 m and released from rest.

(a) What is the speed of the block at the instant the spring returns to its relaxed length (which is when the block loses contact with the spring)? (b) What is the speed of the block when it reaches the lower end of the incline ?



Q.5 A block of mass m = 2.0 kg is dropped from height h = 40 cm onto a spring of spring constant k = 1960 N/s. Find the maximum distance through which the spring is compressed. Take $g = 9.8\ m/s^2$



Answers

Q.1 35 m/s Q.2 (i) 6 m/s (ii) 3 m/s Q.3 (i) $\frac{2F}{k}$ (ii) $\frac{F}{\sqrt{mk}}$

Q.4 (a) 2.4 m/s (b) 4.2 m/s Q.5 10 cm

Consevative and Non-conservertive force

If we throw a body up along smooth incline plane with some speed v_0, then it moves along the incline till it becomes stationary for a moment and then moves down the incline. It is observed the, when it reaches the point of projection, its speed is v_0 again, which proves that during the journey the net work done on the block is zero. Two forces act on the block during its motion. One is Normal force (N) which is continuously perpendicular to the block's motion, so its work for any part of the path is zero. Another one is weight (mg) which does negative work while upward motion and positive work of same magnitude during downward motion, does zero net work when the body reaches the initially position again.

Now we throw the same body on a rough incline plane. When it reaches the initial position, its speed is lesser than the speed of projection since friction does negative work for motion during up and as well as down the incline.

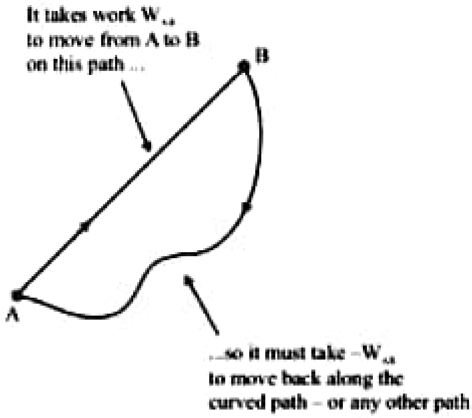
Thus, here we find two categories of force .

Conservative force

When the total work done by a force F acting as an object moves over any closed path is zero, the force is conservative. Mathematically.

$$\oint \vec{F}\cdot d\vec{r} = 0 \qquad \text{(conservative force)}$$

Suppose we move an object along the straight path between point A and B shown in figure, along which a conservative force acts; let the work done by the conservative force be W_{AB}. Since the work done over any closed path is zero, the work W_{BA} done in moving back from B to A must be $-W_{AB}$, whether we return along the straight path or the curved path or any other path. So, going from A to B involves work W_{AB}, regardless of the path taken.



In other words : *The work done by a conservative force in moving between two points is independent of the path taken; mathematically,* $\int_A^B \vec{F}\cdot d\vec{r}$ *depends only on the endpoints A and B, not on the path between them.*

These include force due to gravity (mg), spring force, electrostatic force etc.

Non-conservative force

Work done by non-conservative forces depends on path also. Least work is done for straight line path and any curved path it involves different work. These include frictional force, viscous force etc.

Illustration :

A particle moves in x-y plane from (0,0) to (a,a) and is acted upon by a force $\vec{F} = k(y^2\,\hat{i} + x^2\,\hat{j})\,N$, *where k is a constant and x and y are coordinates in meter. Find work done by this force, if the particle mones along*

(i) Two straight lines first from (0,0) to (a,0) and then from (a,0) to (a,a)

(ii) A single straight line.

Sol. $dW = \vec{F}\cdot d\vec{s} = k(y^2\,\hat{i} + x^2\,\hat{j})\cdot(dx\,\hat{i} + dy\,\hat{j})$

$= ky^2\,dx + k\,x^2\,dy.$

(i) when it moves from (0,0) to (a,0)

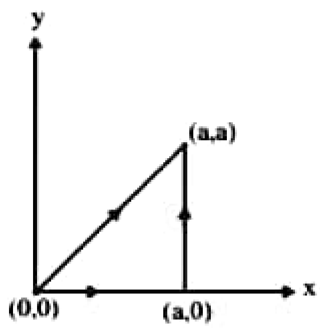
$y = constant = 0$

$\Rightarrow \quad dy = 0$

$\therefore \quad dW_A = k\,(0)\,dx + kx^2\,(0)$

$= 0$

$\Rightarrow \quad W_A = \int dw_A = 0$



When it moves from (a,0) to (a,a)

$x = constant = a \quad \Rightarrow \quad dx = 0$

and y changes from 0 to a

$\therefore \quad dW_B = ky^2\,(0) + ka^2\,dy = ka^2\,dy$

$\therefore \quad W_B = ka^2 \int_0^a dy = ka^3$

$\therefore \quad W = W_A + W_B = ka^3$

(ii) When moves from (0,0) to (a,a) as shown in above figure, along path C which is a straight line for which y = x

$\Rightarrow \quad dy = dx$

$\therefore \quad dW = kx^2\,dx + k\,x^2\,dx = 2k\,x^2\,dx$

$\therefore \quad W = \int dW = 2k \int_0^a x^2 dx = \frac{2ka^3}{3}$

In above illustration, the work done by the force is different for different paths for different paths taken, so it provides an example of non-conservative force.

Potential Energy

When we move an object upto some height against gravity and release, it gains kinetic energy while moving down. It means work done against a conservative force like gravity is somehow stored, in the sense that we can get it back again in the form of kinetic energy. We can consider the "stored work" as potential energy U.

The change ΔU_{AB} in potential energy associated with a conservative force is the negative of the work done by that force as it acts over any path from point A to point B:

$$\Delta U_{AB} = -\int_A^B \vec{F} \cdot d\,\vec{r}$$

If a conservative force does positive work (as does gravity on a falling object), then potential energy must decrease – and that means ΔU must be negative. Conversely, if a conservative force does negative work (as does gravity on a weight being lifted), then energy is stored and ΔU must be positive.

Change in potential energy are all that ever matter physically; the actual value of potential energy is meaningless. All though optenly it is convinient to establish a refernce point at which the potential energy is defined to be zero. When we say "the potential energy U," we really mean the potential-energy difference ΔU between the point we're considering and the reference point.

Gravitational Potential Energy (Near the Earth's Surface)

The work done by gravity on a particle of mass m whose vertical coordinate changes from y_A to y_B is

$$W_g = -\,mg\,(y_B - y_A)$$

From equation, we have $W_g = -\Delta U_g = -(U_B - U_A)$

Thus gravitational potential energy at the point B near the surface of the Earth is given by

$$U_B = U_A + mgh$$

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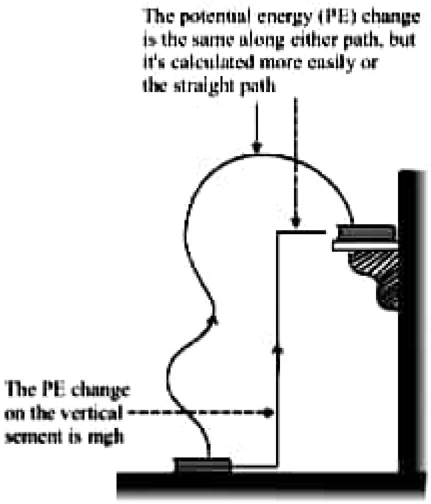
If we assume potential energy at the point A to be zero, then potential energy at the point B is given by

$$U_B = 0 + mgh = mgh$$



Elastic Potential Energy

When you stretch or compress a spring, you do work against the spring force, and that work gets stored as elastic potential energy. For an ideal spring, the force is $F = -kx$, where x is the distance the spring is stretched from equilibrium, and the minus sign shows that the force opposes the stretching or compression.

$$\Delta U = -\int_{x_1}^{x_2} F(x)\,dx = -\int_{x_1}^{x_2} (-kx)\,dx = \frac{1}{2}kx_2^2 - \frac{1}{2}kx_1^2$$

where x_1 and x_2 are the initial and final values of the stretch. If we take $U = 0$ when $x = 0$ –that is, when the spring is neither stretched nor compressed –then we can use this result to write the potential energy $U_2 = U$ at an arbitrary stretch (or compression) $x_2 = x$, $U = \frac{1}{2}kx^2$

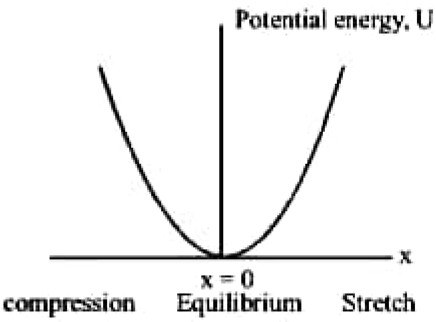
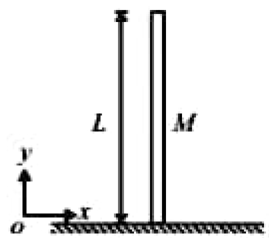


Illustration :

A uniform rod of mass M and length L is held vertically upright on a horizontal surface as shown in the figure. Find the potential energy of the rod if the zero potential energy level is assumed at the horizontal energy level is assumed at the horizontal surface.



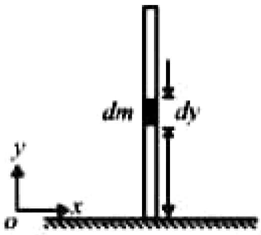
Sol. *Since the parts of the rod are at different level with respect to the horizontal surface, therefore, we have to use the integration to find its potential energy. Consider a small element of length dy at a height y from the horizontal.*

Mass of the element is

$$dm = \frac{M}{L} dy$$

Its potential energy is given by

$$dU = (dm)gy$$



or $$dU = \frac{M}{L} gydy$$

On integration, we get

$$U = \frac{Mg}{L}\int_0^L ydy$$

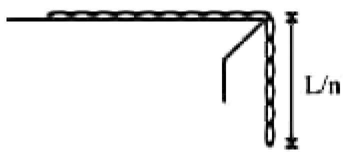
or $$U = \frac{Mg}{L}\left|\frac{y^2}{2}\right|_0^L$$

or $$U = \frac{1}{2} MgL$$

Note that the potential energy of the rod is equal to the product of Mg and height of the center of mass $\left(\frac{L}{2}\right)$ from the surface.

Illustration :

A unifrom chain of mass M and length L lies on a table with n^{th} part of it hanging off the table. Find the work required to slowly pull the hanging part up to the table.



Sol. *Here work is done agains the granity.*

$$\therefore \quad W = \Delta U = U_f - U_{in}$$

Taking table as reference level, i.e. $U_f = 0$

$$\therefore \quad W = -U_{in} = -[-mgh_{C.M.}]$$

$$= Mgh_{C.M.}$$

here mass of hanging part is $m = \frac{M}{n}$

and its centre of mass is $h_{C.M.}$ hight below the table

where $h_{C.M.} = \frac{(L/n)}{2} = \frac{L}{2n}$

$$\therefore \quad W = \left(\frac{M}{n}\right) g \left(\frac{L}{2n}\right) = \frac{MgL}{2n^2}$$

Conservation of Mechanical Energy

The work-energy theorem, shows that the change ΔKE in a body's kinetic energy is equal to the net work done on it :

$$\Delta KE = W_{net}$$

Consider separately the work W_c done by conservative force and the work W_{nc} done by nonconservative forces. Then

$$\Delta KE = W_c + W_{nc}$$

We've defined the change in potential energy ΔU as the negative of the work done by conservative forces. So we can write

$$\Delta KE = -\Delta U + W_{nc}$$

or $$\Delta KE + \Delta U = W_{nc}$$

We define the sum of the kinetic and potential energy as the mechanical energy. Then Equation shown that the change in mechanical energy is equal to the work downe by non-conservative forces.

i.e. $$\Delta E = W_{nc}$$

∴ $$\Delta E = 0 \qquad \text{if } W_{nc} = 0$$

Thus if work done by non-conservative forces is zero the mechanical energy of the system is unchanged. This called law of conservation of mechanical energy. It may also be written as

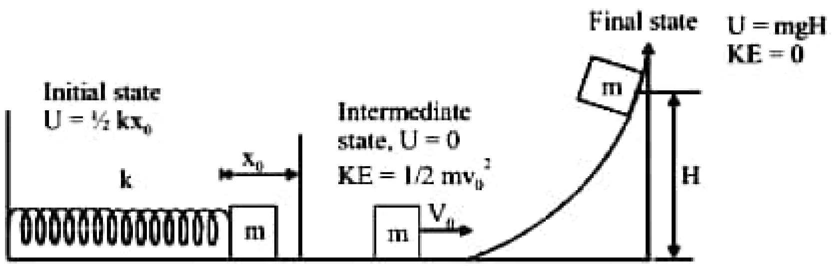
$$\Delta U + \Delta K\,E = 0$$

or $$\Delta U = -\Delta KE$$

or $$U + K\,E = \text{constant}$$

or $$U_{in} + K\,E_{in} = U_f + K\,E_f$$

The surfaces shown in the figure are frictionless and horizontal surface is taken as reference level.

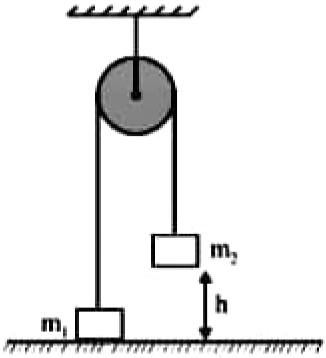


By conservation of mechanical energy

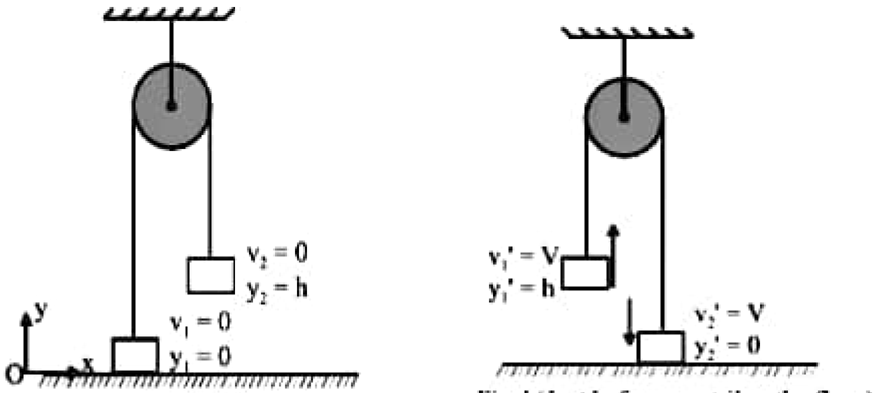
$$\frac{1}{2}kx_0^2 + 0 = 0 + \frac{1}{2}mv_0^2 = mgH + 0$$

Illustration :

Two block with masses $m_1 = 3kg$ and $m_2 = 5$ kg are connected by a light string that slides over a frictionless pulley as shown in figure. Initially, m_2 is held 5 m off the floor while m_1 is on the floor. The system is then released. At what speed does m_2 hit the floor ?



Sol. *The inital and final configurations are shown in the figure. It is convenient to set $U_g = 0$ at the floor. Initially, only m_2 has potential energy. As it falls, it loses potential energy and gains kinetic nergy. At the same time, m_1 gains potential energy and kinetic energy. Just before m_2 lands, it has only kinetic energy. Let v the final speed of each mass. Then, using the law of conservation of mechanical energy.*



Initial

Final (Just before m_2 strikes the floor)

$$K_f + U_f = K_i + U_i$$

$$\frac{1}{2}(m_1 + m_2)v^2 + m_1gh = 0 + m_2gh$$

$$v^2 = \frac{2(m_2 - m_1)gh}{m_1 + m_2}$$

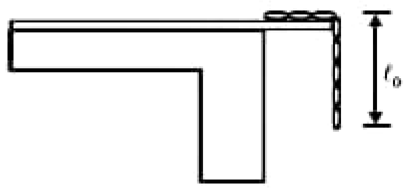
Putting $m_1 = 3$ kg; $m_2 = 5$ kg; $h = 5$ m and $g = 10\ m/s^2$

we get $v^2 = \frac{2(5-3)(10)(5)}{5+3}$

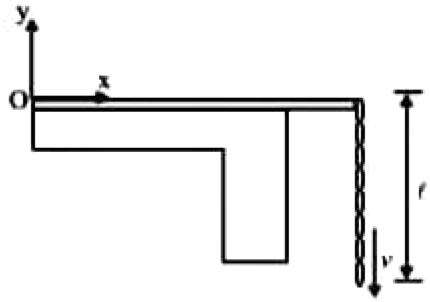
or $v = 5$ m/s.

Illustration :

A chain of length $\ell = 80$ cm and mass $m = 2$ kg is hanging from the end of plane so that the length ℓ_0 of the vertical segment is 50 cm as shown in the figure. The other end of the chain is fixed by a nail. At a certain instant, the nail is pushed out, what is the velocity of the chain at the moment it completely slides off the plane ? Neglect the friction.



Sol. *We assume the zero potential energy level at the horizontal plane. The initial and final configuration of the chain are shown in the figure. Initially,* $KE_{in} = 0$



$$U_{in} = 0 + \left(\frac{m}{l}l_0\right)g\left(-\frac{l_0}{2}\right)$$

or $$U_{in} = -\frac{ml_0^2}{2l}g$$

Note *that the part of chain lying over the table has zero potential energy.*

Finally, $$KE_f = \frac{1}{2}mv^2$$

Where v is the final velocity of the chain.

and $$U_f = -mg\frac{l}{2}$$

Using the law of energy conservation

$$KE_f + U_f = KE_{in} + U_{in}$$

$$\frac{1}{2}mv^2 - mg\frac{\ell}{2} = 0 - \frac{ml_0^2 g}{2l}$$

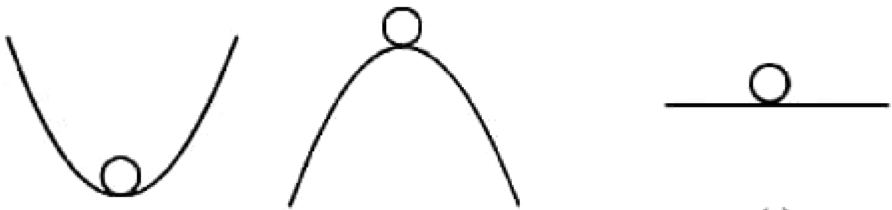
or $$v = \sqrt{\frac{g}{l}\left(l^2 - l_0^2\right)}$$

Putting $l = 0.8$m; $l_0 = 0.5m$; $g = 10\ m/s^2$, *we get*

$$v = \sqrt{5.1}\ m/s$$

or $$v = 2.23\ m/s$$

Types of equilibrium on the basis of stability



(a) (b) (c)

Suppose a small ball is placed on a smooth track under three different situations as shown in figure (a), (b) & (c). In all the three situation, the ball, is in equilibrium

(A) In figure (a), when the ball is slightly displaced from its equilibrium position, it tries to attain the shown position again, such type of equilibrium is called **stable equilibrium**.
Here potential energy in equilibrium position is minimum as compares to its neighbouring points i.e. under stable equilibrium, potential energy is minimum

$$\text{i.e. } \frac{dU}{dr} = 0 \text{ and } \frac{d^2U}{dr^2} > 0$$

(B) In figure (b), when the ball is slightly displaced from its equilibrium position it tends to move farther form the shown equilibrium position. Such type of equilibrium is called **unstable equilibrium.**
Here potential energy in equilibrium is maximum as compared to its near by points

$$\text{i.e. } \frac{dU}{dr} = 0 \text{ and } \frac{d^2U}{dr^2} < 0$$

(C) In figure (c), when the ball is displaced, it accepts the new position as equilibrium position, such type of equilibrium is called **neutral equilibrium.**

Here potential energy remains uniform for the equilibrium position

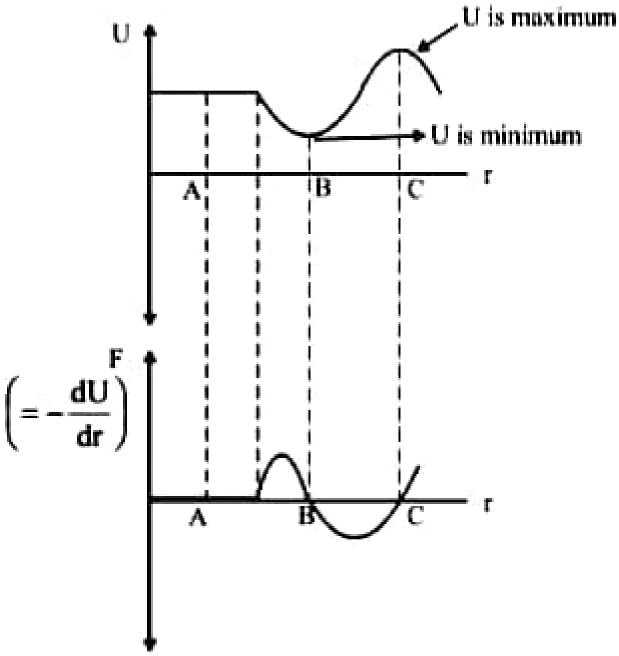
$$\text{i.e. } \frac{dU}{dr} = 0 \text{ and } \frac{d^2U}{dr^2} = 0$$

Although, the above discussion is under the effect of gravity but the results observed is applicable in other situations also where a particle can move under the effect of the conservative force only.

The above result can also be studied with the help of the following graphs.

For a particle whose position (r) varies along a straight line, the graphs below

Show variation of U vs r and F vs r.



At Point A $\quad F = 0;\ \frac{dU}{dr} = 0$, but F = 0 at its nearly points also. So when slightly displaced from A, the new position is also equilibrium. Thus point, A shown is the position of neutral equilibrium.

At Point B $\quad F = 0;\ \frac{dU}{dr} = 0$, Now when it is slightly displaced towards left of B, force is positive i.e. towards right and when it is slightly displaced towards right of B, force is negative i.e. towards right. Thus force tries to bring the particle towards B again. This type of force is called restoring force and the point B is the position of stable equilibrium.

At Point C $\quad F = 0,\ \frac{dU}{dr} = 0$ but when particle is displaced slightly form it towards any direction, force acts in that direction only i.e. to move the particle away from C. Thus point C is the position of unstable equilibrium.

Illustration :

The potential energy of a conservative system is given by

$$U = ax^2 - bx$$

Where a and b are positive constants. Find the equilibrium position and discuss whether the equilibrium is stable unstable or neutral.

Sol. *In a conservative field*

$$F = -\frac{dU}{dx} = -\frac{d}{dx}(ax^2 - bx) = -(2ax - b)$$

$$\therefore \quad F = b - 2ax$$

For equilibrium $F = 0$

or $\quad b - 2ax = 0 \quad \therefore \quad x = \frac{b}{2a}$

From the given equation we can see that $\frac{d^2U}{dx^2} = 2a$ *(positive), i.e., U minimum.*

Therefore, $x = \frac{b}{2a}$ *is the stable equilibrium position.*

Power

Power is the work done per unit time. If ΔW a work done in time Δt, the average power is $P_{av} = \frac{\Delta W}{\Delta t}$

Average power becomes intantaneous power as Δt approches to zero.

$\therefore$ Instantaneous Power $\quad P = \lim\limits_{\Delta t \to 0}\left(\frac{\Delta W}{\Delta t}\right)$

$$\Rightarrow \quad P = \frac{dW}{dt}$$

Due to a particular force on a particle

$$\Delta W = \vec{F} \cdot \Delta\vec{S}$$

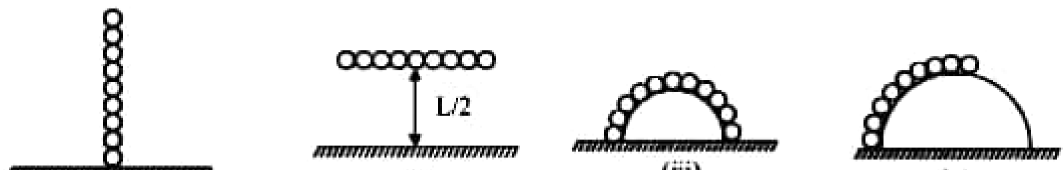
$$\therefore \quad P = \lim_{\Delta t \to 0} \vec{F} \cdot \frac{\Delta \vec{S}}{\Delta t} = \vec{F} \cdot \frac{d\vec{s}}{dt}$$

$\Rightarrow \quad P = \vec{F} \cdot \vec{V}$ (where $\vec{V}$ is velocity)

Centripetal force is always perpendicular to velocity, so power due to it is zero.

Practice Exercise

Q.1 A chain of length L and mass M is arranged as shown in following four cases. Arrange the potential energy (assumed zero at horizontal surface) in decreasing order.



(i) (ii) (iii) (iv)

Q.2 A particle is constrained to move in the region $x \geq 0$ and its potential energy is given by

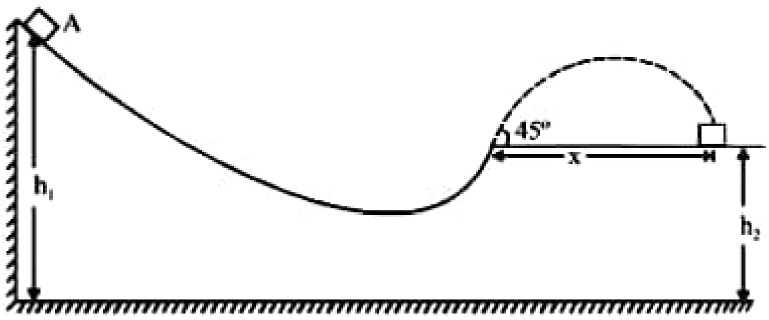
$$U = 2x^4 - 3x^2 \text{ J} \qquad \text{(where x is in m)}$$

For what values of x, the particle is under

(i) Stable equilibrium (ii) Unstable equilibrium

Q.3 A force $\vec{F} = -k(y\hat{i} + x\hat{j})$ (where K is a positive constant) acts on a particle moving in the X–Y plane. Starting from the origin, the particle is taken along the positive x-axis to the point (a, 0) and then parallel to the y-axis to the point (a,a). Find the total work done by the force F on the particle.

Q.4 A block starts from rest at point A, slides down a frictionless incline that terminates in a ramp pointing up at 45° angle, as shown in figure. Find an expression for the horizontanl range x shown in the figure as a function of the heights h_1 and h_2.



Q.5 A horse pulls a cart upto 200m in 5 minutes, exerting a force of 750N along the displacement of the cart. Find its power output measured in watts and in horsepower.

Answers

Where work done by tension, $W_T = 0$ (since tension is continuously perpendicular to motion) and work done by gravity $W_g = -mgh$

$$= -mgR(1-\cos\theta)$$

$\therefore \quad -mgR(1-\cos\theta) = \frac{1}{2}m(v^2 - u^2)$

$\therefore \quad v^2 = u^2 - 2gR(1-\cos\theta) \qquad \text{......(ii)}$

$\therefore$ from equation (i) and (ii)

$$T = mg\cos\theta + \frac{m}{R}[u^2 - 2gR(1-\cos\theta)]$$

$$\therefore \quad T = \frac{mu^2}{R} + 3mg\cos\theta - 2mg \qquad \text{.......(iii)}$$

Thus T is maximum when $\theta = 0^\circ$ i.e. at the lower most position A

$$T_{max} = T_A = mg + \frac{mu^2}{R}$$

Also T is minimum when $\theta = 180^\circ$ i.e. at the top most point B

$$T_{min} = T_B = \frac{mu^2}{R} - 5mg$$

Thus : $T_{max} - T_{min} = 6mg$ (independent of initial speed provided the porticle is able to complete the circular path)

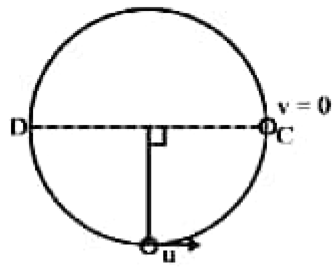
From equations (i) and (iii), following results are obtained

Case I

If $u = \sqrt{2gR}$, V becomes zero exactly at point C

Also when it is at point C tension, $T = m\frac{v^2}{R} = 0$

Now it oscillates between points C and D.



Case II

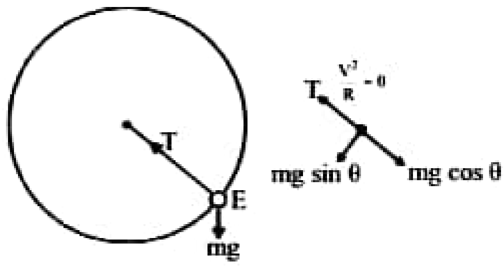
If $u < \sqrt{2gR}$, V becomes zero

even before $\theta = \frac{\pi}{2}$, let at point E

At this point

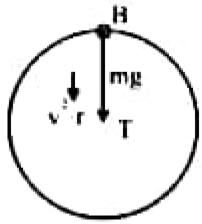
$$T - mg\cos\theta = m\left(\frac{v^2}{R}\right) = 0$$

$\Rightarrow \quad T = mg\cos\theta \neq 0$



Case III

If $u = \sqrt{5gR}$ then at the top most point B where $\theta = 180°$, the speed of the particle is



$v = \sqrt{5gR - 2gR(1-(-1))} = \sqrt{5gR - 4gR} = \sqrt{gR}$

At this point, $mg + T = \dfrac{m(\sqrt{gR})^2}{R}$

$\Rightarrow \quad T = 0$

i.e. string is just slacked, but it still continues to move in circuler path, since it has got speed.

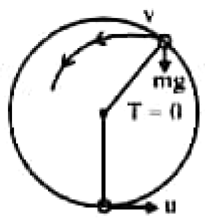
Case IV

If $u > \sqrt{5gR}$ and $T > 0$ even at the topmost point (B) and particle completes circular path

Case V

If $\sqrt{5gR} > u > \sqrt{2gR}$

T becomes zero at some point between C and B i.e. for $\frac{\pi}{2} < \theta < \pi$. And that instant v is non zero [see equation (ii)] and thus particle move in parabolic path after that due to gravity only.

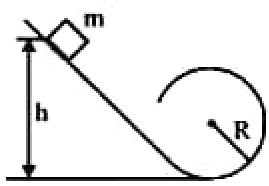


Motion of a particle at the inner surface of a vertical circular track

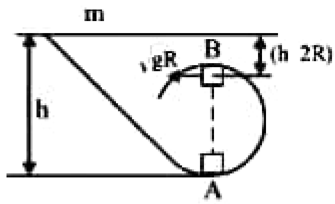
Same results as above are obtained when a body is given some speed u at the bottom most point inside a fixed circular loop as shown. Here instead of tension; normal force due to inner surface of the loop comes into action.

Illustration :

Figure shows a an incline which ends into a circular track of radius R. What should be the minimum value of height (h), so that the small object shown after release, is able to complete the loop. Neglect friction.



Sol. *It completes the loop, if its speed is atleast $\sqrt{gR}$ at B and $\sqrt{5gR}$ at A. Applying work energy theorem, for motion from starting point to the point B,*



$$mg\,(h - 2R) = \frac{1}{2} m \left[\left(\sqrt{gR}\right)^2 - 0 \right]$$

$$\Rightarrow \quad h = \frac{5R}{2}$$

Alternatively : Applying work - energy theorem for motion from starting point to A.

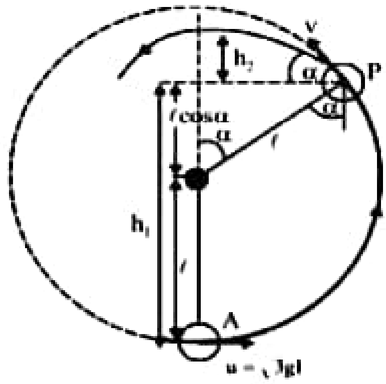
$$mgh = \frac{1}{2} m \left[\left(\sqrt{5gR}\right)^2 - 0 \right]$$

$$\Rightarrow \quad h = \frac{5R}{2}$$

Illustration :

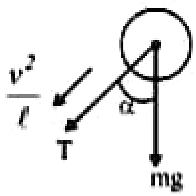
The bob of a simple pendulum of length ℓ is given a sharp hit to impart it a horizontal speed of $\sqrt{3g\ell}$. When it was at its lowermost position. Find (i) angle α shwon of the string from upside of vertical and speed of the particle when the string becomes slack. (ii) maximum height (from the bottom)

Sol. *(i) Since $\sqrt{2gl} < u < \sqrt{5gl}$, the string slacks somewhere between horizontal point and the topmost point.*
Let string slock at P, where speed is say v.



$h_1 = \ell + \ell\cos\alpha$

At point P,



$$\Rightarrow \quad T + mg\cos\alpha = \frac{mv^2}{\ell}$$

As the string slacks, T = 0

$$\Rightarrow \quad mg\cos\alpha = \frac{mv^2}{\ell}$$

$$\Rightarrow \quad v = \sqrt{g\ell\cos\alpha} \qquad \text{........(i)}$$

Applying work energy theorem for motion from A to P

$$-mg\,h_1 = \frac{1}{2}m(v^2 - u^2)$$

$\therefore$ *from equation (i)*

$$-mg\ell(1+\cos\alpha) = \frac{1}{2}\,m\,[g\,(\ell\cos\alpha) - 3g\ell]$$

$$\Rightarrow \quad \cos\alpha = \frac{1}{3}$$

$$\Rightarrow \quad \alpha = \cos^{-1}\left(\frac{1}{3}\right)$$

$\therefore$ *equation (i), $u = \sqrt{\dfrac{g\ell}{3}}$*

(ii) Now, after slackening of the string, the motion of the bob is under gravity only, for which the maximum height from P is given by

$$h_2 = \frac{v^2\sin^2\alpha}{2g}$$

where $v^2 = \dfrac{g\ell}{3}$ *and* $\sin^2\alpha = 1 - \cos^2\alpha = 1 - \left(\dfrac{1}{3}\right)^2 = \dfrac{8}{9}$

$$\therefore \qquad h_2 = \frac{\left(\frac{g\ell}{3}\right)\left(\frac{8}{9}\right)}{2g} = \frac{4\ell}{27}$$

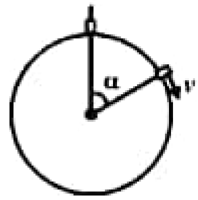
$$\therefore \qquad \text{maximum height from } A \text{ is} = h_1 + h_2 = \ell\left(1+\frac{1}{3}\right) + \frac{4\ell}{27}$$

$$= \frac{40\ell}{27}$$

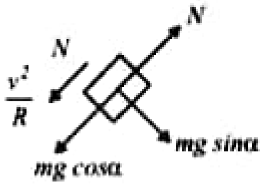
Illustration :

A particle slides on the surface of a fixed smooth sphese starting from the topmost point. Find the angle rotated by radius through the particle, where it leaves contact with the sphere. Also find speed at that instant.

Sol. *Let it rotates by angle* α



$$mg\cos\alpha - N = \frac{mv^2}{R}$$



It looses contact i.e. $N = 0$

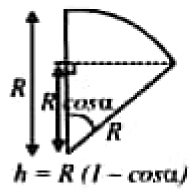
$$\Rightarrow \qquad mg\cos\alpha = \frac{mv^2}{R}$$

$$\Rightarrow \qquad v^2 = gR\cos\alpha \qquad \text{.........(i)}$$

Also applying work energy theorem

$$mgh = \frac{1}{2}m\,(v^2 - 0)$$

$$\Rightarrow \qquad mg\,R\,(1-\cos\alpha) = \frac{1}{2}m\,(gR\cos\alpha - 0)$$



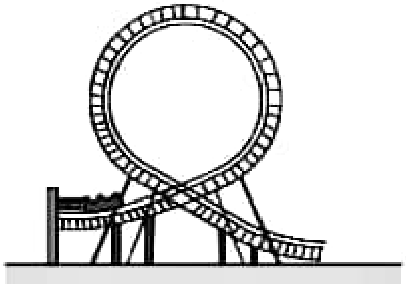
$$\Rightarrow \qquad \cos\alpha = \frac{1}{3}$$

$$\Rightarrow \qquad \alpha = \cos^{-1}\left(\frac{1}{3}\right)$$

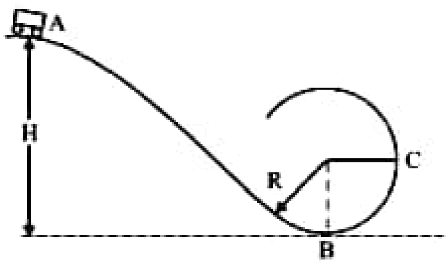
Also from equation (i), $\qquad v = \sqrt{\dfrac{gR}{3}}$

Practice Exercise

Q.1 An 900-kg roller-coaster car is launched from a giant spring of constant $k = 31$ kN/m into a frictionless loop-the-loop track of radius 6.2 m as shown in Figure. What is the minimum amount that the spring must be compressed if the car is to stay on the track ?



Q.2 A small toy car of mass m slides with negligible friction on 'loop' the loop track as shown in figure. The toy car starts from rest at point A which is height $H = 2R$ above level of the lowest point of the track :



(i) What normal force is exerted by the track on the toy car at point B ?

(ii) What are the speed and normal force at point C ?

(iii) At what height will the ball leave the track and to what maximum height will it rise afterwards ?

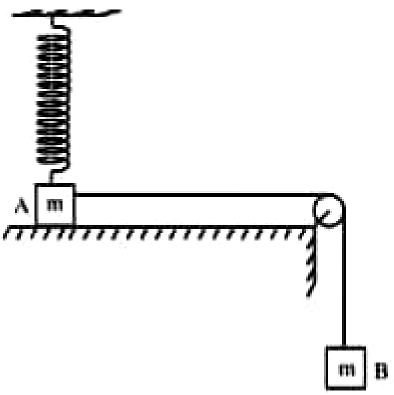
Q.3 A particle attached to a vertical string of length 1 m is projected horizontally with a velocity of $5\sqrt{2}$ m/s.

(a) What is maximum height reached by the particle from the lowermost point of its trajectory.

(b) If the string breaks when it makes an angle of 60° with downward vertical, find maximum height reached by the particle from the lowermost point of its trajectory.

Solved Example

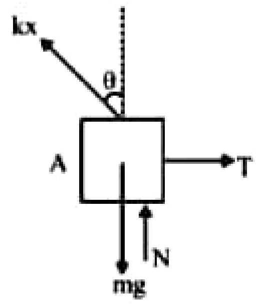
Q.1 Two blocks A and B, each having a mass of 320 g connected by a light string passing over a smooth light pulley. The block A is attached to a spring of spring constant 40 N/m whose other end is fixed to a support 40 cm above the horizontal surface as shown. Initially, the spring is vertical and ustretched when the system is released to move. Find the velocity of the block A at the instant it breaks off
the surface below it . Take $g = 10\ m/s^2$.



Sol. At the instant of break-off ,N = 0

$\Rightarrow \quad kx\cos\theta = mg$

Where extension in the spring



$$x = \ell - 0.4 = \frac{0.4}{\cos\theta} - 0.4$$

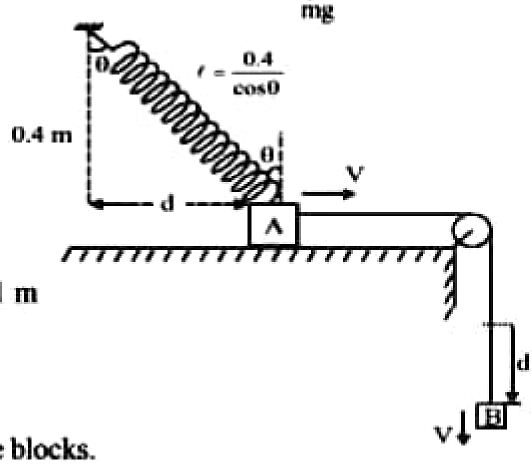
$$\therefore \quad k\left(\frac{0.4}{\cos\theta} - 0.4\right)\cos\theta = mg$$

$$k(0.4)(1-\cos\theta) = mg$$

$$\Rightarrow \quad \cos\theta = 1 - \frac{mg}{0.4k}$$

$$= 1 - \frac{0.320\times 10}{0.4\times 40}$$

$$\cos\theta = \frac{4}{5} \quad \Rightarrow \quad x = \frac{0.4\times 5}{4} - 0.4 = 0.1\ m$$



Also $\tan\theta = \frac{3}{4} \quad \Rightarrow \quad d = 0.4\tan\theta = 0.3\ m$

$\therefore$ By work energy theorem for motion of both the blocks.

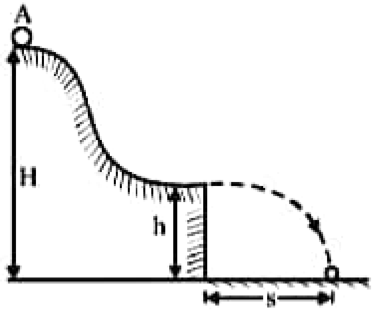
$$W_g + W_s = \Delta KE$$

$$\Rightarrow \quad mgd - \frac{1}{2}kx^2 = \left(\frac{1}{2}mv^2\right)\times 2$$

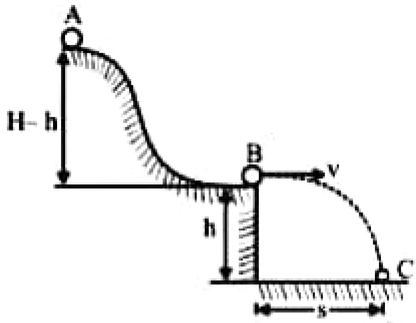
$$\Rightarrow \quad 0.32\times 10\times 0.3 - \frac{1}{2}(40)(0.1)^2 = (0.32)\,v^2$$

$$\Rightarrow \quad v = 1.54\ m/s$$

Q.2 A small body starts sliding from the height H with zero velocity down a smooth hill which has horizontal portion as shown. What must be the height of the horizontal portion 'h' to ensure the maximum distance 's' covered by the body ? What is the maximum value of 's' ?



Sol.



Applying work energy theorem for motion from A to B

$$mg(H-h) = \frac{1}{2}m(V^2 - 0)$$

$\therefore \quad V = \sqrt{2g(H-h)}$

If time taken by the disc to move from point B to the point C on ground is t, then

$$\Delta y = u_y t - \frac{1}{2}gt^2 = h$$

$\therefore \quad 0 - \frac{1}{2}gt^2 = -h \Rightarrow t = \sqrt{\frac{2h}{g}}$

$$S = \Delta x = vt$$

$\therefore \quad S = \sqrt{2g(H-h)\times\frac{2h}{g}} = 2\sqrt{Hh-h^2} \qquad \text{.....(i)}$

for s to be maximum, $(Hh - h^2)$ is maximum

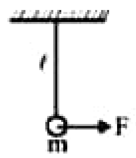
i.e. $\quad \frac{d}{dh}\left[Hh - h^2\right] = 0$

$\therefore \quad H - 2h = 0 \quad \Rightarrow \quad h = \frac{H}{2}$

Putting this value of h in equation (i)

$\therefore \quad S_{max} = 2\sqrt{H\left(\frac{H}{2}\right) - \left(\frac{H}{2}\right)^2} = H$

Q.3 A constant horizontal force F of magnitude equal to $\frac{mg}{2}$ begins to act on the bob of pendulum shown when the bob was at rest. What maximum angle the string makes with the vertical ?



Sol. Let the string makes angle θ with the vertical when it comes to rest (momentarily) as shown

$\therefore \quad x = \ell \sin\theta$

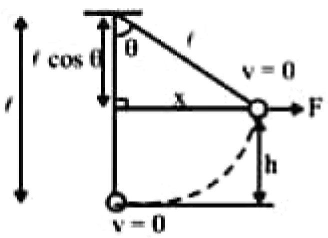
and $\quad h = \ell - \ell\cos\theta = \ell(1-\cos\theta)$

Here work done by gravity $W_g = -mgh = -mg\,\ell(1-\cos\theta)$

Work done by tension, $W_T = 0$ $[\vec{T} \perp \vec{v}]$

Work done by force F, $W_F = F \times$ (displacement in the direction of F) $= Fx$

$$= \frac{mg}{2}\,\ell\sin\theta$$



$\therefore$ Applying work-energy theorem

$$W_g + W_F + W_T = \Delta K \cdot E = 0$$

$$\therefore \quad \frac{1}{2}\sin\theta = 1-\cos\theta$$

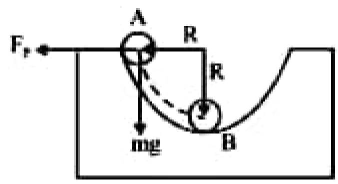
$$\therefore \quad \frac{1}{2}\left[2\times\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)\right] = 2\sin\left(\frac{\theta}{2}\right)$$

$$\therefore \quad \tan\left(\frac{\theta}{2}\right) = \frac{1}{2}$$

$$\therefore \quad \theta = 2\tan^{-1}\left(\frac{1}{2}\right)$$

Q.4 A small body is placed at rest at the bottom B of a smooth hemispherical surface of wedge as shown. If the wedge is shifted horizontally towards right with acceleration $a_0 = 3g$, find speed of the body w.r.t the wedge at the instant the body reaches points A.

Sol. Here we need calculation w.r.t the wedge which is accelerated i.e. non-inertial frame of reference, So we have to consider Pseudo froce (F_P) and work done by the Pseudo force (W_P) also.
Where $F_P = ma_0 = 3mg$ and is towards left i.e. apposite to a_0
By work-energy theorem for motion from B to A

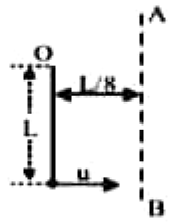


$$F_P R - mg R = \frac{1}{2}m(v^2-0)$$

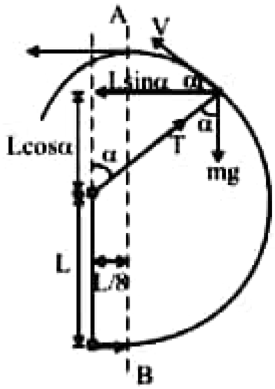
$$\Rightarrow \quad 3\,mgR - mgR = \frac{1}{2}mv^2$$

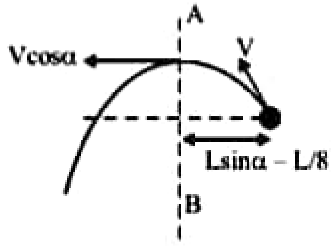
$$\therefore \quad v = 2\sqrt{gR}$$

Q.5 A particle is suspended vertically from a point O by an inextensible massless string of length L. A vertical line AB is at a distance L/8 from O as shown. The object given a horizontal velocity u. At some point, its motion ceases to be circular and eventually the object passes through the line AB. At the instant of crossing AB, its velocity is horizontal. Find u.



Sol Let the string slacks when the particle is at point P as shown





At point P, $T + mg\cos\alpha = \dfrac{mV^2}{L}$

where $T = 0$ (as string slacks)

$\therefore \quad mg\cos\alpha = \dfrac{mV^2}{L}$

$\Rightarrow \quad V^2 = gL\cos\alpha \qquad$ ....(i)

After this it undergoes parabolic path. When it passes through line AB its velocity is horizontal which implies that $(L\sin\alpha - L/8)$ is half of horizontal range.

i.e. $\quad L\sin\alpha - \dfrac{L}{8} = \dfrac{1}{2}\left[\dfrac{V^2\sin(2\alpha)}{g}\right]$

$\therefore$ from equation (i)

$$L\sin\alpha - \frac{L}{8} = \frac{1}{2}\left[\frac{gL\cos\alpha(2\sin\alpha\ \cos\alpha)}{g}\right]$$

$\therefore \quad \sin\alpha - \dfrac{1}{8} = \sin\alpha\cos^2\alpha$

$\Rightarrow \quad \sin\alpha(1 - \cos^2\alpha) = \dfrac{1}{8}$

$\Rightarrow \quad \sin^3\alpha = \dfrac{1}{8} \quad \Rightarrow \quad \sin\alpha = \dfrac{1}{2} \quad \Rightarrow \quad \alpha = 30^\circ$

$\therefore \quad V^2 = \dfrac{gL\sqrt{3}}{2}$

Also by applying work energy theorem

$$-mgL(1 + \cos\alpha) = \frac{1}{2}m(V^2 - u^2)$$

$= \quad -gL\left(1 + \dfrac{\sqrt{3}}{2}\right) = \dfrac{1}{2}\left(\dfrac{gL\sqrt{3}}{2} - u^2\right)$

$\therefore$ On solving, we get $\quad u = \sqrt{\dfrac{gL}{2}\left(4 + 3\sqrt{3}\right)}$