KINEMATICS

Introduction

The branch of physics in which motion and the forces causing motion are studied is called mechanics.

As a first step in studying mechanics, we describe the motion of particles and bodies in terms of space and time without studying the cause of motion. This part of mechanics is called kinematics. We first define displacement, velocity and acceleration. Then, using these concepts, we study the motion of the objects moving under different conditions. The forces causing motion will be discussed later in Dynamics.

From everyday experience, we recognize that motion represents continuous change in position, so we begin our study with change in position i.e. with displacement.

Various quantities used in Kinematics

Displacement ($\vec{S}$ or $\overrightarrow{\Delta r}$) :

Change in position vector is called **displacement**.

Its magnitude is minimum distance between final and initial point, and is directed from initial position to final position.

For a particle moving along x axis, motion from one position x_1 to another position x_2 is displacement, Δx where, $\Delta x = x_2 - x_1$

If the particle moves from $x_1 = 4m$ to $x_2 = 12$ m, then $\Delta x = (12m) - (4m) = +8m$. The positive result indicates that the motion is in the positive direction. If the particle then returns to x = 4m, the displacement for the full trip is zero. The actual number of meters covered for the full trip is irrelevant ***displacement involves only the original and final position.***

In general if initial position vector and final position vector are $\vec{r}_{in}$ and $\vec{r}_f$ respectively, then

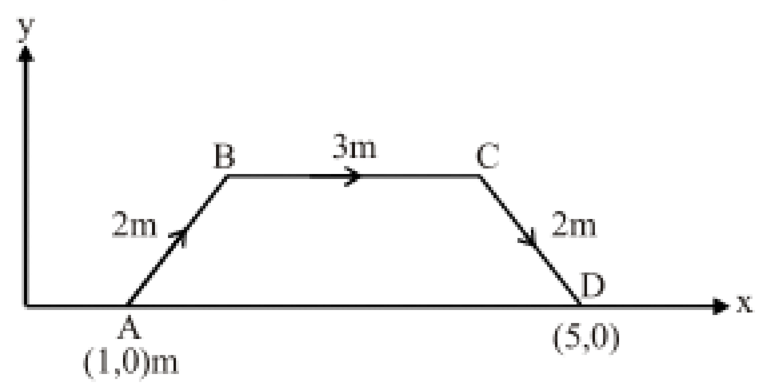
$$\vec{S} = \vec{r}_f - \vec{r}_{in} = \overrightarrow{\Delta r}$$

Distance:

Length of path traversed by a body is called **distance**.

It is dependent on the path chosen, thus for motion between two fixed points A and B we can have many different values of distance traversed. It is a **scalar** quantity, as length of path has no indication of direction in it. Its SI unit is meter (m) and dimensions is (L).

eg. Suppose a particle moves from position A to B as shown after travelling from A to B to C to D.



Here Displacement $\vec{S} = \overrightarrow{AD} = 5\hat{i} - \hat{i} = 4\hat{i}$ m

$\therefore \quad |\text{displacement}| = 4m$

Also distance covered ,

$$l = |\overrightarrow{AB}| + |\overrightarrow{BC}| + |\overrightarrow{CD}| = 2 + 3 + 2 = 7 \text{ m}$$

Note : Here $|\text{Displacement}| < \text{Distance}$

Magnitude of displacement would be equal to distance travelled if there is no change in direction during the whole motion.

In general, $|\text{Displacement}| \le \text{Distance}$

Average Velocity :

The average velocity $\vec{V}_{avg}$ is the ratio of the total displacement $\overrightarrow{\Delta r}$, and total time ($\Delta t$) taken to complete that displacement. It should be noted that $\vec{V}_{avg}$ is independent of path as displacement is independent of path.

$$\vec{V}_{avg} = \frac{\Delta \vec{r}}{\Delta t} = \frac{\vec{r}_f - \vec{r}_{in}}{\Delta t}$$

Unit for $\mathbf{V}_{avg}$ is the meter per second (m/s). The average velocity $\mathbf{V}_{avg}$ always has the same sign as the displacement $\overrightarrow{\Delta r}$.

Average Speed :

$$\text{Average speed} = \frac{\text{Total distance travelled}}{\text{Total time interval}} = \frac{l}{\Delta t}$$

It is a scalar and always has positive sign.

$$|\text{Average velocity}| \le |\text{Average speed}|$$

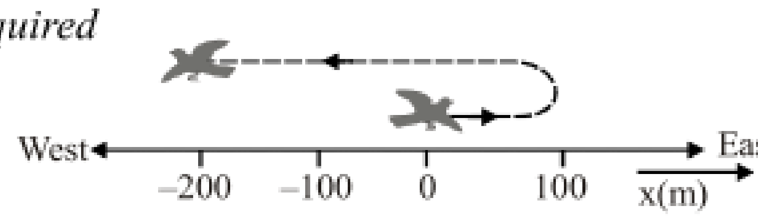
Illustration :

A bird flies east at 10 m/s for 100 m. It then turns around flies at 20 m/s for 15 s. Neglect time taken for turning, find

(a) its average speed

(b) its average velocity

Sol. *Let us take the x axis to point east. A sketch of the path is shown in the figure. To find the required quantities, we need the total time interval.*



The first part of the journey took

$\Delta t_1 = (100 \text{ m}) / (10 \text{ m/s}) = 10 s$,

and we are given $\Delta t_2 = 15$ *s for the second part. Hence the total time interval is*

$$\Delta t = \Delta t_1 + \Delta t_2 = 25 \text{ s}$$

The bird flies 100 m east and then (20 m/s) (15s) = 300 m west

(a) Average speed $= \dfrac{\text{Distance}}{\Delta t} = \dfrac{100\text{m} + 300\text{m}}{25\text{s}} = 16 \text{ m/s}$

(b) The net displacement is

$$\Delta x = -200 \text{ m}$$

So that

$$v_{av} = \frac{\Delta x}{\Delta t} = \frac{-200}{25s} = -8\ m/s$$

The negative sign means that v_{av} is directed toward the west.

Illustration :

A particle moves with speed v_1 along a particular direction. After some time it turns back and reaches the starting point again travelling with speed v_2. Find (for the whole journey)

(a) Average velocity *(b) Average speed*

Sol. *(a) Since the particle reaches the starting point again, its displacement is zero*

$$\therefore \quad \text{Average velocity} = \frac{\text{displacement}}{\text{total time}} = 0$$

(b) Let it travelled distance x while moving away as well as while moving towards the starter point.

Time taken to go away is $t_1 = \dfrac{x}{v_1}$

Time taken while return journey $t_2 = \dfrac{x}{v_2}$

$$\therefore \quad \text{Average speed} = \frac{x+x}{t_1+t_2} = \frac{2x}{\frac{x}{v_1}+\frac{x}{v_2}} = \frac{2v_1v_2}{v_1+v_2}$$

Instantaneous Velocity :

Instantaneous Velocity is defined as the value approached by the average velocity when the time interval for measurement becomes closer and closer to zero, i.e. $\Delta t \to 0$. Mathematically

$$v(t) = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t}$$

Thus Instantaneous velocity function is the derivative of the displacement with respect to time.

$$v(t) = \frac{dx(t)}{dt}$$

Instantaneous Speed

It is measure of how fast a particle or a body is moving at a particular instant. It is the magnitude of instantaneous velocity. Thus particle moving with instantaneous velocity of + 5m/s and another moving with –5m/s will have same instantaneous speed of 5 m/s.

The speedometer in a car measure the instantaneous speed not the instantaneous velocity, because it cannot determine the direction.

Average Acceleration

. For any change in velocity either in its magnitude or direction or both, acceleration must be present. Without acceleration neither direction nor magnitude of velocity can be changed.

When a particle's velocity changes, the particle is said to undergo acceleration (or to accelerate).

The **Average Acceleration ($\vec{a}_{avg}$)** over a time interval Δt is

$$\vec{a}_{avg} = \frac{\vec{v}_2 - \vec{v}_1}{t_2 - t_1} = \frac{\Delta\vec{v}}{\Delta t}$$

where the particle has velocity v_1 at time t_1 and then velocity v_2 at time t_2.

Instantaneous Acceleration

The **Instantaneous Acceleration** (or simply acceleration) is the derivative of the velocity with respect to time.

$$\vec{a} = \frac{d\vec{v}}{dt}$$

In words, the acceleration of a particle at any instant is the rate at which its velocity is changing at that instant.

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d}{dt}\left(\frac{d\vec{r}}{dt}\right) = \frac{d^2\vec{r}}{dt^2}$$

In words, the acceleration of a particle at any instant is the second derivative of its position vector with respect to time.

Acceleration has both magnitude and direction (it is yet another vector quantity). For motion on a straight line its algebraic sign represents its direction on an axis just as for displacement and velocity; that is, acceleration with a positive value is in the positive direction of an axis, and acceleration with a negative value is in the negative direction.

Illustration :

The position of a particle moving along x-axis is given by $x = (5t^2 - 4t + 20)$ meter, where t is in second.

(a) Find average velocity between 1s & 3s

(b) Find velocity as a function of time v(t) and its value at t = 3s

(c) Find acceleration at t = 2 sec.

(d) When is the particle at rest ?

Sol. (a) At $t = 1s$; $x_{in} = 5(1)^2 - 4(1) + 20 = 21\ m$

At $t = 3s$; $x_f = 5(3)^2 - 4(3) + 20 = 53\ m$

$$v_{av} = \frac{\Delta x}{\Delta t} = \frac{53-21}{3-1} = 16\ m/s$$

(b) $v(t) = \frac{dx}{dt} = (10t - 4)\ m/s$

at $t = 3s$, $v = 10(3) - 4 = 26\ m/s$

(c) $a = \frac{dv}{dt} = 10\ m/s^2$ (constant at any instant)

(d) particle at rest i.e. $v = 0 = 10t - 4$

$\Rightarrow t = 0.4s$

Note :

(1) Here we can observe at $t = 0.4s$, particle has zero velocity but acceleration of $10\ m/s^2$. Thus particle having zero velocity need not have zero acceleration.

(2) For $t < 0.4s$, velocity is negative and for $t > 0.4$ s, velocity is in positive direction i.e. its velocity

changes its direction at t = 0.4 sec, when becoming zero.

Illustration :

Position of a particle moving along a straight live is given by

$x = (t^2 - 4t)$ *meters. (t is in sec.)*

Find Displacement and distance travelled between

$t = 0$ *and* $t = 3sec$

Sol. *Displacement* $= \Delta x = x_3 - x_0$

$= [(3)^2 - 4(3)] - [(0)^2 - 4\,(0)]$

$= -\,3m$

Now, velocity $v = \dfrac{dx}{dt} = 2t - 4$

i.e. $v = 0$ *at* $t = 2$ *sec*

for $t < 2$ *sec,* $V = -ve$ & *for* $t >$ *sec,* $v = +ve$

Distance travelled = |*Displacement in –ve direction*| + |*Displacement in +ve direction*|

$= |\Delta x \text{ for } t = 0 \text{ to } t=2s| + |\Delta x \text{ for } t = 2s \text{ to } t = 3s|$

$= |x_2 - x_0| + |x_3 - x_2|$

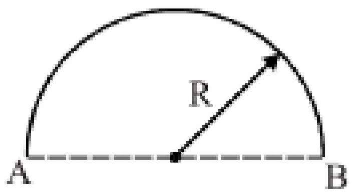
$= |[(2)^2 - 4(2) - (0)]| + |3^2 - 4(3) - 2^2 + 4\,(2)|$

$= |-4| + |1|$

$= 5m$

Practice Exercise

Q.1 If a particle traverses on a semicircular path of radius R from A to B as shown in time T, find average speed and average velocity.



Q.2 A man runs for first 120 m at 6m/s and then next 120m at 3m/s in the same direction. Find

(a) Total time of run (b) Average velocity

Q.3 Position of particle moving along x-axis is given by

$x = (3t^2 - 2t^3)$m (t is in sec)

Find

(a) its average velocity form t = 0 s to t = 2s

(b) v (t) and a (t)

(c) The time at which its acceleration is zero and find velocity at the instant.

Q.4 Position of a particle moving along straight line is given by $x = (-t^2 + 6t + 5)$ m (t is in sec) Find

(a) The time at which velocity of particle is zero.

(b) Average velocity from $t = 0$ to $t = 4$ sec

(c) Average speed from $t = 0$ to $t = 4$sec

Answers

1. Average speed $= \frac{\pi R}{T}$; Average velocity $= \frac{2R}{T}$ (from A to B)

2. (a) 60s (b) 4 m/s 3. (a) – 2m/s (b) $v(t) = 6t - 6t^2$ m/s (c) $\frac{1}{2}$s ; $\frac{3}{2}$m/s

4. (a) 3 sec (b) – 2 m/s (c) 2.5 m/s.

One Dimensional or Rectilinear Motion

We may divide this topic in the following different situations.

(i) Motion with constant velocity

(ii) Motion with variable velocity but constant acceleration

(iii) Motion with variable acceleration.

Motion with constant velocity

$$v = \frac{dx}{dt} \quad \Rightarrow \quad \int_{x_0}^{x} dx = \int_{0}^{t} v\,dt$$

Since velocity is constant, it comes out of the integration

$$\Rightarrow \int_{x_0}^{x} dx = v\int_{0}^{t} dt$$

$$[x]_{x_0}^{x} = v\,[t]_0^t$$

$x - x_0 = vt$ i.e., displacement $\mathbf{\Delta x = vt}$

Motion with variable velocity but constant acceleration

Basic formulae

(i) $a = \frac{dv}{dt}$

(ii) $a = \frac{dv}{dx} \cdot \frac{dx}{dt} = v\frac{dv}{dx}$ (By chain rule)

From formula (i)

$$a = \frac{dv}{dt} \Rightarrow dv = a\,dt; \quad \int_u^v dv = \int_0^t a\,dt$$

Since acceleration is constant so it comes out of the integration

$$[v]_u^v = a \int dt$$

$$\therefore v - u = at$$

$$\Rightarrow \quad \mathbf{v = u + at} \quad \text{............(i)}$$

$$\frac{dx}{dt} = u + at$$

$$dx = udt + at\,dt$$

on further integrating

$$\int_{x_0}^{x} dx = u \int_0^t dt + a \int_0^t t\,dt$$

$$[x]_{x_0}^{x} = ut + \frac{at^2}{2}$$

$$x - x_0 = ut + \frac{1}{2} at^2$$

$$\Rightarrow \quad \mathbf{\Delta x = ut + \frac{1}{2} at^2} \quad \text{.............(ii)}$$

From formula (ii)

$$a = v\frac{dv}{dx}$$

$$\int_u^v vdv = a\int_{x_0}^{x} dx$$

$$\frac{v^2}{2} - \frac{u^2}{2} = a(x - x_0)$$

$$\mathbf{v^2 = u^2 + 2a(\Delta x)} \quad \text{.............(iii)}$$

Taking $a = \frac{v-u}{t}$ from equation (i) and putting it in equation (ii), we get

$$\Delta x = ut + \frac{1}{2}\left(\frac{v-u}{t}\right)t^2$$

$$\Rightarrow \quad \mathbf{\Delta x} = \left(\frac{v+u}{2}\right)t \quad \text{.............(iv)}$$

Displacement in n^{th} second

Displacement in n^{th} second = Displacement in n sec. – Displacement in (n–1) sec.

$$S_{n^{th}} = S_n - S_{n-1^{th}} = [u(n) + \frac{1}{2}\ an^2] - [u\ (n-1) + \frac{1}{2}\ a\ (n-1)^2]$$

$$\therefore \quad \mathbf{S_{n^{th}} = u + \frac{a}{2}\ (2n-1)} \qquad \text{.............(v)}$$

Illustration :

On seeing a board of speed limit, you brake a car from speed of 108 km/h to a speed of 72 km/h. covering a distance of 100m at a constant acceleration.

(a) What is that acceleration ?

(b) How much time is required for the given decrease in speed ?

Sol. Initial speed, $u = 108\ km/h = 108 \times \frac{5}{18} m/s = 30\ m/s$

final speed, $v = 72\ km/h = 72 \times \frac{5}{18} m/s = 20\ m/s$

(a) By 3^{rd} equation of motion

$$v^2 = u^2 + 2as$$

$$\therefore a = \frac{v^2 - u^2}{2s} = \frac{(20)^2 - (30)^2}{2 \times 100}$$

$$\therefore a = -2.5\ m/s^2$$

(b) By 1^{st} equation, $v = u + at$

$$\Rightarrow t = \frac{v-u}{a} = \frac{20-30}{(-2.5)} = 4 sec$$

Illustration :

The time taken between observation of an event and taking action according to that is called reaction time. Suppose a person having reaction time of 0.3 sec is driving the car as stated in above example. Find the distance travelled by him after seeing the board till the car reaches 72 km/h.

Sol. *Till the reaction time i.e. till the brakes are applied speed of car remains uniform. So distance travelled during that time is*

$$S_1 = 30 \times 0.3 = 9\ m$$

Distance travelled after that time is $S_2 = 100$

$\therefore$ *Total distance travelled* $= S_1 + S_2 = 109\ m.$

Illustration :

A train is moving with 108 km/h. On a straight track. receiving red signal its brakes are applied and it retards at the rate of $3m/s^2$. Find its displacement and average velocity for next 15 sec.

Sol. *Initial velocity,* $u = 100\ km/h = 30\ m/s$

Let time reqd. for the velocity to become zero is t.

$$V_{final} = u + at$$

$$\therefore \quad 30 - 3t = 0 \Rightarrow t = 10\ sec. < 15\ sec.$$

i.e., it covers no distance after $t = 10$ *sec.*

$\therefore$ *Displacement till 15 sec = displacement till 10 sec*

$$= 30\ (10) + \frac{1}{2}\ (-3)\ (10)^2$$

$$= 150\ m$$

$$V_{av} = \frac{\text{Total displacement}}{\text{Total time}} = \frac{150}{15} = 10\ m/s$$

Note : In above example, for finding V_{av}*, we have taken total time of 15 sec, which actually was required.*

If we have to find V_{av} *for 10 sec, it would be*

$$V_{av} = \frac{150}{10} = 15\ m/s$$

Although displacement in 15 sec = Displacement in 10 sec., but times are different.

Thus V_{av} *for 15 sec. is not same as* V_{av} *for 10 sec.*

Motion with variable acceleration

Relations :

(i) $\dfrac{dv}{dt} = a \quad \Rightarrow \quad \int_{v_1}^{v_2} dv = \int_{t_1}^{t_2} a dt$

(iii) $\dfrac{dx}{dt} = v \quad \Rightarrow \quad \int_{x_1}^{x_2} dx = \int_{t_1}^{t_2} v dt$

(iii) $a = \dfrac{dv}{dt} = \dfrac{dv}{dx} \cdot \dfrac{dx}{dt}$ (By chain rule)

$$\therefore \quad a = v\frac{dv}{dx}$$

$$\therefore \quad \int_{v_1}^{v_2} V dv = \int_{x_1}^{x_2} a dx$$

Illustration :

The acceleration of a particle is given by $a = 2t^2\ m/s^2$. If it is at rest at the origin at time $t = 0$, find its position, velocity, and acceleration at time $t = 1s$.

Sol. $a = 2t^2$

$\therefore$ $a = 2 \times 1^2 = 2\ m/s^2$ (at $t = 1$ sec.)

Formula for v,

$$\frac{dv}{dt} = 2t^2$$

or, $$\int_0^v dv = \int_0^t 2t^2\, dt$$

or, $$v = \frac{2t^3}{3}$$

At $t = 1$ sec $\quad t = 1,\ v = \dfrac{2 \times 1^3}{3} = \dfrac{2}{3}\ m/s$

Formula for x,

$$\frac{dx}{dt} = \frac{2}{3}t^3 \qquad \text{or,} \qquad \int_0^x dx = \int_0^t \frac{2}{3}t^3 dt$$

or, $$x = \frac{t^4}{6}$$

At $t = 1$ sec, $\quad x = \dfrac{1}{6}\ m$

Illustration :

A particle located at $x = 0$ at time $t = 0$ starts moving along the positive x direction with a velocity v that varies as $v = \alpha\sqrt{x}$. How do the velocity and acceleration of the particle vary with time? What is the average velocity of the particle after the first s meter of its path ?

Sol. $$v = \alpha\sqrt{x} \qquad \text{or,} \qquad \frac{dx}{dt} = \alpha\sqrt{x}$$

or $$\int_0^x \frac{dx}{\sqrt{x}} = \alpha\int_0^t dt \qquad \text{or,} \qquad 2\sqrt{x} = \alpha t \quad ....(i)$$

$\therefore$ $$x = \left(\frac{1}{4}\right)\alpha^2 t^2$$

$\therefore$ $$v = \frac{dx}{dt} = \frac{1}{4}\alpha^2(2t) = \frac{1}{2}\alpha^2 t$$

$\therefore$ $$a = \frac{dv}{dt} = \frac{1}{2}\alpha^2$$

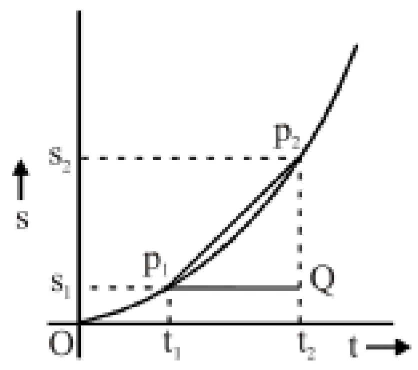
Time to cover a path of length s is obtained from (i) as $t = \dfrac{2\sqrt{s}}{\alpha}$ $(\therefore\ x = z)$

$\therefore$ *The average velocity over the path s is given by* $v_{av} = \dfrac{s}{t}\ \dfrac{s}{\left(\dfrac{2\sqrt{s}}{\alpha}\right)} = \dfrac{1}{2}\alpha\sqrt{s}$

Graphical Representation of Motion in one Dimension

S-t curve :

If we put **s** on y-axis and **t** on x-axis then for every value of **t** we have a specific value of **s**.



The **Average velocity** from time t_1 to t_2 will be

$$V_{avg} = \frac{s_2 - s_1}{t_2 - t_1} = \text{slope of line joining the points } p_1 \text{ and } p_2$$

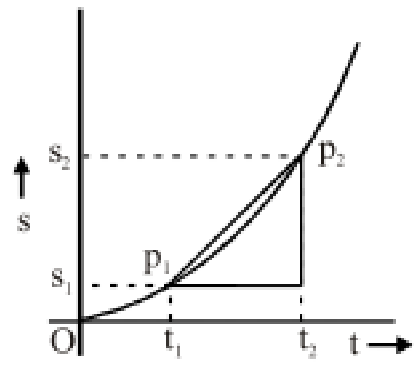
For a particle moving along a straight line when we plot a graph of **s** versus **t**, $\mathbf{V}_{avg}$ is the slope of the straight line that connects two particular points on the **s(t)** curve : one is the point that corresponds to s_2 and t_2, and the other is the point that corresponds to s_1 and t_1. Like displacement, $\mathbf{v_{avg}}$ has both magnitude and direction (it is another vector quantity). Its magnitude is the magnitude of the line's slope. A positive $\mathbf{v_{avg}}$ (and slope) tells us that the line slants upward to the right ; a negative v_{avg} (and slope), that the line slants downward to the right.

Instantaneous velocity

According to defination

$$v = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t}$$

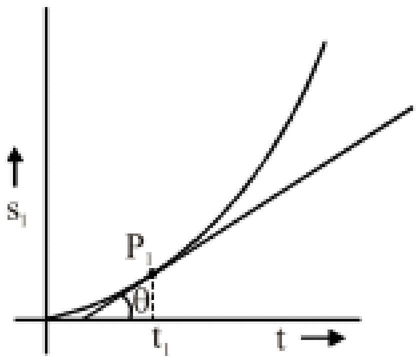
In curve if $\Delta t \to 0$ the point p_2 comes very close to point p_1.



Note : The instantaneous velocity can be found by determining the slope of the tangent to the displacement time graph at that instant.

Velocity at point p_1 or time t_1 is V

$V = \tan\theta$



Cases :

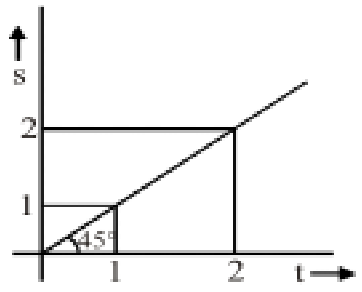
(A) **Uniform velocity :**

If velocity is uniform slope of curve must remain unchanged.

Curve with uniform slope is straight line

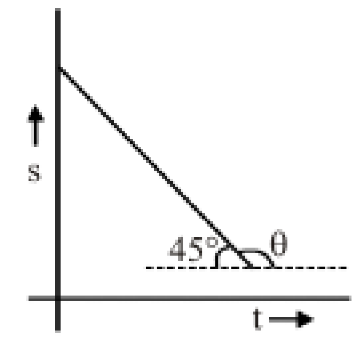
e.g. (i) $S = Vt$, If Velocity is $1 ms^{-1} \Rightarrow S = t$

$\tan\theta = 1$



e.g. (ii) If velocity is –1 m/s ⇒ S = – t

$\tan\theta = -1$



For negative velocity

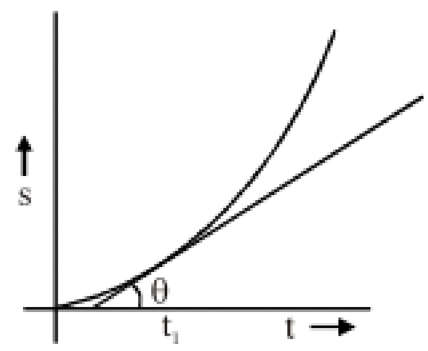
(B) **Uniform acceleration**

We have a particle moving with uniform acceleration a and initial velocity u. Its displacement s at any time t can be represented as

$$s = ut + \frac{1}{2}at^2$$

Curve is parabola

Velocity at t_1 is $\tan\theta$

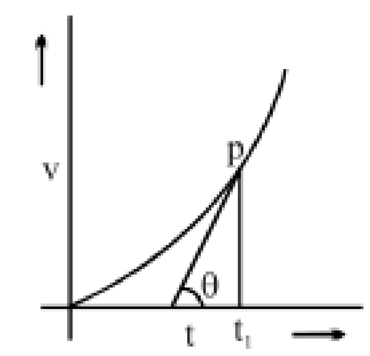


Velocity Vs time curve

By using dependence of v on t we can plot a Vs t graph.

Slope of v Vs t curve at any point represents acceleration at that instant.

$\tan\theta$ = acceleration at time t_1



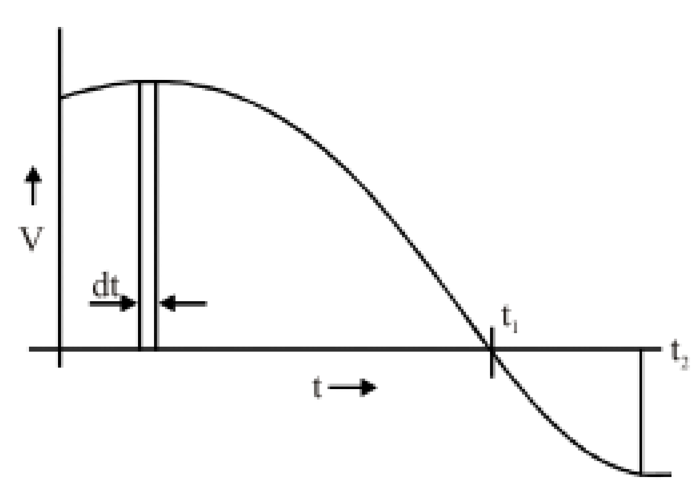
<u>Area under v Vs t graph</u> :

As we know $dx = Vdt$

$$\Rightarrow \int_{x_1}^{x_2} dx = \int_{t_1}^{t_2} vdt$$

$$\Rightarrow \Delta x = \int_{t_1}^{t_2} vdt$$

= Area under v Vs t graph.



Thus area under curve will represent displacement in that time period.

Note : (1) Area above t-axis +ve displacement.

(2) Area below t-axis is –ve displacement.

Thus,

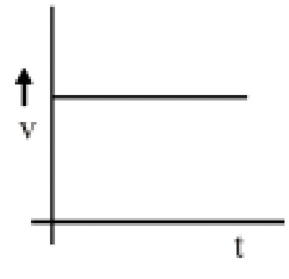
1. Total displacement will be sum of areas with appropriate signs.
2. Total distance travelled will be sum of areas without sign.

Cases :

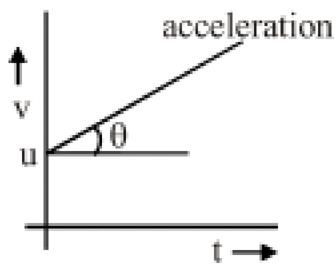
(1) For uniform velocity :

acceleration = 0

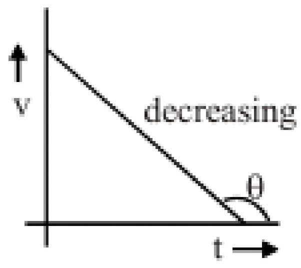
slope = 0



(2) For uniform st. line curve
$\tan\theta$ = acceleration
For increasing velocity



$\tan\theta$ = acceleration
for decreasing velocity
(slope is –ve) i.e. $\theta > 90°$



Note θ is always with + ive x - axis

Table for :

Variation of Displacement (s), velocity (v) and acceleration (a) with respect to time for different type of motion,

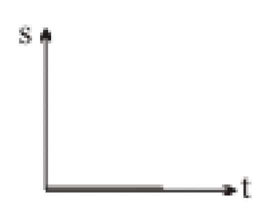
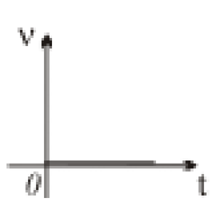
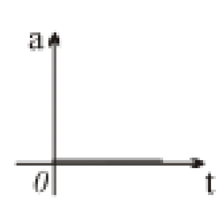
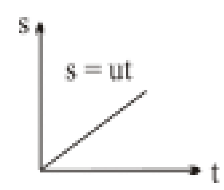
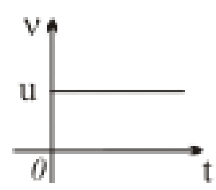
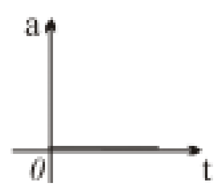
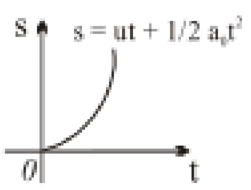
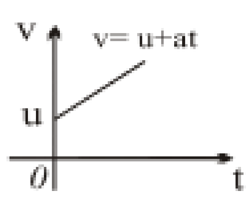
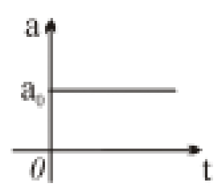
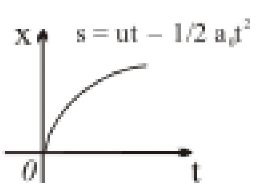
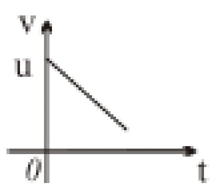
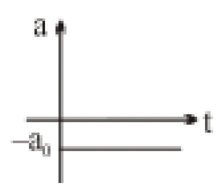
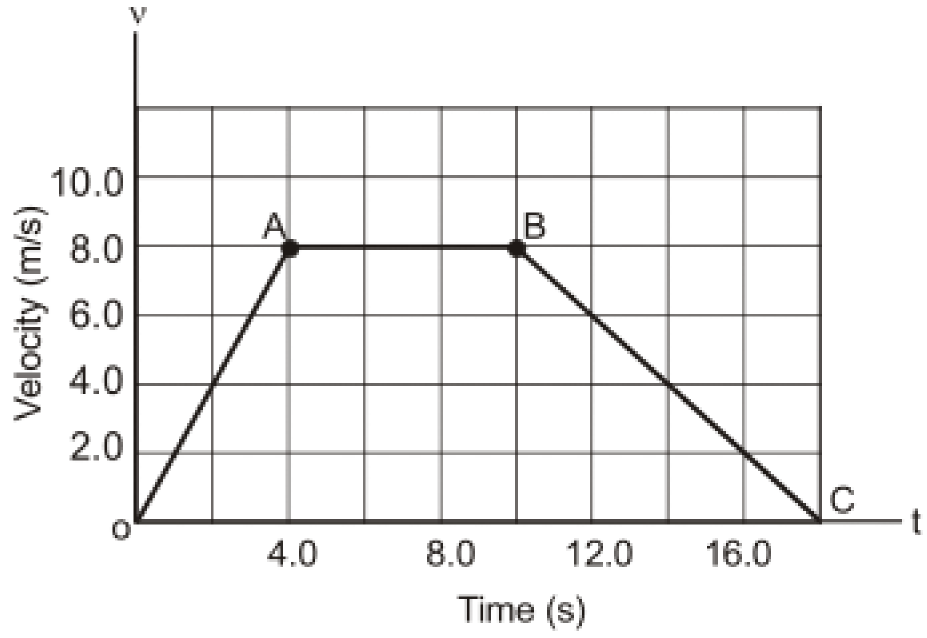
	Displacement	Velocity	Acceleration
1. At rest			
2. Motion with constant velocity			
3. Motion with constant acceleration			
4. Motion with constant deceleration			

Illustration :

What is the average acceleration for each graph segment in figure? Describe the motion of the object over the total time interval. Also calculate displacement.



Sol. Segment OA; $a = \dfrac{8-0}{4-0} = 2\ m/s^2$

Segment AB; graph horizontal i.e., slope zero i.e., $a = 0$

Segment BC; $a = \dfrac{0-8}{18-10} = -1 m/s^2$

The graph is trapezium. Its area between $t = 0$ to $t = 18s$ is displacement

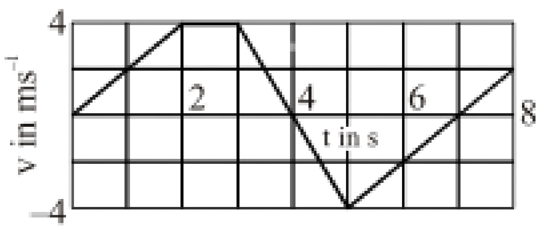
$$\text{Area} = \text{displacement} = \frac{1}{2}(18 + 6) \times 8 = 96m$$

Particle accelerates uniformly for first 4 sec., then moves uniformly for 6 sec. and then retards uniformly to come to rest in next 8 sec.

Illustration :

Figure here gives the velocity time graph for a body. Find the displacement and distance travelled between $t = 0s$ and $t = 7.0$:

Sol Area between $t = 0$ sec. to $t = 4$ sec.



$$= \frac{1}{2} \times (4 + 1) \times 4 = 10\ m$$

Area between $t = 4$ sec. to $t = 7$ sec.

$$= \frac{1}{2} \times 3 \times (-4) = -6$$

Net displacement = total area $= 10 - 6 = 4\ m$

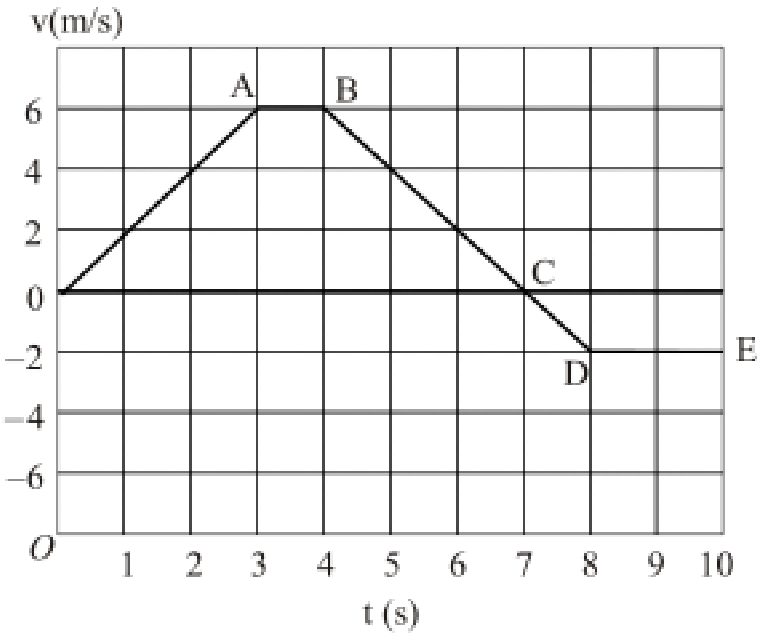
Distance $= |10| + |-6| = 16\ m$

Illustration :

Figure is a graph of V versus t for a particle moving along a straight line. The position of the particle at time $t = 0$ is $x_0 = 0$.

(a) *Find x for various times t and sketch x versus t.*

(b) *Sketch the acceleration a versus t.*



Sol. ***Segment OA***;

Displacement $= x_A - x_0 = x_A - 0 = x_A$

Also, displacement = area between O and A $= \frac{1}{2} \times 6 \times 3 = 9\ m$

$\therefore \qquad X_A = 9\ m$

Again, acceleration = slope of segment OA $= \frac{6}{3} = 2\ m/s^2$

In this segment v and a both are positive, so speed increases.

Segment AB;

Displacement $= x_B - x_A = x_B - 9$

Also, displacement = area between A and B $= 6 \times 1 = 6m$

$\therefore \qquad x_B - 9 = 6 \qquad or, \qquad x_B = 15$

Again, acceleration = slope of segment AB $= 0$

In this segment acceleration is zero, so speed is constant.

Segment BC ;

Displacment $= x_C - x_B = x_C - 15$

Also, displacement = area between B and C $= \frac{1}{2} \times 3 \times 6 = 9m$

$\therefore \qquad x_C - 15 = 9 \qquad or, \qquad x_c = 24\ m$

Again, acceleration = slope of segment BC $= \frac{0-6}{7-4} = -2\ m/s^2$

In this segment velocity is positive but acceleration is negative, so particle decreases its speed.

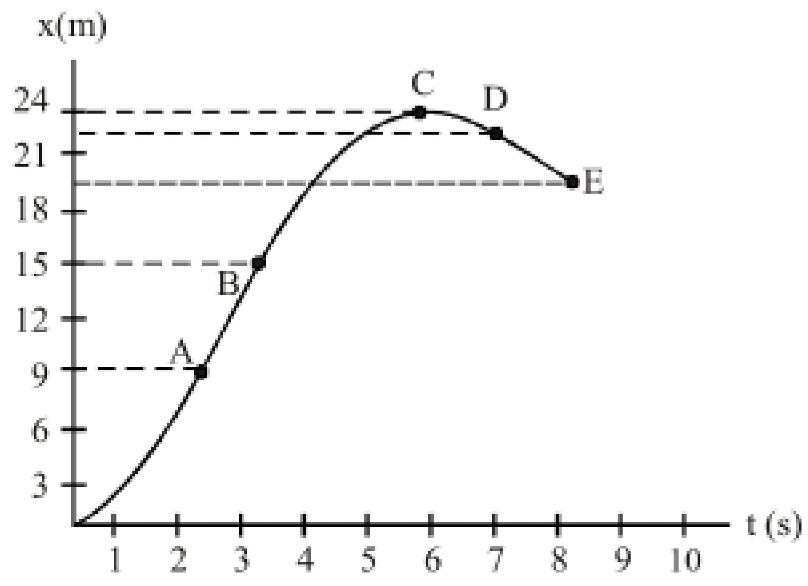
*Similarly, for **segment CD**; we have*

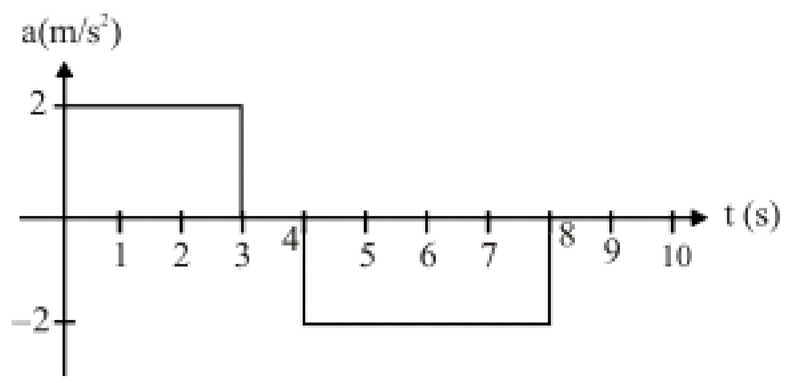
$$x_D = 23\ m \quad \text{and} \quad a = -2\ m/s^2$$

and for segment DE;

$x_E = 19\ m$ *and* $a = 0\ m/s^2$

The graphs are shown below :



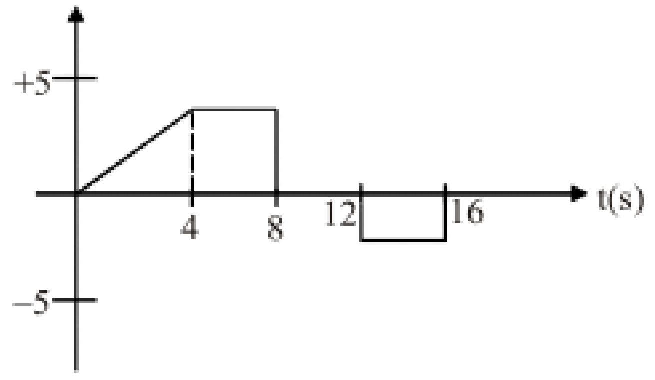


Practice Exercise

Q.1 A particle starts moving with speed 3 m/s and accelerates for 5sec. with acceleration $2m/s^2$. Find the displacement of the particle.

Q.2 A particle has an initial velocity of 9 m/s due east and has a constant acceleration of $2m/s^2$ due west. Find the distance covered by the particle in the 5^{th} second of its motion.

Q.3 The acceleration of a particle traveling along a straight line is shown in the figure. What is the maximum speed of the particle ?



Q.4 A runner is at the position x = 0 m when time t = 0s. One hundred meters away is the finish line. Every ten seconds, this runner runs half the remaining distance to the finish line. During each ten-second segment, the runner has a constant velocity. For the first thirty seconds of the motion, construct

(a) the position-time graph.

(b) the velocity-time graph.

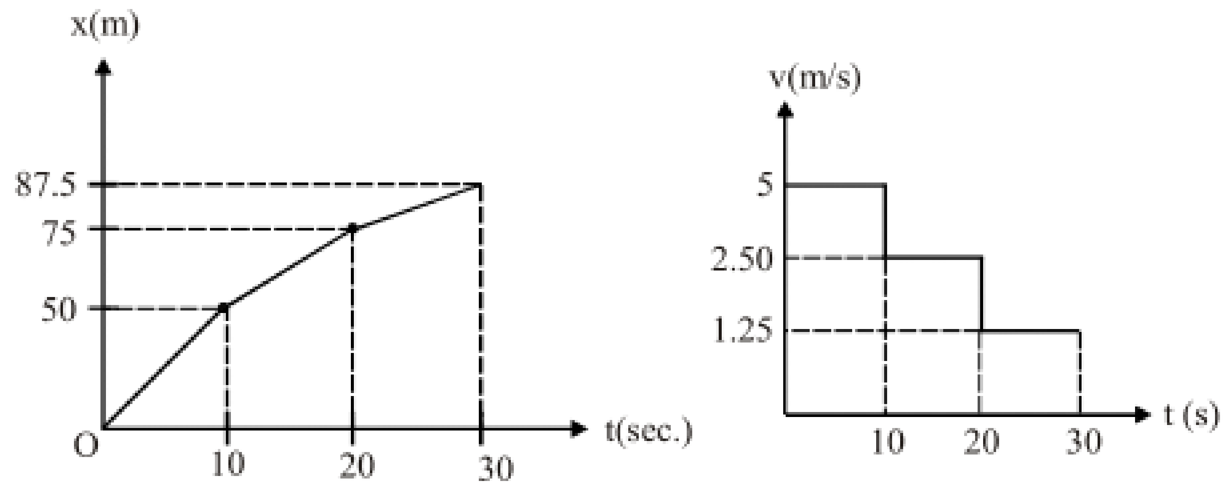
Q.5 Velocity of particle starting from rest varies with position according to equation $v = \sqrt{\alpha x}$. What is distance travelled by particle in t second from start?

Q.6 A body starts from origin and moves along x-axis such that at any instant velocity is $v = 4t^3 - 2t$. Find the acceleration of the particle when it is 2 m from the origin.

Answers

Q.1 2sec   Q.2 0.5m   Q.3 30 m/s

Q.4



Q.5 $\frac{1}{4}\alpha t^2$ Q.6 22 m/s^2

Vertical motion under gravity (Free fall)

Motion that occurs solely under the influence of gravity is called free fall. Thus a body projected upward or downward or released from rest are all under free fall.

In the absence of air resistance all falling bodies have the same acceleration due to gravity, regardless of their sizes or shapes.

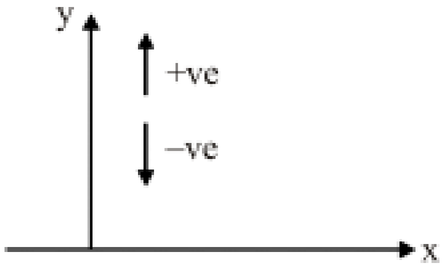
The value of the acceleration due to gravity depends on both latitude and altitude. It is approximately 9.8 m/s^2 near the surface of the earth. For simplicity a value of 10 m/s^2 is used. To do calculations regarding motion under gravity, we follow a proper sign convention. We are taking upward direction as positive and downward as negative, Thus acceleration is taken $a = -g = 10$ m/s^2 no matter whether body is moving upwards or downwards, since **g** always acts downward.

Thus the equation of kinematics may be modified as

$v = u - gt$ .... (i)

$\Delta y = y - y_0 = ut - \frac{1}{2}gt^2$ .... (ii)

$v^2 = u^2 - 2g(y - y_0)$ .... (iii)



These y_0 = position of particle at time $t = 0$

y = position of particle at time t.

u = velocity of particle at time $t = 0$

v = velocity of particle at time t.

Illustration :

A man is standing on the top of a building, throws a ball with speed 5m/s from 30 height above the ground level. How much time it takes to reach the ground.

Sol. $u = 5m/s$

when it reaches the ground, $\Delta y = -30m$

$\therefore$ *from above equation (ii)*

$$-30 = 5t - \frac{1}{2}(10)t^2$$

$\Rightarrow \quad t^2 - t - 6 = 0$

On solving, we get $t = 3$ & -2

Rejecting $t = -2$ *sec, we get* $t = 3$ *sec*

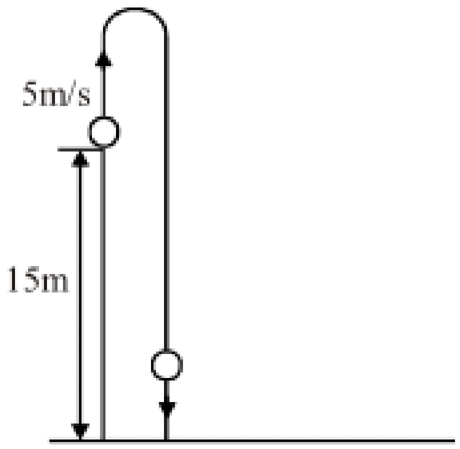
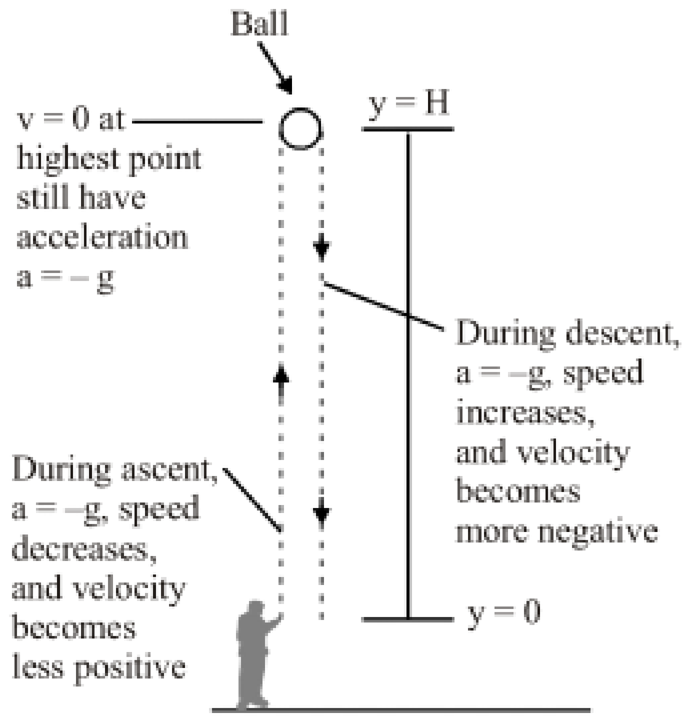


Illustration :

A kid throws a ball up, with some initial speed. Comment on magnitudes and signs of acceleration and velocity of the ball.

Sol.



Here : *(i) During ascent,* $a = -g$*, velocity becomes less positive i.e., speed decreases*

(ii) During descend, $a = -g$*, but now it is in the direction of velocity so it is not retardation. It makes velocity becomes more negative i.e. increases v in negative direction.*

Some results

1. *Maximum Height : -* $\mathbf{H = \frac{u^2}{2g}}$

Derivation : At maximum height $v = 0$

$\therefore$ from equation (iii), $v^2 = u^2 - 2gH = 0 \Rightarrow H = \frac{u^2}{2g}$

2. *Time to reach maximum height : -* $\boldsymbol{t = \frac{u}{g}}$

Derivation : At maximum height $v = o = u - gt$ [equation (i)]

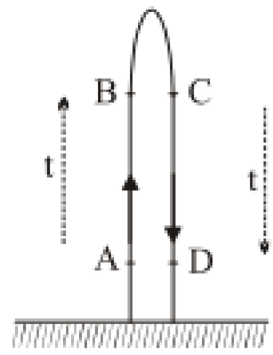
$$\therefore \quad t = \frac{u}{g}$$

3. Total time of flight = time to go up + time to move down (to reach the same horizontal level again)

$$T = 2t$$

$$\mathbf{T = \frac{2u}{g}}$$

4. Time of ascent = Time of descent for motion between two points at same horizontal level for example between A & B and between C & D shown in the figure.



5. If an object is dropped (means initial velocity is zero) from Height h. Its speed on reaching ground is $v = \sqrt{2gh}$ and time taken to reach ground is $t = \sqrt{\frac{2h}{g}}$

Derivation : From equation (iii) $0 - 2g(-h) = v^2$ $\quad [\because \Delta y = -h]$

Also from equation (ii) $\Delta y = -h = 0 - \frac{1}{2}gt^2$

$$\therefore \quad t = \sqrt{\frac{2h}{g}}$$

6. A particle has the same speed at a point on the path. While going vertically up and down.

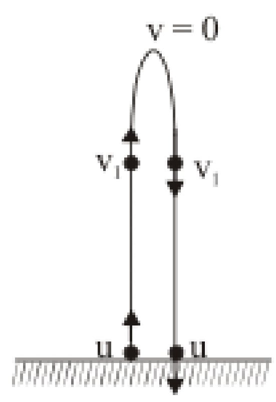


Illustration :

A ball is released from the top of a building. It travels 25 m in last second of its motion before striking the ground. Find height of the building. Take g = 10 m/s^2.

Sol. *Let it takes 't' time to strike the ground.*

$|\Delta y \text{ in } t \text{ sec}| - |\Delta y \text{ in } (t-1) \text{ sec}| = 25$

$$\frac{1}{2}gt^2 - \frac{1}{2}g(t-1)^2 = 25$$

on solving , $t = 3$ *sec*

$\therefore$ *height of the building,* $h = \frac{1}{2}g(3)^2$

$$h = 45 \text{ m}$$

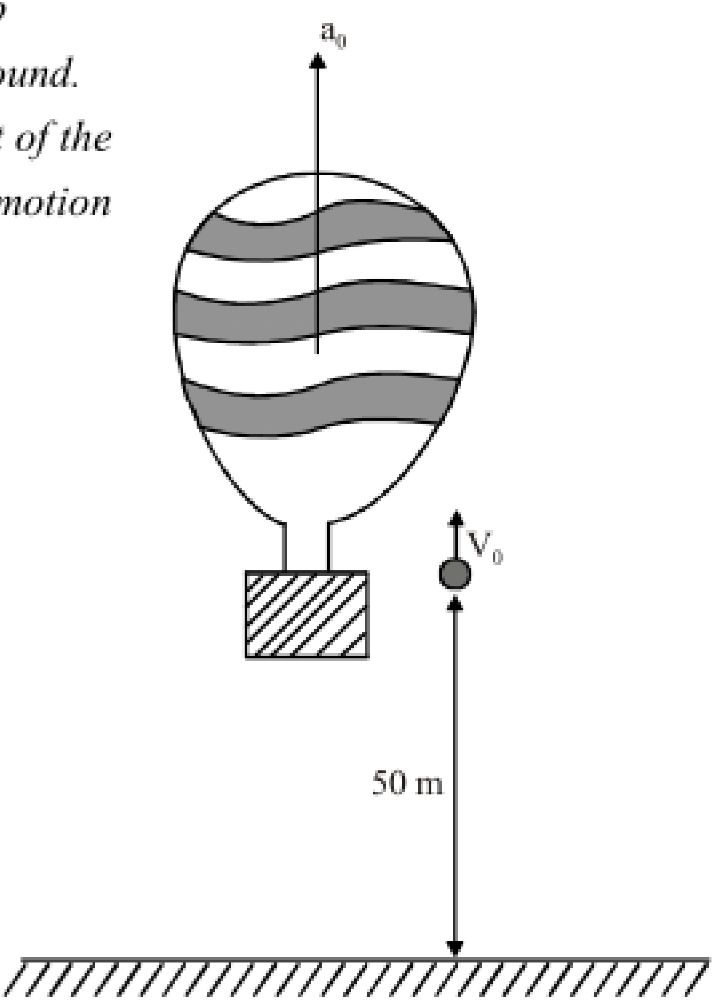
Illustration :

A Baloon is moving up with an acceleration $a_0 = 4$ m/s^2 *starting from rest. A coin is dropped from the baloon 5 sec after the start balloon. Find:*

(a) The initial velocity of the dropped coin.

(b) The height attained by the lift till the time of drop

(c) The time after the drop when the coin reaches ground.



Sol. *Till* $t = 5$*sec, the coin shares the same motion as that of the baloon and for* $t > 5$*sec (after release) the coin has motion under gravity only.*

(a) Velocity of the coin just after it is dropped

V_0 = *velocity of the lift at 5 sec*

$= 0 + a_0(5) = 20$ m/s

(b) The height attained by the lift till 5 sec.

$$h = \frac{1}{2}a_0(5)^2 = 50 \text{ m}$$

(c) Let it takes t_o *time to reach the ground after the drop i.e. for the time* t_0 *its displacement is 50 m in downword direction.*

$$\therefore \quad \Delta y = -50 = 20t_0 - \frac{1}{2}gt_0^2$$

on solving, $t_0 = 5.74$ *sec.*

Motion In Two Dimensions

Whatever we have studied in kinematics of one dimensional motion, we apply the same for motion in two and three dimensional motion, for x, y & Z components separately.

Suppose a particle has position coordinates (x,y) at any instant, then its position vector is

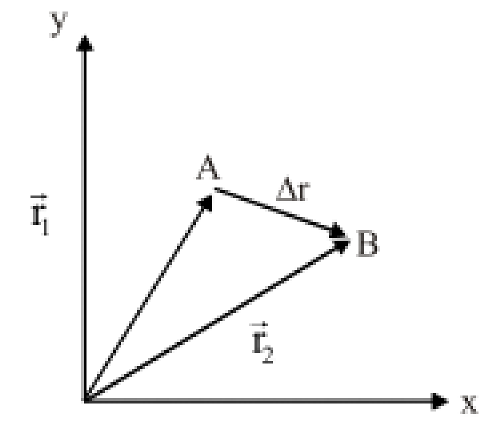
given by, $\vec{r} = x\hat{i} + y\hat{j}$

If particle moves from point A to B, through any path, then its displacement is

$$\Delta\vec{r} = \vec{r}_2 - \vec{r}_1$$

$= \Delta x \hat{i} + \Delta y \hat{j}$

Now at any instant, its velocity is given by



$$\vec{V} = \frac{d\vec{r}}{dt} = \left(\frac{dx}{dt}\right)\hat{i} + \left(\frac{dy}{dt}\right)\hat{j}$$

i.e $\quad V_x = \dfrac{dx}{dt}$ i.e. x - component of velocity.

and $\quad V_y = \dfrac{dy}{dt}$ i.e. y - component of velocity

Similarly $\vec{a} = \dfrac{d\vec{v}}{dt} = a_x \hat{i} + a_y \hat{j}$

Where $a_x = \dfrac{dv_x}{dt}$ & $a_y = \dfrac{dv_y}{dt}$

Projectile Motion

It consists of two independent motions, a horizontal motion at constant velocity and a vertical motion under acceleration due to gravity.

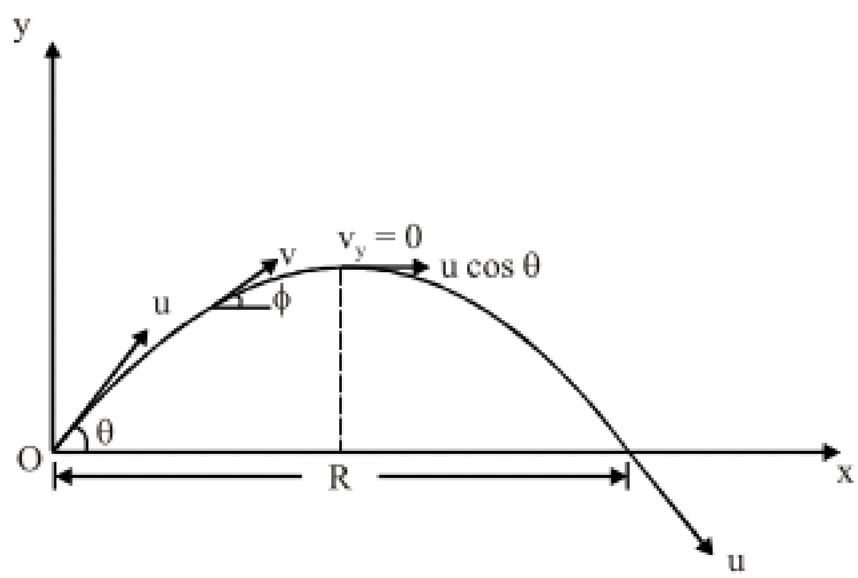
In order to deal with problems in projectile motion, one has to choose a coordinate system. Let's take horizontal as x-axis and vertical upward direction as y-axis, then

$\mathbf{a_x = 0}$ and $\mathbf{a_y = -g}$; since there is only one force "mg" downward (negative air resistance)

Equation along x-axis	**Equations along y-axis**
$v_x = u_x$ (constant)	$v_y = u_y - gt$
$\Delta x = u_x t$	$\Delta y = u_y t - \frac{1}{2} gt^2$
	$v_y^2 = u_y^2 - 2g(\Delta y)$

If an object is dropped from rest or projected up or down, it follows straight line path. If its initial velocity is not along the line of force it follows parabolic path which is proved mathematically in this topic later on.

Projectile Thrown from the Ground Level



A particle is projected from ground level at an angle θ from horizontal with speed u.

$\therefore\ u_x = u\cos\theta$ and $u_y = u\sin\theta$

At any instant, $v_x = u_x = u\cos\theta$ & $v_y = u_y - gt = u\sin\theta - gt$

$$\Delta x = (u\cos\theta)\,t \qquad \& \qquad \Delta y = (u\sin\theta)\,t - \frac{1}{2}gt^2$$

Time of flight (T) : Let it strikes the ground again at time T.

i.e. for $t = T$, $\Delta y = 0$ $u_y = 0 = u_yT - \frac{1}{2}gT^2$

$$\therefore \quad \boxed{T = \frac{2u_y}{g} = \frac{2u\sin\theta}{g}}$$

Horizontal Range (R) It is horizontal displacement till time $t = T$

i.e. $R = u_x T$

$$\boxed{R = \frac{2u_xu_y}{g}}$$

i.e. $$R = \frac{2(u\cos\theta)(u\sin\theta)}{g} = \frac{u^2\sin(2\theta)}{g}$$

Maximum height (H)

$H = \Delta y$ when $v_y = 0$

$\therefore\ v_y^2 = u_y^2 - 2g\,(H) = 0$

$$\therefore \quad \boxed{H = \frac{u_y^2}{2g} = \frac{u^2\sin^2\theta}{2g}}$$

From above formulae, we can observe

(i) $T \propto u_y$ i.e. depends only on vertical component of initial velocity

(ii) $H \propto u_y^2$ i.e. depends only on vertical component of initial velocity

(iii) $R \propto u_xu_y$ i.e. depends both on horizontal and vertical components of initial velocity.

Velocity at any general point

$$v = \sqrt{v_x^2 + v_y^2} = \sqrt{(u\cos\theta)^2 + (u\sin\theta - gt)^2} = \sqrt{u^2 + g^2t^2 - 2(u\sin\theta)gt}$$

If angle which direction of motion makes at an instant is ϕ, then

$$\tan\phi = \frac{v_y}{v_x} = \frac{u\sin\theta - gt}{u\cos\theta}$$

$\tan\phi$ is positive during its upward notion i.e. before reaching highest point and after that $\tan\phi$ is negative.

Illustration :

A particle is projected with 20 m/s at an angle 60° with the horizontal. At what time it is moving at an angle 45° with the horizontal while moving downwards.

Sol. $u_x = 20\cos 60° = 10$ *m/s*

& $u_y = 20\sin 60° = 10\sqrt{3}$ *m/s*

At required instant, $\tan\phi = -1$

i.e. $$\frac{u_y - gt}{u_x} = -1$$

i.e. $\dfrac{10\sqrt{3}-10t}{10} = -1$

on solving, me get $t = \left(\sqrt{3}+1\right)sec$

Illustration :

A particle is projected in the X-Y plane with y-axis along vertical. At 2 sec after projection the velocity of the particle makes an angle 45° with the X - axis and 4 sec after projection, it moves horizontally. Find the velocity of projection.

Sol. At t = 2 sec, $\tan\phi = \dfrac{u_y - 10(2)}{u_x} = 1\ (\because \phi = 45°)$

$\Rightarrow \quad u_y - 20 = u_x \quad$ .... (i)

Also $\dfrac{1}{2}$ (time of flight) = 4 sec

$\Rightarrow \quad \dfrac{1}{2}\left(\dfrac{2u_y}{g}\right) = 4 \quad \Rightarrow \quad u_y = 40\ m/s$

$\therefore$ from equation (i), $u_x = 20\ m/s$

$\therefore \quad u = \sqrt{u_x^2 + u_y^2} = 20\sqrt{5}\ m/s$

Equation of Trajectory :

$$y = u \sin\theta\ t - (1/2)\ gt^2$$

and $\quad x = (u\cos\theta)\ t \quad \Rightarrow \quad t = \dfrac{x}{u\cos\theta}$

From these equations, (eliminating t)

$$\mathbf{y = x \tan\theta} - \frac{g}{2u^2\cos^2\theta}\mathbf{x^2}$$

The above relation between x and y is equation of parabola, which proves that the trajectory i.e. path of projectile is parabolic.

Illustration :

The path followed by a body projected along y axis is given as by $y = \sqrt{3}x - (1/2)x^2$. If $g = 10\ m/s^2$ then the initial velocity of projectile will be- (x and y are in m)

(A) $3\sqrt{10}$ m/s (B) $2\sqrt{10}$ m/s (C) $10\sqrt{3}$ m/s (D) $10\sqrt{2}$ m/s

Sol. Given, that $y = \sqrt{3}\ x - (1/2)\ x^2$ ....(1)

The above equation is similar to equation of trajectory of the projectiles

$y = \tan\theta\ x - 1/2\ \dfrac{g}{u^2\cos^2\theta}\ x^2$ ....(2)

Comparing (1) & (2) we get

$\tan\theta = \sqrt{3}$

$\Rightarrow \quad \theta = 60^o$

and $\quad 1/2 = (1/2)\,\dfrac{g}{u^2\cos^2\theta}$

$\Rightarrow \quad u^2\cos^2\theta = g$

$\Rightarrow \quad u^2\cos^2 60 = 10$

$\Rightarrow \quad u^2\,(1/4) = 10$

$\Rightarrow \quad u = 2\sqrt{10}\ m/s$

Hence correct answer is (B)

Other points of remember

* $$R = \frac{u^2\sin(2\theta)}{g}$$

Range is maximum, when $\theta = 45°$

and $R_{max} = \dfrac{u^2}{g}$

* For two objects projected with **same speed** Range is same for two angles of projection θ & $(90–\theta)$

Proof : Let $R_1 = R_2$ for θ and α

i.e. $\dfrac{u^2\sin(2\theta)}{g} = \dfrac{u^2\sin(2\alpha)}{g}$

i.e. $\sin(2\theta) = \sin(2\alpha)$

i.e. $2\theta = 180° - 2\alpha$

$\Rightarrow \alpha = 90° - \theta$

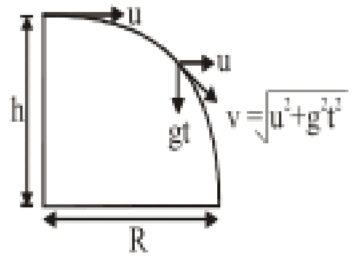
* $H = \dfrac{u^2\sin^2\theta}{g}$ is maximum i.e. $\left(\dfrac{u^2}{2g}\right)$ if projected at $\theta = 90°$

Horizontal Projection :

	In Horizontal Direction		In Vertical Direction
(i)	Initial velocity $u_x = u$	(i)	Initial velocity $u_y = 0$
(ii)	Acceleration $= 0$	(ii)	Acceleration = 'g' downward
(iii)	Horizontal velocity of particle remains same after time t horizontal velocity $= v_x = u$	(iii)	Velocity of particle after time t $v_y = 0 + (-g)t$ $= -gt = gt$ (downward)
(iv)	Range $x = ut$	(iv)	Displacement $y = (1/2)\,gt^2$ (downward)

Velocity at a general point P(x, y):

$v = \sqrt{v_x^2 + v_y^2} \qquad \tan\phi = \dfrac{v_y}{v_x}$



ϕ is angle made by v with horizontal in clockwise direction

Time of flight:

$$-h = u_y t - (1/2)\ gt^2 = 0 - \frac{1}{2}\ gt^2$$

$$t = \pm\sqrt{\frac{2h}{g}}$$

$$t = +\sqrt{\frac{2h}{g}}$$

(negative time is not possible)

Range:

$$R = u_x\, t = u\sqrt{\frac{2h}{g}}$$

Note :

If a projectile is projected with initial velocity u and another particle is dropped from same height at the same time, both the projectile would strike the ground at the same instant velocity. Both will have same vertical components of velocity but their net velocities would be different.

Illustration :

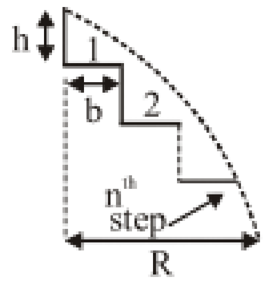
A ball rolls off top of a stair way with a horizontal velocity u m/s. If the steps are h m high and b meters wide, the ball will just hit the edge of n^{th} step if n equals to-

(A) $\dfrac{hu^2}{gb^2}$ (B) $\dfrac{u^2g}{gb^2}$ (C) $\dfrac{2hu^2}{gb^2}$ (D) $\dfrac{2u^2g}{hb^2}$

Sol. *If the ball hits the n^{th} step, the horizontal and vertical distances traversed are nb and nh respectively. Let t be the time taken by the ball for these horizontal and vertical displacement. Then velocity along horizontal direction remains constant = u and initial vertical velocity is zero*

$\therefore$ $nb = ut$ ....(1)

$\&\ nh = 0 + (1/2)\ gt^2$ ....(2)



From (1) & (2) we get

$n\ h = (1/2)\ g\ (nb/u)^2$

$\Rightarrow$ $n = \dfrac{2hu^2}{gb^2}$ *(eliminating t)*

Hence correct answer is (C)

Illustration :

An aeroplane is flying horizontally with a velocity of 720 km/h at an altitude of 490 m. When it is just vertically above the target a bomb is dropped from it. How far horizontally it missed the target? (Take $g = 9.8\ m/s^2$)

(A) 1000 m (B) 2000 m (C) 100 m (D) 200 m

Sol. *Horizontal component of velocity*

$$= 720 \times 5/8 = 200 \text{ m/s}$$

Let t be the time taken for a freely falling body from 490. Then

$$y = (1/2)\, gt^2$$

$$\Rightarrow \quad 490 = (1/2) \times 9.8 \times t^2$$

$$\Rightarrow \quad t = 10 \text{ second}$$

Now horizontal distance

= Velocity × time = 200 × 10 = 2000 m

Hence the bomb missed the target by 2000 m

Hence correct answer is (B)

Projected from some height at some angle

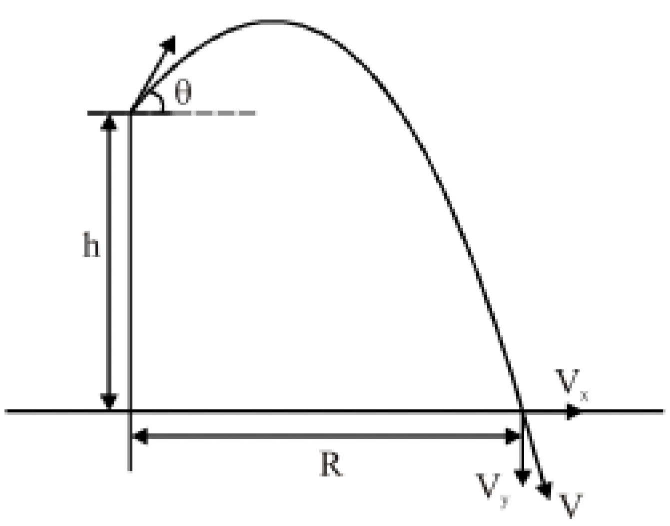
Case I :

When projected at some angle θ with the horizontal towards upward direction.

Let it takes time $\mathbf{t_1}$ (time of flight) to strike the ground

$\Delta y = -h \qquad$ When $t = \mathbf{t_1}$

$$\therefore \quad -h = (u \sin\theta)\, t_1 - \frac{1}{2} g t_1^2$$



$$\Rightarrow \quad t_1^2 - \left(\frac{2u\sin\theta}{g}\right) t_1 - \frac{2h}{g} = 0$$

$$\therefore \quad \mathbf{t_1} = \frac{T + \sqrt{T^2 + 8h/g}}{2}$$

$$\left(\text{where } T = \frac{2u\sin\theta}{g}\right)$$

Also $\quad \mathbf{R} = \Delta x = u_x t_1$ (putting values of t_1 & u_x we can find **R** whenever required)

When it reaches ground $v_x = u\cos\theta$

$$\& \; v_y^2 = u_y^2 - 2g\,(\Delta y) \Rightarrow v_y = \sqrt{(u\sin\theta)^2 - 2g(-h)}$$

$$\therefore \quad v = \sqrt{v_x^2 + v_y^2} = \sqrt{u^2 + 2gh}$$

Case II :

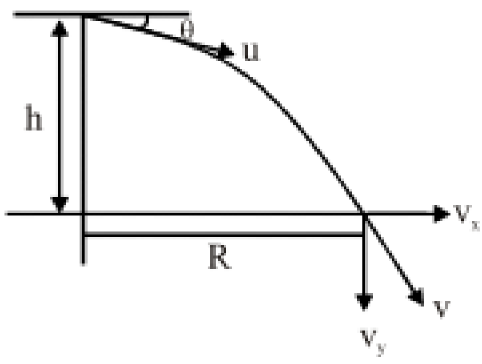
When projected at angle θ with horizontal towards downward direction

Here $\quad u_y = -u\sin\theta$

Thus, $\quad$ if it takes time

$\mathbf{t_2}$ to strike the ground then

$$-h = -(u\sin\theta)\, t_2 - \frac{1}{2} g t_2^2$$



$$\Rightarrow \quad t_2 + \left(\frac{2u\sin\theta}{g}\right) t_2 - h = 0$$

$$\therefore \quad \mathbf{t_2} = \frac{\sqrt{T^2 + 8h/g} - T}{2}$$

Also $\mathbf{R} = u_x t_2$

Here on reaching ground, $v_x = u\cos\theta$

and $v_y = \sqrt{(u\sin\theta)^2 - 2g(-h)}$

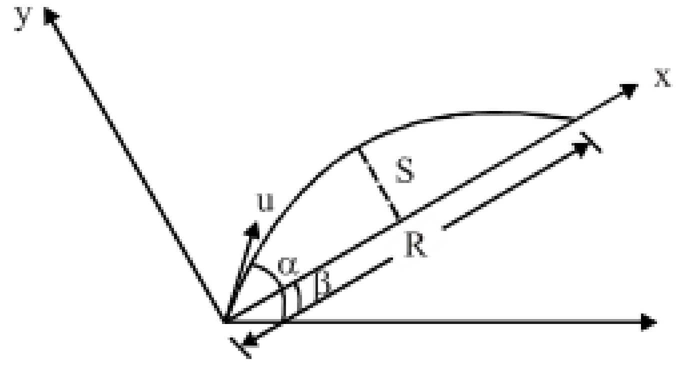
$$\therefore \quad v = \sqrt{v_y^2 + v_x^2} = \sqrt{u^2 + 2gh}$$

Thus we can observe if some particles are projected from same height with same speed, they reach the ground with same speed whatever may be the angles of their projection.

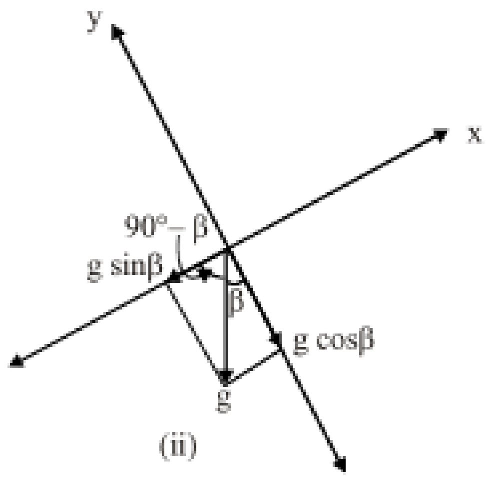
Projectile thrown On An Inclined plane

To deal with problems of projectile thrown along an incline we choose the x-axis along the plane and y-axis perpendicular to the plane.

Let a particle is projected at an angle α with the horizontal on an incline plane which has angle of inclination β with the horizontal



(i)



(ii)

From fig (i) ; $u_x = u\cos(\alpha - \beta)$; $u_y = u\sin(\alpha - \beta)$

[$\therefore$ angle of projection with the incline is $(\alpha - \beta)$]

From fig(ii) ; $a_x = -g\sin\beta$; $a_y = -g\cos\beta$

(a) Time of flight (T) on the incline

At t = T, it strikes the incline

i.e. $\Delta y = 0$

$$u_y T + \frac{1}{2} a_y T^2 = 0$$

$$\Rightarrow \quad u \sin(\alpha - \beta)\, T - \frac{1}{2} g \cos\beta\, T^2 = 0$$

$$\Rightarrow \quad \mathbf{T = \frac{2u\sin(\alpha - \beta)}{g\cos\beta}}$$

(b) Range (R) Along the incline

$R = \Delta x$ till $t = T$

i.e. $R = u_x T + \frac{1}{2} a_x T^2$ (Here $a_x = -g \sin\beta \neq 0$)

Putting values of u_x, a_x & T, we get

$$\mathbf{R = \frac{2u^2 \sin(\alpha - \beta)\cos\alpha}{g\cos^2\beta}}$$

(c) Maximum range

From above formula, $R = \frac{u^2[\sin(2\alpha - \beta) - \sin\beta]}{g\cos^2\beta}$

R is maximum when $\sin(2\alpha - \beta) = 1$

i.e. $2\alpha - \beta = 90°$

$$\Rightarrow \quad \alpha = 45° + \frac{\beta}{2}$$

Also, $R_{max} = \frac{u^2(1 - \sin\beta)}{g\cos^2\beta} = \frac{u^2(1 - \sin\beta)}{g(1 - \sin^2\beta)}$

$$\therefore \quad R_{max} = \frac{u^2}{g(1 + \sin\beta)}$$

(d) Greatest distance from incline

$S = \Delta y$ when $v_y = 0$ i.e. when $u_y + at = 0$

i.e. when $u \sin(\alpha - \beta) - g \cos\beta\, t = 0$

$$\Rightarrow \quad t = \frac{u\sin(\alpha - \beta)}{g\cos\beta}$$

Now, $S = u_y t + \frac{1}{2} a_y t^2$

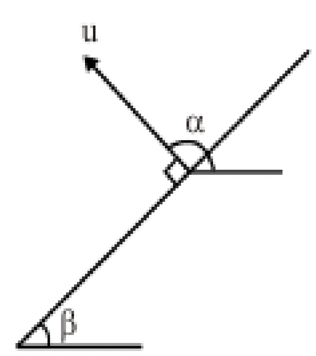
Putting values of u_y, a_y and t, we get

$$S = \frac{u^2 \sin(\alpha - \beta)}{2g\cos\beta}$$

Special cases :

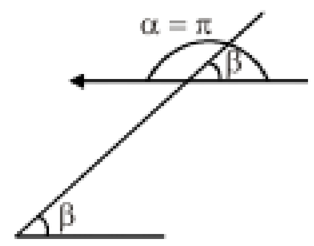
(1) If projected normally (i.e. perpendicular) to the plane, i.e. angle with plane $(\alpha - \beta) = 90°$

i.e. $\alpha = (90° + \beta)$



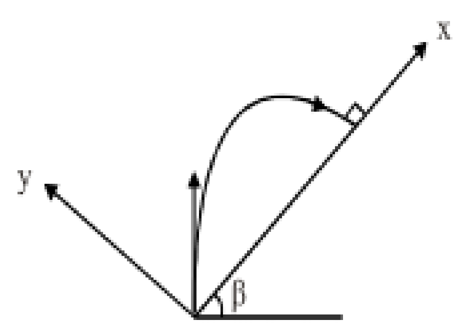
(2) If projected horizontally

i.e. $\alpha = 180°$ and angle with the incline $= \alpha - \beta = 180° - \beta$



(3) If the particle strikes normally to the plane i.e. at the moment of strike, $V_x = 0$

i.e. $u \cos(\alpha - \beta) - g \sin\beta\, t = 0$



Practice Exercise

Q.1 A juggler throws a boll vertically upward and catches it after 6 sec. Determine

(i) the initial velocity of the ball.

(ii) the maximum height attained by the ball.

(iii) the position of ball at t = 2 sec.

(iv) the time at which ball is 20 m below the topmost point

Q.2 A healthy yeoman standing at a distance of 7 m from a 11.8 m high building sees a kid slipping from the top floor. With what uniform speed should he run to catch the kid at his arms height (1.8m) ?

Q.3 A balloon starts rising from the ground with an acceleration of 1.25 m/s^2. After 8s a stone is released from the balloon. Find the time taken by the stone (after its fall) to reach ground.

Q.4 A particle is projected in such a way that its position coordinates vary with time as $y = 8t - 5t^2$ and $x = 6t$ (taking point of projection as origin). What is the range of projectile?

(A) 48 m (B) 4.8 m (C) 9.6 m (D) 24 m

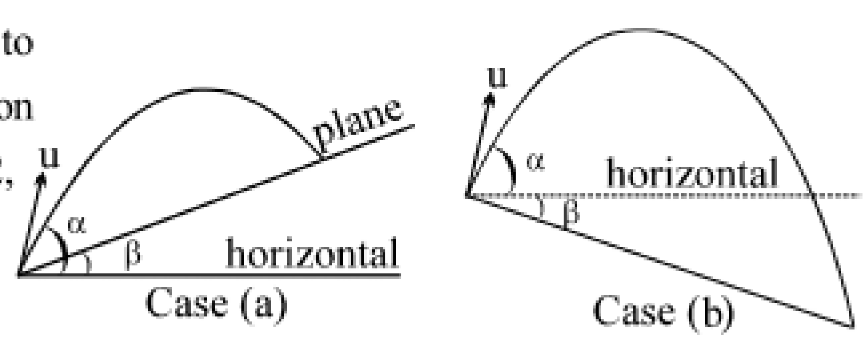
Q.5 A ball is projected upwards from the top of tower with a velocity 50 m/s making an angle 30º with the horizontal. The height of the tower is 70 m. After how many seconds from the instant of throwing will the ball reach the ground ?

(A) 2 s (B) 5 s (C) 7 s (D) 9 s

Q.6 A particle is projected upwards with a velocity of 100 m/s at an angle of 60º, with the vertical. Then time taken by the particle when it will move perpendicular to its initial direction-

(A) 10 sec (B) $\frac{20}{\sqrt{3}}$ sec (C) 5 sec (D) $10\sqrt{3}$ sec

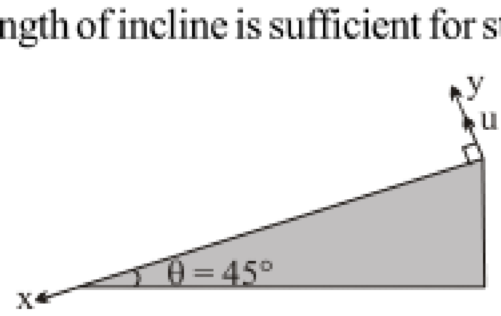
Q.7 A particle is projected at an angle 'α' to the horizontal. Up and down there is a plane in case (a) & case (b), inclined at an angle β to the horizontal. If the ratio of time of flights on these plane in case (a) & case (b) be 1 : 2, then the ratio $\frac{\tan\alpha}{\tan\beta}$ is equal to:



(A) $\frac{2}{1}$ (B) $\frac{3}{1}$ (C) $\frac{4}{1}$ (D) $\frac{5}{3}$

Paragraph for question no. 8 to 10

An inclined plane makes an angle $\theta = 45°$ with horizontal. A stone is projected normally from the inclined plane, with speed u m/s at $t = 0$ x and y axis are drawn from point of projection along and normal to inclined plane as shown. The length of incline is sufficient for stone to land on it and neglect air friction.



Q.8 The instant of time at which velocity of stone is parallel to x-axis

(A) $\frac{2\sqrt{2}\,u}{g}$ (B) $\frac{2u}{g}$ (C) $\frac{\sqrt{2}u}{g}$ (D) $\frac{u}{\sqrt{2}\,g}$

Q.9 The instant of time at which velocity of stone makes an angle $\theta = 45°$ with positive x-axis

(A) $\frac{2\sqrt{2}\,u}{g}$ (B) $\frac{2u}{g}$ (C) $\frac{\sqrt{2}u}{g}$ (D) $\frac{u}{\sqrt{2}\,g}$

Q.10 The instant of time till which (starting from $t = 0$) component of displacement along x-axis is half the range on inclined plane is

(A) $\frac{2\sqrt{2}\,u}{g}$ (B) $\frac{2u}{g}$ (C) $\frac{\sqrt{2}u}{g}$ (D) $\frac{u}{\sqrt{2}\,g}$

Answers

Q.1 (i) 30m/s (ii) 45m (iii) 40 m (iv) 1sec. and 5 sec

Q.2 $\frac{7}{\sqrt{2}}$ m/s Q.3 4sec Q.4 (C) 9.6 m

Q.5 (C) 7 s Q.6 (B) $\frac{20}{\sqrt{3}}$ sec

Q.7 (B) $\frac{3}{1}$ Q.8 (C) $\frac{\sqrt{2}u}{g}$

Q.9 (D) $\frac{u}{\sqrt{2}\,g}$ Q.10 (B) $\frac{2u}{g}$

Q.8 The instant of time at which velocity of stone is parallel to x-axis

(A) $\frac{2\sqrt{2}\,u}{g}$ (B) $\frac{2u}{g}$ (C) $\frac{\sqrt{2}u}{g}$ (D) $\frac{u}{\sqrt{2}\,g}$

Q.9 The instant of time at which velocity of stone makes an angle $\theta = 45°$ with positive x-axis

(A) $\frac{2\sqrt{2}\,u}{g}$ (B) $\frac{2u}{g}$ (C) $\frac{\sqrt{2}u}{g}$ (D) $\frac{u}{\sqrt{2}\,g}$

Q.10 The instant of time till which (starting from $t = 0$) component of displacement along x-axis is half the range on inclined plane is

(A) $\frac{2\sqrt{2}\,u}{g}$ (B) $\frac{2u}{g}$ (C) $\frac{\sqrt{2}u}{g}$ (D) $\frac{u}{\sqrt{2}\,g}$

Answers

Q.1 (i) 30m/s (ii) 45m (iii) 40 m (iv) 1sec. and 5 sec

Q.2 $\frac{7}{\sqrt{2}}$ m/s Q.3 4sec Q.4 (C) 9.6 m

Q.5 (C) 7 s Q.6 (B) $\frac{20}{\sqrt{3}}$ sec

Q.7 (B) $\frac{3}{1}$ Q.8 (C) $\frac{\sqrt{2}u}{g}$

Q.9 (D) $\frac{u}{\sqrt{2}\,g}$ Q.10 (B) $\frac{2u}{g}$

Solved Example

Q.1 The motion of an object falling from rest in a viscous medium can be described by the equation $a = \alpha - \beta v$. Where a and v are the acceleration and velocity of the object and α and β are constants. Find.

(i) the initial acceleration.

(ii) the velocity at which acceleration becomes zero.

(iii) the velocity as a function of time.

Sol. (i) The initial velocity of object $v = 0$

So, initial acceleration $a = \alpha - (\beta \times 0) = \alpha$

(ii) For acceleration to be zero

$$0 = \alpha - \beta v \quad \text{or,} \quad v = \frac{\alpha}{\beta}$$

Note : The velocity at which acceleration reduces to zero is the maximum velocity with which an object will fall in a viscous medium. This velocity is called Terminal velocity.

(iii) $a = \dfrac{dv}{dt}$ or, $\alpha - \beta v = \dfrac{dv}{dt}$ or, $\dfrac{dv}{\alpha - \beta v} = dt$

Integrating the expression with boundary conditions : $t = 0, v = 0$ and $t = t, v = v$

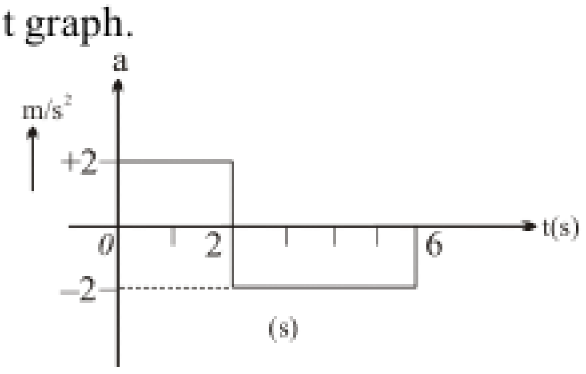
$$\int_0^v \frac{dv}{\alpha - \beta v} = \int_0^t dt \quad \text{or} \quad \left(\frac{-1}{\beta}\right) ln\,(\alpha - \beta v)\big|_0^v = t\,\big|_0^t$$

$$\Rightarrow \quad -\frac{1}{\beta}[\ell n\,(\alpha - \beta v) - \ell n\,(\alpha)] = t - 0 \quad \Rightarrow \ln\left(\frac{\alpha - \beta v}{\alpha}\right) = -\beta t$$

Rearranging the terms, we get

$$v = \frac{\alpha}{\beta}(1 - e^{-\beta t})$$

Q.2 At $t = 0$, a particle is at rest at origin. Its acceleration is $2 m/s^2$ for first 2 sec. and $-2\ m/s^2$ for next 4 sec as shown in a versus t graph.



Plot graphs for

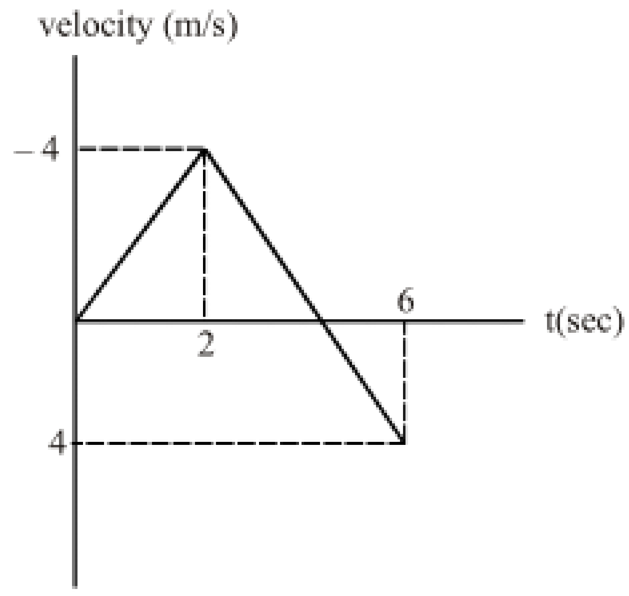
(i) Velocity versus time (ii) speed versus time

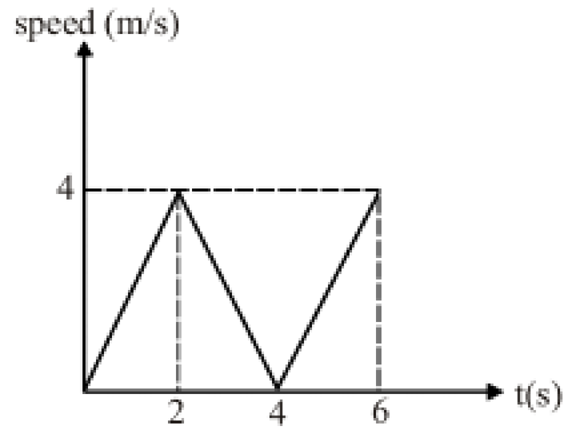
(iii) Displacement versus time (iv) Distance versus time

Sol. (i) $V_2 - V_0$ = Area of a Vs t graph for t = 0 to t= 2 sec

$V_2 - 0 = 2 \times 2 \Rightarrow \quad V_2 = +4\ m/s$

Now $V_6 - V_2 = -2 \times 4 \quad \Rightarrow \quad V_6 = -4$ m/s





(ii) Since slope of a Vs t graph from t = 2 to 6 sec. is constant, we can observe its speed i.e. magnitude of its velocity is zero at = 4 sec. and after that magnitude of velocity increases in negative direction up to 4 m/s at the same rate.

(iii) Displacement (x) Vs t

$x_2 - x_0$ = area of v vst graph for t = 0, t = 2sec

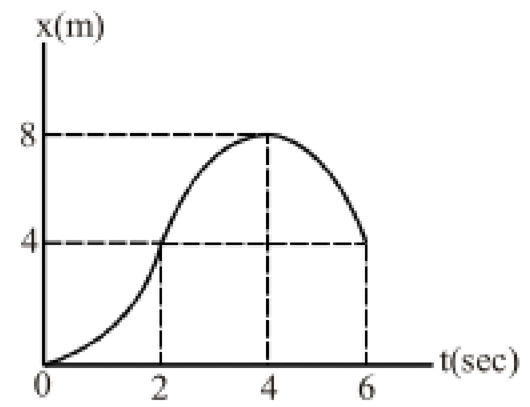
$x_2 - 0 = \frac{1}{2}(2)(4) \Rightarrow x_2 = +4\text{m}$

$x_4 - x_2 = \frac{1}{2}(4)(2)$

$x_4 = 8$ m

also $x_6 - x_4 = \frac{1}{2}(-4)(2) = -4$ m

$\therefore \quad x_6 = +4\text{m}$



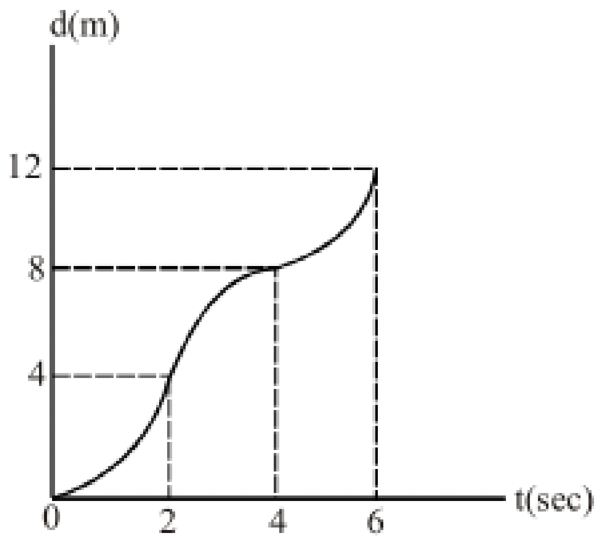
(iv) Distance (d) vs t

$d_2 - d_0 = \frac{1}{2}(2)(4) \quad \Rightarrow \quad d_2 = 4$ m

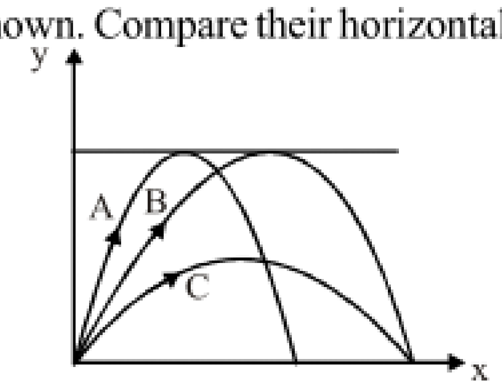
$d_4 - d_2 = \frac{1}{2}(2)(4) \quad \Rightarrow \quad d_4 = 8\text{m}$

Also $d_6 - d_4 = \left|\frac{1}{2}(2)(-4)\right| = 4$

$\Rightarrow d_6 = 12$ m



Q.3 Three particles are projected from same point and their paths are as shown. Compare their horizontal and vertical component of velocities of projection



Sol. $H_A = H_B > H_C \Rightarrow (u_y)_A = (u_y)_B > (u_y)_C$

$\therefore$ $R_B > R_A$

i.e. $(u_x u_y)_B > (u_x u_y)_A$ [$\because$ their u_x are equal]

$\therefore$ $(u_x)_B > (u_x)_A$

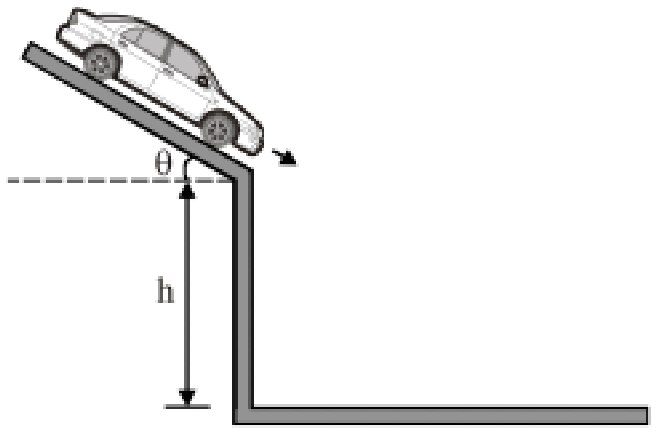
Also $R_B = R_C$

i.e. $(u_x u_y)_B = (u_x u_y)_C$

but $(u_y)_B > (u_y)_C$

$\therefore$ $(u_x)_C > (u_x)_B > (u_x)_C$

Q.4 A car goes out of control and slides off a steep embankment of height h at θ to the horizontal. It lands in a ditch at a distance R from the base. Find the speed at which the car leaves the slope. (Take h = 12.5 m ; R = 10 m ; θ = 45°)



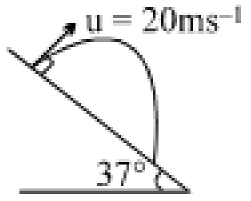
Sol. $\Delta x = 10 = (u \cos 45°)t \quad \Rightarrow t = \dfrac{10\sqrt{2}}{u}$

$$\Delta y = -(u \sin 45°)\, t - \frac{1}{2} gt^2 = -12.5$$

$$\Rightarrow \left(\frac{u}{\sqrt{2}} \times \frac{10\sqrt{2}}{u}\right) + \frac{1}{2}(10)\left(\frac{10\sqrt{2}}{u}\right)^2 = 12.5$$

On solving we get u = 20 m/s

Q.5 Find range of projectile on the inclined plane which is projected perpendicular to the incline plane with velocity 20m/s as shown in figure.



Sol. $\beta = 37°$

$\alpha - \beta = 90°$ & $\alpha = 90° + \beta = 90° + 37°$

$$\therefore \quad \text{Range}, R = \frac{2(20)^2 \sin(90°)\cos(90°+37°)}{10 \times \cos^2(37°)}$$

$$= \frac{2(400)}{10(4/5)^2} \times \left(-\frac{3}{5}\right) \qquad [\because \cos(90° + \theta) = -\sin\theta]$$

$\therefore$ R = – 75 m

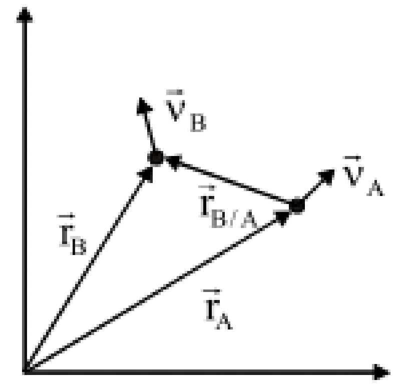
$\therefore$ |R| = 75 m

Here negative sign shown that particle strikes the plane along down the incline

Relative Velocity

It is given by the time rate of change of position of one object w.r.t. another. Relative velocity of a body *B* with respect to some other body A means velocity of B is recorded by an observer sitting on A. Mathematically.

Relative velocity of B w.r.t. A : $\vec{v}_{B/A} = \vec{v}_B - \vec{v}_A$



Proof : $\vec{r}_{B/A} = \vec{r}_B - \vec{r}_A$. Differentiation this equation w.r.t. time, we get

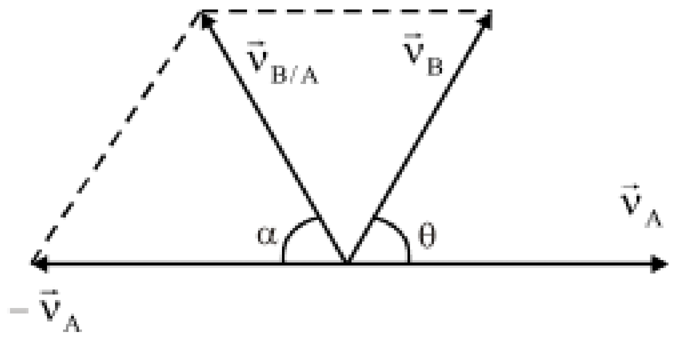
$$\frac{d(\vec{r}_{B/A})}{dt} = \frac{d\vec{r}_B}{dt} - \frac{d\vec{r}_A}{dt} \qquad \text{but} \qquad \frac{d\vec{r}_A}{dt} = \vec{v}_A\ ,\ \frac{d\vec{r}_B}{dt} = \vec{v}_B \text{ and } \frac{d(\vec{r}_{B/A})}{dt} = \vec{v}_{B/A}$$

putting these values we get $\vec{v}_{B/A} = \vec{v}_B - \vec{v}_A$. Hence proved.

Similarly, we can prove that relative velocity of A w.r.t. B :

$\vec{v}_{A/B} = \vec{v}_A - \vec{v}_B$

Graphical Method to find Relative Velocity



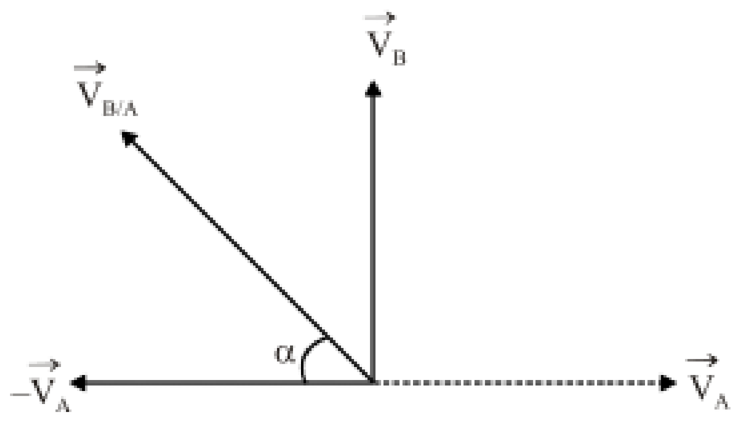
When two bodies move at angle θ with each other then their relative velocity is given by :

Magnitude : $|\vec{v}_{B/A}| = |\vec{v}_B - \vec{v}_A| = \sqrt{v_A^2 + v_B^2 + 2v_A v_B \cos(180-\theta)} = \sqrt{v_A^2 + v_B^2 - 2v_A v_B \cos\theta}$

Direction : $\tan\alpha = \dfrac{v_B \sin(180-\theta)}{v_A + v_B\cos(180-\theta)}$

$\Rightarrow \qquad \tan\alpha = \dfrac{v_B \sin\theta}{v_A - v_B \cos\theta}$

If $\theta = 90°$ then $|\vec{v}_{B/A}| = \sqrt{v_A^2 + v_B^2}$ and $\tan\alpha = \dfrac{v_B}{v_A}$



We can find the velocity of a particle in a frame if we know the particle's velocity in some other frame and the relative velocity of frames w.r.t each other.

In two observers are watching a moving particle P from the origins of reference A and B, while B moves at a constant velocity $\vec{v}_{B/A}$ relative to A.

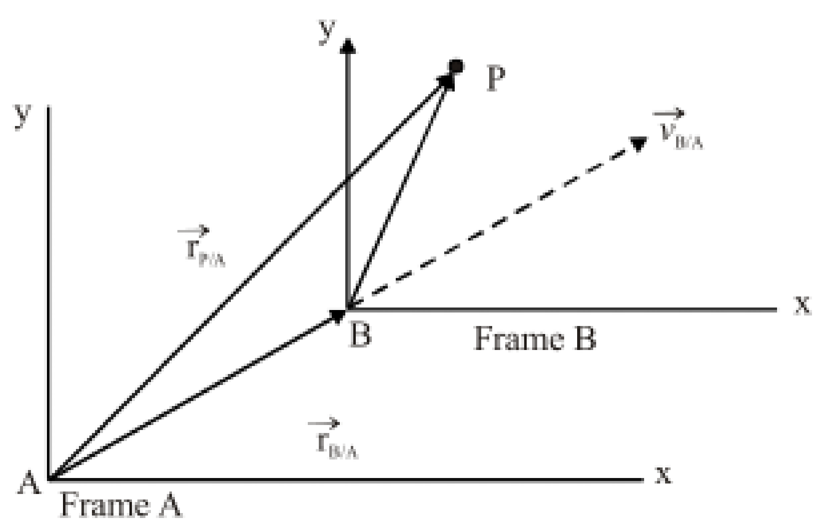


Fig. shows a certain instant during the motion. At this instant, the position vector of B relative to A is $\vec{r}_{BA}$. Also, the position vectors of particle P are $\vec{r}_{P/A}$ relative to A and $\vec{r}_{P/B}$ relative to B. From the arrangement of heads and tails of those three position vectors, we can relate the vectors with $\vec{r}_{P/A} = \vec{r}_{P/B} + \vec{r}_{B/A}$.

By taking the time derivative of this equation, we can relate the velocities $\vec{v}_{P/A} = \vec{v}_{P/B} + \vec{v}_{B/A}$.

We can understand the concept of relative velocity by a simple situation as follows :

Illustration:

Assume two cars A and B each 5 m long. Car A is travelling at 84 km/h and overtakes another car B which travelling at low speed of 12 km/h. Find out the time taken for overtaking.

Sol. *To analyses the motion in case of overtaking we need relative velocity of object which overtakes w.r.t. the other object. Therefore, we need to find relative velocity of car A w.r.t car B which is*

$84 - 12 = 72\ km/h = 20\ ms^{-1}$

Total relative distance covered with this velocity = sum of lengths of car A and car B = 5 + 5 10 m.

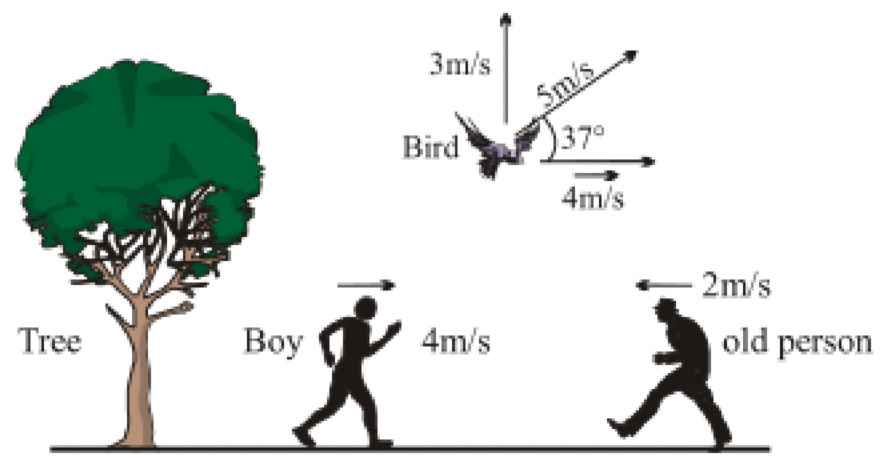
$$\therefore \quad \text{the time taken} = \frac{\text{Distance covered}}{\text{Relative velocity}} = \frac{10}{20} = 0.5\ s$$

Illustration:

(a) *Find velocity of tree, bird and old man as seen by boy.*

(b) *Find velocity of tree, bird, boy as seen by old man*

(c) *Find velocity of tree, boy and old man as seen by bird.*



[Sol. (a) *With respect to boy :*

$V_{tree} = 4\ m/s\ (\leftarrow)$

$V_{bird} = 3\ m/s\ (\uparrow)\ \&\ 0\ m/s\ (\rightarrow)$

$V_{old\ man} = 6\ m/s\ (\leftarrow)$

(b) *With respect to old man :*

$V_{boy} = 6\ m/s\ (\rightarrow)$

$V_{tree} = 2\ m/s\ (\rightarrow)$

$V_{bird} = 6\ m/s\ (\rightarrow)$ *and* $3\ m/s\ (\uparrow)$

(c) *With respect to Bird :*

$V_{tree} = 3\ m/s\ (\downarrow)$ *and* $4\ m/s\ (\leftarrow)$

$V_{old\ man} = 6\ m/s\ (\leftarrow)$ *and* $3\ m/s\ (\downarrow)$

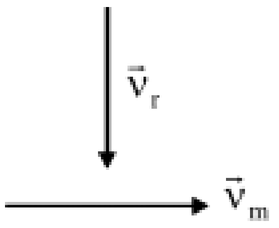
$V_{boy} = 3\ m/s\ (\downarrow)$

Rain - Man Problems

Formula to be applied : $\vec{v}_{r/m} = \vec{v}_r - \vec{v}_m$, where $\vec{v}_{r/m}$ is velocity of rain w.r.t. man, $\vec{v}_r$ is the velocity of rain (w.r.t. ground), and $\vec{v}_m$ is the velocity of man (w.r.t. ground).

If rain is falling vertically downwards with a speed v_r and a man is running horizontally towards east with a speed v_m.

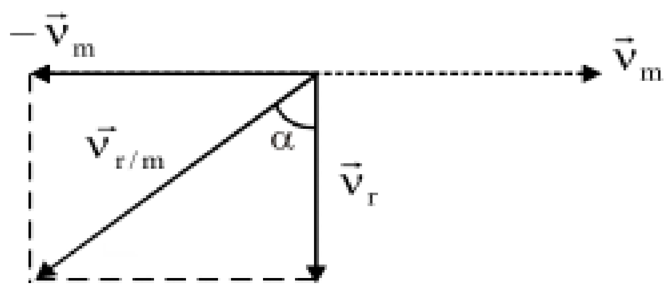
What is the relative velocity of rain w.r.t. man ?



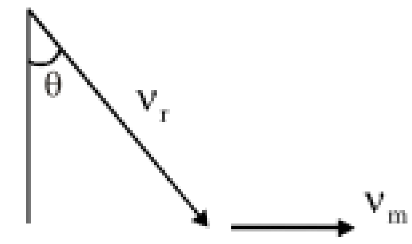
Given : $\vec{v}_r = -v_r\hat{j}$, $\vec{v}_m = v_m\hat{i}$,

Now $\vec{v}_{r/m} = \vec{v}_r - \vec{v}_m = -v_r\hat{j} - v_m\hat{i} \Rightarrow \vec{v}_{r/m} = -v_m\hat{i} - v_r\hat{j}$.

Magnitude : $\sqrt{v_m^2 + v_r^2}$ and direction : $\tan\alpha = \dfrac{v_m}{v_r}$



Example : If rain is already falling at some angle θ with horizontal, then with what velocity the man should travel so that the rain appears vertically downwards to him ?



Here, $\vec{v}_m = v_m\,\hat{i}$, $\vec{v}_r = v_r \sin\theta\,\hat{i} - v_r \cos\theta\,\hat{j}$

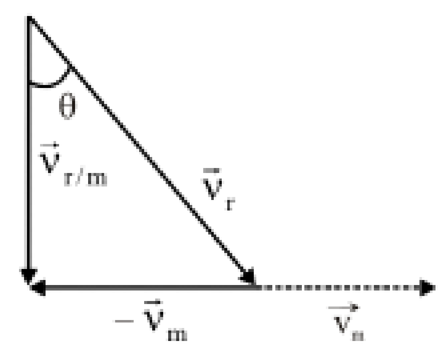
Now, $\vec{v}_{r/m} = \vec{v}_r - \vec{v}_m = (v_r \sin\theta - v_m)\,\hat{i} - v_r \cos\theta\,\hat{j}$

Now for rain to appear falling vertically, the horizontal component of $\vec{v}_{r/m}$ should be zero, i.e.,

$$v_r \sin\theta - v_m = 0 \Rightarrow \sin\theta = \frac{v_m}{v_r} \text{ and } |v_{r/m}| = v_r \cos\theta$$

$$= v_r\sqrt{1-\sin^2\theta} = v_r\sqrt{1-\frac{v_m^2}{v_r^2}}$$

$$\text{or } v_{r/m} = \sqrt{v_r^2 - v_m^2}$$



We can illustrate the whole situation by the diagrams.

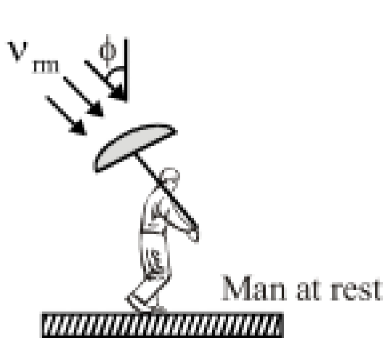
It is quite interesting to notice the steady rainfall sitting in a vehicle such as bus, car, etc. While moving on a straight track the direction of rainfall changes when the vehicle changes its velocity. That means, the velocity of the rain you observe is the velocity of the rain relative to you. Therefore, your observed velocity of rainfall (both magnitude and direction of velocity of rainfall) is the velocity of the rain with respect to the vehicle (you). If you measure the velocity of the rainfall while the vehicle is stationary, that gives actual velocity of rainfall.

Remember following points regarding relative motion :

- If the velocity is mentioned without specifying the frame, assume it is with respect to the ground.
- In many cases, a body travels on water or in air. Depending on the context you will have to figure out whether the velocity is with respect to the water/air or with respect to the ground.

Let us analyse following situation.

The man is stationary and the rain is falling at his back to an angle ϕ with the vertical

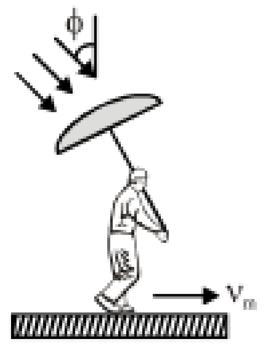


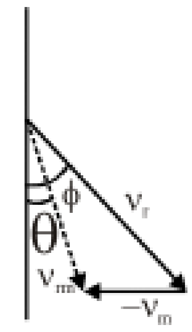
$v_{rm} = v_r \quad v_m = 0$

$\boxed{\theta = \phi}$

here θ = Angle at which rain appears to man

Now man starts moving forward with speed v_m. The relative velocity of rain w.r.t. man shifts towards vertical direction.

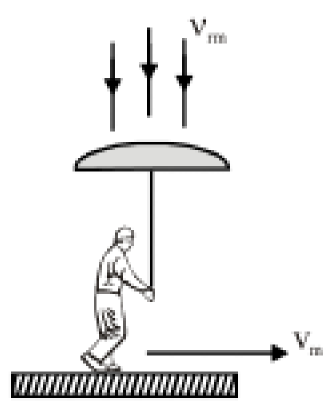


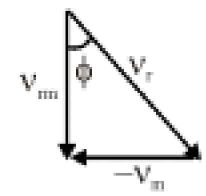


$\boxed{\theta < \phi}$

Velocity vector diagram

As the man further increase his speed, then at a particular value the rain appears to be falling vertically.

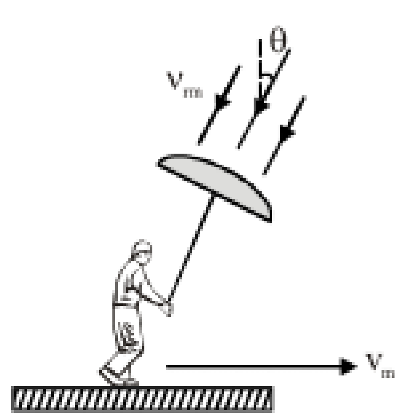


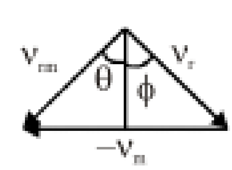


$\boxed{\theta = 0°}$

Velocity vector diagram

If the man increases his speed further more, then rain appear to be falling from the forward direction.





Velocity Vector Diagram

Notice in above figure how man changes orientation of umbrella to prevent himself from rain

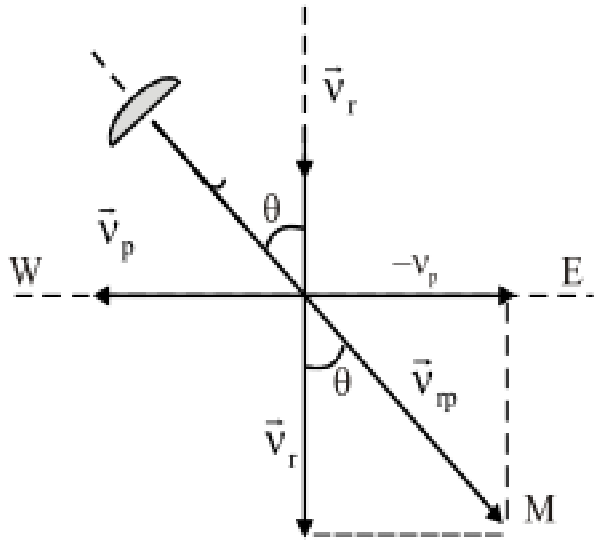
Illustration:

Rain is falling vertically with a speed of 12 ms^{-1}. A cyclist is moving east to west with a speed of $12\sqrt{3}$ ms^{-1}. In order to protect himself from rain at what angle he should hold his umbrella ?

Sol. ***Method 1 :*** *In the case of rain falling vertically with a velocity of tan $\theta = \dfrac{\upsilon_{re}}{\upsilon_{br}}$ and a person (cyclist, bikers, etc) is moving horizontally with a speed $\vec{\upsilon}_m$, the person can protect himself from rain by keeping umbrella in the direction of relative velocity of rain w.r.t. person $\vec{\upsilon}_{rp}$. If θ is the angle that $\vec{\upsilon}_{rp}$ makes with vertical or rain*

$\therefore$ velocity of rain w.r.t. cyclist

$$\vec{\upsilon}_{rp} = \vec{\upsilon}_r - \vec{\upsilon}_p$$



Here, $v_r = 12\ ms^{-1}$ *and* $v_p = 12\sqrt{3}$, $\tan\theta = \left(\dfrac{\upsilon_p}{\upsilon_r}\right)$

and $\theta = \tan^{-1}\left(\sqrt{3}\right) = 60°$

So the cyclist has to hold the umbrella at an angle 60° to the vertical.

Method 2 : $\vec{v}_P = -12\sqrt{3}\ \hat{i}$ *(m/s)*

$\vec{\upsilon}_{rain} = -12\ \hat{j}$ *(m/s);* $\vec{\upsilon}_{rain,person} = \vec{\upsilon}_{rain} - \vec{\upsilon}_{person}$

$= [(-12\ \hat{j})] - (-12\sqrt{3}\,\hat{i})]\ (m/s) = (12\sqrt{3}\,\hat{i} - 12\,\hat{j}\ )\ m/s$

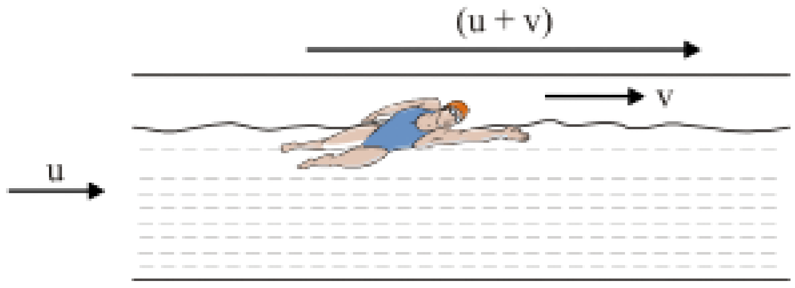
Hence, the direction of orientation of umbrella with vertical is

$$\tan\theta = \frac{12\sqrt{3}}{12} = \sqrt{3} \Rightarrow \qquad \theta = 60°$$

River-Swimmer Problems

When a man or a boat is swimming in water, he generates a velocity relative to water (v m/s) by his own efforts. Actual velocity of man in water will be a resultant of man's effort and the river velocity (u m/s).

Down stream : Man makes efforts in direction of flow, the velocity of man w.r.t. ground is (u + v) m/s as shown below.



Up stream : Man makes efforts opposite to the direction of flow, the velocity of man w.r.t. ground is (v – u) m/s as shown below.

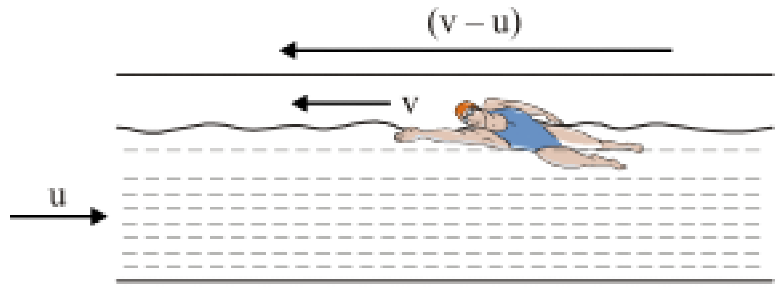


Illustration:

A man whose velocity in still water is 5m/s swims from point A to B (100m downstream of A) and back to A. velocity of river is 3m/s. Find the time taken in going down stream and up stream and the average speed of the man during the motion ?

Sol. *In down stream velocity of man* = $\vec{v}_m = \vec{v}_{m/w} + \vec{v}_w = 3 + 5 = 8\ \text{m/s}$

In down stream time : 100/8 = 12.5 sec

In upstream velocity of man = $\vec{v}_m = \vec{v}_{m/w} + \vec{v}_w$

$= -5 + 3 = -2$ *m/s.*

In up stream time : 100/2 = 50 sec

average speed = 200/62.5 = 3.2 m/s

In a similar manner, when a boat is rowed across a river, the river tries to carry it down stream whereas the boatman makes an effort at an angle to the river bank. The natural consequence is that he reaches somewhere in between. Here also, the velocity of man in still water refers to velocity due to his own efforts. This is fixed in magnitude, but the direction can be changed at will.

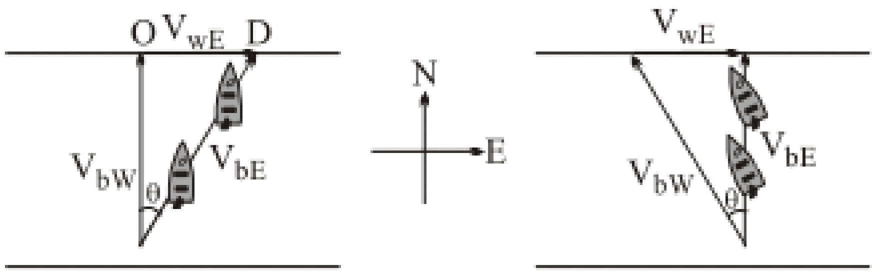


Fig. (a) Fig. (b)

For example, in the figure (a), the boat is rowed directly across in the north direction, but it will reach somewhere in the northeast direction due to the river flow. Similarly in figure (b), the boat is rowed in the north west direction, whereas it will reach in the north direction due to the effect of river flow.

Drift is the distance down stream from the point exactly opposite to the starting point where a person finally reaches. In figure (a) DO = drift. In figure (b) drift = 0

Note following points :

(i) Swimmer keeps himself at an angle of 30° with river flow mean the velocity of swimmer is w.r.t river flow.

(ii) A man swims in water ⇒ velocity of man w.r.t. water.

(iii) A swimmer heads to means (velocity is not w.r.t. ground)

(iv) Person wants to go to destination then direction of velocity is w.r.t. ground.

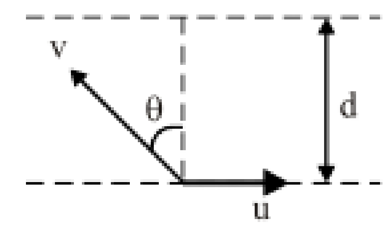
Let discuss a situation when swimmer & river velocity are known

Suppose velocity of river is u and swimmer can swim with a velocity 'v' w.r.t. river flow.

(a) What should be the angle θ with the river flow at which the man should swim so that the time taken to cross the river be minimum ?

Sol. *Let man storts swimming at an angle as shown in figure.*

$$\vec{v}_m = \vec{v}_m + \vec{v}_r$$



$$= (-V \sin\theta\,\hat{i} + v\cos\theta\,\hat{j}) + u\,\hat{i}$$

$$= (u - v\sin\theta)\,\hat{i} + (v\cos\theta)\,\hat{j}$$

= *If width of river is 'd' then time to cross.*

$$t = \frac{d}{v\cos\theta}$$

for t_{min}, $\cos\theta = 1$ *at* $\theta = 0°$

$$t_{min} = \frac{d}{v}$$

So the man should try to swim perpendicular to the river flow to minimize the time in each case.

(b) *What should be the angle* θ *at which the man should swim so that the length of path be minimum ?* *for minimum length of the path, drift x should be minimum.*

Sol. *Drift for given situation = time × {* $\vec{v}_m$ *along the flow}*

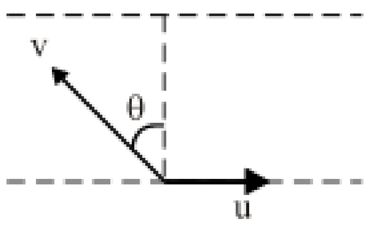
$$x = \frac{d}{v\cos\theta} \times (u - v\sin\theta)$$

$$x = \frac{du}{v}\sec\theta - d\tan\theta \quad ....(A)$$

Case-I

$v > u$ or the river flow is less than the velocity of man's effort.

In such case the minimum possible drift will be zero. So the man should has swim at the angle.



$$x = 0 \Rightarrow u - v\sin\theta = 0$$

$$\sin\theta = \frac{u}{v}$$

Case-II

$v < u$ or the river flow is greater than velocity of man's effort.

In such a case, boat cannot reach the point directly opposite to its starting point. i.e. drift can never be zero.

$\therefore$ for x to be minimum

$$\frac{dx}{d\theta} = 0$$

Differentialing equation (A)

$$\therefore \qquad \frac{dx}{d\theta} = \frac{du}{v}\sec\theta\tan\theta - d\sec^2\theta = 0$$

$$\therefore \qquad \sin\theta = \frac{v}{u} \qquad\qquad \theta = \sin^{-1}\frac{v}{u}$$

Thus, to minimize the drift, boat starts at an angle θ from the river flow.

In this case minimum drift can be calculated by putting value of

$$\theta = \sin^{-1}\left(\frac{v}{u}\right) \text{ in equation (A)}$$

$$x_{min} = \frac{d\sqrt{u^2 - v^2}}{v}$$

Illustration:

A boat heading due north crosses a wide river with a speed of 12 km/h relative to the water. The water in the river has uniform speed of 5 km/h due east relative to the earth.

(a) *Determine the velocity of the boat relative to an observer standing on either bank and the direction of boat.*

(b) *If the boat travels with the speed of 13 kh/h relative to the river and is to travel due north, what should its angle of direction be ?*

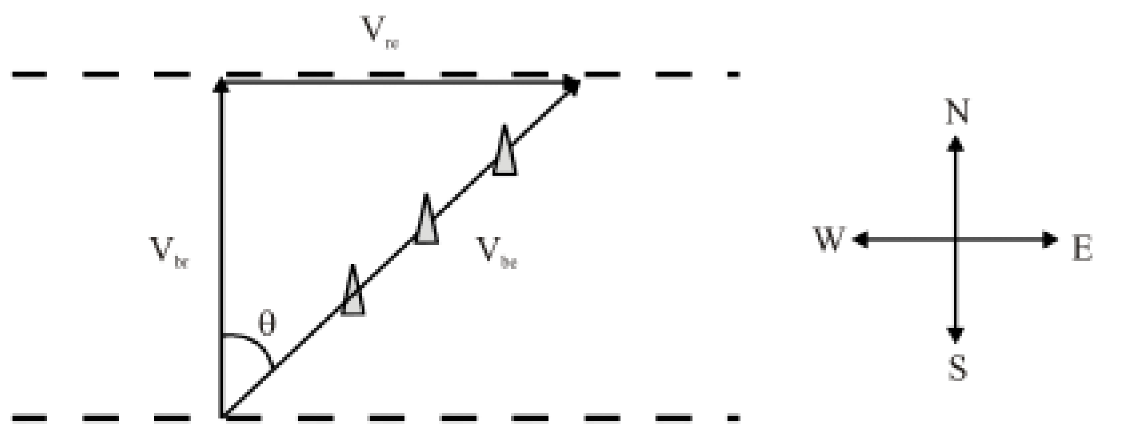
Sol.

(a) *Imagine a situation in your mind of a boat moving across the river. The boat is heading north, which means it wants to go straight, where the current pushes the boat along the direction of current, i.e., east We are given*

Velocity of boat relative to the river $\upsilon_{br} = 10$ *km/h*

Velocity of river = velocity of river relative to earth

$= \upsilon_{re} = 5$ *km/h*



Velocity of the river can be taken as relative to the earth as the velocity measured has only earth as reference. We have to find out the velocity of the boat relative to an observer standing on the bank. Since the observer is stationary with respect to earth, so the velocity of boat relative to observer will be same as the velocity of boat relative to earth.

Let us suppose due to push of current the boat gets drifted by an angle θ form the straight line path.

As seen from velocities in situation from a right angled triangle and we have the values of two sides. Therefore, the third side can be calculated which represents the desired velocity.

From pythagorous theorem, $\upsilon_{be} = \sqrt{\upsilon_{br}^2 + \upsilon_{rE}^2}$

$= \sqrt{(12)^2 + (5)^2} = \sqrt{144+25} = 13$ km/h

To find out the direction, we need to find the angle θ through which boat has deviated.

$$\tan\theta = \frac{\upsilon_{re}}{\upsilon_{br}} \Rightarrow \theta = \tan^{-1}\left(\frac{\upsilon_{re}}{\upsilon_{be}}\right) = \tan^{-1}\left(\frac{5}{12}\right)$$

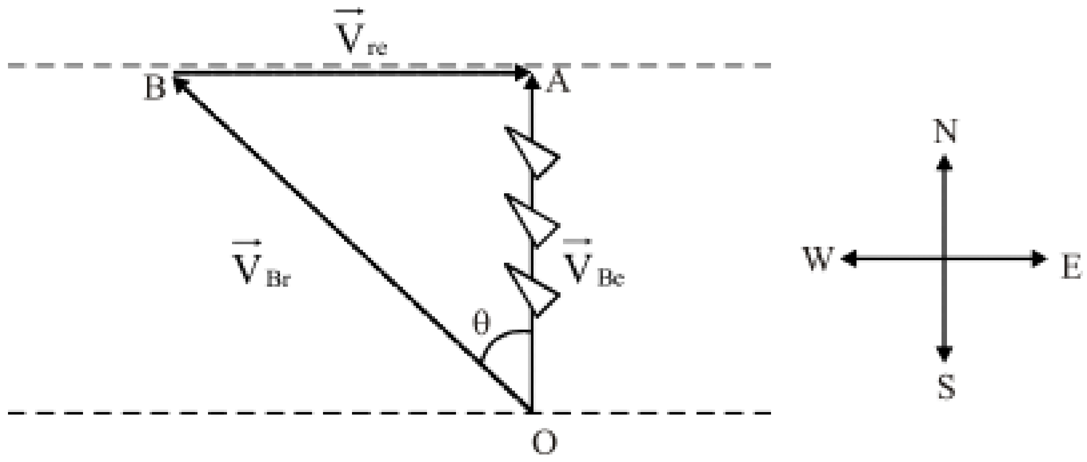
Hence, the boat is moving with a velocity 13 km/h in the direction $\tan^{-1}\left(\frac{5}{12}\right)$ east of north relative to earth.

Sol.

(b) Given $\upsilon_{br} = 10$ km/h

As the boat has to move due north, so it needs to start at an angle θ move upward direction of the river.

This is necessary because the boat during the motion will be drifted downwards due to the push of current.



υ_{be} = velocity of boat w.r.t. earth is along hypotenuse = 13 km/h

υ_{re} = velocity of river w.r.t. earth is along perpendicular 5 km/h

υ_{be} = velocity of boat w.r.t. earth is along base = ?

$$\vec{\upsilon}_{Br} = \vec{\upsilon}_{Be} - \vec{\upsilon}_{re} \Rightarrow \vec{\upsilon}_{Br} = 13 \text{ km/hr}$$

$$\vec{\upsilon}_{Be} = \vec{\upsilon}_{Br} + \vec{\upsilon}_{re} \Rightarrow \vec{\upsilon}_{re} = 5 \text{ km/hr}$$

Using Pythagorous theorem we have, $\upsilon_{br}^2 = \upsilon_{be}^2 + \upsilon_{re}^2$

$$\Rightarrow \upsilon_{be}^2 = \upsilon_{br}^2 + \upsilon_{re}^2 \Rightarrow \upsilon_{be} = \sqrt{\upsilon_{br}^2 - \upsilon_{re}^2}$$

$$\therefore \upsilon_{be} = \sqrt{(13)^2 - (5)^2} = \sqrt{169-25} = \sqrt{144} = 12 \text{ m/s}$$

Now to find the right direction of movement of boat so that it goes straight in north direction, the *angle θ needs to be obtained*

$$\tan\theta = \frac{v_{re}}{v_{be}} \Rightarrow \theta = \tan^{-1}\left(\frac{v_{re}}{v_{be}}\right) = \tan^{-1}\left(\frac{5}{12}\right)$$

Hence the boat has to start at an angle $\tan^{-1}\left(\frac{5}{12}\right)$ *in order to move due north.*

Practice Exercise

Q.1 A man is trying to cross the river 100m wide by a boat. The river is flowing with the velocity 5m/s and the boat's velocity in still water is 3m/s. Find the minimum time in which he can cross the river and the drift in this case?

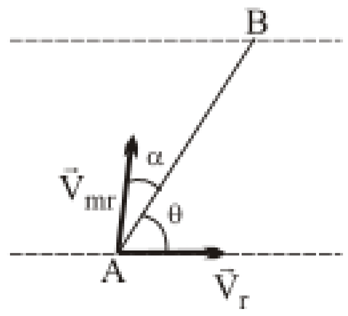
Q.2 Find the direction in which the man (of above illstration) should row so as to have minimum drift. Also find the minimum possible drift and the time taken to cross the river in this case ?

Answers

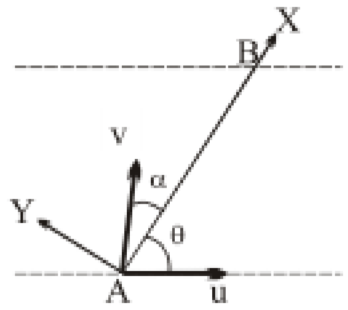
Q.1 $\frac{100}{3}$ sec, $\frac{500}{3}$ m   Q.2 $\sin\theta = \frac{3}{5}$, $x_{min} = \frac{400}{3}$ m, $t = \frac{125}{3}$ sec.

Swimming in a desired direction:

Many times the person is not interested in minimizing the time or drift. But he has to reach a particular place. This is common in the cases of an airplane or motor boat.



The man desires to have this final velocity along AB in other words he has to move from A to B. We wish to find the direction in which he should make an effort so that his actual velocity is along line AB, w.r.t. ground. In this method we assume AB to be the reference line the resultant of v and u is along line AB. Thus the components of v and u in a direction perpendicular to line AB should cancel each other.



So $v \sin\alpha = u \sin\theta$

or $\sin\alpha = \dfrac{u \sin\theta}{v}$

here θ, u & v are given in a problem, so we can calculate α by above relation

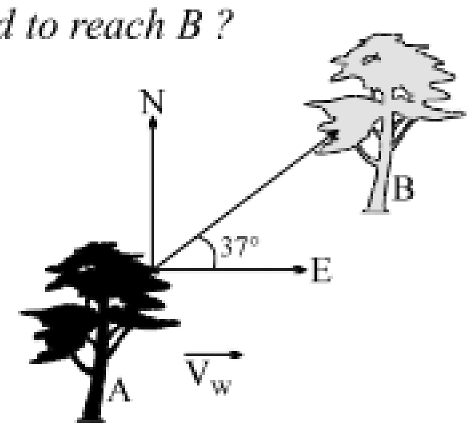
Illustration:

Wind is blowing in the east direction with a speed of 2m/s. A bird wishes to travel from tree A to tree B. Tree B is 100m away from A in a direction 37° north of east the velocity of bird in still air is 4m/s.

(a) Find the direction in which bird should fly so that it can reach from A to B directly.

(b) Find the actual velocity of the bird during the flight ?

(c) Find the time taken by the bird to reach B ?



Sol.

(a) $4 \sin\alpha = 2 \sin 37°. \Rightarrow \quad \alpha = \sin^{-1}\left(\dfrac{3}{10}\right)$

$\Rightarrow 37^o + \sin^{-1}\left(\dfrac{3}{10}\right)$ *with east.*

(b) $\vec{v}_b = \vec{v}_{bw} + \vec{v}_w$

$= v_w \cos 37^o + 4 \cos\alpha.$

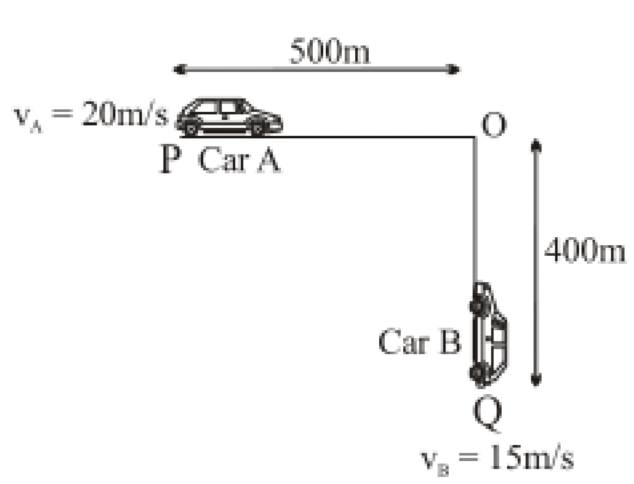
$= 2 \times \dfrac{4}{5} + 4 \times \dfrac{\sqrt{91}}{10} = \dfrac{8+2\sqrt{91}}{5}$

(c) $t = \dfrac{100 \times 5}{8+2\sqrt{91}} = \dfrac{250}{4+\sqrt{91}}$ *sec.*

Closest distance of approach between two bodies

Illustration :

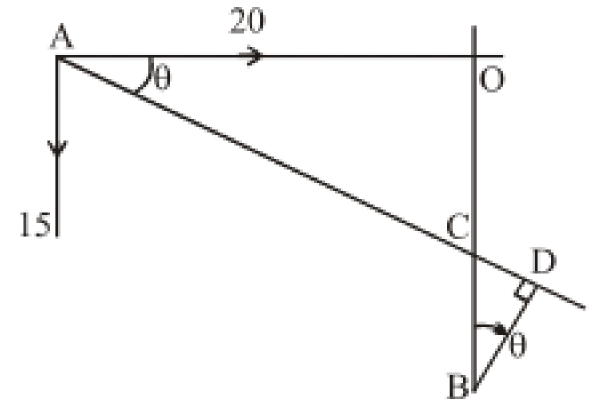
Two roads intersect at right angles. Car A is situated at P which is 500m from the intersection O on one of the roads. Car B is situated at Q which is 400m from the intersection on the other road. They start out at the same time and travel towards the intersection at 20m/s and 15m/s respectively. What is the minimum distance between them ? How long do they take to reach it.



Sol. *First we find out the velocity of car A relative to B*

As can be seen from (fig.), the magnitude of velocity of B with respect to A

$v_A = 20\ m/s,\ \ V_B = 15\ m/s,\ \ OP = 500\ m;\ \ OQ = 400m$

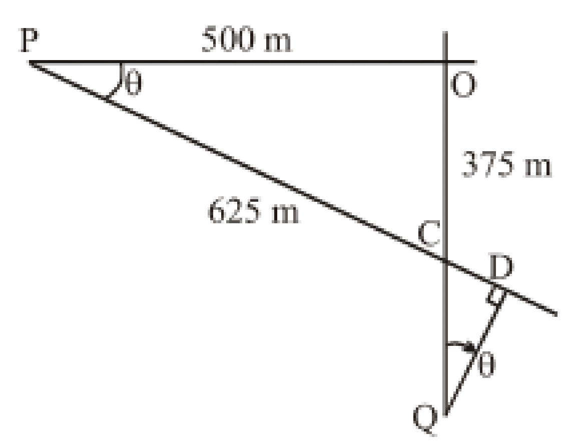


$$\tan\theta = \frac{15}{20} = \frac{3}{4};\qquad \cos\theta = \frac{4}{5};\qquad \sin\theta = \frac{3}{5}$$

$$OC = AO\tan\theta = 500 \times \frac{3}{4} = 375\ m$$

$$BC = OB - OC = 400 - 375 = 25\ m$$

$$BD = BC(\cos\theta) = 25 \times \frac{4}{5} = 20\ m$$



shortest distance = 20 m

$$PD = PC + CD = 625 + 15 = 640$$

$$|\vec{v}_{AB}| = 25\ m/s$$

$$t = \frac{640}{25} = 25.6\ sec.$$

Relative Motion Between two Projectiles

Let us now discuss the relative motion between two projectiles or the path observed by one projectile of the other. Suppose that two particles are projected from the ground with speeds u_1 and u_2 at angles α_1 and α_2 as shown in figure. Acceleration of both the particles is g downwards. So, relative acceleration between them is zero because

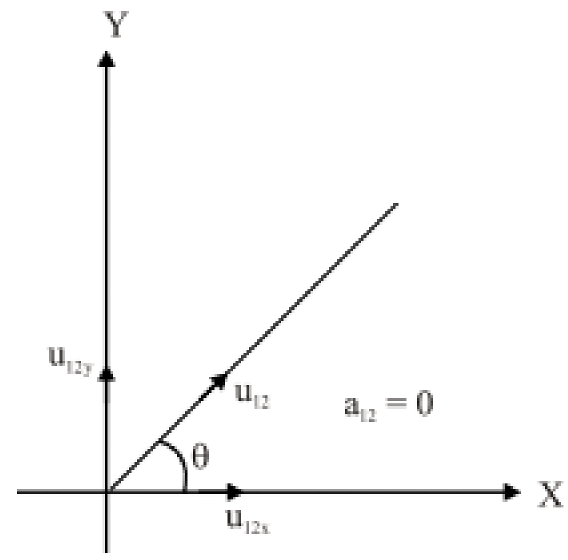
$a_{12} = a_1 - a_2 = g - g = \text{zero}$

i.e., the relative motion between the two particles is uniform. Now

$$u_{1x} = u_1 \cos \alpha_1, \qquad u_{2x} = u_2 \cos \alpha_2$$
$$u_{1y} = u_1 \sin \alpha_1, \text{ and } \quad u_{2y} = u_2 \sin \alpha_2$$

Therefore, $u_{12x} = u_{1x} - u_{2x} = u_1 \cos \alpha_1 - u_2 \cos \alpha_2$

and $u_{12y} = u_{1y} - u_{2y} = u_1 \sin \alpha_1 - u_2 \sin a_2$



u_{12x} and u_{12y} are the x and y components of relative velocity of 1 with respect to 2.

Hence, relative motion of 1 with respect to 2 is a straigh line at an angle $\theta = \tan^{-1}\left(\frac{u_{12y}}{u_{12x}}\right)$ with positive x-axis.

Now, if $u_{12x} = 0$ or $u_1 \cos \alpha_1 = u_2 \cos \alpha_2$, the relative motion is along y-axis or in vertical direction (as $\theta = 90°$). similarly, if $u_{12y} = 0$ or $u_1 \sin \alpha_1 = u_2 \sin \alpha_2$, the relative motion is along x-axis or in horizontal direction (as $\theta = 0°$).

Condition of Collision of two Projectiles

From the above discussion, it is clear that relative motion between two projectiles is uniform and the path of one projectile as observed by the other is a straight line. Now let the particles are projected simultaneously from two different heights h_1 and h_2 with speeds u_1 and u_2 in the directions shown in figure. Then the particles collide in air if relative velocity of 1 with respect to 2 $(\vec{u}_{12})$ is along line AB or the relative velocity of 2 with respect to 1 $(\vec{u}_{12})$ is along the line BA. Thus,

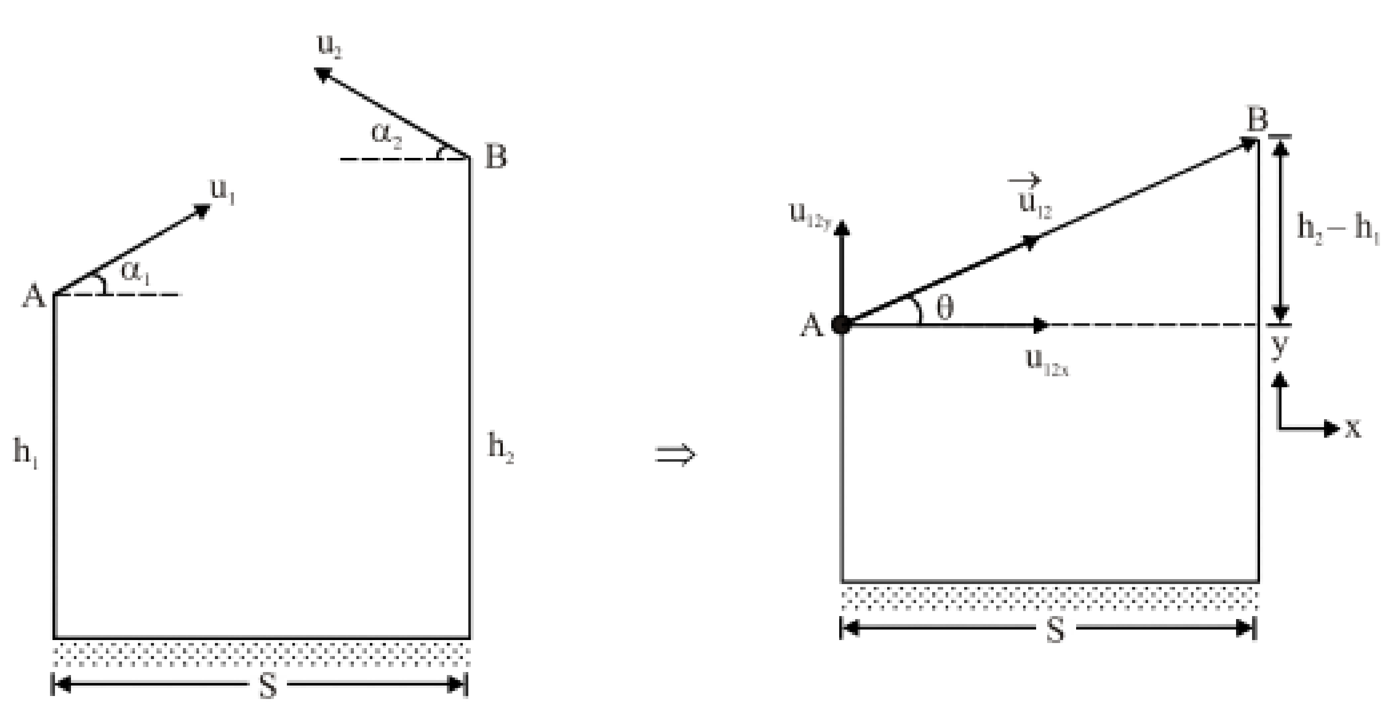
$$\tan \theta = \frac{u_{12y}}{u_{12x}} = \left(\frac{h_2 - h_1}{s}\right)$$

Here $\quad u_{12y} = u_1 \sin \alpha_1 - u_2 \sin a_2$

and $\quad u_{12x} = (u_1 \cos \alpha_1) - (-u_2 \cos \alpha_2) = u_1 \cos \alpha_1 + u_2 \cos \alpha_2$

If both the particles are initially at the same level ($h_1 = h_2$), then for collision

$$u_{12y} = 0 \qquad \text{or} \qquad u_1 \sin \alpha_1 = u_2 \sin \alpha_2$$



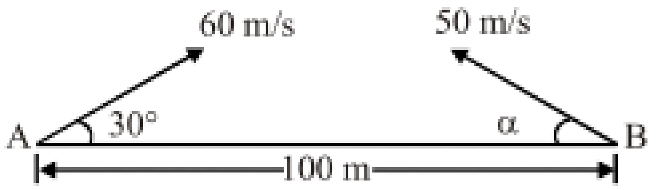
The time of collision of the two particles will be

$$t = \frac{AB}{|\vec{u}_{12}|} = \frac{AB}{\sqrt{(u_{12x})^2 + (u_{12y})^2}}$$

Further, the above conditions are not merely sufficient for collision to takes place. For example, the time of collision discussed above should be less than the time of collision of either of the particles with the ground.

Illustration:

A particle A is projected with an initial velocity of 60 m/s at an angle 30° to the horizontal. At the same time a second particle B is projected in opposite direction with initial speed of 50 m/s from a point at a distance of 100 m from A. If the particles collide in air, find



(a) The angle of projection α of particle B (b) time when the collision takes place and (c) the distance of P from A, where collision occurs. ($g = 10\ m/s^2$)

Sol. *(a) Taking x and y directions as shown in figure.*

Here $$\vec{a}_A = -g\,\hat{j}$$

$$\vec{a}_B = -g\,\hat{j}$$

$$u_{Ax} = 60 \cos 30° = 30\sqrt{3}\ m/s$$

$$u_{Ay} = 60 \sin 30° = 30\ m/s$$

$$u_{Bx} = -50 \cos \alpha$$

and $$u_{By} = 50 \sin \alpha$$

Relative acceleration between the two is zero as $\vec{a}_A = \vec{a}_B$. Hence the relative motion between the two is uniform. Condition of collision is that $\vec{u}_{AB}$ should along AB. This is possible only when

$$u_{Ay} = u_{By}$$

i.e., component of relative velocity along y-axis should be zero.

or $$30 = 50 \sin \alpha$$

$\therefore$ $$\alpha = \sin^{-1}(3/5)$$

(b) Now, $$|\vec{u}_{AB}| = u_{Ax} - u_{Bx}$$

$$= (30\sqrt{3} + 50 \cos \alpha)\ m/s$$

$$= \left(30\sqrt{3} + 50 \times \frac{4}{5}\right) m/s$$

$$= (30\sqrt{3} + 40)\ m/s$$

Solved Example

Q.1 Ram crossing a 2.5m wide conveyor belt moves with a speed of 1.6 m/s. The conveyor belt moves at uniform speed of 1.2 m/s.

(A) If the Ram walks straight across the belt, determine the velocity of the Ram relative to an observer standing on ground.

Sol. If you walk across a conveyor belt while the conveyor belt takes you along the length, you will not be able to move directly across the conveyor belt, but will end up down the length.

Here the velocity of the Ram will be net effect of his own motion and due to motion of conveyor belt

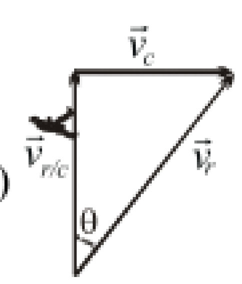
The velocity of the Ram relative to the conveyor belt v_{rc}, is same as velocity of Ram if conveyor belt was still ,

v_c is the velocity of the conveyor belt

we need to find $\mathbf{v}_r$, the velocity of the Ram relative to the Earth.

Writing Equation of net motion $\mathbf{v}_r = \mathbf{v}_{rc} + \mathbf{v}_c$.

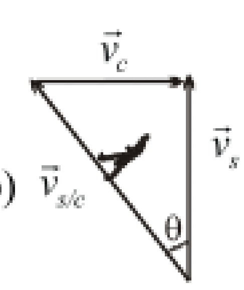
three vectors are shown in Figure (a). The quantity $\mathbf{v}_{rc}$ is due y ; $\mathbf{v}_c$ is due x ; and the vector sum of the two, $\mathbf{v}_r$, is at an angle θ as defined in Figure (a).

(a) 

the speed υ_r of the Ram relative to the Earth is

$$v_r = \sqrt{v_{r/c}^2 + v_c^2}$$

(B) If Shyam has same speed on a still conveyor belt, and is to reach directly across the same moving conveyor belt . At what angle should he walk?

(b) 

Sol. To go straight across the conveyor belt he has to walk at some angle.

Writing Equation of net motion $\mathbf{v}_s = \mathbf{v}_{s/c} + \mathbf{v}_c$.

three vectors are shown in Figure (b)

As in part (b), we know $\mathbf{v}_c$ and the magnitude of the vector $\mathbf{v}_{s/c}$, and we want $\mathbf{v}_c$ to be directed across the conveyor belt.

$$\upsilon_s = \sqrt{v_{s/c}^2 - v_c^2}$$

Note the difference between the triangle in Figure (a) and the one in Figure (b)

Q.2 An aeroplane pilot wishes to fly due west. A wind of 100 km/h is blowing toward the south

(A) What is the speed of the plane with respect to ground ?

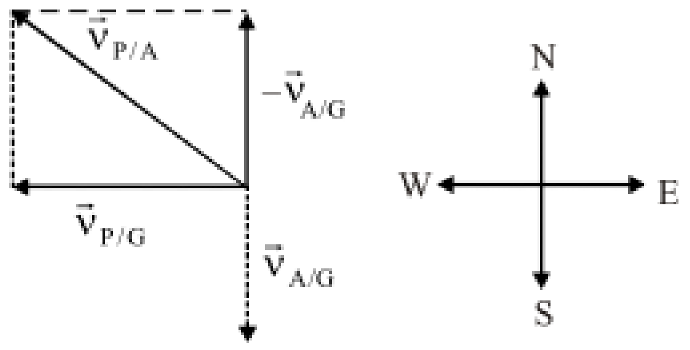
(B) If the airspeed of the plane (its speed in still air) is 300 km/h, in which direction should the pilot head ?

Sol.

(A) Given,

Velocity of air with respect of ground $\vec{v}_{A/G} = 100$ km/hr

Velocity of plane with respect to air $\vec{v}_{P/A} = 300$ km/hr



(B) As the plane is to move towards west, due to air in south direction, air will try drift the plane in south direction., air will try to drift the plane in south direction. Hence, the plane has to make an angle θ towards north-west, south west direction, in order to reach at point on west.

$$\vec{v}_{P/A} = \vec{v}_{P/G} - \vec{v}_{A/G} \text{ and } \mathbf{V}_{P/A} \sin\theta = \mathbf{V}_{AG}$$

Q.3 A river flows due south with a speed of 2.0 m/s. A man steers a motorboat across the diver: his velocity relative to the water is 4 m/s due east. The river is 800 m wide.

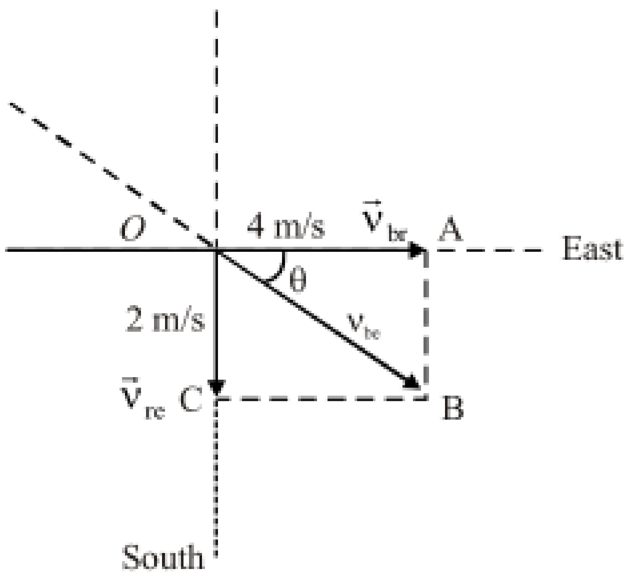
(A) What is his velocity (magnitude direction) relative to the earth ?

(B) How much time is required to cross the river ?

(C) How far south of his starting point will be reach the opposite bank ?

Sol. Velocity of river (i.e., speed of river w.r.t. earth) $\vec{\upsilon}_{re} = 2$ m/s

Width of the river = 800 m



According to the given statement the diagram will be as given

(A) When two vectors are acting at an angle of 90°, their resultant can be obtained by pythagorous theorem,

$$\vec{v}_{be} = \sqrt{\upsilon_{br}^2 + \upsilon_{re}^2} = \sqrt{16+4} = \sqrt{20} = 4.6 \text{ m/s}$$

To find direction , we have

$$\tan\theta = \frac{\upsilon_{re}}{\upsilon_{br}} = \frac{2}{4} = \frac{1}{2} \Rightarrow \qquad \theta = \tan^{-1}\left(\frac{1}{2}\right)$$

(B) Time taken to cross the river $= \dfrac{\text{Displacement of boat w.r.t. river}}{\text{Velocity of boat w.r.t river}}$

$$\Rightarrow \frac{800}{4} = 200 \text{ s}$$

(C) Desired position on other side is A, but due to current of river boat is drifted to position B. To find out this drift we need time taken in all to cross the river (200s) and speed of current (2 ms^{-1})

So the distance AB = Time taken × speed of current = 200 × 2 = 400 m

Hence, the boat is drifted by 400 m away from position A.

Q.4 A person walks up a stationary escalator in t_1 second. If he remains stationary on the escalator, then it can take him up in t_2 second. If the length of the escalator is L, then

(A) Determine the speed of man with respect to the escalator.

(B) Determine the speed of the escalator.

(C) How much time would it take him to walk up the moving escalator ?

Sol.

(A) As the escalator is stationary, so the distance covered in t_1 second is L which is the length of the escalator.

Speed of the man w.r.t. the escalator $\upsilon_{me} = \dfrac{L}{t_1}$

(B) When the man is stationary, by taking man as reference point the distance covered by the escalator is L in time t_2.

Speed of escalator $\upsilon_e = \dfrac{L}{t_2}$

(C) Speed of man w.r.t. the ground

$$v_m = v_{me} + v_e$$

$$\Rightarrow \qquad v_m = \frac{L}{t_1} + \frac{L}{t_2} = L\left[\frac{1}{t_2} + \frac{1}{t_2}\right] = L\left[\frac{t_1 + t_2}{t_1 t_2}\right]$$

$$\Rightarrow \qquad L = \upsilon_m\left[\frac{t_1 t_2}{t_1 + t_2}\right]$$

$\left[\dfrac{t_1 t_2}{t_1 + t_2}\right]$ is the time taken by the man to walk up the moving escalator.

Q.5 A person standing on a road has to hold his umbrella at 60° with the vertical to keep the rain away. He throws the umbrella and starts running at 20 ms^{-1}. He finds that rain drops are falling on him vertically. Find the speed of the rain drops with respect to

(A) the road, and

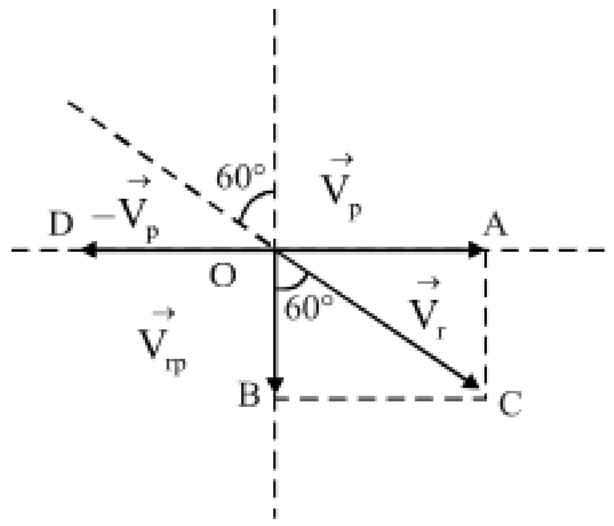
(B) the moving person.

Sol. Given $\theta = 60°$ and velocity person

$\vec{v}_P = \overrightarrow{OA} = 20\ ms^{-1}$.

This velocity is the same as the velocity of person w.r.t ground. First of all let's see how the diagram works out.

$\vec{\upsilon}_{rP} = \overrightarrow{OB}$ = velocity of rain w.r.t. the person.



$\vec{\upsilon}_r = \overrightarrow{OC}$ = velocity of rain w.r.t. earth $\vec{\upsilon}_{rP}$ is along $\overrightarrow{OB}$ as a person has to hold umbrella at an angle with vertical which is the angle between velocity of rain and velocity of rain w.r.t. the person.

Values of $\vec{\upsilon}_r$ and $\vec{\upsilon}_{rP}$ can be obtained by using simple trigonometric relations.

(A) Speed of rain drops w.r.t. earth = $\vec{\upsilon}_r = \overrightarrow{OC}$

From ΔOCM, $\frac{CB}{OC} = \sin 60° \Rightarrow \quad OC = \frac{CB}{\sin 60°}$

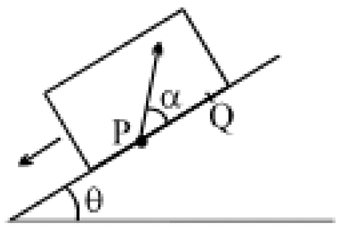
$$= \frac{20}{\sqrt{3}/2} = \frac{40}{\sqrt{3}} = \frac{40\sqrt{3}}{3}\ ms^{-1}$$

(B) Speed of rain w.r.t. the person $\vec{\upsilon}_{rP} = \overrightarrow{OB}$

From ΔOCM, $\frac{OB}{CB} = \cot 60°$

$$\Rightarrow \quad OB = CB \cot 60° = \frac{20}{\sqrt{3}} = \frac{20\sqrt{3}}{3}\ ms^{-1}$$

Q.6 A large heavy box is sliding without friction down a smooth plane of inclination θ. From a point P on the bottom of a box, a particle is projected inside the box. The initial speed of the particle with respect to box is u and the direction of projection makes an angle α with the bottom as shown in figure.



(a) Find the distance along the bottom of the box between the point of projection P and the point Q where the particle lands. (Assume that the particle does not hit any other surface of the box. Neglect air resistance).

(b) If the horizontal displacement of the particle as seen by an observer on the ground is zero, find the speed of the box with respect to the ground at the instant when the particle was projected.

Sol.

(a) u is the relative velocity of the particle with respect ot the box. Resolve u.

u_x is the relative velocity of particle with respect to the box in x - direction.

u_y is the relative velocity with respect to the box in y - direction.

Since, there is no velocity of the box in the y-direction, therefore this is the vertical velocity of the particle with respect to ground also.

Y - direction motion (Taking relative terms w.r.t. box)

$$u_y = +u \sin \alpha$$

$$a_y = -g \cos \theta$$

$$s_y = 0$$

$$s_y = u_y t + \frac{1}{2} a_y t^2 \Rightarrow 0 = (u \sin \alpha)\, t - \frac{1}{2} g \cos \theta \times t^2$$

$$\Rightarrow t = \frac{2u \sin \alpha}{g \cos \theta}$$

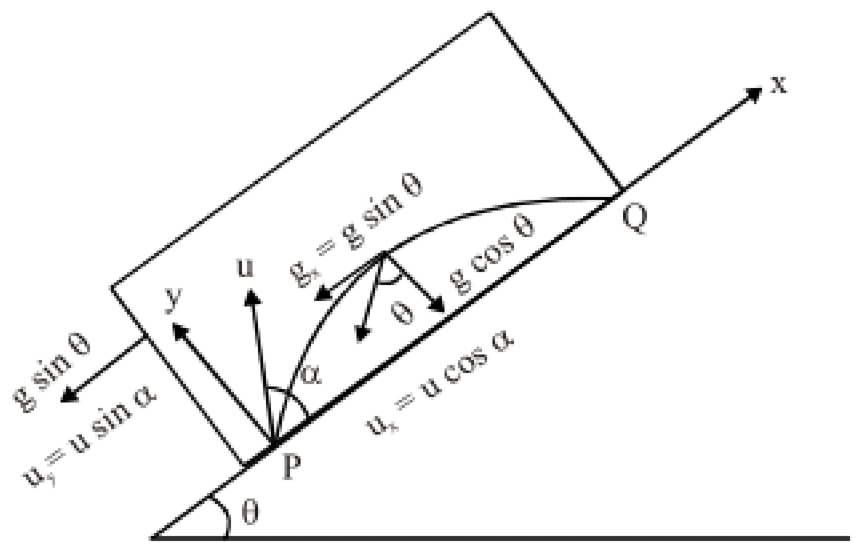
x - direction motion (Taking relative terms w.r.t. box)

$$u_x = +u \cos \alpha \;;\; a_x = 0$$

$$s_x = u_y t + \frac{1}{2} a_y t^2 \Rightarrow s_x = u \cos \alpha \times \frac{2u \sin \alpha}{g \cos \theta} = \frac{u^2 \sin 2\alpha}{g \cos \theta}$$

(b) For the observer (on ground) to see the horizontal displacement to be zero, the distance travelled by box in time $\left(\frac{2u \sin 2\alpha}{g \cos \theta}\right)$ should be equal to the range of the particle w.r.t. box.

Let the speed of the box at the time projection of particle be U. Then for the motion of box with respect to ground.



$$u_x = -U\ ;\ a_x = -g\sin\theta\ ;\ t = \frac{2u\sin\alpha}{g\cos\theta}\ ;\ s_x = \frac{-u^2\sin 2\alpha}{g\cos\theta}$$

$$s_x = u_x t + \frac{1}{2}a_x t^2$$

$$\frac{-u^2\sin 2\alpha}{g\cos\theta} = -U\left(\frac{2u\sin\alpha}{g\cos\theta}\right) - \frac{1}{2}g\sin\theta\left(\frac{2u\sin\alpha}{g\cos\theta}\right)^2$$

on solving we get

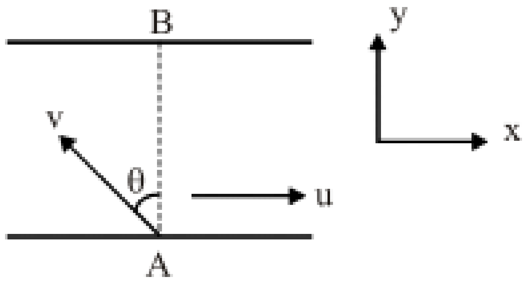
$$U = \frac{u\cos(\alpha+\theta)}{\cos\theta}$$

Q.7 A man wants to cross a river 500 m wide. Rowing speed of the man relative to water is 3 km/hr and river flows at the speed of 2 km/hr. If man's walking speed on the shore is 5 km/hr, then in which direction he should start rowing in order to reach the directly opposite point on the other bank in shortest time.

Sol. Let he sould start at an angle θ with the normal

hence

$$\vec{v}_m = (u - v\sin\theta)\hat{i} + v\cos\hat{j}$$



Here $\vec{v}_m$ = velocity of the man relative to ground.

v = velocity of the man relative to water

u = velocity of water

Hence time taken by the man to cross the river is $t_1 = \frac{0.5}{v\cos\theta}$

∴ Drift of the man along the river is

$$x = (u - v\sin\theta)t_1$$

$$x = (u - v\sin\theta)\frac{0.5}{v\cos\theta}$$

Time taken by the man to cover this distance is

Therefore, time of collision is

$$t = \frac{AB}{|\vec{u}_{AB}|} = \frac{100}{30\sqrt{3}+40}$$

or $$t = 1.09\ s$$

(c) Distance of point P from A where collision takes place is

$$s = \sqrt{(u_{Ax}t)^2 + \left(u_{Ay}t - \frac{1}{2}gt^2\right)^2}$$

$$= \sqrt{(30\sqrt{3}\times 1.09)^2 + \left(30\times 1.09 - \frac{1}{2}\times 10\times 1.09\times 1.09\right)^2}$$

$$s = 62.64\ m$$

$$t_2 = \frac{0.5\left(\frac{u\sec\theta}{v}\tan\theta\right)}{5} = 0.1\left(\frac{u}{v}\sec\theta - \tan\theta\right)$$

Therefore,

total time $T = t_1 + t_2$

$$\Rightarrow \quad T = \frac{0.5}{v}\sec\theta + \frac{0.1u}{v}\sec\theta - 0.1\tan\theta$$

Putting the value of u and v, we get

$$T = \frac{0.5}{3}\sec\theta + \frac{0.1\times 2}{3}\sec\theta - 0.1\tan\theta$$

$$= \frac{0.7}{3}\text{secsec}\,\theta - 0.1\tan\theta$$

$$\Rightarrow \quad \frac{dT}{d\theta} = \frac{0.7}{3}\sec\theta\tan\theta - 0.1\sec^2\theta$$

for T to be minimum

$$\frac{dT}{d\theta} = 0$$

$\Rightarrow \quad \sin\theta = (3/7)$

$\Rightarrow \quad \theta = \sin^{-1}(3/7)$