Lecture Notes in Physics 810

Edward A. Spiegel
*Editor*

# The Theory of Turbulence

## Subrahmanyan Chandrasekhar's 1954 Lectures

Springer

# Lecture Notes in Physics

## The Lecture Notes in Physics

The series Lecture Notes in Physics (LNP), founded in 1969, reports new developments in physics research and teaching – quickly and informally, but with a high quality and the explicit aim to summarize and communicate current knowledge in an accessible way. Books published in this series are conceived as bridging material between advanced graduate textbooks and the forefront of research and to serve three purposes:

- to be a compact and modern up-to-date source of reference on a well-defined topic
- to serve as an accessible introduction to the field to postgraduate students and nonspecialist researchers from related areas
- to be a source of advanced teaching material for specialized seminars, courses and schools

Both monographs and multi-author volumes will be considered for publication. Edited volumes should, however, consist of a very limited number of contributions only. Proceedings will not be considered for LNP.

Volumes published in LNP are disseminated both in print and in electronic formats, the electronic archive being available at springerlink.com. The series content is indexed, abstracted and referenced by many abstracting and information services, bibliographic networks, subscription agencies, library networks, and consortia.

Proposals should be sent to a member of the Editorial Board, or directly to the managing editor at Springer:

Christian Caron
Springer Heidelberg
Physics Editorial Department I
Tiergartenstrasse 17
69121 Heidelberg/Germany
christian.caron@springer.com

For other titles published in this series, go to
www.springer.com/series/5304

Edward A. Spiegel
Editor

# The Theory of Turbulence

## Subrahmanyan Chandrasekhar's 1954 Lectures

Notes prepared by E.A. Spiegel

Springer

*Editor*
Dr. Edward A. Spiegel
Dept. Physics, Pupin Laboratory
Columbia University
West 120th St. 538
10027 New York, NY
USA
eas5@columbia.edu

Spiegel, E.A., *The Theory of Turbulence: Subrahmanyan Chandrasekhar's 1954 Lectures*, Lect. Notes Phys. 810 (Springer, Dordrecht 2011), DOI 10.1007/978-94-007-0117-5

ISSN 0075-8450
e-ISSN 1616-6361
ISBN 978-94-007-0116-8
e-ISBN 978-94-007-0117-5
DOI 10.1007/978-94-007-0117-5
Springer Dordrecht Heidelberg London New York

Library of Congress Control Number: 2010938378



*Cover illustration*: eStudio Calamar S.L.

Printed on acid-free paper

Springer is part of Springer Science+Business Media (www.springer.com)

*DE GUICHE.*

*Car, lorsqu'on les attaque, il arrivent souvent …*
*Qu'un moulinet de leurs grands bras chargés de toiles*
*Vous lance dans la boue !…*

*CYRANO.*

*Ou bien dans les étoiles !*

# Prologue

At a Midwest Neighborhood Astronomy Meeting in the spring of 1953, I gave a fairly naive ten-minute talk about observations of turbulence in a supergiant star. Chandrasekhar was sitting near the front so, of course, I was nervous. After all, I had first heard of fluid dynamical turbulence as an undergraduate just three years before that in a lecture given by Chandrasekhar himself, when he looked about like the photo shown here from the Publications of the Astronomical Society of the Pacific. Hearing that talk was a bit like the butterfly effect that is a popular mantra among students of chaos theory. When I had announced to one of my professors that I wanted to pursue theoretical rather than observational research, he asked what I wanted to work on. I quickly responded with the only explicit theoretical topic whose name I knew: turbulence. And so I had sealed my fate on the basis of a recollection that brought me to my present uncomfortable situation.

Half way into my talk, Chandra, as he shall be known in these notes, blew his nose rather loudly and this caused my knees almost to buckle. But I got through it all and was invited to have lunch with him afterwards. In the course of that meal, Chandra invited me to come and study with him at the Yerkes Observatory in Williams Bay Wisconsin. We compromised on a visit for the following summer. This was in 1954 when I was at the end of my second year as an astronomy graduate student in the University of Michigan and had not had the benefit of any real graduate courses in physics or mathematics. However, I had been studying fluid dynamics on my own and that was perhaps the reason Chandra invited me. He had been at work on fluid dynamics, especially on convective instability, and was homing in on the problem of turbulence.

A few years before that, Bengt Stromgren, the then director of the Yerkes Observatory, had written that convection was the single most important problem in the theory of stellar atmospheres, and stellar atmospheres was then perhaps the most dynamic subtopic on the frontier of astrophysics. On the other hand, the way stellar convection was being treated at the time was such as to make any theoretician with a conscience feel the need to do something to improve the situation. Since convection in stars like the sun is turbulent by the measures that are used to make that judgement, it seemed that the first order of business was to understand turbulence.

Chandra had already begun preparing himself to do this some years earlier. The 1949 Astrophysical Journal contains his “Turbulence: A Physical Theory of Astrophysical Interest,” which was an account of his Russell Lecture, a major annual lecture of the American Astronomical Society. And so, in the summer of 1954, Chandra had set himself the goal of tilting at turbulence. These notes are an account of the series of lectures on the subject he presented at the Observatory that summer.

In order to support myself during my stay at the Observatory, I worked as the research assistant of W.W. Morgan, one of the great observational astronomers of that period. Among other assignments, I was expected to take spectra for Morgan on the 40″ Yerkes refracting telescope. Though I am a klutz and never had any practical skill as an observer, I found this work exhilarating since I was taking spectra of Be stars, hot stars with emission lines, and I found them quite interesting (and still do). Moreover, Morgan liked to teach me things and that was also a great pleasure for me. He was a naturalist and could look at a picture of the astronomical sky and observe things that ordinary mortals did not. He would have me sit down before an astronomical microscope and have me look into it. “What do you see in there?” he would ask. And I would tell him that I saw this or that. To which he would replay, “Don’t you see the XXX?” So I would look till I finally saw the desired object (or thought I did). From such sessions, I came to appreciate an aspect of astronomical discovery that few people are aware of. Great observers really do observe. Like the eye of a good artist who observes things in heir[1] surroundings that many people do not notice, there is also a naturalist’s eye which is a gift that Morgan had and that he used to great effect.

At that time, Yerkes Observatory was a great astronomical center with a staff of renowned astronomers. Heady encounters were a daily occurrence for me from the day I arrived from northern Michigan where I had gone to witness a total solar eclipse. I had presented myself to Barbara Perkins, the main secretary of the observatory, and stood there waiting for her to decide what to do with me when the Director, Bengt Stromgren, came by. He was a most impressive figure who (like Karl Schwarzschild) had written his first paper in his teens. Barbara introduced him to the “newly arrived graduate student” and he greeted me warmly, invited me into his office and asked me to sit by him on a couch with a writing pad between us. He asked me what I had been working on and, after I finished, he asked me to repeat my story so that he could be sure he understood it. I was to encounter such kindness from the leading astronomers of the period but I did not then know that this was normal behavior among them. Still, no one could match Stromgren for gracious behavior, though Paul Ledoux and Albrecht Unsöld were up there.

Somewhat awed, I returned to Barbara Perkins and to the problem of what to do with me when a cheerful Britisher came by and told her to put me in his “room.”[2] So for a summer I became the office mate of Raymond Hide. Though our biological

[1]To avoid such abuses of grammar as substituting ‘their’ for ‘him-or-her’ I simply remove the ‘t’ to create a gender neutral, *singular* possessive pronoun or adjective. Similarly, I use hey and hem for the corresponding nominative and accusative pronouns.

[2]English for the American “office”.

ages were not very different, our academic ages were. Raymond was Chandra's postdoc and I a mere graduate student (in astronomy at that). The arrangement was fine for me since I was poised to learn anything and everything I could and Raymond knew quite a lot even then. The first thing he taught me was drinking. I had left New York City for California at age seventeen where the legal drinking age was eighteen, so I did not have a chance there. In California the drinking age was twenty-one but I left for Michigan just after I turned twenty-one. No one drank at home when I grew up, so I was a novice and was not initiated into that activity until the end of my first day in Raymond's office, when we went for some sherry. That introduction to social drinking was my beginning and I have continued the practice ever since. I am reminded of a remark of W.C. Fields who said "A woman drove me to drink and I never had the courtesy to thank her." Raymond led me to drink and, like Fields, I have been remiss in my obligation to express gratitude until now that his eightieth birthday provides the chance for redress with a paper in his honor.

Another important aspect of my learning experience that summer was that there seemed to be few colleagues then at Yerkes with whom Chandra could discuss the problems he was working on so that he often spent time discussing general scientific matters with me during my stay. This was also part of my extraordinary experience. Of course, the high point of that experience, represented by this little volume, was Chandrasekhar's lecture course on Turbulence.

For me, the first surprise in going to those lectures was the size of the audience. Though I never counted heads, I estimate from the image in my mind's eye that there were more than twenty people in the room and that most of them stayed with the lectures to their conclusion. This was a large audience for mathematical lectures in an astronomy department, but the observatory staff was relatively large and it was, after all, Chandra up in front. I imagine one could go to his lectures, not understand a word and still be enthralled. In his writing, his lecturing, his conversation, and even in his handwriting, Chandra had a pronounced style that was somewhat hypnotic and invited imitation. But he was inimitable, particularly in his mathematical pyrotechnics, so that most of the young people who tried to be like him became mere bad copies. I recognized that danger and (with much embarrassment) declined his invitation to remain at Yerkes as his student and returned to Michigan to fumble along on my own after that summer.

I suppose I should apologize for going on in this way but I wanted to give an idea of the surroundings in which things took place at Yerkes Observatory in that time, as well as my lack of scientific sophistication. That I was a *tabula rasa* is important to understanding the character of these notes.

What was newest to me in Chandra's turbulence lectures was his deft manipulation of mathematical expressions. The mathematical tone I had met in various books on fluid dynamics and other branches of physics had none of his virtuosity, so Chandra's mathematical usage was eye-opening. (I further cured my deficiencies later by taking a one year course in graduate quantum mechanics given by K.M. Case, who was said to be the first person to break the four-minute mile—at the blackboard! His was an entirely different style . . . but I digress.)

After each lecture by Chandra, as I transcribed my notes into a bound notebook (of a kind popular among physics graduate students in Ann Arbor), I filled in details of the derivations that Chandra had not included; they would have been obvious to anyone with a greater knowledge of mathematical physics than mine. This feature will no doubt make these notes seem very elementary to many people. On the other hand, I tried to use Chandra's language as much as I was able to preserve the spirit of his presentation. I even included a few of his asides.

The notes as presented here are as I had set them down in 1954 except that I have proofread them for the first time to remove any obvious typos. Naturally, this far along, I see things that I myself would rather do differently but that would not be in the spirit of this venture. Nor have I added references or included an introductory history of the subject as the referee (Uriel Frisch) proposed. The only attributions in the notes are those made by Chandra who gave very few references. I have added a brief epilogue that does have a reference or two. Though I have been at work on a longer concluding section for some while, Joe Keller convinced me not to delay the publication of these notes till I had completed that epiproject and, in my experience, his advice should always be taken. Moreover, 2010 is the centennial of Chandra's birth, so there was no other reasonable choice.

A further point needs to be explained. S.K. Trehan once took the notes from a course given by Chandra at the University of Chicago and published them with the University of Chicago Press with the title "Plasma Physics" and with the author given as S. Chandrasekhar. Chandra once told me that he was unhappy about this as he had not been consulted about the contents of the book. To avoid having a sense of responsibility for those contents he had never looked into the book. This feeling of Chandra posed a dilemma for me that I have solved in my own way. (Though I am named as editor herein, that is not the "way" I had in mind.) Still, I believe that anyone who reads these notes will sense Chandra's presence behind the naivety of the presentation. I have long agonized over whether to allow myself to make slight changes in the presentation given here. But once that starts it does not end till the whole thing is rewritten and a different book emerges. So despite the deficiencies of my interpolated intermediate steps of 1954, I have left things as they were except for a few slight changes of word order for clarity.

Thus, I reluctantly send off these notes. Though there is much more I should do with them, the time has come and I have only to thank the editorial staff at Springer for their gentle prodding and Steve Lyle for his fine LaTeXing. And yet there is another issue on my mind and that concerns the inclusion of the anecdotes or side remarks with which Chandra would enliven his lectures. Those are not in the notes but I shall mention two right here.

After Chandra produced his deft solution of Heisenberg's equation for the turbulent energy described in the notes, he annunciated this 'theorem': *Given any great man, he has a weakness through which you can associate yourself to him.* Then he smiled and added "Heisenberg can't solve differential equations."

The other story, one that gave him much pleasure to relate, concerned a visit to Fermi's office on the Chicago University campus to which Chandra went once a week to give his courses and to edit the Astrophysical Journal. He had gotten interested in turbulence and wondered what Fermi might know of the subject. On being asked that question, Fermi thought a moment and said he did not know the subject but he thought it should go something like this. Whereupon he went to the blackboard and off the cuff gave a derivation of Kolmogorov's law that Chandra reproduced in the lectures. I think that Chandra felt that Fermi had had no previous knowledge of this law and, magician that he was, pulled Kolmogorov's spectrum right out of a nonexistent hat.

S. Chandrasekhar (1910–1995), originally published in 1952 in the publications of the Astronomical Society of the Pacific, Vol. 64, No. 377, p. 55 and reproduced with kind permission of this Society © 1952

A cascade. Photograph by Antonello Provenzale

# Contents

# Chapter 1
# The Turbulence Problem

## 1.1 The Meaning of 'Turbulence'

In a general way we may say that turbulence is a phenomenon associated with the velocity state of a fluid medium. It exists in incompressible fluids or with small (with respect to the velocity of sound) velocities in a compressible fluid, so that to sufficient accuracy, the problems of turbulence may be discussed in terms of incompressible fluids.

We know, moreover, that the turbulence phenomenon is described by the more usual equations which apply to problems of hydrodynamics. In particular, the Navier–Stokes equation

$$\frac{\partial u_i}{\partial t} + \frac{\partial}{\partial x_j}(u_i u_j) = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + \nu \nabla^2 u_i, \tag{1.1}$$

and the equation of continuity for an incompressible fluid

$$\frac{\partial u_i}{\partial x_i} = 0, \tag{1.2}$$

apply to the turbulence phenomenon. Here, the symbols all carry their usual meaning, and (as in the remainder of these notes) the summation convention holds.

In the 'usual' applications to normal hydrodynamic situations, the velocity $u_i$ is a single-valued, smooth function of $x_i$ and (for laminar flow, at any rate) gives no sign of irregularity. However, there are situations in which irregularities do appear in the velocity field and, as these irregularities grow, the turbulent situation is approached. The following physical example will clarify this last point.

Consider two coaxial circular cylinders, with a fluid contained between them. Let $\omega_\mathrm{o}$ and $\omega_\mathrm{i}$ be the angular velocities of the outer and inner cylinders, respectively. Consider also that $\omega_\mathrm{i} > \omega_\mathrm{o}$. Then there exists a range of values $E = \omega_\mathrm{i} - \omega_\mathrm{o}$, ranging from $E = 0$ to some critical value $E_\mathrm{c}$, in which (1.1) and (1.2) are uniquely satisfied by the time independent (stationary) solution

$$u_\theta = Ar + \frac{B}{r}, \qquad u_r = 0, \qquad u_z = 0, \tag{1.3}$$

E.A. Spiegel (ed.), *The Theory of Turbulence*, Lecture Notes in Physics 810,
DOI 10.1007/978-94-007-0117-5_1, © Springer Science+Business Media B.V. 2011

where $(r, \theta, z)$ are cylindrical coordinates and $A$ and $B$ are constants. But as $E$ is increased beyond $E_c$, the motion breaks up into a cellular pattern, and it appears that other solutions [satisfying the boundary conditions as (1.3) does] can exist for (1.1) and (1.2). At the first stage of this cellular motion only one cell size exists; this is the one which dissipates energy least effectively. But as $E$ increases, more and more cell sizes appear until a spectrum of sizes exists.

These new cellular motions in the fluid represent various new solutions of (1.1) and (1.2). It may be shown that, if the Reynolds number $R = \rho v l / \mu$ is sufficiently large, numerous solutions to (1.1) and (1.2) may be found to satisfy a particular set of boundary conditions. As more and more solutions become possible in any physical situation (and therefore simultaneously manifest themselves) a situation of 'fully developed turbulence' is approached. We see that, from this standpoint, 'turbulence' refers to an idealized situation, and that fully developed turbulence can only be approximated by any physical situation.

## 1.2 Two Fundamental Aspects of Turbulence

(a) **Viscous Dissipation of Energy.** If energy is supplied to a system at a fixed rate, the system is in an essentially stationary state if this energy can be dissipated at the rate (on the average) at which it is supplied. (Of course, it remains to describe the sense in which the system is stationary.) Viscous dissipation is the only mechanism available to a fluid medium to dissipate the energy input and thus maintain an energy balance. Because of this, and because it is physically clear that turbulence cannot exist in an inviscid medium, viscosity and viscous dissipation are necessary aspects of the turbulence phenomenon.

(b) **Interchange of Energy Between the Various States of Motion.** We have seen that the existence of turbulence in a physical situation implies the coexistence of states of motion representing various solutions of (1.1) and (1.2) and subject to the boundary conditions. If these states of motion did not somehow interact, the turbulence problem would resolve (or degenerate) into the problem of describing these several states of motion independently. However, an interaction does occur between the states of motion in the form of a continual interchange of energy, as we shall see in Chap. 3.

The properties (a) and (b) are fundamental to the turbulence problem which may be framed in terms of them, namely: what is the mechanism of (b), and how may we, in view of the effects of (a) and (b), describe the phenomenon that we have called turbulence?

# Chapter 2
# The Net Energy Balance

In this section we see what the terms in the Navier–Stokes equation contribute to the production and dissipation of energy in a fluid. Consider a fluid contained in a volume $V$ whose boundary is the surface $S$. We prescribe that no material shall cross $S$ and we thus have the boundary condition that the velocity component normal to $S$ is zero everywhere on $S$, i.e.,

$$\boldsymbol{u} \cdot \mathrm{d}\boldsymbol{S} = 0 \quad \text{on } S, \tag{2.1}$$

where $\mathrm{d}\boldsymbol{S}$ is a vector normal to the surface and of arbitrary magnitude. If we multiply (1.1) by $\rho u_i$ we obtain, after integrating both sides over $V$,

$$\rho \int_V u_i \frac{\partial u_i}{\partial t} \mathrm{d}V + \rho \int_V u_i u_j \frac{\partial u_i}{\partial x_j} \mathrm{d}V = -\int_V u_i \frac{\partial p}{\partial x_i} \mathrm{d}V + \nu\rho \int_V u_i \nabla^2 u_i \mathrm{d}V. \tag{2.2}$$

Here $\mathrm{d}V$ is the element of volume in $V$, $\rho$ has been treated as constant, and the solenoidal character of $u_i$ will be assumed throughout. We then consider the four terms of (2.2) separately:

(a) **The First Term on the Left-Hand Side:**

$$\rho \int_V u_i \frac{\partial u_i}{\partial t} \mathrm{d}V = \frac{\rho}{2} \int_V \frac{\partial (u_i u_i)}{\partial t} \mathrm{d}V = \frac{\rho}{2} \int_V \frac{\partial |\boldsymbol{u}|^2}{\partial t} \mathrm{d}V = \frac{\partial \mathcal{T}}{\partial t},$$

where $\mathcal{T}$ is the kinetic energy contained in the volume.

(b) **The Non-Linear Term:** In (1.1) the term $\partial(u_i u_j)/\partial x_i$ arises as the changing velocity of a mass element arising from its changing position in the velocity field; it is known as the inertial term. We have,

$$\rho \int_V u_i u_j \frac{\partial u_i}{\partial x_j} \mathrm{d}V = \frac{1}{2}\rho \int_V u_j \frac{\partial (u_i u_j)}{\partial x_j} \mathrm{d}V = \frac{1}{2}\rho \int_V \frac{\partial (u_j |\boldsymbol{u}|^2)}{\partial x_j} \mathrm{d}V,$$

E.A. Spiegel (ed.), *The Theory of Turbulence*, Lecture Notes in Physics 810,
DOI 10.1007/978-94-007-0117-5_2, © Springer Science+Business Media B.V. 2011

since $u_j$ is solenoidal. Then, by the divergence theorem,

$$\begin{aligned}\rho\int_V u_i u_j \frac{\partial u_i}{\partial x_j}\mathrm{d}V &= \frac{1}{2}\rho\int_V \operatorname{div}(\boldsymbol{u}|\boldsymbol{u}|^2)\mathrm{d}V\\ &= \frac{1}{2}\rho\int_S |\boldsymbol{u}|^2\boldsymbol{u}\cdot\mathrm{d}\boldsymbol{S}\\ &= 0 \quad \text{by (2.1)}.\end{aligned}$$

(c) **The Pressure Term:**

$$\int_V u_i\frac{\partial p}{\partial x_i}\mathrm{d}V = \int_V \frac{\partial(pu_i)}{\partial x_i}\mathrm{d}V = \int_S p\boldsymbol{u}\cdot\mathrm{d}\boldsymbol{S} = 0,$$

by (2.1).

(d) **The Dissipation Term:** We have the lemma

$$u_i\nabla^2 u_i = -|\operatorname{curl}\boldsymbol{u}|^2 + \operatorname{div}(\boldsymbol{u}\times\operatorname{curl}\boldsymbol{u}),$$

proved at the end of this section. Then, in view of this identity,

$$\begin{aligned}\rho\nu\int_V u_i\nabla^2 u_i\mathrm{d}V &= \mu\int_V\Big[-|\operatorname{curl}\boldsymbol{u}|^2 + \operatorname{div}(\boldsymbol{u}\times\operatorname{curl}\boldsymbol{u})\Big]\mathrm{d}V\\ &= -\mu\int_V|\operatorname{curl}\boldsymbol{u}|^2\mathrm{d}V + \mu\int_S(\boldsymbol{u}\times\operatorname{curl}\boldsymbol{u})\cdot\mathrm{d}\boldsymbol{S}.\end{aligned}$$

Let $\boldsymbol{\omega} = \operatorname{curl}\boldsymbol{u}$. Then, on gathering the various terms of (2.2) we obtain

$$\frac{\partial\mathcal{T}}{\partial t} = -\mu\int_V|\boldsymbol{\omega}|^2\mathrm{d}V + \mu\int_S(\boldsymbol{u}\times\operatorname{curl}\boldsymbol{u})\cdot\mathrm{d}\boldsymbol{S}. \tag{2.3}$$

The first term on the right-hand side is the viscous dissipation of the vorticity, i.e., $-\mu|\boldsymbol{\omega}|^2$ is the rate of dissipation of energy per unit volume. The stationary state requires $\partial\mathcal{T}/\partial t = 0$, and so the energy input must be balanced by $\int_V(-\mu|\boldsymbol{\omega}|^2)\mathrm{d}V$. This implies that the small scale motion predominates in the dissipation, as will become clear below.

*Proof of the Lemma*

To show

$$u_i\nabla^2 u_i = -|\operatorname{curl}\boldsymbol{u}|^2 + \operatorname{div}(\boldsymbol{u}\times\operatorname{curl}\boldsymbol{u}),$$

consider

$$\operatorname{curl}_i\boldsymbol{u} = \epsilon_{ijk}\frac{\partial u_k}{\partial x_j}, \qquad |\operatorname{curl}\boldsymbol{u}|^2 = \epsilon_{ijk}\epsilon_{imn}u_{k,j}u_{n,m}. \tag{2.4}$$

But

$$\epsilon_{ijk}\epsilon_{imn} = \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}. \tag{2.5}$$

Combining (2.4) and (2.5) leads to

$$\begin{aligned}|\mathrm{curl}\,\boldsymbol{u}|^2 &= (\delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km})u_{k,j}u_{n,m}\\ &= u_{i,j}u_{i,j} - u_{i,j}u_{j,i}.\end{aligned}$$

Further,

$$\begin{aligned}(\boldsymbol{u} \times \mathrm{curl}\,\boldsymbol{u})_i &= \epsilon_{ijk}u_j\,\epsilon_{kmn}u_{n,m}\\ &= (\delta_{im}\delta_{jn} - \delta_{in}\delta_{jm})u_j u_{n,m} = u_j u_{j,i} - u_j u_{i,j},\end{aligned} \tag{2.6}$$

whence

$$\mathrm{div}\,(\boldsymbol{u} \times \mathrm{curl}\,\boldsymbol{u}) = u_{j,i}u_{j,i} + u_j u_{j,ii} - u_{j,i}u_{i,j} - u_j u_{i,ji}.$$

Now,

$$u_j u_{i,ji} = u_j u_{i,ij} = u_j \frac{\partial}{\partial x_j}\left(\frac{\partial u_i}{\partial x_i}\right) = 0,$$

by (1.2). Then,

$$\mathrm{div}\,(\boldsymbol{u} \times \mathrm{curl}\,\boldsymbol{u}) - |\mathrm{curl}\,\boldsymbol{u}|^2 = u_i \nabla^2 u_i,$$

and the lemma is proved. □

Returning to the example of concentric rotating cylinders, we note that the second term on the right of (2.3) must be the energy introduced by the cylinders per unit time and must be balanced by the dissipation term if $\partial\mathcal{T}/\partial t$ is to be zero. Let $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}$ be unit vectors in the $r, \theta, z$ directions, respectively. Then, the velocity (at least in one solution) is given by $v_r = v_z = 0$ and $v_\theta = v$, where $v$ is given by (1.3). Then

$$\mathrm{curl}\,\boldsymbol{u} = \boldsymbol{k}\left[\frac{1}{r}\frac{\partial(rv)}{\partial r}\right] = 2A\boldsymbol{k},$$

whence

$$\boldsymbol{u} \times \mathrm{curl}\,\boldsymbol{u} = \boldsymbol{i}2Av,$$

and

$$(\boldsymbol{u} \times \mathrm{curl}\,\boldsymbol{u}) \cdot \mathrm{d}\boldsymbol{S} = 2Av\mathrm{d}S,$$

so that

$$\begin{aligned}\int (\boldsymbol{u} \times \mathrm{curl}\,\boldsymbol{u}) \cdot \mathrm{d}\boldsymbol{S} &= \int_0^l \int_0^{2\pi} \Big[2Av\Big]_{R_1}^{R_2} r\,\mathrm{d}\theta\mathrm{d}z\\ &= 4\pi l A\left[R_2\left(AR_2 + \frac{B}{R_2}\right) - R_1\left(AR_1 + \frac{B}{R_1}\right)\right]\\ &= 4\pi l A^2(R_2^2 - R_1^2).\end{aligned}$$

Also, since

$$
\begin{aligned}
|\operatorname{curl} \boldsymbol{u}|^2 &= \left[\frac{1}{r}\frac{\partial(rv)}{\partial r}\right]^2 = (2A)^2, \\
\int_V |\operatorname{curl} \boldsymbol{u}|^2 \mathrm{d}V &= 4A^2 \int_V \mathrm{d}V = 4A^2(\pi l R_2^2 - \pi l R_1^2) \\
&= 4\pi l A^2 (R_2^2 - R_1^2),
\end{aligned}
$$

and $\partial \mathcal{T}/\partial t = 0$, which will hold on the average, even when instability occurs.

# Chapter 3
# Interchange of Energy Between States of Motion

That energy is continually interchanged between the various states of motion was stated in Sect. 1.1. We now state this property of turbulence analytically in the language of Fourier analysis.

Maintaining our notion of an incompressible fluid contained in a finite volume, we make a Fourier representation of the velocity field of the medium:

$$\boldsymbol{u}(\boldsymbol{r},t) = \sum_{\boldsymbol{k}} \boldsymbol{u}_{\boldsymbol{k}}(t)\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}}. \tag{3.1}$$

Here $\boldsymbol{k}$ is the wave vector, corresponding to the wavelength

$$\lambda = \frac{\pi}{|\boldsymbol{k}|} \tag{3.2}$$

and the Fourier transform $\boldsymbol{u}_{\boldsymbol{k}}$ is the velocity of the component represented by $\boldsymbol{k}$ in the resolution.

On applying (1.2) to (3.1) we may observe that

$$\operatorname{div}\boldsymbol{u} = \mathrm{i}\sum_{\boldsymbol{k}} (\boldsymbol{u}_{\boldsymbol{k}}\cdot\boldsymbol{k})\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}} = 0. \tag{3.3}$$

Equation (3.3) will be true in general (only) if

$$\boldsymbol{u}_{\boldsymbol{k}}\cdot\boldsymbol{k} = 0. \tag{3.4}$$

The orthogonality of $\boldsymbol{u}_{\boldsymbol{k}}$ and $\boldsymbol{k}$ implies that the time variation of $\boldsymbol{u}_{\boldsymbol{k}}$ is a rotation in the plane orthogonal to $\boldsymbol{k}$.

For the sake of brevity, let

$$\varpi = \frac{p}{\rho}, \tag{3.5}$$

and perform the Fourier resolution of $\varpi$ :

$$\varpi(\boldsymbol{r},t) = \sum_{\boldsymbol{k}} \varpi_{\boldsymbol{k}}(t)\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}}. \tag{3.6}$$

E.A. Spiegel (ed.), *The Theory of Turbulence*, Lecture Notes in Physics 810,
DOI 10.1007/978-94-007-0117-5_3, 

Another convenient step is to rewrite (1.1) in vector form. We then have

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \mathrm{grad})\boldsymbol{u} = -\mathrm{grad}\,\varpi + \nu\nabla^2\boldsymbol{u}. \tag{3.7}$$

Introducing (3.1) and (3.6) into (3.7), we find

$$\sum_{\boldsymbol{k}} \frac{\partial}{\partial t}\left[\boldsymbol{u}_{\boldsymbol{k}}(t)\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}}\right] + \sum_{\boldsymbol{k}} (u_j)_{\boldsymbol{k}}\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}} \frac{\partial}{\partial x_j} \sum_{\boldsymbol{k}'} \boldsymbol{u}_{\boldsymbol{k}'}\mathrm{e}^{\mathrm{i}\boldsymbol{k}'\cdot\boldsymbol{r}}$$

$$= -\mathrm{i}\sum_{\boldsymbol{k}} \varpi_{\boldsymbol{k}}\boldsymbol{k}\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}} - \nu\sum_{\boldsymbol{k}} \boldsymbol{u}_{\boldsymbol{k}}|\boldsymbol{k}|^2\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}}. \tag{3.8}$$

The second term on the left-hand side of (3.8) may be simplified. In this term, the summation indicated by the repeated index $j$ should not be overlooked. That term becomes

$$\sum_{\boldsymbol{k}}\sum_{\boldsymbol{k}'} \boldsymbol{u}_{\boldsymbol{k}'} \cdot (\boldsymbol{k}-\boldsymbol{k}')\boldsymbol{u}_{\boldsymbol{k}-\boldsymbol{k}'}\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}},$$

with the use of the standard formula for the products of infinite series, namely

$$\sum_{n=0}^{\infty} a_n z^n \sum_{n=0}^{\infty} b_n z^n = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} a_k b_{n-k} z^n.$$

Finally, under application of (3.4), the term becomes

$$\mathrm{i}\sum_{\boldsymbol{k}}\sum_{\boldsymbol{k}'} \boldsymbol{u}_{\boldsymbol{k}-\boldsymbol{k}'}(\boldsymbol{u}_{\boldsymbol{k}'} \cdot \boldsymbol{k})\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}}.$$

The resolved form of (3.7) in final form is thus

$$\sum_{\boldsymbol{k}} \frac{\partial \boldsymbol{u}_{\boldsymbol{k}}}{\partial t}\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}} + \mathrm{i}\sum_{\boldsymbol{k}}\sum_{\boldsymbol{k}'} \boldsymbol{u}_{\boldsymbol{k}-\boldsymbol{k}'}(\boldsymbol{u}_{\boldsymbol{k}'} \cdot \boldsymbol{k})\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}}$$

$$= -\mathrm{i}\sum_{\boldsymbol{k}} \varpi_{\boldsymbol{k}}\boldsymbol{k}\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}} - \nu\sum_{\boldsymbol{k}} \boldsymbol{u}_{\boldsymbol{k}}|\boldsymbol{k}|^2\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}}. \tag{3.9}$$

If the $\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}}$ coefficients are equated term by term, there results

$$\frac{\partial \boldsymbol{u}_{\boldsymbol{k}}}{\partial t} + \mathrm{i}\sum_{\boldsymbol{k}'} \boldsymbol{u}_{\boldsymbol{k}-\boldsymbol{k}'}(\boldsymbol{u}_{\boldsymbol{k}'} \cdot \boldsymbol{k}) = -\mathrm{i}\varpi_{\boldsymbol{k}}\boldsymbol{k} - \nu|\boldsymbol{k}|^2\boldsymbol{u}_{\boldsymbol{k}}. \tag{3.10}$$

We may multiply both sides of (3.10) by $\boldsymbol{u}_{\boldsymbol{k}}^*$ to get the rate of change of kinetic energy for a given $\boldsymbol{k}$. Then

$$\frac{1}{2}\frac{\partial|\boldsymbol{u}_{\boldsymbol{k}}|^2}{\partial t} = -\mathrm{i}\sum_{\boldsymbol{k}'} \boldsymbol{u}_{\boldsymbol{k}-\boldsymbol{k}'}(\boldsymbol{u}_{\boldsymbol{k}'} \cdot \boldsymbol{k}) \cdot \boldsymbol{u}_{\boldsymbol{k}}^* - \mathrm{i}\varpi_{\boldsymbol{k}}\boldsymbol{k} \cdot \boldsymbol{u}_{\boldsymbol{k}}^* - \nu|\boldsymbol{k}|^2|\boldsymbol{u}_{\boldsymbol{k}}|^2. \tag{3.11}$$

The rate of change of energy for a given $\boldsymbol{k}$ is thus seen to be the sum of three terms:

(1) The term

$$\sum_{\boldsymbol{k}'} Q(\boldsymbol{k}', \boldsymbol{k}) = -\mathrm{i} \sum_{\boldsymbol{k}'} \boldsymbol{u}_{\boldsymbol{k}-\boldsymbol{k}'}(\boldsymbol{u}_{\boldsymbol{k}'} \cdot \boldsymbol{k}) \cdot \boldsymbol{u}_{\boldsymbol{k}}^*$$

gives the rate of interchange of energy between the Fourier component $\boldsymbol{k}$ and all other components $\boldsymbol{k}'$. The quantity $Q(\boldsymbol{k}', \boldsymbol{k})$ is a sort of transition probability, giving relative strengths of the tendency for energy to be exchanged with $\boldsymbol{k}$ by the various $\boldsymbol{k}'$. This interaction term comes from the inertial term in (1.1), and so (see Sect. 1.2) cannot alter the net energy balance; its average effect over all $\boldsymbol{k}$ must be null. (It should also be remarked that stability problems say nothing of the interaction between components since they depend on the neglect of the inertial term.)

(2) The term $-i\varpi_{\boldsymbol{k}}\boldsymbol{k} \cdot \boldsymbol{u}_{\boldsymbol{k}}^*$ has no interaction properties, but can cause the exchange of energy between the various spatial positions of the same Fourier component, i.e., it will spread the energy spatially, but will maintain it in fixed $\boldsymbol{k}$-space position. It is intuitively likely that this term will tend to promote isotropy. For the moment, we will assume isotropy, and neglect this term.

(3) The term $-\nu|\boldsymbol{k}|^2|\boldsymbol{u}_{\boldsymbol{k}}|^2$ represents the viscous dissipation by the $\boldsymbol{k}$ component. If (1.1) and (1.2) were not present in (3.11), this term would lead to a solution of the form

$$u_k(t) = u_k(0)\mathrm{e}^{-\nu|\boldsymbol{k}|^2 t}.$$

To sum up, only the inertial term contributes to the energy interchange. The balance of energy for any component $\boldsymbol{k}$ is governed, in the case of isotropy, by the equation

$$\frac{1}{2}\frac{\mathrm{d}|\boldsymbol{u}_{\boldsymbol{k}}|^2}{\mathrm{d}t} = \sum_{\boldsymbol{k}'} Q(\boldsymbol{k}', \boldsymbol{k}) - \nu|\boldsymbol{k}|^2|\boldsymbol{u}_{\boldsymbol{k}}|^2. \tag{3.12}$$

# Chapter 4
# Some Remarks

## 4.1 On the Harmonic Analysis

We have heretofore considered finite volumes over which to Fourier-analyze velocities and pressures. Since finite volumes will have only a finite number of modes, Fourier series have been adequate for these pruposes. But for the analysis of an infinite region, the Fourier integral is needed. We introduce the latter at this point.

The number of modes per volume of $\boldsymbol{k}$-space of size $\mathrm{d}V_k = \mathrm{d}k_x\,\mathrm{d}k_y\,\mathrm{d}k_z$ is

$$\mathrm{d}n = \frac{V}{(2\pi)^3}\mathrm{d}k_x\,\mathrm{d}k_y\,\mathrm{d}k_z. \tag{4.1}$$

This may be best visualized in one dimension: a length contains

$$n_x = \frac{l_x}{\lambda_x} \tag{4.2}$$

modes. On use of (4.1), the number of modes in the range $k_x$ to $k_x + \mathrm{d}k_x$ is, therefore,

$$\mathrm{d}n_x = \mathrm{d}\left(\frac{l_x k_x}{2\pi}\right) = \frac{l_x}{2\pi}\mathrm{d}k_x. \tag{4.3}$$

On the basis of (4.1) the operator $\sum_{\boldsymbol{k}}$ becomes

$$\sum_{\boldsymbol{k}} \to \int_{\boldsymbol{k}} \frac{V}{(2\pi)^3}\mathrm{d}k_x\,\mathrm{d}k_y\,\mathrm{d}k_z.$$

## 4.2 On the Concept of Isotropy

It is initially clear that in isotropy we should be interested in absolute values connected with $\boldsymbol{k}$, and not in associated directions. We will, for the moment, content ourselves to make use of this notion of isotropy without exploring it further. The

E.A. Spiegel (ed.), *The Theory of Turbulence*, Lecture Notes in Physics 810,
DOI 10.1007/978-94-007-0117-5_4, 

notion implies that for isotropy we must have spherical symmetry in $\boldsymbol{k}$-space. Then, the number of modes in a volume $V$ in physical space contained in a range $\mathrm{d}k$ of $k$ is (see Richtmeyer and Kennard, p. 188)

$$\mathrm{d}n = \frac{V}{(2\pi)^3} 4\pi k^2 \mathrm{d}k. \tag{4.4}$$

The Fourier integral operator then becomes

$$\int_0^\infty \frac{V}{(2\pi)^3} 4\pi k^2 \mathrm{d}k.$$

## 4.3 On the Possibility of a Universal Theory

We saw in Chap. 2 that in, the simple case of concentric, rotating cylinders, energy was introduced at the boundary and dissipated throughout the volume. Such a situation implies an inherent inhomogeneity which is characteristic of turbulence as it occurs in nature.

The question then arises: Does there exist a volume element, i.e., some physical domain, sufficiently small (but not so small as to be insignificant) and sufficiently far from the boundary that it may be considered independent of the nature of conditions at the boundaries and of the inhomogeneity they introduce? If such a domain can be defined, the laws which apply to the turbulence in it will be of universal character.

# Chapter 5
# The Spectrum of Turbulent Energy

## 5.1 The Spectrum

We may obtain the kinetic energy of motion (the square of the velocity) by evaluating $\boldsymbol{u} \cdot \boldsymbol{u}^*$. We then obtain

$$
\begin{aligned}
|\boldsymbol{u}|^2 &= \sum_{\boldsymbol{k}'} \sum_{\boldsymbol{k}''} \boldsymbol{u}_{\boldsymbol{k}'} \mathrm{e}^{\mathrm{i}\boldsymbol{k}' \cdot \boldsymbol{r}} \cdot \boldsymbol{u}^*_{\boldsymbol{k}''} \mathrm{e}^{-\mathrm{i}\boldsymbol{k}'' \cdot \boldsymbol{r}} \\
&= \sum_{\boldsymbol{k}'} \sum_{\boldsymbol{k}''} \boldsymbol{u}_{\boldsymbol{k}'} \cdot \boldsymbol{u}^*_{\boldsymbol{k}''} \mathrm{e}^{-\mathrm{i}(\boldsymbol{k}'-\boldsymbol{k}'') \cdot \boldsymbol{r}}.
\end{aligned}
\tag{5.1}
$$

The mean-squared velocity in the volume $V$ is

$$
\begin{aligned}
\overline{|\boldsymbol{u}|^2} &= \frac{1}{V} \int_V \sum_{\boldsymbol{k}'} \sum_{\boldsymbol{k}''} \mathrm{e}^{-\mathrm{i}(\boldsymbol{k}'-\boldsymbol{k}'') \cdot \boldsymbol{r}} \boldsymbol{u}_{\boldsymbol{k}'} \cdot \boldsymbol{u}^*_{\boldsymbol{k}''} \mathrm{d}V \\
&= \frac{1}{V} \sum_{\boldsymbol{k}} \sum_{\boldsymbol{k}''} \boldsymbol{u}_{\boldsymbol{k}} \cdot \boldsymbol{u}^*_{\boldsymbol{k}''} \int_V \mathrm{e}^{\mathrm{i}(\boldsymbol{k}-\boldsymbol{k}'') \cdot \boldsymbol{r}} \mathrm{d}V,
\end{aligned}
$$

where we have replaced $\boldsymbol{k}'$ by $\boldsymbol{k}$.

The integral in the last term vanishes for $\boldsymbol{k}' - \boldsymbol{k}'' \neq 0$, so

$$
\overline{|\boldsymbol{u}|^2} = \frac{1}{V} \sum_{\boldsymbol{k}} \boldsymbol{u}_{\boldsymbol{k}} \cdot \boldsymbol{u}^*_{\boldsymbol{k}} \int_V \mathrm{d}V,
$$

or

$$
\overline{|\boldsymbol{u}|^2} = \sum_{\boldsymbol{k}} |\boldsymbol{u}_{\boldsymbol{k}}|^2. \tag{5.2}
$$

The summation of (5.2) can be converted to an integration by the considerations of Sect. 4.2, if $V \to \infty$. The mean-squared velocity, averaged over all space, is then,

E.A. Spiegel (ed.), *The Theory of Turbulence*, Lecture Notes in Physics 810,
DOI 10.1007/978-94-007-0117-5_5, 

for the case of isotropic turbulence,

$$\overline{|\boldsymbol{u}|^2} = \overline{u^2} = \frac{V}{(2\pi)^3} \int_0^\infty 4\pi k^2 |\boldsymbol{u}_{\boldsymbol{k}}|^2 \mathrm{d}k,$$

and the mean kinetic energy per unit mass is

$$\frac{1}{2}\overline{u^2} = \frac{V}{(2\pi)^2} \int_0^\infty k^2 |\boldsymbol{u}_{\boldsymbol{k}}|^2 \mathrm{d}k = \frac{V}{(2\pi)^2} \int_0^\infty k^2 |\boldsymbol{u}_k|^2 \mathrm{d}k. \tag{5.3}$$

The function

$$F(k) = \frac{V}{(2\pi)^2} k^2 |\boldsymbol{u}_k|^2 \tag{5.4}$$

is the spectrum of (isotropic) turbulence.

For the non-isotropic case, a three-dimensional spectrum may be employed. Then,

$$\overline{|\boldsymbol{u}|^2} = \frac{V}{(2\pi)^3} \int_{\boldsymbol{k}} |\boldsymbol{u}_{\boldsymbol{k}}|^2 \mathrm{d}k_x \, \mathrm{d}k_y \, \mathrm{d}k_z, \tag{5.5}$$

and the general turbulence spectrum is

$$F(\boldsymbol{k}) = \frac{V}{(2\pi)^3} |\boldsymbol{u}_{\boldsymbol{k}}|^2. \tag{5.6}$$

## 5.2 An Equation for the Spectrum

Any equation that the energy spectrum satisfies is likely to follow from (3.7), (1.1), and (3.6), which we recall here:

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \mathrm{grad})\boldsymbol{u} = -\frac{1}{\rho} \mathrm{grad}\, p + \nu \nabla^2 \boldsymbol{u}, \tag{5.7}$$

$$\boldsymbol{u}(\boldsymbol{r}, t) = \sum_{\boldsymbol{k}'} \boldsymbol{u}_{\boldsymbol{k}'}(t) \mathrm{e}^{\mathrm{i}\boldsymbol{k}' \cdot \boldsymbol{r}}, \tag{5.8}$$

and

$$\varpi = \sum_{\boldsymbol{k}} \varpi_{\boldsymbol{k}} \mathrm{e}^{\mathrm{i}\boldsymbol{k} \cdot \boldsymbol{r}}. \tag{5.9}$$

Since $\boldsymbol{u}(\boldsymbol{r}, t)$ is real,

$$\boldsymbol{u}(\boldsymbol{r}, t) = \boldsymbol{u}^*(\boldsymbol{r}, t)\,,$$

we have from (5.8) that

$$\boldsymbol{u}_{\boldsymbol{k}} = \boldsymbol{u}^*_{-\boldsymbol{k}}. \tag{5.10}$$

(It might be noted here that the pressure term could simply be eliminated from (3.10). Scalar multiplication by $\boldsymbol{k}$, gives an expression for $\varpi_{\boldsymbol{k}}$ which may be introduced into the equation for $\varpi_{\boldsymbol{k}}$, but this step is unnecessary in what follows.)

From these equations, we found [see (3.10) on p. 8]

$$\frac{\partial \boldsymbol{u}_{\boldsymbol{k}}}{\partial t} = -\mathrm{i}\sum_{\boldsymbol{k}'}(\boldsymbol{u}_{\boldsymbol{k}'}\cdot\boldsymbol{k})\boldsymbol{u}_{\boldsymbol{k}-\boldsymbol{k}'} - \mathrm{i}\varpi_{\boldsymbol{k}}\boldsymbol{k} - \nu|\boldsymbol{k}|^2\boldsymbol{u}_{\boldsymbol{k}}. \tag{5.11}$$

Multiplying both sides of (5.11) by $\boldsymbol{u}^*_{\boldsymbol{k}}$, we obtain, using (3.4),

$$\boldsymbol{u}^*_{\boldsymbol{k}}\frac{\partial \boldsymbol{u}_{\boldsymbol{k}}}{\partial t} = -\mathrm{i}\sum_{\boldsymbol{k}'}(\boldsymbol{u}_{\boldsymbol{k}'}\cdot\boldsymbol{k})(\boldsymbol{u}^*_{\boldsymbol{k}}\cdot\boldsymbol{u}_{\boldsymbol{k}-\boldsymbol{k}'}) - \nu|\boldsymbol{k}|^2|\boldsymbol{u}_{\boldsymbol{k}}|^2. \tag{5.12}$$

The complex conjugate of (5.12) is

$$\boldsymbol{u}_{\boldsymbol{k}}\frac{\partial \boldsymbol{u}^*_{\boldsymbol{k}}}{\partial t} = \mathrm{i}\sum_{\boldsymbol{k}'}(\boldsymbol{u}^*_{\boldsymbol{k}'}\cdot\boldsymbol{k})(\boldsymbol{u}_{\boldsymbol{k}}\cdot\boldsymbol{u}^*_{\boldsymbol{k}-\boldsymbol{k}'}) - \nu|\boldsymbol{k}|^2|\boldsymbol{u}_{\boldsymbol{k}}|^2. \tag{5.13}$$

Upon adding (5.12) and (5.13), we find that

$$\frac{\partial |\boldsymbol{u}_{\boldsymbol{k}}|^2}{\partial t} = -\mathrm{i}\sum_{\boldsymbol{k}'}(\boldsymbol{u}_{\boldsymbol{k}'}\cdot\boldsymbol{k})(\boldsymbol{u}^*_{\boldsymbol{k}}\cdot\boldsymbol{u}_{\boldsymbol{k}-\boldsymbol{k}'}) + \mathrm{i}\sum_{\boldsymbol{k}'}(\boldsymbol{u}^*_{\boldsymbol{k}'}\cdot\boldsymbol{k})(\boldsymbol{u}_{\boldsymbol{k}}\cdot\boldsymbol{u}^*_{\boldsymbol{k}-\boldsymbol{k}'}) - 2\nu|\boldsymbol{k}|^2|\boldsymbol{u}_{\boldsymbol{k}}|^2. \tag{5.14}$$

In the first summation of (5.14), we replace $\boldsymbol{k}'$ by $\boldsymbol{k}-\boldsymbol{k}''$. It then becomes

$$\sum_{\boldsymbol{k}-\boldsymbol{k}''}(\boldsymbol{u}_{\boldsymbol{k}-\boldsymbol{k}''}\cdot\boldsymbol{k})(\boldsymbol{u}^*_{\boldsymbol{k}}\cdot\boldsymbol{u}_{\boldsymbol{k}''}). \tag{5.15}$$

For a fixed $\boldsymbol{k}$, $\sum_{\boldsymbol{k}-\boldsymbol{k}''}$ is equivalent to summation over $-\boldsymbol{k}''$, viz., $\sum_{-\boldsymbol{k}''}$, so (5.15) is equivalent to

$$\sum_{-\boldsymbol{k}''}(\boldsymbol{u}_{\boldsymbol{k}-\boldsymbol{k}''}\cdot\boldsymbol{k})(\boldsymbol{u}^*_{\boldsymbol{k}}\cdot\boldsymbol{u}_{\boldsymbol{k}''}). \tag{5.16}$$

We may replace the index of summation $-\boldsymbol{k}''$ by a more convenient one, say $\boldsymbol{k}'$. Then (5.16) becomes, when (5.10) is introduced,

$$\sum_{\boldsymbol{k}'}(\boldsymbol{u}_{\boldsymbol{k}+\boldsymbol{k}'}\cdot\boldsymbol{k})(\boldsymbol{u}_{\boldsymbol{k}}\cdot\boldsymbol{u}_{\boldsymbol{k}'})^*. \tag{5.17}$$

In a similar way, we may replace $\boldsymbol{k}'$ by $\boldsymbol{k}+\boldsymbol{k}''$ in

$$\sum_{\boldsymbol{k}'}(\boldsymbol{u}^*_{\boldsymbol{k}'}\cdot\boldsymbol{k})(\boldsymbol{u}_{\boldsymbol{k}}\cdot\boldsymbol{u}^*_{\boldsymbol{k}-\boldsymbol{k}'}) \tag{5.18}$$

to obtain

$$\sum_{\boldsymbol{k}''} (\boldsymbol{u}^*_{\boldsymbol{k}+\boldsymbol{k}''} \cdot \boldsymbol{k})(\boldsymbol{u}_{\boldsymbol{k}} \cdot \boldsymbol{u}^*_{-\boldsymbol{k}''}), \tag{5.19}$$

into which the same reasoning has been introduced as in going from (5.15) to (5.16). Replacing $\boldsymbol{k}''$ by $\boldsymbol{k}'$ in (5.19) and introducing (5.10), we find

$$\sum_{\boldsymbol{k}'} (\boldsymbol{u}^*_{\boldsymbol{k}+\boldsymbol{k}'} \cdot \boldsymbol{k})(\boldsymbol{u}_{\boldsymbol{k}} \cdot \boldsymbol{u}_{\boldsymbol{k}'}). \tag{5.20}$$

Then (5.20) and (5.17) are the equivalent of the sums in (5.14), and if we let

$$Q(\boldsymbol{k},\boldsymbol{k}') = \mathrm{i}\Big[(\boldsymbol{u}^*_{\boldsymbol{k}+\boldsymbol{k}'} \cdot \boldsymbol{k})(\boldsymbol{u}_{\boldsymbol{k}} \cdot \boldsymbol{u}_{\boldsymbol{k}'}) - (\boldsymbol{u}_{\boldsymbol{k}+\boldsymbol{k}'} \cdot \boldsymbol{k})(\boldsymbol{u}_{\boldsymbol{k}} \cdot \boldsymbol{u}_{\boldsymbol{k}'})^*\Big], \tag{5.21}$$

we have, for fixed $\boldsymbol{k}$ [see (3.12)],

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\boldsymbol{u}_{\boldsymbol{k}}|^2 = -\frac{1}{2}\sum_{\boldsymbol{k}'} Q(\boldsymbol{k},\boldsymbol{k}') - \nu|\boldsymbol{k}|^2|\boldsymbol{u}_{\boldsymbol{k}}|^2. \tag{5.22}$$

The foregoing equation has been discussed above. It gives the rate of change of kinetic energy in a Fourier component. This rate of change has two components:

(a) **A Damping Term.** This term alone gives rise to an exponential decay in a given Fourier component with a decay time

$$\tau = \frac{1}{2\nu|\boldsymbol{k}|^2}. \tag{5.23}$$

Equation (5.22), as a result of the linearity of the velocity derivative in the Navier–Stokes equation, shows that the damping in one $\boldsymbol{k}$ is independent of all other $\boldsymbol{k}$. Moreover, shorter wavelengths decay most quickly, the decay time being proportional to $\lambda^2$.

The $\lambda^2$ dependence leads to an interesting property of decaying turbulence. If there is turbulence which is stationary and isotropic in some volume $V$, and the source of energy is removed, the isotropy will be destroyed with time. This occurs because the smaller eddies, which lend isotropy to the turbulence, decay first, leaving the larger, boundary-dependent eddies to exist for longer times. The larger eddies are, of course, anisotropic.

(b) **The Transfer Term.** The term $Q(\boldsymbol{k},\boldsymbol{k}')$ is a sort of transition probability for the transfer of energy between the $\boldsymbol{k}$ and $\boldsymbol{k}'$ components. But $Q(\boldsymbol{k},\boldsymbol{k}')$ depends on the component $\boldsymbol{k}+\boldsymbol{k}'$, the intermediary through which the energy must pass.

As would be expected from conservation reasoning, $Q(\boldsymbol{k},\boldsymbol{k}')$ is antisymmetric, i.e.,

$$Q(\boldsymbol{k},\boldsymbol{k}') = -Q(\boldsymbol{k}',\boldsymbol{k}). \tag{5.24}$$

This last remark follows from the definition of $Q(\boldsymbol{k},\boldsymbol{k}')$ and the property (3.4), viz.,

$$\boldsymbol{u}_{\boldsymbol{k}+\boldsymbol{k}'} \cdot (\boldsymbol{k}+\boldsymbol{k}') = 0. \tag{5.25}$$

For if $\boldsymbol{k}$ and $\boldsymbol{k}'$ are interchanged in $Q$ as given by (5.21), $\boldsymbol{u}_{\boldsymbol{k}+\boldsymbol{k}'} \cdot \boldsymbol{k}'$ replaces $\boldsymbol{u}_{\boldsymbol{k}+\boldsymbol{k}'} \cdot \boldsymbol{k}$. But by (5.25),

$$\boldsymbol{u}_{\boldsymbol{k}+\boldsymbol{k}'} \cdot \boldsymbol{k}' = -\boldsymbol{u}_{\boldsymbol{k}+\boldsymbol{k}'} \cdot \boldsymbol{k}, \tag{5.26}$$

and (5.24) follows.

We may now proceed to see how (5.22) provides an equation for the turbulence spectrum. Assuming isotropy, we multiply (5.22) by $k^2$ and, on noting (5.4), we find

$$\frac{\partial F(k,t)}{\partial t} = -\frac{V}{(2\pi)^2} \sum_{\boldsymbol{k}'} k^2 Q(\boldsymbol{k}, \boldsymbol{k}') - 2\nu |\boldsymbol{k}|^2 F(k,t). \tag{5.27}$$

If we pass from the sum to the integral in (5.27), we get

$$\sum_{\boldsymbol{k}'} k^2 Q(\boldsymbol{k}, \boldsymbol{k}') \to \frac{V}{(2\pi)^3} 4\pi \int_0^\infty k^2 k'^2 Q(\boldsymbol{k}, \boldsymbol{k}') \mathrm{d}k,$$

and (5.27) becomes

$$-\frac{\partial F(k,t)}{\partial t} = \int_0^\infty T(\boldsymbol{k}, \boldsymbol{k}') \mathrm{d}\boldsymbol{k}' + 2\nu |\boldsymbol{k}|^2 F(k,t), \tag{5.28}$$

in which

$$T(\boldsymbol{k}, \boldsymbol{k}') = 2 \times \frac{V^2}{(2\pi)^4} k^2 k'^2 Q(\boldsymbol{k}, \boldsymbol{k}'). \tag{5.29}$$

From (5.28), one would hope to develop a theory of turbulence. Two procedures seem possible:

1. Develop some physical description of turbulence and justify the results in terms of (5.28).
2. Carry (5.28) as far as possible mathematically, and give a physical interpretation to the results.

Procedure (1) has been followed in the past, and we will here consider Heisenberg's scheme as it embodies most others which proceed along those lines. Following this, we will show how procedure (2) may be exploited.

# Chapter 6
# Some Preliminaries to the Development of a Theory of Turbulence

We may integrate (5.28) over the wave numbers 0 to $k$:

$$-\frac{1}{2}\frac{\partial}{\partial t}\int_0^k F(k,t)\mathrm{d}k = \int_0^k \mathrm{d}k \int_0^\infty T(\boldsymbol{k},\boldsymbol{k}')\mathrm{d}k' + \nu\int_0^k k^2 F(k)\mathrm{d}k, \qquad (6.1)$$

where a factor of 1/2 has been absorbed into the definition of $T(\boldsymbol{k},\boldsymbol{k}')$. The expression (6.1) gives the rate of passage of energy from the eddies included in the range from 0 to $k$.

The integral over $\boldsymbol{k}'$ on the right-hand side of (6.1) may be simplified by introducing the antisymmetric property of $Q(\boldsymbol{k},\boldsymbol{k}')$. It is in fact clear that $T(\boldsymbol{k},\boldsymbol{k}')$ is also antisymmetric, i.e.,

$$T(\boldsymbol{k},\boldsymbol{k}') = -T(\boldsymbol{k}',\boldsymbol{k}). \qquad (6.2)$$

Hence,

$$\int_0^k \mathrm{d}k \int_0^k T(\boldsymbol{k},\boldsymbol{k}')\mathrm{d}k' = 0, \qquad (6.3)$$

and (6.1) becomes

$$-\frac{1}{2}\frac{\partial}{\partial t}\int_0^k F(k,t)\mathrm{d}k = \int_0^k \mathrm{d}k \int_k^\infty T(\boldsymbol{k},\boldsymbol{k}')\mathrm{d}k' + \nu\int_0^k k^2 F(k,t)\mathrm{d}k. \qquad (6.4)$$

On the right-hand side:

1. The first integral gives the amount of energy flowing per unit time into the eddies in the range $k$ to $\infty$ from those in the range 0 to $k$.
2. The second integral gives the rate at which energy is dissipated by viscosity in the range 0 to $k$. To aid in the understanding of the structure of the dissipation term, we will investigate the way it may arise from the energy balance equation.

We saw that the mean rate of dissipation over the volume $V$ is [see (2.3) on p. 4]

$$\varepsilon = -\mu\frac{1}{V}\int_V |\mathrm{curl}\,\boldsymbol{u}|^2\mathrm{d}V = -\mu\overline{|\mathrm{curl}\,\boldsymbol{u}|^2}. \qquad (6.5)$$

E.A. Spiegel (ed.), *The Theory of Turbulence*, Lecture Notes in Physics 810,
DOI 10.1007/978-94-007-0117-5_6, 

If we put the Fourier expression for $\boldsymbol{u}$ into (6.5), we will discover the contributions to the dissipation by the various Fourier components.

On p. 4, we wrote the relation

$$\mathrm{curl}_i \boldsymbol{u} = \epsilon_{imn} \frac{\partial u_n}{\partial x_m},$$

where $\boldsymbol{u}$ may be replaced in accord with

$$\mathrm{curl}_i \bar{u} = \epsilon_{imn} \frac{\partial}{\partial x_m} \sum_{\boldsymbol{k}} (u_{\boldsymbol{k}})_n \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}}$$

$$= \epsilon_{imn} \mathrm{i} \sum_{\boldsymbol{k}} (u_{\boldsymbol{k}})_n k_m \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}}.$$

But

$$\epsilon_{imn} k_m (u_{\boldsymbol{k}})_n = (\boldsymbol{k} \times \boldsymbol{u}_{\boldsymbol{k}})_i ,$$

so

$$\mathrm{curl}\, \boldsymbol{u} = \mathrm{i} \sum_{\boldsymbol{k}} (\boldsymbol{k} \times \boldsymbol{u}_{\boldsymbol{k}}) \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}}. \tag{6.6}$$

Moreover,

$$(\mathrm{curl}\, \boldsymbol{u})^* = -\mathrm{i} \sum_{\boldsymbol{k}} (\boldsymbol{k} \times \boldsymbol{u}^*_{\boldsymbol{k}}) \mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}}. \tag{6.7}$$

Then

$$|\mathrm{curl}\, \boldsymbol{u}|^2 = \sum_{\boldsymbol{k}} \sum_{\boldsymbol{k}'} (\boldsymbol{k}' \times \boldsymbol{u}_{\boldsymbol{k}'})^* \cdot (\boldsymbol{k} \times \boldsymbol{u}_{\boldsymbol{k}}) \mathrm{e}^{\mathrm{i}(\boldsymbol{k}-\boldsymbol{k}')\cdot\boldsymbol{r}}. \tag{6.8}$$

If we integrate (6.8) over a volume which we will let grow to include all $\boldsymbol{r}$, the $\mathrm{e}^{\mathrm{i}(\boldsymbol{k}-\boldsymbol{k}')\cdot\boldsymbol{r}}$ will go to zero unless $\boldsymbol{k} = \boldsymbol{k}'$, and the Fourier summation will tend to an integral. Then

$$\varepsilon = \frac{V}{(2\pi)^3} \int_0^\infty 4\pi k^2 |\boldsymbol{k} \times \boldsymbol{u}_{\boldsymbol{k}}|^2 \mathrm{d}k. \tag{6.9}$$

But by the orthogonality of $\boldsymbol{k}$ and $\boldsymbol{u}_{\boldsymbol{k}}$, we find

$$|\boldsymbol{k} \times \boldsymbol{u}_{\boldsymbol{k}}|^2 = |\boldsymbol{k}|^2 |\boldsymbol{u}_{\boldsymbol{k}}|^2,$$

and

$$\varepsilon = 2\nu \int_0^\infty F(k) |\boldsymbol{k}|^2 \mathrm{d}k, \tag{6.10}$$

whence the dissipation by just those eddies in the range 0 to $k$ is clearly

$$\varepsilon_k(\mathrm{thermal}) = 2\nu \int_0^k |\boldsymbol{k}|^2 F(k) \mathrm{d}k. \tag{6.11}$$

# Chapter 7
# Heisenberg's Theory of Turbulence

## 7.1 The Fundamental Equation of the Theory

We have seen that the rate at which energy passes from the range 0 to $k$ of wave numbers into the range $k$ to $\infty$ is

$$\varepsilon_k = -\frac{1}{2}\frac{\partial}{\partial t}\int_0^k F(k,t)\mathrm{d}k, \tag{7.1}$$

whence according to (6.4)

$$\varepsilon_k = \int_0^k \mathrm{d}k \int_k^\infty T(\boldsymbol{k},\boldsymbol{k}')\mathrm{d}k' + \nu\int_0^k k^2 F(k)\mathrm{d}k. \tag{7.2}$$

From the physical interpretation of

$$\varepsilon_k(\text{mechanical}) = \int_0^k \mathrm{d}k \int_k^\infty T(\boldsymbol{k},\boldsymbol{k}')\mathrm{d}k', \tag{7.3}$$

namely that the eddies in the range $k$ to $\infty$ take mechanical energy from those in the range 0 to $k$, arises the notion that $\varepsilon_k$(mechanical) must have the form of a dissipation term in which the dissipation in the range 0 to $k$ is provided by the eddies in $k$ to $\infty$. Heisenberg therefore assumes that $\varepsilon_k$(mechanical) is of the form

$$\varepsilon_k(\text{mechanical}) = \nu_k\int_0^k F(k')k'^2\mathrm{d}k', \tag{7.4}$$

in analogy with

$$\varepsilon_k(\text{thermal}) = \nu\int_0^k F(k')k'^2\mathrm{d}k', \tag{7.5}$$

where $\nu_k$ is the 'viscosity' of the eddies $k$ to $\infty$ acting on 0 to $k$.

Another assumption made in this theory is that $T(k,k')$ is separable in $k$ and $k'$, i.e.,

$$T(k,k') = f(k)g(k'). \tag{7.6}$$

E.A. Spiegel (ed.), *The Theory of Turbulence*, Lecture Notes in Physics 810,
DOI 10.1007/978-94-007-0117-5_7, 

By our assumption (7.4), we see that $f(k)$ must be given by

$$f(k) = k^2 F(k). \tag{7.7}$$

It is also plausible that $g(k')$ should be expressible in terms of $k'$ and $F(k')$. Moreover, $g(k')$ will give the $\nu_k$ and it is clear, both by analogy with (7.3) and from the notion that $\nu_k$ is a 'viscosity' produced by the $k'$ in the range $k$ to $\infty$, that

$$\varepsilon_k(\text{mechanical}) = \int_k^\infty g(k')\mathrm{d}k' \int_0^k k^2 F(k)\mathrm{d}k. \tag{7.8}$$

And finally, consistently with the physical ideas we have introduced, we must ensure that $\int_k^\infty g(k')\mathrm{d}k'$ has the same dimensions as $\nu$. If we let [ ] operating on some quantity mean 'dimensions of', we have

$$[F] = \frac{\mathrm{L}^2}{\mathrm{T}^2}\mathrm{L} = \frac{\mathrm{L}^3}{\mathrm{T}^2}, \tag{7.9}$$

and

$$[\nu] = \frac{\mathrm{L}^2}{\mathrm{T}}. \tag{7.10}$$

We must therefore have

$$[g][\mathrm{d}k] = \frac{\mathrm{L}^2}{\mathrm{T}}$$

and

$$[g] = \frac{\mathrm{L}^3}{\mathrm{T}}. \tag{7.11}$$

Since we have assumed that $g$ is expressible in terms of (powers of) $F(k')$ and $k'$, we must also have

$$[F]^\alpha [k]^\beta = \frac{\mathrm{L}^3}{\mathrm{T}}, \tag{7.12}$$

or

$$\mathrm{L}^{3\alpha-\beta}\mathrm{T}^{-2\alpha} = \mathrm{L}^3\mathrm{T}^{-1}, \tag{7.13}$$

whence

$$\begin{cases} 3\alpha - \beta = 3, \\ -2\alpha = -1. \end{cases} \tag{7.14}$$

Then (7.14) has the solutions

$$\alpha = \frac{1}{2}, \qquad \beta = -\frac{3}{2}, \tag{7.15}$$

and we find for $g(k')$ the result

$$g(k') = \sqrt{\frac{F(k')}{k'^3}}. \tag{7.16}$$

At this point we change the definition of the $\varepsilon$ to conform to the literature without altering the notation. In order to make $\varepsilon$ the rate of flow of energy per unit mass, rather than per unit volume, we introduce the factor $\rho$ (density) into all the $\varepsilon$. Moreover, we will bring the factor 1/2 in (7.1) to the other side, thus multiplying the various expressions for $\varepsilon$ by 2. With these considerations and equations (7.5), (7.8) and (7.16), the rate of flow of energy per unit mass of fluid from the eddies in the range 0 to $k$ is

$$\varepsilon_k = 2\rho\left[\nu + K_0\int_k^\infty \sqrt{\frac{F(k')}{k'^3}}\mathrm{d}k'\right]\int_0^k F(k)k^2\mathrm{d}k. \tag{7.17}$$

In the case of stationary turbulence with sufficiently high Reynolds number,

$$\varepsilon_k = \text{constant}, \tag{7.18}$$

while for the decay of turbulence,

$$\varepsilon_k = -\frac{\partial}{\partial t}\int_0^k F(k,t)\mathrm{d}k. \tag{7.19}$$

From (7.17) may be derived the results of Kolmogorov and the rest of this school of similitude. Perhaps the chief contribution of Heisenberg is that he has embodied all the previous attempts along this line in equation (7.17).

## 7.2 Chandrasekhar's Solution of (7.17) for the Case of Stationary Turbulence

In the case of stationary turbulence in which viscosity is assumed to act only on the largest $k$ values, (7.18) applies. Then differentiation of (7.17) with respect to $k$ gives

$$\nu F(k)k^2 + K_0k^2F(k)\int_k^\infty \sqrt{\frac{F(k')}{k'^3}}\mathrm{d}k' - K_0\sqrt{\frac{F(k)}{k^3}}\int_0^k k^2F(k)\mathrm{d}k = 0. \tag{7.20}$$

We may rearrange (7.20) to give

$$\frac{\nu}{K_0} + \int_k^\infty \frac{\mathrm{d}k'}{k'^{3/2}}\sqrt{F(k')} = \frac{1}{k^2\left[F(k)k^3\right]^{1/2}}\int_0^k F(k)k^2\mathrm{d}k. \tag{7.21}$$

We let

$$\begin{cases} g = k^3F(k), \\ y = \displaystyle\int_0^k F(k)k^2\mathrm{d}k. \end{cases} \tag{7.22}$$

Then,

$$\frac{\mathrm{d}y}{\mathrm{d}k} = \frac{g}{k}, \tag{7.23}$$

or

$$\frac{\mathrm{d}y}{g} = \frac{\mathrm{d}k}{k},$$

so that, if we can get $g(y)$, we can find $k$:

$$\log k = \text{constant} + \int_0^y \frac{\mathrm{d}y}{g(y)}. \tag{7.24}$$

Introducing (7.22) and (7.23) into (7.21), we find

$$\frac{\nu}{K_0} + \int_y^\infty \frac{\mathrm{d}y}{g^{1/2}k^2} = \frac{y}{k^2\sqrt{g}}. \tag{7.25}$$

If we differentiate (7.25) and use (7.24), taking $y$ as the independent variable,

$$-\frac{1}{g^{1/2}k^2} - \frac{1}{k^2}\frac{\mathrm{d}}{\mathrm{d}y}\left(\frac{y}{\sqrt{g}}\right) + \frac{2y}{k^3\sqrt{g}}\frac{\mathrm{d}k}{\mathrm{d}y} = 0,$$

or

$$-\frac{1}{g^{1/2}k^2} - \frac{1}{k^2}\frac{\mathrm{d}}{\mathrm{d}y}\left(\frac{y}{\sqrt{g}}\right) + \frac{2y}{k^2g^{3/2}} = 0. \tag{7.26}$$

Carrying out the differentiation, we find

$$-\frac{1}{\sqrt{g}k^2} - \frac{1}{\sqrt{g}k^2} + \frac{1}{2k^2}\frac{y}{g^{3/2}}\frac{\mathrm{d}g}{\mathrm{d}y} + \frac{2y}{k^2g^{3/2}} = 0,$$

or

$$\frac{\mathrm{d}g}{\mathrm{d}y} - 4\frac{g}{y} + 4 = 0. \tag{7.27}$$

The solution of the homogeneous form of (7.27), i.e., of

$$\frac{\mathrm{d}g}{\mathrm{d}y} - 4\frac{g}{y} = 0, \tag{7.28}$$

is

$$g = Ay^4, \tag{7.29}$$

where $A$ is an integration constant, and a particular integral of (7.27) is

$$g = By. \tag{7.30}$$

Putting (7.30) into (7.27), we find that $B$ must satisfy

$$B - 4B + 4 = 0, \tag{7.31}$$

whence

$$B = 4/3. \tag{7.32}$$

The general solution is therefore

$$g = Ay^4 + \frac{4}{3}y,$$

or

$$g = \frac{4}{3} y(1 - Cy^3). \tag{7.33}$$

We may now return to the integral in (7.24). It is

$$\int_0^y \frac{\mathrm{d}y}{g} = \int_0^y \frac{\mathrm{d}y}{\frac{4}{3}y(1 - Cy^3)} = \frac{1}{3} \int_0^y \frac{\mathrm{d}(y^3)}{\frac{4}{3}y^3(1 - Cy^3)}.$$

The foregoing is

$$\frac{1}{4} \int_0^y \frac{\mathrm{d}(y^3)}{y^3(1 - Cy^3)} = \int_0^u \frac{\mathrm{d}u}{u(1 - Cu)}.$$

By the method of partial fractions, we may write

$$\frac{M}{u} + \frac{N}{1 - Cu} = \frac{1}{u(1 - Cu)},$$

whence

$$M(1 - Cu) + Nu = 1.$$

Setting $u = 0$ and $u = 1/C$, we find that

$$M = 1, \qquad N = C.$$

Then

$$\begin{aligned} \int_0^u \frac{\mathrm{d}u}{u(1 - Cu)} &= \int_0^u \frac{\mathrm{d}u}{u} + \int_0^u \frac{\mathrm{d}(Cu)}{1 - Cu} \\ &\propto \log y^3 - \log(1 - Cy^3) \\ &= \log \frac{y^3}{1 - Cy^3}. \end{aligned} \tag{7.34}$$

Combining (7.24) and (7.34), we find

$$k = \alpha \left( \frac{y^3}{1 - Cy^3} \right)^{1/4}, \tag{7.35}$$

where $\alpha$ is an integration constant. On solving (7.35) for $y$, we obtain

$$\frac{k^4}{\alpha^4}(1 - Cy^3) = y^3, \tag{7.36}$$

$$y^3 = \frac{k^4}{\alpha^4 + Ck^4}. \tag{7.37}$$

Or, on substituting $1 - Cy^3$ as found from (7.36) into (7.33), we have

$$g = \frac{4}{3} \left( \frac{\alpha y}{k} \right)^4. \tag{7.38}$$

Then the spectrum is [see (7.22)]

$$\begin{aligned}
F(k) &= \frac{g}{k^3} \\
&= \frac{4}{3}\frac{\alpha^4}{k^7}y^4 \quad \text{[by (7.38)]} \\
&= \frac{4}{3}\frac{\alpha^4}{k^7}\left(\frac{k^4}{\alpha^4 + Ck^4}\right)^{4/3} \quad \text{[by (7.37)]}\,.
\end{aligned}$$

Finally,

$$F(k) = \frac{4}{3}\frac{\alpha^4}{k^{5/3}(\alpha^4 + Ck^4)^{4/3}}. \tag{7.39}$$

This expression may also be conveniently rewritten by factoring out $\alpha^4$:

$$F(k) = \frac{4}{3}\frac{\alpha^{-4/3}}{k^{5/3}}\frac{1}{1 + \frac{C}{\alpha^4}k^4}. \tag{7.40}$$

If we write

$$k_\mathrm{s} = \left(\frac{C^{1/4}}{\alpha}\right)^{-1}, \qquad F(k_0) = \frac{4}{3}\frac{1}{\alpha^{4/3}k_0^{5/3}}, \tag{7.41}$$

equation (7.40) becomes

$$F(k) = F(k_0)\left(\frac{k_0}{k}\right)^{5/3}\frac{1}{[1 + (k/k_\mathrm{s})^4]^{4/3}}. \tag{7.42}$$

In obtaining the solutions of the Heisenberg equation, differentiations were performed which raised the order of the equations. Thus, an additional integration constant was introduced. It is therefore to be expected that $\alpha$ and $C$ are not independent, and in particular, that they must be related through the differential equation (7.25). If (7.39) and (7.22) are introduced into (7.25), or its equivalent (7.21), we find

$$\frac{\nu}{K_0} = \frac{y}{k^2 g^{1/2}} - \frac{\alpha^2}{2\sqrt{3}}\int_\infty^k \frac{1}{(C + \alpha^4/k^4)^{2/3}}\frac{\mathrm{d}}{\mathrm{d}k}\left(\frac{1}{k^4}\right)\mathrm{d}k. \tag{7.43}$$

The integral in (7.43) is, under the substitution $u = 1/k^4$,

$$\begin{aligned}
\int_0^u \frac{\mathrm{d}u}{(C + \alpha^4 u)^{2/3}} &= +\frac{3}{\alpha^4}(C + \alpha^4 u)^{1/3}\Big|_0^u \\
&= +\frac{3}{\alpha^4}\left[\left(C + \frac{\alpha^4}{k^4}\right)^{1/3} - C^{1/3}\right],
\end{aligned}$$

and (7.43) is therefore

$$\frac{\nu}{K_0} = \frac{y}{k^2 g^{1/2}} - \frac{\sqrt{3}}{2\alpha^2}\left[\left(C + \frac{\alpha^4}{k^4}\right)^{1/3} - C^{1/3}\right]. \tag{7.44}$$

By (7.38) we have

$$\frac{y}{k^2 g^{1/2}} = \frac{y}{k^2 \frac{2}{\sqrt{3}} \frac{\alpha^2 y^2}{k^2}} = \frac{\sqrt{3}}{2\alpha^2 y}, \tag{7.45}$$

and by (7.37),

$$\frac{1}{y} = \frac{(\alpha + Ck^4)^{1/3}}{k^{4/3}} = \left(C + \frac{\alpha^4}{k^4}\right)^{1/3}. \tag{7.46}$$

Then (7.44) becomes

$$\frac{\nu}{K_0} = \frac{\sqrt{3}}{2\alpha^2} C^{1/3}. \tag{7.47}$$

Combining (7.41) and (7.45), we find

$$k_s = \frac{1}{\sqrt{\alpha}} \left(\frac{\sqrt{3}}{2} \frac{K_0}{\nu}\right)^{3/4}. \tag{7.48}$$

We note that, as $\nu \to 0$ (and therefore $k \to \infty$), $k_s \to \infty$. We see that in (7.42),

$$F \longrightarrow F(k_0) \left(\frac{k_0}{k}\right)^{5/3} \quad \text{as } \nu \to 0, \tag{7.49}$$

and we have recovered the Kolmogorov $k^{-5/3}$ law. On the other hand, when $\nu$ is not zero, and $k \gg k_s$,

$$F \propto k^{-7}. \tag{7.50}$$

Actually, (7.50) is not verified by experiment.

In conclusion, we have found that the following picture presents itself:

1. There is a $k_0$ beyond which $F(k)$ decreases as $k$ increases, so that beyond $k_0$, i.e., $k > k_0$, the spectrum will not depend on the boundary conditions, it being physically clear that $L_0 = 2\pi/k_0$ is the dimension of the vessel. In the region beyond $k_0$, the spectrum follows a $k^{-5/3}$ power law.
2. There exists a $k = k_s$ at which the viscous term is comparable with the inertial term. As $k$ increases beyond $k_s$, the viscous term predominates and a $k^{-7}$ law obtains.

The region from $k_0$ to $k_s$ is called the universal equilibrium range.

# Chapter 8
# Other Derivations of the $k^{-5/3}$ Law

The methods in this section are based on the assumption that local isotropy exists. Local isotropy may be thought of as the existence of a $k_0$ such that the boundary conditions are irrelevant for $k > k_0$. These methods also require that for $k \to \infty$ there is a finite range $k_0 \ll k \ll k_\mathrm{s}$, where $k_\mathrm{s}$ has the meaning of the previous section.

## 8.1 Fermi's Approach

For an eddy of wave number $k$ [see (6.6) on p. 20],

$$\varepsilon_k = \nu_k |\mathrm{curl}\, \boldsymbol{v}_k|^2 = \nu_k (v_k k)^2.$$

Since the dimensions of $\nu$ are $\mathrm{L}^2/\mathrm{T}$, we must have

$$\nu_k \sim \frac{v_k}{k},$$

so

$$\varepsilon_k \sim v_k^3 k = \text{const.} \quad \text{for } k_0 \ll k \ll k_\mathrm{s}.$$

Now this last expression indicates

$$v_k \sim k^{-1/3}.$$

But we know that $[F(k)] = \mathrm{L}^3/\mathrm{T}^2$ so

$$F(k) \sim \frac{v_k^2}{k} = \frac{1}{k^{5/3}}. \tag{8.1}$$

Since we have been dividing by $k$, especially in the preceding equation, the results must apply to all eddies in the range $\infty$ to $k$, since $(\text{eddy})_k$ contains the rest. (That is, it contains the ones in $k$ to $\infty$.)

Now the inflow from $k < k_\mathrm{s}$ into $k > k_\mathrm{s}$ is

$$\nu_k \overline{|\mathrm{curl}\, \boldsymbol{v}|^2},$$

E.A. Spiegel (ed.), *The Theory of Turbulence*, Lecture Notes in Physics 810,
DOI 10.1007/978-94-007-0117-5_8, 

where curl $\boldsymbol{v}$ refers to the velocity of the eddies 0 to $k$. By the reasoning of the preceding sentences,

$$\nu_k \sim \frac{v_k}{k},$$

where $\nu_k$ refers to all the eddies in 0 to $k$. Moreover, since curl $\boldsymbol{v}$ refers to all these eddies, and since $k \sim k_s$, it is (nearly) the curl of the actual velocity. It must therefore be constant, on the average. So the inflow of energy to the range $k > k_s$ is (especially as $k_s \to \infty$)

$$\text{constant} \times \frac{v_k}{k}.$$

Moreover, the outflow due to dissipation is

$$\nu|\text{curl}\, v_k|^2 \sim \nu v_k^2 k^2 = \text{inflow} = \frac{v_k}{k} \times \text{constant}.$$

Furthermore, $v_k \sim k^{-3}$, whence

$$F(k) \sim k^{-7}. \tag{8.2}$$

## 8.2 Kolmogorov's Theory

The theory of Kolmogorov follows from two principles of similitude:

1. There exists a range of eddy sizes $k > k_0$ in which the spectrum, and other distribution functions of physical character, do not depend on the boundary conditions in any way. In this range, for finite Reynolds numbers, these distribution functions depend only on $\nu$ and $\varepsilon_k$.
2. In the limit as $\nu \to 0$, all these functions will depend only on $\varepsilon_k$. The action of $\nu$ becomes relegated to increasingly larger $k$ values.

It would follow that there exist certain universal functions characterizing the turbulence in those parts of Fourier space unaffected by the boundary conditions of the individual problem. This sort of similarity principle can exist if the lengths involved are expressed in an appropriate dimension. Now, a length $\eta$ can be constructed from the fundamental parameters $\nu$ and $k$ by taking

$$\eta = \left(\frac{\nu^3}{\varepsilon}\right)^{1/4}, \tag{8.3}$$

since $[\nu] = \mathrm{L}^2/\mathrm{T}$ and $[\varepsilon] = \mathrm{L}^2/\mathrm{T}^3$. Moreover, velocities can be constructed from the unit

$$[\text{velocity}] = (\nu\varepsilon)^{1/4}. \tag{8.4}$$

In these units, i.e., (8.3) and (8.4), all functions may have universal character. For example, the spectrum function can be constructed:

$$F(k) = \left(\nu^{5/4}\varepsilon^{1/4}\right) f\left[k\left(\nu^3/\varepsilon\right)^{1/4}\right], \tag{8.5}$$

where $f$ is a universal function (assuming that there is local isotropy).

According to Kolmogorov's second principle, when $\nu \to 0$, $F(k)$ must be independent of $\nu$, so that as $\nu \to 0$, $f$ must be proportional to $\nu^{-5/4}$. That is,

$$f \to Ck^{-5/3}\left(\frac{\nu^3}{\varepsilon}\right)^{-5/12} \qquad \text{as } \nu \to 0, \tag{8.6}$$

where $C$ is a proportionality constant, so that for large Reynolds numbers,

$$F = C\varepsilon^{2/3}k^{-5/3}, \tag{8.7}$$

which is the well-known Kolmogorov spectrum.

Of course, these dimensional arguments are based on the assumption that the various functions describing the spectrum follow power laws. If this is not the case, these arguments break down.

## 8.3 The Method of von Neumann

As we have seen, the two integration constants which arose in the Chandrasekhar solution of the Heisenberg equation are related through the viscosity. Thus $k_0$ and $k_s$ as they arise in Heisenberg's theory should be related through $\nu$. It should also be asked how $k_s$ behaves as the Reynolds number tends to infinity, and how the ratio $k_s/k_0$ depends on the Reynolds number. From the Heisenberg theory, we find (approximately)

$$k_s = 0.2211K_0\left(R_0K_0\right)^{3/4}, \tag{8.8}$$

where $R_0$ is the Reynolds number of the entire motion [see PRS **200**, 20 (1949), equation (27)]. That $k_s/k_0$ should go as $R_0^{3/4}$, as well as the need for definitions of quantities such as $k_0$ and $k_s$ appears in von Neumann's theory, as we shall now see.

The mean energy per unit mass is

$$E = \frac{1}{2}\overline{u^2},$$

which is strictly

$$E = \int_0^\infty F(k)\mathrm{d}k. \tag{8.9}$$

The energy dissipation per unit volume is strictly

$$W = 2\nu\int_0^\infty F(k)k^2\mathrm{d}k. \tag{8.10}$$

If we wish to accept the results of Kolmogorov we must replace $F(k)$ by $k^{-5/3}$. (Proportionality constants are overlooked here.) Then these expressions become

$$E = \int_0^\infty k^{-5/3}\mathrm{d}k \tag{8.11}$$

and

$$W = 2\nu \int_0^\infty k^{1/3} \mathrm{d}k. \tag{8.12}$$

On physical grounds we should expect $E < \infty$ and $W < \infty$, but this is not the case according to these last expressions. The integral (8.11) diverges for small $k$ and the integral (8.12) diverges for large $k$. In order to provide for the convergence of (8.11), we must assume a lower limit for the integration which allows convergence, but which does not alter the accuracy of the expression too greatly. That is, we want a $k_0 \neq 0$, but such that the eddies $k < k_0$ contain little of the energy. (Physically, one feels that $k_0 = 2\pi/L_0$, where $L_0$ is a dimension of the vessel.) If such a $k_0$ exists, then to a tolerable approximation, we have

$$E = \int_{k_0}^\infty k^{-5/3} \mathrm{d}k. \tag{8.13}$$

Similar reasoning applies to the integral (8.12). A suitable finite upper integration limit $k_\mathrm{s}$ must be found to ensure convergence. Then

$$W = 2\nu \int_0^{k_\mathrm{s}} k^{1/3} \mathrm{d}k. \tag{8.14}$$

Now the average energy of an eddy $k$ is the average over the eddies whose wavenumbers we may take in the interval in $[k_0, k_\mathrm{s}]$. This would be given by

$$\overline{u_k^2} = A \int_k^{k_\mathrm{s}} k^{-5/3} \mathrm{d}k = \frac{3}{2} A k^{-2/3},$$

where $A$ is a suitable constant. Then

$$\sqrt{\overline{u_k^2}} = v_k = \left(\frac{3}{2}\right)^{1/2} A^{1/2} k^{-1/3}. \tag{8.15}$$

The Reynolds number appropriate to $k$ is

$$R_k = \frac{\frac{2\pi}{k} v_k}{\nu} = \frac{2\pi \left(\frac{3}{2}\right)^{1/2} A^{1/2} k^{-4/3}}{\nu}. \tag{8.16}$$

Now $k_\mathrm{s}$ will occur when $R_k$ is of order unity. (Not order 1000 because $R = 1000$ is needed to initiate turbulence, not to maintain it; experimentally, it is known that once turbulence occurs, it persists to low Reynolds numbers $\sim 1$.) Arbitrarily, we shall choose

$$R_{k_\mathrm{s}} = 2\pi. \tag{8.17}$$

Then from (8.16), we have

$$k_\mathrm{s}^{4/3} = \frac{(3/2)^{1/2} A^{1/2}}{\nu}. \tag{8.18}$$

From (8.14) we see that

$$W = \frac{3}{2} \nu A k_\mathrm{s}^{4/3}, \tag{8.19}$$

or, on dividing by (8.18) and rearranging, we have

$$A = \frac{2}{3} W^{2/3}. \tag{8.20}$$

From (8.18), we find that

$$k = \left( \frac{W}{\nu^3} \right)^{1/4}. \tag{8.21}$$

On using (8.17) and (8.16), we have

$$\frac{R_{k_0}}{R_{k_s}} = \frac{R_{k_0}}{2\pi} = \frac{k_0^{-4/3}}{k_s^{-4/3}}. \tag{8.22}$$

Hence,

$$k_s = k_0 \left( \frac{R_{k_0}}{2\pi} \right)^{3/4}, \tag{8.23}$$

in agreement with the conclusion from the Heisenberg theory. This 3/4 power law is also verified experimentally. Starting from this as an experimental result, it is reasonable to conclude that, in terms of the Kolmogorov spectrum, viscosity takes over at an $R_k \sim 2\pi$.

## 8.4 Conclusion

The implication of the remarks in this and the last chapter is that you need only specify the spectrum and the turbulence problem is solved. This remark is true for the radiant energy distribution in the thermal enclosure, where phase relations do not enter. However, in the turbulence problem the phase relations are fundamental; energy passes between the Fourier components, so that it will be necessary to specify more than just the spectrum if the problem is to be solved.

# Chapter 9
# An Alternate Approach: Correlations

At the end of the previous chapter it was indicated that description of the spectrum is not necessarily an adequate representation of turbulence. Some kind of statistical representation is what is usually sought; but this sort of search has underlying difficulties.

Consider the statistical mechanics in kinetic theory. The Hamiltonian description of a volume containing $N$ molecules requires a $3N$-dimensional description, to be precise. The problem is, however, solved by a statistical representation in the 3-dimensional physical space.

However, the Navier–Stokes equations give a precise representation in the physical 3-space, though the attempt is made to make a statistical representation in this same space. That is, one is trying to represent statistically what the Navier–Stokes equations represent exactly in the same framework.

One can ask whether it would be worthwhile to develop a new statistical mechanics. Perhaps it would, but in any case it is necessary to describe a fluctuating velocity field, and we have seen that the spectrum is not necessarily fundamental for such an attempt. Such a description might take the form of specifying the way in which the velocity varies from point to point. This would be possible through the correlations of velocities at different points. To do this we consider the velocities at the points $\boldsymbol{r}$ and $\boldsymbol{r}+\boldsymbol{\xi}$:

$$u_i(\boldsymbol{r}) = \frac{1}{(2\pi)^3}\int_{\boldsymbol{k}} u_i(\boldsymbol{k})\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}}\mathrm{d}\boldsymbol{k}, \tag{9.1}$$

$$u_j(\boldsymbol{r}+\boldsymbol{\xi}) = \frac{1}{(2\pi)^3}\int_{\boldsymbol{k}'} u_j(\boldsymbol{k}')\mathrm{e}^{\mathrm{i}\boldsymbol{k}'\cdot(\boldsymbol{r}+\boldsymbol{\xi})}\mathrm{d}\boldsymbol{k}'. \tag{9.2}$$

Then the correlation is

$$u_i(\boldsymbol{r})u_j(\boldsymbol{r}+\boldsymbol{\xi}) = \int_{\boldsymbol{k}'} u_j(\boldsymbol{k}')\mathrm{e}^{\mathrm{i}\boldsymbol{k}'\cdot\boldsymbol{\xi}}\mathrm{d}\boldsymbol{k}'\int_{\boldsymbol{k}} u_i(\boldsymbol{k})\mathrm{e}^{\mathrm{i}(\boldsymbol{k}+\boldsymbol{k}')\cdot\boldsymbol{r}}\mathrm{d}\boldsymbol{k}. \tag{9.3}$$

The average value of $u_i(\boldsymbol{r})u_j(\boldsymbol{r}+\boldsymbol{\xi})$ over space is then proportional to the integral over space. Then (9.3) becomes

$$\overline{u_i(\boldsymbol{r})u_j(\boldsymbol{r}+\boldsymbol{\xi})} = \text{const.}\times\int_{\boldsymbol{k}'} u_j(\boldsymbol{k}')\mathrm{e}^{\mathrm{i}\boldsymbol{k}'\cdot\boldsymbol{\xi}}\mathrm{d}\boldsymbol{k}'\int_{\boldsymbol{k}} u_i(\boldsymbol{k})\mathrm{d}\boldsymbol{k}\int_V \mathrm{e}^{\mathrm{i}(\boldsymbol{k}+\boldsymbol{k}')\cdot\boldsymbol{r}}\mathrm{d}\boldsymbol{r}.$$

E.A. Spiegel (ed.), *The Theory of Turbulence*, Lecture Notes in Physics 810,
DOI 10.1007/978-94-007-0117-5_9, © Springer Science+Business Media B.V. 2011

The volume integral is a $\delta$-function and vanishes unless $k + k' = 0$. Using this condition and a minor change in independent variable to $\boldsymbol{k}$, we find

$$\overline{u_i(\boldsymbol{r})u_j(\boldsymbol{r}+\boldsymbol{\xi})} = \text{const.} \times \int_{\boldsymbol{k}} u_i(\boldsymbol{k})u_j(\boldsymbol{k})\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{\xi}}\mathrm{d}\boldsymbol{k}. \tag{9.4}$$

Or, if we have isotropy,

$$\overline{u_i(\boldsymbol{r})u_j(\boldsymbol{r}+\boldsymbol{\xi})} = \text{const.} \times \int_0^\infty k^2 u_i(\boldsymbol{k})u_j(\boldsymbol{k})\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{\xi}}\mathrm{d}k. \tag{9.5}$$

Finally, if we set $\xi = 0$ and contract, we have

$$\overline{|\boldsymbol{u}(\boldsymbol{r})|^2} = \text{const.} \times \int_0^\infty |\boldsymbol{u}_{\boldsymbol{k}}|^2 \mathrm{d}k, \tag{9.6}$$

and we see that the correlations must be connected with the energy spectrum through the Fourier transform.

Thus, in introducing the correlation function, we have introduced no essentially new functions. However, it will be convenient to think in terms of correlations. Moreover, the velocity correlations are more accessible to measurement than spectral functions.

If we write $u'_j$ for $u_j(\boldsymbol{r}+\boldsymbol{\xi})$, we have $\overline{u_i u'_j}$ for the correlation. We will let

$$Q_{ij}(\boldsymbol{\xi}) = \overline{u_i u'_j}. \tag{9.7}$$

$Q_{ij}$ is a tensor since it is the product of two vectors. Also, if we let

$$\Gamma_{ij}(\boldsymbol{k}) = u_i(\boldsymbol{k})u_j(\boldsymbol{k}). \tag{9.8}$$

equation (9.4) becomes

$$Q_{ij}(\boldsymbol{\xi}) = \text{const.} \times \int \Gamma_{ij}(\boldsymbol{k})\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{\xi}}\mathrm{d}\boldsymbol{k}, \tag{9.9}$$

where $\Gamma_{ij}(\boldsymbol{k})$ is a generalized spectrum.

# Chapter 10
# The Equations of Isotropic Turbulence

For the case of isotropic turbulence, the spectrum specifies the turbulence. Since $Q_{ij}$ and $\Gamma_{ij}$ are Fourier transforms of one another, it is to be expected that any restrictions on one will imply restrictions on the other; but it must not be concluded that one can pass back and forth between the two with ease. It is true that in working with linear equations one can go to the Fourier space by means of an algebraic equation, and in fact solutions are sometimes more readily obtained in this way. But in nonlinear equations, one has to go from the products of transforms to the transform of products, and this passage is usually quite complicated, so that the nonlinearity of the Navier–Stokes equations inhibits the usefulness of the concepts involved in passage to the Fourier space. Because of this difficulty, perhaps, the preoccupation of many workers in turbulence with the Fourier space has been the stumbling block in the way of a general theory.

## 10.1 The Concept of Isotropy

We say that turbulence is isotropic if the mean (over space, in this section) values of quantities which are of scalar character and which involve the physical parameters of the medium (such as velocity, pressure and density) are invariant under the full rotation group, i.e., invariant under rotations as a rigid body and under reflections through the origin.

The turbulence is homogeneous if these quantities are invariant under translations.

We will be interested in two kinds of correlation. We have already defined the double correlation

$$Q_{ij}(\boldsymbol{\xi}) = \overline{u_i u'_j}. \tag{10.1}$$

The triple correlation we will be interested in is

$$T_{ijk}(\boldsymbol{\xi}) = \overline{u_i u_j u'_k}. \tag{10.2}$$

E.A. Spiegel (ed.), *The Theory of Turbulence*, Lecture Notes in Physics 810,
DOI 10.1007/978-94-007-0117-5_10, 

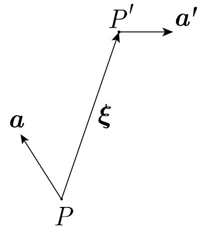


**Fig. 10.1** Notation for double correlation

Let the origin (or point of interest) be at $P$ (see Fig. 10.1); at any rate let it be the origin in the sense that we require invariance under reflection through $P$ for isotropy. And let $P'$ be displaced from $P$ by $\boldsymbol{\xi}$. Let $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \ldots,$ be arbitrary unit vectors at $P$ and $\boldsymbol{a}', \boldsymbol{b}', \boldsymbol{c}', \ldots,$ be arbitrary unit vectors at $P'$. Then for isotropic turbulence it is necessary that

$$\overline{u_{\boldsymbol{a}} u'_{\boldsymbol{a}'}} = a_i a'_j \overline{u_i u'_j} = a_i a'_j Q_{ij} \tag{10.3}$$

and

$$\overline{u_{\boldsymbol{a}} u_{\boldsymbol{b}} u'_{\boldsymbol{a}'}} = a_i b_j a'_k \overline{u_i u_j u'_k} = a_i b_j a'_k T_{ijk} \tag{10.4}$$

should be invariant under the full rotation group. With this restriction on $Q_{ij}$ and $T_{ijk}$ we can discover the forms of these tensors for isotropic turbulence.

**Theorem** *Any invariant function of any number of vectors* $\boldsymbol{\xi}, \boldsymbol{a}, \boldsymbol{b}, \ldots,$ *etc., can be expressed in terms of the fundamental invariants of the following types*:

- *the scalar products such as* $\boldsymbol{\xi} \cdot \boldsymbol{a} = \xi_i a_i$, $\boldsymbol{a} \cdot \boldsymbol{b} = a_i b_i$, *of any two vectors including the scalar squares* $\boldsymbol{\xi} \cdot \boldsymbol{\xi}$, $\boldsymbol{a} \cdot \boldsymbol{a}$, *etc.*,
- *the determinants such as* $[\boldsymbol{a}\boldsymbol{b}\boldsymbol{\xi}] = \epsilon_{ijk} a_i b_j \xi_k$ *of any three of the vectors.*

## 10.2 $Q_{ij}$ as an Isotropic Tensor

In the case of the double correlation, $Q_{ij}$, three vectors are available for the construction of the invariant (10.3), i.e., $\boldsymbol{a}$, $\boldsymbol{a}'$, and $\boldsymbol{\xi}$. Then the fundamental invariants are

$$\boldsymbol{a} \cdot \boldsymbol{a}', \qquad \boldsymbol{a} \cdot \boldsymbol{\xi}, \qquad \boldsymbol{a}' \cdot \boldsymbol{\xi}, \qquad [\boldsymbol{a}\boldsymbol{a}'\boldsymbol{\xi}], \qquad \boldsymbol{a} \cdot \boldsymbol{a}, \qquad \boldsymbol{a}' \cdot \boldsymbol{a}', \qquad \boldsymbol{\xi} \cdot \boldsymbol{\xi}.$$

Since $Q_{ij} a_i a'_j$ is a bilinear form in $\boldsymbol{a}$ and $\boldsymbol{a}'$, the squares of quantities $|\boldsymbol{a}|$ and $|\boldsymbol{a}'|$, etc., will not enter. We write $\boldsymbol{b}$ for $\boldsymbol{a}'$ for consistency with the literature and then have

$$\begin{aligned} Q_{ij} a_i b_j &= Q_1(r)(\boldsymbol{a} \cdot \boldsymbol{\xi})(\boldsymbol{b} \cdot \boldsymbol{\xi}) + Q_2(r)(\boldsymbol{a} \cdot \boldsymbol{b}) + Q_3(r)[\boldsymbol{a}\boldsymbol{b}\boldsymbol{\xi}] \\ &= Q_1(r) a_i b_j \xi_i \xi_j + Q_2(r) a_i b_i + Q_3(r) \epsilon_{ijk} a_i b_j \xi_k. \end{aligned} \tag{10.5}$$

where $r^2 = \boldsymbol{\xi} \cdot \boldsymbol{\xi}$. Since $\boldsymbol{a}$ and $\boldsymbol{b}$ are arbitrary, we must have

$$Q_{ij} = Q_1(r)\xi_i \xi_j + Q_2(r)\delta_{ij} + Q_3(r)\epsilon_{ijk}\xi_k. \tag{10.6}$$

The definition of isotropy requires that $Q_{ij}a_ib_j$ be invariant under reflections through $P$. Since $Q_3\varepsilon_{ijk}\xi_k$ changes sign under such reflections, isotropy requires

$$Q_3(r)\epsilon_{ijk}\xi_k = 0. \tag{10.7}$$

Then,

$$Q_{ij} = Q_1(r)\xi_i\xi_j + Q_2(r)\delta_{ij} \tag{10.8}$$

is an isotropic tensor.

### *10.2.1 Two More Examples*

#### The Isotropic Vector

The problem is to find $L_i$ such that $L_ia_i$ is invariant under the full rotation group, where $a_i$ is an arbitrary unit vector. Only two vectors exist for this purpose: $a_i$ and $\xi_i$, so that the only available fundamental invariant is $\boldsymbol{a}\cdot\boldsymbol{\xi}$. Hence,

$$L_ia_i = L(r)(\boldsymbol{a}\cdot\boldsymbol{\xi}) = L(r)a_i\xi_i, \tag{10.9}$$

and the isotropic vector is

$$L_i = L(r)\xi_i. \tag{10.10}$$

#### The Isotropic Tensor of Third Order

Consider the correlation of two components of $\boldsymbol{u}$ at $P$ with a component of $\boldsymbol{u}'$ at $P'$. The vectors $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c} = \boldsymbol{a}'$ are arbitrary in direction only, being of unit length. Then we wish to find $T_{ijk}$ such that $T_{ijk}a_ib_jc_k$ is invariant under the full rotation group. We have seen that $\epsilon_{ijk}a_ib_j\xi_k$, etc., is not invariant under reflection through $P$, so we need not include such invariants in the present connection. Moreover, since $T_{ijk}a_ib_jc_k$ is trilinear, $(\boldsymbol{a}\cdot\boldsymbol{a})$ will not enter either. Then we must have

$$\begin{aligned} T_{ijk}a_ib_jc_k = {} & T_1(r)(\boldsymbol{a}\cdot\boldsymbol{\xi})(\boldsymbol{b}\cdot\boldsymbol{\xi})(\boldsymbol{c}\cdot\boldsymbol{\xi}) + T_2(r)(\boldsymbol{\xi}\cdot\boldsymbol{a})(\boldsymbol{b}\cdot\boldsymbol{c}) \\ & + T_3(r)(\boldsymbol{b}\cdot\boldsymbol{\xi})(\boldsymbol{c}\cdot\boldsymbol{a}) + T_4(r)(\boldsymbol{\xi}\cdot\boldsymbol{c})(\boldsymbol{a}\cdot\boldsymbol{b}), \end{aligned} \tag{10.11}$$

or

$$T_{ijk}a_ib_jc_k = T_1\xi_i\xi_j\xi_ka_ib_jc_k + T_2\xi_ia_ib_jc_j + T_3\xi_jb_ja_ic_i + T_4\xi_kc_ka_ib_i. \tag{10.12}$$

Since (10.12) must be true for arbitrary $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$, $T_{ijk}$ must be of the form

$$T_{ijk} = T_1(r)\xi_i\xi_j\xi_k + T_2(r)\xi_i\delta_{jk} + T_3(r)\xi_j\delta_{ik} + T_4(r)\xi_k\delta_{ij}. \tag{10.13}$$

But we want to use $T_{ijk}$ as defined above, i.e.,

$$T_{ijk} = \overline{u_iu_ju'_k}, \tag{10.14}$$

so that it must be symmetric in $i$ and $j$. This requires that

$$T_2(r) = T_3(r),$$

and we have

$$T_{ijk} = T_1(r)\xi_i\xi_j\xi_k + T_2(r)\big(\xi_i\delta_{jk} + \xi_j\delta_{ik}\big) + T_4(r)\xi_k\delta_{ij}. \tag{10.15}$$

## 10.3 Solenoidal Isotropic Tensors

The types of quantities we are interested in here are tensors up to order three which are isotropic. The equation of continuity places the additional restriction on such quantities that, when they are related to velocities, they should be solenoidal in at least one index. We will next look briefly into the resulting properties of each of the three orders of tensors.

### *10.3.1 Isotropic Vectors $L_i$*

In order that $L_i$, an isotropic vector [see (10.10)], should be solenoidal, we must have

$$\frac{\partial L_i}{\partial \xi_i} = \frac{\partial}{\partial \xi_i}\big[L(r)\xi_i\big] = 0. \tag{10.16}$$

Now,

$$\frac{\partial}{\partial \xi_i}L(r) = L'(r)\frac{\partial r}{\partial \xi} = L'(r)\frac{\xi}{r}.$$

Then,

$$L(r)\delta_{ii} + L'(r)\frac{\xi_i}{r}\xi_i = 0, \tag{10.17}$$

or

$$3L(r) + rL'(r) = 0. \tag{10.18}$$

Equation (10.18) is equivalent to ($r \neq 0$)

$$\frac{\partial}{\partial r}(r^3 L) = 0, \tag{10.19}$$

which implies

$$r^3 L(r) = \text{constant}, \tag{10.20}$$

or

$$L(r) = \frac{\text{const.}}{r^3}. \tag{10.21}$$

In order that $L(r)$ be continuous as $r \to 0$, we require that the constant be zero, whence

$$r^3 L(r) = 0. \tag{10.22}$$

That is, there exists no non-vanishing, solenoidal isotropic vector.

### Isotropic Second Order Tensor $Q_{ij}$

We have written $Q_{ij}$ in terms of the two scalar functions $Q_1$ and $Q_2$. Yet we know that the transform $\Gamma_{ij}$ of $Q_{ij}$ has been expressed in terms of one scalar function, the energy spectrum. However, our derivation of $F(k)$ employed the equation of continuity in the form $\boldsymbol{u_k} \cdot \boldsymbol{k} = 0$. Therefore, we may expect that, if we now introduce the equation of continuity, we may reduce the number of scalar functions needed to specify $Q_{ij}$.

For the incompressible fluid, the continuity equation is

$$\operatorname{div} \boldsymbol{u} = \frac{\partial u_i}{\partial x_i} = 0. \tag{10.23}$$

If we operate div on $Q_{ij} = \overline{u_i u'_j}$ at $P$, $u'_j$ is not affected by the operation and we have

$$\frac{\partial Q_{ij}}{\partial x_i} = \frac{\partial \overline{u_i u'_j}}{\partial x_i} = 0. \tag{10.24}$$

Now,

$$\xi_i = x'_i - x_i, \tag{10.25}$$

so that

$$\frac{\partial}{\partial x_i} Q_{ij} = -\frac{\partial}{\partial \xi_i} Q_{ij}, \tag{10.26}$$

and

$$\frac{\partial}{\partial \xi_i} Q_{ij} = 0. \tag{10.27}$$

Since $Q_{ij}$ is symmetric in $i$ and $j$, it is solenoidal in both indices. From (10.27) will arise a relation between $Q_1$ and $Q_2$.

If we substitute (10.8) into (10.27), we obtain

$$\begin{aligned}
\frac{\partial}{\partial \xi_i}\Big[Q_1(r)\xi_i\xi_j + Q_2(r)\delta_{ij}\Big] &= Q'_1(r)r\xi_j + 3Q_1(r)\xi_j + Q_1(r)\xi_j + \frac{Q_2(r)}{r}\xi_j \\
&= \xi_j\left[Q'_1(r)r + 4Q_1(r) + \frac{Q_2(r)}{r}\right] \\
&= 0.
\end{aligned}$$

Or, since $\xi_j$ is not in general zero,

$$Q'_1 r + 4Q_1 + \frac{1}{r}Q_2 = 0. \tag{10.28}$$

This gives the relation between $Q_1$ and $Q_2$ on whose account we may conclude that $Q_{ij}$ can be expressed in terms of one defining scalar.

### 10.3.2 Further Manipulations

If we take $\partial T_{ijk}/\partial x_k$, we get an expression for a second order tensor, with symmetry in the indices. There result two equations, so that for a solenoidal isotropic tensor, only one independent defining scalar is required. We shall return to this matter presently. But first we elaborate on the nature of the second order tensor.

As we have seen, the isotropic tensors may be represented by a defining scalar, and so it is clear that the tensor equations governing isotropic turbulence may be transformed into scalar equations governing these defining scalars. The immediate problem is to pass to the scalar equations from the tensor equations. This requires a direct way of specifying the defining scalar. This may be done most directly for solenoidal tensors by introducing a 'tensor potential'. That is, one can express a tensor solenoidal in the index $j$ by the curl of a tensor with respect to that index. Clearly, this potential tensor must be skew-symmetric, as its curl must be an isotropic tensor. In the case of first order tensors, we have seen that there exist no non-vanishing isotropic vectors. We go then to second order tensors.

If $q_{ij}$ is a skew tensor, the isotropic, solenoidal tensor is

$$Q_{ij} = \operatorname{curl} q_{ij}, \tag{10.29}$$

where $q_{ij}$ must also be skew-isotropic. The requirement that that $q_{ij}a_ib_j$ be skew-invariant, where $a_i$ and $b_j$ are arbitrary unit vectors is fulfilled when

$$q_{ij}a_ib_j = Q(r)\epsilon_{ijk}a_ib_j\xi_k. \tag{10.30}$$

Here $Q(r)$ is some arbitrary function of $r$ and $\epsilon_{ijk}a_ib_j\xi_k$ is the only determinant from which $q_{ij}a_ib_j$ can be constructed, which is consistent with the bilinear form of $q_{ij}a_ib_j$. From the foregoing we may conclude that $q_{ij}$ is of the form

$$q_{ij} = Q(r)\epsilon_{ijk}\xi_k. \tag{10.31}$$

It follows from (10.29) that

$$Q_{ij} = \epsilon_{jlm}\frac{\partial q_{im}}{\partial \xi_l}. \tag{10.32}$$

$Q_{ij}$ is symmetric in $i$ and $j$, and so must be solenoidal in $i$ as well as $j$, although the curl is taken in $j$.

Combining (10.31) and (10.32), we find

$$\begin{aligned}
Q_{ij} &= \epsilon_{jlm}\frac{\partial}{\partial \xi_l}(Q\epsilon_{imk}\xi_k) \\
&= -\epsilon_{mjl}\epsilon_{mik}\frac{\partial}{\partial \xi_l}(Q\xi_k) \\
&= (\delta_{jk}\delta_{li} - \delta_{ji}\delta_{lk})\frac{\partial}{\partial \xi_l}(Q\xi_k) \\
&= \frac{\partial}{\partial \xi_i}(Q\xi_j) - \delta_{ij}\frac{\partial}{\partial \xi_k}(Q\xi_k)
\end{aligned}$$

$$= \frac{Q'}{r}\xi_i\xi_j + Q\delta_{ij} - 3Q\delta_{ij} - rQ'\delta_{ij}.$$

Thus

$$Q_{ij} = \frac{Q'}{r}\xi_i\xi_j - (rQ' + 2Q)\delta_{ij}, \tag{10.33}$$

is the most general form of a solenoidal isotropic tensor of second order and $Q$ is the defining scalar of $Q_{ij}$. If we let

$$Q_1 = \frac{Q'}{r} \tag{10.34}$$

and

$$Q_2 = -(rQ' + 2Q), \tag{10.35}$$

we see that (10.33) is identical with (10.8). Moreover, $Q_1$ and $Q_2$ as given in (10.34) and (10.35) are seen to satisfy (10.28), the condition imposed by the equation of continuity.

It may be further seen that the representation of $Q_{ij}$ in terms of $Q$ is unique. For, when $Q_{ij} = 0$, the coefficients in (10.8) vanish and $Q$ must vanish. Also, if $Q = 0$, $Q_{ij} = 0$. In other words,

$$Q = 0 \iff Q_{ij} \equiv 0. \tag{10.36}$$

We are now in a position to see how to pass from a tensor equation to an equation in the defining scalar. Consider, for example, the equation

$$\nabla^2 Q_{ij} = \frac{\partial Q_{ij}}{\partial t}. \tag{10.37}$$

Now

$$\nabla^2 Q_{ij} = \nabla^2\big(\mathrm{curl}_{(j)}\, q_{ij}\big) = \mathrm{curl}_{(j)}\,\big(\nabla^2 q_{ij}\big), \tag{10.38}$$

and our attention is diverted to $\nabla^2 q_{ij}$. We readily find this latter quantity to be given by

$$\begin{aligned}
\nabla^2 q_{ij} &= \frac{\partial^2}{\partial \xi_l^2}\big(Q\epsilon_{ijk}\xi_k\big)\\
&= \frac{\partial}{\partial \xi_l}\left(\frac{Q'}{r}\xi_l\epsilon_{ijk}\xi_k + Q\epsilon_{ijl}\right)\\
&= \left(\frac{Q'}{r}\right)' r\epsilon_{ijk}\xi_k + 3\frac{Q'}{r}\epsilon_{ijk}\xi_k + \frac{Q'}{r}\xi_l\epsilon_{ijl} + \frac{Q'}{r}\epsilon_{ijl}\xi_l\\
&= \left[\left(\frac{Q'}{r}\right)' r + 5\frac{Q'}{r}\right]\epsilon_{ijk}\xi_k.
\end{aligned}$$

That is,

$$\nabla^2 q_{ij} = \left(Q'' + 4\frac{Q'}{r}\right)\epsilon_{ijk}\xi_k. \tag{10.39}$$

Since $\nabla^2 q_{ij}$ is the skew-isotropic tensor from which we construct $\nabla^2 Q_{ij}$, the defining scalar of $\nabla^2 Q_{ij}$ is given by

$$Q''(r) + 4\frac{Q'(r)}{r},$$

where $Q(r)$ is the defining scalar of $Q_{ij}$. Then the tensor equation (10.37) passes over into

$$\frac{\partial^2 Q}{\partial r^2} + \frac{4}{r}\frac{\partial Q}{\partial r} = \frac{\partial Q}{\partial t}. \tag{10.40}$$

The left-hand side of (10.40) is the radial component of the 5-dimensional Laplacian. The strength of this method lies in the uniqueness of the relation between $Q$ and $Q_{ij}$.

### *10.3.3 The Isotropic Third Order Tensor, $T_{ijk}$*

We may now see how to find the defining scalar for $T_{ijk}$. Originally, we wrote $T_{ijk}$ in terms of 4 scalars [see (10.13)] but we will see how, for the special cases we are interested in, one scalar will suffice to express $T_{ijk}$. We set out as before to express $T_{ijk}$ as the curl of a skew-tensor. Here $T_{ijk}$ is solenoidal in $k$, and we write it

$$T_{ijk} = \epsilon_{klm}\frac{\partial}{\partial \xi_l} t_{ijm}, \tag{10.41}$$

where $t_{ijm}$ is skew-isotropic. In order that $t_{ijm}a_i b_j c_k$ be skew-invariant, we must express it in the appropriate combinations of determinants. (A product of two determinants can always be expressed in terms of scalar products.) The available determinants are $[\boldsymbol{abc}]$, $[\boldsymbol{ab\xi}]$, $[\boldsymbol{bc\xi}]$, and $[\boldsymbol{ca\xi}]$. Now, $t_{ijk}a_i b_j c_k$ must be trilinear, so that it will be a linear combination of $[\boldsymbol{abc}]$, $[\boldsymbol{ab\xi}](\boldsymbol{c}\cdot\boldsymbol{\xi})$, $[\boldsymbol{bc\xi}](\boldsymbol{a}\cdot\boldsymbol{\xi})$, and $[\boldsymbol{ac\xi}](\boldsymbol{b}\cdot\boldsymbol{\xi})$. In component form, these invariants are, respectively, to be expressed in terms of the tensors $\epsilon_{ijk}$, $\xi_k\epsilon_{ijk}\xi_l$, $\xi_i\epsilon_{jkl}\xi_l$, and $\xi_i\epsilon_{kil}\xi_l$. In other words, $T_{ijk}$ must be a linear combination of these tensors. This again would imply four fundamental scalars, if the four tensors were independent. However, these tensors are not independent, as we shall now see.

Consider the fourth order tensor

$$\epsilon_{ijk}\xi_l.$$

Among the results of permuting the indices, only 4 possible permutations could possibly be independent because of the properties of $\epsilon_{ijk}$. These four are the ones (any of many) with the factors $\xi_i$, $\xi_j$, $\xi_k$, or $\xi_l$. Since only three possible values of $\xi_i$ exist, it seems that even these four may not be independent. We then ask whether it is possible to write an expression of the form

$$a\epsilon_{ijk}\xi_l + b\epsilon_{ljk}\xi_i + c\epsilon_{ilk}\xi_j + d\epsilon_{ijl}\xi_l = 0, \tag{10.42}$$

that is, whether we may discover appropriate $a, b, c, d$ such that (10.42) holds. We may distinguish several different cases:

- $i = j = k$. In this case, (10.42) is identically true for all $a, b, c, d$.
- Two of $i, j, k$ equal, and only two equal, e.g., $i = j \neq k$. Two possible subcases arise:

  1. $l = i = j$, which satisfies (10.42) for all $a, b, c, d$.
  2. $l \neq i = j$, which admits the two possible subcases: (i) $l = k$ which is satisfied by all $a, b, c, d$; (ii) $l \neq k$ which is satisfied only if $b = c$.

  Similarly one finds, by considering the cases $i = k$ and $j = k$, that $b = d$ and $c = d$. Thus (10.42) can be true only if $b = c = d$.
- $i, j, k$ all different. Then $l$ must be the same as one of them. We will consider $l = i$. Then

$$a\epsilon_{ijk}\xi_l + b\epsilon_{ljk}\xi_i = 0.$$

Since $i = l$, $\xi_l = \xi_i$ and, in a loose way of writing,

$$a\epsilon_{ijk} + b\epsilon_{ijk} = 0,$$

which requires $a = -b$. It follows that, if

$$a = -b = -c = -d,$$

equation (10.42) is true. In particular, we may choose $a = 1$ and write

$$\begin{aligned} \epsilon_{ijk}\xi_l &= \epsilon_{ljk}\xi_i + \epsilon_{ilk}\xi_j + \epsilon_{ijl}\xi_k \\ &= \xi_i\epsilon_{jkl} + \xi_j\epsilon_{kil} + \xi_k\epsilon_{ijl}. \end{aligned} \tag{10.43}$$

If we multiply (10.43) by $\xi_l$, we find

$$r^2\epsilon_{ijk} = \xi_i\epsilon_{jkl}\xi_l + \xi_j\epsilon_{kil}\xi_l + \xi_k\epsilon_{ijl}\xi_l. \tag{10.44}$$

Hence, $t_{ijk}$ can depend on at most three independent scalars.

Another restriction may be placed on $t_{ijk}$ to make it gauge invariant. This latter one is essentially a matter of convention. From (10.44), we see that the form of $t_{ijk}$ will be a linear function of tensors of the sort

$$Q(r) = \epsilon_{ijk}\xi_k\xi_l$$

and, as matters stand, $T_{ijk}$ will be unchanged if a gradient is added to this tensor. Such gradients will have the appropriate form

$$\frac{\partial}{\partial \xi_k}\left(Q\epsilon_{ijl}\xi_l\right) = \frac{Q'}{r}\xi_k\epsilon_{ijl}\xi_l + Q\epsilon_{ijk}. \tag{10.45}$$

In other words, $T_{ijk}$ is unchanged if (10.45) is added to $t_{ijk}$. However, the form of the defining scalar will be affected and so we must establish some condition to make $t_{ijk}$ gauge invariant. The right-hand side of (10.45) is a linear combination of tensors of the forms $\epsilon_{ijk}$ and $\xi_k\epsilon_{ijl}\xi_l$, and rearrangement of (10.45) shows that they differ by the gradient of a tensor. Thus $T_{ijk}$ is equivalently represented by

$$-\frac{Q'}{r}\epsilon_{ijl}\xi_l\xi_k \quad \text{and} \quad Q\epsilon_{ijk},$$

and we will take them equal. That is,

$$Q\epsilon_{ijk} + \frac{Q'}{r}\xi_k\epsilon_{ijl}\xi_l = 0. \tag{10.46}$$

So, with the relations (10.46) and (10.44), we find that only two of $\epsilon_{ijk}$, $\xi_k\epsilon_{ijl}\xi_l$, $\xi_i\epsilon_{jkl}\xi_l$, and $\xi_j\epsilon_{kil}\xi_l$ are independent, and we may completely specify $t_{ijk}$ in terms of any two of them. We choose to employ $\xi_j\epsilon_{ikm}\xi_m$ and $\xi_i\epsilon_{jkm}\xi_m$. Then,

$$t_{ijk} = T_1\xi_i\epsilon_{jkm}\xi_m + T_2\xi_j\epsilon_{ikm}\xi_m. \tag{10.47}$$

Now $T_{ijk} = \overline{u_i u_j u_k'}$ is symmetric in $i$ and $j$, and so $t_{ijk}$ must be. This condition implies

$$T_1 = T_2 = T \quad \text{(say)}, \tag{10.48}$$

and

$$t_{ijk} = T\big(\xi_i\epsilon_{jkm}\xi_m + \xi_j\epsilon_{ikm}\xi_m\big), \tag{10.49}$$

where $T = T(r)$ is said to be the defining scalar of $T_{ijk}$. To find $T_{ijk}$, let us take the curl of the first term on the right of (10.49). We have

$$\begin{aligned}
\epsilon_{klm}\frac{\partial}{\partial\xi_l}\big(T\xi_i\epsilon_{jms}\xi_s\big) &= \epsilon_{klm}\epsilon_{jms}\frac{\partial}{\partial\xi_l}\big(T\xi_i\xi_s\big)\\
&= (\delta_{ks}\delta_{lj} - \delta_{kj}\delta_{ls})\frac{\partial}{\partial\xi_l}\big(T\xi_i\xi_s\big)\\
&= \frac{\partial}{\partial\xi_j}\big(T\xi_i\xi_k\big) - \delta_{kj}\frac{\partial}{\partial\xi_l}\big(T\xi_i\xi_l\big)\\
&= \frac{T'}{r}\xi_j\xi_i\xi_k + T\delta_{ij}\xi_k + T\delta_{jk}\xi_i - \frac{T'}{r}r^2\delta_{jk}\xi_i\\
&\quad - 3T\delta_{kj}\xi_i - T\xi_i\delta_{kj}\\
&= \frac{T'}{r}\xi_i\xi_k\xi_j - (3T + T'r)\xi_i\delta_{jk} + T\delta_{ij}\xi_k.
\end{aligned}$$

The second term from (10.49) is obtained by interchanging $i$ and $j$. The result is

$$T_{ijk} = 2\frac{T'(r)}{r}\xi_i\xi_j\xi_k - \big[T'(r)r + 3T(r)\big](\xi_i\delta_{jk} + \xi_j\delta_{ik}) + 2T\delta_{ij}\xi_k. \tag{10.50}$$

If $T_{ijk} \equiv 0$, then $T'(r)/r = 0$ and $T' = 0$. Thus,

$$rT'(r) + 3T(r) = 0 \quad \text{and} \quad 2T = 0,$$

whence we see that $T = 0$. Moreover, if $T = 0$, then $T_{ijk} \equiv 0$. That is,

$$T = 0 \quad \Longleftrightarrow \quad T_{ijk} \equiv 0,$$

and the representation of $T_{ijk}$ by the defining scalar $T(r)$ is unique.

*Uniqueness Proof* Let two tensors $U_{ijk\ldots}$ and $V_{ijk\ldots}$ be given by

$$U_{ijk\ldots} = \epsilon_{kmn}\frac{\partial}{\partial x_m}q_1 t_{nij\ldots} \tag{10.51}$$

and

$$V_{ijk\ldots} = \epsilon_{kmn} \frac{\partial}{\partial x_m} q_2 t_{nij\ldots}. \tag{10.52}$$

Then

$$U_{ijk\ldots} - V_{ijk\ldots} = \epsilon_{kmn} \frac{\partial}{\partial x_m} (q_1 - q_2) t_{nij\ldots}, \tag{10.53}$$

so that the defining scalar of

$$D_{ijk\ldots} = U_{ijk\ldots} - V_{ijk\ldots}$$

is $q_1 - q_2$. Then

$$q_1 - q_2 = 0 \quad \Longleftrightarrow \quad U_{ijk\ldots} - V_{ijk\ldots} \equiv 0$$

so that $U_{ijk\ldots} \equiv V_{ijk\ldots}$. We may suppose $q_1 = q_2 = q$ (say). But $U_{ijk\ldots}$ and $V_{ijk\ldots}$ cannot have the same defining scalar and be different. Moreover, if $U_{ijk\ldots} \equiv V_{ijk\ldots}$, then $q_1 = q_2$. Finally, $q_1 = 0 \Longleftrightarrow U_{ijk\ldots} \equiv 0$, and so if $V_{ijk\ldots}$ and $q_2$ are chosen $\equiv 0$ and $= 0$, respectively, the conditions on (10.51) and (10.53) become identical. □

As an example of the passage to scalar equations from third order tensor equations, we consider

$$\nabla^2 T_{ijk} = \frac{\partial T_{ijk}}{\partial t}. \tag{10.54}$$

The defining scalar of $\partial T_{ijk}/\partial t$ is $\partial T/\partial t$. The defining scalar of $\nabla^2 T_{ijk}$ is obtained by finding $\nabla^2 t_{ijk}$ (as in the case of $\nabla^2 Q_{ij}$). We consider just one term in $t_{ijk}$ and get the rest of $\nabla^2 t_{ijk}$ by making use of the symmetry:

$$\begin{aligned}
\nabla^2\big(T\xi_i\epsilon_{jkl}\xi_l\big) &= \frac{\partial^2}{\partial \xi_m^2}(T\xi_i\epsilon_{jkl}\xi_l) \\
&= \frac{\partial}{\partial \xi_m}\left(\frac{T'}{r}\xi_m\xi_i\epsilon_{jkl}\xi_l + T\delta_{im}\epsilon_{jkl}\xi_l + T\xi_i\epsilon_{jkm}\right) \\
&= \left(\frac{T'}{r}\right)' r\xi_i\epsilon_{jkl}\xi_l + 3\frac{T'}{r}\xi_i\epsilon_{jkl}\xi_l + \frac{T'}{r}\xi_i\epsilon_{jkl}\xi_l + \frac{T'}{r}\xi_i\epsilon_{jkm}\xi_m \\
&\quad + \frac{T'}{r}\xi_i\epsilon_{jkl}\xi_l + T\epsilon_{jki} + \frac{T'}{r}\xi_i\epsilon_{jkm}\xi_m + T\epsilon_{jki} \\
&= T''\xi_i\epsilon_{jkl}\xi_l + 6\frac{T'}{r}\xi_i\epsilon_{jkl}\xi_l + 2T\epsilon_{jki} \\
&= \left(T'' + 6\frac{T'}{r}\right)\xi_i\epsilon_{jkl}\xi_l + 2T\epsilon_{jki}.
\end{aligned}$$

Then,

$$\nabla^2\big(\xi_j\epsilon_{ikm}\xi_m\big) = \left(T'' + 6\frac{T'}{r}\right)\xi_j\epsilon_{ikl}\xi_l + 2T\epsilon_{ikj},$$

and, on adding, we find

$$\nabla^2 t_{ijk} = \left( T'' + 6\frac{T'}{r} \right) \left( \xi_i \epsilon_{jkl} \xi_l + \xi_j \epsilon_{ikl} \xi_l \right). \tag{10.55}$$

Thus, the defining scalar of $\nabla^2 T_{ijk}$ is $T'' + 6T'/r$ and this must be the same as the defining scalar for $\partial Q_{ij}/\partial t$. Then (10.54) passes over into

$$T'' + \frac{6}{r} T' = \frac{\partial T}{\partial t}. \tag{10.56}$$

The left-hand side of (10.56) is the radial part of the 6-dimensional wave equation.

We will also be interested in the contracted tensor

$$\frac{\partial T_{ikj}}{\partial \xi_k}. \tag{10.57}$$

Since this tensor is isotropic of second order, it must be symmetric in $i$ and $j$. We can get its defining scalar by performing a similar contraction on $t_{ikj}$. Thus

$$\begin{aligned} \frac{\partial t_{ikj}}{\partial \xi_k} &= \frac{\partial}{\partial \xi_k} \left( T \xi_i \epsilon_{kjl} \xi_l + T \xi_k \epsilon_{ijl} \xi_l \right) \\ &= \frac{T'}{r} \xi_k \xi_i \epsilon_{kjl} \xi_l + T \epsilon_{ijl} \xi_l + T' r \epsilon_{ijl} \xi_l + 3T \epsilon_{ijl} \xi_l + T \xi_k \epsilon_{ijk}. \end{aligned}$$

Now,

$$\xi_k \xi_l \epsilon_{kjl} = -\epsilon_{jkl} \xi_k \xi_l = (\xi \times \xi)_j = 0.$$

Hence,

$$\frac{\partial t_{ikj}}{\partial \xi_k} = (rT' + 5T) \epsilon_{ijl} \xi_l,$$

and the defining scalar of (10.57) is

$$rT' + 5T. \tag{10.58}$$

# Chapter 11
# The Karman-Howarth Equations

We may now proceed to the (statistical) dynamical aspects of isotropic turbulence by introducing the equations of motion:

$$\frac{\partial u_i}{\partial t} + \frac{\partial}{\partial x_k}(u_i u_k) = -\frac{\partial \varpi}{\partial x_i} + \nu \nabla^2 u_i. \tag{11.1}$$

If we multiply (11.1) by $u'_j$ and average, we have

$$\overline{u'_j \frac{\partial u_i}{\partial t}} + \frac{\partial}{\partial x_k}\overline{u_i u_k u'_j} = \nu \nabla^2 \overline{u_i u'_j}, \tag{11.2}$$

on observing that $\overline{u_j \varpi}$ is a solenoidal isotropic vector and must vanish. Since

$$\xi_i = x'_i - x_i, \tag{11.3}$$

we have

$$\frac{\partial}{\partial x_i} = -\frac{\partial}{\partial \xi_i}, \qquad \frac{\partial}{\partial x'_i} = \frac{\partial}{\partial \xi_i}. \tag{11.4}$$

Then (11.2) becomes

$$\overline{u'_j \frac{\partial u_i}{\partial t}} - \frac{\partial}{\partial \xi_k} T_{ijk} = \nu \nabla^2 Q_{ij}. \tag{11.5}$$

We may also express (11.1) in $u'_j$. It is

$$\frac{\partial u'_j}{\partial t} + \frac{\partial}{\partial x'_k} u'_j u'_k = -\frac{\partial \varpi'}{\partial x'_i} + \nu \nabla^2 u'_j. \tag{11.6}$$

By multiplying (11.6) by $u_i$, averaging and introducing (11.5), we obtain

$$\overline{u_i \frac{\partial u'_j}{\partial t}} + \frac{\partial}{\partial \xi_k}\overline{u_i u'_j u'_k} = \nu \nabla^2 \overline{u_i u'_j}. \tag{11.7}$$

Recall from (10.50) that

$$T_{ijk} = 2\frac{T'(r)}{r}\xi_i \xi_j \xi_k - \left(T'r + 3T\right)\left(\xi_i \delta_{jk} + \xi_j \delta_{ik}\right) + 2T\delta_{ij}\xi_k.$$

E.A. Spiegel (ed.), *The Theory of Turbulence*, Lecture Notes in Physics 810,
DOI 10.1007/978-94-007-0117-5_11, 

This expression is odd in $\xi_i$. Thus the correlation with $\mathbf{r}$ as origin with the point at $\mathbf{r}+\boldsymbol{\xi}$ is the negative of the correlation between these two points when $\mathbf{r}+\boldsymbol{\xi}$ is taken as origin. Therefore,

$$\overline{u_i u_j u_k'} = -\overline{u_i' u_j' u_k} \tag{11.8}$$

and

$$T_{jki} = -\overline{u_j' u_k' u_i}. \tag{11.9}$$

Then with (11.9), equation (11.7) becomes

$$\overline{u_i \frac{\partial u_j'}{\partial t}} + \frac{\partial}{\partial \xi_k} T_{jki} = \nu \nabla^2 Q_{ij}. \tag{11.10}$$

Adding (11.5) and (11.10), we have

$$\frac{\partial}{\partial t} Q_{ij} = \frac{\partial}{\partial \xi_k}(T_{jki} + T_{ikj}) + 2\nu \nabla^2 Q_{ij}.$$

Since $\partial T_{jki}/\partial \xi_k$ is a second order isotropic tensor, it must be symmetric in $i$ and $j$:

$$\frac{\partial Q_{ij}}{\partial t} = 2\frac{\partial T_{ikj}}{\partial \xi_k} + 2\nu \nabla^2 Q_{ij}. \tag{11.11}$$

By introducing the various defining scalars in these quantities, we find

$$\frac{\partial Q}{\partial t} = 2\left(r\frac{\partial}{\partial r} + 5\right) T + 2\nu \left(\frac{\partial^2}{\partial r^2} + \frac{4}{r}\frac{\partial}{\partial r}\right) Q, \tag{11.12}$$

which is the well-known Karman–Howarth equation. We notice that the inertial term gives rise to the appearance of the triple correlation. If the velocity distributions were normal, the means of odd powers of the velocity would vanish. However, not only does the triple correlation not vanish, but it is associated with the inertial term. We may conclude that non-normality is fundamental to the turbulence phenomenon.

# Chapter 12
# The Meanings of the Defining Scalars

Consider the longitudinal correlation $\overline{u_{\parallel} u'_{\parallel}}$ (see Fig. 12.1 left). In terms of (10.33), with $i = j$,

$$\begin{aligned} \overline{u_{\parallel} u'_{\parallel}} &= rQ' - (2Q + rQ') \\ &= -2Q(r) \\ &= f(r) \quad \text{in the older literature.} \end{aligned} \tag{12.1}$$

The lateral correlation $\overline{u_{\perp} u'_{\perp}}$ may also be found from (10.33) (see Fig. 12.1 right):

$$\begin{aligned} \overline{u_{\perp} u'_{\perp}} &= -(rQ' + 2Q) \\ &= g(r). \end{aligned} \tag{12.2}$$

Accordingly,

$$g(r) = f(r) + \frac{1}{2} r \frac{\partial f(r)}{\partial r}. \tag{12.3}$$

Equation (12.3) has been verified experimentally, thus strengthening confidence in the concept of isotropy and in the use of the equation of continuity.

Now, $Q(r)$ is an even function of $r$. This follows from the fact that

$$Q_{ij}(-\boldsymbol{r}) = Q_{ji}(\boldsymbol{r}) = Q_{ij}(r)$$

and (10.33). Hence we may write

$$Q(r) = Q(0) + Q_2 r^2 + \cdots. \tag{12.4}$$

Moreover, if we let $r \to 0$ in (12.1) and (12.2), we find

$$\overline{u_{\parallel}^2} = -2Q(0) = f(0), \tag{12.5}$$

$$\overline{u_{\perp}^2} = -2Q(0) = g(0), \tag{12.6}$$

so that

$$\overline{u_{\parallel}^2} = \overline{u_{\perp}^2} \tag{12.7}$$

E.A. Spiegel (ed.), *The Theory of Turbulence*, Lecture Notes in Physics 810,
DOI 10.1007/978-94-007-0117-5_12, © Springer Science+Business Media B.V. 2011

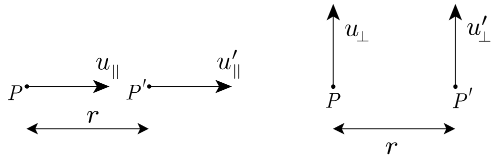


**Fig. 12.1** Longitudinal and Lateral correlations

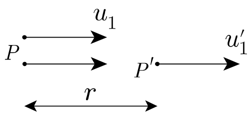


**Fig. 12.2** Longitudinal triple correlation

and

$$Q(0) = -\frac{1}{6}\overline{u^2}. \tag{12.8}$$

If there is a mean flow, we should consider $\overline{(u_\parallel - u'_\parallel)^2}$ instead of $\overline{u_\parallel^2}$. Now

$$\begin{aligned}\overline{(u_\parallel - u'_\parallel)^2} &= \overline{u_\parallel^2} + \overline{u_\parallel'^2} - 2\overline{u_\parallel u'_\parallel} \\ &= -2Q(0) - 2Q(0) + 4Q(r) \\ &= 4\big[Q(r) - Q(0)\big]. \end{aligned} \tag{12.9}$$

Next consider the longitudinal triple correlation $\overline{u_1^2 u'_1}$ (see Fig. 12.2). By (10.50),

$$\begin{aligned}\overline{u_1^2 u'_1} &= 2r^2 T' - 2(rT' + 3T)r + 2rT \\ &= -4rT. \end{aligned} \tag{12.10}$$

Now we have seen that $T_{ijk}$ is odd in $r$, so that, according to (12.10), $T(r)$ must be even. Then we may express $T(r)$ as a power series:

$$T(r) = T_0 + r^2 T_2 + \cdots. \tag{12.11}$$

An alternate means of expression for $r$ small would be to expand $u'$ in a series about $P$:

$$\begin{aligned}\overline{u_1^2 u'_1} &= \overline{u_1^2\left(u_1 + r\frac{\partial u_1}{\partial r} + r^2\frac{\partial^2 u_1}{\partial r^2} + \cdots\right)} \\ &= \overline{u_1^3} + \frac{1}{3}r\frac{\partial}{\partial r}\overline{u_1^3} + r^2\overline{u_1^2\frac{\partial^2 u_1}{\partial r^2}} + \cdots. \end{aligned} \tag{12.12}$$

If there is no mean motion, $\overline{u_1^3} = 0$ and the first term in the series seems to be

$$r^2\overline{u_1^2\frac{\partial^2 u_1}{\partial r^2}}.$$

However, we note that

$$\frac{\partial}{\partial r}\overline{u_1^2\frac{\partial u_1}{\partial r}} = \overline{u_1^2\frac{\partial^2 u_1}{\partial r^2}} + \overline{2u_1\left(\frac{\partial u_1}{\partial r}\right)^2}. \tag{12.13}$$

The term on the left is odd in $\partial u_1/\partial r$ and the second term on the right is odd to the first power in $u_1$. We may conclude that

$$\overline{u_1^2 \frac{\partial^2 u_1}{\partial r^2}} = 0.$$

It follows that the series (12.12) commences with the $r^3$ term. We may also combine (12.10) with (12.11):

$$\begin{aligned} \overline{u_1^2 u_1'} &= -4rT(r) \\ &= -4\big(rT_0 + r^3 T_2 + \cdots\big). \end{aligned} \tag{12.14}$$

Now (12.14) must begin with a term in $r^3$ because, as we have just seen, (12.12) does. Thus, we must have

$$T_0 = 0, \tag{12.15}$$

whence (12.11) becomes

$$T(r) = r^2 T_2 + \cdots, \tag{12.16}$$

or

$$T(r) = r^2 T_2 + O(r^4) \cdots. \tag{12.17}$$

If there is a mean motion, we might consider

$$\begin{aligned} \overline{(u_1 - u_1')^3} &= \overline{u_1^3} - 3\overline{u_1^2 u_1'} + 3\overline{u_1 u_1'^2} - \overline{u_1'^3} \\ &= 3\left(\overline{u_1 u_1'^2} - \overline{u_1^2 u_1'}\right) \\ &= -6\overline{u_1^2 u_1'} = 24rT. \end{aligned} \tag{12.18}$$

# Chapter 13
# Some Results from the Karman–Howarth Equation

## 13.1 The Taylor Microscale

To second order, according to (12.4) and (12.16), we have

$$Q = Q_0 + Q_2 r^2 \tag{13.1}$$

and

$$T = r^2 T_2. \tag{13.2}$$

We may substitute (13.1) and (13.2) into the Karman–Howarth equation:

$$\frac{\partial Q_0}{\partial t} + r^2 \frac{\partial Q_2}{\partial t} = 4r^2 T_2 + 10r^2 T_2 + 16\nu Q_2 + 4\nu Q_2.$$

This may be rearranged to give

$$\left(\frac{\partial Q_0}{\partial t} - 20\nu Q_2\right) + \left(\frac{\partial Q_2}{\partial t} - 14T_2\right) r^2 = 0. \tag{13.3}$$

The coefficients must vanish, and we have

$$\frac{\partial Q_0}{\partial t} = 20\nu Q_2. \tag{13.4}$$

If we introduce (12.8), we find

$$\frac{\partial Q_0}{\partial t} = -\frac{1}{6}\frac{\mathrm{d}}{\mathrm{d}t}\overline{u^2}, \tag{13.5}$$

$$\frac{\partial Q_0}{\partial t} = \frac{1}{3}\varepsilon = 20\nu Q_2, \tag{13.6}$$

where $\varepsilon$ is the rate of dissipation. Thus, it is clear why no constant term arose in the series for $T(r)$: the constant term gives rise to dissipation, while $T$ arises in the inertial term.

We may then write

$$Q(r) = Q_0\left(1 + \frac{Q_2}{Q_0} r^2 + \cdots\right). \tag{13.7}$$

E.A. Spiegel (ed.), *The Theory of Turbulence*, Lecture Notes in Physics 810,
DOI 10.1007/978-94-007-0117-5_13, 

Now a correlation coefficient should equal 1 at $r = 0$, i.e., the correlation should go as

$$1 - \frac{r^2}{\lambda^2} + \cdots,$$

where $\lambda$ is the radius of curvature near the origin. $\lambda$ is known as the microscale (Taylor's microscale) of turbulence.

From the foregoing, it is clear that $Q_0/Q_2$ has the dimensions (length)$^2$. We may readily see how the term microscale arises for $\lambda$. From (12.1) and (13.5), we have

$$\frac{\partial f_0}{\partial t} = 20\nu f_2, \tag{13.8}$$

$$\frac{\mathrm{d}\overline{u_1^2}}{\mathrm{d}t} = 20\nu f_2 \left( \frac{\overline{u_1^2}}{f_0} \right) = 20\nu \overline{u_1^2} \frac{f_2}{f_0}, \tag{13.9}$$

and

$$\frac{\mathrm{d}\overline{u_1^2}}{\mathrm{d}t} = -\frac{20\nu \overline{u_1^2}}{\lambda^2}. \tag{13.10}$$

Multiplying by 3/2 and recalling that $\overline{u^2} = 3\overline{u_1^2}$, we see that (13.5) and (13.6) imply that

$$\varepsilon = -\frac{15\nu \overline{u_1^2}}{\lambda^2}. \tag{13.11}$$

Taylor verified (13.11) experimentally and found $\lambda$. Now $\lambda$ is the size of the eddy which causes the dissipation so it must represent the smallest eddies in the turbulence. Thus it may also be concluded that viscosity causes dissipation and that Stokes was correct on this point.

## 13.2 The Study of the Decay of Turbulence

In the stage of decay, the inertial term vanishes and the Karman–Howarth equation reduces to the equation of diffusion:

$$\frac{\partial Q}{\partial t} = 2\nu \left( \frac{\partial^2 Q}{\partial r^2} + \frac{4}{r} \frac{\partial Q}{\partial r} \right). \tag{13.12}$$

Equation (13.12) admits the asymptotic solution

$$Q \sim \frac{\mathrm{e}^{-r^2/8\nu t}}{(\nu t)^{5/2}}. \tag{13.13}$$

Because of the neglect of $T$, this is a normal function. The dissipating effect of viscosity is again made clear. This solution (13.13) is the sort of thing that happens when the viscosity dominates, i.e., when the Reynolds number is low. One then has no effects from the inertial term, and if a random velocity is introduced, this sort of laminar (noninteracting) turbulence can occur.

But as far as our actual knowledge of turbulence as manifested through the inertial term is concerned, these considerations have led only to the conclusion that the concept of isotropy is a good one (at least as far as we have gone with the series for $f$ and $g$).

## 13.3 The Connection Between the Karman–Howarth Equation and the Kolmogorov Theory

### *13.3.1 The Double Correlation*

We saw that the fundamental length in Kolmogorov theory [see (8.3) on p. 30] is

$$\eta = \left(\frac{\nu^3}{\varepsilon}\right)^{1/4}, \tag{13.14}$$

and that

$$[\text{velocity}] = (\nu\varepsilon)^{1/4}. \tag{13.15}$$

Then if $f$ is a universal function,

$$\overline{(u_\parallel - u'_\parallel)^2} = (\nu\varepsilon)^{1/2} f\left[r\left(\frac{\varepsilon}{\nu^3}\right)^{1/4}\right]. \tag{13.16}$$

As $R \to \infty$, $\overline{(u_\parallel - u'_\parallel)^2}$ should become independent of $\nu$. Thus, $f$ must be of the form $c[r(\varepsilon/\nu^3)^{1/4}]^{2/3}$, where $c$ is a constant. Then, if $\nu \to 0$,

$$\overline{(u_\parallel - u'_\parallel)^2} = c(\nu\varepsilon)^{1/2}\left[r\left(\frac{\varepsilon}{\nu^3}\right)^{1/4}\right]^{2/3} = c(r\varepsilon)^{2/3}. \tag{13.17}$$

It is clear that (13.17) is nearly true only when $r$ is larger than $\lambda$, the Taylor microscale. Of course, as $\nu \to 0$, we have $\lambda \to 0$.

Now we have by (12.9)

$$\overline{(u_\parallel - u'_\parallel)^2} = 4\big[Q(r) - Q(0)\big], \tag{13.18}$$

so that, for $\nu \to 0$, $r > \lambda$, we have

$$Q(r) = Q(0) - cr^{2/3}, \tag{13.19}$$

where $\varepsilon$ has been absorbed into $c$. Alternatively, if $c$ is appropriately modified,

$$Q(r) = Q(0)\big(1 - cr^{2/3}\big). \tag{13.20}$$

Moreover we see that we must have

$$0 \le \left|1 - cr^{2/3}\right| \le 1, \tag{13.21}$$

so that it is reasonable to take

$$c = r_0^{-2/3},$$

where $r_0$ is the size of the largest eddies. Then,

$$Q(r) = Q(0)\left[1 - \left(\frac{r}{r_0}\right)^{2/3}\right]. \tag{13.22}$$

Now in the Kolmogorov theory (see Sect. 8.2)

$$v_k \sim k^{-1/3},$$

so

$$v \sim r^{1/3},$$

and

$$v^2 \sim r^{2/3}.$$

This seems consistent with (13.22).

### 13.3.2 The Triple Correlation

Here we expect $\overline{(u_1 - u_1')^3}$ or $\overline{u_1^2 u_1'}$ to be of the form

$$(\nu\varepsilon)^{3/4} f\left[r\left(\frac{\varepsilon}{\nu^3}\right)^{1/4}\right].$$

As $\nu \to 0$,

$$\overline{(u_1 - u_1')^3} \to (\nu\varepsilon)^{3/4} c\left[r\left(\frac{\varepsilon}{\nu^3}\right)^{1/4}\right] = c\varepsilon r. \tag{13.23}$$

With these considerations regarding the triple correlations, we may now enquire what the Karman–Howarth equations imply in the case of stationary turbulence. The first impulse might be to set $\partial Q/\partial t = 0$. But this does not allow the energy to be introduced into the equations. Instead, one notes that

$$\frac{\partial Q}{\partial t} = \frac{\partial(Q - Q_0)}{\partial t} + \frac{\partial Q_0}{\partial t}. \tag{13.24}$$

Then we assert, for stationary turbulence,

$$\frac{\partial(Q - Q_0)}{\partial t} = 0. \tag{13.25}$$

Also, according to (13.6),

$$\frac{\partial Q_0}{\partial t} = \frac{1}{3}\varepsilon,$$

so that

$$\frac{\partial Q}{\partial t} = \frac{1}{3}\varepsilon \quad \text{(stationary turbulence)}. \tag{13.26}$$

Then the Karman–Howarth equation (11.12) becomes

$$\frac{1}{3}\varepsilon = 2\left(r\frac{\partial T}{\partial r} + 5T\right) + 2\nu\left(\frac{\partial^2 Q}{\partial r^2} + \frac{4}{r}\frac{\partial Q}{\partial r}\right). \tag{13.27}$$

Upon multiplication of (13.27) by $r^4$, we find

$$r^4\frac{\varepsilon}{3} = 2\frac{\partial}{\partial r}(r^5 T) + 2\nu\frac{\partial}{\partial r}\left[r^4\frac{\mathrm{d}(Q-Q_0)}{\mathrm{d}r}\right]. \tag{13.28}$$

The operator $\partial/\partial r$ is equivalent to $\mathrm{d}/\mathrm{d}r$ since all quantities depend upon $r$ alone. Hence integration of (13.28) is a straightforward matter. We have

$$\frac{\varepsilon}{15}r^5 = 2r^5 T + 2\nu r^4\frac{\mathrm{d}}{\mathrm{d}r}(Q-Q_0). \tag{13.29}$$

We can replace $T$ and $Q-Q_0$ according to (12.18) and (12.9):

$$\frac{\varepsilon}{30}r = \frac{1}{24}\overline{(u_\parallel - u'_\parallel)^3} + \frac{\nu}{4}\frac{\mathrm{d}}{\mathrm{d}r}\overline{(u_\parallel - u'_\parallel)^2}. \tag{13.30}$$

We have seen that $Q-Q_0 \sim r^{2/3}$. Hence,

$$\frac{\mathrm{d}}{\mathrm{d}r}(Q-Q_0) \sim \frac{2}{3}r^{-1/3},$$

which tends to zero as $r\to\infty$. Hence,

$$\overline{(u_\parallel - u'_\parallel)^3} = \frac{4}{5}\varepsilon r, \tag{13.31}$$

which agrees with the result from Kolmogorov theory and indicates the value of the coefficient.

# Chapter 14
# The Relation Between the Fourth and Second Order Correlations When the Velocity Follows a Gaussian Distribution

Stewart has measured the quantity

$$\frac{\overline{(u'-u)^4}}{\overline{\overline{(u'-u)^2}}^{2}}, \tag{14.1}$$

and found the value 2.9. The value for a Gaussian distribution is 3.0. This result suggests that the fourth order moments of the velocities at two points are related to the second order moments in the same way as for a normal distribution, except for small values of $r$. Under these circumstances,

$$\overline{u_i u_j u'_l u'_m} = \overline{u_i u_j}\,\overline{u'_l u'_m} + \overline{u_i u'_l}\,\overline{u_j u'_m} + \overline{u_i u'_m}\,\overline{u_j u'_l}. \tag{14.2}$$

We now develop this relation.

## 14.1 Some Properties of the Gaussian Distribution

### *14.1.1 One-Dimensional Gaussian Distribution*

For a one-dimensional distribution,

$$W(y) = \frac{1}{\sqrt{2\pi}\sigma} \mathrm{e}^{-(y-a)^2/2\sigma^2}, \tag{14.3}$$

the mean occurs at $y = a$. If we measure $y$ from its mean,

$$W(y) = \frac{1}{\sqrt{2\pi}\sigma} \mathrm{e}^{-y^2/2\sigma^2}. \tag{14.4}$$

Now,

$$\int_{-\infty}^{+\infty} W(y)\,\mathrm{d}y = \frac{2}{\sigma\sqrt{2\pi}} \int_0^{\infty} \mathrm{e}^{-y^2/2\sigma^2}\,\mathrm{d}y,$$

and on setting $x = y/\sqrt{2}\sigma$, we obtain

E.A. Spiegel (ed.), *The Theory of Turbulence*, Lecture Notes in Physics 810,
DOI 10.1007/978-94-007-0117-5_14, 

$$\begin{aligned}\int_{-\infty}^{+\infty} W(y)\,\mathrm{d}y &= \frac{2}{\sigma\sqrt{2\pi}}\sigma\sqrt{2}\int_0^\infty \mathrm{e}^{-x^2}\,\mathrm{d}x\\ &= \frac{2^{3/2}}{2^{1/2}\pi^{1/2}}\left(\frac{1}{2}\sqrt{\pi}\right)\\ &= 1.\end{aligned}$$

So $W$ is normalized. Further

$$\begin{aligned}\overline{y^2} &= \frac{2}{\sigma\sqrt{2\pi}}\int_0^\infty \mathrm{e}^{-y^2/2\sigma^2}y^2\,\mathrm{d}y\\ &= \frac{2^{5/2}\sigma^3}{\sigma\sqrt{2\pi}}\int_0^\infty \mathrm{e}^{-x^2}x^2\,\mathrm{d}x\\ &= \frac{2^{5/2}\sigma^3}{\sigma\sqrt{2\pi}}\frac{1}{2^2}\sqrt{\pi} = \sigma^2,\end{aligned}$$

so $\sigma$ is the dispersion.

## Characteristic Function

The Fourier transform is

$$\psi(t) = \int_{-\infty}^{+\infty} \mathrm{e}^{+\mathrm{i}ty}W(y)\,\mathrm{d}y. \tag{14.5}$$

For the one-dimensional Gaussian distribution

$$\psi(t) = \frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{+\infty} \mathrm{e}^{\mathrm{i}ty-y^2/2\sigma^2}\,\mathrm{d}y, \tag{14.6}$$

or

$$\begin{aligned}\psi(t) &= \frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{+\infty}\exp\left[-\frac{1}{2\sigma^2}(y^2-2\sigma^2\mathrm{i}ty)\right]\mathrm{d}y\\ &= \frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{+\infty}\exp\left\{-\frac{1}{2\sigma^2}\left[(y-\sigma^2\mathrm{i}t)^2+\sigma^4t^2\right]\right\}\mathrm{d}y\\ &= \frac{\mathrm{e}^{-\sigma^2t^2/2}}{\sigma\sqrt{2\pi}}\int_{-\infty}^{+\infty}\exp\left[-\frac{1}{2\sigma^2}\left(y-\sigma^2\mathrm{i}t\right)^2\right]\mathrm{d}y.\end{aligned} \tag{14.7}$$

Thus,

$$\psi(t) = \mathrm{e}^{-\sigma^2t^2/2}, \tag{14.8}$$

and

$$\psi(t) = \sum_{n=0}^\infty (-1)^n\left(\frac{\sigma^2t^2}{2}\right)^n\frac{1}{n!}. \tag{14.9}$$

### The Importance of the Characteristic Function

First,

$$\psi(0) = \int_{-\infty}^{+\infty} W(y)\,\mathrm{d}y = 1. \tag{14.10}$$

Now an alternate derivation of the moment relations is possible:

$$W(y) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \mathrm{e}^{-\mathrm{i}ty}\psi(t)\,\mathrm{d}t, \tag{14.11}$$

$$\int_{-\infty}^{+\infty} W(y)\,\mathrm{d}y = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \psi(t)\,\mathrm{d}t \int_{-\infty}^{+\infty} \mathrm{e}^{-\mathrm{i}ty}\,\mathrm{d}y. \tag{14.12}$$

Now,

$$\delta(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \mathrm{e}^{-\mathrm{i}ty}\,\mathrm{d}y, \tag{14.13}$$

so

$$\int_{-\infty}^{+\infty} W(y)\,\mathrm{d}y = \psi(0) = 1,$$

thereby showing that $W(y)$ is normalized.

For the $n$th moment,

$$\int_{-\infty}^{+\infty} W(y)y^n\,\mathrm{d}y = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \psi(t)\,\mathrm{d}t \int_{-\infty}^{+\infty} \mathrm{e}^{-\mathrm{i}ty}y^n\,\mathrm{d}y \tag{14.14}$$

$$= \int_{-\infty}^{+\infty} (-\mathrm{i})^{-n}\delta^{(n)}(t)\psi(t)\,\mathrm{d}t. \tag{14.15}$$

Successive integrations by parts give

$$\int_{-\infty}^{+\infty} W(y)y^n\,\mathrm{d}y = \frac{1}{\mathrm{i}^n}\left(\frac{\partial^n\psi}{\partial t^n}\right)_{t=0}. \tag{14.16}$$

We have therefore

$$\begin{aligned}\left(\frac{\partial\psi}{\partial t}\right)_{t=0} &= \mathrm{i}\int_{-\infty}^{+\infty} W(y)y\,\mathrm{d}y = \mathrm{i}\overline{y},\\ \left(\frac{\partial\psi}{\partial t}\right)_{t=0} &= \mathrm{i}y_1,\end{aligned} \tag{14.17}$$

and

$$\left(\frac{\partial^n\psi}{\partial t^n}\right)_{t=0} = \mathrm{i}^n y_n. \tag{14.18}$$

Or, alternatively, the $n$th moment is given by

$$y_n = \frac{1}{\mathrm{i}^n}\left(\frac{\partial^n\psi}{\partial t^n}\right)_{t=0}, \tag{14.19}$$

$$y_{2n} = (-1)^{-n}\left(\frac{\partial^{2n}\psi}{\partial t^{2n}}\right)_{t=0} = (-1)^n\left(\frac{\partial^{2n}\psi}{\partial t^{2n}}\right)_{t=0}. \tag{14.20}$$

Introducing (14.9), we find

$$y_{2n} = (-1)^n \left[ \frac{\partial^{2n}}{\partial t^{2n}} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \left(\frac{\sigma^2}{2}\right)^k t^{2k} \right]_{t=0},$$

$$\frac{\partial^{2n}}{\partial t^{2n}} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \left(\frac{\sigma^2}{2}\right)^k t^{2k}$$
$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \left(\frac{\sigma^2}{2}\right)^k t^{2k-2n} (2k)(2k-1)\cdots(2k-2n+1).$$

Of course, all terms for which $n > k$ will have vanished under differentiation, and this series is equivalent to

$$\frac{\partial^{2n}}{\partial t^{2n}} \psi = \sum_{k=n}^{\infty} \frac{(-1)^k}{k!} \left(\frac{\sigma^2}{2}\right)^k (t^2)^{k-n} (2k)(2k-1)\cdots(2k-2n+1). \tag{14.21}$$

Upon evaluation of (14.20) for $t = 0$, we find that the only non-vanishing term is that for which $k = n$. We thus have

$$\left(\frac{\partial^{2n}}{\partial t^{2n}} \psi\right)_{t=0} = \frac{(-1)^n}{n!} \left(\frac{\sigma^2}{2}\right)^n (2n)! \tag{14.22}$$

and

$$y_n = \frac{(2n)!}{n!} \left(\frac{\sigma^2}{2}\right)^n = \frac{(2n)!}{n! 2^n} \sigma^{2n}. \tag{14.23}$$

We may verify this result:

$$\begin{aligned}
y_{2n} &= \frac{2}{\sigma\sqrt{2\pi}} \int_0^{\infty} \mathrm{e}^{-y^2/2\sigma^2} y^{2n}\, \mathrm{d}y \\
&= \frac{2}{\sigma\sqrt{2\pi}} (\sqrt{2}\sigma)^{2n+1} \int_0^{\infty} \mathrm{e}^{-x^2} x^{2n}\, \mathrm{d}x \\
&= \frac{2}{\sqrt{\pi}} (\sqrt{2}\sigma)^{2n} \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2^{n+1}} \sqrt{\pi} \\
&= 2(2\sigma^2)^n \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2^{n+1}} \\
&= \frac{(2n)!}{n! 2^n} \sigma^{2n},
\end{aligned}$$

which agrees with (14.23).

### *14.1.2 n-Dimensional Gaussian Function*

We have

$$W(y_1, \ldots, y_n) = \frac{1}{(2\pi)^{n/2}} \frac{1}{\sqrt{|B|}} \exp\left(-\frac{1}{2|B|} B_{kl} y_k y_l\right), \tag{14.24}$$

where $B=(b_{kl})$ is the cofactor of $b_{kl}$, i.e., the minor with appropriate sign, $|B|$ is the determinant of $(b_{kl})$, with $b_{kl}=\overline{y_k y_l}$ and $B_{kl}$ a symmetric tensor.

### Characteristic Function

The characteristic function is

$$\psi(t_1,\ldots,t_n)=\exp\left(-\frac{1}{2}b_{kl}t_k t_l\right). \tag{14.25}$$

To prove this we must evaluate the integral

$$\begin{aligned}&\frac{1}{(2\pi)^n}\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty}\mathrm{e}^{\mathrm{i}y_k t_k}\psi(t_1,\ldots,t_n)\,\mathrm{d}t_1\ldots\mathrm{d}t_n\\&=\frac{1}{(2\pi)^n}\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty}\exp\left(\mathrm{i}y_k t_k-\frac{1}{2}b_{kl}t_k t_l\right)\mathrm{d}t_1\ldots\mathrm{d}t_n.\end{aligned}$$

Let

$$t_k=u_k+\frac{\mathrm{i}}{|B|}B_{kr}y_r. \tag{14.26}$$

Then

$$\begin{aligned}\mathrm{i}y_k t_k-\frac{1}{2}b_{kl}t_k t_l&=\mathrm{i}y_k u_k-\frac{B_{kr}}{|B|}y_k y_r-\frac{1}{2}b_{kl}\bigg[u_k u_l+\frac{\mathrm{i}}{|B|}B_{kr}y_r u_l\\&\qquad+\frac{\mathrm{i}}{|B|}B_{ls}y_s u_k-\frac{1}{|B|^2}B_{kr}B_{ls}y_r y_s\bigg]\\&=\mathrm{i}y_k u_k-\frac{B_{kr}}{|B|}y_k y_r-\frac{1}{2}b_{kl}u_k u_l-\frac{\mathrm{i}}{2|B|}b_{kl}B_{kr}y_r u_l\\&\qquad-\frac{\mathrm{i}}{2|B|}b_{kl}B_{ls}y_s u_k+\frac{B_{ls}}{2|B|^2}b_{kl}B_{kr}y_r y_s.\end{aligned}$$

But

$$b_{kl}B_{kr}=|B|\delta_{lr},$$

so continuing from above,

$$\begin{aligned}\mathrm{i}y_k t_k-\frac{1}{2}b_{kl}t_k t_l&=\mathrm{i}y_k u_k-\frac{B_{kr}}{|B|}y_k y_r-\frac{1}{2}b_{kl}u_k u_l\\&\qquad-\frac{1}{2}\mathrm{i}\delta_{lr}y_r u_l-\frac{1}{2}\mathrm{i}\delta_{ks}y_s u_k+\frac{1}{2|B|}\delta_{lr}B_{ls}y_r y_s\\&=\mathrm{i}y_k u_k-\frac{B_{kr}y_k y_r}{|B|}-\frac{1}{2}b_{kl}u_k u_l\\&\qquad-\frac{\mathrm{i}}{2}y_l u_l-\frac{\mathrm{i}}{2}y_k u_k+\frac{1}{2|B|}B_{rs}y_r y_s\\&=-\frac{1}{2|B|}B_{kr}y_k y_r-\frac{1}{2}b_{kl}u_k u_l.\end{aligned}$$

Hence,

$$W = \frac{1}{(2\pi)^n} \exp\left(-\frac{1}{2|B|} B_{kr} y_k y_r\right) \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \exp\left(-\frac{1}{2} b_{kl} u_k u_l\right) \mathrm{d}u_1 \ldots \mathrm{d}u_n.$$

Now, bringing $b_{kl}u_k u_l$ into the form $\lambda_k u_k^2$, we find (with $\lambda_1 \cdots \lambda_n = |B|$)

$$\begin{aligned} W &= \frac{1}{(2\pi)^n} \exp\left(-\frac{1}{2|B|} B_{kr} y_k y_r\right) \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \exp\left(-\frac{1}{2}\lambda_k u_k^2\right) \mathrm{d}u_1 \ldots \mathrm{d}u_n \\ &= \frac{1}{(2\pi)^n} \exp\left(-\frac{1}{2|B|} B_{kr} y_k y_r\right) \sqrt{\frac{(2\pi)^n}{\lambda_1 \cdots \lambda_n}} \\ &= \frac{1}{(2\pi)^{n/2}\sqrt{|B|}} \mathrm{e}^{-B_{kr} y_k y_r / 2|B|}. \end{aligned}$$

Let $F(y_1, \ldots, y_n)$ be a polynomial in $(y_1, \ldots, y_n)$. Then by Fourier's integral theorem,

$$\begin{aligned} \overline{F(y_1, \ldots, y_n)} &= \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} W(y_1, \ldots, y_n) F(y_1, \ldots, y_n)\, \mathrm{d}y_1 \ldots \mathrm{d}y_n \\ &= \frac{1}{(2\pi)^n} \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} F(y_1, \ldots, y_n)\, \mathrm{d}y_1 \ldots \mathrm{d}y_n \\ &\quad \times \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \mathrm{e}^{\mathrm{i} y_k t_k} \psi(t_1, \ldots, t_n)\, \mathrm{d}t_1 \ldots \mathrm{d}t_n \\ &= \frac{1}{(2\pi)^n} \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \psi(t_1, \ldots, t_n)\, \mathrm{d}t_1 \ldots \mathrm{d}t_n \\ &\quad \times \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} F(y_1, \ldots, y_n) \mathrm{e}^{\mathrm{i} y_k t_k}\, \mathrm{d}y_1 \ldots \mathrm{d}y_n \\ &= \left[F\left(\frac{1}{\mathrm{i}}\frac{\partial}{\partial t_1}, \ldots, \frac{1}{\mathrm{i}}\frac{\partial}{\partial t_n}\right) \psi(t_1, \ldots, t_n)\right]_{t_1 = \cdots = t_n = 0}. \end{aligned} \tag{14.27}$$

Hence,

$$\overline{y_k y_l} = \left[-\frac{\partial^2}{\partial t_k \partial t_l} \psi(t_1, \ldots, t_n)\right]_{t_1 = \cdots = t_n = 0}.$$

Since

$$\psi(t_1, \ldots, t_n) = \exp\left(-\frac{1}{2} b_{kl} t_k t_l\right),$$

$$\overline{y_k y_l} = \left[-\frac{\partial^2}{\partial t_k \partial t_l} \exp\left(-\frac{1}{2} b_{rs} t_r t_s\right)\right]_{t_1 = \cdots = t_n = 0}.$$

Now

$$\frac{\partial}{\partial t_l} \exp\left(-\frac{1}{2} b_{rs} t_r t_s\right) = -\frac{1}{2} b_{rs} (t_s \delta_{rl} + t_r \delta_{ls}) \exp\left(-\frac{1}{2} b_{rs} t_r t_s\right)$$

$$= -\frac{1}{2}(b_{ls}t_s + b_{rl}t_r)\exp\left(-\frac{1}{2}b_{rs}t_r t_s\right)$$
$$= -b_{lm}t_m \exp\left(-\frac{1}{2}b_{rs}t_r t_s\right),$$

and

$$\frac{\partial^2}{\partial t_k \partial t_l}\exp\left(-\frac{1}{2}b_{rs}t_r t_s\right) = -\left[b_{lk}\exp\left(-\frac{1}{2}b_{rs}t_r t_s\right) - b_{lm}t_m b_{kn}t_n \exp\left(-\frac{1}{2}b_{rs}t_r t_s\right)\right]$$
$$= (-b_{lk} + b_{lm}b_{kn}t_m t_n)\exp\left(-\frac{1}{2}b_{rs}t_r t_s\right).$$

Hence,

$$\overline{y_k y_l} = b_{lk} = b_{kl}.$$

### *14.1.3 Two-Dimensional Gaussian Function*

In the two-dimensional case,

$$B = \begin{pmatrix} \overline{y_1^2} & \overline{y_1 y_2} \\ \overline{y_1 y_2} & \overline{y_2^2} \end{pmatrix} = \begin{pmatrix} \sigma^2 & \rho\sigma\tau \\ \rho\sigma\tau & \tau^2 \end{pmatrix},$$

where $\rho$ is the correlation coefficient, i.e.,

$$\rho = \frac{\overline{y_1 y_2}}{\sqrt{\overline{y_1^2}}\sqrt{\overline{y_2^2}}} = \frac{\overline{y_1 y_2}}{\sigma\tau},$$

and

$$B_{11} = \tau^2, \qquad B_{12} = -\rho\sigma\tau, \qquad B_{22} = \sigma^2, \qquad |B| = \sigma^2\tau^2(1-\rho^2).$$

Finally,

$$W(y_1, y_2) = \frac{1}{2\pi\sigma\tau\sqrt{1-\rho^2}}\exp\left[-\frac{1}{2(1-\rho^2)}\left(\frac{y_1^2}{\sigma^2} + \frac{y_2^2}{\tau^2} - \frac{2\rho y_1 y_2}{\sigma\tau}\right)\right].$$

## 14.2 Addition Theorem for Gaussian Distributions

Let $x_1, \ldots, x_n$ be distributed according to

$$W(x_1, \ldots, x_n) = \frac{1}{(2\pi)^{n/2}}\frac{1}{\sigma_1\sigma_2\cdots\sigma_n}\exp\left(-\frac{1}{2}\sum\frac{x_i^2}{\sigma_i^2}\right).$$

Let $y_1, \ldots, y_s$ $(s \le m)$ be $s$ linear combinations of the $x_i$, i.e.,

$$y_k = a_{ki} x_i, \quad k = 1, \ldots, s,$$

where the $a_{ki}$ are constants. The $y_k$ are then distributed according to an $s$-dimensional distribution

$$W(y_1, \ldots, y_s) = \frac{1}{(2\pi)^{s/2}\sqrt{|B|}} \exp\left(-\frac{1}{2|B|} B_{kl} y_k y_l\right),$$

where

$$b_{kl} = \overline{y_k y_l} = \sum_{i=1}^{n} a_{ki} a_{li} \sigma_i^2 .$$

*Proof* Clearly,

$$\begin{aligned}
W(y_1, \ldots, y_s) &= \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} W(x_1, \ldots, x_n) \prod_{k=1}^{s} \delta(y_k - a_{ki} x_i)\, \mathrm{d}x_1 \ldots \mathrm{d}x_n \\
&= \frac{1}{(2\pi)^{s+n/2} \sigma_1 \cdots \sigma_n} \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \exp\left(-\frac{1}{2} \frac{x_i^2}{\sigma_i^2}\right) \mathrm{d}x_1 \ldots \mathrm{d}x_n \\
&\quad \times \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \mathrm{e}^{\mathrm{i}(t_k y_k - t_k a_{ki} x_i)}\, \mathrm{d}t_1 \ldots \mathrm{d}t_n \\
&= \frac{1}{(2\pi)^{s+n/2} \sigma_1 \cdots \sigma_n} \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \mathrm{e}^{\mathrm{i} y_k t_k}\, \mathrm{d}t_1 \ldots \mathrm{d}t_s \\
&\quad \times \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \exp\left(-\frac{1}{2} \frac{x_i^2}{\sigma_i^2} - \mathrm{i} a_{ki} t_k x_i\right) \mathrm{d}x_1 \ldots \mathrm{d}x_n .
\end{aligned}$$

Now

$$\begin{aligned}
\frac{1}{2} \sum_i \left[\frac{x_i^2}{\sigma_i^2} + 2\left(\sum_k a_{ki} t_k\right) x_i\right] &= \frac{1}{2} \sum_i \left\{\frac{1}{\sigma_i^2}\left[x_i^2 + 2\sigma_i^2 \left(\sum_k a_{ki} t_k\right) x_i\right]\right\} \\
&= \frac{1}{2} \sum_i \left(\frac{1}{\sigma_i^2}\left\{\left[x_i + \sigma_i^2 \left(\sum_k a_{ki} t_k\right)\right]^2 \right.\right. \\
&\qquad \left.\left. - \sigma_i^4 \left(\sum_k a_{ki} t_k\right)\left(\sum_l a_{li} t_l\right)\right\}\right) \\
&= \sum_i \frac{1}{2\sigma_i^2} \left(x_i + \sigma_i^2 \sum_k a_{ki} t_k\right)^2 \\
&\qquad - \frac{1}{2} \sum_i \sum_k \sum_l \sigma_i^2 a_{ki} a_{li} t_k t_l .
\end{aligned}$$

Let

$$b_{kl} = \sum_i a_{ki} a_{li} \sigma_i^2 .$$

Then

$$\frac{1}{2}\sum_i\left[\frac{x_i^2}{\sigma_i^2}+2\Bigl(\sum_k a_{ki}t_k\Bigr)x_i\right]=\sum_i\frac{1}{2\sigma_i^2}\Bigl(x_i+\sigma_i^2\sum_k a_{ki}t_k\Bigr)^2-\frac{1}{2}b_{kl}t_kt_l,$$

and

$$\begin{aligned}W(y_1,\ldots,y_s)&=\frac{1}{(2\pi)^{s+n/2}\sigma_1\cdots\sigma_n}\int\cdots\int \mathrm{e}^{\mathrm{i}y_kt_k-b_{kl}t_kt_l}\,\mathrm{d}t_1\ldots\mathrm{d}t_n\\&\quad\times\int\cdots\int\exp\Bigl[-\sum_i\frac{1}{2\sigma_i^2}\Bigl(x_i+\sigma_i\sum_k a_{ki}t_k\Bigr)\Bigr]\mathrm{d}x_1\ldots\mathrm{d}x_n,\\W(y_1,\ldots,y_n)&=\frac{1}{(2\pi)^s}\int\cdots\int\exp\Bigl(\mathrm{i}y_kt_k-\frac{1}{2}b_{kl}t_kt_l\Bigr)\mathrm{d}t_1\ldots\mathrm{d}t_n,\end{aligned}$$

as required. □

## 14.3 Proof of (14.2)

Suppose that the six quantities $u_i$ and $u_j'$ are distributed according to a six-dimensional Gaussian distribution. The characteristic function must be of the form

$$\exp\left(-\frac{1}{2}\overline{y_ky_l}t_kt_l\right).$$

Let $\alpha_i\to u_i$, $\beta_j\to u_j'$. We have $\overline{u_iu_j}$, $\overline{u_i'u_j'}$, and $\overline{u_iu_j'}$, and

$$\psi(\boldsymbol{\alpha},\boldsymbol{\beta})=\exp\left[-\frac{1}{2}\Bigl(\overline{u_iu_j}\alpha_i\alpha_j+2\overline{u_iu_j'}\alpha_i\beta_j+\overline{u_i'u_j'}\beta_i\beta_j\Bigr)\right].$$

By the general theorem on moments, viz., (14.27),

$$\overline{u_iu_ju_l'u_m'}=\left[\left(\frac{1}{\mathrm{i}}\right)^4\frac{\partial^4}{\partial\alpha_i\partial\alpha_j\partial\beta_l\partial\beta_m}\psi(\boldsymbol{\alpha},\boldsymbol{\beta})\right]_{\boldsymbol{\alpha}=\boldsymbol{\beta}=0}.$$

Replace $i$ and $j$ in $\psi(\boldsymbol{\alpha},\boldsymbol{\beta})$ by $s$ and $t$. Then

$$\begin{aligned}\frac{\partial}{\partial\beta_m}\psi(\boldsymbol{\alpha},\boldsymbol{\beta})&=-\frac{1}{2}\left(2\overline{u_su_m'}\alpha_s+\overline{u_s'u_m'}\beta_s+\overline{u_t'u_m'}\beta_t\right)\psi(\boldsymbol{\alpha},\boldsymbol{\beta})\\&=-\Bigl(\overline{u_su_m'}+\overline{u_s'u_m'}\Bigr)\psi(\boldsymbol{\alpha},\boldsymbol{\beta}),\\\frac{\partial^2\psi(\boldsymbol{\alpha},\boldsymbol{\beta})}{\partial\beta_l\partial\beta_m}&=\Bigl[\Bigl(\overline{u_su_m'}\alpha_s+\overline{u_s'u_m'}\beta_s\Bigr)\left(\overline{u_su_l'}\alpha_s+\overline{u_s'u_l'}\beta_s\right)-\overline{u_l'u_m'}\Bigr]\psi(\boldsymbol{\alpha},\boldsymbol{\beta})\\&=A\psi(\boldsymbol{\alpha},\boldsymbol{\beta}),\\\frac{\partial^3\psi}{\partial\alpha_j\partial\beta_l\partial\beta_m}&=[-\left(\overline{u_ju_s}\alpha_s+\overline{u_ju_t'}\beta_t\right)A+\overline{u_ju_m'}\left(\overline{u_su_l'}\alpha_s+\overline{u_s'u_l'}\beta_s\right)\\&\qquad+\overline{u_ju_l'}\Bigl(\overline{u_su_m'}\alpha_s+\overline{u_s'u_m'}\beta_s\Bigr)]\psi(\boldsymbol{\alpha},\boldsymbol{\beta})\\&=B\psi(\boldsymbol{\alpha},\boldsymbol{\beta}),\end{aligned}$$

and

$$\frac{\partial^4\psi}{\partial\alpha_j\partial\alpha_j\partial\beta_l\partial\beta_m} = \Big\{ -A\overline{u_i u_j} - \Big(\overline{u_j u_s}\alpha_s + \overline{u_s u_t'}\beta_t\Big)\Big[\Big(\overline{u_i u_m'}\Big)\Big(\overline{u_s u_l'}\alpha_s + u_s' u_l'\beta_s\Big) + \Big(\overline{u_i u_l'}\Big)\Big(\overline{u_s u_m'}\alpha_s + u_s' u_m'\beta_s\Big)\Big] + u_j' u_m' u_i u_l' + \overline{u_j u_l'}\,\overline{u_i u_m'} - B\Big(\overline{u_s u_i}\alpha_s + \overline{u_i u_s'}\beta_s\Big)\Big\}\psi(\boldsymbol{\alpha}, \boldsymbol{\beta}).$$

When $\boldsymbol{\alpha} = \boldsymbol{\beta} = 0$, we then find

$$\overline{u_i u_j u_l' u_m'} = \overline{u_i u_j}\,\overline{u_l' u_m'} + \overline{u_i u_m'}\,\overline{u_j u_l'} + \overline{u_i u_l'}\,\overline{u_j u_m'},$$

which is (14.2).

# Chapter 15
# Chandrasekhar's Theory of Turbulence

In the derivation of the Karman–Howarth equation, we multiplied the equation of motion in terms of the velocity at $x_i$ by the velocity at $x_i'$. But in order to obtain $\partial Q_{ij}/\partial t$, a similar procedure was followed in which the roles of these two velocities were interchanged. The results of these two operations, when combined, gave $\partial Q_{ij}/\partial t$. If, however, the equation of motion had in the first place been multiplied by a velocity at another time, say $t'$, at $x_i'$, only one operation would have been needed to obtain the desired equation in the correlations. Moreover, after $Q(r,t)$ has been introduced by requirements of convenience ($t$ is the interval between $t$ and $t'$), it is seen that just such a description of turbulence is what is needed for a general theory. For we have seen that, both physically and mathematically, one cannot get an equation in $Q(r)$ alone. More importantly, perhaps, $Q(r)$ cannot include a description of the phase relationships that we have seen to be fundamental to a description of turbulence as well as to a study of its dynamical properties.

Having made this change in approach, we pose ourselves the problem of getting an equation in $Q(r,t)$ alone. As an aside note that, from what has been said, it is clear that $Q(r,t)$ is the defining scalar of

$$Q_{ij} = \overline{u_i(x_i, t) u_j(x_i', t')} = \overline{u_i u_j'}. \tag{15.1}$$

Now we might continue the process of multiplying the equation of motion by quantities leading to the introduction of correlations. In particular, we might multiply by $u_l u_m$ and introduce an additional equation relating the third and fourth order correlations. But each new equation introduced in this way brings with it the introduction of another defining scalar, unless some additional condition is imposed. In other words, we can get a new equation without bringing a new incalculable scalar only if the new scalar can be related back to some other known quantity. Such a condition could be an assumption concerning the state of motion of the fluid. It might be assumed that the velocity distribution is Gaussian. In this case the third moment, i.e., the triple correlation, would vanish. Though it is true that the triple correlations are found to be small experimentally, their finiteness is fundamental to the turbulence problem. However, the fourth order correlations behave as if they were nearly Gaus-

E.A. Spiegel (ed.), *The Theory of Turbulence*, Lecture Notes in Physics 810,
DOI 10.1007/978-94-007-0117-5_15, 

sian, and these might be calculated on the assumption of a Gaussian distribution in accordance with (14.2). In this way, a closure to the number of scalars introduced with each correlation might be achieved. Thus, in terms of the notation of correlations, we could make (14.2) a hypothesis as follows:

$$Q_{ij;lm} = \left(\overline{u_1^2}\right)^2 \delta_{ij}\delta_{lm} + Q_{il}Q_{jm} + Q_{im}Q_{jl}. \tag{15.2}$$

With this hypothesis and the introduction of concepts implicit in $Q(r,t)$, we proceed to attempt the derivation of an equation in $Q$ alone.

We multiply

$$\frac{\partial u_i}{\partial t} + \frac{\partial}{\partial x_k} u_k u_i = -\frac{\partial \varpi}{\partial x_i} + \nabla^2 u_i \tag{15.3}$$

by $u'_j = u_j(x'_i, t')$. Of course, $\overline{u_i u'_j}$ is a solenoidal isotropic tensor, and (10.33) applies, i.e.,

$$Q_{ij} = \overline{u_i u'_j} = \frac{Q'}{r}\xi_i\xi_j - (rQ' + 2Q)\delta_{ij}. \tag{15.4}$$

We will adopt the convention

$$t' - t = \tau > 0.$$

Then

$$\frac{\partial}{\partial \tau} Q_{ij} - \frac{\partial}{\partial \xi_k} T_{ikj} = \nu \nabla^2 Q_{ij}, \tag{15.5}$$

since

$$\overline{u'_j \frac{\partial \varpi}{\partial x_i}} = \frac{\partial}{\partial x_i} \overline{\varpi u_j},$$

and $\overline{\varpi u_j}$ is a solenoidal isotropic vector and must vanish.

Now multiply (15.3) by $u'_l u'_m$ and average. We have

$$-\frac{\partial}{\partial \tau} T_{lmi} - \frac{\partial}{\partial \xi_j} Q_{ij;lm} = \frac{\partial}{\partial \xi_i} P_{lm} - \nu \nabla^2 T_{lmi}, \tag{15.6}$$

where

$$P_{lm} = \overline{\varpi u'_l u'_m} \tag{15.7}$$

and we have used

$$\overline{u'_l u'_m u_i} = -\overline{u_l u_m u'_i} = -T_{lmi}.$$

We shall now seek to pass from (15.5) and (15.6) into scalar equations. Let

$$D_n = \frac{\partial^2}{\partial r^2} + \frac{n-1}{r}\frac{\partial}{\partial r}. \tag{15.8}$$

We have seen that the quantities in (15.7) have the following defining scalars:

$$\frac{\partial}{\partial \tau} Q_{ij} \to \frac{\partial}{\partial \tau} Q,$$

$$\frac{\partial}{\partial \xi_k} T_{ikj} \to rT' + 5T \quad \text{[see (10.58)]}, \tag{15.9}$$

$$\nabla^2 Q_{ij} \to D_5 Q \quad \text{[see (10.39)]}. \tag{15.10}$$

Hence (15.5) passes over into

$$\left(\frac{\partial}{\partial \tau} - \nu D_5\right) Q = rT' + 5T. \tag{15.11}$$

We turn to (15.6). From (15.7), we see that $P_{lm}$ must be symmetric in its indices, and hence, by (10.8),

$$P_{lm} = P_1 \xi_l \xi_m + P_2 \delta_{lm}. \tag{15.12}$$

Unfortunately, $P_{lm}$ is not solenoidal. We may rewrite (15.6) in the form

$$\left(\frac{\partial}{\partial \tau} - \nu \nabla^2\right) T_{lmi} = -\left(\frac{\partial}{\partial \xi_i} P_{lm} + \frac{\partial}{\partial \xi_j} Q_{ij;lm}\right). \tag{15.13}$$

In obtaining (15.13) from (15.3) as we had written it, no manipulations involving $i$ were performed. Thus, since (15.3) was solenoidal in $i$, (15.13) must be. Moreover, the left-hand side of (15.13) is solenoidal in $i$. Then

$$\frac{\partial}{\partial \xi_i} P_{lm} + \frac{\partial}{\partial \xi_j} Q_{ij;lm}$$

must also be solenoidal in $i$. This quantity is also symmetric in $l$ and $m$, so that it must have a single defining scalar, say $X$. We write

$$X_{lmi} = \frac{\partial}{\partial \xi_i} P_{lm} + \frac{\partial}{\partial \xi_j} Q_{ij;lm}. \tag{15.14}$$

Then (15.14) must pass over into

$$\left(\frac{\partial}{\partial \tau} - \nu D_7\right) T = -X. \tag{15.15}$$

Now (15.11) is essentially the Karman–Howarth equation for $Q(r, \tau)$, and (15.15) relates the third and fourth order correlations. We now attempt to relate $X$ and $Q$ by means of the statistical hypothesis (15.2). Clearly, an equation in $Q$ and $X$ in addition to (15.11) and (15.15) will be sufficient to provide an equation in $Q$ alone. We have, from (15.2),

$$\frac{\partial}{\partial \xi_j} Q_{ij;lm} = \frac{\partial Q_{il}}{\partial \xi_j} Q_{jm} + \frac{\partial Q_{im}}{\partial \xi_j} Q_{jl}. \tag{15.16}$$

Now

$$
\begin{aligned}
Q_{il} &= \frac{Q'}{r}\xi_i\xi_l - (rQ' + 2Q)\delta_{il} \\
&= Q_1\xi_i\xi_l + Q_2\delta_{il},
\end{aligned}
$$

and

$$
\frac{\partial Q_{il}}{\partial \xi_j} = \frac{Q_1'}{r}\xi_i\xi_j\xi_l + Q_1\big(\delta_{ij}\xi_l + \delta_{lj}\xi_i\big) + \frac{Q_2'}{r}\delta_{il}\xi_j. \tag{15.17}
$$

Hence,

$$
\begin{aligned}
Q_{jm}\frac{\partial Q_{il}}{\partial \xi_j} = &\left(rQ_1'Q_1 + 2Q_1^2 + \frac{Q_1'Q_2}{r}\right)\xi_i\xi_l\xi_m + \left(rQ_1Q_2' + \frac{Q_2Q_2'}{r}\right)\xi_m\delta_{il} \\
&+ Q_1Q_2\xi_l\delta_{im} + Q_1Q_2\xi_i\delta_{lm},
\end{aligned} \tag{15.18}
$$

and

$$
\begin{aligned}
\frac{\partial}{\partial \xi_j}Q_{ij;lm} = 2&\left(rQ_1'Q_1 + 2Q_1^2 + \frac{Q_1'Q_2}{r}\right)\xi_i\xi_l\xi_m \\
&+ \left(rQ_1Q_2' + \frac{Q_2Q_2'}{r}\right)(\xi_m\delta_{il} + \xi_l\delta_{im}) \\
&+ Q_1Q_2(\xi_m\delta_{il} + \xi_l\delta_{im}) + 2Q_1Q_2\xi_i\delta_{lm}.
\end{aligned} \tag{15.19}
$$

Now,

$$
Q_1 = \frac{Q'}{r}
$$

and

$$
Q_2 = -(rQ' + 2Q).
$$

Hence,

$$
Q_1' = \frac{Q''}{r} - \frac{Q'}{r^2}
$$

and

$$
Q_2' = -(rQ'' + 3Q').
$$

We have

$$
rQ_1'Q_1 = \frac{1}{r}\left(Q'Q'' - \frac{Q'^2}{r}\right), \tag{15.20}
$$

$$
2Q_1^2 = 2\frac{Q'^2}{r^2}, \tag{15.21}
$$

$$\frac{Q_1 Q_2}{r} = \frac{1}{r^2}\left(Q'^2 + 2\frac{QQ'}{r} - rQ'Q'' - 2QQ''\right), \tag{15.22}$$

whence

$$(15.20) + (15.21) + (15.22) = 2\left(\frac{Q'^2}{r^2} - \frac{QQ''}{r^2} + \frac{QQ'}{r^3}\right).$$

Further,

$$rQ_1Q_2' = -\left(rQ'Q'' + 3Q'^2\right), \tag{15.23}$$

$$\frac{Q_2Q_2'}{r} = \frac{6}{r}QQ' + 3Q'Q' + 2QQ'' + rQ'Q'', \tag{15.24}$$

$$Q_1Q_2 = -\left(Q'^2 + \frac{2QQ'}{r}\right), \tag{15.25}$$

whence

$$(15.23) + (15.24) + (15.25) = 2QQ'' + \frac{4}{r}QQ' - Q'^2.$$

Hence,

$$\begin{aligned}\frac{\partial}{\partial \xi_j} Q_{ij;lm} = {} & 4\left(\frac{Q'^2}{r^2} - \frac{QQ''}{r^2} + \frac{QQ'}{r^3}\right)\xi_i\xi_l\xi_m \\ & + \left(2QQ'' + \frac{4}{r}QQ' - Q'^2\right)(\xi_m\delta_{il} + \xi_l\delta_{im}) \\ & - 2\left(Q'^2 + 2\frac{QQ'}{r}\right)\xi_i\delta_{lm}.\end{aligned} \tag{15.26}$$

Now, from (15.12), we see that

$$\frac{\partial}{\partial \xi_i} P_{lm} = \frac{P_1'}{r}\xi_i\xi_l\xi_m + P_1(\xi_m\delta_{il} + \xi_l\delta_{im}) + \frac{P_2'}{r}\xi_i\delta_{lm}. \tag{15.27}$$

Combining (15.26) and (15.27), we may write an expression for $X_{lmi}$ according to (15.14). This expression will be equivalent to

$$X_{lmi} = T_1\xi_i\xi_l\xi_m + T_2(\xi_m\delta_{il} + \xi_l\delta_{im}) + T_4\xi_i\delta_{lm}, \tag{15.28}$$

since $X_{lmi}$ is a third order isotropic tensor, symmetric in $l$ and $m$ and solenoidal in $i$ [see (10.50)]. We may associate coefficients of the various tensors. We find

$$\left.\begin{aligned} T_1 &= 4\left(\frac{Q'^2}{r^2} - \frac{QQ''}{r^2} + \frac{QQ'}{r^3}\right) + \frac{P_1'}{r}, \\ T_2 &= 2QQ'' + \frac{4}{r}QQ' - Q'^2 + P_1, \\ T_4 &= \frac{P_2'}{r} - 2\left(Q'^2 + 2\frac{QQ'}{r}\right). \end{aligned}\right\} \tag{15.29}$$

Inspection of (10.50) shows that, if $X$ is the defining scalar of $X_{lmi}$, then

$$\left.\begin{aligned} T_1 &= \frac{2}{r}X', \\ T_2 &= -(rX' + 3X), \\ T_4 &= 2X. \end{aligned}\right\} \tag{15.30}$$

In order to relate $P_1$ and $P_2$, we make use of the knowledge that $X_{lmi}$ is solenoidal in $i$. Thus,

$$\begin{aligned} \frac{\partial X_{lmi}}{\partial \xi_i} &= \left(rT_1' + 5T_1 + \frac{2T_2'}{r}\right)\xi_l\xi_m + (2T_2 + 3T_4 + rT_4')\delta_{lm} \\ &\equiv 0, \end{aligned} \tag{15.31}$$

and we must have

$$\left.\begin{aligned} rT_1' + 5T_1 + \frac{2}{r}T_2' &= 0, \\ 2T_2 + rT_4' + 3T_4 &= 0. \end{aligned}\right\} \tag{15.32}$$

We introduce (15.29) and find, after noting that

$$\begin{aligned} T_1' &= 4\left(\frac{Q'Q''}{r^2} + 3\frac{QQ''}{r^3} - \frac{Q'^2}{r^3} - \frac{Q'Q'''}{r^2} - 3\frac{QQ'}{r^4}\right) + \frac{P_1''}{r} - \frac{P_1'}{r^2}, \\ T_2' &= 2QQ''' + \frac{4}{r}Q'^2 + \frac{4}{r}QQ'' - \frac{4}{r^2}QQ' + P_1', \\ T_4' &= \frac{P_2''}{r} - \frac{P_2'}{r^2} - 2\left(2Q'Q'' + \frac{2Q'^2}{r} + \frac{2QQ''}{r} - \frac{2QQ'}{r^2}\right), \end{aligned}$$

that the conditions (15.32) become

$$\left.\begin{aligned} P_1'' + \frac{6}{r}P_1' &= -4\left(\frac{Q'Q''}{r} + 6\frac{Q'^2}{r^2}\right), \\ P_2'' + \frac{2}{r}P_2' + 2P_1 &= 4(3Q'^2 + rQ'Q''). \end{aligned}\right\} \tag{15.33}$$

We might combine these two foregoing equations, but elect instead to follow an alternate procedure. We note that, if we contract $X_{lmi}$ in $l$ and $m$, the resulting $X_{lli}$ is an isotropic, solenoidal vector and must vanish. That is,

$$X_{lli} \equiv 0.$$

This gives us

$$2T_2 + rT_4' + 3T_4 = 0, \tag{15.34}$$

which clearly must be a linear combination of the two equations (15.33). If we substitute (15.29) into (15.33), we find

$$\frac{3}{r}P_2' + rP_1' + 2P_1 = 4Q'^2. \tag{15.35}$$

We may eliminate $P_1$ between (15.35) and the second of (15.33):

$$\frac{P_2''}{r} - \frac{1}{r^2}P_2' - P_1' = \frac{8}{r}Q'^2 + 4Q'Q''. \tag{15.36}$$

This equation may also be written

$$\frac{\partial}{\partial r}\left(\frac{P_2'}{r} - P_1\right) = 4\left(Q'Q'' + \frac{2}{r}Q'^2\right). \tag{15.37}$$

From (15.30), we note that

$$\begin{aligned} -(5X + rX') &= T_2 - T_4 \\ &= 2QQ'' + 8\frac{QQ'}{r} + Q'^2 - \left(\frac{P_2'}{r} - P_1\right), \end{aligned}$$

and, upon introduction of (15.37), we find

$$-\frac{\partial}{\partial r}(5X + rX') = 2Q\left(Q''' + \frac{4}{r}Q'' - \frac{4}{r^2}Q'\right), \tag{15.38}$$

which may be rewritten

$$-\frac{\partial}{\partial r}(5X + rX') = 2Q\frac{\partial}{\partial r}\left(\frac{\partial^2 Q}{\partial r^2} + \frac{4}{r}\frac{\partial Q}{\partial r}\right). \tag{15.39}$$

Thus (15.38) together with (15.11) and (15.15) form a system of equations from which may be found a single equation in $Q$ alone. The system is

$$\left(\frac{\partial}{\partial \tau} - \nu D_5\right)Q = rT' + 5T, \tag{15.40}$$

$$\left(\frac{\partial}{\partial \tau} - \nu D_7\right)T = -X, \tag{15.41}$$

$$-\frac{\partial}{\partial r}(5X + rX') = 2Q\frac{\partial}{\partial r}D_5 Q. \tag{15.42}$$

It is possible to eliminate $T$ from these equations in a simple way. We note that

$$\left(5 + r\frac{\partial}{\partial r}\right) D_7 = D_5\left(5 + r\frac{\partial}{\partial r}\right). \tag{15.43}$$

We may easily verify (15.43). The left-hand side when expanded is

$$\begin{aligned}\left(5 + r\frac{\partial}{\partial r}\right)\left(\frac{\partial^2}{\partial r^2} + \frac{6}{r}\frac{\partial}{\partial r}\right) &= 5\frac{\partial^2}{\partial r^2} + \frac{30}{r}\frac{\partial}{\partial r} + r\frac{\partial^3}{\partial r^3} + 6\frac{\partial^2}{\partial r^2} - \frac{6}{r}\frac{\partial}{\partial r}\\ &= r\frac{\partial^3}{\partial r^3} + 11\frac{\partial^2}{\partial r^2} + \frac{24}{r}\frac{\partial}{\partial r},\end{aligned}$$

while the right-hand side becomes

$$\begin{aligned}\left(\frac{\partial^2}{\partial r^2} + \frac{4}{r}\frac{\partial}{\partial r}\right)\left(5 + r\frac{\partial}{\partial r}\right) &= 5\frac{\partial^2}{\partial r^2} + \frac{20}{r}\frac{\partial}{\partial r} + 2\frac{\partial^2}{\partial r^2} + r\frac{\partial^3}{\partial r^3} + \frac{4}{r}\frac{\partial}{\partial r} + 4\frac{\partial^2}{\partial r^2}\\ &= r\frac{\partial^3}{\partial r^3} + 11\frac{\partial^2}{\partial r^2} + \frac{24}{r}\frac{\partial}{\partial r},\end{aligned}$$

and (15.43) is verified.

In the light of the above identity, we rewrite (15.41), (15.40) and (15.42) as

$$\left(\frac{\partial}{\partial \tau} - \nu D_5\right) Q = \left(5 + r\frac{\partial}{\partial r}\right) T, \tag{15.44}$$

$$\left(\frac{\partial}{\partial \tau} - \nu D_7\right) Q = -X, \tag{15.45}$$

$$\frac{\partial}{\partial r}\left(5 + r\frac{\partial}{\partial r}\right) X = -2Q\frac{\partial}{\partial r}D_5 Q. \tag{15.46}$$

First, we operate $(\partial/\partial\tau - \nu D_5)$ on (15.44), and introduce (15.45) and (15.46):

$$\begin{aligned}\left(\frac{\partial}{\partial \tau} - \nu D_5\right)^2 Q &= \left(\frac{\partial}{\partial \tau} - \nu D_5\right)\left(5 + r\frac{\partial}{\partial r}\right) T\\ &= \left(5 + r\frac{\partial}{\partial r}\right)\left(\frac{\partial}{\partial \tau} - \nu D_5\right) T\\ &= -\left(5 + r\frac{\partial}{\partial r}\right) X,\end{aligned} \tag{15.47}$$

$$\frac{\partial}{\partial r}\left(\frac{\partial}{\partial \tau}-\nu D_5\right)^2 Q = -\frac{\partial}{\partial r}\left(5+r\frac{\partial}{\partial r}\right)X$$
$$= 2Q\frac{\partial}{\partial r}D_5 Q. \tag{15.48}$$

We have thus seen how an equation in $Q$ alone may be derived, subject to the statistical hypothesis made.

# Chapter 16
# A More Subjective Approach to the Derivation of Chandrasekhar's Equation

The equation in $Q(r,t)$ alone was derived on the basis of three equations, namely (15.44), (15.45) and (15.46). However, the derivation of (15.46) in the form given clearly anticipated the knowledge that a simple elimination of $X$ and $T$ would be possible. This chapter is devoted to showing how (15.46) really arises following a more straightforward line of reasoning.

We have, in the absence of viscosity,

$$\frac{\partial u_i}{\partial t} + \frac{\partial}{\partial x_k} u_i u_k = -\frac{\partial \varpi}{\partial x_i}. \tag{16.1}$$

The usual procedure would be to multiply by $u'_j = u_j(x'_i)$ and average. For the new theory, we multiply by $u'_j(t')$ and average. We find

$$\frac{\partial}{\partial t}\overline{u'_j(t')u_i(t)} = -\frac{\partial}{\partial x_k}|\overline{u'_j(t')u_i(t)u_k(t)}| - \frac{\partial}{\partial x_i}\overline{\varpi(t)u'_j(t')}. \tag{16.2}$$

We note that, if $T < t'$,

$$u'_j(t') = -\int_{t'}^{T} \frac{\partial}{\partial t''} u'_j(t'')\,\mathrm{d}t'' + u'_j(T). \tag{16.3}$$

However, we have from the equations of motion,

$$\frac{\partial}{\partial t} u'_j = -\frac{\partial}{\partial x'_k} u'_i u'_k - \frac{\partial \varpi'}{\partial x'_i}, \tag{16.4}$$

and so

$$u'_j(t') = \int_{t'}^{T} \left[\frac{\partial}{\partial x'_l} u'_j(t'')u'_l(t'') + \frac{\partial \varpi'(t'')}{\partial x'_j}\right] \mathrm{d}t'' + u_j(T). \tag{16.5}$$

If we substitute (16.5) into the right-hand side of (16.2), we find, if $T \to \infty$,

$$\begin{aligned}\frac{\partial}{\partial \tau} Q_{ij} = \int_{t'}^{\infty} &\overline{\frac{\partial}{\partial \xi_k}\left[\frac{\partial}{\partial \xi_l} u'_j(t'')u'_l(t'') + \frac{\partial \varpi'(t'')}{\partial \xi_l}\right] u_i(t)u_k(t)\,\mathrm{d}t''} \\ &+ \lim_{T\to\infty} \frac{\partial}{\partial \xi_k}\overline{u'_j(T)u_k(t)u_i(t)},\end{aligned} \tag{16.6}$$

E.A. Spiegel (ed.), *The Theory of Turbulence*, Lecture Notes in Physics 810,
DOI 10.1007/978-94-007-0117-5_16, © Springer Science+Business Media B.V. 2011

where $\tau = t' - t$ and $\xi_i = x_i' - x_i$.

The term in $\overline{\varpi(t)u_j't'}$ vanishes. As $T \to \infty$, $\tau \to \infty$, and the triple correlation vanishes. Then the last term on the right of (16.6) vanishes and (16.6) becomes

$$\frac{\partial}{\partial \tau} Q_{ij} = \int_{\tau}^{\infty} \frac{\partial}{\partial \xi_k} \left( \frac{\partial}{\partial \xi_l} Q_{ik;jl} + \frac{\partial}{\partial \xi_j} P_{ik} \right) \mathrm{d}\tau. \tag{16.7}$$

Now we saw that

$$X_{ikj} = \frac{\partial}{\partial \xi_l} Q_{ik;jl} + \frac{\partial}{\partial \xi_l} P_{ik} \tag{16.8}$$

is symmetric in $i$ and $k$, and solenoidal in $j$. Hence, according to (10.58), the defining scalar of $\partial X_{ikj}/\partial \xi_k$ is

$$rX' + 5X.$$

Then if we pass to the scalar equation of (16.7), we have

$$\frac{\partial Q}{\partial \tau} = \int_{\tau}^{\infty} (5X + rX')\,\mathrm{d}\tau, \tag{16.9}$$

and

$$\frac{\partial^2 Q}{\partial \tau^2} = -(5X + rX'). \tag{16.10}$$

This is the motivation behind writing the expressions leading to (15.38), and in deriving (15.42). Once we obtain (15.42), we look for an identity such as (15.43) and proceed in a straightforward way.

The simplifications which appeared in the elimination of $X$ and $T$ are in a general way both surprising and encouraging. One could not have expected such simple mathematics, unless the mathematics is a good representation of elementary physical ideas. In this sense, the theory justifies some attention.

# Chapter 17
# The Dimensionless Form of Chandrasekhar's Equation

We introduce

$$f(r,t) = \frac{\overline{u_{\|} u'_{\|}}}{\overline{u_{\|}^2}}, \tag{17.1}$$

and, by (12.1), write

$$\overline{u_{\|}^2} f(r,t) = -2Q(r,t). \tag{17.2}$$

Also, we will use $\ell$ as the unit of length and

$$\frac{\ell}{\sqrt{\overline{u_{\|}^2}}}$$

as the unit of time. Then, using (17.2), we may rewrite (15.48) in dimensionless fashion, introducing the new units. We note that the dimensions of $\nu$ are

$$[\nu] = \frac{\mathrm{L}^2}{\mathrm{T}}$$

and that

$$[D_5] = \left[\frac{\partial^2}{\partial r^2} + \frac{5}{r}\frac{\partial}{\partial r}\right] = \frac{1}{\mathrm{L}^2}.$$

We wish to keep $\nu$ in evidence so, to make it dimensionless, we replace it by $\nu/(\ell\sqrt{\overline{u_1^2}})$, which is the inverse of a Reynolds number based on these units. Further, we bring $D_5$ to dimensionless form by replacing it with $\ell^2 D_5$. By similar manipulations with the remaining operators, we find the dimensionless form of (15.48):

$$\frac{\partial}{\partial r}\left(\frac{\partial}{\partial t} - \nu D_5\right)^2 f = -f\frac{\partial}{\partial r} D_5 f. \tag{17.3}$$

E.A. Spiegel (ed.), *The Theory of Turbulence*, Lecture Notes in Physics 810,
DOI 10.1007/978-94-007-0117-5_17, 

# Chapter 18
# Some Aspects and Advantages of the New Theory

The Karman–Howarth equation relates the two scalars $Q$ and $T$. For a deductive theory of turbulence, we would like to find a way of isolating an equation in one of these scalars. Although the possibility of introducing a normal velocity distribution is appealing, it has the effect of requiring

$$T = 0,$$

and thereby demolishing the effects of the inertial term. Though $T$ is small, it is definitely nonzero in fully developed turbulence.

Thus, we were led to calculating $T$, a small quantity, by considering that the fourth order correlations behave as though the velocity is normally distributed. This is essentially the calculation implied by (16.5). For if we multiply both sides of (16.5) by $u_i u_l$ and average, we obtain a relation between the third and fourth order correlations by letting $T \to \infty$.

There are at least two advantages of the new theory. These are:

- a mathematical justification of the assumptions of the Heisenberg theory,
- compatibility with the Kolmogorov theory.

## 18.1 A Mathematical Justification of the Assumptions of the Heisenberg Theory

Equation (15.48), the equation for $Q$, is

$$\frac{\partial}{\partial r}\left(\frac{\partial}{\partial t} - \nu D_5\right)^2 Q = 2Q\frac{\partial}{\partial r} D_5 Q,$$

and for $\nu = 0$, it becomes

$$\frac{\partial^3}{\partial t^2 \partial r} Q = 2Q\frac{\partial}{\partial r} D_5 Q. \tag{18.1}$$

E.A. Spiegel (ed.), *The Theory of Turbulence*, Lecture Notes in Physics 810,
DOI 10.1007/978-94-007-0117-5_18, © Springer Science+Business Media B.V. 2011

When the inertial term is neglected, we have

$$\frac{\partial u_i}{\partial t} = -\frac{\partial \varpi}{\partial x_i} + \nu \nabla^2 u_i,$$

$$\frac{\partial \overline{u_i u'_j}}{\partial t} = \nu \nabla^2 \overline{u_i u'_j},$$

or

$$\frac{\partial Q}{\partial t} = \nu D_5 Q. \tag{18.2}$$

This becomes

$$\frac{\partial^2 Q}{\partial r \partial t} = \nu \frac{\partial}{\partial r} D_5 Q, \tag{18.3}$$

and we see that the inertial and viscous terms differ in their effects by a time derivative.

Expanding (15.48), we have

$$\frac{\partial}{\partial r}\left(\frac{\partial^2}{\partial t^2} + 2\nu D_5 \frac{\partial}{\partial t} + \nu^2 D_5^2\right) Q = 2Q \frac{\partial}{\partial r} D_5 Q. \tag{18.4}$$

If we neglect $O(\nu^2)$,

$$\frac{\partial^3}{\partial r \partial t^2} Q = 2\left(Q - \nu \frac{\partial}{\partial t}\right) \frac{\partial}{\partial r} D_5 Q. \tag{18.5}$$

We see that the operators $-Q$ and $\nu \partial/\partial t$ are the effects, respectively, of the inertial term [see (18.1)] and of the viscous term [see (18.3)], and that they combine linearly. This similarity in the behaviour of the two operators is the essential assumption of the Heisenberg theory.

## 18.2 Compatibility with the Kolmogorov Theory

We choose our scale so that

$$f(0) = 1.$$

The Kolmogorov theory tells us that in this case [see (13.20)]

$$f(r) = 1 - \left(\frac{r}{r_0}\right)^{2/3}.$$

In the Kolmogorov case of $\nu \to 0$, (17.3) becomes

$$\frac{\partial^3 f}{\partial t^2 \partial r} = -f \frac{\partial}{\partial r} D_5 f. \tag{18.6}$$

If we consider only values of $f$ not far from unity, and substitute

$$f = 1 - g \tag{18.7}$$

in the foregoing, we may neglect $O(g^2)$. We then have

$$\frac{\partial^3 g}{\partial r \partial t^2} = -\frac{\partial}{\partial r} D_5 g. \tag{18.8}$$

Alternatively, we may write

$$\frac{\partial^2 g}{\partial t^2} = -D_5 g. \tag{18.9}$$

Since at $t = 0$, we have $g \sim r^{2/3}$,

$$\left(\frac{\partial^2 g}{\partial t^2}\right)_{t=0} = -\text{const.} \times D_5 r^{2/3} = -\text{const.} \times r^{-4/3}. \tag{18.10}$$

Equation (18.10) indicates a life time of $r^{-2/3}$ for an eddy of size $r$. The Kolmogorov theory predicts a life time of

$$\tau \approx \frac{1}{v_k k}. \tag{18.11}$$

Now (see p. 29)

$$v_k \sim k^{-1/3},$$

so that

$$\tau \sim r^{-2/3},$$

in agreement with the result from (18.10). (This demonstration of compatibility is due to Fermi.)

# Chapter 19
# The Problem of Introducing the Boundary Conditions

Since we are considering isotropic turbulence, the presence of a boundary seems inconsistent with the framework. In particular, it is difficult to see how one could introduce $\varepsilon$ into the problem at all.

If we neglect viscosity, we have

$$\frac{\partial^3 f}{\partial r \partial t^2} = -f \frac{\partial}{\partial r}\left(\frac{\partial^2 f}{\partial r^2} + \frac{4}{r}\frac{\partial f}{\partial r}\right). \tag{19.1}$$

One boundary condition of which we may be certain is

$$f(0,0) = 1. \tag{19.2}$$

It is tempting to say that $f$ should be bounded as $r \to \infty$, but it is not feasible to deal with $r$ greater than the largest eddies present. We do have, though, the following (apparently) reasonable condition:

$$f(r,t) \to 0 \quad \text{as } t \to \infty \quad (\text{for all } r). \tag{19.3}$$

Finally, it may be possible to introduce $\varepsilon$ if we include viscosity. For in that case, i.e., $\nu \neq 0$, we have, according to (13.11),

$$15\nu\overline{u_1^2}\left(\frac{\partial^2 f}{\partial r^2}\right)_{0,0} = -\varepsilon. \tag{19.4}$$

But we refrain from applying this at the present state of the theory.

In the framework of the Kolmogorov theory, we would expect (18.11) to yield only one solution (apart from scale factors), and indeed (18.11) does have certain homologous properties. But, more generally, one feels that a turbulent medium (apart from intrinsic properties like $\nu$) may exhibit states of motion differing by more than scale transformations and differing according to the value of $\varepsilon$ associated with the medium. Accordingly, we may expect (18.11) to yield a one parameter family of solutions, where different values of the parameter correspond to different values of $\varepsilon$.

E.A. Spiegel (ed.), *The Theory of Turbulence*, Lecture Notes in Physics 810,
DOI 10.1007/978-94-007-0117-5_19, 

# Chapter 20
# Discussion of the Case of Negligible Inertial Term

If we ignore the inertial term in the equation of motion, we find that the equation for $f$ is, according to (18.2),

$$\frac{\partial f}{\partial t} = \nu \left( \frac{\partial^2 f}{\partial r^2} + \frac{4}{r} \frac{\partial f}{\partial r} \right). \tag{20.1}$$

In this situation, we may use (19.4) as a boundary condition. Of course, (19.2) is valid too. We have seen that we cannot discuss the behaviour of $f$ as $r \to \infty$, but we can use as a restriction some condition, such as $r$ tends to some $r_0$. Consistently with (19.3), we might impose the condition (for large $t$, at any rate) that

$$f = \mathrm{e}^{-\lambda t} \phi \quad \text{at } r = 0. \tag{20.2}$$

Substitution of (20.2) into (20.1) yields

$$-\lambda \phi = \nu \left( \frac{\partial^2 \phi}{\partial r^2} + \frac{4}{r} \frac{\partial \phi}{\partial r} \right). \tag{20.3}$$

This equation may be rewritten

$$\phi'' + \frac{a}{r} \phi' + k^2 \phi = 0, \tag{20.4}$$

where

$$a = 4, \qquad k^2 = \frac{\lambda}{\nu}, \tag{20.5}$$

and primes denote differentiation with respect to $r$.

We may make the transformation

$$\phi = r^n \theta. \tag{20.6}$$

We note that

$$\phi' = n r^{n-1} \theta + r^n \theta'$$

E.A. Spiegel (ed.), *The Theory of Turbulence*, Lecture Notes in Physics 810,
DOI 10.1007/978-94-007-0117-5_20, 

and

$$\phi'' = n(n-1)r^{n-2}\theta + 2nr^{n-1}\theta + r^n\theta'',$$

so that (20.4) becomes

$$r^n \left\{ \theta'' + \frac{2n+a}{r}\theta' + \left[ \frac{n(n-1+a)}{r^2} + k^2 \right] \theta \right\} = 0. \tag{20.7}$$

If we choose $n$ such that

$$2n + a = 1,$$

we find that

$$n = \frac{1}{2}(1-a). \tag{20.8}$$

With this choice, we conclude that

$$\begin{aligned} n(n-1+a) &= \frac{1-a}{2}\left(\frac{1-a}{2} + a - 1\right) \\ &= -\left(\frac{1-a}{2}\right)^2 \\ &= -n^2. \end{aligned} \tag{20.9}$$

Hence, for $r \neq 0$ at any rate, we find that $\theta$ satisfies

$$\theta'' + \frac{1}{r}\theta' + \left(k^2 - \frac{n^2}{r^2}\right)\theta = 0, \tag{20.10}$$

where $n$ is given by (20.8). For $a = 4$, we have $n = -3/2$. Thus,

$$\theta = J_{-3/2}(kr)$$

is a solution of (20.10). It follows that

$$\theta = J_{3/2}(kr)$$

must also be a solution, and since these two Bessel's functions must be independent, the complete solution of (20.10) may be written

$$\theta = AJ_{3/2}(kr) + BJ_{-3/2}(kr). \tag{20.11}$$

For $r = 0$, $J_{-3/2}(kr)$ is unbounded and so, according to the boundary conditions,

$$B = 0.$$

Thus,

$$\phi = \frac{A}{r^{3/2}} J_{3/2}(kr). \tag{20.12}$$

We may arbitrarily take

$$A = \frac{c}{k^{3/2}},$$

where $c$ is a constant, and may depend on the value of $k$. In general, $k$ will be an eigenvalue determined by the boundary conditions. In particular, if we require that $f = 0$ at $r = r_0$, then $k$ will correspond to values for which $J_{3/2} = 0$.

We now have

$$\phi(r) = c\frac{J_{3/2}(kr)}{(kr)^{3/2}}. \tag{20.13}$$

Of course,

$$f(r,0) = \phi(r), \tag{20.14}$$

and if we require

$$f(0,0) = 1,$$

this is equivalent to requiring

$$\phi(0) = 1. \tag{20.15}$$

But

$$J_{3/2}(kr) = \sum_{m=0}^{\infty}(-1)^m \frac{(kr)^{3/2+2km}}{2^{3/2+2m}m!\Gamma(5/2+m)} \tag{20.16}$$

and

$$\phi(r) = c\sum(-1)^m \frac{(kr)^{2m}}{2^{3/2+2m}m!\Gamma(5/2+m)}. \tag{20.17}$$

According to (20.15), we must have

$$c\frac{1}{2^{3/2}\Gamma(5/2)} = 1,$$

or

$$c = 2^{3/2}\Gamma(5/2). \tag{20.18}$$

The eigenvalues of $k$ will be specified by (19.4). We have

$$f(r,t) = 2^{3/2}\Gamma(5/2)J_{3/2}(kr)\mathrm{e}^{-\lambda t}. \tag{20.19}$$

Clearly,

$$\left(\frac{\partial^2 f}{\partial r^2}\right)_{0,0} = \left(\frac{\partial^2 \phi}{\partial r^2}\right)_0 \equiv \phi''(0). \tag{20.20}$$

Now

$$\phi''(r) = 2^{3/2}\Gamma(5/2)\sum_{m=1}^{\infty}(-1)^m \frac{2m(2m-1)(kr)^{2m-2}k^2}{2^{3/2+2m}m!\Gamma(5/2+m)},$$

and

$$\phi''(0) = -\frac{2k^2}{2^2 \cdot 5/2} = -\frac{k^2}{5}. \tag{20.21}$$

Then by (19.4),

$$k^2 = \frac{\lambda}{\nu} = \frac{\varepsilon}{3\nu\overline{u_1^2}}, \tag{20.22}$$

from which we find the eigenvalues

$$\lambda = \frac{\varepsilon}{3\overline{u_1^2}} = \frac{\varepsilon}{\overline{u^2}}. \tag{20.23}$$

We note that $\lambda$ does not depend on the viscosity. Moreover,

$$\varepsilon = \frac{\mathrm{d}\overline{u^2}}{\mathrm{d}t},$$

and $\varepsilon/\overline{u_2}$ is the life time of an eddy, that is, $\lambda$ is a relaxation time.

Finally, we note that

$$J_{3/2}(kr) = \left(\frac{2}{\pi kr}\right)^{1/2}\left(\frac{\sin kr}{kr} - \cos kr\right), \tag{20.24}$$

and $k$ may thus be determined by the wavelength of the oscillation of $f$ as $r \to \infty$. However, we found $k$ in this case by evaluating the curvature at the origin.

# Chapter 21
# The Case in Which Viscosity Is Neglected

If the viscosity is negligible, the equation in $f$ becomes (19.1), viz.,

$$\frac{\partial^3 f}{\partial t^2 \partial r} = -f \frac{\partial}{\partial r}\left(\frac{\partial^2}{\partial r^2} + \frac{4}{r}\frac{\partial}{\partial r}\right) f. \tag{21.1}$$

We first attempt to discover the existence of separable solutions. Let

$$f = \phi(t)\psi(r). \tag{21.2}$$

Then, upon substitution into (21.1), we find

$$\phi''(t)\psi'(r) = -\phi^2(t)\psi(r)\left[\psi'''(r) + \frac{4}{r}\psi''(r) - \frac{4}{r^2}\psi'(r)\right],$$

where primes denote differentiation with respect to the variable in the argument. We have then,

$$\frac{\phi''(t)}{\phi^2(t)} + \frac{\psi(r)}{\psi'(r)}\left[\psi'''(r) + \frac{4}{r}\psi''(r) - \frac{4}{r^2}\psi'(r)\right] = 0. \tag{21.3}$$

Then we may set

$$\frac{\phi''}{\phi^2} = -\alpha,$$

or

$$\phi'' + \alpha\phi^2 = 0. \tag{21.4}$$

We multiply (21.4) by $\phi'$ to obtain

$$\frac{\mathrm{d}^2\phi}{\mathrm{d}t^2}\frac{\mathrm{d}\phi}{\mathrm{d}t} = -\alpha\phi^2\frac{\mathrm{d}\phi}{\mathrm{d}t}, \tag{21.5}$$

which may be rewritten

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}\phi}{\mathrm{d}t}\right)^2 = -\frac{\alpha}{3}\frac{\mathrm{d}}{\mathrm{d}t}\phi^3. \tag{21.6}$$

E.A. Spiegel (ed.), *The Theory of Turbulence*, Lecture Notes in Physics 810,
DOI 10.1007/978-94-007-0117-5_21, © Springer Science+Business Media B.V. 2011

This may be integrated:

$$\left(\frac{\mathrm{d}\phi}{\mathrm{d}t}\right)^2 = -\frac{2\alpha}{3}\phi^3 + C_1. \tag{21.7}$$

We know that $\phi \to 0$ as $t \to \infty$ and we would like to have $\mathrm{d}\phi/\mathrm{d}t \to 0$ under these circumstances as well. Thus we are led to set

$$C_1 = 0.$$

(If we did not set $C_1 = 0$, we would obtain elliptic integrals as solutions.) Further, we specify that $\alpha$ is negative. Then

$$\frac{\mathrm{d}\phi}{\mathrm{d}t} = C_2\phi^{3/2}, \tag{21.8}$$

and

$$\frac{\mathrm{d}\phi}{\phi^{3/2}} = C_2\mathrm{d}t. \tag{21.9}$$

We then have

$$\phi^{-1/2} = C_2 t + C_3. \tag{21.10}$$

$C_3$ will be, in any case, unimportant for large $t$ and we set

$$C_3 = 0.$$

We thus have

$$\phi = \frac{C}{t^2}, \tag{21.11}$$

and

$$f = \frac{C}{t^2}\psi(r). \tag{21.12}$$

If we put this into (18.11), we find

$$\frac{\mathrm{d}\psi}{\psi} = -\frac{C}{6}\mathrm{d}\left(\frac{\mathrm{d}^2\psi}{\mathrm{d}r^2} + \frac{4}{r}\frac{\mathrm{d}\psi}{\mathrm{d}r}\right),$$

or

$$\frac{C}{6}\left(\frac{\mathrm{d}^2\psi}{\mathrm{d}r^2} + \frac{4}{r}\frac{\mathrm{d}\psi}{\mathrm{d}r}\right) + \ln\psi = C_4. \tag{21.13}$$

If we change scale appropriately, we may write

$$\frac{\mathrm{d}^2\psi}{\mathrm{d}r^2} + \frac{4}{r}\frac{\mathrm{d}\psi}{\mathrm{d}r} + \ln\psi = K. \tag{21.14}$$

Clearly, (21.14) has the solution

$$\psi = \psi_0, \tag{21.15}$$

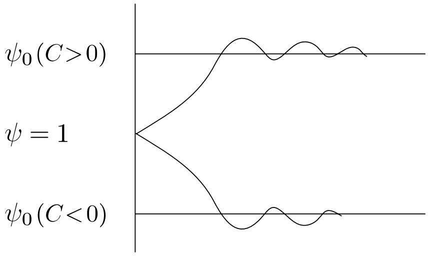


**Fig. 21.1** Solutions of (21.14)

where $\psi_0$ is a constant. In particular, $\psi_0 = \mathrm{e}^K$. We may ask whether other solutions can be found from the intuitive knowledge that they may approach this solution asymptotically. Accordingly, we seek solutions of the form

$$\psi = \psi_0 + \phi, \tag{21.16}$$

where

$$\phi \to 0 \quad \text{as } r \to \infty. \tag{21.17}$$

Upon substitution of (21.16) into (21.14), we have

$$\frac{\mathrm{d}^2\phi}{\mathrm{d}r^2} + \frac{4}{r}\frac{\mathrm{d}\phi}{\mathrm{d}r} + \ln\left[\psi_0\left(1 + \frac{\phi}{\psi_0}\right)\right] = K, \tag{21.18}$$

which, since $\psi_0 = \mathrm{e}^K$, becomes

$$\frac{\mathrm{d}^2\phi}{\mathrm{d}r^2} + \frac{4}{r}\frac{\mathrm{d}\phi}{\mathrm{d}r} + \ln\left(1 + \frac{\phi}{\psi_0}\right) = 0. \tag{21.19}$$

Since

$$\ln(1+u) = \sum_{n=1}^{\infty} \frac{u^n}{n},$$

and because of (21.17), we may write approximately, for large $r$,

$$\frac{\mathrm{d}^2\phi}{\mathrm{d}r^2} + \frac{4}{r}\frac{\mathrm{d}\phi}{\mathrm{d}r} + \frac{1}{\psi_0}\phi = 0. \tag{21.20}$$

We saw in the previous section that (21.20) is equivalent to Bessel's equation and has the solution

$$\phi = D\frac{J_{3/2}(kr)}{(kr)^{3/2}}, \tag{21.21}$$

where $D$ is a constant and

$$k^2 = \frac{1}{\psi_0}. \tag{21.22}$$

Thus we see that, for large $r$, (21.14) admits the solutions shown in Fig. 21.1. We have therefore seen that, for sufficiently large $t$ and $r$, the function $f$ is of the form

$$f = \frac{A}{t^2}\left[\psi_0 + B\frac{J_{3/2}(kr)}{(kr)^{3/2}}\right], \tag{21.23}$$

which is not unlike the form of solution obtained in the pure viscous case.

# Chapter 22
# Solution of the Non-Viscous Case Near $r = 0$

We saw that, in the Kolmogorov theory [see (13.20) on p. 57],

$$f = 1 - \left(\frac{r}{r_0}\right)^{2/3},$$

so that, by letting

$$\phi = 1 - f, \tag{22.1}$$

we find that, for small $r$, and hence small $\phi$,

$$\frac{\partial^3 \phi}{\partial t^2 \partial r} = -\frac{\partial}{\partial r}\left(\frac{\partial^2 \phi}{\partial r^2} + \frac{4}{r}\frac{\partial \phi}{\partial r}\right). \tag{22.2}$$

Upon integration of (22.2), we have

$$\frac{\partial^2 \phi}{\partial t^2} = -\left(\frac{\partial^2 \phi}{\partial r^2} + \frac{4}{r}\frac{\partial \phi}{\partial r}\right) + h(t), \tag{22.3}$$

where $h(t)$ is an arbitrary function of $t$. The effect of $h(t)$ is to add an arbitrary function of time to $\phi$ in the form of a particular integral of (22.3). We will not be interested in such an addition and will take

$$h(t) \equiv 0. \tag{22.4}$$

Then (22.3) becomes

$$\frac{\partial^2 \phi}{\partial t^2} = -\left(\frac{\partial^2 \phi}{\partial r^2} + \frac{4}{r}\frac{\partial \phi}{\partial r}\right), \tag{22.5}$$

which we must solve with the boundary condition

$$\phi = \phi(r, 0), \qquad t = 0. \tag{22.6}$$

It is clear that $D_5$, the 5-dimensional Laplacian operator, is

$$D_5 = \sum_{i=1}^{5} \frac{\partial^2}{\partial x_i^2}, \tag{22.7}$$

E.A. Spiegel (ed.), *The Theory of Turbulence*, Lecture Notes in Physics 810,
DOI 10.1007/978-94-007-0117-5_22, 

and that (22.5) is

$$\frac{\partial^2 \phi}{\partial t^2} = -\sum_{i=1}^{5} \frac{\partial^2 \phi}{\partial x_i^2}. \tag{22.8}$$

We will treat (22.8) as a quasi-wave equation, rather than as Laplace's equation in 6 dimensions, and will proceed to the solution after an illustrative example in the next chapter.

# Chapter 23
# Solution of the Heat Equation

Consider the equation of heat diffusion:

$$\frac{\partial \theta}{\partial t} = \kappa \nabla^2 \theta. \tag{23.1}$$

We will set $\kappa = 1/2$ and treat

$$\frac{\partial \theta}{\partial t} = \frac{1}{2} \nabla^2 \theta. \tag{23.2}$$

If at $t = 0$, $\theta$ is a unit heat source, the temperature distribution at time $t$ is

$$\theta = \frac{1}{\sqrt{2t}} \mathrm{e}^{-x^2/2t}. \tag{23.3}$$

This may be verified by substitution in (23.2).

A more general problem is: given any arbitrary distribution of $\theta$ at $t = 0$, how is $\theta$ distributed at a later time $t$? One can express the distribution at $t = 0$ in terms of unit sources ($\delta$-functions) in the following way:

$$\theta(x) = \int_{-\infty}^{+\infty} \theta(a) \delta(x - a) \mathrm{d}a. \tag{23.4}$$

Then given any $\theta(\boldsymbol{r})$ at $t = 0$, $\theta(\boldsymbol{r}, t)$ may be expressed as

$$\theta(\boldsymbol{r}, t) = \iiint_{-\infty}^{+\infty} \theta(\boldsymbol{r}') \Theta\big(|\boldsymbol{r} - \boldsymbol{r}'|^2, t\big) \mathrm{d}\boldsymbol{r}, \tag{23.5}$$

where

$$\Theta = \frac{A}{t^{3/2}} \mathrm{e}^{-|\boldsymbol{r} - \boldsymbol{r}'|^2/at}. \tag{23.6}$$

E.A. Spiegel (ed.), *The Theory of Turbulence*, Lecture Notes in Physics 810,
DOI 10.1007/978-94-007-0117-5_23, 

# Chapter 24
# Solution of the Quasi-Wave Equation

In order to make the wave equation invariant under translations of the origin, we wrote it in the form (22.8):

$$\frac{\partial\phi^2}{\partial t^2} = -\sum_{i=1}^{5}\frac{\partial^2\phi}{\partial x_i^2}. \tag{24.1}$$

The appropriate $\delta$-function (or unit source) is

$$\frac{2}{\pi^3}\frac{t}{(t^2+|\boldsymbol{r}-\boldsymbol{r}'|^2)^3}. \tag{24.2}$$

To test whether (24.2) is an appropriate $\delta$-function, we merely substitute

$$\frac{2}{\pi^3}\frac{t}{(t^2+r^2)^3} \tag{24.3}$$

into (22.5) and note that it is identically satisfied. In order to test the normalization of (24.2), we investigate the integral

$$\frac{2}{\pi^3}t\iiiint\!\!\int_{-\infty}^{+\infty}\frac{\mathrm{d}V}{(t^2+r^2)^3}, \tag{24.4}$$

where $\mathrm{d}V = \mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}x_3\,\mathrm{d}x_4\,\mathrm{d}x_5$, or in terms of spherical polars:

$$\left.\begin{aligned}
x_1 &= r\cos\theta,\\
x_2 &= r\sin\theta\cos\phi_1,\\
x_3 &= r\sin\theta\sin\phi_1\cos\phi_2,\\
x_4 &= r\sin\theta\sin\phi_1\sin\phi_2\cos\phi_3,\\
x_5 &= r\sin\theta\sin\phi_1\sin\phi_2\sin\phi_3.
\end{aligned}\right\} \tag{24.5}$$

E.A. Spiegel (ed.), *The Theory of Turbulence*, Lecture Notes in Physics 810,
DOI 10.1007/978-94-007-0117-5_24, 

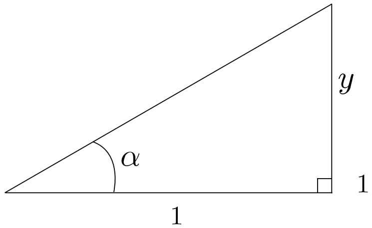


**Fig. 24.1** Transformation for calculating the integral in (24.7)

The integral (24.4) becomes

$$\begin{aligned}&\frac{2}{\pi^3}t\int_0^\infty\int_0^\pi\int_0^\pi\int_0^\pi\int_0^{2\pi}\frac{r^4\sin^3\theta\sin^2\phi_1\sin\phi_2\mathrm{d}\theta\,\mathrm{d}\phi_1\,\mathrm{d}\phi_2\,\mathrm{d}\phi_3\mathrm{d}r}{(t^2+r^2)^3}\\&\quad=\frac{16}{3\pi}t\int_0^\infty\frac{r^4\mathrm{d}r}{(t^2+r^2)^3}.\end{aligned}\tag{24.6}$$

Putting $r=ty$, we find

$$\frac{16}{3\pi}t\int_0^\infty\frac{r^4\mathrm{d}r}{(t^2+r^2)^3}=\frac{16}{3\pi}\int_0^\infty\frac{y^4\mathrm{d}y}{(y^2+1)^3}.\tag{24.7}$$

If we perform the transformation implied by Fig. 24.1, we find that (24.7) is

$$\frac{16}{3\pi}\int_0^\infty\frac{y^4\mathrm{d}y}{(y^2+1)^3}=\frac{16}{3\pi}\int_0^{\pi/2}\sin^4\alpha\,\mathrm{d}\alpha=1.\tag{24.8}$$

Note that it may also be shown that (24.2) is a unique point source in this problem.

As in the last chapter, if we are given an initial distribution

$$\Phi(\boldsymbol{r})\quad\text{at } t=0,$$

we have at time $t$

$$\phi(\boldsymbol{r},t)=\frac{2}{\pi^3}t\iiiint\!\!\!\int_{-\infty}^{+\infty}\frac{\Phi(\boldsymbol{r}')\mathrm{d}x_1'\,\mathrm{d}x_2'\,\mathrm{d}x_3'\,\mathrm{d}x_4'\,\mathrm{d}x_5'}{(t^2+|\boldsymbol{r}-\boldsymbol{r}'|^2)^3}.\tag{24.9}$$

Equation (24.9) is the solution of the boundary value problem of (22.8). For the spherically symmetric case

$$\begin{aligned}\phi(\boldsymbol{r},t)&=\frac{2t}{\pi^3}\int_0^\infty\int_0^\pi\int_0^\pi\int_0^\pi\int_0^{2\pi}\frac{\Phi(r')r'^4\sin^3\theta\sin^2\phi_1\sin\phi_2\mathrm{d}r'\,\mathrm{d}\phi_1\,\mathrm{d}\phi_2\,\mathrm{d}\phi_2\,\mathrm{d}\theta}{(t^2+r^2+r'^2-2rr'\cos\theta)^3}\\&=\frac{4t}{\pi}\int_0^\pi\sin^3\theta\mathrm{d}\theta\int_0^\infty\frac{\Phi(r')r'^4\mathrm{d}r'}{(t^2+r^2+r'^2-2rr'\cos\theta)^3}.\end{aligned}\tag{24.10}$$

This provides a unique solution. The integration over $\theta$ is messy and is not performed here.

In the framework of isotropy, the integration over infinite $r$ is not permissible. On the other hand, if the integral converges, no large contribution will be made at large $r$ and so this integral is approximately correct, even in isotropy, if it converges. We assume convergence, and thus have a restriction on $\Phi(r')$, namely

$$\Phi(r') \lesssim r'. \tag{24.11}$$

In particular, $\Phi(r') \sim (r')^{2/3}$ is permitted by this restriction.

If in (24.10), we replace $r'$ by the complex variable $z$, the integrand has no zeros on the real axis, but its denominator has zeros in the complex plane. The calculus of residues can therefore be used in evaluation of the integral, if we are careful of the difficulty introduced by the fact that the origin is a branch point.

The complex integral is

$$\phi = \frac{4t}{\pi}\int_0^{\pi} \mathrm{d}\theta \sin^3\theta \int_0^{\infty} \frac{z^{4+a}\mathrm{d}z}{(t^2+r^2+z^2-2rz\cos\theta)^3}. \tag{24.12}$$

This becomes

$$\phi = r^a \frac{(4+a)(3+a)}{2\sin a\pi} x(x^2+1)\frac{a}{2}\int_{\arctan x}^{\pi-\arctan x} \left[(1-x^2)\cot^2\psi\right]\sin(2+a)\psi\,\mathrm{d}\psi, \tag{24.13}$$

where

$$x = \frac{t}{r}. \tag{24.14}$$

We see, therefore, that

$$\phi = r^a\psi(x). \tag{24.15}$$

Clearly, $\phi$ as given by (24.15) should satisfy (22.8) and we may substitute (24.15) back in. This procedure should give us an equation in $\psi(x)$.

We note that

$$\frac{\mathrm{d}x}{\mathrm{d}r} = -\frac{t}{r^2} = -\frac{X}{r}. \tag{24.16}$$

In making the substitution into (24.15), we find it necessary to make the following calculations:

$$\frac{\partial\phi}{\partial t} = r^{a-1}\psi'(x), \tag{24.17}$$

$$\frac{\partial^2\phi}{\partial t^2} = r^{a-2}\psi''(x), \tag{24.18}$$

$$\begin{aligned}\frac{\partial\phi}{\partial r} &= ar^{a-1}\psi(x) - r^{a-1}x\psi'(x)\\ &= r^{a-1}\left[a\psi(x) - x\psi'(x)\right],\end{aligned} \tag{24.19}$$

$$\frac{\partial^2\phi}{\partial r^2} = (a-1)r^{a-2}\big[a\psi(x) - x\psi'(x)\big] - r^{a-2}x\big[a\psi'(x) - \psi'(x) - x\psi''(x)\big]$$
$$= r^{a-2}\big[x^2\psi'' - 2(a-1)x\psi' + a(a-1)\psi\big], \tag{24.20}$$
$$\frac{4}{r}\frac{\partial\phi}{\partial r} = 4r^{a-2}\big(-x\psi' + a\psi\big). \tag{24.21}$$

Then (22.5) becomes

$$\psi''(x) = -\Big[x^2\psi''(x) - 2(a+1)x\psi'(x) + a(a+3)\psi(x)\Big], \tag{24.22}$$

or

$$(1+x^2)\psi'' - 2(a+1)x\psi' + a(a+3)\psi = 0. \tag{24.23}$$

Kamke's dictionary of differential equations gives

$$(ax^2+1)y'' + bxy' + cy = 0, \tag{24.24}$$

and adds that (24.24) admits solutions in closed form if

$$(a-b)^2 - 4ac = (2n+1)^2a^2,$$

where $n$ is an integer. In our case,

$$a = 1, \qquad b = -2(a+1), \qquad c = a(a+3),$$

and so

$$\begin{aligned}\big[1+2(a+1)\big]^2 - 4a(a+3) &= (2n+1)^2,\\ (2a+3)^2 - 4a(a+3) &= (2n+1)^2,\\ 9 &= (2n+1)^2,\end{aligned}$$

from which $n = 1$. Hence, (24.24) may be solved explicitly, and Kamke indicates that Forsythe gives the theory of this type of equation.

To solve (24.23), we let

$$x = \mathrm{e}^{\tau}. \tag{24.25}$$

Then

$$\frac{\mathrm{d}\tau}{\mathrm{d}x} = \mathrm{e}^{-\tau}, \tag{24.26}$$
$$\frac{\partial\psi}{\partial x} = \frac{\partial\psi}{\partial\tau}\frac{\partial\tau}{\partial x} = \frac{\partial\psi}{\partial\tau}\mathrm{e}^{-\tau}, \tag{24.27}$$
$$\frac{\partial^2\psi}{\partial x^2} = \frac{\partial}{\partial\tau}\left(\frac{\partial\psi}{\partial\tau}\mathrm{e}^{-\tau}\right)\mathrm{e}^{-\tau} = \left(\frac{\partial^2\psi}{\partial\tau^2} - \frac{\partial\psi}{\partial\tau}\right)\mathrm{e}^{-2\tau}. \tag{24.28}$$

Let

$$D^n = \frac{\partial^n}{\partial \tau^n}. \tag{24.29}$$

Then (24.23) becomes

$$\left(1 - e^{-2\tau}\right)(D^2 - D)\psi - 2(\alpha+1)D\psi + \alpha(\alpha+3)\psi = 0. \tag{24.30}$$

Then

$$\begin{aligned} e^{-2\tau}(D^2 - D)\psi + D^2\psi - (2\alpha+3)D\psi + \alpha(\alpha+3)\psi &= 0, \\ D(D-1)\psi + e^{2\tau}\left[D^2 - (2\alpha+3)D + \alpha(\alpha+3)\right]\psi &= 0, \\ D(D-1)\psi + e^{2\tau}(D-\alpha)(D-\alpha-3)\psi &= 0. \end{aligned} \tag{24.31}$$

Now

$$\phi(D)e^{n\tau} f = e^{n\tau}\phi(D+n) f, \tag{24.32}$$

so (24.31) can be written

$$D(D-1)\psi + (D-\alpha-2)(D-\alpha-5)e^{2\tau}\psi = 0, \tag{24.33}$$

$$\psi + \frac{(D-\alpha-2)(D-\alpha-5)}{D(D-1)}e^{2\tau}\psi = 0. \tag{24.34}$$

Let

$$\psi = (D-\alpha-2)\psi_1. \tag{24.35}$$

Then

$$\psi_1 + \frac{(D-\alpha-5)}{D(D-1)}e^{2\tau}(D-\alpha-2)\psi_1 = 0, \tag{24.36}$$

and

$$\psi_1 + \frac{(D-\alpha-5)(D-\alpha-4)}{D(D-1)}e^{2\tau}\psi_1 = 0, \tag{24.37}$$

$$\psi_1 + \frac{D-\alpha-4}{D}e^{\tau}\frac{D-\alpha-4}{D}e^{\tau}\psi_1 = 0,$$

$$\left[1 + \left(\frac{D-\alpha-4}{D}e^{\tau}\right)^2\right]\psi_1 = 0, \tag{24.38}$$

$$\pm i\psi_1 + \frac{D-\alpha-4}{D}e^{\tau}\psi_1 = 0,$$

$$\pm iD\psi_1 + e^{\tau}(D-\alpha-3)\psi_1 = 0,$$

$$\left(e^{\tau} \pm i\right)\frac{d\psi_1}{d\tau} - (\alpha+3)e^{\tau}\psi_1 = 0,$$

$$\psi_1 = A\left(e^\tau \pm i\right)^{\alpha+3}, \tag{24.39}$$

$$\psi = A\left[\left(e^\tau \pm i\right)^{\alpha+3} \mp (\alpha+3)\left(e^\tau \pm i\right)^{\alpha+2}\right], \tag{24.40}$$

or

$$\psi = A\left[\left(x \pm i\right)^{\alpha+3} \mp i(\alpha+3)\left(x \pm i\right)^{\alpha+2}\right]. \tag{24.41}$$

Now,

$$x \pm i = (x^2+1)^{1/2} \exp\left(\pm i \arctan\frac{1}{x}\right). \tag{24.42}$$

Setting

$$\theta = \arctan(1/x), \tag{24.43}$$

we have

$$\begin{aligned} x \pm i &= (x^2+1)^{1/2} e^{\pm i\theta}, \\ (x \pm i)^{\alpha+3} &= (x^2+1)^{(\alpha+3)/2} e^{\pm i(\alpha+3)\theta} \\ &= (x^2+1)^{(\alpha+3)/2}\left[\cos(\alpha+3)\theta \pm i\sin(\alpha+3)\theta\right]. \end{aligned}$$

Similarly, the second terms become

$$\begin{aligned} \mp i(x \pm i)^{\alpha+2} &= \mp i(x^2+1)^{(\alpha+2)/2}\left[\cos(\alpha+2)\theta \pm i\sin(\alpha+2)\theta\right] \\ &= (x^2+1)^{(\alpha+2)/2}\left[\sin(\alpha+2)\theta \mp i\cos(\alpha+2)\theta\right], \end{aligned}$$

and (24.41) becomes

$$\begin{aligned} \psi = A\Big\{ &(x^2+1)^{(\alpha+2)/2}\left[(x^2+1)^{1/2}\cos(\alpha+3)\theta + (\alpha+3)\sin(\alpha+2)\theta\right] \\ &\pm i(x^2+1)^{(\alpha+2)/2}\left[(x^2+1)^{1/2}\sin(\alpha+3)\theta - (\alpha+3)\cos(\alpha+2)\theta\right]\Big\}. \end{aligned} \tag{24.44}$$

We may take as the linearly independent solutions the real and imaginary parts of (21.7). Thus the solutions are

$$\left.\begin{aligned} \psi_1 &= (x^2+1)^{(\alpha+2)/2}\left[(x^2+1)^{1/2}\cos(\alpha+3)\theta + (\alpha+3)\sin(\alpha+2)\theta\right], \\ \psi_2 &= (x^2+1)^{(\alpha+2)/2}\left[(x^2+1)^{1/2}\sin(\alpha+3)\theta - (\alpha+3)\cos(\alpha+2)\theta\right]. \end{aligned}\right\} \tag{24.45}$$

Now the general solution of (24.23) will be some linear combination of $\psi_1$ and $\psi_2$, as given by boundary conditions. We note that:

- for $r \neq 0$, as $t \to 0$, $x \to 0$ and $\theta \to \pi/2$,
- for $t \neq 0$, as $r \to 0$, $x \to \infty$ and $\theta \to 0$.

Hence, in the neighbourhood of $r = 0$, we may note that

$$\psi_1 \to x^{\alpha+3} \quad \text{as } x \to \infty,$$

and

$$g_1 \to r^a \left(\frac{t}{r}\right)^{a+3} \to \frac{t^{a+3}}{r^3} \to \infty.$$

# Chapter 25
# The Introduction of Boundary Conditions

We had

$$\frac{\partial}{\partial r}\left(\frac{\partial^2}{\partial t^2} - \nu^2 D_5^2\right) f = f \frac{\partial}{\partial r} D_5 f. \tag{25.1}$$

In applying the boundary conditions, we saw that going to $r \to \infty$ is incompatible with local isotropy and we were restricted to $r < r_0$. Moreover, if $\tau$ is the lifetime of the largest eddies, we are similarly restricted to $t < \tau$. Thus the condition $f \to 0$ as $t \to \infty$ cannot be applied. The lack of boundedness of $f$ as $r$ and $t$ tend to infinity is the result of the ordering of the correlation imposed by the largest eddies. Another condition must therefore be sought.

One thing appears now as strange: the Kolmogorov principles as applied to the velocity correlations are incompatible with the new theory. Kolmogorov applies his principles to the relative velocity correlations, i.e., to

$$\psi = \overline{\frac{1}{2}\left(u_1' - u_1''\right)^2}. \tag{25.2}$$

If we now choose to measure quantities in the dimensionless units, we must have, according to the first principle,

$$\psi = (\varepsilon\nu)^{1/2}\Psi\left(r(\varepsilon/\nu^3)^{1/4}\right). \tag{25.3}$$

Moreover, if the second principle is applied, we find that

$$\Psi(x) \to C x^{2/3} \quad \text{as } \nu \to 0. \tag{25.4}$$

Now

$$\psi = \overline{u_1^2} - f(r,t), \tag{25.5}$$

and if we put this into (25.1), there results

$$\frac{\partial}{\partial r}\left(\frac{\partial^2}{\partial t^2} - \nu^2 D_5^2\right)\psi = -\psi \frac{\partial}{\partial r} D_5 \psi + \overline{u_1^2}\frac{\partial}{\partial r} D_5 \psi. \tag{25.6}$$

E.A. Spiegel (ed.), *The Theory of Turbulence*, Lecture Notes in Physics 810,
DOI 10.1007/978-94-007-0117-5_25, © Springer Science+Business Media B.V. 2011

The ordinary application of Kolmogorov's principle must now be extended to the present case of correlation over a time interval. We must choose a suitable time unit. An appropriate choice can be found by recalling that

$$[\varepsilon] = \frac{\mathrm{L}^2}{\mathrm{T}^3}, \qquad [\nu] = \frac{\mathrm{L}^2}{\mathrm{T}}. \tag{25.7}$$

Hence we may express $\psi$ as

$$\psi = (\varepsilon\nu)^{1/2}\Psi\left(r(\varepsilon/\nu^3)^{1/4}, t(\varepsilon/\nu)^{1/2}\right), \tag{25.8}$$

and if we substitute the foregoing into (25.6), we get an appropriate equation for $\Psi$. However, the appearance of $\overline{u_1^2}$ is in violation of the first Kolmogorov principle.

One therefore feels that either the theory or the application of the Kolmogorov principles to $\psi$ is incorrect. The latter supposition seems the more reasonable. This is suggested by the following argument. From the Kolmogorov principles, we found that

$$\psi \to \text{const.} \times (\varepsilon r)^{2/3},$$

and hence is unbounded. But

$$\overline{(u_1' - u_1'')^2}$$

must be bounded, so that the application of Kolmogorov's principles to $\psi$ leads to a contradiction. It is therefore natural to seek some other physical variable which may be more suited to the application of the principles.

Consider

$$\chi = -\frac{\partial \psi}{\partial r}.$$

This quantity goes as $r^{-1/3}$ and so is bounded even in the Kolmogorov theory. Moreover, it contains no reference to $r_0$ and $\overline{u_1}$, which is the aspect of $\psi$ that leads to difficulty. The lack of reference to $r_0$ and $\overline{u_1}$ removes the necessity of having to choose between the two possible fundamental length scales, $r_0$ and the velocity unit $\sqrt{\overline{u_1^2}}$.

We may now see what Chandrasekhar's equation is for $x$. We write (25.6) as

$$\frac{\frac{\partial}{\partial r}(\frac{\partial^2}{\partial t^2} - \nu^2 D_5^2)\psi}{\frac{\partial}{\partial r} D_5 \psi} = \overline{u_1^2} - \psi.$$

Then

$$\frac{\partial}{\partial r}\left[\frac{\frac{\partial}{\partial r}(\frac{\partial^2}{\partial t^2} - \nu^2 D_5^2)\psi}{\frac{\partial}{\partial r} D_5 \psi}\right] = \chi,$$

and we see that the extra term due to the $\overline{u_1^2}$ is lost in the differentiation. Another aspect in looking at the behavior of $\chi$ is to discover its form in the Kolmogorov theory. If we merely differentiate (25.8), we find that

$$\chi = (\varepsilon\nu)^{1/2}\frac{\partial\Psi(x,y)}{\partial x}\frac{\partial x}{\partial r},$$

where $x = (\varepsilon/\nu^3)^{1/4}r$, $y = (\varepsilon/\nu)^{1/2}t$, and clearly $\partial y/\partial r = 0$. Thus,

$$\chi = \left(\frac{\varepsilon^3}{\nu}\right)^{1/4}\frac{\partial\Psi}{\partial x} = \left(\frac{\varepsilon^3}{\nu}\right)^{1/4}X,$$

where $X$, the derivative of a universal function, is itself a universal function. In order to facilitate the transition to $\nu \to 0$, we choose as the unit of time $r^{2/3}/\varepsilon^{1/3}$. Then

$$\chi = \left(\frac{\varepsilon^3}{\nu}\right)^{1/4}X\left(\left(\frac{\varepsilon^3}{\nu}\right)^{1/4}r, \frac{t}{r^{2/3}}\varepsilon^{1/3}\right).$$

Now, as $\nu \to 0$, one argument of $X$ goes to infinity. But if we assume the power law and the second principle of Kolmogorov, we require that as $\nu \to 0$,

$$X\left(x, \frac{t\varepsilon^{1/3}}{r^{2/3}}\right) \to x^{-1/3}\sigma\left(\frac{t\varepsilon^{1/3}}{r^{2/3}}\right).$$

Also, $\chi \to 0$ as $r \to \infty$, or $t \to \infty$.

There is some physical basis for the choice of $\chi$ as a variable. We have

$$f = \int_0^\infty F(k)\frac{J_{3/2}(kr)}{(kr)^{3/2}}\mathrm{d}k.$$

But in local isotropy, $F(k) \to k^{-5/3}$, so that the integral diverges. But on differentiating the foregoing, we obtain (see Chamberlain and Roberts)

$$\frac{1}{r}\frac{\partial}{\partial r}(r^3 D_5 f) = -\left(\frac{2}{\pi}\right)^{1/2}\int_0^\infty kF(k)\sin kr\,\mathrm{d}k,$$

which converges. The way to make $f$ converge would be to cut off the integration at some $k_0$, but this would imply a reference to the largest eddies. $D_5 f$ is essentially the defining scalar of the vorticity. This is a local property whose description requires no reference to the larger eddies, i.e., to $r = 0$.

**Postscript** Here the notes end with a remark that points to the difference of the results of the statistical theories of the era from those of Kolmogorov's theory of 1941. The data at the time did not yet completely confirm K41 but it was the accepted version of things and the problem was to bring conformity with it.

# Epilogue

These notes do not end with a bang, for not only was the summer ending but Chandra was still wrestling with some technical details. I returned to the Yerkes Observatory for a two-week visit in the summer of 1955 and Chandra gave a seminar on the then current state of the theory. He was getting results in conflict with those of Kolmogorov and he had concluded that "either the theory or the application of Kolmogorov's principles to $\Psi$ was inappropriate." As one knows nowadays, a problem arises from the use of Eulerian coordinates in this context.

Still, Chandra pulled things together and published two papers on his approach (in 1955 and 1956). The initial reception of the theory was positive. Indeed, Stanley Corrsin once told me that, back in the mid-fifties, he was so sure that the "turbulence problem" would soon be solved that he bet George Uhlenbeck five dollars that he was right. Afterwards, when Corrsin and Uhlenbeck heard Chandra lecture on his theory, Uhlenbeck came over and handed Corrsin a fiver. It soon appeared that Uhlenbeck should have waited before parting with his money.

Numerical treatments of the theory revealed that the energy spectrum goes negative for some wavenumbers. Kraichnan pointed out that all such theories, which he called quasi-normal, and especially the multi-time ones such as Chandra's and an earlier one by Heisenberg, contained inconsistencies. Of course, the hope of the quasi-normal theory was that, by making approximation on the fourth-order correlation, one does not do too much damage to the energy conserving property of the triple correlation. The problem is that the danger is not merely quantitative. Early in these notes, Chandra remarks that phase relations are important in discussing the statistics of turbulence. In his 1955 paper he wrote:

> A description in terms of $F(k)$ only (or $Q(r)$ only) would be complete only if there were no phase relationships between the different Fourier components of the velocity field. But this is not the case. Phase relationships must exist: without them there would be no exchange of energy between the different Fourier components which is, after all, the essence of the phenomenon of turbulence. A theory, albeit an approximate theory, must incorporate in itself some element which describes these phase relationships; without such an element the theory would lack the means of accounting for the essence of the phenomenon. It would appear that by introducing the correlations in the velocity components at two different points and at two different times, we can incorporate features which are the result of these phase relationships.

E.A. Spiegel (ed.), *The Theory of Turbulence*, Lecture Notes in Physics 810,
DOI 10.1007/978-94-007-0117-5, 

His attempt to come to grips with this aspect of the statistics of turbulence turned out to be a significant source of trouble in his theory. The phase relations cannot be dealt with so easily, though the thought seems a good one. Unfortunately, it is in the multitime formulations that the inconsistencies in the quasi-normal theory show up most vivdly as Kraichnan elucidated in a paper of 1959. That matter is clarified also in the thesis[1] that Gary Deem wrote under J.B. Keller's supervision.[2]

But Chandra was not alone in finding that problems arose in applying the quasi-normal theories in Eulerian coordinates. In May 1955, I heard a lecture by George Batchelor in which he reported difficulties encountered by Proudman and Reid in their version of the approach in the form of a singularity at small wave number. Batchelor explained the problem in terms of the nonlocal behavior caused by the pressure gradient, though he made no mention of the role of incompressibility in such difficulties.

It is surprising that Chandra made no mention of any other quasi-normal theories than his own, though several existed by the time of his lectures, going back to Milliontschikov in 1941 (the same year that Kolmogorov's paper appeared). For better or worse, I have decided to keep to the informal style of the original notes from Chandra's lectures and to not include a set of relevant references, whose information is easily obtained these days. But indeed others had developed quasi-normal approximations for the statistical theory of turbulence and, in a paper on these matters, Kraichnan, gave a fairly complete list of them[3] along with an extensive discussion of the issues they present.

In his thesis, Deem tried to express the distinction among the quasi-normal theories in terms of the different time integration paths taken to develop them in the space of the different times involved. I have wondered whether something like Caratheodory's approach to thermodynamics might be introduced to enhance the understanding that Deem's work suggests, but this can be effective in only certain special cases. A much more direct way to modify Chandra's quasi-normal theory into something usable (called the EDQNM model) was given by Orszag and pursued by the group (GRETPA)[4] in the Nice Observatory. Perhaps the real problem was that none of these statistically motivated approaches made attempts to bring their theories into conformity with the physical imagery behind Kolmogorov's theory, a point well made by Kraichnan. Now there are books discussing all that and this is not the place to write another one. Rather, with Chandra's centennial upon us, it is more appropriate to make Chandra's notes available. Their value is that, besides being a very simple introduction to the theory of homogeneous turbulence, they show us Chandra's approach to science. He was, above all, a fine stylist who lent elegance and clarity to the subjects he treated through purely mathematical efforts.

[1] Kraichan, R.H.: The structure of isotopic turbulence at very high Reynolds number, Journal of Fluid Mechanics, Vol. 5, 1959, p. 497.

[2] A Nearly Normal Theory for Decaying Zero-Mean Turbulence, Thesis, NYU, 1969.

[3] In the *Proceedings of Symposia in Applied Math.* **13** "Hydrodynamic Instability" (Am. Math. Soc., 1962), pp. 199–225.

[4] GRETPA: Groupe de Recherche sur la Turbulence et les Phénomènes Aléatoires.

I believe that his lectures had a twofold purpose. They not only provided a very elementary introduction to some aspects of the subject for novices, they also allowed Chandra to organize his thoughts in preparation to formulating his attack on the statistical problem of homogeneous turbulence. He told me, in that summer of 1954, that he continually said to himself that he must find a way to close the equations for the moment hierarchy. From the notes, we get some flavor of how Chandra worked. His starting thoughts in a subject were not so different from the final forms you find in his books. Above all, he let the mathematics be his guide. This kind of approach seems to be in the spirit advocated by Dirac, though I believe that Dirac kept the physical details more firmly in mind and had the good sense to stick mainly to linear problems.

But, joking aside, I find it intriguing that the ideas that have most influenced the course of turbulence theory in the past fifty years were the thoughts of another mathematician. Komogorov's calculation may seem to be simplicity itself, but it is based on a powerful vision of the physics that continues to inspire the mathematical thinking about turbulence to this day. And in preparing these notes I see more clearly the truth in Kraichnan's criticism that the proposers of the seemingly harmless quasi-normal theories did not make much effort to bring their statistical approaches into line with Kolmogorov's vision. In fact, as Heisenberg noted, this could not be done in Eulerian coordinates.

Thus did Chandra join the list of distinguished physical scientists who had tilted at the turbulence problem and managed only to uncover further difficulties in its analysis. It is interesting that one does not find the name of Einstein on that list, though he turned out to be a grey eminence in the subject. Kraichnan was (what we now call) a postdoc at the IAS in Princeton, where he was associated with Einstein, who advised him to go into fluid dynamics. Thus did Kraichnan go on to clarify the inconsistencies of the quasi-normal theories and, by his own contribution, to leave the imprint of his quantum field theoretical background on current theories of turbulence.