A BOOLEAN ALGEBRA

A. P. Bowran

Σ

MACMILLAN EDUCATION

Introductory Monographs in Mathematics

A BOOLEAN ALGEBRA

Abstract and Concrete

10*s.* 6*d.* *net*

INTRODUCTORY MONOGRAPHS IN MATHEMATICS

General Editor

A. J. Moakes, M.A.

Numerical Mathematics — A. J. Moakes
Exercises in computing with a desk calculating machine

Mathematics for Circuits — W. Chellingsworth

The Core of Mathematics — A. J. Moakes
An introduction to 'modern' mathematics

A Boolean Algebra — A. P. Bowran
Abstract and Concrete

A BOOLEAN ALGEBRA

Abstract and Concrete

BY
A. P. BOWRAN, B.A.
LOUGHBOROUGH TRAINING COLLEGE

Macmillan Education

ISBN 978-0-333-03116-2 ISBN 978-1-349-00216-0 (eBook)
DOI 10.1007/978-1-349-00216-0

MACMILLAN AND COMPANY LIMITED
St Martin's Street London WC2
also Bombay Calcutta Madras Melbourne

THE MACMILLAN COMPANY OF CANADA LIMITED
70 Bond Street Toronto 2

ST MARTIN'S PRESS INC
175 Fifth Avenue New York 10010 NY

PREFACE

This book is written for readers who have some experience of mathematics—up to about University entrance level. To help the less experienced student, the answers to many of the early exercises have been written out in full, and can thus be used as examples. Further exercises are provided in the Supplement.

Experience of teaching Boolean Algebra and its various applications to students from 16 to 20 years old has clearly shown two points:

(i) their desire for a rigorous development of the algebra,

(ii) their need of a solid link with their earlier mathematical experiences.

Chapter 1 is intended to provide the latter, by stressing the way in which the postulates have been used in the algebra of numbers and how the basic methods have been applied. It can be omitted.

A complete grasp of all the algebra of Chapter 2 before going on to its applications is not necessary. Care has been taken in the later chapters to refer back to the paragraph containing any theorem or method used, and the necessary algebra can be done then, if preferred.

Readers who find difficulty in starting with a completely abstract algebra can read Chapter 3, on sets, before tackling the algebra, and then, when reading Chapter 2, they can imagine the elements of the algebra are sets. They could alternatively read something about sets from an elementary textbook (e.g. *Sets for Schools*).

Notation in mathematics is often a difficulty, especially before one form of notation has been generally accepted. The '$+ \cdot$' notation stresses the analogies with the algebra of numbers, but the beginner usually finds $A \cup A = A$ more digestible than $A + A = A$! I have found that most students, starting with the '$\cup \cap$' notation, and given the choice for later working, use the '$+ \cdot$' signs.

It must be stressed that this book is not intended to be a textbook on sets, logic, etc., but merely deals with some of their aspects which provide opportunities of applying the algebra. Chapter 7 suggests further reading on each of these subjects.

We thank Mr. D. P. St. Barnard and *The Observer* for their permission to use some of the 'Braintwisters'.

A. P. Bowran

CONTENTS

1

INTRODUCTION

1.1 What is an algebra? How can we form a new one, and how, then, can we use it? In this introductory chapter we try to answer these questions and to give a general idea of the subject.

An algebra is concerned with *elements* and with *operations*, which we are careful *not* to define. They obey certain laws, called *postulates*, and from them we deduce *theorems*.

Any meanings we may later give to these elements and operations will, if they satisfy the postulates, also satisfy all the theorems of the algebra. So, in Chapter 2 we develop a Boolean algebra and then go on to apply it to

(1) sets, and probability
(2) statements
(3) circuits

1.2 Our elements will be represented by capital letters, A, B, C, . . ., and operations by $\cup$ and $\cap$ or by $+$ and $.$, thus

$$(A \cup B) \quad \text{and} \quad (A \cap B)$$

or

$$(A + B) \quad \text{and} \quad (A.B)$$

These are called *binary* operations, because in each case two elements are involved; we have also a *unary* operation, which derives the element A′ from the one element, A.

For reasons that will be obvious later, we usually read $A \cup B$ as A *or* B, and $A \cap B$ as A *and* B.

1.3 The set of elements of an algebra and its operations are such that, if A and B are elements, so also are $(A \cup B)$, $(A \cap B)$, and A'; the set of elements is then said to be *closed* under these operations.

Exercise. Show that the set of the positive integers, which is closed under addition and multiplication, is not closed under subtraction nor under division. What set of numbers is closed under (i) subtraction (ii) division (iii) both?

1.4 The postulates of an algebra must be self-sufficient; no other assumption about the elements or the operations may be made. We will now consider some of these postulates as applied to arithmetic, and the algebra of numbers.

The commutative law

$$a + b = b + a$$
$$a.b = b.a$$

These statements appear trivial in the algebra of numbers, but they sometimes have their uses.

1.5 *Exercise.* Show that the chord joining the points $(at_1, a/t_1)$ and $(at_2, a/t_2)$ on the rectangular hyperbola $xy = a^2$ is

$$x + t_1.t_2.y - a(t_1 + t_2) = 0$$

That the commutative law tells us that interchanging t_1 and t_2 in this equation does not alter its truth gives us a check on our working, for it follows from the fact that the line joining A to B is the same as the line joining B to A.

1.6 *Exercise.* A, $(at_1^2, 2at_1)$, and B, $(at_2^2, 2at_2)$, are two points on the parabola $y^2 = 4ax$. Apply the commutative law as a check on the equation of the chord AB and also on the co-ordinates of the point where the tangents at A and B meet.

1.7 *Exercise.* Show that the Commutative Law, which is true for addition and for multiplication, does not hold for division nor for subtraction.

1.8 The associative law

$$(a + b) + c = a + (b + c); \qquad (a.b).c = a.(b.c)$$

This law is not a necessary assumption in the Boolean algebra we intend to develop, but it is included here because it seems so fundamental in the algebra of numbers.

1.9 *Exercise.* Is this law true for the operations of (i) subtraction (ii) division?

1.10 *Example*

$$\begin{aligned} (a + b) + c &= (b + a) + c && \mathbf{1.4} \\ &= b + (a + c) && \mathbf{1.8} \\ &= (a + c) + b && \mathbf{1.4} \\ &= \text{etc.} \end{aligned}$$

and obviously we can arrange the letters a, b, c in any order we choose, and bracket which pair we like, without altering the value of the expression. This allows us to write it as

$$a + b + c$$

where addition is still a binary operation; this expression merely emphasises the ability to add the sum of any pair of the numbers to the remaining number.

1.11 *Exercise.* In a manner similar to **1.10**, justify the expression

$$a.b.c$$

1.12 *Exercise.* (i) Prove that

$$(t_1.t_2).(t_3.t_4) = (t_1.t_3).(t_2.t_4)$$

(ii) What is the corresponding statement for the operation of addition?

1.13 From **1.5**, the chord joining the points t_1, t_2, is

$$x + (t_1.t_2).y - a.(t_1 + t_2) = 0$$

and the chord t_3, t_4 is

$$x + (t_3.t_4).y - a.(t_3 + t_4) = 0$$

and they are perpendicular if

$$(t_1.t_2).(t_3.t_4) + 1 = 0$$

1.14 *Exercise.* Show that this and **1.12** (i) give the statement: 'The orthocentre of the triangle formed by three points on a rectangular hyperbola also lies on the same hyperbola.'

1.15 *Exercise.* A, B, C, D are four points on a parabola.

The tangents	at	A, B	meet in	L,	those at	C, D	meet in M
,,	,,	A, C	,,	P,	,,	B, D	,, Q
,,	,,	A, D	,,	R,	,,	B, C	,, S

and X, Y, Z are the midpoints of LM, PQ, RS.

Use **1.6** and **1.12** (ii) to make a statement about XYZ.

1.16 **The distributive law**

$$a.(b + c) = (a.b) + (a.c)$$
$$a + (b.c) = (a + b).(a + c)$$

While the former of these is familiar to us from the algebra of numbers, the latter is not! It comes naturally from the former if we follow the rule that was followed in the previous postulates, namely, to change a statement to its *dual*, we interchange the signs (.) and (+), or their equivalents ($\cap$) and ($\cup$). We must also interchange the elements 0 and 1, if they occur.

1.17 The *theory of duality* is obviously very useful, for if, whenever we assume a postulate, we also assume its dual, then the dual of every theorem we prove will also be true.

1.18 *Exercise.* Write down the duals of the following statements:

(i) $a + a'.b = a + b$
(ii) $(y + z).(z + x).(x + y) = y.z + z.x + x.y$
(iii) $(a + b)' = a'.b'$
(iv) $a.b + b.c + c.a' = a.b + c.a'$

1.19 There exists an important duality between points and straight lines in a plane. If $A, B, C \dots$ are points and $a, b, c \dots$ are lines, we can see that the following pairs of statements are dual:

two points A, B determine a straight line (AB)	two lines a, b, determine a point (ab)
A, B, C are three points on m	a, b, c are three lines through M
C is a point on the line AB	c is a line through the point ab

1.20 *Exercise.* If X is a statement and X_d is its dual, prove that

$$(X_d)_d = X$$

1.21 *Exercise.* State the dual of the following theorem, and draw a figure to illustrate it.

If A, B, C are three points on a straight line, f, and similarly for A', B', C' on f', and also

$$\begin{array}{lll} BC' \text{ meets } B'C \text{ in } X \\ CA' \quad ,, \quad C'A \;,, \; Y \\ AB' \quad ,, \quad A'B \;,, \; Z \end{array}$$

then X, Y, Z lie on a straight line p.

1.22 *Exercise.* Desargues' theorem states that, if two triangles ABC and $A'B'C'$ are such that AA', BB', CC' are concurrent, then the points of intersection of corresponding sides of these triangles are collinear. What is remarkable about its dual?

1.23 To each of the binary operations there corresponds an *identity element*, which is such that the operation by this element leaves the other element unaltered. These are, in the algebra of numbers, zero for the operation of addition and unity for the operation of multiplication, and we use the same symbols, 0 and 1, in other algebras. So these elements are, in fact, defined by the postulates, that there shall exist different elements, 0 and 1, such that $A + 0 = A$ and that $A.1 = A$, for all A.

1.24 The *complementary element* has no corresponding element in arithmetic. It gives A', the complement of A, satisfying $A + A' = 1$ $A.A' = 0$. We shall prove that A defines A' uniquely.

1.25 *Exercise.* Using this and **1.4**, show that

$$(A')' = A$$

1.26 *Exercise.* What are the identity elements for the operations of (i) subtraction (ii) division?

1.27 As nearly all the working we have done so far has been in the algebra of numbers, we have not had cause to examine the function of the = sign in other algebras.

If we (see **1.1**) are deliberately vague about the meaning of our elements and also our operations, 'equality' must be treated in the same way, and we 'define' the meaning of the = sign by some rules it must obey. These are

(I) *The Reflexive Law* $A = A$
(II) *The Symmetric Law* If $A = B$, then $B = A$
(III) *The Transitive Law* If $A = B$ and $B = C$, then $A = C$

This is the law that allows us to 'simplfy' an equation. For if $A = B$, and A, B have simpler forms $A_1 = A$ and $B_1 = B$, then

$$\begin{aligned} A_1 &= A \\ &= B && \text{(III)} \\ \text{and} \quad B &= B_1 && \text{(II)} \\ \text{so} \quad A_1 &= B_1 && \text{(III)} \end{aligned}$$

(IV) If $A = B$, then $A + C = B + C$, and the dual
(IVD) If $A = B$, then $A.C = B.C$

It is very important to note that, unlike the algebra of numbers, the converse is not true—that is to say, that $A + C = B + C$ does not imply that $A = B$, nor does $A.C = B.C$. We shall later prove if *both* $A + C = B + C$ and also $A.C = B.C$, then $A = B$.

(V) If $A = B$, then $A' = B'$

1.28 *Exercise.* From these laws prove that

(i) If $A = B$, then $(A.X) + Y = (B.X) + Y$
(ii) If $A = B$, $C = B$, and $A = Y$, then $Y = C$

1.29 *Exercise.* Rewrite the postulates of **1.27** for the $\geqslant$ sign in the algebra of numbers.

1.30 *Exercise.* Check whether the Commutative and Associative Laws hold for an algebra of four elements 0, 1, B, C, and two operations, @ and &, where results of these operations are given by the tables

@	0	1	B	C
0	0	1	B	C
1	1	B	C	0
B	B	C	0	1
C	C	0	1	B

&	0	1	B	C
0	0	0	0	0
1	0	1	B	C
B	0	B	0	B
C	0	C	B	1

It is usual, in tables of this sort, to take the first element from the top row and the second from the first column, e.g.

$$\begin{aligned} B\ @\ 1 &= C \\ B\ \&\ C &= B \end{aligned}$$

What geometrical property of the table follows from the Commutative Law?

1.31 *Exercise.* An algebra contains only two elements, 0 and 1, and two operators, $\cap$ and $\cup$, and we know that duality applies. From the given table, complete the other.

$\cup$	0	1
0	0	1
1	1	1

$\cap$	1	0
1		
0		

2

A BOOLEAN ALGEBRA

2.1 Before starting on what follows, the reader is advised to turn back to the Preface for suggestions on how much of this chapter to read, and how many of the examples to do.

2.2 We consider a collection of elements represented by A, B, C, . . . , two binary operations represented by the symbols $\cup$ and $\cap$, and a unary operation ($'$), which shall obey the postulates given in the table of **2.3**. The dual of each postulate is also a postulate, so they are listed in pairs, (1), (1D); (2), (2D); etc.

2.3 The postulates

The commutative laws

$$A \cup B = B \cup A \tag{1}$$
$$A \cap B = B \cap A \tag{1D}$$

The distributive laws

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C) \tag{2}$$
$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C) \tag{2D}$$

The identity elements

$$A \cup 0 = A \tag{3}$$
$$A \cap 1 = A \tag{3D}$$

The complementary element

$$A \cup A' = 1 \tag{4}$$
$$A \cap A' = 0 \tag{4D}$$

. . . .

The associate laws

$$(A \cup B) \cup C = A \cup (B \cup C) \tag{5}$$
$$(A \cap B) \cap C = A \cap (B \cap C) \tag{5D}$$

2.4 Also we have the 'equality axioms' as given in **1.27**. When we are using **1.27**, (I), (II), or (III), this will not be stated.

2.5 *Exercise.* Rewrite **2.3** using the + . notation.

2.6 Some useful theorems

$$A \cup 1 = 1 \tag{6}$$
$$A \cap 0 = 0 \tag{6D}$$

$$A \cup A = A \tag{7}$$
$$A \cap A = A \tag{7D}$$

$$A \cup (A \cap B) = A \tag{8}$$
$$A \cap (A \cup B) = A \tag{8D}$$

$$(A')' = A \tag{9}$$

$$(0)' = 1 \tag{10}$$
$$(1)' = 0 \tag{10D}$$

$$A \cup (A' \cap B) = A \cup B \tag{11}$$
$$A \cap (A' \cup B) = A \cap B \tag{11D}$$

$$(A \cap B)' = A' \cup B' \tag{12}$$
$$(A \cup B)' = A' \cap B' \tag{12D}$$

$$\text{If } A \cup B = 0, \text{ then } A = B = 0 \tag{13}$$
$$\text{If } A \cap B = 1, \text{ then } A = B = 1 \tag{13D}$$

$$(A \cap B) \cup (B \cap C) \cup (C \cap A') = (A \cap B) \cup (C \cap A') \tag{14}$$
$$(A \cup B) \cap (B \cup C) \cap (C \cup A') = (A \cup B) \cap (C \cup A') \tag{14D}$$

$$[(A' \cap B) \cup (A \cap B')]' = (A \cap B) \cup (A' \cap B') \tag{15}$$

2.7 The theorems we are about to prove have been listed and numbered in **2.6**, so that we can state, at any step in a proof, our number for the theorem or postulate we are using.

If two steps are taken at once, both theorems used will be stated. For instance, instead of

$$1 \cap (A \cup 1) = (A \cup 1) \cap 1 \qquad (1D)$$
$$= (A \cup 1) \qquad (3D)$$

we write

$$1 \cap (A \cup 1) = A \cup 1 \qquad (1D, 3D)$$

The duals of proved theorems will be stated but not proved; the reader is recommended to do several as exercises.

2.8 To prove

$$A \cup 1 = 1 \tag{6}$$
$$A \cap 0 = 0 \tag{6D}$$

$$\begin{aligned} A \cup 1 &= (A \cup 1) \cap 1 && (3D) \\ &= (A \cup 1) \cap (A \cup A') && (4) \\ &= A \cup (1 \cap A') && (2D) \\ &= A \cup A' && (1D, 3D) \\ &= 1 && (4) \end{aligned}$$

Exercise. Prove (i) $1 \cup 1 = 1$ (ii) $0 \cup 1 = 1$ (iii) $1 \cap 0 = 0$ (iv) $0 \cap 0 = 0$

2.9 The Laws of absorption

$$A \cup (A \cap B) = A \qquad (8)$$
$$A \cap (A \cup B) = A \qquad (8D)$$

$$\begin{aligned} A \cup (A \cap B) &= (A \cap 1) \cup (A \cap B) && (3D) \\ &= A \cap (1 \cup B) && (2) \\ &= A \cap 1 && (1, 6) \\ &= A && (3D) \end{aligned}$$

2.10 The laws of tautology (the names of several of these laws come from logic or other applications of the algebra)

$$A \cup A = A \qquad (7)$$
$$A \cap A = A \qquad (7D)$$

In (8) put $B = 1$,

$$A \cup (A \cap 1) = A$$
$$A \cup A = A \qquad (3D)$$

and similarly put $B = 0$ in (8D) for (7D).

Exercise. Prove that (i) $0 \cup 0 = 0$ (ii) $1 \cap 1 = 1$.

2.11 Our next theorem (see **1.18**) is to prove that each element A has a unique complement, A′. We do this by assuming two elements, B and C, both satisfying the postualtes (4) and (4D), and proving them equal. So, given

(a) $A \cup B = A \cup C = 1$

(b) $A \cap B = A \cap C = 0$

to prove $B = C$

$$\begin{aligned} B &= B \cap 1 && (3D) \\ &= B \cap (A \cup C) && (a) \\ &= (B \cap A) \cup (B \cap C) && (2D) \\ &= 0 \cup (B \cap C) && (1, (b)) \\ &= B \cap C && (1, 3) \end{aligned}$$

similarly we can prove that $C = C \cap B$, and so, by (1D),

$$B = C$$

and A has one and only one complement.

2.12 It is also true that no element is its own complement; the proof of this depends on the assumption that (3) and (3D) define *different* elements, 0 and 1, so that 0 and 1 cannot be equal.

For suppose there exists an element X such that $X = X'$.

Then from (4D) we have $X \cap X = 0$ and also $X \cup X = 1$;

but $$X \cap X = X \cup X = X \qquad (7 \;\&\; 7D)$$

so $X = 0$ and $X = 1$, which is impossible, and so $X \neq X'$.

2.13 *Exercise.* Prove that

$$0' = 1 \qquad (10)$$

2.14 *Exercise.* Prove that the number of elements in a Boolean algebra is always even.

2.15 *Exercise.* Fill in the numbers of the statements that justify each step in the following proof of (11)

$$\begin{aligned} A \cup (A' \cap B) &= (A \cup A') \cap (A \cup B) \\ &= 1 \cap (A \cup B) \\ &= A \cup B \end{aligned}$$

2.16 *Exercise.* (i) Prove **de Morgan's Laws**

$$(A \cup B)' = A' \cap B' \qquad (12)$$
$$(A \cap B)' = A' \cup B' \qquad (12D)$$

Note that **2.11** shows us that it is sufficient to prove that

$$(A \cup B) \cap (A' \cap B') = 0 \quad \text{and also} \quad (A \cup B) \cup (A' \cap B') = 1$$

(ii) Extend (2) to prove that:

(a) $A \cap (P \cup Q \cup R) = (A \cap P) \cup (A \cap Q) \cup (A \cap R)$
(b) $(A \cup B) \cap (P \cup Q) = (A \cap P) \cup (A \cap Q) \cup (B \cap P) \cup (B \cap Q)$

and similarly extend (12D) to

$$(A \cup B \cup C)' = A' \cap B' \cap C'$$

when applying these we shall say we are applying (2), (12), etc.

2.17 Two other identities that are often useful are

$$(A \cap B) \cup (B \cap C) \cup (C \cap A') = (A \cap B) \cup (C \cap A') \qquad (14)$$

and its dual, (14D); for

$$\begin{aligned}
&(A \cap B) \cup (B \cap C) \cup (C \cap A')\\
&= (A \cap B) \cup [(B \cap C) \cap (A \cup A')] \cup (C \cap A')\\
&= (A \cap B) \cup (B \cap C \cap A) \cup (B \cap C \cap A') \cup (C \cap A')\\
&= (A \cap B) \cup [(A \cap B) \cap C] \cup (C \cap A') \cup [(C \cap A') \cap B)]\\
&= (A \cap B) \cup (C \cap A')
\end{aligned}$$

2.18 *Exercise.* Justify each step in **2.17.**

2.19 *Exercise.* Prove that, if both $A \cap B = A \cup C$ and also $A \cap B = A \cap C$, then $B = C$.

2.20 De Morgan's laws show us that we can, if we wish, dispense entirely with one or other of the operators $\cup$ or $\cap$.

Example. Write $A \cup \{B \cap C \cap (D \cup E)\}$:

(i) without using $\cup$
(ii) ,, ,, $\cap$

$$\begin{aligned}
\text{(i)}\quad A \cup \{B \cap C \cap (D \cup E)\} &= A \cup \{B \cap C \cap (D' \cap E')'\}\\
&= [A' \cap \{B \cap C \cap (D' \cap E')'\}']'\\
\text{(ii)}\quad A \cup \{B \cap C \cap (D \cup E)\} &= A \cup \{(B' \cup C')' \cap (D \cup E)\}\\
&= A \cup \{(B' \cup C')'' \cup (D \cup E)'\}'\\
&= A \cup \{(B' \cup C') \cup (D \cup E)'\}'
\end{aligned}$$

2.21 *Exercise.* Justify each step in Example **2.20.**

2.22 *Exercise.* Express:

(i) $A \cup B \cup C$ without using $\cup$, and
(ii) $A \cup (B \cap C) \cup (B \cap C' \cap D)$ without $\cap$

2.23 The major application of a Boolean algebra is made by expressing the idea we want in terms of the algebra, and then using the algebra to simplify the expression. Sometimes, as here,

$$(Y \cup Z) \cap (Z \cup X) \cap (X \cup Y) = (Y \cap Z) \cup (Z \cap X) \cup (X \cap Y)$$
$$(A \cup B)' = (A' \cap B')$$

neither form is obviously simpler than the other, but the choice is easy in

$$A \cup (A' \cap B) = A \cup B$$

and
$$A' \cup B' \cup (A \cap B) = 1$$

2.24 *Examples.* Simplify:

(a) $X.Y.Z + X'.Y.Z + X.Y'.Z + X.Y.Z'$

$$X.Y.Z. + X'.Y.Z = Y.Z.(X + X') \qquad (1)\ \&\ (2)$$
$$= Y.Z.1 \qquad (4)$$
$$= Y.Z \qquad (3D)$$

$$X.Y.Z + X'.Y.Z + X.Y'.Z + X.Y.Z'$$
$$= (X.Y.Z + X'.Y.Z) + (X.Y.Z + X.Y'.Z)$$
$$+ (X.Y.Z + X.Y.Z') \qquad (7)$$
$$= Y.Z + Z.X + X.Y \quad \text{(as above)}$$

(b)
$$A + A'.B.C + A'.B.C'$$
$$= A + A'.B.(C + C') \qquad (2)$$
$$= A + A'.B.(1) \qquad (4)$$
$$= A + A'.B \qquad (3D)$$
$$= A + B \qquad (11)$$

(c)
$$A.B + A'.C + A'.D + B'.C + B'.D$$
$$= A.B + (A' + B').(C + D) \qquad (2)$$
$$= A.B + (A.B)'.(C + D) \qquad (12)$$
$$= A.B + C + D \qquad (11)$$

(See also **2.28**.)

2.25 *Exercise.* Simplify the following:

(a) $A'.B'.(A + B + C)$
(b) $X.Y + X'.Y + X.Y' + X'.Y'$
(c) $A.A'.B.C + A.B.B'.C + A.B.C.C'$
(d) $A.B.C + A'.B.C. + A'.B'.C + A'.B'.C'$

2.26 In discussing expressions, etc., in Boolean algebra we retain many of the terms used in the algebra of numbers. Elements, like numbers, will be 'known', 'unknown', 'constant', 'variable', and so on, and we will talk of sums, products, and *mononomials.* These are elements represented by one letter, A, B, C, . . . or its complement A′, B′, C′, . . . or any product formed by any number of these, for example, $A.B'.C'.D$. Note that $(A.B)'$ is not a mononomial, but that $(A'.B')$ is.

2.27 *Exercise.* Which of the following are mononomials (i) $A \cap B \cap C'$ (ii) $(A')'$ (iii) $P \cap Q' \cap R$ (iv) $P' \cap Q \cap (R \cap S)'$?

2.28 The sum of a number of mononomials is called a *polynomial.* That every expression can be written in this form by using (12), (12D), and (2) is fairly easy to see.

Example. Express as a polynomial

$$(A.B + B.C + B'.X.Y).\{A + B.C + (A.X + B.Y)'\}$$
$$= (A.B + B.C + B'.X.Y).\{A + B.C + (A.X)'.(B.Y)'\} \qquad (12)$$

$$= (A.B + B.C + B'.X.Y)$$
$$.\{A + B.C + (A' + X').(B' + Y')\} \quad (12D)$$
$$= (A.B + B.C + B'.X.Y)$$
$$.(A + B.C + A'.B' + A'.Y' + B'.X' + X'.Y') \quad (2)$$
$$= (A.B + B.C + B'.X.Y)$$
$$.(A + B.C + B' + Y' + B'.X' + X'.Y') \quad (11)$$
$$= (A.B + B.C + B'.X.Y).(A + B.C + B' + Y') \quad (8)$$
$$= (A.B + B.C + B'.X.Y).(A + C + B' + Y') \quad (9, 8)$$
$$= A.A.B + A.B.C. + A.B'.X.Y + A.B.C. + B.C.C.$$
$$+ B'.C.X.Y + A.B.B' + B.B'.C + B'.B'.X.Y$$
$$+ A.B.Y' + B.C.Y' + B'.X.Y.Y' \quad (2)$$
$$= A.B + A.B.C + A.B'.X.Y + A.B.C + B.C + B'.C.X.Y$$
$$+ B'.X.Y + A.B.Y' + B.C.Y' \quad (3), (4D) \& (7D)$$
$$= A.B + B.C + B'.X.Y \quad (8)$$

2.29 *Exercise.* Express as polynomials:

(a) $(X'.Y.Z + X.Y'.Z + X.Y.Z')'$
(b) $(A.B + A'.B')'$
(c) $(A.B + C.D).(A'.B' + C'.D')'$

2.30 The disjunctive normal form. First, we are concerned with the mononomials that can be formed from a given group of elements $A_1, A_2, A_3, \ldots, A_n$; where in each mononomial each of these n elements, or its complement, is a factor.

Exercise. Show that there are 2^n such mononomials.

We proceed to show how to express any Boolean function of given elements as the sum of a number of such terms. This is called the *disjunctive normal form* of the given function. First, as in **2.28** and **2.29** we reduce the expression to an equivalent polynomial, and then

$$X + X'.Y + X'.Y'.Z$$
$$= X.(Y + Y').(Z + Z') + X'.Y.(Z + Z') + X'.Y'.Z \quad (4)$$
$$= X.Y.Z + X.Y.Z' + X.Y'.Z + X.Y'.Z' + X'.Y.Z$$
$$+ X'.Y.Z' + X'.Y'.Z \quad (2)$$

2.31 *Exercise.* Express in their disjunctive normal forms the answers to **2.25** (d) and **2.29** (a) and (b).

2.32 *Exercise.* Simplify the following disjunctive normal forms:

(i) $A.B.C + A.B'.C + A.B.C' + A.B'.C'$
(ii) $A.B'.C + A.B'.C' + A'.B'.C + A'.B'.C'$
(iii) $A.B.C.D + A.B.C.D' + A.B.C'.D + A.B.C'.D'$

2.33 The property of duality leads us at once to the dual of the disjunctive normal form, the *conjunctive normal form*, which is, of course, the product of a number of factors, each of which is the sum of n elements. To obtain the conjunctive normal form, we use an extension of (2D)

$$\begin{aligned} A.B + P.Q.R &= (A.B + P).(A.B + Q).(AB + R) \\ &= (A + P).(B + P).(A + Q).(B + Q) \\ &\quad .(A + R).(B + R) \end{aligned}$$

and then
$$\begin{aligned} L + M &= L + M + O \\ &= L + M + N.N' \\ &= (L + M + N).(L + M + N') \end{aligned}$$

Example. $X.Z + Y.Z' + X'.Y'.Z'$

$$\begin{aligned} &= (X + Y).(X + Z').(Z + Y).(Z + Z') + X'.Y'.Z' \\ &= (X + Y).(X + Z').(Z + Y) + X'.Y'.Z' \\ &= (X + Y + X').(X + Z' + X').(X' + Y + Z) \\ &\quad .(X + Y + Y').(X + Y' + Z').(Z + Y + Y') \\ &\quad .(X + Y + Z').(X + Z' + Z').(Z + Y + Z') \\ &= 1.1.(X' + Y + Z).1.(X + Y' + Z').1.(X + Y + Z) \\ &\quad .1.1 \\ &= (X' + Y + Z).(X + Y' + Z').(X + Y + Z') \end{aligned}$$

2.34 Express similarly:

(a) $A + (B + C).(C + A)$
(b) $Y.Z + Z.X + X.Y$

2.35 Some other useful theorems are:

(i) If $A = 0$ and $B = 0$, then $A \cup B = 0$ and conversely.

$$\begin{aligned} A &= 0 && \text{(given)} \\ A \cup B &= B && \textbf{1.27}\text{ (IV)} \\ &= 0 && \text{(given)} \end{aligned}$$

conversely, if $A \cup B = 0$

$$\begin{aligned} A \cap (A \cup B) &= 0 && \textbf{1.27}\text{ (IVD)} \\ A \cup (A \cap B) &= 0 && \text{(2, 7D)} \\ A &= 0 && \text{(8)} \end{aligned}$$

The dual theorems are 'If $A = 1$, and $B = 1$, then $A \cap B = 1$. and conversely'.

(ii) If $\quad A.B = 0 \quad$ (a)
and $\quad A'.C = 0 \quad$ (b)
then $\quad B.C = 0$

From (a)	$A.B.C = 0$	(IVD)
,, (b)	$A'.B.C = 0$	(IVD)
	$A.B.C + A'.B.C = 0$	**2.35** (i)
	$B.C.(A + A') = 0$	(1D, 2)
	$B.C.1 = 0$	(4)
	$B.C = 0$	(3D)

2.36 'Simplifying' a disjunctive normal form can lead to two answers which are equally 'simple' and not obviously equal. Consider

$$X.Y.Z + X.Y.Z' + X.Y'.Z + X'.Y'.Z + X'.Y'.Z'$$

$$= X.Y.(Z + Z') + X'.Y'.(Z + Z') + X.Y'.Z \qquad (2)$$
$$= X.Y + X'.Y' + X.Y'.Z \qquad (4, 3D)$$
$$= X.(Y + Y'.Z) + X'.Y' \quad \textit{or} \quad X.Y + Y'.(X' + X.Z) \qquad (2)$$
$$= X.(Y + Z) + X'.Y' \qquad = X.Y + Y'.(X' + Z) \qquad (11)$$
$$= X.Y + X'.Y' + X.Z \qquad = X.Y + X'.Y' + Y'.Z \qquad (2)$$
$$= E_1 \text{ (say)} \qquad = E_2 \text{ (say)}$$

2.37 *Exercise.* Show that these two expressions satisfy

$$E_1.E_2 = E_1$$
$$E_1.E_2' = 0$$

2.38 'Solving equations'—a process one performs so often in the early stages of the algebra of numbers—is not often useful in Boolean Algebra. We can only take an equation connecting some known elements, A, B, C, . . . and an unknown, X, and reshape it to describe X as clearly as possible. This is usually done by reducing the equation to the form

$$(A \cap X) \cup (B \cap X') = 0$$

whence $A \cap X = 0$ and $B \cap X' = 0$ **2.35** (i)

Exercise. Prove that this solution is possible

$$\text{iff} \quad A \cap B = 0$$

2.39 *Exercise.* (i) Show that, if $A = B$, then

$$A \cap B' = B \cap B' = 0$$

but that $A \cap B' = 0$ does not imply that $A = B$.

(ii) Prove that $A = B$ does follow from

$$(A \cap B') \cup (A' \cap B) = 0$$

and conversely.

2.40 Our ability (see **2.28**, etc.) to express any Boolean function as a polynomial allows us to write any equation in X in the form

$$(A \cap X) \cup (B \cap X') \cup C = (P \cap X) \cup (Q \cap X') \cup R$$

and we see, by **2.39**, that this can be written

$$\begin{aligned}&(A.X + B.X' + C).(P.X + Q.X' + R)'\\ &\qquad + (A.X + B.X' + C)'.(P.X + Q.X' + R) = 0\end{aligned}$$

which reduces to the form

$$L.X + M.X' + N = 0$$

This can be written $L.X + M.X' + N.(X + X') = 0$ whence

$$(L + N).X = (M + N).X' = 0$$

2.41 *Exercise.* (i) Perform this reduction and so get L, M, N in terms of A, B, C, P, Q, R.

(ii) Solve: (a) $A + X = B$
(b) $A.X + B = 0$

2.42 A very important function in most applicat ons of Boolean algebra is

$$(A' \cap B) \cup (A \cap B')$$

and it is called the *symmetric difference* between A and B (see **2.39** (ii)). In many books that use the $\cup$ $\cap$ notation this is represented by $A + B$, but we must use

$$(A' \cap B) \cup (A \cap B') = A \Delta B$$

Many properties of this function of A and B follow readily.

2.43 *Exercise.* Prove that:

(i) $A \Delta B = B \Delta A = A' \Delta B' = B' \Delta A'$
(ii) $(A \Delta B)' = A' \Delta B = A \Delta B'$ (see **2.29** (b))
(iii) $A \Delta B = (A \cup B) \cap (A' \cup B')$
$= (A \cup B) \cap (A \cap B)'$
$= (A \cup B) \Delta (A \cap B)$
(iv) $A \Delta 1 = A'$
(v) $A \Delta A = 0$
(vi) $A \Delta A' = 1$
(vii) $A \Delta 0 = A$
(viii) $1 \Delta 1 = 0$
(ix) $1 \Delta 0 = 1$
(x) $0 \Delta 0 = 0$
(xi) $(A \cup X) \Delta (A \cup Y) = A' \cap (X \Delta Y)$
(xii) $(A \cap X) \Delta (A \cap Y) = A \cap (X \Delta Y)$

3

SETS

3.1 We frequently think and talk of collections and groups of articles, people, statements, etc., for instance 'students', 'real numbers', 'the positive integers', 'the Ten Commandments'. We will call any such collection a *set* and will represent sets in two ways—by capital letters A, B, C . . ., or by $\{p, q, r, s\}$ where p, q, r, s are the four *members of the set.*

A change in the order of the members of a set makes no difference —we may think of them as arranged in any order we like; the members must all be different and distinguishable; so we can talk of the set of all* days of the week, but not of the set of the twelve pence in a shilling.

Two sets, P and Q, are *equal* (written P = Q) if the two sets consist of exactly the same members, so that

every member of P is also a member of Q
and also ,, ,, ,, Q ,, ,, ,, ,, ,, P

Example. If P = the soa odd numbers between 2 and 8, find an equal set.

$$P = \{3, 5, 7\}$$

and the set $\{3, 5, 7\}$ can be described in a number of ways, e.g. 'the soa odd primes less than 10', 'the soa prime factors of 105'.

Exercise. Which of the following sets are equal:

A = {Tom, Dick, Harry}
B = the soa digits of 2240
C = {Harry, Tom, Harry, Dick}
D = {0, 2, 4}
E = {2, 2, 4, 0}

3.2 The number of members of a set A is written

$$n(A)$$

* Set of all = soa.

and is often important. If two sets, A and B, have the same number of members, i.e. if

$$n(\mathrm{A}) = n(\mathrm{B})$$

we say that the sets A and B are *equivalent.*

Exercise. (i) Which of the sets in Exercise **3.1** are equivalent?

(ii) What is noteworthy in the statement, 'If $\mathrm{P} = \mathrm{Q}$, then $n(\mathrm{P}) = n(\mathrm{Q})$'?

3.3 The number of members of a set may be any integer from 0 to an infinite number. A set with no member, e.g. the soa seaside resorts in Switzerland, the soa kings of the United States is written $\{\ \}$, $\emptyset$, or 0, and is called the *null* or *empty* set. All empty sets are regarded as equal as well as equivalent.

3.4 If we consider E, the soa even numbers, we naturally think about the soa odd numbers, the 'other' set, and we can see a difficulty in the problem, what elements are not members of E? White elephants are not members of E, nor are brown cows. When we are talking of E, we are usually thinking of the soa integers, composed of two sets, the ones that are members of E and the ones that are not.

In general, we have a *universal set,* represented by 1, which contains all the elements under discussion. To every set, A, corresponds a set, A′, whose members are all the members of 1 which are not members of A. A′ is called the *complement* of A.

Exercise. Complete the statements:

(i) $1 =$ the soa human adults, $\mathrm{M} =$ the soa men, $\mathrm{M}' =$?
(ii) $\mathrm{F} =$ the soa fathers, $\mathrm{F}' =$ soa mothers, $1 =$?

3.5 The binary operations for sets are defined by

(i) all the members of $(\mathrm{A} \cap \mathrm{B})$ are members of A *and* members of B, and conversely
(ii) all the members of $(\mathrm{A} \cup \mathrm{B})$ are members of A *or* members of B, or members of both, and conversely.

Exercise.

(a) Describe the set $(\mathrm{A} \cap \mathrm{B})$ in words, when:

(i) A = the soa one's parents' children
B = ,, ,, males
(ii) A = ,, ,, cyclic quadrilaterals
B = ,, ,, parallelograms

(b) Describe the set $(A \cup B)$ in words, when:

(i) A = the soa boys
B = ,, ,, girls
(ii) A = ,, ,, positive odd numbers
B = ,, ,, ,, even ,,

3.6 We must next check that, with these meanings of the elements and operations, sets satisfy the postulates of our algebra.

Example. Translate into words equations (1), (1D), where

A = the soa tall men
B = ,, ,, dark ,,

$A \cup B = B \cup A$ tall, or dark men are dark, or tall men
$A \cap B = B \cap A$, tall, dark men are dark, tall men

3.7 *Exercise.* Check that the postulates are satisfied by giving A, B, C, 1 the following meanings

1 = the soa adult people
A = ,, ,, men
B = ,, ,, married people
C = ,, ,, employed people

With this notation, express as an equation, 'People who are men, or married women, are men or married'.

3.8 If every member of a set A is also a member of B, we say that A is a *subset* of B, and write

$$A \subseteq B$$

When we know that there exist members of B that are not members of A, we write

$$A \subset B$$

and say that A is a *proper subset* of B

If every member of A is a member of B, then no member of A is a member of B′, and we have that

$$A \subseteq B \quad \text{iff} \quad A \cap B' = 0$$

3.9 *Example.* The theorems of our algebra now give us several theorems on sets and subjects.

(i) If $A \subseteq B$, then $A \cup B = B$

For if $\quad A \subseteq B, \quad \text{then} \quad A.B' = 0$ **3.8**

$$\begin{aligned} B &= B + 0 && (3) \\ &= B + A.B' && \text{(given)} \\ &= A + B && (1D, 11, 1) \end{aligned}$$

(ii) The empty set is a subset of every set

$$\begin{aligned} 0 \cap A' &= A' \cap 0 && (1D) \\ &= 0 && (6D) \end{aligned}$$

so $\quad 0 \subseteq A$, for all A **3.8**

(iii) Every set is a subset of the universal set

$$\begin{aligned} A \cap (1)' &= A \cap 0 && (10D) \\ &= 0, && (6D) \end{aligned}$$

so $\quad A \subseteq 1$ **3.8**

(iv) If $A \subseteq B$ and $B \subseteq A$, then $A = B$

For, if $\quad A \subseteq B, \quad \text{then} \quad A \cup B = B$ **3.9** (i)
and if $\quad B \subseteq A, \quad \text{then} \quad B \cup A = A$ **3.9** (i)

so $\quad A = B$ (1), **2.4** (ii)

(v) *The syllogism.* If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$

For, $\quad A.B' = B.C' = 0$ **3.8**

so $\quad A.C' = 0$ **2.35** (ii)

and $\quad A \subseteq C$

3.10 *Exercise*

(i) Prove that, if $(A \cup B) = B$, then $A \cap B' = 0$, and so that this is a sufficient condition for $A \subseteq B$.
(ii) For all A, B, $(A \cap B) \subseteq A$
(iii) ,, ,, $A \subseteq (A \cup B)$
(iv) If $A \subseteq B$, prove that (a) $B' \subseteq A'$
(b) $A' \cup B = 1$
(v) If $X.Y = X$, prove that $X \subseteq Y$, and conversely
(vi) If $A \subseteq A'$, then $A = 0$

3.11 *Exercise.* Show that

$$\{p, q, r, s\} \cup \{t\} = \{p, q, r, s, t\}$$

and that

$$\{p, q, r, s, t\} \cap \{t\}' = \{p, q, r, s\}$$

3.12 *Example.* (With apologies to Lewis Carroll)

'Now let me see,' mused Alice, 'There are only Red Knights and White Knights, and not many White ones now, since the Duchess beheaded all the mounted White Knights for riding across the croquet lawn. That was the morning when the King confiscated the horses of any Knight who couldn't sing "Humpty-Dumpty" and *that* was a silly thing to do, because everybody knows that no Red Knight could ever sing a note.'

She looked up, and exclaimed, 'Look! There's a man on a horse! Now I wonder whether he's a Knight or not.'

Let

$$\begin{array}{rl} 1 = & \text{the soa Knights} \\ W = & \text{,, ,, White Knights} \\ R = & \text{,, ,, Red Knights} \\ M = & \text{,, ,, mounted Knights} \\ S = & \text{,, ,, singing Knights} \end{array}$$

and from the passage above we deduce

$$\begin{array}{llrcl} \text{lines } 1-2 & R = W' & & & \\ \text{,, } 3-4 & M.W = 0 & \text{or} & & M \subseteq W' \\ \text{,, } 6-7 & S' \subseteq M' & & & \\ \text{,, } 9-10 & R \subseteq S' & \text{or} & & W' \subseteq S' \end{array}$$

and so $M \subseteq W' \subseteq S' \subseteq M'$

and, by **3.10** (vi)

$M = 0$ and the man on a horse was not a Knight

3.13 In translation from words to the language of sets, it is important to be ready for different ways of expressing the same idea. We have seen, in **3.8**, **3.9**, **3.10**, that the statement $A \subseteq B$ can also be written

$$B' \subseteq A', \quad A \cap B' = 0, \quad A \cup B = B, \quad A' \cup B = 1$$

Example. Three propositions were carried at a committee meeting:

(I) All eligible candidates must be over 18 years of age, or have obtained grade A in three subjects, or both.

(II) No girl is eligible unless she has at least three grade A passes.

(III) Candidates over 18 years of age who have not obtained three grade A passes cannot be considered.

The Chairman remarked, 'Perhaps for the minutes, Mr. Secretary, you might find some simpler form for these rules.'

$$1 = \text{the soa candidates}$$

If E = the soa candidates who are eligible
G = „ „ „ „ „ girls
S = „ „ „ „ „ over 18 years old
A = „ „ „ „ have three grade A passes

(I) $E \subseteq A + S$ or $E.(A + S)' = 0$ $\quad E.A'.S' = 0$
(II) $G.E \subseteq A$ $\quad G.E.A' = 0$
(III) $E.S \subseteq A$ $\quad E.S.A' = 0$

from (I) & (III) $E.A'.S' + E.A'.S = 0$ **2.35** (1)
$E.A'(S + S') = 0$ (2)
$E.A'.1 = 0$ (4)
$E.A' = 0$ (3D)

from this we can deduce (II), so the three rules can be condensed into

'All candidates must have three grade A passes'

3.14 *Exercise.* With the notation of **3.13**, write down expressions for each of the following

(I) 'Either girl candidates, or boys over 18.'
(II) 'Candidates should have three grade A passes, or be over 18 years old, but not both.'
(III) 'Candidates should be boys over 18, or girls under 18 years of age.'

3.15 *Exercise.* The statements (a) to (e) are to be assumed.

(a) No untrained mind can really concentrate.
(b) Nobody can call himself well educated who has not travelled abroad.
(c) Some mathematics is an essential part of mind-training.
(d) To pass the Driving Test, great powers of concentration are needed.
(e) Only a good education can produce a trained mind.

Which of the following can be deduced?

(i) Foreign travel is a necessary preliminary to passing the Driving Test.
(ii) Concentration is impossible to those who have never studied mathematics.

(iii) Mathematics is an essential part of a good education.

(iv) The man who has travelled abroad, who can concentrate, and has a well-trained and educated mind is the only type that can pass the Driving Test?

3.16 *Exercise.* (An extract from Lewis Carroll)

'Of all the prisoners who were put on their trial at the last Assizes, all against whom the verdict "Guilty" was returned were sentenced to imprisonment; some who were sentenced to imprisonment were sentenced to hard labour. Hence, some against whom the sentence "Guilty" was returned were sentenced to hard labour.'

Is this sound?

3.17 *Exercise.* This 'Braintwister' by D. P. St. Barnard appeared in *The Observer.*

Christmas Party Pudding

Dear Aunt Maud,

What a pity you couldn't join us here for Christmas—we are having such fun! The late arrivals missed out on the skating, but they all got one of Tom's famous brandy-punches. He sampled it very conscientiously himself, too.

On Christmas Eve we played charades, but only the late arrivals took part—something to do with the punch, perhaps. Sally and Walter were a scream as Romeo and Juliet, and Uncle Roger had to stand on his head as a forfeit. Even Aunt Gertrude joined in.

That punch must have been potent. All who tasted it went straight out and kissed under the mistletoe—except Aunt Gertrude, who, I swear, has never been kissed in her life.

As usual, the mince pies were a failure, but everyone who didn't eat one was made to promise that they would stay over for the New Year. They all did promise except Sally, who is off to France on Saturday. Not one of those who kissed under the mistletoe would touch those mince pies, and Vera gave hers to the dog without even looking at it.

Fortunately, all those that are staying for the New Year, came down here by car, so that solves a transport problem. Funnily enough, except for Walter, who cut himself shaving, all who came by car were early for Christmas dinner, which was a real treat—the dinner, I mean.

Ever so many thanks for the Christmas pudding—it was delicious—though only those four wearing paper hats were allowed a second helping. You see, paper hats were given as a reward to all those who were not late for dinner.

Love and best wishes,

Margaret

P.S. Uncle Bertrand was awfully mad when he arrived. His car had a puncture on the way down here, and he had to fix it himself.

P.P.S. I leave you to guess who had second helpings of Christmas pudding.

Hint. Use **3.11**.

3.18 There are several ways of representing sets in diagrams. The first method we use is called the *Venn* or the *Euler* diagram. In it, a rectangle represents the universal set, and any other set, A, is represented by the area of a closed curve inside the rectangle. The rest of the rectangle represents A′.

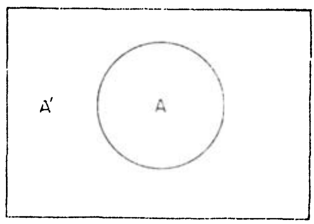


Fig. (i)

It follows that the sets named below are represented by the corresponding shaded areas.

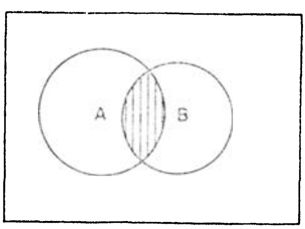


Fig. (ii). $(A \cap B)$

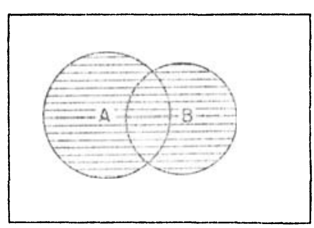


Fig. (iii). $(A \cup B)$

3.19 *Exercise.* (a) Draw the Venn diagrams for three sets A, B, C and on them show the area representing:

(i) $A \cap B \cap C$
(ii) $(A \cup B) \cap C$
(iii) $A' \cap (B \cup C)$

(b) In **3.18** Fig. (ii), if A = the soa points on circle *a* in **3.19** Fig. (i), and similarly for B, what, in **3.19** Fig. (i) is represented by the shaded area of **3.18** Fig. (ii)? When is this an empty set? When is $n(A \cap B) = 1$?

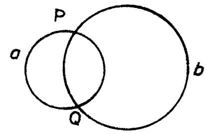


Fig. (i)

3.20 There is a type of problem that can be solved very easily by using a Venn diagram.

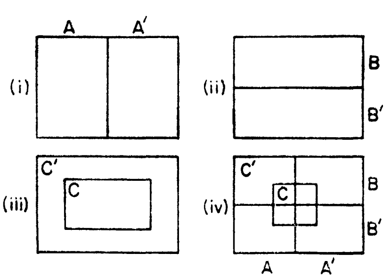
Exercise. Of a group of children it was found that

40	liked	apples
42	,,	bananas
40	,,	cherries
17	,,	bananas and cherries
19	,,	cherries and apples
22	,,	apples and bananas
7	,,	all three

Let A = the soa children who liked apples, etc., and draw a Venn diagram, and write 7 in the area that represents $(A \cap B \cap C)$; we now put 15 ($= 22 - 7$) in the area that represents $(A \cap B \cap C')$, and so on, working backwards through the list of given statements.

How many children liked apples only?

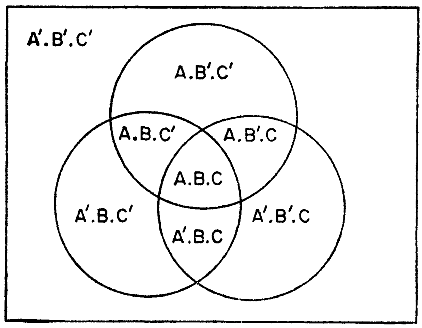
3.21 Another method of representation is the *Carroll diagram*, which can be left to explain itself.



3.22 *Exercise.* Copy **3.21** Fig. (iv) and on it shade the areas that represent:

(i) $A' \cap B' \cap C'$
(ii) $B \cap C'$

3.23 The Venn diagram for the three sets, A, B, C divides the area that represents the universal set into 8 regions, each of which represents one of the 8 terms of the complete disjunctive normal form for



the three elements, A, B, C. That there is no overlap of these regions shows (compare **2.30/1** (i)) that the corresponding sets have no common member. They are then said to be *mutually exclusive* or *disjoint*.

If X, Y are disjoint sets, then $n(X \cap Y) = 0$, and hence

$$n(X \cup Y) = n(X) + n(Y)$$

and, in general, if $A_1, A_2, A_3, \ldots$ are disjoint

$$n(A_1 \cup A_2 \cup A_3 \ldots) = n(A_1) + n(A_2) + n(A_3) \ldots$$

3.24 If P, Q are not disjoint, then

$$n(P \cup Q) = n(P) + n(Q) - n(P \cap Q)$$

Note first that $(P \cap Q)$, $(P' \cap Q)$, and $(P \cap Q')$ are disjoint, also

$$(P \cup Q) = (P \cap Q) \cup (P \cap Q') \cup (P' \cap Q) \qquad (4, 6, 11)$$
$$n(P \cup Q) = n(P \cap Q) + n(P' \cap Q) + n(P \cap Q') \qquad \mathbf{3.23}$$

and $n(P) + n(Q) - n(P \cap Q)$

$$= n\{(P \cap Q') \cup (P \cap Q)\} + n\{(P \cap Q) \cup (P' \cap Q)\} - n(P \cap Q)$$
$$= n(P \cap Q') + n(P \cap Q) + n(P \cap Q) + n(P' \cap Q) - n(P \cap Q)$$
$$= n(P \cap Q') + n(P' \cap Q) + n(P \cap Q)$$
$$= n(P \cup Q)$$

3.25 *Exercise.* (i) Prove that

$$n(\mathrm{P} + \mathrm{Q} + \mathrm{R})$$
$$= n(\mathrm{P}) + n(\mathrm{Q}) + n(\mathrm{R}) - n(\mathrm{Q}.\mathrm{R}) - n(\mathrm{R}.\mathrm{P}) - n(\mathrm{P}.\mathrm{Q}) + n(\mathrm{P}.\mathrm{Q}.\mathrm{R})$$

and find an expression for $n(\mathrm{P} + \mathrm{Q} + \mathrm{R} + \mathrm{S})$.

(ii) Show that if $\mathrm{A} \subset \mathrm{B}$, then

$$n(\mathrm{A}) < n(\mathrm{B})$$

The basic idea of **3.23** was used in **3.20.** This and the results of **3.24** have applications to questions in probability.

3.26 Richard Dedekind (1831–1916) and Georg Cantor (1845–1918) used the concept of sets to get clearer ideas on 'infinity'. We will here give only a brief and intuitive outline of their approach. We start with some apparent paradoxes.

If 1 is the soa positive integers and E of positive even integers
then $\quad \mathrm{E}' =$ soa positive odd integers
and $\quad \mathrm{E} \cap \mathrm{E}' = 0$ and $\mathrm{E} \cup \mathrm{E}' = 1$

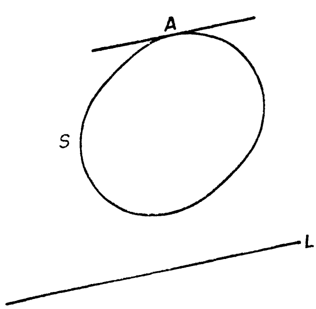
If we are prepared to extend the notation $n(S)$ to include infinite as well as finite sets then

$$n(1) = n(\mathrm{E}) + n(\mathrm{E}') \qquad \text{(i)}$$

But Cantor laid down that for both infinite and finite sets a 1:1 correspondence establishes equivalence; so if x is a positive integer, every even number can be written as $(2x)$ and so there is a one-to-one correspondence between the members of the sets 1 and E; similarly for 1 and E′, and so

$$n(1) = n(\mathrm{E}) = n(\mathrm{E}')$$

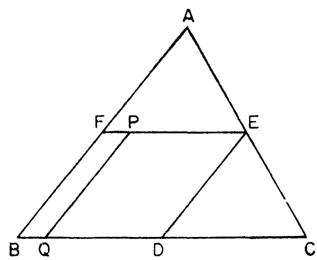
which seems to contradict (i) above.



3.27 *Exercise.* (i) L is a given straight line, and S is a closed curve cut by a straight line in not more than two points. The tangent to S at A is parallel to L. Then if

$$\begin{array}{ll}\{P_1, P_2, P_3, \ldots\} & \text{is the soa points on } S \\ \{Q_1, Q_2, Q_3, \ldots\} & \text{,, ,, ,, ,, ,, } L\end{array}$$

establish a one-to-one relation between the members of these sets.



(ii) D, E, F are the midpoints of BC, CA, AB. PQ is parallel to AB. Establish a one-to-one relation between the members of

the soa points on FE
,, ,, ,, ,, BD
,, ,, ,, ,, BC

(iii) Establish similar relationships between

the soa rational numbers between 0 and 1
,, ,, ,, ,, ,, 1 and 2
,, ,, ,, ,, greater than 1

3.28 In every case here we find the same difficulty—a set which is equivalent to a true subset of itself, and the 'number of members' of such a set cannot be treated in the usual way. Euclid's axiom 'the whole is greater than any of its parts' does not apply to such numbers.

This was taken as the criterion by which to distinguish between 'finite' and 'infinite' numbers. If a set has no proper subset equivalent to itself, it is a finite set, and has a finite number of members. If it has a proper subset equivalent to itself, we say that the number of its members is infinite.

3.29 A simple example of an infinite set is N = the soa natural numbers = 1, 2, 3, 4, 5 . . ., and we call any set whose members have a one-to-one relation with the members of this set a ***denumerable set***.

3.30 *Exercise.* Prove that the following are denumerable sets:

(i) the soa numbers divisible by 3
(ii) ,, ,, ,, which are perfect squares
(iii) ,, ,, ,, ,, ,, primes

3.31 We can see that, provided we can put the members of an infinite set into a definite ordered sequence, we can show that the set is denumerable. That the soa integers, positive or negative, is a denumerable set is shown by the sequence

$$0, 1, -1, 2, -2, 3, -3, \ldots$$

and it is, in fact, possible to prove that the soa algebraic numbers is denumerable.

3.32 This introduction to the concept of sets gives us the first subject of which our algebra is a *mathematical model* (an abstract algebra whose elements, operations, and postulates fit the subject). It enabled us to translate statements into algebra, to obtain simpler statements equivalent to them, to test them for consistency, and to deduce other statements from them. We shall find another application of the properties of sets in Chapter 6, 'Choice and Chance'.

4

TRUTH TABLES

4.1 The problem, '£100 was divided among 100 people so that each man received £10, each woman 10s., and each child 2s. 6d. How many men, women and children were there?' leads to the *diophantine equations*

$$\begin{aligned} m + \quad w + c &= 100 \\ 80.m + 4.w + c &= 800 \end{aligned}$$

and for solution, needs also the fact that the 'variables' or 'unknowns' can take only values that are positive integers.

4.2 We now consider an indefinite number of 'unknown' or 'variable' elements $A_1, A_2, A_3, \ldots$ in a Boolean algebra which can take either of the values 0 or 1, and the corresponding values—also 0 or 1—of any given function of these variables.

Exercise. Prove that if all the variables $A_1, A_2, \ldots$ take the value 0 or 1, then every function of these variables will take one of these values.

4.3 One application of such an algebra is in *symbolic logic*. Here the elements are 'truth values' of statements. If p represents a statement, then 'p is true' is represented by $P = 1$, and 'p is false' by $P = 0$. Ambiguity rarely follows from letting P represent the actual statement as well as its truth value.

The statement $A = B$ means that these two statements, A and B, have the same truth value, not that they are the same statement. So, if A is the statement 'yesterday was Monday', and B is 'tomorrow is Wednesday', then $A = B$ means 'if yesterday was Monday, then tomorrow is Wednesday, and if tomorrow is Wednesday, then yesterday was Monday'.

Exercise.

(i) Show that A' is $\begin{matrix}\text{true}\\\text{false}\end{matrix}$ when A is $\begin{matrix}\text{false}\\\text{true}\end{matrix}$

(ii) ,, ,, if $P = 0$ means that P is true we need only a change in meaning between $\cup$ and $\cap$ to obtain a consistent system.

4.4 The *truth table* of a function, F, of the variables $A_1, A_2, A_3, \ldots, A_n$, gives the value of F for every one of the possible sets of values of the n variables $A_1, A_2, A_3, \ldots, A_n$.

Exercise. Show that there are 2^n such sets of values to be considered.

Example.

(i) Write a truth table for the expression $(A.B' + A'.B)$

A	B	$A.B'$	$A'.B$	$A.B' + A'.B$
1	1	0	0	0
1	0	1	0	1
0	1	0	1	1
0	0	0	0	0

(ii) Check that the truth table of $(X + Y).(X + Y')$ is the same as that of X

X	Y	$X + Y$	$X + Y'$	$(X + Y).(X + Y')$
1	1	1	1	1
1	0	1	1	1
0	1	1	0	0
0	0	0	1	0

and we see that the first and last columns are the same.

4.5 *Exercise.* Write truth tables for the expressions:

(i) $A \cup (A' \cap B \cap C)$
(ii) $(X \cup Y \cup Z) \cap (Y' \cup Z')$

4.6 Note that a mononomial will take the value 1 in one and only one line of its truth table; for if it has the value 1, each of its factors must have this value, and this can be done in one and only one way.

Exercise. What mononomials in P, Q, R, S are represented by X and Y if we have the following values

P	Q	R	S	X	Y
1	0	1	0	1	0
0	0	0	0	0	1

4.7 We have seen (**2.30**, Exercise), that a disjunctive normal form has a maximum of 2^n terms. The function that has all these terms is called the *complete disjunctive normal form*, and will, by **4.4**, have 2^n rows in its truth table, which must, by **4.6**, all have the value 1.

That the complete disjunctive normal form is always equal to 1 can be proved by induction; for let

$$C_n = \text{the complete disjunctive normal form of the } n \text{ variables } A_1, A_2, A_3, \ldots, A_n$$

and since C_n is the *complete* form, we have

$$\begin{aligned} C_n &= (A_n \cap C_{n-1}) \cup (A'_n \cap C_{n-1}) & \\ &= C_{n-1} \cap (A_n \cup A'_n) & (1D, 2) \\ &= C_{n-1} \cap (1) & (4) \\ &= C_{n-1} & (3D) \end{aligned}$$

but

$$\begin{aligned} C_1 &= A_1 \cup A'_1 & \\ &= 1 & (4) \end{aligned}$$

and so C_n is 1 for all positive integral n.

4.8 *Exercise.* (i) Prove that if X is a disjunctive normal form, and Y is that disjunctive normal form in the same variables which contains all the terms of the complete disjunctive normal which are not terms of X, then $Y = X'$.

(ii) Write out the complete disjunctive normal form for three variables, X, Y, Z, and check that it does simplify to the value 1.

(iii) Follow the dual argument to **4.6–4.8** (i), (ii) for the conjunctive normal form.

4.9 *Exercise.* If A, B, C, ... represent statements, and

$A \cap B$	means	'A *and* B'
$A \cup B$	,,	'A *or* B or both'
A'	,,	'not A'
$A = 1$	,,	'A is true'
$A = 0$	,,	'A is false'

check that the postulates given in **2.3** hold.

4.10 *Exercise.* Check that these equations follow from the corresponding statements:

(a) 'If A is true, so is B' $A \cap B' = 0$
(b) 'Neither A nor B is true' $A \cup B = 0$
(c) 'Both the statements A and B are true' $A \cap B = 1$
(d) 'One of A and B is true' $A \cup B = 1$
(e) 'One, and only one of A and B is true'

$$(A \cap B') \cup (A' \cap B) = 1$$
$$\textit{or } A \Delta B = 1$$

(f) 'A and B are both false or both true'

$$(A \cap B) \cup (A' \cap B') = 1$$

4.11 *Exercise.* Show that **4.10** (e) can be written as $A = B'$, and (f) as $A = B$.

4.12 *Example.* The following problem, 'Braintwister No. 138' by D. P. St. Barnard, appeared in *The Observer.*

Fixing The Forecast

'Weather forecasts that cover only the next 24 hours are just not good enough. We need 48-hour forecasts; moreover they should have "built-in" provisions just in case today isn't what we think it's going to be, e.g.:

(a) If fine today, it will be windy tomorrow.
(b) If wet today, it will be fine tomorrow.
(c) If today is cold, humidity today will be high.
(d) If hot today it will be calm tomorrow.
(e) If calm today, it will be hot tomorrow.
(f) If windy today, humidity tomorrow will be low, and tomorrow will be wet.
(g) If fine tomorrow, tomorrow will be cold.
(h) Humidity tomorrow will be the same as today.

'Readers will note the disappearance of all such vague terms as "warm", "changeable", "light to variable". Every day is either Hot or Cold, Wet or Fine, Calm or Windy, and Humidity is either High or Low.

'Assuming that, for once, the forecasters will turn out to have been right, what will be the weather today and tomorrow in terms of temperature, rain, wind and humidity?'

Let H_1 = the statement, 'Today is hot'
H_2 = ,, ,, 'Tomorrow will be hot'
H_1' = ,, ,, 'Today is cold'

and similarly for W, W′, wet, fine; C, C′, calm, windy; M, M′, high, low humidity, and our statements become:

(a) $W_1'.C_2 = 0$	(f_1) $C_1'.M_2 = 0$
(b) $W_1.W_2 = 0$	(f_2) $C_1'.W_2' = 0$
(c) $H_1'.M_1' = 0$	(g) $W_2'.H_2 = 0$
(d) $H_1.C_2' = 0$	(h) $M_1 = M_2$
(e) $C_1.H_2' = 0$	

from (e) and (f_2)	$H_2'.W_2' = 0$	**2.35** (ii)
„ this and (g)	$W_2'.H_2 + W_2'.H_2' = 0$	**2.35** (i)
	$W_2'.(H_2 + H_2') = 0$	(2)
	$W_2'.1 = 0$	(4)
	$W_2' = 0$	(3D)
	$W_2 = 1$	**2.4** (iv), (10)
substitute in (b)	$W_1.1 = 0$	
	$W_1 = 0$	

and similarly substituting in:

(a) $C_2 = 0$	(h) $M_1 = M_2 = 1$
(d) $H_1 = 0$	(f_1) $C_1' = 0$
(c) $M_1' = 0$	(e) $H_2' = 0$

So today is cold, dry, calm, and high in humidity, and tomorrow will be hot, wet, windy, and high in humidity.

4.13 Our algebra contained three operations $\cap$, $\cup$, and ($'$), but we saw in **2.20** that two operations, $\cap$ and ($'$) or else $\cup$ and ($'$) enabled us to express any required function. However, we went on to use a shorthand in **2.42** to write $(A' \cap B) \cap (A \cap B')$ as $A \triangle B$, and in our application to sets we wrote $A \subseteq B$ for $A \cap B' = 0$. Now we meet some operations used in symbolic logic.

4.14 The statement (C), that one state (A) *implies* another statement (B), or that, if A is true, so is B, is obviously incomplete; all that it tells us is given in the truth table.

A	B	C
1	1	1
1	0	0

We extend this, and say that A *implies* B, written

$A \rightarrow B$ has the truth table

A	B	$A \rightarrow B$
1	1	1
1	0	0
0	1	1
0	0	1

hence

$$\begin{aligned} A \rightarrow B &= A.B + A'.B + A'.B' \\ &= B.(A + A') + A'.B' && (1D, 2) \\ &= B + A'.B' && (4, 3D) \\ &= A' + B && (11, 1) \end{aligned}$$

4.15 *Exercise.* Show that:

(a) the operation is not commutative
(b) ,, ,, is analogous to the statement that $A \subseteq B$
(c) if $A \rightarrow B$ and $B \rightarrow C$, then $A \rightarrow C$; (the syllogism)

4.16 *Exercise.* Show that, by the use of $\rightarrow$ we can dispense with $\cup$ and $\cap$, and express any function by the use of just $\rightarrow$ and (′).

If $A \rightarrow B$, then $A' + B = 1$,

so

$$\begin{aligned} (A' + B)' = 1' &= 0 && \mathbf{1.27}\ (v), (10D) \\ A.B' &= 0 && (12) \\ A &\subseteq B && \mathbf{3.8} \end{aligned}$$

It is not, however, true to say that we can express any function using just $\subseteq$ and (′). Why not?

4.17 There are other operations that have this property of giving, with (′), a complete range of functions. Two are called the *Sheffer stroke* functions and are given by the truth tables

A	B	$A \downarrow B$	$A \uparrow B$
1	1	0	0
1	0	1	0
0	1	1	0
0	0	1	1

4.18 *Exercise.* Prove that:

(a) $A \downarrow B = (A \cap B)'$, and can be read as 'not both A and B'
(b) $A \downarrow A = A'$
(c) $A \downarrow B = B \downarrow A$

(d) $(A \downarrow A) \downarrow (A \downarrow A) = A$

(e) $(A \downarrow A) \downarrow (B \downarrow B) = A' \downarrow B'$

$$= (A' \cap B')'$$
$$= A \cup B$$

(f) both $\cup$ and $\cap$ can be eliminated from any expression, and it can be expressed using only $\downarrow$ and (′)

(g) the identity $A \cup (A' \cap B) = A \cup B$ becames $A' \downarrow (A' \downarrow B)' = (A' \downarrow B')$

(h) (b) above now shows us that we can express any function in terms of $\downarrow$ alone, for in (g) we saw how to obtain any expression in terms of $\downarrow$ and (′), and then we can use (b) to eliminate (′). Do this to the answer to (g).

4.19 *Exercise.* Prove that:

(i) $A \uparrow B = A' \cap B'$ (and can be read 'neither A nor B')

$$= (A' \downarrow B')'$$
$$= (A' \downarrow B') \downarrow (A' \downarrow B')$$

(ii) $A \uparrow B = B \uparrow A$

(iii) $A \uparrow A' = 0$

(iv) $A \uparrow A = A'$

(v) $(A \uparrow B) \uparrow C = (A' \uparrow C) \cup (B' \uparrow C)$

(vi) $A \cup B = (A \uparrow B)'$

(vii) (i), (iv), and (vi) show how to obtain a functionally complete set using only $\uparrow$

(viii) Express $A \cap (A' \cup B) = A \cap B$ in terms of A, B, $\uparrow$

4.20 *Exercise.* Andrew stated that Bernard and Charlie are always right, and Charlie was sure that Edward and Fred were always the same—both truthful, or both liars. Donald said that either Andrew or Bernard or both were right and Bernard said that one, and only one, of Edward and Fred was reliable. Edward was certain that that both Andrew and Bernard tell the truth, but Fred said that he knew Bernard and Charlie were not both right.

Who did tell the truth?

4.21 *Exercise.* Prove that the following are inconsistent

$$A' \cap B' = 1, \quad (A \cup B) \cap (A' \cup C') = 1$$

4.22 *Exercise.* 'Braintwister No. 172' by D. P. St. Barnard, in *The Observer.*

Peace Conference

'That something must be done to restore law and order in Nuemania is something on which the peace committee is agreed. The members also agree that an international police force should be set up, consisting of three contingents from the Northern Powers (Atalantia, Battolia, Cornovia, Dubbland, and Empiria) and three from the Southern bloc (Voolubu, Womboland, Xandolia, Yubabi, and Zemberia).

'Unfortunately, the Northern Powers will not agree to a force that includes both Womboland and Yubabi. The Southern bloc has retaliated by saying that it will not agree to contingents from both Atalantia and Dubbland.

'Again, the Northern Powers insist that, if both Voolubu and Womboland are included, then Zemberia shall not be represented on the force. The Southern response to this is that unless Womboland is represented, the bloc will not agree to both Cornovia and Dubbland. Moreover, if Zemberia is excluded, the Southern bloc will not agree to the inclusion of Empiria.

'If Zemberia sends a contingent, then Battolia will refuse to join the force. To this Xandolia has retorted that she will withdraw if Cornovia is represented. The conference is in grave danger of deadlock, and seeks some suggestion as to how the force may be comprised in a way that will satisfy all parties.'

4.23 The reader will see many analogies between sets and symbolic logic, and we have seen the same sorts of uses of algebra, viz.: to clarify and simplify statements, and to reduce a system of them to a smaller system that says the same; to solve some problems, and to reveal any inconsistency that may occur in a group of statements.

5

THE ALGEBRA OF CIRCUITS

5.1 This application of the algebra is very similar to that in Chapter 4. Instead of the elements being statements that are true (A = 1) or false (A = 0), here they represent switches that are 'on' (A = 1) or 'off' (A = 0), and the application to 'electronic brains' etc. becomes easy to see.

We are not concerned here with mechanical or electrical devices for operating the 'switch' but merely with the application of algebra to the circuit.

5.2 We represent an element A in a wiring diagram or a switching circuit thus

——————— A ———————

Fig. (i)

and a current will flow along the wire if A = 1, but not if A = 0. (The elements in this application of the algebra, as in the previous chapter, can take only the values 0 or 1.)

The meanings of ∪ and ∩ are also similar to those met in previous chapters, for we say that (A ∩ B) represents

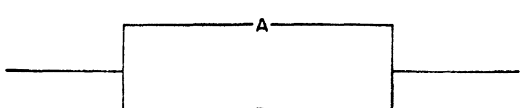
———— A ———— B ————

A ∩ B; A *and* B in *series*

Fig. (ii)

where the current passes through A *and* B, and A and B are in *series*.

Similarly, (A ∪ B) represents



(A ∪ B); A or B; A, B are in *parallel*

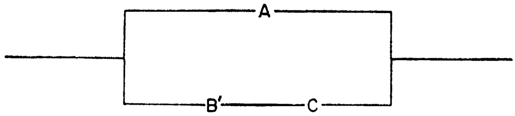
Fig. (iii)

where the current passes through A *or* B or both, and we say that A, B are in *parallel.*

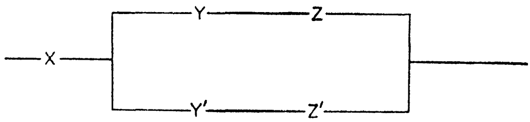
A′ is defined as 'not-A' and so A′ is on/off when A is off/on.

With these definitions we can draw the circuit represented by any expression.

5.3 *Examples.* (i) $A + B'.C$



(ii) $X.(Y.Z + Y'.Z')$

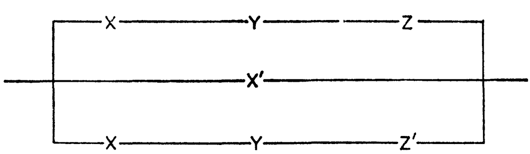


5.4 *Exercise.* Show that, with these meanings of the elements and the operations, the postulates are satisfied.

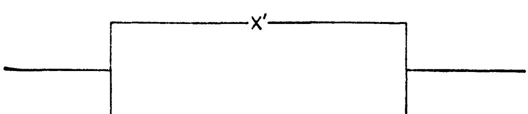
5.5 *Exercise.* Draw diagrams for:

(i) $Y.Z + Z.X + X.Y$
(ii) $(Y + Z).(Z + X).(X + Y)$
(iii) $A + B.(C + D)$

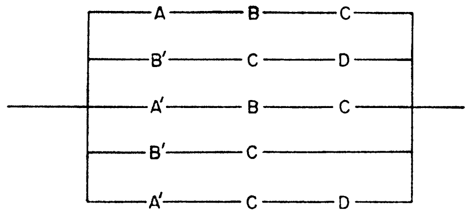
5.6 *Exercise.* (i) Write down the algebraic expression for



simplify it, and so show that this circuit is equivalent to



(ii) Similarly show that



is equal to

5.7 *Exercise.* (i) A circuit is 'on' for the following positions of the switches A, B, C, and D

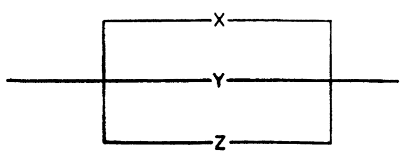
A	on	off	on	off
B	on	on	on	on
C	on	on	off	off
D	on	on	on	on

draw the wiring diagram.

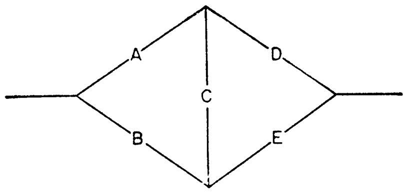
(ii) Show that the simplest circuit satisfied by

X	on	on	on	on	off	off	off
Y	on	on	off	off	on	on	off
Z	on	off	on	off	on	off	on

is



5.8 *Exercise.* Show that the algebraic expression for



is A.D + B.E + A.C.E + B.C.D and draw the equivalent 'series–parallel' diagram.

5.9 We can apply our algebra to obtain circuits which will perform addition and subtraction of numbers, but first the numbers must be expressed in the *binary* scale. An ordinary number in the decimal scale is

$$D_n.10^n + D_{n-1}.10^{n-1} + \cdots + D_1.10 + D_0$$

where the D's are the digits of the number, and they take integral values from 0 to 9.

A number in the binary scale is

$$A_n.2^n + A_{n-1}.2^{n-1} + \cdots + A_1.2 + A_0$$

where the A's are the digits, and have value 0 or 1.

Example. Express the number of days in a year as a binary number, and the binary number 100101011 as a decimal number.

$$\begin{aligned} 365 &= 256 + 64 + 32 + 8 + 4 + 1 \\ &= 1.2^8 + 0.2^7 + 1.2^6 + 1.2^5 + 0.2^4 + 1.2^3 \\ &\qquad\qquad\qquad\qquad + 1.2^2 + 0.2 + 1 \\ &= 101101101 \text{ (binary)} \end{aligned}$$

$$\begin{aligned} 100101011 &= 2^8 + 2^5 + 2^3 + 2 + 1 \\ &= 256 + 32 + 8 + 2 + 1 \\ &= 299 \end{aligned}$$

5.10 *Exercise.* (i) Express 11011011 (binary) as a decimal number, and 195 (decimal) as a binary number.

(ii) Add the binary numbers 1001101 and 111011, and then find the difference between them.

5.11 *Exercise.* In this addition of two binary numbers:

$$\begin{array}{r} A_n.2^n + A_{n-1}.2^{n-1} + \cdots + A_1.2 + A_0 \\ \underline{B_n.2^n + B_{n-1}.2^{n-1} + \cdots + B_1.2 + B_0} \\ \cdots + S_1.2 + S_0 \end{array}$$

where the number 'carried' from the first column to the second is C_1, etc., show that:

(i) All the S's and C's, like the A's and B's, are either 0 or 1
(ii) $A_0 + B_0 = 2.C_1 + S_0$
(iii) $A_1 + B_1 + C_1 = 2.C_2 + S_1$
(iv) $A_r + B_r + C_r = 2.C_{r+1} + S_r$
(v) The addition tables can thus be written

A_0	B_0	C_1	S_0
1	1	1	0
1	0	0	1
0	1	0	1
0	0	0	0

and

A_1	B_1	C_1	C_2	S_1
1	1	1	1	1
0	1	1	1	0
1	0	1	1	0
1	1	0	1	0
1	0	0	0	1
0	1	0	0	1
0	0	1	0	1
0	0	0	0	0

5.12 The resemblence to a truth table is obvious, and if we accept the convention that a flow of current shall represent 1, and an absence of flow represents 0, we can apply our algebra of circuits, and obtain, as in **4.6**,

$$\begin{aligned} \mathrm{C_1} &= \mathrm{A_0.B_0} \\ \mathrm{S_0} &= \mathrm{A_0.B_0' + A_0'.B_0 = A_0 \Delta B_0} \\ \mathrm{C_2} &= \mathrm{A_1.B_1.C_1 + A_1'.B_1.C_1 + A_1.B_1'.C_1 + A_1.B_1.C_1'} \\ &= \mathrm{A_1.B_1.(C_1 + C_1') + C_1.(A_1 \Delta B_1)} \\ &= \mathrm{A_1.B_1 + C_1.(A_1 \Delta B_1)} \end{aligned}$$

$$\begin{aligned} S_1 &= A_1.B_1.C_1 + A_1.B_1'.C_1' + A_1'.B_1.C_1' + A_1'.B_1'.C_1 \\ &= C_1.(A_1.B_1 + A_1'.B_1') + C_1'.(A_1'.B_1 + A_1.B_1') \\ &= C_1.(A_1 \Delta B_1)' + C_1'.(A_1 \Delta B_1) \end{aligned}$$

5.13 To simplify the wiring diagram, we use two units, a 'half-adder' and a 'union element'. The *half-adder* (Fig. 1) controlled by inputs A_0 and B_0, allows a flow of current through the upper wire if $A_0.B_0 = 1$, and through the lower wire if $A_0 \Delta B_0 = 1$

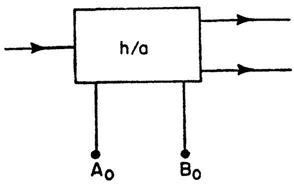


Fig. 1

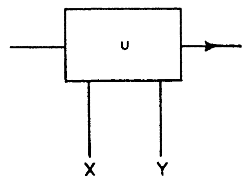


Fig. 2

and similarly the ***union element*** (Fig. 2) allows a flow if $X + Y = 1$.

5.14 *Exercise.* Find the algebraic conditions for a signal from S_0, S_1, in Fig. 3 (p. 44) and compare with the equations of **5.12.**

5.15 The combination of two half-adders and a union element as arranged in Fig. 3, p. 44 to give outputs S_1, C_2, controlled by A_1, B_1, C_1, is called an *adder*. **5.11** (iv) shows us that a series of such adders will enable us to add any two numbers.

5.16 *Exercise.* Express the conditions of **5.12**, viz.,

$$C_1 = A_0.B_0 = 1 \quad \text{and} \quad S_0 = A_0 \Delta B_0 = 1$$

in words, and check their truth.

5.17 *Exercise.* Show that the use of two adders and a union element to form an adder is really saying, 'in order to add x, y, and z, first add x and y, and then add z to the answer'. (To a computer, addition is still a binary operation!)

5.18 *Exercise.* Interchanging the inputs A and B in **5.13** Fig. 1 makes no difference. Why?

5.19 *Exercise.* Use the fact that

$$(A + B) + C = (C + A) + B$$

to re-arrange the wiring diagram of **5.14.**

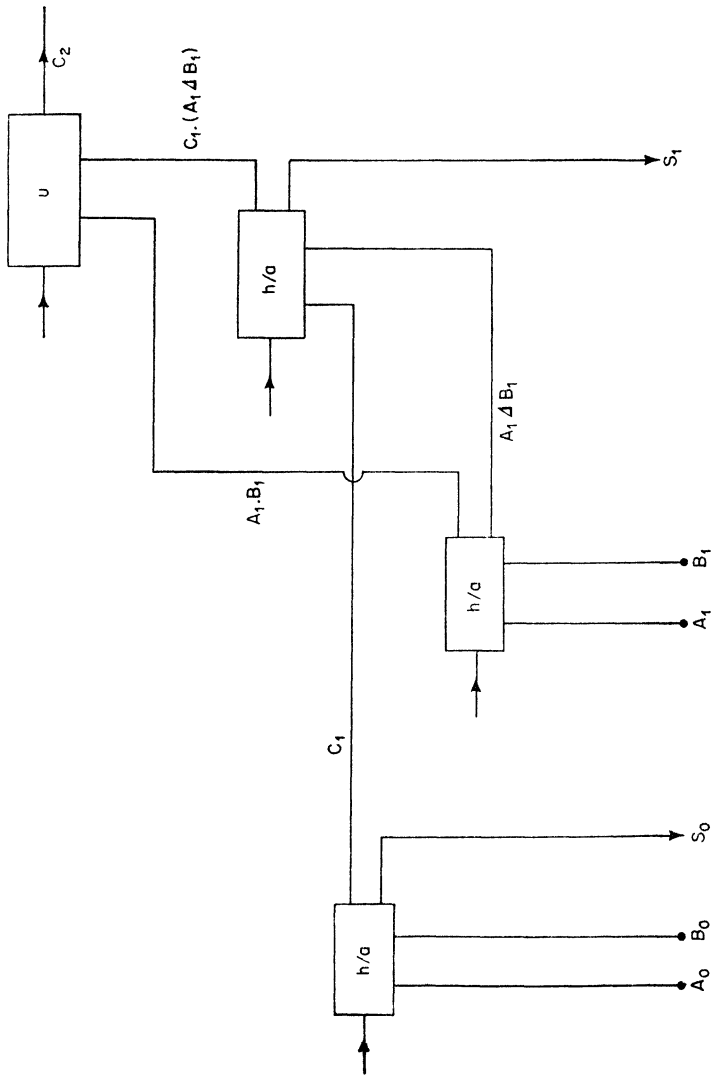


Fig. 3. The half-adder and the adder

The letters ($A_1 \cdot B_1$, etc.) beside the wires show the expressions that must be equal to 1 if a current is to flow.

5.20 *Exercise.* Obtain a wiring diagram to perform the subtraction of one binary number from another, with notation

$$\begin{array}{r} A_n.2^n + A_{n-1}.2^{n-1} + \ldots + A_1.2 + A_0 \\ \underline{B_n.2^n + B_{n-1}.2^{n-1} + \ldots + B_1.2 + B_0} \\ \ldots + D_1.2 + D_0 \end{array}$$

where we 'borrow L' and in general

$$2.L_{r+1} + A_r - B_r - L_r = D_r$$

(Note that **5.17** is still true—there are two binary operations, to subtract B_1 from A_1 and to subtract L_1 from the difference. We can thus expect to need two 'half-subtractors' and a union element as before.)

6

CHOICE AND CHANCE

6.1 The concept of sets is useful when thinking about the number of ways an experimental trial can turn out. If P is the soa possible outcomes and all are equally probable, and F is the subset which we consider favourable, then we can define the *probability* of a favourable event as

$$\frac{n(\text{F})}{n(\text{P})}$$

6.2 *Exercise.* (i) In any leap year, what is the probability that there will be 53 Wednesdays?

(ii) If in a given leap year there are 53 Wednesdays, what is the probability that there will be 53 Thursdays?

(iii) What is the probability that the 13th of any month will be a Friday?

6.3 *Exercise.* A student is equally good at French and German. He was given a list of 100 French nouns and asked to state the gender of each. Those that he did not know, he filled in at random, and he got 85 right. What would his probable score be in a similar test in German?

6.4 *Exercise.* The seats in the compartments of a railway train are numbered as in the diagram. Bookable seats in the train, in 25 com-

17	18	19	20
21	22	23	24

partments, are numbered 1 to 200. A man, booking seats for his wife and for himself, got seats whose numbers were consecutive. What is the probability that they:

(i) sat side by side?

(ii) ,, in different compartments?

6.5 *Exercise.* A car-park has places for N cars. The first car to arrive when the car-park is empty can occupy any place, but from then onwards each car that arrives must fill a place next to one that is already taken. How many ways are there of filling the car-park? (For any given way of filling the places, by reversing this order we obtain a member of the soa ways of emptying the full car-park according to a given rule. What rule? Hence, what is the required number?)

6.6 *Exercise.* Five coins are tossed. Find the probability of having at least four alike by thinking of the expansion of $(H + T)^5$.

6.7 In both **6.5** and **6.6** we obtained the number of members of a set, F, by establishing a one-to-one correspondence between its members and those of another set, F_1, where $n(F)$ and $n(F_1)$ were both finite.

We now extend this to methods of finding $n(F)/n(P)$ by establishing a one-to-one relation between the members of F, F_1; P, P_1, when the number of members of the sets is not finite.

Fig. (i)

A line AB is divided at random by a point X into two parts (Fig. (i)). What is the probability (p) that the part AX shall be less than the part XB? The answer is obviously $\frac{1}{2}$. The event is favourable when X is at any point between A and C, the midpoint of AB and if L, M are the soa points in AC, AB respectively, then

$$p = \frac{n(L)}{n(M)} = \frac{\text{length of } AC}{\text{length of } AB} = \tfrac{1}{2}$$

More generally, the probability that a random point on AB, Fig. (ii), shall lie between X and Y is

$$\frac{\text{length of } XY}{\text{length of } AB}$$

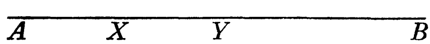


Fig. (ii)

and the probability that a random point inside R, Fig. (iii), shall also lie inside C is

$$\frac{\text{area of } C}{\text{area of } R}$$

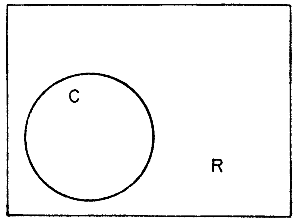


Fig. (iii)

6.8 *Exercise.* The lines of a shove-halfpenny board are $1\frac{1}{4}$ inches apart. What is the probability of a random shot leaving a halfpenny (diameter 1 inch) clear of a line?

6.9 *Exercise.* John's devotion wavered between Mary, who was mathematical, and Dora, who was dumb. They lived on the same bus route, but the girls lived on opposite sides of John.

Mary suggested that, as buses ran every 12 minutes in each direction, it would be fair if John caught the first bus that came, and let that decide which girl he visited. John and Dora agreed that nothing could be fairer than that.

Some time later John remarked that Fate was on Mary's side—he was seeing her about three times as often as he was seeing Dora.

Mary explained. . . .

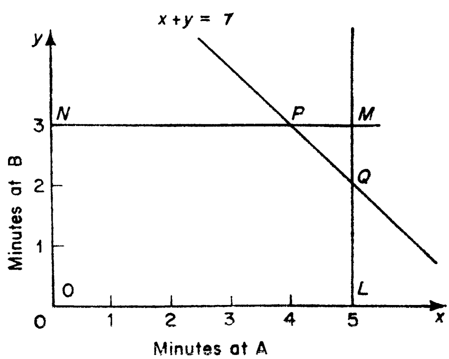
6.10 *Example.* A man has to catch a train at station A, where there is a train every 5 minutes, and change at B to another line, on which there is a train every 3 minutes. For this he allows a total of 7 minutes.

What is the probability that he will arrive in time?

Here we are concerned with the set of all ordered pairs of numbers, x, y, such that

$$0 \geqslant x \geqslant 5 \quad \text{and} \quad 0 \geqslant y \geqslant 3$$

Any such ordered number-pair can represent the times he had to



wait at each station, and all such number-pairs are possible and are equally likely.

The diagram explains itself. There is a one-to-one relation between members of the soa ordered number-pairs and the soa points in the plane. For our question, the soa possible events has a one-to-one relation with the soa points inside the rectangle $LMNO$, and the soa favourable events with those inside the figure $OLQPN$, and the required probability is

$$\frac{\text{area of } OLQPN}{\text{area of } LMNO} = \frac{29}{30}$$

6.11 *Exercise.* A stick is broken at random into three parts. Find the probability that the three pieces can form a triangle.

6.12 *Exercise.* Extend the method of **6.8** to find the probability of an early arrival of a man who catches a train at A (one every a minutes) and has to change at B (one every b minutes) and again at C (one every c minutes), if he allows a total of M minutes, where, of course,

$$M < a + b + c$$

7

MEN AND BOOKS

7.1 To trace the development of the application of an algebra to various branches of thought is difficult.

Plato (429–348 B.C.) was one of the early philosophers who tried to find a basic system for logical thought and his statement 'nothing is not nothing' can still cause thought about $\{\ \}$!

Aristotle (384–322 B.C.) was roughly his contemporary, and also attempted an analysis of deductive logic, and formulated laws which we might express as:

(I)	$A \cup A = A$
(II)	$A \cap A' = 0$
(III)	$A \cup A' = 1$

7.2 G. Boole (1815–1864), born in Lincoln, lectured in mathematics at Queen's College, Cork. His works, *A Mathematical Analysis of Logic* (1847), and *Laws of Thought* (1854) and the works of Frege (1848–1925) give an axiomatic basis for an algebra of statements and also treat mathematics as an application of logic—work that was continued in Whitehead and Russell's *Principia Mathematica* (1913).

7.3 De Morgan (1806–1871) investigated the properties of mathematical operations, and wrote a *Formal Logic*, and E. V. Huntington in 1904 published *Sets of Independent Postulates for the Algebra of Logic*. Meanwhile, the theory of sets was being developed and applied by Richard Dedekind (1831–1916) and Georg Cantor (1845–1918) who used the ideas of sets and of a one-to-one correspondence to investigate transfinite numbers.

7.4 Books recommended for further reading:

ALGEBRA

Boolean Algebra. R. L. Goodstein (Pergamon)
Selections from Modern Abstract Algebras. Andrée (Constable)

Boolean Algebra and its Applications. J. E. Whitesitt (Addison Wesley)
Fundamental Concepts of Mathematics. R. L. Goodstein (Pergamon)

SETS

Introduction to the Theory of Sets. J. Breuer (Prentice Hall)
Sets, Logic, and Axiomatic Theories. R. R. Stoll (Freeman)

LOGIC

Mathematical Logic. R. L. Goodstein (Leicester University Press)
Symbolic Logic, and the Game of Logic. Lewis Carroll (Dover)
The Use of Reason. E. R. Emmet (Longman)

CIRCUITS

Applied Boolean Algebra, an Elementary Introduction. F. E. Hohn (Collier-Macmillan)
Thinking Machines. Irving Adler (Dobson)
Mathematics for Circuits. Chellingsworth (Macmillan)

PROBABILITY

Probability, an Introduction. Goldberg (Prentice-Hall)
Integration, Measure and Probability. Pitt (Oliver and Boyd)
Probability, an Intermediate Textbook (mainly for actuarial work). Bizley (C.U.P.)

8

SUPPLEMENTARY NOTES AND EXERCISES

The numbering of the sections here refers to the earlier paragraph which the section could have followed immediately.

1.3/1 *Exercise.* Is a set of coplanar vectors closed under the operation of (i) addition (ii) subtraction (iii) vector products (iv) scalar products?

1.6/1 *Exercise.* A, $(a.\cos\alpha, b.\sin\alpha)$ and B, $(a.\cos\beta, b.\sin\beta)$ are two points on the ellipse $x^2/a^2 + y^2/b^2 = 1$.

Find the equation of the chord joining them. What fact, other than the Commutative Law, is required to show that the chord AB is the same as the chord BA?

1.6/2 *Exercise.* For line joining A, (x', y') to B, (x'', y'') show that the line BA is the same.

1.16/1 *Exercise.* If a, b, c are non-zero numbers, and satisfy the Distributive Laws, then

$$a + b + c = 1$$

1.22/1 Show that, in the following statements, another true statement is obtained by interchanging the words 'equal' and 'parallel':

'Lines that are parallel to the same line are parallel to each other.'

'A quadrilateral is a parallelogram if it has both pairs of opposite sides equal.'

'The lines joining the extremities of equal and parallel lines are themselves equal.'

Show that this is not a true example of duality, by giving examples of true statements which lead thus to false ones.

1.22/1 *Exercise.* A figure is defined by an ordered series of n vertices $A_1, A_2, \ldots, A_n$, no three of which are collinear. Pairs of consecutive vertices are joined to form the n sides of the figure, and the lines joining the non-consecutive pairs of vertices form the $n(n-3)/2$ diagonals.

Interchanging the words 'sides' and 'vertices', and putting 'intersections' for 'diagonals', write down the dual statement, and draw figures for both when $n = 5$.

1.22/2 *Exercise.* A quadrilateral consists of its four sides a, b, c, d; six vertices $(ab), (ac), \ldots$, and the diagonal triangle formed by joining pairs of the three points (ab, cd), (ac, bd), (ad, bc).

Describe and draw the dual figure.

1.25/1 *Exercise.* Use **1.23, 1.24, 1.4**, to prove that $0' = 1$ and $1' = 0$.

1.25/2 *Exercise.* Write down the duals of the following statements:

(i) $A \cup A = A$
(ii) $A \cup 1 = 1$
(iii) $(A \cup B) \cap (A \cup B') = A$
(iv) $(A \cup B \cup C') \cap (A \cup B \cup C) = A \cup B$
(v) $A \cup B \cup (A' \cap B') = 1$
(vi) $A \cap (A' \cup B) = A \cap B$

1.25/3 *Exercise.* From **1.23** prove

$$1 \cup 0 = 1 \qquad 0 \cup 0 = 0$$
$$0 \cap 1 = 0 \qquad 1 \cap 1 = 1$$

1.27/1 *Exercise.* If $a, b, c \ldots$ are numbers, and we write $a \mathrm{F} b$ to mean 'a is a factor of b', which of the laws of **1.27** are true when we write F in place of $=$?

2.4/1 *Exercise.* Prove what you can from the postulates given in **2.3**.

2.9/1 *Exercise.* Prove that:

(i) $X \cup Y \cup (X \cap A) \cup (Y \cap B) = X \cup Y$
(ii) $X \cup (X \cap Y) \cup (X \cap Y \cap Z) = X$
(iii) $A + A.B + A.B.C + A.B.C.D.E.F.G = A$
(iv) $(A + B).(A + B + C).(A + B + C + D) = A + B$

2.10/1 *Exercise.* Prove (7), $A \cup A = A$, without using (8). (Hint—$A = A \cup 0 = A \cup (A \cap A') = \ldots$)

2.16/1 *Exercise.* Simplify:

(i) $(A.B + C)'$
(ii) $X + Y + X'.Y'.Z$
(iii) $A \cup B \cup C \cup (A' \cap B' \cap C' \cap D)$
(iv) $(X + X.Y + X'.Z)'$

(v) $(A \cap B') \cup (A' \cap B)$
(vi) $[\{(A + B)' + A\}' + B]'$
(vii) $P.Q.R.S' + P'.R + Q'.R + R.S$

2.19/1 *Exercise.*

(i) If $A.X = A.Y$ and $A'.X = A'.Y$, then $X = Y$
(ii) If $A + X = A + Y$ and $A' + X = A' + Y$, then $X = Y$

2.22/1 *Exercise.* Express (a) without the + sign and (b) without the (.) sign, the following:

(i) $A + B.C.(B + D)$
(ii) $(A + B').(B' + C).(C + A')$
(iii) $(X + X.Y + X'.Z)'$

2.29/1 *Exercise.* Express as polynomials:

(i) $(A.C + B.C')'$
(ii) $(X \cap Y')' \cup Y$

2.30/1 *Exercise.* Show that, for the disjunctive normal form of expressions in the n elements $A_1, \ldots, A_n$:

(i) The product of any two different terms is 0.
(ii) If all the A's are given the value 0 or the value 1, show that there is one and only one set of values that will give any one term of F_n the value 1.
(iii) To every expression in the A's there corresponds one and only one disjunctive normal form.
(iv) If a disjunctive normal form P, has m terms, then the m sets of values, 0 or 1, of the A's which give P the value of 1, will determine P uniquely.
(v) From n A's we can form a total of 2^{2^n} different expressions.

2.30/2 *Exercise.* Find the disjunctive normal forms of:

(i) $X \cup Y$
(ii) $X \cup Y \cup Z$ (see **2.30**, exercise)
(iii) $A.B + B'.C'$

2.30/3 The reduction of this form of expression to some simpler equivalent expression is particularly useful in work on wiring diagrams.

Example. Find a simple expression for

$$E = A.B.C + A'.B.C + A.B'.C + A.B'.C' + A'.B'.C. + A'.B'.C'$$

$$\begin{aligned} E' &= A.B.C' + A'.B.C' \qquad \textbf{2.30/1}\text{ (iii)} \\ &= B.C'.(A + A') \\ &= B.C' \\ E &= B' + C \qquad (12) \end{aligned}$$

2.30/4 *Exercise.* Simplify:

(i) $A.B + A'.B + A'.B'$
(ii) $Y.Z + Y'.Z' + Y'.Z + Y.Z'$
(iii) $P.Q.R + P.Q'.R + P.Q'.R' + P'.Q'.R$
(iv) $A'.B.C + A.B.C' + A'.B.C' + A'.B'.C'$

2.43/1 *Exercise.* Prove that:

(i) $(A \Delta B) \Delta C = A \Delta (B \Delta C)$
$= (A \Delta C) \Delta B$
(ii) $(A \Delta B) \Delta (C \Delta D) = (A \Delta C) \Delta (B \Delta D)$
$= (A \Delta D) \Delta (B \Delta C)$
(iii) If $A \Delta B = 0$, then $A = B$
(iv) $(A + X) \Delta (A + Y) = A'.(X \Delta Y)$
(v) $(A + X) \Delta (A + Y) \Delta (A + Z) = A' + (X \Delta Y \Delta Z)$
(vi) $(A.X) \Delta (A.Y) \Delta (A.Z) = A.(X \Delta Y \Delta Z)$

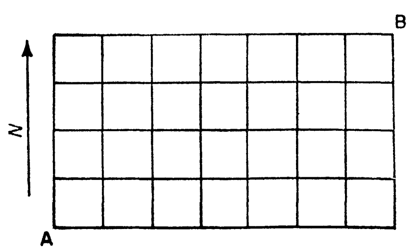
2.43/2 *Exercise.* Four variables, A, B, C, D can each take only the values 0 or 1. Z is a function of A, B, C, D which takes the value 1 if, and only if, an even number of the elements A, B, C, D are equal to 1. Show that

$$Z = (A \Delta B \Delta C \Delta D)'$$

What is the corresponding function if we change the word 'even' to 'odd'?

3.2/1 *Exercise.* Name sets that P could be if $n(P)$ is: (i) 4 (ii) 5 (iii) 11 (iv) 12 (v) a large, but finite number (vi) an infinite number.

3.2/2 *Exercise.* What is $n(R)$ where R is the soa routes from A to B along roads represented by lines in the diagram, if movement is always in a northerly or an easterly direction?



3.4/1 *Exercise.* Name the third set in each of the following:

(i) 1 = the soa Members of Parliament
C = ,, ,, ,, ,, the House of Commons
C' = ,, ,, ?

(ii) R = ,, ,, rational numbers
R' = ,, ,, irrational ,,
1 = ,, ,, ?

(iii) A = {Tom, Dick, and Harry}
A' = {Jack and Jill}
$1 = ?$

(iv) $1 = \{a, b, c, d, e\}$
$P = \{a, b, c, d\}$
$P' = ?$

(v) 1 = the soa prime numbers
Q = ,, ,, odd primes
$Q' = ?$

3.5/1 Care must be taken with the word *or*, which can be translated into the language of sets in various ways; these can be seen by considering the following passages:

(I) 'The girl I marry must be good-looking or a good cook.'
(II) 'I will wear a cap or a trilby hat.'
(III) 'All members of the school, whether boys or girls, must attend on Sports Day.'

In (I) we have an example of the '*inclusive or*'—she must be good-looking, or a good cook, *or both*, i.e. she must be a member of $(L \cup C)$,

where L = the soa good-looking girls
and C = ,, ,, good cooks

(II) differs from this in an obvious way, for he intends to wear one or the other, *but not both.* So, if

D = the soa men wearing a cap
and T = ,, ,, ,, ,, trilby hat

he will be a member of the set

$(D \cup T) \cap (D \cap T)'$ (D *or* T, *and not* D *and* T)
$= (D \cap T') \cup (D' \cap T)$ **2.43** (iii)
$= D \Delta T$

Example (III) is again different. If B = the soa boys, and G = the soa girls, then the set (B $\cap$ G) is an empty set. If here we are regarding the soa members of the school as 1, the universal set, then

$$B \cup G + 1 \quad \text{and} \quad B \cap G = 0$$

and so, by (4) and (4D) we have

$$B = G'$$

Otherwise, if B $\cup$ B $\neq$ 1, we have

$$\begin{aligned} (B \cup G) &= [B \cap (G \cup G')] \cup [G \cap (B \cup B')] && (4, 6) \\ &= (B \cap G) \cup (B \cap G') \cup (G \cap B) \cup (G \cap B') && (2) \\ &= (B \cap G') \cup (B' \cap G) && (3) \end{aligned}$$

since $$B \cap G = G \cap B = 0$$

3.5/2 *Exercise.* Classify the use of *or* as similar to its use in (I), (II), (III), above, as used or implied in the following:

(a) 'Candidates should have passed 'A' level in Mathematics or Physics.'
(b) His wife had bought him two ties for Xmas; he wore one on Xmas Day; she burst into tears and said, 'I felt, somehow, that you didn't like the other one!'
(c) 'Did you cross the Channel by boat or 'plane?'
(d) '£1,000 REWARD for —— ——, dead or alive.'
(e) 'Over-drive is an optional extra.'

3.5/3 *and* also has its dangers. If

B = the soa black minstrels
and W = ,, ,, white ,,

then the 'Black and White Minstrels' are *not* members of the set (B $\cap$ W), for this is the set of minstrels that are both black and white, i.e., an empty set.

3.9/1 *Exercise.* Prove that:

(i) If A $\subseteq$ B for all A, then B = 1
(ii) .... B, then A = 0
(iii) ...., then A.B = A. What can you deduce from Example **2.28**? Prove it.

3.13/1 *Exercise.* From the statements:

(i) All racing motorists are quick-witted.
(ii) Plato was a profound thinker.

(iii) All philosophers are profound thinkers.
(iv) Nobody is both quick-witted, and also a profound thinker.

can it be deduced that:

(a) Plato was a philosopher
(b) ,, ,, not a racing motorist?

Prove your deductions.

3.13/2 Another notation for sets, usually for sets of numbers, involves the use of { } to represent the set, and of | meaning 'such that'; so we have

$$\{n \mid n \text{ is a positive integer}\}$$

read as 'the set of all n, such that n is a positive integer', and then

$$\{x \mid x = 2n - 1\} \text{ is the soa positive odd numbers}$$

and similarly $\{x \mid x = n^2\}$ is the soa squares

3.13/3 *Exercise.* Express in this notation the following sets of numbers:

(i) 1, 2, 4, 8, 16, . . .
(ii) 1, 4, 7, 10, 13, . . .
(iii) 1, 2, 6, 24, 120, . . .

3.13/4 Note that, although the members of a set are not arranged in any order, there is nothing to stop us thinking about them in an order, if that helps us. The soa acute angles (α), can be written

$$\{\alpha \mid 0 < \alpha < \pi/2\}$$

and then the soa obtuse angles can be written as

$$\{\beta \mid \beta = \alpha + \pi/2\}$$

or as

$$\{\beta \mid \beta = \pi - \alpha\}$$

and we see that, if the α's in the soa acute angles are arranged in ascending order of magnitude, then so will the first set of obtuse angles be, but that the second will be in descending order.

3.13/5 *Exercise.* Express in words:

(i) $\{x \mid x = n(n + 1)/2\}$
(ii) $\{x \mid -1 < x < +1\}$
(iii) $\{x \mid x > 0\}$

3.13/6 Another symbol in common use is $\in$, meaning 'is a member of' thus

$$2793 \in \{x \mid x = 3n\}$$

or $$z \in (A \cap B) \text{ iff } z \in A \text{ and also } z \in B$$

Care must be taken to distinguish between the statements

$$a \in \{a, b, c\} \quad \text{and} \quad \{a\} \subset \{a, b, c\}$$

which are both true, and

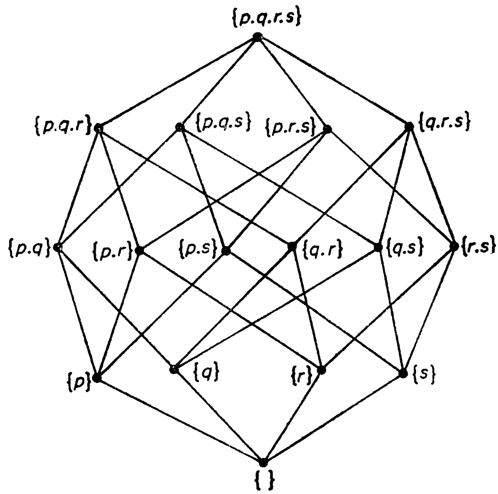
$$a \subset \{a, b, c\}$$

which is not true.

Exercise. Given three sets, A, B, C:

(i) If $A \subset B$ and $B \subset C$, is $A \subset C$?
(ii) ,, $A \in B$ and $B \in C$, is $A \in C$?

3.13/7 The concept of the subsets of a set leads to an easy example of a *lattice*, an arrangement of a set and its subsets, in such a way that a line sloping down the page joins a set to its subsets, thus



3.13/8 *Exercise.* (i) Construct the lattice for $\{a, b, c\}$. How many such figures are contained in the figure of **3.13/7**?

(ii) How many vertices has such a lattice for a set with n members?

3.13/9 *Exercise.* For stage lighting on cycloramas, back-cloths, etc., 'colour addition' is used. Three colours, 'primary red', 'primary green', and 'primary blue', are used together at strengths controlled by dimmers, and often the colour obtained is made paler by adding unfiltered 'white' light. For experiment and demonstration, to a full-strength colour or colours another is gradually added or subtracted. Show that a lattice for a set of four elements shows how to do this completely, and use it to write out a scheme that will explore all possible changes without any repetition.

3.13/10 *Exercise.* C_1 is a given circle, radius $4r$, and L is a straight line whose distance from the centre of C_1 is z. S is the soa real circles, radius r, in the plane of C_1 and L. If

$$X = \{S \mid S \text{ touches } C_1\}$$
$$Y = \{S \mid S \text{ touches } L\}$$

find values of z such that

$$n(X \cap Y) = 0, 1, 2, 4, 6, 7, 8$$

3.13/11 The co-ordinates, x, y, of a point in a Cartesian plane are well known. We vary the notation slightly, and write

$$\langle x, y\rangle$$

to represent an 'ordered number pair' to stress the fact that here the order of the numbers *does* matter.

3.13/12 *Exercise.* (i) S is a set of points in a Cartesian plane such that, if $\langle x, y\rangle \in S$, so is $\langle y, x\rangle$. What geometrical property has the figure? Similarly, describe figures for which

(ii) If $\langle x, y\rangle$, then $\langle -x, -y\rangle$
(iii) ....$\langle -y, -x\rangle$
(iv) ....$\langle -x, y\rangle$

3.13/13 *Exercise.* Give diagrams for the following sets of points:

(i) $A = \{\langle x, y\rangle \mid |x| < 1 \text{ and } |y| < 1\}$
(ii) $B = \{\langle x, y\rangle \mid y > x\}$
(iii) $C = \{\langle x, y\rangle \mid 1 < x^2 + y^2 < 4\}$
(iv) $E = \{\langle x, y\rangle \mid 1\frac{1}{2} < x < 2\frac{1}{2} \text{ and } 1\frac{1}{2} < y < 2\frac{1}{2}\}$
(v) $A \cap B$

3.13/14 *Exercise.* If

$$A = \{\langle x, y\rangle \mid x + y = 7\}$$

and

$$B = \{\langle x, y\rangle \mid 2x - y = 5\}$$

find

$$A \cap B$$

(sometimes, called the *intersection* of A and B).

3.13/15 *Exercise.* Extending this to three dimensions, if

$$\begin{aligned} A &= \{\langle x, y, z\rangle \mid x > 0, y > 0, z > 0\} \\ B &= \{\langle x, y, z\rangle \mid x + y + z = 1\} \\ C &= \{\langle x, y, z\rangle \mid x < \tfrac{1}{2}, y < \tfrac{1}{2}, z < \tfrac{1}{2}\} \end{aligned}$$

draw diagrams of $(A \cap B)$ and $(A \cap B \cap C)$.

3.13/16 *Exercise.* Express in the notation of **3.13/2** the following sets of points:

(i) inside a triangle bounded by the axes, and the line $x + y = 7$
(ii) inside the parabola $y^2 = 4ax$ and between the lines $x = a$ and $x = 2a$
(iii) above the line $x + y = 1$, but less than one unit length from the origin

In each case give a rough sketch.

3.13/17 *Exercise.* Describe in words

$$\{x \mid x \neq x\}$$

3.17/1 *Exercise.* From

All men who are Europeans, or fair, but not both, are good-tempered.
All Europeans are tall, or fair, or both.
All dark Europeans are short.

Prove that all bad-tempered, fair people are Europeans.

3.17/2 Use the language of sets to clarify the following:

(i) Four tailors had shops in the same street of a Chinese town. The first one advertised

'I am the best tailor in town'. The second went one better and announced, 'I am the best tailor in China'. The third said, 'I am the best tailor in the world', and the fourth claimed, 'I am the best tailor in the street'.

(ii) Extract from electioneering speech

'If my party is returned to power, we will see to it that every miner, yes, and every *Welsh* miner, gets full consideration from the government.'

(iii) Conversation

'No nice little girl eats raw fish.'
'Angela eats raw fish.'
'Then Angela is not a nice little girl.'
'On no! Angela is my kitten.'

(iv) Advertisement

'Ninety-nine dentists out of a hundred recommend "Kleenerteef".'

(v) Address

Mr. A. B. Charles,
73, Dover Road,
Eastborough,
Kent.

3.17/3 *Exercise.* Express the normally accepted meaning of the following phrases in the language of sets:

(a) There's no good snake but a dead snake.
(b) No dogs admitted unless led.
(c) Children under 16, unless accompanied by an adult, are not admitted.
(d) There's no smoke without a fire.
(e) All that glisters is not gold.
(f) Social Club Car Park—for members only.

3.21/1 The following is from Lewis Carroll.

'In a very hotly fought battle, at least 70% of the combatants lost an eye, at least 75% lost an ear, at least 80% lost an arm, and at least 85% lost a leg.'

How many lost all four members?

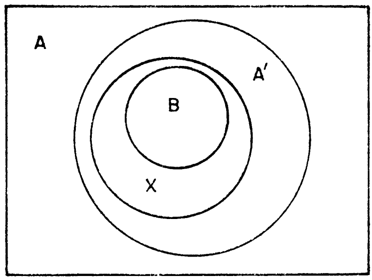
3.21/2 *Exercise.* Of a section of the population of a town, the following was the report.

'The number of immigrants was 87, of whom 51 were married, and 68 were in full employment; the total number of those fully employed was 290, and of them 160 were married. Of the 266 married people, 27 were employed immigrants.'

Show that this is impossible.

3.32/2 We saw, in **2.38–2.41**, that the 'solution' of an equation in X is usually expressed in the form

$$A.X = B.X' = 0$$



provided that $A.B = 0$.

The notation of sets enables us to write this as $B \subseteq X \subseteq A'$, if $B \subseteq A'$, and this Venn diagram makes clear the sort of limitation put on X by these equations. (Note that A′ here is represented by the area *inside* the closed curve.)

3.32/3 *Exercise.* Illustrate the following identities by means of the Venn diagram:

(i) $A + A'.B = A + B$
(ii) $A + B.C = (A + B).(A + C)$
(iii) $(A + B)' = A'.B'$
(iv) $(A' + B') = (A.B)'$
(v) $X.Y + X'.Y' + X.Z = X.Y + X'.Y' + Y'.Z$ (see **2.36**)

3.32/4 *Exercise.* Repeat **3.32/3** for the Carroll diagram.

4.7/1 *Exercise.* Show that:

(i) To each term of the complete disjunctive normal form there corresponds a region of the Venn diagram.

(ii) To each term of the disjunctive normal form of a function there is one row of its truth table for which the function takes the value 1.

(iii) A function is defined by its truth table.

4.7/2 *Exercise.* From these truth tables, find X, Y, Z as functions of A, B, C.

A	B	C	X	Y	Z
1	1	1	1	0	1
0	1	1	0	0	1
1	0	1	1	1	0
1	1	0	1	0	0
1	0	0	0	1	1
0	1	0	0	1	0
0	0	1	0	0	1
0	0	0	1	1	0

4.9/1 *Exercise.* If A is the statement 'men must work' and B is 'women must weep' express in words:

(i) $A \cup B = 1$
(ii) $A \cap B = 0$
(iii) $A' \cup B = 0$
(iv) $A \Delta B = 1$

4.16/1 *Exercise.* Prove that:

If $(X \cup Y) \rightarrow B$, then $X \rightarrow B$ or $Y \rightarrow B$

4.20/1 *Exercise*

Statement A F is true and D is false
" B One and only one of C and D is true.
" C A and E are both true.
" D Either C or F is true, or both.
" E B and F are both true or both false.
" F A, D, and E are all true.

Which are true?

5.17/1 *Exercise.* An adding machine consists of a half-adder and 12 adders. What is the largest total it can register?

6.12/1 *Exercise.* (i) $2 + 2 = 4$. What is the probability of the truth of this if each of the three numbers is correct to the nearest whole number?

(ii) Show that the answer to **3.13/12** (vii) gives the corresponding probability for $2.2 = 4$.

6.12/2 *Exercise.* The angles of a triangle are measured to the nearest degree, and added. What is the probability of an answer of 180°? Show that:

(i) The angles can be written $(A + a)°, (B + b)°, (C + c)°$ or $(A' + a')°, \ldots$ where capital letters are whole numbers and small letters are positive fractions, and

$$\begin{array}{ll} A + B + C = 179 & A' + B' + C' = 178 \\ a + b + c = 1 & a' + b' + c' = 2 \end{array}$$

(ii) If

$$\begin{aligned} X &= \{\langle a, b, c\rangle \mid 0 < a, b, c < 1, \text{ and } a + b + c = 1\} \\ Y &= \{\langle a', b', c'\rangle \mid 0 < a', b', c' < 1, a' + b' + c' = 2\} \end{aligned}$$

there is a one-to-one relation between the members of X and Y. (Put $a' = 1 - x$).

(iii) A member of X, $\langle a, b, c\rangle$, will provide an answer of 180 if one of a, b, c is greater than 1/2.

(iv) The probability of an answer of 180 from X or Y is the same.

(v) The answer to **3.13/15** gives the probability as 3/4, and a probability of 1/8 for 179° or 181°.

(vi) Prove that the sum of the perpendicular distances to the sides of an equilateral triangle from a point inside it is constant and equal to the altitude of the triangle.

(vii) By letting the altitudes of the triangle LMN in Fig. (i) repre-

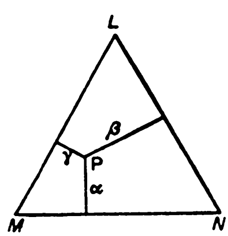


Fig. (i)

sent 180 units, and denoting the perps. by α, β, γ we have

$$\alpha + \beta + \gamma = 180$$

and so points P inside triangle LMN represent triangles whose angles are $\alpha°, \beta°, \gamma°$.

(viii) In Fig. (ii) continuous lines represent integral values of α, β, γ, and dotted lines are for $\frac{1}{2}°$. Comparing with (i) we have

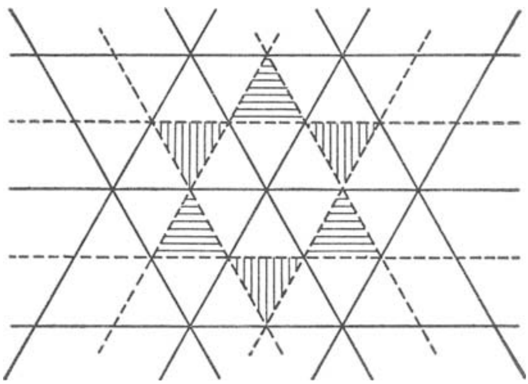


Fig. (ii)

$\alpha = A + a$ or $A' + a'$, etc. Show that points in the areas shaded horizontally represented triangles with an answer of 179°, and in areas shaded vertically, 181°.

(ix) Check that this gives the probabilities of the angle-sums as

179	181/1440
180	3/4
181	179/1440

(x) Explain the difference between the answers given in (v) and (ix).

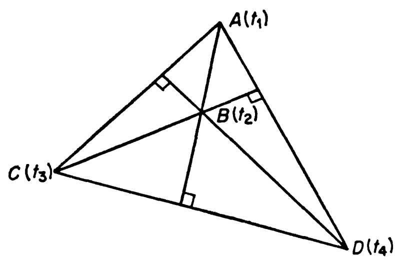
ANSWERS TO SOME EXERCISES

1.3 (i) the integers (ii) positive rationals (iii) rationals.

1.9 (i) no (ii) no

1.12 (ii) $(t_1 + t_2) + (t_3 + t_4) = (t_1 + t_3) + (t_2 + t_4)$

1.14 From **1.13** we see that if AB and CD are perp., so are AC and BD, and also AD and BC, and that any one of the points A, B, C, D, is the orthocentre of the triangle formed by the other three.



1.15 XYZ is a straight line parallel to the axis.

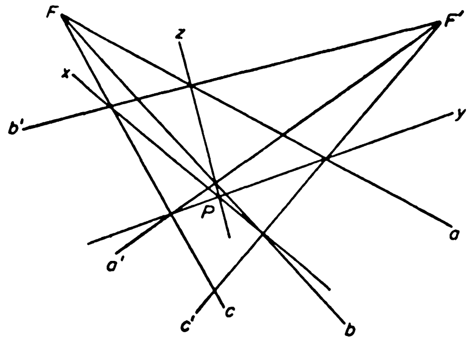
At each point $y = \frac{a}{2}(t_1 + t_2 + t_3 + t_4)$

1.18 (i) $a.(a' + b) = a.b$
(ii) $y.z + z.x + x.y = (y + z).(z + x).(x + y)$
(iii) $(a.b)' = a' + b'$
(iv) $(a + b).(b + c).(c + a') = (a + b).(c + a')$

1.21 If a, b, c are three lines through a point F, and similarly for a' b', c' through F′, and also

$b'c$ and bc' determine the line x
ca' ,, $c'a$,, ,, ,, y
ab' ,, $a'b$,, ,, ,, z

then x, y, z meet in P.



1.22 The dual theorem is also the converse.

1.26 (i) 0 (ii) 1

1.29 (I) no similar form
(II) If $A \geqslant B$, then $B \leqslant A$
(III) If $A \geqslant B$ and $B \geqslant C$, then $A \geqslant C$

1.30 Both laws apply to both operations.
There is symmetry about a diagonal.

1.31

$\cap$	1	0
1	1	0
0	0	0

2.8 (i) In (6) put $A = 1$
(ii) .... $A = 0$
(iii) ...(6D).. $A = 1$
(iv) .... $A = 0$

2.10 (i) In (7) put $A = 0$
(ii) ...(7D).. $A = 1$

2.13 $0 \cup 0' = 1$ (4)
$0' \cup 0 = 1$ (1)
$0' = 1$ (3)

2.14 Follows from **2.12** and **2.13**.

2.15 (2D), (4), (1D) and (3D).

2.16 (i) $(A \cup B) \cap (A' \cap B')$

$$
\begin{aligned}
&= [A \cap (A' \cap B')] \cup [B \cap (A' \cap B')] && (1, 2)\\
&= [(A \cap A') \cap B'] \cup [(B \cap B') \cap A'] && (1, 5D)\\
&= (0 \cap B') \cup (0 \cap A') && (4D)\\
&= 0 \cup 0 && (1D, 6D)\\
&= 0 && (7)
\end{aligned}
$$

$(A \cup B) \cup (A' \cap B')$

$$
\begin{aligned}
&= A \cup B \cup (B' \cap A') && (5, 1D)\\
&= A \cup (B \cup A') && (11)\\
&= (A \cup A') \cup B && (5)\\
&= 1 \cup B && (4)\\
&= 1 && (1, 6)
\end{aligned}
$$

(ii) $A \cap (P \cup Q \cup R)$

$$
\begin{aligned}
&= A \cap [(P \cup Q) \cup R] && (5)\\
&= [A \cap (P \cup Q)] \cup (A \cap R) && (2)\\
&= (A \cap P) \cup (A \cap Q) \cup (A \cap R) && (2)
\end{aligned}
$$

$(A \cup B) \cap (P \cup Q)$

$$
\begin{aligned}
&= [(A \cup B) \cap P] \cup [(A \cup B) \cap Q] && (2)\\
&= (A \cap P) \cup (A \cap Q) \cup (B \cap P) \cup (B \cap Q) && (2, 1, 1D)
\end{aligned}
$$

$(A \cup B \cup C)'$

$$
\begin{aligned}
&= \{(A \cup B) \cup C\}' && (5)\\
&= (A \cup B)' \cap C' && (12D)\\
&= (A' \cap B') \cap C' && (12D)\\
&= A' \cap B' \cap C' && (5D)
\end{aligned}
$$

2.18 (3D) & (4); (2); (1) & (1D); 8.

2.19

$$
\begin{aligned}
A \cup B &= A \cup C\\
A' \cap (A \cup B) &= A' \cap (A \cup C) && (IVD)\\
(A' \cap A) \cup (A' \cap B) &= (A' \cap A) \cup (A' \cap C) && (2)\\
0 \cup (A' \cap B) &= 0 \cup (A' \cap C) && (1D, 4D)\\
A' \cap B &= A' \cap C && (3)\\
\text{but} \qquad A \cap B &= A \cap C\\
(A' \cap B) \cup (A \cap B) &= (A' \cap C) \cup (A \cap C) && \mathbf{1.27}\ (III), (IV)\\
B \cap (A \cup A') &= C \cap (A \cup A') && (1, 1D, 2)\\
B \cap 1 &= C \cap 1 && (4)\\
B &= C && (3D)
\end{aligned}
$$

2.21 (i) (12), (12).
(ii) (12D), (12D), (9).

2.22 (i) $(A' \cap B' \cap C')'$
(ii) $A \cup (B' \cup C')' \cup (B' \cup C \cup D')'$

2.25 (a) $A' \cap B' \cap C$
(b) 1
(c) 0
(d) $(B \cap C) \cup (A' \cap B')$

2.27 (i) (ii) (iii) yes, (iv) no.

2.29 (a) $(X' \cap Y') \cup (Y' \cap Z') \cup (Z' \cap X') \cup (X \cap Y \cap Z)$
(b) $(A \cap B') \cup (A' \cap B)$
(c) $(B \cap C \cap D) \cup (C \cap D \cap A) \cup (D \cap A \cap B)$
$\cup (A \cap B \cap C)$

2.30 There are two ways of choosing the first factor—it is either A_1 or A_1'—and so on.

2.31 (i) $A.B.C + A'.B.C + A'.B'.C + A'.B'.C'$
(ii) $X'.Y'.Z + X'.Y'.Z' + X.Y'.Z' + X'.Y.Z' + X.Y.Z$
(iii) No change is necessary

2.32 (i) A (ii) B' (iii) $A.B$

2.34 (i) $(A + B + C).(A + B' + C)$
(ii) $(X + Y + Z)(X' + Y + Z)(X + Y' + Z)(X + Y + Z')$

2.37
$$\begin{aligned} E_1.E_2 &= (X.Y + X'.Y' + X.Z).(X.Y + X'.Y' + Y'.Z) \\ &= X.Y + X'.Y' + X.Y'.Z \\ &= X'.Y' + X.(Y + Y'.Z) \\ &= X'.Y' + X.(Y + Z) \\ &= X'.Y' + X.Y + X.Z \\ &= E_1 \\ E_1.E_2' &= (X.Y + X'.Y' + X.Z).(X + Y).(X' + Y') \\ &\quad .(Y + Z') \\ &= (X.Y + X'.Y'.Z').(X.Y' + X'.Y) \\ &= 0 \end{aligned}$$

2.39 (ii) If $A.B' + A'.B = 0$,

then
$$A.B' = A'.B = 0$$
$$A + A'.B = A$$
$$A + B = A \quad \text{and similarly} \quad A + B = B$$
so
$$A = B$$

2.41 (i) $L = (A + C).(P + R)' + (P + R).(A + C)'$
$M = (B + C).(Q + R) + (Q + R).(B + C)$
$N = C.P'.Q'.R' + R.A'.B'.C'$
(ii) (a) $B'.X + (A'.B + A.B').X' = 0$ if $AB' = 0$
(b) $B.X' + (A + B)X = 0$ if $A = 0$

3.1 $A = C, \quad B = D = E$

3.2 (i) All are equivalent; $n(A) = n(B) = \cdots = n(E) = 3$
(ii) The different uses of '=' in the two statements.

3.4 (i) M' = the soa women
(ii) 1 = „ „ human parents

3.5 (a) (i) the soa one's brothers, and oneself, if a male
(ii) „ „ rectangles
(b) (i) „ „ children
(ii) „ „ positive integers

3.7 $A + A'.B = A + B$ (see **2.6**, theorem 11)

3.10 (i)

$A + B = B$	
$(A + B).B' = B.B'$	**1.27** (IVD)
$A.B' + B.B' = B.B'$	(2)
$A.B' + 0 = 0$	(4D)
$A.B' = 0$ and $A \subseteq B$	(3), **3.8**

(ii)

$(A.B).A' = (A.A').B$	(5D)
$= 0.B$	(4D)
$= 0$ so $(A.B) \subseteq A$	(6D), **3.8**

(iii)

$A.(A + B)' = A.A'.B'$	(12D)
$= 0$	(4D, 6D)

(iv)

If $A \subseteq B$, $A.B' = 0$	**3.8**
$B'.(A')' = 0$	(9, 1D)
$B' \subseteq A'$	**3.8**

(v)

$X.Y = X$, dual of **3.9** (i) or	
$X.Y' = X.Y.Y' = 0$	(IVD, 4D, 6D)
so $X \subseteq Y$	
conversely $X = X.1$	(3D)
$= X.(Y + Y')$	(4)
$= X.Y + X.Y'$	(2)
$= X.Y + 0$ since $X \subseteq Y$	
$= X.Y$	(3)

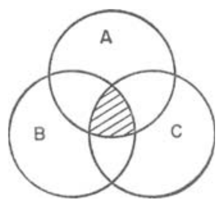
3.14 (I) $G + G'.S = G + S$
(II) $A.S' + A'.S$ or $(A + S).(A.S)'$
(III) $G'.S + G.S'$

3.15 (i) (ii) and (iv) yes, (iii) no.

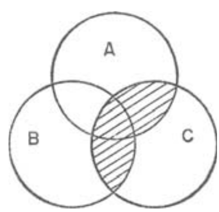
3.16 No.

3.17 Uncle Bertrand, Roger, Tom, and Vera.

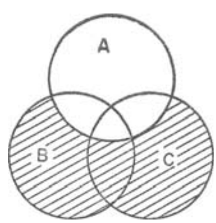
3.19 (a) (i)



(ii)



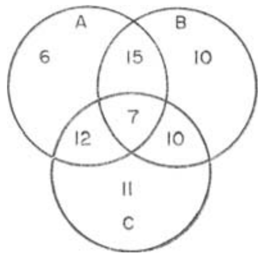
(iii)



(b) The points P, Q.

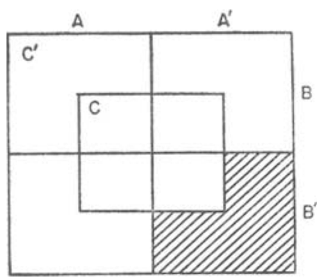
$n(A.B) = 0$ when a and b do not meet
$n(A.B) = 1$,, ,, ,, touch one another

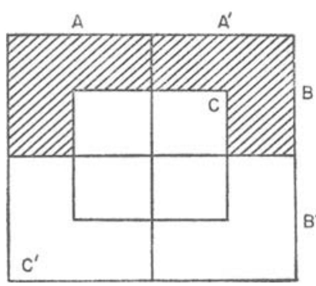
3.20 6 liked apples only



3.22 (i) A′B′C′

(ii) B.C′





3.25 (i) $n(P + Q + R + S) = n(P) + n(Q) + n(R) + n(S)$
$- n(P.Q) - n(P.R) - n(P.S) - n(Q.R) - n(Q.S)$
$- n(R.S) + n(Q.R.S) + n(R.S.P) + n(S.P.Q)$
$+ n(P.Q.R)$
$- n(P.Q.R.S)$

3.27 (i) Draw APQ through A

(ii) P on FE to Q on BD by lines parallel to AB
P „ „ „ R „ BC „ „ through A

(iii) $0 < x < 1; 1 < y < 2; 1 < z$

correspondence established by

$$y = 1 + x$$
$$z = 1/x$$

3.30 $n \in N$, then:

(i) $\{x \mid x = 3n\}$
(ii) $\{y \mid y = n^2\}$
(iii) there is no greatest prime, and then arrange them in ascending order.

4.2 Every function can be expressed as a polynomial; each constituent mononomial will be the product of a number of 1's and 0's, etc.

4.3 (i) Use (10) (10D).

4.5

(i)

A	B	C	A′.B.C	A + A′.B.C
1	1	1	0	1
0	1	1	1	1
1	0	1	0	1
1	1	0	0	1
1	0	0	0	1
0	1	0	0	0
0	0	1	0	0
0	0	0	0	0

(ii)

X	Y	Z	X + Y + Z	Y′ + Z′	(X + Y + Z) .(Y′ + Z′)
1	1	1	1	0	0
0	1	1	1	0	0
1	0	1	1	1	1
1	1	0	1	1	1
1	0	0	1	1	1
0	1	0	1	1	1
0	0	1	1	1	1
0	0	0	0	1	0

4.6 $X = P \cap Q' \cap R \cap S'$
$Y = P' \cap Q' \cap R' \cap S'$

4.8 (i) Show that $X + Y = 1$, and $X.Y = 0$
(ii) $1 = X.Y.Z + X'.Y.Z + X.Y'.Z + X.Y.Z' + X.Y'.Z' + X'.Y.Z' + X'.Y'.Z + X'.Y'.Z'$

and use $A.B.C + A.B.C' = A.B.(C + C') = A.B$, etc.

4.11 $A.B' + A'.B = 1$

$$(A.B' + A'.B)' = 1' = 0 \qquad \textbf{2.4}\ \text{(iv)}$$
$$(A' + B).(A + B') = 0$$
$$A.B + A'.B' = 0$$
$$A.(B')' + A'.(B') = 0$$
$$A = B' \qquad \textbf{2.39}\ \text{(ii)}$$

similarly for **4.10** (f).

4.15 (a) $A \cup B' \neq B \cup A'$
(b) If $A \rightarrow B$ then $A' \cup B = 1$ **4.14**
$A \cap B' = 0$ (12, 9, 10D)
and if $P \subseteq Q$,, $P \cap Q' = 0$

the algebraic conditions for material implication and for a subset are the same.

(c) Use **4.15** (b) and **3.9** (v), or

$$\begin{array}{cr} A' \cup B = 1 \qquad B' \cup C = 1 & \mathbf{4.14} \\ B' \cap (A' \cup B) = B', \qquad B \cap (B' \cup C) = B & (IVD) \\ A' \cap B' = B', \qquad B \cap C = B & (2, 4D, 6D) \\ (A' \cap B') \cup (B \cap C) = B \cup B' = 1 & (IV, 4) \\ (A' \cup B) \cap (A' \cup C) \cap (B' \cup B) \cap (B' \cup C) = 1 & \mathbf{2.33} \\ 1.1.1.(A' \cup C) = 1 & \mathbf{4.14}\ (4) \\ A' \cup C = 1 & (3D) \\ A \rightarrow C & \mathbf{4.14} \end{array}$$

4.16 $(A \rightarrow B) = A' \cup B$ **4.14**

so $P \cup Q = (P' \rightarrow Q)$

and $R \cap S = (R' \cup S')'$

$= (R \rightarrow S')'$

The whole argument is false. $(A \rightarrow B)$ is a statement which has a meaning only when A and B are *statements.* $A \subseteq B$ is a statement which has meaning if A and B are *sets.* We can regard $\rightarrow$ as an operation, but not $\subseteq$. For in the application of our algebra to logic, the elements are statements, and $A \rightarrow B$ is a statement, and so the set of elements is closed (see **1.3**) under this operation. $\subseteq$ does not qualify, as $P \subseteq Q$ is also a statement, not a set.

4.18 (a)
$$\begin{array}{rlr} A \downarrow B &= A.B' + A'.B + A'.B' & \mathbf{4.17} \\ &= A.B' + A' & (2, 4, 3D) \\ &= A' + B' & (1, 9, 11) \\ &= (A.B)' & (12) \end{array}$$

or use $(A \downarrow B)' = A.B$ from **4.17.**

(b)
$$\begin{array}{rlr} A \downarrow A &= (A \cap A)' & (a) \\ &= A' & (7D) \end{array}$$

(c)
$$\begin{array}{rlr} A \downarrow B &= (A.B)' & (a) \\ &= (B.A)' & (1D) \\ &= B \downarrow A & (a) \end{array}$$

(d)
$$\begin{array}{rlr} (A \downarrow A) \downarrow (A \downarrow A) &= (A') \downarrow (A') & (b) \\ &= (A')' & (b) \\ &= A & (9) \end{array}$$

(e) Use (b), (a), (12), (9).

(f) $\cup$ eliminated by using (e)

$\cap$,, ,, ,, $A \cap B = (A \downarrow B)'$ (a)

(h) $A' \downarrow (A' \downarrow B) = A' \downarrow B'$

$(A \downarrow A) \downarrow \{(A' \downarrow B) \downarrow (A' \downarrow B)\} = (A \downarrow A) \downarrow (B \downarrow B)$

$(A \downarrow A) \downarrow [\{(A \downarrow A) \downarrow B\} \downarrow \{(A \downarrow A) \downarrow B\}]$

$$= (A \downarrow A) \downarrow (B \downarrow B)$$

4.19 (v)
$$\begin{aligned}(A \uparrow B) \uparrow C &= (A' \cap B') \uparrow C\\ &= (A' \cup B') \cap C'\\ &= (A \cup B) \cap C'\\ &= (A \cap C') \cup (B \cap C')\\ &= (A' \uparrow C) \cup (B' \uparrow C)\end{aligned}$$

(vi)
$$\begin{aligned}A \cup B &= (A' \cap B')'\\ &= (A \uparrow B)'\end{aligned}$$

(viii) $A \cap (A' \cup B) = A \cap B$

becomes $(A \uparrow A) \uparrow \{(A \uparrow A) \uparrow B\} = (A \uparrow A) \uparrow (B \uparrow B)$

4.20 If A is the statement 'Andrew tells the truth', etc.

$$\begin{aligned}A &= B.C & C &= E.F + E'.F'\\ D &= A + B & E &= A.B\\ B &= E.F' + E'.F & F &= (B.C)'\end{aligned}$$

$$B = D = F = 1$$
$$A = C = E = 0$$

4.21
$$\begin{aligned}1 &= (A \cup B) \cap (A' \cup C')\\ &= (A' \cap B')' \cap (A' \cup C')\\ &= (1)' \cap (A' \cup C')\\ &= 0 \cap (A' \cup C')\\ &= 0\end{aligned}$$

and the statements are inconsistent.

4.22 Let A be the statement 'Atlantia sends a contingent', etc., and we have

$$W.Y = 0 \quad \text{(i)}$$
$$A.D = 0 \quad \text{(ii)}$$
$$V.W.Z = 0 \quad \text{(iii)}$$
$$W'.C.D = 0 \quad \text{(iv)}$$
$$Z'.E = 0 \quad \text{(v)}$$
$$Z.B = 0 \quad \text{(vi)}$$
$$X.C = 0 \quad \text{(vii)}$$

from (v) and (vi) $\quad B.(Z'.E) + E.(B.Z) = 0$

$$B.E = 0$$

,, (ii) $\quad A.D = 0$

3 from among A, B, C, D, E = 1,

so $C = 1$

from (vii) $X = 0$

" (i) and (iii) $V.Y.Z = 1$

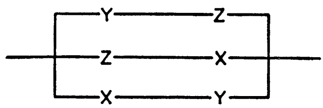
$$V = Y = Z = 1 \qquad (13D)$$

" (vi) $B = 0$ so $E = 1$

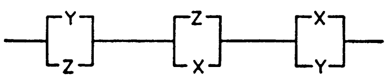
" (iv) $D = 0$ so $A = 1$

5.5

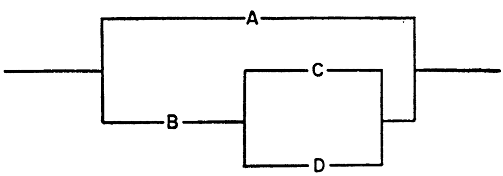
(i)



(ii)



(iii)



5.6 (i) $X.Y.Z + X' + X.Y.Z'$

$= X' + X.Y.(Z + Z')$

$= X' + X.Y$

$= X' + (X')'.Y$

$= X' + Y \qquad (11)$

(ii) $A.B.C + B'.C.D + A'.B.C + B.C' + A.C.D'$

$= C.(B' + B'.D + A.B + A'.B + A.D')$

$= C.\{B' + B.(A + A') + B'.D + A.D'\}$

$= C.(B' + B + B'.D + A.D')$

$= C.(1 + B'.D + A.D')$

$= C(1)$

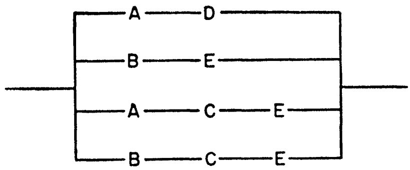
$= C$

5.7 (i) $A.B.C.D + A'.B.C.D + A.B.C'.D + A'.B.C'.D$
$= (A + A').B.C.D + (A + A').B.C'.D$
$= B.C.D + B.C'.D$
$= B.D.(C + C')$
$= B.D$

———— B ——— D ————

(ii) $X.Y.Z + X.Y.Z' + X.Y'.Z + X.Y'.Z' + X'.Y.Z$
$+ X'.Y.Z' + X'.Y'.Z$
$= X.Y.(Z + Z') + X.Y'.(Z + Z') + X'.Y.(Z + Z')$
$+ X.Y'.Z$
$= X.Y + X.Y' + X'.Y + X'.Y'.Z$
$= X.(Y + Y') + X'.(Y + Y'.Z)$
$= X + X'.(Y + Z)$
$= X + (Y + Z)$
$= X + Y + Z$

5.8

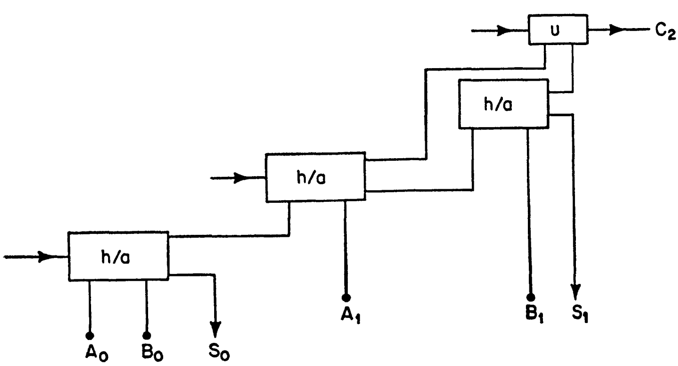


5.10 (i) 219 (ii) 11000011

5.16 'There is "one to carry" if both the digits are 1 (if $A_0.B_0 = 1$, then $A_0 = B_0 = 1$), but if, of A_0 and B_0, one and only one is equal to 1, then their sum is 1.'

5.18 Because the addition of numbers is commutative.

5.19



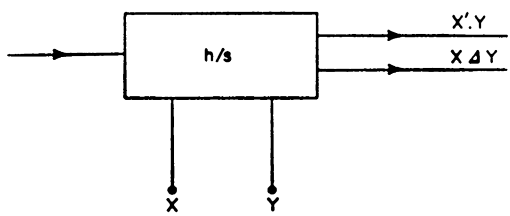
5.20

A_0	B_0	L_1	D_0
1	1	0	0
0	1	1	1
1	0	0	1
0	0	0	0

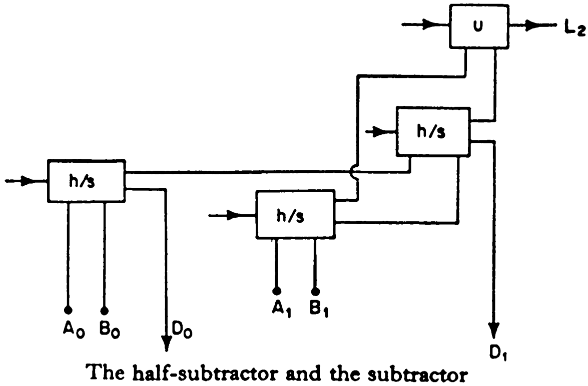
$L_1 = A_0'.B_0, \qquad D_0 = A_0 \Delta B_0$

A_1	B_1	L_1	L_2	D_1
1	1	1	1	1
0	1	1	1	0
1	0	1	0	0
1	1	0	0	0
1	0	0	0	1
0	1	0	1	1
0	0	1	1	1
0	0	0	0	0

$$L_2 = A_1'.B_1 + L_1.(A_1 \Delta B_1)$$
$$D_1 = L_1.(A_1 \Delta B_1)' + L_1'.(A_1 \Delta B_1)$$



The 'half-subtractor'



The half-subtractor and the subtractor

6.2 (i) 2/7 (ii) 1/2 (iii) 1/7

6.3 80 (there are two genders in French and three in German).

6.4 150/199, 24/199.

6.5 2^{N-1}.

6.6 The expression

$$(H + T).(H + T).(H + T).(H + T).(H + T)$$

can be regarded as a diagram of five coins, of which each must be a head or a tail. We can also regard H and T as numbers, and the array as representing $(H + T)^5$.

To every member of the soa ways of getting four heads and a tail, there is one and only one term in $H^4.T$ in the expansion of $(H + T)^5$, and conversely. But

$$(H + T)^5 = H^5 + 5.H^4.T + 10.H^3.T^2 + 10.H^2.T^3 + 5.H.T^4 + T^5$$

and so the probability of having at least four coins alike is

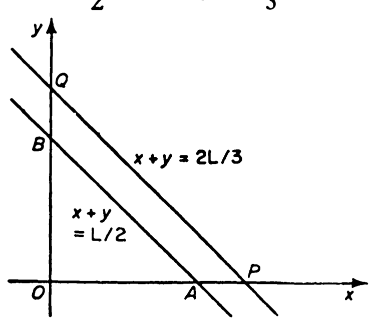
$$\frac{1 + 5 + 5 + 1}{1 + 5 + 10 + 10 + 5 + 1} = \frac{3}{8}$$

6.9 A bus in Mary's direction goes from John's stop 9 minutes after a bus in Dora's direction, then 3 minutes later is the next bus in Dora's direction; probability of visiting Mary is then

$$\frac{9 \text{ minutes}}{12 \text{ minutes}} = \frac{3}{4}$$

6.11 If x, y are the two shorter pieces of the stick, total length L, then

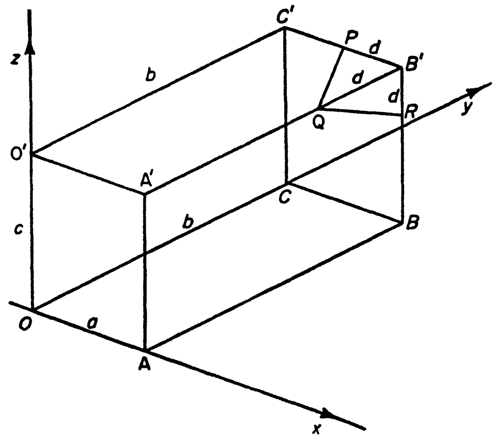
$$\frac{L}{2} < x + y < \frac{2L}{3}$$



so, by the method of **6.10**, the required probability is

$$\frac{\text{area of APQB}}{\text{area of OPQ}} = \frac{7}{16}$$

6.12 Axes as in diagram; there is a one-to-one relation between the times waited at stations and the points of the figure OABCC′B′A′D′.



For early arrival the point must lie below the plane PQR, i.e.

$$x + y + z = \mathrm{M}$$
$$a + b + c - \mathrm{M} = d$$

and probability is

$$\frac{\text{volume of OABRQPC'O'}}{\text{volume of OABCC'B'A'O'}} = \frac{6\,abc - d^3}{6\,abc}$$

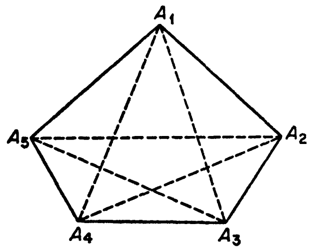
1.3/1 (i) yes (ii) yes (iii) no (iv) no

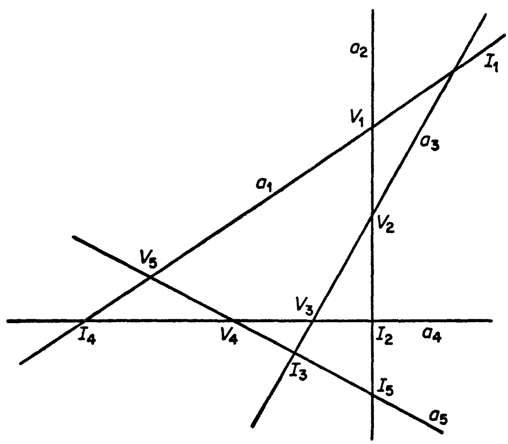
1.6/1 $\frac{x}{a}.\cos\frac{\alpha+\beta}{2} + \frac{y}{b}.\sin\frac{\alpha+\beta}{2} = \cos\frac{\alpha-\beta}{2}$

$\cos(-\mathrm{A}) = \cos \mathrm{A}$

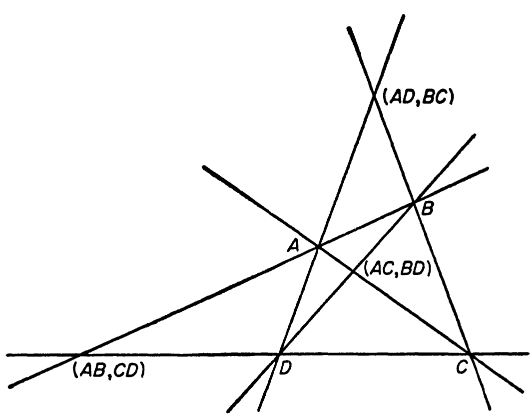
1.16/1 $(a + b).(a + c) = a + b.c$
$a^2 + a.(b + c) = a$ and $a \neq 0$
$a + b + c = 1$

1.22/1


A_1
A_2
A_3
A_4
A_5


a_2
I_1
V_1
a_3
a_1
V_2
V_5
V_3
I_4
V_4
I_2
a_4
I_3
I_5
a_5

1.22/2


(AD, BC)
B
A
(AC, BD)
(AB, CD)
D
C

1.25/1

$$
\begin{aligned}
0' &= 0' + 0 && \textbf{1.23}\\
&= 0 + 0' && \textbf{1.4}\\
&= 1 && \textbf{1.24}\\
1' &= 1'.1 && \textbf{1.23}\\
&= 1.1' && \textbf{1.4}\\
&= 0 && \textbf{1.24}
\end{aligned}
$$

1.25/2 (i) $A \cap A = A$
(ii) $A \cap 0 = 0$
(iii) $(A \cap B) \cup (A \cap B') = A$
(iv) $(A \cup B \cup C') \cap (A \cup B \cup C) = A \cup B$
(v) $A \cap B \cap (A' \cup B') = 0$
(vi) $A \cup (A' \cap B) = A \cup B$

1.27/1

(I)	$a \mathrm{F} a$			true
(II)	If $a \mathrm{F} b$,	then	$b \mathrm{F} a$	false
(III)	If $a \mathrm{F} b$,	and	$b \mathrm{F} c$, then $a \mathrm{F} c$	true
(IV)	If $a \mathrm{F} b$,	then	$(a + c) \mathrm{F} (b + c)$	false
(IVD)	If $a \mathrm{F} b$,	then	$a.c \mathrm{F} b.c$	true

2.9/1 (i), (ii), (iii) use $P \cup (P \cap Q) = P$

2.16/1 (i) $C'.(A' + B')$
(ii) $X + Y + Z$
(iii) $A \cup B \cup C \cup D$
(iv) $X' \cap Y' \cap Z'$
(v) no simpler form
(vi) R

2.19/1 (i)

$$
\begin{aligned}
A.X &= A.Y\\
A'.X &= A'.Y\\
A.X + A'.X &= A.Y + A'.Y && \text{(IV)}\\
1.X &= 1.Y\\
X &= Y
\end{aligned}
$$

(ii)

$$
\begin{aligned}
A + X &= A + Y\\
A' + X &= A' + Y\\
(A + X).(A' + X) &= (A + Y).(A' + Y) && \text{(IVD)}\\
X + A.A' &= Y + A.A' && \text{(2D)}\\
X + 0 &= Y + 0\\
X &= Y
\end{aligned}
$$

2.29/1 (i) $B'.C' + C.A' + A'.B'$
(ii) $X' \cup Y$

2.30/2 (i) $X.Y + X.Y' + X'.Y$
(ii) same answer as **2.30**
(iii) $A.B.C + A.B.C' + A.B'.C' + A'.B'.C'$

2.30/4 (i) $A' + B$
(ii) 1
(iii) $P.R + Q'.(P'.R + P.R')$
(iv) $A'.B + C'.(A.B + A'.B')$

2.43/1 (i) $(A \Delta B) \Delta C$
$= (A.B' + A'.B).C' + (A.B + A'.B').C$
$= A.(B.C + B'.C') + A'.(B'.C + B.C')$
$=$ etc.
(ii) similar to (i)
(iii) see **2.39** (ii)

2.43/2 $Z' = A \Delta B \Delta C \Delta D$

3.2/1 (i) the soa strings of a violin
(ii) ,, ,, toes on a human foot
(iii) ,, ,, members of a hockey team
(iv) ,, ,, months of the year
(v) ,, ,, fish in the sea
(vi) ,, ,, prime numbers

3.2/2 $\dfrac{9!}{4!\,5!}$

3.4/1 (i) The soa members of the House of Lords
(ii) ,, ,, real numbers
(iii) {Tom, Dick, Harry, Jack, Jill}
(iv) $\{e\}$
(v) $\{2\}$

3.5/2 (a) I; (b) II; (c) II; (d), (e) III.

3.13/1 M = the soa racing motorists
T = ,, ,, profound thinkers
P = ,, ,, philosophers
Q = ,, ,, quick-witted men

$M \subseteq Q$, $\{\text{Plato}\} \subseteq T$, $P \subseteq T$, $Q.T = 0$, or $Q \subseteq T'$,
$\{\text{Plato}\} \subseteq T \subseteq Q' \subseteq M'$, and $P \subseteq T \subseteq \ldots$.

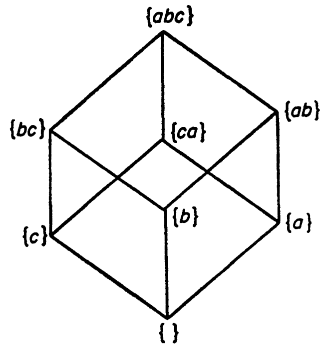
Plato was not a racing motorist, but it cannot be deduced that he was a philosopher.

3.13/3 (i) $\{x \mid x = 2^{n-1}\}$
(ii) $\{x \mid x = 3n - 2\}$
(iii) $\{x \mid x = n!\}$

3.13/5 (i) the soa triangular numbers
(ii) „ „ numbers numerically less than 1
(iii) „ „ positive numbers

3.13/6 (i) yes (ii) no

3.13/8 (i)



There are 8; from $\{pqrs\}$ to $\{p\}$, $\{q\}$, $\{r\}$, $\{s\}$, and from $\{pqr\}$ $\{pqs\}$ $\{prs\}$, $\{qrs\}$ to $\{\ \}$.
(ii) 2^n

3.13/9 In the diagram of **3.13/7**, for $\{p, q, r, s\}$ write (r, g, b, w) etc., and the point (r, b) represents red and blue full on; a line down the page, say from (r, g, b) to (r, b) thus represents a gradual subtraction of green from a combination of red, green, and blue; similarly (g) to (g, w) represents a gradual addition of white to green. A way of covering the whole figure without tracing any line twice is

0, r, rg, g, 0, b, bg, g, gw, w, wr, r, rb, b, bw, rbw, rw, rgw, rg, rgb, rb, rbw, rbwg, rbg, bg, bgw, gw, rgw, rbgw, gbw, bw, w, 0

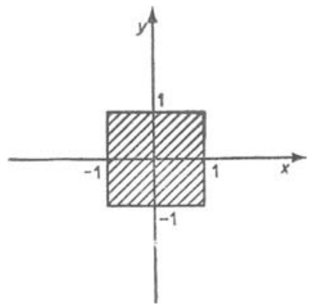
3.13/10
$$
\begin{aligned}
n(X \cap Y) &= 0, \quad z > 6r \\
&= 1, \quad z = 6r \\
&= 2, \quad 4r < z < 6r \\
&= 4, \quad z = 4r \\
&= 6, \quad 3r < z < 4r \\
&= 7, \quad z = 3r \\
&= 8, \quad z < 3r
\end{aligned}
$$

3.13/12 Symmetry about $y = x$

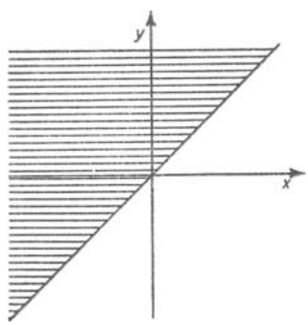
(i)	,,	,,	origin
(ii)	,,	,,	$x + y = 0$
(iii)	,,	,,	y-axis

3.13/13

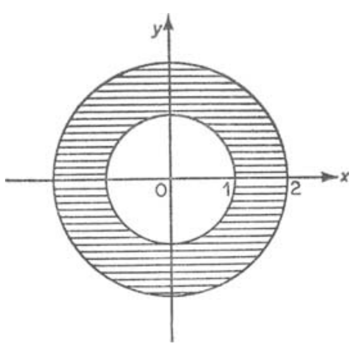
(i)



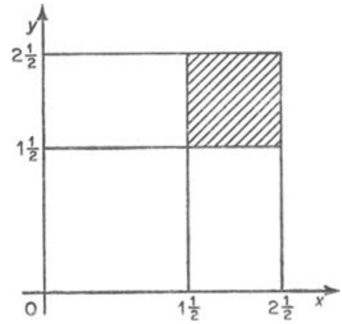
(ii)



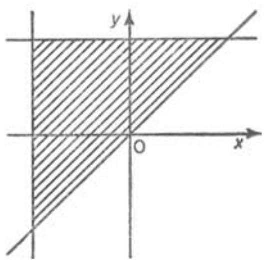
(iii)



(v)

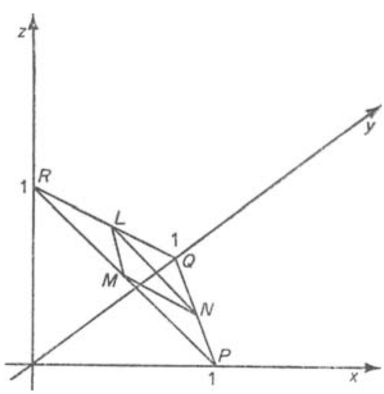


(vi)



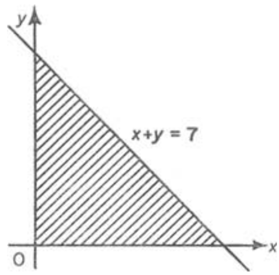
3.13/14 The point 4, 3

3.13/15

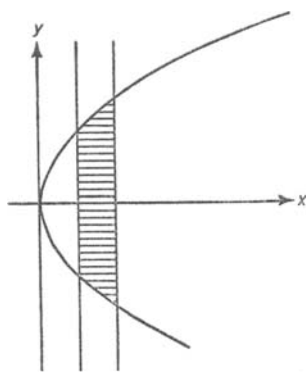


(A ∩ B) is the soa points in triangle PQR.
(A ∩ B ∩ C) is the soa points in the triangle LMN.

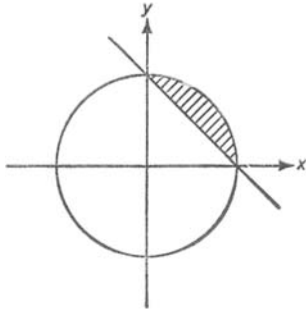
3.13/16 (i) $\{\langle x, y\rangle \mid x > 0, y > 0, x + y < 7\}$



(ii) $\{\langle x, y\rangle \mid a < x < 2a, y^2 < 4ax\}$



(iii) $\{\langle x, y\rangle \mid x^2 + y^2 < 1, x + y > 1\}$



3.13/17 The empty set.

3.17/1 E, F, G, T = the soa Europeans, fair, good-tempered, tall men, respectively.

$$\begin{aligned} E.F' + E'.F &\subseteq G & \qquad G'.(E.F' + E'.F) &= 0 \\ E &\subseteq (T + F) & E.F'.T' &= 0 \\ E.F' &\subseteq T' & E.F'.T &= 0 \end{aligned}$$

so

$$\begin{aligned} E.F'.(T + T') = E.F' &= 0 \\ G'.E'.F &= 0 \\ G'.F &\subseteq E \end{aligned}$$

3.17/2 (i) All four tailors were members of each of the four sets mentioned.

(ii) An example of $A + A.B = A$.

(iii) The first two lines imply a universal soa little girls.

(iv) 99 out of a set of 100 selected dentists, or 99% of the soa dentists?

(v) C = the soa men called A. B. Charles
D = „ „ „ at 73, Dover Road
E = „ „ „ at Eastborough
F = „ „ „ living in Kent

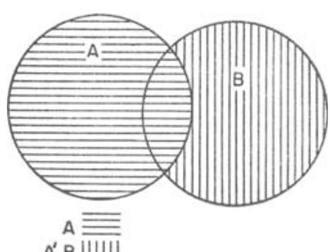
the addressee is a member of $C \cap D \cap E \cap F$

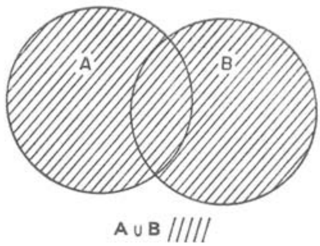
3.21/1 10

3.21/2 Use Venn diagram; the number of single, unemployed immigrants is -5.

3.32/3

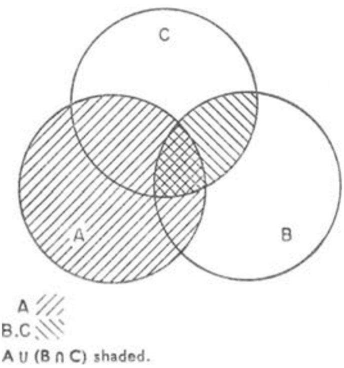
(i)

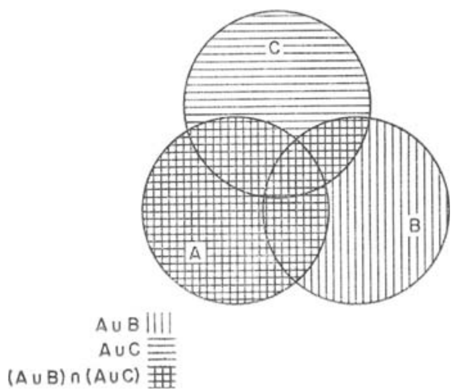




3.32/3

(ii)





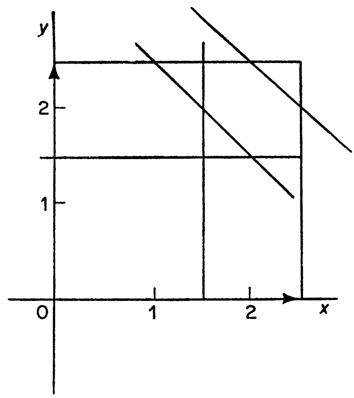
4.7/2 $X = A \Delta (B'.C')$
$Y = A.B' + A'.C'$
$Z = (A.B') \Delta C$

4.20/1 $E = 1, \quad A = B = C = D = F = 0$

5.27/1 $2^{13} - 1 = 8{,}191$

6.12/1 $\{\langle x, y\rangle \mid 3/2 < x, y < 5/2, 7/2 < x + y < 9/2\}$
$\{\langle x, y\rangle \mid 3/2 < x, y < 5/2\}$

probability $= 3/4$



INDEX