Brane gases in the early universe: thermodynamics and cosmology

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We consider the thermodynamic and cosmological properties of brane gases in the early universe. Working in the low energy limit of M-theory we assume the universe is a homogeneous but anisotropic 10-torus containing wrapped 2-branes and a supergravity gas. We describe the thermodynamics of this system and estimate a Hagedorn temperature associated with excitations on the branes. We investigate the cross-section for production of branes from the thermal bath and derive Boltzmann equations governing the number of wrapped branes. A brane gas may lead to decompactification of three spatial dimensions. To investigate this possibility we adopt initial conditions in which we fix the volume of the torus but otherwise assume all states are equally likely. We solve the Einstein-Boltzmann equations numerically, to determine the number of dimensions with no wrapped branes at late times; these unwrapped dimensions are expected to decompactify. Finally we consider holographic bounds on the initial volume, and find that for allowed initial volumes all branes typically annihilate before freeze-out can occur.

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1 Introduction

One of the few firm predictions of string or M-theory is the existence of extra spatial dimensions. The conventional scenario is that these dimensions are unobservably small today. Understanding how the universe got into such an asymmetric state is necessarily a problem for cosmology. One intriguing possibility is that extended objects play a vital role in explaining the asymmetry. This is the idea behind *brane gas cosmology* [1, 2, 3, 4, 5, 6].

Brane gas cosmology rests on the assumption that in string or M-theory branes will be present as one component of a heat bath that fills the early universe. In a universe with compact directions that can be wrapped by branes, the dynamics of the wrapped branes may play a significant role in the overall evolution of the universe. In particular one can explore analogues of the Brandenberger-Vafa scenario, which proposes that the three large spatial dimensions we see at present arose from a thermal fluctuation in a primordial gas of winding strings [7, 8, 9, 10, 11].

In a recent paper we considered the late time behavior of a brane gas model, arising from M-theory compactified on T^{10}, in which the universe contains a supergravity gas and 2-branes wrapped on the various cycles of the torus [12]. We could safely ignore 5-branes, which are also part of the M-theory spectrum, since they will quickly intersect and annihilate in ten spatial dimensions. The key conclusion of [12] was that the directions which were not wrapped by 2-branes expanded faster than those that were, and that the overall expansion rate of the wrapped and unwrapped subspaces depended only on their dimensionality. In the present paper we turn our attention to the early time behavior of this model, and include two crucial ingredients which are unimportant at late times: the fluctuations on the branes themselves, and the annihilation and creation of branes out of the thermal bath provided by the supergravity gas.

This paper is laid out as follows. In the following section we derive the analogs of the Friedmann equations for this cosmology. Section 3 examines the statistical mechanics of a universe which contains 2-branes and radiation (the supergravity gas), and presents a simple and, to our knowledge, novel derivation of a limiting (Hagedorn) temperature for the 2-brane gas. In section 4 we investigate the cosmology of the "Hagedorn phase," showing that in this phase the negative pressure of the brane tension cancels the positive pressure of the brane fluctuations, so the universe expands almost as if it were filled with pressureless dust. In section 5 we investigate the cross-section for

brane-antibrane annihilation and write down Boltzmann equations governing the number density of the different brane wrapping modes. We show that the effective interaction rate drops to zero at a finite time, producing a *freeze-out* analogous to that leading to a relic abundance of dark matter species in conventional cosmology, so that some directions can remain wrapped at late times. In Section 6 we numerically evolve the combined Einstein-Boltzmann equations for a variety of different initial conditions, and show that the number of directions which are not wrapped by branes depends on the initial volume of the universe. Section 7 describes holographic constraints on the initial conditions for the universe, and we conclude in Section 8. Throughout this paper we scale the M-theory Planck length to unity, so that Newton's constant is given by $16\pi G = (2\pi)^8$ and the 2-brane tension is $T_2 = 1/(2\pi)^2$.

2 Gravitational dynamics

We consider a universe whose spatial topology is a d-dimensional torus T^d. The case of interest for M-theory is $d = 10$, but we frequently write d as a parameter, to clarify the origin of the numerical constants appearing in our equations. The universe is thus spatially flat but has finite volume, a fact which will be of crucial importance in what follows. We use the metric

$$ds^2 = -dt^2 + \sum_{i=1}^{d} e^{2\lambda_i(t)} dx_i^2 \qquad 0 \le x_i \le 1 \tag{1}$$

and add matter in two forms: M2-branes and a supergravity gas. The stress tensor for the massless supergravity gas is

$$T^\mu{}_\nu = \mathrm{diag}(-\rho_S, p_S, \ldots, p_S) \tag{2}$$

where $\rho_S = c_S T^{11}$ is the energy density of the supergravity gas. The coefficient c_S is computed in appendix A, and the equation of state fixes $p_S = \frac{1}{d}\rho_S$.

The M2-brane gas consists of 2-dimensional membranes, each of which is wrapped around a 2-cycle inside the T^{10}. These wrapped branes are thus topologically stable. There are 45 ($= 10 \times 9/2$) independent ways to wrap the 2-branes, so we effectively have 45 distinct species of branes in our model. The universe is spatially compact, so to satisfy Gauss' law we must have equal numbers of branes and antibranes on each 2-cycle.

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Note that we are ignoring the possibility of having "diagonally wound" branes, which wrap on non-trivial linear combinations of the basic (ij) homology cycles. This is required by our metric ansatz (1) which describes a rectangular torus. Under time evolution such a torus is compatible with having branes wound in the (ij) directions, but not with having diagonally wound branes, whose tension would cause the torus to tilt.

The brane energy density gets contributions from two sources: the brane tension and the energy in transverse brane fluctuations. We ignored the latter contribution in Ref. [12] since it is negligible at late times, but we must include it here. We describe the wrapped branes using the leading long-wavelength approximation to the Nambu-Goto action, and thus model the transverse fluctuations as a non-interacting gas of massless particles living on the brane. The entire brane can also move in the transverse directions, but we assume that this motion is non-relativistic. The brane is thus effectively at rest, so its kinetic energy is negligible compared to its rest mass and can be ignored.

For a brane at rest wrapped once around the (12) cycle and smeared over the eight transverse dimensions, the contribution to the stress tensor from the brane tension is

$$T^\mu{}_\nu = -\frac{T_2}{\mathrm{vol}_\perp}\mathrm{diag}(1,1,1,0,\ldots,0), \tag{3}$$

where T_2 is the brane tension and $\mathrm{vol}_\perp = \exp\sum_{i=3}^{10}\lambda_i$ denotes the volume perpendicular to the brane. Similarly, for the worldvolume gas on the brane

$$T^\mu{}_\nu = \frac{1}{\mathrm{vol}_\perp}\mathrm{diag}(-\rho_X, p_X, p_X, 0, \ldots, 0), \tag{4}$$

where $\rho_X = c_X T^3$ is the energy density in the fluctuation gas. The coefficient c_X is computed in appendix A. The equation of state for a gas in two spatial dimensions fixes $p_X = \frac{1}{2}\rho_X$.

Combining these expressions we can write down the energy density for this universe:

$$\rho = c_S T^{11} + \frac{1}{V}\sum_{i\neq j}N_{ij}(T_2 + c_X T^3)e^{\lambda_i + \lambda_j}\,. \tag{5}$$

Here N_{ij} for $i > j$ is the number of branes wrapped on the (ij) cycle, $N_{ji} = N_{ij}$ is the number of antibranes, and $V = e^{\sum \lambda_i}$ is the total volume of the

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torus. Likewise the pressure on the ith dimension is

$$p_i = \frac{1}{d}c_S T^{11} + \frac{1}{V}\sum_{k \neq i}(N_{ik} + N_{ki})\left(-T_2 + \frac{1}{2}c_X T^3\right)e^{\lambda_i + \lambda_j}. \tag{6}$$

The relevant Einstein equations are derived in appendix B. They take the form of a Hamiltonian constraint (the analog of the Friedmann equation)

$$\frac{1}{16\pi G}\sum_{i \neq j}\dot{\lambda}_i\dot{\lambda}_j = \rho \tag{7}$$

along with a set of dynamical equations of motion

$$\ddot{\lambda}_i + (\sum_k \dot{\lambda}_k)\dot{\lambda}_i = 8\pi G\left(\frac{1}{d-1}\rho + p_i - \frac{1}{d-1}\sum_k p_k\right). \tag{8}$$

3   Brane gas thermodynamics

In this section we work out the statistical distribution of scale parameters λ_i, velocities $\dot{\lambda}_i$ and wrapping numbers N_{ij} when the system is in thermal equilibrium.

3.1   Empty universes

We start with the trivial case of an empty universe, with no supergravity gas and no branes, and begin by setting up the canonical formalism. For the metric (1), the curvature scalar is

$$R = -2\sum_i \ddot{\lambda}_i - 2\sum_i \dot{\lambda}_i^2 - \sum_{i \neq j}\dot{\lambda}_i\dot{\lambda}_j \tag{9}$$

and the Einstein-Hilbert action is

$$\begin{aligned} S &= -\frac{1}{16\pi G}\int d^{11}x\sqrt{-g}\,R \\ &= -\frac{1}{16\pi G}\int dt\,V\sum_{i \neq j}\dot{\lambda}_i\dot{\lambda}_j \end{aligned} \tag{10}$$

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where $V = e^{\sum_i \lambda_i}$ and we have integrated by parts in the second line. The canonical momenta π_i are given by

$$\pi_i = \frac{\partial \mathcal{L}}{\partial \dot{\lambda}_i} = -\frac{V}{8\pi G} \sum_{j \neq i} \dot{\lambda}_j \tag{11}$$

and the Hamiltonian is

$$H_{\text{gravity}} = \sum_i \pi_i \dot{\lambda}_i - \mathcal{L} = -\frac{V}{16\pi G} \sum_{i \neq j} \dot{\lambda}_i \dot{\lambda}_j. \tag{12}$$

The equations of motion one obtains from the action (10) or the Hamiltonian (12) do not completely reproduce the Einstein equations, since our metric (1) fixes a choice of gauge $g_{tt} = -1$. Varying g_{tt} gives the time-time component of the Einstein equations, which is a constraint that must be imposed on the initial conditions. One can check that this constraint is equivalent to requiring that the Hamiltonian vanishes, $H_{\text{gravity}} = 0$. This condition is, of course, expected in a spatially compact universe. With this constraint the equations of motion that follow from the Hamiltonian (12) are equivalent to the usual Einstein equations. This constraint has been interpreted to mean that the "wave function of the universe" should satisfy $H_{\text{gravity}} \Psi = 0$ [13, 14, 15].

Now consider the equilibrium distribution of states for an empty universe. Quantizing the system semiclassically, and assuming that all zero-energy states are equally likely, the volume of phase space available to the system is

$$\Gamma = \int \frac{d^d\pi \, d^d\lambda}{(2\pi)^d} \, \delta(H_{\text{gravity}}). \tag{13}$$

This is nothing but the microcanonical ensemble of classical statistical mechanics. It is more transparently written in terms of the radii $R_i = \frac{1}{2\pi} e^{\lambda_i}$ as

$$\Gamma = \frac{1}{4\pi G} \int d^d\pi \, d^d R \, \delta\left(\sum_i \pi_i^2 - \frac{1}{d-1}(\sum_i \pi_i)^2\right). \tag{14}$$

Note that the radii are uniformly distributed from zero to infinity. That is, in equilibrium the typical universe has very large volume and is very anisotropic.

3.2 Adding matter

We consider three different matter contributions to the energy of the system, namely

$$E_{\text{matter}} = E_S + E_T + E_X \tag{15}$$

arising from the supergravity gas, brane tension, and excitations on the branes, respectively (note the slight abuse of the term 'matter' to include the radiation-like supergravity gas). At temperature T the energy and entropy of the supergravity gas are given by

$$
\begin{aligned}
E_S &= c_S V T^{11} \\
S_S &= \frac{11}{10} c_S V T^{10} \, .
\end{aligned}
\tag{16}
$$

The coefficient c_S is worked out in appendix A. For branes at rest the energy due to brane tension is

$$E_T = T_2 \sum_{i \neq j} N_{ij} e^{\lambda_i + \lambda_j} \tag{17}$$

while the energy and entropy due to a massless gas of excitations on the branes are given by

$$
\begin{aligned}
E_X &= \sum_{i \neq j} N_{ij} c_X e^{\lambda_i + \lambda_j} T^3 \\
S_X &= \sum_{i \neq j} N_{ij} \frac{3}{2} c_X e^{\lambda_i + \lambda_j} T^2 \, .
\end{aligned}
\tag{18}
$$

The coefficient c_X is worked out in Appendix A.

The above expressions for the energy and entropy of massless particles are only exact in the thermodynamic limit. But fortunately there are enough massless quanta in the early universe for these expressions to be precise. The thermodynamics of the branes, on the other hand, is more subtle, because the universe may contain only a small number of branes at early times, and moreover the branes that we do have are divided into 45 different subpopulations, labelled by the directions (ij) on which they are wrapped. Thus we need to allow for thermal fluctuations in the brane wrapping numbers N_{ij}.

To do this we study the probability distribution for the combined matter–gravity system. The volume of phase space is

$$\Gamma = \int \frac{d^d \pi \, d^d \lambda}{(2\pi)^d} \sum_{N_{ij}} \int dE_{\text{matter}} e^{S_{\text{matter}}} \delta(H_{\text{gravity}} + E_{\text{matter}}) \, . \tag{19}$$

Note that this expression corresponds to the microcanonical ensemble of statistical mechanics, since the δ-function enforces the constraint that the total energy in the universe must be zero. Given that $E_{\mathrm{matter}} > 0$, we must have $H_{\mathrm{gravity}} < 0$, which is possible since the gravitational Hamiltonian (12) is unbounded below. To avoid a possible confusion, note that in (19) we are regarding S_{matter} as a function of E_{matter}, as appropriate when working in the microcanonical ensemble. That is, in (16) and (18) we regard T merely as a convenient control parameter, which we will determine shortly in terms of E_{matter}.

Using the δ-function to evaluate the integral in (19) gives our final result for the distribution of radii, velocities and wrapping numbers. Up to inessential numerical factors

$$\Gamma = \int d^d\lambda \, d^d\dot{\lambda} \sum_{N_{ij}} V^d e^{S_{\mathrm{matter}}} \,. \tag{20}$$

In this expression the matter entropy is

$$S_{\mathrm{matter}} = \frac{11}{10} c_S V T^{10} + \sum_{i \neq j} N_{ij} \frac{3}{2} c_X e^{\lambda_i + \lambda_j} T^2 \tag{21}$$

where temperature is fixed by the Hamiltonian constraint

$$H_{\mathrm{gravity}} + E_{\mathrm{matter}} = 0 \,.$$

Written out explicitly this constraint reads

$$-\frac{V}{16\pi G} \sum_{i \neq j} \dot{\lambda}_i \dot{\lambda}_j + c_S V T^{11} + \sum_{i \neq j} N_{ij} \left(T_2 + c_X T^3 \right) e^{\lambda_i + \lambda_j} = 0 \,. \tag{22}$$

Incidentally, it is easy to show that the volume of the universe increases monotonically, by rewriting the Hamiltonian constraint as

$$\left(\frac{\dot{V}}{V} \right)^2 = \sum_i (\dot{\lambda}_i)^2 + \frac{16\pi G E_{\mathrm{matter}}}{V} \geq 0 \,. \tag{23}$$

3.3 Maximum entropy configurations

The distribution (20) is dominated by configurations which maximize S_{matter}. We now turn to the problem of identifying these equilibrium configurations.

We will hold the radii of the torus fixed, so the quantities we can vary are the wrapping matrix N_{ij} and the temperature T. Introducing a Lagrange multiplier μ to enforce the Hamiltonian constraint, we wish to extremize

$$S_{\text{matter}} - \mu \left(E_{\text{matter}} - \frac{V}{16\pi G} \sum_{i \neq j} \dot{\lambda}_i \dot{\lambda}_j \right) \tag{24}$$

with respect to T, N_{ij} and μ. Extremizing with respect to T yields

$$\mu = \left. \frac{\partial S_{\text{matter}}}{\partial E_{\text{matter}}} \right|_{\lambda_i} \equiv \frac{1}{T} \tag{25}$$

so that (24) is proportional to the free energy. Extremizing with respect to N_{ij} yields an equation that fixes the temperature of the universe:

$$c_X \left(\frac{3}{2} T^2 - \mu T^3 \right) - \mu T_2 = 0$$

$$\Rightarrow T = T_H \equiv \left(\frac{2T_2}{c_X} \right)^{1/3} . \tag{26}$$

We will refer to T_H as the M-theory Hagedorn temperature, for reasons we discuss in more detail in the next section. Finally extremizing with respect to μ enforces the Hamiltonian constraint (7), which fixes the equilibrium total area in membranes to be

$$\mathcal{N}_{\text{eq}} \equiv \sum_{i \neq j} N_{ij} e^{\lambda_i + \lambda_j} = \frac{V}{3T_2} \left(\frac{1}{16\pi G} \sum_{i \neq j} \dot{\lambda}_i \dot{\lambda}_j - c_S T_H^{11} \right) . \tag{27}$$

The velocities $\dot{\lambda}_i$ decrease as the universe expands. At some point the equilibrium area in membranes goes to zero. Beyond this point the right hand side of (27) becomes negative, which simply means that no branes are present in equilibrium.[1] In this regime only the supergravity gas remains in thermal equilibrium, with a temperature that monotonically drops below T_H.

To move from the equilibrium total area (27) to the equilibrium wrapping matrix itself we assume that, on average, the membrane area gets equally

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[1]Of course branes which have dropped out of equilibrium may be present at arbitrarily late times.

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distributed among all the 2-cycles. Thus the equilibrium number of branes wrapped on the (ij) cycle is

$$\{N_{\text{eq}}\}_{ij} = \frac{1}{d(d-1)} \mathcal{N}_{\text{eq}} e^{-\lambda_i - \lambda_j} \, . \tag{28}$$

Note that in equilibrium large dimensions are wrapped by fewer branes than small dimensions.

4 Limiting temperatures in M-theory

We denoted the critical temperature found in the previous section by $T_H = (2T_2/c_X)^{1/3}$. Using $T_2 = 1/(2\pi)^2$ and the value of c_X from appendix A, this temperature has the numerical value (we usually set $M_{11} = 1$)

$$
\begin{aligned}
T_H &= (28\pi\zeta(3))^{-1/3} \, M_{11} \tag{29} \\
&\approx 0.211473 \, M_{11} \, .
\end{aligned}
$$

This is very close to the M-theory critical temperature found by Russo [16] after a much less heuristic calculation. Our result differs from Russo's by a factor of $2^{1/3} \approx 1.2599$. We discuss this discrepancy in more detail in appendix C. In any case we interpret T_H as the *M2-brane limiting temperature*, in the same sense that the Hagedorn temperature is the string limiting temperature.

The physics behind this Hagedorn behavior is simply that branes can be created from the thermal bath. Suppose we start at low temperature and increase the matter energy density. The temperature of the universe will rise to T_H then stay there.[2] As the matter energy density increases further, the extra energy will be pumped into the creation of M2-branes. Thus the brane number density will increase while the energy density of the supergravity gas stays fixed. Conversely, if the universe starts out in this Hagedorn phase, the matter energy density decreases as the universe expands. A point will be reached at which the equilibrium N_{ij} drop to zero and the configuration consists entirely of supergravity gas. Beyond this point $\rho \sim \rho_S \sim T^{11}$, so for the energy density to drop further the temperature must drop.

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[2]We are implicitly assuming that the specific heat does not diverge too rapidly as $T \to T_H$. In string theory this assumption was studied in [7].

To be complete, we mention *another* limiting temperature, associated with the presence of M5-branes. These are not important for the late-time dynamics of [12] because they intersect and annihilate quickly in the 10 spatial dimensions of M-theory, but their presence at early times implies a limiting temperature ($T_5 = M_{11}^6/(2\pi)^5$):

$$
\begin{aligned}
T_{M5} &= \left(\frac{5T_5}{c_X'}\right)^{1/6} \\
&= \left(\frac{15}{16\pi^8}\right)^{1/6} M_{11} \\
&\approx 0.215012 \; M_{11}
\end{aligned}
\tag{30}
$$

Note that $T_H \approx T_{M5}$. We do not believe that there are two limiting temperatures, one for M2-branes and one for M5-branes, and it is just coincidence that they happen to be nearly identical. Rather we conjecture that these are indications of a *single* M-theory limiting temperature $T_M \approx 0.2M_{11}$. We believe that a full understanding of M-theory (including higher-order corrections to the supergravity action we are studying here) will produce the corrections necessary for the two temperatures to coincide. Indeed, since both temperatures were computed only at lowest order, it is remarkable they agree this well. We leave this as a significant open problem.

Finally, it is interesting to compare our M-theory results to the limiting temperature expected in string theory. In general some of the M2-branes could be wrapped around directions whose scale factors e^{λ_i} and e^{λ_j} differ significantly from one another. When both scale factors are large we expect (29) to be a reasonable estimate for the limiting temperature. But when one scale factor becomes small the membrane can be modelled as a string, with the small dimension playing the role of the dilaton. For type II strings the limiting Hagedorn temperature is (expressed in M-theory units)

$$
T_H = \left(\pi\sqrt{8\alpha'}\right)^{-1} = \frac{1}{\pi\sqrt{8}}\sqrt{R_{10}}M_{11}^{3/2}
\tag{31}
$$

where R_{10} is the radius of the small dimension. The true limiting temperature should interpolate between these two extremes; it would be interesting to study this in more detail.

4.1 Thermodynamics and cosmology in the Hagedorn phase

We now consider the evolution of a universe in the Hagedorn phase. For simplicity we specialize to the case of an isotropic torus (all λ_i equal). The "brane area density" is

$$n = \frac{\text{total area in wrapped branes}}{\text{total volume of the universe}} = \frac{\mathcal{N}_{\text{eq}}}{V} \tag{32}$$

where $\mathcal{N}_{\text{eq}}$ is given by equation (27). This must be positive, which requires

$$\dot{\lambda} > \left(\frac{16\pi G c_S T_H^{11}}{d(d-1)} \right)^{1/2} \approx 0.502 \quad \text{in Planck units.} \tag{33}$$

Thermodynamics in the Hagedorn phase is straightforward. As always, the energy density is fixed by the Hamiltonian constraint,

$$\rho = c_S T^{11} + \frac{1}{V} \sum_{i \neq j} N_{ij}(T_2 + c_X T^3) e^{\lambda_i + \lambda_j} = \frac{d(d-1)}{16\pi G} \dot{\lambda}^2 \tag{34}$$

In general the pressure is given by (6). At the Hagedorn temperature the positive pressure due to excitations on the branes exactly cancels the negative pressure due to brane tension.[3] Thus the pressure

$$p = \frac{1}{d} c_S T_H^{11} \tag{35}$$

is isotropic and comes only from the supergravity gas. Finally the entropy density is given by

$$s = \frac{1}{T_H} \left(\frac{1}{d} c_S T_H^{11} + \frac{d(d-1)}{16\pi G} \dot{\lambda}^2 \right) . \tag{36}$$

To determine the evolution of the scale factor we proceed in the usual way. For an isotropic universe energy conservation requires

$$d\left(\rho e^{d\lambda} \right) = -p \, d\left(e^{d\lambda} \right) . \tag{37}$$

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[3]This can be understood by noting that for an extensive thermodynamic system the pressure is (minus) the free energy density. The latter quantity vanishes for branes at the Hagedorn temperature.

The pressure is constant, so this implies

$$\rho + p = \frac{\text{const.}}{e^{d\lambda}} \, . \tag{38}$$

Plugging this result into the Hamiltonian constraint (34) gives a differential equation for the scale factor. The general solution is

$$e^{\lambda(t)} = \text{const.} \ \sin^{2/d} \left(\frac{t}{2} \sqrt{\frac{16\pi Gdp}{d-1}} \right) \tag{39}$$

where we have fixed initial conditions $\lambda \to -\infty$ as $t \to 0$.

This result looks a little odd, but it is only valid when the universe is in the Hagedorn phase $(T = T_H)$, so it cannot be interpreted as an oscillatory universe. Having found the exact solution, it's actually an excellent approximation to neglect the pressure. Recall that the brane gas does not contribute to the pressure, so in the Hagedorn phase we have an inequality

$$p = p_{\text{SUGRA}} = \frac{1}{d}\rho_{\text{SUGRA}} \leq \frac{1}{d}\rho \tag{40}$$

In this approximation the universe is filled with pressureless dust, and the scale factor has the usual matter-dominated form $e^{\lambda(t)} = \text{const.} \, t^{2/d}$.

5   Brane annihilation

We now look at interactions between the branes and the supergravity gas, which communicate via the reaction

$$M2-\text{brane} + \overline{M2}-\text{brane} \leftrightarrow \ \text{SUGRA particles} \, . \tag{41}$$

For thermodynamic equilibrium the interaction rate must be sufficiently high, which means the branes must be able to meet in the transverse dimensions. Hence interactions will be suppressed if the transverse dimensions are big, as discussed in [7, 8]. When the interaction ceases, the branes are "frozen in," and will remain wound for the remainder of the cosmological evolution. This process is exactly analogous to the freeze-out of dark matter in standard cosmology.

To describe this process quantitatively we need the cross-section for brane-antibrane annihilation. It is not clear how to calculate this from first principles in M-theory. For inspiration we turn to an analogous process in string

theory, namely the annihilation of two fundamental strings wound on a torus with opposite orientations. The basic process was studied by Polchinski [17]; for more details on the following calculation the reader should consult his paper. For two strings moving in the x^1 direction and wrapped with opposite orientations on x^2 the center-of-mass momentum and winding vectors are

$$p_1^\mu = (E, Ev, 0) \qquad \ell_1^\mu = (0, 0, L_2)$$
$$p_2^\mu = (E, -Ev, 0) \qquad \ell_2^\mu = (0, 0, -L_2)$$

where L_i denotes the size of the torus in the x^i direction. Thus

$$s_R = -(p_{1R} + p_{2R})^2 = \frac{L_2^2}{4\pi^2(1 - v^2)} \tag{42}$$

where $p_R^\mu = p^\mu + \frac{1}{4\pi}\ell^\mu$. Following Polchinski [17], the annihilation probability during a collision is given by the optical theorem

$$\text{prob.} = \frac{1}{v}\text{Im}T_{ii} = \frac{1}{4E^2v}\left(\frac{\kappa_2}{2\pi}\right)^4 \frac{16\pi^3}{\kappa_2^2}\text{Im}I(s_R, t_R = 0) \tag{43}$$

where κ_2^2 is the 1+1 dimensional gravitational coupling and the imaginary part of the Shapiro-Virasoro amplitude is $\text{Im}I(s_R, t_R = 0) = 2\pi^2 s_R^2$. The strings collide repeatedly, since x^1 direction is periodic, so it is more convenient to work in terms of the annihilation probability per unit time

$$\frac{\text{prob.}}{\text{time}} = \frac{2T_1\kappa_{10}^2}{L_1 \cdots L_9}L_2^2 f(v) \qquad f(v) = \frac{2}{1 - v^2}. \tag{44}$$

We inserted a factor of $4\pi T_1$ on dimensional grounds, where T_1 is the fundamental string tension, and expressed the result in terms of the 9+1 dimensional gravitational coupling $\kappa_{10}^2 = \kappa_2^2 L_2 \cdots L_9$. The key qualitative features are that the annihilation rate is proportional to the gravitational coupling, inversely proportional to the volume of the torus, and proportional to the square of the length of the wound strings.

We assume that the annihilation rate for two oppositely-oriented membranes has similar qualitative features. We write it as

$$\frac{\text{prob.}}{\text{time}} = \frac{2T_2^{4/3}\kappa_{11}^2}{V}A^2 f(v) \tag{45}$$

where A is the area of the wrapped membranes and V is the volume of the torus. The peculiar fractional power of the membrane tension is required on

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dimensional grounds. The string result suggests that $f(v) = 2/(1-v^2)$; since we are interested in slowly moving membranes we will take $f(v) \approx \text{const.} \approx 2$.

It is straightforward to promote the two-membrane annihilation rate (45) to a Boltzmann equation governing the evolution of the brane wrapping matrix N_{ij}.

$$\frac{d}{dt}N_{ij} = -\frac{16\pi G\,T_2^{4/3}\,f\,e^{2(\lambda_i+\lambda_j)}}{V}\left((N_{ij})^2 - (N_{ij}^{\text{eq}})^2\right).\qquad(46)$$

Here N_{ij}^{eq} is the equilibrium wrapping matrix (28), and we have used $2\kappa_{11}^2 = 16\pi G$. Branes wrapped on the (ij) directions will freeze out when their annihilation rate

$$\Gamma_{ij} = \frac{16\pi G\,T_2^{4/3}\,f\,e^{2(\lambda_i+\lambda_j)}\,N_{ij}}{V}\qquad(47)$$

is small compared to the Hubble rate, $\Gamma_{ij} \ll H$. Note that we take Γ_{ij} to be proportional to N_{ij}, not N_{ij}^{eq}, so that we get a sensible annihilation rate even when $N_{ij}^{\text{eq}} = 0$. In practice we say that freeze-out occurs when the largest $\Gamma_{ij} < 0.01H$.

We conclude with a few comments on these results. First, note that both the string and membrane annihilation rates (44), (45) are compatible with the dimension-counting arguments of Brandenberger and Vafa [7]. For example, if three dimensions of the torus become large then strings wrapped on the large dimensions will still be able to annihilate: due to the factor of L_2^2 upstairs in (44), the wound strings effectively behave like point particles moving in one large spatial dimension. Likewise, if five dimensions become large membranes wrapped on the large dimensions will behave like point particles moving in one big dimension, and thus will still be able to annihilate.

Also note that we have ignored diagonally-wound membranes. Membranes wrapped on the (ij) and (kl) cycles could interact, and indeed could lower their energy, by merging to form a single membrane wrapped on the linear combination $(ij) \oplus (kl)$. Such diagonally-wound membranes are not compatible with our metric ansatz, for reasons discussed in section 2. Moreover, reactions such as $(ij)+(kl) \rightarrow (ij)\oplus(kl)$ do not get rid of any conserved winding numbers, unlike the annihilation to supergravity particles which we considered above. So we do not expect that including diagonally wound membranes would qualitatively affect the nature of our results.

6 Numerical simulation

We are now in a position to solve the combined Einstein-Boltzmann equations (8) and (46). We have implemented the equations in a FORTRAN code, allowing us to consider the evolution of λ_i and N_{ij} for many different sets of initial conditions.

At the level of supergravity, we might expect initial conditions to be drawn at random from a probability distribution corresponding to the phase space volume derived in section 3.2.

$$\Gamma = \int d^d\lambda \, d^d\dot{\lambda} \sum_{N_{ij}} V^d e^{S_{\text{matter}}} \tag{48}$$

This amounts to assuming that no state in the early universe is *a priori* special. In practice, however, we must place some restrictions on the states we consider. The first restriction is that we must fix the initial volume of the universe. That is, we sample from the distribution (48) on a hypersurface with a fixed value of $\log V = \sum_i \lambda_i$. In principle this might not seem like a serious restriction. The volume increases monotonically with time, as shown in (23), so this is equivalent to choosing an initial instant of time. However in practice our results will depend rather sensitively on the instant of time when we first assume that semiclassical M-theory is valid and that the universe is in thermal equilibrium.

A second restriction arises because our action is only a low energy approximation to M-theory, so it only makes sense to begin studying the evolution at a moment when this approximation is reasonable. The low-energy approximation is valid when all length scales in the problem are larger than the Planck scale. We actually have two length scales associated with each direction – the physical size e^{λ_i}, and the "Hubble length" $1/\dot{\lambda}_i$.

The choice of a minimum physical size is not particularly crucial; for simplicity we will assume that we can trust our action when all $\lambda_i > 0$. The choice of a minimum Hubble length is somewhat more subtle. In sampling from the distribution (48) our results will be dominated by configurations which maximize S_{matter}. Given the entropy density in the Hagedorn phase (36), note that S_{matter} is proportional to the volume (which we are holding fixed) but is also an increasing function of $\sum_{i \neq j} \dot{\lambda}_i \dot{\lambda}_j$. Thus for entropic reasons our results will be dominated by configurations in which all $\dot{\lambda}_i$ are equal and as large as possible (equal to the maximum allowed value). In a

way this is very encouraging, since it means it is natural for the universe to start out in the Hagedorn phase. We will study the dependence on the initial velocities below, and find that the exact choice of cut-off doesn't make a significant difference, provided the initial $\dot{\lambda}_i$ are large enough that the universe begins at the Hagedorn temperature.

For given values of λ_i and $\dot{\lambda}_i$ we also need to specify the matter content. Assuming we begin in the Hagedorn phase, the behavior discussed in section 4 means the energy density of the supergravity gas is equal to $c_S T_H^{11}$. Any additional contribution to the energy budget of the universe will be supplied by branes. The equilibrium total area in branes is given in (27), so the only remaining question is how to distribute this area across the different wrapping modes. We do this by assuming a uniform distribution for the wrapping numbers N_{ij}, subject only to the constraint (27).

We now look at two different sets of solutions. In the first set we start with all $\dot{\lambda}_i = 1$, and vary the initial volume of the universe. In the second set we fix the initial volume and vary the initial velocities. In both cases we are interested in determining the number of directions that are unwrapped at late times. Our prescription is that we round the wrapping numbers N_{ij} to the nearest integer at freeze-out. Thus we say the i-th direction is unwrapped if $N_{ij} < 0.5$ for all $j \neq i$ at the time of freeze-out.

6.1   Volume dependence

We begin by studying how the number of unwrapped dimensions at freeze-out depends on the initial volume. To do this we select the λ_i at random, subject to the constraint that $\lambda_i > 0$ and that $\log V = \sum_i \lambda_i = $ constant. We take all initial velocities $\dot{\lambda}_i$ equal to unity, and distribute the wrapping numbers as described above.

As can be seen in Figure 1, larger initial volumes mean fewer unwrapped dimensions at late times. This can be understood as follows. For fixed $\dot{\lambda}$ the total area in branes at the start of the simulation, given by (27), is proportional the volume V. If we assume a roughly isotropic universe then the expected number of branes in each wrapping state $N_{ij} \sim V^{4/5}$, as can be seen in (28). Thus larger initial universes will have larger initial wrapping numbers. At the start of the simulation the annihilation rate (47) scales like

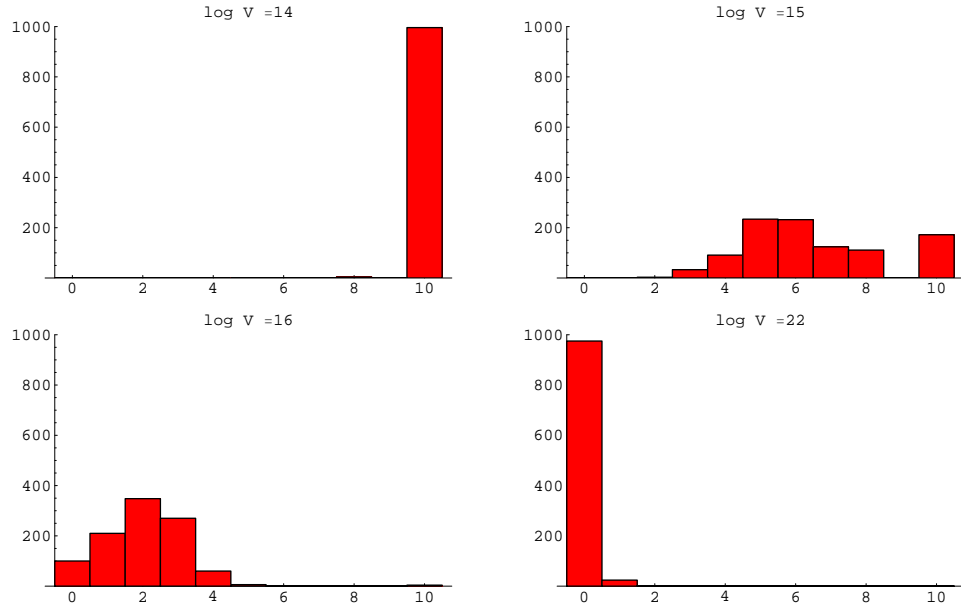$$\Gamma_{ij} \sim \frac{1}{V^{3/5}} N_{ij} \sim V^{1/5} \,. \tag{49}$$

Figure 1: Probability distribution for the number of unwrapped dimensions at freeze out, for four different choices of the initial volume. The number of unwrapped dimensions is indicated on the horizontal axis. Each histogram is a Monte Carlo based on 10^3 different sets of initial conditions. Note that it's impossible to have nine unwrapped dimensions, since the wrapping matrix is symmetric.
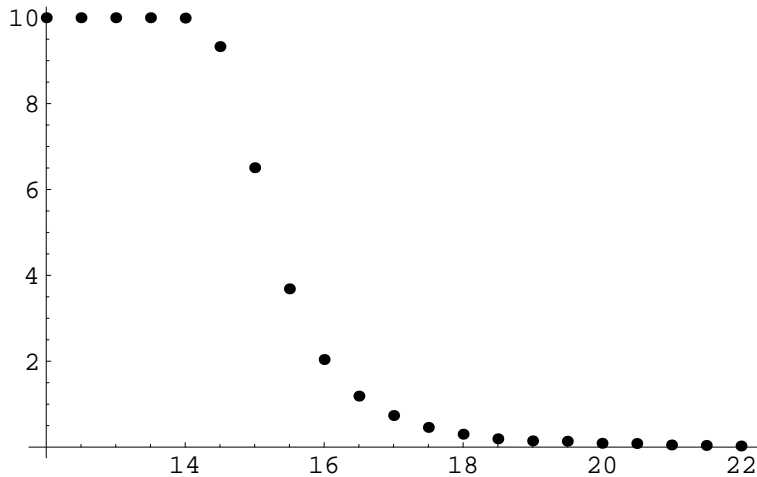
Figure 2: Mean number of unwrapped dimensions at freeze-out (y-axis) versus log of the initial volume (x-axis).

Thus larger universes are initially more efficient at getting rid of their branes. But as the wrapping numbers drop the $V^{-3/5}$ prefactor in the annihilation rate wins out, and larger universes ultimately find it more difficult to get rid of their branes before freeze-out.[4] Conversely, if the universe starts with a small initial volume the initial wrapping numbers will be small. One could easily have all $N_{ij} < 0.5$, in which case we would regard the initial state as having no branes present.

Looking at Figure 1, we see that the initial volume determines the distribution of dimensionality. For very small volumes, the branes always annihilate before freeze-out and all ten dimensions unwrap. For large volumes all directions tend to be wrapped at freeze-out. Figure 2 shows the transition between these two extremes, by plotting the mean number of unwrapped directions at freeze-out as a function of the initial volume.

6.2 Velocity dependence

The dependence on initial velocity is much weaker than the dependence on initial volume, provided the universe is expanding fast enough (and thus has

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[4]The time to freeze-out is roughly independent of the initial volume and thus does not affect this conclusion.
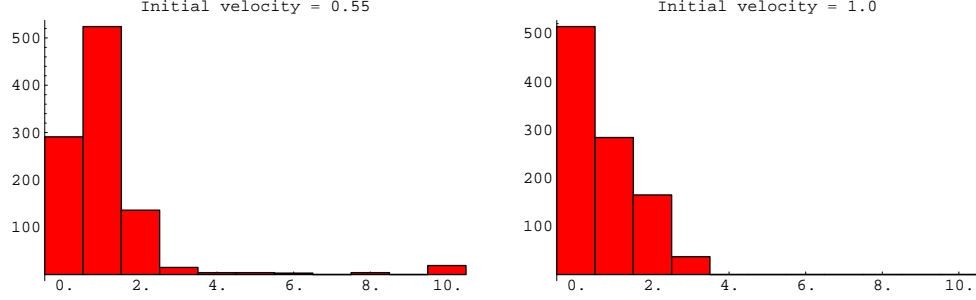
Figure 3: Probability distribution for the number of unwrapped dimensions at freeze out for two different choices of the initial velocity. The initial conditions are all $\dot{\lambda}_i = 0.55$ (left plot) and all $\dot{\lambda}_i = 1$ (right plot). In both plots the initial volume is fixed to $\log V = 20$. The plots are Monte Carlos based on 10^3 different sets of initial conditions. There is relatively weak dependence on the initial velocity, as long as $\dot{\lambda}$ is large enough to start in the Hagedorn phase.
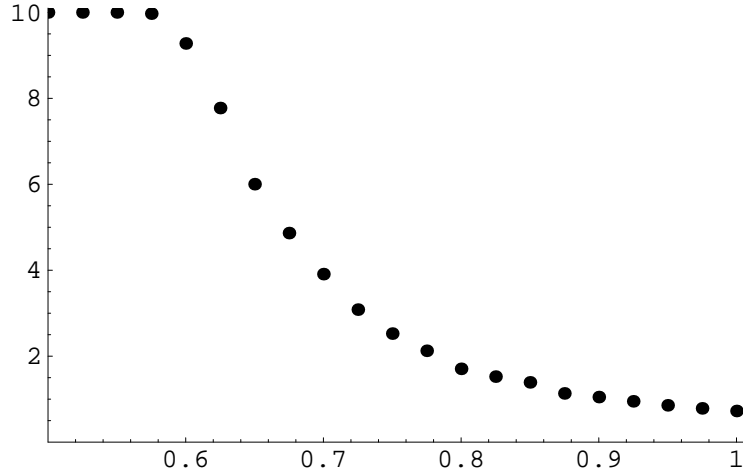

Figure 4: Mean number of unwrapped dimensions at freeze-out (y-axis) versus initial velocity (x-axis).

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sufficient energy) to be in the Hagedorn phase when the simulation begins. We take all $\dot{\lambda}_i$ to be identical at the outset, and choose the values of λ_i randomly, subject only to the volume constraint. Figures 3 and 4 show that the distribution in the number of unwrapped dimensions at freeze-out only depends weakly on the initial velocity. Note that the left hand panel in Figure 3 shows the distribution for initial velocities which are only marginally above the value (33) needed to ensure we start in the Hagedorn phase. In Figure 5 we show the dependence of the mean number of unwrapped dimensions on both the initial radii and velocities. One can see both the onset of a Hagedorn phase at $\dot{\lambda} \approx 0.5$, and the volume dependence of the final number of unwrapped dimensions.

6.3   Summary

From these numerical results we see no evidence for a preferred number of unwrapped dimensions at late times. Rather the wrapping numbers at freeze-out depend on the initial conditions. We therefore cannot uniquely predict the effective dimensionality of the universe at late times, although we can assign a probability to different final states.

We can ask what parameters determine the final state. Provided the initial volume is large enough, the most important factor determining the number of unwrapped directions at freeze-out is the anisotropy in the initial values of the λ_i. We verified this by running the code for the same set of λ_i and different values of the N_{ij} (chosen randomly, as outlined above), and found that for sufficiently large $\log V$ the same set of λ_i typically produced the same number of the unwrapped directions, independent of the values of the N_{ij}.

Consequently, if we want to sharpen our prediction for the number of unwrapped dimensions, we must constrain the initial conditions. A similar problem arose in the original work of Brandenberger and Vafa [7], where it was argued that a thermal fluctuation could produce a universe with either one, two, or at most three large dimensions. In the Brandenberger-Vafa scenario one could imagine deploying an anthropic argument to argue against observation of less than three dimensions.[5] Rather than pursue anthropic arguments, in the next section we turn our attention to a rationale for restricting the space of initial conditions, namely insisting that the initial state
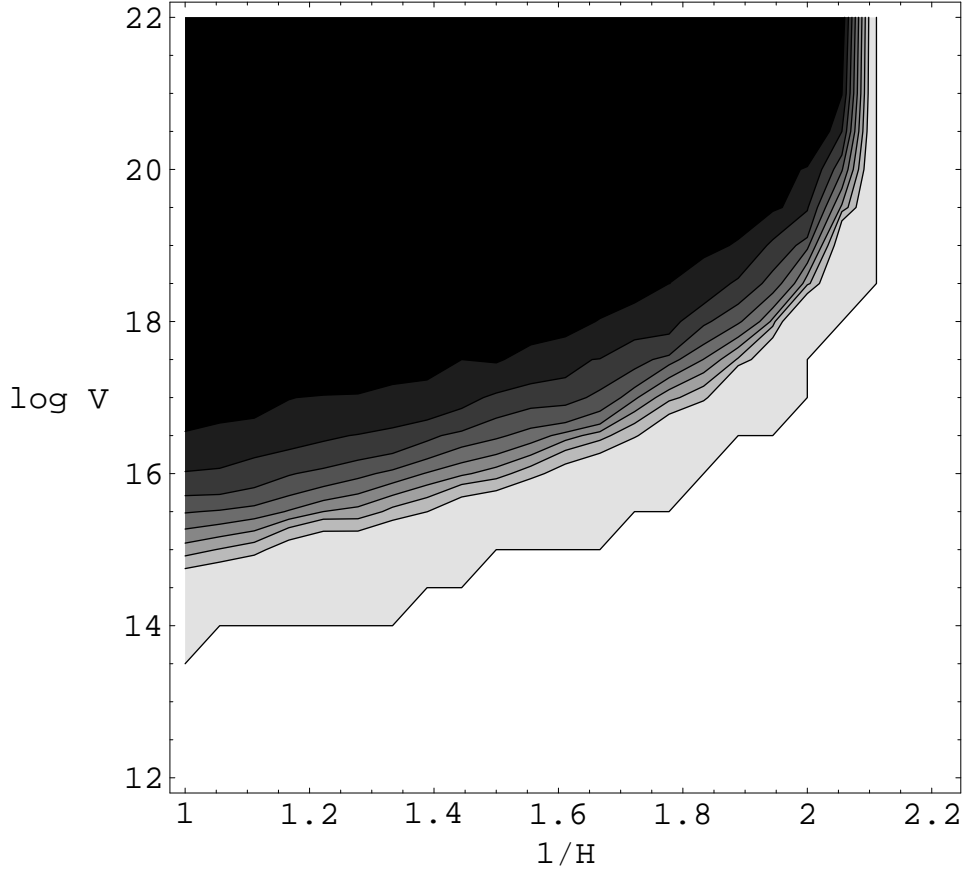
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[5]For a dynamical approach see [4].

Figure 5: This contour plot shows the mean number of unwrapped dimensions as a function of both the log of the initial volume (y-axis) and the inverse initial velocity $1/\dot{\lambda}_i \equiv 1/H$ (x-axis). For each run the initial λ_i are chosen randomly, but the initial $\dot{\lambda}_i$ are all identical. There is little dependence on $1/H$, provided we are in the Hagedorn phase to begin with ($\dot{\lambda} > 0.502$). The darkest shading corresponds to a mean number of unwrapped dimensions less than unity, while the lightest shading corresponds to a mean of 10 (fully unwrapped).

of the universe be consistent with holography.

7 Holography and initial conditions

The holographic principle [18, 19] is thought to be a fundamental property of quantum gravity. Loosely speaking, it requires that the number of degrees of freedom in a given volume scale like the surface area. But so far we have treated M-theory semi-classically. In this approximation the number of degrees of freedom is extensive in the volume, so we run the risk of violating holography.

We now apply the holographic principle to brane gas cosmology. For simplicity we specialize to the case of a square torus ($\lambda_1 = \cdots = \lambda_{10} = \lambda$) with uniform wrapping ($N_{ij} = N$ for all $i \neq j$). We will argue that holography is satisfied provided we put restrictions on the initial conditions. By combining holographic bounds with entropy arguments, we will argue for a preferred set of initial conditions for the universe. Moreover these preferred initial conditions have the right qualitative features to drive the brane gas scenario.

7.1 Holographic bounds

The holographic principle was first applied to cosmology in [20, 21, 22, 23]. We will use the covariant form of the entropy bound developed in [24, 25] to obtain limits on the initial size of the universe. Our analysis closely follows section 3.4 of [25].

In the brane gas scenario one expects that at early times the universe is in a Hagedorn phase, with scale factor $e^\lambda \sim t^{2/d}$. At intermediate times there could be a radiation-dominated phase, with scale factor $\sim t^{2/(d+1)}$. Finally at late times the universe is dominated by brane tension; for uniform wrapping this means $e^\lambda \sim t^{2/(d-2)}$ [12]. Thus to a good approximation throughout its history the universe has a flat FRW metric with a power-law scale factor.

A flat FRW universe has an apparent horizon at a proper radius $d_{AH} = 1/\dot{\lambda}$ [23]. Holography requires that the entropy inside a spherical volume of radius $R < d_{AH}$ be bounded by $A/4G$. That is, for a given entropy density s the radius must satisfy

$$sV \leq A/4G \tag{50}$$

or equivalently

$$R \leq R_{\max} = \frac{d}{4Gs} \tag{51}$$

where we have used the relation $V = RA/d$ appropriate to a sphere in d dimensions. If the sphere is larger than the apparent horizon $R > d_{AH}$ then holography puts no restrictions on the allowed entropy [25].

The entropy density in the Hagedorn phase is given in (36), while in the radiation-dominated phase the entropy density comes just from the supergravity gas:

$$s = \frac{d+1}{d}c_S T^{10} = \frac{d+1}{d}c_S \left(\frac{d(d-1)}{16\pi G c_S}\dot{\lambda}^2\right)^{10/11}. \tag{52}$$

Thus the holographic bound on the radius is

$$R_{\max} = \begin{cases} \frac{d^2}{4G(d+1)c_S}\left(\frac{d(d-1)}{16\pi G c_S}\dot{\lambda}^2\right)^{-10/11} & \dot{\lambda} < 0.502 \\ \frac{dT_H}{4G}\left(\frac{1}{d}c_S T_H^{11} + \frac{d(d-1)}{16\pi G}\dot{\lambda}^2\right)^{-1} & \dot{\lambda} > 0.502 \end{cases} \tag{53}$$

For $\dot{\lambda} < 0.234$ it turns out that $R_{\max}$ is larger than the radius of the apparent horizon d_{AH}, so holography puts no restriction on the physical volume of the universe. For $\dot{\lambda} > 0.234$, on the other hand, $R_{\max}$ is smaller than d_{AH} and we must limit the size of the universe to satisfy $e^{\lambda} \leq 2R_{\max}$. These bounds are illustrated in Fig. 6.

7.2 Holography and initial conditions

We conclude with some speculation about holography and the choice of initial conditions for the universe. The basic point is very simple. Fischler and Susskind have shown that the holographic principle is satisfied in the universe today [20]. Moreover Flanagan, Marolf and Wald have shown that if holography is satisfied at some instant of time then it will be satisfied both in the future and (by time reversal) in the past, up to the point where semiclassical general relativity breaks down [26]. Evolving our universe backwards in time, this means the holographic bound must be satisfied until general relativity breaks down. This would occur when either the volume is too small ($e^{\lambda} \sim 1$) or the Hubble parameter is too large ($\dot{\lambda} \sim 1$). These bounds are indicated by the dotted lines in Fig. 6.
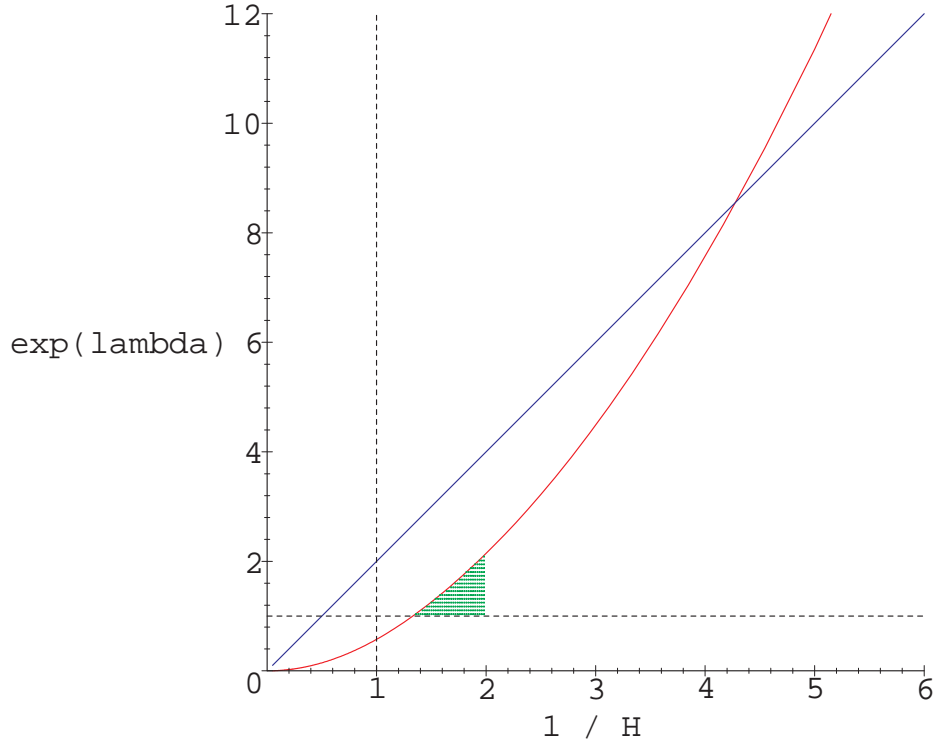
Figure 6: Bounds on the size of the universe as a function of $1/H \equiv 1/\dot{\lambda}$. The red curve is the holographic bound $2R_{\max}$, the straight blue line is the diameter of the apparent horizon $2/\dot{\lambda}$, and semiclassical gravity breaks down at the dotted black lines. In the green shaded region the universe satisfies the holographic bound and is in the Hagedorn phase.

Thus the universe must have originated from the horizontal dotted line in Fig. 6, somewhere to the right of the holographic bound. Now consider the expression for the entropy density in the Hagedorn phase (36). The entropy density is an increasing function of $\dot{\lambda}$, and the total volume is fixed. Thus on entropic grounds *the preferred initial conditions for the universe saturate the holographic bound at the point where general relativity breaks down.* This behavior was first noted in [20]. Here we are arguing that it is a general feature.

If this argument is correct, the preferred initial conditions for the universe have the right qualitative features to drive brane gas cosmology: the universe begins with a small initial volume and large initial Hubble parameter (or equivalently a large initial energy density). The numerical values suggested by our analysis are discouraging, unfortunately: the preferred initial conditions are roughly $\lambda = 0$ and $\dot{\lambda} = 3/4$. Comparing Figures 5 and 6, we see that the the Hagedorn region identified in Figure 6 is corresponds to a volume too low to even be plotted in Figure 5. Given such small initial volumes very few branes are present in the initial state. Thus the most likely evolution of the universe leads to ten unwrapped and roughly isotropic dimensions.

On the face of it, this is an extremely discouraging result for the brane gas scenario, as it appears to imply that the initial number of wrapped branes is very small – effectively one is starting from a brane gas without branes. However there are several reasons for qualifying this conclusion. First, the comparison of initial volumes is sensitive to $\mathcal{O}(1)$ numerical coefficients. Our use of a low-energy supergravity action seems to capture the right qualitative behavior of the Hagedorn phase, but we do not expect it to precisely capture all numerical coefficients. For example our estimate of the M-theory Hagedorn temperature is only an estimate, which surely receives $\mathcal{O}(1)$ corrections. Indeed we expect such corrections in order to get the M2 and M5 critical temperatures to agree. Likewise our estimate for the Hagedorn equation of state presumably receives $\mathcal{O}(1)$ corrections.

A more fundamental issue is that our discussion of holography assumed an isotropic torus, while the brane gas scenario relies on an initial anisotropy to seed the asymmetric growth of dimensions. It would thus be interesting to study holographic bounds on an anisotropic torus. Indeed there is reason to think that in the limit of extreme anisotropy, where M-theory reduces to IIA string theory, the holographic bound could be less restrictive, simply because the holographic bound involves the Planck length while the entropy density

in the stringy Hagedorn phase is set by the string scale.

We should also consider the impact of inhomogeneity on our analysis. Following Easther and Lowe [21], we can regard the holographic bound as a manifestation of the generalized second law of thermodynamics. From this perspective, which takes a more limited view of holography than positing it as a key feature of some underlying fundamental theory, violations of the holographic bound are only important if they can be exploited to form a black hole that contains less entropy than the material used to create it – thus violating the generalized second law. While investigating an inhomogeneous 11 dimensional spacetime is a forbidding prospect, we can perform a rough check by asking whether a Schwarzschild black hole with a mass equal to the entire energy budget of the universe would fit neatly inside our torus. For a universe in the Hagedorn phase the answer is "no," which suggests that homogeneity is not such a bad assumption.[6]

Even if a more detailed analysis of holography made it possible to decompactify three dimensions, we would still be faced with a fine-tuning problem. That is, our analysis shows that in the M-theory context the initial volume must fall within a fairly narrow window in order to have a significant probability of decompactifying three dimensions. Let us be optimistic and suppose that by obtaining the correct numerical coefficients and including the effects of anisotropy and inhomogeneity we would find that this window overlapped with the holographically allowed range of initial conditions. We would still face the difficulty that small changes in the initial volume significantly affect the probability of decompactifying three dimensions. Part of the appeal of the brane gas scenario was the hope that a brane gas in the early universe would automatically lead to decompactification of three dimensions. In the M-theory context this hope is not realized.

8   Conclusions

In this paper we extended the brane gas scenario in several directions. We gave a simple estimate of the Hagedorn temperature for 2-branes, and investigated the properties of a universe dominated by a Hagedorn gas of 2-branes. We estimated the cross-section for interactions between the branes and the SUGRA gas, showing that annihilation of branes becomes less efficient as the universe expands. Thus the branes eventually freeze out, leading to a

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[6]We are grateful to Erick Verlinde for a valuable discussion on this point.

relic density of winding branes at late times. We numerically solved the corresponding Boltzmann equations, and found that the number of "wrapped" dimensions at late times was essentially determined by the initial volume, provided the universe starts out in the Hagedorn phase.

We then looked more closely at constraints on the initial conditions, identifying regions of initial condition space that are compatible with the holographic bound. With this cut, we found that in order to be consistent with holography the initial volume of the universe had to be relatively small, implying that the equilibrium number of branes was also small. Comparing this bound with the numerical work, we saw that the holographically allowed region of initial condition space typically leads to a universe in which all branes annihilate before freeze-out, thereby leaving all ten dimensions free to expand isotropically.

This result is noteworthy for two reasons. Firstly, it suggests that the M-theoretic version of the brane-gas scenario cannot produce a universe with anisotropic distributions of winding branes, and thus does not provide a mechanism for ensuring that the universe contains a small number of macroscopic dimensions. Secondly, it is – to our knowledge – the first time a holographic bound has been applied to a toroidal cosmology and, more importantly, it is the first time that holographic arguments have been used to successfully put new constraints on cosmological models.

These results come with the caveat that while the dynamics were analysed in a fully anisotropic spacetime, the holographic argument was formulated for an isotropic universe. It is not clear to us whether a moderate level of anisotropy could modify the holographic constraint to the point where decompactification of three dimensions becomes possible. Moreover, the current analysis involves constants of order unity that are not reliably determined, providing another possible loophole in our conclusions. Finally, we have assumed that the universe is homogeneous, with branes that are "smeared out" in the transverse directions. This should be a good approximation when dealing with a large number of branes, but the number of branes we see are small.

At this point, however, our inclination is to take these results seriously, and to explore their consequences. One promising possibility is to posit that one direction of the torus is small compared to the Planck scale. This reduces the mass of branes wrapped around the small direction – thus increasing their equilibrium number density – while holding the overall volume fixed. In effect, this scenario describes the stringy limit of M-theory. Our results do

not directly apply to this regime, since our underlying supergravity action only makes sense if all ten directions are large compared to the Planck scale. But the technology we have developed should be applicable to string gas cosmology, a subject we intend to analyze in the future.

Acknowledgements

A   Equations of state

Expressions for the energy densities associated with massless particles in thermal equilibrium can be found in any statistical mechanics text. For a relativistic gas in d spatial dimensions each degree of freedom has an energy density

$$\text{BOSON}: \qquad \rho_b \;=\; \frac{1}{(2\pi)^d} S_d d! \zeta(d+1) T^{d+1}$$

$$\text{FERMION}: \qquad \rho_f \;=\; \frac{1}{(2\pi)^d} S_d (1 - 2^{-d}) d! \zeta(d+1) T^{d+1}$$

There are three instances where this will be used in our work: the supergravity gas, the M2-brane worldvolume, and the M5-brane worldvolume. In each application, we must remember to sum the energy densities from each degree of freedom.

A.1   Supergravity gas

For the supergravity gas we have 128 bosonic and 128 fermionic degrees of freedom, all massless, in 10 dimensions. So:

$$\rho_b = \frac{4725\zeta(11)}{16\pi^5} T^{11} \qquad\qquad \rho_f = \frac{4833675\zeta(11)}{16384\pi^5} T^{11} \qquad (54)$$

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producing

$$\rho_S = 128\rho_b + 128\rho_f \equiv c_S T^{11}, \qquad c_S = \frac{9672075\zeta(11)}{128\pi^5} \qquad (55)$$

A.2 $M2$-brane gas

The M2-brane has transverse fluctuations which we model as a gas composed of massless particles residing on the brane. There are 8 bosonic and 8 fermionic degrees of freedom. Thus

$$\rho_b = \frac{\zeta(3)}{\pi}T^3 \qquad\qquad \rho_f = \frac{3\zeta(3)}{4\pi}T^3 \qquad (56)$$

producing

$$\rho_X = 8\rho_b + 8\rho_f \equiv c_X T^3, \qquad c_X = \frac{14\zeta(3)}{\pi}. \qquad (57)$$

A.3 M5-brane gas

M5-branes have 5 transverse coordinates, a 2-form whose field strength is self-dual, and all their superpartners. These produce (again) 8 bosonic and 8 fermionic degrees of freedom, but now in five dimensions. Thus

$$\rho_{M5} = 8\rho_b + 8\rho_f \equiv c'_X T^6, \qquad c'_X = \frac{\pi^3}{6}. \qquad (58)$$

B Einstein equations

We begin with the metric ansatz used in [12],

$$ds^2 = -dt^2 + \sum_{i=1}^{d} R_i(t)^2 d\theta_i^2 \qquad (59)$$

where the angular coordinates θ_i run from 0 to 2π. The Einstein tensor has the following non-zero components

$$G^t{}_t = \frac{1}{2}\sum_{k\neq l}\frac{\dot{R}_k\dot{R}_l}{R_k R_l} \qquad (60)$$

$$G^i{}_i = \sum_{k\neq i}\frac{\ddot{R}_k}{R_k} + \frac{1}{2}\sum_{k\neq l}\frac{\dot{R}_k\dot{R}_l}{R_k R_l} - \sum_{k\neq i}\frac{\dot{R}_k\dot{R}_i}{R_k R_i} \qquad (61)$$

29

(no sum on i on the second line). For future reference,

$$\sum_i G^i{}_i = (d-1) \sum_k \frac{\ddot{R}_k}{R_k} + \frac{d-2}{2} \sum_{k \neq l} \frac{\dot{R}_k \dot{R}_l}{R_k R_l} \tag{62}$$

The Einstein equations are

$$G^t{}_t = 8\pi G \rho \qquad G^i{}_i = -8\pi G p_i \tag{63}$$

where the energy density and pressures are given in (5), (6). At this point it is convenient to set $\theta_i = 2\pi x_i$ and $e^{\lambda_i(t)} = 2\pi R_i(t)$. That is, we write the metric as

$$ds^2 = -dt^2 + \sum_{i=1}^{d} e^{2\lambda_i(t)} dx_i^2 \qquad 0 \leq x_i \leq 1. \tag{64}$$

In terms of these variables the Einstein equations are

$$\frac{1}{2} \sum_{i \neq j} \dot{\lambda}_i \dot{\lambda}_j = 8\pi G \rho \tag{65}$$

$$\ddot{\lambda}_i + \frac{\dot{V}}{V} \dot{\lambda}_i = 8\pi G \left(\frac{1}{d-1} \rho + p_i - \frac{1}{d-1} \sum_k p_k \right) \tag{66}$$

To obtain the second line it is useful to take the sum of the space-space equations and use (62).

C   Hagedorn temperatures

There is a distressing factor of $2^{1/3} \approx 1.26$ between our limiting M2-brane temperature and that found by Russo [16]. While such a factor would not affect the qualitative features of our analysis, it is important to determine the reason for the discrepancy.

In studying the membrane energy, we used the following large-winding approximation to the membrane mass:

$$m \sim T_2 A + (\text{ideal gas of transverse fluctuations}).$$

This expression is valid at low temperature (and thus at low excitation number), but there is no reason to believe it is valid near the Hagedorn temperature. Russo identifies the critical temperature at which a membrane wound

on the Euclidean time dimension becomes tachyonic. But unlike the present work Russo does not expand the membrane action for large winding. We believe this is the origin of the discrepancy.[7] The complications of studying membranes prevent us from showing this directly, but an analogous numerical discrepancy can be seen in the following string calculation.

The (exact) bosonic string spectrum is ($\alpha' = 1$)

$$m^2 = w^2 R^2 + N - 4$$

where we let N refer to all oscillations. If we consider only unwound strings, this means $m \sim \sqrt{N}$ at high oscillation number, and the partition function behaves roughly like

$$
\begin{aligned}
Z &= \sum_N d(N) e^{-E/T} \\
&\sim \int_0^\infty dN \ e^{\sqrt{N}(\beta_H - \beta)}
\end{aligned}
$$

which diverges for $T \geq T_H$. Now consider winding strings, for which a mass approximation *valid at low excitation number* would be

$$m \sim wR + \frac{N}{2wR} \, . \tag{67}$$

The partition function is now roughly

$$
\begin{aligned}
Z &= \sum_{N, w \neq 0} d(N) e^{-E/T} \\
&\sim \sum_{w \neq 0} \int_0^\infty dN \ \exp\left[\sqrt{N}\beta_H - (wR + N/2wR)\beta\right] \\
&\sim \sum_{w \neq 0} e^{wR(\beta_H^2/2\beta - \beta)}
\end{aligned}
$$

which diverges for $T \geq \sqrt{2} T_H$. This calculation is incorrect because we used the approximation (67) for arbitrarily large N; at any fixed w there will eventually be an excitation number $N \gg w^2 R^2$ at which it is better to use

$$m \sim \sqrt{N} + \frac{wR}{2\sqrt{N}}$$

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[7]Both Russo and the present work ignore interactions on the membrane worldvolume.

which will reproduce the old divergence at $T = T_H$. Since each w term in the partition function diverges towards positive infinity at this temperature, the total partition function will also diverge. Thus the $2^{1/2}$ discrepancy is merely an artifact of the large winding approximation to the mass.

References

[1] S. Alexander, R. H. Brandenberger and D. Easson, *Brane gases in the early universe,* Phys. Rev. D **62**, 103509 (2000) [arXiv:hep-th/0005212].

[2] R. Brandenberger, D. A. Easson and D. Kimberly, *Loitering phase in brane gas cosmology,* Nucl. Phys. B **623**, 421 (2002) [arXiv:hep-th/0109165].

[3] D. A. Easson, *Brane gases on K3 and Calabi-Yau manifolds,* arXiv:hep-th/0110225.

[4] S. Watson and R. H. Brandenberger, *Isotropization in brane gas cosmology,* arXiv:hep-th/0207168.

[5] T. Boehm and R. Brandenberger, *On T-duality in brane gas cosmology,* arXiv:hep-th/0208188.

[6] S. H. Alexander, *Brane gas cosmology, M-theory and little string theory,* arXiv:hep-th/0212151.

[7] R. H. Brandenberger and C. Vafa, *Superstrings in the early universe,* Nucl. Phys. B **316**, 391 (1989).

[8] A. A. Tseytlin and C. Vafa, *Elements of string cosmology,* Nucl. Phys. B **372**, 443 (1992) [arXiv:hep-th/9109048].

[9] G. B. Cleaver and P. J. Rosenthal, *String cosmology and the dimension of space-time,* Nucl. Phys. B **457**, 621 (1995) [arXiv:hep-th/9402088].

[10] M. Sakellariadou, *Numerical experiments in string cosmology,* Nucl. Phys. B **468**, 319 (1996) [arXiv:hep-th/9511075].

[11] R. Easther, B. R. Greene and M. G. Jackson, *Cosmological string gas on orbifolds,* Phys. Rev. D **66** (2002) 023502 [arXiv:hep-th/0204099].

[12] R. Easther, B. R. Greene, M. G. Jackson and D. Kabat, *Brane gas cosmology in M-theory: late time behavior,* arXiv:hep-th/0211124.

[13] B. S. Dewitt, *Quantum theory of gravity. I. The canonical theory,* Phys. Rev. **160**, 1113 (1967).

[14] J. A. Wheeler and B. S. Dewitt, *Battelle Rencontres*, edited by C. Dewitt and J. A. Wheeler. (Benjamin, New York, 1968).

[15] J. B. Hartle and S. W. Hawking, *Wave function of the universe,* Phys. Rev. D **28**, 2960 (1983).

[16] J. G. Russo, *Free energy and critical temperature in eleven dimensions,* Nucl. Phys. B **602**, 109 (2001) [arXiv:hep-th/0101132].

[17] J. Polchinski, *Collision of macroscopic fundamental strings,* Phys. Lett. B **209**, 252 (1988).

[18] G. 't Hooft, *Dimensional reduction in quantum gravity,* arXiv:gr-qc/9310026.

[19] L. Susskind, *The world as a hologram,* J. Math. Phys. **36**, 6377 (1995) [arXiv:hep-th/9409089].

[20] W. Fischler and L. Susskind, *Holography and cosmology,* arXiv:hep-th/9806039.

[21] R. Easther and D. A. Lowe, *Holography, cosmology and the second law of thermodynamics,* Phys. Rev. Lett. **82**, 4967 (1999) [arXiv:hep-th/9902088].

[22] G. Veneziano, *Pre-bangian origin of our entropy and time arrow,* Phys. Lett. B **454**, 22 (1999) [arXiv:hep-th/9902126].

[23] D. Bak and S. J. Rey, *Cosmic holography,* Class. Quant. Grav. **17**, L83 (2000) [arXiv:hep-th/9902173].

[24] R. Bousso, *A covariant entropy conjecture,* JHEP **9907**, 004 (1999) [arXiv:hep-th/9905177].

[25] R. Bousso, *Holography in general space-times,* JHEP **9906**, 028 (1999) [arXiv:hep-th/9906022].

[26] E. E. Flanagan, D. Marolf and R. M. Wald, *Proof of classical versions of the Bousso entropy bound and of the generalized second law*, Phys. Rev. D **62**, 084035 (2000) [arXiv:hep-th/9908070].