Dark Energy and Stabilization of Extra Dimensions

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Abstract

We discuss the role Casimir energies may play in addressing issues of moduli stabilization and dark energy. In particular, we examine a (non-supersymmetric) brane world scenario with toroidal extra dimensions in which Casimir energies of bulk fields generate a stabilizing potential for the toroidal volume while driving accelerated expansion in the non-compact directions. We speculate that such a scenario might establish a link between asymmetric topology and asymmetric geometry; that is, asymmetric topology could be linked to the hierarchy between large and small dimensions.

I. INTRODUCTION

Extra dimensions were proposed almost a century ago in an attempt to unify gravity with electromagnetism [1, 2]. Although that particular model of unification has fallen away, the proposal did raise a simple question: Why are precisely three spatial dimensions observably large? Modern attempts at unification, notably supergravity and string theory, revived the idea of extra dimensions in a more promising context, but have left the question of "why three?" unanswered. What's more, these theories, together with recent astronomical measurements, have highlighted significant subtleties such as the need to stabilize the size of the extra dimensions and to drive periods of accelerated expansion of the large dimensions.

One approach to these issues is to consider them part of the larger problem of string/M-theory vacuum degeneracy, and invoke anthropically motivated arguments. However, a more satisfying and convincing solution would be to find, for example, a dynamical mechanism that ensures three spatial dimensions grow while the others remain unobservably small. There have been a number of attempts along these lines within string/M-theory [3, 4] but to date these proposals have fallen short of the mark.

We suggest here that Casimir energies may play an important role in addressing these questions. Related observations have been pursued in various guises in earlier works [5, 6, 7, 8, 9, 10, 11]. As is well known, whenever there are dimensions whose spatial extent is finite, there are contributions to a purely quantum mechanical energy density [12] that results from the boundary conditions imposed on a finite space. We argue that if this Casimir energy has certain features, then it is possible, at least in toy models, to (1) stabilize the extra dimensions, (2) allow three dimensions to grow large, and (3) provide an effective dark energy in the large dimensions.

It is also intriguing that the lightest field fluctuating in the compact space will set the size of the dark energy and the size of the extra dimensions. We would like to draw a connection between the scale of this dark energy and a neutrino mass and thereby reduce the number of small parameters in need of explanation. Although our toy model does not manage to forge this connection (the "neutrino" we will use is not the standard model neutrino for reasons discussed below), we will mention possible scenarios that could.

By way of brief summary, note that the Casimir energies arising from field fluctuations in large dimensions are insignificant since the contributions are inversely proportional to

some power of the size of the space. For small extra dimensions, on the other hand, Casimir energies can be correspondingly large. In fact, by balancing the Casimir energy contributions of fields with different masses and spins, the total Casimir energy, as a function of the radius of the extra dimensions, can develop a non-trivial minimum that stabilizes their size [1]. And with the size of the extra dimensions fixed, the large dimensions feel ever-more of this Casimir energy density as they expand, which is the hallmark behavior of dark energy. [2]

As we discuss below, there are substantial difficulties to realizing a phenomenologically viable version of this scenario, but it is intriguing to note that Casimir energies have the potential of fruitfully linking the issues of moduli stabilization, large/small dimensions, and dark energy.

II.   ASYMMETRIC COSMOLOGIES

There are many ways in which familiar features of isotropic cosmologies differ significantly when considered in anisotropic settings. We briefly review two examples that play a key role in what follows.

First is the famous conclusion that accelerated expansion arises from a cosmological source with a sufficiently negative pressure ($p_a < -\rho/3$):

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p_a) \qquad , \tag{1}$$

As this familiar equation assumes isotropy, it is illuminating to consider an anisotropic spacetime with metric $ds^2 = -dt^2 + a_i^2 dx_i^2$. Then, from the Einstein equations we can make the following comparison:

$$\dot{H}_i - \dot{H}_k + \frac{\dot{V}}{V}(H_i - H_k) = 8\pi G(p_i - p_k) \qquad , \tag{2}$$

where the volume is $V = \prod_i a_i$ and the $H_i = \dot{a}_i/a_i$ are the Hubble factors. For a growing overall volume, the $\dot{V}/V$ is a friction term and the pressure differential is a driving force. In

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[1] In a different context, this observation was utilized in [13]. See also [14].

[2] One could invoke an additional mechanism to stabilize the compact dimensions. However, any energy source capable of stabilizing the extra dimension, such as a potential for b due to fluxes in string theory, may well dominate the energy density of the universe and drive inflation on its own, in which case the Casimir energy is not providing any fundamentally new physics, although it will alter the details by lifting the field out of the minimum of the potential or by changing it's overall shape in a possibly important way.

contrast to our intuition from the isotropic case, we can see that the role of negative pressure is more nuanced than one would expect. Namely, negative pressure p_i in the ith direction acts to *decelerate* expansion in that direction relative to the others, while negative pressure in directions transverse to the ith dimension act to accelerate expansion relative to the other dimensions. In an isotropic cosmology, all the pressures have equal magnitudes and, as can be seen from eqn. (1), combine to yield a net acceleration in each spatial dimension. But, fundamentally speaking, negative pressure does not necessarily entail acceleration.

Second, consider the nature of Casimir energy in an $N + 1$ dimensional spacetime with all of the spatial sections compact with characteristic size b. The associated stress tensor, $T_{\mu\nu}$ takes the form

$$\langle T_{\mu\nu} \rangle = diag(-\rho, \vec{p}) \tag{3}$$

where the energy density has the form $\rho \propto b^{-(N+1)}$, and the pressure in each dimension, defined as $p = -\partial(\rho V)/\partial V$ where $V \propto b^N$ is the volume, is $p_b = \rho/N$. If we now relax isotropy, say, by taking n of the spatial dimensions to be an isotropic torus whose radii are set by b, while the remaining $N - n$ spatial dimensions comprise $\mathbf{R}^{N-n}$ (or at least are very large) and evolve according to the scale a, then

$$\langle T_{\mu\nu} \rangle = diag(-\rho, \vec{p}_a, \vec{p}_b) \tag{4}$$

where, as before, $\rho \propto b^{-(N+1)}$ but now the pressure in the large directions is $p_a = -\partial(\rho V_a)/\partial V_a = -\rho$ where $V_a \propto a^{N-n}$. The equation of state $p_a = -\rho$ follows since ρ does not depend on a and will hold in the large directions regardless of the topology on the compact space. By contrast, the pressure in the small directions is $p_b = -\partial(\rho V_b)/\partial V_b = (N-n+1)\rho/n$, where $V_b \propto b^n$. Consistent with our first observation, it is straightforward to see that this form of the stress energy tensor gives a negative contribution of $-(N+1)\rho/n$ to the right hand side in eqn. (2), despite the negative pressure in the large directions. Furthermore, the extra dimensions will expand or contract in response to the internal pressure so that H_b cannot be neglected in analyzing the dynamics. (Shear from contracting directions can even drive inflation in vacuum [15].)

In the same spirit of examining features of less symmetric cosmologies, it is worthwhile to consider Casimir contributions of not only massless fields (implicit in the expressions above) but also massive fields. And as we show below, the Casimir energy from massive fields in

a $(3 + n + 1)$-dimensional universe can lead to an equation of state that drives a de Sitter epoch in 3 directions while simultaneously stabilizing the additional n compact dimensions.

In particular, we show on general grounds that this dynamics is achieved with an equation of state $p_a = -\rho$ and $p_b = -2\rho$. If the Casimir energy is due to a massless field in a toroidal compactification in n flat directions, then $\rho \propto b^{-(N+1)}$ and $p_b = (N - n + 1)\rho/n$ as noted. However, we show that if there are massive fields and/or there are other scales at play – curvature, warp factors, branes, specific topologies – then $p_b = -2\rho$ becomes possible.

While this might seem like a strange equation of state, it is not pathological as can be confirmed, for example, by looking at an effective field theory description in which we integrate over the compact directions. This folds the size of the space b into a radion field Ψ in an Einstein frame and the Casimir energy into a potential $U(\Psi)$. As we will see in §IV, the anisotropic equation of state with $p_b = -2\rho$ is equivalent to a stable minimum of the potential $U(\Psi)$. The potential, it should be emphasized, is solely due to the Casimir effect. There is no Λ, nor any other effects, added.

A sense of the size of b can be gained by setting the dimensionally reduced Casimir energy (i.e. the 4-dimensional energy density $\rho^{(4)}$) equal to the current cosmological constant. By dimensional analysis, the dimensionally reduced Casimir energy from a massless field – we will later consider massive modifications – in a flat spacetime must be of the form

$$\rho^{(4)} \propto b^{-4} \tag{5}$$

since b is the only scale in the problem. Given the measured value of the dark energy [16], a comparison gives

$$\frac{\alpha}{b^4} = \rho_{DE}^{(4)} \sim (2.3 \times 10^{-3}\text{eV})^4 \qquad , \tag{6}$$

which yields a scale of $b \sim \mathcal{O}\left(10^{-5}m\right)$ for $\alpha \sim 0.1$, or, in terms of a mass $b \sim 1/(\text{few} \times 10^{-3}\text{eV})$. A well known approach for making such large extra dimensions phenomenologically viable is to consider standard model fields to be confined to a brane and only gravitational fields to propogate in the bulk [17, 18, 19]. It is interesting that $b \sim \text{few} \times 10^{-5}m$ is just below the experimental bounds on deviations of Newton's law [8]. Moreover, the mass scale associated with b^{-1} is in the range of recent neutrino mass measurements, an observation which will motivate the appearance of a small scale in our analysis below but one which, as will become clear, we've yet to incorporate in a fully realistic model.

Before moving on to the details, we note that nothing in our discussion singles out 3 for the number of large, non-compact spatial dimensions. Instead, the point is that should a mechanism establish a sufficient asymmetry between large and small dimensions, the effects we describe here can maintain–and in fact augment–that asymmetry. Moreover, note too that because Casimir energies are sensitive to global topology, not every choice of topology will stabilize the compact dimensions. This leads to the intriguing possibility that if all of the spatial dimensions are compact [20], the reason some expand and others stay small may be due to the topological form of the large vs. small dimensions. Whatever physical law selects topology may thus have inadvertently fated three dimensions to grow large.

III.   COSMOLOGICAL CASIMIR DYNAMICS

We begin with the action for general relativity in $(3 + n + 1)$ spacetime dimensions,

$$S = \int d^{4+n}x\sqrt{-g}\left(\frac{M^{2+n}}{16\pi}\mathcal{R}\right) \qquad , \tag{7}$$

(with $\mathcal{G} = M^{-(2+n)}$) coupled to a source action whose explicit form we will specify shortly. We take the metric to be homogeneous but anisotropic,

$$ds^2 = -dt^2 + a^2(t)d\vec{x}^2 + b^2(t)d\vec{y}^2 \tag{8}$$

where a is the scale factor of the 3 large directions and b is the scale factor of the n small directions, and we allow for the possibility that the large dimensions are finite but large so that any Casimir effect due to those dimensions is negligibly small. We take the small dimensions to be compactified on an n-torus for simplicity. In the future it should be interesting to consider the impact of other topologies.

The Einstein equations can be written as

$$3H_a^2 + \frac{n}{2}(n-1)H_b^2 + 3nH_aH_b = 8\pi\mathcal{G}\rho \tag{9}$$

$$\dot{H}_a + 3H_a^2 + nH_aH_b = \frac{8\pi\mathcal{G}}{(2+n)}\left[\rho + (n-1)p_a - np_b\right] \tag{10}$$

$$\dot{H}_b + nH_b^2 + 3H_aH_b = \frac{8\pi\mathcal{G}}{(2+n)}\left[\rho + 2p_b - 3p_a\right] \tag{11}$$

and the conservation of energy equation is

$$\dot{\rho} + 3H_a(\rho + p_a) + nH_b(\rho + p_b) = 0 \qquad . \tag{12}$$

Using eqn. (9) to eliminate $3H_a^2$ in eqn. (10) and requiring that $\dot{H}_a = 0$ when $H_b, \dot{H}_b = 0$ gives the two conditions:

$$- (n+1) + (n-1)w_a - nw_b = 0 \tag{13}$$

$$1 + 2w_b - 3w_a = 0 \quad , \tag{14}$$

where $p_a = w_a\rho$ and $p_b = w_b\rho$. Using (14) in (13), the extra dimensions are constant and the large dimensions are inflating if $w_a = -1$ and $w_b = -2$. (We could relax this condition and require only that $\dot{H}_a + H_a^2 = \ddot{a}/a > 0$ to get a weaker condition on w_a. However, we'll see in the next paragraph that $w_a = -1$ is also ensured by conservation of energy.) For an arbitrary number of large dimensions $n_a > 1$, the conditions $\dot{H}_a = \dot{H}_b = H_b = 0$ give $w_a = -1$ and $w_b = -(n_a+1)/(n_a-1)$. It is interesting to note that w_b is independent of the number of small dimensions and that it approaches -1 as the number of large dimensions gets big.

For the extra dimensions to be stably constant requires an additional condition. The right hand side of eqn. (11), looks like the negative of the slope of an effective potential for H_b. Stability requires that effective potential to be concave up. In other words, the derivative of the right hand side of eqn. (11) with respect to b needs to be negative for stability.

The full quantum energy momentum tensor for the Casimir effect from a massless field can be calculated from the n-dimensional Green's function using a method of images technique to sum over the infinite copies in the compact space. But, since b is the only scale in the problem, we can also see dimensionally that $\rho \propto b^{-(n+4)}$. If ρ is to be independent of a, the conservation equation (12) requires the second term vanish and this requires $w_a = -1$, as did conditions (13) and (14). (This also follows directly from $p_a = -\frac{\partial(\rho V_a)}{\partial V_a}$, and the requirement that ρ is independent of a.) The conservation equation then reduces to $\dot{\rho} = -nH_b(\rho + p_b)$, which can be reexpressed as

$$p_b = -\frac{\partial(\rho V_b)}{\partial V_b} \quad , \tag{15}$$

where $V_b = b^n$. For a massless field then $w_b = 4/n$. (In the case of $n = 1$, $w_b = N = 4$, where N is the total number of spatial dimensions, as discussed in §II.) This equation of state does not stabilize the dimensions nor does it act as a dark energy.

However, suppose we consider an energy density of the form

$$\rho = \frac{\alpha}{b^{4+n}} \left(1 - \beta b^2 + \gamma b^4\right) \quad . \tag{16}$$
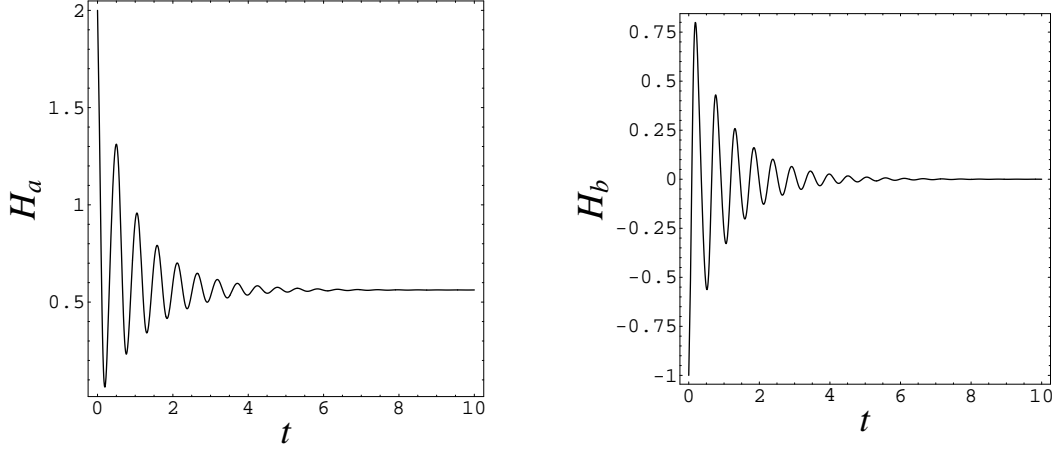
FIG. 1: Left Hand Side: The Hubble constant for the large dimensions, H_a as a function of time. Notice the Hubble constant oscillates indicating periods of deceleration and acceleration until it settles down to a constant value during a de Sitter phase. Right Hand Side: The Hubble constant for the compact directions as a function of time. H_b oscillates between periods of expansion and contraction until it settles down to zero and the dimensions stabilize.

This, in fact, is the basic form of the Casimir energy for a light field ($mb << 1$) with one flat compact direction with periodic boundary conditions [13, 21]. For scalar fields $\alpha < 0$ and for fermions $\alpha > 0$. We will show below that the form (16) results from the sum of Casimir contributions from a spectrum of massive particles that, for definiteness, we can imagine to be neutrino-like species. Other mechanisms to generate a ρ with the properties needed could involve fields from a hidden sector of string theory with masses of specific relative magnitudes, and/or certain conditions on the topology. Since a supersymmetric theory would have equal and opposite contributions from fermions and bosons, the total Casimir energy would vanish, so ρ also depends on the specific manner in which supersymmetry is broken. If, for example, supersymmetry is only broken on a brane [7], then the boundaries set up by a configuration of branes will entirely shape the Casimir energy. The existence of branes will also generate a different metric, which might be difficult to determine. Although we'll return to this discussion below, we show here that a Casimir energy of this form can realize the scenario described above. Hereafter we consider $\beta, \gamma > 0$.

Using eqn. (15) gives an expression for w_b,

$$w_b = \frac{4}{n} + \frac{2}{n} \frac{(\beta b^2 - 2\gamma b^4)}{(1 - \beta b^2 + \gamma b^4)} \qquad . \tag{17}$$

The dimensions will be stabilized when $w_b = -2$, which gives the condition on b:

$$b^2_{max,min} = \frac{\beta(n+1) \pm \sqrt{(n+1)^2\beta^2 - 4n(n+2)\gamma}}{2n\gamma} \qquad . \tag{18}$$

Requiring

$$\frac{\beta^2}{4} < \gamma < \frac{(n+1)^2}{4n(n+2)}\beta^2 \qquad , \tag{19}$$

gives two positive roots (right hand side of (19)) and a positive energy density at these roots (left hand side of (19)). This condition can be satisfied for all n although the window gets pretty narrow as n gets large.

The critical value b_{min} with the negative sign in eqn. (18), corresponds to a stabilization of the extra dimensions while the critical value b_{max} with the positive sign in eqn. (18) is an unstable fixed point of eqn. (11).

Fig. 1 shows the dynamical evolution of the Hubble factors under the influence of the energy density (16) with $n = 2$, $\alpha = \beta = 1$ and $\gamma = 0.9 \times ((n+1)^2/(4n(n+2)))\beta^2 = 0.9 \times (9/32)$. The Hubble factor for the large dimensions alternates in a series of accelerations and decelerations as the scale factor of the extra dimensions oscillates about the value b_{min}. Eventually b settles at the value b_{min}, as shown on the right hand side of fig. 2, the dimension stabilizes and the energy density is constant. The Hubble factor then slides into a constant value and the large dimensions accelerate as shown on the left hand side of fig. 2.

From our perspective in the large dimensions, the universe would appear to be $(3+1)$-dimensional and to be dark energy dominated. To reproduce the strength of gravity today, $M^{2+n}b^n_{min} = m^2_p$, where m_p is the Planck mass today.

IV.   THE RADION PICTURE

Since our interest, ultimately, is in how this scenario appears to a four dimensional observer, it is worthwhile rephrasing our analysis in the language of four-dimensional effective field theory. Toward this end, let's augment the action (7) by a source term, $-V(b)$, capturing the Casimir energy contribution discussed in the last section, and thus start from

$$S = \int d^{4+n}x\sqrt{-g}\left(\frac{M^{2+n}}{16\pi}\mathcal{R} - V(b)\right). \tag{20}$$

We then integrate over the extra dimensions, and to put the resulting effective action in canonical form we perform a conformal transformation to the Einstein frame $g^E_{\mu\nu} = \Omega g_{\mu\nu}$
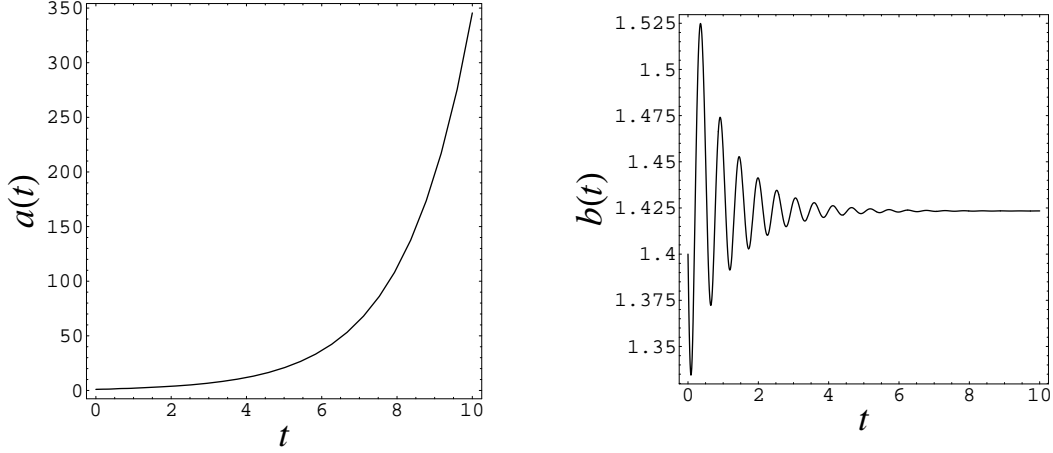
FIG. 2: Left Hand Side: The scale factor for the large directions, a, as a function of time. Right Hand Side: The scale factor for the compact directions, b, oscillates about the critical value b_{min} until it stabilizes.

$(\mu, \nu = 0...3)$, redefine the time variable $dt_E = \Omega^{1/2}dt$, with $\Omega = \left(M^{2+n}b^n/m_p^2 \right)$. Notice that when $b = b_{min}$, $\Omega = 1$. Under this conformal transformation, the metric is in standard FRW form with scale factor $a_E(t) = a\Omega^{1/2}$. Finally, change field variables to $d\Psi = \frac{m_p}{\sqrt{16\pi}}\sqrt{n(n+2)}\, db/b$. Collectively, these transformations yield

$$S^{\text{eff}} = \int d^4x \sqrt{-g_E} \left(\frac{m_p^2}{16\pi}R[g_E] - \frac{1}{2}g_E^{\mu\nu}D_\mu\Psi D_\nu\Psi - U(\Psi) \right) \tag{21}$$

from which we derive the equations of motion

$$H_E^2 = \frac{8\pi}{3m_p^2}\left(\frac{1}{2}\left(\frac{d\Psi}{dt_E} \right)^2 + U(\Psi) \right)$$
$$\frac{d^2\Psi}{dt_E} + 3H_E\frac{d\Psi}{dt_E} = -\frac{\partial U(\Psi)}{\partial\Psi} \tag{22}$$

where $U(\Psi) = Vb^n\Omega^{-2}$ and $H_E = (da_E/dt_E)/a_E$.

To make contact with the last section, note that the energy momentum tensor associated with action (20) yields the relations

$$\mathcal{G}\rho = -\mathcal{G}p_a = \frac{\Omega}{m_p^2}U \quad , \quad \mathcal{G}p_b = -\frac{\Omega}{m_p^2}\left(2U + \frac{b}{n}\frac{\partial U}{\partial b} \right) \quad . \tag{23}$$

From this we see that $p_b = -2\rho$ corresponds to $(\partial U/\partial\Psi) \propto (\partial U/\partial b) = 0$. In other words, $p_b = -2\rho$ in the spacetime description corresponds to being at an extremum of the potential
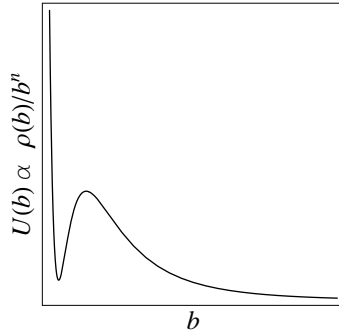
FIG. 3: The potential for the radion field as a function of b.

in the radion description. The extrema of U are at $b_{min,max}$, as must be the case. The potential is drawn as a function of b in fig. 3.

The radion picture is particularly useful for examining forms of ρ that could stabilize the extra dimensions and create the dark energy. Previously we mentioned that the sum of the Casimir energies due to massive fields on a compact torus might have the polynomial form needed. Here we present a graphical argument for how one can build a total ρ of the desired form. In a $(3+n+1)$-dimensional spacetime with n dimensions compactified to T^n, the Casimir energy density per degree of freedom for massive fields with periodic boundary conditions [3] is known to be a sum of Bessel functions [21, 22]:

$$\rho = \frac{m^{N+1}}{(2\pi)^{(N+1)/2}} \sum_{j_1=-\infty}^{\infty} ... \sum_{j_n=-\infty}^{\infty} \frac{K_{(N+1)/2}(bm\sqrt{j_1^2 + ... + j_n^2})}{(bm\sqrt{j_1^2 + ... + j_n^2})^{(N+1)/2}} \tag{24}$$

where $N = 3+n$ is the total number of spatial dimensions, m is the mass of the contributing field. The $j_1 = ... = j_n = 0$ term is infinite and is subtracted in the usual spirit of renormalizing away the infinite Minkowski space contribution. The energy density ρ is negative for bosons and positive for fermions.

Now, let's consider the Casimir energies due to two fermion fields and one scalar field, all having different masses (not necessarily light) as drawn in fig. 4. By adjusting the masses and the amplitudes through the number of species with a given (similar) mass, these three graphs can be summed to give a potential with the general shape of fig. 3 as indicated by the dotted line in fig. 4.

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[3] Different topologies can lead to boundary conditions other than simply periodic. Interaction of bulk fields with branes will also change the modes [23, 24, 25].
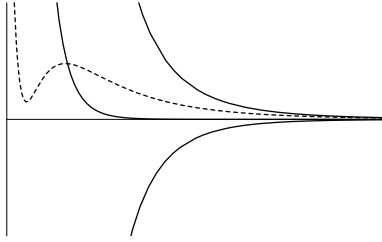
FIG. 4: Different contributions to the Casimir energy that sum graphically to the desired form. The positive solid curves are fermions of different masses while the negative solid curve is a scalar contribution of intermediate mass. The dashed line is the resultant sum.

As an illustration, consider a $(4+1)$-dimensional spacetime with one direction compactified on a circle of radius b. Just for the sake of argument consider confining all standard model fields to a flat brane as in the ADD model of Ref. [18, 19]. We neglect any brane tension as well as any bulk cosmological constant. (Alternatively, they can be tuned to cancel.) If the only field propogating in the bulk is the gravition, then there is nothing to superpose to create a minimum. However, if we posit the existence of sterile fields in the bulk (a possibility pursued for other, phenomenologically motivated reasons in [26, 27]), we can arrange for a minimum to emerge with careful choices of masses and numbers of degrees of freedom. We do not justify the existence of these fields phenomenologically, as our intention here is simply to demonstrate that if such fields are present it is possible to stabilize the radion. We limit ourselves to flat internal dimensions but there is reason to suppose that curvature or a warp factor could provide the additional scale needed to generate a minimum and thereby remove the dependence on superposing masses. We leave that for a future investigation.

We take the particle spectrum in the bulk to consist of sterile bulk Majorana neutrinos with masses $m_{\nu 1} \neq 0$, and $m_{\nu 2} = \lambda m_{\nu 1}$ with $\lambda > 1$. In this toy model we imagine that there are equal and opposite numbers of massless fermions and bosons. Other combinations of massess and spins can be used so this is just in way of illustration. We want $p_b = -2$, which corresponds to a minimum of $U \propto \rho b^{-1}$. For simply periodic boundary conditions, each term (24) in the sum of the contributions is positive and it is impossible to add them to get a minimum. To create a minimum, we add $N_{m_s} = 4$ fermion degrees of freedom with anti-periodic boundary conditions and mass in between the two light neutrinos, $m_s = 2m_{\nu 1}$. The Casimir energy in $(4+1)$ dimensions compactified on T^1 with antiperiodic boundary
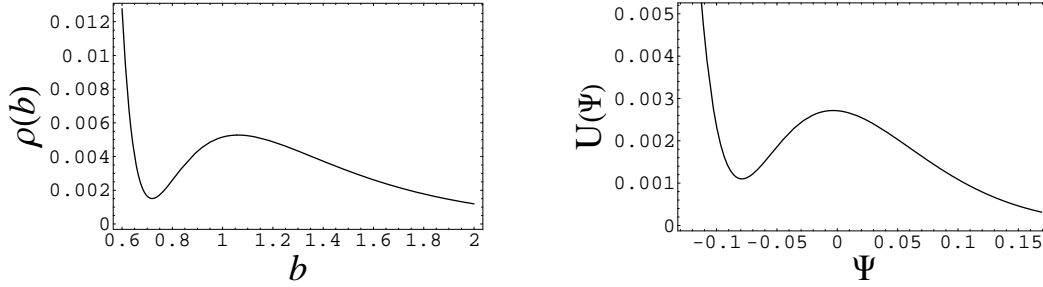
FIG. 5: Left: The x-axis is b in units of $m_{\nu 1}$ and the y-axis is ρ in units of $m_{\nu 1}^5$. Right: The x-axis is $\Psi \propto \ln(b\, m_{\nu 1})$ in units of m_p and the y-axis is $U = (b_{min}^2/b(\Psi))\rho(\Psi)$ in units of $m_{\nu 1}^4$.

conditions is [13, 21]:

$$\rho = \frac{2m^5}{(2\pi)^{5/2}} \sum_{j_1=1}^{\infty} \frac{K_{5/2}(bmj_1)}{(bmj_1)^{5/2}} \cos(j_1\theta) \tag{25}$$

with $\theta = \pi$. We can interpret this as the addition of a sterile Dirac neutrino with mass, m_s. This example establishes that we can find positive minima, as shown in fig. 5, for a very tight region, narrower than $3 < \lambda < 3.2$.

Returning to the expansion in terms of γ and β, we can see why the masses must be so finely tuned. The Bessel functions in eqns. (24) and (25) can be expanded about small bmj_1 and summed so that all of the contributions together can be written in the form

$$\rho = \frac{\alpha}{b^5}\left[1 - \beta b^2 + \gamma b^4\right] \qquad . \tag{26}$$

This approximation is somewhat suspect since the tendency in practice has been to sum over all j_1, long after the condition $bmj_1 \ll 1$ breaks down. We only remark that such an expansion shows clearly that γ/β^2 is very tightly constrained; Using condition (19) for $n = 1$,

$$1/4 < \gamma\beta^{-2} < 1/3 \qquad . \tag{27}$$

This translates directly into the statement that the relative masses and relative degrees of freedom have to be tuned to create a positive energy density minimum. For any quantitative results we continue to use the full Bessel functions.

For comparison with the ADD braneworld scenario proposed to address the mass hierarchy problem [18, 19] – that scenario favored $n = 2$ extra dimensions – we can consider a $(5 + 1)$-dimensional spacetime with two directions compactified on a 2-torus of radius b and a particle spectrum in the bulk consisting of a massless spin 2 boson (2 degrees of
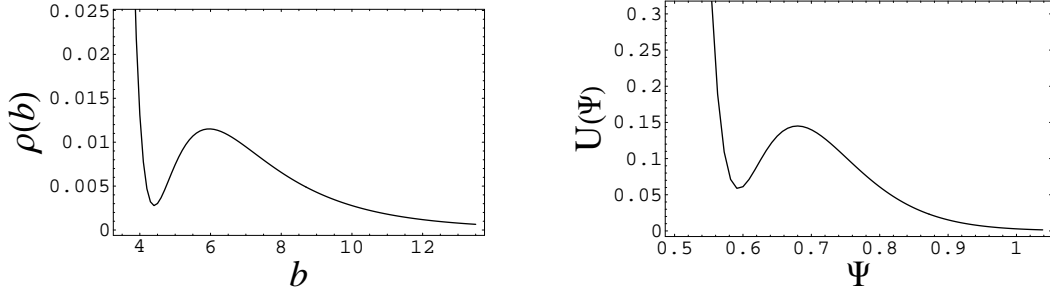
FIG. 6: Left: The x-axis is b in units of m_ν and the y-axis is ρ in units of m_ν^6. Right: The x-axis is $\Psi \propto \ln(b\, m_\nu)$ in units of m_p and the y-axis is $U = (b_{min}^4/b^2)\rho = (b_{min}/b)^2 \rho^{(4)}$ in units of m_ν^4.

freedom) and a massless fermion that we can think of as a bulk Dirac neutrino (4 degrees of freedom). Also needed is a light fermion – another sterile Dirac neutrino (4 degrees of freedom) – living in the bulk with mass $m_\nu \neq 0$. We want $p_b = -2$, which corresponds to a minimum of $U \propto \rho b^{-2}$. These fields give a net positive contribution to the Casimir energy. To get the negative contribution needed we add 4 bulk scalar degrees of freedom with mass $m_s = \lambda m_\nu$. (Alternatively, we could interpret this as another sterile Dirac neutrino with anti-periodic boundary conditions.) Numerically, we find positive minima for a very tight region, narrower than $0.4 < \lambda < 0.42$.

Again, in terms of γ and β, condition (19) for $n = 2$, imposes the restriction

$$1/4 < \gamma\beta^{-2} < 9/32 \qquad , \tag{28}$$

or equivalently $0.25 < \gamma\beta^{-2} < 0.28125$. The range gets tighter as more additional dimensions are invoked.

For $m_s = 0.406m_\nu$, the resultant ρ and U from the full Bessel functions is shown in fig. 6. At the minimum, $b_{min} \sim 4.5/m_\nu$, the reduced energy density is roughly $\rho_{DE}^{(4)} = U(b_{min}) = \rho b_{min}^2 \sim (0.47 \times m_\nu)^4$. Compare this to the observed value $\rho_{DE}^{(4)obs} \sim (2.3 \times 10^{-3}\text{eV})^4$. We can choose $m_\nu \sim 5 \times 10^{-3}\text{eV}$ from which it follows that $b_{min} \sim 0.2mm$. For comparison, the ADD solution to the hierarchy problem exploits the observation that the Planck mass on the brane is

$$m_p^2 = M^{2+n}b^n \quad . \tag{29}$$

Turning this around, for $n = 2$ and $b = b_{min} \sim 1/(10^{-3}\text{eV})$, the natural scale for M is ~ 3 TeV. The size of the space needed to fix the dark energy at the observed value is therefore consistent with the size needed to address the hierarchy problem. The cosmological constant

problem would then be directly mappable to the hierarchy problem. Even if the vaccum energy is naturally zero, the boundary conditions on the finite extra dimensions can create a dark energy of the observed magnitude.

Since the dark energy is really set by the neutrino mass in the bulk, the picture would feel more complete if such light bulk fields could also find some justification. It should not go without mention that the mass $m_\nu \sim 5 \times 10^{-3}$eV is a reasonable value for a neutrino mass and we have been able to express the small dark energy naturally in these units. The shortcoming is that these are, again, bulk neutrino fields and not standard model fields. Thus, from the perspective of our toy models, the parameters in the effective Lagrangian are simply chosen to accommodate dark energy. If these ideas could be incorporated into a more realistic scenario, they might suggest a mechanism linking neutrino masses and dark energy. Although we've yet to achieve that link, viable directions to pursue include consideration of internal manifolds whose geometry and topology separates the Casimir and Kaluza-Klein scales allowing standard model fields to have fluctuations that live in the bulk. If this were successful, then the field setting the scale of dark energy could be a light standard model neutrino thereby reducing the number of small parameters in need of explanation.

It is also noteworthy that the hierarchy between the Planck scale and the electroweak scale $m_p/M \sim 10^{16}$ is repeated in the hierarchy between the observed neutrino mass splittings of $\mathcal{O}(10^{-2}-10^{-3})$ eV and the electroweak scale: $M/m_\nu \sim 10^{14}-10^{15}$. Attempts to connect the two have invoked sterile neutrinos in the bulk [26]. However, these models require massless neutrinos in the bulk and do not offer an explanation for a light massive fermion in the bulk.

The numerical coincidence between the magnitude of a Casimir term from macroscopic extra dimensions and the magnitude of the dark energy has been noted before (see for instance [7, 8, 9]). Here we're able to generate a proper dark energy with the correct equation of state ($p_a = -\rho$ and $p_b = -2\rho$, not $p_a = -\rho$, $p_b = 0$) so that the dimensions are fixed and a de Sitter expansion is sourced. Furthermore, this example hints at a connection between the dark energy, a light sterile neutrino in the bulk, and the hierarchy problem. The ability to sweep up so many seemingly disparate mysteries into one framework is obviouly appealing. The possibility that these sterile fields could have heavier Kaluza-Klein excitations that might be connected with the dark matter adds another layer of interest to this approach [28].

Nevertheless, it is important to stress that the confining potential is generally quite

shallow $\Delta U/U \sim 3$ in fig. 6, implying that the most minor of perturbations to the energy density could unravel the extra dimensions. The effective mass of the radion field, $m_\Psi^2 = \frac{\partial^2 U}{\partial \Psi^2}$, would have to be big enough to avoid distending the extra dimension unacceptably. (For additional discussion of bounds on a light radion field in the ADD model, see Ref. [29].) A bolder move to extend the particle spectrum in the bulk would be required to address this problem. It is also possible that intrinsic curvature in the small space could create a deeper, more stable potential.

To demonstrate the destabilizing effect of matter [30], consider the energy density in non-relativistic matter today. We allow the matter to live in the bulk,

$$S = \int d^{4+n}x\sqrt{-g}\left[\frac{M^{2+n}}{16\pi}\mathcal{R} + \mathcal{L}_{matter}\right] \qquad , \qquad (30)$$

although we'll see in the next paragraph that the effect is ultimately the same if matter is confined to a brane. For matter living in the bulk, $\rho_m = M/(a^3 b^n)$ and $p_M = 0$, where M is the total mass of all non-relativistic matter including baryonic matter as well as any non-relativistic dark matter. For the sake of argument, suppose all of the dark matter is cold so that roughly 26% of the energy density in the universe is non-relativistic. The other 74% is in dark energy, which we take to be due to the Casimir energy density today. In other words, $\rho_{M0} = (2.6/7.4)\rho_{DE}$. Using $(a_0/a) = 1 + z$, we can express ρ_M as

$$\rho_M = (2.6/7.4)\rho_{DE}^{(4)}(1 + z)^3\left(\frac{1}{b^n}\right) \qquad , \qquad (31)$$

where $\rho_{DE}^{(4)} = \rho_{DE}b_{min}^n$ is the dimensionally reduced dark energy we measure in $(3 + 1)$ dimensions today. The right hand side of eqn. (11) becomes $\propto \rho_m + \rho + 2p_b - 3p_a$ where ρ is still the Casimir energy density and so it is still the case that $p_a = -\rho$. Stability now requires

$$p_b = -2\rho - \frac{1}{2}\rho_m \qquad . \qquad (32)$$

Using eqn. (32) in the weaker requirement that $\dot{H}_a + H_a^2 = \ddot{a}/a > 0$ gives the condition $\rho_m < 2\rho$. This is the same constraint one gets for a $(3 + 1)$-dimensional Λ dominated universe. That's to be expected. When b is stable as required by (32), the model is an effective $(3 + 1)$-dimensional Λ dominated universe.

In the radion picture when non-relativistic matter is included, the field is driven to the minimum of

$$U + \frac{m_p^2 \mathcal{G}}{\Omega}\frac{\rho_m}{4} \qquad , \qquad (33)$$

where $U = (m_p^2 \mathcal{G}/\Omega)\rho$. Using the dimensionally reduced $\rho_m^{(4)} = \rho_m b^n$, and the definition of Ω, this can be reexpressed as

$$U + \left(\frac{b_{min}}{b}\right)^{2n} \frac{\rho_m^{(4)}}{4} \qquad . \tag{34}$$

Notice that if the non-relativistic matter does not live in the bulk but is rather confined to a brane,

$$S = \int d^{4+n}x \sqrt{-g} \left[\frac{M^{2+n}}{16\pi}\mathcal{R}\right] + \int d^4x \sqrt{-g^{(4)}}\mathcal{L}_{matter} \qquad , \tag{35}$$

then there is no dimensional reduction of the matter action. Under the conformal transformation, the 4-dimensional matter is shifted, $\rho_m^{(4)}/\Omega^2$, and the radion is again driven to the minimum of eqn. (34). Today, the presence of non-relativistic matter would only lift the minimum slightly. However, at earlier times, the presence of non-relativistic matter dominates the shape and eventually destroys the presence of extrema altogether, as shown in fig. 7. The effect is to force b to expand so that even if a minimum exists today, b has already sped past it with no way back to its stable position. This may be curable if there are meta-stable minima to keep b hovering near its stable value, allowing it to roll gently into today's minimum when it appears. The point is that it is not enough to understand the shape of the potential today. The early universe has to be investigated as well to determine if b can actually make it to its minimum value.

An early universe cosmology that begins supersymmetric would have no Casimir energy – except that due to thermal effects. A conformal anomaly contribution also arises in an expanding space [31]. After supersymmetry is broken, any field with $m < 1/b$ will contribute to the Casimir energy and thereby shape the potential for b. As the universe cools, different species contribute to the energy density in non-relativistic matter leading to an alteration in the shape of the potential. These issues are currently being investigated [32] to see if the extra dimensions could drive inflation and then smoothly evolve to generate the dark energy today.

It is perhaps also worth noting that if we are willing to accept some radical fine tuning, the potential can be made arbitrarily deep. As m_s decreases to some critical value, the energy density at the minimum will be pushed closer to zero. Instead of interpreting these bulk fields as neutrinos, we can suppose they are associated with electroweak scale physics and have mass $M \sim \text{TeV}$. Then $b \sim \text{TeV}^{-1}$ and we need not even be on a brane. We could imagine this in a model of Universal Extra Dimensions (UED) where all standard model
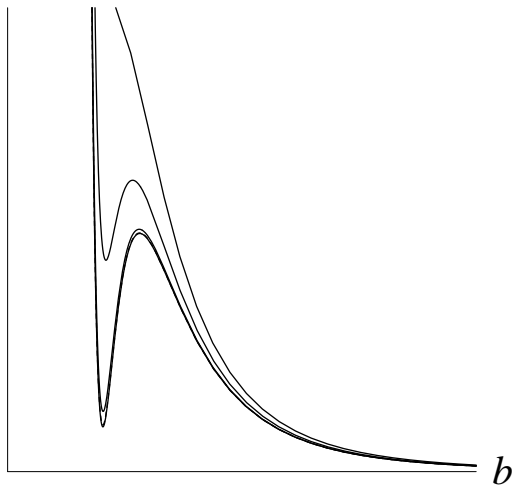
FIG. 7: The potential of figure 3 with non-relativistic matter included. The lowermost black line is without non-relativistic matter for comparison. The line that includes non-relativistic matter today is dashed but can't be picked out by eye as it essentially overlaps the lowermost line. Successive lines as one moves up in the figure include matter at a redshift of 1, 3.5, and finally 5.5, for which there is no minimum.

fields propagate in all dimensions [28]. In units of M, ρ could be arranged to be exceedingly small relative to a TeV, $\rho \sim (10^{-15}M)^4$ by choosing the relative masses so carefully that a near cancellation occurs. The potential would be quite stable, even to the effects of ordinary matter, although one would have to investigate if tunneling out of the minimum was viable. But, of course, this extreme fine tunning would be none other than the usual cosmological constant problem recast. Turning this around, in a model of UED, Casimir energy contributions to the energy density of the universe would naturally be huge, with no obvious mechanism for cancellation – other than a fortuitous near cancellation between masses.

V.   CONCLUSION

We've shown that Casimir energies–which unavoidably arise in non-supersymmetric theories with compact extra dimensions–can act to stabilize the size of the extra dimensions while also sourcing accelerated expansion in the familiar four spacetime dimensions. While we've only worked in the context of toy models, we find it intriguing that there might be a

link between issues of moduli stabilization and dark energy that is a direct consequence of quantum mechanics on compact spaces.

We've also noted how this approach may link geometrical asymmetries to topological asymmetries. Namely, because the topology of spacetime's spatial sections determines the form of the resulting Casimir energy, different topologies will result in different moduli potentials. Some potentials will stabilize all but three spatial dimensions, as in the example discussed herein, while others will stabilize different numbers of spatial dimensions. In an ensemble of finite $(3+n+1)$-dimensional universes, the number of dimensions that evolve to be large or small in a given universe will depend on the topology. Explaining the existence of 3 large dimensions may thus reduce to explaining the topology of the universe.

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