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József Lőrinczi, Fumio Hiroshima, Volker Betz

FEYNMAN-KAC-TYPE THEOREMS AND GIBBS MEASURES ON PATH SPACE

WITH APPLICATIONS TO RIGOROUS QUANTUM FIELD THEORY

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Preface

For a long time disciplines such as astronomy, celestial mechanics, the theory of heat and the theory of electromagnetism counted among the most important sources in the development of mathematics, continually providing new problems and challenges. The main mathematical facts resulting from these fields typically crystallized in linear ordinary or partial differential equations.

In the early 20th century Heisenberg's discovery of a rule showing that the measurement of the position and momentum of a quantum particle gives different answers dependent on the order of their measurement led to the concept of commutation relations, and it made matrix theory find an unexpected application in the natural sciences. This resulted in an explosion of interest in linear structures developed by what we call today linear algebra and linear analysis, among them functional analysis, operator theory, C^*-algebras, and distribution theory. Similarly, quantum mechanics inspired much of group theory, lattice theory and quantum logic.

Following this initial boom Wigner's beautiful phrase talking of the "unreasonable effectiveness of mathematics" marks a point of reflection on the fact that it is by no means obvious why this kind of abstract and sophisticated mathematics can be expected at all to make faithful descriptions and reliable predictions of natural phenomena. If indeed this can be appreciated as a miracle, as he said, it is perhaps at least a remarkable fact the degree of flexibility that mathematics allows in its uses. Feynman has pointed to the different cultures of using mathematics when he remarked that "if all of mathematics disappeared, physics would be set back by exactly one week." While mathematics cannot compete with physics in discovering new phenomena and offering explanations of them, physics continues to depend on the terminology, arsenal and discipline of mathematics. Mathematical physics, which is part of mathematics and therefore operates by its rules and standards, has set the goal to understand the models of physics in a rigorous way. Mathematical physicists thus find themselves at the borderline, listening to physics and speaking mathematics, at best able to use these functionalities interchangeably.

If one of the imports of early quantum mechanics has been the realization that the basic laws on the atomic scale can be formulated by linear superposition rules, another was a probabilistic interpretation of the wave function. This connection with chance, amplified and strongly advocated by the Copenhagen school, was no less revolutionary than the commutation relations. Richard Feynman has discovered a second connection with probability when he offered a representation of the state of a particle in terms of averages over all of its possible histories from one point in space-time to another. While this was an instance seriously questioning his dictum quoted above and his use of mathematics was in this case very problematic, the potential lying in the description advanced in his work turned out to be far reaching. It took the efforts of the mathematician Mark Kac to be the first to show that this method could be made sense of and was eminently viable.

Our project to write the present monograph has grown out of the thematic programme *At the Interface of PDE, Self-Adjoint Operators and Stochastics: Models with Exclusion* organized by the first named author at the Wolfgang Pauli Institute, Vienna, in 2006. The initial concept was a smaller scale but up-to-date account of Feynman–Kac-type formulae and their uses in quantum field theory, which we wanted to dedicate to the person all three of us have much to thank both scientifically and personally, Herbert Spohn. At that time we secretly meant this as a present for his 60th birthday, and we are glad that we are now able to make this hopefully more mature tribute on the occasion of his 65th birthday!

Apart from the inspiring and pleasant environment at Zentrum Mathematik, Munich University of Technology, where our collaboration has started, and Wolfgang Pauli Institut, University of Vienna, hosting the thematic programme, we are indebted to a number of other individuals and institutions. We are thankful for the joint work and friendship of our closest collaborators in this direction, Asao Arai, Massimiliano Gubinelli, Masao Hirokawa, and Robert Adol'fovich Minlos. We also thank Cristian Gérard, Takashi Ichinose, Kamil Kaleta, Jacek Małecki, Annalisa Panati, Akito Suzuki for ongoing related joint work. It is furthermore a pleasure to thank Norbert Mauser, Director of the Wolfgang Pauli Institute, for supporting the programme, and Sergio Albeverio, Rodrigo Bañuelos, Krzysztof Bogdan, Carlo Boldrighini, Thierry Coulhon, Laure Coutin, Cécile DeWitt-Morette, Aernout van Enter, Bill Faris, Gero Friesecke, Jürg Fröhlich, Mohammud Foondun, Hans-Otto Georgii, Alexander Grigor'yan, Martin Hairer, Takeru Hidaka, Wataru Ichinose, Keiichi Itô, Niels Jacob, Hiroshi Kawabi, Tadeusz Kulczycki, Kazuhiro Kuwae, Terry Lyons, Tadahiro Miyao, Hirofumi Osada, Habib Ouerdiane, Sylvie Roelly, Itaru Sasaki, Tomoyuki Shirai, Toshimitsu Takaesu, Setsuo Taniguchi, Josef Teichmann, Daniel Ueltschi, Jakob Yngvason for many useful discussions over the years. We thank especially Erwin Schrödinger Institut, Vienna, Institut des Hautes Études Scientifiques, Bures-sur-Yvette, and Università La Sapienza, Rome, for repeatedly providing excellent environments to further the project and where parts of the manuscript have been written. Also, we thank the hospitality of Kyushu University, Loughborough University and Warwick University, for accommodating mutual visits and extended stays.

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József Lőrinczi, Fumio Hiroshima, Volker Betz February 2011

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Chapter 1
Heuristics and history

1.1 Feynman path integrals and Feynman–Kac formulae

In Feynman's work concepts such as paths and actions, just discarded by the new-born quantum mechanics, witnessed a comeback in the description of the time-evolution of quantum particles, proposing an alternative to Heisenberg's matrix mechanics and Schrödinger's wave mechanics. It is worthwhile briefly to review how this turned out to be a viable project and led to a mathematically meaningful approach to quantum mechanics and quantum field theory.

A basic tool in the hands of the quantum physicist is the *Schrödinger equation*

$$i\hbar \frac{\partial}{\partial t}\psi(t,x) = -\frac{\hbar^2}{2m}\Delta\psi(t,x) + V(x)\psi(t,x), \qquad (1.1.1)$$
$$\psi(0,x) = \phi(x), \quad (t,x) \in \mathbb{R}\times\mathbb{R}^3$$

with *Hamilton operator*

$$H = -\frac{\hbar^2}{2m}\Delta + V, \qquad (1.1.2)$$

and a complex valued *wave function* ψ describing a quantum particle of mass m at position x in space at time t. Here $\hbar$ is Planck's constant, V is a potential in whose force field the particle is moving, and ϕ describes the initial state of the quantum particle. The Schrödinger equation can be solved only under exceptional choices of V. However, formally the solution of the equation can be written as

$$\psi(t,x) = (e^{-(it/\hbar)H}\phi)(x)$$

for any V. The difficulty is that for a general choice of V we do not know how to calculate the right-hand side of the equality.

The action of the complex exponential operator on ϕ can be evaluated by using an *integral kernel* G allowing to write the wave function of the particle at position y at time t based on its knowledge in x at $s < t$, i.e.,

$$\psi(t,y) = \int_{\mathbb{R}^3} G(x,y;s,t)\psi(s,x)dx.$$

This representation can then in principle be iterated from one interval $[s,t]$ to another, and thus our knowledge of the wave function "propagates" from the initial time to any time point in the future. This observation suggests that at least a qualitative analysis of the solution might be possible for a wider choice of V by using this integral kernel, and the key to this approach is the information content of the kernel.

Feynman proposed a method of constructing this integral kernel in the following way. Consider first a classical *Hamiltonian* function

$$\mathscr{H}(p,q) = \frac{1}{2m}p^2 + V(q), \quad (p,q) \in \mathbb{R}^3 \times \mathbb{R}^3, \tag{1.1.3}$$

describing a particle at position q moving with momentum p under the same potential V as above. The corresponding *Lagrangian* is obtained through Legendre transform,

$$\mathscr{L}(q,\dot{q}) = \dot{q}p - \mathscr{H}(p,q) = \frac{m}{2}\dot{q}^2 - V(q),$$

where

$$\dot{q} = \frac{\partial \mathscr{H}}{\partial p} = \frac{p}{m}.$$

Classical mechanics says that the particle is equivalently described by the *action functional*

$$\mathscr{A}(q,\dot{q};0,t) = \int_0^t \mathscr{L}(q(s),\dot{q}(s))ds,$$

where $q(s)$ is the path of the particle as a function of the time s. In particular, the equation of motion can be derived by the *principle of least action* by using variational calculus, and it is found that the particle follows a route from a fixed initial to a fixed endpoint which minimizes the action.

In the late 1920s Dirac worked on a Lagrangian formulation of quantum mechanics and arrived at the conclusion that the ideas of the classical Lagrangian theory can be taken over, though not its equations. Specifically, he arrived at a rule he called "transformation theory" allowing to connect the positions at s and t of the quantum particle through the phase factors $e^{(i/\hbar)\mathscr{A}(q,\dot{q};s,t)}$. By dividing up the interval $[s,t]$ into small pieces, Dirac noticed [128, 129, 130] that due to the fact that $\hbar$ is of the order of 10^{-34} in standard units, this is so small that the integrands in the exponent are rapidly oscillating functions around zero. Therefore, as a result of the many cancellations the only significant contribution into the integral is given by those q whose large variations over the subintervals involve small differences in the Lagrangian.

Feynman, while a Princeton research assistant during 1940–41, turned Dirac's somewhat vague observations into a more definite formulation. He argued that the propagators $G(x,y;s,t)$, which were Dirac's "contact transformations," can be obtained as limits over products of propagators at intermediary time points. This means that with a partition of $[s,t]$ to subintervals $[t_j,t_{j+1}]$ with $t_j = s + jb$, $j = 0,\dots,n-1$, $b = (t-s)/n$, the expression

$$G(x,y;s,t) = \lim_{n\to\infty} \prod_{j=0}^{n-1} \int_{\mathbb{R}^3} G(x_j,x_{j+1};t_j,t_{j+1}) \prod_{j=1}^{n-1} dx_j$$

holds, where

$$G(x_j, x_{j+1}; t_j, t_{j+1}) = \sqrt{\frac{m}{2b\pi i\hbar}} \exp\left(\frac{ib}{\hbar}\mathscr{L}\left(x_{j+1}, \frac{x_{j+1}-x_j}{b}\right)\right).$$

From here it was possible to conclude that

$$(e^{-(it/\hbar)H})(x,y) = G(x,y;0,t) = \text{const} \times \int_{\Omega_{xy}} \exp\left(\frac{i}{\hbar}\mathscr{A}(q,\dot{q};0,t)\right) \prod_{0\le s\le t} dq(s) \tag{1.1.4}$$

giving a formal expression of the integral kernel of the operator $e^{-(it/\hbar)H}$ for all t. Here $dq(s)$ is Lebesgue measure, and Ω_{xy} is the set of continuous paths $q(s)$ such that $q(0) = x$ and $q(t) = y$. The right-hand side of the expression is *Feynman's integral*. These results have been obtained first in his 1942 thesis, which remained unpublished; the paper [161] published in 1948, however, reproduces the crucial statements (see also [164]). Feynman was able to apply his approach also to quantum field theory [162, 163].

For all its intuitive appeal and potential, Feynman's method did not win immediate acclaims, not even in the ranks of theoretical physicists. Mathematically viewed, the object Feynman proposed was by no means easy to define. Mathematicians hitherto have not found a satisfactory meaning of Feynman integrals covering a sufficiently wide class of choices of V. For instance, there is no countably additive measure on Ω_{xy} allocating equal weight to every path, and $\mathscr{A}(q,\dot{q};0,t)$ fails to be defined on much of Ω_{xy} since it involves $\dot{q}(s)$. Various ways of trying to cope with these difficulties have been explored. Several authors considered purely imaginary potentials V and masses m leading to a complex measure to define Feynman's integral, but these attempts had to face the problem of infinite total variation [79, 102, 372]. Another suggestion was to define the Feynman integral in terms of Fresnel distributions [126, 3, 127, 80, 85, 474]. Other attempts constructing solutions of Schrödinger's equation from Feynman's integral through piecewise classical paths or broken lines or polygonal paths [177, 279, 283, 273, 476, 292].

In Feynman's days integration theory on the space of continuous functions has already been in existence due to Wiener's work on Brownian motion initiated in 1923. It was Kac in 1949 who first realized that this was a suitable framework, however, not for the path integral (1.1.4) directly, see also [294, 295]. In contrast to Feynman's expression, the *Feynman–Kac formula* offers an integral representation on path space for the semigroup $e^{-(1/\hbar)tH}$ instead of $e^{-(i/\hbar)tH}$, obtained on replacing t by $-it$. Why should this actually improve on the situation can be seen by the following argument. Analytic continuation $s \to -is$, $ds \to -ids$ and the replacement $\dot{q}(s)^2 \to -\dot{q}(s)^2$

in (1.1.4) leads to the kernel

$$(e^{-tH})(x,y) = \text{const} \times \int_{\Omega_{xy}} e^{-(1/\hbar)\int_0^t V(q(s))ds} e^{-(m/2\hbar)\int_0^t \dot{q}(s)^2 ds} \prod_{0\le s\le t} dq(s). \tag{1.1.5}$$

Since defining the latter factor as a measure remains to be a problem as

$$\int_{\Omega_{xy}} \prod_{0\le s\le t} dq(s) = \infty \tag{1.1.6}$$

for every t, one might boldly hope that its support $\mathcal{C}_{xy}$ is a set on which the exponential weights typically vanish. If $\dot{q}$ were almost surely plus or minus infinite on this set, then the situation

$$\exp\left(-\frac{1}{\hbar}\int_0^t \frac{m}{2}\dot{q}(s)^2 ds\right)\Big\lceil_{\mathcal{C}_{xy}} = 0 \tag{1.1.7}$$

would occur. $\mathcal{C}_{xy}$ would thus be required to consist of continuous but nowhere differentiable paths. Once this can be allowed, zero in (1.1.7) can possibly cancel infinity in (1.1.6), and it would in principle be possible that

$$\exp\left(-\frac{1}{\hbar}\int_0^t \frac{m}{2}\dot{q}(s)^2 ds\right) \prod_{0\le s\le t} dq(s) = 0 \times \infty$$

is a well-defined measure. As we will explain below, the miracle happens: The paths of a random process $(B_t)_{t\ge 0}$ called *Brownian motion* just have the required properties and the *Feynman–Kac formula*

$$(e^{-tH})(x,y) = \int_{C(\mathbb{R},\mathbb{R}^3)} e^{-\int_0^t V(B_s(\omega))ds} d\mathcal{W}^{x,y}_{[0,t]}(\omega), \quad \forall t \ge 0,\ x,y \in \mathbb{R}^3 \tag{1.1.8}$$

rigorously holds provided some technical but highly satisfying conditions hold on V. In particular, there is a measure supported on the space $\mathcal{C}_{xy}$ identified with the space $C(\mathbb{R},\mathbb{R}^3)$ of continuous functions $\mathbb{R} \to \mathbb{R}^3$, and it can be identified as *Wiener measure* $\mathcal{W}$ conditional on paths leaving from x at time 0 and ending in y at time t. In other words, the kernel of the semigroup at the left-hand side (the object corresponding to $G(x,y;0,t)$ above) can be studied by running a Brownian motion driven by the potential V.

Kac has actually proved [293] that the *heat equation with dissipation*

$$\frac{\partial f}{\partial t} = \frac{1}{2}\Delta f - Vf, \quad \psi(x,0) = \phi(x), \tag{1.1.9}$$

is solved by the function

$$f(x,t) = \int_{C(\mathbb{R},\mathbb{R}^3)} e^{-\int_0^t V(B_s(\omega))\,ds} \phi(B_t(\omega))\, d\mathcal{W}^x(\omega). \tag{1.1.10}$$

This equation is actually the same as the Schrödinger equation (1.1.1) when analytic continuation $t \mapsto -it$ is made, a fact that has been noticed by Ehrenfest as early as 1927. In particular, on setting $V = 0$ and formally replacing ϕ with a δ-distribution, the fundamental solution $\Pi(x,t)$ of the standard heat equation is obtained. The relation of this so called *heat kernel*

$$\Pi(x,t) = \frac{1}{(2\pi t)^{3/2}} \exp\left(-\frac{|x|^2}{2t}\right), \quad t > 0, \tag{1.1.11}$$

is once again strong with random processes as it gives the transition probabilities of Brownian motion. The Feynman–Kac formula makes thus a link between an operator semigroup, the heat equation, and Brownian motion. This observation has opened up a whole new conceptual and technical framework pointing to and exploring the rich interplay between self-adjoint operators, probability and partial differential equations.

1.2 Plan and scope

The Feynman–Kac formula for the operator $H = -\frac{1}{2}\Delta + V$ derived from the heat equation with dissipation can be written in any dimension d as

$$(e^{-tH} f)(x) = \int_{C(\mathbb{R},\mathbb{R}^d)} e^{-\int_0^t V(B_s(\omega))\,ds} f(B_t(\omega))\, d\mathcal{W}^x(\omega)$$

where $(B_t)_{t\geq 0}$ is accordingly an $\mathbb{R}^d$-valued Brownian motion. Due to the connections outlined above, the operator H is called a *Schrödinger operator* with potential V. If the potential is such that H admits an eigenfunction ψ at eigenvalue E, then $e^{-tH}\psi = e^{-Et}\psi$ and thus this eigenfunction can be represented in terms of an average over the paths of Brownian motion given by the right-hand side. This makes possible to obtain information on the spectral properties of H by probabilistic means. In particular, the Feynman–Kac formula can be used to study the *ground states* (i.e., eigenfunctions at the bottom of the spectrum) of Schrödinger operators.

It is natural to ask the question whether it is possible to derive representations similar to the Feynman–Kac formula for any other operator. In the pages below we will show that also other operators, derived from quantum field theory, allow to derive Feynman–Kac-type formulae with appropriately modified random processes at the right-hand side. On the one hand, this relation will allow us to ask and answer questions relevant to quantum field theory in a rigorous way. On the other hand, we will face situations in which new concepts and methods need to be developed, which thus generates new mathematics.

We split the material of this monograph in two parts of comparable size, each focussing on one aspect explained above. In Part I we discuss the mathematical framework and in Part II we present the applications in rigorous quantum field theory.

Part I is called *Feynman–Kac-type theorems and Gibbs measures* and it consists of four chapters.

In Chapter 1 we outline the origins of the idea of a path integral applied in quantum mechanics and quantum field theory, due to the physicist Richard P Feynman, and formulated in a mathematically rigorous way by the mathematician Mark Kac.

In Chapter 2 we offer background material from stochastic analysis starting from a review of Brownian motion, Markov processes and martingales. We devote a section to basic concepts of Itô integration theory including some examples, and we conclude by a review of Lévy processes and subordinate Brownian motion.

In Chapter 3 we start by a review of background material of the theory of self-adjoint operators and their semigroups, with a special view on Schrödinger operators. As in the previous chapter we present established material in the light of our goals. The remainder of the chapter deals with a systematic presentation of versions of the Feynman–Kac formula. We cover cases of increasing complexity starting from an external potential V, through including vector potentials describing interactions with magnetic fields, spin, and relativistic effects. Purely mathematically this amounts to saying that we proceed from operators consisting of the sum of the negative Laplacian and a multiplication operator of varying regularity, through including gradient operator terms, and pseudo-differential operators such as the square root of the Laplacian and related operators. From the perspective of random processes involved, we proceed from Brownian motion to diffusions ($P(\phi)_1$-processes and Itô diffusions), and subordinate Brownian motion (relativistic stable processes). Apart from establishing the variants of the Feynman–Kac formula for these choices of operators and processes, we also address their applications in the study of analytic and spectral properties of these operators (properties of integral kernels, positivity improving properties, L^p-smoothing properties, intrinsic utracontractivity, uniqueness and multiplicity of eigenfunctions, decay of eigenfunctions, number of eigenfunctions with negative eigenvalues, diamagnetic inequalities, spectral comparison inequalities etc). From the perspective of quantum mechanics, these results translate into the ground state properties of non-relativistic or relativistic quantum particles with or without spin.

In Chapter 4 we focus on the "right-hand side" of the Feynman–Kac formula and its variants. In all cases that we consider the kernel of the operator semigroup can be expressed in terms of an expectation over the paths of a random process with respect to a measure weighted by exponential densities dependent on functionals additive in the time intervals. This structure suggests to view this as a new probability measure which can be interpreted as a Gibbs measure on path space for the potential(s) appearing directly in or derived from the generator of the process. Gibbs measures are well understood in discrete stochastic models, and originate from statistical mechanics where they have been used to model thermodynamic equilibrium states. In our context this interpretation is not relevant, however, the so obtained Gibbs measures provide a tool for studying analytic and spectral properties of the operators standing at the left-hand side of the Feynman–Kac formula. In this chapter we define and prove the existence and properties of Gibbs measures on path space. First we present this for the case of external potentials and define the framework of $P(\phi)_1$-processes. Then we establish existence and other results (uniqueness, almost sure path behaviour, mixing properties etc) for the case when the densities of the Gibbs measure also contain a pair interaction potential. Finally, we present the special case when the external potential is zero, and discuss Gibbs measures on the Brownian increment process, showing an emerging diffusive behaviour under proper scaling.

Part II of the monograph is called *Rigorous quantum field theory* and consists of three chapters.

In Chapter 5 we discuss the representations of the free boson field Hamilton operator. First we discuss it in Fock space, which is the natural setting for the description of quantum Hamiltonians in terms of self-adjoint operators. Then we introduce suitable function spaces in which the operator semigroups will be mapped by using the Wiener–Itô–Segal isomorphism, and discuss the basic theorems of Bochner and of Minlos on the existence of an infinite Gaussian measure on these spaces. Then we establish the functional integral representation of the free field Hamiltonian on this space. Also, we establish the Feynman–Kac–Nelson formula, which is a representation of the free quantum field in terms of the path measure of an infinite dimensional Ornstein–Uhlenbeck process, and we discuss the Markov property which is crucial in this context.

In Chapter 6 we make a detailed study in terms of path measures of Nelson's model of an electrically charged spinless quantum particle coupled to a scalar boson field. First we formulate the Nelson model in Fock space and discuss the infrared and ultraviolet cutoffs which we need to impose to define the model rigorously. Then we formulate the Nelson model in function space by using the same Wiener–Itô–Segal isomorphism as for the free quantum field previously. We present a proof of existence and uniqueness of a ground state. Next we establish a path measure representation of the Nelson Hamiltonian and identify a Gibbs measure on path space which can be associated to it. This Gibbs measure is defined for a given external potential, and a pair interaction potential arising as an effective potential obtained after integrating over the quantum field. Using the Gibbs measure we derive and prove detailed properties of the ground state, in particular, we discuss ground state expectations for the field operators, establish exponential spatial localization of the particle etc. We furthermore discuss the case of the translation invariant Nelson model obtained when the external potential is zero. We conclude the chapter by discussing infrared divergence manifested in the absence of a ground state, and ultraviolet divergence appearing in the point charge limit.

In Chapter 7 we carry out a similar study in terms of path measures of the Pauli–Fierz model. This model is obtained from the quantization of the Maxwell (vector) field. As before, we start by a formulation of the model in terms of Fock space operators. Next we derive and prove a functional integral representation of the operator semigroup generated by the Pauli–Fierz Hamiltonian. As in the Nelson model, also in this case we obtain a pair interaction potential through the functional integral, however, the double Riemann integrals seen previously are replaced by a double stochastic integral. We define this integral and will be able to extend the functional integral representation to a large class of external potentials. As applications of functional integration first of all we establish self-adjointness of the Pauli–Fierz operator. Also, we prove uniqueness of its ground state by exploring the positivity properties that the integral yields, and discuss its spatial decay. We will use the functional integral formula in order to define the Pauli–Fierz operator for Kato-decomposable potentials as a self-adjoint operator. Next we discuss the translation invariant case when the external potential is zero. In a next step by using the appropriate function spaces we establish a path integral representation of the Pauli–Fierz operator, in which the pair interaction potential becomes a twice iterated stochastic integral. Then we turn to the case of the relativistic Pauli–Fierz model, whose functional integral representation involves subordinate Brownian motion. Finally we discuss the Pauli–Fierz model with spin, and the spin-boson model as an application.

In a concluding section we offer a commented guide to the selected bibliography. Undoubtedly, our list of references can be further increased, however, in lack of space we need to defer this to the reader.

Chapter 2
Probabilistic preliminaries

2.1 An invitation to Brownian motion

A random process is a mathematical model of a phenomenon evolving in time in a way which cannot be predicted with certainty. Brownian motion is a basic random process playing a central role throughout below. We offer here a brief summary of the mathematical theory of Brownian motion as background material for a reasonably self-contained presentation of what follows.

Let $(\Omega, \mathscr{F})$ be a measurable space, where Ω is a set and $\mathscr{F}$ a σ-field. When Ω is equipped with a topology, its *Borel σ-field* will be denoted by $\mathcal{B}(\Omega)$ and then $(\Omega, \mathcal{B}(\Omega))$ is called a *Borel measurable space*. A measurable space $(\Omega, \mathscr{F})$ is *separable* if and only if $\mathscr{F}$ is generated by a countable collection of subsets in Ω. In particular, if Ω is a locally compact space with a countable basis, then $(\Omega, \mathcal{B}(\Omega))$ is separable. Throughout we will assume that Borel measurable spaces are separable.

With measurable spaces $(\Omega, \mathscr{F})$ and $(S, \mathscr{S})$ we write $f \in \mathscr{F}/\mathscr{S}$ to indicate that the function $f : \Omega \to S$ is *measurable* with respect to $\mathscr{F}$ and $\mathscr{S}$, i.e., $f^{-1}(E) \in \mathscr{F}$ for all $E \in \mathscr{S}$. When $f \in \mathscr{F}/\mathcal{B}(\mathbb{R}^d)$, f is called a *Borel measurable function*. A measurable space equipped with a measure μ is a *measure space* denoted by $(\Omega, \mathscr{F}, \mu)$. One can define the integral of a Borel measurable function f on $(\Omega, \mathscr{F}, \mu)$ which will be denoted by $\int_\Omega f d\mu$.

A *probability space* is a measurable space $(\Omega, \mathscr{F}, P)$ in which Ω is called *sample space*, $\mathscr{F}$ is called *algebra of events* and P is a *probability measure* on $(\Omega, \mathscr{F})$. In this case the integral with respect to P of a Borel measurable function f is called *expectation* of f and will be denoted by any of the symbols

$$\int_\Omega f(\omega) dP(\omega) = \int_\Omega f dP = \mathbb{E}_P[f].$$

The probability of event $\{\omega \in \Omega \,|\, \text{conditions}\}$ is simply denoted by

$$P(\text{conditions}).$$

A measurable function X from a probability space $(\Omega, \mathscr{F}, P)$ to a measurable space $(S, \mathscr{S})$ is called an S-valued *random variable*. The inverse map identifies the sample points in Ω on which the observation of event A depends. Let X be an S-valued random variable. The measure P_X on $(S, \mathscr{S})$ defined by

$$P_X(A) = P(X^{-1}(A)), \quad A \in \mathscr{S},$$

is called the *image measure* or the *distribution* of P under X. By the definition of the distribution of $X : \Omega \to S$ we have

$$\mathbb{E}_P[f(X)] = \int_S f(x) dP_X(x) \tag{2.1.1}$$

for a bounded Borel measurable function $f : S \to \mathbb{R}^d$. The *covariance* of two $\mathbb{R}^d$-valued random variables X and Y on the same probability space is defined by

$$\operatorname{cov}(X, Y) = \mathbb{E}_P[(X - m_X)(Y - m_Y)], \tag{2.1.2}$$

where $m_X = \mathbb{E}_P[X]$ and $m_Y = \mathbb{E}_P[Y]$ are the expectations of X and Y, respectively. For two S-valued random variables X and Y, not necessarily defined on the same probability space, we use the notation $X \stackrel{\mathrm{d}}{=} Y$ whenever they are *identically distributed*, i.e., $P_X = P_Y$.

A random variable $X : \Omega \to \mathbb{R}$ with distribution

$$dP_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-m)^2}{2\sigma^2}} dx$$

is called a *Gaussian random variable* and the notation $X \stackrel{\mathrm{d}}{=} N(m, \sigma^2)$ is used. An $N(0, 1)$-distributed random variable is a *standard Gaussian random variable*. A random vector $X = (X_1, \ldots, X_d) : \Omega \to \mathbb{R}^d$ is a *multivariate Gaussian random variable* if its distribution is given by

$$dP_X(x) = \frac{1}{(2\pi)^{d/2} (\det C)^{1/2}} e^{-\frac{1}{2}(x-m)\cdot C^{-1}(x-m)} dx,$$

where $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$, $C = (C_{ij})$ is a $d \times d$ symmetric, strictly positive definite matrix and $m = (m_1, \ldots, m_d) \in \mathbb{R}^d$ is a vector. The matrix element C_{ij} gives the covariance $\operatorname{cov}(X_i, X_j)$ of the multivariate Gaussian random variable, and $m_j = \mathbb{E}_P[X_j]$.

We summarize the notions of convergence of random variables next.

Definition 2.1 (Convergence of random variables). Let $(X_n)_{n \in \mathbb{N}}$ be a sequence of real-valued random variables and X another real-valued random variable, all on a given probability space $(\Omega, \mathscr{F}, P)$.

(1) $X_n \stackrel{\text{a.s.}}{\to} X$, i.e., X_n is convergent *almost surely* to X if

$$P(\lim_{n \to \infty} X_n(\omega) = X(\omega)) = 1.$$

(2) $X_n \stackrel{L^p}{\to} X$, i.e., X_n is convergent *in L^p-sense* to X for $1 \le p < \infty$ if

$$\lim_{n \to \infty} \mathbb{E}_P[|X_n - X|^p] = 0.$$

(3) $X_n \xrightarrow{P} X$, i.e., X_n is convergent *in probability* to X if for every $\varepsilon > 0$,

$$\lim_{n\to\infty} P\left(|X_n(\omega) - X(\omega)| \geq \varepsilon\right) = 0.$$

(4) $X_n \xrightarrow{\mathrm{d}} X$, i.e., X_n is convergent *in distribution* or *weakly* to X if

$$\lim_{n\to\infty} \mathbb{E}_P[f(X_n)] = \mathbb{E}_P[f(X)]$$

for any bounded continuous function f.

The relationships of the various types of convergence are given by

Theorem 2.1. *Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of real-valued random variables and X another real-valued random variable, all on $(\Omega, \mathscr{F}, P)$.*

(1) *If $X_n \xrightarrow{\text{a.s.}} X$ or $X_n \xrightarrow{L^p} X$ for some $p \geq 1$, then $X_n \xrightarrow{P} X$ as $n \to \infty$.*

(2) *If $X_n \xrightarrow{P} X$, then $X_n \xrightarrow{\mathrm{d}} X$ as $n \to \infty$.*

(3) *If $X_n \xrightarrow{P} X$, then there exists a subsequence $X_{n_k}, k \geq 1$, such that $X_{n_k} \xrightarrow{\text{a.s.}} X$ as $k \to \infty$.*

(4) (*Dominated convergence theorem*) *Let $X_n \xrightarrow{P} X$ as $n \to \infty$ and suppose there is a random variable Y such that $|X_n| \leq Y$, $n = 1, 2, \ldots$, and $\mathbb{E}[Y] < \infty$. Then $X \in L^1(\Omega, dP)$ and $X_n \xrightarrow{L^1} X$.*

(5) (*Monotone convergence theorem*) *Let $X_n \xrightarrow{\text{a.s.}} X$ as $n \to \infty$. Suppose the sequence $(X_n)_{n\geq 1}$ is monotone and there exists $M > 0$ such that $\mathbb{E}_P[X_n] < M$, for all n. Then $X \in L^1(\Omega, dP)$ and $\mathbb{E}_P[X_n] \to \mathbb{E}_P[X]$ as $n \to \infty$.*

Next we review the convergence properties of probability measures.

Definition 2.2. Let S be a metric space and P a probability measure on $(S, \mathcal{B}(S))$.

(1) A sequence of probability measures $(P_n)_{n\in\mathbb{N}}$ on $(S, \mathcal{B}(S))$ is *weakly convergent* to P whenever

$$\lim_{n\to\infty} \mathbb{E}_{P_n}[f] = \mathbb{E}_P[f] \tag{2.1.3}$$

for every bounded and continuous real-valued function f on S.

(2) A family $\mathscr{P}$ of probability measures on $(S, \mathcal{B}(S))$ is *relatively compact* whenever every sequence of elements of $\mathscr{P}$ contains a weakly convergent subsequence.

(3) A family $\mathscr{P}$ of probability measures on $(S, \mathcal{B}(S))$ is *tight* whenever for every $\varepsilon > 0$ there exists a compact set $K_\varepsilon \subset S$ such that $P(K_\varepsilon) > 1 - \varepsilon$, for every $P \in \mathscr{P}$.

Note that for any given sequence $(X_n)_{n\in\mathbb{N}}$ of random variables on $(\Omega, \mathscr{F}, P)$ with values in a metric space $(S, \mathcal{B}(S))$, the convergence $X_n \xrightarrow{\mathrm{d}} X$ is equivalent to weak convergence of $P \circ X_n^{-1}$ to $P \circ X^{-1}$.

We will be mainly interested in the cases $S = C([0, \infty))$ and $S = C(\mathbb{R})$. Since both are Polish (i.e., complete separable metric) spaces, a family of probability measures $(S, \mathcal{B}(S))$ has a weakly convergent sequence if it is tight. Take $S = C(\mathbb{R})$. Recall that for $T, \delta > 0$ the *modulus of continuity* of $\omega \in S$ is defined by

$$m^T(\omega, \delta) = \max_{\substack{|s-t|\le\delta \\ -T\le s,t\le T}} |\omega(s) - \omega(t)|. \tag{2.1.4}$$

For $S = C([0, \infty))$ the modulus of continuity only modifies by $-T$ being replaced by 0.

Theorem 2.2. *Let S be a Polish space.*

(1) *Prokhorov's theorem: A family $\mathscr{P}$ of probability measures on $(S, \mathcal{B}(S))$ is relatively compact if and only if it is tight.*

(2) *A sequence $(P_n)_{n\in\mathbb{N}}$ of probability measures on $(C(\mathbb{R}), \mathcal{B}(C(\mathbb{R})))$ is tight if and only if*

$$\lim_{\lambda\uparrow\infty} \sup_{n\ge1} P_n(\omega(0) > \lambda) = 0, \tag{2.1.5}$$

$$\lim_{\delta\downarrow0} \sup_{n\ge1} P_n(m^T(\omega, \delta) > \varepsilon) = 0, \quad \forall T > 0, \forall\varepsilon > 0. \tag{2.1.6}$$

Two events $A, B \in \mathscr{F}$ are *independent* if and only if $P(A \cap B) = P(A)P(B)$. Two sub-σ-fields $\mathscr{F}_1, \mathscr{F}_2$ of $\mathscr{F}$ are independent if every $A \in \mathscr{F}_1$ and $B \in \mathscr{F}_2$ are pairwise independent. Let X be a random variable on $(\Omega, \mathscr{F}, P)$. $\sigma(X)$ denotes the minimal σ-field such that X is measurable, i.e.,

$$\sigma(X) = \{X^{-1}(U) | U \in \mathscr{S}\}. \tag{2.1.7}$$

Let Y be also a random variable on $(\Omega, \mathscr{F}, P)$. If the sub-σ-fields $\sigma(X)$ and $\sigma(Y)$ are independent, then X, Y are said to be independent. If X, Y are independent, then $\mathrm{cov}(f(X), g(Y)) = 0$ for all bounded Borel measurable functions f, g. For two Gaussian random variables the converse is also true, i.e., if their covariance is zero, then they are independent.

Let $(\Omega, \mathscr{F}, P)$ be a probability space and J be a given set. A family of S-valued random variables $(X_t)_{t\in J}$ is called a *random process with index set J*. When $S =$

$\mathbb{R}^d$ we refer to it as a real-valued random process no matter the dimension. If J is uncountable, then $(X_t)_{t\geq 0}$ is called a *continuous time random process*. For any fixed $t \in J$ the map $\omega \mapsto X_t(\omega)$ is an S-valued random variable on Ω, while for any fixed $\omega \in \Omega$ the map $t \mapsto X_t(\omega)$ is a function called *path*. This distinction justifies for any random process to address distributional properties on the one hand, and sample path properties on the other.

Definition 2.3 (Brownian motion). A real-valued random process $(B_t)_{t\geq 0}$ on a probability space $(\Omega, \mathscr{F}, P)$ is called *Brownian motion* or *Wiener process* starting at $x \in \mathbb{R}^d$, whenever the following conditions are satisfied:

(1) $P(B_0 = x) = 1$

(2) the increments $(B_{t_i} - B_{t_{i-1}})_{1\leq i\leq n}$ are independent Gaussian random variables for any $0 = t_0 < t_1 < \cdots < t_n$, $n \in \mathbb{N}$, with $B_{t_i} - B_{t_{i-1}} \stackrel{\mathrm{d}}{=} N(0, t_i - t_{i-1})$

(3) the function $t \mapsto B_t(\omega)$ is continuous for almost every $\omega \in \Omega$.

In particular $(B_t)_{t\geq 0}$ is called *standard Brownian motion* when $x = 0$. A d-component vector of independent Brownian motions $(B_t^1, \dots, B_t^d)_{t\geq 0}$ is called *d-dimensional Brownian motion*.

Brownian motion is thus a real-valued continuous time random process with index set $\mathbb{R}^+ = [0, \infty)$. Consider

$$\mathscr{X} = C(\mathbb{R}^+; \mathbb{R}^d), \tag{2.1.8}$$

the space of $\mathbb{R}^d$-valued continuous functions on $\mathbb{R}^+$, equipped with the metric

$$\operatorname{dist}(f, g) = \sum_{k\geq 0} \frac{1}{2^k}\Big(\sup_{0\leq |x|\leq k} |f(x) - g(x)|\Big) \wedge 1.$$

This metric induces the *locally uniform topology* in which $f_n \to f$ if and only if $f_n \to f$ uniformly on every compact set in $\mathbb{R}^+$. The so obtained topology gives rise to the Borel σ-field $\mathscr{B}(\mathscr{X})$.

Proposition 2.3. *The Borel σ-field $\mathscr{B}(\mathscr{X})$ coincides with the σ-field generated by the cylinder sets of the form*

$$A_{t_1,\dots,t_n}^{E_1,\dots,E_n} = \{\omega \in \mathscr{X} \,|\, (\omega(t_1), \dots, \omega(t_n)) \in E_1 \times \cdots \times E_n\}$$

with $E_1, \dots, E_n \in \mathscr{B}(\mathbb{R}^d)$, $0 \leq t_1 < t_2 < \cdots < t_n$ and $n \in \mathbb{N}$.

Next we consider the canonical representation of Brownian motion on the measure space $(\mathscr{X}, \mathscr{B}(\mathscr{X}), \mathcal{W}^x)$, where $\mathcal{W}^x$ is the *Wiener measure* defined below. In this representation a realization of the process $(B_t)_{t\geq 0}$ is given by

$$B_t(\omega) = \omega(t) \in \mathbb{R}^d, \quad \omega \in \mathscr{X} \tag{2.1.9}$$

called *coordinate process*. For each fixed $\omega \in \mathscr{X}$ we think of $\mathbb{R}^+ \ni t \mapsto B_t \in \mathbb{R}^d$ as a *Brownian path*. Write

$$\Pi_t(x) = \frac{1}{(2\pi t)^{d/2}} \exp\left(-\frac{|x|^2}{2t}\right), \quad t > 0, \ x \in \mathbb{R}^d \tag{2.1.10}$$

called d-dimensional *heat kernel*.

Definition 2.4 (Wiener measure). Let $0 = t_0 < t_1 < \cdots < t_n$, $E_j \in \mathscr{B}(\mathbb{R}^d)$, $j = 1, \ldots, n$, $n \in \mathbb{N}$, and 1_E be the indicator function of $E \in \mathscr{B}(\mathbb{R}^d)$. The unique measure $\mathcal{W}^x$ on $(\mathscr{X}, \mathscr{B}(\mathscr{X}))$ whose finite dimensional distributions are given by

$$\mathcal{W}^x(B_0 = x) = 1 \tag{2.1.11}$$

and

$$\begin{aligned} &\mathcal{W}^x(B_{t_1} \in E_1, \ldots, B_{t_n} \in E_n) \\ &= \int_{\mathbb{R}^{dn}} \Big(\prod_{j=1}^n 1_{E_j}(x_j)\Big)\Big(\prod_{j=1}^n \Pi_{t_j - t_{j-1}}(x_{j-1} - x_j)\Big) \prod_{j=1}^n dx_j \end{aligned} \tag{2.1.12}$$

is called *Wiener measure* starting at x. For expectation with respect to Wiener measure we will use the notations

$$\mathbb{E}^x[F] = \mathbb{E}_{\mathcal{W}^x}[F], \quad \mathbb{E}[F] = \mathbb{E}^0[F]. \tag{2.1.13}$$

By the definition of Wiener measure we have

$$\mathbb{E}^x\Big[\prod_{j=1}^n f_j(B_{t_j})\Big] = \int_{\mathbb{R}^{nd}} \Big(\prod_{j=1}^n f_j(x_j)\Big)\Big(\prod_{j=1}^n \Pi_{t_j - t_{j-1}}(x_{j-1} - x_j)\Big) \prod_{j=1}^n dx_j \tag{2.1.14}$$

for all bounded Borel measurable functions $f_1, \ldots, f_n$ with $x_0 = x$. In particular,

$$\mathbb{E}^x\left[f(B_t)\right] = \int_{\mathbb{R}^d} f(y)\Pi_t(x-y)dy. \tag{2.1.15}$$

Recall that the real-valued random processes $(X_t)_{t \in J}$ and $(Y_t)_{t \in J}$ on a probability space $(\Omega, \mathscr{F}, P)$ are *versions* of each other if for each $t \geq 0$ there is a null set $N_t \subset \Omega$ such that $X_t(\omega) = Y_t(\omega)$ for all $\omega \notin N_t$. When $t \mapsto Y_t(\omega)$ is continuous in t for almost every $\omega \in \Omega$, $(Y_t)_{t \in J}$ is called a *continuous version* of $(X_t)_{t \in J}$.

Theorem 2.4. *The Wiener measure $\mathcal{W}^x$ exists on $(\mathscr{X}, \mathscr{B}(\mathscr{X}))$ and the coordinate process $B_t(\omega) = \omega(t)$, $t \geq 0$, on $(\mathscr{X}, \mathscr{B}(\mathscr{X}), \mathcal{W}^x)$ is Brownian motion starting at $x \in \mathbb{R}^d$.*

Proof. The first part of the argument is to prove that there is a unique measure μ on $(\mathbb{R}^{[0,\infty)}, \mathscr{G})$ satisfying (2.1.11) and (2.1.12), where $\mathscr{G}$ is the σ-field generated by the cylinder sets in $\mathbb{R}^{[0,\infty)}$. This can be done by using the Kolmogorov extension theorem quoted in Proposition 2.5 below. The second part is to prove that there exists a continuous version $(\tilde{B}_t)_{t\geq 0}$ of the coordinate process $(B_t)_{t\geq 0}$ on $\mathbb{R}^{[0,\infty)}$ by using the simple criterion given by the Kolmogorov–Čentsov theorem, see Proposition 2.6 below. Note that $(\tilde{B}_t)_{t\geq 0}$ is not the coordinate process, and while $\tilde{B}_t(\omega) \neq \omega(t)$ for every $\omega \in \mathbb{R}^{[0,\infty)}$, it is true that $\tilde{B}_t(\omega) = \omega(t)$, for $\omega \notin N_t$, with a null set N_t depending on t. With the so constructed Brownian motion $(\tilde{B}_t)_{t\geq 0}$ on $(\mathbb{R}^{[0,\infty)}, \mathscr{G}, \mu)$, $\tilde{B}_0 = 0$, let

$$X : (\mathbb{R}^{[0,\infty)}, \mathscr{G}, \mu) \to (\mathscr{X}, \mathscr{B}(\mathscr{X}))$$

be defined by $X_t(\omega) = \tilde{B}_t(\omega)$. Since $\tilde{B}_t(\omega)$ is continuous in t, it can be checked that $X^{-1}(E) \in \mathscr{B}(\mathbb{R}^{[0,\infty)})$ for any cylinder set E. Thus $X \in \mathscr{G}/\mathscr{B}(\mathscr{X})$, since $\mathscr{B}(\mathscr{X})$ coincides with the σ-field generated by the cylinder sets. The image measure μ_X of μ under X on $(\mathscr{X}, \mathscr{B}(\mathscr{X}))$ is denoted by $\mathcal{W}^0$. It is seen that $\mathcal{W}^0$ is Wiener measure with $x = 0$, and the coordinate process $B_t : \omega \in \mathscr{X} \mapsto \omega(t)$ is Brownian motion on $(\mathscr{X}, \mathscr{B}(\mathscr{X}), \mathcal{W}^0)$ starting at 0. Define

$$Y : (\mathscr{X}, \mathscr{B}(\mathscr{X}), \mathcal{W}^0) \to (\mathscr{X}, \mathscr{B}(\mathscr{X}))$$

by $Y_t(\omega) = B_t(\omega) + x$. It can be seen that $Y \in \mathscr{B}(\mathscr{X})/\mathscr{B}(\mathscr{X})$. Then the image measure of $\mathcal{W}^0$ under Y on $(\mathscr{X}, \mathscr{B}(\mathscr{X}))$ is denoted by $\mathcal{W}^x$. This is Wiener measure starting at x, and the coordinate process $B_t(\omega) = \omega(t)$, $t \geq 0$, under $\mathcal{W}^x$ is Brownian motion starting at x. $\square$

Let Ω be a Polish space and J be an index set, $\Lambda \subset J$, $|\Lambda| < \infty$, with $|\cdot|$ denoting the cardinality of a finite set. Consider the family of probability spaces $(\Omega^\Lambda, \mathscr{B}(\Omega^\Lambda), \mu_\Lambda)$ indexed by the finite subsets of J, where $\Omega^\Lambda = \times_{k=1}^{|\Lambda|}\Omega$. An element $\omega \in \Omega^\Lambda$ is regarded as a map $\Lambda \to \Omega$. Take now $\Lambda_1 \subset \Lambda_2 \subset \Lambda_3 \subset J$ and let $\pi_{\Lambda_1\Lambda_2} : \Omega^{\Lambda_2} \to \Omega^{\Lambda_1}$ denote the projection defined by $\pi_{\Lambda_1\Lambda_2}(\omega) = \omega\lceil_{\Lambda_1}$. The relation

$$\mu_{\Lambda_1}(E) = \mu_{\Lambda_2}(\pi^{-1}_{\Lambda_1\Lambda_2}(E)) \tag{2.1.16}$$

for $E \in \mathscr{B}(\Omega^{\Lambda_1})$ and $\Lambda_1 \subset \Lambda_2$ with $|\Lambda_1|, |\Lambda_2| < \infty$, is called *Kolmogorov consistency relation.*

Proposition 2.5 (Kolmogorov extension theorem). *Let* $(\Omega^\Lambda, \mathscr{B}(\Omega^\Lambda), \mu_\Lambda)$, $\Lambda \subset J$ *with* $|\Lambda| < \infty$, *be a family of probability spaces with underlying Polish space* Ω. *Suppose that* μ_Λ *satisfies the consistency relation* (2.1.16)*, and define*

$$\mathscr{A} = \{\pi^{-1}_{\Lambda J}(E) \in \Omega^J | \Lambda \subset J,\ |\Lambda| < \infty,\ E \in \mathscr{B}(\Omega^{|\Lambda|})\},$$

where $\pi_{\Lambda J}^{-1}(E) = \{\omega \in \Omega^J | \omega\lceil_\Lambda \in E\}$. *Then there exists a unique probability measure* $\tilde{\mu}$ *on* $(\Omega^J, \sigma(\mathcal{A}))$ *such that* $\mu_\Lambda(E) = \tilde{\mu}(\pi_\Lambda^{-1}(E))$, *for* $E \in \mathcal{B}(\Omega^\Lambda)$ *with* $\Lambda \subset J$ *and* $|\Lambda| < \infty$, *where* $\sigma(\mathcal{A})$ *denotes the minimal* σ*-field containing* $\mathcal{A}$.

To address the second part of the existence theorem first we need the following concept.

Definition 2.5 (Hölder continuity). A function $f : \mathbb{R}^+ \to \mathbb{R}$ is called *locally* γ*-Hölder continuous* at $t \geq 0$ if there exist $\varepsilon, \gamma > 0$ and $C_{s,t} > 0$ such that

$$|f(t) - f(s)| \leq C_{s,t}|t - s|^\gamma,$$

for every $s \geq 0$ such that $|t - s| < \varepsilon$. The number γ is called *Hölder exponent* and $C_{s,t}$ is the *Hölder constant*.

Proposition 2.6 (Kolmogorov–Čentsov theorem). *Let* $(\Omega, \mathcal{F}, P)$ *be a probability space and* $(X_t)_{t\geq 0}$ *a real-valued random process on it. Suppose there exist* $\alpha \geq 1$, $\beta, C > 0$ *such that*

$$\mathbb{E}_P[|X_t - X_s|^\alpha] \leq C|t - s|^{\beta+1}. \tag{2.1.17}$$

Then there exists a version $(Y_t)_{0\leq t\leq T}$ *of* $(X_t)_{0\leq t\leq T}$ *for every* $T \geq 0$ *with* γ*-Hölder continuous paths, with* $0 \leq \gamma \leq \beta/\alpha$.

Using a version copes with the fact that for a continuous time process path continuity is not a measurable event since, intuitively, to know about a path that it is continuous would need uncountably many observations and the countably generated field of cylinder sets cannot accommodate them.

By the definition of Brownian motion it is straightforward to see that standard Brownian motion is a multivariate Gaussian process with

$$\mathbb{E}[B_t] = 0,$$
$$\mathbb{E}[B_s B_t] = s \wedge t,$$

and an easy computation gives

$$\mathbb{E}[(B_t - B_s)^{2n}] = \frac{(2n)!}{n!2^n}|t - s|^n, \quad n \in \mathbb{N}, \tag{2.1.18}$$

while all odd order moments of the increments of Brownian motion vanish. When applying Proposition 2.6 to Brownian motion one can make use of (2.1.18) and take $\alpha = 2k$, $\beta = k - 1$, $k = 1, 2, \ldots$, to construct a γ-Hölder continuous version for every $0 < \gamma < (k - 1)/(2k)$. Optimizing over k implies Hölder continuity for any $\gamma = (0, 1/2)$.

In the following we briefly review some basic properties of Brownian motion starting with the distributional properties.

Proposition 2.7 (Symmetry properties). *Let $(B_t)_{t\geq 0}$ be standard Brownian motion. The following symmetry properties hold:*

(1) (*spatial homogeneity*) *if $(B_t^x)_{t\geq 0}$ is Brownian motion starting at $x \in \mathbb{R}$, then* $B_t + x \overset{\mathrm{d}}{=} B_t^x$

(2) (*reflection symmetry*) $-B_t \overset{\mathrm{d}}{=} B_t$

(3) (*self-similarity*) $\sqrt{c}B_{t/c} \overset{\mathrm{d}}{=} B_t$, *for all* $c > 0$

(4) (*time inversion*) *let* $Z_t(\omega) = \begin{cases} 0 & \textit{if } t = 0, \\ tB_{1/t}(\omega) & \textit{if } t > 0, \end{cases}$ *then* $Z_t \overset{\mathrm{d}}{=} B_t$

(5) *(time reversibility)* $B_{t-s} - B_t \overset{\mathrm{d}}{=} B_s$, *for all* $0 \leq s \leq t$.

Proof. These equalities easily follow from (2.1.14). The fact that $Z_0 = 0$ a.s. is an application of the strong law of large numbers quoted below. □

Next we discuss sample path properties of Brownian motion without proof.

Proposition 2.8 (Global path properties). *Let $(B_t)_{t\geq 0}$ be standard Brownian motion. Then the following hold almost surely:*

(1) $\liminf_{t\to\infty} B_t = -\infty$ *and* $\limsup_{t\to\infty} B_t = \infty$

(2) $\liminf_{t\to\infty} B_t/\sqrt{t} = -\infty$ *and* $\limsup_{t\to\infty} B_t/\sqrt{t} = \infty$

(3) (*strong law of large numbers*) $B_t/t \overset{\text{a.s.}}{\to} 0$ *as* $t \to \infty$

(4) (*law of the iterated logarithm*)

$$\limsup_{t\to\infty} B_t/\sqrt{2t\log\log t} = 1 \quad \textit{and} \quad \liminf_{t\to\infty} B_t/\sqrt{2t\log\log t} = -1,$$

$$\limsup_{t\downarrow 0} B_t/\sqrt{2t\log\log(1/t)} = 1 \quad \textit{and} \quad \liminf_{t\downarrow 0} B_t/\sqrt{2t\log\log(1/t)} = -1.$$

Proposition 2.9 (Local path properties). *Let $(B_t)_{t\geq 0}$ be standard Brownian motion. Then the following hold:*

(1) (*Hölder continuity*) *For every $T > 0$, $\gamma \in (0, 1/2)$ and almost every $\omega \in \mathscr{X}$ there is $\kappa(T,\gamma,\omega) > 0$ such that*

$$|B_t(\omega) - B_s(\omega)| \leq \kappa(T,\gamma,\omega)|t-s|^\gamma, \quad 0 \leq s,t \leq T.$$

(2) (*Nowhere differentiability*) $t \mapsto B_t$ *is almost surely at no point Hölder continuous for any $\gamma > 1/2$, hence nowhere differentiable.*

(3) (*p-variation*) *Let $p > 0$ and $\Delta_n = \{t_0 < t_1 < \cdots < t_n\}$ be a partition of a bounded interval $I \subset \mathbb{R}$ such that t_0 and t_n coincide with the two endpoints of I, with mesh $m(\Delta_n) = \max_{k=0,\dots,n-1} |t_{k+1} - t_k|$. Define*

$$V^p_{I,\Delta_n}(\omega) = \sum_{k=0}^{n-1} |B_{t_{k+1}}(\omega) - B_{t_k}(\omega)|^p.$$

Then for every bounded interval I and d-dimensional Brownian motion

(a) $\limsup_{m(\Delta_n)\to 0} V^p_{I,\Delta_n}(\omega) = \begin{cases} d|I|, & p = 2, \\ 0, & p > 2, \end{cases}$ *in $L^P(\mathscr{X}, d\mathcal{W}^x)$-sense*

(b) $\limsup_{m(\Delta_n)\to 0} V^p_{I,\Delta_n}$ *is almost surely bounded if $p > 2$, and almost surely unbounded if $p \le 2$.*

By a similar construction as above Brownian motion can be extended over the whole time-line $\mathbb{R}$ instead of defining it only on $\mathbb{R}^+$. This can be done on the measurable space $(\mathscr{Y}, \mathcal{B}(\mathscr{Y}))$, with $\mathscr{Y} = C(\mathbb{R}; \mathbb{R}^d)$. Consider $\tilde{\mathscr{X}} = \mathscr{X} \times \mathscr{X}$ and $\mu^x = \mathcal{W}^x \times \mathcal{W}^x$. Let $\omega = (\omega_1, \omega_2) \in \tilde{\mathscr{X}}$ and define

$$\tilde{B}_t(\omega) = \begin{cases} \omega_1(t) & \text{if } t \ge 0 \\ \omega_2(-t) & \text{if } t < 0. \end{cases}$$

Since $\tilde{B}_t(\omega)$ is continuous in $t \in \mathbb{R}$ under μ^x, $X : (\tilde{\mathscr{X}}, \mathcal{B}(\tilde{\mathscr{X}})) \to (\mathscr{Y}, \mathcal{B}(\mathscr{Y}))$ can be defined by $X_t(\omega) = \tilde{B}_t(\omega)$. It can be seen that $X \in \mathcal{B}(\tilde{\mathscr{X}})/\mathcal{B}(\mathscr{Y})$ by showing that $X^{-1}(E) \in \mathcal{B}(\tilde{\mathscr{X}})$, for any cylinder set $E \in \mathcal{B}(\mathscr{Y})$. Thus X is a $\mathscr{Y}$-valued random variable on $\tilde{\mathscr{X}}$. Denote the image measure of μ^x under X on $(\mathscr{Y}, \mathcal{B}(\mathscr{Y}))$ by $\tilde{\mathcal{W}}^x$. The coordinate process denoted by the same symbol

$$\tilde{B}_t : \omega \in \mathscr{Y} \mapsto \omega(t) \in \mathbb{R}^d \tag{2.1.19}$$

is *Brownian motion over* $\mathbb{R}$ on $(\mathscr{Y}, \mathcal{B}(\mathscr{Y}), \tilde{\mathcal{W}}^x)$ with $\tilde{B}_0 = x$ almost surely. We will denote the path space of two sided Brownian motion by $\mathfrak{X} = C(\mathbb{R}, \mathbb{R}^d)$. The properties of Brownian motion on the whole real line can be summarized as follows.

Proposition 2.10 (Brownian motion on $\mathbb{R}$). (1) $\mathcal{W}^x(\tilde{B}_0 = x) = 1$

(2) *the increments $(\tilde{B}_{t_i} - \tilde{B}_{t_{i-1}})_{1\le i\le n}$ are independent Gaussian random variables for any $0 = t_0 < t_1 < \cdots < t_n$, $n \in \mathbb{N}$, with $\tilde{B}_t - \tilde{B}_s \stackrel{\mathrm{d}}{=} N(0, t-s)$, for $t > s$*

(3) *the increments $(\tilde{B}_{-t_{i-1}} - \tilde{B}_{-t_i})_{1\le i\le n}$ are independent Gaussian random variables for any $0 = -t_0 > -t_1 > \cdots > -t_n$, $n \in \mathbb{N}$, with $\tilde{B}_{-t} - \tilde{B}_{-s} \stackrel{\mathrm{d}}{=} N(0, s-t)$, for $-t > -s$*

(4) *the function $\mathbb{R} \ni t \mapsto \tilde{B}_t(\omega) \in \mathbb{R}$ is continuous for almost every ω*

(5) *B_t and B_s for $t > 0$ and $s < 0$ are independent.*

It can be checked directly through the finite dimensional distributions (2.1.14) that the joint distribution of $\tilde{B}_{t_0}, \ldots, \tilde{B}_{t_n}$, $-\infty < t_0 < t_1 < \cdots < t_n < \infty$, with respect to $dx \otimes d\tilde{\mathcal{W}}^x$ is invariant under time shift, i.e.,

$$\int_{\mathbb{R}^d} dx \mathbb{E}_{\tilde{\mathcal{W}}^x}\Big[\prod_{i=0}^{n} f_i(\tilde{B}_{t_i})\Big] = \int_{\mathbb{R}^d} dx \mathbb{E}_{\tilde{\mathcal{W}}^x}\Big[\prod_{i=0}^{n} f_i(\tilde{B}_{t_i+s})\Big] \tag{2.1.20}$$

for all $s \in \mathbb{R}$. Moreover, the left-hand side of (2.1.20) can be expressed in terms of Wiener measure on $\mathscr{X}$ as

$$\int_{\mathbb{R}^d} dx \mathbb{E}_{\tilde{\mathcal{W}}^x}\left[\prod_{i=0}^{n} f_i(\tilde{B}_{t_i})\right] = \int_{\mathbb{R}^d} dx \mathbb{E}_{\mathcal{W}^x}\left[\prod_{i=0}^{n} f_i(B_{t_i-t_0})\right]. \tag{2.1.21}$$

2.2 Martingale and Markov properties

2.2.1 Martingale property

Let $\mathscr{G} \subset \mathscr{F}$ be a sub-σ-field and X a random variable on $(\Omega, \mathscr{F}, P)$. Recall that the *conditional expectation* $\mathbb{E}_P[X|\mathscr{G}]$ is defined as the unique $\mathscr{G}$-measurable random variable such that

$$\mathbb{E}_P[1_A X] = \mathbb{E}_P[1_A \mathbb{E}_P[X|\mathscr{G}]], \quad A \in \mathscr{G}. \tag{2.2.1}$$

The left-hand side of (2.2.1) defines a measure $\tilde{P}(A) = \mathbb{E}_P[1_A X]$ on $\mathscr{G}$ and $\mathbb{E}_P[X|\mathscr{G}]$ is in fact the *Radon–Nikodým derivative* $d\tilde{P}/dP$. The next lemma is of key importance.

Lemma 2.11. *Let $(\Omega, \mathscr{F}, P)$ be a probability space and $(S, \mathscr{S})$ a measurable space. Let, moreover, X be an S-valued random variable on Ω and Y a real-valued and $\sigma(X)$-measurable random variable on Ω. Then there exists an $\mathscr{S}$-measurable function f on S such that $Y = f \circ X$ a.e.*

Proof. It suffices to show the lemma for non-negative Y. Let $\mathscr{L}$ be the set of real-valued $\sigma(X)$-measurable random variables on Ω satisfying the assumption of the lemma. If $A \in \sigma(X)$, then $1_A \in \mathscr{L}$ and all $\sigma(X)$-measurable step functions belong to $\mathscr{L}$. Since Y is $\sigma(X)$-measurable, there exists a sequence of increasing step functions ϕ_n such that $\phi_n \to Y$ almost surely. Since $\phi_n \in \mathscr{L}$, there exists a sequence of $\mathscr{S}$-measurable functions f_n such that $\phi_n = f_n \circ X$. Define the function f on S by $f(x) = \limsup_{n\to\infty} f_n(x)$ for $x \in S$. Then f is $\mathscr{S}$-measurable and $f(X(\omega)) = \limsup_{n\to\infty} f_n \circ X(\omega) = \lim_{n\to\infty} f_n \circ X(\omega) = Y(\omega)$, and hence $Y \in \mathscr{L}$. $\square$

Let $(X_t)_{t\geq 0}$ be a random process on $(\Omega, \mathscr{F}, P)$. Then $\mathbb{E}_P[f|\sigma(X_t)]$ is a $\sigma(X_t)$-measurable function and by Lemma 2.11 it can be expressed as $\mathbb{E}_P[f|\sigma(X_t)] = h(X_t)$ with a Borel measurable function h. This function is usually denoted by

$$h(x) = \mathbb{E}_P[f|X_t = x]. \tag{2.2.2}$$

Note that if $h(X_t) = f(X_t) = \mathbb{E}_P[f|\sigma(X_t)]$ a.s., then $h(x) = f(x)$ for $P \circ X_t^{-1}$-a.e. $x \in \mathbb{R}$, so the ambiguity in the expression (2.2.2) has measure zero under $P \circ X_t^{-1}$.

Since $\mathbb{E}_P[f(X_0)g(X_t)] = \mathbb{E}_P[f(X_0)\mathbb{E}_P[g(X_t)|\sigma(X_0)]]$, using the notation (2.2.2) we have

$$\mathbb{E}_P[f(X_0)g(X_t)] = \int_{\mathbb{R}^d} \mu_0(dx) f(x) \mathbb{E}_P[g(X_t)|X_0 = x], \tag{2.2.3}$$

where $\mu_0(dx)$ denotes the distribution of X_0 on $\mathbb{R}$. The conditional expectation $\mathbb{E}_P[1_A|\mathscr{G}]$ is in particular denoted by $P(A|\mathscr{G})$.

It is not obvious that $P(\cdot|\mathscr{G})(\omega)$ defines a probability measure for almost every ω, since $P(A|\mathscr{G})(\cdot)$ is defined almost surely for each A. Let $P(A|\mathscr{G})(\omega)$ be defined on $\omega \in \Omega \setminus N_A$ with $P(N_A) = 0$. Then $P(\cdot|\mathscr{G})(\omega) : \mathscr{F} \to [0, 1]$ is defined only for $\omega \in \Omega \setminus \bigcup_{A\in\mathscr{F}} N_A$. It is, however, in general not clear that $P(\bigcup_{A\in\mathscr{F}} N_A) = 0$. This justifies the following definition.

Definition 2.6 (Regular conditional probability measure). Let $(\Omega, \mathscr{F}, P)$ be a probability space, and $\mathscr{G}$ a sub-σ-field of $\mathscr{F}$. A function $Q : \Omega \times \mathscr{F} \to [0, 1]$ is called a *regular conditional probability measure* for $\mathscr{F}$ given $\mathscr{G}$ if

(1) $Q(\omega, \cdot)$ is a probability measure on $(\Omega, \mathscr{F})$ for every $\omega \in \Omega$

(2) $Q(\cdot, A)$ is $\mathscr{G}$-measurable for every $A \in \mathscr{F}$

(3) $Q(\omega, A) = P(A|\mathscr{G})(\omega)$, P-a.s. $\omega \in \Omega$, for every $A \in \mathscr{F}$.

We will be interested in the case when the sub-σ-field is generated by a random variable X, which makes worthwhile to rephrase this definition. We use the notation $\mathbb{E}[1_A|X = x] = P(A|X = x)$.

Definition 2.7 (Regular conditional probability measure). Let $(\Omega, \mathscr{F}, P)$ be a probability space and X be a random variable on it with values in a measure space $(S, \mathscr{S})$. A function $Q : S \times \mathscr{F} \to [0, 1]$ is called a *regular conditional probability measure* for $\mathscr{F}$ given X if

(1) $Q(x, \cdot)$ is a probability measure on $(\Omega, \mathscr{F})$ for every $x \in S$

(2) $Q(\cdot, A)$ is $\mathscr{S}$-measurable for every $A \in \mathscr{F}$

(3) $Q(x, A) = P(A|X = x)$, $P \circ X^{-1}$-a.s., for every $A \in \mathscr{F}$.

Theorem 2.12. *Suppose that Ω is a Polish space and $\mathscr{F} = \mathcal{B}(\Omega)$.*

(1) *Let P be a probability measure on $(\Omega, \mathscr{F})$ and $\mathscr{G}$ a sub-σ-field of $\mathscr{F}$. Then there exists a regular conditional probability measure for $\mathscr{F}$ given $\mathscr{G}$, and it is unique.*

(2) *Let X be random variable on $(\Omega, \mathscr{F})$ with values in a measurable space $(S, \mathscr{S})$. Then there exists a regular conditional probability measure for $\mathscr{F}$ given X. Moreover, if Q' is another regular conditional probability measure, then there exists a set $N \in \mathscr{S}$ with $P \circ X^{-1}(N) = 0$, such that $Q(x, A) = Q'(x, A)$, for all $A \in \mathscr{F}$ and every $x \in S \setminus N$.*

(3) *If S is a Polish space and $S = \mathcal{B}(S)$, then the null set $N \in \mathscr{S}$ above can be chosen such that $Q(x, X^{-1}(B)) = 1_B(x)$, for $B \in \mathscr{S}$, $x \in S \setminus N$. In particular, $Q(x, X^{-1}(x)) = 1$, $P \circ X^{-1}$-a.e. $x \in S$, holds.*

In what follows the regular conditional probability measures $Q(\omega, A)$ and $Q(x, A)$ will be denoted by $P(A|\mathscr{G})$ and $P(A|X = x)$, respectively, unless confusion may arise.

Definition 2.8 (Filtration). Let $(\Omega, \mathscr{F}, P)$ be a probability space and $(\mathscr{F}_t)_{t\geq 0}$ a family of sub-σ-fields of $\mathscr{F}$. The collection $(\mathscr{F}_t)_{t\geq 0}$ is called a *filtration* whenever $\mathscr{F}_s \subset \mathscr{F}_t$ for $s \leq t$. The given measurable space endowed with a filtration is called a *filtered space*.

Intuitively, $\mathscr{F}_t$ contains the information known to an observer at time t. Let $(X_s)_{s\geq 0}$ be an S-valued random process. A convenient choice of filtration is the *natural filtration* given by $\mathscr{F}_t^X = \sigma(X_s, 0 \leq s \leq t)$, containing the information accumulated by observing X up to time t.

Definition 2.9 (Adapted process). Let $(\Omega, \mathscr{F})$ be a filtered space, $(\mathscr{F}_t)_{t\geq 0}$ a given filtration and $(X_t)_{t\geq 0}$ a random process. The process is called $(\mathscr{F}_t)_{t\geq 0}$-*adapted* whenever X_t is $\mathscr{F}_t$-measurable for every $t \geq 0$.

If a process $(X_t)_{t\geq 0}$ is $(\mathscr{F}_t)_{t\geq 0}$-adapted it means that the process does not carry more information at time t than $\mathscr{F}_t$. Obviously, $(X_t)_{t\geq 0}$ is adapted to its natural filtration $(\mathscr{F}_t^X)_{t\geq 0}$, or equivalently, the natural filtration $(\mathscr{F}_t^X)_{t\geq 0}$ is the smallest filtration making $(X_t)_{t\geq 0}$ adapted.

Definition 2.10 (Martingale). Let $(\Omega, \mathscr{F}, P)$ be a filtered space with a given filtration $(\mathscr{F}_t)_{t\geq 0}$. The random process $(X_t)_{t\geq 0}$ is an $(\mathscr{F}_t)_{t\geq 0}$-*martingale* whenever

(1) $\mathbb{E}_P[|X_t|] < \infty$, for every $t \geq 0$

(2) $(X_t)_{t\geq 0}$ is $(\mathscr{F}_t)_{t\geq 0}$-adapted

(3) $\mathbb{E}_P[X_t|\mathscr{F}_s] = X_s$, for every $s \leq t$.

We will make use of the next result on martingales which will be quoted here with no proof.

Theorem 2.13 (Martingale inequality). *Let $(X_t)_{t\geq 0}$ be an $(\mathscr{F}_t)_{t\geq 0}$-martingale on $(\Omega, \mathscr{F}, P)$ and suppose that $X_t(\omega)$ is continuous in t, for almost every $\omega \in \Omega$. Then for all $p \geq 1$, $T \geq 0$ and all $\lambda > 0$,*

$$P\Big(\sup_{0\leq t\leq T} |X_t| \geq \lambda\Big) \leq \frac{1}{\lambda^p}\mathbb{E}_P[|X_T|^p]. \tag{2.2.4}$$

Proposition 2.14. *Let $(B_t)_{t\geq 0}$ be Brownian motion on $(\Omega, \mathscr{F}, P)$ and consider its natural filtration $(\mathscr{F}_t^B)_{t\geq 0}$. Then both $(B_t)_{t\geq 0}$ and $(B_t^2 - t)_{t\geq 0}$ are $(\mathscr{F}_t^B)_{t\geq 0}$-martingales.*

Proof. We have that

$$\mathbb{E}_P[B_t|\mathscr{F}_s^B] = \mathbb{E}_P[B_t - B_s|\mathscr{F}_s^B] + \mathbb{E}_P[B_s|\mathscr{F}_s^B] = \mathbb{E}_P[B_t - B_s] + B_s = B_s \tag{2.2.5}$$

by independence of $B_t - B_s$ and B_r for $r \leq s$. By a similar calculation it is easy to show also the second claim. □

Remarkably, there is a strong converse of this proposition.

Proposition 2.15 (Lévy martingale characterization theorem). *Let $(X_t)_{t\geq 0}$ be a random process and $(\mathscr{F}_t^X)_{t\geq 0}$ be its natural filtration. Suppose that*

(1) *$X_0 = 0$ almost surely,*

(2) *the paths $t \mapsto X_t$ are almost surely continuous,*

(3) *both $(X_t)_{t\geq 0}$ and $(X_t^2 - t)_{t\geq 0}$ are $(\mathscr{F}_t^X)_{t\geq 0}$-martingales.*

Then $(X_t)_{t\geq 0}$ is a standard Brownian motion.

There is a useful weakening of the martingale property which removes the condition of integrability in Definition 2.10. To introduce and apply it below, we first need a concept of random times.

Definition 2.11 (Stopping time). Let $(\Omega, \mathscr{F}, P)$ be a filtered space with filtration $(\mathscr{F}_t)_{t\geq 0}$. A random variable $\tau : \Omega \to [0, \infty]$ such that $\{\tau \leq t\} \in \mathscr{F}_t$, for all $t \geq 0$, is called a *stopping time*.

Whenever the filtration is right-continuous, τ is a stopping time if and only if $\{\tau < t\} \in \mathscr{F}_t$, for every $t \geq 0$. An example of a stopping time is the first hitting time of Brownian motion $(B_t)_{t\geq 0}$ of an open (or a closed) set E, i.e., $\tau = \inf\{t \geq 0 : B_t \in E\}$. If $(\tau_n)_{n\in\mathbb{N}}$ is a sequence of stopping times and $\tau_n \downarrow \tau$, then since $\{\tau < t\} = \bigcup_{n\geq 1}\{\tau_n < t\}$, we have that τ also is a stopping time. Similarly, if a sequence of stopping times $(\tau_n)_{n\in\mathbb{N}}$ is such that $\tau_n \uparrow \tau$, then since $\{\tau \leq t\} = \bigcap_{n\geq 1}\{\tau_n < t\}$, again τ is a stopping time.

Notice that $t \wedge \tau$ is a stopping time whenever τ is. We make use of this in the following property.

Definition 2.12 (Local martingale). A random process $(X_t)_{t\geq 0}$ on a filtered probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, P)$ is said to be a *local martingale* if there exists a sequence of stopping times $(\tau_n)_{n\in\mathbb{N}}$ such that $(X_{t\wedge\tau_n})_{t\geq 0}$ is a martingale with respect to the filtration $(\mathscr{F}_{t\wedge\tau_n})_{t\geq 0}$.

Whenever $(X_t)_{t\geq 0}$ has a continuous version, one way of choosing the stopping times is $\tau_n = \inf\{t \geq 0 \,|\, |X_n| > n\}$.

2.2.2 Markov property

In addition to the martingale property Brownian motion possesses the equally important Markov property. Intuitively, this means that the future values of the random process only depend on its present value and do not on its past history.

Definition 2.13 (Markov process). Let $(X_t)_{t\geq 0}$ be an adapted process on a filtered space $(\Omega, \mathscr{F}, P)$ with a filtration $(\mathscr{F}_t)_{t\geq 0}$. $(X_t)_{t\geq 0}$ is a *Markov process* with respect to $(\mathscr{F}_t)_{t\geq 0}$ whenever

$$\mathbb{E}_P\left[f(X_t)|\mathscr{F}_s\right] = \mathbb{E}_P\left[f(X_t)|\sigma(X_s)\right], \quad 0 \leq s \leq t, \tag{2.2.6}$$

for all bounded Borel measurable functions f.

Markov processes can be characterized by their probability transition kernels.

Definition 2.14 (Probability transition kernel). A map $p : \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^d \times \mathscr{B}(\mathbb{R}^d) \to \mathbb{R}$, $(s, t, x, A) \mapsto p(s, t, x, A)$, is called a *probability transition kernel* if

(1) $p(s, t, x, \cdot)$ is a probability measure on $\mathscr{B}(\mathbb{R}^d)$

(2) $p(s, t, \cdot, A)$ is Borel measurable

(3) for every $0 \leq r \leq s \leq t$, the *Chapman–Kolmogorov identity*

$$\int_{\mathbb{R}^d} p(s, t, y, A)\, p(r, s, x, dy) = p(r, t, x, A) \tag{2.2.7}$$

holds.

Let $(X_t)_{t\geq 0}$ be a Markov process. We define

$$p_X(s, t, x, A) = \mathbb{E}_P[1_A(X_t)|X_s = x] \tag{2.2.8}$$

for $A \in \mathscr{B}(\mathbb{R}^d)$, $0 \leq s \leq t < \infty$ and $x \in \mathbb{R}^d$.

Lemma 2.16. *Let $(X_t)_{t\geq 0}$ be a Markov process with a filtration $(\mathscr{F}_t)_{t\geq 0}$. Then p_X is a transition probability kernel.*

Proof. Properties (1) and (2) of Definition 2.13 are straightforward to check, therefore we only show (3). Since $\int_{\mathbb{R}^d} 1_A(y)p_X(r,s,x,dy) = \mathbb{E}_P[1_A(X_s)|X_r = x]$, we have $\int_{\mathbb{R}^d} f(y)p_X(r,s,x,dy) = \mathbb{E}_P[f(X_s)|X_r = x]$, for bounded Borel functions f. Hence

$$\int_{\mathbb{R}^d} p_X(s,t,y,A)p_X(r,s,x,dy) = \mathbb{E}_P[p_X(s,t,X_s,A)|X_r = x]. \tag{2.2.9}$$

Here $p_X(s,t,X_s,A) = p_X(s,t,y,A)$ evaluated at $y = X_s$. Thus the identity

$$p_X(s,t,X_s,A) = \mathbb{E}_P[1_A(X_t)|\sigma(X_s)]$$

follows, and furthermore $p_X(s,t,X_s,A) = \mathbb{E}_P[1_A(X_t)|\mathscr{F}_s]$ by the Markov property. Inserting this into (2.2.9) we obtain

$$\int_{\mathbb{R}^d} p_X(s,t,y,A)p_X(r,s,x,dy) = \mathbb{E}_P[\mathbb{E}_P[1_A(X_t)|\mathscr{F}_s]|X_r = x]. \tag{2.2.10}$$

Since $\sigma(X_r) \subset \mathscr{F}_s$, we have that $\mathbb{E}_P[\mathbb{E}_P[1_A(X_t)|\mathscr{F}_s]|\sigma(X_r)] = \mathbb{E}_P[1_A(X_t)|\sigma(X_r)]$, and therefore

$$\mathbb{E}_P[\mathbb{E}_P[1_A(X_t)|\mathscr{F}_s]|X_r = x] = p_X(r,t,x,A).$$

Combining this with (2.2.10), $\int_{\mathbb{R}^d} p_X(s,t,y,A)p_X(r,s,x,dy) = p_X(r,t,x,A)$ is obtained. □

The measure

$$P^0(A) = P(X_0 \in A) \tag{2.2.11}$$

is the *initial distribution* of the Markov process describing the random variable at $t = 0$. Then the finite dimensional distributions of $(X_t)_{t\geq 0}$ are determined by its probability transition kernel and initial distribution through the formula

$$\begin{aligned} &P(X_{t_0} \in A_0, \dots, X_{t_n} \in A_n) \\ &= \int_{\mathbb{R}^d} \Big(\prod_{i=1}^n 1_{A_i}(x_i)\Big)\Big(\prod_{i=1}^n p(t_{i-1}, t_i, x_{i-1}, dx_i)\Big) 1_{A_0}(x_0) P^0(dx_0), \end{aligned} \tag{2.2.12}$$

for $n = 1, 2, \dots$, and where $t_0 = 0$.

Proposition 2.17. *Let $(X_t)_{t\geq 0}$ be an $\mathbb{R}^d$-valued random process on a probability space $(\Omega, \mathscr{F}, P)$, P^0 a probability measure on $\mathbb{R}^d$ and $p(s,t,x,A)$ a probability transition kernel. Suppose that the finite dimensional distributions of the process are* *given by* (2.2.12). *Then* $(X_t)_{t\geq 0}$ *is a Markov process under the natural filtration with probability transition kernel* $p(s,t,x,A)$.

Proof. It suffices to show that

$$\mathbb{E}_P\Big[1_A(X_{t+s})\prod_{j=0}^{n}1_{A_j}(X_{t_j})\Big]=\mathbb{E}_P\Big[\mathbb{E}_P\left[1_A(X_t)|\sigma(X_s)\right]\prod_{j=0}^{n}1_{A_j}(X_{t_j})\Big] \quad (2.2.13)$$

for every $0=t_0<t_1\leq\cdots\leq t_n\leq t\leq r$ and every $A_j\in\mathcal{B}(\mathbb{R}^d)$, $j=0,1,\ldots,n$. From the identity $\mathbb{E}_P[1_A(X_t)\mathbb{E}_P[f(X_r)|\sigma(X_t)]]=\mathbb{E}_P[1_A(X_t)f(X_r)]$, it follows that

$$\int_{\mathbb{R}^d}1_A(y)\mathbb{E}_P[f(X_r)|X_t=y]P_t(dy)=\int_{\mathbb{R}^d}1_A(y)f(y')p(t,r,y,dy')P_t(dy),$$

where we denote $P_t(dy)=\int_{\mathbb{R}^d}P^0(dy_0)p(0,t,y_0,dy)$ and note that $P_t(dy)$ is the distribution of X_t on $\mathbb{R}^d$. Thus it is seen that

$$\mathbb{E}_P[f(X_r)|X_t=y]=\int_{\mathbb{R}^d}f(y')p(t,r,y,dy'). \quad (2.2.14)$$

By the definition of $\mathbb{E}_P[f(X_r)|X_t=y]$ we have

$$\mathbb{E}_P[f(X_r)|\sigma(X_t)]=\int_{\mathbb{R}^d}f(y)p(t,r,X_t,dy) \quad (2.2.15)$$

and thus

$$\mathbb{E}_P[1_A(X_r)|\sigma(X_t)]=p(t,r,X_t,A). \quad (2.2.16)$$

Using (2.2.16) and the Chapman–Kolmogorov identity, (2.2.13) follows. □

Furthermore we have the following strong description of the converse situation.

Proposition 2.18. *Let* $p(s,t,x,A)$ *be a probability transition kernel and* P^0 *a probability distribution on* $\mathbb{R}^d$. *Then there exists a filtered probability space* $(\Omega,\mathcal{F},P)$ *and a Markov process* $(X_t)_{t\geq 0}$ *on it such that* $\mathbb{E}_P[1_A(X_t)|X_s=x]=p(s,t,x,A)$ *and* $P\circ X_0^{-1}=P^0$.

When the transition probability kernel $p(s,t,X,A)$ of a Markov process satisfies $p(s,t,x,A)=p(0,t-s,x,A)$, the process is said to be *stationary Markov process*. In this case by a minor abuse of notation we write $p(t,x,A)$ for $p(0,t,x,A)$, and the

Chapman–Kolmogorov identity reduces to

$$\int_{\mathbb{R}^d} p(s, y, A)p(t, x, dy) = p(s+t, x, A). \tag{2.2.17}$$

Theorem 2.19. *Let $(B_t)_{t\geq 0}$ be Brownian motion adapted to a filtration $(\mathscr{F}_t)_{t\geq 0}$. Then $(B_t)_{t\geq 0}$ is a stationary Markov process and*

$$\mathbb{E}^x[f(B_{s+t_0}, \dots, B_{s+t_n})|\mathscr{F}_s] = \mathbb{E}^{B_s}\left[f(B_{t_0}, \dots, B_{t_n})\right], \quad \forall x \in \mathbb{R}^d, \tag{2.2.18}$$

holds for every bounded Borel function f. Moreover, its initial distribution $P^0 = \delta_x$ is Dirac measure with mass at x, while its probability transition kernel is given by

$$P_{\mathrm{BM}}(t, x, A) = \int_{\mathbb{R}^d} 1_A(z)\Pi_t(x-z)dz \tag{2.2.19}$$

or equivalently,

$$P_{\mathrm{BM}}(t, x, dy) = \Pi_t(x-y)dy. \tag{2.2.20}$$

Proof. The Markov property with transition probability kernel (2.2.20) follows by the definition of the finite dimensional distributions of Brownian motion and Proposition 2.17. Note that $P_{\mathrm{BM}}(s, t, x, A) = \int_{\mathbb{R}^d} 1_A(z)\Pi_{t-s}(x-z)dz$. We show (2.2.18). Let $h(y) = \mathbb{E}^x[\prod_{j=1}^n f(B_{s+t_j})|B_s = y]$. By the identity

$$\mathbb{E}^x\Big[1_A(B_s)\prod_{j=1}^n f_j(B_{s+t_j})\Big] = \mathbb{E}^x\Big[1_A(B_s)\mathbb{E}^x\Big[\prod_{j=1}^n f_j(B_{s+t_j})|\sigma(B_s)\Big]\Big]$$

we have

$$\int_{\mathbb{R}^{(n+1)d}} 1_A(y)\Pi_s(y-x)\Big(\prod_{j=1}^n f_j(x_j)\Big)\Pi_{t_1}(x_1-y)\Big(\prod_{j=2}^n \Pi_{t_j-t_{j-1}}(x_j-x_{j-1})\Big)dy\prod_{j=1}^n dx_j$$
$$= \int_{\mathbb{R}^d} 1_A(y)h(y)\Pi_s(y-x)dy.$$

Comparing the two sides above

$$h(y) = \int_{\mathbb{R}^{nd}} \Big(\prod_{j=1}^n f_j(x_j)\Big)\Pi_{t_1}(x_1-y)\Big(\prod_{j=2}^n \Pi_{t_j-t_{j-1}}(x_j-x_{j-1})\Big)\prod_{j=1}^n dx_j$$

is obtained, while the right-hand side equals $\mathbb{E}^y[\prod_{j=1}^n f_j(B_{t_j})]$. Hence

$$\mathbb{E}^x\Big[\prod_{j=1}^n f(B_{s+t_j})|\sigma(B_s)\Big] = \mathbb{E}^{B_s}\Big[\prod_{j=1}^n f_j(B_{t_j})\Big]$$

and (2.2.18) follows. □

From the proof of Theorem 2.19 it follows that the Markov property (2.2.6) can be reformulated as

$$\mathbb{E}^x[f(B_t)|\mathscr{F}_s] = \mathbb{E}^{B_s}[f(B_{t-s})], \quad t \geq s. \tag{2.2.21}$$

In what follows, the Markov property of a random process will be considered with respect to its natural filtration, unless otherwise stated.

Corollary 2.20. *Let $\alpha \in \mathbb{R}$. Then the random process $(e^{\alpha B_t - \frac{1}{2}\alpha^2 t})_{t\geq 0}$ is a martingale with respect to the natural filtration of Brownian motion.*

Proof. By (2.2.21) we have

$$\mathbb{E}^x[e^{\alpha B_t - \frac{1}{2}\alpha^2 t}|\mathscr{F}_s^B] = \mathbb{E}^{B_s}[e^{\alpha B_{t-s} - \frac{1}{2}\alpha^2 t}].$$

The right-hand side above can be computed as

$$\mathbb{E}^{B_s}[e^{\alpha B_{t-s} - \frac{1}{2}\alpha^2 t}] = e^{-\frac{1}{2}\alpha^2 t}\int_{\mathbb{R}^d} e^{\alpha y}\Pi_{t-s}(B_s - y)dy = e^{\alpha B_s - \frac{1}{2}\alpha^2 s}$$

and the corollary follows. □

Brownian motion has a yet stronger Markov property than (2.2.18) since the same equality holds even for random times. Given a stopping time τ, we define the σ-field $\mathscr{F}_\tau$ by saying that $A \in \mathscr{F}_\tau$ if and only if $A \in \mathscr{F}$ and $A \cap \{\tau \leq t\} \in \mathscr{F}_t$, for all $t \geq 0$. Intuitively, $\mathscr{F}_\tau$ contains the events occurring before or at the given stopping time.

Theorem 2.21 (Strong Markov property). *Let $(B_t)_{t\geq 0}$ be a Brownian motion and τ a bounded stopping time with respect to the filtration $(\mathscr{F}_t)_{t\geq 0}$. Then*

$$\mathbb{E}^x[B_{t+\tau}|\mathscr{F}_\tau] = \mathbb{E}^{B_\tau}[B_t], \quad t \geq 0,\ x \in \mathbb{R}^d. \tag{2.2.22}$$

The theorem is valid also for unbounded stopping times but we have stated it in this form for simplicity.

2.2.3 Feller transition kernels and generators

Markov processes are intimately related to semigroups, which intuitively describe the time evolution of the probability distributions. Here we need to use operator semigroup theory; for essential background material see Section 3.1.8 below.

Definition 2.15 (Feller transition kernel). A probability transition kernel $p(t,x,A)$ is said to be a *Feller transition kernel* whenever the following conditions hold:

(1) $p(t,x,\mathbb{R}^d) = 1$

(2) $\int_{\mathbb{R}^d} p(t,\cdot,dy) f(y) \in C_\infty(\mathbb{R}^d)$ for $f \in C_\infty(\mathbb{R}^d)$

(3) $\lim_{t\downarrow 0} \int_{\mathbb{R}^d} p(t,x,dy) f(y) = f(x)$, for every $f \in C_\infty(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$.

Consider

$$(P_t f)(x) = \int_{\mathbb{R}^d} p(t,x,dy) f(y). \tag{2.2.23}$$

When $P_t f \geq 0$ for $f \geq 0$, for all $t \geq 0$, the semigroup $\{P_t, t \geq 0\}$ is called *non-negative*. The next proposition gives the relationship between Markov processes and C_0-semigroups.

Proposition 2.22. *Let $p(t,x,A)$ be a Feller transition kernel. Then $\{P_t : t \geq 0\}$ is a non-negative C_0-semigroup on $C_\infty(\mathbb{R}^d)$.*

Proof. It is easy to see that $P_0 = 1$ and P_t is non-negative. $P_t P_s = P_{t+s}$ follows by the Chapman–Kolmogorov identity. Moreover, $\|P_t f\|_\infty \leq \|f\|_\infty$ can also easily be seen. The difficult part is to show the strong continuity, i.e.,

$$\lim_{t\to 0} P_t f = f \tag{2.2.24}$$

in the sup-norm. Let $\alpha > 0$ and define $f_\alpha = \int_0^\infty \alpha e^{-\alpha t} P_t f(x) dt$. Then $f_\alpha \in C_\infty(\mathbb{R}^d)$, since $f \in C_\infty(\mathbb{R}^d)$. Let $\mathscr{M}$ be the closed subspace of $C_\infty(\mathbb{R}^d)$ spanned by f such that (2.2.24) holds. Take $f \geq 0$ and define $f_{\alpha,h}(x) = \int_h^\infty \alpha e^{-\alpha t} P_t f(x) dx$; then $f_{\alpha,h}(x) \uparrow f_\alpha(x)$ as $h \downarrow 0$. This convergence is uniform as the sequence of continuous monotone functions $(f_{\alpha,h})_h$ converges to $f_\alpha(x)$ pointwise on the compact set $\mathbb{R}^d \cup \{\infty\}$. Since $\mathscr{M}$ is closed, $f_\alpha \in \mathscr{M}$ follows for any non-negative $f \in C_\infty(\mathbb{R}^d)$ and therefore for all $f \in C_\infty(\mathbb{R}^d)$. Let $l \in C_\infty(\mathbb{R}^d)^*$ be a linear functional on $C_\infty(\mathbb{R}^d)$. Since $\|f_\alpha\|_\infty \leq \|f\|_\infty$ and $\lim_{\alpha\to\infty} f_\alpha(x) = f(x)$ pointwise, it follows that $f_\alpha \to f$ weakly as $\alpha \to \infty$ and hence $l(f_\alpha) \to l(f)$ as $\alpha \to \infty$. Suppose $l(g) = 0$ for all $g \in \mathscr{M}$. Then for any $f \in C_\infty(\mathbb{R}^d)$, $l(f) = \lim_{\alpha\to\infty} l(f_\alpha) = 0$. Thus the Hahn–Banach theorem yields that $C_\infty(\mathbb{R}^d) = \mathscr{M}$. □

The converse statement of Proposition 2.22 is also true.

Proposition 2.23. *For any non-negative C_0-semigroup there exists a unique Feller transition kernel such that* (2.2.23) *holds.*

Let $(X_t)_{t\geq 0}$ be a Markov process and consider its Feller transition kernel. P_t in (2.2.23) is by Proposition 2.22 a C_0-semigroup and there exists a closed operator L on $C_\infty(\mathbb{R}^d)$ such that

$$\lim_{t\to 0} \frac{1}{t}(P_t f - f) = Lf.$$

L is called the *generator* of the Markov process and is formally written as $P_t = e^{tL}$.

Proposition 2.24. *The probability transition kernel P^{BM} of Brownian motion is a Feller transition kernel and its generator is $-(1/2)\Delta$ on $C_0^\infty(\mathbb{R}^d)$.*

Proof. We have already shown that $p^{\mathrm{BM}}(t,x,A) = \int_{\mathbb{R}^d} 1_A(z)\Pi_t(x-z)dz$ in (2.2.19). Thus $p^{\mathrm{BM}}(t,x,\mathbb{R}^d) = 1$ holds. Since $P_t^{\mathrm{BM}} f(x) = (2\pi t)^{-d/2}\int_{\mathbb{R}^d} e^{-|z|^2/2t} f(x+z)dz$ for $f \in C_\infty(\mathbb{R}^d)$, by dominated convergence it is clear that $P_t^{\mathrm{BM}} f(x)$ is continuous in x and $\lim_{|x|\to\infty} P_t^{\mathrm{BM}} f(x) = 0$. Thus $P_t^{\mathrm{BM}} f \in C_\infty(\mathbb{R}^d)$. Next we check that

$$\lim_{t\to 0} P_t^{\mathrm{BM}} f(x) = f(x). \tag{2.2.25}$$

Notice that f is uniformly continuous on the compact set $\mathbb{R}^d \cup \{\infty\}$. Thus for any $\varepsilon > 0$ there exists $\delta > 0$ such that $|f(x+y) - f(x)| \leq \varepsilon$ for all $x \in \mathbb{R}^d$ and $|y| < \delta$. Hence

$$\begin{aligned}
&|P_t^{\mathrm{BM}} f(x) - f(x)| \\
&\leq \int_{|y|<\delta} |f(x+y) - f(x)|\Pi_t(y)dy + \int_{|y|\geq\delta} |f(x+y) - f(x)|\Pi_t(y)dy \\
&\leq \varepsilon \int_{|y|<\delta} \Pi_t(y)dy + M\int_{|y|\geq\delta} \Pi_t(y)dy,
\end{aligned}$$

where $M = \sup_x |f(x)|$. Thus $\lim_{t\to 0} |P_t^{\mathrm{BM}} f(x) - f(x)| \leq \varepsilon$, whence (2.2.25) follows. Let L_{BM} denote the generator of Brownian motion and take $f \in C_0^\infty(\mathbb{R}^d)$. The Fourier transform of $P_t^{\mathrm{BM}} f$ is equal to $e^{-t|k|^2/2}\hat{f}(k)$. This gives

$$\lim_{t\to 0} \left\| \frac{1}{t}(\widehat{P_t^{\mathrm{BM}} f} - \hat{f}) - \frac{1}{2}|k|^2 \hat{f} \right\| = 0 \tag{2.2.26}$$

and

$$\lim_{t\to 0} \frac{1}{t}(P_t^{\mathrm{BM}} f - f) = -\frac{1}{2}\Delta f \tag{2.2.27}$$

in $L^2(\mathbb{R}^d)$. By taking a subsequence t', (2.2.27) stays valid for almost every $x \in \mathbb{R}^d$ while the right-hand side (2.2.27) is equal to $L_{\mathrm{BM}} f$. Hence $L_{\mathrm{BM}} f = -(1/2)\Delta f$ for almost every x. $\square$

We will see in Section 2.4.2 below that Feller transition kernels form a rich class and further examples of generators will be given.

2.2.4 Conditional Wiener measure

We conclude this section by discussing pinned Brownian motion.

Definition 2.16 (Brownian bridge measure). Given a division $T_1 \le t_1 < \cdots < t_n \le T_2$ and $x, y \in \mathbb{R}^d$, the probability measure on $(\mathscr{X}, \mathscr{B}(\mathscr{X}))$ defined by

$$\begin{aligned} &\mathcal{W}^{x,y}_{[T_1,T_2]}(B_{t_1} \in A_1, \dots, B_{t_n} \in A_n) \qquad (2.2.28)\\ &= \int_{\mathbb{R}^{nd}} \frac{\Pi_{t_1-T_1}(x-x_1)\Pi_{t_2-t_1}(x_1-x_2)\cdots\Pi_{T_2-t_n}(x_n-y)}{\Pi_{T_2-T_1}(x-y)} \Big(\prod_{i=1}^n 1_{A_i}(x_i)\Big)\prod_{i=1}^n dx_i \end{aligned}$$

is called *Brownian bridge measure* starting in x at $t = T_1$ and ending in y at $t = T_2$. Furthermore, the measure

$$\Pi_{T_2-T_1}(x-y)\mathcal{W}^{x,y}_{[T_1,T_2]} \qquad (2.2.29)$$

is called *conditional Wiener measure*.

Note that the Brownian bridge measure is a probability measure, while conditional Wiener measure is not.

Write $\mathbb{E}^{x,y}_{[T_1,T_2]}$ for expectation with respect to $\mathcal{W}^{x,y}_{[T_1,T_2]}$. From the definition it is directly seen that $\mathcal{W}^x$ and $\mathcal{W}^{x,y}_{[T_1,T_2]}$ are related through

$$\mathbb{E}^x\Big[\prod_{j=1}^n f_j(B_{t_j})\Big] = \int_{\mathbb{R}^d} dy \Pi_{T_2-T_1}(x-y)\mathbb{E}^{x,y}_{[T_1,T_2]}\Big[\prod_{j=1}^n f_j(B_{t_j})\Big], \qquad (2.2.30)$$

with $T_1 \le t_1 \le \cdots \le t_n \le T_2$ and any bounded Borel function f_j. In particular,

$$\begin{aligned} &\mathbb{E}^x\Big[f(B_{T_1})\Big(\prod_{j=1}^n f_j(B_{t_j})\Big)g(B_{T_2})\Big] \qquad (2.2.31)\\ &= \int_{\mathbb{R}^d} dy \Pi_{T_2-T_1}(x-y) f(x) g(y) \mathbb{E}^{x,y}_{[T_1,T_2]}\Big[\prod_{j=1}^n f_j(B_{t_j})\Big]. \end{aligned}$$

Definition 2.17 (Full Wiener measure). We refer to the measure

$$d\mathcal{W} = d\mathcal{W}^x \otimes dx \quad \text{on } \mathscr{X} \times \mathbb{R}^d \qquad (2.2.32)$$

as *full Wiener measure*, carrying infinite mass.

Although full Wiener measure is not a probability measure on $\mathscr{X} \times \mathbb{R}^d$, we use the notation $\mathbb{E}_{\mathcal{W}}[\cdots]$ for $\int \cdots d\mathcal{W}$. The full Wiener measure $d\mathcal{W}$ is expressed in terms of the Brownian bridge measure as

$$d\mathcal{W} = \Big(\int_{\mathbb{R}^d} \Pi_{T_2-T_1}(x-y) d\mathcal{W}^{x,y}_{[T_1,T_2]} dy\Big) dx,$$

or in other words,

$$\int_{\mathscr{X}\times\mathbb{R}^d}\prod_{i=1}^{n} f_i(B_{t_i})d\mathcal{W} = \int_{\mathbb{R}^d\times\mathbb{R}^d} dxdy\Pi_{T_2-T_1}(x-y)\mathbb{E}^{x,y}_{[T_1,T_2]}\Big[\prod_{i=1}^{n} f_i(B_{t_i})\Big]. \tag{2.2.33}$$

For $T_1 \le t_1 \le \cdots \le t_n \le T_2$ define $\mathbb{E}_{\mathcal{W}}[\cdots|B_{T_2}=x, B_{T_1}=y]$ by

$$\mathbb{E}_{\mathcal{W}}\Big[\prod_{j=1}^{n} f_j(B_{t_j})\Big|B_{T_2}=x, B_{T_1}=y\Big] = \Pi_{T_2-T_1}(x-y)\mathbb{E}^{x,y}_{[T_1,T_2]}\Big[\prod_{j=1}^{n} f_j(B_{t_j})\Big]. \tag{2.2.34}$$

Thus for $T_1 \le t_1 \le \cdots \le t_n \le T_2$ we have

$$\begin{aligned}&\mathbb{E}_{\mathcal{W}}\Big[f(B_{T_1})\Big(\prod_{j=1}^{n} f_j(B_{t_j})\Big)g(B_{T_2})\Big]\\ &= \int_{\mathbb{R}^d\times\mathbb{R}^d} f(x)g(y)\mathbb{E}_{\mathcal{W}}\Big[\prod_{j=1}^{n} f_j(B_{t_j})\Big|B_{T_1}=x, B_{T_2}=y\Big]dxdy.\end{aligned}$$

2.3 Basics of stochastic calculus

2.3.1 The classical integral and its extensions

Classical analysis provides a working notion of integral called after Riemann. In this concept a real-valued function f on an interval $I=[a,b]$ is considered, I is divided into subintervals $\Delta_n=\{a=x_0<\cdots<x_n=b\}$ and limits of sums of the type

$$\sum_{j=0}^{n-1} f(\xi_j)(x_{j+1}-x_j), \quad \xi_j\in[x_j,x_{j+1})$$

are considered when $\lim_{n\to\infty} m(\Delta_n)=0$, where $m(\Delta_n)$ is the mesh of the partition. When such a limit exists independent of the choice of the partition and the values ξ_j, we call it the *Riemann integral* $\int_a^b f(x)dx$ of f. A sufficient condition for the Riemann integral of f to exist is that f is continuous.

If for some purposes one is interested in integrating a function against the increments of another, it is possible to extend the concept of Riemann integral to cover this case. This extension is achieved by the *Stieltjes integral* $\int_a^b f(x)dg(x)$, obtained as the limit of sums of the type

$$\sum_{j=0}^{n-1} f(\xi_j)(g(x_{j+1})-g(x_j)), \quad \xi_j\in[x_j,x_{j+1}).$$

Riemann integral then follows when the integrator is the identity function, $g(x) = x$. Two sufficient conditions for the Stieltjes integral of f with respect to g to exist are (1) f and g have no discontinuities at the same point x, (2) f is continuous and g is of bounded variation. However, (2) is not a necessary condition.

Surprisingly, there are few results on weaker conditions on the existence of Riemann–Stieltjes integral. Young has shown as early as in 1936 that if f is ρ-Hölder continuous and g is γ-Hölder continuous such that $\gamma + \rho > 1$, then $\int_a^b f(x)dg(x)$ exists. A nearly optimal (sufficient and "very nearly" also necessary) condition is that

(1) f and g are not both discontinuous at the same point x,

(2) f has bounded p-variation and g has bounded q-variation for some $p, q > 0$ such that $1/p + 1/q = 1$.

Our interest is to define integrals of the type $\int_a^b X_t dB_t$ where we would use Brownian motion as integrator and a random process X_t as integrand. Clearly, due to nowhere differentiability of Brownian motion, the usual Riemann–Stieltjes integral will not work. However, since Brownian motion is of bounded p-variation for $p > 2$, see Proposition 2.9, provided the integrand is of bounded q-variation with suitable $q < 2$, Young's criteria can be applied. This is possible, in particular, when $(X_t)_{t \geq 0}$ has bounded variation, i.e., $q = 1$. A sufficient condition for this is that $t \mapsto X_t$ is differentiable and has bounded derivative. However, the approach offered by Young or the conditions above quickly show their limitations as they would fail to cover the simple integral $\int_a^b B_t dB_t$. In order to include this and more interesting cases a genuinely new type of integral is needed.

2.3.2 Stochastic integrals

Recall the basic theorem of calculus: If $f \in C^1(\mathbb{R}^d; \mathbb{R})$ and $g \in C^1(\mathbb{R}; \mathbb{R}^d)$, then

$$f(g(t)) - f(g(0)) = \int_0^t \nabla f(g(s)) \cdot (dg(s)/ds)\, ds.$$

Since Brownian paths are nowhere differentiable, we cannot substitute B_s for $g(s)$ in the formula above. In spite of that, a similar formula still exists with an extra term appearing as a correction. Indeed, we will see that almost surely we have

$$f(B_t) - f(B_0) = \int_0^t \nabla f(B_s) \cdot dB_s + \frac{1}{2} \int_0^t \Delta f(B_s)\, ds,$$

with an expression formally written as dB_s "=" $(dB_s/ds)ds$ which will be explained below. The first term on the right-hand side above leads to a new concept of integral.

Following the conventional concept of Riemann–Stieltjes integrals, in order to construct the integral with respect to Brownian motion we divide the interval into n subintervals and look at approximating sums

$$S_n = \sum_{j=1}^{n} f(B_{s_j})(B_{t_{j+1}} - B_{t_j}), \quad s_j \in [t_j, t_{j+1}).$$

The pathwise (that is, in $\mathscr{X}$ pointwise) limit of the random variable S_n does not exist due to the unbounded variation of Brownian paths, thus it cannot be defined as a Stieltjes integral. However, as it will be seen below, when S_n is regarded as an element of $L^2(\mathscr{X}, d\mathcal{W})$, it does have a limit. But even in this situation we face the peculiar feature that, unlike in the case of Riemann–Stieltjes integral, the limit depends on how the point s_j is chosen and a different object is obtained when $s_j = t_j$ or when $s_j = t_{j+1}$.

First we consider one dimensional standard Brownian motion.

Example 2.1. Take a division of the interval $[0, T]$ into subintervals of equal length, i.e., choose $t_j = jT/n$. Then in L^2-sense $\lim_{n\to\infty} \sum_{j=0}^{n-1} B_{t_j}(B_{t_{j+1}} - B_{t_j}) = \frac{1}{2}(B_T^2 - T)$ and $\lim_{n\to\infty} \sum_{j=0}^{n-1} B_{t_{j+1}}(B_{t_{j+1}} - B_{t_j}) = \frac{1}{2}(B_T^2 + T)$. Indeed, a rearrangement of the terms gives for the first sum

$$\sum_{j=0}^{n-1} B_{t_j}(B_{t_{j+1}} - B_{t_j}) = \frac{1}{2}B_T^2 - \frac{1}{2}\sum_{j=0}^{n-1}(B_{t_{j+1}} - B_{t_j})^2.$$

By Proposition 2.9 the latter sum converges in L^2 sense to the quadratic variation $V_T^2 = T$, thus the first claim follows. The other limit can be computed similarly.

To remove the problem of dependence on the intermediary point, we restrict here to the choice $s_j = t_j$, i.e., choose the integrand to be adapted to the natural filtration $(\mathscr{F}_t^B)_{t\geq 0}$ of standard Brownian motion. We call $(\phi(t))_{t\geq 0}$ a *random step process* if there is a sequence $0 = t_0 < \cdots < t_n$ and random variables $f_0, \ldots, f_n \in L^2(\mathscr{X}, d\mathcal{W})$ such that f_j are $\mathscr{F}_{t_j}^B$-adapted and

$$\phi(t, \omega) = \sum_{j=0}^{n-1} f_j(\omega) 1_{[t_j, t_{j+1})}(t). \tag{2.3.1}$$

The notation M_{step}^2 will be used for the space of random step processes.

Definition 2.18. We denote by $M^2(S, T)$ the class of functions $f : \mathbb{R}^+ \times \mathscr{X} \to \mathbb{R}$ such that

(1) $f \in \mathscr{B}(\mathbb{R}^+) \times \mathscr{B}(\mathscr{X}) / \mathscr{B}(\mathbb{R})$

(2) $f(t, \omega)$ is $\mathscr{F}_t^B$-adapted

(3) $\mathbb{E}[\int_S^T |f(t, \omega)|^2 dt] < \infty$.

Let $\phi(t,\omega)$ be (2.3.1) with $S = t_0 < t_1 < \cdots < t_n = T$. Define

$$\int_S^T \phi(t,\omega)dB_t = \sum_{j=0}^{n} f_j(\omega)\cdot(B_{t_{j+1}}(\omega) - B_{t_j}(\omega)). \tag{2.3.2}$$

Lemma 2.25. *Let $\phi(t,\omega)$ be as in* (2.3.2). *Then*

$$\mathbb{E}^x\left[\left|\int_S^T \phi(t,\omega)dB_t\right|^2\right] = \mathbb{E}^x\left[\int_S^T |\phi(t,\omega)|^2 dt\right]. \tag{2.3.3}$$

Proof. Denote for a shorthand $I(\phi) = \int_S^T \phi(t,\omega)dB_t$, $\Delta B_j = B_{t_{j+1}} - B_{t_j}$ and $\Delta t_j = t_{j+1} - t_j$. Then clearly $\mathbb{E}^x[\Delta B_j] = 0$, $\mathbb{E}^x[(\Delta B_j)^2] = \Delta t_j$. We have

$$|I(\phi)|^2 = \sum_{j=0}^{n-1} f_j^2(\Delta B_j)^2 + 2\sum_{j<k} f_j f_k \Delta B_j \Delta B_k.$$

By independence, if $j < k$, then $\mathbb{E}^x[f_j f_k \Delta B_j \Delta B_k] = 0$. On the other hand, for $j = k$, $\mathbb{E}^x[f_j^2(\Delta B_j)^2] = \mathbb{E}^x[f_j^2]\Delta t_j$. Hence

$$\mathbb{E}^x[|I(\phi)|^2] = \sum_{j=0}^{n-1} \mathbb{E}^x[f_j^2]\Delta t_j. \tag{2.3.4}$$

Moreover, $|\phi(t)|^2 = \sum_{j=0}^{n-1} f_j^2 1_{[t_j,t_{j+1})}(t)$, i.e.,

$$\mathbb{E}^x\left[\left|\int_S^T |\phi(t)|^2 dt\right|\right] = \sum_{j=0}^{n-1} \mathbb{E}^x[f_j^2]\Delta t_j \tag{2.3.5}$$

follows. A comparison of the right-hand sides of (2.3.4) and (2.3.5) above completes the proof. □

Equality (2.3.3) is called *Itô isometry*. Using Itô isometry we can extend the definition of the stochastic integral $\int_S^T f(t,\omega)dB_t$ to $f \in M^2(S,T)$. We first construct in the lemma below a sequence of random step functions $\phi_n \in M^2_{\text{step}}$ convergent to a given $f \in M^2(S,T)$, that is,

$$\mathbb{E}^x\left[\int_S^T |f(t,\omega) - \phi_n(t,\omega)|^2 dt\right] \to 0 \quad \text{as } n \to \infty. \tag{2.3.6}$$

Lemma 2.26. *Let $f \in M^2(S,T)$. Then there exits a sequence $\phi_n \in M^2_{\text{step}}$ satisfying* (2.3.6).

Proof. We divide the proof into four steps.

Step 1*:* Let $g \in M^2(S,T)$ be bounded and $g(\cdot,\omega)$ be continuous for every $\omega \in \mathscr{X}$. Then there exists a sequence of random step functions $\phi_n \in M^2_{\mathrm{step}}$ such that

$$\int_S^T |g(t,\omega) - \phi_n(t,\omega)|^2 dt \to 0 \tag{2.3.7}$$

as $n \to \infty$. In fact, we can take $\phi_n = \sum_{j=0}^{n-1} g(t_j,\omega) 1_{[t_j,t_{j+1})}(t)$.

Step 2*:* Let $h \in M^2(S,T)$ be bounded. Then there exists a bounded sequence $g_n \in M^2(S,T)$ of functions $g_n(\cdot,\omega)$ continuous for every ω, with

$$\int_S^T |h(t,\omega) - g_n(t,\omega)|^2 dt \to 0 \tag{2.3.8}$$

as $n \to \infty$. We can take $g_n(t,\omega) = \int_0^t \psi_n(s-t)h(s,\omega)ds$, where ψ_n is (1) a non-negative and continuous function, (2) $\mathrm{supp}\psi_n \subset [-1/n, 0]$, and (3) $\int_{\mathbb{R}} \psi_n(x)dx = 1$.

Step 3*:* Let $f \in M^2(S,T)$. Then there exists a sequence $h_n \in M^2(S,T)$ of bounded functions and

$$\int_S^T |f(t,\omega) - h_n(t,\omega)|^2 dt \to 0 \tag{2.3.9}$$

as $n \to \infty$. Here we can take

$$h_n(t,\omega) = \begin{cases} -n, & f(t,\omega) < -n \\ f(t,\omega), & -n \le f(t,\omega) \le n \\ n, & f(t,\omega) > n. \end{cases}$$

Step 4*:* If $f \in M^2(S,T)$, by (2.3.7)–(2.3.9) we can choose a sequence of random step functions $\phi_n \in M^2_{\mathrm{step}}$ such that $\mathbb{E}^x[\int_S^T |f(t,\omega) - \phi_n(t,\omega)|^2 dt] \to 0$ as $n \to \infty$. $\square$

Hence by Lemma 2.3.6 and the Itô isometry $(\int_S^T \phi_n dB_t)_{n\in\mathbb{N}}$ is a Cauchy sequence in $L^2(\mathscr{X}, d\mathcal{W}^x)$. Thus $\int_S^T \phi_n dB_t$ converges to a random variable as $n \to \infty$ in $L^2(\mathscr{X}, d\mathcal{W}^x)$.

Now we are in the position to define the Itô integral.

Definition 2.19 (Itô integral/stochastic integral). Let $f \in M^2(S,T)$ and $(\phi_n)_{n\in\mathbb{N}} \subset M^2_{\mathrm{step}}$ such that $\mathbb{E}^x[\int_S^T |f(t,\omega) - \phi_n(t,\omega)|^2 dt] \to 0$ as $n \to \infty$. We define the *Itô integral* of f by

$$\int_S^T f(t,\omega)dB_t = \lim_{n\to\infty} \int_S^T \phi_n(t,\omega)dB_t. \tag{2.3.10}$$

Note that the definition of $\int_S^T f(t,\omega)dB_s$ is independent of the choice of random step functions ϕ_n.

Example 2.2. We compute $\int_0^t B_s dB_s$ as follows. Take the division $0 = s_0 < \cdots < s_n = t$ of the interval, with $s_j = jt/n$. Write $\phi_n(t) = \sum_{j=0}^{n-1} B_{s_j} 1_{[s_j, s_{j+1})}$. Then $\int_0^t \phi_n(s,\omega)dB_s$ approximates $\int_0^t B_s dB_s$ in L^2 and we see by Example 2.1 that

$$\int_0^t B_s dB_s = \lim_{n\to\infty}\int_0^t \phi_n(t)dB_s = \lim_{n\to\infty}\sum_{j=0}^{n-1} B_{s_j}(B_{s_{j+1}} - B_{s_j}) = \frac{1}{2}B_t^2 - \frac{t}{2}.$$

Here are some basic properties of the Itô integral.

Theorem 2.27 (Itô isometry). *Let $f \in M^2(S,T)$. Then the expectation of any Itô integral is*

$$\mathbb{E}^x\left[\int_S^T f(s,\omega)dB_s\right] = 0,$$

and its covariance is given by

$$\mathbb{E}^x\left[\left|\int_S^T f(s,\omega)dB_s\right|^2\right] = \mathbb{E}^x\left[\int_S^T |f(s,\omega)|^2 ds\right].$$

The latter relationship is called Itô isometry for this class of functions.

Proof. This follows from the definition of $\int_S^T f(t,\omega)dB_t$ and (2.3.3). □

Theorem 2.28 (Martingale property). *Let $f \in M^2(0,t)$ for all $t \geq 0$. The random process $(I_t)_{t\geq 0}$, $I_t = \int_0^t f(s,\omega)dB_s$, is a martingale with respect to the natural filtration of Brownian motion, $(\mathscr{F}_t^B)_{t\geq 0}$, i.e.,*

(1) *I_t is $\mathscr{F}_t^B$-measurable,*

(2) *$\mathbb{E}^x[|I_t|] < \infty$,*

(3) *$\mathbb{E}^x[I_s|\mathscr{F}_t^B] = I_t$, for $s \geq t$.*

Proof. (1) and (2) are immediate; we show (3). Let $I_t(n) = \int_0^t \phi_n(s,\omega)dB_s$, where $\phi_n = \sum_{j=0}^{n-1} f_j^{(n)}(\omega)1_{[t_{j+1}^{(n)} - t_j^{(n)})}(t)$. $I_t(n)$ is an approximating sequence of I_t. We have

$$\mathbb{E}^x[I_s(n)|\mathscr{F}_t^B] = \int_0^t \phi_n(s,\omega)dB_s + \mathbb{E}^x\left[\int_t^s \phi_n(r,\omega)dB_r|\mathscr{F}_t^B\right].$$

The claim follows by showing that the second term on the right-hand side above vanishes. We have

$$\mathbb{E}^x\left[\int_t^s \phi_n(r,\omega)dB_r\,\middle|\,\mathscr{F}_t^B\right] = \sum_{t\le t_j^{(n)}\le t_{j+1}^{(n)}\le s} \mathbb{E}^x\left[f_j^{(n)}(B_{t_{j+1}^{(n)}} - B_{t_j^{(n)}})\,\middle|\,\mathscr{F}_t^B\right].$$

$f_j^{(n)}(B_{t_{j+1}^{(n)}} - B_{t_j^{(n)}})$ is independent of $\mathscr{F}_t^B$ and thus $\mathbb{E}^x[f_j^{(n)}(B_{t_{j+1}^{(n)}} - B_{t_j^{(n)}})|\mathscr{F}_t^B] = \mathbb{E}^x[f_j^{(n)}(B_{t_{j+1}^{(n)}} - B_{t_j^{(n)}})] = 0$. Therefore

$$\mathbb{E}^x\left[\int_t^s \phi_n(r,\omega)dB_r\,\middle|\,\mathscr{F}_t^B\right] = 0. \qquad \square$$

Theorem 2.29 (Continuous version). *Let $f \in M^2(0,T)$. Then the random process $(I_t)_{0\le t\le T}$, $I_t = \int_0^t f(s,\omega)dB_s$, has a continuous version $(\tilde{I}_t)_{0\le t\le T}$, i.e., $\tilde{I}_t(\omega)$ is continuous in t for almost every $\omega \in \mathscr{X}$ and $\mathcal{W}^x(\tilde{I}_t = I_t) = 1$, for all $0 \le t \le T$.*

Proof. Without restricting generality we set $x = 0$. Let $\phi_n(t) \in M^2_{\text{step}}$ be such that $\mathbb{E}[\int_0^t |f(s,\omega) - \phi_n(s,\omega)|^2 ds] \to 0$ as $n \to \infty$. Write $I_t(n) = \int_0^t \phi_n(s,\omega)dB_s$. Note that $I_t(n)$ is continuous in t for all n and is an $\mathscr{F}_t^B$-martingale. This implies that $I_t(n) - I_t(m)$ is also $\mathscr{F}_t^B$-martingale, thus by the martingale inequality (see Theorem 2.13) we have

$$\begin{aligned}\mathcal{W}\Big(\sup_{0\le t\le T} |I_t(n) - I_t(m)| > \varepsilon\Big) &\le \frac{1}{\varepsilon^2}\mathbb{E}[|I_T(n) - I_T(m)|^2] \\ &= \frac{1}{\varepsilon^2}\mathbb{E}\left[\int_0^T |\phi_n(s) - \phi_m(s)|^2 ds\right] \to 0\end{aligned}$$

as $m,n \to \infty$. Hence we can choose a subsequence $n_k \to \infty$ such that $\mathcal{W}(A_k) \le 2^{-k}$, where

$$A_k = \{\omega \in \mathscr{X} \,|\, \sup_{0\le t\le T} |I_t(n_{k+1},\omega) - I_t(n_k,\omega)| > 2^{-k}\}.$$

Since $\sum_{k=1}^\infty \mathcal{W}(A_k) < \infty$, we get $\mathcal{W}(\cap_{N=1}^\infty \cup_{k=N}^\infty A_k) = 0$ by the Borel–Cantelli lemma. Thus for almost every $\omega \in \mathscr{X}$ it follows that

$$\sup_{0\le t\le T} |I_t(n_{k+1},\omega) - I_t(n_k,\omega)| \le 2^{-k}$$

for all $k > N(\omega)$, with some $N(\omega)$. Therefore $(I_t(n_k,\omega))_{k\in\mathbb{N}}$ is a Cauchy sequence and so it converges uniformly in $t \in [0,T]$ as $k \to \infty$ for almost every $\omega \in \mathscr{X}$, which is denoted by $\tilde{I}_t(\omega)$. $\tilde{I}_t(\omega)$ is continuous in t for almost every $\omega \in \mathscr{X}$ since the convergence is uniform in t. Since $I_t(n_k) \to I_t$ in $L^2(\mathscr{X})$, we conclude that $\tilde{I}_t = I_t$ almost surely. $\square$

From now on we will consider the continuous version of the Itô integral of a random process without explicitly stating it.

Next we consider d-dimensional Brownian motion $(B_t)_{t\ge 0}$ and $f = (f_1, \dots, f_d)$ such that $f_\mu \in M^2(0,t)$, $\mu = 1, \dots, d$. We define the Itô integral

$$\int_0^t f(s,\omega)\cdot dB_s = \sum_{\mu=1}^d \int_0^t f_\mu(s,\omega) dB_s^\mu. \tag{2.3.11}$$

It is easy to see that the basic properties $\mathbb{E}^x[\int_0^t f(s,\omega)\cdot dB_s] = 0$ and

$$\mathbb{E}^x\left[\int_0^t f_\mu(s,\omega)dB_s^\mu \int_0^t f_\nu(s,\omega)dB_s^\nu\right] = \delta_{\mu\nu}\mathbb{E}^x\left[\int_0^t f_\mu(s,\omega)f_\mu(s,\omega)ds\right]$$

similarly hold.

Let $C_{\mathrm{b}}^n(\mathbb{R}^d)$ denote the set of bounded and n times continuously differentiable functions on $\mathbb{R}^d$. Consider the following specific cases.

Corollary 2.30. *Let B_t be d-dimensional Brownian motion. Write $t_j = tj/2^m$ and $\Delta B_j^\mu = B_{t_j}^\mu - B_{t_{j-1}}^\mu$, $1 \le j \le 2^m$.*

(1) *Let $f \in C_{\mathrm{b}}^1(\mathbb{R}^d)$. Then*

$$\lim_{n\to\infty} \sum_{j=1}^{2^n} f(B_{t_{j-1}})\Delta B_j^\mu = \int_0^t f(B_s) dB_s^\mu. \tag{2.3.12}$$

(2) *Let $f \in C_{\mathrm{b}}^2(\mathbb{R}^d)$. Then*

$$\lim_{n\to\infty} \sum_{j=1}^{2^n} \frac{1}{2}(f(B_{t_{j-1}}) + f(B_{t_j}))\Delta B_j^\mu = \int_0^t f(B_s)dB_s^\mu + \frac{1}{2}\int_0^t \partial_\mu f(B_s)ds. \tag{2.3.13}$$

The limits in both expressions above are understood in $L^2(\mathscr{X}, d\mathcal{W}^x)$ sense.

Proof. (1) is directly obtained from the definition of Itô integral. For (2) we write

$$\sum_{j=1}^{2^n} \left| f(B_{t_j}) - f(B_{t_{j-1}}) - \sum_\mu \partial_\mu f(B_{t_{j-1}})\Delta B_j^\mu - \frac{1}{2}\sum_{\mu,\nu}\partial_\mu\partial_\nu f(B_{t_{j-1}})\Delta B_j^\mu \Delta B_j^\nu \right|$$

$$\le C\sum_{j=1}^{2^n} |B_{t_j} - B_{t_{j-1}}|^3 \overset{n\to\infty}{\longrightarrow} 0$$

in $L^2(\mathscr{X}, d\mathcal{W}^x)$, where C is a constant independent of ω. By writing $f(B_{t_{j-1}}) + f(B_{t_j}) = 2f(B_{t_{j-1}}) + (f(B_{t_j}) - f(B_{t_{j-1}}))$ and making a Taylor expansion we have

$$\begin{aligned}
&\text{l.h.s. (2.3.13)}\\
&= \lim_{n\to\infty} \sum_{j=1}^{2^n} f(B_{t_{j-1}})\Delta B_j^\mu + \frac{1}{2} \lim_{n\to\infty} \sum_{j=1}^{2^n} \sum_{\nu=1}^{d} \partial_\nu f(B_{t_{j-1}})\Delta B_j^\nu \Delta B_j^\mu\\
&\quad + \frac{1}{2\cdot 3!} \lim_{n\to\infty} \sum_{j=1}^{2^n} \sum_{\nu,\lambda=1}^{d} \partial_\lambda \partial_\nu f(B_{t_{j-1}})\Delta B_j^\nu \Delta B_j^\lambda \Delta B_j^\mu\\
&= \int_0^t f(B_s) dB_s^\mu + \frac{1}{2} \int_0^t \partial_\mu f(B_s) ds.
\end{aligned}$$

yielding (2.3.13). □

Definition 2.20 (Stratonovich integral). The right-hand side of (2.3.13) defines the *Stratonovich integral*, i.e., for $f = (f_1, \ldots, f_d) \in (C_b^2(\mathbb{R}^d))^d$,

$$\int_0^t f(B_s) \circ dB_s = \int_0^t f(B_s) \cdot dB_s + \frac{1}{2} \int_0^t \nabla \cdot f(B_s) ds. \tag{2.3.14}$$

Using random processes given by Itô integrals we have the following important class.

Definition 2.21 (Itô process). A random process $(X_t)_{t\geq 0}$ on $(\mathscr{X}, \mathcal{B}(\mathscr{X}), \mathcal{W})$ is called *Itô process* if

(1) $t \mapsto X_t(\omega)$ is continuous for almost every $\omega \in \mathscr{X}$

(2) there exist a random process $(b_t)_{t\geq 0}$ with $\int_0^t |b_s| ds < \infty$, and an $\mathbb{R}^d$-valued random process $(\sigma_t)_{t\geq 0} \in M^2(0,t)$, for all $t \geq 0$, such that

$$X_t = X_0 + \int_0^t b_s ds + \int_0^t \sigma_s \cdot dB_s. \tag{2.3.15}$$

In concise differential notation the above integral equality is written as

$$dX_t = b_t dt + \sigma_t \cdot dB_t. \tag{2.3.16}$$

The process $(b_t)_{t\geq 0}$ is called *drift term* and $(\sigma_t)_{t\geq 0}$ *diffusion term*.

It should be stressed that the *stochastic differentials* dX_t resp. dB_t have no meaning other than the shorthand explained above.

Finally, we define a more general stochastic integral for a wider class of integrands by weakening the conditions of $M^2(0,T)$.

Definition 2.22. Let $\mathbb{M}^2(S,T)$ be the class of $f : \mathbb{R}^+ \times \mathscr{X} \to \mathbb{R}$ such that

(1) $f \in \mathcal{B}(\mathbb{R}^+) \times \mathcal{B}(\mathscr{X})/\mathcal{B}(\mathbb{R})$

(2) $f(t,\omega)$ is $(\mathscr{F}_t^B)_{t\geq 0}$-adapted

(3) $\mathcal{W}^x(\int_0^t |f(s,\omega)|^2 ds < \infty) = 1$.

For this class it is not necessarily true that $\mathbb{E}^x[\int_0^t |f(s,\omega)|^2 ds] < \infty$. However, for $f \in \mathbb{M}^2(0,t)$ there exists a sequence $(f_n)_{n\in\mathbb{N}} \subset M^2(0,t)$ such that

$$\int_0^t |f_n(s,\omega) - f(s,\omega)|^2 ds \to 0 \tag{2.3.17}$$

as $n \to \infty$, for almost every $\omega \in \mathscr{X}$. The sequence $\int_0^t f_n(s,\omega)dB_s$ converges to a random variable for almost every $\omega \in \mathscr{X}$, and the limit only depends on f while it does not on f_n.

Definition 2.23. Let $f \in \mathbb{M}^2(0,t)$ and $(f_n)_{n\in\mathbb{N}} \subset M^2(0,t)$ be an approximating sequence as in (2.23). Define the stochastic integral

$$\int_0^t f(s,\omega)dB_s = \lim_{n\to\infty} \int_0^t f_n(s,\omega)dB_s \quad \text{in probability.} \tag{2.3.18}$$

Note that the so defined integral is not an $\mathscr{F}_t^B$-martingale and the Itô isometry is also not automatically satisfied.

Example 2.3. Let $f \in L^2_{\mathrm{loc}}(\mathbb{R})$. It is not necessarily true that $f(B_s) \in M^2(0,t)$, however, it is true that $f(B_s) \in \mathbb{M}^2(0,t)$. Therefore $\int_0^t f(B_s)dB_s$ is well defined by (2.3.18).

2.3.3 Itô formula

Another landmark result of stochastic analysis is the Itô formula which is the counterpart of the fundamental change of variable rule in classical analysis. In particular, this provides a tool for computing Itô integrals without directly having to use the definition. Recall that $C^{1,2}(\mathbb{R}^+ \times \mathbb{R})$ denotes the space of functions on $\mathbb{R}^+ \times \mathbb{R}$ which are C^1 with respect to the first variable and C^2 with respect to the second.

Theorem 2.31 (Itô formula for Brownian motion). *Let $h \in C^{1,2}(\mathbb{R}^+ \times \mathbb{R})$ and define $X_t = h(t,B_t)$. Then $(X_t)_{t\geq 0}$ is an Itô process and we have*

$$X_t - X_0 = \int_0^t \left(\dot{h}(s,B_s) + \frac{1}{2}\partial_x^2 h(s,B_s)\right) ds + \int_0^t \partial_x h(s,B_s)dB_s, \tag{2.3.19}$$

or in equivalent differential notation (and with $\dot{h} = \partial_s h(s,x)$)

$$dX_t = \left(\dot{h}(t,B_t) + \frac{1}{2}\partial_x^2 h(t,B_t)\right) dt + \partial_x h(t,B_t)dB_t. \tag{2.3.20}$$

Remark 2.1 (Rules of Itô differential calculus). For practical purposes the following formal rules of Itô differential calculus are useful:

$$dt\,dt = 0, \quad dt\,dB_t^\mu = 0, \quad dB_t^\mu dt = 0, \quad dB_t^\mu \cdot dB_t^\nu = \delta_{\mu\nu} dt \tag{2.3.21}$$

for all $\mu, \nu = 1, \dots, d$. The last property intuitively corresponds to the fact that the quadratic variation of Brownian motion grows linearly in time, that is, B_t has large fluctuations on small scale, $dB_t = O(\sqrt{dt})$.

By using the rules of formal Itô differential calculus the following more general formula is readily obtained.

Theorem 2.32 (Itô formula for Itô process). *Let $(X_t^i)_{t\geq 0, 1\leq i\leq n}$ be the $\mathbb{R}^d$-valued Itô process*

$$dX_t^i = b_t^i dt + \sigma_t^i \cdot dB_t, \quad i = 1, \dots, n, \tag{2.3.22}$$

on $(\mathscr{X}, \mathcal{B}(\mathscr{X}), \mathcal{W}^x)$, where

$$b_t^i(\omega) \in L^1_{\mathrm{loc}}(\mathbb{R}^+, dt), \quad \forall \omega \in \mathscr{X},$$
$$\sigma_t^i(\omega) = (\sigma_t^{i\mu}(\omega))_{1\leq \mu\leq d} \subset \bigcup_{T\geq 0} \mathbb{M}^2(0,T).$$

Let $h = (h^1, \dots, h^m) \in C^{1,2}(\mathbb{R}^+ \times \mathbb{R}^n; \mathbb{R}^m)$. Then also $Y_t = h(t, X_t)$ is an Itô process with

$$dY_t^k = \dot{h}^k dt + \sum_{i=1}^n h_i^k dX_t^i + \frac{1}{2} \sum_{i,j=1}^n h_{ij}^k dX_t^i dX_t^j, \quad 1 \leq k \leq m. \tag{2.3.23}$$

Here $\dot{h} = \partial_t h$, $h_i^k = \partial_{x_i} h^k$ and $h_{ij}^k = \partial_{x_i}\partial_{x_j} h^k$, evaluated at (t, X_t). In differential notation,

$$dY_t^k = \Big(\dot{h}^k + \sum_{i=1}^n h_i^k b_t^i + \frac{1}{2} \sum_{i,j=1}^n h_{ij}^k (\sigma_t \sigma_t^T)_{ij}\Big) dt + \sum_{i=1}^n h_i^k \sigma_t^i \cdot dB_t. \tag{2.3.24}$$

An application of the Itô formula gives the following useful result.

Corollary 2.33 (Product formula). *Let $(X_t)_{t\geq 0}$ and $(Y_t)_{t\geq 0}$ be two Itô processes as in* (2.3.22)*. Then the product formula*

$$d(X_t \cdot Y_t) = dX_t \cdot Y_t + X_t \cdot dY_t + dX_t \cdot dY_t \tag{2.3.25}$$

holds.

Proof. Choose $g(x, y) = x \cdot y$, $x = X_t$ and $y = Y_t$. Then the formula follows directly from the Itô formula. □

Example 2.4. When the integrand $f(s)$ is continuous and of bounded variation but non-random, i.e., independent of ω, an application of the Itô formula gives

$$\int_0^t f(s)dB_s = f(t)B_t - \int_0^t B_s df(s).$$

Example 2.5. Let $Z_t = g(t, X_t)$ with $g(t, x) \in C^{1,2}(\mathbb{R}^+ \times \mathbb{R})$, where

$$dX_t = V(B_t)dt + a(B_t) \cdot dB_t$$

and $a = (a_1, \dots, a_d)$. Then

$$dZ_t = \left(\dot{g} + \frac{1}{2}(a \cdot a)\Delta_x g + V \partial_x g\right) dt + (\partial_x g) a \cdot dB_t. \tag{2.3.26}$$

This formula shows the relationship between Stratonovich and Itô integrals. In the particular case $g(t, x) = g(x)$ and $X_t = B_t$

$$g(B_t) = g(B_0) + \int_0^t (\partial_x g)(B_s) \circ dB_s. \tag{2.3.27}$$

We conclude this section by two further applications of the Itô formula (2.3.23).

Example 2.6 (Wick exponential). Put $h(t, x) = e^{\alpha x - \frac{1}{2}\alpha^2 t}$; notice that it satisfies $(1/2)\Delta h + \dot{h} = 0$. The Itô formula yields $dh(t, B_t) = \alpha h dB_t$. Hence

$$\int_0^t h(s, B_s)dB_s = \frac{1}{\alpha}(h(t, B_t) - 1). \tag{2.3.28}$$

Define the *Wick exponential* of $e^{\alpha B_t}$ by

$$:\!\exp(\alpha B_t)\!: = \exp\left(\alpha B_t - \frac{1}{2}\alpha^2 t\right), \tag{2.3.29}$$

giving

$$:\!\exp\left(\int_0^t f(s, B_s)dB_s\right)\!: = \exp\left(\int_0^t f(s, B_s)dB_s - \frac{1}{2}\mathbb{E}^x\left[\int_0^t |f(s, B_s)|^2 ds\right]\right).$$

This definition is useful for stating the Cameron–Martin formula (2.3.57) below. The Wick product of B_t^n can then be written as

$$:\!B_t^n\!: = \left(\frac{d^n h}{d\alpha^n}\right)(t, B_t)\lceil_{\alpha=0},$$

and as a result we obtain by (2.3.28)

$$\int_0^t :\!B_s^n\!: dB_s = \frac{1}{n+1} :\!B_t^{n+1}\!:.$$

The following inequality offers useful bounds on the moments of the martingale $\int_0^T f(s, B_s) \cdot dB_s$.

Proposition 2.34 (Burkholder–Davis–Gundy (BDG) inequality). *Suppose that for* $m \in \mathbb{N}$ *we have* $\mathbb{E}[\int_0^T |f(t, B_t)|^{2m} dt] < \infty$. *Then*

$$\mathbb{E}\left[\left|\int_0^T f(t, B_t) \cdot dB_t\right|^{2m}\right] \le (m(2m-1))^m T^{m-1} \mathbb{E}\left[\int_0^T |f(t, B_t)|^{2m} dt\right]. \tag{2.3.30}$$

Proof. To see this, apply the Itô formula to $|X_t|^{2m}$, where $dX_t = f(t, B_t) \cdot dB_t$. We obtain

$$d|X_t|^{2m} = m(2m-1)|X_t|^{2m-2}|f(t, B_t)|^2 dt + 2m|X_t|^{2m-1} f(t, B_t) \cdot dB_t.$$

Hence

$$\mathbb{E}[|X_t|^{2m}] \le m(2m-1) \int_0^T \mathbb{E}[|X_s|^{2m-2}|f(s, B_s)|^2] ds. \tag{2.3.31}$$

By the Jensen inequality we have $\mathbb{E}[|X_t|^{2m}|\mathscr{F}_s] \ge |X_s|^{2m}$ for $t \ge s$. Thus

$$\mathbb{E}[|X_s|^{2m}] \le \mathbb{E}[|X_T|^{2m}].$$

By (2.3.31) and the Hölder inequality it follows that

$$\mathbb{E}[|X_T|^{2m}] \le m(2m-1)(\mathbb{E}[|X_T|^{(2m-2)p}])^{1/p}\left(\mathbb{E}\left[\left(\int_0^T |f(s, B_s)|^2 ds\right)^q\right]\right)^{1/q}$$

for $1/p + 1/q = 1$. Putting $p = m/(m-1)$ and $q = m$ we obtain

$$(\mathbb{E}[|X_T|^{2m}])^{1/m} \le m(2m-1)\mathbb{E}\left[\left(\int_0^T |f(s, B_s)|^2 ds\right)^m\right]^{1/m}.$$

Taking the mth power of both sides, by Schwarz inequality

$$\mathbb{E}[|X_T|^{2m}] \le [m(2m-1)]^m T^{m-1} \mathbb{E}\left[\int_0^T |f(s, B_s)|^{2m} ds\right]. \qquad \square$$

Since $(\int_0^t f(s, B_s) \cdot dB_s)_{t \ge 0}$ is a continuous martingale provided that $f \in M^2(0, t)$ for all $t \ge 0$, combining the BDG inequality and the martingale inequality (2.2.6) further useful inequalities can be derived.

2.3.4 Stochastic differential equations and diffusions

In this section we consider a class of stochastic differential equations (SDE) and their solutions. An SDE can be thought of arising from an ordinary differential equation $\dot{X}_t = a_t X_t$ in which a_t is replaced by a factor consisting of a non-random and a noise term. The solution of such an equation is a random process. The theory of SDE makes possible to implement the noise term mathematically and allows to show that under sufficient regularity a solution exists, it is continuous in t and has the Markov property.

Consider the SDE

$$X_t^i = \int_0^t b_s^i(X_s)ds + \int_0^t \sigma_s^i(X_s) \cdot dB_s \tag{2.3.32}$$

or in differential form

$$dX_t^i = b_t^i(X_t)dt + \sigma_t^i(X_t) \cdot dB_t. \tag{2.3.33}$$

Theorem 2.35 (Solution of SDE). *Suppose that for $t \in [0, T]$*

$$|b_t(x)| + |\sigma_t(x)| \leq C(1 + |x|), \tag{2.3.34}$$

$$|b_t(x) - b_t(y)| + |\sigma_t(x) - \sigma_t(y)| \leq D|x - y|, \tag{2.3.35}$$

with some constants C, D independent of $x, y \in \mathbb{R}^d$ and $t \in [0, T]$, where we set $|\sigma_t(x)| = (\sum_{i,j=1}^n |\sigma_t^{ij}(x)|^2)^{1/2}$ and $|b_t(x)| = (\sum_{i=1}^n |b_t^i(x)|^2)$. Then there exists a unique solution $X^x = (X_t^x)_{t\in[0,T]}$ on $(\mathscr{X}, \mathcal{B}(\mathscr{X}), \mathcal{W}^0)$ of

$$dX_t^i = b_t^i(X_t)dt + \sigma_t^i(X_t) \cdot dB_t, \quad X_0^i = x^i \in \mathbb{R}, \quad i = 1, \ldots, n, \tag{2.3.36}$$

such that

(1) *$(X_t^x)_{t\geq 0}$ is adapted to the natural filtration of Brownian motion $(\mathscr{F}_t^{\mathrm{BM}})_{t\geq 0}$*

(2) *$\mathbb{E}[\int_0^T |X_t^x|^2 dt] < \infty$*

(3) *X_t^x is almost surely continuous in $t \in [0, T]$.*

An important class of solutions of SDE is the following.

Definition 2.24 (Diffusion process). A random process $(X_t)_{t\geq 0}$ on a filtered space $(\mathscr{X}, \mathcal{B}(\mathscr{X}), \mathcal{W}^0)$ with filtration $(\mathscr{F}_t)_{t\geq 0}$ is a *diffusion process* whenever

(1) $(X_t)_{t\geq 0}$ is a Markov process,

(2) $X_t(\omega)$ is continuous in t for every $\omega \in \mathscr{X}$.

Now consider (2.3.36) with b_t^i and σ_t^i replaced by a time-independent b^i and σ^i, respectively.

Theorem 2.36 (Itô diffusion). *Suppose that*

$$|b(x) - b(y)| + |\sigma(x) - \sigma(y)| \leq D|x - y|,$$

with some constant D independent of $x, y \in \mathbb{R}^d$. Then the unique solution $(X_t^x)_{t \in [0,T]}$ of

$$dX_t^i = b^i(X_t)dt + \sigma^i(X_t) \cdot dB_t, \quad X_0^i = x^i \in \mathbb{R}, \quad i = 1, \dots, n, \tag{2.3.37}$$

satisfies the Markov property

$$\mathbb{E}[f(X_{s+t}^x)|\mathscr{F}_s^{\mathrm{BM}}] = \mathbb{E}[f(X_t^{X_s^x})]$$

for every Borel measurable bounded function f, where $\mathbb{E}[f(X_t^{X_s^x})] = \mathbb{E}[f(X_t^y)]$ evaluated at $y = X_s^x$. In particular, $\mathbb{E}[f(X_{s+t}^x)|\mathscr{F}_s^{\mathrm{BM}}] = \mathbb{E}[f(X_{s+t}^x)|\sigma(X_s^x)]$ holds.

The diffusion process in Theorem 2.36 is called *Itô diffusion*. Next we compute the transition probability kernel of the Itô diffusion $(X_t^x)_{t \geq 0}$. The Markov property $\mathbb{E}[1_A(X_t^x)|\sigma(X_s^x)] = \mathbb{E}[1_A(X_{t-s}^{X_s^x})]$ implies the equality $\mathbb{E}[1_A(X_t^x)|X_s^x = y] = \mathbb{E}[1_A(X_{t-s}^y)]$. Thus the probability transition kernel of $(X_t^x)_{t \geq 0}$ satisfies

$$p(s, t, y, A) = \mathbb{E}[1_A(X_{t-s}^y)]. \tag{2.3.38}$$

Moreover, by (2.3.38) the Itô diffusion $(X_t^x)_{t \geq 0}$ is stationary.

Let $(X_t^x)_{t \geq 0}$ be an Itô diffusion and $\mathscr{U}_t : L^\infty(\mathbb{R}^d) \to L^\infty(\mathbb{R}^d)$ be given by

$$\mathscr{U}_t f(x) = \mathbb{E}[f(X_t^x)]. \tag{2.3.39}$$

Then by the definition of $\mathscr{U}_t$,

$$\mathscr{U}_s \mathscr{U}_t f(x) = \mathbb{E}[\mathscr{U}_t f(X_s^x)] = \mathbb{E}[\mathbb{E}[f(X_t^{X_s^x})]],$$

and by the Markov property we furthermore have

$$= \mathbb{E}[\mathbb{E}[f(X_{t+s}^x)|\mathscr{F}_s^{\mathrm{BM}}]] = \mathbb{E}[f(X_{t+s}^x)] = \mathscr{U}_{t+s} f(x).$$

Thus $\{\mathscr{U}_t : t \geq 0\}$ defines a semigroup on $L^\infty(\mathbb{R}^d)$.

Definition 2.25 (Generator of Itô diffusion). The *generator* H_X of an Itô diffusion $(X_t^x)_{t \geq 0}$ is defined by

$$H_X f(x) = \lim_{t \downarrow 0} \frac{1}{t}(\mathscr{U}_t f(x) - f(x)). \tag{2.3.40}$$

Theorem 2.37 (Generator of Itô diffusion). *Let $(X_t^x)_{t\geq 0}$ be an Itô diffusion obtained as the solution of* (2.3.37). *Its generator is given by the second order partial differential operator*

$$H_X = \frac{1}{2}\sum_{i,j=1}^{n}(\sigma\sigma^T)_{ij}(x)\partial_i\partial_j + \sum_{i=1}^{n} b^i(x)\partial_i \tag{2.3.41}$$

acting on $C_0^2(\mathbb{R}^d)$.

Proof. Let $(Y_t^i)_{t\geq 0}$ be the Itô process $dY_t^i = u^i dt + v^i \cdot dB_t$ with $Y_0^i = x^i$. The Itô formula (2.3.24) yields

$$\begin{aligned} f(Y_t) - f(Y_0) = &\int_0^t \Big(\frac{1}{2}\sum_{i,j=1}^{n}(vv^T)_{ij}\partial_i\partial_j f(Y_s) + \sum_{i=1}^{n} u^i \partial_i f(Y_s)\Big) ds \\ &+ \int_0^t \sum_{i=1}^{n}(\partial_i f(Y_s))v^i \cdot dB_s. \end{aligned}$$

Taking expectation, the martingale part $\int_0^t \sum_{i=1}^n (\partial_i f)v^i \cdot dB_s$ vanishes and we are left with

$$\mathbb{E}[f(Y_t)] = f(x) + \mathbb{E}\Big[\int_0^t \Big(\frac{1}{2}\sum_{i,j=1}^{n}(vv^T)_{ij}\partial_i\partial_j f(Y_s) + \sum_{i=1}^{n} u^i \partial_i f(Y_s)\Big) ds\Big].$$

Hence

$$\begin{aligned} &\mathscr{U} f(x) - f(x) \\ &= \mathbb{E}\Big[\int_0^t \Big(\frac{1}{2}\sum_{i,j=1}^{n}(\sigma\sigma^T)_{ij}(X_s^x)\partial_i\partial_j f(X_s^x) + \sum_{i=1}^{n} b^i(X_s^x)\partial_i f(X_s^x)\Big) ds\Big]. \end{aligned} \tag{2.3.42}$$

Since X_s^x is continuous in s, the lemma now follows directly. □

The last equality is in fact a simple version of *Dynkin's formula*.

Example 2.7 (Linear SDE). Assume that the $n \times n$, $n \times 1$ and $n \times d$ matrices A_t, a_t and σ_t, respectively, are non-random and locally bounded. Let Φ_t be the $n \times n$ matrix function solving the non-random equation $\dot{\Phi}_t = A_t \Phi_t$. Then the solution of the linear stochastic differential equation

$$dX_t = (A_t \cdot X_t + a_t)\,dt + \sigma_t \cdot dB_t \tag{2.3.43}$$

is given by

$$X_t = \Phi_t \left(X_0 + \int_0^t \Phi_s{}^{-1} a_s ds + \int_0^t \Phi_s{}^{-1} \sigma_s \cdot dB_s \right), \tag{2.3.44}$$

where Φ_t^{-1} is the inverse matrix of Φ_t. This is easily checked by an application of the Itô formula.

Example 2.8 (Langevin equation). Let $d = 1$ and $a, \sigma > 0$. Consider the stochastic differential equation

$$dX_t = -aX_t dt + \sigma dB_t, \quad X_0 = x \in \mathbb{R}. \tag{2.3.45}$$

By (2.3.44) the solution of (2.3.45) is given by

$$X_t^x = \mathrm{e}^{-at} x + \sigma \int_0^t \mathrm{e}^{-a(t-s)} dB_s. \tag{2.3.46}$$

The random process $(X_t^x)_{t \geq 0}$ is called *Ornstein–Uhlenbeck process*. We have by (2.3.46)

$$\mathbb{E}[X_t^x] = \mathrm{e}^{-at} x, \quad \mathrm{cov}[X_t^x X_s^x] = \frac{\sigma^2}{2a}(1 - \mathrm{e}^{-2a(s \wedge t)})\mathrm{e}^{-a|t-s|}. \tag{2.3.47}$$

Moreover, the generator of X_t^x is given by

$$\frac{\sigma^2}{2}\Delta - ax \cdot \nabla. \tag{2.3.48}$$

Definition 2.26 (Brownian bridge). A *Brownian bridge* over the interval $[T_1, T_2]$ starting in $a \in \mathbb{R}^d$ and ending in $b \in \mathbb{R}^d$ is a multivariate Gaussian random process $(X_t)_{T_1 \leq t \leq T_2}$ on $(\Omega, \mathscr{F}, P)$ such that

(1) $X_{T_1} = a$ a.s.

(2) $\mathbb{E}_P[X_t] = a\left(1 - \dfrac{t}{T_2 - T_1}\right) + b\dfrac{t}{T_2 - T_1}$

(3) $\mathrm{cov}(X_s, X_t) = s \wedge t - \dfrac{st}{T_2 - T_1}$.

From this it follows that $\mathbb{E}_P[X_{T_2}] = b$, $\mathrm{cov}(X_s, X_{T_2}) = 0$, thus actually $X_{T_2} = b$ almost surely since $\mathbb{E}_P[|X_{T_2} - b|^2] = 0$, justifying the name of the process. For $t \in [T_1, T_2]$ the distributional relation

$$X_t \stackrel{\mathrm{d}}{=} \left(1 - \frac{t}{T_2 - T_1}\right) a + \frac{t}{T_2 - T_1}(b - B_{T_2}) + B_t \tag{2.3.49}$$

holds.

Example 2.9 (SDE for Brownian bridge). The Brownian bridge starting in a at $t = 0$ and ending in b at $t = T$ solves the stochastic differential equation

$$dX_t = \frac{b - X_t}{T - t} dt + dB_t, \quad 0 \leq t < T, \quad X_0 = a. \tag{2.3.50}$$

As a linear SDE, it can be solved exactly through (2.3.44) giving

$$X_t = a\left(1 - \frac{t}{T}\right) + b\frac{t}{T} + (T - t)\int_0^t \frac{1}{T - s} dB_s, \quad 0 \leq t < T. \tag{2.3.51}$$

Let

$$\alpha_t = (T - t)\int_0^t \frac{1}{T - r} dB_r, \quad 0 \leq t < T. \tag{2.3.52}$$

Then α_t is a Gaussian random variable with $\mathbb{E}[\alpha_t] = 0$ and

$$\mathbb{E}[\alpha_s \alpha_t] = (T - t)(T - s)\int_0^{t \wedge s} \frac{1}{(T - r)^2} dr = s \wedge t - \frac{st}{T}.$$

Thus

$$\mathbb{E}[X_t] = a\left(1 - \frac{t}{T}\right) + b\frac{t}{T}, \quad \operatorname{cov}(X_t, X_s) = s \wedge t - \frac{st}{T}.$$

It is easily seen that $X_t \to b$ as $t \to T$ in $L^2(\mathscr{X}, d\mathcal{W})$. Furthermore, it can also be shown that $X_t(\omega) \to b$ for almost every $\omega \in \mathscr{X}$.

2.3.5 Girsanov theorem and Cameron–Martin formula

In this section we add a pathwise shift to Brownian paths and show that this results in a Brownian motion on a suitable probability space.

Fix $T > 0$ and suppose that $a_t = a_t(\omega) = (a_t^1, \ldots, a_t^d)$ is such that

$$\sup_{\omega \in \mathscr{X}} \int_0^T |a_t(\omega)|^2 dt < \infty.$$

Define the random process $(Z_t)_{0 \leq t \leq T}$ by

$$dZ_t = a_t \cdot dB_t - \frac{1}{2} a_t^2 dt. \tag{2.3.53}$$

Next, define the measure $\hat{P}_T$ on $(\mathscr{X}, \mathscr{F}_T)$, with $\mathscr{F}_T = \mathscr{F}_T^B = \sigma(B_s, 0 \leq s \leq T)$, and the d-dimensional random process $(\hat{B}_t)_{0 \leq t \leq T}$ by putting

$$\hat{P}_T(A) = \mathbb{E}[e^{Z_T} 1_A], \quad A \in \mathscr{F}_T,$$

$$\hat{B}_t(\omega) = B_t(\omega) - \int_0^t a_s(\omega) ds, \quad t \in [0, T].$$

Note that $\hat{P}_T(\mathscr{X}) = 1$.

Proposition 2.38 (Girsanov theorem). *The random process $(e^{Z_t})_{0\le t\le T}$ is an $(\mathscr{F}_t)_{0\le t\le T}$-martingale and $(\hat{B}_t)_{0\le t\le T}$ is a d-dimensional Brownian motion on the space $(\mathscr{X}, \mathscr{F}_T, \hat{P}_T)$.*

Proof. Since $de^{Z_t} = Z_t a_t \cdot dB_t$, e^{Z_t} is an $(\mathscr{F}_t)_{0\le t\le T}$-martingale (already seen in Corollary 2.20). For $\phi \in C^2(\mathbb{R}^d)$ a combination of the Itô formula and the product formula (2.3.25) gives

$$d(e^{Z_t}\phi(\hat{B}_t)) = e^{Z_t}(\nabla\cdot\phi(\hat{B}_t) + a_t)\cdot dB_t + \frac{1}{2}e^{Z_t}\Delta\phi(\hat{B}_t)dt.$$

From this and by the martingale property of e^{Z_t} it follows that for $A \in \mathscr{F}_s$, $s \le t$,

$$\begin{aligned}\mathbb{E}_{\hat{P}_T}[\phi(\hat{B}_t)1_A] &= \mathbb{E}\left[\left(\phi(\hat{B}_0) + \int_0^s e^{Z_r}(\nabla\cdot\phi(\hat{B}_r) + a_r)\cdot dB_r\right)1_A\right] \\ &\quad + \frac{1}{2}\int_0^t \mathbb{E}[1_A e^{Z_r}\Delta\phi(\hat{B}_r)]dr \\ &= \mathbb{E}[1_A e^{Z_s}\phi(\hat{B}_s)] + \frac{1}{2}\int_s^t \mathbb{E}[1_A e^{Z_r}\Delta\phi(\hat{B}_r)]dr \\ &= \mathbb{E}_{\hat{P}_T}\left[\left(\phi(\hat{B}_s) + \frac{1}{2}\int_s^t \Delta\phi(\hat{B}_r)dr\right)1_A\right].\end{aligned}$$

Hence it can be seen that $Y_t = \phi(\hat{B}_t) - \phi(\hat{B}_0) - \frac{1}{2}\int_0^t \Delta\phi(\hat{B}_s)ds$ is an $(\mathscr{F}_t)_{0\le t\le T}$-martingale on $(\mathscr{X}, \mathscr{F}_T, \hat{P}_T)$. Set $\phi(x) = e^{i\xi\cdot x}$, $\xi \in \mathbb{R}^d$, and substitute this into Y_t. Then the random process

$$e^{i\xi\cdot\hat{B}_t} + \frac{|\xi|^2}{2}\int_0^t e^{i\xi\cdot\hat{B}_s}ds, \quad 0 \le t \le T,$$

is an $(\mathscr{F}_t)_{0\le t\le T}$-martingale on $(\mathscr{X}, \mathscr{F}_T, \hat{P}_T)$. Since $\frac{d}{dt}\mathbb{E}_{\hat{P}_T}[Y_t|\mathscr{F}_s] = \frac{d}{dt}Y_s = 0$, we have

$$\frac{d}{dt}\mathbb{E}_{\hat{P}_T}[e^{i\xi\cdot\hat{B}_t}|\mathscr{F}_s] = -\frac{|\xi|^2}{2}\mathbb{E}_{\hat{P}_T}[e^{i\xi\cdot\hat{B}_t}|\mathscr{F}_s].$$

Solving this equation yields

$$\mathbb{E}_{\hat{P}_T}[e^{i\xi(\hat{B}_t-\hat{B}_s)}|\mathscr{F}_s] = e^{-|\xi|^2(t-s)/2}. \tag{2.3.54}$$

Thus $(\hat{B}_t^2 - t)_{0\le t\le T}$ is an $(\mathscr{F}_t)_{0\le t\le T}$-martingale, therefore by Proposition 2.15 $(\hat{B}_t)_{0\le t\le T}$ is a Brownian motion on $(\mathscr{X}, \mathscr{F}_T, \hat{P}_T)$, as claimed. □

We conclude by two applications of Girsanov's theorem.

Example 2.10 (Drift transformation). Consider the process $(X_t)_{t\geq 0}$ given by the solution of

$$dX_t = dB_t + a_t(X_t)dt. \tag{2.3.55}$$

By the Girsanov theorem $\tilde{B}_t = B_t - \int_0^t a_s(B_s)ds$, $t \in [0,T]$, is a Brownian motion on $(\mathscr{X}, \mathscr{F}_T, e^{Z_T}\mathcal{W})$, where Z_T is given by (2.3.53). From the definition of $\tilde{B}_t$ we have

$$dB_t = d\tilde{B}_t + a_t(B_t)dt. \tag{2.3.56}$$

Put

$$d\tilde{Z}_t = a_t(B_t)\cdot dB_t - \frac{1}{2}a_t(B_t)^2dt.$$

We regard B_t as a solution of the stochastic differential equation (2.3.56) on $(\mathscr{X}, \mathscr{F}_T, e^{\tilde{Z}_T}\mathcal{W})$. Its generator is $\frac{1}{2}\Delta + a\cdot\nabla$, which coincides with that of X_t of the solution of (2.3.55) on $(\mathscr{X}, \mathscr{B}(\mathscr{X}), \mathcal{W})$. Therefore

$$\mathbb{E}[f(X_t)] = \mathbb{E}[f(B_t)e^{\tilde{Z}_T}].$$

Let Q be the distribution of $\mathcal{W}$ under the random variable $\mathscr{X} \ni \omega \mapsto X.(\omega) \in \mathscr{X}$. Then

$$\frac{dQ}{d\mathcal{W}} = \exp\left(\int_0^T a_s(B_s)\cdot dB_s - \frac{1}{2}\int_0^T a_s(B_s)^2ds\right).$$

Example 2.11 (Cameron–Martin formula). Here we consider the effect of pathwise shifts applied to Brownian motion. Let

$$dY_t = dB_t + a_t dt,$$
$$d\bar{Z}_t = a_t dB_t - \frac{1}{2}a_t^2 dt.$$

Note that a_t is independent of the path $\omega \in \mathscr{X}$ and

$$Y_t(\omega) = B_t(\omega) + \int_0^t a_s ds = \omega(t) + \int_0^t a_s ds$$

is a shift of ω by the path $\int_0^{\cdot} a_s ds$. Since $\bar{B}_t = B_t - \int_0^t a_s ds$ is a Brownian motion on $(\mathscr{X}, \mathscr{F}_T, e^{\bar{Z}_T}\mathcal{W})$ by the Girsanov theorem, we have that

$$\mathbb{E}[f(Y_t)] = \mathbb{E}\left[f\left(\bar{B}_t + \int_0^t a_s ds\right)e^{\bar{Z}_T}\right] = \mathbb{E}[f(B_t)e^{\bar{Z}_T}]. \tag{2.3.57}$$

Here we used that $\int_0^t a_s ds$ is independent of path ω.

Let Q be the distribution of $\mathcal{W}$ under the random variable $\mathscr{X} \in \omega \to Y.(\omega) \in \mathscr{X}$ on $(\mathscr{X}, \mathcal{B}(\mathscr{X}))$. By (2.3.57)

$$\frac{dQ}{d\mathcal{W}} = \exp\left(\int_0^t a_s \cdot dB_s - \frac{1}{2}\int_0^t a_s^2 ds\right) \tag{2.3.58}$$

follows, known as the *Cameron–Martin formula*.

2.4 Lévy processes

2.4.1 Lévy process and Lévy–Khintchine formula

Brownian motion is a specific case of a wider class of random processes which itself is of interest for our purposes. It is a simple fact of elementary probability that the distributions of sums of independent, identically distributed random variables are obtained as convolutions of their distributions. A natural question is then whether one can break up a given random process into sums of others, simpler or canonical ones. This leads to considering the following concept.

Definition 2.27 (Infinite divisibility). An $\mathbb{R}^d$-valued random variable X on a probability space $(\Omega, \mathscr{F}, P)$ is *infinitely divisible* if for every $n \in \mathbb{N}$ there exist independent, identically distributed random variables $X_1^{(n)}, \ldots, X_n^{(n)}$ such that

$$X \stackrel{\mathrm{d}}{=} X_1^{(n)} + \cdots + X_n^{(n)}. \tag{2.4.1}$$

Equivalently, the probability distribution μ on $\mathbb{R}^d$ of X is infinitely divisible if there exists a probability distribution μ_n common to all $X_j^{(n)}$ such that

$$\mu = \underbrace{\mu_n * \cdots * \mu_n}_{n\text{-fold}}. \tag{2.4.2}$$

Recall that the convolution of distributions μ_1 and μ_2 on $\mathbb{R}^d$ is defined by the formula

$$(\mu_1 * \mu_2)(A) = \int_{\mathbb{R}^d \times \mathbb{R}^d} 1_A(x+y)\mu_1(dx)\mu_2(dy)$$

for $A \in \mathcal{B}(\mathbb{R}^d)$. In terms of the characteristic function $\hat{\mu}$ of μ, i.e.,

$$\hat{\mu}(u) = \mathbb{E}_P[e^{iu\cdot X}] = \int_{\mathbb{R}^d} e^{iu\cdot x}\mu(dx), \quad u \in \mathbb{R}^d,$$

we can alternatively say that a random variable is infinitely divisible if and only if for every $n \in \mathbb{N}$ there is a characteristic function $\hat{\mu}_n$ satisfying

$$\hat{\mu}(u) = (\hat{\mu}_n(u))^n. \tag{2.4.3}$$

Let μ be a probability measure on $\mathbb{R}^d$. Then the characteristic function of μ has the properties (1) $\hat{\mu}(0) = 1$ and $|\hat{\mu}(u)| \leq 1$; (2) $\hat{\mu}$ is uniformly continuous; (3) $\hat{\mu}$ is non-negative definite, i.e.,

$$\sum_{k,l=1}^{n} \hat{\mu}(u_k - u_l) z_k \bar{z}_l \geq 0 \tag{2.4.4}$$

for $u_j \in \mathbb{R}^d$, $z_i \in \mathbb{C}$, $i, j = 1, \ldots, n$, $n \geq 1$. A remarkable fact is that the converse of this statement is also true: If a function $\varphi : \mathbb{R}^d \to \mathbb{C}$ satisfies properties (1)-(3) above, then there exists a probability measure μ on $\mathbb{R}^d$ such that the characteristic function of μ is $\hat{\mu} = \varphi$. This is Bochner's theorem, discussed in Theorem 5.10 below.

Example 2.12. Examples of infinitely divisible processes include:

(1) *Gaussian distribution:* Take a Gaussian random variable $X \stackrel{\mathrm{d}}{=} N(m, \sigma^2)$ over $\mathbb{R}$ with Gaussian distribution

$$\mu(dx) = (2\pi\sigma^2)^{-1/2} \exp\left(-\frac{(x-m)^2}{2\sigma^2}\right) dx.$$

Since by direct computation

$$\hat{\mu}(u) = e^{ium-(1/2)u^2\sigma^2} = \left(\exp\left(iu\frac{m}{n} - \frac{1}{2}u^2\left(\frac{\sigma}{\sqrt{n}}\right)^2\right)\right)^n,$$

we have $X_1^{(n)}, \ldots, X_n^{(n)} \stackrel{\mathrm{d}}{=} N\left(m/n, (\sigma/\sqrt{n})^2\right)$.

(2) *Poisson distribution:* Since for a Poisson random variable $X \stackrel{\mathrm{d}}{=} \mathrm{Poi}\,(\lambda)$ over $\mathbb{R}$ with the distribution

$$\mu(\{k\}) = e^{-\lambda}\frac{\lambda^k}{k!}, \quad k \in \mathbb{N} \cup \{0\},$$

we have

$$\hat{\mu}(u) = \exp\left(\lambda(e^{iu} - 1)\right) = \left(\exp\left(\frac{\lambda}{n}(e^{iu} - 1\right)\right)^n,$$

we can identify $X_1^{(n)}, \ldots, X_n^{(n)} \stackrel{\mathrm{d}}{=} \mathrm{Poi}\ (\lambda/n)$.

(3) *Cauchy distribution:* Let $X \stackrel{\mathrm{d}}{=} C(\alpha, \beta)$, $\alpha > 0$, $\beta \in \mathbb{R}^d$, be a random variable with the Cauchy distribution on $\mathbb{R}^d$

$$\mu(dx) = \frac{\Gamma((d+1)/2)}{\pi^{(d+1)/2}} \frac{\alpha}{(|x-\beta|^2 + \alpha^2)^{(d+1)/2}} dx.$$

Then

$$\hat{\mu}(u) = \exp\left(-\alpha|u| + i\beta \cdot u\right) = \left(\exp\left(-\frac{\alpha}{n}|u| + i\frac{\beta}{n} \cdot u\right)\right)^n.$$

Thus we can identify $X_1^{(n)}, \dots, X_n^{(n)} \stackrel{\mathrm{d}}{=} C(\alpha/n, \beta/n)$.

Two important counterexamples are the uniform and binomial distributions.

Next we will see that there is a general formula describing the characteristic function of any infinitely divisible random variable.

Definition 2.28 (Lévy measure). A σ-finite Borel measure ν on $\mathcal{B}(\mathbb{R}^d \setminus \{0\})$ satisfying

$$\int_{\mathbb{R}^d\setminus\{0\}} (|y|^2 \wedge 1)\nu(dy) < \infty \tag{2.4.5}$$

is called *Lévy measure*.

Theorem 2.39 (*Lévy–Khintchine formula*). *If a probability distribution μ on $\mathbb{R}^d$ is infinitely divisible, then there exist a vector $b \in \mathbb{R}^d$, a non-negative definite symmetric $d \times d$ matrix A and a Lévy measure ν such that $\hat{\mu}(u) = e^{\eta(u)}$, for all $u \in \mathbb{R}^d$, where*

$$\eta(u) = ib \cdot u - \frac{1}{2}u \cdot Au + \int_{\mathbb{R}^d\setminus\{0\}} (e^{iu\cdot y} - 1 - iu \cdot y\, 1_{\{|y|<1\}})\, \nu(dy). \tag{2.4.6}$$

The triplet (b, A, ν) is uniquely given. Conversely, given a triplet (b, A, ν) with the above properties there exists an infinitely divisible probability distribution μ on $\mathbb{R}^d$ whose characteristic function is $\hat{\mu} = e^{\eta(u)}$.

Remark 2.2. The integrand in (2.4.6) satisfies

$$|e^{iu\cdot y} - 1 - iu \cdot y\, 1_{\{|y|<1\}}| \le \begin{cases} \frac{1}{2}|u|^2|y|^2 + O(|y|^3), & |y| < 1, \\ 2, & |y| \ge 1 \end{cases},$$

ensuring that the integral is well defined. Furthermore, if $\int |y| 1_{\{|y|<1\}}\nu(dy)$ is bounded, then (2.4.6) becomes

$$\eta(u) = ib' \cdot u - \frac{1}{2}u \cdot Au + \int_{\mathbb{R}^d\setminus\{0\}} (e^{iu\cdot y} - 1)\nu(dy)$$

with $b' = b - \int_{\{|y|<1\}} y\nu(dy)$.

Now we consider the following large class of random processes.

Definition 2.29 (Lévy process). A random process $(X_t)_{t\geq 0}$ on a probability space $(\Omega, \mathscr{F}, P)$ is called a *Lévy process* if

(1) $X_0(\omega) = 0$ for almost every $\omega \in \Omega$

(2) the increments $(X_{t_i} - X_{t_{i-1}})_{1\leq i\leq n}$ are independent random variables for any $0 = t_0 < t_1 < \cdots < t_n$

(3) $X_t - X_s \stackrel{\mathrm{d}}{=} X_{t-s} - X_0$ for $0 \leq s \leq t$

(4) X_t is stochastically continuous, i.e., for all $t \geq 0$ and $\varepsilon > 0$

$$\lim_{s\to t} P(|X_s - X_t| > \varepsilon) = 0.$$

Remark 2.3 (Càdlàg paths). It can be shown that every Lévy process has a version with paths $t \mapsto X_t$ which are right continuous and have left limits; functions, and thereby random processes, of this type are called *càdlàg* retaining the original French terminology "continue à droite, limitée à gauche." The space of $\mathbb{R}^d$-valued càdlàg functions on an interval $I \subset \mathbb{R}$ is denoted by $D(I; \mathbb{R}^d)$; it is a linear space with respect to pointwise addition and multiplication by real numbers.

There is a close relationship between infinitely divisible distributions and Lévy processes.

Theorem 2.40.

(1) *Let $(X_t)_{t\geq 0}$ be a Lévy process. Then for every t the distribution μ_t of X_t is infinitely divisible.*

(2) *Conversely, for every infinitely divisible distribution μ on $\mathbb{R}^d$ with a triplet (b, A, ν), where $b \in \mathbb{R}^d$, A is a $d \times d$ non-negative definite symmetric matrix and ν a Lévy measure, there is a Lévy process $(X_t)_{t\geq 0}$ on $(\Omega, \mathscr{F}, P)$ such that the distribution of X_1 is μ and the characteristic triplet of X_t is $(tb, tA, t\nu)$, i.e.,*

$$\mathbb{E}_P[e^{iuX_t}] = e^{t\eta(u)}, \tag{2.4.7}$$

where η is given by (2.4.6).

The vector b is called *drift term*, the matrix A *diffusion matrix* and the triplet (b, A, ν) the *characteristics* of the Lévy process $(X_t)_{t\geq 0}$, while η given by (2.4.6) is called *Lévy symbol*.

Example 2.13. By Theorem 2.40 the processes in Example 2.12 are Lévy processes. The Lévy triplet of Brownian motion with drift (Itô process) is $(b, A, 0)$ and its Lévy symbol is $\eta(u) = ib \cdot u - \frac{1}{2}u \cdot Au$. For a Poisson process with intensity $\lambda > 0$ the Lévy triplet is $(0, 0, \lambda\delta_1)$, where δ_1 is one dimensional Dirac measure, and the Lévy symbol is $\eta(u) = \lambda(e^{iu} - 1)$. The Cauchy process is a specific case of the class in the next example.

Example 2.14 (Stable processes). A Lévy process $(X_t)_{t\geq 0}$ is called a *stable process* if for any $a > 0$ there exist $b > 0$ and $c \in \mathbb{R}$ such that $X_{at} \stackrel{\mathrm{d}}{=} bX_t + ct$, and a *strictly stable process* if $c = 0$. This definition originates from an extension of the scaling property of Brownian motion, see Proposition 2.7 (3). From the definition it can be seen that for a stable process there exist $0 < \alpha \leq 2$ and $b_t \in \mathbb{R}^d$ such that $X_t \stackrel{\mathrm{d}}{=} t^{1/\alpha}X_1 + b_t$; for a strictly stable process one can take $b_t = 0$. α is called the *index* of the stable process. For a stable process either $\alpha = 2$ and its Lévy triplet is $(b, A, 0)$, i.e., it is a Gaussian process with mean b and covariance matrix A, or $0 < \alpha < 2$ and its Lévy triplet is $(b, 0, \nu)$ with Lévy measure

$$\nu(E) = \int_{S^{d-1}} \lambda(d\theta) \int_0^\infty 1_E(r\theta)\frac{dr}{r^{1+\alpha}}, \tag{2.4.8}$$

where S^{d-1} is the $d-1$-dimensional unit sphere centered in the origin. The above integral can be computed; for $d = 1$ the calculation gives

$$\nu(dy) = \left(\frac{c_1}{y^{1+\alpha}}1_{(0,\infty)} + \frac{c_2}{(-y)^{1+\alpha}}1_{(-\infty,0)}\right) dy, \quad c_1, c_2 \geq 0. \tag{2.4.9}$$

A stable process with index $0 < \alpha \leq 2$ is rotation invariant if $\mathscr{R}X_t \stackrel{\mathrm{d}}{=} X_t$, for all $\mathscr{R} \in O(\mathbb{R}^d)$, the orthogonal group. In this case the Lévy symbol of the process becomes $\eta(u) = -c|u|^\alpha$, with some $c > 0$, and for $0 < \alpha < 2$ the Lévy measure has uniform distribution on S_{d-1}. In particular, the rotation invariant 1-stable process is the Cauchy process.

2.4.2 Markov property of Lévy processes

In this section we show that Lévy processes are Markov processes possessing Feller transition kernels, and derive their generators.

Let $(X_t)_{t\geq 0}$ be a Lévy process on $(\Omega, \mathscr{F}, P)$. Let μ_t be the infinitely divisible distribution of X_t and define $p(s,t,x,A)$ by

$$p(s,t,x,A) = \mu_{t-s}(A - x) \tag{2.4.10}$$

for $t \geq s \geq 0$, $x \in \mathbb{R}^d$ and $A \in \mathscr{B}(\mathbb{R}^d)$, where $A - x = \{y - x | y \in A\}$. Note that $\mu_t(A - x) = P(X_t + x \in A)$. By the definition of a Lévy process we have $p(s,t,x,A) = P(X_t - X_s \in A - x)$ and $p(s,t,x,A)$ is a probability transition kernel. The Chapman–Kolmogorov identity results from the semigroup property

$$\mu_{t-s} * \mu_{u-t} = \mu_{u-s}, \quad s < t < u. \tag{2.4.11}$$

Set

$$p(s,t,x,A) = p(t-s,x,A). \tag{2.4.12}$$

Proposition 2.41. *A Lévy process $(X_t)_{t \geq 0}$ is a stationary Markov process and its probability transition kernel is given by $p(t, x, A)$ under* (2.4.12).

Proof. By the independent increment property of the process its finite dimensional distributions can be written as

$$P(X_0 \in A_0, \ldots, X_n \in A_n) = \int_{\mathbb{R}^{(n+1)d}} \prod_{i=0}^{n} 1_{A_i}(x_i) \prod_{i=1}^{n} p(t_i - t_{i-1}, x_{i-1}, dx_i) P^0(dx_0), \tag{2.4.13}$$

where $P^0(dx) = \delta(x)$ is Dirac measure with mass at $x = 0$. Thus by Proposition 2.17, $(X_t)_{t \geq 0}$ is a Markov process with probability transition kernel $p(t, x, A)$. □

Theorem 2.42 (C_0-semigroup on $C_\infty(\mathbb{R}^d)$). *The probability transition kernel $p(t, x, A)$ of a Lévy process is a Feller transition kernel. In particular, $P_t f(x) = \int_{\mathbb{R}^d} p(t, x, dy) f(y)$ defines a C_0-semigroup on $C_\infty(\mathbb{R}^d)$.*

Proof. The fact $p(t, x, A) \leq 1$ is trivial. We show that $P_t f \in C_\infty(\mathbb{R}^d)$ for $f \in C_\infty(\mathbb{R}^d)$. Let $x_n \to x$ as $n \to \infty$. By the dominated convergence theorem we have

$$\lim_{n \to \infty} P_t f(x_n) = \int_{\mathbb{R}^d} \lim_{n \to \infty} p(t, 0, dy) f(x_n + y) = P_t f(x)$$

and

$$\lim_{|x| \to \infty} P_t f(x) = \int_{\mathbb{R}^d} \lim_{|x| \to \infty} p(t, 0, dy) f(x + y) = 0.$$

Thus $P_t f \in C_\infty(\mathbb{R}^d)$. To conclude, we show that $\lim_{t \to 0} P_t f = f$ for $f \in C_\infty(\mathbb{R}^d)$ in the sup-norm. Since f is uniformly continuous, for any $\varepsilon > 0$ there exists an $r > 0$ such that $\sup_{x \in \mathbb{R}^d} |f(x + y) - f(x)| < \varepsilon$ for all $|y| < r$. Moreover, by stochastic continuity of the Lévy process $\int_{|y|>r} p(t, 0, dy) = P(|X_t| > r) \to 0$ as $t \to 0$. Thus there exists t_0 such that $\int_{|y|>r} p(t, 0, dy) < \varepsilon$ for $0 < t < t_0$. Hence

$$\begin{aligned}
\sup_{x \in \mathbb{R}^d} |P_t f(x) - f(x)| &= \sup_{x \in \mathbb{R}^d} \left| \int_{\mathbb{R}^d} (f(x + y) - f(x)) p(t, 0, dy) \right| \\
&= \sup_{x \in \mathbb{R}^d} \left| \int_{|y| \leq r} (f(x + y) - f(x)) p(t, 0, dy) + \int_{|y|>r} (f(x + y) - f(x)) p(t, 0, dy) \right| \\
&\leq \varepsilon \int_{|y|<r} p(t, 0, dy) + \varepsilon \| f \|_\infty
\end{aligned}$$

for $0 < t < t_0$. Thereby $\lim_{t \to 0} P_t f = f$ follows. □

This amply illustrates the fact that the class of Feller transition kernels is far from empty. When the infinitely divisible distribution μ_t of $(X_t)_{t\geq 0}$ satisfies

$$\mu_t(-A) = \mu_t(A), \tag{2.4.14}$$

P_t can be extended to a C_0-semigroup on $L^2(\mathbb{R}^d)$.

Theorem 2.43 (C_0-semigroup on $L^2(\mathbb{R}^d)$). *Suppose that* (2.4.14) *holds. Then P_t can be extended to a symmetric C_0-semigroup on $L^2(\mathbb{R}^d)$.*

Proof. Suppose that $f, g \geq 0$ and $\int_{\mathbb{R}^d} g(x)(P_t f)(x)dx < \infty$. Since

$$\begin{aligned}\int_{\mathbb{R}^d} dx g(x)(P_t f)(x) &= \int_{\mathbb{R}^d} dx g(x) \int_{\mathbb{R}^d} f(x+y)p(t,0,dy) \\ &= \int_{\mathbb{R}^d} dx g(x-y) \int_{\mathbb{R}^d} f(x)p(t,0,dy),\end{aligned}$$

(2.4.14) implies that

$$\begin{aligned}\int_{\mathbb{R}^d} dx g(x)(P_t f)(x) &= \int_{\mathbb{R}^d} dx g(x+y) \int_{\mathbb{R}^d} f(x)p(t,0,dy) \\ &= \int_{\mathbb{R}^d} dx (P_t g)(x) f(x).\end{aligned} \tag{2.4.15}$$

Let $f \in L^2(\mathbb{R}^d) \cap C_\infty(\mathbb{R}^d)$. Thus by Schwarz inequality and the symmetry of equality (2.4.15) we have

$$\begin{aligned}\int_{\mathbb{R}^d} |(P_t f)(x)|^2 dx \leq \int_{\mathbb{R}^d} P_t(|f|^2)(x)dx &= \int_{\mathbb{R}^d} 1 \cdot P_t(|f|^2)(x)dx \\ &= \int_{\mathbb{R}^d} P_t 1 \cdot |f|^2(x)dx.\end{aligned}$$

$P_t 1 = 1$ implies that $\|P_t f\| \leq \|f\|$. Since $L^2(\mathbb{R}^d) \cap C_\infty(\mathbb{R}^d)$ is dense, the inequality $\|P_t f\| \leq \|f\|$ can be extended to $f \in L^2(\mathbb{R}^d)$, and the theorem follows. □

By the Hille–Yoshida theorem (see Proposition 3.25) the semigroup $\{P_t : t \geq 0\}$ associated with a Lévy process is of the form $P_t = e^{tL}$ with a generator L on $C_\infty(\mathbb{R}^d)$. The operator L is closed on $C_\infty(\mathbb{R}^d)$ endowed with the sup-norm. The semigroup $\{P_t : t \geq 0\}$ can be also viewed as a pseudo-differential operator with symbol $e^{t\eta(k)}$ (see below). Denote by $\mathscr{S}(\mathbb{R}^d)$ Schwartz space over $\mathbb{R}^d$, i.e., the space of infinitely differentiable rapidly decreasing functions on $\mathbb{R}^d$.

Proposition 2.44 (Generator of a Lévy process). *Let $(X_t)_{t\geq 0}$ be a Lévy process with characteristics (b, A, ν) and symbol η. Let $\{P_t : t \geq 0\}$ be the associated C_0-semigroup on $C_\infty(\mathbb{R}^d)$ and L its generator, i.e., $P_t = e^{tL}$. Then the following hold:*

(1) *For* $t \geq 0$, $f \in \mathscr{S}(\mathbb{R}^d)$ *and* $x \in \mathbb{R}^d$,

$$(P_t f)(x) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{ikx} e^{t\eta(k)} \hat{f}(k) dk. \tag{2.4.16}$$

(2) *For* $f \in \mathscr{S}(\mathbb{R}^d)$ *and* $x \in \mathbb{R}^d$,

$$Lf(x) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{ikx} \eta(k) \hat{f}(k). \tag{2.4.17}$$

In particular

$$L = \frac{1}{2} \nabla \cdot A \nabla + b \cdot \nabla + L_\nu, \tag{2.4.18}$$

where

$$L_\nu f(x) = \int_{\mathbb{R}^d \setminus \{0\}} (f(x+y) - f(x) - y \cdot \nabla f(x) 1_{\{|y|<1\}}) \nu(dy).$$

Proof. (1) By the definition of P_t we have

$$(P_t f)(x) = (2\pi)^{-d/2} \mathbb{E}_P \left[\int_{\mathbb{R}^d} e^{ik(x+X_t)} \hat{f}(k) dk \right].$$

Since $\mathbb{E}_P[e^{ik \cdot X_t}] = e^{t\eta(k)}$, we have (1). To prove (2) consider the generator L given by

$$(Lf)(x) = \lim_{t \to 0} (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{ikx} \hat{f}(k) \left(\frac{e^{t\eta(k)} - 1}{t} \right) dk.$$

Since $|e^{ikx} \hat{f}(k) (\frac{e^{t\eta(k)}-1}{t})| \leq (1 + |k|^2) \hat{f} \in L^2(\mathbb{R}^d)$, the result follows by the dominated convergence theorem. □

Example 2.15. The generators of some specific Lévy process are as follows:

(1) Brownian motion with drift:

$$L = \frac{1}{2} \nabla \cdot A \nabla + b \cdot \nabla. \tag{2.4.19}$$

(2) Poisson process with intensity λ:

$$Lf(x) = \lambda (f(x+1) - f(x)). \tag{2.4.20}$$

(3) α-stable process:

$$L = -(-\Delta)^{\alpha/2}. \tag{2.4.21}$$

2.4.3 Random measures and Lévy–Itô decomposition

Lévy processes make a class large enough to contain random processes with jumps. One would like to know if there is a canonical way of separating the continuous part from the part with discontinuous paths. As we will see now, any Lévy process can be decomposed into a sum of four independent random processes accounting for these components.

Before discussing the decomposition of Lévy processes we briefly review *random measures*. Let $(S, \mathscr{S})$ be a measurable space and $(\Omega, \mathscr{F}, P)$ a probability space. Furthermore, let $M = (M(A))_{A \in \mathscr{S}}$ be a collection of random variables $M(A)(\cdot) : \Omega \to \mathbb{R}$ indexed by $A \in \mathscr{S}$.

Definition 2.30 (Random measure). $M = (M(A))_{A \in \mathscr{S}}$ is a *random measure* whenever

(1) $M(\emptyset) = 0$

(2) $M(\cup_n A_n) = \sum_{n=1}^{\infty} M(A_n)$ a.s., provided that $A_n \cap A_m = \emptyset$ for $n \neq m$

(3) $M(A)$ and $M(B)$ are independent whenever $A \cap B = \emptyset$.

Definition 2.31 (Poisson random measure). If in addition to (1)–(3) of Definition 2.30 each $M(A)$ has a Poisson distribution with intensity $\lambda(A)$, i.e.,

$$P(M(A) = n) = e^{-\lambda(A)} \frac{\lambda(A)^n}{n!}, \tag{2.4.22}$$

then M is called a *Poisson random measure*.

From (2.4.22) it follows that $\lambda(A) = \mathbb{E}_P[M(A)]$ and it is a measure on $\mathscr{S}$ since M is a random measure. Conversely, given a σ-finite measure λ on $\mathscr{S}$, there exists a probability space $(\Omega, \mathscr{G}, P)$ and a Poisson random measure M such that $\lambda(A) = \mathbb{E}_P[M(A)]$ for $A \in \mathscr{S}$.

Next we define the *counting measure* associated with a Lévy process $(X_t)_{t \geq 0}$ on $(\Omega, \mathscr{F}, P)$. Every càdlàg function is bounded on compact intervals and attains its extrema there, moreover, every such function is Borel measurable. Denote $X_{t-} = \lim_{s \uparrow t} X_s$ and define the *jump* at t by

$$\Delta X_t = X_t - X_{t-}. \tag{2.4.23}$$

A càdlàg path can only have jump discontinuities, and the set

$$\mathscr{J} = \{t \in \mathbb{R}^+ | \Delta X_t \neq 0\} \tag{2.4.24}$$

of time points where a jump occurs, is at most countable but does not have accumulation points for each path. Lévy processes can be classified by their sample path properties:

(1) *type* I*:* $A = 0$ and $\nu(\mathbb{R}^d \setminus \{0\}) < \infty$;

(2) *type* II*:* $A = 0$, $\nu(\mathbb{R}^d \setminus \{0\}) = \infty$ and $\int_{0<|y|\le 1} |y|\nu(dy) < \infty$;

(3) *type* III*:* $A \neq 0$ or $\int_{0<|y|\le 1} |y|\nu(dy) = \infty$.

A process of type III has a finite number of jumps in every bounded interval. If a process is of type I or II, then the paths are of bounded variation on every bounded interval, while if it is of type III, then paths are of unbounded variation on any interval. The *counting measure* $N(t,\cdot) : \mathcal{B}(\mathbb{R}^d \setminus \{0\}) \to \mathbb{N} \cup \{0\}$ adds up the jumps of specific size, i.e., for $0 \le t$ and $A \in \mathcal{B}(\mathbb{R}^d \setminus \{0\})$,

$$N(t, A) = |\{0 \le s \le t | \Delta X_s \in A\}|. \tag{2.4.25}$$

It is seen that $N(t,\cdot)(\omega)$ is a measure on $\mathcal{B}(\mathbb{R}^d \setminus \{0\})$ and $\mathbb{E}_P[N(t,\cdot)]$ is a Borel measure on $\mathcal{B}(\mathbb{R}^d \setminus \{0\})$, while

$$\mu(\cdot) = \mathbb{E}_P[N(1,\cdot)] \tag{2.4.26}$$

is called *intensity measure*. In Theorem 2.47 below we will see that μ coincides with the Lévy measure ν of a Lévy process. When the expectation of the number of small jumps at $t = 1$ of a Lévy process is zero, the Lévy symbol is simply

$$\eta(u) = ib \cdot u - \frac{1}{2} u \cdot Au + \int_{|y|\ge 1} (e^{iu\cdot y} - 1)\nu(dy).$$

Proposition 2.45. *We have the following properties:*

(1) *Let* $A \in \mathcal{B}(\mathbb{R}^d \setminus \{0\})$ *and* $0 \notin \bar{A}$. *Then* $(N(t, A))_{t\ge 0}$ *is a Poisson process with intensity* $\mu(A)$.

(2) $N(t, A_1), \dots, N(t, A_n)$ *for pairwise disjoint* $A_1, \dots, A_n \in \mathcal{B}(\mathbb{R}^d \setminus \{0\})$ *are independent.*

To see the relationship between the Lévy measure ν associated with a Lévy process $(X_t)_{t\ge 0}$ and the intensity measures μ given by (2.4.26) we note that for $f = \sum_{j=1}^n c_j 1_{A_j}$ with $A_j, A \in \mathcal{B}(\mathbb{R}^d \setminus \{0\})$, $A_j \subset A$ and $A_i \cap A_j = \emptyset$ for $i \neq j$,

$$\begin{aligned}\mathbb{E}_P[e^{iu\int_A f(x)N(t,dx)}] &= \prod_{j=1}^n \mathbb{E}_P[e^{iuc_j N(t,A_j)}] \\ &= e^{t\sum_{j=1}^n (e^{iuc_j}-1)\mu(A_j)} = e^{t\int_A (e^{iuf(x)}-1)\mu(dx)}.\end{aligned}$$

The following result is obtained by an approximation procedure.

Lemma 2.46. *Let* $A \in \mathcal{B}(\mathbb{R}^d \setminus \{0\})$ *and* $f \in L^1(A, \mu)$. *Then*

$$\mathbb{E}_P[e^{i\int_A f(x)N(t,dx)}] = e^{t\int_A (e^{if(x)}-1)\mu(dx)}. \tag{2.4.27}$$

By replacing f by sf with $s \in \mathbb{R}$ in (2.4.27) and differentiating both sides above at $s = 0$, we obtain $\mathbb{E}[\int_A f(x)N(t,dx)] = t\int_A f(x)\mu(dx)$, and the covariance $\mathbb{E}[|\int_A f(x)N(t,dx)|^2] = t\int_A |f(x)|^2\mu(dx)$. Thus for a general $f \in L^1(A, d\mu)$ the random process

$$\left(\int_A f(x)N(t,dx)\right)_{t\geq 0} \tag{2.4.28}$$

is well defined. In order to obtain the decomposition of a Lévy process $(X_t)_{t\geq 0}$, we define the total sum of jumps over path X_t. For $A \subset \mathcal{B}(\mathbb{R}^d \setminus \{0\})$,

$$\int_0^t \int_A zN(dsdz) = \int_A zN(t,dz) = \sum_{0\leq u\leq t} \Delta X_u 1_A(\Delta X_u)$$

is the sum of all the jumps taking values in set A up to time t. Since the paths of $(X_t)_{t\geq 0}$ are càdlàg, this is a finite sum of random variables. Note that the random processes $(\int_A zN(t,dz))_{t\geq 0}$ and $(\int_B zN(t,dz))_{t\geq 0}$ are independent as soon as $A \cap B = \emptyset$. The difference between the counting measure $N(t,\cdot)$ and the intensity measure μ

$$\tilde{N}(t,A) = N(t,A) - t\mu(A) \tag{2.4.29}$$

defines a *compensated Poisson random measure*. It can be checked that $\tilde{N}(t,A)$ is martingale. By Lemma 2.46 we can compute the characteristic functions of

$$Z_t^{(1)} = \int_{|z|\geq 1} zN(t,dz) \quad \text{and} \quad Z_t^{(2)} = \int_{0<|z|<1} z\tilde{N}(t,dz)$$

as $\mathbb{E}_P[e^{iuZ_t^{(1)}}] = e^{t\eta^{(1)}(u)}$ and $\mathbb{E}_P[e^{iuZ_t^{(2)}}] = e^{t\eta^{(2)}(u)}$, respectively, where

$$\eta^{(1)}(u) = \int_{|\lambda|\geq 1} (e^{iu\cdot y} - 1)\mu(dy), \tag{2.4.30}$$

$$\eta^{(2)}(u) = \int_{0<|\lambda|<1} (e^{iu\cdot y} - 1 - u\cdot y1_{\{|\lambda|<1\}})\mu(dy). \tag{2.4.31}$$

Then the characteristic function of the process

$$X_t = bt + \sqrt{A}\cdot B_t + Z_t^{(1)} + Z_t^{(2)}, \quad t \geq 0, \tag{2.4.32}$$

with $b \in \mathbb{R}^d$, a $d \times d$ non-negative matrix A such that $A = \sqrt{A}\sqrt{A}^T$, is given by $\mathbb{E}_P[e^{iuX_t}] = e^{t\eta(u)}$, where

$$\eta(u) = iu\cdot b - \frac{1}{2}u\cdot Au + \int_{\mathbb{R}^d\setminus\{0\}} (e^{iu\lambda} - 1 - u\cdot\lambda 1_{\{|\lambda|<1\}})\mu(d\lambda). \tag{2.4.33}$$

Finally we can state the Lévy–Itô decomposition theorem saying that every Lévy process can be represented as the sum of four independent components of which a part has almost surely continuous paths and the other part can be expressed as a compensated sum of jumps.

Theorem 2.47 (*Lévy–Itô decomposition*). *Let $(X_t)_{t\geq 0}$ be a Lévy process with characteristics (b, A, ν) and N the counting measure on $\mathcal{B}(\mathbb{R}^+) \times \mathcal{B}(\mathbb{R}^d \setminus \{0\})$ in* (2.4.25). *Write*

$$\tilde{N}(dsdz) = N(dsdz) - ds\nu(dz). \tag{2.4.34}$$

Then

(1) *N is a Poisson random measure with intensity measure $ds\nu(dz)$.*

(2) *X_t can be written as the sum of four independent Lévy processes, i.e.,*

$$X_t = X_t^{(1)} + X_t^{(2)} + X_t^{(3)} + X_t^{(4)}, \tag{2.4.35}$$

where

$$X_t^{(1)} = bt, \qquad X_t^{(2)} = \sqrt{A} \cdot B_t,$$
$$X_t^{(3)} = \int_0^t \int_{|z|\geq 1} zN(dsdz), \quad X_t^{(4)} = \int_0^t \int_{0<|z|<1} z\tilde{N}(dsdz).$$

$X_t^{(1)}$ is a constant drift with Lévy symbol $\eta_1(u) = iu \cdot b$, $X_t^{(2)}$ a Brownian motion with $\eta_2(u) = -(1/2)u \cdot Au$, where $A = \sqrt{A}\sqrt{A}^T$, $X_t^{(3)}$ a compound Poisson process with $\eta_3(u) = \int_{|y|\geq 1}(e^{iu\cdot y} - 1)\nu(dy)$, and $X_t^{(4)}$ a square integrable martingale with $\eta_4(u) = \int_{0<|y|<1}(e^{iu\cdot y} - 1 - iu \cdot y)\nu(dy)$.

2.4.4 Itô formula for semimartingales

In view of the Lévy–Itô decomposition a Lévy process can be split into four integrals with respect to ds, dB_s, $N(dsdz)$ and $\tilde{N}(dsdz)$. This justifies to consider a general stochastic integral of the form

$$\begin{aligned} X_t = X_0 &+ \int_0^t a(s,\omega)ds + \int_0^t b(s,\omega) \cdot dB_s \\ &+ \int_0^t \int_{\mathbb{R}^d\setminus\{0\}} c(s,z,\omega)N(dsdz) + \int_0^t \int_{\mathbb{R}^d\setminus\{0\}} d(s,z,\omega)\tilde{N}(dsdz), \end{aligned} \tag{2.4.36}$$

where the integrands satisfy suitable conditions for the integrals to exist. Furthermore, we want to consider the process $Y_t = F(X_t)$ for a function F and are interested whether Y_t is again of the form as (2.4.36).

Let $(\Omega, \mathscr{F}, P)$ be a probability space with filtration $(\mathscr{F}_t)_{t\geq 0}$. Fix a Brownian motion $(B_t)_{t\geq 0}$ and a Lévy process $(X_t)_{t\geq 0}$ on $(\Omega, \mathscr{F}, P)$ assuming them to be independent. $N(t, A)$ is the Poisson random measure associated with the Lévy process, and $\tilde{N}(dtdz)$ is given by (2.4.34) with ν being its Lévy measure.

Let $\mathscr{P}$ be the smallest σ-field on $\mathbb{R}^+ \times \mathbb{R}^d \times \Omega$ such that all g having the properties below are measurable:

(1) for every $t \geq 0$, the mapping $(z, \omega) \mapsto g(t, z, \omega)$ is $\mathcal{B}(\mathbb{R}^d \setminus \{0\}) \times \mathscr{G}_t / \mathcal{B}(\mathbb{R})$;

(2) for every $(z, \omega) \in \mathbb{R}^d \times \Omega$, the mapping $t \mapsto g(t, z, \omega)$ is left continuous.

$\mathscr{P}$ is called a *predictable σ-field*, while a $\mathscr{P}$-measurable mapping $h : \mathbb{R}^+ \times \mathbb{R}^d \times \Omega \to \mathbb{R}$ is called *$(\mathscr{F}_t)_{t\geq 0}$-predictable*. Note that if g is $(\mathscr{F}_t)_{t\geq 0}$-predictable, then the process $(g(t, x, \cdot)_t)_{t\geq 0}$ is $(\mathscr{F}_t)_{t\geq 0}$-adapted for each x.

Definition 2.32. Define $\mathbb{P}$ and $\mathbb{P}_2$ by the following conditions:

(1) $f \in \mathbb{P}$ if f is $\mathscr{P}$-predictable and $\int_0^{t+} \int_{\mathbb{R}^d \setminus \{0\}} |f(s, z, \omega)| N(dsdz) < \infty$ a.s., $t > 0$,

(2) $f \in \mathbb{P}_2$ if f is $\mathscr{P}$-predictable and $\mathbb{E}_P[\int_0^t \int_{\mathbb{R}^d \setminus \{0\}} |f(s, z, \omega)|^2 ds \nu(dz)] < \infty$, $t > 0$.

Next we define semimartingales which are extensions of the Itô processes seen before.

Definition 2.33 (Semimartingale). Let $(\Omega, \mathscr{F}, P)$ be a probability space with a filtration $(\mathscr{F}_t)_{t\geq 0}$. Assume that for each $i = 1, \dots, n$ and $\mu = 1, \dots, d$,

(1) $f^i_\mu \in \mathbb{M}^2(0, t)$ for all $t \geq 0$, i.e., $(f^i_\mu(t, \omega))_{t\geq 0}$ is $(\mathscr{F}_t)_{t\geq 0}$-adapted and we have $P(\int_0^t |f^i_\mu(s, \omega)|^2 ds < \infty) = 1$, for all $t \geq 0$;

(2) $(g^i(t, \omega))_{t\geq 0}$ is $(\mathscr{F}_t)_{t\geq 0}$-adapted and $g^i(t, \omega) \in L^1_{\mathrm{loc}}(\mathbb{R}; dt)$

(3) $h^i_1 \in \mathbb{P}$ and $h^i_2 \in \mathbb{P}_2$.

The random process $(X^i_t)_{t\geq 0}$, $i = 1, \dots, n$, on $(\Omega, \mathscr{F}, P)$ defined by

$$\begin{aligned} X^i_t = X^i_0 &+ \int_0^t f^i(s, \omega) \cdot dB_s + \int_0^t g^i(s, \omega) ds \\ &+ \int_0^{t+} \int_{\mathbb{R}^d \setminus \{0\}} h^i_1(s, z, \omega) N(dsdz) + \int_0^{t+} \int_{\mathbb{R}^d \setminus \{0\}} h^i_2(s, z, \omega) \tilde{N}(dsdz) \end{aligned} \tag{2.4.37}$$

is called a *semimartingale*.

Proposition 2.48 (Itô formula for semimartingales). *Let $F \in C^2(\mathbb{R}^d)$ and $(X_t)_{t\geq 0} = (X^i_t)_{t\geq 0, 1\leq i\leq n}$ be a semimartingale given by (2.4.37). Suppose $h^i_1 h^j_2 \equiv 0$, $i, j =$*

$1, \dots, n$. *Then* $(F(X_t))_{t\geq 0}$ *is also a semimartingale and the following formula holds:*

$$
\begin{aligned}
F(X_t) = F(X_0) &+ \sum_{i=1}^{n} \int_0^t F_i(X_s) f^i \cdot dB_s \\
&+ \int_0^t \Big(\sum_{i=1}^{n} F_i(X_s) g^i + \frac{1}{2} \sum_{i,j=1}^{n} \int_0^t F_{ij}(X_s) f^i \cdot f^j \Big) ds \\
&+ \int_0^{t+} \int_{\mathbb{R}^d \setminus \{0\}} (F(X_{s-} + h_1) - F(X_{s-})) N(dsdz) \\
&+ \int_0^{t+} \int_{\mathbb{R}^d \setminus \{0\}} (F(X_{s-} + h_2) - F(X_{s-})) \tilde{N}(dsdz) \\
&+ \int_0^{t} \int_{\mathbb{R}^d \setminus \{0\}} \Big(F(X_s + h_2) - F(X_s) - \sum_{i=1}^{n} h_2^i F_i(X_s) \Big) ds \nu(dz),
\end{aligned}
$$

where $F_i = \partial_{x_i} F$ *and* $F_{ij} = \partial_{x_i} \partial_{x_j} F$.

Example 2.16. An application of the Itô formula above is the variation formula of a Lévy process $(X_t)_{t\geq 0}$ such as $F(X_t) - F(X_0)$. Consider the Lévy process $(X_t)_{t\geq 0}$ given by the Lévy–Itô decomposition

$$
X_t = bt + A \cdot B_t + \int_0^t \int_{|z|\geq 1} z N(dsdz) + \int_0^t \int_{|z|<1} z \tilde{N}(dsdz).
$$

Since $1_{\{|z|\geq 1\}} \cdot 1_{\{0<|z|<1\}} = 0$, for $F \in C^2(\mathbb{R}^n)$, $Y_t = F(X_t)$ is also a semimartingale of the form

$$
\begin{aligned}
Y_t = Y_0 &+ \sum_{\mu=1}^{d} \sum_{i=1}^{n} \int_0^t F_i(X_s) A_{i\mu} dB_s^{\mu} \\
&+ \int_0^t \Big(\sum_{i=1}^{n} b_i F_i(X_s) + \frac{1}{2} \sum_{i,j=1}^{n} \int F_{ij}(X_s)(AA^T)_{ij} \Big) ds \\
&+ \int_0^{t+} \int_{\mathbb{R}^d \setminus \{0\}} (F(X_{s-} + z 1_{\{|z|\geq 1\}}) - F(X_{s-})) N(dsdz) \\
&+ \int_0^{t+} \int_{\mathbb{R}^d \setminus \{0\}} (F(X_{s-} + z 1_{\{|z|<1\}}) - F(X_{s-})) \tilde{N}(dsdz) \\
&+ \int_0^{t} \int_{\mathbb{R}^d \setminus \{0\}} \Big(F(X_s + z 1_{\{|z|<1\}}) - F(X_s) - \sum_{i=1}^{n} z_i 1_{\{|z|<1\}} F_i(X_s) \Big) ds \nu(dz).
\end{aligned}
$$

Finally, we show the product formula for semimartingales. Write (2.4.37) as

$$dX_t^i = g^i(t)dt + f^i(t) \cdot dB_t + \int_{\mathbb{R}^d \setminus \{0\}} h_1^i dN_t + \int_{\mathbb{R}^d \setminus \{0\}} h_2^i d\tilde{N}_t$$

in differential notation. Take $d = 1$ and let $(Z_t)_{t \geq 0}$ and $(Y_t)_{t \geq 0}$ be semimartingales given by

$$dZ_t = u_t^{(1)} dt + v_t^{(1)} \cdot dB_t + \int_{\mathbb{R}^d \setminus \{0\}} f_t^{(1)} dN_t + \int_{\mathbb{R}^d \setminus \{0\}} g_t^{(1)} d\tilde{N}_t,$$

$$dY_t = u_t^{(2)} dt + v_t^{(2)} \cdot dB_t + \int_{\mathbb{R}^d \setminus \{0\}} f_t^{(2)} dN_t + \int_{\mathbb{R}^d \setminus \{0\}} g_t^{(2)} d\tilde{N}_t.$$

Then by Proposition 2.48 we have the following product formula.

Corollary 2.49 (Product formula). *Let* $f_t^{(1)} g_t^{(1)} = 0$, $f_t^{(1)} g_t^{(2)} = 0$, $f_t^{(2)} g_t^{(2)} = 0$ *and* $f_t^{(2)} g_t^{(1)} = 0$. *Then*

$$\begin{aligned} d(Z_t Y_t) = {} & Z_t u_t^{(2)} dt + Z_t v_t^{(2)} \cdot dB_t + \int_{\mathbb{R}^d \setminus \{0\}} Z_{t-} f_t^{(2)} dN_t + \int_{\mathbb{R}^d \setminus \{0\}} Z_{t-} g_t^{(2)} d\tilde{N}_t \\ & + Y_t u_t^{(1)} dt + Y_t v_t^{(1)} \cdot dB_t + \int_{\mathbb{R}^d \setminus \{0\}} Y_{t-} f_t^{(1)} dN_t + \int_{\mathbb{R}^d \setminus \{0\}} Y_{t-} g_t^{(1)} d\tilde{N}_t \\ & + v_t^{(1)} \cdot v_t^{(2)} dt + \int_{\mathbb{R}^d \setminus \{0\}} (f_t^{(1)} f_t^{(2)} + g_t^{(1)} g_t^{(2)}) dN_t. \end{aligned}$$

This formula is written as $d(Z_t Y_t) = dZ_t \cdot Y_t + Z_t \cdot dY_t + dZ_t \cdot dY_t$ in concise notation.

2.4.5 Subordinators

We conclude this section by discussing a special class of Lévy processes.

Definition 2.34 (Subordinator). A one-dimensional Lévy process $(T_t)_{t \geq 0}$ is called a *subordinator* whenever $s \leq t$ implies $T_s \leq T_t$ almost surely.

Since $T_{t+s} - T_s \stackrel{\mathrm{d}}{=} T_t$, for all $s, t \geq 0$, the definition readily implies that $T_t \geq 0$ almost surely, for every $t > 0$. A subordinator can be regarded as a random time since it is non-decreasing and non-negative.

Theorem 2.50 (Characterization of subordinators). *The Lévy symbol of a subordinator is*

$$\eta(u) = ibu + \int_0^\infty (e^{iuy} - 1)\nu(dy), \tag{2.4.38}$$

where $b \geq 0$ and the Lévy measure satisfies that

$$\nu((-\infty, 0)) = 0 \quad \textit{and} \quad \int (y \wedge 1)\nu(dy) < \infty.$$

Conversely, for every $\eta(u)$ given by (2.4.38)*, there exists a subordinator $(T_t)_{t\geq 0}$ whose Lévy symbol is of the form $\eta(u)$.*

By Theorem 2.50 it is seen that the map $u \mapsto \mathbb{E}_P[e^{iuT_t}] = e^{t\eta(u)}$ can be analytically continued into the region $\{iu \in \mathbb{C} \mid u > 0\}$. This implies that the Laplace transform of T_t is well-defined and given by

$$\mathbb{E}_P[e^{-uT_t}] = e^{-t\psi(u)}, \tag{2.4.39}$$

where

$$\psi(u) = bu + \int_0^\infty (1 - e^{-uy})\nu(dy) \tag{2.4.40}$$

for every $u > 0$.

Example 2.17 ($\alpha/2$-stable subordinator). Let $b = 0$, $0 < \alpha < 2$ and using (2.4.9) take

$$\nu(dx) = \frac{\alpha}{2\Gamma(1-\alpha/2)} \frac{1_{(0,\infty)}(x)}{x^{1+\alpha/2}} dx, \tag{2.4.41}$$

where Γ denotes the Gamma function. This is called $\alpha/2$-*stable subordinator* and

$$\mathbb{E}_P[e^{-uT_t}] = e^{-tu^{\alpha/2}}$$

holds.

Example 2.18 (Inverse Gaussian subordinator). Let $(\Omega, \mathscr{F}, P)$ be a probability space, $m \geq 0$, $\delta > 0$ be given constants, and define the first hitting time of one-dimensional standard Brownian motion by

$$T_t(m, \delta) = \inf\{s > 0 | B_s + ms = \delta t\}. \tag{2.4.42}$$

$(T_t(m,\delta))_{t\geq 0}$ is called *inverse Gaussian subordinator* for $m > 0$. We have

$$\mathbb{E}_P[e^{-uT_t(m,\delta)}] = \exp(-t\delta(\sqrt{2u + m^2} - m)). \tag{2.4.43}$$

Moreover, the distribution of $T_t(m,\delta)$ is given by

$$\rho(s) = \frac{\delta t}{\sqrt{2\pi}} e^{\delta m t} \frac{1}{s^{3/2}} \exp\left(-\frac{1}{2}\left(t^2\delta^2\frac{1}{s} + m^2 s\right)\right).$$

To see this, recall that by Corollary 2.20, the random process $(e^{\alpha B_t - \frac{1}{2}\alpha^2 t})_{t\geq 0}$ is a martingale. In addition, $(e^{\alpha B_{T_t\wedge n} - \frac{1}{2}\alpha^2 T_t\wedge n})_{t\geq 0}$ is also a martingale. Thus

$$\mathbb{E}_P[e^{\alpha B_{T_t\wedge n} - \frac{1}{2}\alpha^2 T_t\wedge n}] = \mathbb{E}_P[e^{\alpha B_0}] = 1.$$

Let $A_n = \{T_t \leq n\}$ and

$$\mathbb{E}_P[e^{\alpha B_{T_t\wedge n} - \frac{1}{2}\alpha^2 T_t\wedge n}] = \mathbb{E}_P[e^{\alpha B_{T_t\wedge n} - \frac{1}{2}\alpha^2 T_t\wedge n} 1_{A_n}] + \mathbb{E}_P[e^{\alpha B_{T_t\wedge n} - \frac{1}{2}\alpha^2 T_t\wedge n} 1_{A_n^c}].$$

It is easy to see that

$$\mathbb{E}_P[e^{\alpha B_{T_t\wedge n} - \frac{1}{2}\alpha^2 T_t\wedge n} 1_{A_n^c}] \leq e^{-\frac{1}{2}\alpha^2 n}\mathbb{E}_P[1_{A_n^c} e^{\alpha B_n}]$$

and since $B_n < \delta t - mn$ on A_n^c, it follows that $\lim_{n\to\infty} \mathbb{E}_P[e^{\alpha B_{T_t\wedge n} - \frac{1}{2}\alpha^2 T_t\wedge n} 1_{A_n^c}] = 0$. Furthermore,

$$\lim_{n\to\infty} \mathbb{E}_P[e^{\alpha B_{T_t\wedge n} - \frac{1}{2}\alpha^2 T_t\wedge n} 1_{A_n}] = \mathbb{E}_P[e^{\alpha(\delta t - mT_t)} e^{-\frac{1}{2}\alpha^2 T_t}].$$

This implies that $1 = \mathbb{E}_P[e^{\alpha(\delta t - mT_t)} e^{-\frac{1}{2}\alpha^2 T_t}]$, and hence

$$e^{-\alpha\delta t} = \mathbb{E}_P[e^{-\frac{1}{2}\alpha(\alpha+2m)T_t}]$$

follows. Setting $\alpha = \sqrt{2u + m^2} - m$ proves the claim.

2.4.6 Bernstein functions

Subordinators and Bernstein functions have a deep connection with each other. We will make use of this in the path integral representations below.

Definition 2.35 (Bernstein function). Let $f \in C^\infty((0,\infty))$ with $f \geq 0$. The function f is called a *Bernstein function* whenever $(-1)^n \frac{d^n f}{dx^n}(x) \leq 0$, for all $n \in \mathbb{N}$. We denote the set of Bernstein functions by $\mathcal{B}$.

It follows from the definition that Bernstein functions are positive, increasing and concave. Define a subset of $\mathcal{B}$ by

$$\mathcal{B}_0 = \{\Psi \in \mathcal{B} | \lim_{u\to 0+} \Psi(u) = 0\}. \tag{2.4.44}$$

Examples of functions in $\mathcal{B}_0$ include $\Psi(u) = cu^{\alpha/2}$ with $c > 0$ and $0 < \alpha \leq 2$, and $\Psi(u) = 1 - e^{-au}$ with $a \geq 0$.

It can be seen that sums $\sum_j (1 - e^{-a_j u})$ of Bernstein functions are also Bernstein functions, and u^α can be expressed as $u^\alpha = \int_0^\infty (1 - e^{-yu})\nu(dy)$ by using the measure ν in (2.4.41). The right-hand side is derived from the analytic continuation of the Lévy symbol of subordinator (2.4.38). We will see next that $\mathcal{B}_0$ can be characterized by subordinators.

We define the following subset of Lévy measures.

Definition 2.36 (Lévy measures for Bernstein functions). Let $\mathscr{L}$ be the set of Borel measures λ on $\mathbb{R} \setminus \{0\}$ such that $\lambda((-\infty, 0)) = 0$ and $\int_0^\infty (y \wedge 1)\lambda(dy) < \infty$.

Note that $\lambda \in \mathscr{L}$ satisfies $\int_0^\infty (y^2 \wedge 1)\lambda(dy) < \infty$. The important fact connecting $\mathscr{L}$ and $\mathscr{B}_0$ is given by the next proposition.

Proposition 2.51 (Characterization of Bernstein functions). *For every $\Psi \in \mathscr{B}_0$ there exists $(b, \lambda) \in \mathbb{R}_+ \times \mathscr{L}$ such that*

$$\Psi(u) = bu + \int_{(0,\infty)} (1 - e^{-uy})\lambda(dy). \tag{2.4.45}$$

Conversely, the right-hand side of (2.4.45) *is in $\mathscr{B}_0$ for every $(b, \lambda) \in \mathbb{R}_+ \times \mathscr{L}$.*

For a given $\Psi \in \mathscr{B}_0$ the constant b is uniquely determined by

$$b = \lim_{u \to \infty} \Psi(u)/u.$$

Note that $f = d\Psi/du$ is a completely monotone function. For a completely monotone function g it is well known that there exists a unique measure μ on $[0, \infty)$ such that $g(u) = \int_0^\infty e^{-us} d\mu(s)$, for $u > 0$. Since

$$d\Psi/du = b + \int_0^\infty e^{-uy}\lambda(dy),$$

the measure $\lambda \in \mathscr{L}$ for the Bernstein function Ψ is also uniquely determined.

Proposition 2.52. *The map $\mathscr{B}_0 \to (\mathbb{R}_+, \mathscr{L})$, $\Psi \mapsto (b, \lambda)$, is a bijection.*

Denote by $\mathscr{S}$ the set of subordinators. From the one-to-one correspondence between $\mathscr{B}_0$ and $(\mathbb{R}_+, \mathscr{L})$ we have also the one-to-one correspondence between $\mathscr{S}$ and $\mathscr{B}_0$.

Proposition 2.53. *Let $\Psi \in \mathscr{B}_0$. Then there exists a unique $(T_t)_{t \geq 0} \in \mathscr{S}$ such that*

$$\mathbb{E}_P[e^{-uT_t}] = e^{-t\Psi(u)}. \tag{2.4.46}$$

Conversely, let $(T_t)_{t \geq 0} \in \mathscr{S}$. Then there exists $\Psi \in \mathscr{B}_0$ such that (2.4.46) *holds.*

Proof. Let $\Psi \in \mathscr{B}_0$ be given. Then there exists $(b, \lambda) \in \mathbb{R}_+ \times \mathscr{L}$ such that (2.4.45) holds. This means that there exists a subordinator $(T_t)_{t \geq 0}$ with characteristic $(b, 0, \lambda)$. Moreover, (2.4.46) follows by analytic continuation of $\mathbb{E}_P[e^{iuT_t}] = e^{t\eta(u)}$ with $\eta(u) = ibu + \int_0^\infty (e^{iuy} - 1)\lambda(dy)$. Conversely, let $(T_t)_{t \geq 0} \in \mathscr{S}$ be given. By $(b, \lambda) \in \mathbb{R}_+ \times \mathscr{L}$ associated with $(T_t)_{t \geq 0}$ we define the Bernstein function by (2.4.45), and the proposition follows. $\square$

Chapter 3
Feynman–Kac formulae

3.1 Schrödinger semigroups

3.1.1 Schrödinger equation and path integral solutions

As we have explained in Chapter 1 a basic motivation of the interest in Feynman–Kac formulae is that of analyzing the solutions of the Schrödinger equation

$$i\partial_t \varphi = -\frac{1}{2}\Delta\varphi + V\varphi \tag{3.1.1}$$

or its imaginary-time counterpart, heat equation with spatially non-homogeneous dissipation

$$-\partial_t \varphi = -\frac{1}{2}\Delta\varphi + V\varphi. \tag{3.1.2}$$

In fact, the scope of Feynman–Kac formulae in the context of partial differential equations goes far beyond these two cases, however, in this book we have no space to discuss the PDE aspect separately.

Let $f : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ be a solution of the heat equation, i.e.

$$\partial_t f(t,x) = \frac{1}{2}\Delta f(t,x), \quad f(0,x) = g(x). \tag{3.1.3}$$

Recall the formula known from the theory of partial differential equations:

$$f(t,x) = \int_{\mathbb{R}^d} \Pi_t(x-y) g(y)\, dy. \tag{3.1.4}$$

A comparison with

$$\mathbb{E}^x[g(B_t)] = \int_{\mathbb{R}^d} \Pi_t(x-y) g(y)\, dy, \tag{3.1.5}$$

shows that, as we have said above, the solution of (3.1.3) can be obtained by running a Brownian motion, and

$$f(t,x) = \mathbb{E}^x[g(B_t)] \tag{3.1.6}$$

is obtained. Using operator semigroup notation (3.1.3) implies $f = e^{(t/2)\Delta}g$. In what follows we ask whether a similar representation is possible when $-\frac{1}{2}\Delta$ is replaced by $-\frac{1}{2}\Delta + V$, with suitable V. Let now $f : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ be a solution of

$$\partial_t f(t,x) = \frac{1}{2}\Delta_x f(t,x) - V(x)f(t,x). \tag{3.1.7}$$

In semigroup notation, this problem translates then to studying $f = e^{-t(-\frac{1}{2}\Delta+V)}g$. In what follows, instead of equations in our considerations the basic input will be linear operators and one-parameter semigroups generated by them as above, although we will remember the deep relations with partial differential equations.

3.1.2 Linear operators and their spectra

In what follows we will be interested in various spectral properties of sums of self-adjoint operators. In the present section we only consider Schrödinger operators

$$-\frac{1}{2}\Delta + V \tag{3.1.8}$$

defined on a suitable space as the operator sum of the Laplacian and the multiplication operator V viewed in a specific sense as a perturbation of the Laplacian. As we progress to take magnetic fields and spin into account, further operator terms will appear, and when we discuss relativistic quantum mechanics, the Laplacian will be replaced by its square root. In Part II of the book we then turn to quantum field theory where moreover different types of operators will be added to Schrödinger operators.

We start by briefly reviewing key concepts and facts of general linear operator theory used throughout in this book, and next focus on first Laplacians and then on their perturbations.

Let $\mathcal{K}$ be a Hilbert space over the complex field $\mathbb{C}$ with inner product $(\cdot,\cdot)$ and norm $\|\cdot\|$. When inner products or norms of several distinct Hilbert spaces are compared, we indicate the specific space in the subscript whenever necessary; for L^p spaces we use the standard notation $\|\cdot\|_p$. Let $\mathcal{H}$ be also a Hilbert space. The domain of a linear operator $A : \mathcal{K} \to \mathcal{H}$ will be denoted by $D(A)$, its range by $\mathrm{Ran}(A)$ and its kernel by $\mathrm{Ker}(A)$. Recall that for two linear operators A and B we have $A = B$ if and only if $D(A) = D(B)$ and $A\varphi = B\varphi$ for all $\varphi \in D(A)$, and $A \subset B$ if and only if $D(A) \subset D(B)$ and $A\varphi = B\varphi$ for all $\varphi \in D(A)$; in the latter case B is said to be an *extension* of A, or from the opposite point of view, A is the *restriction* of B to $D(A)$.

An operator A on $\mathcal{K}$ is *densely defined* if $D(A)$ is a dense subset of $\mathcal{K}$.

Definition 3.1 (Bounded/unbounded operators). Let $A : \mathcal{K} \to \mathcal{H}$ be a linear operator. Whenever there exists $c \geq 0$ such that $\|A\varphi\|_{\mathcal{H}} \leq c\|\varphi\|_{\mathcal{K}}$ for all $\varphi \in D(A)$, the operator A is called *bounded*, otherwise it is called *unbounded*.

A linear operator A is called *strongly continuous at* $\varphi \in D(A)$ when there is a sequence $(\varphi_n)_{n\in\mathbb{N}} \subset D(A)$ such that $\varphi_n \to \varphi$ in strong convergence sense implies $A\varphi_n \to A\varphi$ also in strong convergence sense. If A is continuous for all $\varphi \in D(A)$, A is called *strongly continuous*. A linear operator is bounded if and only if it is strongly continuous. The norm of a bounded operator A is defined as

$$\|A\| = \|A\|_{\mathcal{K},\mathcal{H}} = \sup_{\varphi\in D(A),\varphi\neq 0} \frac{\|A\varphi\|_{\mathcal{H}}}{\|\varphi\|_{\mathcal{K}}}. \tag{3.1.9}$$

A bounded operator is a *contraction* if $\|A\| \leq 1$. When $\mathcal{K}$ is finite dimensional, any linear operator on it is bounded.

Next we give the notions of bounded operator topology we will use in what follows.

Definition 3.2 (Topologies of bounded operators). Let $(A_n)_{n\in\mathbb{N}} : \mathcal{K} \to \mathcal{H}$ be a sequence of bounded operators such that $D(A_n) = D, n \in \mathbb{N}$, and A another bounded operator. Then $(A_n)_{n\in\mathbb{N}}$ is

(1) *weakly convergent* to A, denoted $A = w - \lim_{n\to\infty} A_n$, if $D(A) = D$ and $\lim_{n\to\infty}(\psi, A_n\varphi - A\varphi) = 0$, for all $\varphi \in D$ and for all $\psi \in \mathcal{H}$

(2) *strongly convergent* to A, denoted $A = \text{s-}\lim_{n\to\infty} A_n$, if $D(A) = D$ and $\lim_{n\to\infty} \|A_n\varphi - A\varphi\|_{\mathcal{H}} = 0$, for all $\varphi \in D$

(3) *uniformly convergent* to A if $D(A) = D$ and $\lim_{n\to\infty} \|A_n - A\| = 0$.

If A is a densely defined bounded operator with $\|A\varphi\|_{\mathcal{H}} \leq c\|\varphi\|_{\mathcal{K}}$, then it can be uniquely extended to an operator B such that $A \subset B$, $D(B) = \mathcal{K}$ and $\|B\varphi\|_{\mathcal{H}} \leq c\|\varphi\|_{\mathcal{K}}$ hold. In what follows we assume that the domain of a bounded operator is the whole space.

Recall that the *graph* of a linear operator $A : \mathcal{K} \to \mathcal{H}$ is the subset

$$G(A) = \{(\varphi, A\varphi) |\, \varphi \in D(A)\} \subset \mathcal{K} \oplus \mathcal{H}. \tag{3.1.10}$$

Definition 3.3 (Closed/closable operator). Let $A : \mathcal{K} \to \mathcal{H}$ be a linear operator.

(1) A is a *closed* operator if $G(A)$ is closed in $\mathcal{K} \oplus \mathcal{H}$, in other words, if the sequence $(\varphi_n)_{n\in\mathbb{N}}$ satisfies that $A\varphi_n \to \phi$ and $\varphi_n \to \varphi$ as $n \to \infty$, then $\varphi \in D(A)$ and moreover $\phi = A\varphi$ holds.

(2) A is a *closable* operator if there exists a closed operator $B : \mathcal{K} \to \mathcal{H}$ such that $A \subset B$.

(3) The *closure* of A, denoted by $\bar{A}$, is the smallest closed extension of A, i.e., $\bar{A}$ is a closed extension of A and if B is another closed extension, then $\bar{A} \subset B$.

(4) A subspace $\mathcal{D} \subset \mathcal{K}$ is a *core* of A if $\overline{A\lceil_{\mathcal{D}}} = A$, in other words, if for every $\varphi \in D(A)$ there is a sequence $(\varphi)_{n\in\mathbb{N}} \subset \mathcal{D}$ such that $\varphi \to \varphi$ and $A\varphi_n \to A\varphi$ strongly as $n \to \infty$.

The bounded operators are closed. The next proposition ensures the existence of the closure of closable operators.

Proposition 3.1. *Let A be closable. Then the closure of A exists and $\overline{G(A)} = G(\bar{A})$ holds.*

A linear operator A is *injective* if and only if $A\varphi = 0$ implies $\varphi = 0$. In this case the *inverse* A^{-1} of A is defined by

$$D(A^{-1}) = \mathrm{Ran}(A), \quad A^{-1}\varphi = \phi \tag{3.1.11}$$

for $A\phi = \varphi$. From this definition it follows that for $\varphi \in \mathrm{Ran}(A)$ and $\phi \in D(A)$,

$$AA^{-1}\varphi = \varphi, \quad A^{-1}A\phi = \phi. \tag{3.1.12}$$

Definition 3.4 (Invertibility). A linear operator A is called *invertible* if it is injective and has a bounded inverse on Ran A.

Note that invertibility of an operator and the existence of an inverse are properties that do not necessarily coincide.

Definition 3.5 (Addition and multiplication of operators). The *sum* $A + B$ of two linear operators A and B is defined by

$$D(A + B) = D(A) \cap D(B), \quad (A + B)\varphi = A\varphi + B\varphi. \tag{3.1.13}$$

The *product* AB is defined by

$$D(AB) = \{\varphi \in D(B) | B\varphi \in D(A)\}, \quad (AB)\varphi = A(B\varphi). \tag{3.1.14}$$

Note that in general it may happen that $D(A) \cap D(B)$ is not a dense subset in $D(A)$ or $D(B)$.

Let $B(\mathcal{K}, \mathcal{H})$ denote the set of bounded operators from $\mathcal{K}$ to $\mathcal{H}$ with the whole space $\mathcal{K}$ as their domain. For $A, B \in B(\mathcal{K}, \mathcal{H})$, $A + B \in B(\mathcal{K}, \mathcal{H})$ and $\|A + B\| \leq \|A\| + \|B\|$ hold, and for $B \in B(\mathcal{K}, \mathcal{H})$ and $A \in B(\mathcal{H}, \mathcal{L})$, $AB \in B(\mathcal{K}, \mathcal{L})$ and $\|AB\| \leq \|A\|\|B\|$ hold.

In the following a central issue will be the spectral properties of some linear operators.

Definition 3.6 (Resolvent and spectrum). Let $A : \mathcal{K} \to \mathcal{H}$ be closed. The *resolvent set* of A is

$$\rho(A) = \{\lambda \in \mathbb{C} |\ \mathrm{Ran}(\lambda - A) = \mathcal{H},\ \lambda - A \text{ is injective and } (\lambda - A)^{-1} \text{ is bounded}\} \tag{3.1.15}$$

and its *spectrum* is defined as

$$\mathrm{Spec}\, A = \mathbb{C} \setminus \rho(A). \tag{3.1.16}$$

For $\lambda \in \rho(A)$ the bounded operator $R_A(\lambda) = (\lambda - A)^{-1}$ is the *resolvent* of A.

In the definition of the resolvent A must be assumed to be closed. The reason is as follows. If $\lambda \in \rho(A)$, then $R_A(\lambda) = (\lambda - A)^{-1}$ is bounded and hence closed. Let $j : \mathcal{K} \oplus \mathcal{H} \to \mathcal{H} \oplus \mathcal{K}$ be defined by $j(\varphi, \phi) = (\phi, \varphi)$. This is an isomorphism between $\mathcal{K} \oplus \mathcal{H}$ and $\mathcal{H} \oplus \mathcal{K}$, and $G(B^{-1}) = jG(B)$ follows. Let B be closed. Then $G(B)$ is a closed set and hence $G(B^{-1})$ is also closed. We conclude that B^{-1} is closed. Thus we proved that in general if B is closed and injective, then B^{-1} is also closed. In particular $\lambda - A$ is closed since so is $(\lambda - A)^{-1}$, from which follows that A must be closed.

Furthermore, it can be also seen that if A is closable and injective, then $(\lambda - A)^{-1}$ is also closable and $\overline{(\lambda - A)^{-1}} = (\lambda - \overline{A})^{-1}$ follows. For a closable operator $A : \mathcal{K} \to \mathcal{H}$ its *resolvent set* is defined by

$$\rho(A) = \{\lambda \in \mathbb{C} \mid \operatorname{Ran}(\lambda - A) \text{ is dense, } \lambda - A \text{ is injective, } (\lambda - A)^{-1} \text{ is bounded}\}.$$

For a closed operator A the conditions (1) $\operatorname{Ran}(\lambda - A)$ is dense, (2) $\lambda - A$ is injective and (3) $(\lambda - A)^{-1}$ is bounded automatically imply that $\operatorname{Ran}(\lambda - A) = \mathcal{H}$.

The spectrum of A can be decomposed into three parts:

(1) The *point spectrum* $\operatorname{Spec}_p(A)$ of A consists of the elements $\lambda \in \operatorname{Spec}(A)$ such that $\lambda - A$ is not injective; the elements of the point spectrum are called *eigenvalues*, i.e., for every such λ there exists $\varphi_\lambda \in D(A) \setminus \{0\}$, such that $A\varphi_\lambda = \lambda\varphi_\lambda$, and φ_λ is an *eigenvector* (or *eigenfunction* when the vector space is a set of functions), while the number $\dim \operatorname{Ker}(\lambda - A)$ is the *multiplicity* of eigenvalue λ.

(2) The *continuous spectrum* $\operatorname{Spec}_c(A)$ of A consists of all $\lambda \in \operatorname{Spec} A$ for which $\lambda - A$ has an inverse with a dense domain but it is not a bounded operator.

(3) The *residual spectrum* $\operatorname{Spec}_r(A)$ of A is given by $\lambda \in \operatorname{Spec}(A)$ such that $\lambda - A$ has an inverse, i.e., injective, but its domain is not dense.

Thus the complex field $\mathbb{C}$ is decomposed into the disjoint sets

$$\mathbb{C} = \rho(A) \cup \operatorname{Spec}_p(A) \cup \operatorname{Spec}_c(A) \cup \operatorname{Spec}_r(A). \tag{3.1.17}$$

The resolvent set of a bounded operator A contains $\{z \in \mathbb{C} \,|\, |z| > \|A\|\}$ since

$$(\lambda - A)^{-1} = \frac{1}{\lambda} \sum_{n=0}^{\infty} \lambda^{-n} A^n$$

holds in the uniform topology and the right-hand side of the equality is bounded for $\lambda > \|A\|$. Thus the spectrum of bounded operators is a closed non-empty subset of $\{z \in \mathbb{C} \,|\, |z| \le \|A\|\}$, however, in the case of unbounded operators both the spectrum and the resolvent set may be empty.

The operators we will consider below are self-adjoint, therefore we do not need to address spectral theory in its full generality.

Definition 3.7 (Adjoint operator). Let $A : \mathcal{K} \to \mathcal{H}$ be a densely defined linear operator. The *adjoint operator* A^* of A is a linear operator from $\mathcal{H}$ to $\mathcal{K}$ defined by

$$D(A^*) = \{\varphi \in \mathcal{H} \,|\, \exists \psi \in \mathcal{H} \text{ such that } (\varphi, A\phi) = (\psi, \phi),\ \forall \phi \in D(A)\}$$

and $A^*\varphi = \psi$.

Since A is densely defined, A^* is well defined. The adjoint operator A^* is always closed, and whenever A is closed, then moreover $A = (A^*)^*$. If A is closable, then $\bar{A} = (A^*)^*$ holds. It can be also checked that if $D(A^*)$ is dense, then A is closable.

If $A, B \in B(\mathcal{K}, \mathcal{K})$, then $A^*, B^* \in B(\mathcal{K}, \mathcal{K})$ and the properties $\|A^*\| = \|A\|$, $(A^*)^* = A$, $(aA + bB)^* = \bar{a}A^* + \bar{b}B^*$, and $(AB)^* = B^*A^*$ hold.

Definition 3.8 (Symmetric operator and self-adjoint operator). Let $A : \mathcal{K} \to \mathcal{K}$ be a densely defined linear operator. A is said to be a *symmetric operator* if $A \subset A^*$, a *self-adjoint operator* if $A^* = A$, and an *essentially self-adjoint operator* if $\overline{A}$ is self-adjoint.

Whenever $D(A^*)$ is dense, A is closable, moreover, $(A^*)^* = \bar{A}$ for a closable operator. If A is symmetric, $D(A^*)$ is dense by the definition. Hence A is closable and $\bar{A}$ is a closed symmetric operator.

In general, infinitely many self-adjoint extensions of a given symmetric operator exist. Note that if $A \subset B$, then $B^* \subset A^*$. Let A be symmetric and B a self-adjoint extension of A. Then $A \subset B = B^* \subset A^*$, thus $B = A^*\lceil_D$ with some domain D. Next let A be essentially self-adjoint and B a self-adjoint extension of A. Since $A \subset B$, it follows that $(A^*)^* \subset B$. Thus $B = B^* \subset ((A^*)^*)^* = (A^*)^*$ and therefore $B = (A^*)^* = \bar{A}$. We summarize this in the proposition below.

Proposition 3.2 (Self-adjoint extensions). (1) *Let A be symmetric and B a self-adjoint extension of A. Then B is a restriction of A^*.*

(2) *Let A be essentially self-adjoint. Then $\bar{A}$ is the only self-adjoint extension of A, i.e. if B is self-adjoint and $A \subset B$, then $\bar{A} = B$.*

(3) *Let A be a symmetric operator and suppose it has only one self-adjoint extension. Then A is essentially self-adjoint.*

A bounded operator A is symmetric if and only if it is self-adjoint. However, for an unbounded operator these two properties are in general different, indeed, a symmetric operator need not be closed and a closed symmetric operator need not be self-adjoint. Equivalent conditions to self-adjointness and essential self-adjointness of a symmetric operator are as follows.

Proposition 3.3 (Equivalent conditions to self-adjointness). *Let A be a symmetric operator in $\mathcal{K}$. Then properties* (1)–(4) *below are equivalent.*

(1) *A is self-adjoint.*

(2) *A is closed and* $\operatorname{Ker}(A^* \pm i) = \{0\}$.

(3) $\operatorname{Ran}(A \pm i) = \mathcal{K}$.

(4) *A is closed and* $\overline{\operatorname{Ran}(A \pm i)} = \mathcal{K}$.

Proposition 3.4 (Equivalent conditions to essential self-adjointness). *Let A be a symmetric operator in* $\mathcal{K}$. *Then properties* (1)–(3) *below are equivalent.*

(1) *A is essentially self-adjoint.*

(2) $\operatorname{Ker}(A^* \pm i) = \{0\}$.

(3) $\overline{\operatorname{Ran}(A \pm i)} = \mathcal{K}$.

Self-adjointness removes some of the complications of the structure of the spectrum. The following facts are fundamental on the spectrum and resolvent of self-adjoint operators.

Proposition 3.5 (Spectral properties of self-adjoint operators). *Let A be a self-adjoint operator. Then*

(1) $\operatorname{Spec}(A) \subset \mathbb{R}$.

(2) *The residual spectrum* $\operatorname{Spec}_{\mathrm{r}}(A)$ *is empty.*

(3) *The eigenvectors for distinct eigenvalues are orthogonal.*

(4) *For* $\lambda \in \operatorname{Spec}(A)$ *there exists* $(\varphi_n)_{n\in\mathbb{N}} \subset D(A)$ *such that* $\|\varphi_n\| = 1$ *and* $\lim_{n\to\infty} \|(\lambda - A)\varphi_n\| = 0$.

(5) *If there exists* $c > 0$ *such that* $\|(\lambda - A)\varphi\| \geq c\|\varphi\|$ *for all* $\varphi \in D(A)$, *then* $\lambda \in \rho(A)$ *and* $\{z \in \mathbb{C} \,|\, |z - \lambda| < c\} \subset \rho(A)$.

(6) $\|R_A(\lambda)\| \leq 1/|\Im\lambda|$ *holds.*

A self-adjoint operator A with the property that $(\varphi, A\varphi) \geq 0$, for all $\varphi \in D(A)$, is called a *positive* operator and denoted by $A \geq 0$. Positivity of A is equivalent with $\operatorname{Spec}(A) \subset [0, \infty)$.

Definition 3.9 (Unitary operator). A bounded operator $U : \mathcal{K} \to \mathcal{H}$ is called a *unitary operator* if it is an isometry, i.e., $\|U\varphi\|_{\mathcal{H}} = \|\varphi\|_{\mathcal{K}}$ for all $\varphi \in \mathcal{K}$, and $\operatorname{Ran}(U) = \mathcal{H}$.

The isometry condition can be equivalently formulated as $U^*U = 1$. Unitary operators can be used to define a *conjugate* operator to any given operator by keeping its spectrum unchanged. More precisely, if A is a closed operator and U is a unitary operator, then UAU^{-1} is well defined on $UD(A)$ and closed, moreover, $\operatorname{Spec}(UAU^{-1}) = \operatorname{Spec}(A)$. For instance, Fourier transform is a unitary operator on $L^2(\mathbb{R}^d)$ and this equality can be used to compute the spectrum of the Laplacian, see Example 3.2 below. Furthermore, if A is self-adjoint, then UAU^{-1} also is self-adjoint on $UD(A)$.

Definition 3.10. Let $(A_n)_{n\in\mathbb{N}}$ be a sequence of unbounded operators.

(1) $A_n \to A$ is said to converge in the sense of *strong resolvent convergence* if $R_{A_n}(\lambda)$ strongly converges to $R_A(\lambda)$, for all $\lambda \in \rho(A) \cap \bigcap_n \rho(A_n)$.

(2) $A_n \to A$ is said to converge in the sense of *uniform resolvent convergence* if $R_{A_n}(\lambda)$ uniformly converges to $R_A(\lambda)$, for all $\lambda \in \rho(A) \cap \bigcap_n \rho(A_n)$.

Convergence in strong and uniform resolvent sense are the proper generalizations to unbounded operators of strong and uniform convergence of bounded operators, respectively.

3.1.3 Spectral resolution

The *spectral theorem* establishes a fundamental relationship between a linear operator and a measure on its spectrum. Let $\mathscr{K}$ be a Hilbert space and the set of projections on $\mathscr{K}$ be denoted by $P(\mathscr{K})$. Consider the map $\mathscr{B}(\mathbb{R}) \ni B \mapsto E(B) \in P(\mathscr{K})$ with the properties

(1) $E(\emptyset) = 0$ and $E(\mathbb{R}) = 1$

(2) if $B = \bigcup_{n\in\mathbb{N}} B_n$ with $B_m \cap B_n = \emptyset$, $m \neq n$, then

$$E(B) = \operatorname*{s-lim}_{n\to\infty} \sum_{k=1}^{n} E(B_k)$$

(3) $E(B_1 \cap B_2) = E(B_1)E(B_2)$.

A family of projections indexed by $\mathscr{B}(\mathbb{R})$ satisfying properties (1)–(3) above is called a *projection-valued measure*. For $\varphi \in \mathcal{K}$ the Borel measure $(\varphi, E(B)\varphi)$ on $\mathbb{R}$ can be defined, and by the polarization identity it gives rise to the complex measure $(\varphi, E(B)\psi)$, $\varphi, \psi \in \mathcal{K}$ through

$$(\varphi, E(B)\psi) = \frac{1}{4}\sum_{n=1}^{4} i^n((\varphi + i^n\psi), E(B)(\varphi + i^n\psi)). \tag{3.1.18}$$

The spectral theorem says that there is a one-to-one correspondence between self-adjoint operators and projection-valued measures. For a self-adjoint operator A and the corresponding $E_A(\lambda)$ we define $f(A)$ for a Borel measurable function f by

$$(\varphi, f(A)\psi)_{\mathcal{K}} = \int_{\mathbb{R}} f(\lambda)d(E_A(\lambda)\varphi, \psi), \tag{3.1.19}$$

with domain

$$D(f(A)) = \left\{\varphi \in \mathcal{K} \,\middle|\, \int_{\mathbb{R}} |f(\lambda)|^2 d\|E_A(\lambda)\varphi\|^2 < \infty\right\}.$$

Note that the support of the measure E_A coincides with $\mathrm{Spec}(A)$. Equality (3.1.19) is formally written as $f(A) = \int_{\mathbb{R}} f(\lambda) dE_A(\lambda)$ and is called the *spectral resolution* of $f(A)$.

The following algebraic relations hold.

Proposition 3.6. *Let A be a self-adjoint operator. Then for Borel measurable functions f and g,*

(1) $D(f(A)+g(A)) = D((f+g)(A)) \cap D(g(A)) = D((f+g)(A)) \cap D(f(A))$ *and* $f(A) + g(A) \subset (f+g)(A)$

(2) $D(f(A)g(A)) = D((fg)(A)) \cap D(g(A))$ *and* $f(A)g(A) \subset (fg)(A)$.

We will use the terminology of also another classification of the spectrum.

Definition 3.11 (Discrete spectrum and essential spectrum). Let A be self-adjoint. If $\lambda \in \mathrm{Spec}(A)$ is such that there is $\varepsilon > 0$ for which $\dim \mathrm{Ran}\, E_A(\lambda-\varepsilon, \lambda+\varepsilon) < \infty$, then λ is said to be an element of the *discrete spectrum*, denoted $\mathrm{Spec}_{\mathrm{d}}(A)$. If $\lambda \in \mathrm{Spec}(A)$ is such that $\dim \mathrm{Ran}\, E_A(\lambda - \varepsilon, \lambda + \varepsilon) = \infty$ for all $\varepsilon > 0$, then λ is in the *essential spectrum*, denoted $\mathrm{Spec}_{\mathrm{ess}}(A)$.

Note that the discrete and the essential spectra form a partition of the spectrum. Moreover, $\mathrm{Spec}_{\mathrm{ess}}(A)$ is a closed set, while $\mathrm{Spec}_{\mathrm{d}}(A)$ need not be so. Furthermore, the following facts hold.

Proposition 3.7. $\lambda \in \mathrm{Spec}_{\mathrm{d}}(A)$ *if and only if both of the following properties hold:*

(1) λ *is an isolated point of* $\mathrm{Spec}(A)$*, i.e., there exists* $\varepsilon > 0$ *such that* $\mathrm{Spec}(A) \cap (\lambda - \varepsilon, \lambda + \varepsilon) = \{\lambda\}$

(2) λ *is an eigenvalue of finite multiplicity, i.e.,* $\dim\{\varphi \in \mathcal{K} | A\varphi = \lambda\varphi\} < \infty$.

An eigenvalue of arbitrary multiplicity which is not an isolated point in the spectrum is called an *embedded eigenvalue*.

Example 3.1 (Multiplication operator). Let V be a Lebesgue measurable real function and define a *multiplication operator* by $(V\varphi)(x) = V(x)\varphi(x)$ and domain $D(V) = \{\varphi \in L^2(\mathbb{R}^d) | Vf \in L^2(\mathbb{R}^d)\}$. Note that it is densely defined. Since V is real-valued, it is symmetric, and in fact it is a self-adjoint operator.

Example 3.2 (Laplace operator). Recall that the *weak partial derivative* $\partial_j \varphi$ of $\varphi \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$ is a function $h \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$ such that

$$\int_{\mathbb{R}^d} h(x) f(x) dx = -\int_{\mathbb{R}^d} \varphi(x) \partial_j f(x) dx, \quad f \in C_0^\infty(\mathbb{R}^d),$$

and is the jth component of the gradient operator in weak sense. This notion reduces to the partial derivative (gradient) known from elementary calculus whenever $\varphi \in C^1(\mathbb{R}^d)$. The *Laplacian* Δ in $L^2(\mathbb{R}^d)$ is the symmetric operator defined by

$$\Delta : \varphi \mapsto \sum_{j=1}^{d} \partial_j^2 \varphi \tag{3.1.20}$$

for $\varphi \in C_0^\infty(\mathbb{R}^d)$. It is seen that $T = \Delta\lceil_{C_0^\infty(\mathbb{R}^d)}$ is essentially self-adjoint. We denote the closure of T by the same symbol, i.e., $\Delta = \overline{T}$. The domain of Δ is the Sobolev space

$$D(-(1/2)\Delta) = H^2(\mathbb{R}^d) = \{f \in L^2(\mathbb{R}^d) | |k|^2 \hat{f} \in L^2(\mathbb{R}^d)\},$$

where $\hat{f}$ and $\check{f}$ denote the Fourier transform and inverse Fourier transform of f, respectively. Moreover, $\widehat{-\Delta f}(k) = |k|^2 \hat{f}(k)$ holds for every $f \in H^2(\mathbb{R}^d)$. Since the range of the multiplication operator $|k|^2$ is the positive semi-axis, we have that $-\Delta$ has continuous spectrum and $\mathrm{Spec}(-\Delta) = \mathrm{Spec}_{\mathrm{ess}}(-\Delta) = [0, \infty)$, in particular, $-\Delta$ is a positive operator.

Example 3.3 (Fractional Laplacian). The operator with domain $H^\alpha(\mathbb{R}^d) = \{f \in L^2(\mathbb{R}^d) \big| |k|^\alpha \hat{f} \in L^2(\mathbb{R}^d)\}$, $0 < \alpha < 2$, defined by $\widehat{(-\Delta)^{\alpha/2} f}(k) = |k|^\alpha \hat{f}(k)$, is called *fractional Laplacian* with exponent $\alpha/2$. It is essentially self-adjoint on $C_0^\infty(\mathbb{R}^d)$, and its spectrum is $\mathrm{Spec}((-\Delta)^{\alpha/2}) = \mathrm{Spec}_{\mathrm{ess}}((-\Delta)^{\alpha/2}) = [0, \infty)$. This is a case of a pseudo-differential operator. A related operator is the *relativistic fractional Laplacian* $(-\Delta + m^{2/\alpha})^{\alpha/2} - m$, with $m > 0$.

3.1.4 Compact operators

Finally we discuss compact operators.

Definition 3.12 (Compact operator). A bounded operator $A : \mathcal{K} \to \mathcal{H}$ is a *compact operator* whenever A takes bounded sets in $\mathcal{K}$ into pre-compact sets in $\mathcal{H}$. Equivalently, A is compact if and only if for every bounded sequence $(\varphi_n)_{n\in\mathbb{N}}$, $(A\varphi_n)_{n\in\mathbb{N}}$ has a subsequence convergent in $\mathcal{H}$.

Example 3.4. Let $k \in L^2(\mathbb{R}^d \times \mathbb{R}^d)$. Then $A : f \mapsto \int_{\mathbb{R}^d} k(x,y) f(y) dy$ is a compact operator on $L^2(\mathbb{R}^d)$.

Example 3.5. Let $k(x,y) = V^{1/2}(x)|x-y|^{2-d} V^{1/2}(y)$ for $d \geq 3$ with $0 \leq V \in L^{d/2}(\mathbb{R}^d)$. Then the integral operator $A : f \mapsto \int k(x,y) f(y) dy$ is compact. In the case of $d = 3$ this is proven by the Hardy–Littlewood–Sobolev inequality

$$\left| \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(x)|x-y|^{-\lambda} g(y) dx dy \right| \leq C \|f\|_p \|g\|_q \tag{3.1.21}$$

for $\lambda < d$ and $\frac{1}{p} + \frac{\lambda}{d} + \frac{1}{q} = 2$, with some constant C. See Lemma 3.47.

Some important properties of compact operators are summarized below.

Proposition 3.8. *Let A be a compact operator.*

(1) $\mathrm{Spec}(A) \setminus \{0\} \subset \mathrm{Spec}_{\mathrm{d}}(A)$.

(2) $\mathrm{Spec}(A)$ *is a finite set or a countable set without accumulation points.*

(3) *Let $C(\mathcal{K})$ be the set of compact operators from $\mathcal{K}$ to itself. Then $C(\mathcal{K})$ is an ideal of $B(\mathcal{K}, \mathcal{K})$, i.e., $AB, BA \in C(\mathcal{K})$ for $A \in C(\mathcal{K})$ and $B \in B(\mathcal{K}, \mathcal{K})$. Furthermore, $K(\mathcal{K}) = B(\mathcal{K}, \mathcal{K})/C(\mathcal{K})$ is simple, i.e., $K(\mathcal{K})$ has no non-trivial ideal.*

(4) *$C(\mathcal{K})$ is closed in the uniform operator topology.*

The space $K(\mathcal{K})$ is called *Calkin algebra*. It is useful to have criteria guaranteeing that the spectrum of a self-adjoint operator is purely discrete.

Proposition 3.9. *Let A be a self-adjoint operator and bounded from below. Then properties* (1)–(4) *below are equivalent:*

(1) $(\lambda - A)^{-1}$ *is a compact operator for some $\lambda \in \rho(A)$.*

(2) $(\lambda - A)^{-1}$ *is a compact operator for all $\lambda \in \rho(A)$.*

(3) *The set $\{\varphi \in D(A) | \|\varphi\| < 1, \|A\varphi\| \leq b\}$ is compact for all b.*

(4) *The set $\{\varphi \in D(|A|^{1/2}) | \|\varphi\| < 1, \||A|^{1/2}\varphi\| \leq b\}$ is compact for all b.*

We will consider perturbations of given operators, and address the question of invariance of the essential spectrum of a self-adjoint operator under perturbations.

Definition 3.13 (Relative compactness). Let A be self-adjoint. An operator B such that $D(A) \subset D(B)$ is called *relatively compact* with respect to A if and only if $C(i + A)^{-1}$ is compact.

Proposition 3.10. *Let A be self-adjoint and B be relatively compact with respect to A. Then $A + B$ with domain $D(A)$ is closed and* $\mathrm{Spec}_{\mathrm{ess}}(A + B) = \mathrm{Spec}_{\mathrm{ess}}(A)$.

3.1.5 Schrödinger operators

Definition 3.14 (Schrödinger operator). Let V be a real-valued multiplication operator in $L^2(\mathbb{R}^d)$. The linear operator

$$H = -\frac{1}{2}\Delta + V \tag{3.1.22}$$

with the dense domain

$$D(H) = D(-(1/2)\Delta) \cap D(V) \tag{3.1.23}$$

is called a *Schrödinger operator*.

By the definition of Δ it is readily seen that every $\varphi \in D(-(1/2)\Delta)$ is a bounded continuous function whenever $d \le 3$. Indeed, by the estimate

$$\|\hat{\varphi}\|_1 = \|(1+|k|^2)^{-1}(1+|k|^2)\hat{\varphi}\|_1 \le \|(1+|k|^2)^{-1}\|_2 \|(1+|k|^2)\hat{\varphi}\|_2 < \infty,$$

the Fourier transform of φ lies in $L^1(\mathbb{R}^d)$. Thus φ is a continuous bounded function by the Riemann-Lebesgue lemma. Moreover, by the scaling $\hat{\varphi}_r(k) = r^d \hat{\varphi}(rk)$ we have $\|\hat{\varphi}\|_1 \le cr^{\frac{d-4}{2}} \||k|^2\hat{\varphi}\|_2 + cr^{d/2}\|\hat{\varphi}\|_2$, with a constant c independent of r. Since $d \le 3$, for arbitrary $a > 0$ there exists b such that

$$\|\varphi\|_\infty \le \|\hat{\varphi}\|_1 \le a\|\Delta\varphi\|_2 + b\|\varphi\|_2. \tag{3.1.24}$$

This bound is useful in estimating the relative bound of a given potential V with respect to the Laplacian.

We are interested in defining Schrödinger operators as self-adjoint operators. One reason is that then the spectrum is guaranteed to be real, in match with the uses of quantum mechanics. Secondly, since the solution of the Schrödinger equation (3.1.1) is given by $e^{-itH}\varphi_0$ for $\varphi(x,0) = \varphi_0(x)$, to make sure that e^{-itH} is a unitary map the question has to be addressed whether the symmetric operator H is self-adjoint or essentially self-adjoint. From the point of view of quantum mechanics, distinct self-adjoint extensions lead to different time-evolutions, however, since several self-adjoint extensions of H may exist, a particular one must be chosen to describe the evolution of the system. Since for any self-adjoint extension K of H it is true that $K \subset H^*$, an important step is to find a core of H as then $T = \overline{H\lceil_{\mathcal{D}}}$ is the unique self-adjoint extension of $H\lceil_{\mathcal{D}}$.

In general it is difficult to show self-adjointness or essential self-adjointness of a Schrödinger operator. Here we review some basic criteria, however, as we are interested in defining and studying the properties of perturbations of the Laplacian, first we give a notion of perturbation.

Definition 3.15 (Relative boundedness). Given an operator A, the operator B is called *A-bounded* whenever $D(A) \subseteq D(B)$ and

$$\|Bf\| \le a\|Af\| + b\|f\| \tag{3.1.25}$$

with some numbers $a, b \ge 0$, for all $f \in D(A)$. In this case a is called a *relative bound*. Moreover, if the relative bound a can be chosen arbitrarily small, B is said to be *infinitesimally small* with respect to A.

A classic result of self-adjoint perturbations of a given self-adjoint operator is offered by the following theorem.

Theorem 3.11 (Kato–Rellich). *Suppose that K_0 is a self-adjoint operator, and K_{I} is a symmetric K_0-bounded operator with relative bound strictly less than* 1. *Then $K = K_0 + K_{\mathrm{I}}$ is self-adjoint on $D(K_0)$ and bounded from below. Moreover, K is essentially self-adjoint on any core of K_0.*

Applied to Schrödinger operators, the natural class of V for the Kato–Rellich theorem is $L^p(\mathbb{R}^d) + L^\infty(\mathbb{R}^d)$.

Example 3.6. Let $d = 3$ and $V = V_1 + V_2 \in L^2(\mathbb{R}^3) + L^\infty(\mathbb{R}^3)$. Since $\|V\varphi\|_2 \le \|V_1\|_2\|\varphi\|_\infty + \|V_2\|_\infty\|\varphi\|_2$ and $\|\varphi\|_\infty \le a\|\Delta\varphi\|_2 + b\|\varphi\|_2$ for arbitrary $a > 0$ and some $b > 0$ by (3.1.24), we have

$$\|V\varphi\|_2 \le a\|V_1\|_2\,\|\Delta\varphi\|_2 + (b + \|V_2\|_\infty)\|\varphi\|_2.$$

Thus by Theorem 3.11 the Schrödinger operator $-(1/2)\Delta + V$ is self-adjoint on $D(-(1/2)\Delta)$ and bounded from below. Moreover, it is essentially self-adjoint on any core of $-\Delta$. In particular, $-(1/2)\Delta - |x|^{-1}$ is self-adjoint on $D(-(1/2)\Delta)$ for $d = 3$.

Example 3.7. The previous example can be extended to arbitrary d-dimension. Let

$$V \in L^p(\mathbb{R}^d) + L^\infty(\mathbb{R}^d) \begin{cases} p = 2, & d \le 3 \\ p > d/2, & d \ge 4. \end{cases} \tag{3.1.26}$$

In the case $d \ge 4$ consider $V \in L^p(\mathbb{R}^d)$. Set $q = 2p/(p-2)$ and $r = 2p/(p+2)$. By the Hölder and the Hausdorff–Young inequalities it follows that

$$\|Vf\|_2 \le \|V\|_p\|f\|_q \le C\|V\|_p\|\hat{f}\|_r.$$

Thus

$$\begin{aligned}\|\hat{f}\|_r &= \|(1+\varepsilon|k|^2)^{-1}(1+\varepsilon|k|^2)\hat{f}\|_r \le \|(1+\varepsilon|k|^2)^{-1}\|_p\|(1+\varepsilon|k|^2)\hat{f}\|_2 \\ &\le C'\varepsilon^{-d/(2p)}(\varepsilon\|\Delta f\|_2 + \|f\|_2).\end{aligned}$$

For a given $\delta > 0$ by taking a sufficiently small ε we see that

$$\|Vf\|_2 \le \delta\|(-1/2)\Delta f\| + C_\delta\|f\|_2.$$

Thus V is infinitesimally small with respect to $-\Delta$. In particular, H is self-adjoint on $D(-(1/2)\Delta)$ and essentially self-adjoint on any core of $-\Delta$.

From the discussion in Example 3.7 we conclude the following result.

Proposition 3.12. *Let $V \in L^p(\mathbb{R}^d)$, with $p = 2$ for $d \le 3$, and $p > d/2$ for $d \ge 4$. Then there exists a constant C such that*

$$\|Vf\|_2 \le C\|V\|_p\|(-(1/2)\Delta + 1)f\| \tag{3.1.27}$$

for $f \in D(-(1/2)\Delta)$.

The Kato–Rellich theorem fails to be useful in investigating the self-adjointness of Schrödinger operators with a polynomial potential such at $V(x) = |x|^2$. In this case the following can be applied.

Theorem 3.13 (Kato's inequality). *Let $u \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$ and suppose the distributional Laplacian $\Delta u \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$. Then the inequality*

$$\Delta|u| \geq \Re[(\operatorname{sgn} u)\Delta u] \tag{3.1.28}$$

holds in distributional sense.

Above we have the sign function $\operatorname{sgn} u = u(x)/|u(x)|$ whenever $u(x) \neq 0$, and $\operatorname{sgn} u = 0$ for $u(x) = 0$, defined as usually. The natural class for Kato's inequality is $L^2_{\mathrm{loc}}(\mathbb{R}^d)$.

Corollary 3.14. *Let $V \in L^2_{\mathrm{loc}}(\mathbb{R}^d)$ be bounded from below. Then H is essentially self-adjoint on $C_0^\infty(\mathbb{R}^d)$.*

Proof. By (3) of Proposition 3.3 it suffices to show that $\overline{\mathrm{Ran}(H\lceil_{C_0^\infty(\mathbb{R}^d)}+1)} = L^2(\mathbb{R}^d)$, or equivalently, $(u, (-\Delta + V + 1)f) = 0$ for $\forall f \in C_0^\infty(\mathbb{R}^d)$ implies that $u = 0$. Suppose that $(-\Delta + V + 1)u = 0$ in distribution sense. Then $\Delta u = Vu + u$ follows. By Kato's inequality $\Delta|u| \geq \Re([\operatorname{sgn} u]\Delta u) = (V + 1)|u| \geq 0$. In particular, $\Delta|u| \geq 0$. Let $v_\delta(x) = v(x/\delta)\delta^{-d}$, where $v \geq 0$, $v \in C_0^\infty(\mathbb{R}^d)$ and $\int_{\mathbb{R}^d} v(x)dx = 1$. Consider $w_\delta = |u| * v_\delta$. Then $w_\delta \in D(-(1/2)\Delta)$, $\Delta w^\delta = |u| * \Delta v_\delta = \Delta|u| * v_\delta \geq 0$ and $(w_\delta, \Delta w_\delta) \leq 0$ is trivial, thus $w_\delta = 0$. Since $w_\delta \to |u|$ as $\delta \to 0$ in $L^2(\mathbb{R}^d)$, we have $|u| = 0$. Hence the statement follows. □

Corollary 3.14 can be extended to more general cases.

Corollary 3.15. *Let $V = V_1 + V_2$ be such that $V_1 \geq 0$, $V_1 \in L^2_{\mathrm{loc}}(\mathbb{R}^d)$ and V_2 is $-(1/2)\Delta$-bounded with a relative bound less than 1. Then H is essentially self-adjoint on $C_0^\infty(\mathbb{R}^d)$.*

Example 3.8. By Corollary 3.14 the operator $-(1/2)\Delta + P(x)$ with a polynomial $P(x)$ bounded from below is essentially self-adjoint on $C_0^\infty(\mathbb{R}^d)$ and bounded from below.

Example 3.9. By combining the Kato–Rellich theorem with Kato's inequality it is seen that H with potential $V = V_1 + V_2$ such that $V_1 \in L^2_{\mathrm{loc}}(\mathbb{R}^d)$, $V_1 \geq 0$ and $(-(1/2)\Delta)^{1/2}$-bounded V_2 is self-adjoint on $D(-(1/2)\Delta + V_1))$, and essentially self-adjoint on $C_0^\infty(\mathbb{R}^d)$. To see this first note that $-(1/2)\Delta + V_1$ is essentially self-adjoint by Kato's inequality. Secondly, V_2 is $-(1/2)\Delta + V_1$-bounded with infinitesimally small relative bound. Hence H is self-adjoint on $D(-(1/2)\Delta + V_1)$ by the Kato–Rellich theorem.

3.1.6 Schrödinger operators by quadratic forms

When the potential is too singular and $D(-\Delta) \cap D(V)$ is too small (in the extreme case, it may reduce to $\{0\}$), then a Schrödinger operator cannot be defined as an operator sum, however, it still can be defined in terms of a form sum. Recall that a *quadratic form* q on a Hilbert space $\mathcal{K}$ is a map $q : Q(q) \times Q(q) \to \mathbb{C}$ such that $q(f, g)$ is linear in g and antilinear in f for $f, g \in Q(q)$, where $Q(q)$ denotes the *form domain* of q. The form is *symmetric* if $\overline{q(f, g)} = q(g, f)$. If $q(f, f) \geq -M\|f\|^2$ for some M, then q is said to be *semibounded*. Notice that a semibounded quadratic form is automatically symmetric if $\mathcal{K}$ is defined over the complex field $\mathbb{C}$. A semibounded quadratic form q is closed if $Q(q)$ is complete in the norm $\|f\|_{+1} = (q(f, f) + (M + 1)\|f\|^2)^{1/2}$ with M as above. Moreover, if q is closed and $D \subset Q(q)$ is dense in $Q(q)$ in the norm $\|\cdot\|_{+1}$, then D is called a *form core* for q.

By Riesz's theorem there is a one-to-one correspondence between semibounded quadratic forms and self-adjoint operators bounded from below. Let A be a self-adjoint operator bounded from below. Then it is clear that A defines the semibounded quadratic form

$$q_A(f, g) = \int_{\mathbb{R}} \lambda d(E_A(\lambda) f, g)$$

with form domain $Q(q_A) = D(|A|^{1/2})$.

Proposition 3.16. *If q is a closed semibounded quadratic form, then there exists a unique self-adjoint operator A such that $q = q_A$.*

Using this fact we can define the Schrödinger operator with a singular potential such as $V(x) = \delta(x)$.

Let A and B be self-adjoint operators bounded from below, and suppose that $Q(q_A) \cap Q(q_B)$ is dense. Then the quadratic form $q(f, g) = q_A(f, g) + q_B(f, g)$ with $Q(q) = Q(q_A) \cap Q(q_B)$ defines a closed semibounded quadratic form. Its associated self-adjoint operator is denoted by $C = A \dot{+} B$. For a non-positive self-adjoint operator B, $A \dot{+} (-B)$ is written as $A \dot{-} B$. Here is next a form analogue of the Kato–Rellich theorem.

Theorem 3.17 (KLMN theorem). *Let A be a positive self-adjoint operator and suppose that β is a symmetric quadratic form on $Q(q_A)$ such that*

$$|\beta(f, f)| \leq a q_A(f, f) + b(f, f)$$

with some $a < 1$ and $b \geq 0$. Then there exists a unique self-adjoint operator C such that $Q(q_C) = Q(q_A)$ and $q_C(f, g) = q_A(f, g) + \beta(f, g)$ for $f, g \in Q(q_C)$.

Also the concept of relative boundedness of operators can be extended to forms.

Definition 3.16 (Relative form boundedness). Let A be a positive self-adjoint operator and B a self-adjoint operator such that (1) $Q(q_B) \supset Q(q_A)$ and (2) $|q_B(f, f)| \leq aq_A(f, f) + b(f, f)$, for some $a > 0$ and $b \geq 0$. Then B is said to be *relatively form-bounded* with respect to A.

If B is A-bounded, then B is relatively form-bounded with respect to A with the same relative bound as the A-bound.

Example 3.10. Let $V(x) = |x|^{-\alpha}$, $0 \leq \alpha < 2$ and $d = 3$. In quantum mechanics the relation $\||x|^{-1} f\| \leq 4\|\Delta f\|$ is known as the *uncertainty principle*. It can be seen that V is infinitesimally small form bounded with respect to $-\Delta$, i.e., for any ε there exists $b_\varepsilon > 0$ such that $\|V^{1/2} f\| \leq \varepsilon\|(-\Delta)^{1/2} f\| + b_\varepsilon\|f\|$ for $f \in Q(-\Delta)$. Therefore there exists a self-adjoint operator K such that $(f, Kg) = (f, (-\Delta + V)g)$ for $f, g \in C_0^\infty(\mathbb{R}^3)$.

Example 3.11. Let $q(f, g) = (\nabla f, \nabla g) + \int_{\mathbb{R}^d} V(x)\bar{f}(x)g(x)dx$. The natural class of V such that $\inf\{q(f, f) | \|f\|_2 = 1, f \in H^1(\mathbb{R}^d)\} > -\infty$ is

$$V \in \begin{cases} L^{d/2}(\mathbb{R}^d) + L^\infty(\mathbb{R}^d), & d \geq 3 \\ L^{1+\varepsilon}(\mathbb{R}^2) + L^\infty(\mathbb{R}^2), \varepsilon > 0, & d = 2 \\ L^1(\mathbb{R}^1) + L^\infty(\mathbb{R}^1), & d = 1. \end{cases} \tag{3.1.29}$$

Note that in the case of (3.1.29) the fact that $f \in H^1(\mathbb{R}^d)$ implies by the Sobolev inequality that $\int_{\mathbb{R}^d} |V(x)| f(x)|^2 dx \leq \|V\|_{d/2}\|f\|^2_{2d/(d-2)} \leq C\|V\|_{d/2}\|\nabla f\|_2^2 < \infty$. Since q is semibounded on the form domain $H^1(\mathbb{R}^d)$, there exists a self-adjoint operator associated with q.

The relationship between the convergence of sequences of quadratic forms and associated self-adjoint operators is the following. Let q_1 and q_2 be symmetric quadratic forms bounded from below. $q_1 \geq q_2$ means that $Q(q_1) \subset Q(q_2)$ and $q_1(f, f) \geq q_2(f, f)$, for all $f \in Q(q_1)$. A sequence $(q_n)_{n\in\mathbb{N}}$ of symmetric quadratic forms bounded from below is non-increasing (resp. non-decreasing) if $q_n \geq q_{n+1}$ (resp. $q_n \leq q_{n+1}$), for all n.

Theorem 3.18 (Monotone convergence theorem for forms). (1) *Case of non-increasing sequence: Let $(q_n)_{n\in\mathbb{N}}$ be a non-increasing sequence of densely defined, closed symmetric quadratic forms uniformly bounded from below; $q_n \geq \gamma$ with some constant γ, and let H_n be the self-adjoint operator associated with q_n. Let q_∞ be the symmetric quadratic form defined by*

$$q_\infty(f, f) = \lim_{n\to\infty} q_n(f, f), \quad Q(q_\infty) = \bigcup_n Q(q_n). \tag{3.1.30}$$

Let H be the self-adjoint operator associated with $\bar{q}_\infty$, the closure of q_∞. Then it follows that s-$\lim_{n\to\infty}(H_n - \zeta)^{-1} = (H - \zeta)^{-1}$, *for $\Re\zeta < \gamma$.*

(2) *Case of non-decreasing sequence: Let* $(q_n)_{n\in\mathbb{N}}$ *be a non-decreasing sequence of densely defined, closed symmetric quadratic forms, and* H_n *be the self-adjoint operator associated with* q_n*. Let* q_∞ *be the symmetric quadratic form defined by*

$$q_\infty(f,f) = \lim_{n\to\infty} q_n(f,f), \quad Q(q_\infty) = \{f \mid \lim_{n\to\infty} q_n(f,f) < \infty\}. \tag{3.1.31}$$

Then q_∞ *is a closed symmetric quadratic form. Suppose furthermore that* $Q(q_\infty)$ *is dense. Let* H *be the self-adjoint operator associated with* q_∞*. Then it follows that* $\text{s-lim}_{n\to\infty}(H_n - \zeta)^{-1} = (H-\zeta)^{-1}$*, for* $\operatorname{Im}\zeta \neq 0$.

3.1.7 Confining potential and decaying potential

A natural question is how the spectrum of the Laplacian changes under a perturbation V. It is not our scope to discuss spectral theory in this book, however, we make a few remarks by means of examples which are relevant in our considerations below.

Theorem 3.19 (Rellich's criterion). *Let* F *and* G *be functions on* $\mathbb{R}^d$ *such that* $\lim_{|x|\to\infty} F(x) = \infty$ *and* $\lim_{|x|\to\infty} G(x) = \infty$*. Then the set*

$$\Big\{ f \in L^2(\mathbb{R}^d) \Big| \int_{\mathbb{R}^d} |f(x)|^2 dx \le 1, \\ \int_{\mathbb{R}^d} F(x)|f(x)|^2 dx \le 1, \int_{\mathbb{R}^d} G(p)|\hat{f}(p)|^2 dp \le 1 \Big\}$$

is compact.

Using this criterion the following interesting result is obtained.

Theorem 3.20 (Confining potentials). *Let* $V \in L^1_{\text{loc}}(\mathbb{R}^d)$ *be bounded from below such that* $V(x) \to \infty$ *as* $|x| \to \infty$*. Then* $H = (-1/2)\Delta \dot{+} V$ *has compact resolvent. In particular, it has a purely discrete spectrum, i.e.,* $\operatorname{Spec}(H) = \operatorname{Spec}_{\mathrm{d}}(H)$.

Proof. By Proposition 3.9 it suffices to show that

$$S = \{f \in D(|H|^{1/2}) \mid \|f\| \le 1, \||H|^{1/2} f\|^2 \le b\}$$

is compact for all b. Since S is closed, we need only prove that S is contained in a compact set. Indeed,

$$S \subset \Big\{ f \in L^2(\mathbb{R}^d) \Big| \int_{\mathbb{R}^d} |f(x)|^2 dx \le 1, \\ \int_{\mathbb{R}^d} V(x)|f(x)|^2 dx \le b, \int_{\mathbb{R}^d} \frac{1}{2} p^2 |\hat{f}(p)|^2 dp \le b \Big\}$$

and the right-hand side is compact by Rellich's criterion. □

From this theorem we see that Schrödinger operators with confining potentials have a purely discrete spectrum. Next we consider Schrödinger operators with decaying potentials. The spectral properties of this class are markedly different from those with confining potentials. Let

$$L^{\infty,0}(\mathbb{R}^d) = \{f \in L^\infty(\mathbb{R}^d) | f(x) \to 0 \text{ as } |x| \to \infty\}$$

Define the operator $P(-i\nabla)$ by $P(-i\nabla)f = (2\pi)^{-d/2}\int P(k)e^{ik\cdot x}\hat{f}(k)dk$.

Proposition 3.21 (Compactness). *Let $P, Q \in L^{\infty,0}(\mathbb{R}^d)$. Then $Q(x)P(-i\nabla)$ is compact.*

Proof. Let $1_\Lambda \in C_0^\infty(\mathbb{R}^d)$ be given by

$$1_\Lambda(x) = \begin{cases} 1, & |x| < \Lambda, \\ 0, & |x| > \Lambda + 1. \end{cases}$$

Then the product $1_\Lambda(x)1_\Lambda(-i\nabla)$ is an integral operator with kernel $k(x,y) = (2\pi)^{-d/2}1_\Lambda(x)\hat{1}_\Lambda(y-x) \in L^2(\mathbb{R}^d \times \mathbb{R}^d)$. Thus $1_\Lambda(x)1_\Lambda(-i\nabla)$ is compact by Example 3.4. Since $Q(x)1_\Lambda(x) \to Q(x)$ and $1_\Lambda(-i\nabla)P(-i\nabla) \to P(-i\nabla)$ as $\Lambda \to \infty$ uniformly, it is seen that

$$Q(x)P(-i\nabla) = \lim_{\Lambda\to\infty} Q(x)1_\Lambda(x)1_\Lambda(-i\nabla)P(-i\nabla).$$

Since $Q(x)P(-i\nabla)$ is a uniform limit of a sequence of compact operators, it is compact by Proposition 3.8. □

Theorem 3.22 (Decaying potentials). *Let V be such that*

$$V \in L^p(\mathbb{R}^d) + L^{\infty,0}(\mathbb{R}^d) \begin{cases} p = 2, & d \le 3, \\ p > d/2, & d \ge 4. \end{cases}$$

Then $H = (-1/2)\Delta + V$ is essentially self-adjoint on $C_0^\infty(\mathbb{R}^d)$ and $\mathrm{Spec}_{\mathrm{ess}}(H) = \mathrm{Spec}_{\mathrm{ess}}(-\Delta) = [0,\infty)$.

Proof. Set $H_0 = -(1/2)\Delta$. Essential self-adjointness follows by the Kato–Rellich theorem. By Proposition 3.21, $V(H_0+1)^{-1}$ is compact for $V \in L^{\infty,0}(\mathbb{R}^d)$. Thus it suffices to show that $V \in L^p(\mathbb{R}^d)$ is relatively compact with respect to H_0. Let $V \in L^p(\mathbb{R}^d)$. For each $\varepsilon > 0$ we decompose $V = V_\varepsilon + W_\varepsilon$, where $\|V_\varepsilon\|_p < \varepsilon$ and $W_\varepsilon \in L^{\infty,0}(\mathbb{R}^d)$. Indeed let

$$1_\Lambda(x) = \begin{cases} 1, & |x| \le \Lambda \\ 0, & |x| > \Lambda \end{cases} \quad \text{and} \quad g^{(N)} = \begin{cases} g, & |g| < N \\ N, & g \ge N \\ -N, & g \le -N. \end{cases}$$

Then V is decomposed as

$$V = \underbrace{(V1_\Lambda)^{(N)}}_{=V_\Lambda^N} + \underbrace{V - (V1_\Lambda)^{(N)}}_{=W_\Lambda^N}. \tag{3.1.32}$$

Thus $V_\Lambda^N \in L^{\infty,0}(\mathbb{R}^d)$, $W_\Lambda^N \in L^p(\mathbb{R}^d)$ and $\|W_\Lambda^N\|_p < \varepsilon$ for sufficiently large Λ and N. Furthermore, $V_\Lambda^N \to V$ in L^p as $\Lambda, N \to \infty$. We have by (3.1.27)

$$\|(V(1+H_0)^{-1} - V_\Lambda^N(1+H_0)^{-1})f\| \le C\|V - V_\Lambda^N\|_p\|f\|_2.$$

Thus $V_\Lambda^N(1+H_0)^{-1} \to V(1+H_0)^{-1}$ in the uniform topology and $V_\Lambda^N(1+H_0)^{-1}$ is compact since $V_\Lambda^N \in L^{\infty,0}(\mathbb{R}^d)$. Hence $V(1+H_0)^{-1}$ is compact, and the proof is complete. □

Example 3.12. Two notable examples of spectra of Schrödinger operators are:

(1) *Harmonic oscillator:* Choose, for simplicity, $d = 1$, and let $\omega > 0$. The Schrödinger operator H with $V(x) = \frac{\omega^2}{2}x^2 - \frac{\omega}{2}$ describes the harmonic oscillator. In this case the spectrum is purely discrete $\mathrm{Spec}(H) = \mathrm{Spec}_{\mathrm{d}}(H) = \{\omega n | n \in \mathbb{N} \cup \{0\}\}$.

(2) *Hydrogen atom:* Choose $d = 3$, and let $\gamma > 0$. The Schrödinger operator H with $V(x) = -\frac{\gamma}{|x|}$ describes the hydrogen atom. In this case $\mathrm{Spec}_{\mathrm{ess}}(H) = [0,\infty)$, and $\mathrm{Spec}_{\mathrm{p}}(H) = \mathrm{Spec}_{\mathrm{d}}(H) = \{-\frac{\gamma^2}{2n^2} | \, n \in \mathbb{N}\}$.

3.1.8 Strongly continuous operator semigroups

We conclude this section by a summary of basic facts on operator semigroups.

Definition 3.17 (C_0-semigroup). The one-parameter family of bounded operators $\{S_t : t \ge 0\}$ on a Banach space $\mathscr{B}$ is said to be a *strongly continuous one-parameter semigroup* or *C_0-semigroup* whenever

(1) $S_0 = 1$

(2) $S_s S_t = S_{s+t}$

(3) S_t is strongly continuous in t, i.e., $\text{s-lim}_{t\to 0} S_t = S_0 = 1$.

Every C_0-semigroup is generated by a uniquely associated operator.

Definition 3.18 (Generator). Let $\{S_t : t \ge 0\}$ be a C_0-semigroup on a Banach space $\mathscr{B}$. Its *generator* is defined by

$$Tf = \operatorname*{s-lim}_{t\to 0} \frac{1}{t}(S_t f - f)$$

with domain $D(T) = \left\{f \in \mathscr{B} \,\middle|\, \text{s-lim}_{t\to 0} \frac{1}{t}(S_t f - f) \text{ exists}\right\}$.

Note that T is densely defined, closed, and determines S_t uniquely.

Proposition 3.23. *Let $\{S_t : t \geq 0\}$ be a C_0-semigroup with generator T. The following properties hold for all $t \geq 0$:*

(1) *$S_t D(T) \subset D(T)$ and $\frac{d}{dt} S_t \varphi = T S_t \varphi$*
(2) *there exist $M \geq 1$ and $a > 0$ such that $\|S_t\| \leq M e^{at}$*
(3) *$\int_0^t S_r \varphi dr \in D(T)$ for all $\varphi \in \mathscr{B}$*
(4) *$S_t \varphi - \varphi = T \int_0^t S_r \varphi dr$, moreover $S_t \varphi - \varphi = \int_0^t S_r T \varphi dr$ if $\varphi \in D(T)$*
(5) *if T is a bounded operator, then $D(T) = \mathscr{B}$ and S_t is uniformly continuous.*

With a given self-adjoint operator A, the semigroups e^{-itA} and e^{-tA} can be defined through the spectral resolution (3.1.19). In particular, $\{e^{-itA} : t \in \mathbb{R}\}$ is a strongly continuous one-parameter unitary group, and $\{e^{-tA} : t \geq 0\}$ is a strongly continuous one-parameter semigroup whenever A is bounded from below.

Example 3.13 (Heat kernels). Some specific choices of interest of generators giving rise to strongly continuous semigroups are as follows:

(1) $T = -\frac{1}{2}\Delta$ gives

$$e^{(1/2)t\Delta}(x, y) = (2\pi t)^{-d/2} \exp(-|x - y|^2/2t).$$

(2) $T = (-\Delta)^{\alpha/2}$, $0 < \alpha < 2$, has the integral kernel

$$e^{-t(-\Delta)^{\alpha/2}}(x, y) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-t|k|^\alpha + ik\cdot(x-y)} dk,$$

moreover for $\alpha = 1$ the explicit formula

$$e^{-t(-\Delta)^{1/2}}(x, y) = \Gamma\left(\frac{d+1}{2}\right) \frac{1}{\pi^{(d+1)/2}} \frac{t}{(t^2 + |x - y|^2)^{(d+1)/2}}$$

holds.

(3) $T = (-\Delta + m^2)^{1/2} - m$, $m > 0$, has the integral kernel

$$\begin{aligned} &e^{-t((-\Delta+m^2)^{1/2}-m)}(x, y) \\ &= \frac{1}{(2\pi)^d} \frac{t}{\sqrt{t^2 + |x - y|^2}} \int_{\mathbb{R}^d} e^{mt - \sqrt{(t^2+|x-y|^2)(|k|^2+m^2)}} dk. \end{aligned}$$

The cases above are examples of Feller semigroups associated with Brownian motion, α-stable processes and relativistic stable processes, respectively.

Conversely, given a semigroup here are some basic results on their generators, of which we will often make use.

Proposition 3.24 (Stone's Theorem). *Let $\{U_t : t \in \mathbb{R}\}$ be a strongly continuous one-parameter unitary group on a Hilbert space. Then there exists a unique self-adjoint operator A such that $U_t = e^{itA}$, $t \in \mathbb{R}$.*

Proposition 3.25 (Hille–Yoshida theorem). *A linear operator T is the generator of a C_0-semigroup $\{S_t : t \geq 0\}$ on a Banach space if and only if*

(1) *T is a densely defined closed operator,*

(2) *there exists $a \in \mathbb{R}$ such that $(a, \infty) \subset \rho(T)$ and*

$$\|(a - T)^{-m}\| \leq \frac{M}{(\lambda - a)^m}, \quad \lambda > a, \quad m = 1, 2, \ldots,$$

where $\rho(T)$ denotes the resolvent set of T.

Proposition 3.26 (Semigroup version of Stone's theorem). *Let S_t be a symmetric C_0-semigroup on a Hilbert space. Then there exists a self-adjoint operator A bounded from below such that $S_t = e^{-tA}$, $t \geq 0$.*

Proof. Define the operator A by

$$-Af = \operatorname*{s-lim}_{t \to 0} \frac{1}{t}(S_t - 1)f, \quad D(A) = \left\{ f \;\middle|\; \operatorname*{s-lim}_{t \to 0} \frac{1}{t}(S_t - 1)f \text{ exists} \right\}.$$

By the Hille–Yoshida theorem and the fact that S_t is symmetric, A is a closed symmetric operator and for some $a \in \mathbb{R}$ the resolvent $\rho(A)$ contains $(-\infty, a)$. It is a standard fact that the spectrum of a closed symmetric operator is one of the following cases: (a) the closed upper half-plane, (b) the closed lower-half plane, (c) the entire plane, (d) a subset of $\mathbb{R}$. Hence A must be (d), which implies that A is self-adjoint and bounded from below. Since a given generator gives rise to a unique C_0-semigroup, we have that $S_t = e^{-tA}$. □

In case when S_t is not a semigroup yet $\text{s-lim}_{t \to 0} t^{-1}(S_t - 1)$ does exist, one can use the following result.

Proposition 3.27. *Let A be a positive self-adjoint operator and $\varrho(t)$ a family of self-adjoint operators with $0 \leq \varrho(t) \leq 1$. Suppose that $S_t = t^{-1}(1 - \varrho(t))$ converges to A as $t \downarrow 0$ in strong resolvent sense. Then*

$$\operatorname*{s-lim}_{n \to \infty} \varrho(t/n)^n = e^{-tA}.$$

Proof. $\text{s-lim}_{n \to \infty} \exp\left(-tS_{t/n}\right) = e^{-tA}$ follows by strong resolvent convergence of $S_{t/n}$ to A. Then it suffices to show that

$$\operatorname*{s-lim}_{n \to \infty}(\exp(-tS_{t/n}) - \varrho(t/n)^n) = \operatorname*{s-lim}_{n \to \infty} G_n(tS_{t/n}) = 0,$$

where

$$G_n(x) = \begin{cases} e^{-x} - (1 - x/n)^n, & 0 \le x \le n, \\ e^{-x}, & x \ge n. \end{cases}$$

It is easily seen that $\lim_{n\to\infty} \|G_n\|_\infty = 0$ and the proposition follows. □

We now define semigroups generated by Schrödinger operators.

Definition 3.19 (Schrödinger semigroup). The one-parameter operator semigroup $\{e^{-tH} : t \ge 0\}$ defined by a Schrödinger operator H is a *Schrödinger semigroup*.

Due to the additive structure of Schrödinger operators the following result will also be useful.

Proposition 3.28 (Trotter product formula). *Let A and B be positive self-adjoint operators such that $A + B$ is essentially self-adjoint on $D(A) \cap D(B)$. Then*

$$\operatorname*{s-lim}_{n\to\infty} (e^{-(t/n)A} e^{-(t/n)B})^n = e^{-t(\overline{A+B})}, \tag{3.1.33}$$

where $\overline{A+B}$ denotes the closure of $(A + B)\lceil_{D(A)\cap D(B)}$.

Proof. Let $\varrho(t) = e^{-(t/2)A} e^{-tB} e^{-(t/2)A}$. Then

$$(e^{-(t/n)A} e^{-(t/n)B})^n = e^{-(t/2n)A} \varrho(t/n)^{n-1} e^{-(t/2n)A} e^{-(t/n)B}$$

holds. Since e^{-tA} and e^{-tB} are uniformly bounded in $t \ge 0$ and strongly converge to identity as $t \to 0$, it suffices to show that $s\text{–}\lim_{n\to\infty} \varrho(t/n)^{n-1} = e^{-tC}$, where $C = \overline{A+B}$. We have that $t^{-1}(1 - \varrho(t))$ is a bounded self-adjoint operator and

$$\begin{aligned} t^{-1}(1 - \varrho(t))\phi &= e^{-(t/2)A} e^{-tB} t^{-1}(1 - e^{-(t/2)A})\phi + e^{-(t/2)A} t^{-1}(1 - e^{-tB})\phi \\ &\quad + t^{-1}(1 - e^{-(t/2)A})\phi \to C\phi \end{aligned}$$

as $t \to \infty$, for $\phi \in D(A) \cap D(B)$, which implies that $t^{-1}(1 - \varrho(t)) \to C$ in strong resolvent sense. Thus $\text{s-lim}_{n\to\infty} \varrho(t/n)^n = e^{-tC}$ follows by Proposition 3.27. Furthermore,

$$\begin{aligned} &\operatorname*{s-lim}_{n\to\infty} (\varrho(t/n)^{n-1} - e^{-tC}) \\ &= \operatorname*{s-lim}_{n\to\infty} \varrho(t/n)^{n-1}(1 - \varrho(t/n)) + \operatorname*{s-lim}_{n\to\infty} (\varrho(t/n)^n - e^{-tC}) = 0. \end{aligned}$$

Hence the proposition follows. □

The Trotter product formula can be extended to more general cases.

Proposition 3.29. *Let A and B be positive self-adjoint operators in a Hilbert space $\mathscr{H}$ with form domains $Q(q_A)$ and $Q(q_B)$. Denote by P_{AB} the projection to $\overline{Q(q_A) \cap Q(q_B)}$. Then*

$$\operatorname*{s-lim}_{n\to\infty}(e^{-(t/n)A}e^{-(t/n)B})^n = e^{-t(A\,\dot{+}\,B)}P_{AB}, \tag{3.1.34}$$

where $A \dot{+} B$ is the quadratic form sum of A and B, i.e., the self-adjoint operator in $P_{AB}\mathscr{H}$ associated with the densely defined closed quadratic form $f \mapsto \|A^{1/2}f\|^2 + \|B^{1/2}f\|^2$.

3.2 Feynman–Kac formula for external potentials

3.2.1 Bounded smooth external potentials

We begin discussing Feynman–Kac formulae by assuming that $V \in C_0^\infty(\mathbb{R}^d)$, which is the simplest case. For such a choice of potential the Schödinger operator H is self-adjoint on $D(H) = H^2(\mathbb{R}^d)$ and bounded from below by the Kato–Rellich theorem. Therefore the semigroup e^{-tH} can be defined by the spectral resolution and the solution of (3.1.7) with initial condition g is given by $f(x,t) = (e^{-tH}g)(x)$. We will denote Brownian motion throughout by $(B_t)_{t\geq 0}$ and its natural filtration $(\mathscr{F}_t)_{t\geq 0}$ as discussed in Chapter 2.

Theorem 3.30 (Feynman–Kac formula for Schrödinger operator with smooth external potentials). *Let $V \in C_0^\infty(\mathbb{R}^d)$. Then for $f, g \in L^2(\mathbb{R}^d)$,*

$$(f, e^{-tH}g) = \int_{\mathbb{R}^d} \mathbb{E}^x[\overline{f(B_0)}e^{-\int_0^t V(B_s)\,ds}g(B_t)]dx. \tag{3.2.1}$$

In particular,

$$(e^{-tH}g)(x) = \mathbb{E}^x[e^{-\int_0^t V(B_s)\,ds}g(B_t)]. \tag{3.2.2}$$

Proof. Define the map K_t on $L^2(\mathbb{R}^d)$ by

$$(K_t f)(x) = \mathbb{E}^x[e^{-\int_0^t V(B_s)\,ds}f(B_t)]. \tag{3.2.3}$$

K_t is bounded on $L^2(\mathbb{R}^d)$ for each t as $\|K_t f\|^2 \leq e^{-2\inf V}\|f\|^2$. First we show that $\{K_t : t \geq 0\}$ is a symmetric C_0-semigroup on $L^2(\mathbb{R}^d)$. Define $\tilde{B}_s = B_{t-s} - B_t$, $s < t$, for a fixed $t > 0$. By Proposition 2.7 we have $\tilde{B}_s \stackrel{\mathrm{d}}{=} B_s$. Thus

$$\begin{aligned}(f, K_t g) &= \int_{\mathbb{R}^d} \mathbb{E}[\overline{f(x)}e^{-\int_0^t V(B_s+x)ds}g(B_t + x)]dx \\ &= \mathbb{E}\Big[\int_{\mathbb{R}^d} \overline{f(x)}e^{-\int_0^t V(\tilde{B}_s+x)ds}g(\tilde{B}_t + x)dx\Big].\end{aligned}$$

Changing the variable x to $y = \tilde{B}_t + x$, we obtain

$$\begin{aligned}(f, K_t g) &= \mathbb{E}\left[\int_{\mathbb{R}^d} \overline{f(y - \tilde{B}_t)} e^{-\int_0^t V(\tilde{B}_s - \tilde{B}_t + y) ds} g(y) dy\right] \\ &= \int_{\mathbb{R}^d} \mathbb{E}[\overline{f(y + B_t)} e^{-\int_0^t V(B_{t-s} + y) ds} g(y)] dy \\ &= (K_t f, g), \end{aligned} \tag{3.2.4}$$

i.e., K_t is symmetric. Write now $Z_t = e^{-\int_0^t V(B_s) ds}$. The semigroup property follows directly from the Markov property of Brownian motion:

$$\begin{aligned}(K_s K_t f)(x) &= \mathbb{E}^x[Z_s \mathbb{E}^{B_s}(Z_t f(B_t))] \\ &= \mathbb{E}^x[\mathbb{E}^x[Z_s e^{-\int_0^t V(B_{s+u}) du} f(B_{s+t}) | \mathscr{F}_s]] \\ &= \mathbb{E}^x\, [Z_{s+t} f(B_{s+t})] \\ &= K_{s+t} f(x).\end{aligned}$$

Strong continuity is implied by

$$\|K_t f - f\| \le \mathbb{E}^0[\|e^{-\int_0^t V(\cdot + B_s) ds} f(\cdot + B_t) - f\|] \to 0$$

as $t \to 0$, by using the dominated convergence theorem. We now know that on both sides of (3.2.1) there is a C_0-semigroup. What remains to show is that their generators are equal, i.e., $\frac{1}{t}(K_t - 1)$ converges to $-H$ strongly in L^2 on $D(H)$. We will use Itô calculus for this. Put $Y_t = f(B_t)$; then the Itô formula gives $dZ_t = -V Z_t dt$ and $dY_t = \nabla f \cdot dB_t + (1/2)\Delta f dt$. By the product rule (2.3.25),

$$\begin{aligned} d(Z_t Y_t) &= Z_t \left(-V f(B_t) + \frac{1}{2}\Delta f(B_t)\right) dt + Z_t \nabla f(B_t) \cdot dB_t \\ &= -Z_t (Hf)(B_t) dt + Z_t \nabla f(B_t) \cdot dB_t. \end{aligned} \tag{3.2.5}$$

Taking expectation with respect to Wiener measure on both sides of the integrated form of (3.2.5) and using that $\mathbb{E}^x[\int_0^t Z_s \nabla f \cdot dB_s] = 0$, we obtain

$$(K_t f)(x) = f(x) - \int_0^t \mathbb{E}^x[Z_s (Hf)(B_s)] ds = f(x) - \int_0^t K_s H f(x) ds. \tag{3.2.6}$$

For $f \in D(H)$ and $\varepsilon > 0$ there exists $s_0 > 0$ with $\|K_s H f - H f\| \le \varepsilon$, for all $s < s_0$. Thus by (3.2.6), with $t < s_0$

$$\left\| \frac{1}{t}(K_t f - f) + Hf \right\| \le \frac{1}{t} \int_0^t \| - K_s H f + H f \| ds \le \varepsilon$$

is obtained. This shows strong convergence of $\frac{1}{t}(K_t - 1) f$ to $-Hf$ for $f \in D(H)$ as $t \to 0$, completing the proof. □

Definition 3.20 (Feynman–Kac semigroup). The symmetric C_0-semigroup $\{K_t : t \geq 0\}$ defined by (3.2.3) is called *Feynman–Kac semigroup* for the given V.

It is seen that the solution of (3.1.7), like the solution of the heat equation, can be obtained by running a Brownian motion with an additional exponential weight applied. This gives a probabilistic representation of the solution or, equivalently, of the kernel of the semigroup e^{-tH}. Using this picture we see that Brownian paths spending a long time in a region where V is large are exponentially penalized and thus their contribution to the expected value will be accordingly small. The dominant contribution will therefore come from paths that spend most of their time near the lowest values of V. This is a property which will be explored in more detail in Part II of the book.

3.2.2 Derivation through the Trotter product formula

By using the Trotter product formula we can give another proof of Theorem 3.30.

Alternate proof of Feynman–Kac formula for Schrödinger operator with smooth potentials. Write $H_0 = -(1/2)\Delta$, let $f_1, \ldots, f_{n-1} \in L^\infty(\mathbb{R}^d)$ and $f_0, f_n \in L^2(\mathbb{R}^d)$. By the finite dimensional distributions of Brownian motion, for $0 = t_0 \leq t_1 \leq \cdots \leq t_n$ we have

$$\Big(\prod_{j=1}^{n} e^{-(t_j - t_{j-1})H_0} f_j\Big)(x) = \mathbb{E}^x\Big[\prod_{j=1}^{n} f_j(B_{t_j})\Big] \tag{3.2.7}$$

and

$$\Big(\bar{f}_0, \prod_{j=1}^{n} e^{-(t_j - t_{j-1})H_0} f_j\Big) = \int_{\mathbb{R}^d} \mathbb{E}^x\Big[\prod_{j=0}^{n} f_j(B_{t_j})\Big] dx. \tag{3.2.8}$$

The Trotter product formula combined with (3.2.8) gives

$$\begin{aligned}(f, e^{-tH} g) &= \lim_{n\to\infty} (f, (e^{-(t/n)H_0} e^{-(t/n)V})^n g)\\ &= \lim_{n\to\infty} \int_{\mathbb{R}^d} \mathbb{E}^x[\overline{f(B_0)} g(B_t) e^{-(t/n)\sum_{j=1}^{n} V(B_{tj/n})}] dx.\end{aligned}$$

Since V is continuous and B_s is almost surely continuous in s,

$$\frac{t}{n}\sum_{j=1}^{n} V(B_{tj/n}) \to \int_0^t V(B_s) ds$$

almost surely as $n \to \infty$. Thus we have

$$(f, e^{-tH} g) = \int_{\mathbb{R}^d} \mathbb{E}^x[\overline{f(B_0)} g(B_t) e^{-\int_0^t V(B_s) ds}] dx. \qquad \square$$

Having the Feynman–Kac formula for smooth bounded potentials at hand, we are now interested in extending it to wider classes of V. We present here one immediate extension covering potentials as singular as $V(x) = -1/|x|^3$ (see Section 3.4.4).

Theorem 3.31 (Feynman–Kac formula for Schrödinger operator with singular external potentials). *Assume that for $V = V_+ - V_-$ we have $V_+ \in L^1_{\mathrm{loc}}(\mathbb{R}^d \setminus S)$, where S is a closed set of measure zero, and V_- is relatively form bounded with respect to $-(1/2)\Delta$ with a relative bound strictly less than* 1. *Define $H = -(1/2)\Delta \dot{+} V_+ \dot{-} V_-$. Then* (3.2.1) *holds.*

Proof. Suppose that $V \in L^\infty$ and $V_n = \phi(x/n)(V * j_n)$, where $j_n = n^d\phi(xn)$ with $\phi \in C_0^\infty(\mathbb{R}^d)$ such that $0 \leq \phi \leq 1$, $\int_{\mathbb{R}^d} \phi(x)dx = 1$ and $\phi(0) = 1$. Then $V_n(x) \to V(x)$ almost everywhere. Thus there is a set $\mathscr{N} \subset \mathbb{R}^d$ of Lebesgue measure zero such that $V_n(x) \to V(x)$ for $x \notin \mathscr{N}$, and

$$\mathbb{E}^x[1_{\{B_s \in \mathscr{N}\}}] = \int_{\mathbb{R}^d} 1_{\mathscr{N}}(y)\Pi_t(y-x)dy = 0$$

for every $x \in \mathscr{N}$. Then for every $x \in \mathscr{N}$,

$$0 = \int_0^t ds\mathbb{E}^x[1_{\{B_s \in \mathscr{N}\}}] = \mathbb{E}^x\left[\int_0^t ds 1_{\{B_s \in \mathscr{N}\}}\right]$$

by Fubini's theorem. Thus for every $x \in \mathscr{N}$, the measure of $\{t \in \mathbb{R}^+ | B_t(\omega) \in \mathscr{N}\}$ is zero, for almost every $\omega \in \mathscr{X}$. Hence $\int_0^t V_n(B_s)ds \to \int_0^t V(B_s)ds$ almost surely under $\mathcal{W}^x$, for $x \notin \mathscr{N}$. Therefore by the dominated convergence theorem,

$$\int_{\mathbb{R}^d} dx\mathbb{E}^x[\overline{f(B_0)}g(B_t)e^{-\int_0^t V_n(B_s)ds}] \to \int_{\mathbb{R}^d} dx\mathbb{E}^x[\overline{f(B_0)}g(B_t)e^{-\int_0^t V(B_s)ds}].$$

On the other hand, $e^{-t(-(1/2)\Delta+V_n)} \to e^{-t(-(1/2)\Delta+V)}$ strongly, since $-(1/2)\Delta+V_n$ converges to $-(1/2)\Delta + V$ on a common core. Thus (3.2.1) holds for $V \in L^\infty(\mathbb{R}^d)$. Now assume that $V_+ \in L^1_{\mathrm{loc}}(\mathbb{R}^d \setminus S)$ and V_- is relatively form bounded with respect to $-(1/2)\Delta$. Let

$$V_{+n}(x) = \begin{cases} V_+(x), & V_+(x) < n, \\ n, & V_+(x) \geq n, \end{cases} \qquad V_{-m}(x) = \begin{cases} V_-(x), & V_-(x) < m, \\ m, & V_-(x) \geq m. \end{cases}$$

Then

$$(f, e^{-t(-(1/2)\Delta+V_{n,m})}g) = \int_{\mathbb{R}^d} dx\mathbb{E}^x[\overline{f(B_0)}e^{-\int_0^t V_{n,m}(B_s)\,ds}g(B_t)]. \tag{3.2.9}$$

Set $h = -(1/2)\Delta$. Define the closed quadratic forms

$$\mathfrak{q}_{n,m}(f, f) = (h^{1/2} f, h^{1/2} f) + (V_{+n}^{1/2} f, V_{+n}^{1/2} f) - (V_{-m}^{1/2} f, V_{-m}^{1/2} f),$$
$$\mathfrak{q}_{n,\infty}(f, f) = (h^{1/2} f, h^{1/2} f) + (V_{+n}^{1/2} f, V_{+n}^{1/2} f) - (V_{-}^{1/2} f, V_{-}^{1/2} f),$$
$$\mathfrak{q}_{\infty,\infty}(f, f) = (h^{1/2} f, h^{1/2} f) + (V_{+}^{1/2} f, V_{+}^{1/2} f) - (V_{-}^{1/2} f, V_{-}^{1/2} f),$$

whose form domains are respectively $Q(\mathfrak{q}_{n,m}) = Q(h)$, $Q(\mathfrak{q}_{n,\infty}) = Q(h)$ and $Q(\mathfrak{q}_{\infty,\infty}) = Q(h) \cap Q(V_+)$. Clearly, $\mathfrak{q}_{n,m} \geq \mathfrak{q}_{n,m+1} \geq \mathfrak{q}_{n,m+2} \geq \cdots \geq \mathfrak{q}_{n,\infty}$ and $\mathfrak{q}_{n,m} \to \mathfrak{q}_{n,\infty}$ in the sense of quadratic forms on $\cup_m Q(\mathfrak{q}_{n,m}) = Q(h)$. Since $\mathfrak{q}_{n,\infty}$ is closed on $Q(h)$, by Theorem 3.18 the associated positive self-adjoint operators satisfy $h + V_{+n} - V_{-m} \to h + V_{+n} \dot{-} V_-$ in strong resolvent sense, which implies that for all $t \geq 0$,

$$\exp\left(-t\left(h + V_{+n} - V_{-m}\right)\right) \to \exp\left(-t\left(h + V_{+n} \dot{-} V_-\right)\right) \tag{3.2.10}$$

strongly as $m \to \infty$. Similarly, we have $\mathfrak{q}_{n,\infty} \leq \mathfrak{q}_{n+1,\infty} \leq \mathfrak{q}_{n+2,\infty} \leq \cdots \leq \mathfrak{q}_\infty$ and $\mathfrak{q}_{n,\infty} \to \mathfrak{q}_{\infty,\infty}$ in form sense on $\{f \in \cap_n Q(\mathfrak{q}_{n,\infty}) \mid \sup_n \mathfrak{q}_{n,\infty}(f, f) < \infty\} = Q(h) \cap Q(V_+)$. Hence by Theorem 3.18 again we obtain

$$\exp(-t(h \dot{+} V_{+n} \dot{-} V_-)) \to \exp(-t(h \dot{+} V_+ \dot{-} V_-)), \quad t \geq 0, \tag{3.2.11}$$

in strong sense as $n \to \infty$. By taking first $n \to \infty$ and then $m \to \infty$ it can be proven that both sides of (3.2.9) converge; the left-hand side of (3.2.9) converges by monotone convergence for forms, (3.2.10) and (3.2.11), and the right-hand side by monotone convergence for integrals. □

3.3 Feynman–Kac formula for Kato-class potentials

3.3.1 Kato-class potentials

The Feynman–Kac formula we have derived so far covers only a limited range of interesting models of mathematical physics. First, the potential was not allowed to grow at infinity. This excludes cases when the particle is trapped in some region of $\mathbb{R}^d$, like under the harmonic potential $V(x) = |x|^2$. Secondly, it is desirable to allow for local singularities, especially the case of the Coulomb potential $V(x) = -1/|x|$ in dimension 3. To include these and further cases of interest, we introduce here a large class of potentials and use it as a space of reference as far as we can. This space is Kato-class which on several counts is a natural function space as it will be made clear below. However, occasionally we will have to leave this space in order to draw stronger conclusions or because the conditions of Kato-class are too strong themselves for a particular statement we want to prove.

Our objective in this section is to introduce Kato-class and discuss its basic properties, then define Schrödinger operators with Kato-class potentials as self-adjoint operators, and finally derive a corresponding Feynman–Kac formula.

Definition 3.21 (Kato-class). (1) $V : \mathbb{R}^d \to \mathbb{R}$ is called a *Kato-class potential* whenever

$$\lim_{r\to 0} \sup_{x\in\mathbb{R}^d} \int_{B_r(x)} |g(x-y)V(y)|\, dy = 0 \tag{3.3.1}$$

holds, where $B_r(x)$ is the closed ball of radius r centered at x, and

$$g(x) = \begin{cases} |x|, & d = 1, \\ -\log|x|, & d = 2, \\ |x|^{2-d}, & d \geq 3. \end{cases} \tag{3.3.2}$$

We denote this linear space by $\mathcal{K}(\mathbb{R}^d)$.

(2) V is *Kato-decomposable* whenever $V = V_+ - V_-$ with $V_+ \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$ and $V_- \in \mathcal{K}(\mathbb{R}^d)$.

In the case of $d = 1$ we have an equivalent definition of Kato-class which we will use below. V is of Kato-class if and only if

$$\sup_{x\in\mathbb{R}} \int_{B_1(x)} |V(y)|dy < \infty. \tag{3.3.3}$$

Suppose (3.3.3) holds. Then $\sup_x \int_{B_r(x)} |x-y||V(y)|dy \leq r \sup_x \int_{B_1(x)} |V(y)|dy$, for $0 < r < 1$, thus V is of Kato-class. Next suppose that V is of Kato-class. For any $\varepsilon > 0$ there exists $r > 0$ such that $\sup_x \int_{B_r(x)} |x-y||V(y)|dy < \varepsilon$. Then it follows that

$$\frac{r}{2} \sup_x \int_{r/2\leq|x-y|\leq r} |V(y)|dy \leq \sup_x \int_{r/2\leq|x-y|\leq r} |x-y||V(y)|dy \leq \varepsilon$$

and therefore

$$\sup_x \int_{r/2\leq|x-y|\leq r} |V(y)|dy \leq \frac{2\varepsilon}{r}.$$

Since $B_1(0)$ can be covered by a finite number of annuli, say N, we have that $A_j = \{y | r/2 \leq |x_j - y| \leq r\}$ with some point x_j. Then $\sup_x \int_{B_1(x)} |V(y)|dy \leq \sum_j^N \sup_x \int_{A_j+x} |V(y)|dy \leq \frac{2N\varepsilon}{r} < \infty$ holds.

Before deriving a Feynman–Kac formula for such potentials it is useful to discuss some basic features of Kato-class. We begin by explaining the definition. Let Π_t

be the transition kernel of Brownian motion given by (2.1.10), and f a non-negative function. By Fubini's theorem we have

$$\mathbb{E}^x\left[\int_0^\infty f(B_t)dt\right] = \int_{\mathbb{R}^d} H(x,y)f(y)dy, \tag{3.3.4}$$

with integral kernel $H(x,y) = \int_0^\infty \Pi_t(x,y)dt$. In the limit $t \to \infty$, $\Pi_t(x)$ behaves like $(2\pi t)^{-d/2}$, thus H diverges for $d = 1$ and 2. For $d \geq 3$, on making the change of variable $|x-y|^2/(2t) \mapsto t$,

$$H(x,y) = \frac{\Gamma(d/2-1)}{2\pi^{d/2}} \frac{1}{|x-y|^{d-2}} \tag{3.3.5}$$

is obtained, where Γ is the Gamma-function. For lower dimensions we proceed by compensation with a suitable additive term. Let $\tilde{H}(x,y) = \int_0^\infty (\Pi_t(x,y) - h(t))\,dt$. For $d = 1$ we choose $h(t) = \Pi_t(0,0) = (2\pi t)^{-1/2}$ yielding

$$\tilde{H}(x,y) = \frac{1}{\sqrt{2\pi}} \int_0^\infty \frac{e^{-|x-y|^2/(2t)} - 1}{\sqrt{t}} dt = -|x-y|$$

by making the same change of variable as above. For $d = 2$ write $h(t) = \Pi_t(0,v)$, with unit vector $v = \binom{1}{0}$. Then

$$\tilde{H}(x,y) = \frac{1}{2\pi} \int_0^\infty \frac{e^{-|x-y|^2/(2t)} - e^{-1/(2t)}}{t} dt = -\frac{1}{\pi} \log|x-y|.$$

This shows that g in Definition 3.21 is basically given by the potential kernels of the Laplacian.

The conditions in (3.3.2) limit the growth at infinity ($\mathcal{K}(\mathbb{R}^d)$) and the severity of local singularities ($L^1_{\mathrm{loc}}(\mathbb{R}^d)$). These are illustrated by the following examples.

Example 3.14. Let $d = 3$. Kato-class allows singularities of the type $|x|^{-(2-\varepsilon)}$ with any $\varepsilon > 0$, including in particular Coulomb potential $V(x) = 1/|x|$, as well as infinitely many isolated singularities of the same type are allowed. In contrast, the function $V(x) = |x|$ is not contained in $\mathcal{K}(\mathbb{R}^3)$.

Example 3.15. An application of the Hölder inequality gives an idea about the growth at infinity allowed. Let $d = 3$; then

$$\left|\int_{B_r(x)} \frac{1}{|x-y|}|V(y)|\,dy\right| \leq \left(\int_{B_r(x)} \frac{1}{|x-y|^p}\,dy\right)^{1/p} \left(\int_{B_r(x)} |V(y)|^q\,dy\right)^{1/q},$$

with $1/p + 1/q = 1$. The first factor is independent of x and when $p < 3$, it goes to zero as $r \to 0$. Thus $V \in \mathcal{K}(\mathbb{R}^3)$ if $\sup_x \int_{B_r(x)} |V(y)|^q dy < \infty$ for some $r > 0$ and some $q > 3/2$. A direct calculation shows, for instance, that

$$V(x,y,z) = |x|^\alpha e^{-|x|^\beta(|y|^2+|z|^2)}$$

is in $\mathcal{K}(\mathbb{R}^3)$ if $\beta > 3\alpha/2$. This shows that Kato-class is essentially an integrability condition; pointwise divergence is allowed as long as the given integrals can be controlled.

Example 3.16. Let $V \in L^p(\mathbb{R}^d) + L^\infty(\mathbb{R}^d)$ with $p = 1$ for $d = 1$, and $p > d/2$ for $d \geq 2$. Then $V \in \mathcal{K}(\mathbb{R}^d)$.

If V is Kato-decomposable, the positive part of V is in $L^1_{\mathrm{loc}}(\mathbb{R}^d)$. We have the following property.

Lemma 3.32. *Suppose that* $0 \leq V \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$. *Then* $\mathcal{W}^x(\int_0^t V(B_s)ds = \infty) = 0$ *for almost every* $x \in \mathbb{R}^d$. *In particular,* $\mathbb{E}^x[e^{-\int_0^t V(B_s)ds}] > 0$, *for almost every* $x \in \mathbb{R}^d$.

Proof. Let $N \in \mathbb{N}$ and

$$1_N(x) = \begin{cases} 1, & |x| < N, \\ 0, & |x| \geq N. \end{cases}$$

Since $1_N V \in L^1(\mathbb{R}^d)$, we have

$$\int_{\mathbb{R}^3} dx \mathbb{E}^x \left[\int_0^t 1_N V(B_s) ds \right] \leq t \|1_N V\|_{L^1}.$$

Hence for almost every fixed $x \in \mathbb{R}^d$, $\mathcal{W}^x(\int_0^t 1_N V(B_s)ds < \infty) = 1$, and then there exits $\mathcal{N}_N$, dependent on x, such that $\mathcal{W}^x(\mathcal{N}_N) = 0$ and $\int_0^t 1_N V(B_s(\omega))ds < \infty$, for every $\omega \in \mathcal{X} \setminus \mathcal{N}_N$. Let $\mathcal{N} = \cup_N \mathcal{N}_N$. Note that $\mathcal{N}$ depends on x. Thus $\int_0^t 1_N V(B_s(\omega))ds < \infty$, for every $N \in \mathbb{N}$ and $\omega \in \mathcal{X} \setminus \mathcal{N}$. Since $B_s(\omega)$ is continuous in s almost surely, for every $\omega \in \mathcal{X} \setminus \mathcal{N}$ there exists $N = N(\omega) \in \mathbb{N}$ such that $V(B_s(\omega)) = 1_N V(B_s(\omega))$, for all $0 \leq s \leq t$. Thus $V(B_s(\omega)) = 1_N V(B_s(\omega))$ and hence

$$\int_0^t V(B_s(\omega))ds = \int_0^t 1_N V(B_s(\omega))ds < \infty, \quad \omega \in \mathcal{X} \setminus \mathcal{N}.$$

□

Next we present a first equivalent characterization of Kato-class.

Proposition 3.33. *A non-negative function* V *is in* $\mathcal{K}(\mathbb{R}^d)$ *if and only if*

$$\lim_{t \downarrow 0} \sup_{x \in \mathbb{R}^d} \mathbb{E}^x \left[\int_0^t V(B_s)\, ds \right] = 0. \tag{3.3.6}$$

Proof. As in (3.3.4), write for $t \in \mathbb{R}$

$$\mathbb{E}^x \left[\int_0^t V(B_s)\, ds \right] = \int_0^t ds \int_{\mathbb{R}^d} dy\, \Pi_s(x-y)V(y).$$

By the same change of variable as above,

$$\int_0^t \Pi_s(x-y)ds = \frac{1}{(2\pi)^{d/2}} \frac{1}{|x-y|^{d-2}} \int_{|x-y|^2/(2t)}^{\infty} e^{-u} u^{d/2-2}\, du = H(x,y;t)$$

readily follows. The key observation now is that $H(x,y;t)$ behaves like $g(x-y)$ in Definition 3.21 whenever $|x-y|^2/2t$ is small, and vanishes rapidly when $|x-y|^2/2t$ goes to infinity. This is obvious in the cases $d \geq 3$ and is seen through integration by parts in $d = 1$ and 2. We give the details only in the simplest case $d = 4$; the others go by the same arguments, though require further estimates.

The convenient feature of the case $d = 4$ is that then we have an explicit formula for $H(x,y;t)$ and

$$\sup_{x\in\mathbb{R}^4} \mathbb{E}^x \left[\int_0^t V(B_s)\, ds \right] = \sup_{x\in\mathbb{R}^4} \int_{\mathbb{R}^4} \frac{1}{(2\pi|x-y|)^2} e^{-\frac{|x-y|^2}{2t}} V(y)dy \tag{3.3.7}$$

follows. We split up the domain of integration on the right-hand side above into $A_q = \{y \mid |x-y| < t^{1/q}\}$ and $B_q = \mathbb{R}^4 \setminus A_q$ for $q > 0$. Notice that A_q shrinks to $\{x\}$ when $t \to 0$. If (3.3.6) holds, then the integral over A_2 on the right-hand side of (3.3.7) vanishes with $t \to 0$, uniformly in x. On A_2, $\exp(-|x-y|^2/2t)$ is bounded away from zero, hence $V \in \mathcal{K}(\mathbb{R}^4)$ follows. Conversely, assume $V \in \mathcal{K}(\mathbb{R}^4)$. Then by definition and the fact that the negative exponential is bounded from above, we obtain $\lim_{t\downarrow 0} \sup_{x\in\mathbb{R}^4} \int_{A_3} \frac{1}{(2\pi|x-y|)^2} e^{-\frac{|x-y|^2}{2t}} V(y)dy = 0$. On the other hand, on B_3 we have $\exp(-|x-y|^2/2t) \leq \exp(-|x-y|/t^{1/6})$, thus the right-hand side of (3.3.7) goes to zero on this set by dominated convergence. □

We present a second equivalent characterization of Kato-class recalling the asymptotic properties of the resolvent of the Laplacian. The kernel of the resolvent operator $(-\Delta+\lambda)^{-1}$ is explicitly given by

$$(-\Delta+\lambda)^{-1}(x,y) = C_\lambda(x-y), \tag{3.3.8}$$

where

$$C_\lambda(x) = (2\pi)^{-d/2} \left(\frac{\sqrt{\lambda}}{|x|} \right)^{(d-1)/2} K_{(d-2)/2}(\sqrt{\lambda}|x|), \tag{3.3.9}$$

with $K_d(z)$ being the modified Bessel function of the third kind. As $|x-y| \to 0$,

$$C_\lambda(x-y) \sim \begin{cases} |x-y|^{2-d}, & d \ge 3, \\ -\log|x-y|, & d = 2, \\ C, & d = 1, \end{cases}$$

moreover for any $\delta > 0$,

$$\lim_{\lambda \to \infty} \sup_{|x-y|>\delta} e^{|x-y|} C_\lambda(x-y) = 0.$$

Then we readily have $\sup_x \int_{\mathbb{R}^d} C_E(x-y)|V(y)|dy \to 0$ as $E \to \infty$ if and only if $V \in \mathcal{K}(\mathbb{R}^d)$, and by comparison with Definition 3.21 the proposition below follows. We see that $(-\Delta+E)^{-1}|V|(x) = \int_{\mathbb{R}^d} C_E(x-y)|V(y)|dy$, and we use that

$$\|(-\Delta+E)^{-1}|V|\|_\infty = \sup_{x \in \mathbb{R}^d} \left|(-\Delta+E)^{-1}|V|(x)\right|. \tag{3.3.10}$$

Proposition 3.34. $V \in \mathcal{K}(\mathbb{R}^d)$ *if and only if*

$$\lim_{E \to \infty} \|(-\Delta+E)^{-1}|V|\|_\infty = 0. \tag{3.3.11}$$

For the sake of completeness, we present a proof of equivalence between (3.3.11) and (3.3.6).

Proof. Take τ in a suitable neighbourhood of 0 and denote $H_0 = -(1/2)\Delta$. Clearly, $\mathbb{E}^x[|V(B_s)|] = (e^{-sH_0}|V|)(x)$. Then by Laplace transform,

$$((H_0+E)^{-1}|V|)(x) = \sum_{n \ge 0} \int_{nT}^{(n+1)T} e^{-sE}(e^{-sH_0}|V|)(x)ds.$$

By a change of variable we furthermore have

$$((H_0+E)^{-1}|V|)(x) = \sum_{n \ge 0} e^{-nTE} \int_0^T e^{-sE}(e^{-(s+nT)H_0}|V|)(x)ds.$$

By the semigroup property and using the integral kernel $e^{-tH_0}(x,y)$ we derive that

$$\begin{aligned} &((H_0+E)^{-1}|V|)(x) \\ &= \sum_{n \ge 0} e^{-nTE} \int_{\mathbb{R}^d} dy (e^{-nTH_0})(x,y) \int_0^T e^{-sE}(e^{-sH_0}|V|)(y)ds. \end{aligned}$$

Take the supremum with respect y. Then

$$\begin{aligned}&((H_0+E)^{-1}|V|)(x)\\&\leq \Big(\sup_{y\in\mathbb{R}^d}\int_0^T e^{-sE}(e^{-sH_0}|V|)(y)ds\Big)\sum_{n\geq 0}e^{-nTE}\int_{\mathbb{R}^d}(e^{-nTH_0})(x,y)dy.\end{aligned}$$

Since $\int_{\mathbb{R}^d}(e^{-nTH_0})(x,y)dy = 1$, we have

$$\begin{aligned}\int_{\mathbb{R}^d}(e^{-nTH_0})(x,y)dy &\leq \frac{1}{1-e^{-TE}}\sup_{y\in\mathbb{R}^d}\int_0^T e^{-sE}(e^{-sH_0}|V|)(y)ds\\&\leq \frac{1}{1-e^{-TE}}\|(H_0+E)^{-1}|V|\|_\infty.\end{aligned}$$

Taking the supremum, this yields

$$\begin{aligned}&(1-e^{-TE})\|(H_0+E)^{-1}|V|\|_\infty\\&\leq \sup_{y\in\mathbb{R}^d}\int_0^T e^{-sE}(e^{-sH_0}|V|)(y)ds \;\leq\; \|(H_0+E)^{-1}|V|\|_\infty.\end{aligned}$$

Furthermore, trivially we have

$$e^{-TE}\int_0^T(e^{-sH_0}|V|)(y)ds \;\leq\; \int_0^T e^{-sE}(e^{-sH_0}|V|)(y)ds \;\leq\; \int_0^T(e^{-sH_0}|V|)(y)ds.$$

Then $\sup_{y\in\mathbb{R}^d}\int_0^T(e^{-sH_0}|V|)(y)ds$ is bounded by $e^{TE}\|(H_0+E)^{-1}|V|\|_\infty$, for all $T\geq 0$, hence

$$\lim_{T\downarrow 0}\sup_{x\in\mathbb{R}^d}\int_0^T(e^{-sH_0}|V|)(x)ds \leq \|(H_0+E)^{-1}|V|\|_\infty,$$

for all E. Therefore, if $\lim_{E\to\infty}\|(H_0+E)^{-1}|V|\|_\infty = 0$, then the right-hand side decreases with E, and optimizing over E implies (3.3.6). Conversely, if we assume (3.3.6), then by the above

$$\lim_{E\to\infty}\|(H_0+E)^{-1}|V|\|_\infty \leq \lim_{T\downarrow 0}\lim_{E\to\infty}\frac{1}{1-e^{-TE}}\sup_{x\in\mathbb{R}^d}\int_0^T(e^{-sH_0}|V|)(x)ds = 0,$$

i.e., (3.3.11) holds. □

The latter characterization is particularly interesting in view of the following property, which through the KLMN theorem implies that if V is Kato-class, then H is self-adjoint at least in form sense.

Proposition 3.35. *If there exist $a, b > 0$ and $0 < \delta < 1$ such that for all $0 < \varepsilon < 1$*

$$(f, |V|f) \le \varepsilon \|\sqrt{-\Delta} f\|^2 + a \exp(b\epsilon^{-\delta})\|f\|^2, \quad \forall f \in D(\sqrt{-\Delta}), \tag{3.3.12}$$

then $V \in \mathcal{K}(\mathbb{R}^d)$. Conversely, if V is Kato-class, then $D(\sqrt{-\Delta}) \subset D(V^{1/2})$ and V is $-\Delta$-form bounded with infinitesimally small relative bound.

Proof. Suppose (3.3.12). Since

$$(-\Delta + E)^{-1}|V|(x) = \int_0^\infty dt e^{-tE} \int_{\mathbb{R}^d} \Pi_t(x-y)|V|(y)dy,$$

we estimate $\Pi_t(x-y)|V(y)|$ for a fixed x. Let $\phi(\cdot) = \sqrt{\Pi_t(x-\cdot)}$. By the assumption

$$\begin{aligned}(\phi, |V|\phi) &\le \varepsilon\|\sqrt{-\Delta}\phi\|^2 + a\exp(b\varepsilon^{-\delta})\|\phi\|^2 \\ &= (\phi, -\Delta\phi) + a\exp(b\varepsilon^{-\delta}) \le ct^{-2} + a\exp(b\varepsilon^{-\delta})\end{aligned}$$

with a constant c independent of x. Hence

$$\left|(-\Delta + E)^{-1}|V|(x)\right| \le \left(\int_0^1 + \int_1^\infty\right) e^{-tE}(ct^{-2} + a\exp(b\varepsilon^{-\delta}))dt. \tag{3.3.13}$$

Taking $\varepsilon = (1 + \log|t|)^{2/(1+\delta)}$, we have $a\exp(b\varepsilon^{-\delta}) \le t^b a e^b$, thus (3.3.11) follows.

Conversely, suppose that V is of Kato-class. By duality it is seen that $\|(-\Delta + E)^{-1}|V|\|_{\infty,\infty} = \||V|(-\Delta + E)^{-1}\|_{1,1}$, where $\|\cdot\|_{p,p}$ denotes bounded operator norm on L^p. Notice that $\|(-\Delta + E)^{-1}|V|\|_{\infty,\infty} = \|(-\Delta + E)^{-1}|V|\|_\infty$. For the operator-valued function $F(z) = |V|^z(-\Delta + E)^{-1}|V|^{1-z}$ on the strip $\{z \in \mathbb{C} | \Re z \in [0,1]\}$, by duality $\||V|^{1/2}(-\Delta + E)^{-1}|V|^{1/2}\|_{2,2} \le \|(-\Delta + E)^{-1}|V|\|_{1,1} = \|(-\Delta + E)^{-1}|V|\|_{\infty,\infty}$ by the Stein interpolation theorem. Hence

$$\||V|^{1/2}(-\Delta + E)^{-1/2}\|_{2,2}^2 \le \|(-\Delta + E)^{-1}|V|\|_{\infty,\infty} \to 0$$

as $E \to \infty$. Since

$$\||V|^{1/2} f\| \le \|(-\Delta + E)^{-1}|V|\|_{\infty,\infty}^{1/2} \|(-\Delta + E)^{1/2} f\|,$$

V is $-\Delta$-form bounded with an infinitesimally small relative bound. □

Remark 3.1 (Stummel-class). Kato-class defined above can be regarded as the quadratic form analogue of Stummel-class. A potential V is in *Stummel class* whenever

$$\lim_{r\to 0} \sup_{x\in\mathbb{R}^d} \int_{B_r(x)} h(x-y)|V(y)|^2 \, dy = 0 \tag{3.3.14}$$

holds, where

$$h(x) = \begin{cases} 1, & \text{if } d \le 3, \\ -\log|x|, & \text{if } d = 4, \\ |x|^{4-d}, & \text{if } d > 4. \end{cases} \tag{3.3.15}$$

The counterparts of Propositions 3.34 and 3.35 hold similarly. An equivalent condition to (3.1) is

$$\lim_{E\to\infty} \|(-\Delta + E)^{-2}|V|^2\|_\infty = 0.$$

Suppose that there are $a, b > 0$ and a δ with $0 < \delta < 1$ such that for all $\varepsilon < 1$ and all $f \in D(-\Delta)$,

$$\|Vf\|^2 \le \varepsilon\| - \Delta f\|^2 + a\exp(b\varepsilon^{-\delta})\|f\|^2.$$

Then $V \in S(\mathbb{R}^d)$.

Remark 3.2. From the perspective of self-adjointness Kato-class is not a natural space, certainly not the largest possible. A well-known example is $V(x) = |x|^{-2}|\log|x||^{-\varepsilon}$ which is $-\Delta$-form bounded with relative bound 0 exactly when $\varepsilon = 0$, while it is of Kato-class if and only if $\varepsilon > 1$.

Proposition 3.33 says that given $V \in \mathcal{K}(\mathbb{R}^d)$ one can make $\mathbb{E}^x[\int_0^t V(B_s)ds]$ arbitrarily small by taking t small uniformly in x. We will now see that this implies exponential integrability of the random variable $\int_0^t V(B_s)\,ds$. The proof of this fact is based on Khasminskii's lemma which follows directly from the Markov property of Brownian motion but is so useful that we state it as a separate result. First we show a preliminary result.

Lemma 3.36. *Let $f, g : \mathscr{X} \to \mathbb{R}$ be functions such that f is $\mathscr{F}_t^B$-measurable, $\mathbb{E}^x[|f|] < \infty$ and $\sup_y \mathbb{E}^y[|g|] < \infty$. Let $\theta_t : \mathscr{X} \to \mathscr{X}$ be time shift given by $(\theta_t\omega)(\cdot) = \omega(t + \cdot)$ and define $\theta_t^* g(\omega) = g \circ \theta_t(\omega) = g(\omega(\cdot + t))$. Then*

$$\left|\mathbb{E}^x[f\theta_t^* g]\right| \le \mathbb{E}^x[|f|]\sup_y \mathbb{E}^y[|g|].$$

Proof. Notice that $|\mathbb{E}^x\left[g(B_{\cdot+t})|\mathscr{F}_t^B\right]| = |\mathbb{E}^{B_t}[g(B_\cdot)]| \le \sup_y \mathbb{E}^y[|g|]$. Since f is $\mathscr{F}_t^B$-measurable,

$$|\mathbb{E}^x[f\theta_t^* g]| = |\mathbb{E}^x[f\mathbb{E}^x[\theta_t^* g|\mathscr{F}_t^B]]| \le \mathbb{E}^x[|f|\sup_y \mathbb{E}^y[|g|]] = \sup_y \mathbb{E}^y[|g|]\mathbb{E}^x[|f|],$$

where we used the Markov property of Brownian motion. $\square$

Lemma 3.37 (Khasminskii's lemma). *Let $V \geq 0$ be a measurable function on $\mathbb{R}^d$ with the property that for some $t > 0$ and some $\alpha < 1$*

$$\sup_{x\in\mathbb{R}^d} \mathbb{E}^x \left[\int_0^t V(B_s)\, ds \right] = \alpha. \tag{3.3.16}$$

Then

$$\sup_{x\in\mathbb{R}^d} \mathbb{E}^x [e^{\int_0^t V(B_s)\, ds}] \leq \frac{1}{1-\alpha}. \tag{3.3.17}$$

Proof. By expanding the exponential in (3.3.17), it suffices to show that

$$\begin{aligned}\alpha^n &\geq \sup_x \mathbb{E}^x \left[\frac{1}{n!} \left[\int_0^t V(B_s)\, ds \right]^n \right] \\ &= \frac{1}{n!} \sup_x \mathbb{E}^x \left[\int_0^t ds_1 \cdots \int_0^t ds_n\, V(B_{s_1}) \cdots V(B_{s_n}) \right].\end{aligned}$$

Since there are $n!$ ways of ordering $\{s_1, \ldots, s_n\}$ it suffices to prove

$$\alpha^n \geq \sup_x \mathbb{E}^x \left[\int_{\Delta_n} V(B_{s_1}) \cdots V(B_{s_n}) ds_1 \cdots ds_n \right], \tag{3.3.18}$$

where $\Delta_n = \{(s_1, \ldots, s_n) \in \mathbb{R}^n | 0 < s_1 < \cdots < s_n\}$. For fixed $s_1 < \cdots < s_{n-1}$, Lemma 3.36 gives

$$\begin{aligned}&\mathbb{E}^x \left[V(B_{s_1}) \cdots V(B_{s_{n-1}}) \int_{s_{n-1}}^t V(B_{s_n})\, ds_n \right] \\ &\leq \mathbb{E}^x \left[V(B_{s_1}) \cdots V(B_{s_{n-1}}) \right] \sup_x \mathbb{E}^x \left[\int_{s_{n-1}}^t V(B_{s_n})\, ds_n \right] \\ &\leq \alpha \mathbb{E}^x \left[V(B_{s_1}) \cdots V(B_{s_{n-1}}) \right].\end{aligned}$$

The result follows by induction and an application of Fubini's theorem. □

Now we are in the position to prove exponential integrability of the integral over the potential. As it will be seen below this is essentially all we need to prove the Feynman–Kac formula for Kato-decomposable potentials. When $V_- \in \mathscr{K}(\mathbb{R}^d)$, it can be seen that the exponent $e^{\int_0^t V(B_s) ds}$ is integrable with respect to Wiener measure so that $\mathbb{E}^x[e^{\int_0^t V(B_s) ds}]$ is bounded for all x.

Lemma 3.38. *Let $0 \leq V \in \mathcal{K}(\mathbb{R}^d)$. Then there exist $\beta, \gamma > 0$ such that*

$$\sup_x \mathbb{E}^x [e^{\int_0^t V(B_s)}] < \gamma e^{\beta t}. \tag{3.3.19}$$

Furthermore, if $V \in L^p(\mathbb{R}^d)$ with $p = 1$ for $d = 1$, and $p > d/2$ for $d \geq 2$, then there exists C such that

$$\beta \leq C\|V\|_p. \tag{3.3.20}$$

Proof. There exists $t^* > 0$ such that $\alpha_t = \sup_x \mathbb{E}^x[\int_0^t V(B_s)] < 1$, for all $t \leq t^*$, and $\alpha_t \to 0$ as $t \to 0$. By Khasminskii's lemma we have

$$\sup_x \mathbb{E}^x[e^{\int_0^t V(B_s)}] < \frac{1}{1-\alpha_t} \tag{3.3.21}$$

for all $t \leq t^*$. By means of Lemma 3.36 we obtain

$$\mathbb{E}^x[e^{\int_0^{2t^*} V(B_s)}] = \mathbb{E}^x[e^{\int_0^{t^*} V(B_s)}\mathbb{E}^{B_{t^*}}[e^{\int_0^{t^*} V(B_s)}]] \leq \left(\frac{1}{1-\alpha_{t^*}}\right)^2.$$

Repeating this procedure we see that

$$\sup_x \mathbb{E}^x[e^{\int_0^t V(B_s)}] \leq \left(\frac{1}{1-\alpha_{t^*}}\right)^{[t/t^*]+1} \tag{3.3.22}$$

for all $t > 0$, where $[z] = \max\{w \in \mathbb{Z} | w \leq z\}$. Set $\gamma = (\frac{1}{1-\alpha_{t^*}})$ and $\beta = \log(\frac{1}{1-\alpha_{t^*}})^{1/t^*}$. This proves (3.3.19). Next we prove (3.3.20). Suppose $V \in L^p(\mathbb{R}^d)$. When $d = 1$ it follows directly that

$$\alpha_t = \int_0^t \mathbb{E}^x\,[V(B_s)]\,ds \leq \int_0^t (2\pi s)^{-1/2} ds \|V\|_1. \tag{3.3.23}$$

Next we let $d \geq 2$ and q be such that $\frac{1}{p} + \frac{1}{q} = 1$. Fix an arbitrary $\varepsilon > 0$. We have

$$\begin{aligned}
&\int_0^t \mathbb{E}^x[V(B_s)]ds = \int_0^t \mathbb{E}^x[V(B_s)1_{\{|B_s-x|\geq\varepsilon\}} + 1_{\{|B_s-x|<\varepsilon\}}]ds \\
&\leq t\int_{|y|\geq\varepsilon} \Pi_t(y)V(x+y)dy + e^t\int_0^\infty \mathbb{E}^x[e^{-s}V(B_s)1_{\{|B_s-x|<\varepsilon\}}].
\end{aligned}$$

It is easy to see that

$$\int_{|y|\geq\varepsilon} \Pi_t(y)V(x+y)dy \leq (2\pi)^{-d/2}\left(\int_{\mathbb{R}^d} e^{-q|y|^2/2}dy\right)^{1/q} \|V\|_p. \tag{3.3.24}$$

Let f be the integral kernel of $(-(1/2)\Delta + 1)^{-1}$. Then

$$\int_0^\infty ds\mathbb{E}^x[e^{-s}V(B_s)1_{\{|B_s-x|<\varepsilon\}}] \leq \int_{|x-y|<\varepsilon} f(x-y)V(y)dy.$$

Since $|f(z)| \leq Cg(z)$ for $|z| \leq \frac{1}{2}$ with some constant C and g given by Definition 3.21, we have

$$\int_0^\infty ds\mathbb{E}^x[e^{-s}V(B_s)1_{\{|B_s-x|<\varepsilon\}}] \leq C\int_{|x-y|<\varepsilon} g(x-y)V(y)dy$$

and then

$$\int_0^\infty ds\mathbb{E}^x[e^{-s}V(B_s)1_{\{|B_s-x|<\varepsilon\}}] \leq C\left(\int_{|z|<\varepsilon} g(z)^q dy\right)^{1/q} \|V\|_p \qquad (3.3.25)$$

by the Hölder inequality. Hence from (3.3.23),(3.3.24) and (3.3.25) there exists $C_t(\varepsilon)$ such that $\alpha_t \leq C_t(\varepsilon)\|V\|_p$ and $\lim_{t\to 0} C_t(\varepsilon) = C(\int_{|z|<\varepsilon} g(z)^q dy)^{1/q}$. Thus for sufficiently small T and ε we have $\beta \leq (\frac{1}{1-C_T(\varepsilon)\|V\|_p})^{1/T}$ and then there exists D_T such that $\beta \leq D_T\|V\|_p$. This proves (3.3.20). □

3.3.2 Feynman–Kac formula for Kato-decomposable potentials

When V is Kato-decomposable the operator H is not always an easy object to define. We have seen in the previous section that much functional analytic machinery is needed to decide whether it is essentially self-adjoint and on which domain. The strategy we will adopt here is to define the Feynman–Kac semigroup as given in Definition 3.20 for Kato-decomposable potentials, and identify Schrödinger operators for such potentials as generators of this semigroup.

By Lemma 3.38 the map $K_t : L^\infty \to L^\infty$ is a bounded and linear whenever V is Kato-decomposable. In fact, a yet stronger property holds.

Definition 3.22. Let (M, dm) be a measurable space. T is said to be $L^p - L^q$ *bounded* if there exists $D \subset L^p(M)$ such that it is dense for all $1 \leq p \leq \infty$, and $T\lceil_D: L^p(M) \to L^q(M)$ is bounded for all $1 \leq p \leq q \leq \infty$. In other words, $T_p = \overline{T\lceil_D}^{L^p}$ is bounded from $L^p(M)$ to $L^q(M)$ with domain $L^p(M)$.

Theorem 3.39 (L^p–L^q boundedness). *Let V be Kato-decomposable. Then for every $1 \leq p \leq q \leq \infty$, the semigroup $\{K_t : t \geq 0\}$ as defined by* (3.2.3) *exists and maps bounded operators from $L^p(\mathbb{R}^d)$ to $L^q(\mathbb{R}^d)$.*

In the proof we will rely on the Riesz–Thorin theorem quoted below.

Lemma 3.40 (Riesz–Thorin theorem). *Let $T : L^{p_j}(\mathbb{R}^d) \to L^{q_j}(\mathbb{R}^d)$ be a bounded operator, $j = 1, 2$ and $1 \leq p_1, p_2, q_1, q_2 \leq \infty$. Then T is bounded from $L^r(\mathbb{R}^d)$ to $L^s(\mathbb{R}^d)$ for every r, s such that*

$$\left\{(r,s)\,\middle|\, \left(\frac{1}{r},\frac{1}{s}\right) = \left(\frac{1-\theta}{p_1}+\frac{\theta}{p_2}, \frac{1-\theta}{q_1}+\frac{\theta}{q_2}\right), 0<\theta<1\right\}.$$

Proof of Theorem 3.39. By making use of the Riesz–Thorin theorem it suffices to prove that K_t is bounded from $L^1(\mathbb{R}^d)$ to $L^1(\mathbb{R}^d)$, from $L^1(\mathbb{R}^d)$ to $L^\infty(\mathbb{R}^d)$, and from $L^\infty(\mathbb{R}^d)$ to $L^\infty(\mathbb{R}^d)$. Boundedness from $L^\infty(\mathbb{R}^d)$ to $L^\infty(\mathbb{R}^d)$ has already been shown in Lemma 3.38.

We prove boundedness from $L^1(\mathbb{R}^d)$ to $L^1(\mathbb{R}^d)$ by using duality. For $f \in L^1(\mathbb{R}^d)$ and $g \in L^\infty(\mathbb{R}^d)$ we have

$$\int_{\mathbb{R}^d} f(x) K_t g(x)\, dx = \int_{\mathbb{R}^d} dx \mathbb{E}^x[g(B_0) e^{-\int_0^t V(B_s)\, ds} f(B_t)]. \tag{3.3.26}$$

This follows from (3.2.4) in the proof of Theorem 3.30. By taking $g = 1$ and reading (3.3.26) from right to left we find

$$\|K_t f\|_1 \le \|f\|_1 \|K_t 1\|_\infty,$$

hence K_t is bounded from $L^1(\mathbb{R}^d)$ to $L^1(\mathbb{R}^d)$. Since K_t is bounded from $L^\infty(\mathbb{R}^d)$ to $L^\infty(\mathbb{R}^d)$ and $L^1(\mathbb{R}^d)$ to $L^1(\mathbb{R}^d)$, the Riesz–Thorin theorem implies that K_t is bounded from $L^p(\mathbb{R}^d)$ to $L^p(\mathbb{R}^d)$ for all $p \ge 1$, and by the Markov property of Brownian motion it is a semigroup on all of these spaces.

It remains to show that K_t is bounded from $L^1(\mathbb{R}^d)$ to $L^\infty(\mathbb{R}^d)$. Consider the diagram

$$L^1(\mathbb{R}^d) \xrightarrow{K_t} L^2(\mathbb{R}^d) \xrightarrow{K_t} L^\infty(\mathbb{R}^d). \tag{3.3.27}$$

Assume first that $f \in L^2(\mathbb{R}^d)$. Then by Schwarz inequality

$$\|K_t f\|_\infty^2 \le \sup_x (\mathbb{E}^x[e^{-2\int_0^t V(B_s)\, ds}] \mathbb{E}^x[f(B_t)^2]).$$

The first factor above is bounded by the Khasminskii lemma, while the second is bounded due to $\mathbb{E}^x[f(B_t)^2] = (2\pi t)^{-d/2} \int_{\mathbb{R}^d} f(y)^2 \mathrm{e}^{-|x-y|^2/2t} dy$. Hence $K_t : L^2(\mathbb{R}^d) \to L^\infty(\mathbb{R}^d)$ and $\|K_t f\|_\infty \le C\|f\|_2$. Now for $0 \le f \in L^1(\mathbb{R}^d)$ choose $g \in L^2(\mathbb{R}^d)$, and by a similar reasoning as in (3.3.26) we find

$$\int_{\mathbb{R}^d} (K_t f)(x) g(x)\, dx = \int_{\mathbb{R}^d} f(x) (K_t g)(x)\, dx \le \|K_t g\|_\infty \|f\|_1.$$

This can be extended to any $f \in L^2(\mathbb{R}^d)$ by separating $f = (\Re f)_+ - (\Re f)_- + i(\Im f)_+ - i(\Im f)_-$, where $f_+ = \max\{f(x), 0\}$, $f_- = -\min\{f(x), 0\}$, and

$$\left| \int_{\mathbb{R}^d} (K_t f)(x) g(x)\, dx \right| \le 4\|K_t g\|_\infty \|f\|_1.$$

Hence $K_t f \in L^2(\mathbb{R}^d)$ for $f \in L^1(\mathbb{R}^d)$, and for $f \in L^1(\mathbb{R}^d)$ we have

$$\|K_t f\|_\infty = \|K_{t/2} K_{t/2} f\|_\infty \le C\|K_{t/2} f\|_2 \le \tilde{C}\|f\|_1.$$

Therefore K_t maps $L^1(\mathbb{R}^d)$ to $L^\infty(\mathbb{R}^d)$ and the proof is complete. $\square$

Theorem 3.41. *Let V be Kato-decomposable and $f \in L^p(\mathbb{R}^d)$ for $1 \le p \le \infty$. Then for every $t > 0$, $K_t f$ is a continuous function.*

Proof. First we make a preliminary observation. In the proof of Lemma 3.37 we have

$$\sup_x \mathbb{E}^x[|1 - e^{-\int_0^t V(B_s)\,ds}|] \le \sum_{k=1}^{\infty} \frac{1}{k!} \sup_x \mathbb{E}^x\Big[\Big(\int_0^t |V(B_s)|\,ds\Big)^k\Big]$$
$$\le \frac{\sup_x \mathbb{E}^x[\int_0^t |V(B_s)|ds]}{1 - \sup_x \mathbb{E}^x[\int_0^t |V(B_s)|ds} \xrightarrow{t\to 0} 0 \qquad (3.3.28)$$

by Proposition 3.33. Notice that the Khasminskii lemma implies that for small enough T,

$$\sup_x \sup_{0\le t\le T} \mathbb{E}^x[e^{\int_0^t |V(B_s)|\,ds}] < \infty. \qquad (3.3.29)$$

We can now proceed to prove the theorem. Since $K_t f = K_{t/2}K_{t/2}f$ and $K_{t/2}f \in L^\infty(\mathbb{R}^d)$ whenever $f \in L^p(\mathbb{R}^d)$, $1 \le p \le \infty$, it suffices to consider $f \in L^\infty(\mathbb{R}^d)$. First assume $V \in \mathcal{K}(\mathbb{R}^d)$. For every $\tau > 0$, $g_\tau(x) = \mathbb{E}^x[e^{-\int_\tau^t V(B_s)\,ds} f(B_t)]$ is a continuous function with respect to x. In fact, since

$$g(x_n) - g(x) = \mathbb{E}^0[e^{-\int_\tau^t V(x_n+B_s)\,ds} f(x_n + B_t)] - \mathbb{E}^0[e^{-\int_\tau^t V(x+B_s)\,ds} f(x + B_t)],$$

and, since f and $e^{-\int_0^t V(\cdot+B_s)ds}$ are bounded it suffices to show that

$$\lim_{n\to\infty} \mathbb{E}^0[e^{-\int_\tau^t V(x_n+B_s)\,ds} - e^{-\int_\tau^t V(x+B_s)\,ds}] = 0, \qquad (3.3.30)$$

and

$$\lim_{n\to\infty} \mathbb{E}^0[f(x_n + B_t) - f(x + B_t)]$$
$$= \lim_{n\to\infty} (2\pi t)^{-d/2} \int_{\mathbb{R}^d} e^{-t|y|^2} |f(x_n + y) - f(x + y)|dy = 0. \qquad (3.3.31)$$

Here (3.3.30) follows from the dominated convergence theorem and (3.3.31) can be easily proven. We have

$$\|g_\tau - K_t f\|_\infty = \sup_x \mathbb{E}^x[(1 - e^{-\int_0^\tau V(B_s)\,ds}) e^{-\int_\tau^t V(B_s)\,ds} f(B_t)]$$
$$\le \|f\|_\infty \sup_x \sup_{r\le t} \mathbb{E}^x[e^{-\int_0^r V(B_s)\,ds}] \sup_x \mathbb{E}^x[1 - e^{-\int_0^\tau V(B_s)\,ds}] \xrightarrow{\tau\to 0} 0.$$

In the last line we have used Lemma 3.36 and (3.3.28)–(3.3.31). Thus we find that $K_t f$ is continuous as a uniform limit of continuous functions.

Next, let V be Kato-decomposable, and consider for $R > 0$ the function V_R with $V_R(x) = V(x)$ if $|x| \le R$ and $V(x) = 0$ otherwise. Then $V_R \in \mathcal{K}(\mathbb{R}^d)$ and the above reasoning applies. Let $M \subset B_R(0)$ be compact, where $B_R(0)$ denotes the ball of radius R centered at the origin. Then $\int_0^t V_R(B_s)\,ds = \int_0^t V(B_s)\,ds$ for each path $\omega \in \mathscr{X}$ starting in M and not leaving the ball up to time t. Hence

$$\begin{aligned}
&\sup_{x\in M} |\mathbb{E}^x[e^{-\int_0^t V(B_s)\,ds} f(B_t)] - \mathbb{E}^x[e^{-\int_0^t V_R(B_s)\,ds} f(B_t)]| \\
&\le \sup_{x\in M} \mathbb{E}^x[e^{-\int_0^t V(B_s)\,ds} f(B_t) 1_{\{\sup_{0\le s\le t}|B_s|\ge R\}}] \\
&\le \sup_{x\in M} (\mathbb{E}^x[e^{-2\int_0^t V(B_s)\,ds} |f(B_t)|^2])^{1/2} (\mathbb{E}^x[1_{\{\sup_{0\le s\le t}|B_s|\ge R\}}])^{1/2}.
\end{aligned}$$

The first factor above is bounded since V is Kato decomposable, the second goes to zero as $R \to \infty$ by Lévy's maximal inequality

$$\mathbb{E}^x[1_{\{\sup_{0\le s\le t}|B_s|\ge R\}}] \le 2\mathbb{E}^x[1_{\{|B_t|>R\}}].$$

Hence $K_t f$ is a locally uniform limit of continuous functions. $\square$

The last part of the above argument is of some independent interest, so we single it out.

Corollary 3.42. *Let V be Kato-decomposable and $f \in L^p(\mathbb{R}^d)$ with $1 \le p \le \infty$. Then $e^{-tH} f$ is the local uniform limit of a sequence $e^{-tH_n} f$ with $H_n = -\frac{1}{2}\Delta + V_n$ and V_n with compact support.*

One possible choice for V_n in the corollary above is $1_{\{x:|x|\le n\}}V$. In particular, each V_n is in $\mathcal{K}(\mathbb{R}^d)$ as opposed to being only Kato-decomposable.

We are now ready to define H with Kato-class potential as a self-adjoint operator and give the Feynman–Kac formula for e^{-tH}.

Theorem 3.43 (Feynman–Kac formula with Kato-decomposable potentials). *Assume that V is Kato-decomposable. Then $\{K_t : t \ge 0\}$ as defined by* (3.2.3) *is a symmetric C_0-semigroup on $L^2(\mathbb{R}^d)$. Moreover, there exists a unique self-adjoint operator K bounded from below such that $K_t = e^{-tK}$, for every $t \ge 0$.*

Proof. We already know that K_t is a semigroup of bounded operators on $L^2(\mathbb{R}^d)$, and $(g, K_t f) = (K_t g, f)$ can be proven in the same as (3.2.4). To see strong continuity, by denseness it suffices to prove $\|K_t f - f\| \to 0$ as $t \to 0$ for all bounded L^2-functions f with compact support. Let $Q_t = e^{t\Delta/2}$. Then $\|Q_t f - f\| \to 0$ as $t \to 0$. We only need to show that

$$x \mapsto K_t f(x) - Q_t f(x) = \mathbb{E}^x[(e^{-\int_0^t V(B_s)\,ds} - 1) f(B_t)]$$

converges to 0 in $L^2(\mathbb{R}^d)$ as $t \to 0$. Using the proof of Theorem 3.41 and Corollary 3.42 it follows that $K_t f - Q_t f \to 0$ as $t \to 0$ pointwise in $\mathbb{R}^d$. Moreover,

$$(K_t f(x) - Q_t f(x))^2 \le \mathbb{E}^x[(e^{-\int_0^t V(B_s)\,ds} - 1)^2]\mathbb{E}^x[f(B_t)^2].$$

Here, the first factor is bounded uniformly in t and x by Lemma 3.38, while the t-supremum over the second one is bounded and exponentially decaying at infinity, thus integrable. Thus $K_t f - Q_t f \to 0$ in $L^2(\mathbb{R}^d)$ by the dominated convergence theorem. Hence by the triangle inequality $\|K_t f - f\| \le \|K_t f - Q_t f\| + \|Q_t f - f\|$ the statement follows. Furthermore, by the Hille–Yoshida theorem there exists a unique self-adjoint operator K bounded from below such that $K_t = e^{-tK}$, for every $t \ge 0$. □

Definition 3.23 (Schrödinger operator for Kato-decomposable potentials). We call the self-adjoint operator K in Theorem 3.43 *Schrödinger operator for Kato-decomposable potentials* V.

Remark 3.3 (Fractional Schrödinger operator). Using the fractional Laplacian (3.3) it is possible similarly to define fractional Schrödinger operators $(-\Delta)^{\alpha/2} + V$. Similarly to Kato-class one can define also fractional Kato-class by using the potential kernels of the fractional Laplacian. A Feynman–Kac-type formula similarly holds for this class replacing Brownian motion with symmetric α-stable processes.

3.4 Properties of Schrödinger operators and semigroups

3.4.1 Kernel of the Schrödinger semigroup

From the perspective of functional analysis and the theory of partial differential equations it is a crucial point that for every $t > 0$ the operator e^{-tH} has an integral kernel given by the Feynman–Kac formula. This is addressed in the next theorem.

Theorem 3.44. *Let V be a Kato-decomposable potential. Then e^{-tK} is an integral operator in $L^p(\mathbb{R}^d)$ for every $1 \le p \le \infty$. Moreover, the integral kernel $\mathbb{R}^d \times \mathbb{R}^d \ni (x, y) \mapsto K_t(x, y)$ is jointly continuous in x and y, and is given by*

$$K_t(x, y) = \Pi_t(x - y) \int e^{-\int_0^t V(B_s)\,ds} d\mathcal{W}^{x,y}_{[0,t]}. \tag{3.4.1}$$

Here $\mathcal{W}^{x,y}_{[0,t]}$ is Brownian bridge measure defined in (2.2.28).

Proof. By Theorem 3.43 we have $(e^{-tK} f)(x) = \mathbb{E}^x[e^{-\int_0^t V(B_s)\,ds} f(B_t)]$, for every $f \in L^p(\mathbb{R}^d)$, and by Theorem 3.41 the above equality holds pointwise as the right-hand side is a continuous function. Thus

$$(e^{-tK} f)(x) = \int_{\mathbb{R}^d} \Pi_t(x - y) f(y) dy \int e^{-\int_0^t V(B_s)\,ds} d\mathcal{W}^{x,y}_{[0,t]},$$

which shows that (3.4.1) holds for all x and almost all y. Next we show continuity. Let $s = t/3$. Then

$$\begin{aligned} K_t(x,y) &= \int_{\mathbb{R}^d} K_s(x,v) K_s(v,w) K_s(w,y)\, dw\, dv \\ &= \int_{\mathbb{R}^d} K_s(x,v) K_s(y,w) K_s(v,w)\, dw\, dv. \end{aligned}$$

The product

$$K_s(x,v) K_s(y,w) = \mathbb{E}^{(x,v),(y,w)}_{[0,s]}[e^{-\int_0^s (V(B_r^{(1)}) + V(B_r^{(2)}))\, dr}]$$

is the kernel of the Schrödinger operator with Kato-class potential

$$\tilde{K} = -\frac{1}{2}\Delta_x - \frac{1}{2}\Delta_y + V(x) + V(y),$$

and $(x,y) \mapsto V(x) + V(y)$ is Kato-decomposable in $\mathbb{R}^{2d}$ since V is in $\mathbb{R}^d$, where $B_t^{(i)}$, $i = 1,2$, denote two independent d-dimensional Brownian motions. On the other hand, the function $(v,w) \mapsto K_s(v,w)$ is bounded. One way to see this is to recall that e^{-sK} is bounded from $L^1(\mathbb{R}^d)$ to $L^\infty(\mathbb{R}^d)$ and that $\sup_{v,w} |K_s(v,w)| = \sup_{f \in L^1(\mathbb{R}^d), \|f\|_1 = 1} \|e^{-tK} f\|_\infty < \infty$. Hence

$$K_t(x,y) = (e^{-t\tilde{K}} K_s)(x,y)$$

with $K_s \in L^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ and is thus jointly continuous in x and y by Theorem 3.41.

□

The above proof has the virtue of simplicity; using more sophisticated arguments, it can be shown that $K_t(x,y)$ is even jointly continuous in x, y and t for $t > 0$.

3.4.2 Number of eigenfunctions with negative eigenvalues

In this section we will prove the *Lieb–Thirring inequality* giving an estimate on the number of eigenfunctions in the level sets of negative eigenvalues of a Schrödinger operator. From the point of view of quantum mechanics these eigenfunctions describe bound states. Let $1_{(\cdot)}(T)$ be the spectral resolution of self-adjoint operator T, and for $-E \le 0$ define

$$N_E(V) = \dim 1_{(-\infty,-E]}(H) = \#\{\text{eigenvalues of } -(1/2)\Delta + V \text{ below } -E\}.$$

Theorem 3.45 (Lieb–Thirring inequality). *Suppose $d \ge 3$ and $V \in L^{d/2}(\mathbb{R}^d)$. Then there exists a constant a_d independent of V such that*

$$N_0(V) \le a_d \int_{\mathbb{R}^d} |V_-(x)|^{d/2} dx.$$

Before proving Theorem 3.45 we briefly discuss the *Birman–Schwinger principle*. Recall the kernel of $-\Delta + m^2$ introduced in (3.3.9). Let $C_m(x, y)$ denote the integral kernel of the operator $-\Delta + m^2$, which is the Green function defined by

$$(-\Delta + m^2)C_m(x, y) = \delta(x - y).$$

Alternatively, by Fourier transform

$$C_m(x, y) = C_m(x - y) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} \frac{e^{-ip(x-y)}}{|p|^2 + m^2} dp.$$

As it was seen in (3.3.9) this can be computed explicitly:

$$C_m(x) = (2\pi)^{-d/2} \left(\frac{m}{|x|}\right)^{(d-2)/2} K_{(d-2)/2}(m|x|), \tag{3.4.2}$$

where $K_\nu(x) > 0$ is the modified Bessel function of the third kind. The fact

$$C_m(x) = \begin{cases} \frac{e^{-m|x|}}{2m}, & d = 1, \\ \frac{e^{-m|x|}}{4\pi|x|}, & d = 3 \end{cases}$$

is well known. Many properties of the covariance operator $C_m(x, y)$ can be deduced from (3.4.2) and the known properties of Bessel functions such as a singularity in the $|x - y| \to 0$ limit and exponential decay as $m|x - y| \to \infty$. Here we summarize the properties of $C_m(x, y)$.

Proposition 3.46. *The covariance $C_m(x, y) = C_m(x - y)$ has the following properties:*

(1) $C_m(x, y) > 0$

(2) *for $m|x - y| > 0$, $C_m(x, y) \le \mathrm{const} \times m^{(d-3)/2} \frac{e^{-m|x-y|}}{|x-y|^{(d-1)/2}}$, $d \ge 3$*

(3) *for $m|x - y| \sim 0$, $C_m(x, y) \sim \begin{cases} |x - y|^{-d+2}, & d \ge 3, \\ -\log(m|x - y|), & d = 2 \end{cases}$*

(4) $\lim_{m\to 0} C_m(x, y) = \frac{1}{4}\frac{\Gamma(d/2-1)}{\pi^{d/2}}\frac{1}{|x-y|^{d-2}}$, $d \ge 3$.

We now look at potentials V for which $V^{1/2}(-\Delta + m^2)^{-1}V^{1/2}$ is a compact operator, i.e., when V is form compact with respect to $-\Delta$.

Lemma 3.47. *Let $0 \le V \in L^q(\mathbb{R}^d)$ with $q = d/2$ for $d \ge 3$, $q > 1$ for $d = 2$ and $q = 1$ for $d = 1$. Then $K = V^{1/2}(-\Delta + m^2)^{-1}V^{1/2}$ is a compact operator with $m \ge 0$ for $d \ge 3$ and $m > 0$ for $d = 1, 2$.*

Proof. In the case of $d = 1, 2$ it is trivial to see that

$$\int_{\mathbb{R}^d \times \mathbb{R}^d} |V(x)|C_m(x-y)^2|V(y)|dxdy \le C\|V\|_p^2.$$

Then K is a Hilbert–Schmidt operator. In case when $d = 3$

$$C_m(x-y) \le C_0(x-y) = \frac{1}{4\pi}\frac{1}{|x-y|}.$$

Then we have

$$\int_{\mathbb{R}^3 \times \mathbb{R}^3} |V(x)|C_m(x-y)^2|V(y)|dxdy \le C\|V\|_{3/2}^2$$

by the Hardy–Littlewood–Sobolev inequality.

In the case of $d \ge 4$ we show an outline of the proof. It suffices to show compactness for $m = 0$. Let $L_w^p(\mathbb{R}^d)$ be the set of Lebesgue measurable functions u such that $\sup_{\beta>0} \beta|\{x \in \mathbb{R}^d||u(x) > \beta\}|^{1/p} < \infty$, where $|E|$ denotes Lebesgue measure of E. Let $g \in L^p(\mathbb{R}^d)$ and $u \in L_w^p(\mathbb{R}^d)$ for $2 < p < \infty$. Define the operator $B_{u,g}$ by

$$B_{u,g}h(k) = (2\pi)^{-d/2}\int_{\mathbb{R}^d} e^{ikx}u(k)g(x)h(x)dx.$$

It can be shown that $B_{u,g}$ is a compact operator on $L^2(\mathbb{R}^d)$ and $u(k) = 2|k|^{-1} \in L_w^d(\mathbb{R}^d)$ for $d \ge 3$. Let F denote Fourier transform on $L^2(\mathbb{R}^d)$, and suppose that $V \in L^{d/2}(\mathbb{R}^d)$. Then $B_{u,|V|^{1/2}}$ is compact and hence so is $(-\Delta)^{-1/2}|V|^{1/2} = FB_{u,V^{1/2}}$. Thus also $|V|^{1/2}(-\Delta)^{-1}|V|^{1/2}$ is compact. □

We assume in what follows that $V(x) = -W(x) \le 0$ and V is form compact with respect to $-\Delta$. Let $H(\lambda) = H_0 - \lambda W$, where $H_0 = -(1/2)\Delta$. It can be seen that for $\gamma > 0$ we have $H(\lambda)\psi = -\gamma\psi$ if and only if $W^{1/2}(H_0+\gamma)^{-1}W^{1/2}\varphi = (1/\lambda)\varphi$, where $\varphi = W^{1/2}\psi$. Put

$$K_\gamma = W^{1/2}(H_0+\gamma)^{-1}W^{1/2}. \tag{3.4.3}$$

This is the so called *Birman–Schwinger kernel*, for which

$$-\gamma \in \mathrm{Spec}(H(\lambda)) \iff 1/\lambda \in \mathrm{Spec}(K_\gamma) \tag{3.4.4}$$

holds. Since $-W$ is $-(1/2)\Delta$-form compact, $\mathrm{Spec}_{\mathrm{ess}}(H(\lambda)) = \mathbb{R}^+$. Let $e_j(\lambda) < 0$ be the jth negative eigenvalue of $H(\lambda)$, and $e_i(\lambda) = 0$ if $N_E(\lambda V) \le i-1$. Then $\lambda \mapsto e_j(\lambda)$ is continuous and strictly decreasing in the region $\{\lambda|e_j(\lambda) < 0\}$. Hence for $-E < 0$,

$$N_E(V) = \#\{i \mid e_i(1) \le -E\} = \#\{i \mid e_i(\lambda) = -E, \exists\lambda \le 1\} = \dim 1_{[1,\infty)}(K_E). \tag{3.4.5}$$

Proposition 3.48 (Birman–Schwinger principle). *Suppose that $V \leq 0$ and it is relatively form compact. Then*

$$\begin{aligned} N_E(V) &= \dim 1_{[1,\infty)}(K_E), \quad -E < 0, \quad d \geq 1, \\ N_0(V) &\leq \dim 1_{[1,\infty)}(K_0), \qquad E = 0, \quad d \geq 3. \end{aligned} \tag{3.4.6}$$

Proof. For $-E < 0$ relation (3.4.6) follows from (3.4.5). To obtain the result for $E = 0$ note that $K_E \leq K_0$ and $N_0(V) = \lim_{E\downarrow 0} N_E(V)$. □

The Birman–Schwinger principle has the following immediate implication.

Corollary 3.49. *Let $d \geq 3$. Suppose that V is $-(1/2)\Delta$-form compact and the operator norm of $K_{\gamma=0}$ is $\|K_0\| < 1$. Then $-(1/2)\Delta + V$ has no ground state.*

Proof. By the Birman–Schwinger principle we have $N_0(V) \leq \dim 1_{[1,\infty)}(K_0) = 0$, and $\mathrm{Spec}_{\mathrm{ess}}(-(1/2)\Delta + V) = [0, \infty)$. Then the corollary follows. □

We will estimate $N_E(V)$ from above. It can be seen directly that

$$W^{1/2}(H_0 + \lambda W + \gamma)^{-1} W^{1/2} = K_\gamma (1 + \lambda K_\gamma)^{-1}. \tag{3.4.7}$$

On the other hand, the integral kernel of

$$W^{1/2}(H_0 + \lambda W + \gamma)^{-1} W^{1/2} = W^{1/2} \int_0^\infty e^{-t\gamma} e^{-t(H_0 + \lambda W)} W^{1/2} dt$$

is given by using the conditional Wiener measure,

$$\begin{aligned} &(W^{1/2}(H_0 + \lambda W + \gamma)^{-1} W^{1/2})(x, y) \\ &= W^{1/2}(x) W^{1/2}(y) \int_0^\infty e^{-t\gamma} dt \int d\mathcal{W}^{x,y}_{[0,t]} e^{-\lambda \int_0^t W(B_s) ds} \Pi_t(x - y). \end{aligned} \tag{3.4.8}$$

Let $g(y) = e^{-\lambda y}$ and $F(x) = x(1 + \lambda x)^{-1}$, i.e.,

$$F(x) = x \int_0^\infty e^{-y} g(xy) dy. \tag{3.4.9}$$

Then together with (3.4.7), (3.4.8) can be reformulated as

$$\begin{aligned} &F(K_\gamma)(x, y) \\ &= W^{1/2}(x) W^{1/2}(y) \int_0^\infty e^{-t\gamma} dt \int d\mathcal{W}^{x,y}_{[0,t]} g\left(\int_0^t W(B_s) ds\right) \Pi_t(x - y). \end{aligned} \tag{3.4.10}$$

Moreover, when W is continuous, the right-hand side of (3.4.8) is continuous in x and y. Then by a general result $\mathrm{Tr}\, F(K_\gamma)$ can be evaluated by setting $x = y$. Hence

$$\mathrm{Tr}\, F(K_\gamma) = \int_0^\infty e^{-t\gamma} dt \int_{\mathbb{R}^d} dx W(x) \int d\mathcal{W}_{[0,t]}^{x,x} g\left(\int_0^t W(B_s) ds\right) \Pi_t(0). \tag{3.4.11}$$

Lemma 3.50. *Suppose that W is $-(1/2)\Delta$-form compact and continuous, and let $f(y) = yg(y) = ye^{-\lambda y}$. Then it follows that*

$$\mathrm{Tr}\, F(K_\gamma) = \int_0^\infty \frac{dt}{t} e^{-t\gamma} \int_{\mathbb{R}^d} dx \int d\mathcal{W}_{[0,t]}^{x,x} f\left(\int_0^t W(B_s) ds\right) \Pi_t(0). \tag{3.4.12}$$

Proof. It suffices to show that

$$\begin{aligned} &\frac{1}{t} \int_{\mathbb{R}^d} dx \int d\mathcal{W}_{0,t}^{x,x} \left(\int_0^t W(B_s) ds\right) e^{-\lambda \int_0^t W(B_s) ds} \Pi_t(0) \\ &= \int_{\mathbb{R}^d} dx W(x) \int d\mathcal{W}_{0,t}^{x,x} \left(\int_0^t W(B_s) ds\right) e^{-\lambda \int_0^t W(B_s) ds} \Pi_t(0). \end{aligned} \tag{3.4.13}$$

Let $U_r = e^{-rH(-\lambda)} W e^{-(t-r)H(-\lambda)}$. Note that U_r is compact. Its integral kernel can be computed as

$$\begin{aligned} (U_r f)(x) &= \mathbb{E}^x[e^{-\lambda \int_0^r W(B_s) ds} W(B_r) \mathbb{E}^{B_r}[e^{-\lambda \int_0^{t-r} W(B_s) ds} f(B_{t-r})]] \\ &= \mathbb{E}^x[e^{-\lambda \int_0^t W(B_s) ds} W(B_r) f(B_t)] \\ &= \int_{\mathbb{R}^d} dy \Pi_t(x-y) \int d\mathcal{W}_{0,t}^{x,y} e^{-\lambda \int_0^t W(B_s) ds} W(B_r) f(y). \end{aligned}$$

Hence $U_r(x,y) = \Pi_t(x-y) \int d\mathcal{W}_{0,t}^{x,y} e^{-\lambda \int_0^t W(B_s) ds} W(B_r)$ and thus

$$\mathrm{Tr}\, U_r = \int_{\mathbb{R}^d} dx \int d\mathcal{W}_{0,t}^{x,x} e^{-\lambda \int_0^t W(B_s) ds} W(B_r) \Pi_t(0). \tag{3.4.14}$$

From the identity $\mathrm{Tr}\, U_r = \mathrm{Tr}\, U_0$ it follows that

$$\int_0^t \frac{1}{t} \mathrm{Tr}\, U_r dr = \mathrm{Tr}\, U_0. \tag{3.4.15}$$

Inserting (3.4.14) into (3.4.15) yields thus (3.4.13). □

Furthermore, a more general formula holds.

Lemma 3.51. *Let $W \in L^p(\mathbb{R}^d)$ with $p = d/2$ for $d \geq 3$, $p > 1$ for $d = 2$, and $p = 1$ for $d = 1$ with $W > 0$. Suppose $\gamma > 0$ for $d = 1, 2$, and $\gamma \geq 0$ for $d \geq 3$. Let f be a non-negative lower semicontinuous function on $\mathbb{R}^+$ with $f(0) = 0$ and let f, g, F be related by* (3.4.9) *and $f(y) = g(y)y$. Then*

$$\operatorname{Tr} F(K_\gamma) = \int_0^\infty \frac{dt}{t} e^{-t\gamma} \int_{\mathbb{R}^d} dx \int d\mathcal{W}_{[0,t]}^{x,x} f\left(\int_0^t W(B_s)ds\right) \Pi_t(0) \tag{3.4.16}$$

$$= \int_0^\infty dt e^{-t\gamma} \int_{\mathbb{R}^d} dx W(x) \int d\mathcal{W}_{[0,t]}^{x,x} g\left(\int_0^t W(B_s)ds\right) \Pi_t(0). \tag{3.4.17}$$

Proof. Note that $W \in L^p(\mathbb{R}^d)$ implies that $-\frac{1}{2}\Delta \dot{+} W$ is bounded from below. See Example 3.11. We divide the proof into six steps. In Steps 1–3 we suppose that $\gamma > 0$ and f is continuous with a compact support $\subset [a, b]$ with $a > 0$. In this proof we assume that $d \geq 3$. In the case where $d = 1, 2$ the proof is similar.

Step 1*:* Suppose first that $W \in C_0^\infty(\mathbb{R}^d)$. Since $F(x) = \int_0^\infty e^{-z/x} g(z)dz$, F is linear with respect to g. The sum of functions of the form $g(y) = e^{-\lambda y}$ is dense in the space of continuous functions on $[0, \infty)$ vanishing at infinity. By a limiting argument (3.4.10) follows for any continuous g with compact support. Since the right-hand side of (3.4.10) is jointly continuous in x and y, $\operatorname{Tr}(F(K_\gamma))$ is given by (3.4.17) or (3.4.16) by Lemma 3.50.

Step 2*:* Suppose $W \in L^\infty(\mathbb{R}^d)$ with compact support. Then there exists a sequence $W_n \in C_0^\infty(\mathbb{R}^d)$ and a bounded set $S \subset \mathbb{R}^d$ such that $W_n \leq a 1_S$ with some a, $W_n \to W$ in $L^{d/2}(\mathbb{R}^d)$ and $W_n \to W$ almost everywhere as $n \to \infty$. Thus $K_\gamma(W_n) \to K_\gamma(W)$ in the operator norm, so that $F(K_\gamma(W_n)) \to F(K_\gamma(W))$ also in the operator norm, which implies that $\operatorname{Tr} F(K_\gamma(W_n)) \to \operatorname{Tr} F(K_\gamma(W))$. We next show the convergence of the right-hand side of (3.4.17). Let h be defined by $g(x) = x^m h(x)$. Note that $\operatorname{supp} g \subset [a, b]$ and $a > 0$. Then $\|h\|_\infty < \infty$. Note also that

$$e^{-t\gamma} W_n(x) g\left(\int_0^t W_n(B_s)ds\right) \Pi_t(0) \leq e^{-t\gamma} a 1_S(x) g\left(\int_0^t a 1_S(B_s)ds\right) \Pi_t(0).$$

We have

$$\int_0^t dt e^{-t\gamma} \int_{\mathbb{R}^d} dx a 1_S(x) \int d\mathcal{W}_{0,t}^{x,x} \left(\int_0^t a 1_S(B_s)ds\right)^m h\left(\int_0^t a 1_S(B_s)ds\right) \Pi_t(0)$$

$$\leq \|h\|_\infty \int_0^t dt e^{-t\gamma} \int_{\mathbb{R}^d} dx a^{m+1} 1_S(x) t^m \Pi_t(0)$$

$$\leq \|h\|_\infty a^{m+1} (2\pi)^{-d/2} \int_0^t dt e^{-t\gamma} t^m t^{-d/2} < \infty.$$

Hence $e^{-t\gamma} a 1_S(x) g\left(\int_0^t a 1_S(B_s) ds\right) \Pi_t(0)$ is $L^1(\mathbb{R}_+ \times \mathbb{R}^d \times \mathscr{X})$. Thus (3.4.17) follows by the dominated convergence theorem. By a similar limiting argument (3.4.16) also follows.

Step 3*:* Suppose that $W \in L^{d/2}(\mathbb{R}^d)$. Let $W_n \in L^\infty(\mathbb{R}^d)$ with compact support such that $W_n \uparrow W$. Since F is a monotone function and the eigenvalues $e_j(n)$, $j = 1, 2, \ldots$, of $K_\gamma(W_n)$ are monotone with respect to n, we have $\operatorname{Tr} F(K_\gamma(W_n)) \to \operatorname{Tr} F(K_\gamma(W))$ by the monotone convergence theorem. Let $h(x) = x^m k(x)$ such that $h \le g$ and $\|k\|_\infty < \infty$, and $H(x) = x \int_0^\infty e^{-y} h(xy) dy$. Hence (3.4.16) holds with g and F replaced by h and H, respectively. Note that $\operatorname{Tr} K_\gamma^m < \infty$ as long as $m > d/2$. Since $H(x) \le x^{m+1} \|k\|_\infty \int_0^\infty e^{-y} y^m dy$,

$$\operatorname{Tr} H(K_\gamma) \le \|k\|_\infty \left(\int_0^\infty e^{-y} y^m dy \right) \operatorname{Tr} K_\gamma^m < \infty$$

follows. Then (3.4.17) follows by the dominated convergence theorem, and (3.4.16) follows in the same way as (3.4.17).

Step 4*:* Suppose that W is as in Step 3 and f is continuous. Let $f_n = f 1_{[-n,n]}$. By the monotone convergence theorem again it follows that $\lim_n \operatorname{Tr} F_n(K_\gamma) = \operatorname{Tr} F(K_\gamma)$ and

$$\begin{aligned}
&\lim_n \int_0^\infty \frac{dt}{t} e^{-t\gamma} \int_{\mathbb{R}^d} dx \int d\mathcal{W}_{[0,t]}^{x,x} f_n \left(\int_0^t W(B_s) ds \right) \\
&= \int_0^\infty \frac{dt}{t} e^{-t\gamma} \int_{\mathbb{R}^d} dx \int d\mathcal{W}_{[0,t]}^{x,x} f \left(\int_0^t W(B_s) ds \right).
\end{aligned}$$

Then (3.4.16) follows and (3.4.17) is similarly proven.

Step 5*:* Suppose that W is as in Step 3 and f is a non-negative lower semicontinuous function on $\mathbb{R}^+$ with $f(0) = 0$, and f, g and F are related by (3.4.9). Since any lower semicontinuous function is a monotone limit of continuous function, $F_n = \int_0^\infty e^{-y} f_n(xy) dy/y$ also converges to $F = \int_0^\infty e^{-y} f(xy) dy/y$ monotonously. Hence $\lim_n \operatorname{Tr} F_n(K_\gamma) = \operatorname{Tr} F(K_\gamma)$ by the monotone convergence theorem, and also $\lim_n \int_0^\infty \frac{dt}{t} e^{-t\gamma} \int_{\mathbb{R}^d} dx \int d\mathcal{W}_{[0,t]}^{x,x} f_n(\int_0^t W(B_s) ds)$ converges to the right-hand side of (3.4.16) by the monotone convergence theorem. (3.4.17) can be also proven in the same way as (3.4.16).

Step 6*:* Make the same assumptions as in Step 5, moreover suppose that $\gamma \ge 0$. It suffices to check the case of $\gamma = 0$, which is readily given by the monotone convergence theorem. This completes the proof. □

Proof of Theorem 3.45. Notice that F is monotone increasing, thus $F(x)/F(1) \ge 1$, for $x \ge 1$. Let $f(y) = y g(y)$ and F be related by (3.4.9), and f be convex. Then by

Lemma 3.51 we have

$$\begin{aligned} N_0(V) &\le \operatorname{Tr} F(K_0)/F(1) \\ &= F(1)^{-1}\int_0^\infty \frac{dt}{t}\int_{\mathbb{R}^d} dx \int d\mathcal{W}^{x,x}_{[0,t]} f\left(\int_0^t tW(B_s)\frac{ds}{t}\right)\Pi_t(0) \\ &\le F(1)^{-1}\int_0^\infty \frac{dt}{t}\int_{\mathbb{R}^d} dx \int_0^t \frac{ds}{t}\int d\mathcal{W}^{x,x}_{[0,t]} f\,(tW(B_s))\,\Pi_t(0), \end{aligned}$$

where we used Jensen's inequality. By using the definition of Brownian bridge

$$\int d\mathcal{W}^{x,x}_{[0,t]} h(B_s) = \int_{\mathbb{R}^d} dy h(y)\Pi_s(x-y)\Pi_{t-s}(y-x)/\Pi_t(0)$$

the right-hand side above can be computed as

$$\begin{aligned} &F(1)^{-1}\int_0^\infty \frac{dt}{t}\int_{\mathbb{R}^d} dx\int_0^t \frac{ds}{t}\int_{\mathbb{R}^d} dy f(tW(y))\Pi_s(x-y)\Pi_{t-s}(y-x) \\ &= F(1)^{-1}\int_0^\infty \frac{dt}{t}\int_0^t \frac{ds}{t}\int_{\mathbb{R}^d} dy f(tW(y))\Pi_t(0) \\ &= F(1)^{-1}(2\pi)^{-d/2}\int_0^\infty t^{-1}t^{-d/2}dt\int_{\mathbb{R}^d} dy f(tW(y)). \end{aligned}$$

Changing the variable t to $s/W(y)$, we furthermore have

$$F(1)^{-1}(2\pi)^{-d/2}\int_0^\infty f(s)s^{-1}s^{-d/2}ds\int_{\mathbb{R}^d} W(y)^{d/2}dy = a_d\int_{\mathbb{R}^d} W(y)^{d/2}dy,$$

where $a_d = F(1)^{-1}(2\pi)^{-d/2}\int_0^\infty f(s)s^{-1}s^{-d/2}ds$. □

Remark 3.4. In the case of $d = 1$ or 2 the operator $H = -(1/2)\Delta + V$ with $V \le 0$ (not identically zero) has at least one negative eigenvalue. The operator norm of the compact operator K_γ goes to infinity as $\gamma \uparrow 0$. This may be compared with the case of $d \ge 3$ where by the Lieb–Thirring inequality there are no negative eigenvalues if $\|V\|_{d/2}$ is sufficiently small.

3.4.3 Positivity improving and uniqueness of ground state

We have seen in the preceding section how to use the Feynman–Kac formula to derive the properties of the semigroup e^{-tH}. Here we give some examples of another application. We explain below how to use it to study eigenfunctions of e^{-tH} which are obviously also eigenfunctions of H. This last fact makes them important for quantum physics through the Schrödinger eigenvalue equation $H\psi = E\psi$. A detailed study of ground states in models of quantum field theory through Feynman–Kac formulae will be further done in Part II of the book.

Definition 3.24 (Ground state). Let $E(H) = \inf \mathrm{Spec}(H)$. An eigenfunction satisfying $H\Psi_{\mathrm{p}} = E(H)\Psi_{\mathrm{p}}$ is called a *ground state* of H, and the bottom of the spectrum $E(H)$ is called *ground state energy*. The non-negative integer

$$m(H) = \dim \mathrm{Ker}(H - E(H)) \tag{3.4.18}$$

is called the *multiplicity* of the ground state. If $m(H) = 1$, the ground state is said to be *unique*.

The simplest but very useful fact is that $e^{-tH}(x, y)$ is strictly positive. This is readily implied by (3.4.1) whenever $e^{-\int_0^t V(B_s)ds}$ is strictly positive for almost every $\omega \in \mathscr{X}$. In particular, e^{-tH} is positivity improving in the sense of the following definition.

Definition 3.25 (Positivity preserving/improving operator). Let (M, μ) be a σ-finite measure space.

(1) A non-zero function $f \in L^2(M, d\mu)$ is called *positive* if $f \geq 0$ μ-almost everywhere. Moreover, f is called *strictly positive* if $f > 0$ μ-almost everywhere.

(2) A bounded operator A on $L^2(M, d\mu)$ is called a *positivity preserving operator* if $(f, Ag) \geq 0$, for all positive $f, g \in L^2(M, d\mu)$. A is called a *positivity improving operator* if $(f, Ag) > 0$, for all positive $f, g \in L^2(M, d\mu)$.

Example 3.17. Since $e^{t\Delta} f(x) = \int_{\mathbb{R}^d} \Pi_t(x) f(x) dx$, the operator $e^{t\Delta}$ is positivity improving. Moreover, $e^{-ia\cdot(-i\nabla)}$ is a positivity preserving operator, as it produces the shift $e^{-ia(-i\nabla)} f(x) = f(x - a)$.

Positivity preserving operators of the form $e^{-tE(-i\nabla)}$ have a deep connection with Lévy processes. As it was seen in Example 2.15, the square root of the Laplacian generates a stable Lévy process. The Lévy–Khintchine formula (2.4.6) has the following important relationship with positivity preserving operators.

Proposition 3.52. *Let* $E : \mathbb{R}^d \to \mathbb{C}$.

(1) *Suppose that the real part of* E *is bounded from below. Then* (a)–(d) *below are equivalent:*

(a) $e^{-tE(-i\nabla)}$ *is a positivity preserving semigroup.*

(b) $e^{-tE(u)}$ *is a positive definite distribution, i.e., for all* $\phi \in C_0^\infty(\mathbb{R}^d)$ *we have* $\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} e^{-tE(u)} \overline{\phi(x - u)} \phi(x) du dx \geq 0$.

(c) $E(-u) = \overline{E(u)}$, *and it is conditionally negative definite, i.e.,*

$$\sum_{i,j=1}^{n} E(p_i - p_j) \bar{z}_i z_j \leq 0,$$

for any $n \in \mathbb{N}$, $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$ *with* $\sum_{i=1}^n z_i = 0$.

(d) *There exists a triplet* (b, A, ν), *where* $b \in \mathbb{R}^d$, *A is a positive definite symmetric matrix, and* ν *is a Lévy measure such that*

$$-E(u) = a + iu \cdot b - \frac{1}{2}u \cdot Au + \int_{\mathbb{R}^d \setminus \{0\}} (e^{iu\cdot y} - 1 - iu \cdot y 1_{\{|y|\le 1\}})\nu(dy) \tag{3.4.19}$$

with some $a \in \mathbb{R}$.

(2) *Suppose that* E *is a spherically symmetric and polynomially bounded continuous function with* $E(0) = 0$ *and real part bounded from below. Suppose, moreover, that* ΔE *in distributional sense is positive definite. Then* E *is conditionally negative definite.*

Write

$$E(u) = \sqrt{u^2 + m^2} - m, \quad m \ge 0;$$

then by a straightforward calculation

$$\Delta_u E(u) = (d-1)(|u|^2 + m^2)^{-1/2} + m^2(|u|^2 + m^2)^{-3/2}.$$

Since $(|u|^2 + m^2)^{-\beta} = d_\beta \int_0^\infty t^{\beta-1} e^{-t(|u|^2+m^2)} dt$ and $e^{-t|u|^2}$ is positive definite, we conclude by Proposition 3.52 that $E(u)$ is conditionally negative definite, which implies that $-E(u)$ can be represented as in (3.4.19). Since the left-hand side is real and $E(u)/|u|^2 \to 0$ as $|u| \to 0$, its Lévy triplet is $(b, A, \nu) = (0, 0, \nu)$. Thus we have

$$-\sqrt{|u|^2 + m^2} + m = \int_{|y|>0} (e^{iu\cdot y} - 1 - iu \cdot y 1_{\{|y|<1\}})\nu(dy). \tag{3.4.20}$$

The following result gives the Feynman–Kac formula for $e^{-E(-i\nabla)}$.

Proposition 3.53. *Let* $(X_t)_{t\ge 0}$ *be a Lévy process with characteristic triplet* $(0, 0, \nu)$ *in* (3.4.20). *Then*

$$(g, e^{-tE(-i\nabla)} f) = \int \mathbb{E}_\nu[\overline{g(x)} f(x + X_t)] dx.$$

The Lévy measure ν in (3.4.20) can be determined exactly. The integral kernel of the operator $e^{-tE(-i\nabla)}$ is given by the Fourier transform $Fe^{-tE(u)}F^{-1}$ and thus

$$e^{-tE(-i\nabla)}(x, y) = 2\left(\frac{m}{2\pi}\right)^{(d+1)/2} \frac{tK_{(d+1)/2}(m\sqrt{|x-y|^2 + t^2})}{(t^2 + |x-y|^2)^{(d+1)/4}}, \tag{3.4.21}$$

where K_d is the modified Bessel function of the third kind. Note that $K_d(z) \sim \frac{1}{2}\Gamma(d)(\frac{1}{2}z)^{-d}$ as $|z| \downarrow 0$. Hence

$$(g, E(-i\nabla)f) = \lim_{t\to 0} \frac{1}{t}[(g, f) - (g, e^{-tE(-i\nabla)} f)]$$

$$= \left(\frac{m}{2\pi}\right)^{(d+1)/2} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{\overline{(g(x) - g(y))}(f(x) - f(y))}{|x-y|^{(d+1)/2}} K_{(d+1)/2}(m\sqrt{|x-y|^2}) dx dy$$

In the case of $m = 0$, the integral form of $(g, E(-i\nabla)f)$ is given by

$$(g, E(-i\nabla)f) = \frac{\Gamma((d+1)/2)}{2\pi^{(d+1)/2}} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{\overline{(g(x)-g(y))}(f(x)-f(y))}{|x-y|^{d+1}} dxdy.$$

Hence it is seen that the Lévy measure associated with $E(u)$ for $m = 0$ is

$$\nu_{-1}(dy) = \frac{1}{\pi^{(d+1)/2}} \Gamma\left(\frac{d+1}{2}\right) \frac{1}{|y|^{d+1}} dy$$

and for $m > 0$

$$\nu_{-1}^m(dy) = 2\left(\frac{m}{2\pi}\right)^{(d+1)/2} \frac{1}{|y|^{(d+1)/2}} K_{(d+1)/2}(m|y|)dy. \tag{3.4.22}$$

Remark 3.5. Let

$$\nu_{-\alpha}(dy) = \frac{2^\alpha}{\pi^{d/2}} \frac{\Gamma(\frac{d+\alpha}{2})}{|\Gamma(-\frac{\alpha}{2})|} \frac{1}{|y|^{d+\alpha}} dy, \quad 0 < \alpha < 2,$$

where $\Gamma(-x) = -\frac{1}{x}\Gamma(1-x)$ for $x > 0$. Then the Lévy process $(X_t)_{t\geq 0}$ with triplet $(0, 0, \nu_{-\alpha})$ is a stable process with index α and its generator is $(-\Delta)^{\alpha/2}$.

A useful application of the operator positivity properties is to show uniqueness of the ground state of self-adjoint operators.

Theorem 3.54 (Perron–Frobenius). *Let (M, μ) be a σ-finite measure space and K be a self-adjoint operator bounded from below in $L^2(M, d\mu)$. Suppose that* $\inf \mathrm{Spec}(K)$ *is an eigenvalue and e^{-K} is positivity improving. Then* $\inf \mathrm{Spec}(K)$ *is non-degenerate and the eigenfunction corresponding to the eigenvalue* $\inf \mathrm{Spec}(K)$ *is strictly positive.*

Proof. Put $\lambda = \inf \mathrm{Spec}(K)$. Then $\|e^{-K}\| = e^{-\lambda}$. Let f be an eigenvector corresponding to the eigenvalue λ. Since e^{-K} maps real-valued functions to real-valued functions, $\Re f$ is also an eigenvector and therefore we may assume f to be real-valued from the outset. Since e^{-K} is positivity improving and linear, we find

$$(g, e^{-K}f) = (g, e^{-K}f^+ - e^{-K}f^-) \leq (g, e^{-K}f^+ + e^{-K}f^-) = (g, e^{-K}|f|)$$

for all $g \geq 0$ and all $f = f^+ - f^- \in L^2$ with $f^+ > 0$ and $f^- > 0$. Thus

$$e^{-\lambda}\|f\|^2 = (e^{-K}f, f) \leq (e^{-K}|f|, |f|) \leq \|e^{-K}\|\|f\|^2 = e^{-\lambda}\|f\|^2,$$

implying $(e^{-K}f, f) = (e^{-K}|f|, |f|)$. This further implies

$$(e^{-K}f^+, f^-) = -(e^{-K}f^-, f^+). \tag{3.4.23}$$

Since f^+, f^-, $e^{-K} f^+$ and $e^{-K} f^-$ are all nonnegative, both sides of (3.4.23) must vanish, and this can only happen if either f^+ or f^- vanish almost everywhere as e^{-K} improves positivity. Hence f has a definite sign and, for instance, $f \geq 0$ can be assumed. However, since $f = e^{-K} e^{+\lambda} f$, f is even strictly positive.

The above reasoning applies to any eigenvector corresponding to the eigenvalue λ. This shows that there cannot be two linearly independent such eigenvectors since then they could be chosen orthogonal which is impossible as they both would have to be strictly positive. Hence f is unique. □

Theorem 3.55 (Uniqueness of ground state). *Suppose that there exists $t > 0$ such that*

$$\int_{\mathbb{R}^d} \mathcal{W}^x (e^{-\int_0^t V(B_s)ds} = 0) dx = 0. \tag{3.4.24}$$

Then the ground state of H is unique. In particular, for $V_+ \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$, H has the unique ground state whenever it exists.

Proof. Since $e^{(t/2)\Delta}$ is positivity improving, $\int_{\mathbb{R}^d} dx \mathbb{E}^x[f(B_0)g(B_t)] > 0$ for any positive f and g. Then the measure of $\mathcal{M} = \{(x,\omega) \in \mathbb{R}^d \times \mathcal{X} \mid f(x)g(B_t(\omega) + x) > 0\}$ is positive. By (3.4.24), $e^{-\int_0^t V(B_s+x)ds} > 0$ for almost every $(x,\omega) \in \mathbb{R}^d \times \mathcal{X}$. Then

$$\begin{aligned} (f, e^{-tH} g) &= \int_{\mathbb{R}^d} dx \mathbb{E}[f(x)g(B_t + x)e^{-\int_0^t V(B_s+x)ds}] \\ &\geq \int_{\mathcal{M}} dx d\mathcal{W} f(x)g(B_t + x)e^{-\int_0^t V(B_s+x)ds} > 0, \end{aligned}$$

and the theorem follows from the Perron–Frobenius theorem, Theorem 3.54. For $V_+ \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$, it is seen in Lemma 3.32 that $\int_{\mathbb{R}^d} dx \mathcal{W}^x(\int_0^t V_+(B_s)ds = \infty) = 0$, which implies (3.4.24). □

3.4.4 Degenerate ground state and Klauder phenomenon

Now we turn to presenting an example where e^{-tH} is not positivity improving. To break positivity down we construct a potential V such that

$$\int_{\mathbb{R}^d} dx \mathcal{W}^x (e^{-\int_0^t V(B_s)ds} = 0) > 0.$$

This requires of the potential V to be singular in a suitable form. Let

$$\begin{aligned} \mathcal{M}_1 &= \{x = (x_1, x_2, x_3) \in \mathbb{R}^3 | x_3 > 0\}, \\ \mathcal{M}_2 &= \{x = (x_1, x_2, x_3) \in \mathbb{R}^3 | x_3 < 0\}, \\ \mathcal{M} &= \mathcal{M}_1 \cup \mathcal{M}_2. \end{aligned}$$

Define

$$H(\nu) = -\frac{1}{2}\Delta + \frac{1}{2}|x|^2 + \frac{\nu}{|x - \partial\mathscr{M}|^3}. \tag{3.4.25}$$

Here $\partial\mathscr{M}$ is the boundary of $\mathscr{M}$ and $|x - \partial\mathscr{M}|$ denotes the distance between x and $\partial\mathscr{M}$. Since the potential in (3.4.25) is singular, self-adjointness of $H(\nu)$, $\nu \neq 0$, is unclear. To see this we use the following general result.

Proposition 3.56. *Let D be an open set in $\mathbb{R}^d$ and $V \geq 0$ on D. Suppose that there is a uniform Lipshitz continuous function h on every compact subset of D such that (1) $\sum_{i=1}^d (\partial_i h)^2 \leq e^{2h}$ on D, (2) $\lim_{x\to\partial D} h(x) = \infty$ (if D is not bounded, ∞ is regarded as a point of ∂D), (3) there exists $\delta > 0$ such that $(f, (-\Delta + V)f) \geq (1+\delta)(f, e^{2h} f)$ for $f \in C_0^\infty(D)$. Then $-\Delta + V$ is essentially self-adjoint on $C_0^\infty(D)$.*

Lemma 3.57. *H is essentially self-adjoint on $C_0^\infty(\mathscr{M})$, for all $\nu \in \mathbb{R}$.*

Proof. Let

$$V_\varepsilon(x) = \begin{cases} \nu/|x - \partial\mathscr{M}|^3 + \frac{1}{2}|x|^2, & |x - \partial\mathscr{M}| < \varepsilon \\ \nu/\varepsilon^3 + \frac{1}{2}|x|^2, & \text{otherwise.} \end{cases}$$

Since $V = V_\varepsilon + T$, where T is a bounded operator, it suffices to show that $-\Delta + 2V_\varepsilon$ is essentially self-adjoint on $C_0^\infty(\mathscr{M})$. Define $h = \log V_\varepsilon^{1/2}$. h can be directly shown to satisfy (2) and (3) in Proposition 3.56. We shall check (1). (1) is equivalent to $0 \leq 4V_\varepsilon^3 - \sum_{i=1}^3 (\partial_i V_\varepsilon)^2$. For $|x - \partial\mathscr{M}| \geq \varepsilon$, it is immediate to see that $4V_\varepsilon^3 - \sum_{i=1}^3 (\partial_i V_\varepsilon)^2 > 0$ for sufficiently small ε. We have for $|x - \partial\mathscr{M}| < \varepsilon$,

$$4V_\varepsilon^3 - \sum_{i=1}^3 (\partial_i V_\varepsilon)^2 \geq 4\left(\frac{\nu}{|x - \partial\mathscr{M}|^3} + \frac{1}{2}|x|^2\right)^3 - \frac{1}{2}\left(\frac{9\nu^2}{|x - \partial\mathscr{M}|^8} + |x|^2\right).$$

Take sufficiently small ε. Then the right hand side is positive, and the lemma follows. □

Since $H(0) \leq H(\nu)$, we have $\mu_n(H(0)) \leq \mu_n(H(\nu))$, where

$$\mu_n(K) = \sup_{\phi_1,\ldots,\phi_{n-1}} \inf_{\substack{\psi\in D(K),\|\psi\|=1 \\ \psi\in[\phi_1,\ldots,\phi_{n-1}]^\perp}} (\psi, K\psi).$$

By the fact that $\mu_n(H(0)) = n + \frac{1}{2}$ it is seen that $\mu_n(H(\nu)) \to \infty$ as $n \to \infty$. In particular, by the min-max principle $H(\nu)$ has a purely discrete spectrum. Further by Theorem 3.31 we have

$$(f, e^{-tH(\nu)} g) = \int_{\mathbb{R}^3} \mathbb{E}^x\left[\overline{f(x)} g(B_t) \exp\left(-\int_0^t \frac{\nu}{|B_s - \partial\mathscr{M}|^3} + |B_s|^2 ds\right)\right] dx. \tag{3.4.26}$$

Lemma 3.57 gives that $H(\nu) = H_D(\nu)$, where $H_D(\nu)$ denotes $H(\nu)$ with Dirichlet boundary condition. We see that

$$\begin{aligned}&(f, e^{-tH_D(\nu)} g)\\ &= \int_{M_1 \cup M_2} \overline{f(x)} g(B_t) \exp\Big(-\int_0^t \frac{\nu}{|B_s - \partial\mathscr{M}|^3} + |B_s|^2 ds\Big) d\mathcal{W}^x dx, \end{aligned}\tag{3.4.27}$$

with $M_j = \{(x,\omega) \in \mathbb{R}^3 \times \mathscr{X} | x + B_s \in \mathscr{M}_j, s \in [0,t]\}$ denoting the set of paths starting at x and confined to $\mathscr{M}_j$. Comparing (3.4.26) with (3.4.27), we can see that

$$\int_0^t \frac{1}{|B_s(\omega) - \partial\mathscr{M}|^3} ds = \infty$$

for $\omega \in \mathbb{R}^3 \times \mathscr{X} \setminus (M_1 \cup M_2)$. Thus such paths do not contribute to (3.4.26) at all. Hence $(f, e^{-tH(\nu)} g) = 0$ for f and g for which $\operatorname{supp} f \subset \mathscr{M}_1$ and $\operatorname{supp} g \subset \mathscr{M}_2$. Thus $e^{-tH(\nu)}$ is not positivity improving. Moreover we have

$$\lim_{\nu \to 0} (f, e^{-tH(\nu)} g) = \int_{M_1 \cup M_2} \overline{f(x)} g(B_t) d\mathcal{W}^x dx \neq (f, e^{-tH(0)} g).$$

This vestigial effect of singular potentials is called *Klauder phenomenon* and shows that once a singular potential is turned on, its effect is felt after the potential is turned off.

We have seen that for $e^{-tH(\nu)}$ positivity improving breaks down. This offers the possibility of constructing a two-fold degenerate ground state.

Corollary 3.58. *The ground state of $H(\nu)$ is two-fold degenerate.*

Proof. It can be seen that $H(\nu)$ can be reduced by $L^2(\mathscr{M}_j)$. With $H_j(\nu) = H(\nu)\lceil_{L^2(\mathscr{M}_j)}$ we have that $H(\nu) = H_1(\nu) \oplus H_2(\nu)$. Let $E_j = \inf \operatorname{Spec}(H_j(\nu))$; then $E_1 = E_2$ by symmetry. Thus the corollary follows. □

By replacing the potential $\nu|x - \partial\mathscr{M}|^{-3}$ with $\nu|x - \partial\mathscr{M}|^{-2-\varepsilon}$, $\varepsilon > 0$, we can show the same results as in Lemma 3.57 and Corollary 3.58.

3.4.5 Exponential decay of the eigenfunctions

A sufficient condition for the existence of a ground state of a Schrödinger operator is that $\liminf_{|x| \to \infty} V(x) = \infty$. However, ground states exist also for many other choices of V such as Coulomb potential in three dimensions. In many cases these ground states decay exponentially at infinity, while for potentials growing at infinity they decay even faster. An intuitive derivation of the order of decay $|x|^{m+1}$ of the eigenvectors is obtained as follows.

Suppose the potential is of the form $V(x) = |x|^{2m}$ and the ground state of the form $\psi(x) = e^{-|x|^n}$. By the eigenvalue equation $-\frac{1}{2}\Delta\psi + |x|^{2m}\psi = E\psi$ we roughly have

$$-\frac{1}{2}(n(n-1)|x|^{n-2} + n^2|x|^{2n-2})\psi + |x|^{2m}\psi = E\psi.$$

Comparing the two sides in the leading order for large $|x|$ we expect that $n = m + 1$. This can in fact be proved by using the Feynman–Kac formula. Suppose $\psi \in L^2(\mathbb{R}^d)$ with $H\psi = \lambda\psi$ for some $\lambda \in \mathbb{R}$. By adding a constant to V we may assume $\lambda = 0$. Thus

$$\psi(x) = e^{-tH}\psi(x) = \mathbb{E}^x[e^{-\int_0^t V(B_s)\,ds}\psi(B_t)].$$

Therefore the value of $\psi(x)$ is obtained by running a Brownian motion from x under a potential V. This suggests that starting in regions where V is large, the effect will be quite small as most of the Brownian paths take a long time to leave a finite region. Formally,

$$|\psi(x)|^2 \le \|\psi\|_\infty^2 \mathbb{E}^x[e^{-2\int_0^t V_+(B_s)\,ds}]\mathbb{E}^x[e^{-2\int_0^t V_-(B_s)\,ds}]$$

by Schwarz inequality. By Khasminskii's lemma the expectation involving V_- grows at most exponentially in t with a constant that can be estimated conveniently.

We now present Carmona's argument in detail. This will work for any eigenfunction of H and not just the ground state; therefore we assume $\Psi_{\mathrm{p}} \in L^2(\mathbb{R}^d)$ with

$$H\Psi_{\mathrm{p}} = E_{\mathrm{p}}\Psi_{\mathrm{p}}.$$

For the present purposes we introduce two further classes of potentials V.

Definition 3.26. $\mathbb{V}^{\mathrm{upper}}$ and $\mathbb{V}^{\mathrm{lower}}$ are defined as follows.

($\mathbb{V}^{\mathrm{upper}}$) $V \in \mathbb{V}^{\mathrm{upper}}$ if and only if $V = W - U$ such that

(1) $U \ge 0$ and $U \in L^p(\mathbb{R}^d)$ for $p = 1$ if $d = 1$, and $p > d/2$ if $d \ge 2$

(2) $W \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$ and $W_\infty = \inf_{x\in\mathbb{R}^d} W(x) > -\infty$.

($\mathbb{V}^{\mathrm{lower}}$) $V \in \mathbb{V}^{\mathrm{lower}}$ if and only if $V = W - U$ such that

(1) $U \ge 0$ and $U \in L^p(\mathbb{R}^d)$ for $p = 1$ if $d = 1$, and $p > d/2$ if $d \ge 2$

(2) $W \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$.

Let $V = W - U \in \mathbb{V}^{\mathrm{upper}}$. Then $W \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$ and $0 \ge -U \in L^p(\mathbb{R}^d)$ implies that $-U \in \mathcal{K}(\mathbb{R}^d)$. Thus V is Kato decomposable. Hence $e^{-tH} : L^2(\mathbb{R}^d) \to L^\infty(\mathbb{R}^d)$ by Theorem 3.39, in particular, $\Psi_{\mathrm{p}} \in L^\infty(\mathbb{R}^d)$.

A fundamental estimate which we will make use of in showing the spatial exponential decay of eigenfunctions is given in the lemma below.

Lemma 3.59 (Carmona's estimate). *Let $V \in \mathbb{V}^{\mathrm{upper}}$. Then for any $t, a > 0$ and every $0 < \alpha < 1/2$, there exist constants $D_1, D_2, D_3 > 0$ such that*

$$|\Psi_{\mathrm{p}}(x)| \le D_1 e^{D_2 \|U\|_p t} e^{E_{\mathrm{p}} t} (D_3 e^{-\frac{\alpha}{4}\frac{a^2}{t}} e^{-tW_\infty} + e^{-tW_a(x)}) \|\Psi_{\mathrm{p}}\|, \tag{3.4.28}$$

where $W_a(x) = \inf\{W(y) | |x - y| < a\}$.

Proof. By Schwarz inequality

$$|\Psi_{\mathrm{p}}(x)| \le e^{tE_{\mathrm{p}}} (\mathbb{E}^x[e^{-4\int_0^t W(B_s)ds}])^{1/4} (\mathbb{E}^x[e^{+4\int_0^t U(B_s)ds}])^{1/4} \mathbb{E}[|\Psi_{\mathrm{p}}(x + B_t)|^2]^{1/2}. \tag{3.4.29}$$

Note that

$$\begin{aligned}\mathbb{E}[|\Psi_{\mathrm{p}}(x + B_t)|^2] &= \int_{\mathbb{R}^d} \Pi_t(y) |\Psi_{\mathrm{p}}(x + y)|^2 dy \\ &= \int_{\mathbb{R}^d} e^{-\pi|z|^2} |\Psi_{\mathrm{p}}(x + \sqrt{2\pi t} z)|^2 dz \le \|\Psi_{\mathrm{p}}\|^2.\end{aligned}$$

Let $A = \{\omega \in \mathscr{X} | \sup_{0 \le s \le t} |B_s(\omega)| > a\}$. Then it follows by Lévy's maximal inequality that

$$\mathbb{E}[1_A] \le 2\mathbb{E}[1_{\{|B_t| \ge a\}}] = 2(2\pi)^{-d/2} S_{d-1} \int_{a/\sqrt{t}}^\infty e^{-r^2/2} r^{d-1} dx \le \xi_\alpha e^{-\alpha a^2/t}$$

with some ξ_α, for every $0 < \alpha < 1/2$, and where S_{d-1} is the area of the $d - 1$ dimensional unit sphere.

The first factor in (3.4.29) is estimated as

$$\begin{aligned}\mathbb{E}^x[e^{-4\int_0^t W(B_s)ds}] &= \mathbb{E}^0[1_A e^{-4\int_0^t W(B_s + x)ds}] + \mathbb{E}^x[1_{A^c} e^{-4\int_0^t W(B_s)ds}] \\ &\le e^{-4tW_\infty} \mathbb{E}^0[1_A] + e^{-4tW_a(x)} \\ &\le \xi_\alpha e^{-\alpha a^2/t} e^{-4tW_\infty} + e^{-4tW_a(x)}.\end{aligned} \tag{3.4.30}$$

Next we estimate the second factor. Since U is in Kato-class, there exist constants $D_1, D_2 > 0$ such that

$$\mathbb{E}^x[e^{-4\int_0^t U(B_s)ds}] \le D_1 e^{D_2 \|U\|_p t} \tag{3.4.31}$$

by Lemma 3.38. Setting $D_3 = \xi_\alpha{}^{1/4}$, we obtain the lemma by using the inequality $(a + b)^{1/4} \le a^{1/4} + b^{1/4}$ for $a, b \ge 0$. □

Corollary 3.60 (Confining potential). *Let $V = W - U \in \mathbb{V}^{\mathrm{upper}}$. Suppose that $W(x) \ge \gamma |x|^{2n}$ outside a compact set K, for some $n > 0$ and $\gamma > 0$. Take $0 < \alpha < 1/2$. Then there exists a constant $C_1 > 0$ such that*

$$|\Psi_{\mathrm{p}}(x)| \le C_1 \exp\left(-\frac{\alpha c}{16} |x|^{n+1}\right) \|\Psi_{\mathrm{p}}\|, \tag{3.4.32}$$

where $c = \inf_{x \in \mathbb{R}^d \setminus K} W_{|x|/2}(x)/|x|^{2n}$.

Proof. Since $\sup_x |\Psi_p(x)| < \infty$, it suffices to show all the statements for sufficiently large $|x|$. Note that $W_{|x|/2}(x) \geq c|x|^{2n}$ for $x \in \mathbb{R}^d \setminus K$. Then we have the following bounds for $x \in \mathbb{R}^d \setminus K$:

$$|x|W_{|x|/2}(x)^{1/2} \geq c|x|^{n+1},$$
$$|x|W_{|x|/2}(x)^{-1/2} \leq c|x|^{1-n}.$$

Inserting $t = t(x) = |x|W_{|x|/2}(x)^{-1/2}$ and $a = a(x) = \frac{|x|}{2}$ in (3.4.28), we have

$$|\Psi_p(x)| \leq e^{-\frac{\alpha}{16}c|x|^{n+1}} D_1 e^{(D_2\|U\|_p + E_p)c|x|^{1-n}} \times (D_3 e^{c|x|^{1-n}|W_\infty|} + e^{-(1-\frac{\alpha}{16})c|x|^{n+1}})\|\Psi_p\|$$

for $x \in \mathbb{R}^d \setminus K$. Hence (3.4.32) follows. □

For $V = W - U \in \mathbb{V}^{\mathrm{upper}}$ define $\Sigma = \liminf_{|x|\to\infty} V(x)$. Since $-U \in L^p(\mathbb{R}^d)$, $\liminf_{|x|\to\infty}(-U(x)) = 0$ and hence $\Sigma = \liminf_{|x|\to\infty} W(x)$. Moreover $\Sigma \geq W_\infty$ holds.

Corollary 3.61. *Let* $V = W - U \in \mathbb{V}^{\mathrm{upper}}$.

(1) *Decaying potential: Suppose Suppose that* $\Sigma > E_p$, $\Sigma > W_\infty$, *and Let* $0 < \beta < 1$. *Then there exists a constant* $C_2 > 0$ *such that*

$$|\Psi_p(x)| \leq C_2 \exp\left(-\frac{\beta}{8\sqrt{2}}\frac{(\Sigma - E_p)}{\sqrt{\Sigma - W_\infty}}|x|\right)\|\Psi_p\|. \tag{3.4.33}$$

(2) *Confining potential: Suppose that* $\lim_{|x|\to\infty} W(x) = \infty$. *Then there exist constants* $C, \delta > 0$ *such that*

$$|\Psi_p(x)| \leq C \exp(-\delta|x|)\|\Psi_p\|. \tag{3.4.34}$$

Proof. Since $\sup_x |\Psi_p(x)| < \infty$, it is again sufficient to show the statements for large enough $|x|$.

Decaying case: Rewrite formula (3.4.28) as

$$|\Psi_p(x)| \leq D_1 e^{D_2\|U\|_p t}(D_3 e^{-\frac{\alpha}{4}\frac{a^2}{t}} e^{-t(W_\infty - E_p)} + e^{-t(W_a(x)-E_p)})\|\Psi_p\|. \tag{3.4.35}$$

Then on flipping signs, with $\Sigma = \liminf_{|x|\to\infty}(-W_-(x))$ and $\Sigma > W_\infty$ it is possible to choose a decomposition $V = W - U \in \mathbb{V}^{\mathrm{upper}}$ such that $\|U\|_p \leq (\Sigma - E_p)/2$ since $\liminf_{|x|\to\infty}(-U(x)) = 0$. Inserting $t = t(x) = \varepsilon|x|$ and $a = a(x) = \frac{|x|}{2}$ in (3.4.35), we have

$$\begin{aligned}|\Psi_p(x)| &\leq D_1 e^{\|U\|_p \varepsilon|x|}(D_3 e^{-\frac{\alpha}{16\varepsilon}|x|} e^{-\varepsilon|x|(W_\infty - E_p)} + e^{-\varepsilon|x|(W_{|x|/2}(x)-E_p)})\|\Psi_p\| \\ &\leq D_1(D_3 e^{-(\frac{\alpha}{16\varepsilon} + \varepsilon(W_\infty - E_p) - \frac{1}{2}\varepsilon(\Sigma - E_p))|x|} \\ &\qquad + e^{-\varepsilon((W_{|x|/2}(x)-E_p) - \frac{1}{2}(\Sigma - E_p))|x|})\|\Psi_p\|.\end{aligned}$$

Choosing $\varepsilon = \sqrt{\alpha/16}/\sqrt{\Sigma - W_\infty}$, the exponent in the first term above becomes

$$\frac{\alpha}{16\varepsilon} + \varepsilon(W_\infty - E_{\mathrm{p}}) - \frac{1}{2}\varepsilon(\Sigma - E_{\mathrm{p}}) = \frac{1}{2}\varepsilon(\Sigma - E_{\mathrm{p}}).$$

Moreover, we see that $\liminf_{|x|\to\infty} W_{|x|/2}(x) = \Sigma$, and obtain

$$|\Psi_{\mathrm{p}}(x)| \le C_2 e^{-\frac{\varepsilon}{2}(\Sigma - E_{\mathrm{p}})|x|}\|\Psi_{\mathrm{p}}\|$$

for sufficiently large $|x|$. Thus (3.4.33) follows.

Confining case: In this case for any $c > 0$ there exists $N > 0$ such that $W_{|x|/2}(x) \ge c$, for all $|x| > N$. Inserting $t = t(x) = \varepsilon|x|$ and $a = a(x) = \frac{|x|}{2}$ in (3.4.28), we obtain that

$$\begin{aligned} |\Psi_{\mathrm{p}}(x)| &\le D_1 e^{\|U\|_p \varepsilon|x|}(D_3 e^{-\frac{\alpha}{16\varepsilon}|x|} e^{-\varepsilon|x|(W_\infty - E_{\mathrm{p}})} + e^{-\varepsilon|x|(W_{|x|/2}(x) - E_{\mathrm{p}})})\|\Psi_{\mathrm{p}}\| \\ &\le D_1(D_3 e^{-(\frac{\alpha}{16\varepsilon} - \varepsilon\|U\|_p + \varepsilon(W_\infty - E_{\mathrm{p}}))|x|} + e^{-\varepsilon|x|(c - E_{\mathrm{p}} - \|U\|_p)})\|\Psi_{\mathrm{p}}\| \end{aligned}$$

for $|x| > N$. Choosing sufficiently large c and sufficiently small ε such that

$$\frac{\alpha}{16\varepsilon} - \varepsilon\|U\|_p + \varepsilon(W_\infty - E_{\mathrm{p}}) > 0 \quad \text{and} \quad c - E_{\mathrm{p}} - \|U\|_p > 0,$$

we obtain $|\Psi_{\mathrm{p}}(x)| \le C' e^{-\delta'|x|}$ for large enough $|x|$. Thus (3.4.34) follows. □

By using the Feynman–Kac formula we can also establish a lower bound on positive eigenfunctions. Recall that eigenfunctions corresponding to $E(H)$ are strictly positive.

Lemma 3.62. *Let $V = W - U \in \mathbb{V}^{\mathrm{lower}}$ and suppose $\Psi_{\mathrm{p}}(x) \ge 0$. Also, let $x \in \mathbb{R}^d \setminus \{0\}$, and $\alpha_1, \dots, \alpha_d$, $a_1, \dots, a_n$, $b_1, \dots, b_d$ and $t > 0$ satisfy for $j = 1, \dots, d$,*

$$\alpha_j^2 > t, \quad \alpha_j < a_j/2, \quad |[-a_j, a_j] \cap [-x_j - b_j, -x_j + b_j]| > \alpha_j, \tag{3.4.36}$$

where $|\cdot|$ denotes Lebesgue measure. Then

$$\begin{aligned} -\log\Psi_{\mathrm{p}}(x) &\le -E_{\mathrm{p}}t - \log\varepsilon(b) + t\tilde{W}_a(x) \\ &\quad + d\log(2t\sqrt{2\pi t}) - \sum_{j=1}^d \log(\alpha_j^2 a_j) + \frac{9}{8t}\sum_{j=1}^d a_j^2, \end{aligned} \tag{3.4.37}$$

where

$$\tilde{W}_a(x) = \sup\{W(y) \,|\, |y_j - x_j| < a_j,\ j = 1, \dots, d\}$$

and

$$\varepsilon(b) = \inf\{\Psi_{\mathrm{p}}(y) \,|\, |y_j| \le b_j,\ j = 1, \dots, d\} < \infty.$$

Proof. Let

$$A = \left\{\omega \in \mathscr{X} \;\middle|\; |B_t^j(\omega)| \le b_j,\ \sup_{0\le s\le t} |B_s^j(\omega) - x_j| \le a_j,\ j = 1,\dots,d\right\}.$$

(3.4.36) yields that

$$\mathbb{E}^x[1_A] \ge (2t\sqrt{2\pi t})^{-d}\Big(\prod_{j=1}^d \alpha_j^2 a_j\Big) e^{-9|a|^2/(8t)}. \tag{3.4.38}$$

Hence

$$\begin{aligned}
\Psi_{\mathrm{p}}(x) &= e^{tE_{\mathrm{p}}}\mathbb{E}^x[\Psi_{\mathrm{p}}(B_t)e^{-\int_0^t W(B_s)ds}e^{\int_0^t U(B_s)ds}] \\
&\ge e^{tE_{\mathrm{p}}}\mathbb{E}^x[\Psi_{\mathrm{p}}(B_t)e^{-\int_0^t W(B_s)ds}1_A] \\
&\ge \varepsilon(b)e^{tE_{\mathrm{p}}}e^{-t\tilde{W}_a(x)}\mathbb{E}^x[1_A].
\end{aligned}$$

Combining this with (3.4.38) we obtain (3.4.37). □

Corollary 3.63 (Lower bound of ground state). *Let $V = W - U \in \mathbb{V}^{\mathrm{lower}}$. Suppose that $W(x) \le \gamma|x|^{2m}$ outside a compact set with $\gamma > 0$ and $m > 1$, and $\Psi_{\mathrm{p}}(x) \ge 0$. Then there exist constants $\delta, D > 0$ such that $\Psi_{\mathrm{p}}(x) \ge De^{-\delta|x|^{m+1}}$.*

Proof. Since $\|\Psi_{\mathrm{p}}\|_\infty < \infty$ it suffices to prove the corollary for large enough $|x|$. Set $t = |x|^{-(m-1)}$, $a_j = 1 + |x_j|$, $\alpha_j = 1/2$ and $b_j = 1$ for $j = 1,\dots,d$. It can be directly checked that with these choices (3.4.36) is satisfied. Notice that $\tilde{W}_a(x) \le \gamma 2^{4m}|x|^{2m}$. Upon insertion to (3.4.37) we obtain

$$\begin{aligned}
-\log \Psi_{\mathrm{p}}(x) \le{} & -E_{\mathrm{p}}|x|^{-(m-1)} - \log\varepsilon(b) + |x|^{m+1}\gamma 2^{4m} + d\log(2\sqrt{2\pi}) \\
& + d\log(|x|^{-(m-1)3/2}) + 2d\log 2 - \sum_{j=1}^d \log(1+|x_j|) \\
& + \frac{9}{8}|x|^{m-1}\sum_{j=1}^d (1 + 2|x_j| + |x_j|^2),
\end{aligned}$$

from which

$$-\log\Psi_{\mathrm{p}}(x) \le \log(2(2^2\sqrt{2\pi})^d/\varepsilon(b)) + \Big(\gamma 2^{4m} + \frac{9}{8}d + \frac{9}{8}d|x|^{-2} + \frac{9}{4}d|x|^{-1}\Big)|x|^{m+1}.$$

Then for sufficiently large $|x|$ and $\delta > \gamma 2^{4m} + 9d/8$ the result follows. □

3.5 Feynman–Kac–Itô formula for magnetic field

3.5.1 Feynman–Kac–Itô formula

The Schrödinger operator (3.1.22) gives the energy of a quantum particle in a force field $-\nabla V$. In some cases of electrodynamics there is an interaction with a magnetic field that has to be taken into account. This is done by adding a vector potential $a = (a_1, \dots, a_d)$ to the Schrödinger operator. A formal definition of the *Schrödinger operator with vector potentials* is then given by

$$H(a) = \frac{1}{2}(-i\nabla - a)^2 + V. \tag{3.5.1}$$

Lemma 3.64. *Suppose that $a \in (L^{2p}(\mathbb{R}^d) + L^\infty(\mathbb{R}^d))^d$, $V \in L^p(\mathbb{R}^d) + L^\infty(\mathbb{R}^d)$ and $(\nabla \cdot a) \in L^p(\mathbb{R}^d) + L^\infty(\mathbb{R}^d)$, where $p = 2$ for $d \leq 3$ and $p > d/2$ for $d \geq 4$. Then $H(a)$ is self-adjoint on $D(-(1/2)\Delta)$ and essentially self-adjoint on any core of $-(1/2)\Delta$.*

Proof. Under the assumption

$$H(a) = (-1/2)\Delta + (1/2)(i\nabla \cdot a) + a \cdot (i\nabla) + (1/2)a \cdot a + V \tag{3.5.2}$$

follows. It is easy to see that $a \cdot a$, $\nabla \cdot a$ and V are infinitesimally small multiplication operators with respect to $-(1/2)\Delta$. Furthermore, $a \cdot \nabla$ is also infinitesimally small. Indeed, we have

$$\|a_\mu \partial_\mu f\| \leq \|a_\mu\|_{2p} \|\partial_\mu f\|_q \leq \|a_\mu\|_{2p}(\varepsilon \|\Delta f\|_2 + b_\varepsilon \|f\|)$$

for any $\varepsilon > 0$, where $q = 2p/(p-1)$. The second inequality follows from the Hausdorff–Young inequality as

$$\|\partial_\mu f\|_q \leq \|k_\mu \hat{f}\|_{q'} \leq \|(1+|k|)^{-\alpha}\|_{2p} \|(1+|k|)^\alpha k_\mu \hat{f}\|_2,$$

where $q' = 2p/(p+1)$ and $\alpha \in (\frac{d}{2p}, 1)$. Then by the Kato–Rellich theorem $H(a)$ is self-adjoint on $D(-(1/2)\Delta)$, and essentially self-adjoint on any core of $-(1/2)\Delta$. □

In this section our goal is to construct a Feynman–Kac formula of $e^{-tH(a)}$. A drift transformation applied to Brownian motion gives

$$\mathbb{E}^x[f(B_t)e^{\int_0^t a(B_s)\cdot dB_s - \frac{1}{2}\int_0^t a(B_s)^2 ds}] = e^{-t(-(1/2)\Delta - a\cdot\nabla)} f(x).$$

On replacing a by $-ia$, formally

$$\mathbb{E}^x[f(B_t)e^{-i\int_0^t a(B_s)\cdot dB_s + \frac{1}{2}\int_0^t a(B_s)^2 ds}] = e^{-t(-(1/2)\Delta + ia\cdot\nabla)} f(x)$$

is obtained. Let $W = V + (1/2)a \cdot a + (1/2)(i\nabla \cdot a)$. By (3.5.2) the "imaginary" drift transformation yields

$$\begin{aligned}
&\mathbb{E}^x\Big[f(B_t)\exp\Big(-i\Big(\int_0^t a(B_s)\cdot dB_s + \frac{1}{2}\int_0^t \nabla \cdot a(B_s)ds\Big) - \int_0^t V(B_s)ds\Big)\Big] \\
&= \mathbb{E}^x\Big[f(B_t)\exp\Big(-i\int_0^t a(B_s)\cdot dB_s + \frac{1}{2}\int_0^t a(B_s)^2 ds - \int_0^t W(B_s)ds\Big)\Big] \\
&= (e^{-tH(a)})f(x).
\end{aligned}$$

We turn this rigorous in the next theorem.

Theorem 3.65 (Feynman–Kac–Itô formula). *Suppose* $a \in (C_b^2(\mathbb{R}^d))^d$ *and* $V \in L^\infty(\mathbb{R}^d)$. *Then for* $f, g \in L^2(\mathbb{R}^d)$,

$$(f, e^{-tH(a)}g) = \int_{\mathbb{R}^d} dx \mathbb{E}^x[\overline{f(B_0)} e^{-i\int_0^t a(B_s)\circ dB_s} e^{-\int_0^t V(B_s)ds} g(B_t)]. \tag{3.5.3}$$

In particular,

$$(e^{-tH(a)}f)(x) = \mathbb{E}^x[e^{-i\int_0^t a(B_s)\circ dB_s} e^{-\int_0^t V(B_s)ds} f(B_t)]. \tag{3.5.4}$$

Recall that the notation $\circ$ in the stochastic integral above stands for Stratonovich integral.

Proof. Write $Z_t = e^{X_t}$, $X_t = -i\int_0^t a(B_s) \circ dB_s - \int_0^t V(B_s)ds$ and $Y_t = f(B_t)$. Set

$$(Q_t f)(x) = \mathbb{E}^x[Z_t f(B_t)].$$

Note that $V(B_t)$ and $(\nabla \cdot a)(B_t)$ are $L^1_{\text{loc}}(\mathbb{R}, dt)$ almost surely, and $a_j(B_t) \in M^2(0,t)$. Thus, since

$$dX_t = -ia \cdot dB_t - \frac{i}{2}(\nabla \cdot a)dt - Vdt,$$

we arrive at

$$dZ_t = Z_t dX_t + \frac{1}{2}Z_t(dX_t)^2 = Z_t\Big(-\frac{i}{2}(\nabla \cdot a) - \frac{1}{2}a\cdot a - V\Big)dt + Z_t(-ia)\cdot dB_t$$

by the Itô formula. Then by $dY_t = (1/2)\Delta f dt + \nabla f \cdot dB_t$ and the product formula

$$\begin{aligned} d(Z_tY_t) = Z_t &\left(-\frac{i}{2}(\nabla \cdot a)dt - \frac{1}{2}(a\cdot a)dt + (-ia)\cdot dB_t - Vdt\right) Y_t \\ &+ Z_t\left(\frac{1}{2}\Delta f dt + \nabla f \cdot dB_t\right) + Z_t(-ia)\cdot \nabla f dt \\ &- H(a) f Z_t dt + Z_t(\nabla f + (-ia)f)\cdot dB_t \end{aligned}$$

is obtained. It gives

$$(Q_t f)(x) - f(x) = \int_0^t (Q_s H(a) f)(x) ds.$$

Then in a similar manner as in Theorem 3.30 we see that

$$\operatorname*{s-lim}_{t\to 0} \frac{1}{t}(Q_t f - f) = -H(a) f.$$

Then it is enough to show that Q_t is a symmetric C_0 semigroup. We directly see that

$$\begin{aligned}Q_sQ_tf(x) &= \mathbb{E}^x[e^{-i\int_0^s a(B_r)\circ dB_r}\mathbb{E}^{B_s}[f(B_t)e^{-i\int_0^t a(B_s)\circ dB_s}]]\\ &= \mathbb{E}^x[e^{-i\int_0^s a(B_r)\circ dB_r}\mathbb{E}^x[f(B_{s+t})e^{-i\int_s^{s+t} a(B_r)\circ dB_r}|\mathscr{F}_s]]\\ &= Q_{s+t}f(x).\end{aligned}$$

Recall that $(\mathscr{F}_t)_{t\geq 0}$ is the natural filtration of Brownian motion. Hence the semigroup property follows. Let $\tilde{B}_s = B_{t-s} - B_t$, $0 \leq s \leq t$. We have

$$\begin{aligned}(f, Q_tg) &= \int_{\mathbb{R}^d} dx\overline{f(x)}\mathbb{E}^x[e^{-i\int_0^t a(\tilde{B}_s)\circ d\tilde{B}_s}e^{-\int_0^t V(\tilde{B}_s)}g(\tilde{B}_t)]\\ &= \mathbb{E}^0\left[\int_{\mathbb{R}^d} dx\overline{f(x)}e^{-i\int_0^t a(x+\tilde{B}_s)\circ d\tilde{B}_s}e^{-\int_0^t V(x+\tilde{B}_s)}g(x+\tilde{B}_t)\right].\end{aligned}$$

Let

$$I_j = \frac{1}{2}(a(x+\tilde{B}_{tj/n}) + a(x+\tilde{B}_{t(j-1)/n}))(\tilde{B}_{tj/n} - \tilde{B}_{t(j-1)/n}).$$

Then $\sum_{j=1}^n I_j \to \int_0^t a(x+\tilde{B}_s)\circ d\widetilde{B}_s$ in $L^2(\mathscr{X})$ as $n\to\infty$. Thus a subsequence $(\sum_{j=1}^{n'} I_j)_{n'}$ converges to $\int_0^t a(x+\tilde{B}_s)\circ d\widetilde{B}_s$ almost surely. We reset n' as n. Changing the variable x to $y - \tilde{B}_t$, we have

$$(f, Q_tg) = \lim_{n\to\infty}\mathbb{E}^0\left[\int_{\mathbb{R}^d} dy\overline{f(y-\tilde{B}_t)}e^{-i\sum_{j=1}^n \tilde{I}_j}e^{-\int_0^t V(y-\tilde{B}_t-\tilde{B}_s)}g(y)\right],$$

where

$$\tilde{I}_j = \frac{1}{2}(a(y-\tilde{B}_t+\tilde{B}_{jt/n}) + a(y-\tilde{B}_t+\tilde{B}_{t(j-1)/n}))(\tilde{B}_{tj/n} - \tilde{B}_{t(j-1)/n}).$$

Recall that $a \in (C_{\mathrm{b}}^2(\mathbb{R}^d))^d$. Then in $L^2(\mathscr{X})$ sense it follows that

$$\lim_{n\to\infty}\sum_{j=1}^n \tilde{I}_j = -\int_0^t a(y+B_s)\circ dB_s.$$

This gives

$$(f, Q_tg) = \int_{\mathbb{R}^d} dx\overline{\mathbb{E}^x[f(B_t)e^{-i\int_0^t a(B_s)\circ dB_s}e^{-\int_0^t V(B_s)ds}]}g(x) = (Q_tf, g). \tag{3.5.5}$$

Hence Q_t is symmetric. The strong continuity of Q_t with respect to t is derived in the same way as in the proof of Theorem 3.30. □

3.5.2 Alternate proof of the Feynman–Kac–Itô formula

As before, the Trotter product formula can once again be employed to prove the Feynman–Kac–Itô formula. Here we give this proof, which will be useful in constructing the functional integral representation of the Pauli–Fierz model in Chapter 7.

Suppose $a \in (C_{\mathrm{b}}^2(\mathbb{R}^d))^d$ and $V = 0$. Write

$$K_s(x, y) = \Pi_s(x - y)e^{ih}, \tag{3.5.6}$$

where

$$h = h(x, y) = -(1/2)(a(x) + a(y)) \cdot (x - y).$$

Define the family of integral operators $\varrho_s : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ by

$$\varrho_s f(x) = \int_{\mathbb{R}^d} K_s(x, y) f(y)dy, \quad s \geq 0, \tag{3.5.7}$$

where $\varrho_0 f(x) = f(x)$. Notice that ϱ_s is symmetric and $\|\varrho_s\| \leq 1$. Let $f, g \in C_0^\infty(\mathbb{R}^d)$. It is directly seen that

$$\frac{d}{ds}(g, \varrho_s f) = \int_{\mathbb{R}^d} \overline{g(x)}dx \int_{\mathbb{R}^d} \Pi_s(x - y) \left(\frac{1}{2}\Delta_y e^{ih} f(y)\right) dy.$$

Here we used that $(d\Pi_s/ds)(x - y) = (1/2)\Delta_y \Pi_s(x - y)$. Since

$$\begin{aligned}
\partial_j e^{ih} f &= (i\partial_j h \cdot f + \partial_j f)e^{ih}, \\
\partial_j^2 e^{ih} f &= \{\partial_j^2 f + 2i\partial_j h \cdot \partial_j f + (i\partial_j^2 h + (i\partial_j h)^2) f\}e^{ih}, \\
\partial_j h &= -\frac{1}{2}\{\partial_j a(y) \cdot (x - y) - (a_j(x) + a_j(y))\}, \\
\partial_j^2 h &= -\left\{\frac{1}{2}\partial_j^2 a(y) \cdot (x - y) - \partial_j a_j(y)\right\},
\end{aligned}$$

where $\partial_j = \partial_{y_j}$, $j = 1, \ldots, d$, we have

$$\begin{aligned}
\lim_{s\to 0} \int_{\mathbb{R}^d} \Pi_s(x - y)(\Delta_y e^{ih} f(y))dy &= -(-\Delta f - 2a \cdot (-i\nabla f) - (-i\nabla a - a \cdot a) f) \\
&= -2H(a) f.
\end{aligned}$$

Thus $\lim_{s\to 0}(d/ds)(g, \varrho_s f) = (g, -H(a) f)$ and hence

$$\lim_{t\to 0}(g, t^{-1}(1 - \varrho_t) f) = (g, H(a) f). \tag{3.5.8}$$

We show next that $\varrho^{2^n}_{t/2^n}$ converges to a symmetric semigroup. For $f, g \in L^2(\mathbb{R}^d)$,

$$(g, \varrho^{2^n}_{t/2^n} f) = \int_{\mathbb{R}^d} \overline{g(x)} e^{i\sum_{j=1}^{2^n} h(x_j, x_{j-1})} f(x_n) \Big(\prod_{j=1}^{2^n} \Pi_{t/2^n}(x_j - x_{j-1}) \Big) \prod_{j=0}^{n} dx_j$$

with $x_0 = x$. Therefore

$$\begin{aligned} \lim_{n\to\infty} (g, \varrho^{2^n}_{t/2^n} f) &= \lim_{n\to\infty} \int_{\mathbb{R}^d} dx \mathbb{E}^x \Bigg[\overline{g(B_0)} f(B_t) \\ &\quad \times \exp\left(-i \sum_{j=1}^{2^n} \frac{1}{2} (a(B_{tj/2^n}) + a(B_{t(j-1)/2^n}))(B_{tj/2^n} - B_{t(j-1)/2^n}) \right) \Bigg] \\ &= \int_{\mathbb{R}^d} dx \mathbb{E}^x \left[\overline{g(B_0)} f(B_t) \exp\left(-i \int_0^t a(B_s) \circ dB_s \right) \right] \end{aligned} \tag{3.5.9}$$

by Corollary 2.30. Hence $|\lim_{n\to\infty}(g, \varrho^{2^n}_{t/2^n} f)| \leq \|g\| \, \|f\|$. By the Riesz theorem there is a bounded operator S_t such that $\lim_{n\to\infty}(g, \varrho^{2^n}_{t/2^n} f) = (g, S_t f)$. It is symmetric since ϱ_s is symmetric. (3.5.9) implies that

$$S_t f(x) = \mathbb{E}^x [f(B_s) e^{-i\int_0^t a(B_s)\circ dB_s}].$$

Hence $S_0 = 1$, s-$\lim_{t\to 0} S_t = 1$ and $S_s S_t = S_{s+t}$ follow. Thus S_t is a symmetric C_0-semigroup, so there exists a self-adjoint operator A such that $e^{-tA} = S_t$ by Proposition 3.26. It suffices to show that $A = H(a)$. This can be checked without using the Itô formula. We have

$$\begin{aligned} \left(\frac{1}{t}(e^{-tA} - 1)g, f \right) &= \lim_{n\to\infty} \left(\frac{1}{t}(\varrho^{2^n}_{t/2^n} - 1)g, f \right) \\ &= \lim_{n\to\infty} \sum_{j=0}^{2^n-1} \frac{1}{2^n} \left(\frac{2^n}{t}(\varrho_{t/2^n} - 1)g, \varrho^{2^n(j/2^n)}_{t/2^n} f \right) \\ &= \lim_{n\to\infty} \int_0^1 \left(\frac{2^n}{t}(\varrho_{t/2^n} - 1)g, \varrho^{[2^n s]}_{t/2^n} f \right) ds, \end{aligned}$$

where the brackets denote integer part in this formula. Since for $g \in D(-(1/2)\Delta)$,

$$w - \lim_{n\to\infty} \frac{2^n}{t}(\varrho_{t/2^n} - 1)g = -H(a)g,$$

the norm $\|(2^n/t)(\varrho_{t/2^n} - 1)g\|$ is uniformly bounded in n. Moreover, we see that s-$\lim_{n\to\infty} \varrho^{[2^n s]}_{t/2^n} = e^{-stA}$, $s \in [0, 1]$, by Corollary 3.67 below. Thus we conclude that

$$\left(\frac{1}{t}(e^{-tA} - 1)g, f \right) = \int_0^1 (-H(a)g, e^{-tsA} f) ds. \tag{3.5.10}$$

As $t \to 0$ on both sides above, we get $(g, Af) = (H(a)g, f)$, implying that $Ag = H(a)g$. Hence $A = H(a)$ and the proposition follows for $V = 0$.

Let now V be continuous and bounded. Note that for $f_0, f_n \in L^2(\mathbb{R}^d)$ and $f_j \in L^\infty(\mathbb{R}^d)$ for $j = 1, \dots, n-1$, it follows that for $0 = t_0 < t_1 < \cdots < t_n$,

$$\Big(f_0, \prod_{j=1}^n e^{-(t_j - t_{j-1})H(a)} f_j\Big) = \int_{\mathbb{R}^d} dx \Big[\overline{f_0(B_0)}\Big(\prod_{j=1}^n f_j(B_{t_j})\Big) e^{-i\int_0^{t_n} a(B_s)\circ dB_s}\Big]. \tag{3.5.11}$$

This can be proven by using the Markov property of Brownian motion as

$$\begin{aligned}
&\Big(f_0, \prod_{j=1}^n e^{-(t_j - t_{j-1})H(a)} f_j\Big) \\
&= \int_{\mathbb{R}^d} dx \mathbb{E}^x[\overline{f_0(B_0)} f_1(B_{t_1}) e^{-i\int_0^{t_1} a(B_s)\circ dB_s} \\
&\qquad\qquad \times \mathbb{E}^{B_{t_1}}[f_1(B_{t_1}) e^{-i\int_0^{t_2 - t_1} a(B_s)\circ dB_s} G_2(B_{t_2 - t_1})]] \\
&= \int_{\mathbb{R}^d} dx \mathbb{E}^x[\overline{f_0(B_0)} f_1(B_{t_1}) e^{-i\int_0^{t_2} a(B_s)\circ dB_s} f_1(B_{t_1}) G_2(B_{t_2 - t_1})],
\end{aligned}$$

where $G_j(x) = f_j(x)(\prod_{i=j+1}^n e^{-(t_i - t_{i-1})H(a)} f_i)(x)$. Repeating this procedure we arrive at (3.5.11). From this and the Trotter product formula we have

$$\begin{aligned}
(f, e^{-tH(a)} g) &= \lim_{n\to\infty} (f, (e^{-(t/n)H(a)} e^{-(t/n)V})^n g) \\
&= \lim_{n\to\infty} \int_{\mathbb{R}^d} dx \mathbb{E}^x[\overline{f(B_0)} e^{-i\int_0^t a(B_s)\circ dB_s} e^{-\sum_{j=1}^n V(B_{jt/n})}] \\
&= \int_{\mathbb{R}^d} dx \mathbb{E}^x[\overline{f(B_0)} e^{-i\int_0^t a(B_s)\circ dB_s} e^{-\int_0^t V(B_s)ds}].
\end{aligned}$$

Finally, we can extend the result for smooth V in a similar way as done in Theorem 3.31. □

Now we prove the statements used in the proof above.

Lemma 3.66. *Let A_n be a positive and contracting self-adjoint operator such that* s-$\lim_{n\to\infty} A_n = A$. *Then for all $s \ge 0$,* s-$\lim_{n\to\infty} A_n^s = A^s$.

Proof. Suppose that $0 < s \le 1/2$. Since A_n is positive, we have

$$A_n^{2s} f = \frac{\sin(2\pi s)}{\pi} \int_0^\infty \lambda^{2s-1}(A_n + \lambda)^{-1} A_n f d\lambda.$$

From this follows that $A_n^{2s} \to A^{2s}$ weakly as $n \to \infty$, thus $A_n^s \to A^s$ strongly. For $0 \le s \le 1$, s-$\lim_{n\to\infty} A_n^s =$ s-$\lim_{n\to\infty}(A_n^{s/2})^2 = A^s$, since A_n is a contraction. Repeating this procedure the result follows. □

Corollary 3.67. s-$\lim_{n\to\infty} \varrho_{t/n}^{[ns]} = e^{-stA}$ *for* $s \geq 0$.

Proof. Set $A_n = \varrho_{t/n}^{2n}$. Then A_n is a positive and a contracting self-adjoint operator such that s-$\lim_{n\to\infty} A_n = e^{-2tA}$. We have $A_n^{(ns+1)/n} \leq \varrho_{t/n}^{2[ns]} \leq A_n^s$ in form sense. From this inequality, Lemma 3.66 and the fact that s-$\lim_{n\to\infty} A_n^{1/n} = 1$, we obtain that $(f, \varrho_{t/n}^{2[ns]} f) \to (f, e^{-2tsA} f)$ as $n \to \infty$. Hence s-$\lim_{n\to\infty} \varrho_{t/n}^{[ns]} = e^{-tsA}$ follows. □

3.5.3 Extension to singular external potentials and vector potentials

The Feynman–Kac–Itô formula derived in the previous subsection can further be extended to singular vector potentials $a = (a_1, \ldots, a_d)$. By using form technique, we are able to define $H(a)$ as a self-adjoint operator. Let

$$D_\mu = -i\partial_\mu - a_\mu : L^2(\mathbb{R}^d) \to \mathscr{D}'(\mathbb{R}^d), \tag{3.5.12}$$

with Schwartz distribution space $\mathscr{D}'(\mathbb{R}^d)$ over $\mathbb{R}^d$. Define the quadratic form

$$q_a(f, g) = \sum_{\mu=1}^{d} (D_\mu f, D_\mu g) + (V^{1/2} f, V^{1/2} g) \tag{3.5.13}$$

with domain $Q(q_a) = \bigcap_{\mu=1}^d \{f \in L^2(\mathbb{R}^d) | D_\mu f \in L^2(\mathbb{R}^d)\} \cap D(V^{1/2})$.

Lemma 3.68. *Suppose that* $a \in (L^2_{\mathrm{loc}}(\mathbb{R}^d))^d$ *and* $V \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$ *with* $V \geq 0$. *Then* q_a *is a closed symmetric form.*

Proof. Let $(f_n)_n \subset Q(q_a)$ be a Cauchy sequence with respect to the norm $\|\cdot\|_0 = (q_a(\cdot,\cdot) + \|\cdot\|^2)^{1/2}$; then $V^{1/2} f_n$, $D_\mu f_n$ are Cauchy sequences in $L^2(\mathbb{R}^d)$. Thus there exist $g, f \in L^2(\mathbb{R}^d)$ such that $D_\mu f_n \to g$ and $V^{1/2} f_n \to V^{1/2} f$ as $n \to \infty$ in $L^2(\mathbb{R}^d)$. This gives $(f, D_\mu \phi) = (g, \phi)$, for all $\phi \in C_0^\infty(\mathbb{R}^d)$. Therefore $g = D_\mu f$ in $\mathscr{D}'(\mathbb{R}^d)$ distribution sense, hence $Q(q_a)$ is complete with respect to $\|\cdot\|_0$. □

By Lemma 3.68 there exists a self-adjoint operator h such that

$$(f, hg) = q_a(f, g), \quad f \in Q(q_a),\ g \in D(h). \tag{3.5.14}$$

Definition 3.27 (Schrödinger operator with a singular external potential and a vector field). Assume that $a \in (L^2_{\mathrm{loc}}(\mathbb{R}^d))^d$ and $V \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$. We denote the self-adjoint operator h in (3.5.14) by $H(a)$ and call it *Schrödinger operator with singular external potentials and vector potentials*.

A sufficient condition for $C_0^\infty(\mathbb{R}^d)$ to be a core of $H(a)$ is the proposition below.

Proposition 3.69. (1) *Let $V \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$ with $V \geq 0$. If $a \in (L^2_{\mathrm{loc}}(\mathbb{R}^d))^d$, then $C_0^\infty(\mathbb{R}^d)$ is a form core of $H(a)$.*

(2) *Let $V \in L^2_{\mathrm{loc}}(\mathbb{R}^d)$ with $V \geq 0$. If $a \in (L^4_{\mathrm{loc}}(\mathbb{R}^d))^d$ and $\nabla \cdot a \in L^2_{\mathrm{loc}}(\mathbb{R}^d)$, then $C_0^\infty(\mathbb{R}^d)$ is an operator core.*

We can also construct a Feynman–Kac–Itô formula for the form-defined $H(a)$.

Lemma 3.70. *Let $0 \leq V$ and $V \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$. Suppose that $a^{(n)}, a \in (L^2_{\mathrm{loc}}(\mathbb{R}^d))^d$ and $a^{(n)}_\mu \to a_\mu$ in $L^2_{\mathrm{loc}}(\mathbb{R}^d)$ as $n \to \infty$. Then $H(a^{(n)})$ converges to $H(a)$ in strong resolvent sense.*

Proof. Put $H_n = H(a^{(n)})$, $H = H(a)$ and $D^{(n)}_\mu = -i\partial_\mu - a^{(n)}_\mu$. Notice that $C_0^\infty(\mathbb{R}^d)$ is a form core of $H(a)$ by Proposition 3.69. Set $u_n = (H_n + i)^{-1} f$ for $f \in L^2(\mathbb{R}^d)$. Thus $\|u_n\| \leq \|f\|$ and

$$\sum_{\mu=1}^d \|D^{(n)}_\mu u_n\|^2 + \|V^{1/2} u_n\|^2 \leq \|f\|^2.$$

Since u_n, $D^{(n)}_\mu u_n$ and $V^{1/2} u_n$ are bounded, there exist a subsequence $u_{n'}$ and some vectors $u, v, w_\mu \in L^2(\mathbb{R}^d)$ such that $u_{n'} \to u$, $V^{1/2} u_{n'} \to v$ and $D^{(n')}_\mu u_{n'} \to w_\mu$ weakly. Since, furthermore, $V^{1/2} u_{n'} \to V^{1/2} u$ and $D^{(n')}_\mu u_{n'} \to D_\mu u$ in $\mathscr{D}'(\mathbb{R}^d)$, we have $v = V^{1/2} u$ and $D_\mu u = w_\mu$, and hence $u \in Q(q_a)$. For $\phi \in C_0^\infty(\mathbb{R}^d)$

$$q_a(u, \phi) = \lim_{n\to\infty} (u_n, H_n \phi) = (f - iu, \phi). \tag{3.5.15}$$

As $C_0^\infty(\mathbb{R}^d)$ is a form core of H, (3.5.15) implies that $u \in D(H)$ and $f = (H + i)u$. Hence $u = (H + i)^{-1} f$ follows. Therefore, $(H_n + i)^{-1} \to (H + i)^{-1}$ in weak sense, and similarly, $(H_n - i)^{-1} \to (H - i)^{-1}$ weakly. Since

$$\|(H_n + i)^{-1} f\|^2 = (i/2)(f, (H_n + i)^{-1} f - (H_n - i)^{-1} f) \to \|(H + i)^{-1} f\|^2,$$

strong resolvent convergence of H_n follows. □

Lemma 3.71. *Let $V \geq 0$ be in $L^1_{\mathrm{loc}}(\mathbb{R}^d)$, $a \in (L^2_{\mathrm{loc}}(\mathbb{R}^d))^d$ and $\nabla \cdot a \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$. Then $(f, e^{-tH(a)} g)$ has the same functional integral representation as in* (3.5.3).

Proof. First suppose that $V \in L^\infty(\mathbb{R}^d)$. Notice that since $a_\mu(B_s) \in \bigcup_{T>0} \mathbb{M}^2_T$ by the assumption that $a \in (L^2_{\mathrm{loc}}(\mathbb{R}^d))^d$ we have $\mathcal{W}^x(|\int_0^t a(B_s) \cdot dB_s| < \infty) = 1$, moreover by Lemma 3.32, $\mathcal{W}^x(|\int_0^t \nabla \cdot a(B_s) ds| < \infty) = 1$ also holds. By using a mollifier we can choose a sequence $a^{(n)} \in C_0^\infty(\mathbb{R}^d)$, $n = 1, 2, \ldots$, such that $a^{(n)}_\mu \to a_\mu$ in $L^2_{\mathrm{loc}}(\mathbb{R}^d)$ and $\nabla \cdot a^{(n)} \to \nabla \cdot a$ in $L^1_{\mathrm{loc}}(\mathbb{R}^d)$ as $n \to \infty$. Let $1_R = \chi(x_1/R) \cdots \chi(x_d/R)$, $R \in \mathbb{N}$, where $\chi \in C_0^\infty(\mathbb{R})$ such that $0 \leq \chi \leq 1$, $\chi(x) = 1$

for $|x| < 1$ and $\chi(x) = 0$ for $|x| \geq 2$. Since $1_R a^{(n)} \to a^{(n)}$ as $R \to \infty$ in $(L^2_{\mathrm{loc}}(\mathbb{R}^d))^d$ and $a^{(n)} \to a$ as $n \to \infty$ in $(L^2_{\mathrm{loc}}(\mathbb{R}^d))^d$, it follows from Lemma 3.70 that $e^{-tH(1_R a^{(n)})} \to e^{-tH(1_R a)}$, as $n \to \infty$ and $e^{-tH(1_R a)} \to e^{-tH(a)}$ as $R \to \infty$ in strong sense. Furthermore, (3.5.3) remains true for a replaced by $1_R a^{(n)} \in C_0^\infty(\mathbb{R}^d)$. Since $1_R a^{(n)} \in (C_0^\infty(\mathbb{R}^d))^d$ and $1_R a^{(n)} \to 1_R a$ in $(L^2_{\mathrm{loc}}(\mathbb{R}^d))^d$ as $n \to \infty$, it follows that

$$\int_0^t 1_R(B_s) a^{(n)}(B_s) \cdot dB_s \to \int_0^t 1_R(B_s) a(B_s) \cdot dB_s \tag{3.5.16}$$

and since $\nabla \cdot (1_R a^{(n)}) = (\nabla 1_R) \cdot a^{(n)} + 1_R(\nabla \cdot a^{(n)}) \to (\nabla 1_R) \cdot a + 1_R(\nabla \cdot a)$ in $L^1(\mathbb{R}^d)$ it follows that

$$\int_0^t \nabla \cdot (1_R(B_s) a^{(n)}(B_s)) ds \to \int_0^t \{(\nabla 1_R(B_s)) \cdot a(B_s) + 1_R(B_s)(\nabla \cdot a(B_s))\} ds \tag{3.5.17}$$

strongly in $L^1(\mathscr{X}, d\mathcal{W}^x)$. Thus there is a subsequence n' such that (3.5.16) and (3.5.17) hold almost surely for n replaced by n', therefore (3.5.3) follows by a limiting argument for a replaced by $1_R a$. Let

$$\Omega_+(R) = \left\{\omega \in \mathscr{X} \mid \max_{0 \leq s \leq t, 1 \leq \mu \leq d} B_s^\mu(\omega) \leq R\right\},$$
$$\Omega_-(R) = \left\{\omega \in \mathscr{X} \mid \min_{0 \leq s \leq t, 1 \leq \mu \leq d} B_s^\mu(\omega) \geq -R\right\}$$

and

$$I(R) = \left| \int_0^t 1_R(B_s) a(B_s) \cdot dB_s - \int_0^t a(B_s) \cdot dB_s \right|.$$

It is a well-known fact that

$$\begin{aligned} \mathcal{W}^x(\Omega_-(R)) = \mathcal{W}^x(\Omega_+(R)) &= \prod_{\mu=1}^d \mathcal{W}^x(|B_t^\mu| \leq R) \\ &= \left(\frac{2}{\sqrt{2\pi t}} \int_0^R e^{-y^2/(2t)} dy \right)^d. \end{aligned}$$

Since $I(R) = 0$ on $\Omega_+ \cap \Omega_-$, we have

$$\begin{aligned} \mathcal{W}^x (I(R) \geq \varepsilon) &= \mathcal{W}^x \left(I(R) \geq \varepsilon,\ \Omega_+(R)^c \cup \Omega_-(R)^c \right) \\ &\leq 2 \left(\frac{2}{\sqrt{2\pi t}} \int_R^\infty e^{-y^2/(2t)} dy \right)^d. \end{aligned}$$

Hence $\lim_{R\to\infty} \mathcal{W}^x(I(R) \geq \varepsilon) = 0$. Thus there exists a subsequence R' such that $\int_0^t 1_{R'}(B_s)a(B_s)\cdot dB_s \to \int_0^t a(B_s)\cdot dB_s$ as $R' \to \infty$, almost surely. It can be seen in a similar way that with a subsequence (3.5.17) converges to $\int_0^t \nabla\cdot a(B_s)ds$ almost surely. Thus $\int_0^t 1_{R''}(B_s)a(B_s)\circ dB_s \to \int_0^t a(B_s)\circ dB_s$, almost surely. Since

$$|\overline{f(B_0)}g(B_t)e^{-\int_0^t V(B_s)ds}e^{-i\int_0^t 1_{R''}(B_s)a(B_s)\circ dB_s}| = |\overline{f(B_0)}g(B_t)|e^{-\int_0^t V(B_s)ds}$$

and the right-hand side is integrable, (3.5.3) holds for a satisfying $a \in (L^2_{\mathrm{loc}}(\mathbb{R}^d))^d$ and $\nabla\cdot a \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$ by the dominated convergence theorem. Finally, by a similar argument as in the proof of Theorem 3.31, we can extend (3.5.3) to $V \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$. □

Define $H^0(a)$ by $H(a)$ with $V = 0$.

Lemma 3.72. *Let $a \in (L^2_{\mathrm{loc}}(\mathbb{R}^d))^d$, $\nabla\cdot a \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$ and V be a real multiplication operator.*

(1) *Suppose $|V|$ is relatively form bounded with respect to $-(1/2)\Delta$ with relative bound b. Then $|V|$ is also relatively form bounded with respect to $H^0(a)$ with a relative bound not larger than b.*

(2) *Suppose $|V|$ is relatively bounded with respect to $-(1/2)\Delta$ with relative bound b. Then $|V|$ is also relatively bounded with respect to $H^0(a)$ with a relative bound not larger than b.*

Proof. By virtue of Lemma 3.71 we have

$$|(f, e^{-tH^0(a)}g)| \leq (|f|, e^{t(1/2)\Delta}|g|). \tag{3.5.18}$$

Since $(H^0(a)+E)^{-1/2} = \frac{1}{\sqrt{\pi}}\int_0^\infty t^{-1/2}e^{-t(H^0(a)+E)}dt$, $E > 0$, (3.5.18) implies that

$$|(H^0(a)+E)^{-1/2}f|(x) \leq (-(1/2)\Delta + E)^{-1/2}|f|(x) \tag{3.5.19}$$

for almost every $x \in \mathbb{R}^d$. Hence we have

$$|V(x)|^{1/2}|(H^0(a)+E)^{-1/2}f|(x) \leq |V(x)|^{1/2}(-(1/2)\Delta + E)^{-1/2}|f|(x)$$

and

$$\frac{\| |V|^{1/2}(H^0(a)+E)^{-1/2}f\|}{\|f\|} \leq \frac{\| |V|^{1/2}(-(1/2)\Delta+E)^{-1/2}|f|\|}{\|f\|}. \tag{3.5.20}$$

Similarly, by using $(H^0(a)+E)^{-1} = \int_0^\infty e^{-t(H^0(a)+E)}dt$, $E > 0$, we have

$$\frac{\| |V|(H^0(a)+E)^{-1}f\|}{\|f\|} \leq \frac{\| |V|(-(1/2)\Delta+E)^{-1}|f|\|}{\|f\|}. \tag{3.5.21}$$

On taking the limit $E \to \infty$, the right-hand sides of (3.5.20) and (3.5.21) converge to b. Hence (1) follows by (3.5.20) and (2) by (3.5.21). □

Let V be such that $V_+ \in L^1_{\rm loc}(\mathbb{R}^d)$ and V_- is relatively form bounded with respect to $-(1/2)\Delta$ with a relative bound strictly smaller than one. Then the KLMN theorem and Lemma 3.72 allow to define $H^0(a) \dot{+} V_+ \dot{-} V_-$ as a self-adjoint operator.

Definition 3.28 (Schrödinger operator with singular external potentials and vector potentials). Let $a \in (L^2_{\rm loc}(\mathbb{R}^d))^d$ and $\nabla \cdot a \in L^1_{\rm loc}(\mathbb{R}^d)$. Let V be such that $V_+ \in L^1_{\rm loc}(\mathbb{R}^d)$ and V_- is relatively form bounded with respect to $-(1/2)\Delta$ with a relative bound strictly smaller than one. Then the *Schrödinger operator with V and a* is defined by

$$H^0(a) \dot{+} V_+ \dot{-} V_-. \tag{3.5.22}$$

This is also denoted by $H(a)$.

Theorem 3.73 (Feynman–Kac formula for Schrödinger operator with singular external potentials and vector potentials). *Let $a \in (L^2_{\rm loc}(\mathbb{R}^d))^d$ and $\nabla \cdot a \in L^1_{\rm loc}(\mathbb{R}^d)$. Suppose that $V_+ \in L^1_{\rm loc}(\mathbb{R}^d)$ and V_- is relatively form bounded with respect to $-(1/2)\Delta$ with a relative bound strictly smaller than one. Then the Feynman–Kac formula of $(f, e^{-tH(a)}g)$ is given by* (3.5.3).

Proof. The proof is similar to the limiting argument in Theorem 3.31. □

We used inequality (3.5.18) in the proof of Lemma 3.72. This can be extended to operators with singular external potentials.

Corollary 3.74 (Diamagnetic inequality). *Under the assumptions of Theorem* 3.73 *we have*

$$|(f, e^{-tH(a)}g)| \le (|f|, e^{-tH(0)}|g|). \tag{3.5.23}$$

3.5.4 Kato-class potentials and L^p–L^q boundedness

We can also add Kato-class potentials for Schrödinger operators with vector potentials a. Let V be Kato decomposable, $a \in (L^2_{\rm loc}(\mathbb{R}^d))^d$ and $\nabla \cdot a \in L^1_{\rm loc}(\mathbb{R}^d)$. Define

$$K_t f(x) = \mathbb{E}^x[e^{-i\int_0^t a(B_s)\circ dB_s} e^{-\int_0^t V(B_s)ds} f(B_t)]. \tag{3.5.24}$$

Since the absolute value of the first exponential is 1, K_t is well defined as a bounded operator on $L^2(\mathbb{R}^d)$.

Theorem 3.75. *Let V be Kato decomposable, and $a \in (L^2_{\rm loc}(\mathbb{R}^d))^d$, $\nabla \cdot a \in L^1_{\rm loc}(\mathbb{R}^d)$. Then $\{K_t : t \ge 0\}$ is a symmetric C_0 semigroup and there exists the unique self-adjoint operator $K(a)$ bounded from below such that $K_t = e^{-tK(a)}$, $t \ge 0$.*

Proof. In a similar way to Theorem 3.65 it can be proven that $\{K_t : t \geq 0\}$ is a symmetric C_0 semigroup. □

Definition 3.29 (Schrödinger operator with vector potentials and Kato-decomposable potentials). Let $a \in (L^2_{\mathrm{loc}}(\mathbb{R}^d))^d$, $\nabla \cdot a \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$ and V be Kato decomposable. Then we call the self-adjoint operator $K(a)$ defined by Theorem 3.75 *Schrödinger operator with vector potentials and Kato-decomposable potentials*.

Corollary 3.76 (L^p–L^q boundedness). *Let V be Kato decomposable, $a \in (L^2_{\mathrm{loc}}(\mathbb{R}^d))^d$ and $\nabla \cdot a \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$. Then $e^{-tK(a)}$ is L^p–L^q bounded, for every $t \geq 0$.*

Proof. By the diamagnetic inequality

$$|e^{-tK(a)} f(x)| \leq e^{-tK(0)} |f|(x). \tag{3.5.25}$$

In Theorem 3.39 we have shown that $e^{-tK(0)}$ is hypercontractive. By (3.5.25) it follows that $e^{-tK(a)}$ is bounded from $L^1(\mathbb{R}^d)$ to $L^1(\mathbb{R}^d)$, from $L^\infty(\mathbb{R}^d)$ to $L^\infty(\mathbb{R}^d)$, and from $L^1(\mathbb{R}^d)$ to $L^\infty(\mathbb{R}^d)$. □

3.6 Feynman–Kac formula for relativistic Schrödinger operators

3.6.1 Relativistic Schrödinger operator

Relativity theory says that in the high energy regime the kinetic energy of a body in motion should be proportional to the absolute value of its momentum p rather than its square. In fact, the classical energy of a relativistic particle of mass m is $\sqrt{|p|^2 + m^2c^2}$, where c is the speed of light, and this justifies the definition of the *relativistic Schrödinger operator*

$$\sqrt{-\Delta + m^2} - m$$

(in units for which $c = 1$). The constant m is subtracted to set the bottom of its spectrum to zero. In this section we consider Feynman–Kac formulae for relativistic Schrödinger operators of a more general form.

Definition 3.30 (Relativistic Schrödinger operator with vector potentials). Suppose that $a \in (L^2_{\mathrm{loc}}(\mathbb{R}^d))^d$ and $V \in L^\infty(\mathbb{R}^d)$. The operator

$$H_{\mathrm{R}}(a) = (2H^0(a) + m^2)^{1/2} - m + V, \tag{3.6.1}$$

is called *relativistic Schrödinger operator*, where $H^0(a)$ is the self-adjoint operator given in Definition 3.27 with $V = 0$.

Note that the square-root term is defined through the spectral projection of the self-adjoint operator $H^0(a)$. Denote by $H_{\mathrm{R}}^0(a)$ the operator $H_{\mathrm{R}}(a)$ with $V = 0$.

Proposition 3.77. (1) *Suppose that* $a \in (L^4_{\mathrm{loc}}(\mathbb{R}^d))^d$ *and* $\nabla \cdot a \in L^2_{\mathrm{loc}}(\mathbb{R}^d)$. *Then* $C_0^\infty(\mathbb{R}^d)$ *is an operator core of* $H_{\mathrm{R}}^0(a)$.

(2) *Suppose that* $a \in (L^2_{\mathrm{loc}}(\mathbb{R}^d))^d$. *Then* $C_0^\infty(\mathbb{R}^d)$ *is a form core of* $H_{\mathrm{R}}^0(a)$.

Proof. For simplicity, in this proof we set $H_{\mathrm{R}}^0(a) = H_{\mathrm{R}}$ and $H^0(a) = H$.

(1) There exist non-negative constants c_1 and c_2 such that

$$\|H_{\mathrm{R}} f\| \le c_1 \|Hf\| + c_2 \|f\| \tag{3.6.2}$$

for all $f \in D(H)$. Hence $C_0^\infty(\mathbb{R}^d)$ is contained in $D(H_{\mathrm{R}})$. Since H_{R} is a non-negative self-adjoint operator, $H_{\mathrm{R}}+1$ has a bounded inverse, and we use that $C_0^\infty(\mathbb{R}^d)$ is a core of H_{R} if and only if $H_{\mathrm{R}} C_0^\infty(\mathbb{R}^d)$ is dense in $L^2(\mathbb{R}^d)$. Let $g \in L^2(\mathbb{R}^d)$ and suppose that $(g, (H_{\mathrm{R}} + 1) f) = 0$, for all $f \in C_0^\infty(\mathbb{R}^d)$. Then $C_0^\infty(\mathbb{R}^d) \ni f \mapsto (g, H_{\mathrm{R}} f) = -(g, f)$ defines a continuous functional which can be extended to $L^2(\mathbb{R}^d)$. Thus $g \in D(H_{\mathrm{R}})$ and $0 = ((H_{\mathrm{R}} + 1)g, f)$. Since $C_0^\infty(\mathbb{R}^d)$ is dense, we have $(H_{\mathrm{R}} + 1)g = 0$, and hence $g = 0$ since $H_{\mathrm{R}} + 1$ is one-to-one, proving the assertion.

(2) Note that $\|H_{\mathrm{R}}^{1/2} f\|^2 \le c_1 \|H^{1/2} f\|^2 + c_2 \|f\|^2$ for $f \in Q(h) = D(H^{1/2})$, and $C_0^\infty(\mathbb{R}^d)$ is contained in $Q(H_{\mathrm{R}}) = D(H_{\mathrm{R}}^{1/2})$. Since $H_{\mathrm{R}}^{1/2} + 1$ has also bounded inverse, it is seen by the same argument as above that $C_0^\infty(\mathbb{R}^d)$ is a core of $H_{\mathrm{R}}^{1/2}$ or a form core of H_{R}. □

A key element in the construction of the Feynman–Kac formula is to use the Lévy subordinator studied in Example 2.18. In that notation we write $\delta = 1$, $\gamma = m$ so that we have the subordinator $T_t = T_t(1, m)$ on a suitable probability space $(\mathcal{T}, \mathcal{B}_{\mathcal{T}}, \nu)$ such that

$$\mathbb{E}_\nu^0[e^{-uT_t}] = \exp(-t(\sqrt{2u + m^2} - m)). \tag{3.6.3}$$

For simplicity, we write $\mathbb{E}^{x,0}$ for $\mathbb{E}^{x,0}_{\mathcal{W}\times\nu}$.

Theorem 3.78 (Feynman–Kac formula for relativistic Schrödinger operator with vector potentials). *Suppose that* $V \in L^\infty(\mathbb{R}^d)$, $a \in (L^2_{\mathrm{loc}}(\mathbb{R}^d))^d$ *and* $\nabla \cdot a \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$. *Then*

$$(f, e^{-tH_{\mathrm{R}}(a)} g) = \int_{\mathbb{R}^d} dx \mathbb{E}^{x,0}[\overline{f(B_0)} g(B_{T_t}) e^{-i \int_0^{T_t} a(B_s) \circ dB_s} e^{-\int_0^t V(B_s) ds}]. \tag{3.6.4}$$

Proof. We divide the proof into four steps.

Step 1*:* Suppose $V = 0$. Then

$$(f, e^{-tH_{\mathrm{R}}^0(a)}g) = \int_{\mathbb{R}^d} dx \mathbb{E}^{x,0}[\overline{f(B_0)}g(B_{T_t})e^{-i\int_0^{T_t} a(B_s)\circ dB_s}]. \tag{3.6.5}$$

Denote by E_λ the spectral projection of the self-adjoint operator $H^0(a)$. Then

$$(f, e^{-tH_{\mathrm{R}}^0(a)}g) = \int_{[0,\infty)} e^{-t(\sqrt{2\lambda+m^2}-m)} d(f, E_\lambda g). \tag{3.6.6}$$

By (3.6.3)

$$(f, e^{-tH_{\mathrm{R}}^0(a)}g) = \int_0^\infty \mathbb{E}_\nu^0[e^{-T_t\lambda}]d(f, E_\lambda g) = \mathbb{E}_\nu^0[(f, e^{-T_t H^0(a)}g)].$$

Then (3.6.5) follows from inserting the Feynman–Kac–Itô formula of $e^{-T_t H^0(a)}$:

$$(f, e^{-T_t H_{\mathrm{R}}^0(a)}g) = \int_{\mathbb{R}^d} dx \mathbb{E}^{x,0}[\overline{f(B_0)}g(B_{T_t})e^{-i\int_0^{T_t} a(B_s)\circ dB_s}] \tag{3.6.7}$$

developed in Section 3.5.3.

Step 2*:* Let $0 = t_0 < t_1 < \cdots < t_n$ and $f_0, f_n \in L^2(\mathbb{R}^d)$ and $f_j \in L^\infty(\mathbb{R}^d)$ for $j = 1, \ldots, n-1$. We prove next that

$$\begin{aligned}&\Big(f_0, \prod_{j=1}^n e^{-(t_j-t_{j-1})H_{\mathrm{R}}^0(a)} f_j\Big)\\ &= \int_{\mathbb{R}^d} dx \mathbb{E}^{x,0}\Big[\overline{f(B_0)}\Big(\prod_{j=1}^n f_j(B_{T_{t_j}})\Big)e^{-i\int_0^{T_t} a(B_s)\circ dB_s}\Big].\end{aligned} \tag{3.6.8}$$

We use the shorthand $G_j(x) = f_j(x)(\prod_{i=j+1}^n e^{-(t_i-t_{i-1})H_{\mathrm{R}}^0(a)} f_i)(x)$. The left-hand side of (3.6.8) can be written as

$$\int_{\mathbb{R}^d} dx \mathbb{E}^{x,0}[\overline{f(B_0)}e^{-i\int_0^{T_{t_1-t_0}} a(B_s)\circ dB_s} G_1(B_{T_{t_1-t_0}})].$$

By the Markov property of $(B_t)_{t\geq 0}$ we have

$$\begin{aligned}&\Big(f_0, \prod_{j=1}^n e^{-(t_j-t_{j-1})H_{\mathrm{R}}^0(a)} f_j\Big)\\ &= \int_{\mathbb{R}^d} dx \mathbb{E}^{x,0}[\overline{f(B_0)}e^{-i\int_0^{T_{t_1}} a(B_s)\circ dB_s}\\ &\qquad \times \mathbb{E}_\nu^0 \mathbb{E}_{\mathcal{W}}^{B_{T_{t_1}}}[f_1(B_0)e^{-i\int_0^{T_{t_2-t_1}} a(B_s)\circ dB_s} G_2(B_{T_{t_2-t_1}})]]\\ &= \int_{\mathbb{R}^d} dx \mathbb{E}^{x,0}[\overline{f(B_0)}e^{-i\int_0^{T_{t_1}} a(B_s)\circ dB_s}\\ &\qquad \times \mathbb{E}_\nu^0[f_1(B_{T_{t_1}})e^{-i\int_{T_{t_1}}^{T_{t_2-t_1}+T_{t_1}} a(B_s)\circ dB_s} G_2(B_{T_{t_1}+T_{t_2-t_1}})]].\end{aligned}$$

The right-hand side above can be rewritten as

$$\int_{\mathbb{R}^d} dx \mathbb{E}^{x,0}[\overline{f(B_0)} e^{-i\int_0^{T_{t_1}} a(B_s)\circ dB_s} f_1(B_{T_{t_1}}) \\ \times \mathbb{E}_\nu^{T_{t_1}}[e^{-i\int_0^{T_{t_2-t_1}} a(B_s)\circ dB_s} G_2(B_{T_{t_2-t_1}})]].$$

By the Markov property of $(T_t)_{t\geq 0}$ again we can see that

$$= \int_{\mathbb{R}^d} dx \mathbb{E}^{x,0}[\overline{f(B_0)} e^{-i\int_0^{T_{t_1}} a(B_s)\circ dB_s} f_1(B_{T_{t_1}}) e^{-i\int_{T_{t_1}}^{T_{t_2}} a(B_s)\circ dB_s} G_2(B_{T_{t_2}})]$$

$$= \int_{\mathbb{R}^d} dx \mathbb{E}^{x,0}[\overline{f(B_0)} e^{-i\int_0^{T_{t_2}} a(B_s)\circ dB_s} f_1(B_{T_{t_1}}) G_2(B_{T_{t_2}})].$$

Continuing by the above procedure (3.6.8) is obtained.

Step 3*:* Suppose now that $V \in L^\infty$ and it is continuous. We show now (3.6.4) under this condition. By the Trotter product formula,

$$\begin{aligned}
&(f, e^{-tH_{\mathrm{R}}(a)} g) \\
&= \lim_{n\to\infty} (f, (e^{-(t/n)H_{\mathrm{R}}^0(a)} e^{-(t/n)V})^n g) \\
&= \lim_{n\to\infty} \int_{\mathbb{R}^d} dx \mathbb{E}^{x,0}[\overline{f(B_0)} g(B_{T_t}) e^{-i\int_0^{T_t} a(B_s)\circ dB_s} e^{-\sum_{j=1}^n (t/n) V(B_{T_{tj/n}})}] \\
&= \text{r.h.s. } (3.6.8).
\end{aligned}$$

Here we used that for each path (ω, τ), $V(B_{T_s(\tau)}(\omega))$ is continuous in s except for at most finitely many points. Note that T_s almost surely has no accumulation points in any compact interval. Thus

$$\sum_{j=1}^n \frac{t}{n} V(B_{T_{tj/n}}) \to \int_0^t V(B_{T_s}) ds$$

as $n \to \infty$, for each path in the sense of Riemann integral. This completes the proof of the claim.

Step 4*:* Finally, to complete the proof of the theorem define V_n in the same manner as in the proof of Theorem 3.86. V_n is bounded and continuous, moreover $V_n(x) \to V(x)$ as $n \to \infty$ for $x \notin \mathcal{N}$, where $\mathcal{N}$ is a set of Lebesgue measure zero. Notice that

$$\mathbb{E}^{x,0}[1_{\mathcal{N}}(B_{T_s})] = \mathbb{E}^0\left[1_{\{T_s>0\}} \int_{\mathbb{R}^d} 1_{\mathcal{N}}(x+y) k_s(y) dy\right] + 1_{\mathcal{N}}(x)\mathbb{E}^0[1_{\{T_s=0\}}] = 0$$

for $x \in \mathcal{N}$, where $k_s(x)$ is the distribution of the random variable B_{T_s} given by (3.4.21). Hence

$$0 = \int_0^t \mathbb{E}^{x,0}[1_{\mathcal{N}}(B_{T_s})] ds = \mathbb{E}^{x,0}\left[\int_0^t 1_{\mathcal{N}}(B_{T_s}) ds\right].$$

Then for almost surely (ω, τ) the Lebesgue measure of $\{t \in [0, \infty) \mid B_{T_t(\tau)}(\omega) \in \mathscr{N}\}$ is zero. Therefore $\int_0^t V_n(B_{T_s})ds \to \int_0^t V(B_{T_s})ds$ as $n \to \infty$ for almost every (ω, τ). Moreover,

$$\int_{\mathbb{R}^d} dx \mathbb{E}^{x,0}[\overline{f(B_0)} g(B_{T_t}) e^{-i\int_0^{T_t} a(B_s)\circ dB_s} e^{-\int_0^t V_n(B_s)ds}]$$
$$\to \int_{\mathbb{R}^d} dx \mathbb{E}^{x,0}[\overline{f(B_0)} g(B_{T_t}) e^{-i\int_0^{T_t} a(B_s)\circ dB_s} e^{-\int_0^t V(B_s)ds}]$$

as $n \to \infty$. On the other hand, $e^{-t(H_{\mathrm{R}}^0(a)+V_n)} \to e^{-t(H_{\mathrm{R}}^0(a)+V)}$ strongly as $n \to \infty$, since $H_{\mathrm{R}}^0(a) + V_n$ converges to $H_{\mathrm{R}}^0(a) + V$ on the common domain $D(H_{\mathrm{R}}^0(a))$. This proves the theorem. □

The Feynman–Kac formula of $e^{-tH_{\mathrm{R}}(a)}$ can be extended to more singular external potentials. The key idea is similar to that of applied in the case of $e^{-tH(a)}$ discussed in the previous section.

Lemma 3.79. *Let $a \in (L^2_{\mathrm{loc}}(\mathbb{R}^d))^d$, $\nabla \cdot a \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$ and V be a real multiplication operator.*

(1) *Suppose $|V|$ is relatively form bounded with respect to $\sqrt{-\Delta + m^2} - m$ with relative bound b. Then $|V|$ is also relatively form bounded with respect to $H_{\mathrm{R}}^0(a)$ with a relative bound not larger than b.*

(2) *Suppose $|V|$ is relatively bounded with respect to $\sqrt{-\Delta + m^2} - m$ with relative bound b. Then $|V|$ is also relatively bounded with respect to $H_{\mathrm{R}}^0(a)$ with a relative bound not larger than b.*

Proof. By virtue of Theorem 3.78 we have

$$|(f, e^{-tH_{\mathrm{R}}^0(a)} g)| \le (|f|, e^{-t(\sqrt{-\Delta+m^2}-m)}|g|). \tag{3.6.9}$$

The lemma can be proven in the parallel way to that of Lemma 3.72. □

Let V be such that $V_+ \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$ and V_- is relatively form bounded with respect to $\sqrt{-\Delta + m^2} - m$ with a relative bound strictly smaller than one. Then the KLMN theorem and Lemma 3.79 can be used to define $H_{\mathrm{R}}^0(a) \dotplus V_+ \dot{-} V_-$ as a self-adjoint operator.

Definition 3.31 (Relativistic Schrödinger operator with singular external potentials and vector potentials). Let $a \in (L^2_{\mathrm{loc}}(\mathbb{R}^d))^d$ and $\nabla \cdot a \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$. Let V be such that $V_+ \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$ and V_- is relatively form bounded with respect to $\sqrt{-\Delta + m^2} - m$ with a relative bound strictly smaller than one. Then the relativistic Schrödinger operator with a and V is defined by

$$H_{\mathrm{R}}(a) = H_{\mathrm{R}}^0(a) \dotplus V_+ \dot{-} V_-. \tag{3.6.10}$$

Theorem 3.80 (Feynman–Kac formula for relativistic Schrödinger operator with singular external potentials and vector potentials). *Suppose that $a \in (L^2_{\text{loc}}(\mathbb{R}^d))^d$ and $\nabla \cdot a \in L^1_{\text{loc}}(\mathbb{R}^d)$. Let V be such that $V_+ \in L^1_{\text{loc}}(\mathbb{R}^d)$ and V_- is relatively form bounded with respect to $\sqrt{-\Delta + m^2} - m$ with a relative bound strictly smaller than one. Then the Feynman–Kac formula of $(f, e^{-tH_{\mathrm{R}}(a)}g)$ is given by* (3.5.3).

Proof. The proof is similar to the limiting argument in that of Theorem 3.31. □

Corollary 3.81 (Diamagnetic inequality). *With the same assumptions as in Theorem* 3.80 *we have*

$$|(f, e^{-tH_{\mathrm{R}}(a)}g)| \le (|f|, e^{-tH_{\mathrm{R}}(0)}|g|) \tag{3.6.11}$$

Remark 3.6 (Bernstein functions of the Laplacian). Theorems 3.78 and 3.80 can be generalized to fractional Schrödinger operators as in Definition 3.3. $(T_t)_{t\ge 0}$ is then an $\alpha/2$-stable subordinator, $0 < \alpha < 2$, and thus $(B_{T_t})_{t\ge 0}$ is a symmetric α-stable process. In fact, the relativistic and fractional Laplacians are cases of a larger class of operators obtained as Bernstein functions of the Laplacian.

Let $\Phi \in \mathscr{B}_0$ be a Bernstein function with vanishing right limit as in (2.4.44), and consider its uniquely associated subordinator $(T_t^\Phi)_{t\ge 0}$. Then we define a generalized Schrödinger operator

$$H_\Phi(a) = \Phi\left(\frac{1}{2}(-i\nabla - a)^2\right) + V. \tag{3.6.12}$$

Using similar arguments than before we can show the Feynman–Kac formula

$$(f, e^{-tH_\Phi(a)}g) = \int_{\mathbb{R}^d} dx \mathbb{E}^{x,0}[\overline{f(B_0)}g(B_{T_t^\Phi})e^{-i\int_0^{T_t^\Phi} a(B_s)\circ dB_s} e^{-\int_0^t V(B_{T_s^\Phi})ds}]. \tag{3.6.13}$$

Setting $a = 0$, we obtain

$$(f, e^{-tH_\Phi(0)}g) = \int_{\mathbb{R}^d} dx \mathbb{E}^{x,0}[\overline{f(X_0)}g(X_t)e^{-\int_0^t V(X_s)ds}] \tag{3.6.14}$$

where $(X_t)_{t\ge 0} = (B_{T_t^\Phi})_{t\ge 0}$ is subordinate Brownian motion with respect to the given subordinator. This is a Lévy process with the property

$$\mathbb{E}^{0,0}[e^{-iu\cdot X_s}] = \mathbb{E}^0[e^{-u\cdot u T_s^\Phi/2}] = e^{-s\Phi(u\cdot u/2)}. \tag{3.6.15}$$

3.6.2 Relativistic Kato-class potentials and L^p–L^q boundedness

In the previous section we constructed the Feynman–Kac formula of $e^{-tH_{\mathrm{R}}(a)}$. Similarly to $K(a)$, we can define the relativistic Schrödinger operator $K_{\mathrm{R}}(a)$ as the generator of a C_0-semigroup defined through this Feynman–Kac formula with general external potentials.

Definition 3.32 (Relativistic Kato-class). (1) V is a *relativistic Kato-class potential* whenever it satisfies

$$\lim_{t\downarrow 0} \sup_{x\in\mathbb{R}^d} \int_0^t \mathbb{E}^{x,0}[V(B_{T_s})]ds = 0. \tag{3.6.16}$$

(2) $V = V_+ - V_-$ is *relativistic Kato-decomposable* whenever $V_+ \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$ and V_- is a relativistic Kato-class.

Similarly to the non-relativistic case we can show an exponential bound of relativistic Kato-decomposable potentials by using Khasminskii's lemma and (3.6.16).

Lemma 3.82. *Suppose that V is of relativistic Kato-decomposable and write $(X_t)_{t\geq 0} = (B_{T_t})_{t\geq 0}$. Then for any $t \geq 0$,*

$$\sup_{x\in\mathbb{R}^d} \mathbb{E}^{x,0}[e^{-\int_0^t V(X_s)\,ds}] < \infty. \tag{3.6.17}$$

Define the semigroup $K_t^{\mathrm{R}} : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ by

$$(K_t^{\mathrm{R}} f)(x) = \mathbb{E}^{x,0}[e^{-i\int_0^{T_t} a(B_s)\circ dB_s} e^{-\int_0^t V(B_{T_s})ds} f(B_{T_t})]. \tag{3.6.18}$$

Theorem 3.83. *Let $a \in (L^2_{\mathrm{loc}}(\mathbb{R}^d))^d$ and $\nabla\cdot a \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$. Suppose that V is relativistic Kato-decomposable. Then the family of operators $\{K_t^{\mathrm{R}} : t \geq 0\}$ is a symmetric C_0-semigroup.*

Since subordinate Brownian motion is a Markov process, the proof of this result is similar to that of Theorem 3.65. By the above and the Hille–Yoshida theorem we can define the corresponding operator.

Definition 3.33 (Relativistic Schrödinger operator with vector potential and Kato-decomposable potential). Let $a \in (L^2_{\mathrm{loc}}(\mathbb{R}^d))^d$ and $\nabla\cdot a \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$. Suppose that V is relativistic Kato-decomposable. We call the self-adjoint operator $K^{\mathrm{R}}(a)$ defined by

$$K_t^{\mathrm{R}} = e^{-tK_{\mathrm{R}}(a)}, \quad t \geq 0 \tag{3.6.19}$$

relativistic Schrödinger operator with vector potentials and relativistic Kato-class potentials.

Some immediate consequences of the definition are stated below.

Corollary 3.84. *Suppose the same assumptions as in Theorem* 3.83. *Then we have:*

(1) *Diamagnetic inequality:*

$$|(f, e^{-tK_{\mathrm{R}}(a)}g)| \le (|f|, e^{-K_{\mathrm{R}}(0)}|g|). \tag{3.6.20}$$

(2) L^p–L^q *boundedness:* $e^{-tK_{\mathrm{R}}(a)}$ *is* L^p–L^q *bounded.*

These statements can be proven in the same manner as Theorem 3.39 with Brownian motion $(B_t)_{t\ge 0}$ replaced by subordinate Brownian motion $(X_t)_{t\ge 0} = (B_{T_t})_{t\ge 0}$.

3.7 Feynman–Kac formula for Schrödinger operator with spin

3.7.1 Schrödinger operator with spin

In this section we set $d = 3$ and we consider the path integral representation of the Pauli operator, which includes spin $1/2$. The spin will be described in terms of a $\mathbb{Z}_2$-valued jump process and the Schrödinger operator is defined on $L^2(\mathbb{R}^3 \times \mathbb{Z}_2; \mathbb{C})$ instead of $L^2(\mathbb{R}^3; \mathbb{C}^2)$. This construction has the advantage to use scalar valued functions rather than $\mathbb{C}^2$-valued ones, so in the Feynman–Kac formula of Schrödinger operators with spin a scalar valued integral kernel appears.

Let $\sigma_1, \sigma_2, \sigma_3$ be the 2×2 *Pauli matrices* given by

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{3.7.1}$$

Each σ_μ is symmetric and traceless. They satisfy the anticommutation relations

$$\{\sigma_\mu, \sigma_\nu\} = 2\delta_{\mu\nu} \tag{3.7.2}$$

and thereby

$$\sigma_\mu \sigma_\nu = i \sum_{\lambda=1}^{3} \varepsilon^{\mu\nu\lambda} \sigma_\lambda, \tag{3.7.3}$$

where $\varepsilon^{\alpha\beta\gamma}$ denotes the antisymmetric tensor given by

$$\varepsilon^{\alpha\beta\gamma} = \begin{cases} 1, & \alpha\beta\gamma = \text{even permutation of } 123, \\ -1, & \alpha\beta\gamma = \text{odd permutation of } 123, \\ 0, & \text{otherwise.} \end{cases}$$

Specifically, we have $\sigma_1\sigma_2 = i\sigma_3$, $\sigma_2\sigma_3 = i\sigma_1$ and $\sigma_3\sigma_1 = i\sigma_2$ and therefore, alternatively, $i\sigma_1, i\sigma_2, i\sigma_3$ can be seen as the infinitesimal generators of the Lie group SU(2).

Definition 3.34 (Schrödinger operator with spin $1/2$). Let $a \in (C_{\mathrm{b}}^2(\mathbb{R}^3))^3$ and $V \in L^\infty(\mathbb{R}^3)$. We call the operator

$$H_{\mathrm{S}}(a) = \frac{1}{2}(\sigma \cdot (-i\nabla - a))^2 + V \tag{3.7.4}$$

on $L^2(\mathbb{R}^3; \mathbb{C}^2)$ *Schrödinger operator with spin* $1/2$.

On expanding (3.7.4) by using the anticommutation relations (3.7.3) we obtain

$$H_{\mathrm{S}}(a) = H(a) - \frac{1}{2}\sigma \cdot b, \tag{3.7.5}$$

where

$$b = (b_1, b_2, b_3) = \nabla \times a \tag{3.7.6}$$

is a magnetic field. Let $a \in (C_{\mathrm{b}}^2(\mathbb{R}^3))^3$, $b \in (L^\infty(\mathbb{R}^3))^3$ and $V \in L^\infty(\mathbb{R}^3)$. Then $H_{\mathrm{S}}(a)$ is self-adjoint on $D(-(1/2)\Delta)$ and bounded from below, moreover it is essentially self-adjoint on any core of $-(1/2)\Delta$ as a consequence of the Kato–Rellich theorem.

Now we transform $H_{\mathrm{S}}(a)$ into an operator on the set of $\mathbb{C}$-valued functions on a suitable space. Let $\mathbb{Z}_2$ be the set of the square roots of identity, i.e.,

$$\mathbb{Z}_2 = \{\theta_1, \theta_2\}, \tag{3.7.7}$$

where

$$\theta_\alpha = (-1)^\alpha = \begin{cases} -1, & \alpha = 1, \\ 1, & \alpha = 2. \end{cases} \tag{3.7.8}$$

By the identification

$$L^2(\mathbb{R}^3; \mathbb{C}^2) \ni \begin{pmatrix} f(x, +1) \\ f(x, -1) \end{pmatrix} \mapsto f(x, \theta) \in L^2(\mathbb{R}^3 \times \mathbb{Z}_2; \mathbb{C}) \tag{3.7.9}$$

and (3.7.5), $H_{\mathrm{S}}(a)$ can be reduced to the self-adjoint operator $H_{\mathbb{Z}_2}(a)$ defined in $L^2(\mathbb{R}^3 \times \mathbb{Z}_2)$ to obtain

$$H_{\mathbb{Z}_2}(a) f(x, \theta) = \left(H(a) - \frac{1}{2}\theta b_3(x) \right) f(x, \theta) - \frac{1}{2}(b_1(x) - i\theta b_2(x)) f(x, -\theta), \tag{3.7.10}$$

where $x \in \mathbb{R}^3$ and $\theta \in \mathbb{Z}_2$. Thus $H_{\mathrm{S}}(a)$ can be regarded as the operator $H_{\mathbb{Z}_2}(a)$ on the space of $\mathbb{C}$-valued functions with the configuration space $\mathbb{R}^3 \times \mathbb{Z}_2$.

3.7.2 A jump process

In order to derive a Feynman–Kac formula for $e^{-tH_{\mathbb{Z}_2}(a)}$, in addition to Brownian motion we need a Lévy process counting for the spin. We give this 3-dimensional Lévy process $(X_t)_{t\geq 0}$ on a probability space $(\mathscr{S}, \mathscr{B}_{\mathscr{S}}, P)$ with characteristics (b, A, ν) such that $\nu(\mathbb{R}^3 \setminus \{0\}) = 1$. For $I \in \mathscr{B}(\mathbb{R}^3)$ let

$$N(t, I) = |\{0 \leq s \leq t \mid \Delta X_s \in I\}| \tag{3.7.11}$$

be the counting measure associated with $(X_t)_{t\geq 0}$. Define the random process $(N_t)_{t\geq 0}$ by

$$N_t = N(t, \mathbb{R}^3 \setminus \{0\}). \tag{3.7.12}$$

Note that $(N_t)_{t\geq 0}$ is a Poisson process with intensity 1. Consider the measure

$$dN_t = \int_{\mathbb{R}^3\setminus\{0\}} N(dtdz) \tag{3.7.13}$$

on $\mathbb{R}^+$. The compensator of N_t is given by t and $\mathbb{E}_P[e^{-\alpha N_t}] = e^{t(e^{-\alpha}-1)}$. Since

$$\mathbb{E}_P\left[\int_0^{t+}\int_{\mathbb{R}^3\setminus\{0\}} f(s,\tau,z)N(dsdz)\right] = \mathbb{E}_P\left[\int_0^{t}\int_{\mathbb{R}^3\setminus\{0\}} f(s,\tau,z)ds\nu(dz)\right],$$

we have for $f = f(s, \tau)$, $(s, \tau) \in \mathbb{R}^+ \times \mathscr{S}$, independently of $z \in \mathbb{R}^3 \setminus \{0\}$, that

$$\mathbb{E}_P\left[\int_0^{t+} f(s,\tau)dN_s\right] = \mathbb{E}_P\left[\int_0^{t} f(s,\tau)ds\right]. \tag{3.7.14}$$

For each $\tau \in \mathscr{S}$ there exist $n = n(\tau) \in \mathbb{N}$ and points of discontinuity of $t \mapsto N_t$, $0 < s_1 = s_1(\tau), \ldots, s_n = s_n(\tau) \leq t$, such that

$$\int_0^{t+} f(s, N_s)dN_s = \sum_{j=1}^{n} f(s_j, N_{s_j}) = \sum_{j=1}^{n} f(s_j, j). \tag{3.7.15}$$

Since $\mathbb{E}_P[N_t] = t$ and $P(N_t = N) = t^N e^{-t}/N!$, the expectation of (3.7.15) reduces to Lebesgue integral,

$$\mathbb{E}_P\left[\int_0^{t+} f(s,N_s)dN_s\right] = \mathbb{E}_P\left[\int_0^{t} f(s,N_s)ds\right] = \int_0^t \sum_{n=0}^{\infty} f(s,n)\frac{s^n}{n!}e^{-s}ds.$$

Furthermore, Proposition 2.48 gives

Proposition 3.85. *Suppose $h^i(t) = h^i(t,\tau) \in \mathbb{F}$, $i = 1,\dots,d$, are independent of $z \in \mathbb{R}^d \setminus \{0\}$. Let $X^i = (X^i_t)_{t\geq 0}$ be the semimartingale given by*

$$dX^i_t = f^i \cdot dB_t + g^i dt + h^i dN_t, \quad i = 1,\dots,d,$$

and $F \in C^2(\mathbb{R}^d)$, where $f^i = f^i(t,\omega)$ and $g^i = g^i(t,\omega)$. Then

$$F(X_t) = F(X_0) + \sum_{i=1}^n \int_0^t F_i(X_s)(f^i \cdot dB_s) + \sum_{i=1}^n \int_0^t F_i(X_s) g^i ds$$
$$+ \frac{1}{2}\sum_{i,j=1}^n \int_0^t F_{ij}(X_s)(f^i \cdot f^j) ds + \int_0^{t+} (F(X_{s-} + h(s-)) - F(X_{s-})) dN_s.$$

Here $F_i = \partial_i F$ and $F_{ij} = \partial_i \partial_j F$.

Proof. Note that $\int_0^{t+} h^i(s,\omega) dN_s = \int_0^{t+}\int_{\mathbb{R}^d\setminus\{0\}} h^i(s,\omega) N(dsdz)$ by the definition of dN_s. The proposition is obtained by a direct application of the Itô formula. □

Let $N + \alpha = (N_t + \alpha)_{t\geq 0}$ and $\mathbb{E}^\alpha_P[f(N)] = \mathbb{E}_P[f(N+\alpha)]$. The particle and spin processes together yield the $\mathbb{R}^3 \times \mathbb{Z}_2$-valued random process below.

Definition 3.35. Define the $\mathbb{R}^3 \times \mathbb{Z}_2$-valued stochastic process $(q_t)_{t\geq 0}$ on the probability space $(\mathscr{X} \times \mathcal{S}, \mathcal{B}(\mathscr{X}) \times \mathcal{B}_{\mathcal{S}}, \mathcal{W}^x \times P)$ by

$$q_t = (B_t, \theta_{N_t}) \tag{3.7.16}$$

where $\theta_{N_t} = (-1)^{N_t}$.

For simplicity, we write $\mathbb{E}^{x,\alpha}_{\mathcal{W}\times P} = \mathbb{E}^{x,\alpha}$. Next we compute the generator of the process q_t and derive a version of the Feynman–Kac formula. Let σ_{F} be the fermionic harmonic oscillator defined by

$$\sigma_{\mathrm{F}} = \frac{1}{2}(\sigma_3 + i\sigma_2)(\sigma_3 - i\sigma_2) - \frac{1}{2}. \tag{3.7.17}$$

Notice that actually, $\sigma_{\mathrm{F}} = -\sigma_1$. A direct computation yields

$$(f, e^{-t(-(1/2)\Delta + \varepsilon\sigma_{\mathrm{F}})} g) = \sum_{\alpha=1,2} \int_{\mathbb{R}^3} \mathbb{E}^{x,\alpha}[\overline{f(q_0)} g(q_t) \varepsilon^{N_t}] dx. \tag{3.7.18}$$

Thus setting $\epsilon = 1$, we see that the generator of q_t is $(-1/2)\Delta + \sigma_{\mathrm{F}}$ under the identification $L^2(\mathbb{R}^3 \times \mathbb{Z}_2) \cong L^2(\mathbb{R}^3; \mathbb{C}^2)$.

3.7.3 Feynman–Kac formula for the jump process

In this section we derive the Feynman–Kac formula of $e^{-tH_{\mathbb{Z}_2}(a)}$ by making use of the random process $(q_t)_{t\geq 0}$ defined in (3.7.16).

Theorem 3.86 (Feynman–Kac formula for Schrödinger operator with spin 1/2). *Let $a \in (C_\mathrm{b}^2(\mathbb{R}^3))^3$, $b \in (L^\infty(\mathbb{R}^3))^3$ and $V \in L^\infty(\mathbb{R}^3)$. Suppose that*

$$\int_0^t ds \int_{\mathbb{R}^3} \left| \log \frac{1}{2}\sqrt{b_1(y)^2 + b_2(y)^2} \right| \Pi_s(y-x)dy < \infty \tag{3.7.19}$$

for all $(x,t) \in \mathbb{R}^3 \times \mathbb{R}^+$, the Feynman–Kac-type formula

$$(f, e^{-tH_{\mathbb{Z}_2}(a)}g) = e^t \sum_{\alpha=1,2} \int_{\mathbb{R}^3} \mathbb{E}^{x,\alpha}[\overline{f(q_0)}g(q_t)e^{Z_t}]dx \tag{3.7.20}$$

holds. Here

$$\begin{aligned} Z_t = &-i\int_0^t a(B_s)\circ dB_s - \int_0^t V(B_s)ds \\ &- \int_0^t \left(-\frac{1}{2}\right)\theta_{N_s} b_3(B_s)ds + \int_0^{t+} \log\left(\frac{1}{2}(b_1(B_s) - i\theta_{N_s}b_2(B_s))\right) dN_s. \end{aligned}$$

Before turning to the proof of Theorem 3.86 here is a heuristic argument. Since the diagonal part $-(1/2)\theta b_3(x)$ of $H_{\mathbb{Z}_2}(a)$ acts as an external potential up to the sign $\theta = \pm$, we have formally the integral $\int_0^t(-1/2)\theta_{N_s}b_3(B_s)ds$ in Z_t. This explains why the off-diagonal part $\int_0^t \log\left((1/2)(b_1(B_s) - i\theta_{N_s}b_2(B_s))\right) dN_s$ appears in Z_t. Let

$$W(x,-\theta) = \log\left(\frac{1}{2}(b_1(x) - i\theta b_2(x))\right). \tag{3.7.21}$$

Consider

$$(K_t^S f)(x,\sigma) = e^t \mathbb{E}^{x,\alpha}[f(B_t,\theta_{N_t})e^{\int_0^{t+} W(B_s,-\theta_{N_{s-}})dN_s}].$$

Take first, for simplicity, that W has no zeroes. The generator $-K^S$ of K_t^S can be computed by Itô's formula for Lévy processes yielding

$$d(e^{\int_0^{t+} W(B_s,-\theta_{N_s})dN_s}) = e^{\int_0^{t+} W(B_s,-\theta_{N_{s-}})dN_s}(e^{W(B_t,-\theta_{N_t})} - 1)dN_t. \tag{3.7.22}$$

On the other hand

$$d(e^{-\int_0^t V(B_s)ds}) = e^{-\int_0^t V(B_s)ds}(-V(B_t))dt, \tag{3.7.23}$$

implying that $(e^{-t(-(1/2)\Delta+V)}f)(x) = \mathbb{E}^x[e^{-\int_0^t V(B_s)ds} f(B_t)]$. On comparison of (3.7.22) and (3.7.23) it is clear that the Itô formula gives the differential for continuous processes and the difference for discontinuous ones. From (3.7.22) it follows that the generator K^S of K_t^S is

$$K^S f(x,\theta) = \left(-\frac{1}{2}\Delta - e^{W(x,-\theta)} + 1\right) f(x,-\theta).$$

Thus $e^{-tK^S} f(x,\theta_\alpha) = e^t \mathbb{E}^{B_t,\alpha}[f(B_s,\theta_{N_t}) e^{\int_0^t W(x,-\theta_{N_s-})dN_s}]$ giving rise to the special form of the off-diagonal part. We rewrite this term as

$$-\frac{1}{2}(b_1(x) - i\theta b_2(x)) = -e^{W(x,-\theta)} + 1 - 1.$$

Hence (3.7.21) follows.

Remark 3.7. We prove Proposition 3.86 by making use of the Itô formula. In order that Itô's formula applies, however, the integrand in $\int_0^{t+} \dots dN_s$ must be predictable with respect to the given filtration. θ_{N_s} is right continuous in s for each $(\omega,\tau) \in \mathscr{X} \times \mathscr{S}$, so we define $\theta_{N_{s-}} = \lim_{\varepsilon\uparrow 0} \theta_{N_{s-\varepsilon}}$. Then $\theta_{N_{s-}}$ is left continuous and $W(B_s, -\theta_{N_{s-}})$ is predictable, i.e., $W(B_s,-\theta_{N_{s-}})$ is $\sigma(B_r, N_r; 0 \le r \le s)$ measurable and left continuous in s, for every $(\omega,\tau) \in \mathscr{X} \times \mathscr{S}$. This allows then an application of the Itô formula to $\int_0^{t+} W(B_s,-\theta_{N_{s-}})dN_s$.

Proof of Theorem 3.86. Write $U(B_s,\theta_{N_s}) = -(1/2)\theta_{N_s} b_3(B_s) + V(B_s)$ for the diagonal part. The off-diagonal part $W(B_s,-\theta_{N_{s-}})$ is predictable and we have to check that $|\int_0^{t+} W(B_s,-\theta_{N_{s-}})dN_s|$ is bounded almost surely in order to apply Itô's formula. Indeed,

$$\begin{aligned}
&\left|\mathbb{E}^{x,\alpha}\left[\int_0^{t+} W(B_s,-\theta_{N_{s-}})dN_s\right]\right| \\
&\le \mathbb{E}^{x,\alpha}\left[\int_0^t \left|\log\left(\frac{1}{2}\sqrt{b_1(B_s)^2 + b_2(B_s)^2}\right)\right| dN_s\right] \\
&= \int_0^t ds \int_{\mathbb{R}^3} \Pi_s(y-x) \left|\log\left(\frac{1}{2}\sqrt{b_1(y)^2+b_2(y)^2}\right)\right| dy
\end{aligned}$$

is bounded by the assumption, hence $|\int_0^{t+} W(B_s,-\theta_{N_{s-}})dN_s| < \infty$, almost surely.

Define $K_t^S : L^2(\mathbb{R}^3 \times \mathbb{Z}_2) \to L^2(\mathbb{R}^3 \times \mathbb{Z}_2)$ by

$$K_t^S g(x,\theta_\alpha) = \mathbb{E}^{x,\alpha}[e^{Z_t} g(B_t,\theta_{N_t})].$$

It can be seen that $\|K_t^S g\| \le V_M^{1/2} e^{M't} e^{(M-1)t/2}\|g\|$, where $M' = \sup_{x\in\mathbb{R}^3}|b_3(x)/2|$, $M = \sup_{x\in\mathbb{R}^3}(b_1^2(x) + b_2^2(x))/4$ and $V_M = \sup_{x\in\mathbb{R}^3}\mathbb{E}^x[e^{-2\int_0^t V(B_s)ds}]$ which are finite. Thus K_t^S is bounded. Since Z_t is continuous at $t = 0$ for each $(\omega, \tau) \in \mathscr{X} \times \mathscr{S}$, dominated convergence yields

$$\|K_t^S g - g\| \leq \sum_{\alpha=1,2} \int_{\mathbb{R}^3} \mathbb{E}^{x,\alpha}[|g(x,\theta) - g(B_t, \theta_{N_t}) e^{Z_t}|] dx \to 0$$

as $t \to 0$. Then, since $(B_t)_{t \geq 0}$ and $(N_t)_{t \geq 0}$ are independent Markov processes, using $\sigma(B_r, N_r; 0 \leq r \leq s)$ it follows similarly as in the proof of Theorem 3.65 that $(K_s^S K_t^S g)(x, \theta_\alpha) = K_{s+t}^S g(x, \theta_\alpha)$, i.e., $\{K_t^S : t \geq 0\}$ is a C_0-semigroup. Denote the generator of K_t^S by the closed operator h. We will see below that $K_t^S = e^{-th} = e^{-t(H_{\mathbb{Z}_2}(a)+1)}$. By Proposition 3.85 it follows that

$$\theta_{N_t} - \theta_{N_0} = -2 \int_0^{t+} \theta_{N_{s-}} dN_s,$$

$$\begin{aligned} g(B_t, \theta_{N_t}) - g(x, \theta_{N_0}) &= \int_0^t \nabla g(B_s, \theta_{N_s}) \cdot dB_s + \frac{1}{2} \int_0^t \Delta g(B_s, \theta_{N_s}) ds \\ &\quad + \int_0^{t+} \big(g(B_s, -\theta_{N_{s-}}) - g(B_s, \theta_{N_{s-}}) \big) dN_s, \end{aligned}$$

and

$$\begin{aligned} e^{Z_t} - 1 = &\int_0^t e^{Z_s} (-ia(B_s)) \circ dB_s \\ &+ \int_0^t e^{Z_s} \left(-V(B_s) + \frac{1}{2} (-ia(B_s))^2 - U(B_s, \theta_{N_s}) \right) ds \\ &+ \int_0^{t+} e^{Z_{s-}} (e^{W(B_s, -\theta_{N_{s-}})} - 1) dN_s. \end{aligned}$$

By the product rule, $d(e^{Z_t} g) = de^{Z_t} \cdot g + e^{Z_t} \cdot dg + de^{Z_t} \cdot dg$ and the two identities above we have

$$\begin{aligned} &e^{Z_t} g(B_t, \theta_{N_t}) - g(x, \theta_{N_0}) \\ &= \int_0^t e^{Z_s} \left(\frac{1}{2} \Delta - ia(B_s) \cdot \nabla - \frac{1}{2} a(B_s)^2 - V(B_s) - U(B_s, \theta_{N_s}) \right) g(B_s, \theta_{N_s}) ds \\ &\quad + \int_0^t e^{Z_s} (\nabla g(B_s, \theta_{N_s}) - ia(B_s) \cdot g(B_s, \theta_{N_s})) \cdot dB_s \\ &\quad + \int_0^{t+} e^{Z_{s-}} (g(B_s, -\theta_{N_{s-}}) e^{W(B_s, -\theta_{N_{s-}})} - g(B_s, \theta_{N_{s-}})) dN_s. \end{aligned}$$

Take expectation on both sides above. The martingale part vanishes and by (3.7.14) we obtain that

$$\mathbb{E}^{x,\alpha}[e^{Z_t} g(B_t, \theta_{N_t}) - g(x, \theta_\alpha)] = \int_0^t \mathbb{E}^{x,\alpha}[G(s)] ds,$$

where

$$\begin{aligned}G(s) = e^{Z_s}\Big(&\frac{1}{2}\Delta - ia(B_s)\cdot\nabla - i\frac{1}{2}\nabla\cdot a(B_s)\\&- \frac{1}{2}a(B_s)^2 - V(B_s) - U(B_s,\theta_{N_s})\Big)g(B_s,\theta_{N_s})\\&+ e^{Z_{s-}}(g(B_s,-\theta_{N_{s-}})e^{W(B_s,-\theta_{N_{s-}})} - g(B_s,\theta_{N_{s-}})),\end{aligned}$$

with $s > 0$, and

$$\begin{aligned}G(0) &= \Big(\frac{1}{2}\Delta - ia(x)\cdot\nabla - i\frac{1}{2}\nabla\cdot a(x) - \frac{1}{2}a(x)^2 - V(x) - U(x,\theta_\alpha) - 1\Big)g(x,\theta_\alpha)\\&\quad + e^{W(x,-\theta_\alpha)}g(x,-\theta_\alpha)\\&= -(H_{\mathrm{S}}(a)+1)g(x,\theta_\alpha).\end{aligned}$$

$G(s)$ is continuous at $s = 0$ for each $(\omega,\tau)\in\mathscr{X}\times\mathscr{S}$, whence

$$\begin{aligned}\lim_{t\to 0}\frac{1}{t}(f,(K_t^S - 1)g) &= \lim_{t\to 0}\frac{1}{t}\int_0^t ds\sum_{\alpha=1,2}\int_{\mathbb{R}^3} dx\,\overline{f(x,\theta_\alpha)}\mathbb{E}^{x,\alpha}[G(s)]\\&= (f,-(H_{\mathbb{Z}_2}(a)+1)g).\end{aligned}$$

Since $C_0^\infty(\mathbb{R}^3\times\mathbb{Z}_2)$ is a core of $H_{\mathbb{Z}_2}(a)$, (3.7.20) follows. □

3.7.4 Extension to singular potentials and vector potentials

Next we extend Theorem 3.86 to singular vector potentials a and external potentials V as in the spinless case, following a similar strategy as in Section 3.5.3.

Define the quadratic form

$$q_a^{\mathrm{S}}(f,g) = (\sigma_\mu D_\mu f, \sigma_\mu D_\mu g) + (V^{1/2}f, V^{1/2}g) \tag{3.7.24}$$

with domain $Q(q_a^{\mathrm{S}}) = \bigcap_{\mu=1}^3\{f\in L^2(\mathbb{R}^3)|D_\mu f\in L^2(\mathbb{R}^3)\}\cap D(V^{1/2})$.

Lemma 3.87. *Let $V\geq 0$ be $L^1_{\mathrm{loc}}(\mathbb{R}^3)$ and $a\in(L^2_{\mathrm{loc}}(\mathbb{R}^3))^3$. Then q_a^{S} is a closed symmetric form.*

The proof of this is very similar to Lemma 3.68. From here it follows that there exists a self-adjoint operator h_S such that

$$(f, h_S g) = q_a^S(f,g),\quad f\in Q(q_a^S),\ g\in D(h_S). \tag{3.7.25}$$

A sufficient condition for $C_0^\infty(\mathbb{R}^3)$ to be a core of h_S is also derived in a similar way to the spinless case, see Proposition 3.69.

Proposition 3.88.

(1) *Let* $V \in L^1_{\rm loc}(\mathbb{R}^3)$ *with* $V \geq 0$. *If* $a \in (L^2_{\rm loc}(\mathbb{R}^3))^3$, *then* $C_0^\infty(\mathbb{R}^3)$ *is a form core of* h_S.

(2) *Let* $V \in L^2_{\rm loc}(\mathbb{R}^3)$ *with* $V \geq 0$. *If* $a \in (L^4_{\rm loc}(\mathbb{R}^3))^3$, $\nabla \cdot a \in L^2_{\rm loc}(\mathbb{R}^3)$ *and* $b \in (L^2_{\rm loc}(\mathbb{R}^3))^3$, *then* $C_0^\infty(\mathbb{R}^3)$ *is an operator core of* h_S.

In case (2) above the operator h_S can be realized as

$$h_S f = -\frac{1}{2}\Delta f - a \cdot (-i\nabla) f + \left(-\frac{1}{2}a \cdot a - (-i\nabla \cdot a) - \frac{1}{2}\sigma \cdot b\right) f + Vf. \tag{3.7.26}$$

Instead of h_S we define the Schrödinger operator with spin by the sum of the kinetic term and spin.

Assumption 3.1. The magnetic field $-\frac{1}{2}b_j$, $j = 1, 2, 3$, is bounded.

Let $H^0(a)$ be given by Definition 3.27 with $V = 0$. Under Assumption 3.1 the operator $H^0(a) - \frac{1}{2}\sigma \cdot b$ is self-adjoint on $D(H^0(a))$. We give the definition of $H_{\rm S}(a, b)$ below.

Definition 3.36 (Schrödinger operator with spin, singular potential and vector potential). Assume that $a \in (L^2_{\rm loc}(\mathbb{R}^3))^3$, $0 \leq V \in L^1_{\rm loc}(\mathbb{R}^3)$, and that the magnetic field b satisfies Assumption 3.1. We regard b to be independent of a, and denote the self-adjoint operator $H^0(a) \dot{+} V - \frac{1}{2}\sigma \cdot b$ by $H_{\rm S}(a, b)$. We call $H_{\rm S}(a, b)$ *Schrödinger operator with spin, singular potential and vector potential.*

Let $H_{\mathbb{Z}_2}(a, b)$ be the unitary transformed operator to $L^2(\mathbb{R}^3 \times \mathbb{Z}_2)$ of $H_{\rm S}(a, b)$, and let $H^0_{\mathbb{Z}_2}(a, b)$ denote $H_{\mathbb{Z}_2}(a, b)$ with $V = 0$.

Next consider general vector potentials having possible zeroes. Note that (3.7.19) is a sufficient condition making sure that

$$\int_0^{t+} |W(B_s, -\theta_{N_{s-}})| dN_s < \infty, \quad \text{a.e. } (\omega, \tau) \in \mathscr{X} \times \mathscr{S}. \tag{3.7.27}$$

When, however, $b_1(x) - i\theta b_2(x)$ vanishes for some (x, θ), (3.7.27) is not clear. This case is relevant and Theorem 3.86 must be improved since we have to construct the path integral representation of $e^{-tH_{\mathbb{Z}_2}(a,b)}$ in which the off-diagonal part $b_1 - i\theta b_2$ of $H_{\mathbb{Z}_2}(a, b)$ has zeroes or a compact support. The off-diagonal part of $H_{\rm S}(a, b)$, however, in general may have zeroes. For instance, a_μ for all $\mu = 1, 2, 3$ have a compact support, and so does the off-diagonal part of $b = \nabla \times a$. Therefore, in order to avoid that the off-diagonal part vanishes, we introduce

$$H^{\varepsilon}_{\mathbb{Z}_2}(a,b)f(x,\theta) = \left(H(a,b) - \frac{1}{2}\theta b_3(x)\right) f(x,\theta) \\ - \Psi_\varepsilon\left(\frac{1}{2}(b_1(x) - i\theta b_2(x))\right) f(x,-\theta), \tag{3.7.28}$$

where

$$\Psi_\varepsilon(z) = z + \varepsilon\psi_\varepsilon(z) \tag{3.7.29}$$

for $z \in \mathbb{C}$ and $\varepsilon > 0$, with the indicator function

$$\psi_\varepsilon(z) = \begin{cases} 1, & |z| < \varepsilon/2, \\ 0, & |z| \geq \varepsilon/2. \end{cases} \tag{3.7.30}$$

Thus

$$\left|\Psi_\varepsilon\left(\frac{1}{2}(b_1 - i\theta b_2)\right)\right| > \varepsilon/2 > 0.$$

Theorem 3.89 (Feynman–Kac formula for Schrödinger operator with spin, singular external potential and vector potential). *Suppose that* $V \in L^\infty(\mathbb{R}^3)$, $a \in (L^2_{\mathrm{loc}}(\mathbb{R}^3))^3$, $\nabla \cdot a \in L^1_{\mathrm{loc}}(\mathbb{R}^3)$, *Assumption* 3.1 *and*

$$\int_0^t ds \int_{\mathbb{R}^3} \left|\log\left(\frac{1}{2}\sqrt{b_1(y)^2 + b_2(y)^2} + \varepsilon\right)\right| \Pi_s(x-y)dy < \infty \tag{3.7.31}$$

for all $(x,t) \in \mathbb{R}^3 \times \mathbb{R}^+$ *and* $\varepsilon > 0$. *Then*

$$(f, e^{-tH^{\varepsilon}_{\mathbb{Z}_2}(a,b)}g) = e^t \sum_{\alpha=1,2} \int_{\mathbb{R}^3} \mathbb{E}^{x,\alpha}[\overline{f(q_0)}g(q_t)e^{Z^{\varepsilon}_t}]dx, \tag{3.7.32}$$

and

$$(f, e^{-tH_{\mathbb{Z}_2}(a,b)}g) = \lim_{\varepsilon\to 0} e^t \sum_{\alpha=1,2} \int_{\mathbb{R}^3} \mathbb{E}^{x,\alpha}[\overline{f(q_0)}g(q_t)e^{Z^{\varepsilon}_t}]dx, \tag{3.7.33}$$

where

$$Z^{\varepsilon}_t = -i\int_0^t a(B_s)\circ dB_s - \int_0^t V(B_s)ds \\ - \int_0^t \left(-\frac{1}{2}\right)\theta_{N_s} b_3(B_s)ds + \int_0^{t+} \log\left(\Psi_\varepsilon\left(\frac{1}{2}(b_1(B_s) - i\theta_{N_s}b_2(B_s))\right)\right)dN_s.$$

Proof. We show this theorem through a similar limiting argument to Lemma 3.71. Suppose first that $a^{(n)} \in (C_0^\infty(\mathbb{R}^3))^3$, $n = 1, 2, \ldots$, such that $a_\mu^{(n)} \to a_\mu$ in $L^2_{\mathrm{loc}}(\mathbb{R}^3)$ as $n \to \infty$. Take $1_R(x) = \chi(x_1/R)\chi(x_2/R)\chi(x_3/R)$, $R > 0$, where $\chi \in C_0^\infty(\mathbb{R})$ such that $0 \le \chi \le 1$, $\chi(x) = 1$ for $|x| < 1$ and $\chi(x) = 0$ for $|x| \ge 2$. In the same way as in Lemma 3.70 we can show that

$$e^{-tH^\varepsilon_{\mathbb{Z}_2}(1_R a^{(n)}, b)} \to e^{-tH^\varepsilon_{\mathbb{Z}_2}(1_R a, b)} \quad \text{as } n \to \infty, \tag{3.7.34}$$

$$e^{-tH^\varepsilon_{\mathbb{Z}_2}(1_R a, b)} \to e^{-tH^\varepsilon_{\mathbb{Z}_2}(a, b)} \quad \text{as } R \to \infty \tag{3.7.35}$$

in the strong sense. In the proof of Lemma 3.71 we have already shown that there exists a subsequence n' such that $\int_0^t 1_R a^{(n')}(B_s) \circ dB_s \to \int_0^t 1_R a(B_s) \circ dB_s$ almost surely as $n' \to \infty$, and R' such that $\int_0^t 1_{R'} a(B_s) \circ dB_s \to \int_0^t a(B_s) \circ dB_s$. We reset n' and R' as n and R, respectively. Let $Z_t^\varepsilon(n, R)$ be Z_t with a replaced by $1_R a^{(n)}$. Hence we conclude that

$$\lim_{R\to\infty} \lim_{n\to\infty} \int_{\mathbb{R}^3} \mathbb{E}^{x,\alpha}[\overline{f(q_0)} g(q_t) e^{Z_t^\varepsilon(n,R)}] dx = \int_{\mathbb{R}^3} \mathbb{E}^{x,\alpha}[\overline{f(q_0)} g(q_t) e^{Z_t^\varepsilon}] dx.$$

Combining this with (3.7.34) and (3.7.35), we obtain (3.7.32). Since $e^{-tH^\varepsilon_{\mathbb{Z}_2}(a,b)}$ is strongly convergent to $e^{-tH_{\mathbb{Z}_2}(a,b)}$ as $\varepsilon \to 0$, (3.7.33) also follows. □

In case when the off-diagonal part identically vanishes, i.e. $b = (0, 0, b_3)$, we have

$$\Psi_\varepsilon\left(\frac{1}{2}(b_1(x) - i\theta_{N_s} b_2(x))\right) = \varepsilon$$

and

$$\int_0^{t+} \log\left(\Psi_\varepsilon\left(\frac{1}{2}(b_1(B_s) - i\theta_{N_s} b_2(B_s))\right)\right) dN_s = \log \varepsilon^{N_t}.$$

Then it can be seen that

$$\lim_{\varepsilon\downarrow 0} \mathbb{E}^x[e^{Z_t^\varepsilon} g(q_t)] = \lim_{\varepsilon\downarrow 0} \mathbb{E}^x[e^{-i\int_0^t a(B_s)\circ dB_s - \int_0^t V(B_s)ds - \int_0^t (-\frac{1}{2})\theta_{N_s} b_3(B_s) ds} \varepsilon^{N_t} g(q_t)].$$

Notice that as $\varepsilon \to 0$ the functions on $\mathscr{K}_t = \{\tau \in \mathscr{S} \mid N_t(\tau) \ge 1\}$ vanish and those on $\mathscr{K}_t^c = \{\tau \in \mathscr{S} \mid N_t(\tau) = 0\}$ stay different from zero. Note also that for $\tau \in \mathscr{K}_t^c$, $N_s(\tau) = 0$ whenever $0 \le s \le t$, as N_t is counting measure. Since

$$\lim_{\varepsilon\downarrow 0} \varepsilon^{N_t(\tau)} = \begin{cases} 0, & \tau \in \mathscr{K}_t, \\ 1, & \tau \in \mathscr{K}_t^c, \end{cases}$$

we obtain that

$$\begin{aligned} &\lim_{\varepsilon\downarrow 0} e^t \mathbb{E}^{x,\alpha}[e^{Z_t^\varepsilon} g(q_t)] \\ &= e^t \mathbb{E}^x[e^{-i\int_0^t a(B_s)\circ dB_s - \int_0^t V(B_s)ds - \int_0^t (-\frac{1}{2})\theta_\alpha b_3(B_s) ds} g(B_t, \theta_\alpha)]. \end{aligned}$$

Clearly, the right-hand side in the expression above describes the diagonal operator.

By Lemma 3.71 we have a diamagnetic inequality for $H_{\mathbb{Z}_2}(a,b)$.

Corollary 3.90 (Diamagnetic inequality). *Under the assumptions of Lemma* 3.88,

$$|(f, e^{-tH_{\mathbb{Z}_2}(a,b)}g)| \le (|f|, e^{-tH_{\mathbb{Z}_2}(0,b_0)}|g|), \tag{3.7.36}$$

where $b_0 = (\sqrt{b_1^2 + b_2^2}, 0, b_3)$.

Proof. This follows from the fact that $|\Psi_\varepsilon(z)| \le |z| + \varepsilon$ for all $z \in \mathbb{C}$,

$$|e^{\int_0^{t+} \log \Psi_\varepsilon(\frac{1}{2}(b_1(B_s) - i\theta_{N_s} b_2(B_s)))dN_s}| \le e^{\int_0^{t+} \log(\frac{1}{2}\sqrt{b_1^2(B_s) + b_2(B_s)^2} + \varepsilon)dN_s}$$

and

$$|(f, e^{-tH_{\mathbb{Z}_2}(a,b)}g)| \le \lim_{\varepsilon \downarrow 0} \sum_{\alpha=1,2} \int_{\mathbb{R}^3} \mathbb{E}^{x,\alpha}[|f(q_0)|g(q_t)|e^{\hat{Z}_t^\varepsilon}]dx,$$

where the exponent $\hat{Z}_t^\varepsilon$ is defined by

$$\begin{aligned} &-\int_0^t V(B_s)ds - \int_0^t \left(-\frac{1}{2}\right) \theta_{N_s} b_3(B_s) ds \\ &+ \int_0^{t+} \log\left(\frac{1}{2}\sqrt{b_1(B_s)^2 + b_2(B_s)^2} + \varepsilon\right) dN_s. \end{aligned}$$

□

Theorem 3.91. *Suppose that* $a \in (L^2_{\mathrm{loc}}(\mathbb{R}^3))^3$, $\nabla \cdot a \in L^1_{\mathrm{loc}}(\mathbb{R}^3)$, *and b satisfies Assumption* 3.1 *and* (3.7.31).

(1) *Let V be a real-valued multiplication operator relatively bounded (resp. form bounded) with respect to $-(1/2)\Delta$ with a relative bound c. Then V is also relatively bounded (resp. form bounded) with respect to $H^0_{\mathbb{Z}_2}(a,b)$ with a relative bound not exceeding c.*

(2) *Let V be such that $V_+ \in L^1_{\mathrm{loc}}(\mathbb{R}^3)$ and V_- is relatively form bounded with respect to $-(1/2)\Delta$ with a relative bound less than* 1. *Then*

$$H^0_{\mathbb{Z}_2}(a,b) \dotplus V_+ \dot{-} V_- \tag{3.7.37}$$

can be defined as a self-adjoint operator. This also will be denoted by $H_{\mathbb{Z}_2}(a,b)$.

(3) *Let V satisfy the same assumptions as in* (2). *Then the Feynman–Kac formula of $(f, e^{-tH_{\mathbb{Z}_2}(a,b)}g)$ is given by* (3.7.32) *and* (3.7.33).

Proof. By the diamagnetic inequality

$$\frac{\||W|^\alpha (H^0_{\mathbb{Z}_2}(a,b) + E)^{-\alpha} f\|}{\|f\|} \le \frac{\||W|^\alpha (H^0_{\mathbb{Z}_2}(0,b_0) + E)^{-\alpha} f\|}{\|f\|} \tag{3.7.38}$$

for $\alpha = 1$ and $1/2$. (1) follows by the same argument as Lemma 3.72 and the boundedness of the magnetic field b_j. (2) is implied by the KLMN theorem. Thus by Theorem 3.86 and the same limiting argument as in Theorem 3.31 also (3) follows. □

3.8 Feynman–Kac formula for relativistic Schrödinger operator with spin

By the methods developed above it is possible to derive a Feynman–Kac formula also for relativistic Schrödinger operators with spin $1/2$. This operator is formally given by

$$\sqrt{(-i\nabla - a)^2 - \sigma \cdot b + m^2} - m + V. \tag{3.8.1}$$

We have seen that $C_0^\infty(\mathbb{R}^3)$ is a form core of $(\sigma\cdot(-i\nabla-a))^2$ under suitable conditions on a in the previous section. Here we have the following counterpart to this, similarly proven as Proposition 3.77.

Proposition 3.92. *Let $a \in (L^2_{\mathrm{loc}}(\mathbb{R}^3))^3$ and suppose that b satisfies Assumption* 3.1. *Let*

$$H^0_{\mathrm{RS}}(a) = (2H^0_{\mathrm{S}}(a,b) + m^2)^{1/2} - m. \tag{3.8.2}$$

Then (1) *und* (2) *below follow.*

(1) *$C_0^\infty(\mathbb{R}^3)$ is a form core of $H^0_{\mathrm{RS}}(a)$.*

(2) *Suppose that $a \in (L^4_{\mathrm{loc}}(\mathbb{R}^3))^3$ and $\nabla \cdot a \in L^2_{\mathrm{loc}}(\mathbb{R}^3)$. Then $C_0^\infty(\mathbb{R}^3)$ is an operator core of $H^0_{\mathrm{RS}}(a)$.*

In particular, from Proposition 3.92 it follows that $Q(H^0_{\mathrm{RS}}(a)) \supset C_0^\infty(\mathbb{R}^3)$ if $a \in (L^2_{\mathrm{loc}}(\mathbb{R}^3))^3$. Thus $Q(V) \cap Q(H^0_{\mathrm{RS}}(a))$ is dense whenever $V \in L^1_{\mathrm{loc}}(\mathbb{R}^3)$, and then the form sum of $H^0_{\mathrm{RS}}(a)$ and V can be densely defined.

Definition 3.37 (Relativistic Schrödinger operator with spin $1/2$). Let $V \geq 0$ be in $L^1_{\mathrm{loc}}(\mathbb{R}^3)$, $a \in (L^2_{\mathrm{loc}}(\mathbb{R}^3))^3$, and Assumption 3.1 hold. We call

$$H_{\mathrm{RS}}(a,b) = (2H^0_{\mathrm{S}}(a,b) + m^2)^{1/2} - m \dotplus V \tag{3.8.3}$$

relativistic Schrödinger operator with spin $1/2$, where m is assumed to be sufficiently large such that $2H^0_{\mathrm{S}}(a,b) + m^2 \geq 0$.

As it was done for $H_{\mathrm{S}}(a,b)$, we transform $H_{\mathrm{RS}}(a,b)$ by a unitary map to the self-adjoint operator

$$H_{\mathrm{R}\mathbb{Z}_2}(a,b) = (2H^0_{\mathbb{Z}_2}(a,b) + m^2)^{1/2} - m \dotplus V \tag{3.8.4}$$

on $L^2(\mathbb{R}^3 \times \mathbb{Z}_2)$. In order to derive the Feynman–Kac formula for $e^{-tH_{\mathrm{R}\mathbb{Z}_2}(a,b)}$ we need the random process

$$(\tilde{q}_{T_t})_{t\geq 0} = (B_{T_t}, \theta_{T_t})_{t\geq 0} \tag{3.8.5}$$

on $\mathscr{X} \times \mathscr{S} \times \mathscr{T}$ under the probability measure $\mathcal{W}^x \times P \times \nu$, where $(T_t)_{t\geq 0}$ is the subordinator given by $\mathbb{E}_\nu[e^{-uT_t}] = e^{-t(\sqrt{2u+m^2}-m)}$. Write $\mathbb{E}^{x,\alpha,0} = \mathbb{E}^{x,\alpha,0}_{\mathcal{W}\times P\times\nu}$, and let $\rho(\cdot,t)$ be the distribution of T_t on $\mathbb{R}^+$, one of the rare cases which can be described explicitly by a formula:

$$\rho(r,t) = \frac{t}{(2\pi r^3)^{1/2}} \exp\left(-\frac{1}{2}\left(\frac{t^2}{r} + m^2 r\right) + mt\right), \quad r \in \mathbb{R}^+. \tag{3.8.6}$$

To avoid zeroes of the off-diagonal part of the spin we introduce the cutoff function Ψ_ε as defined in (3.7.29). We write

$$H^\varepsilon_{\mathrm{R}\mathbb{Z}_2}(a,b) = (2H^{0,\varepsilon}_{\mathbb{Z}_2}(a,b) + m^2)^{1/2} - m \dotplus V,$$

where $H^{0,\varepsilon}_{\mathbb{Z}_2}(a,b)$ is defined by $H^0_{\mathbb{Z}_2}(a,b)$ with the off-diagonal part $(1/2)(b_1 - i\theta b_2)$ of the spin replaced by $\Psi_\varepsilon((1/2)(b_1 - i\theta b_2))$.

Theorem 3.93 (Feynman–Kac formula for relativistic Schrödinger operator with spin 1/2). *Let $V \geq 0$ be in $L^1_{\mathrm{loc}}(\mathbb{R}^3)$, $a \in (L^2_{\mathrm{loc}}(\mathbb{R}^3))^3$ and $\nabla \cdot a \in L^1_{\mathrm{loc}}(\mathbb{R}^3)$. Suppose that b satisfies Assumption 3.1 and*

$$\int_0^\infty dr\rho(r,t)\int_0^r ds \int_{\mathbb{R}^3} dy \left|\log\left(\frac{1}{2}\sqrt{b_1(y)^2 + b_2(y)^2} + \varepsilon\right)\right| \Pi_s(y-x) < \infty \tag{3.8.7}$$

for all $\varepsilon > 0$ and all $(x,t) \in \mathbb{R}^3 \times \mathbb{R}^+$. Then

$$(f, e^{-tH^\varepsilon_{\mathrm{R}\mathbb{Z}_2}(a,b)} g) = \sum_{\alpha=1,2} \int_{\mathbb{R}^3} \mathbb{E}^{x,\alpha,0}[e^{T_t} \overline{f(\tilde{q}_0)} g(\tilde{q}_{T_t}) e^{\tilde{Z}^\varepsilon_t}] dx \tag{3.8.8}$$

and

$$(f, e^{-tH_{\mathrm{R}\mathbb{Z}_2}(a,b)} g) = \lim_{\varepsilon\downarrow 0} \sum_{\alpha=1,2} \int_{\mathbb{R}^3} \mathbb{E}^{x,\alpha,0}[e^{T_t} \overline{f(\tilde{q}_0)} g(\tilde{q}_{T_t}) e^{\tilde{Z}^\varepsilon_t}] dx, \tag{3.8.9}$$

where

$$\begin{aligned}\tilde{Z}^\varepsilon_t = &-i\int_0^{T_t} a(B_s)\circ dB_s - \int_0^t V(B_{T_s})ds - \int_0^{T_t}\left(-\frac{1}{2}\right)\theta_{N_s} b_3(B_s)ds \\ &+ \int_0^{T_t+} \log\left(\Psi_\varepsilon\left(\frac{1}{2}(b_1(B_s) - i\theta_{N_{s-}} b_2(B_s))\right)\right) dN_s.\end{aligned}$$

Proof. The proof is a combination of the formulae already established for $H_{\mathrm{R}}(a)$ and $H_{\mathbb{Z}_2}(a,b)$. We only give an outline. Note that

$$\begin{aligned}
&\mathbb{E}^{x,\alpha,0}\Big[\Big|\int_0^{T_t+}\log\Big(\Psi_\varepsilon\Big(\frac{1}{2}(b_1(B_s)-i\theta_{N_{s-}}b_2(B_s))\Big)\Big)dN_s\Big|\Big]\\
&\le \mathbb{E}_\nu\Big[\int_0^{T_t}ds\int_{\mathbb{R}^3}dy\Pi_s(y-x)\Big|\log\Big(\frac{1}{2}\sqrt{b_1(y)^2+b_2(y)^2}+\varepsilon\Big)\Big|\Big]\\
&=\int_0^\infty dr\rho(r,t)\int_0^r ds\int_{\mathbb{R}^3}dy\Pi_s(y-x)\Big|\log\Big(\frac{1}{2}\sqrt{b_1(y)^2+b_2(y)^2}+\varepsilon\Big)\Big|
\end{aligned}$$

which is bounded by (3.8.7). Thus $\int_0^{T_t+}\log(\Psi_\varepsilon(\frac{1}{2}(b_1(B_s)-i\theta_{N_{s-}}b_2(B_s))))dN_s$ is almost surely finite for every (x,α).

First consider $V=0$. The Feynman–Kac formula for

$$(f,e^{-tH^{0,\varepsilon}_{\mathrm{R}\mathbb{Z}_2}(a,b)}g)=\mathbb{E}_\nu[(f,e^{-T_tH^{0,\varepsilon}_{\mathbb{Z}_2}(a,b)}g)]$$

is given by Theorem 3.89, namely the time t in the Feynman–Kac formula of $(f,e^{-tH^{0,\varepsilon}_{\mathbb{Z}_2}(a,b)}g)$ is replaced by a random time given by the subordinator T_t, and it is integrated against ν, the measure associated with $(T_t)_{t\ge 0}$. Next, for a continuous bounded V, using the Trotter product formula

$$\lim_{n\to\infty}(e^{-(t/n)H^{0,\varepsilon}_{\mathrm{R}\mathbb{Z}_2}(a,b)}e^{-(t/n)V})^n=e^{-tH^{\varepsilon}_{\mathrm{R}\mathbb{Z}_2}(a,b)},$$

we arrive at (3.8.8). A limiting arguments on V and for $\varepsilon\to 0$ give then (3.8.9). □

Remark 3.8. Recall that $\rho(r,t)$ is the distribution of T_t. Then clearly

$$\begin{aligned}
&\int_0^\infty dr\rho(r,t)\int_0^r ds\int_{\mathbb{R}^3}dy\Big|\log\Big(\frac{1}{2}\sqrt{b_1(y)^2+b_2(y)^2}+\varepsilon\Big)\Big|\Pi_s(y-x)\\
&\le\int_0^\infty ds\int_{\mathbb{R}^3}dy\Big|\log\Big(\frac{1}{2}\sqrt{b_1(y)^2+b_2(y)^2}+\varepsilon\Big)\Big|\Pi_s(y-x). \qquad (3.8.10)
\end{aligned}$$

Since the right-hand side of (3.8.10) bounded, this is sufficient for condition (3.8.7) to hold.

From the above Feynman–Kac formula we can derive a diamagnetic inequality for $H_{\mathrm{R}\mathbb{Z}_2}(a,b)$, however, this expression is not as simple as in the previous cases. In the previous section we obtained the diamagnetic inequality $|(f,e^{-tH^0_{\mathbb{Z}_2}(a,b)}g)|\le(|f|,e^{-tH^0_{\mathbb{Z}_2}(0,b_0)}|g|)$, with $b_0=(\sqrt{b_1^2+b_2^2},0,b_3)$. Note that $H^0_{\mathbb{Z}_2}(0,b_0)$ can be negative, however, in virtue of the diamagnetic inequality $\inf\mathrm{Spec}(H^0_{\mathbb{Z}_2}(0,b_0))\le\inf\mathrm{Spec}(H^0_{\mathbb{Z}_2}(a,b))$. The difficulty here is that the formula $(f,e^{-t\sqrt{2H+m^2}-m}g)=$

$\mathbb{E}_\nu[(f, e^{-T_t H} g)]$, $m \geq 0$, only holds for positive self-adjoint operators H. To derive the diamagnetic inequality for $H_{\mathrm{R}\mathbb{Z}_2}(a,b)$ we define

$$\tilde{H}_{\mathrm{R}\mathbb{Z}_2}(a,b) = \sqrt{2(H^0_{\mathbb{Z}_2}(a,b) - E) + m^2} - m \dotplus V, \tag{3.8.11}$$

where $E = \inf \mathrm{Spec}\left(H^0_{\mathbb{Z}_2}(0,b_0)\right)$ and $H^0_{\mathbb{Z}_2}(a,b) - E \geq 0$ follows. Thus

$$\tilde{H}_{\mathrm{R}\mathbb{Z}_2}(0,b_0) = \sqrt{2(H_{\mathbb{Z}_2}(0,b_0) - E) + m^2} - m \dotplus V. \tag{3.8.12}$$

From the Feynman–Kac formula in Theorem 3.93 we immediately derive

Corollary 3.94 (Diamagnetic inequality). *Under the assumptions of Theorem* 3.93

$$|(f, e^{-t\tilde{H}_{\mathrm{R}\mathbb{Z}_2}(a,b)} g)| \leq (|f|, e^{-t\tilde{H}_{\mathrm{R}\mathbb{Z}_2}(0,b_0)}|g|) \tag{3.8.13}$$

holds.

Using this inequality we can give a similar results to Theorem 3.91.

Theorem 3.95. *Suppose that $a \in (L^2_{\mathrm{loc}}(\mathbb{R}^3))^3$, $\nabla \cdot a \in L^1_{\mathrm{loc}}(\mathbb{R}^3)$. Let b satisfy Assumption* 3.1 *and* (3.8.7).

(1) *Let V be a real-valued multiplication operator relatively bounded (resp. form bounded) with respect to $\sqrt{-\Delta + m^2}$ with a relative bound c. Then V is also relatively bounded (resp. form bounded) with respect to $H^0_{\mathrm{RS}}(a,b)$ with a relative bound not exceeding c.*

(2) *Let V be such that $V_+ \in L^1_{\mathrm{loc}}(\mathbb{R}^3)$ and V_- is relatively form bounded with respect to $\sqrt{-\Delta + m^2}$ with a relative bound less than* 1. *Then*

$$H^0_{\mathrm{RS}}(a,b) \dotplus V_+ \dot{-} V_- \tag{3.8.14}$$

can be defined as a self-adjoint operator. This also will be denoted by $H_{\mathrm{RS}}(a,b)$.

(3) *Let V satisfy the same assumptions as in* (2). *Then the Feynman–Kac formula of $(f, e^{-tH_{\mathrm{RS}}(a,b)} g)$ is given by* (3.8.8) *and* (3.8.9).

Proof. Set $S = -(1/2)\sigma \cdot b_0$ and $E = \inf \mathrm{Spec}(H^0_{\mathrm{S}}(0,b_0))$. Notice that

$$\|\sqrt{-\Delta + m^2} f\|^2 = \|\sqrt{2H^0_{\mathrm{S}}(0,b_0) - 2E + m^2} f\|^2 + (f, -2(S - E) f).$$

Since $|(f, 2Sf)| \leq \kappa' \|f\|^2$ with a constant κ', we have

$$\|\sqrt{-\Delta + m^2} f\|^2 \leq \|\sqrt{2H^0_{\mathrm{S}}(0,b_0) - 2E + m^2} f\|^2 + (|2E| + \kappa')\|f\|^2.$$

Together with $\|Vf\| \le c\|\sqrt{-\Delta + m^2} f\| + \varepsilon' \|f\|^2$ with some constant ε', we have

$$\|Vf\| \le c\|\sqrt{2H_{\mathrm{S}}^0(0, b_0) - 2E + m^2} f\| + C\|f\| \tag{3.8.15}$$

with some constant C. By (3.8.15) and the diamagnetic inequality in Corollary 3.94, V is also relatively bounded with respect to $\sqrt{2H_{\mathrm{S}}^0(a,b) + m^2 - 2E}$ with the same relative bound c. Moreover, since the difference

$$T = \sqrt{2H_{\mathrm{S}}^0(a,b) + m^2 - 2E} - \sqrt{2H_{\mathrm{S}}^0(a,b) + m^2}$$

is a bounded operator with $\|T\| \le \sqrt{|2E|}$, V is also relatively bounded with respect to $\sqrt{2H_{\mathrm{S}}^0(a,b) + m^2}$ with relative bound c. Thus (1) follows. The form version is similarly proven. The proofs of (2) and (3) are the same as Theorem 3.91. □

The essential self-adjointness of $\sqrt{(\sigma \cdot (-i\nabla - a))^2 + m^2} - m + V$ on $\mathbb{C}^2 \otimes C_0^\infty(\mathbb{R}^3)$ can also be obtained from Proposition 3.88 and (1) of Theorem 3.95.

Theorem 3.96 (Essential self-adjointness of relativistic Schrödinger operator with spin). *Let V be relatively bounded with respect to $\sqrt{-\Delta + m^2}$ with a relative bound < 1. Suppose that $a \in (L^4_{\mathrm{loc}}(\mathbb{R}^3))^3$, $\nabla \cdot a \in L^2_{\mathrm{loc}}(\mathbb{R}^3)$, and $\nabla \times a = b$ satisfies Assumption* 3.1 *and* (3.8.7). *Then $\sqrt{(\sigma \cdot (-i\nabla - a))^2 + m^2} - m + V$ is essentially self-adjoint on $\mathbb{C}^2 \otimes C_0^\infty(\mathbb{R}^3)$.*

Proof. By Proposition 3.92, $\sqrt{(\sigma \cdot (-i\nabla - a))^2 + m^2}$ is essentially self-adjoint on $\mathbb{C}^2 \otimes C_0^\infty(\mathbb{R}^3)$. Note that $\sqrt{(\sigma \cdot (-i\nabla - a))^2 + m^2} - m = H_{\mathrm{RS}}^0(a,b)$, since $(\sigma \cdot (-i\nabla - a))^2$ can be expanded and equals $H_{\mathrm{S}}^0(a,b)$. Then (1) of Theorem 3.95 yields that V is relatively bounded with respect to $\sqrt{(\sigma \cdot (-i\nabla - a))^2 + m^2}$ with a relative bound < 1. Hence the theorem follows by the Kato–Rellich theorem. □

3.9 Feynman–Kac formula for unbounded semigroups and Stark effect

Throughout the previous sections the Schrödinger operator H was required to be bounded from below, so that e^{-tH} is bounded for all $t \ge 0$. In what follows we derive a Feynman–Kac formula for an unbounded semigroup e^{-tH} with interesting applications.

Definition 3.38 (Stark Hamiltonian). Let $E \in \mathbb{R}^3$. The operator

$$H(E) = -(1/2)\Delta + E \cdot x + V(x)$$

on $L^2(\mathbb{R}^3)$ is called *Stark Hamiltonian*.

The Stark Hamiltonian describes the interaction between a charged quantum particle and an external electrostatic field E, leading to a shift and split-up of the particle's spectral lines. In case when $V(x) = -1/|x|$, $H(0)$ has a point spectrum. In contrast, the spectrum of $H(E)$ is the whole of $\mathbb{R}$ when $E \neq 0$, in particular, it has no point spectrum at all.

Lemma 3.97 (Faris–Levine). *Let V and W be real-valued measurable functions on $\mathbb{R}^d$ such that $V(x) \geq -c|x|^2 - d$, with $c, d \geq 0$, and $V \in L^2_{\mathrm{loc}}(\mathbb{R}^d)$. Suppose that*

(1) *there exists a dense subset D such that $D \subset D(-(1/2)\Delta) \cap D(V) \cap D(W)$ and $x_j D \subset D$ and $\partial_j D \subset D$ so that $-(1/2)\Delta + V + W + 2c|x|^2$ is essentially self-adjoint on D,*

(2) *$-(a/2)\Delta + W$ is bounded from below on D for some $a < 1$.*

Then $-(1/2)\Delta + V + W$ is essentially self-adjoint on D.

Two immediate corollaries of Lemma 3.97 follow.

Corollary 3.98. *Let V_1 and V_2 be such that*

(1) *$V_2 < -(1/2)\Delta$ with a relative bound less than* 1,

(2) *$V_1 \geq -c|x|^2 - d$ for some c and d and $V_1 \in L^2_{\mathrm{loc}}(\mathbb{R}^d)$.*

Then $-(1/2)\Delta + V_1 + V_2$ is essentially self-adjoint on $C_0^\infty(\mathbb{R}^d)$.

Proof. We apply Lemma 3.97. Let $V = -c|x|^2 - d$ and $W = V_1 + V_2 + c|x|^2 + d$. Then $V \geq -c|x|^2 - d$ and $C_0^\infty \subset D(-(1/2)\Delta) \cap D(V) \cap D(W)$ hold trivially. $-(1/2)\Delta + V + W + 2c|x|^2 = -(1/2)\Delta + (V_1 + 2c|x|^2) + V_2$ is essentially self-adjoint on $C_0^\infty(\mathbb{R}^d)$, since by assumption (1) above $V_1 + 2c|x|^2 + d > 0$ and $V_1 + 2c|x|^2 \in L^2_{\mathrm{loc}}(\mathbb{R}^d)$; see Corollary 3.15. Moreover, $-(a/2)\Delta + W$ is bounded from below. Hence by Lemma 3.97 the statement follows. □

The following corollary covers the Stark Hamiltonian.

Corollary 3.99. *Let $V < -(1/2)\Delta$ with a relative bound less than* 1. *Then the Stark Hamiltonian $H(E)$ is essentially self-adjoint on $C_0^\infty(\mathbb{R}^3)$.*

Lemma 3.100. *Choose Coulomb potential $V(x) = -1/|x|$. Then for all $E \in \mathbb{R}^3$,* $\inf \mathrm{Spec}(H(E)) = -\infty$. *In particular, $e^{-tH(E)}$ is unbounded.*

Proof. Let $H_0(E) = -(1/2)\Delta + E \cdot x$. Since $V < 0$, it suffices to show that $\mathrm{Spec}(H_0(E)) = \mathbb{R}$. By rotation symmetry this can be reduced to addressing the case $E = (\varepsilon, 0, 0)$. Let $x = (x_0, x_\perp) \in \mathbb{R} \times \mathbb{R}^2$ and $p = (p_0, p_\perp)$ be its conjugate momentum. The Hamiltonian becomes $H_0(E) = (1/2)(p_0^2 + p_\perp^2) + \varepsilon x_0$. Let $U_\parallel =$

$e^{i(1/6\varepsilon)p_0^3}$ and $F_\perp$ denote Fourier transform with respect to $x_\perp$. Then $U = F_\perp U_\parallel : L^2(\mathbb{R}_x \times \mathbb{R}^2_{k_\perp}) \to L^2(\mathbb{R}^3)$ maps $\mathscr{S}(\mathbb{R}^3)$ onto itself. Furthermore, we have

$$U^{-1}H_0(E)U = (1/2)k_\perp^2 + \varepsilon x_0 \tag{3.9.1}$$

on $\mathscr{S}(\mathbb{R}^3)$. Since $\mathscr{S}(\mathbb{R}^3)$ is a core of both $H_0(E)$ and the multiplication operator on the right-hand side of (3.9.1), $H_0(E)$ and $T = (1/2)k_\perp^2 + \epsilon x_0$ are unitary equivalent as self-adjoint operators. Since $\mathrm{Spec}(T) = \mathbb{R}$, the lemma follows. □

Next we establish a Feynman–Kac formula for a class of unbounded semigroups including in particular the Stark Hamiltonian. Recall that $H = -(1/2)\Delta + V$. Our assumption on V is that for any $\varepsilon > 0$ there exists C_ε such that $V(x) \geq -\varepsilon|x|^2 - C_\varepsilon$. It is useful to rewrite the Feynman–Kac formula by using Brownian bridge starting from a at $t = 0$ and ending in b at $t = T$, i.e., the solution of the stochastic differential equation

$$dX_t = \frac{b - X_t}{T - t}dt + dB_t, \quad 0 \leq t < T, \quad X_0 = a. \tag{3.9.2}$$

This equation can be solved exactly to obtain

$$X_t = a\left(1 - \frac{t}{T}\right) + b\frac{t}{T} + (T - t)\int_0^t \frac{1}{T - s}dB_s, \quad t < T, \tag{3.9.3}$$

see Example 2.9 and (2.3.49). Note that $X_t \to b$ almost surely as $t \to T$. Hence, for continuous V bounded from below,

$$(f, e^{-tH}g) = \int_{\mathbb{R}^d \times \mathbb{R}^d} dxdy\overline{f(x)}g(y)Q(x, y; V, t)\Pi_t(x - y), \tag{3.9.4}$$

where by (3.9.3)

$$Q(x, y; V, t) = \mathbb{E}\left[\exp\left(-\int_0^t V\left(\left(1 - \frac{s}{t}\right)x + \frac{s}{t}y + \alpha_s\right)ds\right)\right], \tag{3.9.5}$$

and α_s is the Gaussian random variable defined by

$$\alpha_s = \begin{cases} (t - s)\int_0^s \frac{1}{t-r}dB_r, & s < t \\ 0, & s = t. \end{cases}$$

Note that $\mathbb{E}[\alpha_s] = 0$ and $\mathbb{E}[\alpha_s\alpha_{s'}] = s(1 - \frac{s'}{t})$ for $s \leq s' \leq t$. In the remainder of this section we prove the theorem below.

Theorem 3.101 (Feynman–Kac formula for unbounded semigroup). *Assume that V is continuous and for any $\varepsilon > 0$ there exists C_ε such that $V(x) \geq -\varepsilon|x|^2 - C_\epsilon$. Let $f, g \in L^2(\mathbb{R}^d)$ have compact support. Then $f, g \in D(e^{-tH})$ and (3.9.4) holds.*

In order to prove this theorem we first need the following lemma.

Lemma 3.102. *Let A_n and A be self-adjoint operators such that $A_n \to A$ in strong resolvent sense. Also, let f be a continuous function and $\psi \in D(f(A_n))$ for all n. Then*

(1) *If* $\sup_n \|f(A_n)\psi\| < \infty$, *then* $\psi \in D(f(A))$.

(2) *If* $\sup_n \|f(A_n)^2\psi\| < \infty$, *then* $f(A_n)\psi \to f(A)\psi$ *in strong sense as* $n \to \infty$.

Proof. (1) Let

$$f_m(x) = \begin{cases} m, & f(x) \geq m, \\ f(x), & |f(x)| \leq m, \\ -m, & f(x) \leq -m. \end{cases}$$

As f_m is bounded, $f_m(A_n) \to f_m(A)$ strongly as $n \to \infty$. Since

$$\|f_m(A)\psi\| = \lim_{n\to\infty} \|f_m(A_n)\psi\| \leq \sup_n \|f_m(A_n)\| \leq \sup_n \|f(A_n)\|,$$

we have $\sup_m \|f_m(A)\psi\| < \infty$. This implies $\psi \in D(f(A))$.

(2) Since $\|(f(A_n) - f_m(A_n))\psi\| \leq m^{-1}\|f(A_n)^2\psi\|$, it follows that $f_m(A_n)\psi \to f(A_n)\psi$, uniformly in n. Then $f(A_n)\psi \to f(A)\psi$ since $f_m(A_n)\psi \to f_m(A)\psi$. $\square$

Proof of Theorem 3.101. Let $V_n = \max\{V(x), -n\}$ and $H_n = -(1/2)\Delta + V_n$. Since $C_0^\infty(\mathbb{R}^d)$ is a common core of H_n, $H_n \to H$ as $n \to \infty$ in strong resolvent sense. For $f \in L^2(\mathbb{R}^d)$ with compact support we prove

$$\sup_n \|e^{-tH_n} f\| < \infty. \tag{3.9.6}$$

Note that

$$\begin{aligned} &\exp\left(-\int_0^t V_n\left(\left(1-\frac{s}{t}\right)x + \frac{s}{t}y + \alpha_s\right)\right) \\ &\leq \exp\left(\int_0^t \varepsilon\left(\left(1-\frac{s}{t}\right)x + \frac{s}{t}y + \alpha_s\right)^2 + C_\varepsilon ds\right) \\ &\leq \exp\left(C_\varepsilon t + 2\varepsilon t(x^2+y^2) + 2\varepsilon t\int_0^t \alpha_s^2 \frac{ds}{t}\right). \end{aligned}$$

By Jensen's inequality

$$e^{2\varepsilon t\int_0^t \alpha_s^2 \frac{ds}{t}} \leq \int_0^t e^{2\varepsilon t\alpha_s^2}\frac{ds}{t},$$

and therefore

$$\mathbb{E}[e^{2\varepsilon t\int_0^t \alpha_s^2 \frac{ds}{t}}] \leq \int_0^t \mathbb{E}[e^{2\varepsilon t\alpha_s^2}]\frac{ds}{t}.$$

A direct computation gives

$$\mathbb{E}[e^{2\varepsilon t\alpha_s^2}] = \frac{1}{\sqrt{2\pi\kappa_s}}\int e^{2\varepsilon t y^2} e^{-y^2/2\kappa_s} dy = \frac{1}{\sqrt{\pi(1-4\varepsilon t\kappa_s)}},$$

where $\kappa_s = s(1-\frac{s}{t})$ is the covariance of α_s and we suppose that $\varepsilon < 1$. Set $\varepsilon = \delta/t^2$ with some $\delta < 1$. Since $\kappa_s \le \kappa_{t/2} = t/4$, we have

$$Q(x,y;V,t) \le \frac{e^{C_\epsilon t}}{\sqrt{\pi(1-\delta)}} e^{2\delta(x^2+y^2)/t} \tag{3.9.7}$$

Hence

$$\|e^{-tH_n} f\|^2 \le \frac{e^{2C_\epsilon t}}{\sqrt{\pi(1-\delta)}} \int_{\mathbb{R}^d\times\mathbb{R}^d} dxdy|f(x)f(y)|\Pi_{2t}(x-y)e^{(\delta/t)(x^2+y^2)} < \infty,$$

uniformly in n. Thus (3.9.6) follows. By Lemma 3.101 (1), $f \in D(e^{-tH})$ and $e^{-tH_n} f \to e^{-tH} f$ strongly as $n \to \infty$. Thus the left-hand side of (3.9.4) converges and so does the right-hand side by (3.9.7) and the dominated convergence theorem. □

3.10 Ground state transform and related diffusions

3.10.1 Ground state transform and the intrinsic semigroup

The Feynman–Kac semigroup $\{K_t : t \ge 0\}$ has the particularity that in general $K_t 1_{\mathbb{R}^d} \ne 1_{\mathbb{R}^d}$. In this section we introduce a semigroup related with the Feynman–Kac semigroup given by Definition 3.20 which, however, does keep unit mass. This is called the *intrinsic Feynman–Kac semigroup*.

We assume that V is Kato-decomposable and the corresponding Schrödinger operator is defined through the Feynman–Kac formula, as in Definition 3.23. We suppose, moreover, that H has a normalized ground state $\Psi_p > 0$, i.e.,

$$H\Psi_p = E(H)\Psi_p, \quad E(H) = \inf \mathrm{Spec}(H), \quad \|\Psi_p\|_2 = 1. \tag{3.10.1}$$

We set $\bar{H} = H - E(H)$ so that $\bar{H}\Psi_p = 0$.

Definition 3.39 (Ground state transform). Let H be a Schrödinger operator with Kato-decomposable potential V, and $\Psi_p > 0$ be the normalized unique ground state of H. Let, moreover,

$$d\mathsf{N}_0 = \Psi_p^2(x)dx \tag{3.10.2}$$

be a probability measure on $\mathbb{R}^d$ and consider $L^2(\mathbb{R}^d, d\mathsf{N}_0)$. The *ground state transform* is a unitary map

$$\mathscr{U} : L^2(\mathbb{R}^d, d\mathsf{N}_0) \to L^2(\mathbb{R}^d, dx), \quad f \mapsto \Psi_p f.$$

In fact, this is a case of Doob's h-transform familiar from the theory of random processes.

Define the operator

$$L = \mathscr{U}^{-1} \bar{H} \mathscr{U} \tag{3.10.3}$$

with domain

$$D(L) = \{f \in L^2(\mathbb{R}^d, d\mathsf{N}_0) \mid \mathscr{U} f \in D(H)\}.$$

Also, let

$$\widetilde{K}_t(x, y) = \frac{e^{E(H)t} K_t(x, y)}{\Psi_p(x)\Psi_p(y)}, \tag{3.10.4}$$

where $K_t(x, y)$ denotes the integral kernel of K_t.

Definition 3.40 (Intrinsic Feynman–Kac semigroup). The one-parameter semigroup $\{\widetilde{K}_t : t \geq 0\}$ acting on $L^2(\mathbb{R}^d, d\mathsf{N}_0)$ with integral kernel $\widetilde{K}_t(x, y)$, i.e.,

$$\widetilde{K}_t f(x) = \int_{\mathbb{R}^d} f(y) \widetilde{K}_t(x, y) \Psi_p^2(y) dy \tag{3.10.5}$$

is called *intrinsic Feynman–Kac semigroup* for the potential V.

The generator of the intrinsic semigroup is the operator L given by (3.10.3). Notice that even though the L^p-norms of the operators K_t can be larger than 1, the operators $\widetilde{K}_t$ are always contractions. Moreover, the intrinsic Feynman–Kac semigroup is more natural than $\{K_t : t \geq 0\}$ since $\widetilde{K}_t 1_{\mathbb{R}^d} = 1_{\mathbb{R}^d}$, for every $t > 0$.

In view of Examples 3.20 and 3.22 it can be expected that there are important differences between Schrödinger (and therefore Feynman–Kac) semigroups depending on the behaviour of V at infinity. Here we only discuss one particular regularizing property called *ultracontractivity*; for further properties such as hypercontractivity and supercontractivity we refer to the literature.

Definition 3.41 (Ultracontractivity properties). Let $\{e^{-tH} : t \geq 0\}$ be a Schrödinger semigroup on $L^2(\mathbb{R}^d)$.

(1) $\{e^{-tH} : t \geq 0\}$ is said to be *ultracontractive* if e^{-tH} is contractive from $L^2(\mathbb{R}^d)$ to $L^\infty(\mathbb{R}^d)$, for all $t > 0$.

(2) Suppose that H has a strictly positive ground state. Then $\{e^{-tH} : t \geq 0\}$ is called *intrinsically ultracontractive* (IUC) if e^{-tL} is ultracontractive on $L^2(\mathbb{R}^d, d\mathsf{N}_0)$, for all $t > 0$.

(3) We call $\{e^{-tH} : t \geq 0\}$ *asymptotically intrinsically ultracontractive* (AIUC) if there exists $t_0 > 0$ such that e^{-tL} is ultracontractive on $L^2(\mathbb{R}^d, d\mathsf{N}_0)$, for all $t > t_0$.

Theorem 3.103 (Intrinsic ultracontractivity). *Let H be a Schrödinger operator with a Kato-decomposable potential V and ground state Ψ_{p}. Consider its Feynman–Kac semigroup $\{K_t : t \geq 0\}$. Then $\{e^{-tH} : t \geq 0\}$ is IUC (resp. AIUC) if and only if for every $t > 0$ (resp. if for every $t > t_0$, for some $t_0 > 0$) there is a constant $C_{V,t} > 0$ such that*

$$K_t(x, y) \leq C_{V,t}\, \Psi_{\mathrm{p}}(x)\Psi_{\mathrm{p}}(y), \quad x, y \in \mathbb{R}^d, \tag{3.10.6}$$

or equivalently,

$$\widetilde{K}_t(x, y) \leq C_{V,t}, \quad x, y \in \mathbb{R}^d. \tag{3.10.7}$$

A consequence of intrinsic ultracontractivity is that a similar lower bound on the kernel also holds, i.e., for every $t > 0$ there is a constant $C^{(1)}_{V,t} > 0$ such that

$$K_t(x, y) \geq C^{(1)}_{V,t}\, \Psi_{\mathrm{p}}(x)\Psi_{\mathrm{p}}(y), \quad x, y \in \mathbb{R}^d. \tag{3.10.8}$$

An immediate consequence of this is that if the semigroup is intrinsically ultracontractive, then $\Psi_{\mathrm{p}} \in L^1(\mathbb{R}^d)$.

The classic result for the Feynman–Kac semigroup generated by Schrödinger operators $H = -(1/2)\Delta + V$ identifying the borderline case for IUC is the following fact.

Theorem 3.104. *If $V(x) = |x|^\beta$, then the Schrödinger semigroup $\{e^{-tH} : t \geq 0\}$ is intrinsically ultracontractive if and only if $\beta > 2$. Moreover, if $\beta > 2$, then $c\varphi(x) \leq \Psi_{\mathrm{p}}(x) \leq C\varphi(x)$, $|x| > 1$, holds with some $C, c > 0$ and where*

$$\varphi(x) = |x|^{-\beta/4+(d-1)/2} \exp(-2|x|^{1+\beta/2}/(2 + \beta)).$$

For our purposes below an important consequence of AIUC is the following property.

Lemma 3.105. *The following two conditions are equivalent.*

(1) *The semigroup $\{K_t : t \geq 0\}$ is AIUC.*

(2) *The property*

$$\widetilde{K}_t(x, y) \stackrel{t\to\infty}{\longrightarrow} 1, \tag{3.10.9}$$

holds, uniformly in $(x, y) \in \mathbb{R}^d \times \mathbb{R}^d$.

Proof. The implication (2) $\Rightarrow$ (1) is immediate, we only show the converse statement. We have for every $x, y \in \mathbb{R}^d$ and $t > t_0$

$$
\begin{aligned}
&\big|\widetilde{K}_t(x,y) - 1\big| \\
&= \left| \int_{\mathbb{R}^d\times\mathbb{R}^d} \frac{K_{t_0}(x,z)K_{t-2t_0}(z,w)K_{t_0}(w,y)}{e^{-E(H)t}\Psi_{\mathrm{p}}(x)\Psi_{\mathrm{p}}(y)} dz dw - \frac{e^{-E(H)t}\Psi_{\mathrm{p}}(x)\Psi_{\mathrm{p}}(y)}{e^{-E(H)t}\Psi_{\mathrm{p}}(x)\Psi_{\mathrm{p}}(y)} \right| \\
&= \left| \int_{\mathbb{R}^d\times\mathbb{R}^d} \frac{K_{t_0}(x,z)\Psi_{\mathrm{p}}(z)(K_{t-2t_0}(z,w) - e^{-E(H)(t-2t_0)}\Psi_{\mathrm{p}}(z)\Psi_{\mathrm{p}}(w))K_{t_0}(w,y)\Psi_{\mathrm{p}}(w)}{e^{-E(H)t}\Psi_{\mathrm{p}}(x)\Psi_{\mathrm{p}}(z)\Psi_{\mathrm{p}}(w)\Psi_{\mathrm{p}}(y)} dz dw \right| \\
&\le e^{E(H)t} \left\| \frac{K_{t_0}(x,y)}{\Psi_{\mathrm{p}}(x)\Psi_{\mathrm{p}}(y)} \right\|_\infty^2 \\
&\quad \times \int_{\mathbb{R}^d\times\mathbb{R}^d} \left| K_{t-2t_0}(z,w) - e^{-E(H)(t-2t_0)}\Psi_{\mathrm{p}}(z)\Psi_{\mathrm{p}}(w) \right| \Psi_{\mathrm{p}}(z)\Psi_{\mathrm{p}}(w) dz dw \\
&\le Ce^{E(H)t} \left(\int_{\mathbb{R}^d\times\mathbb{R}^d} |K_{t-2t_0}(z,w) - e^{-E(H)(t-2t_0)}\Psi_{\mathrm{p}}(z)\Psi_{\mathrm{p}}(w)|^2 dz dw \right)^{1/2}.
\end{aligned}
$$

The last factor on the right-hand side is the Hilbert–Schmidt norm of the operator $K_{t-2t_0} - e^{-E(H)(t-2t_0)}1_{\mathrm{p}}$, where $1_{\mathrm{p}} : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ is the projection onto the one dimensional subspace of $L^2(\mathbb{R}^d)$ spanned by Ψ_{p}. This gives

$$
\begin{aligned}
|\widetilde{K}_t(x,y) - 1| &\le Ce^{tE(H)} \Big(\sum_{k=1}^\infty e^{-2E_k(H)(t-2t_0)} \Big)^{1/2} \\
&= Ce^{2t_0E_1(H)} e^{-(E_1(H)-E(H))t} \Big(\sum_{k=1}^\infty e^{-2(E_k(H)-E_1(H))(t-2t_0)} \Big)^{1/2},
\end{aligned}
$$

where $E(H) < E_1(H) \le \cdots$ are the eigenvalues of H. By the Lebesgue dominated convergence theorem the last sum converges to the multiplicity of $E_1(H)$ as $t \to \infty$. Since $E_1(H) > E(H)$, (3.10.9) follows. □

Remark 3.9. Ultracontractivity properties can be investigated also for fractional Schrödinger operators as in Definition 3.3. In that case the borderline case of IUC is $V(x) = \log|x|$, indicating that it is “easier” for a càdlàg process to be intrinsically ultracontractive. This can be explained by an efficient strong mixing mechanism due to the possibility of jumps in contrast with the continuous paths of Brownian motion. Also, with

$$
V(x) = \log|x| 1_{\{|x|>1\}}(x) - \frac{1}{|x|^{\alpha/2}} 1_{\{|x|\le 1\}}(x),
$$

the fractional Feynman–Kac semigroup $\{K_t : t \ge 0\}$ corresponding to $(-\Delta)^{\alpha/2} + V$ is AIUC but it is not IUC. This contrasts the case of Schrödinger operators and Brownian motion where the semigroup for the borderline potential is neither IUC nor AIUC.

3.10.2 Feynman–Kac formula for $P(\phi)_1$-processes

Whenever a Schrödinger operator has a ground state, a Feynman–Kac formula of the related Schrödinger semigroup can be constructed on a path space in terms of a diffusion process. The so obtained process is a $P(\phi)_1$-*process* for the given V.

We will construct a diffusion process associated with the operator L. In general, the differentiability of the ground state Ψ_p of H is not clear. However, whenever $\nabla \log \Psi_p$ exists, we have

$$Lf = -\frac{1}{2}\Delta f - \nabla \log \Psi_p \cdot \nabla f. \tag{3.10.10}$$

If Ψ_p is such that $\nabla \log \Psi_p$ is not well-defined, the above SDE is a formal description only with no rigorous meaning in this form. The three-dimensional Coulomb potential $V = -\frac{1}{|x|}$ is a case in point: then $\Psi_p(x) = C\exp(-|x|)$, which is not $C^1(\mathbb{R}^3)$. A solution $(X_t^x)_{t\geq 0}$ of the SDE

$$dX_t = \nabla \log \Psi_p dt + dB_t, \quad X_0 = x \tag{3.10.11}$$

satisfies that

$$\mathbb{E}[f(X_t^x)] = (e^{-tL} f)(x). \tag{3.10.12}$$

It is also not clear that (3.10.11) is a well defined stochastic differential equation and when does it have a solution. A sufficient condition for the existence of a solution of (3.10.11) is the Lipshitz condition (2.3.34)–(2.3.35) in Theorem 2.35 on $\log \Psi_p$, however, in general this is difficult to check. Therefore in a next section we adopt a different way of constructing the process for which (3.10.12) is satisfied.

Consider two-sided Brownian motion as defined at the end of Section 2.1, and denote by

$$\mathscr{X} = C(\mathbb{R}, \mathbb{R}^d)$$

the space of continuous paths on the real line $\mathbb{R}$ as before. In this section we construct a probability measure $\mathscr{N}_0^x$, $x \in \mathbb{R}^d$, on $(\mathscr{X}, \mathscr{B}(\mathscr{X}))$ such that the coordinate process $(X_t)_{t\in\mathbb{R}}$ is a diffusion process generated by L.

Theorem 3.106. *Suppose that V is Kato-decomposable and H has a ground state Ψ_p. Let $L = \mathscr{U}^{-1}\bar{H}\mathscr{U}$ be the ground state transformation of H, and $X_t(\omega) = \omega(t)$ be the coordinate process on $(\mathscr{X}, \mathscr{B}(\mathscr{X}))$. Then there exists a probability measure $\mathscr{N}_0^x$ on $(\mathscr{X}, \mathscr{B}(\mathscr{X}))$ satisfying the following properties:*

(1) *(Initial distribution)* $\mathscr{N}_0^x(X_0 = x) = 1$.

(2) *(Reflection symmetry)* $(X_t)_{t\geq 0}$ *and* $(X_s)_{s\leq 0}$ *are independent, and*

$$X_{-t} \stackrel{\mathrm{d}}{=} X_t, \quad t \in \mathbb{R}.$$

(3) *(Diffusion property) Let*

$$\mathscr{F}_t^+ = \sigma(X_s, 0 \le s \le t), \quad \mathscr{F}_t^- = \sigma(X_s, t \le s \le 0)$$

be given filtrations. Then $(X_t)_{t\ge 0}$ and $(X_s)_{s\le 0}$ are diffusion processes with respect to $(\mathscr{F}_t^+)_{t\ge 0}$ and $(\mathscr{F}_t^-)_{t\le 0}$, respectively, i.e.,

$$\mathbb{E}_{\mathcal{N}_0^x}[X_{t+s}|\mathscr{F}_s^+] = \mathbb{E}_{\mathcal{N}_0^x}[X_{t+s}|\sigma(X_s)] = \mathbb{E}_{\mathcal{N}_0^{X_s}}[X_t],$$
$$\mathbb{E}_{\mathcal{N}_0^x}[X_{-t-s}|\mathscr{F}_{-s}^-] = \mathbb{E}_{\mathcal{N}_0^x}[X_{-t-s}|\sigma(X_{-s})] = \mathbb{E}_{\mathcal{N}_0^{X_{-s}}}[X_{-t}]$$

for $s, t \ge 0$, and $\mathbb{R} \ni t \mapsto X_t \in \mathbb{R}^d$ is a.s. continuous, where $\mathbb{E}_{\mathcal{N}_0^{X_s}}$ means $\mathbb{E}_{\mathcal{N}_0^y}$ evaluated at $y = X_s$.

(4) *(Shift invariance) Let $-\infty < t_0 \le t_1 \le \cdots \le t_n < \infty$. Then*

$$\int_{\mathbb{R}^d} \mathbb{E}_{\mathcal{N}_0^x}\Big[\prod_{j=0}^n f_j(X_{t_j})\Big] d\mathsf{N}_0 = (f_0, e^{-(t_1-t_0)L} f_1 \cdots e^{-(t_n-t_{n-1})L} f_n)_{L^2(\mathbb{R}^d, d\mathsf{N}_0)} \tag{3.10.13}$$

for $f_j \in L^\infty(\mathbb{R}^d)$, $j = 1, \dots, n$, and the finite dimensional distributions of the process are shift invariant, i.e.,

$$\int_{\mathbb{R}^d} \mathbb{E}_{\mathcal{N}_0^x}\Big[\prod_{j=1}^n f_j(X_{t_j})\Big] d\mathsf{N}_0 = \int_{\mathbb{R}^d} \mathbb{E}_{\mathcal{N}_0^x}\Big[\prod_{j=1}^n f_j(X_{t_j+s})\Big] d\mathsf{N}_0, \quad s \in \mathbb{R}.$$

Definition 3.42 ($P(\phi)_1$-process). Suppose H has a ground state $\Psi_{\rm p}$ and consider the probability space $(\mathscr{X}, \mathscr{B}(\mathscr{X}), \mathcal{N}_0^x)$. The coordinate process $X_t(\omega) = \omega(t)$ on this space is called a *$P(\phi)_1$-process* for H.

We proceed now to prove existence and some properties of $P(\phi)_1$-processes in several steps. Define the set function $\nu_{t_0,\dots,t_n} : \times_{j=0}^n \mathscr{B}(\mathbb{R}^d) \to \mathbb{R}$ for $0 \le t_0 \le t_1 \le \cdots \le t_n$ by

$$\nu_{t_0,\dots,t_n}\left(\times_{i=0}^n A_i\right) = (1_{A_0}, e^{-(t_1-t_0)L} 1_{A_1} \cdots e^{-(t_n-t_{n-1})L} 1_{A_n})_{L^2(\mathbb{R}^d, d\mathsf{N}_0)} \tag{3.10.14}$$

and for $t \ge 0$, $\nu_t : \mathscr{B}(\mathbb{R}^d) \to \mathbb{R}$ by

$$\nu_t(A) = (1, e^{-tL} 1_A)_{L^2(\mathbb{R}^d, d\mathsf{N}_0)} = (1, 1_A)_{L^2(\mathbb{R}^d, d\mathsf{N}_0)}. \tag{3.10.15}$$

Step 1*:* The family of set functions $\{\nu_\Lambda\}_{\Lambda\subset\mathbb{R}, |\Lambda|<\infty}$ given by (3.10.14) and (3.10.15) satisfies the consistency condition

$$\nu_{t_0,\dots,t_{n+m}}((\times_{i=0}^n A_i) \times (\times_{i=n+1}^{n+m} \mathbb{R}^d)) = \nu_{t_0,\dots,t_n}(\times_{i=0}^n A_i).$$

Hence by the Kolmogorov extension theorem there exists a probability measure ν_∞ on $((\mathbb{R}^d)^{[0,\infty)}, \sigma(\mathcal{A}))$, where

$$\mathcal{A} = \{\omega : \mathbb{R} \to \mathbb{R}^d \mid \omega\lceil_\Lambda \in E,\ E \in (\mathscr{B}(\mathbb{R}^d))^{|\Lambda|},\ \Lambda \subset \mathbb{R},\ |\Lambda| < \infty\} \quad (3.10.16)$$

such that

$$\nu_t(A) = \mathbb{E}_{\nu_\infty}[1_A(Y_t)], \quad (3.10.17)$$

$$\nu_{t_0,\dots,t_n}(\times_{i=0}^n A_i) = \mathbb{E}_{\nu_\infty}\Big[\prod_{j=0}^n 1_{A_j}(Y_{t_j})\Big], \quad n \ge 1, \quad (3.10.18)$$

where $Y_t(\omega) = \omega(t)$, $\omega \in (\mathbb{R}^d)^{[0,\infty)}$, is the coordinate process. Then the process $(Y_t)_{t\ge 0}$ on the probability space $((\mathbb{R}^d)^{[0,\infty)}, \sigma(\mathcal{A}), \nu_\infty)$ satisfies

$$(f_0, e^{-(t_1-t_0)L} f_1 \cdots e^{-(t_n-t_{n-1})L} f_n)_{L^2(\mathbb{R}^d, d\mathsf{N}_0)} = \mathbb{E}_{\nu_\infty}\Big[\prod_{j=0}^n f_j(Y_{t_j})\Big] \quad (3.10.19)$$

$$(1, f)_{L^2(\mathbb{R}^d; d\mathsf{N}_0)} = (1, e^{-tL} f)_{L^2(\mathbb{R}^d, d\mathsf{N}_0)} = \mathbb{E}_{\nu_\infty}[f(Y_t)] = \mathbb{E}_{\nu_\infty}[f(Y_0)] \quad (3.10.20)$$

for $f_j \in L^\infty(\mathbb{R}^d)$, $j = 0, 1, \dots, n$ and $0 \le t_0 < t_1 < \cdots < t_n$.

Step 2*:* Next we show that this process has a continuous version.

Lemma 3.107. $(Y_t)_{t\ge 0}$ *has a continuous version.*

Proof. By the Kolmogorov–Čentsov theorem (see Proposition 2.6) it suffices to show that $\mathbb{E}_{\nu_\infty}[|Y_t - Y_s|^{2n}] \le |t-s|^n$ for $n \ge 2$. Let $s < t$. We have

$$\mathbb{E}_{\nu_\infty}[|Y_t - Y_s|^{2n}] = \sum_{\mu=1}^d \sum_{k=0}^{2n} \binom{2n}{k} (-1)^k (\Psi_\mathrm{p}, x^\mu e^{-(t-s)\bar{H}} x^\mu \Psi_\mathrm{p}).$$

The last term can be represented in terms of Brownian motion as

$$\begin{aligned}&(\Psi_\mathrm{p}, x^\mu e^{-(t-s)\bar{H}} x^\mu \Psi_\mathrm{p})\\ &= \int_{\mathbb{R}^d} dx \mathbb{E}^x[B_0^\mu \Psi_\mathrm{p}(B_0) B_{t-s}^\mu \Psi_\mathrm{p}(B_{t-s}) e^{-\int_0^{t-s} V(B_r)dr}] e^{(t-s)E(H)}\end{aligned}$$

By Schwarz inequality,

$$
\begin{aligned}
&\mathbb{E}_{\nu_\infty}[|Y_t - Y_s|^{2n}] \\
&= \int_{\mathbb{R}^d} dx \mathbb{E}^x[|B_{t-s} - B_0|^{2n} \Psi_{\mathrm{p}}(B_0)\Psi_{\mathrm{p}}(B_{t-s}) e^{-\int_0^{t-s} V(B_r)dr}] e^{(t-s)E(H)} \\
&\le \mathbb{E}\Bigg[|B_{t-s} - B_0|^{2n} \left(\int_{\mathbb{R}^d} dx e^{-2\int_0^t V(B_r+x)dr} \Psi_{\mathrm{p}}(x)^2 \right)^{1/2} \\
&\qquad \times \left(\int_{\mathbb{R}^d} dx \Psi_{\mathrm{p}}(B_{t-s} + x)^2 \right)^{1/2} \Bigg] e^{(t-s)E(H)} \\
&\le \mathbb{E}[|B_{t-s} - B_0|^{4n}]^{1/2} \mathbb{E}\left[\left(\int_{\mathbb{R}^d} dx e^{-2\int_0^t V(B_r+x)dr} \Psi_{\mathrm{p}}(x)^2 \right) \right] \|\Psi_{\mathrm{p}}\| e^{(t-s)E(H)}.
\end{aligned}
$$

Since V is Kato-decomposable, we furthermore have

$$
\begin{aligned}
&\le \mathbb{E}[|B_{t-s} - B_0|^{4n}]^{1/2} \sup_x \mathbb{E}^x[e^{-2\int_0^t V(B_s)ds}] \|\Psi_{\mathrm{p}}\|^2 e^{-(t-s)E(H)} \\
&\le \sqrt{C_{2n}} |t-s|^n \sup_x \mathbb{E}^x[e^{-2\int_0^t V(B_s)ds}] \|\Psi_{\mathrm{p}}\|^2 e^{-(t-s)E(H)}.
\end{aligned}
$$

Here we used that $\mathbb{E}[|B_t - B_s|^{2n}] = C_n |t-s|^n$. □

Let now $(\overline{Y}_t)_{t\ge 0}$ be the continuous version of $(Y_t)_{t\ge 0}$ on $((\mathbb{R}^d)^{[0,\infty)}, \sigma(\mathscr{A}), \nu_\infty)$. Denote the image measure of ν_∞ on $(\mathscr{X}, \mathscr{B}(\mathscr{X}))$ with respect to $\overline{Y}$ by

$$
\mathscr{M} = \nu_\infty \circ \overline{Y}^{-1}. \tag{3.10.21}
$$

We identify the coordinate process by $\widetilde{Y}_t(\omega) = \omega(t)$, for $\omega \in \mathscr{X}$. We thus constructed a random process $(\widetilde{Y}_t)_{t\ge 0}$ on $(\mathscr{X}, \mathscr{B}(\mathscr{X}), \mathscr{M})$ such that $\overline{Y} \stackrel{\mathrm{d}}{=} \widetilde{Y}$. Then (3.10.19) and (3.10.20) can be expressed in terms of $(\widetilde{Y}_t)_{t\ge 0}$ as

$$
\begin{aligned}
(f_0, e^{-(t_1-t_0)L} f_1 \cdots e^{-(t_n-t_{n-1})L} f_n)_{L^2(\mathbb{R}^d, d\mathsf{N}_0)} &= \mathbb{E}_{\mathscr{M}}\Big[\prod_{j=0}^n f_j(\widetilde{Y}_{t_j}) \Big], \\
(1, f)_{L^2(\mathbb{R}^d, d\mathsf{N}_0)} = (1, e^{-tL} f)_{L^2(\mathbb{R}^d, d\mathsf{N}_0)} &= \mathbb{E}_{\mathscr{M}}[f(\widetilde{Y}_t)] = \mathbb{E}_{\mathscr{M}}[f(\widetilde{Y}_0)].
\end{aligned}
$$

Step 3*:* Define a probability measure on $\mathscr{X}$ by

$$
\mathscr{M}^x(\cdot) = \mathscr{M}(\cdot | \widetilde{Y}_0 = x) \tag{3.10.22}
$$

for all $x \in \mathbb{R}^d$. Since the distribution of $\widetilde{Y}_0$ is $d\mathsf{N}_0$, note that

$$
\mathscr{M}(A) = \int_{\mathbb{R}^d} \mathbb{E}_{\mathscr{M}^x}[1_A] d\mathsf{N}_0.
$$

Then the random process $(\widetilde{Y}_t)_{t\geq 0}$ on $(\mathscr{X}, \mathscr{B}(\mathscr{X}), \mathcal{M}^x)$ satisfies

$$(f_0, e^{-(t_1-t_0)L} f_1 \cdots e^{-(t_n-t_{n-1})L} f_n)_{L^2(\mathbb{R}^d;d\mathsf{N}_0)} = \int_{\mathbb{R}^d} \mathbb{E}_{\mathcal{M}^x}\Big[\prod_{j=0}^n f_j(\widetilde{Y}_{t_j})\Big] d\mathsf{N}_0,$$

$$\begin{aligned}(1, e^{-tL} f)_{L^2(\mathbb{R}^d,d\mathsf{N}_0)} = (1, f)_{L^2(\mathbb{R}^d,d\mathsf{N}_0)} &= \int_{\mathbb{R}^d} \mathbb{E}_{\mathcal{M}^x}[f(\widetilde{Y}_0)] d\mathsf{N}_0 \\ &= \int_{\mathbb{R}^d} f(x) d\mathsf{N}_0. \end{aligned} \tag{3.10.23}$$

Lemma 3.108. *$(\widetilde{Y}_t)_{t\geq 0}$ is a Markov process on $(\mathscr{X}, \mathscr{B}(\mathscr{X}), \mathcal{M}^x)$ with respect to the natural filtration $\sigma(\widetilde{Y}_s, 0 \leq s \leq t)$, $t \geq 0$.*

Proof. Let

$$p_t(x, A) = (e^{-tL} 1_A)(x), \quad A \in \mathscr{B}(\mathbb{R}^d), \quad t \geq 0. \tag{3.10.24}$$

Notice that $p_t(x, A) = \mathbb{E}[1_A(X_t^x)]$. Thus by (3.10.23) the finite dimensional distributions of $(\widetilde{Y}_t)_{t\geq 0}$ are

$$\mathbb{E}_{\mathcal{M}^x}\Big[\prod_{j=1}^n 1_{A_j}(\widetilde{Y}_{t_j})\Big] = \int_{\mathbb{R}^{nd}} \prod_{j=1}^n 1_{A_j}(x_j) \prod_{j=1}^n p_{t_j - t_{j-1}}(x_{j-1}, dx_j) \tag{3.10.25}$$

with $t_0 = 0$ and $x_0 = x$. We show that $p_t(x, A)$ is a probability transition kernel. Note that e^{-tL} is positivity improving. Thus $0 \leq e^{-tL} f \leq 1$, for all function f such that $0 \leq f \leq 1$, and $e^{-tL}1 = 1$ follows, so $p_t(x, \cdot)$ is a probability measure on $\mathbb{R}^d$ with $p_t(x, \mathbb{R}^d) = 1$. Secondly, $p_t(\cdot, A)$ is trivially Borel measurable with respect to x. Thirdly, by the semigroup property $e^{-sL}e^{-tL}1_A = e^{-(s+t)L}1_A$ the Chapman–Kolmogorov equality (2.2.7) follows directly, and hence $p_t(x, A)$ is a probability transition kernel. Thus $(\widetilde{Y}_t)_{t\geq 0}$ is Markov process by (3.10.25) and Proposition 2.17. □

Step 4*:* We extend $(\widetilde{Y}_t)_{t\geq 0}$ to a process on the whole real line $\mathbb{R}$. Consider $\tilde{\mathscr{X}} = \mathscr{X} \times \mathscr{X}$, $\tilde{\mathscr{M}} = \mathscr{B}(\mathscr{X}) \times \mathscr{B}(\mathscr{X})$ and $\tilde{\mathcal{M}}^x = \mathcal{M}^x \times \mathcal{M}^x$. Let $(\tilde{X}_t)_{t\in\mathbb{R}}$ be the random process defined by

$$\tilde{X}_t(\omega) = \begin{cases} \widetilde{Y}_t(\omega_1), & t \geq 0, \\ \widetilde{Y}_{-t}(\omega_2), & t < 0. \end{cases} \tag{3.10.26}$$

for $\omega = (\omega_1, \omega_2) \in \tilde{\mathscr{X}}$, on the product space $(\tilde{\mathscr{X}}, \tilde{\mathscr{M}}, \tilde{\mathcal{M}}^x)$. Note that $\tilde{X}_0 = x$ almost surely under $\tilde{\mathcal{M}}^x$, and $\tilde{X}_t$ is continuous in t almost surely. It is trivial to see that $\tilde{X}_t$, $t \geq 0$, and $\tilde{X}_s$, $s \leq 0$, are independent, and $\tilde{X}_t \stackrel{\mathrm{d}}{=} \tilde{X}_{-t}$.

Step 5: We can now prove the theorem.

Proof of Theorem 3.106. Denote the image measure of $\tilde{\mathcal{M}}^x$ on $(\mathfrak{X}, \mathscr{B}(\mathfrak{X}))$ with respect to $(\tilde{X}_t)_{t\geq 0}$ by

$$\mathcal{N}_0^x = \tilde{\mathcal{M}}^x \circ \tilde{X}^{-1}. \tag{3.10.27}$$

Let $X_t(\omega) = \omega(t)$, $t \in \mathbb{R}$, $\omega \in \mathfrak{X}$, be the coordinate process. Then

$$X_t \stackrel{\mathrm{d}}{=} \widetilde{Y}_t \quad (t \geq 0), \qquad X_t \stackrel{\mathrm{d}}{=} \widetilde{Y}_{-t} \quad (t \leq 0). \tag{3.10.28}$$

Since by Step 3 above $(\widetilde{Y}_t)_{t\geq 0}$ and $(\widetilde{Y}_{-t})_{t\leq 0}$ are Markov processes with respect to the natural filtrations $\sigma(\widetilde{Y}_s, 0 \leq s \leq t)$ and $\sigma(\widetilde{Y}_s, -t \leq s \leq 0)$, respectively, $(X_t)_{t\geq 0}$ and $(X_t)_{t\leq 0}$ are also Markov processes with respect to $(\mathscr{F}_t^+)_{t\geq 0}$ resp. $(\mathscr{F}_t^-)_{t\leq 0}$, where $\mathscr{F}_t^+ = \sigma(X_s, 0 \leq s \leq t)$ and $\mathscr{F}_t^- = \sigma(X_s, -t \leq s \leq 0)$. Thus the diffusion property follows. We also see that $(X_s)_{s\leq 0}$ and $(X_t)_{t\geq 0}$ are independent and $X_{-t} \stackrel{\mathrm{d}}{=} X_t$ by (3.10.28) and Step 4 above. Hence reflection symmetry follows. Shift invariance follows by the lemma below; for any $s \in \mathbb{R}$,

$$\int_{\mathbb{R}^d} \mathbb{E}_{\mathcal{N}_0^x}\Big[\prod_{j=0}^n f_j(X_{t_j})\Big] d\mathsf{N}_0 = \int_{\mathbb{R}^d} \mathbb{E}_{\mathcal{N}_0^x}\Big[\prod_{j=0}^n f_j(X_{t_j+s})\Big] d\mathsf{N}_0. \qquad \square$$

Lemma 3.109. *Let* $-\infty < t_0 \leq t_1 \leq \cdots \leq t_n$. *Then*

$$\int_{\mathbb{R}^d} \mathbb{E}_{\mathcal{N}_0^x}\Big[\prod_{j=0}^n f_j(X_{t_j})\Big] d\mathsf{N}_0 = (f_0, e^{-(t_1-t_0)L} f_1 \cdots e^{-(t_n-t_{n-1})L} f_n)_{L^2(\mathbb{R}^d, d\mathsf{N}_0)}. \tag{3.10.29}$$

Proof. Let $t_0 \leq \cdots \leq t_n \leq 0 \leq t_{n+1} \leq \cdots \leq t_{n+m}$. Then by independence of $(X_s)_{s\leq 0}$ and $(X_t)_{t\geq 0}$,

$$\int_{\mathbb{R}^d} \mathbb{E}_{\mathcal{N}_0^x}\Big[\prod_{j=0}^{n+m} f_j(X_{t_j})\Big] d\mathsf{N}_0 = \int_{\mathbb{R}^d} \mathbb{E}_{\mathcal{N}_0^x}\Big[\prod_{j=0}^{n} f_j(X_{t_j})\Big] \mathbb{E}_{\mathcal{N}_0^x}\Big[\prod_{j=n+1}^{n+m} f_j(X_{t_j})\Big] d\mathsf{N}_0.$$

We furthermore have

$$\begin{aligned} &\mathbb{E}_{\mathcal{N}_0^x}\Big[\prod_{j=n+1}^{n+m} f_j(X_{t_j})\Big] \\ &= (e^{-t_{n+1}L} f_{n+1} e^{-(t_{n+2}-t_{n+1})L} f_{n+2} \cdots e^{-(t_{n+m}-t_{n+m-1})L} f_{n+m})(x) \end{aligned} \tag{3.10.30}$$

and

$$\mathbb{E}_{\mathcal{N}_0^x}\Big[\prod_{j=0}^{n+m} f_j(X_{t_j})\Big] = \mathbb{E}_{\mathcal{N}_0^x}\Big[\prod_{j=0}^{n+m} f_j(X_{-t_j})\Big]$$
$$= (e^{+t_n L} f_n e^{-(t_n-t_{n-1})L} f_{n-1}\cdots e^{-(t_1-t_0)L} f_1)(x). \tag{3.10.31}$$

By (3.10.30) and (3.10.31) we obtain

$$\int_{\mathbb{R}^d} \mathbb{E}_{\mathcal{N}_0^x}\Big[\prod_{j=0}^{n+m} f_j(X_{t_j})\Big] d\mathsf{N}_0$$
$$= (e^{+t_n L} f_n \cdots e^{-(t_1-t_0)L} f_1, e^{-t_{n+1}L} f_{n+1}\cdots e^{-(t_{n+m}-t_{n+m-1})L} f_{n+m})_{L^2(\mathbb{R}^d, d\mathsf{N}_0)}$$
$$= (f_1, e^{-(t_1-t_0)L} f_2 \cdots e^{-(t_{n+m}-t_{n+m-1})L} f_{n+m})_{L^2(\mathbb{R}^d, d\mathsf{N}_0)}.$$

Hence (3.10.29) follows. □

Remark 3.10. By Theorem 3.106 we have

$$(e^{-tL} f)(x) = \mathbb{E}_{\mathcal{N}_0^x}[f(X_t)]. \tag{3.10.32}$$

The right-hand side of (3.10.32) can be also represented in terms of Brownian motion resulting in

$$\mathbb{E}_{\mathcal{N}_0^x}[f(X_t)] = \frac{1}{\Psi_{\mathrm{p}}(x)} \mathbb{E}^x[e^{-\int_0^t V(B_s)ds} f(B_t)\Psi_{\mathrm{p}}(B_t)]e^{tE(H)} \tag{3.10.33}$$

as a vector in $L^2(\mathbb{R}^d)$.

Define the probability measure $\mathcal{N}_0$ on $\mathfrak{X}\times\mathbb{R}^d$ by

$$d\mathcal{N}_0 = d\mathcal{N}_0^x \otimes d\mathsf{N}_0. \tag{3.10.34}$$

Using this measure we have the following representation.

Theorem 3.110 (Feynman–Kac formula for $P(\phi)_1$-processes). *If* $f, g \in L^2(\mathbb{R}^d, d\mathsf{N}_0)$ *then*

$$(f, e^{-tL} g)_{L^2(\mathbb{R}^d, d\mathsf{N}_0)} = (f\Psi_{\mathrm{p}}, e^{-t(H-E(H))} g\Psi_{\mathrm{p}})_{L^2(\mathbb{R}^d)} = \mathbb{E}_{\mathcal{N}_0}[\bar{f}(X_0)g(X_t)]. \tag{3.10.35}$$

In Chapter 6 we will make a detailed study of a scalar quantum field model in terms of the path measure $\mathcal{N}_0$.

3.10.3 Dirichlet principle

In this subsection we consider some path properties of the $P(\phi)_1$-process $(X_t)_{t\geq 0}$ which will be used in Chapter 6 below.

Proposition 3.111. *Let $\Lambda > 0$ and suppose that $f \in C(\mathbb{R}^d) \cap D(L^{1/2})$. Then it follows that*

$$\mathcal{N}_0\Big(\sup_{0\leq s\leq T} |f(X_s)| \geq \Lambda\Big) \leq \frac{e}{\Lambda}((f,f)_{L^2(\mathbb{R}^d, d\mathsf{N}_0)} + T(L^{1/2}f, L^{1/2}f)_{L^2(\mathbb{R}^d, d\mathsf{N}_0)})^{1/2}. \tag{3.10.36}$$

(Here e is the base of natural logarithm.)

Proof. Set $T_j = Tj/2^n$, $j = 0, 1, \ldots, 2^n$, and fix T and n. Let

$$G = \{x \in \mathbb{R}^d \,||\, f(x)| \geq \Lambda\}, \tag{3.10.37}$$

and define the stopping time

$$\tau = \inf\{T_j \geq 0 |\, X_{T_j} \in G\}. \tag{3.10.38}$$

Then the identity

$$\mathcal{N}_0\Big(\sup_{j=0,\ldots,2^n} |f(X_{T_j})| \geq \Lambda\Big) = \mathcal{N}_0(\tau \leq T)$$

follows. We estimate the right-hand side above. Let $0 < \varrho < 1$ be fixed and choose a suitable ϱ later. We see that by Schwarz inequality with respect to $d\mathsf{N}_0$,

$$\begin{aligned}\mathcal{N}_0(\tau \leq T) &= \mathbb{E}_{\mathcal{N}_0}[1_{\{\tau\leq T\}}] \leq \mathbb{E}_{\mathcal{N}_0}[\varrho^{\tau - T}] \\ &\leq \varrho^{-T}\mathbb{E}_{\mathcal{N}_0}[\varrho^\tau] \leq \varrho^{-T}\left(\int_{\mathbb{R}^d} (\mathbb{E}^x_{\mathcal{N}_0}[\varrho^\tau])^2 d\mathsf{N}_0\right)^{1/2},\end{aligned} \tag{3.10.39}$$

where $\mathbb{E}^x_{\mathcal{N}_0} = \mathbb{E}_{\mathcal{N}_0^x}$. Let $0 \leq \psi$ be any function such that $\psi(x) \geq 1$ on G. Then the Dirichlet principle (shown in the lemma below) implies that

$$\begin{aligned}&\int_{\mathbb{R}^d} (\mathbb{E}^x_{\mathcal{N}_0}[\varrho^\tau])^2 d\mathsf{N}_0 \\ &\leq (\psi,\psi)_{L^2(\mathbb{R}^d, d\mathsf{N}_0)} + \frac{\varrho^{T/2^n}}{1-\varrho^{T/2^n}}(\psi, (1 - e^{-(T/2^n)L})\psi)_{L^2(\mathbb{R}^d, d\mathsf{N}_0)}.\end{aligned} \tag{3.10.40}$$

Inserting

$$|f(x)|/\Lambda \begin{cases} \geq 1, & x \in G, \\ = |f(x)|/\Lambda, & x \in \mathbb{R}^d \setminus G,\end{cases}$$

into ψ in (3.10.40), we have

$$\int_{\mathbb{R}^d} (\mathbb{E}^x_{\mathscr{N}_0}[\varrho^\tau])^2 d\mathsf{N}_0 \le \frac{1}{\Lambda^2}(f, f) + \frac{\varrho^{T/2^n}}{1-\varrho^{T/2^n}} \frac{1}{\Lambda^2}(|f|, (1-e^{-(T/2^n)L})|f|). \tag{3.10.41}$$

Since $e^{-(T/2^n)L}$ is positivity improving and thus

$$(|f|, (1-e^{-(T/2^n)L})|f|) \le (f, (1-e^{-(T/2^n)L})f),$$

we have by (3.10.39),

$$\begin{aligned} &\mathscr{N}_0\big(\sup_{j=0,\dots,2^n} |f(X_{T_j})| \ge \Lambda\big) \\ &\le \frac{\varrho^{-T}}{\Lambda}\left((f, f) + \frac{\varrho^{T/2^n}}{1-\varrho^{T/2^n}}(f, (1-e^{-(T/2^n)L})f)\right)^{1/2}. \end{aligned}$$

Set $\varrho = e^{-1/T}$. Then by $\frac{\varrho^{T/2^n}}{1-\varrho^{T/2^n}} \le 2^n$, we have

$$\mathscr{N}_0\big(\sup_{j=0,\dots,2^n} |f(X_{T_j})| \ge \Lambda\big) \le \frac{e}{\Lambda}((f, f) + 2^n(f, (1-e^{-(T/2^n)L})f))^{1/2}. \tag{3.10.42}$$

Since $(f, (1-e^{-(T/2^n)L})f) \le (T/2^n)(L^{1/2}f, L^{1/2}f)$, we obtain that

$$\mathscr{N}_0\big(\sup_{j=0,\dots,2^n} |f(X_{T_j})| \ge \Lambda\big) \le \frac{e}{\Lambda}((f, f) + T(L^{1/2}f, L^{1/2}f))^{1/2}. \tag{3.10.43}$$

Take $n \to \infty$ on both sides of (3.10.43). By the Lebesgue dominated convergence theorem

$$\lim_{n\to\infty} \mathscr{N}_0\big(\sup_{j=0,\dots,2^n} |f(X_{T_j})| \ge \Lambda\big) = \mathscr{N}_0\big(\lim_{n\to\infty} \sup_{j=0,\dots,2^n} |f(X_{T_j})| \ge \Lambda\big).$$

Since $f(X_t)$ is continuous in t, $\lim_{n\to\infty} \sup_{j=0,\dots,2^n} |f(X_{T_j})| = \sup_{0\le s\le T} |f(X_s)|$ follows. $\square$

It remains to show the Dirichlet principle (3.10.40).

Lemma 3.112 (Dirichlet principle). *Let* $0 < \varrho < 1$. *Fix* n *and set* $T_j = Tj/2^n$, $j = 0, 1, \dots, 2^n$. *Let* $G \subset \mathbb{R}^d$ *be measurable and* $\tau = \inf\{T_j \ge 0 | X_{T_j} \in G\}$. *Then for any function* $\psi \ge 0$ *such that* $\psi(x) \ge 1$ *on* G, *it follows that*

$$\begin{aligned} &\int_{\mathbb{R}^d} (\mathbb{E}^x_{\mathscr{N}_0}[\varrho^\tau])^2 d\mathsf{N}_0 \\ &\le (\psi, \psi)_{L^2(\mathbb{R}^d, d\mathsf{N}_0)} + \frac{\varrho^{T/2^n}}{1-\varrho^{T/2^n}}(\psi, (1-e^{-(T/2^n)L})\psi)_{L^2(\mathbb{R}^d, d\mathsf{N}_0)}. \end{aligned}$$

Proof. Define the function ψ_ϱ by $\psi_\varrho(x) = \mathbb{E}^x_{\mathcal{N}_0}[\varrho^\tau]$. By the definition of τ we can see that

$$\psi_\varrho(x) = 1, \quad x \in G, \tag{3.10.44}$$

since $\tau = 0$ when X_s starts from the inside of G. Let $\mathscr{F}_t = \sigma(X_s, 0 \le s \le t)$ be the natural filtration of $(X_t)_{t\ge 0}$. By the Markov property of X it is directly seen that

$$\begin{aligned} e^{-(T/2^n)L}\psi_\varrho(x) &= \mathbb{E}^x_{\mathcal{N}_0}[\mathbb{E}^{X_{T/2^n}}_{\mathcal{N}_0}[\varrho^\tau]] \\ &= \mathbb{E}^x_{\mathcal{N}_0}[\mathbb{E}^x_{\mathcal{N}_0}[\varrho^{\tau\circ\theta_{T/2^n}} | \mathscr{F}_{T/2^n}]] = \mathbb{E}^x_{\mathcal{N}_0}[\varrho^{\tau\circ\theta_{T/2^n}}], \end{aligned} \tag{3.10.45}$$

where θ_t is the shift on $\mathfrak{X}$ defined by $(\theta_t\omega)(s) = \omega(s+t)$ for $\omega \in \mathfrak{X}$. Note that

$$(\tau \circ \theta_{T/2^n})(\omega) = \tau(\omega) - T/2^n \ge 0, \tag{3.10.46}$$

when $x = X_0(\omega) \in \mathbb{R}^d \setminus G$. Hence by (3.10.45) and (3.10.46) we have the identity

$$\varrho^{T/2^n} e^{-(T/2^n)L}\psi_\varrho(x) = \psi_\varrho(x), \quad x \in \mathbb{R}^d \setminus G. \tag{3.10.47}$$

It is trivial to see that

$$\int_{\mathbb{R}^d} (\mathbb{E}^x_{\mathcal{N}_0}[\varrho^\tau])^2 d\mathsf{N}_0 = (\psi_\varrho, \psi_\varrho) \le (\psi_\varrho, \psi_\varrho) + \frac{\varrho^{T/2^n}}{1-\varrho^{T/2^n}}(\psi_\varrho, (1-e^{-(T/2^n)L})\psi_\varrho).$$

By using relation (3.10.47) we can compute the right-hand side above as

$$(\psi_\varrho 1_G, \psi_\varrho 1_G) + \frac{\varrho^{T/2^n}}{1-\varrho^{T/2^n}}(\psi_\varrho 1_G, (1-e^{-(T/2^n)L})\psi_\varrho). \tag{3.10.48}$$

Since

$$\begin{aligned} &(\psi_\varrho 1_G, (1-e^{-(T/2^n)L})\psi_\varrho) \\ &= (\psi_\varrho 1_G, (1-e^{-(T/2^n)L})\psi_\varrho 1_G) + (\psi_\varrho 1_G, (1-e^{-(T/2^n)L})\psi_\varrho 1_{\mathbb{R}^d\setminus G}) \\ &= (\psi_\varrho 1_G, (1-e^{-(T/2^n)L})\psi_\varrho 1_G) - (\psi_\varrho 1_G, e^{-(T/2^n)L}\psi_\varrho 1_{\mathbb{R}^d\setminus G}) \\ &\le (\psi_\varrho 1_G, (1-e^{-(T/2^n)L})\psi_\varrho 1_G), \end{aligned}$$

we have

$$\int_{\mathbb{R}^d} (\mathbb{E}^x_{\mathcal{N}_0}[\varrho^\tau])^2 d\mathsf{N}_0 \le (\psi_\varrho 1_G, \psi_\varrho 1_G) + \frac{\varrho^{T/2^n}}{1-\varrho^{T/2^n}}(\psi_\varrho 1_G, (1-e^{-(T/2^n)L})\psi_\varrho 1_G). \tag{3.10.49}$$

Note that $\psi_\varrho 1_G(x) \le \psi(x)$ for all $x \in \mathbb{R}^d$. Then

$$\begin{aligned} &(\psi_\varrho 1_G, \psi_\varrho 1_G) + \frac{\varrho^{T/2^n}}{1-\varrho^{T/2^n}}(\psi_\varrho 1_G, (1-e^{-(T/2^n)L})\psi_\varrho 1_G) \\ &\le (\psi, \psi) + \frac{\varrho^{T/2^n}}{1-\varrho^{T/2^n}}(\psi, (1-e^{-(T/2^n)L})\psi). \end{aligned} \tag{3.10.50}$$

A combination of (3.10.49) and (3.10.50) then proves the lemma. □

3.10.4 Mehler's formula

In this section we consider the special case $V(x) = (\omega^2/2)|x|^2 - \omega/2$ describing the harmonic oscillator, which is one of the few models for which an explicit formula is available for the kernel of the Schrödinger semigroup. This kernel is given by *Mehler's formula*.

Let $d = 1$ and consider the Schrödinger operator

$$H_{\mathrm{osc}} = -\frac{1}{2m}\frac{d^2}{dx^2} + \frac{m\omega^2}{2}x^2 - \frac{\omega}{2} \tag{3.10.51}$$

on $L^2(\mathbb{R})$. Here m is the mass of the oscillator, and ω is the traditional notation for its frequency and should not be confused with its meaning before. H_{osc} is essentially self-adjoint on $C_0^\infty(\mathbb{R})$ by Corollary 3.14, and from Example 3.12 we know that $\mathrm{Spec}(H_{\mathrm{osc}}) = \{m\omega n | n \in \mathbb{N} \cup \{0\}\}$. The ground state is explicitly given by

$$\Psi_{\mathrm{osc}}(x) = (m\omega/\pi)^{1/4} e^{-m\omega x^2/2} \tag{3.10.52}$$

Furthermore, the complete set of eigenfunctions is given by

$$\Psi_n(x) = (m\omega/(4^n(n!)^2\pi))^{1/4}\, e^{-\frac{x^2}{2}} H_n(x), \tag{3.10.53}$$

where H_n, $n = 1, 2, \ldots$, denote the Hermite polynomials. We present two derivations of Mehler's formula, first by using diffusion theory and next by a direct analytic calculation.

Proposition 3.113 (Mehler's formula). *Let*

$$\kappa_t = \frac{1 - e^{-2m\omega t}}{2m\omega}, \quad t \geq 0.$$

Then

$$e^{-tH_{\mathrm{osc}}}(x, y) = \frac{1}{\sqrt{2\pi\kappa_t/m}} \exp\left(-\frac{2xye^{-m\omega t} - (\omega\kappa_t + e^{-2m\omega t})x^2 - (e^{-2m\omega t} - \omega\kappa_t)y^2}{2\kappa_t}\right). \tag{3.10.54}$$

Proof. Let $m, \omega, \sigma > 0$ and consider the stochastic differential equation

$$dX_t = -m\omega X_t dt + \sigma dB_t, \quad B_0 = x. \tag{3.10.55}$$

As seen in Example 2.3.45, its solution is the Ornstein–Uhlenbeck process

$$X_t^x = e^{-m\omega t}x + \sigma \int_0^t e^{-m\omega(t-s)} dB_s. \tag{3.10.56}$$

By a direct calculation

$$\mathbb{E}[X_t^x] = e^{-m\omega t}x, \quad \mathbb{E}[X_t^x X_s^x] = e^{-m\omega(t+s)}x^2 + \frac{\sigma^2}{2m\omega}(1 - e^{-2m\omega(s\wedge t)}) \tag{3.10.57}$$

follow from (3.10.56). The generator of X_t^x is

$$\frac{\sigma^2}{2}\frac{d^2}{dx^2} - m\omega x\frac{d}{dx}. \tag{3.10.58}$$

Now consider (3.10.51) above. Write $X_t^\cdot = X_t$ and $\mathbb{E}_\mu[\cdots] = \int_{-\infty}^\infty \mathbb{E}[\cdots]\Psi_{\rm osc}(x)^2 dx$. Note that

$$\mathbb{E}_\mu[X_t] = 0 \quad \text{and} \quad \mathbb{E}_\mu[X_s X_t] = \frac{\sigma^2}{2m\omega}e^{-|t-s|m\omega}. \tag{3.10.59}$$

Clearly, $\Psi_{\rm osc} \in C^2(\mathbb{R})$. The ground state transform $\mathscr{U}_{\rm g} : L^2(\mathbb{R}, \Psi_{\rm osc}^2 dx) \to L^2(\mathbb{R})$, $\mathscr{U}_{\rm g} f = \Psi_{\rm osc} f$, gives

$$L_{\rm osc} = \mathscr{U}_{\rm g}^{-1} H_{\rm osc} \mathscr{U}_{\rm g} = -\frac{1}{2m}\frac{d^2}{dx^2} + m\omega x\frac{d}{dx},$$

which coincides with (3.10.58) up to sign on putting $1/m = \sigma^2$. We have

$$\mathbb{E}_\mu\Big[\prod_{j=0}^n f_j(X_{t_j})\Big] = \Big(1, f_0\Big(\prod_{j=1}^n e^{-(t_j - t_{j-1})L_{\rm osc}} f_j\Big)1\Big)_{L^2(\mathbb{R}, \Psi_{\rm osc}^2 dx)}$$

for $f_0, \ldots, f_n \in L^\infty(\mathbb{R})$. Noting that $H_{\rm osc} x\Psi_{\rm osc} = m\omega\Psi_{\rm osc}$, we can also check (3.10.59) by the formal calculation

$$\begin{aligned}\mathbb{E}_\mu\,[X_t X_s] &= (\Psi_{\rm osc}, xe^{-(t-s)H_{\rm osc}} x\Psi_{\rm osc}) \\ &= e^{-m\omega|t-s|}\int_{-\infty}^\infty |x|^2 \Psi_{\rm osc}^2(x) dx = \frac{\sigma^2}{2m\omega}e^{-m\omega|t-s|}.\end{aligned}$$

Next we view $e^{-tH_{\rm osc}}$ as an operator in $L^2(\mathbb{R})$ and derive its kernel. By (3.10.56),

$$\mathbb{E}[f(X_t^x)] = \mathbb{E}\left[f\left(e^{-m\omega t}x + \sigma\int_0^t e^{-m\omega(t-s)} dB_s\right)\right].$$

For any fixed t,

$$r_t = \sigma\int_0^t e^{-m\omega(t-s)} dB_s = \sigma e^{-m\omega t}\left(e^{m\omega t} B_t - (m\omega)^{-1}\int_0^t B_s e^{m\omega s} ds\right)$$

is a Gaussian random variable with zero mean and covariance

$$\mathbb{E}[r_t r_t] = (\sigma^2/2m\omega)(1 - e^{-2m\omega t}) = \frac{\sigma^2}{2}\kappa_t.$$

Thus

$$e^{-tH_{\rm osc}} f(x) = \mathscr{U}_{\rm g} e^{-tL_{\rm osc}} \mathscr{U}_{\rm g}^{-1} f(x) = \Psi_{\rm osc}(x)\mathbb{E}[(\Psi_{\rm osc}^{-1} f)(X_t^x)].$$

Since $X_t^x = e^{-m\omega t}x + r_t$ with the Gaussian random variable r_t, the right-hand side above can be computed as

$$\left(2\pi\frac{\sigma^2}{2}\kappa_t\right)^{-1/2} \Psi_{\rm osc}(x) \int_{-\infty}^{\infty} \frac{f(e^{-m\omega t} + y)}{\Psi_{\rm osc}(e^{-m\omega t} + y)} \exp\left(\frac{-|y|^2}{\frac{\sigma^2}{2}\kappa_t}\right) dy. \tag{3.10.60}$$

Changing the variable $e^{-m\omega t} + y$ to z and setting $\sigma^2 = 1/m$, we further obtain that

$$= \int_{-\infty}^{\infty} f(z)\,(\pi\kappa_t/m)^{-1/2} \frac{\Psi_{\rm osc}(x)}{\Psi_{\rm osc}(z)} \exp\left(\frac{-|z - e^{-m\omega t}|^2}{\frac{\kappa_t}{2m}}\right) dz.$$

From this (3.10.54) follows directly. □

Alternative proof: Mehler's formula can also be obtained through analytic continuation $t \mapsto -it$ as a corollary of the theorem below. From now on we simply set $m = \omega = 1$. Then

$$e^{-tH_{\rm osc}}(x, y) = \frac{1}{\sqrt{\pi(1 - e^{-2t})}} \exp\left(-\frac{4xye^{-t} - (x^2 + y^2)(1 + e^{-2t})}{2(1 - e^{-2t})}\right).$$

Let K_t be the integral kernel of $e^{-itH_{\rm osc}}$, and denote $T = \{k\pi | k \in \mathbb{Z}\}$.

Theorem 3.114. *The kernel $K_t : \mathbb{R} \times \mathbb{R} \to \mathbb{C}$ has the following expressions:*

(1) *for every $t \in \mathbb{R} \setminus T$,*

$$K_t(x, y) = \frac{1}{\sqrt{2\pi i \sin t}}\, e^{-i(\frac{x^2+y^2}{2}\cot t - \frac{xy}{\sin t})} \tag{3.10.61}$$

(2) *for every $t \in T$, a.e. $x \in \mathbb{R}$ and $f \in D(H_{\rm osc})$*

$$\lim_{s \to t} \int_{-\infty}^{\infty} K_s(x, y) f(y) = f(x). \tag{3.10.62}$$

Proof. By eigenfunction expansion the solution of the initial value problem

$$-i\partial_t \Psi = H_{\text{osc}}\Psi, \quad \Psi(x,0) = \phi(x)$$

can be written as

$$\Psi(x,t) = \int_{-\infty}^{\infty} K_t(x,y)\phi(y)dy, \tag{3.10.63}$$

with integral kernel

$$K_t(x,y) = \sum_{n=0}^{\infty} \Psi_n(x)\Psi_n(y)e^{-iE_n t}. \tag{3.10.64}$$

A combination of (3.10.53) and (3.10.64) leads to

$$K_t(x,y) = \frac{1}{\sqrt{\pi}}\, e^{-\frac{x^2+y^2}{2}-\frac{it}{2}} \sum_{n=0}^{\infty} \frac{1}{2^n n!} H_n(x)H_n(y)e^{-int}. \tag{3.10.65}$$

Next, the formula $H_n(y) = (-1)^n e^{y^2} \frac{d^n}{dy^n} e^{-y^2}$ known for Hermite polynomials allows to replace the sum appearing at the right-hand side of (3.10.65) by

$$e^{y^2} \sum_{n=0}^{\infty} \frac{1}{n!} \left(-\frac{1}{2}e^{-it}\frac{d}{dy}\right)^n H_n(x)\, e^{-y^2}.$$

The generating function $e^{2sz-z^2} = \sum_{n=0}^{\infty} \frac{1}{n!} H_n(s) z^n$ of Hermite polynomials gives then for the above the formal expression

$$e^{y^2} \exp\left(-xe^{-it}\frac{d}{dy} - \frac{1}{4}e^{-2it}\frac{d^2}{dy^2}\right) e^{-y^2}. \tag{3.10.66}$$

In the following we will argue that this expression actually makes sense.

Clearly, the exponential operator acting on e^{-y^2} factorizes. Both operators are unbounded, however, if the second can be defined, the first is well defined on the same domain. The first operator is a shift by the prefactor of $\frac{d}{dy}$. We consider

$$\mathcal{K}_z = \exp\left(z\frac{d^2}{dy^2}\right), \quad z \in \mathbb{C},$$

and make sense of it on a set including $z = \tau = -(1/4)e^{-2it}$.

Take the open disc $D = \{z \in \mathbb{C} \,|\, |z| < \frac{1}{4}\}$. Note that for every $z \in D$ the function $f(y) = e^{-y^2}$ is an analytic vector of $\mathcal{K}_z$, i.e., $\mathcal{K}_z e^{-y^2} \in L^2(\mathbb{R})$ and is analytic with respect to z:

$$\sum_{n=0}^{\infty}\left\|\frac{z^n}{n!}\frac{d^n}{dy^n}e^{-y^2}\right\| \leq \sum_{n=0}^{\infty}\frac{|z|^n}{n!}\left\|\frac{d^n}{dy^n}e^{-y^2}\right\|$$

$$= \frac{1}{\sqrt{2}}\sum_{n=0}^{\infty}\frac{|z|^n}{n!}\|k^{2n}e^{-k^2/4}\|_2 = \sum_{n=0}^{\infty}c_n|z|^n,$$

where $c_n = (\pi/2)^{1/4}\sqrt{(4n-1)!!}/n!$. It is seen that $c_n/c_{n+1} \to 1/4$ as $n \to \infty$ and this explains the way D was defined.

For every $z \in D$ define $\mathcal{A}_z = \{f \in D(\mathcal{K}_z) | f$ is analytic vector of $\mathcal{K}_z\}$. This set contains, for instance, e^{-y^2}. For every $z \in D$ and $f \in \mathcal{A}_z$ the derivative with respect to z of $\mathcal{K}_z$ exists. Denote $g(y,z) = (\mathcal{K}_z f)(y)$; then we have

$$\partial_z g = \partial_y^2 g, \quad g(y,0) = f(y)$$

since $z = 0 \in D$, i.e., g satisfies the heat equation with the initial condition above, for every $z \in D$. This observation allows to express $g(y,z)$. Define

$$D_f = \left\{ z \in \bar{D} \,\middle|\, \left| \frac{1}{\sqrt{z}}\int_{-\infty}^{\infty} f(u)e^{-\frac{(y-u)^2}{4z}}du \right| < \infty, \ \ \forall y \in \mathbb{R} \right\}$$

as a subset of the closure of D; in particular, we write $D_{\exp}$ for $f(y) = e^{-y^2}$. An easy calculation shows that $D_{\exp}$ lies in D and coincides with the complement in D of the closed disk of radius $1/8$ centered in $(-1/8, 0)$.

It is checked directly that for every $z \in D_f$

$$F_f(y,z) = \frac{1}{2\sqrt{\pi z}}\int_{-\infty}^{\infty} f(u)e^{-\frac{(y-u)^2}{4z}}du, \tag{3.10.67}$$

where the square root is the one with the smallest argument, is a solution of the heat equation above. Moreover, F_f is analytic since $|\partial F_f/\partial z| < \infty$. Take now $z = \zeta \in (0, \frac{1}{4})$. We have in L^2 sense that $F_f(y,\zeta) = g(y,\zeta)$, hence by uniqueness of the analytic continuation

$$g(y,z) = (\mathcal{K}_z f)(y), \quad z \in D_f. \tag{3.10.68}$$

This representation can be extended to the circle ∂D. Pick a number ε such that $|\Re\ \tau| \leq \varepsilon < 1/4$ and define $\mathcal{A}_z^{\varepsilon} = \mathcal{K}_{\varepsilon}L^2(\mathbb{R}) \subset \mathcal{A}_z$. Take a sequence $(z_n)_{n\in\mathbb{N}} \subset D$ such that $z_n \to \tau$ for some $\tau \in D_h$ and $h \in \bigcap_n \mathcal{A}_{z_n}^{\varepsilon}$. Then it is easily seen that pointwise $\mathcal{K}_{z_n}h \to 1/(2\sqrt{\pi\tau})\int_{-\infty}^{\infty} h(u)\exp(-\frac{(\cdot-u)^2}{4\tau})du$ as $n \to \infty$. By the Lebesgue dominated convergence theorem it follows then for every $\varphi \in C_0^{\infty}$ that

$$(\varphi, \mathcal{K}_{z_n}h) \longrightarrow \left(\varphi, \frac{1}{2\sqrt{\pi\tau}}\int_{-\infty}^{\infty} h(u)\exp\left(-\frac{(\cdot-u)^2}{4\tau}\right)du\right)_{L^2}, \quad \text{as } n \to \infty.$$

On the other hand, $\mathcal{K}_\tau h$ is well defined for every $h \in \bigcap_n \mathcal{A}^\varepsilon_{z_n}$ since $\mathcal{K}_\tau h = \mathcal{K}_{\tau+\varepsilon} f_h$, with some $f_h \in \bigcap_n \mathcal{A}_{z_n}$, and $\varepsilon + \Re\,\tau \geq 0$. Moreover, $\mathcal{K}_{z_n} h \to \mathcal{K}_\tau h$ in strong sense. This follows again through dominated convergence by using the continuity of the exponential function, the equalities

$$\begin{aligned}\|\mathcal{K}_\tau h - \mathcal{K}_{z_n} h\|_2^2 &= \|\mathcal{K}_{\tau+\frac{1}{4}} f_h - \mathcal{K}_{z_n+\frac{1}{4}} f_h\|_2^2 \\ &= \int_0^\infty |e^{-(\tau+\frac{1}{4})\lambda} - e^{-(z_n+\frac{1}{4})\lambda}|^2 dE_{f_h}(\lambda)\end{aligned}$$

and the estimate $|e^{-(\tau+\frac{1}{4})\lambda} - e^{-(z_n+\frac{1}{4})\lambda}|^2 \leq 4$, uniform in λ, and making use of the spectral theorem. Hence also $(\varphi, \mathcal{K}_{z_n} h) \to (\varphi, \mathcal{K}_\tau h)$ as $n \to \infty$, $\forall \varphi \in C_0^\infty$, and by comparison

$$(\mathcal{K}_\tau h)(y) = \frac{1}{2\sqrt{\pi\tau}} \int_{-\infty}^{\infty} h(u) e^{-\frac{(y-u)^2}{4\tau}} du \tag{3.10.69}$$

follows, for every $h \in \bigcap_n \mathcal{A}^\varepsilon_{z_n}$, $\tau \in D_h$ and almost every $y \in \mathbb{R}$.

Take an $f_h \in \bigcap_n \mathcal{A}_{z_n}$ and $\varepsilon \geq |\Re\,\tau|$, and look at the solution of $\exp(\varepsilon \partial_y^2) f_h = e^{-y^2}$. On taking Fourier transforms, $e^{-\varepsilon k^2} \hat{f}_h(k) = e^{-k^2/4}$ follows. $\hat{f}_h$, and thus also f_h, will be L^2 whenever $\varepsilon < 1/4$, and then $e^{-y^2} \in \cap_n \mathcal{A}^\varepsilon_{z_n}$. This also explains the way $\mathcal{A}^\varepsilon_z$ was defined. By putting now $h(y) = e^{-y^2}$ in (3.10.69) and using (3.10.66), (3.10.61) comes about by easy manipulations for every τ with $|\Re\,\tau| \leq \varepsilon < 1/4$.

This argument excludes the values of time corresponding to $\tau = (-1/4, 0)$, i.e., the set T The reason is that this is the only value of τ that falls outside of $D_{\exp}$. Informally, for these values the kernel degenerates into a δ-distribution. Take $t \in T$ and a sequence $(t_n)_{n\in\mathbb{N}} \subset \mathbb{R} \setminus T$ such that $t_n \to t$. Since $e^{-itH_{\mathrm{osc}}}$ is a strongly continuous unitary group, we have with some subsequence $(t_m)_{m\in\mathbb{N}}$ for any L^2 function φ that $e^{-it_m H_{\mathrm{osc}}}\varphi(x) \to e^{-itH_{\mathrm{osc}}}\varphi(x)$, for almost every $x \in \mathbb{R}$. Moreover, for almost every $x \in \mathbb{R}$ we have $\int_{-\infty}^{\infty} K_{t_m}(x, y)\varphi(y) dy \to \varphi(x)$, which is seen directly. □

Chapter 4

Gibbs measures associated with Feynman–Kac semigroups

4.1 Gibbs measures on path space

4.1.1 From Feynman–Kac formulae to Gibbs measures

In this chapter we take a different view on the Feynman–Kac formula. The point of departure is the proof of the Feynman–Kac formula using the Trotter product formula, given in Section 3.2.2. There we have seen that for a Schrödinger operator $H = -\frac{1}{2}\Delta + V$ with smooth and bounded potential V, and for $f, g \in L^2(\mathbb{R}^d)$, we have

$$(f, e^{-tH} g) = \lim_{n\to\infty} (f, (e^{-(t/n)H_0} e^{-(t/n)V})^n g),$$

where $H_0 = -(1/2)\Delta$.

By Theorem 3.44 the integral kernel $K_t(x, y)$ of e^{-tH} is continuous in x and y. Writing out the integral kernels of $e^{-(t/n)H_0}$ and letting f and g approach δ distributions yields

$$K_t(x, y) = \lim_{n\to\infty} \int_{\mathbb{R}^{dn}} \frac{1}{(2\pi t/n)^{d/2}} e^{-\sum_{j=0}^{n} \frac{1}{2t/n}|x_{j+1}-x_j|^2 - \sum_{j=1}^{n} V(x_j)} dx_1 \cdots dx_n, \tag{4.1.1}$$

where $x_0 = x$ and $x_{n+1} = y$. For a finite n, the right-hand side above has an interpretation in the context of statistical mechanics. Consider a one-dimensional chain of length $n + 2$, consisting of particles placed in $\mathbb{R}^d$. Suppose that the chain is pinned at both ends, at x and y, respectively. Furthermore, let neighbouring particles be coupled by a quadratic potential of strength $n/2t$, while each particle is subject to an external potential V. The total potential energy of the chain is then just the exponent in (4.1.1). The energetically most optimal position of the chain is a configuration that minimizes the potential energy. In this case it comes about as a compromise between a straight line through x and y, and a configuration in which all x_j are in the global minimum of V (assuming there is one). On the other hand, entropy makes the position of the chain fluctuate around this energetically optimal state. The expression in (4.1.1) defines a finite measure on $(\mathbb{R}^d)^n$, which can be normalized to a probability measure. The latter gives the probability distribution of finding the chain in a subset of positions, and it describes its equilibrium state in which energy and entropy balance each other in an optimal way. Measures such as this are known as Gibbs measures, and have been studied intensely over the last decades, in various contexts.

A peculiarity of the model (4.1.1) is that the limit $n \to \infty$ is taken. Different from the infinite volume limit, which is standard in the classical theory of Gibbs measures and will be discussed below, this is a type of continuum limit for the particle system: while the number of particles grows like n, the pair interaction energy $\frac{n}{2t}|x_{j+1} - x_j|^2$ increases at the same rate. This couples nearby particles increasingly strongly, forcing them to occupy the same position with large probability, and eventually leads to a Brownian path perturbed by the density $\exp(-\int_0^t V(X_s)ds)$. Thus one can view the right-hand side of the Feynman–Kac formula

$$K_t(x, y) = \Pi_t(x - y) \int_{\mathscr{X}} e^{-\int_0^t V(B_s)ds} d\mathcal{W}^{x,y}_{[0,t]} \tag{4.1.2}$$

appearing in Theorem 3.44 as a model of an elastic string fixed at both ends, under the effect of an external potential V.

This view leads to a new intuition of the right-hand side of (4.1.2). Instead of viewing the parameter t of the process B_t as time and thinking of the process as a time evolution, one can think of t as a spatial variable. The following questions then make sense: How does the elastic string B_t behave when we make it longer without stretching it, i.e., increase t? Is there a sensible limit of a string of infinite length? Is that limit unique and which properties does the infinite string have? All of these questions have been asked, and some answered in a variety of models.

Before we present some examples, we informally discuss a few general concepts related to Gibbs measures. From here on we use the notation $(X_t)_{t\geq 0}$ for a random process under investigation extended to the full time line $\mathbb{R}$, which occasionally can be Brownian motion, Brownian motion with a drift, an Itô process or a Lévy process with càdlàg paths. The basic object of interest is

$$d\mu_{[-t,t]}(\omega|x, y) = e^{-U_t(\omega)} d\beta^{x,y}_{[-t,t]}(\omega), \quad t > 0, \tag{4.1.3}$$

where $\beta^{x,y}_{[-t,t]}$ is the appropriate bridge measure for $(X_t)_{t\in\mathbb{R}}$. Note that in our case it is more natural to consider a two-sided process running from $-t$ to t, instead of starting at $t = 0$. $U_t(\omega)$ is the "energy" associated with path $(X_s(\omega))_{s\in[-t,t]}$, which in the case of (4.1.2) is $U_t(\omega) = -\int_0^t V(X_s(\omega))ds$.

In the classic theory of Gibbs measures the *thermodynamic limit* is the passage from microscopic to macroscopic level. In the context of statistical mechanics and thermodynamics the basic problem is to derive the large-scale observable behaviour in equilibrium of a large collection of components from their small-scale interactions. Mathematically this translates in our context into appropriately extending $\mu_{[-t,t]}(\cdot|x, y)$ from a path space over bounded intervals to a path space over the full line $\mathbb{R}$ by taking $t \to \infty$. A basic difficulty is that for the non-trivial types of potentials V encountered U_t almost surely diverges in this limit. Therefore we follow the *Dobrushin–Lanford–Ruelle (DLR) construction* originally applied in statistical mechanics, and look for probability measures on a path space over $\mathbb{R}$ whose family of probability measures conditional on paths outside $[-t, t]$ match with $\mu_{[-t,t]}$, for every $t > 0$. This problem differs essentially from the Kolmogorov extension problem in that here the extension is made by starting from prescribed conditional measures rather than marginals. In practice, solutions to the DLR problem are constructed by taking suitable $t \to \infty$ limits over $x = X_{-t}(\omega)$ and $y = X_t(\omega)$ in $\mu_{[-t,t]}(\cdot|x, y)$, which can be regarded as *boundary conditions*.

We emphasize that although it is possible to find microscopic models as seen above whose continuum limit may be a probability measure such as appearing at the right-hand side of (4.1.2), our present focus is different. Here we are not interested in such models or interpretations. We retain the language of Gibbs measures because the measures we automatically gain from Feynman–Kac semigroups have the structure of a Gibbs measure, i.e., they come about as modifications of an underlying measure by densities dependent on additive functionals indexed by a family of bounded sets. Indeed, this is roughly the extent to which the measures we study and the measures appearing in statistical mechanics can be parallelled. For a start, the class of sample space on which we operate is that of Brownian paths, which differs essentially from the classic cases where the sample space is often a countable product of a finite or countable state space. This rules out the handy tools using compactness of the spaces. Also, our index set is the family of bounded intervals instead of finite subsets of a countable set. The reference measure in our case is Brownian bridge and not a product measure. These differences necessitate to develop specific methods for constructing limit measures on path space since the classic tools are not applicable. A second point we need to make is that even though the parallels between our and the classic cases run mostly on a general level, the concepts and intuitions of the original Gibbsian theory are relevant and useful in our context. While a thermodynamic limit does not interest us for the reason it does practitioners of statistical mechanics and related fields, we do have an interest in limit Gibbs measures. As it will turn out and will be explained below, Gibbs measures on path space can be used to study analytic and spectral properties of the generators of the random processes these measures are associated with. Some of these relationships can only be seen under Gibbs measures on path space extended over the full index set.

Although it is possible to consider Gibbs measures for continuous time random processes in greater generality, our primary interest here is to use them in studying ground state properties of specific quantum field models and therefore we restrict attention to certain classes of functionals U_t.

Example 4.1. Various choices of the energy functional considered below are as follows, some of which will be considered in more detail in the following sections. (For simplicity of notation we do not explicitly feature ω.)

(1) The case

$$U_t = \int_0^t V(X_s)ds \tag{4.1.4}$$

relates with the path measure μ of a Schrödinger operator with Kato decomposable potential V, as seen in Chapter 3. Moreover, for sufficiently regular potentials μ describes in this case the path measure of the $P(\phi)_1$-process

$$dX_t = dB_t + (\nabla \log \Psi)(X_t)dt, \tag{4.1.5}$$

where Ψ is the ground state of the Schrödinger operator.

(2) Let

$$U_t = \int_0^t V(X_s)ds + \int_0^t \int_0^t W^\varphi(X_s - X_r, s - r)dsdr, \tag{4.1.6}$$

where

$$W^\varphi(x,s) = -\frac{1}{4}\int_{\mathbb{R}^d} \frac{|\widehat{\varphi}(k)|^2}{|k|} e^{-ik\cdot x - |k||s|} dk,$$

with a sufficiently fast decaying, spherical symmetric real-valued function φ. Gibbs measures for this U will be discussed in the present chapter, and in Part II of the book we will see how by using them it is possible to derive and prove the ground state properties of the *Nelson model* of an electrically charged spinless quantum particle linearly coupled to a scalar boson field.

(3) Changing $W^\varphi(x,s)$ above for

$$W_{\mathrm{pol}}(x,s) = -\frac{1}{4|x|} e^{-|s|} \tag{4.1.7}$$

describes the *polaron model*. The *bipolaron model* differs by the fact that it consists of two dressed electrons coupled to the same phonon field, which repel each other by Coulomb interaction. In this case

$$U_t = \alpha^2 \int_0^t \int_0^t \mathcal{E}(X_s, X_r, Y_s, Y_r, s - r)dsdr - g\int_0^t \frac{1}{|X_s - Y_s|} ds,$$

where

$$\mathcal{E}(X_s, X_r, Y_s, Y_r, u) = W_{\mathrm{pol}}(X_s - X_r, u) + 2W_{\mathrm{pol}}(X_s - Y_r, u) + W_{\mathrm{pol}}(Y_s - Y_r, u)$$

with $\alpha < 0$ being the polaron-phonon coupling parameter, and $g > 0$ is the strength of the Coulomb repulsion between the two polarons. In this case the reference measure is a product of two Wiener measures of the independent Brownian motions $(X_t)_{t\in\mathbb{R}}$ and $(Y_t)_{t\in\mathbb{R}}$.

(4) Let

$$U_t = \int_0^t \int_0^t \delta(X_s - X_r)dsdr. \tag{4.1.8}$$

This gives the *intersection local time* of Brownian motion describing a polymer model with short-range soft-core interaction encouraging to avoid self-intersections.

(5) Let

$$U_t = \int_0^t V(X_s)ds + \int_0^t \int_0^t W(X_s - X_r, s - r)dX_s \cdot dX_r \tag{4.1.9}$$

with some choices of W which we do not write down explicitly here. This describes the *Pauli–Fierz model* of a charged particle interacting with a vector boson field. The difference from the cases above consists in having double stochastic integrals here instead of double Riemann integrals. Gibbs measures of this type will be discussed when we study the Pauli–Fierz model in Part II.

(6) Let

$$U_t = \int_0^t \int_0^t \frac{1}{|X_t - X_s|} \, dX_t \cdot dX_s. \tag{4.1.10}$$

This is used in describing *turbulent fluids*. The above formal expression gives the total energy of a vorticity field concentrated along Brownian curves $X_t \in \mathbb{R}^3$ obtained from a given divergence-free velocity field.

(7) Let

$$U_t = \int_0^t V(X_s)ds \tag{4.1.11}$$

where $(X_t)_{t\geq 0}$ is a symmetric α-stable process is obtained for a fractional Schrödinger operator $(-\Delta)^{\alpha/2} + V$, $\alpha \in (0, 2)$. The case $\alpha = 1$ describes a *massless relativistic quantum* particle. A similar expression can be obtained when $(X_t)_{t\geq 0}$ is a relativistic α-stable process generated by the operator $(-\Delta - m^{2/\alpha})^{\alpha/2} + V$, $m > 0$, and $\alpha = 1$ describes a *massive relativistic quantum* particle.

Our main concern here will be to construct infinite volume Gibbs measures covering Examples (1) and (2). A main benefit will be a study, through the appropriate Feynman–Kac-type formulae, of ground state properties of quantum models in terms of averages of suitable functions on path space with respect to these Gibbs measures, which will be carried out in Part II below.

4.1.2 Definitions and basic facts

Let $(\Omega, \mathscr{F}, P)$ be a given probability space, and $(X_t)_{t\in\mathbb{R}}$ be an $\mathbb{R}^d$-valued reversible two-sided Markov process on $(\Omega, \mathscr{F}, P)$ with almost surely continuous paths. For $I \subset \mathbb{R}$ we use the notation $\mathscr{F}_I$ for the sub-σ-field generated by $(X_t)_{t\in I}$, and $\mathscr{T}_I = \mathscr{F}_{I^c}$, where $I^c = \mathbb{R} \setminus I$, as well as $\mathscr{F}_T$ and $\mathscr{T}_T$ for $I = [-T, T]$, $T > 0$. The tail-field will be denoted by $\mathscr{T} = \bigcap_{N\in\mathbb{N}} \mathscr{T}_N$. Also, $|I|$ stands for the Lebesgue measure of I.

Definition 4.1 (Index set and state space). The collection $\{I \subset \mathbb{R} \,|\, |I| < \infty\}$ of bounded intervals of the real line is the *index set*, and the *state space* is $\mathbb{R}^d$.

Let ν be the path measure of the given Markov process $(X_t)_{t\in\mathbb{R}}$, which we use as *reference measure*. Recall that $\mathscr{X} = C(\mathbb{R}, \mathbb{R}^d)$.

Definition 4.2 (Potentials). (1) A Borel measurable function $V : \mathbb{R}^d \to \mathbb{R}$ is called an *external potential*. We call V a *ν-admissible external potential* whenever

$$0 < \int_{\mathscr{X}} e^{-\int_I V(X_s(\omega))\,ds}\, \nu(d\omega) < \infty \tag{4.1.12}$$

for every bounded interval $I \subset \mathbb{R}$.

(2) A function $W : \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is called a *pair interaction potential*. We say that W is admissible whenever

$$\sup_{t\in\mathbb{R}} \int_{-\infty}^{\infty} \sup_{x,y\in\mathbb{R}^d} |W(x,y,s,t)|\, ds < \infty. \tag{4.1.13}$$

If $W(x,y,s,t) = W(x,y,s-t,0)$, we use the simpler notation $W(x,y,t)$; similarly, we write $W(x,t)$ in case $W(x,y,s,t) = W(x-y,0,s-t,0)$.

For brevity we will henceforth refer to the potentials jointly satisfying the conditions above as admissible. The condition on the pair potential is rather strong, however, as we move on we will introduce weaker regularity conditions. In the Feynman–Kac formula considered at the beginning of this chapter, (4.1.12) holds for Kato-decomposable potentials V by Lemma 3.38. While n-body potentials can be defined in a straightforward way, in what follows we will not need more general potentials than the single and two-body potentials V and W.

Definition 4.3. For fixed $\bar\omega \in \mathscr{X}$ denote by $\nu_T^{\bar\omega}$ the unique measure on $(\mathscr{X}, \mathscr{F})$ such that for all f bounded and measurable with respect to $\mathscr{F}_T$, and all g bounded and measurable with respect to $\mathscr{T}_T$

$$\mathbb{E}_{\nu_T^{\bar\omega}}[fg] = \mathbb{E}_\nu[f|\mathscr{T}_T](\bar\omega)g(\bar\omega) \tag{4.1.14}$$

holds.

For general measurable f, $\int f\, d\nu_T^{\bar\omega}$ is obtained from (4.1.14) by approximation of f. Put differently, $\nu_T^{\bar\omega}$ is the product measure on $C([-T,T];\mathbb{R}^d) \times C([-T,T]^c;\mathbb{R}^d)$ of the regular conditional probability of ν given $\omega(t) = \bar\omega(t)$, for all $|t| > T$, and the Dirac measure concentrated at $\bar\omega$ on $C([-T,T]^c;\mathbb{R}^d)$. Thus, intuitively, $\nu_T^{\bar\omega}$ is the measure obtained from ν by fixing the path to $\bar\omega$ outside $[-T,T]$ and allowing it to fluctuate inside $[-T,T]$.

It is clear that $\nu_T^{\bar{\omega}}$ is indeed a version of $\nu(\cdot|\mathscr{T}_T)$, and it is probably the most natural one. We emphasize that we had many choices of possible versions here given that conditional expectations are only defined up to sets of measure zero. By a law of the iterated logarithm-type argument, measures of Markov processes are usually concentrated on a set of paths that is characterized by the asymptotic behaviour of $X_t(\omega)$ as $|t| \to \infty$. Thus had we defined $\nu(\cdot|\mathscr{T}_T)$ arbitrarily on the set $N \subset \mathfrak{X}$ of measure zero consisting of all paths not having this precise asymptotic behavior, we still would have obtained a version of the conditional expectation. Our definition of $\nu_T^{\bar{\omega}}$ reduces this element of arbitrariness considerably, and we will need this when defining Gibbs measures.

Without restricting generality, we choose symmetric bounded intervals from the index set. Write

$$\begin{aligned}\Lambda(S,T) &= ([-S,S]\times[-T,T]) \cup ([-T,T]\times[-S,S]),\\ \Lambda(T) &= (\mathbb{R}\times[-T,T]) \cup ([-T,T]\times\mathbb{R}),\end{aligned}$$

where $S, T > 0$. For $S \le T$ and admissible potentials V, W we use the notation

$$\mathscr{H}_{\Lambda(S,T)}(X) = \int_{-T}^{T} V(X_s)\,ds + \iint_{\Lambda(S,T)} W(X_t, X_s, |t-s|)\,ds\,dt, \tag{4.1.15}$$

and define $\mathscr{H}_{\Lambda(T)}$ by replacing $\Lambda(S,T)$ with $\Lambda(T)$ in (4.1.15). We will also write $\mathscr{H}_T$ instead of $\mathscr{H}_{\Lambda(T,T)}$. To ease the notation we do not explicitly indicate the ω-dependence of these functionals.

Definition 4.4 (Gibbs measure). A probability measure μ on $(\mathfrak{X}, \mathscr{F})$ is called a *Gibbs measure* with respect to reference measure ν and admissible potentials V, W if for every $T > 0$,

(1) $\mu|_{\mathscr{F}_T} \ll \nu|_{\mathscr{F}_T}$ for all $T > 0$,

(2) for every bounded $\mathscr{F}$-measurable function f,

$$\mathbb{E}_\mu[f|\mathscr{T}_T](\bar{\omega}) = \frac{\mathbb{E}_{\nu_T^{\bar{\omega}}}[f e^{-\mathscr{H}_{\Lambda(T)}}]}{\mathbb{E}_{\nu_T^{\bar{\omega}}}[e^{-\mathscr{H}_{\Lambda(T)}}]}, \quad \mu\text{-a.s.} \tag{4.1.16}$$

A probability measure μ_T on $(\mathfrak{X}, \mathscr{F})$ is called *finite volume Gibbs measure* for the interval $[-T, T]$ if

(1′) $\mu|_{\mathscr{F}_S} \ll \nu|_{\mathscr{F}_S}$ for all $S < T$

(2′) for all $0 < S < T$ and every bounded $\mathscr{F}$-measurable function f,

$$\mathbb{E}_{\mu_T}[f|\mathscr{T}_S](\bar{\omega}) = \frac{\mathbb{E}_{\nu_S^{\bar{\omega}}}[f e^{-\mathscr{H}_{\Lambda(S,T)}}]}{\mathbb{E}_{\nu_S^{\bar{\omega}}}[e^{-\mathscr{H}_{\Lambda(S,T)}}]}, \quad \mu_T\text{-a.s.} \tag{4.1.17}$$

In our terminology "Gibbs measure" corresponds to "infinite volume Gibbs measure" in the classic theory. We refer to the path $\bar{\omega}$ as a *boundary condition* or *boundary path*. The normalization factor

$$Z_T(\bar{\omega}) = \mathbb{E}_{\nu_T^{\bar{\omega}}}[e^{-\mathcal{H}_{\Lambda(T)}}]$$

is traditionally called *partition function*.

Remark 4.1 (DLR equations). Conditions (2) and (2′) in the above definition are traditionally called *Dobrushin–Lanford–Ruelle (DLR) equations*. These are in fact equalities and can be viewed as counterparts for conditional measures of the Kolmogorov consistency relations for marginal measures.

Remark 4.2 (Gibbs specification). To make it more visible that a Gibbs measure is a probability measure which has prescribed conditional probability kernels we give the following equivalent definition of a Gibbs measure: A probability measure μ on $(\mathfrak{X}, \mathscr{F})$ is a *Gibbs measure* for the admissible potentials V, W if for every $A \in \mathscr{F}$ and $T > 0$ the function $\bar{\omega} \mapsto \mu_T(A, \bar{\omega})$ is a version of the conditional probability $\mu(A|\mathscr{T}_T)$, i.e.,

$$\mu(A|\mathscr{T}_T)(\bar{\omega}) = \mu_T(A, \bar{\omega}), \quad A \in \mathscr{F},\ T > 0,\ \text{a.e. } \bar{\omega} \in \mathfrak{X} \tag{4.1.18}$$

where

$$\mu_T(A, \bar{\omega}) = \frac{1}{Z_T(\bar{\omega})} \int_{\mathfrak{X}} 1_A e^{-\mathcal{H}_{\Lambda(T)}} d\nu_T^{\bar{\omega}}, \quad A \in \mathscr{F},\ \bar{\omega} \in \mathfrak{X}. \tag{4.1.19}$$

Remark 4.3. To understand why (1) in Definition 4.4 is necessary consider the special version $\nu_T^{\bar{\omega}}$ of $\mathbb{E}_\nu[\cdot|\mathscr{T}_T](\bar{\omega})$. Writing the (on a first sight more natural) expression

$$\mathbb{E}_\mu[f|\mathscr{T}_T] = \frac{\mathbb{E}_\nu[fe^{-\mathcal{H}_{\Lambda(T)}}|\mathscr{T}_T]}{\mathbb{E}_\nu[e^{-\mathcal{H}_{\Lambda(T)}}|\mathscr{T}_T]} \tag{4.1.20}$$

instead of (4.1.16) is in most cases meaningless. The problem is that the left-hand side of the above equality is only defined uniquely outside a set N of μ-measure zero, while the right-hand side is only defined uniquely outside a set M of ν-measure zero. In many cases of interest, ν and μ are mutually singular on $\mathfrak{X}$, and then (4.1.20) is no condition at all. Thus in our case sets of measure zero do matter and extra care is needed. The reason why we require (1) explicitly is the same. Without it the two sides of (2) may refer to functions that are uniquely defined only on disjoint subsets of the probability space. In case of a finite dimensional state space and the measure of a Feller Markov process as reference measure, this inconvenience can be avoided and then (1) is unnecessary.

Constructing a Gibbs measure for given potentials V and W can be quite difficult. However, constructing a finite volume Gibbs measure under the given regularity conditions is straightforward.

Proposition 4.1. *Let ν be a given reference measure, $T > 0$, and V, W be admissible potentials. If $0 < \mathbb{E}_\nu[e^{-\mathcal{H}_T}] < \infty$, then*

$$d\mu_T = \frac{e^{-\mathcal{H}_T}}{\mathbb{E}_\nu[e^{-\mathcal{H}_T}]}\, d\nu$$

is a finite volume Gibbs measure for $[-T, T]$.

Proof. By assumption $e^{-\mathcal{H}_T}$ is ν-integrable. Thus $\mu_T \ll \nu$, and in particular $\mu_T|_{\mathscr{F}_T} \ll \nu|_{\mathscr{F}_T}$. To check DLR consistency let f and g be bounded, and suppose g is $\mathscr{T}_S$-measurable. Then for $T > S$

$$\begin{aligned}
&\mathbb{E}_\nu[e^{-\mathcal{H}_T}]\mathbb{E}_{\mu_T}\left[g\frac{\mathbb{E}_{\nu_S^{\cdot}}[fe^{-\mathcal{H}_{\Lambda(S,T)}}]}{\mathbb{E}_{\nu_S^{\cdot}}[e^{-\mathcal{H}_{\Lambda(S,T)}}]}\right] = \mathbb{E}_\nu\left[\mathbb{E}_\nu\left[g\frac{\mathbb{E}_{\nu_S^{\cdot}}[fe^{-\mathcal{H}_{\Lambda(S,T)}}]}{\mathbb{E}_{\nu_S^{\cdot}}[e^{-\mathcal{H}_{\Lambda(S,T)}}]}e^{-\mathcal{H}_T}\middle|\mathscr{T}_S\right]\right] \\
&= \mathbb{E}_\nu\left[g\frac{\mathbb{E}_{\nu_S^{\cdot}}[fe^{-\mathcal{H}_{\Lambda(S,T)}}]}{\mathbb{E}_{\nu_S^{\cdot}}[e^{-\mathcal{H}_{\Lambda(S,T)}}]}e^{-\mathcal{H}_{[-T,T]^2\setminus\Lambda(S,T)}}\mathbb{E}_\nu[e^{-\mathcal{H}_{\Lambda(S,T)}}|\mathscr{T}_S]\right] \\
&= \mathbb{E}_\nu[e^{-\mathcal{H}_T}]\mathbb{E}_{\mu_T}[fg].
\end{aligned}$$

The last equality is due to the fact that $\mathbb{E}_{\nu_S^{\cdot}}[e^{-\mathcal{H}_{\Lambda(S,T)}}]$ is a version of the conditional expectation $\mathbb{E}_\nu[e^{-\mathcal{H}_{\Lambda(S,T)}}|\mathscr{T}_S]$. Dividing by $\mathbb{E}_\nu[e^{-\mathcal{H}_T}]$ shows that condition (2) in Definition 4.4 is verified. □

Remark 4.4 (Stochastic and sharp boundary conditions). The finite volume Gibbs measure that we constructed above corresponds to free boundary conditions. Virtually the same proof as above shows that for every $\bar{\omega} \in \mathfrak{X}$, the measure

$$d\mu_T^{\bar{\omega}} = \frac{e^{-\mathcal{H}_{\Lambda(T)}}}{\mathbb{E}_{\nu_T^{\bar{\omega}}}[e^{-\mathcal{H}_{\Lambda(T)}}]} d\nu_T^{\bar{\omega}} \tag{4.1.21}$$

is a finite volume Gibbs measure. We call it a finite volume Gibbs measure with *sharp boundary condition $\bar{\omega}$*. Occasionally, instead of pinning down path positions in a sharp boundary condition, it is more convenient to give only their probability distributions. Let $h : \mathfrak{X} \to \mathbb{R}$ be a $\mathscr{T}_T$-measurable non-negative function such that

$$0 < \mathbb{E}_\nu[he^{-\mathcal{H}_T}] < \infty.$$

Then as in Proposition 4.1 we can show that

$$d\mu_T^h = \frac{he^{-\mathcal{H}_{\Lambda(T)}}}{\mathbb{E}_\nu[he^{-\mathcal{H}_{\Lambda(T)}}]}\, d\nu$$

is a finite volume Gibbs measure for $[-T, T]$. The density h can then be regarded as a *stochastic boundary condition.*

Next we show that limits of finite volume Gibbs measures yield Gibbs measures in the sense of Definition 4.4. The following notion of convergence will be used.

Definition 4.5 (Local convergence). A sequence $(m_n)_{n\in\mathbb{N}}$ of probability measures on $\mathfrak{X}$ is said to be *locally convergent* to a probability measure m if for every $0 < T < \infty$ and every $A \in \mathscr{F}_T$, $m_n(A) \to m(A)$ as $n \to \infty$.

Remark 4.5. (1) In many cases Gibbs measures are mutually singular with respect to their reference measures, while finite volume Gibbs measures are absolutely continuous with respect to them. Thus the topology of local convergence is the strongest reasonable for such Gibbs measures. In particular, in many cases it is possible to find a tail measurable function f such that $\int f\, d\nu_N = 0$ for all N, while $\int f\, d\nu = 1$.

(2) Convergence on sets is usually a very strong requirement for the convergence of measures. Indeed, the situation discussed in the previous remark indicates that convergence practically never holds for all sets of $\mathscr{F}$. Since, however, we only need to check convergence on elements of $\mathscr{F}_T$ for finite T, the special structure of Gibbs measures allows to show local convergence of finite volume Gibbs measures in all cases we consider below.

Proposition 4.2. *Let ν be a reference measure and V, W be admissible potentials. Let $(\mu_n)_{n\in\mathbb{N}}$ be a sequence of finite volume Gibbs measures for V, W and $[-T_n, T_n]$, with $T_n \to \infty$ as $n \to \infty$. Suppose that there exists a probability measure μ on $\mathfrak{X}$ such that $\mu_n \to \mu$ in the topology of local convergence. Suppose, moreover, that μ satisfies* (1) *in Definition* 4.4 *with respect to ν. Then μ is a Gibbs measure for V, W.*

Proof. As (1) in Definition 4.4 holds by assumption, only DLR consistency remains to be shown. Each μ_n is a finite volume Gibbs measure, thus for bounded f, g with $\mathscr{T}_S$-measurable g we have

$$\mathbb{E}_{\mu_n}\left[g \frac{\mathbb{E}_{\nu_S^{\cdot}}[f e^{-\mathscr{H}_{\Lambda(S,T_n)}}]}{\mathbb{E}_{\nu_S^{\cdot}}[e^{-\mathscr{H}_{\Lambda(S,T_n)}}]} \right] = \mathbb{E}_{\mu_n}[fg] \tag{4.1.22}$$

for every $S < T_n$. We show now that (4.1.22) remains valid in the $n \to \infty$ limit. By a monotone class argument we may assume that both f and g are $\mathscr{F}_R$-measurable for some $R > 0$. In this case, the right-hand side of (4.1.22) converges by definition of local convergence. At the left-hand side (4.1.13) implies that

$$\iint_{\Lambda(S,T_n)} W(X_s, X_t, |t-s|)\, ds\, dt \stackrel{n\to\infty}{\longrightarrow} \iint_{\Lambda(S)} W(X_s, X_t, |t-s|)\, ds\, dt$$

uniformly in path, and thus

$$F_n(X(\omega)) = \frac{\mathbb{E}_{\nu_S^\omega}[fe^{-\mathcal{H}_{\Lambda(S,T_n)}}]}{\mathbb{E}_{\nu_S^\omega}[e^{-\mathcal{H}_{\Lambda(S,T_n)}}]} \overset{n\to\infty}{\longrightarrow} \frac{\mathbb{E}_{\nu_S^\omega}[fe^{-\mathcal{H}_{\Lambda(S)}}]}{\mathbb{E}_{\nu_S^\omega}[e^{-\mathcal{H}_{\Lambda(S)}}]} = F(X(\omega))$$

uniformly in $\omega \in \mathfrak{X}$. Thus for every $\varepsilon > 0$ there is an $N \in \mathbb{N}$ such that $\|F_m - F\|_\infty < \varepsilon$, whenever $m \geq N$. By the triangle inequality,

$$|\mathbb{E}_{\mu_n}[gF_n] - \mathbb{E}_\mu[gF]| \leq |\mathbb{E}_{\mu_n}[gF_N] - \mathbb{E}_\mu[gF_N]| + 3\|g\|_\infty\varepsilon \overset{n\to\infty}{\longrightarrow} 3\|g\|_\infty\varepsilon.$$

Since ε is arbitrary, convergence of the left-hand side in (4.1.22) follows. □

The previous proposition shows that suitable limits of finite volume Gibbs measures over boundary conditions result in Gibbs measures. The next statement shows that this procedure actually yields all Gibbs measures supported on a specific set of paths.

Proposition 4.3. *Let $\mathfrak{X}^* \subset \mathfrak{X}$ be measurable and μ be a Gibbs measure supported on $\mathfrak{X}^*$, with reference measure ν. For $N \in \mathbb{N}$, $\omega \in \mathfrak{X}$, define*

$$\mathbb{E}_{\mu_N^\omega}[f] = \frac{\mathbb{E}_{\nu_N^\omega}[fe^{-\mathcal{H}_{\Lambda(N)}}]}{\mathbb{E}_{\nu_N^\omega}[e^{-\mathcal{H}_{\Lambda(N)}}]}$$

as in (4.1.16). *Suppose that for every $T > 0$, $A \in \mathscr{F}_T$ and $\omega \in \mathfrak{X}^*$ we have $\mu_N^\omega(A) \to \mu(A)$ as $N \to \infty$. Then μ is the unique Gibbs measure associated with $\mathcal{H}$, supported on $\mathfrak{X}^*$.*

Proof. Let $\tilde{\mu}$ be a Gibbs measure supported by $\mathfrak{X}^*$. For every $T < N$ and $A \in \mathscr{F}_T$, $\bar{\omega} \mapsto \tilde{\mu}(A|\mathscr{T}_N)(\bar{\omega})$ is a backward martingale in N, thus convergent almost everywhere to $\tilde{\mu}(A|\mathscr{T})(\bar{\omega})$. By the Gibbs property $\tilde{\mu}(A|\mathscr{T}_N)(\bar{\omega}) = \mu_N^{\bar{\omega}}(A)$ $\tilde{\mu}$-a.s., and thus for $\tilde{\mu}$-a.e. $\bar{\omega} \in \mathfrak{X}^*$ there exists $\tilde{\mu}(A|\mathscr{T})(\bar{\omega}) = \lim_{N\to\infty} \tilde{\mu}(A|\mathscr{T}_N)(\bar{\omega}) = \lim_{N\to\infty} \mu_N^{\bar{\omega}}(A) = \mu(A)$. Taking $\tilde{\mu}$-expectations on both sides of the above equality shows $\tilde{\mu}(A) = \mu(A)$, and since this holds for every $A \in \mathscr{F}_T$ and $T > 0$, we obtain $\tilde{\mu} = \mu$. □

Finally, we give a useful criterion for a sequence of finite volume Gibbs measures to converge along a sequence of boundary conditions.

Definition 4.6. We say that a family of probability measures $(\mu_T)_{T>0}$ is *locally uniformly dominated* by a family of probability measures $(\tilde{\mu}_S)_{S>0}$ whenever

(1) $\tilde{\mu}_S$ is a probability measure on $\mathscr{F}_{[-S,S]}$,

(2) for every $\varepsilon > 0$ and $S > 0$ there exists $\delta_S > 0$ such that $\limsup_{T\to\infty} \mu_T(A) < \varepsilon$ when $A \in \mathscr{F}_{[-S,S]}$ with $\tilde{\mu}_S(A) < \delta_S$.

The significance of uniform local domination is made clear by the following result.

Proposition 4.4. *Let $(T_n)_{n\in\mathbb{N}}$ be an increasing and divergent sequence of positive numbers and assume that $(\mu_{T_n})_{n\in\mathbb{N}}$ is a locally uniformly dominated family of finite volume Gibbs measures. Then $(\mu_{T_n})_{n\in\mathbb{N}}$ has a convergent subsequence in the local strong topology and the limit is a Gibbs measure.*

Proof. Let $S > 0$ and write $\tilde{\mu}_S$ for the measure dominating μ_n, i.e., suppose that for $\varepsilon > 0$ there exists $\delta_S > 0$ such that $\limsup_{n\to\infty} \mu_n(A) < \varepsilon$, for all A satisfying $\tilde{\mu}_S(A) < \delta_S$. Since $\tilde{\mu}_S$ is a measure we have $\lim_{m\to\infty} \mu_S(A_m) = 0$. In particular, $\tilde{\mu}_S(A_m) < \delta_S$, for all $m > M$ with some M. Thus $\limsup_{n\to\infty} \mu_n(A_m) < \varepsilon$ for all $m > M$ by the local uniform domination property. This means that $\mu_\infty(A_m) < \varepsilon$ for all $m > M$ and hence $\lim_{m\to\infty} \mu_\infty(A_m) = 0$. Thus a limit measure on $\mathscr{F}_S$ exists. For $\mathscr{F}_{S+1}$ we take a further subsequence of the subsequence μ_n and find a measure μ'_∞ on $\mathscr{F}_{S+1}$ whose restriction to $\mathscr{F}_S$ is μ_∞. Continuing inductively, we find a sequence of measures μ_S for each $S \in \mathbb{N}$; they form a consistent family of probability measures and thus by the Kolmogorov extension theorem there is a unique probability measure μ on $\mathscr{F}$ such that all $\mu|_{\mathscr{F}_S} = \mu_S$. This proves existence of a convergent subsequence. Since the DLR property holds in finite volume, due to convergence along a subsequence it carries over in the limit. □

4.2 Existence and uniqueness by direct methods

4.2.1 External potentials: existence

In this section we present results on the existence and uniqueness of Gibbs measures on Brownian paths, using only probabilistic techniques. A powerful alternative, yielding additional insight, is the method of cluster expansion, which will be discussed in Section 4.3.

We consider admissible potentials in the sense of Definition 4.2, and start with the case $W = 0$. Moreover, we restrict attention to the case where the reference measure is Brownian bridge measure.

Recall the notation

$$\Pi_t(x) = \frac{1}{(2\pi t)^{d/2}} \exp\left(-\frac{|x|^2}{2t}\right).$$

For any interval $I = [s,t] \subset \mathbb{R}$ define

$$\mathscr{W}_I^{x,y} = \Pi_{t-s}(y-x)\mathcal{W}_I^{x,y}, \tag{4.2.1}$$

where $\mathcal{W}_I^{x,y}$ denotes the Brownian bridge measure. Although $\mathscr{W}_I^{x,y}$ is a finite measure on $C(I,\mathbb{R}^d)$, it is not a probability measure. However, it makes no difference in the definition of finite volume Gibbs measures below whether we use $\mathscr{W}_I^{x,y}$ or $\mathcal{W}_I^{x,y}$, and using the former has some advantages in calculations.

Definition 4.7 (Itô bridge). Let V be an admissible potential. For any bounded $I \subset \mathbb{R}$ and $x, y \in \mathbb{R}^d$ define

$$\nu_I^{x,y}(A) = \frac{1}{Z_I(x,y)} \int_{\mathfrak{X}} 1_A e^{-\int_I V(X_t)dt} d\mathscr{W}_I^{x,y}, \tag{4.2.2}$$

for every $A \in \mathcal{F}_I$, where

$$Z_I(x,y) = \int_{\mathfrak{X}} e^{-\int_I V(X_t)dt} d\mathscr{W}_I^{x,y}. \tag{4.2.3}$$

We call $\nu_I^{x,y}$ *Itô bridge measure* for potential V.

The following result identifies a large class of admissible potentials V.

Lemma 4.5. *Let $V \in \mathcal{K}(\mathbb{R}^d)$. Then for any bounded interval $I \subset \mathbb{R}$*

$$\sup_{x,y\in\mathbb{R}^d} \int e^{-\int_I V(X_t)dt} d\mathscr{W}_I^{x,y} < \infty. \tag{4.2.4}$$

Proof. This follows from Khasminskii's Lemma 3.37. □

Let V be a potential such that H is a Schrödinger operator with a unique strictly positive ground state Ψ_{p}. By adding a constant, if necessary, we choose V such that $\inf \mathrm{Spec}(H) = 0$. Take any $I \subset \mathbb{R}$, any division $t_1 < \cdots < t_n \in I$, and functions $f_1, \ldots, f_n \in L^2(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$. Recall that the $P(\phi)_1$-process $(X_t)_{t\in\mathbb{R}}$ associated with H satisfies

$$\mathbb{E}_{\mathcal{N}_0}\Big[\prod_{j=1}^n f_j(X_{t_j})\Big] = (f_1\Psi_{\mathrm{p}}, e^{-(t_2-t_1)L} f_2 \cdots e^{-(t_n-t_{n-1})L} f_n\Psi_{\mathrm{p}})_{L^2(\mathbb{R}^d)} \tag{4.2.5}$$

where L is the ground state transform defined in Section 3.10.

From now on we rewrite

$$\nu = \mathcal{N}_0 \tag{4.2.6}$$

and refer to ν as *Itô measure* for potential V. The Feynman–Kac formula then gives

$$\nu(A) = \int_{\mathbb{R}^d} dx \Psi_{\mathrm{p}}(x) \int_{\mathbb{R}^d} dy \Psi_{\mathrm{p}}(y) \int 1_A e^{-\int_I V(X_t)dt} d\mathscr{W}_I^{x,y} \tag{4.2.7}$$

for every $A \in \mathscr{F}_I$.

Lemma 4.6. *For $V \in \mathcal{K}(\mathbb{R}^d)$ and any $I = [a, b] \subset \mathbb{R}$ the restriction $\nu|_{\mathcal{F}_I}$ is absolutely continuous with respect to $\mathcal{W}|_{\mathcal{F}_I}$, and the density is*

$$h_I(\omega) = \Psi_\mathrm{p}(X_a) \exp\left(-\int_I V(X_t)dt\right) \Psi_\mathrm{p}(X_b). \tag{4.2.8}$$

Proof. This follows directly from (4.2.7) since by the definition of Wiener measure

$$\mathbb{E}_\mathcal{W}[f(X_a)g(X_b)h] = \int_{\mathbb{R}^d} dx \int_{\mathbb{R}^d} dy f(x)g(y) \int h d\mathscr{W}_I^{x,y}$$

holds for bounded measurable $f, g : \mathbb{R}^d \to \mathbb{R}$ and bounded, $\mathscr{F}_I$-measurable h. □

Write

$$K_t(x, y) = \int e^{-\int_0^t V(X_s)ds} d\mathscr{W}_{[0,t]}^{x,y} \tag{4.2.9}$$

for the integral kernel of e^{-tH}, and

$$\pi_t(x, y) = \frac{K_t(x, y)}{\Psi_\mathrm{p}(x)\Psi_\mathrm{p}(y)} \tag{4.2.10}$$

for the transition density of the $P(\phi)_1$-process. Since

$$(e^{-tL} f)(x) = \int_{\mathbb{R}^d} \pi_t(x, y) f(y) \Psi_\mathrm{p}^2(y) dy$$

holds, π_t is the probability transition kernel of e^{-tL} in $L^2(\mathbb{R}^d, \Psi_\mathrm{p}^2 dx)$. Recall that L is given by (3.10.3) and (3.10.10). Then

$$\nu_I^{x,y}(A) = \frac{\Pi_{T_2 - T_1}(x - y)}{\Psi_\mathrm{p}(x)\Psi_\mathrm{p}(y)\pi_{T_2 - T_1}(x, y)} \mathbb{E}_{\mathscr{W}_I^{x,y}}[1_A e^{-\int_I V(X_t)dt}]. \tag{4.2.11}$$

We prove now that an Itô measure indeed is a Gibbs measure in the sense of Definition 4.4.

Theorem 4.7 (Existence of Gibbs measure). *Let V be Kato-decomposable and ν be the Itô-measure for potential V. Then ν is a Gibbs measure for V.*

Proof. We need to check is that (4.2.2) is a conditional probability $\nu(A|\mathscr{T}_I)$. However, this follows immediately from (4.2.11) and the representation (4.2.7) of ν. □

We end this section by pointing out that any two Itô measures are distinguished by their potentials.

Proposition 4.8. *Take $V_1, V_2 \in \mathcal{K}(\mathbb{R}^d)$ and their Itô measures ν_1, ν_2, respectively. If $V_1 - V_2$ is different from a constant outside of a set of Lebesgue measure zero, then ν_1 and ν_2 are mutually singular measures.*

Proof. If V_1 and V_2 differ by a non-constant term, then the ground state ψ_1 of $H_1 = -\frac{1}{2}\Delta + V_1$ is different from the ground state ψ_2 of $H_2 = -\frac{1}{2}\Delta + V_2$. As both are continuous functions, there is $A \subset \mathbb{R}^d$ such that

$$a_1 = \nu_1(X_0 \in A) = \int_A \psi_1^2(x)dx \neq \int_A \psi_2^2(x)dx = \nu_2(X_0 \in A) = a_2.$$

Both ν_1 and ν_2 are stationary, and thus the ergodic theorem below implies

$$\lim_{N\to\infty} \frac{1}{n} \sum_{n=0}^{N} 1_A(X_n) = a_i \quad \text{for } \nu_i\text{-almost all paths } X,$$

for $i = 1, 2$. In other words, the set $\{\omega \in \mathfrak{X} | \lim_{N\to\infty} \frac{1}{n} \sum_{n=0}^{N} 1_A(X_n(\omega)) = a_1\}$ is of ν_1-measure 1 and of ν_2-measure 0, i.e., ν_1 and ν_2 are mutually singular. □

Definition 4.8 (Ergodic map). A map T on a probability space $(\Omega, \mathscr{F}, P)$ is called *ergodic* whenever T is measure preserving and $TA = A$ for $A \in \mathscr{F}$ implies $P(A) = 1$ or 0.

Proposition 4.9 (Ergodic theorem). *Let T be a measure preserving map on a probability space $(\Omega, \mathscr{F}, P)$. Take $f \in L^1(\Omega)$. Then the limit*

$$g(\omega) = \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} f(T^j \omega)$$

exists for almost every $\omega \in \Omega$, and $\mathbb{E}_P[g] = \mathbb{E}_P[f]$. If, moreover, T is ergodic, then $g = \mathbb{E}_P[f]$ is constant.

4.2.2 Uniqueness

Having settled the existence problem, the next natural question is whether only a single Gibbs measure exists for given V. We start by an illuminating example.

Example 4.2 (Non-uniqueness). Take $V(x) = \frac{1}{2}(x^2 - 1)$, $x \in \mathbb{R}$, and consider the corresponding Schrödinger operator $H_{\text{osc}} = -\frac{1}{2}\Delta + V$. Its ground state is $\Psi_{\text{p}}(x) = \pi^{-1/4} e^{-x^2/2}$, and the process generated by H_{osc} is the one-dimensional Ornstein–Uhlenbeck process. Fix $\alpha, \beta \in \mathbb{R}$ and define for $s, x \in \mathbb{R}$

$$\psi_s^{\text{l}}(x) = \frac{1}{\pi^{1/4}} \exp\left(-\frac{1}{2}(x + \alpha e^{-s})^2\right) \exp\left(\frac{\alpha e^{-s}}{2}\right)^2,$$

$$\psi_s^{\text{r}}(x) = \frac{1}{\pi^{1/4}} \exp\left(-\frac{1}{2}(x + \beta e^{s})^2\right) \exp\left(\frac{\beta e^{s}}{2}\right)^2.$$

A simple calculation yields

$$e^{-tH_{\rm osc}}\psi_s^{\rm l} = \psi_{s+t}^{\rm l}, \quad e^{-tH_{\rm osc}}\psi_s^{\rm r} = \psi_{s-t}^{\rm r}, \quad (\psi_s^{\rm l}, \psi_s^{\rm r}) = e^{\alpha\beta/2}.$$

Hence

$$\int \prod_{j=1}^{n} f_j(X_{t_j}) d\nu_{\alpha,\beta} = e^{-\alpha\beta/2}(f_1\psi_{t_1}^{l}, e^{-(t_2-t_1)H_{\rm osc}} f_2 \cdots e^{-(t_n-t_{n-1})H_{\rm osc}} f_n\psi_{t_n}^{\rm r})$$

gives rise to the finite dimensional distributions of the probability measure $\nu_{\alpha,\beta}$ on $C(\mathbb{R}, \mathbb{R})$ of a Gaussian Markov process. This is stationary if and only if $\alpha = \beta = 0$. That $\nu_{\alpha,\beta}$ is a Gibbs measure for every $\alpha, \beta \in \mathbb{R}$ can be checked by straightforward computation. In this case thus uncountably many Gibbs measures exist for the same potential.

The above example shows that a Gibbs measure for a Kato-decomposable potential may not be unique. However, Lemma 4.3 gives a simple criterion allowing to check if a Gibbs measure is the only one supported on a given subset. In order to apply Lemma 4.3, we need to know for which $\bar{\omega} \in \mathfrak{X}$ is it true that $\nu_N^{x,y}(A) \to \nu(A)$ (with $\bar{\omega}(-N) = x, \bar{\omega}(N) = y$). The next lemma gives a sufficient condition for this in terms of the transition densities π_T of ν.

Lemma 4.10. *Let V be Kato-decomposable, and assume that the Schrödinger operator $-1/2\Delta + V$ has a ground state $\Psi_{\rm p}$. Suppose that for some $\bar{\omega} \in \mathfrak{X}$*

$$\lim_{N\to\infty} \sup_{x,y\in\mathbb{R}^d} \left| \frac{\pi_{N-T}(\bar{\omega}(-N), x)\pi_{N-T}(y, \bar{\omega}(N))}{\pi_N(\bar{\omega}(-N), \bar{\omega}(N))} - 1 \right| \Psi_{\rm p}(x)\Psi_{\rm p}(y) = 0 \quad (4.2.12)$$

for all $T > 0$. Then for every $T > 0$ and $A \in \mathscr{F}_T$, $\nu_N^{x,y}(A) \to \nu(A)$ as $N \to \infty$.

Proof. Let $A \in \mathscr{F}_T$, and take $A \subset \{\omega \in \mathfrak{X} | |\omega(\pm T)| < M\}$ for some $M > 1$. Fix $N > T$ and $\bar{\omega} \in \mathfrak{X}$, and put $\bar{\omega}(-N) = \xi$ and $\bar{\omega}(N) = \eta$. The Markov property of Brownian motion and the Feynman–Kac formula give

$$\begin{aligned}
\nu_N^{\xi,\eta}(A) &= \frac{1}{Z_N(\bar{\omega})} \int_{\mathbb{R}^d} dx \int_{\mathbb{R}^d} dy\, \mathbb{E}_{[-N,-T]}^{\xi,x}[e^{-\int_{-N}^{-T} V(X_s)ds}] \\
&\quad \times \mathbb{E}_{[-T,T]}^{x,y}[e^{-\int_{-T}^{T} V(X_s)ds} 1_A]\mathbb{E}_{[T,N]}^{y,\eta}[e^{-\int_T^N V(X_s)ds}] \\
&= \int_{\mathbb{R}^d} dx \int_{\mathbb{R}^d} dy\, \frac{K_{N-T}(\xi, x)K_{N-T}(y, \eta)}{K_{2N}(\xi, \eta)} \mathbb{E}_{[-T,T]}^{x,y}[e^{-\int_{-T}^{T} V(X_s)ds} 1_A].
\end{aligned}$$

By having restricted to A and by boundedness of $K_{2T}(x, y)$, the last factor in the above formula is a bounded function of x and y with compact support, thus integrable over $\mathbb{R}^d \times \mathbb{R}^d$. The claim follows for A once we show that

$$K_{N-T}(\bar{\omega}(-N), x)K_{N-T}(y, \bar{\omega}(N))/K_{2N}(\bar{\omega}(-N), \bar{\omega}(N)) \overset{N\to\infty}{\longrightarrow} \Psi_{\rm p}(x)\Psi_{\rm p}(y),$$

uniformly in $x, y \in \mathbb{R}^d$. This is equivalent to (4.2.12). For general $A \in \mathscr{F}_T$, consider $B_M = \{\omega \in \mathfrak{X} | \max\{|\omega(T)|, |\omega(-T)|\} < M\}$ with $M \in \mathbb{N}$, and $A_M = A \cap B_M$. Since $B_M \nearrow \mathfrak{X}$ as $M \to \infty$, for given $\varepsilon > 0$ we may take $M \in \mathbb{N}$ with $\nu(B_M^c) < \varepsilon$. Moreover, for A_M and B_M we find $N_0 \in \mathbb{N}$ such that for all $N > N_0$ we have $|\nu_N(A_M) - \nu(A_M)| < \varepsilon$ resp. $|\nu_N(B_M) - \nu(B_M)| < \varepsilon$. Hence $\nu_N(B_M^c) < 2\varepsilon$ for all $N > N_0$, and thus

$$\begin{aligned}|\nu_N(A) - \nu(A)| &= |\nu_N(A_M) + \nu_N(A \setminus B_M) - \nu(A_M) - \nu(A \setminus B_M)| \\ &\leq |\nu_N(A_M) - \nu(A_M)| + \nu_N(B_M^c) + \nu(B_M^c) \leq 4\varepsilon.\end{aligned}$$

This shows that $\nu_N(A) \to \nu(A)$. □

Having these results at hand, we can now state and prove our results on uniqueness.

Theorem 4.11 (Case of intrinsic ultracontractivity). *Suppose V is such that the Schrödinger semigroup is intrinsically ultracontractive in the sense of Definition* 3.41. *Then the Itô measure ν corresponding to it is the unique Gibbs measure for V supported on $\mathfrak{X}$.*

Proof. By Theorem 3.105, $\lim_{N\to\infty} |\pi_N(x, y) - 1| = 0$ uniformly in x and y. Thus (4.2.12) holds. □

The proof of Theorem 3.105 has the following consequence which we state separately for later use.

Corollary 4.12. *Let π_t be the probability transition kernel of a $P(\phi)_1$-process generated by an intrinsically ultracontractive Schrödinger semigroup. Then there exist $a_1, a_2 > 0$ such that for large enough $t > 0$, $a_1 \leq \pi_t(x, y) \leq a_2$ uniformly in $x, y \in \mathbb{R}^d$.*

Recall that for an operator H the non-negative number

$$\Lambda = \inf\{\mathrm{Spec}(H) \setminus \inf \mathrm{Spec}(H)\} - \inf \mathrm{Spec}(H)$$

is called *spectral gap* of H.

Theorem 4.13 (Case of Kato-class). *Suppose V is Kato-decomposable and $\Psi_\mathrm{p} \in L^2(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$. Put*

$$\mathfrak{X}_0 = \left\{ \omega \in \mathfrak{X} \,\middle|\, \lim_{N\to\pm\infty} \frac{e^{-\Lambda|N|}}{\Psi_\mathrm{p}(\omega(N))} = 0 \right\}. \tag{4.2.13}$$

Then ν is the unique Gibbs measure for V supported on $\mathfrak{X}_0$.

Proof. With P_{Ψ_p} the projection used in the proof of Theorem 4.11 write $L_t = e^{-tH} - P_{\Psi_p}$. This is an integral operator with kernel $\tilde{K}_t(x,y) = K_t(x,y) - \Psi_p(x)\Psi_p(y)$. As $\Lambda > 0$ by assumption, we have $\|L_t\|_{2,2} = e^{-\Lambda t}$. In order to estimate $\tilde{K}_t$ note that

$$\sup_{x,y\in\mathbb{R}^d} |\tilde{K}_t(x,y)| = \sup_{f\in L^1,\|f\|_1=1} \left\| \int_{\mathbb{R}^d} \tilde{K}_t(x,y)f(y)dy \right\|_\infty = \|L_t\|_{1,\infty},$$

and since $e^{-tH}P_{\Psi_p} = P_{\Psi_p}e^{-tH} = P_{\Psi_p}$ for all $t > 0$, we get

$$\|L_t\|_{1,\infty} = \|e^{-H}(e^{-(t-2)H} - P_{\Psi_p})e^{-H}\|_{1,\infty} \le \|e^{-H}\|_{2,\infty}\|L_{t-2}\|_{2,2}\|e^{-H}\|_{1,2}.$$

By Theorem 3.39, both $\|e^{-H}\|_{2,\infty}$ and $\|e^{-H}\|_{1,2}$ are finite, thus it follows that for every $t \le N$,

$$|K_{N-t}(x,y) - \Psi_p(x)\Psi_p(y)| \le C_t e^{-\Lambda N},$$

where $C_t = \|e^{-H}\|_{2,\infty}\|e^{-H}\|_{1,2}e^{\Lambda(2+t)}$ is independent of x, y and N. In terms of π_N this implies that for all $\bar{\omega} \in \mathcal{X}_0$,

$$\begin{aligned}
|\pi_{N-T}(\bar{\omega}(-N),x) - 1|\Psi_p(x) &\le C_T e^{-\Lambda N}/\Psi_p(\bar{\omega}(-N)) \to 0,\\
|\pi_{N-T}(y,\bar{\omega}(N)) - 1|\Psi_p(y) &\le C_T e^{-\Lambda N}/\Psi_p(\bar{\omega}(N)) \to 0,\\
|\pi_{2N}(\bar{\omega}(-N),\bar{\omega}(N)) - 1| &\le C_0 e^{-2\Lambda N}/(\Psi_p(\bar{\omega}(-N))\Psi_p(\bar{\omega}(N))) \to 0
\end{aligned}$$

as $N \to \infty$. From this (4.2.12) can be deduced. It remains to show that ν is actually supported on $\mathcal{X}_0$. By time reversibility it suffices to show that

$$\nu\left(\limsup_{N\to\infty} \frac{e^{-\Lambda N}}{\Psi_p(\omega(N))} > n\right) = 0, \quad \forall n > 0.$$

To prove this, note that by stationarity of ν,

$$\nu\left(\frac{e^{-\Lambda N}}{\Psi_p(\omega(N))} > n\right) = \int_{\mathbb{R}^d} 1_{\{\Psi_p<\exp(-\Lambda N)/n\}}\Psi_p^2 dx \le \frac{\exp(-\Lambda N)}{n}\|\Psi_p\|_{L^1}.$$

The right-hand side of the last expression is summable in N for each n, thus an application of the Borel–Cantelli lemma completes the proof. □

Remark 4.6. The additional assumption that $\Psi_p \in L^1$ is very weak. In fact, for many potentials V it is known that Ψ_p decays exponentially at infinity. A large class with $\Psi_p \in L^1$ is dealt with in Lemma 3.59.

In Theorem 4.13 $\mathcal{X}_0$ is seen to depend directly on the decay of Ψ_p at infinity. Thus for given conditions on V it is interesting to know the asymptotic behaviour of Ψ_p. Suppose that

$$C_2|x|^{2\alpha} \le V(x) \le C_1|x|^{2\alpha}, \quad \alpha > 1 \tag{4.2.14}$$

outside of a compact set of $\mathbb{R}^d$; we say in this case that V is a *super-quadratic* potential. Then there exist constants b_1, b_2, D_1, D_2 such that

$$D_2 \exp(-b_2|x|^{\alpha+1}) \le \Psi_{\mathrm{p}}(x) \le D_1 \exp(-b_1|x|^{\alpha+1}), \tag{4.2.15}$$

see Corollaries 3.60 and 3.63. Also, for super-quadratic potentials V the corresponding Schrödinger semigroup is intrinsically ultracontractive (see Theorem 3.104). It can be checked directly that all potentials considered in Lemma 3.59 are Kato-decomposable.

Example 4.3. Let V be a non-constant polynomial bounded from below. This in particular implies that the degree of V is even, so H has a unique ground state and a positive spectral gap. From Theorem 4.11 and Example 4.2 it follows that a Gibbs measure for V is unique if and only if the degree of V is greater than 2.

Example 4.4. Let $V(x) = |x|^{2\alpha}$ with $\alpha > 0$. Again, H has a unique ground state Ψ_{p} and a spectral gap Λ. In case $\alpha \ge 1$, Theorem 4.11 and Example 4.2 completely solve the question of uniqueness. In case $0 < \alpha < 1$, (3.4.32) implies that Theorem 4.13 applies, and thus uniqueness holds on a set $\mathcal{X}_0 = \mathcal{X}_0(\alpha)$ given by (4.2.13). By Corollaries 3.61 and 3.63 sharp estimates can be obtained on this supporting set. Indeed,

$$\begin{aligned}\left\{\omega \in \mathcal{X} \,\middle|\, \limsup_{T\to\pm\infty} \frac{|\omega(\pm T)|^{\alpha+1}}{T} < \frac{\Lambda}{b_1}\right\} &\subset \mathcal{X}_0(\alpha) \\ &\subset \left\{X \in \mathcal{X} \,\middle|\, \limsup_{T\to\pm\infty} \frac{|\omega(\pm T)|^{\alpha+1}}{T} < \frac{\Lambda}{b_2}\right\}\end{aligned} \tag{4.2.16}$$

with constants b_1, b_2. Note that, counterintuitively, $\mathcal{X}_0(\alpha)$ becomes smaller as α increases to 1. Indeed, a fast growing potential should bring the path back to stationarity more quickly, as it is seen in the case $\alpha > 1$. What happens for $\alpha < 1$ results from (4.2.13), which turns out to be too crude for this range of exponents.

4.2.3 Gibbs measure for pair interaction potentials

In this section we discuss the more difficult situation when the pair interaction potential W is non-zero.

Let $\mathcal{X}'$ be the space of functions $\omega : \mathbb{R} \to \mathbb{R}^d$, $t \mapsto \omega(t)$ that are continuous in every $t \neq 0$, and for which a left and a right limit exist in $t = 0$. For $\tau > 0$ define the *shrinking* operator $\theta_\tau : \mathcal{X} \to \mathcal{X}'$ by

$$(\theta_\tau(\omega))(t) = \begin{cases} \omega(t-\tau) & \text{if } t \ge 0 \\ \omega(t+\tau) & \text{if } t < 0. \end{cases} \tag{4.2.17}$$

Let $V : \mathbb{R}^d \to \mathbb{R}$ be an external potential, and $W : \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}$ be a pair interaction potential in the sense of Definition 4.2. Write

$$W_T(X(\omega)) = \int_{-T}^{T} ds \int_{-T}^{T} W(X_t(\omega), X_s(\omega), t-s)dt \tag{4.2.18}$$

whenever the right-hand side is well defined.

We consider the following regularity conditions on the potentials V and W.

Assumption 4.1. (1) V is Kato-decomposable and is such that the Schrödinger operator $H = -\frac{1}{2}\Delta + V$ has a unique ground state $\Psi_{\mathrm{p}} \in L^2(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$ with the property

$$H\Psi_{\mathrm{p}} = 0. \tag{4.2.19}$$

(2) There exists $C_\infty \in R^+$ such that

$$\sup_{\omega\in\mathfrak{X}} \int_{-\infty}^{+\infty} |W(X_t(\omega), X_s(\omega), t)|dt \leq C_\infty. \tag{4.2.20}$$

(3) There exist $C, D \geq 0$ such that with

$$C < \liminf_{|x|\to\infty} V(x), \tag{4.2.21}$$

the bound

$$-W_T(X(\omega)) \leq -W_T((X(\theta_\tau\omega))) + C\tau + D \tag{4.2.22}$$

holds for all $T, \tau > 0$ and all $\omega \in \mathfrak{X}$.

(4.2.19) can be achieved by adding a constant to V, if necessary. Recall that $\mathcal{N}_0$ denotes the path measure of the $P(\phi)_1$-process with potential V and stationary measure $\Psi_{\mathrm{p}}^2(x)dx$. We then have the following result.

Proposition 4.14. *With Assumption* 4.1*, let $d\mu_T^h$ be the finite volume Gibbs measure for the potentials V and W, with stochastic boundary condition $h = h_T = \Psi_{\mathrm{p}}(X_{-T})\Psi_{\mathrm{p}}(X_T)$. Then on $\mathscr{F}_T$ we have*

$$d\mu_T^h = \frac{1}{Z_T}\exp\left(-\int_{-T}^{T} ds \int_{-T}^{T} W(X_t, X_s, t-s)dt\right) d\mathcal{N}_0,$$

with $Z_T = \mathbb{E}_{\mathcal{N}_0}[\exp(-\int_{-T}^{T} ds \int_{-T}^{T} W(X_t, X_s, t-s)dt)]$.

Proof. By the Feynman–Kac formula in its version (3.10.35) we find for all $-T < s_1 < \cdots < s_n < T$ that

$$\begin{aligned}&\mathbb{E}_{\mathcal{N}_0}[f_1(X_{s_1})\cdots f_n(X_{s_n})]\\&= \int_{\mathbb{R}^d} dx \mathbb{E}^x[\Psi_{\mathrm{p}}(X_{-T}) f_1(X_{s_1})\cdots f_n(X_{s_n})\Psi_{\mathrm{p}}(X_T) e^{-\int_{-T}^T V(X_s)ds}].\end{aligned}$$

By an approximation argument the above equality can be extended to all bounded $\mathscr{F}_T$-measurable functions. Since $\exp(-\int_{-T}^T ds \int_{-T}^T W(X_t, X_s, t-s)dt)$ is such a function, we conclude that

$$\begin{aligned}&\mathbb{E}_{\mu_T^h}[g_1(X_{s_1})\cdots g_n(X_{s_n})]\\&= \mathbb{E}_{\mathcal{N}_0}\Big[\exp\Big(-\int_{-T}^T ds \int_{-T}^T W(X_t, X_s, t-s)dt\Big) g_1(X_{s_1})\cdots g_n(X_{s_n})\Big],\end{aligned}$$

for all bounded functions $g_1, \ldots, g_n$, and all $-T < s_1 < \cdots < s_n < T$, and the claim follows. □

In order to prove the existence of a Gibbs measure for the potentials V and W, we apply Proposition 4.2. To be able to use this result we need to prove that the family

$$d\mu_T = \frac{1}{Z_T}\exp\Big(-\int_{-T}^T ds \int_{-T}^T W(X_t, X_s, t-s)dt\Big) d\mathcal{N}_0 \tag{4.2.23}$$

from Proposition 4.14 is relatively compact. By Proposition 4.4 it will suffice to show local uniform domination. We start by showing that the one-point distributions μ_T are tight as $T \to \infty$.

Proposition 4.15. *Define μ_T as in* (4.2.23)*, and suppose Assumption* 4.1 *holds. Then for every $\varepsilon > 0$ and $S > 0$ there exists $R > 0$ such that*

$$\sup_{T>0} \mu_T(|X_u| > R) \le \varepsilon$$

for all $|u| \le S$.

Proof. First we relate $\mu_T(|X_u| > R)$ to $\mu_T(|X_0| > R)$. Assume that $u > 0$, the case $u < 0$ can be treated similarly. Using a formal, though suggestive, notation we obtain

$$\begin{aligned}\int_{-T}^T\int_{-T}^T &= \Big(\int_{-T}^{-T+u} + \int_{-T+u}^{T+u} - \int_T^{T+u}\Big)^2\\&= \int_{-T+u}^{T+u}\int_{-T+u}^{T+u} + \int_{-T}^{-T+u}\int_{-T}^{-T+u} + \int_T^{T+u}\int_T^{T+u} - 2\int_{-T}^{-T+u}\int_T^{T+u}\\&\quad + 2\int_{-T}^{-T+u}\int_{-T+u}^{T+u} - 2\int_T^{T+u}\int_{-T+u}^{T+u}.\end{aligned}$$

Inserting the pair interaction potentials in the double integrals, we see that the last integral in the first line is bounded in absolute value by $\|W\|_\infty u^2$, and the same is true for the first two terms in the second line, with a factor 2 in front of the second term. The last two terms in the second line are bounded by $2C_\infty u$, where we used (4.2.20). Thus,

$$\sup_{\omega\in\mathfrak{X}} |W_T(X(\omega)) - W_{T+u}(X(\omega))| = 4\|W\|_\infty u^2 + 4C_\infty u = C_1(u) \tag{4.2.24}$$

and we find

$$\begin{aligned} Z_T\mu_T(|X_u| > R) &= \mathbb{E}_{\mathcal{N}_0}[1_{\{|X_u|>R\}}e^{-W_T(X)}] \leq e^{C_1(u)}\mathbb{E}_{\mathcal{N}_0}[1_{\{|X_u|>R\}}e^{-W_{T+u}(X)}] \\ &= e^{C_1(u)}\mathbb{E}_{\mathcal{N}_0}[1_{\{|X_0|>R\}}e^{-W_T(X)}] = e^{C_1(u)}Z_T\mu_T(|X_0| > R). \end{aligned}$$

The equality of the first and second lines above follows from the stationarity of the measure $\mathcal{N}_0$. Thus the claim will be shown once we show it for $u = 0$. We proceed in several steps.

Step 1*:* By Theorem 3.106, we have

$$\mu_T(|X_0| > R) = \frac{1}{Z_T}\int_{|y|>R} \Psi_{\mathrm{p}}^2(y)\mathbb{E}_{\mathcal{N}_0^y}[e^{-W_T(X)}]dy. \tag{4.2.25}$$

In the next steps we will show that there exist $K, r > 0$ such that for every $T > 0$ and $y \in \mathbb{R}^d$,

$$\mathbb{E}_{\mathcal{N}_0^y}[e^{-W_T(X)}] \leq \frac{K}{\Psi_{\mathrm{p}}(y)} \inf_{|z|\leq r} \mathbb{E}_{\mathcal{N}_0^z}[e^{-W_T(X)}]. \tag{4.2.26}$$

Assume for the moment that (4.2.26) holds. We can estimate

$$\begin{aligned} 1 \geq \mu_T(|X_0| \leq r) &= \frac{1}{Z_T}\int_{|y|\leq r} \mathbb{E}_{\mathcal{N}_0^y}[e^{-W_T}]\Psi_{\mathrm{p}}^2(y)dy \\ &\geq \frac{1}{Z_T}C_r \inf_{|z|\leq r} \mathbb{E}_{\mathcal{N}_0^z}[e^{-W_T}] \inf_{|z|\leq r} \Psi_{\mathrm{p}}^2(z), \end{aligned}$$

where C_r is the volume of a ball with radius r. Thus $\inf_{|z|\leq r} \mathbb{E}_{\mathcal{N}_0^z}[e^{-W_T}] \leq \tilde{C}_r Z_T$, where the constant only depends on r but not on T. Using (4.2.25) and (4.2.26) gives

$$\mu_T(|X_0| > R) \leq K\tilde{C}_r \int_{|y|>R} \Psi_{\mathrm{p}}(y)dy. \tag{4.2.27}$$

Since $\Psi_{\mathrm{p}} \in L^1$, the right-hand side above vanishes when $R \to \infty$, and the claim follows.

Step 2: In order to prove (4.2.26) we change the probability space. Recall the definition of $\mathcal{X}'$ at the beginning of the section, and consider

$$J : \mathcal{X}' \to \mathcal{X}_+^2 = C([0,\infty), \mathbb{R}^d \times \mathbb{R}^d),$$
$$(\omega(t))_{t\in\mathbb{R}} \mapsto \begin{cases} (\omega(t), \omega(-t)) & t > 0, \\ (\lim_{t\to 0+} \omega(t), \lim_{t\to 0-} \omega(t)) & t = 0. \end{cases} \tag{4.2.28}$$

On fixing the value of $\omega \in \mathcal{X}'$ at $t = 0$, J becomes a bijection; this value can be chosen, for instance, equal to the left limit. Write $\boldsymbol{\omega} = (\boldsymbol{\omega}', \boldsymbol{\omega}'')$ for the elements of $\mathcal{X}_+^2$. When we omit the variable $\boldsymbol{\omega}$, we write $\boldsymbol{X} = (\boldsymbol{X}', \boldsymbol{X}'')$, where $\boldsymbol{X}'_t(\boldsymbol{\omega}) = \boldsymbol{X}_t(\boldsymbol{\omega}')$, and $\boldsymbol{X}''_t(\boldsymbol{\omega}) = \boldsymbol{X}_t(\boldsymbol{\omega}'')$.

The image measure of $\mathcal{N}_0^z$ under J can be described explicitly. For $\boldsymbol{z} = (z', z'') \in \mathbb{R}^d \times \mathbb{R}^d$ denote by $\mathcal{N}_0^{\boldsymbol{z}}$ the measure of the $\mathbb{R}^d \times \mathbb{R}^d$-valued $P(\phi)_1$-process with potential $\tilde{V}(x,y) = V(x) + V(y)$, conditional on $\boldsymbol{\omega}_0 = \boldsymbol{z}$. Write $\tilde{\mathscr{F}}_T$ for the σ-field over $\mathcal{X}_+^2$ generated by point evaluations at points inside $[0,T]$. Then for every $\tilde{\mathscr{F}}_T$-measurable bounded function f

$$\mathbb{E}_{\mathcal{N}_0^{\boldsymbol{z}}}[f] = \frac{1}{\Psi_{\mathrm{p}}(z')\Psi_{\mathrm{p}}(z'')} \int e^{-\int_0^T (V(\boldsymbol{X}'_s) + V(\boldsymbol{X}''_s))ds} f(\boldsymbol{\omega}) \Psi_{\mathrm{p}}(\boldsymbol{X}'_T)\Psi_{\mathrm{p}}(\boldsymbol{X}''_T)\, d\mathcal{W}^{\boldsymbol{z}}(\boldsymbol{\omega}). \tag{4.2.29}$$

Here, $\mathcal{W}^{\boldsymbol{z}}$ denotes $2d$-dimensional Wiener measure starting at $\boldsymbol{z}$. (4.2.29) follows from the fact that the ground state of the Schrödinger operator

$$H \otimes 1 + 1 \otimes H = -\frac{1}{2}\Delta_{z'} - \frac{1}{2}\Delta_{z''} + V(z') + V(z'') \tag{4.2.30}$$

is given by $\Psi_{\mathrm{p}}(z')\Psi_{\mathrm{p}}(z'')$, combined with (3.10.33). Note that the bottom of the spectrum of (4.2.30) is zero by shifting by a constant.

The Markov property and time reversibility of $\boldsymbol{X}$ imply that for every $z \in \mathbb{R}^d$, $\mathcal{N}_0^{(z,z)}$ is the image of $\mathcal{N}_0^z$ under J, i.e.

$$\mathbb{E}_{\mathcal{N}_0^z}[f] = \mathbb{E}_{\mathcal{N}_0^{(z,z)}}[f \circ J^{-1}],$$

for all bounded functions f. Write $\tilde{W}_T(\boldsymbol{X}) = W_T(J^{-1}(\boldsymbol{X}))$. Then, in particular,

$$\mathbb{E}_{\mathcal{N}_0^z}[e^{-W_T(\boldsymbol{X})}] = \mathbb{E}_{\mathcal{N}_0^{(z,z)}}[e^{-\tilde{W}_T(\boldsymbol{X})}]. \tag{4.2.31}$$

From (4.2.26) it can be seen that the quantity we want to estimate is the expectation of $e^{-\tilde{W}_T(\boldsymbol{X})}$ with respect to $\mathcal{N}_0^{(z,z)}$. It is easy to check that

$$\begin{aligned} \tilde{W}_T(\boldsymbol{X}) = -\int_0^T ds \int_0^T dt\, (&W(\boldsymbol{X}'_t - \boldsymbol{X}'_s, |s-t|) + W(\boldsymbol{X}''_t - \boldsymbol{X}''_s, |s-t|) \\ &+ W(\boldsymbol{X}'_t - \boldsymbol{X}''_s, |s+t|) + W(\boldsymbol{X}''_t - \boldsymbol{X}'_s, |s+t|)). \end{aligned} \tag{4.2.32}$$

Step 3*:* Next notice that the shrinking operator is mapped to the time shift on $\mathfrak{X}_+^2$ under J. In other words, $\tilde{\theta}_\tau = J\theta_\tau J^{-1}$ is the usual time shift that maps $(\boldsymbol{\omega}(t))_{t\geq 0}$ to $(\boldsymbol{\omega}(t+\tau))_{t\geq 0}$. Thus in the new representation (4.2.22) becomes

$$-\tilde{W}_T(\boldsymbol{X}(\boldsymbol{\omega})) \leq -\tilde{W}_T(\boldsymbol{X}(\tilde{\theta}_\tau \boldsymbol{\omega})) + C\tau + D \tag{4.2.33}$$

for all $\boldsymbol{X} \in \mathfrak{X}_+^2$ and all $T, \tau > 0$. Our strategy is to use (4.2.33) together with the strong Markov property of $\mathcal{N}_0^{\boldsymbol{z}}$. For $r > 0$ let

$$\tau_r = \inf\{t \geq 0 \,|\, |\boldsymbol{X}_t| \leq r\}$$

be the hitting time of the centered ball with radius r. This is a stopping time and we write $\mathscr{F}_{\tau_r}$ for the associated σ-field. Then for every $\boldsymbol{z} \in \mathbb{R}^d \times \mathbb{R}^d$,

$$\begin{aligned}
&\mathbb{E}_{\mathcal{N}_0^{\boldsymbol{z}}}[e^{-\tilde{W}_T(\boldsymbol{X})}] \\
&= \mathbb{E}_{\mathcal{N}_0^{\boldsymbol{z}}}[\mathbb{E}_{\mathcal{N}_0^{\boldsymbol{z}}}[e^{-\tilde{W}_T(\boldsymbol{X})}|\mathscr{F}_{\tau_r}]] \leq \mathbb{E}_{\mathcal{N}_0^{\boldsymbol{z}}}[\mathbb{E}_{\mathcal{N}_0^{\boldsymbol{z}}}[e^{-\tilde{W}_T(\boldsymbol{X}\circ\tilde{\theta}_{\tau_r})}e^{C\tau_r+D}|\mathscr{F}_{\tau_r}]] \\
&= \mathbb{E}_{\mathcal{N}_0^{\boldsymbol{z}}}[e^{C\tau_r+D}\mathbb{E}_{\mathcal{N}_0^{\boldsymbol{z}}}[e^{\tilde{W}_T(\boldsymbol{X}\circ\tilde{\theta}_{\tau_r})}|\mathscr{F}_{\tau_r}]] = \mathbb{E}_{\mathcal{N}_0^{\boldsymbol{z}}}[e^{C\tau_r+D}\mathbb{E}_{\mathcal{N}_0^{\boldsymbol{X}_{\tau_r}}}[e^{-\tilde{W}_T}]] \\
&\leq \sup_{|\boldsymbol{y}|\leq r} \mathbb{E}_{\mathcal{N}_0^{\boldsymbol{y}}}[e^{-\tilde{W}_T}]\; \mathbb{E}_{\mathcal{N}_0^{\boldsymbol{z}}}[e^{C\tau_r+D}].
\end{aligned} \tag{4.2.34}$$

In order to obtain (4.2.26) we need a good estimate of the second factor at the right-hand side of (4.2.34), and estimate the supremum in the first factor against an infimum. This will be done in Steps 4 and 5 below.

Step 4*:* We show that there exist $r, \gamma > 0$, such that for all $\boldsymbol{z} \in \mathbb{R}^d \times \mathbb{R}^d$

$$\mathbb{E}_{\mathcal{N}_0^{\boldsymbol{z}}}[e^{C\tau_r}] \leq 1 + \frac{C\|\Psi_{\mathrm{p}}\|_\infty}{\gamma}\left(\frac{1}{\Psi_{\mathrm{p}}(z')} + \frac{1}{\Psi_{\mathrm{p}}(z'')}\right). \tag{4.2.35}$$

Above C is the constant from (4.2.22). To show (4.2.35), we choose $0 < \gamma < \liminf_{|x|\to\infty} V(x) - C$, and take r so large that $V(x) > C + \gamma$ for all $x \in \mathbb{R}^d$ with $|x| > r/\sqrt{2}$. This is possible by the definition of C. Clearly,

$$\begin{aligned}
&\{\boldsymbol{z} \in \mathbb{R}^d \times \mathbb{R}^d \,|\, |\boldsymbol{z}| > r\} \\
&\subset \{\boldsymbol{z} \in \mathbb{R}^d \times \mathbb{R}^d \,|\, |z'| > r/\sqrt{2}\} \cup \{\boldsymbol{z} \in \mathbb{R}^d \times \mathbb{R}^d \,|\, |z''| > r/\sqrt{2}\},
\end{aligned}$$

and with (4.2.29) it follows that

$$
\begin{aligned}
&\Psi_{\mathrm{p}}(z')\Psi_{\mathrm{p}}(z'')\mathbb{E}_{\mathcal{N}_0^z}[\tau_r > t] \\
&= \int_{\mathfrak{X}_+^2} e^{-\int_0^t (V(X'_s)+V(X''_s))ds} 1_{\{|X_s|>r \ \ \forall s\le t\}} \Psi_{\mathrm{p}}(X'_t)\Psi_{\mathrm{p}}(X''_t)\, d\mathcal{W}^{z}(\omega) \\
&\le \int_{\mathfrak{X}_+^2} e^{-\int_0^t V(X'_s)ds} e^{-\int_0^t V(X''_s)ds} (1_{\{|(X'_s)|>r/\sqrt{2} \ \ \forall s\le t\}} + 1_{\{|(X''_s)|>r/\sqrt{2} \ \ \forall s\le t\}}) \\
&\qquad \times \Psi_{\mathrm{p}}(X'_t)\Psi_{\mathrm{p}}(X''_t)\, d\mathcal{W}^{z'}(\omega')\, d\mathcal{W}^{z''}(\omega'') \\
&= \Psi_{\mathrm{p}}(z'') \int e^{-\int_0^t V(X'_s)ds} 1_{\{|(X'_s)|>r/\sqrt{2} \ \ \forall s\le t\}} \Psi_{\mathrm{p}}(X'_t)\, d\mathcal{W}^{z'}(\omega') \\
&\quad + \Psi_{\mathrm{p}}(z') \int e^{-\int_0^t V(X''_s)ds} 1_{\{|(X''_s)|>r/\sqrt{2} \ \ \forall s\le t\}} \Psi_{\mathrm{p}}(X''_t)\, d\mathcal{W}^{z''}(\omega'') \\
&\le (\Psi_{\mathrm{p}}(z') + \Psi_{\mathrm{p}}(z''))\|\Psi_{\mathrm{p}}\|_\infty e^{-(C+\gamma)t}.
\end{aligned}
$$

The second equality above is due the eigenvalue equation $e^{-tH}\Psi_{\mathrm{p}} = \Psi_{\mathrm{p}}$ and the Feynman–Kac formula. It follows that

$$
\mathbb{E}_{\mathcal{N}_0^z}[\tau_r > t] \le \left(\frac{1}{\Psi_{\mathrm{p}}(z')} + \frac{1}{\Psi_{\mathrm{p}}(z'')}\right) \|\Psi_{\mathrm{p}}\|_\infty e^{-(C+\gamma)t},
$$

and using the equality

$$
\mathbb{E}_{\mathcal{N}_0^z}[e^{C\tau_r}] = 1 + \int_0^\infty C e^{Ct} \mathbb{E}_{\mathcal{N}_0^z}[\tau_r > t] dt, \tag{4.2.36}
$$

we arrive at (4.2.35).

Step 5: Let $r > 0$ be as in Step 4. We show that there exists $M > 0$ such that

$$
\sup_{|z|\le r} \mathbb{E}_{\mathcal{N}_0^z}[e^{-\tilde{W}_T}] \le M \inf_{|z|\le r} \mathbb{E}_{\mathcal{N}_0^z}[e^{-\tilde{W}_T}] \tag{4.2.37}
$$

uniformly in $T > 0$. Denote by $p_t(z, y)$ the transition density from y to z in time t under $\mathcal{N}_0$, i.e., define p_t through the relation

$$
\mathcal{N}_0^z(X_t \in A) = \int_{\mathbb{R}^d \times \mathbb{R}^d} p_t(z, y) 1_A(y) dy
$$

for all $z \in \mathbb{R}^d \times \mathbb{R}^d$ and measurable $A \subset \mathbb{R}^d \times \mathbb{R}^d$. By (4.2.29) and the Feynman–Kac formula we have

$$
p_t(z, y) = \frac{\Psi_{\mathrm{p}}(z')\Psi_{\mathrm{p}}(z'')}{\Psi_{\mathrm{p}}(y')\Psi_{\mathrm{p}}(y'')} K_t(y', z') K_t(y'', z''), \tag{4.2.38}
$$

where K_t is the integral kernel of e^{-tH}. Ψ_{p} and K_t are both strictly positive and uniformly bounded; the latter statement follows from the boundedness from L^1 to L^∞ of the operator e^{-tH}, see Theorem 3.39. Thus they are bounded and non-zero on compact sets, and for every $R > 0$

$$S_t(R,r) = \sup\left\{ \frac{p_t(\boldsymbol{x},\boldsymbol{z})}{p_t(\boldsymbol{y},\boldsymbol{z})} \,\middle|\, |\boldsymbol{x}| \le r, |\boldsymbol{y}| \le r, |\boldsymbol{z}| \le R \right\}$$

is finite. Defining $\tilde{W}_T^1$ like in (4.2.32) with the integrals starting at 1 rather than at 0, we see that there is a constant C_1 independent of T such that

$$-\tilde{W}_T(X(\omega)) - C_1 \le -\tilde{W}_T^1(X(\omega)) \le -\tilde{W}_T(X(\omega)) + C_1$$

for all $\omega \in \mathfrak{X}_+^2$ and $T > 0$. Define $B = \{|X_1| < R\}$. Then for every $\boldsymbol{x}$ with $|\boldsymbol{x}| < r$ we have

$$\mathbb{E}_{\mathcal{N}_0^{\boldsymbol{x}}}[e^{-\tilde{W}_T}] \le e^{C_1}\mathbb{E}_{\mathcal{N}_0^{\boldsymbol{x}}}[1_B e^{-\tilde{W}_T^1}] + e^{C+D}\mathbb{E}_{\mathcal{N}_0^{\boldsymbol{x}}}[1_{B^c} e^{-\tilde{W}_T \circ \tilde{\theta}_1}]. \tag{4.2.39}$$

Defining $\bar{W}_T$ as in (4.2.32) with $|s+t+2|$ appearing instead of $|s+t|$ everywhere, in the first term at the right-hand side of (4.2.39) we find

$$\begin{aligned}\mathbb{E}_{\mathcal{N}_0^{\boldsymbol{x}}}[1_B e^{-\tilde{W}_T^1}] &= \int_{|\boldsymbol{z}|<R} p_1(\boldsymbol{x},\boldsymbol{z})\mathbb{E}_{\mathcal{N}_0^{\boldsymbol{z}}}[e^{-\bar{W}_{T-1}}]d\boldsymbol{z}\\ &\le S_1(R,r)\int_{|\boldsymbol{z}|\le R} p_1(\boldsymbol{y},\boldsymbol{z})\mathbb{E}_{\mathcal{N}_0^{\boldsymbol{z}}}[e^{-\bar{W}_{T-1}}]d\boldsymbol{z}\\ &= S_1(R,r)\mathbb{E}_{\mathcal{N}_0^{\boldsymbol{y}}}[1_B e^{-\tilde{W}_T^1}] \le S_1(R,r)e^{C_0}\mathbb{E}_{\mathcal{N}_0^{\boldsymbol{y}}}[e^{-\tilde{W}_T}]\end{aligned} \tag{4.2.40}$$

for every $\boldsymbol{y}$ with $|\boldsymbol{y}| \le r$. Turning to the second term at the right-hand side of (4.2.39), (4.2.34) and (4.2.35) give

$$\begin{aligned}&\mathbb{E}_{\mathcal{N}_0^{\boldsymbol{z}}}[1_{B^c} e^{-\tilde{W}_T\circ\tilde{\theta}_1}] = \int_{|\boldsymbol{y}|>R} p_1(\boldsymbol{z},\boldsymbol{y})\mathbb{E}_{\mathcal{N}_0^{\boldsymbol{y}}}[e^{-\tilde{W}_T}]d\boldsymbol{y}\\ &\le \sup_{|\boldsymbol{x}|\le r}\mathbb{E}_{\mathcal{N}_0^{\boldsymbol{x}}}[e^{-\tilde{W}_T}]\int_{|\boldsymbol{y}|>R} p_1(\boldsymbol{z},\boldsymbol{y})\mathbb{E}_{\mathcal{N}_0^{\boldsymbol{y}}}[e^{C\tau_r+D}]d\boldsymbol{y}\\ &\le \sup_{|\boldsymbol{x}|\le r}\mathbb{E}_{\mathcal{N}_0^{\boldsymbol{x}}}[e^{-\tilde{W}_T}]e^D\int_{|\boldsymbol{y}|>R} p_1(\boldsymbol{z},\boldsymbol{y})\left(1+\frac{C\|\Psi_{\mathrm{p}}\|_\infty}{\gamma}\left(\frac{1}{\Psi_{\mathrm{p}}(y')}+\frac{1}{\Psi_{\mathrm{p}}(y'')}\right)\right)d\boldsymbol{y}.\end{aligned}$$

By (4.2.38) and the eigenvalue equation we have

$$\begin{aligned}&\int_{\mathbb{R}^d\times\mathbb{R}^d} p_1(\boldsymbol{z},\boldsymbol{y})\left(\frac{1}{\Psi_{\mathrm{p}}(y')}+\frac{1}{\Psi_{\mathrm{p}}(y'')}\right)d\boldsymbol{y}\\ &= \frac{1}{\Psi_{\mathrm{p}}(z')}\int_{\mathbb{R}^d} K_1(z'',y'')dy'' + \frac{1}{\Psi_{\mathrm{p}}(z'')}\int_{\mathbb{R}^d} K_1(z',y')dy'.\end{aligned} \tag{4.2.41}$$

We know by Theorem 3.39 that e^{-tH} is bounded from L^∞ to L^∞. In particular, the image of the constant function $f(x) = 1$ is bounded, and we conclude that $\sup_{z''\in\mathbb{R}^d} |\int_{\mathbb{R}^d} K_1(z'', y'')dy''| < \infty$. Thus the right-hand side of (4.2.41) is uniformly bounded on $\{\boldsymbol{z} \in \mathbb{R}^d \times \mathbb{R}^d \,|\, |\boldsymbol{z}| < r\}$. This implies that for any $\delta < 1$ there is $\bar{R} > 0$ large enough such that

$$\sup_{|\boldsymbol{z}|<r} \int_{|\boldsymbol{y}|>\bar{R}} p_1(\boldsymbol{z}, \boldsymbol{y}) \left(1 + \frac{C\|\Psi_{\mathrm{p}}\|_\infty}{\gamma}\left(\frac{1}{\Psi_{\mathrm{p}}(y')} + \frac{1}{\Psi_{\mathrm{p}}(y'')}\right)\right) d\boldsymbol{y} \le e^{-(C+2D)}\delta.$$

Inserting this and (4.2.40) into (4.2.39) we arrive at

$$\mathbb{E}_{\mathcal{N}_0^{\boldsymbol{z}}}[e^{-\tilde{W}_T}] \le S_1(\bar{R}, r)e^{C_1}\mathbb{E}_{\mathcal{N}_0^{\boldsymbol{x}}}[e^{-\tilde{W}_T}] + \delta \sup_{|\boldsymbol{y}|\le r} \mathbb{E}_{\mathcal{N}_0^{\boldsymbol{y}}}[e^{-\tilde{W}_T}], \tag{4.2.42}$$

which holds for all $\boldsymbol{x}, \boldsymbol{z}$ with $|\boldsymbol{x}|, |\boldsymbol{z}| \le r$. By taking the supremum over $\boldsymbol{z}$ and the infimum over $\boldsymbol{x}$ in (4.2.42), after rearranging we find

$$\sup_{|\boldsymbol{x}|\le r} \mathbb{E}_{\mathcal{N}_0^{\boldsymbol{x}}}[e^{-\tilde{W}_T}] \le \frac{S_1(\bar{R}, r)e^{C_1}}{1-\delta} \inf_{|\boldsymbol{z}|\le r} \mathbb{E}_{\mathcal{N}_0^{\boldsymbol{z}}}[e^{-\tilde{W}_T}],$$

which concludes Step 5 and completes the proof of the theorem. □

The next step is to show local uniform domination.

Proposition 4.16. *Let Assumption* 4.1 *hold, and define* μ_T *as in* (4.2.23)*. Then for every* $S > 0$ *the restrictions of the family* $(\mu_T)_{T\ge 0}$ *to* $\mathcal{F}_S$ *are uniformly dominated by the restrictions of* $\mathcal{N}_0$ *to* $\mathcal{F}_S$.

Proof. Fix $S > 0$ and $\varepsilon > 0$. Using Proposition 4.15, choose $R > 0$ so large that $\mu_T(|X_s| > R) < \varepsilon/8$ for all $|s| < S + 1$ and $T > 0$. Put

$$B = \{|X_{-S-1}| < R, |X_{-S}| < R, |X_S| < R, |X_{S+1}| < R\}.$$

Then we have $\mu_T(B^c) < \varepsilon/2$ for all $T > 0$. For any $A \in \mathcal{F}_S$

$$\mu_T(A) = \mu_T(A \cap B^c) + \mu_T(A \cap B) \le \varepsilon/2 + \mathbb{E}_{\mu_T}[\mu_T(A \cap B|\mathcal{T}_{S+1})]. \tag{4.2.43}$$

We are only interested in the lim sup of this as $T \to \infty$, thus we choose $T > S + 1$. Since μ_T is a finite volume Gibbs measure, (4.1.17) gives

$$\begin{aligned}\mu_T(A \cap B|\mathcal{T}_{S+1})(\bar{\omega}) &= \frac{1}{\mathbb{E}_{(\mathcal{N}_0)^{\bar{\omega}}_{S+1}}[e^{-\mathcal{H}_{\Lambda(S+1,T)}}]}\mathbb{E}_{(\mathcal{N}_0)^{\bar{\omega}}_{S+1}}[e^{-\mathcal{H}_{\Lambda(S+1,T)}}1_{A\cap B}] \\ &\le e^{8C(S+1)^2}\mathbb{E}_{(\mathcal{N}_0)^{\bar{\omega}}_{S+1}}[1_{A\cap B}].\end{aligned}$$

The inequality above follows from (4.2.20) by a similar reasoning as the one leading to (4.2.24), and the constant C above does not depend on T. Moreover, by the choice of B we have

$$\mathbb{E}_{(\mathcal{N}_0)^{\bar\omega}_{S+1}}[1_{A\cap B}] = 1_{\{|\bar\omega(-S-1)|\le R\}} 1_{\{|\bar\omega(S+1)|\le R\}} \int_{|z|\le R}\int_{|y|\le R} \frac{K_1(\bar\omega(-S-1),y)}{\Psi_{\rm p}(y)}$$
$$\times \mathbb{E}_{\mathcal{N}_0}[1_A|\bar\omega(-S)=y,\bar\omega(S)=z]\frac{K_1(z,\bar\omega(S+1))}{\Psi_{\rm p}(z)} dy\,dz,$$

where, as in (4.2.38), K_t is the kernel of e^{-tH}. As in the argument following (4.2.38), it can be seen that there exists $D > 0$ such that $\frac{K_1(\bar\omega(-S-1),y)}{\Psi_{\rm p}(y)} \le D\Psi_{\rm p}(y)$ for all $|y| \le R$ and $|\bar\omega(-S-1)| \le R$. Thus,

$$\begin{aligned}
&\mathbb{E}_{(\mathcal{N}_0)^{\bar\omega}_{S+1}}[1_{A\cap B}] \\
&\le D^2 \int_{|z|\le R}\int_{|y|\le R} \Psi_{\rm p}(y)\mathbb{E}_{\mathcal{N}_0}[1_A|\bar\omega(-S)=y,\bar\omega(S)=z]\Psi_{\rm p}(z)dy\,dz \\
&\le D^2 \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \Psi_{\rm p}(y)\mathbb{E}_{\mathcal{N}_0}[1_A|\bar\omega(-S)=y,\bar\omega(S)=z]\Psi_{\rm p}(z)dy\,dz = D^2\mathbb{E}_{\mathcal{N}_0}[1_A].
\end{aligned}$$

We have obtained that

$$\mu_T(A) \le \varepsilon/2 + D^2 e^{C(S+1)^2}\mathcal{N}_0(A)$$

for every $A \in \mathscr{F}_S$. Thus for given $\varepsilon > 0$, we can choose $\delta < D^{-2}e^{-C(S+1)^2}\varepsilon/2$, and then for every $A \in \mathscr{F}_S$ with $\mathcal{N}_0(A) < \delta$ we have $\mu_T(A) < \varepsilon$ uniformly in T. This shows the claim. □

We can now formulate the main theorem of this section.

Theorem 4.17 (Existence of Gibbs measure). *Under Assumption* 4.1 *there exists a Gibbs measure for the potentials V and W.*

Proof. By Propositions 4.16 and 4.14 the family μ_T^h of finite volume Gibbs measures is locally uniformly dominated. By Proposition 4.4, the sequence $(\mu_n^h)_{n\in\mathbb{N}}$ then has a convergent subsequence, and the limit point is a Gibbs measure. □

4.3 Existence and properties by cluster expansion

4.3.1 Cluster representation

Beside existence a number of properties of Gibbs measures for densities dependent on a pair interaction W can be derived by using the special technique called *cluster expansion*. Some of these properties have an immediate application to the understanding of cut-off behaviours as discussed above. In lack of space we can only summarize the key elements of this method and the main results. We refer the reader to the literature for a more detailed discussion.

Cluster expansion is a technique developed originally in classical statistical mechanics. It relies on the existence of a small parameter so that a measure with the interaction switched on can be constructed in terms of a convergent perturbation series around the interaction free case. In our case the Gibbs measure for the $W \equiv 0$ case is a path measure of a Markov process, while the Gibbs measure we want to construct for $W \not\equiv 0$ is a path measure of a random process which is not Markovian. The cluster expansion method has two distinct aspects. One is a combinatorial framework allowing to make series indexed by graphs and derive bounds on them by series indexed by trees. The other part is analytic which is more dependent on the details of the model, and consists of basic estimates on the entries of these sums leading to the convergence of the expansion.

In this section we make the following assumptions on the potentials.

Assumption 4.2. (1) The external potential $V : \mathbb{R}^d \to \mathbb{R}$ is

(a) Kato-decomposable and the Schrödinger operator $H = -\frac{1}{2}\Delta + V$ has a unique ground state $\Psi_{\mathrm{p}} \in L^2(\mathbb{R}^d)$ with the property $H\Psi_{\mathrm{p}} = 0$

(b) the Schrödinger semigroup generated by $H = -\frac{1}{2}\Delta + V$ is intrinsically ultracontractive (see Definition 3.41).

(2) The pair interaction potential $W : \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}$ has the symmetry properties $W(\cdot,\cdot,t-s) = W(\cdot,\cdot,|t-s|)$, $W(x,y,\cdot) = W(y,x,\cdot)$, and satisfies either of the following regularity conditions:

(W1) There exist $R > 0$ and $\beta > 2$ such that

$$|W(x,y,t-s)| \le R\,\frac{|x|^2 + |y|^2}{1 + |t-s|^\beta} \tag{4.3.1}$$

for every $x, y \in \mathbb{R}^d$ and $t, s \in \mathbb{R}$.

(W2) There exist $R > 0$ and $\beta > 1$ such that

$$|W(x,y,t-s)| \;\le\; \frac{R}{1 + |t-s|^\beta} \tag{4.3.2}$$

for every $x, y \in \mathbb{R}^d$ and $t, s \in \mathbb{R}$.

In particular, if V is a super-quadratic potential with exponent $\alpha > 1$ as defined by (4.2.14), then condition (1) above holds.

Let ν be the Itô measure for V, which we will use as reference measure. Similarly to (4.2.23) we define

$$d\mu_T = \frac{1}{Z_T} \exp\left(-\lambda \int_{-T}^{T} ds \int_{-T}^{T} W(X_t, X_s, t-s)dt\right) d\nu \tag{4.3.3}$$

where $\lambda > 0$ is a parameter.

Theorem 4.18 (Existence of Gibbs measure). *Take any increasing sequence $(T_n)_{n\in\mathbb{N}}\subset \mathbb{R}$ such that $T_n \to \infty$, and suppose $0 < |\lambda| \le \lambda^*$ with λ^* small enough. Then under Assumption 4.2 the local limit $\lim_{n\to\infty}\mu_{[-T_n,T_n]} = \mu$ exists and is a Gibbs measure on $(\mathscr{X},\mathscr{F})$. Moreover, μ does not depend on the sequence $(T_n)_{n\in\mathbb{N}}$.*

In order to prove existence of the Gibbs measure μ we use a cluster expansion controlled by the parameter λ which we choose to be suitably small. In what follows we outline the main steps of the proof of this theorem.

For simplicity and without restricting generality we consider the family of symmetric bounded intervals of $\mathbb{R}$ for the index set of measures. We partition $[-T,T]$ into disjoint intervals $\tau_k = (t_k, t_{k+1})$, $k = 0,\dots,N-1$, with $t_0 = -T$ and $t_N = T$, each of length b, i.e. fix $b = 2T/N$; for convenience we choose N to be an even number so that the origin is endpoint to some intervals. We break up a path X into pieces X_{τ_k} by restricting it to τ_k. The total energy contribution of the pair interaction can be written in terms of the sum

$$W_T = \int_{-T}^{T} ds \int_{-T}^{T} W(X_t, X_s, s-t)dt = \sum_{0\le i<j\le N-1} W_{\tau_i,\tau_j} \tag{4.3.4}$$

where with the notation $\mathcal{J}_{ij} = \int_{\tau_i} ds \int_{\tau_j} W(X_s, X_t, s-t)dt$ we have

$$W_{\tau_i,\tau_j} = \begin{cases} \mathcal{J}_{ij} + \mathcal{J}_{ji} & \text{if } |i-j| \ge 2 \\ \frac{1}{2}(\mathcal{J}_{ii} + \mathcal{J}_{jj}) + \mathcal{J}_{ij} + \mathcal{J}_{ji} & \text{if } |i-j| = 1, \text{ and } i \ne 0,\ j \ne N-1 \\ \mathcal{J}_{ij} + \mathcal{J}_{ji} + \frac{1}{2}\mathcal{J}_{00} & \text{if } i = 0 \text{ and } j = 1 \\ \mathcal{J}_{ij} + \mathcal{J}_{ji} + \frac{1}{2}\mathcal{J}_{N-1\,N-1} & \text{if } i = N-1 \text{ and } j = N-2 \end{cases} \tag{4.3.5}$$

In order to keep the notation simple we do not make explicit the X_t dependence of these objects.

A simple calculation gives

$$\begin{aligned} e^{-\lambda \int_{-T}^{T} ds \int_{-T}^{T} W(X_t,X_s,t-s)dt} &= \prod_{0\le i<j\le N-1} (e^{-\lambda W_{\tau_i,\tau_j}} + 1 - 1) \\ &= 1 + \sum_{\mathcal{R}_T \ne \emptyset} \prod_{(\tau_i,\tau_j)\in\mathcal{R}_T} (e^{-\lambda W_{\tau_i,\tau_j}} - 1). \end{aligned} \tag{4.3.6}$$

Here the summation is performed over all non-empty sets of different pairs of bounded intervals $\mathcal{R}_T = \{(\tau_i,\tau_j) | (\tau_i,\tau_j) \ne (\tau_{i'},\tau_{j'}) \text{ whenever } (i,j) \ne (i',j')\}$.

Since the reference measure is the measure of a Markov process with transition density $\pi_t(x,y)$, a break-up of the paths involves a corresponding factorization of the reference measure into piecewise pasted conditional measures. Put $X_{t_k} = x_k$ for the positions at the time-points of the division, $\forall k = 0,\dots,N$, and $\nu_{\tau_k}^{x_k,x_{k+1}}$ for the corresponding bridge measures. We have thus

$$\nu_T(\cdot \,|X_{t_0} = x_0, \dots, X_{t_N} = x_N) = \bigotimes_{k=0}^{N-1} \nu_{\tau_k}^{x_k, x_{k+1}}(\cdot). \tag{4.3.7}$$

Denote by $d\mathrm{m} = \Psi_{\mathrm{p}}^2 dx$ be the stationary distribution of the Itô measure, and let $p_{t_0,\dots,t_N}(x_0, \dots, x_N)$ be the density with respect to $\bigotimes_{k=0}^{N} d\mathrm{m}(x_k)$ of the joint distribution of positions of the path X_t recorded at the time-points of the division. By the Markov property it follows that

$$\begin{aligned} p_{t_0,\dots,t_N}(x_0, \dots, x_N) &= \prod_{k=0}^{N-1} \pi_b(x_{k+1}, x_k) = \prod_{k=0}^{N-1} (\pi_b(x_{k+1}, x_k) - 1 + 1) \\ &= 1 + \sum_{\mathcal{S}_T \neq \emptyset} \prod_{k: \tau_k \in \mathcal{S}_T} (\pi_b(x_{k+1}, x_k) - 1), \end{aligned}$$

where π_b is given by (4.2.10). The summation runs over all non-empty sets $\mathcal{S}_T = \{\tau_k | \tau_k = (t_k, t_{k+1})\}$ of different pairs of consecutive time-points.

We have thus the product of two sums over not unrelated collections of subsets as the net result. To keep a systematic control over these sums we introduce the following objects.

(1) *Contours.* Two distinct pairs of intervals (τ_i, τ_j) and $(\tau_{i'}, \tau_{j'})$ will be called *joint* and denoted $(\tau_i, \tau_j) \sim (\tau_{i'}, \tau_{j'})$ if one interval of the pair (τ_i, τ_j) coincides with one interval of the pair $(\tau_{i'}, \tau_{j'})$. A set of *connected* pairs of intervals is a collection $\{(\tau_{i_1}, \tau_{j_1}), \dots, (\tau_{i_n}, \tau_{j_n})\}$ in which each pair of intervals is connected to another through a sequence of joint pairs, i.e., for any $(\tau_i, \tau_j) \neq (\tau_{i'}, \tau_{j'})$ there exists $\{(\tau_{k_1}, \tau_{l_1}), \dots, (\tau_{k_m}, \tau_{l_m})\}$ such that $(\tau_i, \tau_j) \sim (\tau_{k_1}, \tau_{l_1}) \sim \cdots \sim (\tau_{k_m}, \tau_{l_m}) \sim (\tau_{i'}, \tau_{j'})$. A maximal set of connected pairs of intervals is called a *contour*, denoted by γ. We denote by $\bar{\gamma}$ the set of all intervals that are elements of the pairs of intervals belonging to contour γ, and by γ^* the set of time-points of intervals appearing in $\bar{\gamma}$. Two contours γ_1, γ_2 are *disjoint* if they have no intervals in common, i.e. $\bar{\gamma}_1 \cap \bar{\gamma}_2 = \emptyset$. Clearly, $\mathcal{R}_T$ can be decomposed into sets of pairwise disjoint contours: $\mathcal{R}_T = \bigcup_{r \geq 1} \{\gamma_1, \dots, \gamma_r | \bar{\gamma}_i \cap \bar{\gamma}_j = \emptyset, i \neq j; i, j = 1, \dots, r\}$.

(2) *Chains.* A collection of consecutive intervals $\{\tau_j, \tau_{j+1} \dots, \tau_{j+k}\}$, $j \geq 0$, $j+k \leq N-1$ is called a *chain*. As in the case of contours, $\bar{\varrho}$ and ϱ^* mean the set of intervals belonging to the chain ϱ and the set of time-points in ϱ, respectively. Two chains ϱ_1, ϱ_2 are called *disjoint* if they have no common time-points, i.e. $\varrho_1^* \cap \varrho_2^* = \emptyset$. Denote by $\partial^- \varrho$ resp. $\partial^+ \varrho$ the leftmost resp. rightmost time-points belonging to ϱ.

(3) *Clusters.* Take a (non-ordered) set of disjoint contours and disjoint chains, $\Gamma = \{\gamma_1, \dots, \gamma_r; \varrho_1, \dots, \varrho_s\}$, with some $r \geq 1$ and $s \geq 0$. Note that such contours and chains may have common time-points. The notation $\Gamma^* = (\bigcup_i \gamma_i^*) \cup (\bigcup_j \varrho_j^*)$ means the set of all time-points appearing as beginnings or ends of intervals belonging to some contour or chain in Γ. Also, we put $\bar{\Gamma} = (\bigcup_i \bar{\gamma}_i) \cup (\bigcup_j \bar{\varrho}_j)$ for the set of intervals appearing in Γ through entering some contours or chains. Γ is called a *cluster* if Γ^* is a connected collection of sets (in the usual sense), and for every $\varrho \in \Gamma$ we have that $\partial^- \varrho, \partial^+ \varrho \in \bigcup_{j=1}^r \gamma_j^*$. This means that in a cluster chains have no loose ends. We denote by $\mathbb{K}_N$ the set of all clusters for a given N.

With these notations the sum in (4.3.6) is then further written as

$$\sum_{\mathcal{R}_T \neq \emptyset} \prod_{(\tau_i, \tau_j) \in \mathcal{R}_T} (e^{-\lambda W_{\tau_i, \tau_j}} - 1) = \sum_{r \geq 1} \sum_{\{\gamma_1, \dots, \gamma_r\}} \prod_{k=1}^{r} \prod_{(\tau_i, \tau_j) \in \gamma_k} (e^{-\lambda W_{\tau_i, \tau_j}} - 1), \tag{4.3.8}$$

where now summation goes over collections $\{\gamma_1, \dots, \gamma_r\}$ of disjoint contours. In a similar way (4.3.7) appears in the form

$$\sum_{\mathcal{S}_T \neq \emptyset} \prod_{k : \tau_k \in \mathcal{S}_T} (\pi_b(x_{k+1}, x_k) - 1) = \sum_{s \geq 1} \sum_{\{\varrho_1, \dots, \varrho_s\}} \prod_{j=1}^{s} \prod_{k : \tau_k \in \varrho_j} (\pi_b(x_{k+1}, x_k) - 1) . \tag{4.3.9}$$

Here $\{\varrho_1, \dots, \varrho_s\}$ is a collection of disjoint chains, and this formula justifies how we defined them.

For every cluster $\Gamma = \{\gamma_1, \dots, \gamma_r; \varrho_1, \dots, \varrho_s\} \in \mathbb{K}_N$ define the function

$$\kappa_\Gamma = \prod_{l=1}^{r} \prod_{(\tau_i, \tau_j) \in \gamma_l} (e^{-\lambda W_{\tau_i, \tau_j}} - 1) \prod_{m=1}^{s} \prod_{k : \tau_k \in \varrho_m} (\pi_b(x_{k+1}, x_k) - 1) . \tag{4.3.10}$$

Also, we define the auxiliary probability measure

$$d\chi_N(X) = \bigotimes_{k=0}^{N-1} d\nu_{\tau_k}^{x_k, x_{k+1}}(X_{\tau_k}) \bigotimes_{k=0}^{N} d\mathrm{m}(x_k). \tag{4.3.11}$$

It is straightforward to check that $\{\chi_N\}_{N \in \mathbb{N}}$ is a family of consistent probability measures, thus by the Kolmogorov extension theorem it has a unique extension to $\mathbb{R}$ which we denote by χ. Note that

$$\int_{\mathbb{R}^d} (\pi_b(x_{k+1}, x_k) - 1) d\mathrm{m}(x_{k+1}) = \int_{\mathbb{R}^d} (\pi_b(x_{k+1}, x_k) - 1) d\mathrm{m}(x_k) = 0.$$

This is why in our definition a cluster does not contain chains having loose ends as for any such chain $\mathbb{E}_{\chi_N}[\kappa_\Gamma] = 0$.

Finally, we define the function

$$K_\Gamma = \mathbb{E}_\chi[\kappa_\Gamma], \tag{4.3.12}$$

A combination of (4.3.8), (4.3.7), (4.3.9), (4.3.10) and (4.3.12) then gives the cluster representation of the partition function

$$Z_T = 1 + \sum_{n \geq 1} \sum_{\substack{\{\Gamma_1,\dots,\Gamma_n\} \in \mathbb{K}_N \\ \Gamma_i^* \cap \Gamma_j^* = \emptyset, i \neq j}} \prod_{l=1}^n K_{\Gamma_l} \tag{4.3.13}$$

for every $T = Nb/2 > 0$.

Let G be a graph with vertex set $\{1, \dots, n\}$, in which an edge ij joins vertex i with vertex j whenever $\Gamma_i^* \cup \Gamma_j^* \neq \emptyset$. Denote by $\mathscr{G}_n$ the set of connected graphs G defined in this way, and define

$$\phi^T(\Gamma_1, \dots, \Gamma_n) = \begin{cases} 1 & \text{if } n = 1 \\ \sum_{G \in \mathscr{G}_n} \prod_{ij \in G} (-1_{\{\Gamma_i^* \cap \Gamma_j^* \neq \emptyset\}}) & \text{if } n > 1. \end{cases}$$

By using a combinatorial function calculus it is possible to show that

$$\log Z_T = \sum_{n \geq 1} \sum_{\substack{\{\Gamma_1,\dots,\Gamma_n\} \in \mathbb{K}_N \\ 0 \in \Gamma_1^*}} \phi^T(\Gamma_1, \dots, \Gamma_n) \prod_{l=1}^n K_{\Gamma_l}. \tag{4.3.14}$$

Also, consider for any subset $A \subset \mathbb{R}$ the set function

$$Z_T^A = 1 + \sum_{n \geq 1} \sum_{\substack{\Gamma_1,\dots,\Gamma_n \in \mathbb{K}_N \\ \Gamma_i^* \cap \Gamma_j^* = \emptyset, i \neq j \\ A \cap (\cup_i \Gamma_i^*) = \emptyset}} \prod_{i=1}^n K_{\Gamma_i}$$

and write $Z_T^\Gamma = Z_T^{\overline{\Gamma} \cup \Gamma^*}$. Then by combinatorial function calculus it is furthermore possible to derive the expressions

$$\log Z_T^\Gamma = 1 + \sum_{n \geq 1} \sum_{\substack{\Gamma_1,\dots,\Gamma_n \in \mathbb{K}_N \\ \Gamma^* \cap (\cup_i \Gamma_i^*) = \emptyset}} \phi^T(\Gamma_1, \dots, \Gamma_n) \prod_{i=1}^n K_{\Gamma_i}$$

and

$$f_T^\Gamma = \frac{Z_T^\Gamma}{Z_T} = \exp\Big(- \sum_{n \geq 1} \sum_{\substack{\Gamma_1,\dots,\Gamma_n \in \mathbb{C}_N \\ \Gamma^* \cap (\cup_i \Gamma_i^*) \neq \emptyset}} \phi^T(\Gamma_1, \dots, \Gamma_n) \prod_{i=1}^n K_{\Gamma_i} \Big). \tag{4.3.15}$$

The function f_T^Γ is a *correlation function* associated with cluster Γ.

4.3.2 Basic estimates and convergence of cluster expansion

There are two crucial estimates on which the cluster expansion depends. The first estimate is on the energy terms coming from the contributions of the intervals in the partition. Recall the notation $\mathcal{J}_{ij}$ in (4.3.5).

Lemma 4.19 (Energy estimates). *We have*

$$|\mathcal{J}_{ij}| \leq \begin{cases} C_1 b \, \frac{\int_{\tau_i} X^2_{\tau_i}(t)dt + \int_{\tau_j} X^2_{\tau_j}(s)ds}{(|j-i-1|b)^\beta + 1} & \text{(W1) } \textit{case} \\ \frac{C_2 b^2}{(|j-i-1|b)^\beta + 1} & \text{(W2) } \textit{case} \end{cases} \tag{4.3.16}$$

with some $C_1, C_2 > 0$ in each case respectively.

As a consequence, W_{τ_i,τ_j} is bounded from below for all i, j, which is a stability condition ensuring that the variables indexed by the subintervals are well defined and there are no short range ($i = j$) singularities.

The second estimate for the cluster expansion is given by

Lemma 4.20 (Cluster estimates). *There exist some constants $c_1, c_2 > 0$ and $\delta > 1$ such that for every cluster $\Gamma \in \mathbb{K}_N$*

$$|K_\Gamma| \leq \prod_{\varrho \in \Gamma} (c_1 |\lambda|^{1/3})^{|\bar{\varrho}|} \prod_{\gamma \in \Gamma} \prod_{(\tau_i,\tau_j) \in \gamma} \frac{c_2 |\lambda|^{1/3}}{(|i-j-1|b)^\delta + 1} \tag{4.3.17}$$

where $|\bar{\varrho}|$ denotes the number of intervals contained in ϱ.

In estimate (4.3.17) the factor accounting for the contribution of chains comes from the upper bound $Ce^{-\Lambda b}$ on $|\pi_b(x, y) - 1|$ uniform in x, y (see second factor in (4.3.10)), where Λ is the spectral gap of the Schrödinger operator H of the underlying $P(\phi)_1$-process, and $C > 0$. This bound results from intrinsic ultracontractivity of e^{-tH}, see Lemma 3.105. The factor accounting for the contribution of contours comes from a suitable use of the Hölder inequality and optimization applied to the products over $e^{-\lambda W_{\tau_i \tau_j}} - 1$ (see first factor in (4.3.10)), taken together with Lemma 4.19. b is then chosen in such a combination with λ and Λ that the expression (4.3.17) results.

Proposition 4.21 (Convergence of cluster expansion). *If there exists $\eta(\lambda) > 0$ such that* $\lim_{\lambda \to 0} \eta(\lambda) = 0$ *and for every $n \in \mathbb{N}$*

$$\sum_{\substack{\Gamma \in \mathbb{K}_N \\ \Gamma^* \ni 0, |\Gamma| = n}} |K_\Gamma| \leq c\, \eta^n, \tag{4.3.18}$$

then the series at the right-hand sides of (4.3.13) *and* (4.3.14)*, respectively, are absolutely convergent as $N \to \infty$.*

In order to prove this statement we consider the generating function

$$H(z,\lambda) = \lim_{N\to\infty} \sum_{\substack{\Gamma\in\mathbb{K}_N \\ \Gamma_1^*\ni 0}} K_\Gamma z^{|\overline{\Gamma}|}$$

in which λ is considered a parameter and $|\overline{\Gamma}|$ is the number of intervals contained by Γ. We show that $H(z,\lambda)$ is analytic in z in a circle of radius $r(\lambda)$, and that inside this circle $H(z,\lambda)$ is uniformly bounded in λ. By choosing $\eta(\lambda) = 1/r(\lambda)$ gives then the bound in (4.3.18).

This estimate follows through a procedure of translating the sums over $\mathbb{K}_N$ at the left-hand side of (4.3.18) to sums indexed by connected graphs. For every $\Gamma \in \mathbb{K}_N$ we define a graph G_Γ whose vertex set are the contours $\{\gamma_1,\dots,\gamma_r\}$ in Γ, and whose edge set are those pairs of contours which are appropriately matched with the chains $\rho_1,\dots,\rho_s$ of Γ. For $r=1$ we have $s=0$ and therefore the empty graph G_{Γ_1}. For $r\geq 2$ we have $s\geq 1$ and the sums can be rewritten by sums over connected graphs following a specific enumeration. By using a graph-tree bound, the graph-indexed sums can be bounded by tree-indexed sums, which can be better further estimated. An inductive procedure and some non-trivial combinatorics yields then (4.3.18).

By an application of Proposition 4.21 to (4.3.15) it can be shown that $f^\Gamma = \lim_{T\to\infty} f_T^\Gamma$ exists, and $|f_T^\Gamma| \leq 2^{|\overline{\Gamma}|}$ holds uniformly in T for λ small enough. This can be used to prove the following statement.

Proposition 4.22. *The local limit $\mu = \lim_{T\to\infty}\mu_T$ exists and satisfies the equality*

$$\mathbb{E}_\mu[F_S] = \mathbb{E}_\chi[F_S]f^S + \sum_{n\geq 1} \sum_{\substack{\Gamma_1,\dots,\Gamma_n\in\mathbb{C} \\ \Gamma_i^*\cap\Gamma_j^*=\emptyset,\ i\neq j \\ i:S\cap\Gamma_i^*\neq\emptyset}} \mathbb{E}_\chi\Big[F_S\prod_{i=1}^n \kappa_{\Gamma_i}\Big] f^{\cup\overline{\Gamma}} \tag{4.3.19}$$

for any bounded $\mathscr{F}_S$-measurable function F_S, where S is a finite union of intervals of the partition considered in the cluster expansion. Moreover, the measure μ is invariant with respect to time shift.

Theorem 4.18 follows directly from this proposition. Equality (4.3.19) implies that μ satisfies the DLR-equations, in particular, it is a Gibbs measure.

4.3.3 Further properties of the Gibbs measure

An important aspect in understanding the Gibbs measure is to characterize almost sure properties of paths under this measure. This is answered by the following theorem.

Theorem 4.23 (Typical path behaviour). *If V is a super-quadratic potential with exponent $\alpha > 1$, then*

$$|X_t| \leq C\,(\log(|t|+1))^{1/(\alpha+1)}, \quad \mu\text{-a.s.} \tag{4.3.20}$$

with a suitable number $C > 0$.

The strategy of proving Theorem 4.23 goes by comparing the typical behaviours of the reference process and of the process with the pair interaction potential (which is not a Markov process).

Lemma 4.24. *Suppose there exist some numbers $C, \theta > 0$ such that for any $x > 0$*

$$\nu(\max_{0 \le t \le 1} |X_t| \ge x) \le C e^{-\theta x^{\alpha+1}}. \tag{4.3.21}$$

Then there exist $C' > 0$ and $\theta' > 0$ such that for any $x > 0$

$$\mu(\max_{0 \le t \le 1} |X_t| \ge x) \le C' e^{-\theta' x^{\alpha+1}}. \tag{4.3.22}$$

The proof of this lemma requires once again the use of cluster expansion. The assumption of the lemma can be verified by using (3.111) for the underlying $P(\phi)_1$-process.

Another important property of μ is its uniqueness in DLR sense. This means that for any increasing sequence of real numbers $(T_n)_{n\in\mathbb{N}}$, $T_n \to \infty$, and any corresponding sequence of boundary conditions $(Y_n)_{n\in\mathbb{N}} \subset \mathcal{X}_0$, $\lim_{n\to\infty} \mathbb{E}_{\mu_{T_n}}[F_B|Y_n] = \mathbb{E}_\mu[F_B]$, for every bounded $B \subset \mathbb{R}$, and every bounded $\mathscr{F}_B$-measurable function F_B. Here $\mathcal{X}_0$ is the subspace given by (4.2.13). In other words, DLR uniqueness means that the limit measure μ is independent of the boundary paths.

Theorem 4.25 (Uniqueness of Gibbs measure). *Suppose that V satisfies the conditions in Assumption* 4.2 *and W satisfies* (W2). *Then the following cases occur:*

(a) *If $\beta > 2$, then whenever the Gibbs measure μ exists, it is unique in DLR sense.*

(b) *If $\beta > 1$, then for sufficiently small $|\lambda|$ the limiting Gibbs measure μ is unique in DLR sense whenever the reference measure is unique.*

If $\beta > 2$ the energy functional given by (4.1.15) is uniformly bounded in T, and in the restrictions of paths over $[-T, T]$ and $R \setminus [-T, T]$, respectively. This then implies that only one Gibbs measure can exist as the limit does not depend on the choice of the boundary conditions. This argument requires no restriction on the values of λ. For $1 < \beta \le 2$ this uniform boundedness does not hold any longer and cluster expansion is to date the only method allowing to derive these results.

To conclude, we list some additional properties of Gibbs measures for (W2)-type pair interaction potentials, useful in various contexts. This case in particular covers the Nelson model.

Theorem 4.26. *Let μ be a Gibbs measure for W satisfying* (W2)*, and suppose V satisfies Assumption* 4.2. *Then the following hold:*

(a) *Invariance properties:* μ *is invariant with respect to time shift and time reflection.*

(b) *Univariate distributions: The distributions* ϖ_T *under* μ_T *of positions* x *at time* $t = 0$ *are equivalent to* ν, *i.e. there exist* $C_1, C_2 \in \mathbb{R}$, *independent of* T *and* x *such that*

$$C_1 \leq \frac{d\varpi_T}{d\nu}(x) \leq C_2 \tag{4.3.23}$$

for every $x \in \mathbb{R}^d$ *and* $T > 0$. *Moreover* $\lim_{T\to\infty} \frac{d\varpi_T}{d\nu}(x) = \frac{d\varpi}{d\nu}(x)$ *exists pointwise.*

(c) *Univariate conditional probability distributions: The conditional distributions* $\mu_T(\,\cdot\,|X_0 = x)$ *converge locally weakly to* $\mu(\,\cdot\,|X_0 = x)$, *for all* $x \in \mathbb{R}^d$.

(d) *Mixing properties: For any bounded functions* F, G *on* $\mathbb{R}^d$

$$|\operatorname{cov}_\mu(F_s; G_t)| \leq \text{const}\,\frac{\sup \|F\|_\infty \sup \|G\|_\infty}{1 + |t-s|^\gamma} \tag{4.3.24}$$

where $\gamma > 0$ *and the constant prefactor is independent of* s, t *and* F, G.

The invariance properties are inherited trivially from the underlying $P(\phi)_1$-process. The proof of the remaining properties makes extensive use of cluster expansion.

4.4 Gibbs measures with no external potential

4.4.1 Gibbs measure

Here we consider the case when the external potential $V = 0$. We further assume that the pair interaction potential has the form $W(X_t - X_s, t-s)$. Then the energy $\mathcal{H}_{\Lambda(S,T)} = \iint_{\Lambda(S,T)} W(X_t, X_s, |t-s|)dsdt$, is invariant under path shifts $\omega(t) \mapsto \omega(t) + c$, which makes it reasonable to expect that no Gibbs measure exists. However, when the path is fixed at $t = 0$ we can in certain cases construct a Gibbs measure with interesting properties.

In contrast to the previous sections, in this section we use Wiener measure $\mathcal{W}^0$ on path space over the whole time line as reference measure. We assume that

$$W(x,t) = -\frac{1}{2}\int_{\mathbb{R}^d} \frac{|\widehat{\rho}(k)|^2}{2\omega(k)} e^{ik\cdot x} e^{-\omega(k)|t|} dk \tag{4.4.1}$$

with

$$\omega(k) \geq 0, \quad \omega(k) = \omega(-k), \quad \widehat{\rho}(k) = \overline{\widehat{\rho}(-k)} \tag{4.4.2}$$

$$\widehat{\rho}/\omega^\alpha \in L^2(\mathbb{R}^d), \quad \alpha = 1/2, 1, 3/2. \tag{4.4.3}$$

The reason for this special choice will become clear later. For the moment, we note that by (4.4.3) W is bounded as a function of x and t,

$$\int_{-\infty}^{\infty} \sup_{\omega' \in \mathfrak{X}} |W(B_t(\omega') - B_s(\omega'), t-s)| ds = \frac{1}{2} \int_{\mathbb{R}^d} \frac{|\widehat{\rho}(k)|^2}{\omega(k)^2} dk < \infty, \tag{4.4.4}$$

and

$$\int_{-\infty}^{0} ds \int_{0}^{\infty} dt \sup_{\omega' \in \mathfrak{X}} |W(B_t(\omega') - B_s(\omega'), t-s)| = \frac{1}{4} \int_{\mathbb{R}^d} \frac{|\widehat{\rho}(k)|^2}{\omega(k)^3} dk < \infty. \tag{4.4.5}$$

We define

$$d\mathcal{P}_T^0 = \frac{1}{Z_T} \exp\left(- \int_{-T}^{T} \int_{-T}^{T} W(B_t - B_s, t-s) dt ds \right) d\mathcal{W}^0, \tag{4.4.6}$$

where

$$Z_T = \mathbb{E}^0 \left[\int \exp\left(- \int_{-T}^{T} \int_{-T}^{T} W(B_t - B_s, t-s) dt ds \right) \right].$$

Proposition 4.27 (Finite volume Gibbs measure). *The measure $\mathcal{P}_T^0$ is a finite volume Gibbs measure for reference measure $\mathcal{W}^0$ and pair interaction potential W.*

Proof. This follows directly from Proposition 4.1. □

For the existence of a Gibbs measure we need to refer to the *infinite dimensional Ornstein–Uhlenbeck process* which will be introduced in Section 5.6; see Theorem 5.26 for the details of definitions and notation. Here we quickly give a brief overview of what we need in this subsection.

Let $\mathcal{M}$ be a Hilbert space with scalar product

$$(f, g)_{\mathcal{M}} = \int_{\mathbb{R}^d} \overline{\hat{f}(k)} \hat{g}(k) \frac{1}{2\omega(k)} dk \tag{4.4.7}$$

and $\mathcal{M}_{+2} \subset \mathcal{M} \subset \mathcal{M}_{-2}$ be a triplet, where $\mathcal{M}_{+2}^* = \mathcal{M}_{-2}$. Let $\mathfrak{Y} = C(\mathbb{R}, \mathcal{M}_{-2})$. The process $\mathbb{R} \ni t \mapsto \xi_t \in \mathcal{M}_{-2}$ is the $\mathcal{M}_{-2}$-valued Ornstein–Uhlenbeck process on $(\mathfrak{Y}, \mathcal{G})$ with path measure $\mathcal{G}$. Let $\xi_t(f) = \langle\langle \xi_t, f \rangle\rangle$ for $f \in \mathcal{M}_{+2}$, where $\langle\langle \cdot, \cdot \rangle\rangle$ denotes the pairing between $\mathcal{M}_{-2}$ and $\mathcal{M}_{+2}$. Note that $\xi_t(f)$ can be uniquely extended for $f \in \mathcal{M}$, which for simplicity we keep denoting by $\xi_t(f)$. Then $\xi_t(f)$ has mean zero and covariance

$$\mathbb{E}_{\mathcal{G}}[\xi_s(f)\xi_t(g)] = \int_{\mathbb{R}^d} \overline{\hat{f}(k)} \hat{g}(k) e^{-|t-s|\omega(k)} \frac{1}{2\omega(k)} dk. \tag{4.4.8}$$

Furthermore, G is a stationary measure on $\mathscr{M}_{-2}$ such that $\xi(f)$, $\xi \in \mathscr{M}_{-2}$ and $f \in \mathscr{M}$, with mean zero and the covariance given by (4.4.7). Let $\mathscr{G}^{\xi}(\cdot) = \mathscr{G}(\cdot|\xi_0 = \xi)$ be a regular conditional probability measure; then $\int \mathsf{G}(d\xi)\mathbb{E}_{\mathscr{G}^{\xi}}[\cdots] = \mathbb{E}_{\mathscr{G}}[\cdots]$.

We write $L^2(\mathscr{G})$ and $L^2(\mathsf{G})$ for $L^2(\mathfrak{Y}, d\mathscr{G})$ and $L^2(\mathscr{M}_{-2}, d\mathsf{G})$, respectively. For $q \in \mathbb{R}^d$ let τ_q be the shift by q on $\mathscr{M}_{-2}$ such that $(\tau_q\xi)(f) = \xi(\theta_q f)$ for $\xi \in \mathscr{M}_{-2}$, where $(\theta_q f)(\cdot) = f(\cdot - q)$. It can be seen that the linear hull of $\{e^{i\xi(f)} | f \in \mathscr{M}\}$ is dense in $L^2(\mathsf{G})$. By $(\tau_q e^{i\xi(f)}) = e^{i\xi(\theta_q f)}$, τ_q can be extended to whole $L^2(\mathsf{G})$.

Lemma 4.28. *For every q the map τ_q is unitary on $L^2(\mathsf{G})$, and $q \mapsto \tau_q$ is a strongly continuous group on $L^2(\mathsf{G})$.*

Proof. First note that for $f, g \in \mathscr{M}$,

$$\begin{aligned}(f, \theta_q g)_{\mathscr{M}} &= \int_{\mathbb{R}^d} \overline{\hat{f}(k)}\widehat{\theta_q g}(k)\frac{1}{2\omega(k)}dk = \int_{\mathbb{R}^d} \overline{\hat{f}(k)}e^{ikq}\hat{g}(k)\frac{1}{2\omega(k)}dk \\ &= \int_{\mathbb{R}^d} \overline{\widehat{\theta_{-q} f}(k)}\hat{g}(k)\frac{1}{2\omega(k)}dk = (\theta_{-q} f, g)_{\mathscr{M}}.\end{aligned}$$

Similarly, $(\theta_q f, \theta_q g)_{\mathscr{M}} = (f, g)_{\mathscr{M}}$. By Theorem 5.26 (d),

$$\begin{aligned}&(e^{i\xi(f)}, \tau_q e^{i\xi(g)})_{L^2(\mathsf{G})} \\ &= \exp\left(-\frac{1}{2}((f, f)_{\mathscr{M}} + (f, \theta_q g)_{\mathscr{M}} + (\theta_q g, f)_{\mathscr{M}} + (\theta_q g, \theta_q g)_{\mathscr{M}})\right) \\ &= \exp\left(-\frac{1}{2}((\theta_{-q} f, \theta_{-q} f)_{\mathscr{M}} + (\theta_{-q} f, g)_{\mathscr{M}} + (g, \theta_{-q} f)_{\mathscr{M}} + (g, g)_{\mathscr{M}})\right) \\ &= (\theta_{-q} e^{i\xi(f)}, e^{i\xi(g)})_{L^2(\mathsf{G})}.\end{aligned}$$

Since the functions of the form $e^{i\xi_s(f)}$ span $L^2(\mathsf{G})$, the equality of the first two lines above shows strong continuity of the map $q \mapsto \tau_q$, and the group property $\tau_{q+r} = \tau_q\tau_r$. Moreover, the equality of the first and last lines shows that the adjoint of τ_q is given by τ_{-q}. Since clearly $\tau_q\tau_{-q} = 1$, τ_q is a unitary map. □

For $T > 0$ define

$$\mathscr{P}_0^W = \mathscr{W}^0 \otimes \mathscr{G} \tag{4.4.9}$$

and

$$d\mathscr{P}_T^W = \frac{1}{Z_T}\exp\left(-\int_{-T}^{T} \tau_{B_s}\xi_s(\varrho)ds\right)d\mathscr{P}_0^W, \tag{4.4.10}$$

where Z_T is the normalizing constant

$$Z_T = \mathbb{E}_{\mathscr{P}_0^W}\left[\exp\left(-\int_{-T}^{T} \tau_{B_s}\xi_s(\varrho)ds\right)\right].$$

Proposition 4.29. *For bounded $\mathscr{F}_T$-measurable functions $f : \mathfrak{X} \to \mathbb{R}$ we have* $\mathbb{E}_{\mathscr{P}_T^W}[f] = \mathbb{E}_{\mathscr{P}_T^0}[f]$.

Proof. Since $\int_{-T}^{T} \tau_{B_s}\xi_s(\varrho)ds$ is a Gaussian process with mean zero and covariance $\mathbb{E}_{\mathscr{G}}[|\int_{-T}^{T} \tau_{B_s}\xi_s(\varrho)ds|^2] = \int_{-T}^{T} ds \int_{-T}^{T} dt W(B_t - B_s, t-s)$, the proposition follows. □

In the light of Proposition 4.29, we will now study the family of measures $\mathscr{P}_T^W$ instead of $\mathscr{P}_T^0$. On a first sight this seems more difficult as the state space is now infinite dimensional. However, the great advantage is that $\mathscr{P}_T^W$ is tightly connected to a Markov process, as we will now see. Define the operator P_t on $L^2(\mathbb{R}^d) \otimes L^2(\mathsf{G})$ by

$$(F, P_t G)_{L^2(\mathbb{R}^d)\otimes L^2(\mathsf{G})} = \int_{\mathbb{R}^d} dx \mathbb{E}_{\mathcal{W}^x \otimes \mathscr{G}}[F(B_0, \xi_0) e^{-\int_0^t \tau_{B_s}\xi_s(\varrho)ds} G(B_t, \xi_t)]. \tag{4.4.11}$$

It is not trivial to see that (4.4.11) is well defined for all $F, G \in L^2(\mathbb{R}^d) \otimes L^2(\mathsf{G})$, since the exponent $-\int_0^t \tau_{B_s}\xi_s(\varrho)ds$ is not necessarily negative. However,

$$\left| \int_{\mathbb{R}^d} dx \mathbb{E}_{\mathcal{W}^x \otimes \mathscr{G}}[F(B_0, \xi_0) e^{-\int_0^t \tau_{B_s}\xi_s(\varrho)ds} G(B_t, \xi_t)] \right| \le \|F\|\|G\| e^c, \tag{4.4.12}$$

where $c = t \int_{\mathbb{R}^d} |\hat{\rho}(k)|^2/\omega(k)^2 dk < \infty$. This will be shown in Corollary 7.8 later on; see also (6.5.8). We will need to work with the following additional function spaces:

$$L^\infty(\mathbb{R}^d; L^2(\mathsf{G})) = \{f : \mathbb{R}^d \times \mathscr{M}_{-2} \to \mathbb{C} \mid \operatorname*{ess\,sup}_{q\in\mathbb{R}^d} \|f(q,\cdot)\|_{L^2(\mathsf{G})} < \infty\},$$

$$L^\infty(\mathbb{R}^d \times \mathscr{M}_{-2}) = \{f : \mathbb{R}^d \times \mathscr{M}_{-2} \to \mathbb{C} \mid \operatorname*{ess\,sup}_{q\in\mathbb{R}^d, \xi\in\mathscr{M}_{-2}} |f(q,\xi)| < \infty\},$$

$$L^2(\mathbb{R}^d; L^2(\mathsf{G})) = \left\{ f : \mathbb{R}^d \times \mathscr{M}_{-2} \to \mathbb{C} \,\middle|\, \int |f(q,\xi)|^2 dq d\mathsf{G} < \infty \right\}.$$

Clearly, $L^\infty(\mathbb{R}^d \times \mathscr{M}_{-2}) \subset L^\infty(\mathbb{R}^d; L^2(\mathsf{G}))$, and $L^2(\mathbb{R}^d; L^2(\mathsf{G}))$ is isomorphic to $L^2(\mathbb{R}^d) \otimes L^2(\mathsf{G})$. We use this identification without further notice. Moreover, we write

$$C(\mathbb{R}^d; L^2(\mathsf{G})) = \{f : \mathbb{R}^d \times \mathscr{M}_{-2} \to \mathbb{C} \mid \mathbb{R}^d \ni q \mapsto f(q,\cdot) \in L^2(\mathsf{G}) \text{ is continuous}\},$$

$$C_{\mathrm{b}}(\mathbb{R}^d; L^2(\mathsf{G})) = C(\mathbb{R}^d; L^2(\mathsf{G})) \cap L^\infty(\mathbb{R}^d; L^2(\mathsf{G})).$$

We need to study P_t on two closed subspaces of $C_{\mathrm{b}}(\mathbb{R}^d, L^2(\mathsf{G}))$. The first one is

$$C_0(\mathbb{R}^d; L^2(\mathsf{G})) = \left\{ f \in C_{\mathrm{b}}(\mathbb{R}^d; L^2(\mathsf{G})) \,\middle|\, \lim_{|q|\to\infty} \|f(q,\cdot)\|_{L^2(\mathsf{G})} = 0 \right\}.$$

The second subspace $\mathcal{T}$ is the image of $L^2(\mathsf{G})$ under the operator

$$U : L^2(\mathsf{G}) \to C(\mathbb{R}^d; L^2(\mathsf{G})), \quad (Uf)(q,\cdot) = (\tau_q f)(\cdot).$$

τ_q is an isometry on $L^2(\mathsf{G})$ for every q. Thus, $\mathcal{T}$ equipped with the scalar product

$$\langle f, g\rangle_{\mathcal{T}} = \langle U^{-1} f, U^{-1} g\rangle_{L^2(\mathsf{G})} \tag{4.4.13}$$

is a Hilbert space. Notice that $\|f\|_{\mathcal{T}} = \|f\|_{L^\infty(\mathbb{R}^d;L^2(\mathsf{G}))}$, and U is an isometry from $L^2(\mathsf{G})$ onto $\mathcal{T}$.

Theorem 4.30. *P_t is a strongly continuous semigroup on $\mathcal{T}$ and on $C_0(\mathbb{R}^d; L^2(\mathsf{G}))$. The generator of P_t on both spaces is given by $-H$, with*

$$(Hf)(q,\xi) = -\frac{1}{2}\Delta_q f(q,\xi) + H_{\mathrm{f}} f(q,\xi) + V_\varrho(q,\xi) f(q,\xi), \quad (q,\xi) \in \mathbb{R}^d \times \mathscr{M}_{-2}, \tag{4.4.14}$$

where $V_\varrho(q,\xi) = \tau_q \xi(\varrho)$ and H_{f} is the infinitesimal generator of the infinite dimensional Ornstein–Uhlenbeck process $(\xi_t)_{t\in\mathbb{R}}$. On $\mathcal{T}$, H is a self-adjoint operator.

The proof of this theorem depends on a number of auxiliary results.

Proposition 4.31. *For every $\varepsilon > 0$ there exists a constant C independent of q such that*

$$\|V_\varrho(q,\cdot) f\|^2_{L^2(\mathsf{G})} \le \varepsilon \|H_{\mathrm{f}} f\|^2_{L^2(\mathsf{G})} + C\|f\|^2_{L^2(\mathsf{G})} \tag{4.4.15}$$

for all $f \in D(H_{\mathrm{f}})$. Consequently, $H_{\mathrm{f}} + V_\varrho(q)$ is self-adjoint on $D(H_{\mathrm{f}})$ and bounded from below uniformly in q.

Proof. This is a reformulation of Proposition 6.1, and the proof is given there. The final statement follows from the Kato–Rellich theorem. □

Proposition 4.32. *We have on $L^2(\mathbb{R}^d; L^2(\mathsf{G}))$*

$$P_t = e^{-tH}, \quad t \ge 0, \tag{4.4.16}$$

where H is given by (4.4.14). *Moreover,*

$$\|P_t f\|^2_{L^2(\mathsf{G})}(x) = \mathbb{E}_{\mathcal{W}^x \otimes \mathcal{G}}[f(B_{-t}, \xi_{-t}) e^{\int_{-t}^t \tau_{B_s}\xi_s(\varrho) ds} f(B_t, \xi_t)] \tag{4.4.17}$$

for $f \in L^\infty(\mathbb{R}^d \times \mathscr{M}_{-2}) \cap L^2(\mathbb{R}^d; L^2(\mathsf{G}))$.

Proof. (4.4.16) is the Feynman–Kac formula, see Theorems 6.2 and 6.3. For (4.4.17), note that for $g \in L^2(\mathbb{R}^d)$ with compact support, the definition of P_t, self-adjointness and the semigroup property give

$$\begin{aligned}
&(P_t f, g P_t f) = (f, P_t g P_t f) \\
&= \int_{\mathbb{R}^d} dx \mathbb{E}_{\mathcal{W}^x \otimes \mathcal{G}}[f(B_0, \xi_0) e^{-\int_0^t \tau_{B_s} \xi_s(\varrho) ds} g(B_t) e^{-\int_t^{2t} \tau_{B_s} \xi_s(\varrho) ds} f(B_{2t}, \xi_{2t})] \\
&= \mathbb{E}_{\mathcal{W} \otimes \mathcal{G}}[f(B_0, \xi_0) e^{-\int_0^{2t} \tau_{B_s} \xi_s(\varrho) ds} g(B_t) f(B_{2t}, \xi_{2t})],
\end{aligned}$$

where $\mathcal{W}$ denotes the full Wiener measure. Since both $\mathcal{W}$ and $\mathcal{G}$ are invariant under time shift, we can replace B_s by B_{s-t} and ξ_s with ξ_{s-t} above, and find

$$(P_t f, g P_t f)_{L^2(\mathbb{R}^d; L^2(\mathsf{G}))} = \mathbb{E}_{\mathcal{W} \otimes \mathcal{G}}[f(B_{-t}, \xi_{-t}) e^{-\int_{-t}^t \tau_{B_s} \xi_s(\varrho) ds} g(B_0) f(B_t, \xi_t)].$$

Now taking for g a sequence of functions approximating a Dirac measure at q, we obtain the claim. □

Note that the content of Theorem 4.30 is about extending (4.4.16) to the spaces $C_0(\mathbb{R}^d; L^2(\mathsf{G}))$ and $\mathcal{T}$ in the sup-norm. We need a further auxiliary result.

Proposition 4.33. (1) *P_t is a C_0-semigroup on $L^\infty(\mathbb{R}^d; L^2(\mathsf{G}))$.*

(2) *If $f \in L^\infty(\mathbb{R}^d \times \mathscr{M}_{-2}) \cap C(\mathbb{R}^d; L^2(\mathsf{G}))$, then $P_t f \in C_{\mathrm{b}}(\mathbb{R}^d; L^2(\mathsf{G}))$,*

(3) *If $f \in L^\infty(\mathbb{R}^d \times \mathscr{M}_{-2}) \cap C(\mathbb{R}^d; L^2(\mathsf{G}))$, then*

$$\lim_{t \to 0} \|P_t f(x, \cdot) - f(x, \cdot)\|_{L^2(\mathsf{G})} = 0$$

uniformly in x on compact subsets of $\mathbb{R}^d$.

Proof. (1) Once we have shown that P_t is bounded on $L^\infty(\mathbb{R}^d; L^2(\mathsf{G}))$, the semigroup property follows from the Markov property of $\mathcal{G}$ and $\mathcal{W}$. To show boundedness, note that $|P_t f| \le P_t |f|$ pointwise. Therefore it suffices to consider non-negative functions. First let $f \in L^\infty(\mathbb{R}^d \times \mathscr{M}_{-2})$.

We fix a not necessarily continuous path $X : [-t, t] \to \mathbb{R}^d$ and define the bilinear form

$$\langle f, g \rangle_X = \int f(X_{-t}, \xi_{-t}) e^{-\int_{-t}^t \tau_{X_s} \xi_s(\varrho) ds} g(X_t, \xi_t) d\mathcal{G}(\xi).$$

For two paths X and $\tilde{X}$,

$$|\langle f, g \rangle_X - \langle f, g \rangle_{\tilde{X}}| \le \|f\|_\infty \|g\|_\infty \mathbb{E}_{\mathcal{G}}[(e^{-\int_{-t}^t \tau_{X_s} \xi_s(\varrho) ds} - e^{-\int_{-t}^t \tau_{\tilde{X}_s} \xi_s(\varrho) ds})^2]. \tag{4.4.18}$$

Gaussian integration with respect to the measure $\mathscr{G}$ then shows that $X \mapsto \langle f, g\rangle_X$ is continuous from $L^\infty([-t,t],\mathbb{R})$ to $\mathbb{R}$. Now let X be a continuous path, and define

$$X_s^{(n)} = \sum_{j=-n}^{n-1} X_{tj/n} 1_{[tj/n, t(j+1)/n)}(s).$$

Then

$$|\langle f, g\rangle_{X^{(n)}}| \le e^c \|f(X_{-t},\cdot)\|_{L^2(\mathsf{G})} \|g(X_t,\cdot)\|_{L^2(\mathsf{G})}$$

for bounded f and g. $X^{(n)} \to X$ uniformly on $[-t,t]$ as $n \to \infty$, where $c = (1/4)\int_{\mathbb{R}^d} |\hat\rho(k)|^2/\omega(k)^3 dk$, and thus the above inequality remains valid when we replace $X^{(n)}$ by X. For $f, g \in L^2(\mathbb{R}^d; L^2(\mathsf{G}))$ the monotone convergence theorem implies

$$|\langle f, g\rangle_X| \le e^c \|f(X_{-t},\cdot)\|_{L^2(\mathsf{G})} \|g(X_t,\cdot)\|_{L^2(\mathsf{G})} \tag{4.4.19}$$

for all $X \in \mathfrak{X}$. Using this in (4.4.17) shows (1).

(2) Choose $x, y \in \mathbb{R}^d$, $f, g \in L^\infty(\mathbb{R}^d \times \mathscr{M}_{-2})$, and write $f_y(x,\xi) = f(x+y,\xi)$. Then

$$\begin{aligned}
&\|P_t f(x,\cdot) - P_t f(x+y,\cdot)\|_{L^2(\mathsf{G})}^2 \\
&= \mathbb{E}_{\mathsf{G}}[(\mathbb{E}_{\mathscr{W}^x\otimes\mathscr{G}^\xi}[e^{-\int_0^t \tau_{B_s}\xi_s(\varrho)ds} f(B_t,\xi_t) - e^{-\int_0^t \tau_{(B_s+y)}\xi_s(\varrho)ds} f_y(B_t,\xi_t)])^2] \\
&= \mathbb{E}_{\mathscr{W}^x}[\langle f, f\rangle_B + \langle f_y, f_y\rangle_{B+y} - 2\langle f, f_y\rangle_{B+1_{\{t\ge 0\}}y}].
\end{aligned}$$

Using (4.4.18), (4.4.19) and continuity of $B \mapsto f(B,\cdot)$, we see that each term in the last line above converges to $\langle f, f\rangle_B$ as $r \to 0$. It follows that the integrand in the first line converges to zero pathwise, and the second claim follows by the Lebesgue dominated convergence theorem.

(3) Define

$$(Q_t f)(x,\xi) = \mathbb{E}_{\mathscr{W}^x\otimes\mathscr{G}^\xi}[f(B_t,\xi_t)], \quad (x,\xi).$$

If f is bounded, a calculation similar to (4.4.18) shows

$$\|Q_t f - P_t f\|_{L^\infty(\mathbb{R}^d; L^2(\mathsf{G}))} \to 0$$

as $t \to 0$. Therefore we only need to show that $\|Q_t f - f\|_{L^2(\mathsf{G})}$ vanishes uniformly on compact sets as $t \to 0$. Write $f_t(x,\xi) = f(x,\xi_t)$. Note that f_t is a function on $\mathbb{R}^d \times \mathfrak{Y}$ while f is a function on $\mathbb{R}^d \times \mathscr{M}_{-2}$. By reversibility of $\mathscr{W}$ and $\mathscr{G}$, and Schwarz inequality we get

$$\|Q_t f(x,\cdot) - f(x,\cdot)\|_{L^2(\mathsf{G})}^2 \le \mathbb{E}_{\mathscr{W}^x}[\|f_t(B_t,\cdot) - f_0(B_0,\cdot)\|_{L^2(\mathscr{G})}^2].$$

Moreover,

$$\begin{aligned}
&\|f_t(B_t,\cdot) - f_0(B_0,\cdot)\|_{L^2(\mathscr{G})} \\
&\leq \|f_t(B_t,\cdot) - f_t(B_0,\cdot)\|_{L^2(\mathscr{G})} + \|f_t(B_0,\cdot) - f_0(B_0,\cdot)\|_{L^2(\mathscr{G})} \\
&= \|f(B_t,\cdot) - f(B_0,\cdot)\|_{L^2(\mathsf{G})} + \|f_t(B_0,\cdot) - f_0(B_0,\cdot)\|_{L^2(\mathscr{G})}.
\end{aligned}$$

It thus suffices to show that

$$\lim_{t\to 0} \|f_t(x,\cdot) - f_0(x,\cdot)\|_{L^2(\mathscr{G})} = 0 \tag{4.4.20}$$

uniformly in x on compact sets, and

$$\lim_{t\to 0} \mathbb{E}_{\mathcal{W}^x}[\|f(B_t,\cdot) - f(B_0,\cdot)\|^m_{L^2(\mathsf{G})}] = 0 \tag{4.4.21}$$

uniformly on compact sets for $m = 1, 2$. For proving (4.4.20), assume the contrary. Then there exist bounded sequences $(x_n)_{n\in\mathbb{N}} \subset \mathbb{R}^d$, $(t_n)_{n\in\mathbb{N}} \subset \mathbb{R}^+$ with $t_n \to 0$, and $\|f_{t_n}(x_n,\cdot) - f_0(x_n,\cdot)\|_{L^2(\mathscr{G})} > \delta$ for all n. By compactness, we may assume that x_n converges to $x \in \mathbb{R}^d$. Then

$$\begin{aligned}
\delta &< \|f_{t_n}(x_n,\cdot) - f_0(x_n,\cdot)\|_{L^2(\mathscr{G})} \\
&\leq \|f_{t_n}(x_n,\cdot) - f_{t_n}(x,\cdot)\|_{L^2(\mathscr{G})} \\
&\quad + \|f_{t_n}(x,\cdot) - f_0(x,\cdot)\|_{L^2(\mathscr{G})} + \|f_0(x,\cdot) - f_0(x_n,\cdot)\|_{L^2(\mathscr{G})} \\
&= \|f_{t_n}(x,\cdot) - f_0(x,\cdot)\|_{L^2(\mathscr{G})} + 2\|f(x_n,\cdot) - f(x,\cdot)\|_{L^2(\mathsf{G})}.
\end{aligned}$$

We have assumed continuity of f. Thus we can choose n_0 large enough such that $\|f(x_n,\cdot) - f(x,\cdot)\|_{L^2(\mathsf{G})} < \delta/3$, for all $n > n_0$. Then the above calculation shows that $\|f_{t_n}(x,\cdot) - f_0(x,\cdot)\|_{L^2(\mathscr{G})} > \delta/3$ for all such n. Since

$$\|f_{t_n}(x,\cdot) - f_0(x,\cdot)\|_{L^2(\mathscr{G})} = \|e^{-t_n H_{\mathrm{f}}} f(x,\cdot) - f(x,\cdot)\|_{L^2(\mathscr{G})} \tag{4.4.22}$$

must converge to zero by the strong continuity of the semigroup $e^{-tH_{\mathrm{f}}}$, this is a contradiction and thus (4.4.20) holds. For (4.4.21), note that $x \mapsto f(x,\cdot)$ is uniformly continuous on compact sets. Thus for $\varepsilon > 0$ we can choose $\delta > 0$ such that $\|f(x,\cdot) - f(\tilde{x},\cdot)\|_{L^2(\mathsf{G})} < \varepsilon/2$ whenever $|x - \tilde{x}| < \delta$. By the properties of Brownian motion there exists $t_0 > 0$ such that for $0 \leq t < t_0$,

$$\mathcal{W}^x(|B_t - x| > \delta) < \varepsilon/(2^{m+1}\|f\|^m_{L^\infty(\mathbb{R}^d;L^2(\mathsf{G}))}).$$

Then $\mathbb{E}_{\mathcal{W}^x}[\|f(x_t,\cdot) - f(x_0,\cdot)\|^m_{L^2(\mathsf{G})}] < \varepsilon$ for these t, uniformly in x. This proves (4.4.21). □

Proof of Theorem 4.30. We first show that P_t is a strongly continuous one parameter semigroup on $C_0 = C_0(\mathbb{R}^d; L^2(\mathsf{G}))$. We use Proposition 4.33 (2) and (3) for an approximation and therefore must first show that $L^\infty(\mathbb{R}^d \times \mathscr{M}_{-2}) \cap C_0$ is dense in C_0. Let $f \in C_0$ and take $f_R = (f \wedge -R) \vee R$ for $R \geq 0$. $L^\infty(\mathscr{M}_{-2})$ is dense in $L^2(\mathsf{G})$, and thus for each $x \in \mathbb{R}^d$ and all $\varepsilon > 0$,

$$R_x(\varepsilon) = \inf\{R \geq 0 | \|f(x,\cdot) - f_R(x,\cdot)\|_{L^2(\mathsf{G})} \leq \varepsilon\} < \infty. \tag{4.4.23}$$

$R_x(\varepsilon)$ is bounded on compact subsets of $\mathbb{R}^d$. To see this, assume the contrary. Then there exists $(x_n)_{n\in\mathbb{N}} \subset \mathbb{R}^d$ with $x_n \to x$, and $R_n = R_{x_n}(\varepsilon) > n$ for all n. Choosing n_0 large enough such that $\|f(x_n,\cdot) - f(x,\cdot)\|_{L^2(\mathsf{G})} < \varepsilon/3$ for all $n > n_0$, gives

$$\begin{aligned}
\varepsilon &= \|f(x_n,\cdot) - f_{R_n}(x_n,\cdot)\|_{L^2(\mathsf{G})} \\
&\leq \|f(x_n,\cdot) - f(x,\cdot)\|_{L^2(\mathsf{G})} \\
&\quad + \|f(x,\cdot) - f_{R_n}(x,\cdot)\|_{L^2(\mathsf{G})} + \|f_{R_n}(x,\cdot) - f_{R_n}(x_n,\cdot)\|_{L^2(\mathsf{G})} \\
&\leq \|f(x,\cdot) - f_{R_n}(x,\cdot)\|_{L^2(\mathsf{G})} + 2\varepsilon/3 \leq \|f(x,\cdot) - f_n(x,\cdot)\|_{L^2(\mathsf{G})} + 2\varepsilon/3
\end{aligned}$$

for each $n > n_0$. This implies $R_x(\varepsilon/3) = \infty$, in contradiction to (4.4.23). Thus $R_x(\varepsilon)$ is bounded on compact sets. Since $f \in C_0$, $R_x(\varepsilon) = 0$ for $|x|$ large enough, and thus $R_x(\varepsilon)$ is bounded on all $\mathbb{R}^d$. Thus bounded functions are dense in C_0.

Next we show that P_t leaves C_0 invariant. Let $f \in C_0$. From (4.4.17) and (4.4.19) we have

$$\begin{aligned}
\|P_t f(x,\cdot)\|_{L^2(\mathsf{G})} &\leq e^c \mathbb{E}_{\mathcal{W}^x}[\|f(B_{-t},\cdot)\|_{L^2(\mathsf{G})} \|f(B_t,\cdot)\|_{L^2(\mathsf{G})}] \\
&= e^c \mathbb{E}_{\mathcal{W}^x}[1_{\{|B_t| \leq |x|/2\}} \|f(B_{-t},\cdot)\|_{L^2(\mathsf{G})} \|f(B_t,\cdot)\|_{L^2(\mathsf{G})}] \\
&\quad + e^c \mathbb{E}_{\mathcal{W}^x}[1_{\{|B_t| > |x|/2\}} \|f(B_{-t},\cdot)\|_{L^2(\mathsf{G})} \|f(B_t,\cdot)\|_{L^2(\mathsf{G})}].
\end{aligned}$$

$\mathcal{W}^0(|B_t| \geq R)$ decays exponentially in R for all t, and thus the term in the second line above also decays exponentially since $f \in C_0 \subset L^\infty(\mathbb{R}^d, L^2(\mathsf{G}))$. The last line is bounded by a constant times $\sup_{|y|>|x|/2} \|f(x,\cdot)\|_{L^2(\mathsf{G})}$, and this term decays to zero as $|x| \to \infty$, by the assumption $f \in C_0$. In a similar way, we obtain strong continuity from Proposition 4.33 (3). Thus P_t is a strongly continuous semigroup of bounded operators on $C_0 = C_0(\mathbb{R}; L^2(\mathsf{G}))$.

Now we show that P_t is a strongly continuous one-parameter semigroup on $\mathcal{T}$. Choose $f \in \mathcal{T}$. The measure $\mathcal{G}$ is invariant under the map $\xi \mapsto \tau_x \xi$ for every $x \in \mathbb{R}^d$. Using this, and invariance of the full Wiener measure under translations $B \mapsto B + x$, we find

$$\begin{aligned}
(P_t f)(x,\xi) &= \mathbb{E}_{\mathcal{W}^0 \otimes \mathcal{G}^{\tau_x \xi}}[e^{-\int_0^t \tau_{B_s} \xi_s(\varrho) ds} \tau_{B_t} U^{-1} f(\xi_t)] \\
&= U(\mathbb{E}_{\mathcal{W}^0 \otimes \mathcal{G}^{\xi}}[e^{-\int_0^t \tau_{B_s} \xi_s(\varrho) ds} \tau_{B_t} U^{-1} f(\xi_t)]).
\end{aligned}$$

We conclude that for each $f \in \mathcal{T}$, also $P_t f \in \mathcal{T}$. It is seen that $L^\infty(\mathbb{R}^d, L^2(\mathsf{G})) \cap \mathcal{T}$ is dense in $\mathcal{T}$. Strong continuity then follows directly from Proposition 4.33 (3) and (4.4.13).

Finally, we show that the generator of P_t is given by $-H$, with H from (4.4.14). W $-H_{C_0}$ for the generator in C_0, and $-H_{\mathcal{T}}$ for the generator in $\mathcal{T}$. For $f \in D(H_{\mathcal{T}})$ or $f \in D(H_{C_0})$, we have $fg \in D(H_{C_0})$ for every $g \in C^2(\mathbb{R}^d, \mathbb{C})$ with compact support, and $fg \in D(H_{L^2})$, where H_{L^2} is the generator of P_t as a semigroup on $L^2(\mathbb{R}^d, L^2(\mathsf{G}))$. By the Feynman–Kac formula we know that $H_{L^2} fg = Hfg$ almost everywhere. Therefore $H_{C_0} fg = -Hfg$ in $L^\infty(\mathbb{R}^d, L^2(\mathsf{G}))$. The operators $H_{\mathcal{T}}$, H_{C_0} and H are local in q in the sense that whenever $f = g$ on an open set, then also $H_{\mathcal{T}} f = H_{\mathcal{T}} g$ on this set, and the same is true for H_{C_0} and H. Thus we can use a smooth partition of unity to conclude that $H_{\mathcal{T}} f = H_{C_0} f = Hf$. H is symmetric in $\mathcal{T}$, and is thus self-adjoint as the generator of a strongly continuous semigroup. This completes the proof of the theorem. □

We can now get some spectral information on the generator H of P_t.

Theorem 4.34 (Ground state of H). *Let H be the generator of the semigroup P_t acting on $\mathcal{T}$. Then* $\inf \mathrm{Spec}(H)$ *is an eigenvalue of multiplicity one. The corresponding eigenfunction $\Psi \in \mathcal{T}$ can be chosen strictly positive.*

Proof. For a bounded interval $I \subset \mathbb{R}$ write

$$S_I = -\int_I \int_I W(B_s - B_t, |s-t|) ds dt. \tag{4.4.24}$$

Using Proposition 4.29, we find

$$\langle 1, P_T 1 \rangle_{\mathcal{T}} = \mathbb{E}_{\mathcal{W}^0}[e^{S_{[0,T]}}],$$

and

$$\| P_T 1 \|_{\mathcal{T}}^2 = \mathbb{E}_{\mathsf{G}}[(\mathbb{E}_{\mathcal{W}^0 \otimes \mathcal{G}^\xi}[e^{-\int_0^T \tau_{B_s} \xi_s(\varrho)}])^2].$$

By a similar calculation as the one leading to equation (4.4.17), we obtain

$$\| P_T 1 \|_{\mathcal{T}}^2 = \mathbb{E}_{\mathcal{W}^0}[e^{S_{[-T,T]}}]. \tag{4.4.25}$$

This is a key formula. By assumption (4.4.3) there exists $C_\varrho > 0$ such that

$$S_{[-T,T]} \le S_{[-T,0]} + S_{[0,T]} + C_\varrho$$

uniformly in the path. We use this in (4.4.25), apply the Markov property of Brownian motion in the resulting term and reverse time in one of the factors to get

$$\| P_T 1 \|_{\mathcal{T}}^2 \le e^{C_\varrho} (\mathbb{E}_{\mathcal{W}^0}[e^{S_{[0,T]}}])^2 = e^{C_\varrho} \langle 1, P_T 1 \rangle_{\mathcal{T}}^2,$$

and thus

$$\gamma(T) = \frac{(1, P_T 1)^2_{\mathcal{T}}}{\|P_T 1\|^2_{\mathcal{T}}} \geq e^{-C_\varrho}. \tag{4.4.26}$$

Thus $\lim_{T\to\infty} \gamma(T) > 0$ and hence $\inf \mathrm{Spec}(H)$ is an element in the point spectrum by Proposition 6.8 in Chapter 6. Since $e^{-tH} = P_t$ is positivity improving, Ψ can be chosen strictly positive by the Perron–Frobenius theorem, and the multiplicity of $\inf \mathrm{Spec}(H)$ is one. □

We denote by Ψ the eigenvector corresponding to $\inf \mathrm{Spec}(H)$, which we understand to be the strictly positive and normalized version. In the context of the Nelson model discussed below Ψ is the ground state of the dressed electron for total momentum zero, see Section 6.6.

Next we consider the limit of the families $(\mathcal{P}^W_T)_{T>0}$ and $(\mathcal{P}^0_T)_{T>0}$. For a bounded interval $I \subset \mathbb{R}$ recall that $\mathscr{F}_I = \sigma(X_t, t \in I)$, and write $E_0 = \inf \mathrm{Spec}(H)$.

Definition 4.9 (Limit measure). We define $\mathcal{P}^W_\infty$ to be the unique probability measure on $(C(\mathbb{R}; \mathbb{R}^d \times \mathscr{M}_{-2}), \mathscr{F})$, such that for every $N > 0$ and $A \in \mathscr{F}_{[-N,N]}$

$$\mathcal{P}^W_\infty(A) = e^{2TE_0} \mathbb{E}_{\mathcal{P}^W_0}[\Psi(B_{-T}, \xi_{-T}) e^{-\int_{-T}^T \tau_{B_s} \xi_s(\varrho) ds} \Psi(B_T, \xi_T) 1_A]. \tag{4.4.27}$$

By Theorems 4.30 and 4.34, $\mathcal{P}^W_\infty$ is the path measure of a Markov process with generator L acting as

$$Lf = -\frac{1}{\Psi}(H - E_0)(\Psi f). \tag{4.4.28}$$

This is shown in the same way as in Theorem 3.106.

Proposition 4.35. *$\mathcal{P}^W_T \to \mathcal{P}^W_\infty$ as $T \to \infty$ in the topology of local convergence, i.e. $\mathbb{E}_{\mathcal{P}^W_T}[f] \to \mathbb{E}_{\mathcal{P}^W_\infty}[f]$ for every bounded, $\mathcal{F}_{[-t,t]}$-measurable function f and every $t > 0$.*

Proof. Define $\Psi_T = e^{TE_0} P_T 1$. By Theorem 4.34 and the spectral theorem we have

$$\Psi_T \to \langle 1, \Psi \rangle_{\mathcal{T}} \Psi \quad \text{as } T \to \infty \tag{4.4.29}$$

in $\mathcal{T}$ and thus in $L^\infty(\mathbb{R}^d; L^2(\mathsf{G}))$. Choose $f \in \mathcal{F}_{[-t,t]}$ and $t < T$. We have

$$\mathbb{E}_{\mathcal{P}^W_T}[f] = \frac{e^{2tE_0}}{\|\Psi_T\|^2_{\mathcal{T}}} \mathbb{E}_{\mathcal{P}^W_0}[\Psi_{T-t}(B_{-t}, \xi_{-t}) e^{-\int_{-t}^t \tau_{B_s} \xi_s(\varrho) ds} f \Psi_{T-t}(B_t, \xi_t)]. \tag{4.4.30}$$

(4.4.29) and the fact $\langle 1, \Psi\rangle_{\mathcal{T}} > 0$ guarantee that $\Psi_{T-t}/\|\Psi_T\|_{\mathcal{T}} \to \Psi$ in $\mathcal{T}$. For bounded $f \in \mathcal{F}_{[0,T]}$, the map Q on $L^\infty(\mathbb{R}^d, L^2(\mathsf{G}))$ defined through

$$(Qg)(x,\xi) = \mathbb{E}_{\mathcal{W}^x \otimes \mathcal{G}^\xi}\left[\exp\left(-\int_0^t \tau_{B_s}\xi_s(\varrho)ds\right) fg(B_t, \xi_t)\right]$$

is a bounded linear operator on $L^\infty(\mathbb{R}^d, L^2(\mathsf{G}))$. This follows from the fact $|Qg| \le \|f\|_\infty P_t|g|$, and the boundedness of P_t. We conclude

$$Q(\Psi_{T-t}/\|\Psi_T\|_{\mathcal{T}}) \stackrel{T\to\infty}{\to} Q\Psi$$

in $L^\infty(\mathbb{R}^d; L^2(\mathsf{G}))$, and the claim follows. □

We are now ready to state and prove the main result of this section.

Theorem 4.36 (Gibbs measure). *The family $(\mathcal{P}^0_T)_{T>0}$ converges to a probability measure $\mathcal{P}^0_\infty$ in the topology of local convergence. Moreover, if $f \in \mathcal{F}_{[-t,t]}$ satisfies $\mathbb{E}_{\mathcal{W}^0}[|f|] < \infty$, then also $\mathbb{E}_{\mathcal{P}^0_\infty}[|f|] < \infty$, and in this case we have $\mathbb{E}_{\mathcal{P}^0_T}[f] \to \mathbb{E}_{\mathcal{P}^0_\infty}[f]$ as $T \to \infty$.*

Proof. The first statement follows from Proposition 4.35 and Proposition 4.29. All the other statements will be proved once we show that there exists $C > 0$ such that

$$\sup_{T>0} \mathbb{E}_{\mathcal{P}^0_T}[|f|] \le C\mathbb{E}_{\mathcal{W}^0}[|f|] \tag{4.4.31}$$

for all $f \in \mathcal{F}_{[-t,t]}$. To see (4.4.31), note that

$$\mathbb{E}_{\mathcal{P}^0_T}[|f|] \le \frac{e^{2C_\varrho}}{Z_T}\mathbb{E}_{\mathcal{W}^0}[e^{S_{[-T,-t]}}e^{S_{[-t,t]}}e^{S_{[t,T]}}|f|],$$

where $S_{[a,b]}$ is defined in (4.4.24). By using the Markov property, stationarity of increments and time reversal invariance of two-sided Brownian motion it follows that the right-hand side above equals

$$e^{2C_\varrho}\mathbb{E}_{\mathcal{W}^0}[R_T(B_{-t})e^{S_{[-t,t]}}fR_T(B_t)],$$

where

$$R_T(x) = \frac{1}{\sqrt{Z_T}}\mathbb{E}_{\mathcal{W}^x}[e^{S_{[0,T-t]}}] = \frac{\langle 1, P_{T-t}1\rangle_{\mathcal{T}}}{\|P_T 1\|_{\mathcal{T}}}.$$

Therefore, $R_T(x)$ is in fact independent of x and converges as $T \to \infty$. Since the pair interaction potential W is bounded, it follows that also $S_{[-t,t]}$ is uniformly bounded, and (4.4.31) holds. □

4.4.2 Diffusive behaviour

By Theorem 4.36 we know that a limit measure $\mathcal{P}^0_\infty$ exists. In this section we discuss the peculiar property of this measure that on a large scale under this measure paths behave like Brownian motion, however, with a reduced diffusion constant.

We use the same notations as in the previous section. Notice that for every $g \in C^k(\mathbb{R})$ it follows that $g \in C^k(\mathbb{R}^d, L^2(\mathsf{G}))$, since the constant function 1 is square integrable with respect to G. We will not distinguish between these two meanings of g in the notation. Let L be as in (4.4.28).

Proposition 4.37. *If $g \in C^2(\mathbb{R}^d)$, then $g \in D(L)$, $\Psi L g \in C(\mathbb{R}^d; L^2(\mathsf{G}))$, and*

$$Lg(x,\xi) = \frac{1}{2}\Delta_x g(x) + \nabla_x g(x) \cdot \nabla_x \ln \Psi(x,\xi).$$

Proof. With $\alpha \in \{1,\dots,d\}$, $\partial_\alpha = \frac{\partial}{\partial x_\alpha}$ is the generator of the unitary group $f \mapsto \tau_{tx_\alpha} f$ on $\mathcal{T}$, and $i\partial_\alpha$ is self-adjoint on $\mathcal{T}$. By Proposition 4.31, $H_{\mathrm{f}} + V_\varrho$ is bounded from below on $\mathcal{T}$ by $-a \in \mathbb{R}$. Thus

$$\begin{aligned} E_0 = \langle \Psi, H\Psi\rangle_{\mathcal{T}} &= -\frac{1}{2}\sum_{\alpha=1}^d \langle \Psi, \partial_\alpha^2 \Psi\rangle_{\mathcal{T}} + \langle \Psi, (H_{\mathrm{f}} + V_\varrho)\Psi\rangle_{\mathcal{T}} \\ &\geq \frac{1}{2}\sum_{\alpha=1}^d \|\partial_\alpha \Psi\|^2_{\mathcal{T}} - a. \end{aligned} \tag{4.4.32}$$

This shows $\partial_\alpha \Psi \in \mathcal{T}$. By $(H - E_0)\Psi = 0$ and $(H_{\mathrm{f}} + V_\varrho) g\Psi = g(H_{\mathrm{f}} + V_\varrho)\Psi$, we find

$$(H - E_0) g\Psi = -\frac{1}{2}\Psi \Delta_x g - \nabla_x g \cdot \nabla_x \Psi,$$

and this completes the proof. □

Choosing $g(x) = h_\gamma(x) = \gamma \cdot x$ above gives $(Lh_\gamma) = \gamma \cdot \nabla_x \ln \Psi$. Thus $Lh_\gamma \in \mathcal{T}$. With

$$j = U^{-1}(\gamma \cdot \nabla_x \ln \Psi) \in L^2(\mathsf{G}) \tag{4.4.33}$$

we have $L(\gamma \cdot x) = j(\eta)$ with $\eta(x, \xi_t) = \tau_x \xi_t$, and

$$Lh_\gamma(B_t, \xi_t) = j(\eta_t), \qquad \text{with} \quad \eta_t = \tau_{B_t}\xi_t.$$

Proposition 4.38. *The random process $(\eta_t)_{t\in\mathbb{R}}$ is a reversible Markov process under $\mathcal{P}$. Its reversible measure is given by $(U^{-1}\Psi)^2 d\mathsf{G}$.*

Proof. Take $f, g \in L^2(\mathsf{G})$. Then

$$\mathbb{E}_{\mathscr{P}_\infty^W}[f(\eta_s)g(\eta_t)] = e^{|t-s|E_0}(\Psi U f, P_{|t-s|}(\Psi U g))_{\mathcal{T}}.$$

Thus the generator of the η_t-process is unitary equivalent to the operator L on the Hilbert space $(\mathcal{T}, \|\cdot\|_\Psi)$, where $\|f\|_\Psi = \|\Psi f\|_{\mathcal{T}}$. L is self-adjoint on this Hilbert space, $L1 = 0$, and $\|1\|_\Psi = 1$. This proves reversibility. □

We are interested in proving a central limit theorem for the process $g(B_t, \xi_t)_{t\in\mathbb{R}} = \gamma \cdot B_t$ under $\mathscr{P}_\infty^W$, or equivalently the process $(\gamma \cdot B_t)_{t\geq 0}$ under $\mathscr{P}_\infty^0$. The fundamental tool will be the martingale central limit theorem. Write

$$\gamma \cdot B_t = M_t + \int_0^t j(\eta_s)ds, \tag{4.4.34}$$

with $M_t = \gamma \cdot B_t - \int_0^t j(\eta_s)ds$. $(M_t)_{t\geq 0}$ is a martingale. This follows from the following more general

Theorem 4.39. *Let $(X_t)_{t\geq 0}$ be a Markov process associated with semigroup $P_t = e^{tL}$, and f a continuous function from the state space of the process to the real numbers. Assume that $f \in D(L)$, and that $\mathbb{E}[f^2(X_t)] < \infty$ and $\mathbb{E}[(Lf)^2(X_t)] < \infty$, for all t. Then*

$$M_t = f(X_t) - f(X_0) - \int_0^t Lf(X_s)ds$$

is a martingale, and

$$\mathbb{E}[M_t^2] = \mathbb{E}[f^2(X_t)] - \mathbb{E}[f^2(X_0)] - 2\int_0^t \mathbb{E}[f(X_s)(Lf)(X_s)]ds.$$

Proof. We have almost surely

$$\mathbb{E}[M_t|\mathscr{F}_s] = M_s + \mathbb{E}\left[f(X_t) - f(X_0) - \int_s^t (Lf)(X_r)dr \,\middle|\, \mathscr{F}_s \right]. \tag{4.4.35}$$

Our aim is to show that the last term above is almost surely zero. We know that

$$\frac{d}{dr}P_r f(X_s) = P_r Lf(X_s) = (Lf)(X_{r+s}),$$

where the equality is understood to hold when integrated against a bounded $\mathscr{F}_s$-measurable test function. Integrating over (s,t) we obtain

$$\int_s^t (Lf)(X_r)dr = P_{t-s}f(X_s) - f(X_s) = f(X_t) - f(X_s),$$

in the same sense as above. This shows that the right-hand side of (4.4.35) is zero when integrated against an $\mathscr{F}_s$-measurable function, thus $(M_t)_{t\geq 0}$ is a martingale. Write $g(X_s) = (Lf)(X_s)$. We have

$$\begin{aligned}\mathbb{E}[M_t^2] = \mathbb{E}[f^2(X_t)] + \mathbb{E}[f^2(X_0)] + 2\int_0^t ds \int_s^t dr \mathbb{E}[g(X_s)g(X_r)] \\ - 2\mathbb{E}[f(X_0)f(X_t)] + 2\int_0^t dr \mathbb{E}[f(X_0)g(X_r)] \\ - 2\int_0^t dr \mathbb{E}[f(X_t)g(X_r)]. \end{aligned} \tag{4.4.36}$$

Since $\mathbb{E}[f(X_0)\int_0^t g(X_r)dr] = \mathbb{E}[f(X_0)(f(X_t) - f(X_0))]$, the second, fourth and fifth terms of (4.4.36) combine to give $-\mathbb{E}[f^2(X_0)]$. Also, since

$$\int_s^t \mathbb{E}[g(X_s)g(X_r)]dr = \mathbb{E}[g(X_s)(f(X_t) - f(X_s))],$$

the third term of (4.4.36) decomposes into two expressions, one of which cancels the last term of (4.4.36), while the other gives the last term in the formula for $\mathbb{E}[M_t^2]$ above. □

Using that $M_0 = B_0 = 0$ $\mathscr{P}_\infty^W$-a.s. we now show that $(M_t)_{t\geq 0}$ is a martingale. Consider the decomposition (4.4.34). We have $\mathbb{E}_{\mathscr{P}_\infty^0}[(\gamma \cdot B_t)^2] < \infty$ for all t by Proposition 4.36. By (4.4.33), we have $Lh_\gamma = j$, and

$$\mathbb{E}_{\mathscr{P}_\infty^W}[j(\eta_t)^2] = \|\gamma \cdot \nabla_x \ln \Psi\|^2_{L^2(\mathsf{G})} = \|\gamma \cdot \nabla_x \Psi\|_{L^2(\mathsf{G})} = \|\gamma \cdot \nabla_x \Psi\|_{\mathcal{T}} < \infty,$$

where the last assertion follows from the proof of Proposition 4.37. Thus the conditions of Theorem 4.39 are satisfied, and $(M_t)_{t\geq 0}$ is a martingale. We show that also the second term of (4.4.34) is also a martingale up to a correction term that is negligible in the diffusive limit. We will need the following result.

Theorem 4.40. *Let $(Y_t)_{t\geq 0}$ be a Markov process with respect to a filtration $\mathscr{F}_t$, with state space A, where A is a measure space. Assume that $(Y_t)_{t\geq 0}$ is reversible with respect to a probability measure μ_0, and that the reversible stationary process μ with invariant measure μ_0 is ergodic. Let $F : A \to \mathbb{R}$ be a μ_0 square integrable function with $\int F d\mu_0 = 0$. Suppose in addition that F is in the domain of $L^{-1/2}$, where L is the generator of the process Y_t. Let*

$$L_t = \int_0^t F(Y_s)ds.$$

Then there exists a square integrable martingale $(N_t, \mathscr{F}_t)$, with stationary increments, such that

$$\lim_{t\to\infty} \frac{1}{\sqrt{t}} \sup_{0\leq s\leq t} |L_s - N_s| = 0$$

in probability with respect to μ, where $L_0 = N_0 = 0$. Moreover,

$$\lim_{t\to\infty} \frac{1}{t}\mathbb{E}_\mu[|L_t - N_t|^2] = 0.$$

The above theorem can be used to prove the following result.

Proposition 4.41. *There exists an $\mathscr{F}_t$-martingale $(Z_t)_{t\geq 0}$ such that*

$$\lim_{t\to\infty} \frac{1}{\sqrt{t}} \sup_{0\leq s\leq t} |B_s - Z_s| = 0$$

in probability with respect to $\mathscr{P}^0_\infty$, where $B_0 = Z_0 = 0$. Moreover,

$$\lim_{t\to\infty} \frac{1}{t}\mathbb{E}_{\mathscr{P}^0_\infty}[|B_t - Z_t|^2] = 0,$$

and

$$\lim_{t\to\infty} \frac{1}{t}\mathbb{E}_{\mathscr{P}^0_\infty}[(\gamma\cdot B_t)^2] = |\gamma|^2 - 2\langle \gamma\cdot\nabla_x\Psi, (H-E_0)^{-1}\gamma\cdot\nabla_x\Psi\rangle_{\mathscr{T}}, \tag{4.4.37}$$

for all $\gamma \in \mathbb{R}^d$.

Proof. By Proposition 4.37, we have

$$(Lh_\gamma^2)(x,\xi) = |\gamma|^2 + 2h_\gamma(x)(\gamma\cdot\nabla_x \ln\Psi(x,\xi)) = |\gamma|^2 + 2h_\gamma(x)(Lh_\gamma)(x,\xi).$$

By Theorem 4.39, it then follows that

$$\mathbb{E}_{\mathscr{P}^W_\infty}(M_t^2) = \int_0^t \mathbb{E}_{\mathscr{P}^W_\infty}[(Lh_\gamma^2)(B_s,\xi_s) - 2h_\gamma(B_s)(Lh_\gamma)(B_s,\xi_s)]ds = |\gamma|^2 t.$$

Next we calculate $\mathbb{E}_{\mathscr{P}^W_\infty}[(\gamma\cdot B_t)^2]$. By (4.4.34),

$$\begin{aligned}\mathbb{E}_{\mathscr{P}^W_\infty}[(\gamma\cdot B_t)^2] = \mathbb{E}_{\mathscr{P}^W_\infty}[M_t^2] - \mathbb{E}_{\mathscr{P}^W_\infty}\left[\left(\int_0^t j(\eta_s)ds\right)^2\right] \\ + 2\mathbb{E}_{\mathscr{P}^W_\infty}\left[(\gamma\cdot B_t)\int_0^t j(\eta_s)ds\right].\end{aligned} \tag{4.4.38}$$

The third term in (4.4.38) is zero. This can be seen as follows. We have

$$\mathbb{E}_{\mathscr{P}^W_\infty}\left[(\gamma\cdot B_t)\int_0^t j(\eta_s)ds\right] = \mathbb{E}_{\mathscr{W}^0}[(\gamma\cdot B_t)I(B)],$$

where

$$I(B) = \mathbb{E}_{\mathscr{G}}\left[\Psi(\eta_0)e^{-\int_0^t \eta_s(\varrho)ds}\left(\int_0^t j(\eta_s)ds\right)\Psi(\eta_t)\right].$$

Put $\tilde{B}_s = B_{t-s} - B_t$. Then by the reversibility of $\mathscr{G}$ and the fact that $\mathscr{G}$ is invariant under the constant shift by τ_{q_t}, we have $I(\tilde{X}) = I(X)$. Moreover, $\tilde{X}_t = -B_t$, $\mathcal{W}^0$-a.s. and $\mathcal{W}^0$ is invariant under the transformation $B \mapsto \tilde{B}$. Thus

$$\mathbb{E}_{\mathcal{W}^0}[(\gamma \cdot B_t)I(X)] = -\mathbb{E}_{\mathcal{W}^0}[(\gamma \cdot B_t)I(X)] = 0.$$

From the above we know that $\mathbb{E}_{\mathcal{P}_\infty^W}[M_t^2] = |\gamma|^2 t$. Denote by Q_t the transition semigroup of $\mathcal{P}_\infty^W$. Then

$$\begin{aligned}\frac{1}{t}\mathbb{E}_{\mathcal{P}_\infty^W}\left[\left(\int_0^t j(\eta_s)ds\right)^2\right] &= \frac{1}{t}\int_0^t ds\int_0^t dr\langle\gamma\cdot\nabla_x \ln\Psi, Q_{|r-s|}\gamma\cdot\nabla_x\ln\Psi\rangle_\Psi \\ &\overset{t\to\infty}{\to} -2\langle\gamma\cdot\nabla_x\ln\Psi, L^{-1}\gamma\cdot\nabla_x\ln\Psi\rangle_\Psi \\ &= 2\langle\gamma\cdot\nabla_x\Psi, (H-E_0)^{-1}\gamma\cdot\nabla_x\Psi\rangle_{\mathcal{T}}.\end{aligned}$$

Note that the last quantity is automatically finite; this follows from (4.4.38). So we have shown (4.4.37), but since

$$\langle\gamma\cdot\nabla_x\Psi, (H-E_0)^{-1}\gamma\cdot\nabla_x\Psi\rangle_{\mathcal{T}} = \langle j, L^{-1}j\rangle_{L^2(\Psi^2\mathrm{G})},$$

we have also seen that $j \in D(L^{-1/2})$. We have already seen earlier in the proof that $j(\eta_s)$ is square integrable for all s. Moreover, $\mathbb{E}_{\mathcal{P}_\infty^W}[\eta_t] = \sum_{l=1}^d \gamma_l\langle\Psi, \partial_l\Psi\rangle_{\mathcal{T}} = 0$ since $i\partial_l$ is self-adjoint on $\mathcal{T}$ and Ψ is real-valued. Finally, η_s is a reversible Markov process with respect to $\mathscr{F}_t$ by Proposition 4.38, and is ergodic, since P_t is positivity improving. Thus we can apply Theorem 4.40 to see that $\int_0^t j(\eta_s)ds = M_1 + R$ where M_1 is a martingale and R is a process that disappears in the scaling limit in the sense specified in Theorem 4.40. Since the sum of the two $\mathscr{F}_t$-martingales $Z_t = M_t + (M_1)_t$ is again a martingale, the proof is complete. □

The last ingredient that we need is a version of the martingale central limit theorem. We first specify the topology for the convergence of processes that we need.

Definition 4.10 (Skorokhod topology). Let Λ_T be the set of all strictly increasing, continuous functions $\lambda : [0,T] \to [0,T]$ such that $\lambda(0) = 0$ and $\lambda(T) = T$. For bounded functions $f, g : [0,T] \to [0,T]$ we define

$$d(f,g) = \inf_{\lambda\in\Lambda_T}\max\{\|\lambda - 1\|_\infty, \|f - g\circ\lambda\|_\infty\}.$$

It is obvious that d is a metric on the set of bounded function on $[0,T]$, and the induced topology is the *Skorokhod topology*. The space of bounded functions on $[0,T]$ with the Skorokhod topology is denoted by $D([0,T])$.

Theorem 4.42 (Martingale convergence theorem). *Let $(M_n)_{n\in\mathbb{N}}$ be a martingale with $\sigma^2 = \lim_{n\to\infty} \frac{1}{n}\mathbb{E}[M_n^2] < \infty$, and assume that $(M_n)_{n\in\mathbb{N}}$ has stationary increments. Then*

$$\lim_{n\to\infty} \frac{1}{\sqrt{n}} M_{[tn]} = \sigma^2 B_t,$$

in the sense of weak convergence of measures on $D([0,T])$, for all $T > 0$.

Our main result is as follows.

Theorem 4.43 (Central limit theorem). *For the process $(B_t)_{t\geq 0}$ under $\mathcal{P}_\infty^W$, a functional central limit theorem holds, i.e. as $n \to \infty$, the process $t \mapsto \frac{1}{\sqrt{n}} B_{tn}$, $t \geq 0$, converges in distribution to Brownian motion with diffusion matrix D given by*

$$D_{ik} = \delta_{ik} - 2\langle \partial_i \Psi, (H - E_0)^{-1} \partial_k \Psi\rangle_{\mathcal{T}}, \quad i,k = 1,\dots,d.$$

Proof. Let $(Z_t)_{t\geq 0}$ be the martingale from Proposition 4.41. By that proposition, the difference $B_t - Z_t$ vanishes in the diffusive limit. The martingale $(Z_t)_{t\geq 0}$ is a sum of the martingales from Theorems 4.39 and 4.40. Both of these martingales have stationary increments, the first one due to the fact that $(B_t)_{t\geq 0}$ itself has stationary increments. Since $(B_t)_{t\geq 0}$ has ergodic increments, so does $(Z_t)_{t\geq 0}$. Using (4.4.37), we see that Theorem 4.42 applies for each coordinate direction, and also find the diffusion matrix. □

Part II

Rigorous quantum field theory

Chapter 5

Free Euclidean quantum field and Ornstein–Uhlenbeck processes

5.1 Background

A free quantum field describes a large collection of quantum particles of the same type in which there is no interaction between any of these particles nor with any other particles in the environment. In this section we introduce basic notions of the mathematical theory of free quantum fields. There are various formulations of this theory of which we will discuss two. One uses field operators acting on a Hilbert space, providing a framework in which problems of spectral theory, scattering theory and others are addressed directly. Another is the so called Euclidean quantum field theory which uses an L^2 space-valued random process to address similar problems by using probabilistic methods. Our goal here is to explain the equivalence of the theory of free boson fields with the theory of infinite dimensional Gaussian random processes. The essential link between the two descriptions is a Feynman–Kac-type formula.

First we define the quantum field in (boson) Fock space. While there are many advantages to the Fock space picture, what is needed for an approach using a Feynman–Kac-type formula is that the Hamiltonian is given in Schrödinger representation, i.e. acting on an $L^2(\mathscr{Q}, d\mu)$ space. Fock space is not of this form, and thus the Gaussian space introduced in the previous section and the Wiener–Itô–Segal isomorphism are needed.

Secondly, we discuss the quantum field in terms of Euclidean quantum field theory. In order to obtain the integral representation of $(F, e^{-tH_\mathrm{f}}G)$ for $F, G \in L^2(\mathscr{Q}, d\mu)$ with its Hamiltonian H_f one way is to changing e^{itH} to e^{-tH} regarding it as the analytic continuation of t to it. In this procedure the Minkowski metric turns into a Euclidean metric: $-t^2 + \mathbf{x}^2 \to t^2 + \mathbf{x}^2$, $(t, \mathbf{x}) \in \mathbb{R} \times \mathbb{R}^3$. In the Lorentz covariant formulation of scalar quantum field theory, the so called *two point function* is given by

$$(1, \phi(f)\phi(g)1) = \int f(x)g(y)W(x-y)dxdy$$

with the *Wightman distribution* $W(x-y) = (1, \phi(x)\phi(y)1)$. Here $x = (t, \mathbf{x})$ and $\phi(x) = e^{-itH}\phi((0, \mathbf{x}))e^{itH}$. The Wightman distribution of the free field Hamiltonian is

$$W((t, \mathbf{x})) = \frac{1}{2}\frac{1}{(2\pi)^3}\int_{\mathbb{R}^3} \frac{e^{it\sqrt{|k|^2+\nu^2} - ik\cdot\mathbf{x}}}{\sqrt{|k|^2+\nu^2}} dk.$$

Then the *field at time zero* is given by

$$\int \phi(f)\phi(g)d\mu = \frac{1}{2}\int_{\mathbb{R}^3} \overline{\hat{f}(k)}\hat{g}(k)\frac{dk}{(2\pi)^3\sqrt{|k|^2+\nu^2}}. \tag{5.1.1}$$

Due to the singularities of the so called *Feynman propagator*

$$(k_0^2 - |k|^2 - \nu^2)^{-1},$$

it is natural to make an analytic continuation into the region $k_0^2 < 0$. The depth and utility of this point of view has been long appreciated, and Nelson discovered a method of recovering a Minkowski region field theory from Euclidean field theory. The constructive quantum field theory developed initially by Glimm and Jaffe made use of the Euclidean method.

The analytic continuation $t \mapsto it$ of $W((t,\mathbf{x}))$ yields the *Schwinger function*

$$W((it,\mathbf{x})) = \frac{1}{2}\frac{1}{(2\pi)^4}\int_{\mathbb{R}^4}\frac{e^{itk_0 - ik\cdot\mathbf{x}}}{|k|^2+\nu^2+k_0^2}d\mathbf{k}, \quad \mathbf{k} = (k_0,k) \in \mathbb{R}\times\mathbb{R}^3.$$

Then the Euclidean time zero quantum field $\phi_{\mathrm{E}}(f)$ has covariance

$$\int_{\mathscr{Q}_{\mathrm{E}}} \phi_{\mathrm{E}}(f)\phi_{\mathrm{E}}(g)d\mu_{\mathrm{E}} = \frac{1}{2}\int_{\mathbb{R}^4}\overline{\hat{f}(\mathbf{k})}\hat{g}(\mathbf{k})\frac{d\mathbf{k}}{|k|^2+\nu^2+k_0^2}.$$

To establish the connection between $L^2(\mathscr{Q})$ and the Euclidean space $L^2(\mathscr{Q}_{\mathrm{E}}, d\mu_{\mathrm{E}})$, a family of operators $\{\mathrm{I}_t\}_{t\in\mathbb{R}}$ is introduced. The functional integral representation of $e^{-t(H_{\mathrm{f}}+H_{\mathrm{I}})}$ with some perturbation H_{I} can be also constructed by using the Trotter product formula, the Markov property of $\mathrm{I}_s\mathrm{I}_s^*$ and the identity $\mathrm{I}_t^*\mathrm{I}_s = e^{-|t-s|H_{\mathrm{f}}}$, we have

$$(F, e^{-tH}G) = \int_{\mathscr{Q}_{\mathrm{E}}} \overline{F_0}G_t e^K d\mu_E, \tag{5.1.2}$$

where $F_0 = \mathrm{I}_0 F$, $G_t = \mathrm{I}_t G$ and e^K denotes an integral kernel. The right-hand side of (5.1.2) is the integral over $\mathscr{Q}_{\mathrm{E}}$, e.g., $\mathscr{S}'_{\mathrm{real}}(\mathbb{R}^{d+1})$, which call it *functional integral* in distinction to *path integral*.

An alternative approach is based on a path measure constructed on $C(\mathbb{R}, \mathscr{M}_{-2})$, i.e., the set of $\mathscr{M}_{-2}$-valued continuous paths, where $\mathscr{M}_{-2}$ is the dual of a Hilbert space $\mathscr{M}_{+2}$. For path integral representations of the heat semigroup $(f, e^{-tH}g)$ in quantum mechanics a path measure is given on $C([0,\infty);\mathbb{R}^d)$. In quantum field theory the same procedure yields to $\mathscr{M}_{-2}$-valued paths. In the finite mode $a_k = (1/\sqrt{2})(x_k + \partial_k)$ and $a_k^* = (1/\sqrt{2})(x_k - \partial_k)$, $k = 1,\dots,n$, and the free field Hamiltonian H_{f} becomes

$$H_{\mathrm{f}} \to \sum_{k=1}^n \sqrt{|k|^2+\nu^2}\,a_k^* a_k = \sum_{k=1}^n \frac{\sqrt{|k|^2+\nu^2}}{2}(-\Delta_k + |x_k|^2 - 1).$$

■

In this representation H_{f} has the form of an infinite dimensional harmonic oscillator. It is known that the generator of the d-dimensional Ornstein–Uhlenbeck process on $C(\mathbb{R};\mathbb{R}^d)$ is the harmonic oscillator on $L^2(\mathbb{R}^d)$. This suggests that it might be possible to construct an infinite dimensional Ornstein–Uhlenbeck process $(\xi_t)_{t\in\mathbb{R}}$ on $\mathfrak{Y} = C(\mathbb{R}, \mathscr{M}_{-2})$ whose generator is the free field Hamiltonian H_{f}. This can be done by way of Kolmogorov's consistency theorem through the finite dimensional distributions giving

$$(F, e^{-tH}G) = \int_{\mathfrak{Y}} \overline{F(\xi_0)} G(\xi_t) e^{K(\xi)} d\mathscr{G} \tag{5.1.3}$$

with a suitable kernel $K(\xi)$.

5.2 Boson Fock space

5.2.1 Second quantization

In this section we start the above outlined programme by explaining the Fock space-based formulation of quantum field theory. Instead of physical space-time dimension we work in general d dimension.

Let $\mathscr{W}$ be a separable Hilbert space over $\mathbb{C}$. Consider the operation $\otimes_{\mathrm{s}}^n$ of n-fold symmetric tensor product defined through the symmetrization operator

$$S_n(f_1 \otimes \cdots \otimes f_n) = \frac{1}{n!} \sum_{\pi\in\wp_n} f_{\pi(1)} \otimes \cdots \otimes f_{\pi(n)}, \quad n \geq 1, \tag{5.2.1}$$

where $f_1, \ldots, f_n \in \mathscr{W}$ and $\wp_n$ denotes the permutation group of order n. Define

$$\mathscr{F}_{\mathrm{b}}^{(n)} = \mathscr{F}_{\mathrm{b}}^{(n)}(\mathscr{W}) = \otimes_{\mathrm{s}}^n \mathscr{W} = S_n(\otimes^n \mathscr{W}), \tag{5.2.2}$$

where $\otimes_{\mathrm{s}}^0 \mathscr{W} = \mathbb{C}$. The space

$$\mathscr{F}_{\mathrm{b}} = \mathscr{F}_{\mathrm{b}}(\mathscr{W}) = \bigoplus_{n=0}^{\infty} \mathscr{F}_{\mathrm{b}}^{(n)}(\mathscr{W}), \tag{5.2.3}$$

where $\bigoplus_{n=0}^{\infty}$ is understood to be completed (rather than simple algebraic) direct sum, is called *boson Fock space* over $\mathscr{W}$. The Fock space $\mathscr{F}_{\mathrm{b}}$ can be identified with the space of ℓ_2-sequences $(\Psi^{(n)})_{n\in\mathbb{N}}$ such that $\Psi^{(n)} \in \mathscr{F}_{\mathrm{b}}^{(n)}$ and

$$\|\Psi\|_{\mathscr{F}_{\mathrm{b}}}^2 = \sum_{n=0}^{\infty} \|\Psi^{(n)}\|_{\mathscr{F}_{\mathrm{b}}^{(n)}}^2 < \infty. \tag{5.2.4}$$

$\mathscr{F}_{\mathrm{b}}$ is a Hilbert space endowed with the scalar product

$$(\Psi, \Phi)_{\mathscr{F}_{\mathrm{b}}} = \sum_{n=0}^{\infty} (\Psi^{(n)}, \Phi^{(n)})_{\mathscr{F}_{\mathrm{b}}^{(n)}}. \tag{5.2.5}$$

The vector

$$\Omega_{\mathrm{b}} = (1, 0, 0, \dots) \tag{5.2.6}$$

is called *Fock vacuum*. Write P_n for the projection from $\mathscr{F}_{\mathrm{b}}$ onto $\mathscr{F}_{\mathrm{b}}^{(n)}$. The subspace $P_n\mathscr{F}_{\mathrm{b}}$ can be interpreted as consisting of the states of the quantum field having exactly n boson particles. Functions in $P_n\mathscr{F}_{\mathrm{b}}$ then determine the exact behaviour of these particles, while the permutation symmetry corresponds to the fact that the particles are indistinguishable. In a model of an electron coupled to its radiation field which will be studied in Chapter 7, the particles will be photons. If $\|\Phi\|_{\mathscr{F}_{\mathrm{b}}} = 1$, then $(\Phi, P_n\Phi)_{\mathscr{F}_{\mathrm{b}}}$ represents the probability of having exactly n bosons in the state described by Φ. Note that by (5.2.4) this probability must decay faster than $1/\sqrt{n}$ for large n in order to have $\Phi \in \mathscr{F}_{\mathrm{b}}$. The reason for this constraint is mathematical convenience rather than physical necessity, a fact that we will encounter when discussing the infrared divergence of a specific quantum field model.

In the description of the free quantum field the following operators acting in $\mathscr{F}_{\mathrm{b}}$ are used. There are two fundamental boson particle operators, the *creation operator* denoted by $a^*(f)$, $f \in \mathscr{W}$, and the *annihilation operator* by $a(f)$, both acting on $\mathscr{F}_{\mathrm{b}}$, defined by

$$(a^*(f)\Psi)^{(n)} = \sqrt{n} S_n(f \otimes \Psi^{(n-1)}), \quad n \geq 1, \tag{5.2.7}$$

$$(a^*(f)\Psi)^{(0)} = 0 \tag{5.2.8}$$

with domain

$$D(a^*(f)) = \Big\{ (\Psi^{(n)})_{n\geq 0} \in \mathscr{F}_{\mathrm{b}} \;\Big|\; \sum_{n=1}^{\infty} n \|S_n(f \otimes \Psi^{(n-1)})\|^2_{\mathscr{F}_{\mathrm{b}}^{(n)}} < \infty \Big\}, \tag{5.2.9}$$

and

$$a(f) = (a^*(\bar{f}))^*. \tag{5.2.10}$$

As the terminology suggests, the action of $a^*(f)$ increases the number of bosons by one, while $a(f)$ decreases it by one. Since one is the adjoint operator of the other, the relation $(\Phi, a(f)\Psi) = (a^*(\bar{f})\Phi, \Psi)$ holds. Furthermore, since both operators are closable by the dense definition of their adjoints, we will use and denote their closed extensions by the same symbols. Let $D \subset \mathscr{W}$ be a dense subset. It is known that

$$\text{L.H.}\{a^*(f_1)\cdots a^*(f_n)\Omega_{\mathrm{b}}, \Omega_{\mathrm{b}} |\, f_j \in D, j = 1, \dots, n, n \geq 1\} \tag{5.2.11}$$

is dense in $\mathscr{F}_{\mathrm{b}}(\mathscr{W})$, where L.H. is a shorthand for the linear hull.

The space

$$\mathscr{F}_{\mathrm{b,fin}} = \{(\Psi^{(n)})_{n\geq 0} \in \mathscr{F}_{\mathrm{b}} |\; \Psi^{(m)} = 0 \text{ for all } m \geq M \text{ with some } M\} \tag{5.2.12}$$

is called *finite boson subspace*. The field operators a, a^* leave $\mathscr{F}_{\mathrm{b,fin}}$ invariant and satisfy the *canonical commutation relations*

$$[a(f), a^*(g)] = (\bar{f}, g)1, \quad [a(f), a(g)] = 0, \quad [a^*(f), a^*(g)] = 0 \tag{5.2.13}$$

on $\mathscr{F}_{\mathrm{b,fin}}$.

Given a bounded operator T on $\mathscr{W}$, the *second quantization* of T is the operator $\Gamma(T)$ on $\mathscr{F}_{\mathrm{b}}$ defined by

$$\Gamma(T) = \bigoplus_{n=0}^{\infty} (\otimes^n T). \tag{5.2.14}$$

Here it is understood that $\otimes^0 T = 1$. In most cases $\Gamma(T)$ is an unbounded operator resulting from the fact that it is given by a countable direct sum. However, for a contraction operator T, the second quantization $\Gamma(T)$ is also a contraction on $\mathscr{F}_{\mathrm{b}}$, or equivalently, Γ is a functor

$$\Gamma : \mathscr{C}(\mathscr{W} \to \mathscr{W}) \to \mathscr{C}(\mathscr{F}_{\mathrm{b}} \to \mathscr{F}_{\mathrm{b}}), \tag{5.2.15}$$

of the set $\mathscr{C}(X \to Y)$ of contraction operators from X to Y. The functor Γ has the semigroup property, while $\mathscr{C}(\mathscr{W} \to \mathscr{W})$ is a $*$-algebra with respect to operator multiplication and conjugation $*$ (i.e., taking adjoints). The map Γ pulls this structure over to $\mathscr{F}_{\mathrm{b}}$ so that

$$\Gamma(S)\Gamma(T) = \Gamma(ST), \quad \Gamma(S)^* = \Gamma(S^*), \quad \Gamma(1) = 1, \tag{5.2.16}$$

for $S, T \in \mathscr{C}(\mathscr{W} \to \mathscr{W})$. For a self-adjoint operator h on $\mathscr{W}$ the structure relations (5.2.16) imply in particular that

$$\{\Gamma(e^{ith}) : \ t \in \mathbb{R}\} \tag{5.2.17}$$

is a strongly continuous one-parameter unitary group on $\mathscr{F}_{\mathrm{b}}$. Then by Stone's theorem (Proposition 3.24) there exists a unique self-adjoint operator $d\Gamma(h)$ on $\mathscr{F}_{\mathrm{b}}$ such that

$$\Gamma(e^{ith}) = e^{itd\Gamma(h)}, \quad t \in \mathbb{R}. \tag{5.2.18}$$

The operator $d\Gamma(h)$ is called the *differential second quantization* of h or simply *second quantization* of h. Since

$$d\Gamma(h) = -i \frac{d}{dt} \Gamma(e^{ith}) \lceil_{t=0}$$

on some domain, we have

$$d\Gamma(h) = 0 \oplus \Big[\bigoplus_{n=1}^{\infty} \overline{\Big(\sum_{j=1}^{n} 1 \otimes \cdots \otimes \overset{j}{h} \otimes \cdots \otimes 1 \Big)} \Big], \tag{5.2.19}$$

where the overline denotes closure, and j on top of h indicates its position in the product. Thus the action of $d\Gamma(h)$ on each $\mathscr{F}_{\mathrm{b}}^{(n)}$ is given by

$$d\Gamma(h)\Omega_{\mathrm{b}} = 0 \tag{5.2.20}$$

and

$$d\Gamma(h)a^*(f_1)\cdots a^*(f_n)\Omega_{\mathrm{b}} = \sum_{j=1}^{n} a^*(f_1)\cdots a^*(hf_j)\cdots a^*(f_n)\Omega_{\mathrm{b}}. \tag{5.2.21}$$

It can be also seen by (5.2.19) that

$$\mathrm{Spec}(d\Gamma(h)) = \overline{\{\lambda_1 + \cdots + \lambda_n | \lambda_j \in \mathrm{Spec}(h), j = 1, \ldots, n, n \geq 1\} \cup \{0\}},$$
$$\mathrm{Spec}_{\mathrm{p}}(d\Gamma(h)) = \{\lambda_1 + \cdots + \lambda_n | \lambda_j \in \mathrm{Spec}_{\mathrm{p}}(h), j = 1, \ldots, n, n \geq 1\} \cup \{0\}.$$

If $0 \notin \mathrm{Spec}_{\mathrm{p}}(h)$, the multiplicity of 0 in $\mathrm{Spec}_{\mathrm{p}}(d\Gamma(h))$ is one.

A crucial operator in quantum field theory is the *boson number operator* defined by the second quantization of the identity operator on $\mathscr{W}$:

$$N = d\Gamma(1). \tag{5.2.22}$$

Since

$$\begin{aligned} N\Omega_{\mathrm{b}} &= 0, \\ Na^*(f_1)\cdots a^*(f_n)\Omega_{\mathrm{b}} &= na^*(f_1)\cdots a^*(f_n)\Omega_{\mathrm{b}}, \end{aligned} \tag{5.2.23}$$

it follows that

$$\mathrm{Spec}(N) = \mathrm{Spec}_{\mathrm{p}}(N) = \mathbb{N} \cup \{0\}. \tag{5.2.24}$$

Let ρ be a real-valued measurable function on $\mathbb{R}$. The operator $\rho(N)$ is self-adjoint on the dense domain

$$D(\rho(N)) = \Big\{(\Phi^{(n)})_{n\geq 0} \in \mathscr{F}_{\mathrm{b}} \Big| \sum_{n=0}^{\infty} \rho(n)^2 \|\Phi^{(n)}\|^2_{\mathscr{F}_{\mathrm{b}}^{(n)}} < \infty\Big\}.$$

In the case of $\rho(x) = e^{\beta x}$ with $\beta > 0$ (note the positive exponent), $\Psi \in D(e^{\beta N})$ implies that the probability of finding n bosons decays exponentially as $n \to \infty$.

We will use the following facts below. The superscript in $a^\sharp$ indicates that either of the creation or annihilation operators is meant.

Proposition 5.1 (Relative bounds). *Let h be a positive self-adjoint operator, and $f \in D(h^{-1/2})$, $\Psi \in D(d\Gamma(h)^{1/2})$. Then $\Psi \in D(a^\sharp(f))$ and*

$$\|a(f)\Psi\| \leq \|h^{-1/2}f\|\|d\Gamma(h)^{1/2}\Psi\|, \tag{5.2.25}$$

$$\|a^*(f)\Psi\| \leq \|h^{-1/2}f\|\|d\Gamma(h)^{1/2}\Psi\| + \|f\|\|\Psi\|. \tag{5.2.26}$$

In particular, $D(d\Gamma(h)^{1/2}) \subset D(a^\sharp(f))$, whenever $f \in D(h^{-1/2})$.

While we do not give the proof of this proposition, we will prove it for a special case below.

Example 5.1. (1) A first application of the above estimates is that $a^*(f)$ and $a(f)$ are well defined on $D(N^{1/2})$ for all $f \in \mathscr{W}$. For $f \in \mathscr{W}$,

$$\|a(f)\Psi\| \le \|f\|\|N^{1/2}\Psi\|, \tag{5.2.27}$$

$$\|a^*(f)\Psi\| \le \|f\|(\|N^{1/2}\Psi\| + \|\Psi\|). \tag{5.2.28}$$

(2) Another application is considering a perturbation of $d\Gamma(h)$ by adding the operator $\Phi(f) = (1/\sqrt{2})(a^*(f) + a(\bar{f}))$. Let $f \in D(h^{-1/2})$. Then $d\Gamma(h) + \alpha\Phi(f)$ is self-adjoint on $D(d\Gamma(h))$ for all $\alpha \in \mathbb{R}$. This follows from (5.2.25), (5.2.26) and the Kato–Rellich theorem.

To obtain the commutation relations between $a^\sharp(f)$ and $d\Gamma(h)$, suppose that $f \in D(h^{-1/2}) \cap D(h)$. Then

$$[d\Gamma(h), a^*(f)]\Psi = a^*(hf)\Psi, \quad [d\Gamma(h), a(f)]\Psi = -a(hf)\Psi, \tag{5.2.29}$$

for $\Psi \in D(d\Gamma(h)^{3/2}) \cap \mathscr{F}_{\mathrm{b,fin}}$. By a limiting argument (5.2.29) can be extended to $\Psi \in D(d\Gamma(h)^{3/2})$, and it is seen that $a^\sharp(f)$ maps $D(d\Gamma(h)^{3/2})$ into $D(d\Gamma(h))$. In general

$$a^\sharp(f) : D(d\Gamma(h)^{n+1/2}) \to D(d\Gamma(h)^n)$$

for all $n \ge 1$, when $f \in \bigcap_{n=1}^\infty \left(D(h^n) \cap D(h^{n-1/2})\right)$. In particular, $a^\sharp(f)$ maps $\bigcap_{n=1}^\infty D(d\Gamma(h)^n)$ into itself.

Take now $\mathscr{W} = L^2(\mathbb{R}^d)$ and consider the boson Fock space $\mathscr{F}_{\mathrm{b}}(L^2(\mathbb{R}^d))$. In this case, for $n \in \mathbb{N}$ the space $\mathscr{F}_{\mathrm{b}}^{(n)}$ can be identified with the set of symmetric functions on $L^2(\mathbb{R}^{dn})$ through

$$\otimes_{\mathrm{s}}^n L^2(\mathbb{R}^d) \cong \{f \in L^2(\mathbb{R}^{dn}) | f(k_1, \ldots, k_n) = f(k_{\pi(1)}, \ldots, k_{\pi(n)}), \forall \pi \in \wp_n\}. \tag{5.2.30}$$

The creation and annihilation operators are realized as

$$(a(f)\Psi)^{(n)}(k_1, \ldots, k_n) = \sqrt{n+1} \int_{\mathbb{R}^d} f(k)\Psi^{(n+1)}(k, k_1, \ldots, k_n) dk, \quad n \ge 0, \tag{5.2.31}$$

$$(a^*(f)\Psi)^{(n)}(k_1, \ldots, k_n) = \frac{1}{\sqrt{n}} \sum_{j=1}^n f(k_j)\Psi^{(n-1)}(k_1, \ldots, \hat{k}_j, \ldots, k_n), \quad n \ge 1, \tag{5.2.32}$$

$$(a^*(f)\Psi)^{(0)} = 0. \tag{5.2.33}$$

Here $\Psi \in \mathscr{F}_{\mathrm{b}}^{(n)}$ is denoted as a pointwise defined function for convenience, however, all of these expressions are to be understood in L^2-sense. Formally, we also use a notation of kernels $a(k)$ and $a^*(k)$ to write

$$(a(k)\Psi)^{(n)}(k_1,\dots,k_n) = \sqrt{n+1}\Psi^{(n+1)}(k,k_1,\dots,k_n), \tag{5.2.34}$$

$$(a^*(k)\Psi)^{(n)}(k_1,\dots,k_n) = \frac{1}{\sqrt{n}}\sum_{j=1}^{n}\delta(k-k_j)\Psi^{(n-1)}(k_1,\dots,\hat{k}_j,\dots,k_n), \tag{5.2.35}$$

and

$$[a(k),a^*(k')] = \delta(k-k'), \quad [a(k),a(k')] = 0 = [a^*(k),a^*(k')]. \tag{5.2.36}$$

Let $\omega_\nu : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ be the multiplication operator called *dispersion relation* given by

$$\omega_\nu(k) = \sqrt{|k|^2+\nu^2}, \quad k \in \mathbb{R}^d, \tag{5.2.37}$$

with $\nu \geq 0$. Here ν describes the *boson mass*. The second quantization of the dispersion relation is

$$(d\Gamma(\omega_\nu)\Psi)^{(n)}(k_1,\dots,k_n) = \Big(\sum_{j=1}^{n}\omega_\nu(k_j)\Big)\Psi^{(n)}(k_1,\dots,k_n). \tag{5.2.38}$$

Definition 5.1 (Free field Hamiltonian). The self-adjoint operator $d\Gamma(\omega_\nu)$ is called *free field Hamiltonian* on $\mathscr{F}_{\mathrm{b}}(L^2(\mathbb{R}^d))$ and we use the notation

$$H_{\mathrm{f}} = d\Gamma(\omega_\nu). \tag{5.2.39}$$

The spectrum of the free field Hamiltonian is $\mathrm{Spec}(H_{\mathrm{f}}) = [0,\infty)$, with component $\mathrm{Spec}_{\mathrm{p}}(H_{\mathrm{f}}) = \{0\}$, which is of single multiplicity with $H_{\mathrm{f}}\Omega_{\mathrm{b}} = 0$. Then formally we may write the free field Hamiltonian as

$$H_{\mathrm{f}} = \int_{\mathbb{R}^d}\omega_\nu(k)a^*(k)a(k)dk. \tag{5.2.40}$$

Physically, this describes the total energy of the free field since $a^*(k)a(k)$ gives the number of bosons carrying momentum k, multiplied with the energy $\omega_\nu(k)$ of a single boson, and integrated over all momenta. The commutation relations are

$$[H_{\mathrm{f}},a(f)] = -a(\omega_\nu f), \quad [H_{\mathrm{f}},a^*(f)] = a^*(\omega_\nu f) \tag{5.2.41}$$

and in formal expression they can be written as

$$[d\Gamma(\omega_\nu),a(k)] = -\omega_\nu(k)a(k), \quad [d\Gamma(\omega_\nu),a^*(k)] = \omega_\nu(k)a^*(k). \tag{5.2.42}$$

The relative bound of $a^\sharp(f)$ with respect to the free Hamiltonian H_f can be seen from (5.2.43) and (5.2.44). If $f/\sqrt{\omega_\nu} \in L^2(\mathbb{R}^d)$, then

$$\|a(f)\Psi\| \le \|f/\sqrt{\omega_\nu}\| \|H_\mathrm{f}^{1/2}\Psi\|, \tag{5.2.43}$$

$$\|a^*(f)\Psi\| \le \|f/\sqrt{\omega_\nu}\| \|H_\mathrm{f}^{1/2}\Psi\| + \|f\|\|\Psi\| \tag{5.2.44}$$

hold, following directly from the lemma below.

Lemma 5.2. *Let $h : \mathbb{R}^d \to \mathbb{C}$ be measurable, and $g \in D(h)$. Then for every $\Psi \in D(d\Gamma(|h|^2)^{1/2})$ we have $\Psi \in D(a^\sharp(hg))$, and it follows that*

$$\|a(hg)\Psi\|^2 \le \|g\|^2 \|d\Gamma(|h|^2)^{1/2}\Psi\|, \tag{5.2.45}$$

$$\|a^*(hg)\Psi\|^2 \le \|g\|^2 \|d\Gamma(|h|^2)^{1/2}\Psi\| + \|hg\|^2\|\Psi\|^2. \tag{5.2.46}$$

Proof. Let $\Psi \in (d\Gamma(|h|^2))$. First note that

$$\begin{aligned}(\Psi^{(n)}, (d\Gamma(|h|^2)\Psi)^{(n)}) &= \sum_{i=1}^n \int_{\mathbb{R}^{nd}} |\Psi^{(n)}(k_1,\dots,k_n)|^2 |h(k_i)|^2 dk_1\cdots dk_n \\ &= n\int_{\mathbb{R}^{nd}} |\Psi^{(n)}(k_1,\dots,k_n)|^2 |h(k_1)|^2 dk_1\cdots dk_n.\end{aligned}$$

Thus

$$\begin{aligned}\|(a(hg)\Psi)^{(n-1)}\|^2 &= n\int_{\mathbb{R}^{(n-1)d}} dk_2\cdots dk_n \left| \int_{\mathbb{R}^d} g(k_1)h(k_1)\Psi^{(n)}(k_1,\dots,k_n)dk_1\right|^2 \\ &\le n\|g\|^2 \int_{\mathbb{R}^{nd}} |h(k_1)|^2|\Psi^{(n)}(k_1,\dots,k_n)|^2 dk_1\cdots dk_n \\ &= \|g\|^2(\Psi^{(n)}, (d\Gamma(|h|^2)\Psi)^{(n)}),\end{aligned}$$

and summation over n gives (5.2.45). By the closedness of both $d\Gamma(|h|^2)$ and $a(hg)$ we can extend to $\Psi \in D(d\Gamma(|h|^2)^{1/2})$. The other inequality can be derived from the canonical commutation relation $[a(hg), a^*(hg)] = \|hg\|^2$ and (5.2.45). □

5.2.2 Segal fields

The creation and annihilation operators are not symmetric and do not commute. Roughly speaking, a creation operator corresponds to $\frac{1}{\sqrt{2}}(x + \partial_x)$ and an annihilation operator to $\frac{1}{\sqrt{2}}(x - \partial_x)$ in $L^2(\mathbb{R}^d)$. We can, however, construct symmetric and commutative operators by combining the two field operators and this leads to *Segal fields*. As it will be seen below, Segal fields linearly span Fock space, which suggests that $\mathscr{F}_\mathrm{b}$ may be realized conveniently as a space $L^2(\mathscr{Q})$ with infinitely many variables on a suitable $\mathscr{Q}$ instead of necessarily using operators. This is due to the fact that Segal fields are commutative. In this representation Segal fields translate into real-valued multiplication operators.

Definition 5.2 (Segal field). The *Segal field* $\Phi(f)$ on the boson Fock space $\mathscr{F}_{\mathrm{b}}(\mathscr{W})$ is defined by

$$\Phi(f) = \frac{1}{\sqrt{2}}(a^*(\bar{f}) + a(f)), \quad f \in \mathscr{W}, \tag{5.2.47}$$

and its *conjugate momentum* by

$$\Pi(f) = \frac{i}{\sqrt{2}}(a^*(\bar{f}) - a(f)), \quad f \in \mathscr{W}. \tag{5.2.48}$$

Here $\bar{f}$ denotes the complex conjugate of f.

By the above definition both $\Phi(f)$ and $\Pi(g)$ are symmetric, however, not linear in f and g over $\mathbb{C}$. Note that, in contrast, they are linear operators over $\mathbb{R}$. It is straightforward to check that

$$[\Phi(f), \Pi(g)] = i\mathrm{Re}(f, g), \tag{5.2.49}$$

and

$$[\Phi(f), \Phi(g)] = i\mathrm{Im}(f, g), \quad [\Pi(f), \Pi(g)] = i\mathrm{Im}(f, g). \tag{5.2.50}$$

In particular, for real-valued f and g the canonical commutation relations become

$$[\Phi(f), \Pi(g)] = i(f, g), \quad [\Phi(f), \Phi(g)] = [\Pi(f), \Pi(g)] = 0. \tag{5.2.51}$$

Applying the inequalities (5.2.43) and (5.2.44) to $h = 1$, we see that $\mathscr{F}_{\mathrm{b,fin}}$ is the set of analytic vectors of $\Phi(f)$, i.e.,

$$\operatorname*{s-lim}_{m\to\infty} \sum_{n=0}^{m} \frac{\|\Phi(f)^n \Psi\| t^n}{n!} < \infty$$

for $\Psi \in \mathscr{F}_{\mathrm{b,fin}}$ and $t \geq 0$. The following is a general result.

Proposition 5.3 (Nelson's analytic vector theorem). *Let K be a symmetric operator on a Hilbert space. Assume that there exists a dense subspace $\mathscr{D} \subset D(K)$ such that $\sum_{n=0}^{\infty} \|K^n f\| t^n/n! < \infty$, for $f \in \mathscr{D}$ and $t > 0$. Then K is essentially self-adjoint on $\mathscr{D}$.*

By Nelson's analytic vector theorem $\Phi(f)$ and $\Pi(g)$ are essentially self-adjoint on $\mathscr{F}_{\mathrm{b,fin}}$. We keep denoting the closures of $\Phi(f)\lceil_{\mathscr{F}_{\mathrm{b,fin}}}$ and $\Pi(g)\lceil_{\mathscr{F}_{\mathrm{b,fin}}}$ by the same symbols.

5.2.3 Wick product

Loosely speaking, the so-called Wick product $:a^\sharp(f_1)\cdots a^\sharp(f_n):$ is defined in a product of creation and annihilation operators by moving the creation operators to the left and the annihilation operators to the right without taking the commutation relations into account. For example, $:a(f_1)a^*(f_2)a(f_3)a^*(f_4): = a^*(f_2)a^*(f_4)a(f_1)a(f_3)$.

Definition 5.3 (Wick product). The *Wick product* $:\prod_{i=1}^n \Phi(g_i):$ of $\prod_{i=1}^n \Phi(g_i)$ is recursively defined by the equalities

$$:\Phi(f): = \Phi(f),$$
$$:\Phi(f)\prod_{i=1}^n \Phi(f_i): = \Phi(f):\prod_{i=1}^n \Phi(f_i): - \frac{1}{2}\sum_{j=1}^n (f, f_j):\prod_{i\neq j} \Phi(f_i):.$$

By the above definition we have

$$:\Phi(f)^n: = 2^{-n/2}\sum_{m=0}^n \binom{n}{m} a^*(f)^m a(f)^{n-m} \tag{5.2.52}$$

or

$$:\Phi(f)^n: = \sum_{k=0}^{[n/2]} \frac{n!}{k!(n-2k)!}\Phi(f)^{n-2k}\left(-\frac{1}{4}\|f\|^2\right)^k. \tag{5.2.53}$$

Note that $:\Phi(f_1)\cdots\Phi(f_n):\Omega_\mathrm{b} = 2^{-n/2}a^*(f_1)\cdots a^*(f_n)\Omega_\mathrm{b}$. From this the orthogonality property

$$\left(:\prod_{i=1}^n \Phi(f_i):\Omega_\mathrm{b}, :\prod_{i=1}^m \Phi(g_i):\Omega_\mathrm{b}\right) = \delta_{nm}2^{-n/2}\sum_{\pi\in\mathscr{P}_n}\prod_{i=1}^n (g_i, f_{\pi(i)}) \tag{5.2.54}$$

follows. Hence

$$\overline{\mathrm{L.H.}\left\{:\prod_{i=1}^n \Phi(f_i):\Omega_\mathrm{b} \,\middle|\, f_j\in\mathscr{W}, j = 1,\ldots,n\right\}} = \mathscr{F}_\mathrm{b}^{(n)}. \tag{5.2.55}$$

The Wick product of the exponential can be computed directly to yield

$$:e^{\alpha\Phi(f)}:\Omega_\mathrm{b} = \operatorname*{s-lim}_{m\to\infty}\sum_{n=0}^m \frac{\alpha^n}{n!}:\Phi(f)^n:\Omega_\mathrm{b} = e^{-(1/4)\alpha^2\|f\|^2}e^{\alpha\Phi(f)}\Omega_\mathrm{b}. \tag{5.2.56}$$

Hence for real-valued f and g,

$$(\Omega_\mathrm{b}, \Phi(f)\Omega_\mathrm{b}) = 0, \quad (\Omega_\mathrm{b}, \Phi(f)\Phi(g)\Omega_\mathrm{b}) = \frac{1}{2}(f, g), \tag{5.2.57}$$

and

$$(\Omega_{\mathrm{b}}, e^{\alpha\Phi(f)}\Omega_{\mathrm{b}})_{\mathscr{F}_{\mathrm{b}}} = e^{(1/4)\alpha^2\|f\|^2}. \tag{5.2.58}$$

The commutator $[\Phi(f), \Phi(g)] = 0$, and (5.2.57)–(5.2.58) suggest that $\Phi(f)$ with a real-valued test function f can thus be realized as a Gaussian random variable, as it will be further explored below.

5.3 $\mathscr{Q}$-spaces

5.3.1 Gaussian random processes

Now we define a family of Gaussian random variables indexed by a given real vector space $\mathscr{E}$ on a probability space $(\mathscr{Q}, \Sigma, \mu)$, and study $L^2(\mathscr{Q}) = L^2(\mathscr{Q}, \Sigma, \mu)$. Note that $L^2(\mathscr{Q})$ is a vector space over $\mathbb{C}$. Our main goal here is to show unitary equivalence of $L^2(\mathscr{Q})$ and $\mathscr{F}_{\mathrm{b}}(\mathscr{E}_{\mathbb{C}})$, where $\mathscr{E}_{\mathbb{C}}$ is the complexification of $\mathscr{E}$.

Definition 5.4 (Gaussian random variables indexed by $\mathscr{E}$). We say that $\phi(f)$ is a *family of Gaussian random variables* on a probability space $(\mathscr{Q}, \Sigma, \mu)$ indexed by a real vector space $\mathscr{E}$ whenever

(1) $\phi : \mathscr{E} \ni f \mapsto \phi(f)$ is a map from $\mathscr{E}$ to a Gaussian random variable on $(\mathscr{Q}, \Sigma, \mu)$ with zero mean and covariance

$$\int_{\mathscr{Q}} \phi(f)\phi(g)d\mu = \frac{1}{2}(f, g)_{\mathscr{E}} \tag{5.3.1}$$

(2) $\phi(\alpha f + \beta g) = \alpha\phi(f) + \beta\phi(g), \alpha, \beta \in \mathbb{R}$

(3) Σ is the minimal σ-field generated by $\{\phi(f) | f \in \mathscr{E}\}$.

Property (3) in Definition 5.4 is expressed by saying that the σ-field Σ is *full*. The existence of such a family of Gaussian random variables will be discussed in the next section, here we first look at some properties. Define

$$\mathscr{S}_{\mathscr{Q}} = \{F(\phi(f_1), \ldots, \phi(f_n)) \mid F \in \mathscr{S}(\mathbb{R}^n), f_j \in \mathscr{E}, j = 1, \ldots, n, n \geq 1\}. \tag{5.3.2}$$

Lemma 5.4. *The following statements are equivalent:*

(1) *$\mathscr{S}_{\mathscr{Q}}$ is dense in $L^2(\mathscr{Q})$.*

(2) *Σ is the minimal σ-field generated by $\{\phi(f) | f \in \mathscr{E}\}$.*

Proof. (1) $\Rightarrow$ (2) is fairly standard and we omit the proof. To prove (2) $\Rightarrow$ (1), we regard $\mathscr{S}_{\mathscr{Q}}$ as the set of bounded multiplication operators on $L^2(\mathscr{Q})$. Hence the closure $\overline{\mathscr{S}_{\mathscr{Q}}}$ taken in strong topology is a von Neumann algebra. It follows that $\overline{\mathscr{S}_{\mathscr{Q}}} \subset L^\infty(\mathscr{Q})$, since a strongly convergent bounded operator is bounded. Let $\mathscr{P}$ be the set of all projections in $\overline{\mathscr{S}_{\mathscr{Q}}}$. Every projection is the characteristic function of a measurable subset in $\mathscr{Q}$. Let $\Sigma_{\mathscr{P}}$ denote the family of measurable subsets of $\mathscr{Q}$ associated with $\mathscr{P}$, satisfying $\Sigma_{\mathscr{P}} \subset \Sigma$. It is easily seen by a limiting argument that $\phi(f)$ is measurable with respect to $\Sigma_{\mathscr{P}}$. Hence $\Sigma = \Sigma_{\mathscr{P}}$, as Σ is the minimal σ-field generated by $\{\phi(f) | f \in \mathscr{E}\}$. Since a function in $L^\infty(\mathscr{Q})$ can be approximated by a finite linear sum of characteristic functions and $\Sigma = \Sigma_P$, $\overline{\mathscr{S}_{\mathscr{Q}}} = L^\infty(\mathscr{Q})$ follows as sets of bounded operators. It is clear that $L^\infty(\mathscr{Q}) = \overline{\mathscr{S}_{\mathscr{Q}}} 1 \subset \overline{\mathscr{S}_{\mathscr{Q}}}^{\|\cdot\|_{L^2(\mathscr{Q})}}$. Since $L^\infty(\mathscr{Q})$ is dense in $L^2(\mathscr{Q})$, it follows that $\overline{\mathscr{S}_{\mathscr{Q}}}^{\|\cdot\|_{L^2(\mathscr{Q})}} = L^2(\mathscr{Q})$. $\square$

We define Wick product in $L^2(\mathscr{Q})$ in the same way as in the case of the boson Fock space.

Definition 5.5 (Wick product). The *Wick product* of $\prod_{i=1}^n \phi(f_i)$ is recursively defined by

$$:\phi(f): = \phi(f),$$

$$:\phi(f) \prod_{i=1}^n \phi(f_i): = \phi(f) :\prod_{i=1}^n \phi(f_i): - \frac{1}{2} \sum_{j=1}^n (f, f_j) :\prod_{i \neq j} \phi(f_i):. \tag{5.3.3}$$

Since $d\mu$ is a Gaussian measure, any Wick-ordered polynomial

$$:\prod_{i=1}^n \phi(f_i): \in L^2(\mathscr{Q}, d\mu). \tag{5.3.4}$$

Definition 5.5 has the virtue of allowing explicit calculations, and we will need especially (5.3.3) later on. In particular, for $f_i, g_j \in \mathscr{E}$,

$$\left(:\prod_{i=1}^n \phi(f_i):, :\prod_{i=1}^m \phi(g_i): \right) = \delta_{mn} \sum_{\pi \in \wp_n} 2^{-n} \prod_{i=1}^n (f_i, g_{\pi(i)}). \tag{5.3.5}$$

Furthermore,

$$:e^{\alpha\phi(f)}: = \operatorname*{s-lim}_{m \to \infty} \sum_{n=0}^m \frac{\alpha^n}{n!} :\phi(f)^n: = e^{(1/4)\alpha^2 \|f\|^2} e^{\alpha\phi(f)}. \tag{5.3.6}$$

Write

$$L^2_n(\mathscr{Q}) = \overline{\text{L.H.} \left\{ :\prod_{i=1}^n \phi(f_i): \middle| f_i \in \mathscr{E}, i = 1, \dots, n \right\} \cup \{1\}}. \tag{5.3.7}$$

Then $L^2_n(\mathscr{Q}) \subset L^2(\mathscr{Q})$ and $L^2_m(\mathscr{Q}) \perp L^2_n(\mathscr{Q})$ if $n \neq m$.

Proposition 5.5 (Wiener–Itô decomposition). *We have that*

$$L^2(\mathscr{Q}) = \bigoplus_{n=0}^{\infty} L_n^2(\mathscr{Q}). \tag{5.3.8}$$

Proof. Set $\mathscr{G} = \bigoplus_{n=0}^{\infty} L_n^2(\mathscr{Q})$. It is clear that $\mathscr{G} \subset L^2(\mathscr{Q})$. By (5.3.6), $e^{i\phi(f)} \in \mathscr{G}$, which means that $F(\phi(f_1), \ldots, \phi(f_n)) \in \mathscr{G}$ for $F \in \mathscr{S}(\mathbb{R}^n)$. Since $\mathscr{S}_{\mathscr{Q}}$ is dense in $L^2(\mathscr{Q})$ by Lemma 5.4, $L^2(\mathscr{Q}) = \mathscr{G}$ follows. □

The following result says that the family of Gaussian random variables $\phi(f)$ is unique up to unitary equivalence.

Proposition 5.6 (Uniqueness of Gaussian random variables). *Let $\phi(f)$ and $\phi'(f)$ be Gaussian random variables indexed by a real vector space $\mathscr{E}$ on $(\mathscr{Q}, \Sigma, \mu)$ and $(\mathscr{Q}', \Sigma', \mu')$, respectively. Then there exists a unitary operator $U : L^2(\mathscr{Q}) \to L^2(\mathscr{Q}')$ such that $U1 = 1$ and $U\phi(f)U^{-1} = \phi'(f)$.*

Proof. Define $U : L^2(\mathscr{Q}) \to L^2(\mathscr{Q}')$ by $U{:}\phi(f_1)\cdots\phi(f_n){:} = {:}\phi'(f_1)\cdots\phi'(f_n){:}$. By Proposition 5.5, U can be extended to a unitary operator. □

5.3.2 Wiener–Itô–Segal isomorphism

Define the complexification $\mathscr{E}_{\mathbb{C}}$ of the real Hilbert space $\mathscr{E}$ by

$$\mathscr{E}_{\mathbb{C}} = \{\{f, g\} | f, g \in \mathscr{E}\} \tag{5.3.9}$$

such that for $\lambda \in \mathbb{R}$,

$$\lambda\{f, g\} = \{\lambda f, \lambda g\}, \quad i\lambda\{f, g\} = \{-\lambda g, \lambda f\}, \quad \{f, g\} + \{f', g'\} = \{f + f', g + g'\}.$$

The scalar product of $\mathscr{E}_{\mathbb{C}}$ is defined by

$$(\{f, g\}, \{f', g'\})_{\mathscr{E}_{\mathbb{C}}} = (f, f') + (g, g') + i((f, g') - (g, f')).$$

With these definitions $(\mathscr{E}_{\mathbb{C}}, (\cdot, \cdot)_{\mathscr{E}_{\mathbb{C}}})$ is a Hilbert space over $\mathbb{C}$.

Proposition 5.7 (Wiener–Itô–Segal isomorphism). *There exists a unitary operator $\theta_{\mathrm{W}} : \mathscr{F}_{\mathrm{b}}(\mathscr{E}_{\mathbb{C}}) \to L^2(\mathscr{Q})$ such that*

(1) $\theta_{\mathrm{W}}\Omega_{\mathrm{b}} = 1$,

(2) $\theta_{\mathrm{W}}\mathscr{F}_{\mathrm{b}}^{(n)}(\mathscr{E}_{\mathbb{C}}) = L_n^2(\mathscr{Q})$,

(3) $\theta_{\mathrm{W}}\Phi(f)\theta_{\mathrm{W}}^{-1} = \phi(f)$, *where $\Phi(f)$ is a Segal field.*

The unitary operator θ_{W} is called *Wiener–Itô–Segal isomorphism*, and it allows to identify $\mathcal{F}_{\mathrm{b}}(\mathscr{E}_{\mathbb{C}})$ with $L^2(\mathscr{Q})$.

Proof. Define $\theta_{\mathrm{W}} : \mathcal{F}_{\mathrm{b}}(\mathscr{E}_{\mathbb{C}}) \to L^2(\mathscr{Q})$ by

$$\theta_{\mathrm{W}}{:}\prod_{i=1}^{n} \Phi(f_i){:}\Omega_{\mathrm{b}} = {:}\prod_{i=1}^{n} \phi(f_i){:}, \quad f_1, \ldots, f_n \in \mathscr{E}, \tag{5.3.10}$$

$$\theta_{\mathrm{W}}\Omega_{\mathrm{b}} = 1. \tag{5.3.11}$$

Notice that L.H.$\{{:}\prod_{i=1}^{n} \Phi(f_i){:}\Omega_{\mathrm{b}} | f_i \in \mathscr{E}, i = 1, \ldots, n, n \geq 1\} \cup \{\Omega_{\mathrm{b}}\}$ is dense in the Fock space $\mathcal{F}_{\mathrm{b}}(\mathscr{E}_{\mathbb{C}})$ over the complexification of $\mathscr{E}$. It is straightforward to check that θ_{W} can be extended to a unitary operator from $\mathcal{F}_{\mathrm{b}}(\mathscr{E}_{\mathbb{C}})$ to $L^2(\mathscr{Q})$ by Proposition 5.5, and the lemma follows. □

Let $T : \mathscr{E} \to \mathscr{E}$ be a contraction operator. Then T can be linearly extended to a contraction on $\mathscr{E}_{\mathbb{C}}$. The operator $\theta_{\mathrm{W}}\Gamma(T)\theta_{\mathrm{W}}^{-1} : L^2(\mathscr{Q}) \to L^2(\mathscr{Q})$ is called the *second quantization* of T and is denoted by the same symbol $\Gamma(T)$. It is seen directly that $\Gamma(T)$ acts as

$$\Gamma(T){:}\prod_{i=1}^{n} \phi(f_i){:} = {:}\prod_{i=1}^{n} \phi(Tf_i){:} \tag{5.3.12}$$

and

$$\Gamma(T)1 = 1. \tag{5.3.13}$$

For a self-adjoint operator h, the *differential second quantization* $\theta_{\mathrm{W}} d\Gamma(h)\theta_{\mathrm{W}}^{-1}$ is also denoted by the same symbol. Then

$$d\Gamma(h){:}\phi(f_1)\cdots\phi(f_n){:} = \sum_{j=1}^{n} {:}\phi(f_1)\cdots\phi(hf_j)\cdots\phi(f_n){:} \tag{5.3.14}$$

with

$$d\Gamma(h)1 = 0 \tag{5.3.15}$$

for $f_j \in D(h)$, $j = 1, \ldots, n$, follows.

Proposition 5.8 (Positivity preserving). *Let T be a contraction operator on a real Hilbert space $\mathscr{E}$. Then $\Gamma(T)$ is positivity preserving on $L^2(\mathscr{Q})$.*

Proof. Note that $\Gamma(T){:}\exp(\alpha\phi(f)){:} = {:}\exp(\alpha\phi(Tf)){:}$ for every $\alpha \in \mathbb{C}$. Thus

$$\Gamma(T)e^{\alpha\phi(f)} = {:}e^{\alpha\phi(Tf)}{:}e^{\frac{1}{4}\alpha^2\|f\|^2} = e^{\alpha\phi(Tf)}e^{\frac{1}{4}\alpha^2(f,(1-T^*T)f)}.$$

Hence for $F(\phi(f_1),\dots,\phi(f_n)) \in \mathscr{S}_{\mathscr{Q}}$,

$$\Gamma(T)F(\phi(f_1),\dots,\phi(f_n)) = (2\pi)^{-n/2}\int_{\mathbb{R}^n} dk_1\cdots dk_n \hat{F}(k_1,\dots,k_n)$$
$$\times \exp\Big(-\frac{1}{4}\sum_{i,j=1}^n (f_i,(1-T^*T)f_j)k_ik_j\Big)\exp\Big(i\sum_{j=1}^n k_j\phi(Tf_j)\Big). \tag{5.3.16}$$

Since $\|T\|\le 1$, $\{(f_i,(1-T^*T)f_j)\}_{i,j}$ is positive semi-definite. Then the right-hand side of (5.3.16) is expressed by the convolution of F and a Gaussian kernel D_T, i.e.,

$$\Gamma(T)F(\phi(f_1),\dots,\phi(f_n)) = (2\pi)^{-n/2}(F*D_T)(\phi(Tf_1),\dots,\phi(Tf_n)).$$

Thus $F\ge 0$ implies that $\Gamma(T)F\ge 0$. Let $\Psi\in L^2(\mathscr{Q})$ be a non-negative function. By Lemma 5.4 we can construct a sequence $(F_n)_{n\in\mathbb{N}}\subset\mathscr{S}_{\mathscr{Q}}$ such that $F_n\to\Psi$ as $n\to\infty$ in $L^2(\mathscr{Q})$ and $F_n\ge 0$. Hence $\Gamma(T)\Psi\ge 0$ follows. □

5.3.3 Lorentz covariant quantum fields

Following the conventions in physics a quantum field is required to have the so called Lorentz covariance property, i.e., to transform under the Lorentz group. The measure $\delta(k_0^2-k^2-\nu^2)d\mathbf{k}$, $\mathbf{k}=(k_0,k)\in\mathbb{R}\times\mathbb{R}^3$, on the 4-dimensional space-time $\mathbb{R}^4$ supported on the light cone with mass ν is Lorentz-covariant and

$$\int_{\mathbb{R}^3}\frac{1}{2\omega_\nu(k)}\frac{d^3k}{(2\pi)^3} = \int_{\mathbb{R}^4}(2\pi)\delta(k_0^2-|k|^2-\nu^2)\lceil_{\{k_0>0\}}\frac{d\mathbf{k}}{(2\pi)^4}$$

is a Lorentz invariant integral. On normalization, the Lorentz covariant scalar field is formally given by

$$\phi(t,x) = \int_{\mathbb{R}^d}\frac{1}{\sqrt{2\omega_\nu(k)}}(a(k)e^{-it\omega_\nu(k)+ikx}+a^*(k)e^{it\omega_\nu(k)-ikx})dk. \tag{5.3.17}$$

Then instead of a Segal field the conventional *time-zero scalar field* is obtained,

$$\frac{1}{\sqrt{2}}(a^*(\hat{f}/\sqrt{\omega_\nu})+a(\widetilde{\hat{f}}/\sqrt{\omega_\nu})). \tag{5.3.18}$$

Here $\widetilde{\hat{f}}(k)=\hat{f}(-k)$. Note that $\widetilde{\hat{f}}=\bar{\hat{f}}$ for real f, and (5.3.18) is linear in f. In Chapter 6 we will discuss the Nelson model of the scalar quantum field given by (5.3.18).

Note that (5.3.18) can be generalized by using a covariance operator C. Then a quantum field can be defined in the form $\Phi(C\cdot)$ satisfying

$$(\Phi(Cf),\Phi(Cg)) = \frac{1}{2}(Cf,Cg). \tag{5.3.19}$$

Let $\mathscr{E}, \mathscr{E}'$ be real Hilbert spaces and $C : \mathscr{E} \to \mathscr{E}'$ be a linear operator such that $\operatorname{Ran} C$ is dense in $\mathscr{E}'$, which we apply as covariance operator. Define

$$\phi_C(f) = \phi(Cf). \tag{5.3.20}$$

For $\phi_C(f)$ the Wick product can be defined in the same way as for $\phi(f)$, i.e., $:\!\phi_C(f)\!: = \phi_C(f)$ and

$$:\!\phi_C(f)\prod_{i=1}^n \phi_C(f_i)\!: = \phi_C(f)\!:\!\prod_{i=1}^n \phi_C(f_i)\!: - \frac{1}{2}\sum_{j=1}^n (Cf, Cf_j)\!:\!\prod_{i\neq j} \phi_C(f_i)\!:.$$

The Wiener–Itô–Segal isomorphism $\theta_{\mathrm{W}} : L^2(\mathscr{Q}; d\mu) \to \mathscr{F}_{\mathrm{b}}(\mathscr{E}'_{\mathbb{C}})$ acts as

$$\theta_{\mathrm{W}}\!:\!\prod_{i=1}^n \phi_C(f_i)\!: = a^*(Cf_1)\cdots a^*(Cf_n)\Omega_{\mathrm{b}}.$$

The map θ_{W} can be extended by linearity and using the fact that RanC is dense.

5.4 Existence of $\mathscr{Q}$-spaces

5.4.1 Countable product spaces

In the previous section we have seen that a $\mathscr{Q}$-space is unique up to unitary equivalence. On the other hand, the $\mathscr{Q}$-space is not canonically given. In this section we consider explicit choices of a measure space $(\mathscr{Q}, \Sigma, \mu)$ on which we will be able to construct a family of Gaussian random variables indexed by a real Hilbert space $\mathscr{E}$.

Theorem 5.9. *Let $\mathscr{E}$ be a real Hilbert space. Then there exists a probability space $(\mathscr{Q}, \Sigma, \mu)$ and a family of Gaussian random variables $(\phi(f), f \in \mathscr{E})$ with mean zero and covariance*

$$\int_{\mathscr{Q}} \phi(f)\phi(g)\mu(d\phi) = \frac{1}{2}(f, g)_{\mathscr{E}}.$$

Proof. We show that $\mathscr{Q}$ can be obtained as a countable product space and, roughly, the measure μ may be regarded as the infinite product of Gaussian distributions

$$d\mu = \prod_{i=1}^\infty \pi^{-1/2} e^{-x_i^2} dx_i \quad \text{on} \quad \prod_{i=1}^\infty \mathbb{R}.$$

We establish this next.

Let $\dot{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ be the one-point compactification of $\mathbb{R}$, and set $\mathscr{Q} = \times_{n=1}^\infty \dot{\mathbb{R}}$. By the Tychonoff theorem $\mathscr{Q}$ is a compact Hausdorff space. Let $C(\mathscr{Q})$ be the set of continuous functions on $\mathscr{Q}$, and $C_{\mathrm{fin}}(\mathscr{Q}) \subset C(\mathscr{Q})$ the set of continuous functions on

$\mathcal{Q}$ depending on a finite number of variables. For $F = F(x_1, \dots, x_n) \in C_{\text{fin}}(\mathcal{Q})$ define $\ell : C_{\text{fin}}(\mathcal{Q}) \to \mathbb{C}$ by

$$\ell(F) = \pi^{-n/2} \int_{\mathbb{R}^n} F(x_1, \dots, x_n) e^{-\sum_{i=1}^n x_i^2} dx_1 \cdots dx_n.$$

Since $|\ell(F)| \le \|F\|_\infty$, ℓ is a bounded linear functional on $C_{\text{fin}}(\mathcal{Q})$. On the other hand, $C_{\text{fin}}(\mathcal{Q})$ is a dense subset in $C(\mathcal{Q})$ by the Stone–Weierstrass theorem, thus ℓ can be uniquely extended to a bounded linear functional $\tilde{\ell}$ on the Banach space $(C(\mathcal{Q}), \|\cdot\|_\infty)$. It is clear that $\tilde{\ell}(F) \ge 0$, for $F \ge 0$. The Riesz–Markov theorem furthermore implies that there exists a probability space $(\mathcal{Q}, \Sigma, \mu)$ such that $\tilde{\ell}(F) = \int_{\mathcal{Q}} F(q) d\mu(q)$. Let $\{e_n\}_{n=1}^\infty$ be a complete orthonormal system of a real Hilbert space $\mathscr{E}$. For each e_n we define $e_n(q) = x_n$ for $q = \{x_k\}_{k=1}^\infty \in \mathcal{Q}$. Note that

$$\int_{\mathcal{Q}} |e_n(q)|^2 d\mu(q) = \tilde{\ell}(e_n^2) = \pi^{-1/2} \int_{\mathbb{R}} x^2 e^{-x^2} dx = \frac{1}{2}.$$

Since $C(\mathcal{Q})$ is dense in $L^2(\mathcal{Q})$ in $\|\cdot\|_{L^2(\mathcal{Q})}$ norm, $C_{\text{fin}}(\mathcal{Q})$ is dense in $L^2(\mathcal{Q})$. Then the minimal σ-field generated by $\{e_n(\cdot)\}_{n=1}^\infty$ is Σ by Lemma 5.4. Expand $f \in \mathscr{E}$ as $f = \sum_{n=1}^\infty \alpha_n e_n$, and set $\phi_n(f) = \sum_{k=1}^n \alpha_k e_k(\cdot)$. It is seen that

$$\|\phi_m(f) - \phi_n(f)\|_{L^2(\mathcal{Q})}^2 = \frac{1}{2} \sum_{k=m+1}^n \alpha_k^2 \to 0 \quad m, n \to \infty.$$

Thus $\phi(f) = \text{s-lim}_{n\to\infty} \phi_n(f)$ exists in $L^2(\mathcal{Q})$. The equality $\int_{\mathcal{Q}} e^{i\phi(f)} d\mu = e^{-(1/4)\|f\|^2}$ easily follows. From this we obtain $\int_{\mathcal{Q}} \phi(f) d\mu = 0$ and

$$\int_{\mathcal{Q}} \phi(f)\phi(g) d\mu = (1/2)(f, g),$$

and thereby we have constructed the desired family of Gaussian random variables $\phi(f)$ indexed by $\mathscr{E}$ on $(\mathcal{Q}, \Sigma, \mu)$. □

5.4.2 Bochner theorem and Minlos theorem

The $\mathcal{Q}$ space constructed in Theorem 5.9 was seen to be the countable product $\times_{n=1}^\infty \dot{\mathbb{R}}$, however, the Gaussian random variables $\{\phi(f) | f \in \mathscr{E}\}$, depend on the base of the given Hilbert space $\mathscr{E}$, and therefore the construction is not canonical. The $\mathcal{Q}$-space introduced in this section is obtained through a generalization of Bochner's theorem, known as Minlos' theorem. In this construction the measure on $\mathcal{Q}$ is given on an abstract nuclear space such as $\mathscr{S}'(\mathbb{R}^d)$.

Let X be a random variable on $(\Omega, \mathscr{F}, P)$, and consider its characteristic function

$$C(z) = \mathbb{E}_P[e^{izX}], \quad z \in \mathbb{R}^d. \tag{5.4.1}$$

Clearly, $C(z)$ has the properties

$$(1)\ \sum_{i,j=1}^{n} \alpha_i \bar{\alpha}_j C(z_i - z_j) \geq 0, \quad (2)\ C \text{ is uniformly continuous}, \quad (3)\ C(0) = 1. \tag{5.4.2}$$

The converse statement is known as the Bochner theorem.

Theorem 5.10 (Bochner). *Suppose that function $C : \mathbb{R}^d \to \mathbb{C}$ satisfies condition* (5.4.2). *Then there exists a probability space* $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d), \mu_{\mathrm{B}})$ *such that*

$$C(z) = \int_{\mathbb{R}^d} e^{izx} \mu_{\mathrm{B}}(dx). \tag{5.4.3}$$

Proof. Let $C_t(z) = e^{-t|z|^2/2}$. It is seen that $C_t(z)C(z)$ is positive definite, which implies that

$$\rho_t(x) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{-izx} C_t(z)C(z)dz \geq 0$$

and

$$\int_{\mathbb{R}^d} \rho_t(x)dx = (2\pi)^{d/2} C_t(0)C(0) = (2\pi)^{d/2}.$$

Put $\mu_t(dx) = (2\pi)^{-d/2}\rho_t(x)dx$. Thus $\mu_t(dx)$ is a probability measure on $\mathbb{R}^d$ with characteristic function $C_t(z)C(z)$. Since $\lim_{t\downarrow 0} C_t(z)C(z) = C(z)$ pointwise and $C(z)$ is continuous, there exists a measure μ such that μ_t is weakly convergent to μ and its characteristic function is $C(z)$. □

The Bochner theorem can be extended to infinite dimension, however, as it will be illustrated in Remark 5.1 below, the Hilbert space $\mathscr{E}$ is too small to support the corresponding measure μ. To replace $\mathscr{E}$ we will introduce a suitable nuclear space.

Let $\mathscr{E}$ be a complete topological vector space with a countable family of norms $\{\|\cdot\|_n\}_{n=0}^{\infty}$ such that $\|\cdot\|_0 < \|\cdot\|_1 < \|\cdot\|_2 < \cdots$, and there exists a scalar product $(\cdot,\cdot)_n$ generating these norms so that $\|f\|_n^2 = (f, f)_n$ for each n. Define

$$\mathscr{E} = \bigcap_{n=0}^{\infty} E_n,$$

where $E_n = \overline{\mathscr{E}}^{\|\cdot\|_n}$ and E_n is a Hilbert space with the scalar product $(\cdot,\cdot)_n$. Identifying E_0 with E_0^* gives rise to the nested inclusions

$$\cdots \subset E_2 \subset E_1 \subset E_0 \cong E_0^* \subset E_1^* \subset E_2^* \subset \cdots.$$

Furthermore, let $\mathscr{E}^* = \bigcup_{n=0}^{\infty} E_n^*$, and denote its norm by $\|x\|_{-n} = \sup_{\xi \in E_n} |x(\xi)|$.

Definition 5.6 (Nuclear space). $\mathscr{E}$ is a called a *nuclear space* whenever for any $m > 0$ there exists $n > m$ such that the injection $\iota_{nm} : E_n \to E_m$ is a Hilbert–Schmidt operator.

Example 5.2. Let $H_{\rm osc} = \frac{1}{2}(-\Delta + |x|^2 - 1)$ be the harmonic oscillator on $L^2(\mathbb{R}^d)$ with purely discrete spectrum $\{n_1 + \cdots + n_d \mid n_j \in \mathbb{N} \cup \{0\}, j = 1, \ldots, d\}$. Using a suitable orthonormal basis $\{g_{i_1,\ldots,i_d}\}$, $H_{\rm osc}$ is diagonalized and $(g_{i_1,\ldots,i_d}, Hg_{i_1,\ldots,i_d}) = \sum_{j=1}^d i_j$. It is easily seen that $(H_{\rm osc} + 1)^{-(d+\varepsilon)/2}$ is a Hilbert–Schmidt operator for any $\varepsilon > 0$ and

$$\mathrm{Tr}[(H_{\rm osc} + 1)^{-(d+\varepsilon)}] = \sum_{i_1=0}^{\infty} \cdots \sum_{i_d=0}^{\infty} \Big(\sum_{\nu=1}^{d} (i_\nu + 1) \Big)^{-(d+\varepsilon)}.$$

Let $(\Psi, \Phi)_n = (\Psi, (H_{\rm osc} + 1)^n \Phi)$. Then $\|\Psi\|_n < \|\Psi\|_{n+1}$ and denote the completion of $L^2(\mathbb{R}^d)$ with respect to $\|\cdot\|_n$ by $\mathscr{H}_n$. Then $\cdots \subset \mathscr{H}_{n+1} \subset \mathscr{H}_n \subset \cdots$, and it can be proven that

$$\mathscr{S} = \bigcap_{n=-\infty}^{\infty} \mathscr{H}_n, \quad \mathscr{S}' = \bigcup_{n=-\infty}^{\infty} \mathscr{H}_n.$$

Let $1_{n+1} = 1 : \mathscr{H}_{n+1} \to \mathscr{H}_n$ be the identity map. Thus $(H_{\rm osc} + 1)^{1/2} 1_{n+1}$ is unitary and so $1 = 1_{n+1} \cdots 1_{n+D+1} : \mathscr{H}_{n+D+1} \to \mathscr{H}_n$ is a Hilbert–Schmidt operator for sufficiently large D since $1 = (H_{\rm osc} + 1)^{-D/2} (H_{\rm osc} + 1)^{D/2} 1$ is a product of a Hilbert–Schmidt operator with a unitary operator.

Theorem 5.11 (Minlos). *Let $\mathscr{E}$ be a nuclear space, and suppose that $C : \mathscr{E} \to \mathbb{C}$ satisfies*

$$(1) \sum_{i,j=1}^{n} \alpha_i \bar{\alpha}_j C(\xi_i - \xi_j) \geq 0, \quad (2)\ C(\cdot) \textit{ is continuous in } \mathscr{E}, \quad (3)\ C(0) = 1. \tag{5.4.4}$$

Then there exists a probability space $(\mathscr{E}^*, \mathscr{B}, \mu)$ such that

$$C(\xi) = \int_{\mathscr{E}^*} e^{i\langle\langle x, \xi \rangle\rangle} d\mu(x), \quad \xi \in \mathscr{E}, \tag{5.4.5}$$

where $\langle\langle x, \xi \rangle\rangle$ denotes the pair $x \in \mathscr{E}^*$ and $\xi \in \mathscr{E}$.

Remark 5.1. Let $\mathscr{E}$ be a Hilbert space and $C(\xi) = \exp(-(1/4)\|\xi\|^2)$ for $\xi \in \mathscr{E}$. Then $C(\xi)$ satisfies conditions (5.4.4) in Minlos' theorem. Let $\{\xi_n\}_{n=1}^{\infty}$ be a complete orthonormal system of $\mathscr{E}$. If a measure μ on $\mathscr{E}$ satisfying (5.4.5) existed, then

$$\int_{\mathscr{E}} e^{iz\langle\langle x, \xi_n \rangle\rangle} d\mu(x) = e^{-(1/4)z^2}. \tag{5.4.6}$$

Since $\langle\langle x, \xi_n \rangle\rangle = (x, \xi_n) \to 0$ as $n \to \infty$, the left-hand side of (5.4.6) converges to 1, which is in contradiction with the right-hand side. Moreover, from the fact that $C(\sum_{k=1}^n z_k \xi_k) = \exp\left(-(1/4)\sum_{k=1}^n z_k^2\right)$ and the law of large numbers, it would follow that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \langle\langle x, \xi_n \rangle\rangle^2 = 1$$

which is also a contradiction.

Let $\mathscr{S}_{\mathrm{real}}(\mathbb{R}^d)$ denote the set of real-valued, rapidly decreasing and infinitely differentiable functions on $\mathbb{R}^d$. $\mathscr{S}_{\mathrm{real}}(\mathbb{R}^d)$ is a nuclear space.

Corollary 5.12. *Suppose that $\mathscr{S}_{\mathrm{real}}(\mathbb{R}^d)$ is dense in a real Hilbert space $\mathscr{H}$. Then there exists a family of Gaussian random variables $\phi(f)$ indexed by $\mathscr{H}$ on a probability space $(\mathscr{S}'_{\mathrm{real}}(\mathbb{R}^d), \mathscr{B}, \mu)$.*

Proof. Write $C(\xi) = \exp(-(1/4)\|\xi\|^2)$, for $\xi \in \mathscr{S}_{\mathrm{real}}(\mathbb{R}^d)$. Then C satisfies (5.4.4). Thus the Minlos theorem yields that a probability space $(\mathscr{S}'_{\mathrm{real}}(\mathbb{R}^d), \mathscr{B}, \mu)$ exists such that $C(\xi) = \int_{\mathscr{S}'_{\mathrm{real}}(\mathbb{R}^d)} e^{i\langle\langle \phi, \xi\rangle\rangle} d\mu(\phi)$. Define

$$\phi(\xi) = \langle\langle \phi, \xi \rangle\rangle, \quad \phi \in \mathscr{S}'_{\mathrm{real}}(\mathbb{R}^d) \text{ and } \xi \in \mathscr{S}_{\mathrm{real}}(\mathbb{R}^d).$$

Then we have

$$\int_{\mathscr{S}'_{\mathrm{real}}(\mathbb{R}^d)} \phi(\xi) d\mu = 0, \quad \int_{\mathscr{S}'_{\mathrm{real}}(\mathbb{R}^d)} \phi(\xi)\phi(\eta) d\mu = \frac{1}{2}(\xi, \eta), \quad \xi, \eta \in \mathscr{S}_{\mathrm{real}}(\mathbb{R}^d). \tag{5.4.7}$$

Since $\mathscr{B}$ is the minimal σ-field generated by cylinder sets, $\mathscr{B}$ is the minimal σ-field generated by $\{\phi(\xi), \xi\}$. For $f \in \mathscr{H}$, there is a sequence $(\xi_n)_{n\in\mathbb{N}} \subset \mathscr{S}_{\mathrm{real}}(\mathbb{R}^d)$ such that $\xi_n \to f$ as $n \to \infty$ in $\mathscr{H}$. Thus

$$\int_{\mathscr{S}'_{\mathrm{real}}(\mathbb{R}^d)} |\phi(\xi_n) - \phi(\xi_m)|^2 d\mu = \frac{1}{2}\|\xi_n - \xi_m\|^2_{\mathscr{H}} \to 0$$

as $n, m \to \infty$. Hence $\{\phi(\xi_n)\}_{n=1}^\infty$ is a Cauchy sequence in $L^2(\mathscr{S}'_{\mathrm{real}}(\mathbb{R}^d))$. Define $\phi(f) = \text{s-lim}_{n\to\infty} \phi(\xi_n)$. Then $\{\phi(f) | f \in \mathscr{H}\}$ is a family of Gaussian random variables on $(\mathscr{S}'_{\mathrm{real}}(\mathbb{R}^d), \mathscr{B}, \mu)$ indexed by $\mathscr{H}$. □

Example 5.3. Suppose that $B(f, g)$ is a weakly continuous positive semi-definite quadratic form on $\mathscr{S}_{\mathrm{real}}(\mathbb{R}^d)$. Then by the Minlos theorem we can construct a measure μ on $\mathscr{S}'_{\mathrm{real}}(\mathbb{R}^d)$ and $\phi(f) = \langle\langle \phi, f \rangle\rangle$ such that $\int_{\mathscr{S}'_{\mathrm{real}}(\mathbb{R}^d)} e^{i\phi(f)} d\mu = e^{-(1/4)B(f,f)}$.

5.5 Functional integration representation of Euclidean quantum fields

5.5.1 Basic results in Euclidean quantum field theory

We use the notations

$$\mathscr{H}_\alpha(\mathbb{R}^d) = \overline{\mathscr{S}(\mathbb{R}^d)}^{\|\cdot\|_\alpha}, \quad \|f\|_\alpha^2 = \int_{\mathbb{R}^d} |\hat{f}(k)|^2 \omega_\nu(k)^{2\alpha} dk, \quad \alpha \in \mathbb{R},$$

and the shorthands

$$\mathscr{H}_{\mathrm{M}} = \mathscr{H}_{\mathrm{real},-1/2}(\mathbb{R}^d), \quad \mathscr{H}_{\mathrm{E}} = \mathscr{H}_{\mathrm{real},-1}(\mathbb{R}^{d+1}). \tag{5.5.1}$$

Let $\phi(f)$ be a family of Gaussian random variables indexed by $f \in \mathscr{H}_{\mathrm{M}}$ on a probability space $(\mathscr{Q}, \Sigma, \mu)$ and $\phi_{\mathrm{E}}(F)$ by $F \in \mathscr{H}_{\mathrm{E}}$ on $(\mathscr{Q}_{\mathrm{E}}, \Sigma_{\mathrm{E}}, \mu_{\mathrm{E}})$. One possible choice of $\mathscr{Q}$ and $\mathscr{Q}_{\mathrm{E}}$ is $\mathscr{S}'_{\mathrm{real}}(\mathbb{R}^d)$ and $\mathscr{S}'_{\mathrm{real}}(\mathbb{R}^{d+1})$, respectively, however, we do not fix any particular space here. For notational convenience we write $(f,g)_{-1/2}$ and $(f,g)_{-1}$ for the scalar products in $\mathscr{H}_{\mathrm{M}}$ and $\mathscr{H}_{\mathrm{E}}$, respectively. Note that $\phi(f)$ and $\phi_{\mathrm{E}}(F)$ are Gaussian random variables with mean zero and covariance

$$\int_{\mathscr{Q}} \phi(f)\phi(g) d\mu = \frac{1}{2}(f,g)_{-1/2}, \quad \int_{\mathscr{Q}_{\mathrm{E}}} \phi_{\mathrm{E}}(F)\phi_{\mathrm{E}}(G) d\mu_{\mathrm{E}} = \frac{1}{2}(F,G)_{-1}. \tag{5.5.2}$$

We use the identifications $\mathscr{H}_{-1/2}(\mathbb{R}^d) \cong [\mathscr{H}_{\mathrm{M}}]_{\mathbb{C}}$ and $L^2(\mathscr{Q}) \cong \mathscr{F}_{\mathrm{b}}(\mathscr{H}_{-1/2}(\mathbb{R}^d))$ by the Wiener–Itô–Segal isomorphism, as well as

$$\theta_{\mathrm{W}} \phi(f) \theta_{\mathrm{W}}^{-1} = \frac{1}{\sqrt{2}}(a^*(\hat{f}/\sqrt{\omega_\nu}) + a(\tilde{\hat{f}}/\sqrt{\omega_\nu})). \tag{5.5.3}$$

We will define a family of transformations I_t from $L^2(\mathscr{Q})$ to $L^2(\mathscr{Q}_{\mathrm{E}})$ through the second quantization of a specific transformation τ_t from $\mathscr{H}_{\mathrm{M}}$ to $\mathscr{H}_{\mathrm{E}}$. Define

$$\tau_t : \mathscr{H}_{\mathrm{M}} \to \mathscr{H}_{\mathrm{E}} \tag{5.5.4}$$

by

$$\tau_t : f \mapsto \delta_t \otimes f. \tag{5.5.5}$$

Here $\delta_t(x) = \delta(x-t)$ is the delta function with mass at t. Note that for $f \in \mathscr{H}_{\mathrm{M}}$,

$$\overline{\widehat{\delta_t \otimes f}}(k,k_0) = \overline{\widehat{\delta_t \otimes f}}(-k,-k_0) = \widehat{\delta_t \otimes f}(k,k_0).$$

Thus $\overline{\delta_t \otimes f} = \delta_t \otimes f$, which implies that τ_t preserves realness. Define

$$\hat{\omega}_\nu = \omega_\nu(-i\nabla). \tag{5.5.6}$$

Lemma 5.13. *It follows that*

$$\tau_s^* \tau_t = e^{-|t-s|\hat{\omega}_\nu}, \quad s, t \in \mathbb{R}. \tag{5.5.7}$$

In particular, τ_t is isometry for each $t \in \mathbb{R}$.

Proof. We see that

$$\begin{aligned}(\delta_t \otimes f, \delta_s \otimes g)_{-1} &= \frac{1}{\pi} \int_{\mathbb{R}^d} \overline{\hat{f}(k)} \hat{g}(k) dk \int_{\mathbb{R}} e^{-ik_0(s-t)} \frac{1}{\omega_\nu(k)^2 + |k_0|^2} dk_0 \\ &= \int_{\mathbb{R}^d} \overline{\hat{f}(k)} \hat{g}(k) \frac{e^{-|t-s|\omega_\nu(k)}}{\omega_\nu(k)} dk \\ &= (\hat{f}, e^{-|t-s|\omega_\nu} \hat{g})_{-1/2}. \end{aligned} \tag{5.5.8}$$

Then the lemma follows. □

Definition 5.7 (Contraction operators). (1) Let $T \in \mathscr{C}(\mathscr{H}_{\mathrm{M}} \to \mathscr{H}_{\mathrm{M}})$. Define $\Gamma(T) \in \mathscr{C}(L^2(\mathscr{Q}) \to L^2(\mathscr{Q}))$ by

$$\Gamma(T){:}\phi(f_1)\cdots\phi(f_n){:} = {:}\phi(Tf_1)\cdots\phi(Tf_n){:} \tag{5.5.9}$$

with $\Gamma(T)1_{\mathrm{M}} = 1_{\mathrm{M}}$, where 1_{M} denotes the identity function in $L^2(\mathscr{Q})$.

(2) Let $T \in \mathscr{C}(\mathscr{H}_{\mathrm{E}} \to \mathscr{H}_{\mathrm{E}})$. Define $\Gamma_{\mathrm{E}}(T) \in \mathscr{C}(L^2(\mathscr{Q}_{\mathrm{E}}) \to L^2(\mathscr{Q}_{\mathrm{E}}))$ by (5.5.9) with ϕ replaced by ϕ_{E}.

(3) Let $T \in \mathscr{C}(\mathscr{H}_{\mathrm{M}} \to \mathscr{H}_{\mathrm{E}})$. Define $\Gamma_{\mathrm{Int}}(T) \in \mathscr{C}(L^2(\mathscr{Q}) \to L^2(\mathscr{Q}_{\mathrm{E}}))$ by

$$\Gamma_{\mathrm{Int}}(T){:}\phi(f_1)\cdots\phi(f_n){:} = {:}\phi_{\mathrm{E}}(Tf_1)\cdots\phi_{\mathrm{E}}(Tf_n){:} \tag{5.5.10}$$

with $\Gamma_{\mathrm{Int}}(T)1_{\mathrm{M}} = 1_{\mathrm{E}}$, where 1_{E} denotes the identity function in $L^2(\mathscr{Q}_{\mathrm{E}})$.

Definition 5.8 (Isometries). Define the family of isometries $\{\mathrm{I}_t\}_{t\in\mathbb{R}}$ by $\mathrm{I}_t = \Gamma_{\mathrm{Int}}(\tau_t) : L^2(\mathscr{Q}) \to L^2(\mathscr{Q}_{\mathrm{E}})$, i.e.,

$$\mathrm{I}_t 1_{\mathrm{M}} = 1_{\mathrm{E}}, \quad \mathrm{I}_t{:}\phi(f_1)\cdots\phi(f_n){:} = {:}\phi_{\mathrm{E}}(\delta_t \otimes f_1)\cdots\phi_{\mathrm{E}}(\delta_t \otimes f_n){:}. \tag{5.5.11}$$

From the identity $\tau_s^*\tau_t = e^{-|t-s|\hat{\omega}_\nu}$ it follows that

$$\mathrm{I}_t^* \mathrm{I}_s = e^{-|t-s|\hat{H}_{\mathrm{f}}}, \quad s, t \in \mathbb{R}, \tag{5.5.12}$$

where $\hat{H}_{\mathrm{f}} = \theta_{\mathrm{W}}^{-1} H_{\mathrm{f}} \theta_{\mathrm{W}}$ is the free field Hamiltonian in $L^2(\mathscr{Q})$.

Proposition 5.14 (Functional integral representation for free field Hamiltonian). *Let $F, G \in L^2(\mathscr{Q})$ and $t \geq 0$. Then*

$$(F, e^{-tH_\mathrm{f}}G)_{L^2(\mathscr{Q})} = \int_{\mathscr{Q}_\mathrm{E}} \overline{F_0} G_t d\mu_\mathrm{E}, \tag{5.5.13}$$

where $F_0 = \mathrm{I}_0 F$ and $G_t = \mathrm{I}_t G$.

In order to extend (5.5.13) to Hamiltonians of the form $\hat{H}_\mathrm{f}$ + perturbation, we will use the Markov property of projections

$$\mathrm{E}_t = \mathrm{I}_t \mathrm{I}_t^*, \quad t \in \mathbb{R}, \tag{5.5.14}$$

which will be proven below. The following argument shows why the Markov property is useful. Let

$$\hat{H} = \hat{H}_\mathrm{f} + H_\mathrm{I}$$

with some interaction H_I. By the Trotter product formula we have

$$e^{-t\hat{H}} = \operatorname*{s-lim}_{n\to\infty} (e^{-(t/n)\hat{H}_\mathrm{f}} e^{-(t/n)H_\mathrm{I}})^n. \tag{5.5.15}$$

Substituting $e^{-|t-s|\hat{H}_\mathrm{f}} = \mathrm{I}_t^* \mathrm{I}_s$ into (5.5.15), we have

$$e^{-t\hat{H}} = \operatorname*{s-lim}_{n\to\infty} \mathrm{I}_0^* \Big(\prod_{j=1}^n \mathrm{I}_{tj/n} e^{-(tj/n)H_\mathrm{I}} \mathrm{I}_{tj/n}^* \Big) \mathrm{I}_t, \tag{5.5.16}$$

where $\prod_{j=1}^n T_j = T_1 T_2 \cdots T_n$. Thus it is obtained that

$$e^{-t\hat{H}} = \operatorname*{s-lim}_{n\to\infty} \mathrm{I}_0^* \Big(\prod_{j=1}^n \mathrm{E}_{tj/n} e^{-(tj/n)H_\mathrm{I}(tj/n)} \mathrm{E}_{tj/n} \Big) \mathrm{I}_t \tag{5.5.17}$$

by the definition of the projection E_s. Here $H_\mathrm{I}(t/n)$ denotes an operator acting on $L^2(\mathscr{Q}_\mathrm{E})$. Applying the Markov property of E_s implies that all the E_s in (5.5.17) can be disregarded, thus net result is

$$\begin{aligned}
(F, e^{-t\hat{H}}G) &= \lim_{n\to\infty} \Big(F_0, \Big(\prod_{j=1}^n \exp(-(tj/n)H_\mathrm{I}(tj/n)) \Big) G_t \Big) \\
&= \lim_{n\to\infty} \Big(F_0, \exp\Big(-\sum_{j=1}^\infty (tj/n) H_\mathrm{I}(tj/n) \Big) G_t \Big) \\
&= \int_{\mathscr{Q}_\mathrm{E}} \overline{F_0} G_t \exp\Big(-\int_0^t H_\mathrm{I}(s) ds \Big) d\mu_\mathrm{E}.
\end{aligned}$$

In the next section we investigate the Markov property of E_t and establish the functional integral representation of $(F, e^{-t\hat{H}}G)$.

5.5.2 Markov property of projections

Let $\mathscr{O} \subset \mathbb{R}$. Put

$$\mathscr{H}_{\mathrm{E}}(\mathscr{O}) = \{f \in \mathscr{H}_{\mathrm{E}} | \operatorname{supp} f \subset \mathscr{O} \times \mathbb{R}^d\} \tag{5.5.18}$$

and the projection $\mathscr{H}_{\mathrm{E}} \to \mathscr{H}_{\mathrm{E}}(\mathscr{O})$ is denoted by $e_{\mathscr{O}}$. Let $\Sigma_{\mathscr{O}}$ be the minimal σ-field generated by $\{\phi_{\mathrm{E}}(f) | f \in \mathscr{H}_{\mathrm{E}}(\mathscr{O})\}$. Define

$$\mathscr{E}_{\mathscr{O}} = \{\Phi \in L^2(\mathscr{Q}_{\mathrm{E}}) | \Phi \text{ is } \Sigma_{\mathscr{O}}\text{-measurable}\}. \tag{5.5.19}$$

Let $e_t = \tau_t \tau_t^*$, $t \in \mathbb{R}$. Then $\{e_t\}_{t \in \mathbb{R}}$ is a family of projections from $\mathscr{H}_{\mathrm{E}}$ to $\operatorname{Ran}(\tau_t)$. Let Σ_t, $t \in \mathbb{R}$, be the minimal σ-field generated by $\{\phi_{\mathrm{E}}(f) | f \in \operatorname{Ran}(e_t)\}$. Define

$$\mathscr{E}_t = \{\Phi \in L^2(\mathscr{Q}_{\mathrm{E}}) | \Phi \text{ is } \Sigma_t\text{-measurable}\}, \quad t \in \mathbb{R}. \tag{5.5.20}$$

We will see below that $F \in \mathscr{E}_{[a,b]}$ can be characterized by $\operatorname{supp} F \subset [a,b] \times \mathbb{R}^d$.

Lemma 5.15. (1) $\mathscr{H}_{\mathrm{E}}(\{t\}) = \operatorname{Ran}(e_t)$ *and any* $f \in \operatorname{Ran}(e_t)$ *can be expressed as* $f = \delta_t \otimes g$ *for some* $g \in \mathscr{H}_{\mathrm{M}}$. *In particular,* $e_{\{t\}} = e_t$.

(2) $\mathscr{H}_{\mathrm{E}}([a,b]) = \overline{\text{L.H.}\{f \in \mathscr{H}_{\mathrm{E}} | f \in \operatorname{Ran}(e_t), a \le t \le b\}}^{\|\cdot\|_{-1}}$ *holds.*

Proof. (1) The key fact is that any $f \in \mathscr{S}'_{\mathrm{real}}(\mathbb{R}^{d+1})$ with support $\{t\} \times \mathbb{R}^d$ can be expressed as

$$f(x_0, x) = \sum_{j=0}^{n} \delta^{(j)}(x_0 - t) \otimes g_j(x), \quad (x_0, x) \in \mathbb{R} \times \mathbb{R}^d,$$

where $g_j \in \mathscr{S}'_{\mathrm{real}}(\mathbb{R}^d)$ and $\delta^{(j)}$ is the jth derivative of the delta function δ. Then

$$\hat{f}(k_0, k) = \frac{1}{\sqrt{2\pi}} \sum_{j=0}^{n} (ik_0)^j e^{-ik_0 t} \hat{g}_j(k), \quad (k_0, k) \in \mathbb{R} \times \mathbb{R}^d,$$

and hence $n = 0$ and $g_0 \in \mathscr{H}_{\mathrm{M}}$ is necessary and sufficient for $f \in \mathscr{H}_{\mathrm{E}}$. Then $f \in \mathscr{H}_{\mathrm{E}}(\{t\})$ if and only if there exists $g \in \mathscr{H}_{\mathrm{M}}$ such that $f = \delta_t \otimes g$, which gives that $f \in \operatorname{Ran}(e_t)$. In particular, $\mathscr{H}_{\mathrm{E}}(\{t\}) \subset \operatorname{Ran}(e_t)$. The inclusion $\operatorname{Ran}(e_t) \subset \mathscr{H}_{\mathrm{E}}(\{t\})$ is trivial.

(2) Let $A_1 = \mathscr{H}_{\mathrm{E}}((a,b))$, and $f \in \mathscr{H}_{\mathrm{E}}([a,b])$. Then for $\lambda \in (1, \infty)$ we have $f_\lambda(x_0, x) = f(\lambda x_0 + (1-\lambda)(a+b)/2, x) \in A_1$, and $\|f_\lambda - f\|_{-1} \to 0$ as $\lambda \to 1$. Hence A_1 is dense in $\mathscr{H}_{\mathrm{E}}([a,b])$.

Let $A_2 = C^\infty(\mathbb{R}^d) \cap A_1$, and define $f_\varepsilon = \rho_\varepsilon * f$, where $\rho_\varepsilon(x) = \rho(x/\varepsilon)/\varepsilon^d$, $\rho \in C_0^\infty(\mathbb{R}^d)$, $\rho(x) \ge 0$, $\int \rho(x)dx = 1$ and $\operatorname{supp} \rho \subset \{x \in \mathbb{R}^d | |x| \le 1\}$. Then

$f_\varepsilon \in A_2$ for sufficiently small ε and $\|f_\varepsilon - f\|_{-1} = 0$ as $\varepsilon \to 0$. Thus A_2 is dense in $\mathscr{H}_{\mathrm{E}}([a,b])$.

Next we will show that $A_3 = C_0^\infty(\mathbb{R}^{d+1}) \cap A_1$ is dense in $\mathscr{H}_{\mathrm{E}}([a,b])$. In the case of $\nu > 0$ let

$$\chi_n(x) = \begin{cases} 1, & |x| \le n, \\ 0, & |x| > n+1, \end{cases} \qquad \chi_n \in C_0^\infty(\mathbb{R}^d) \text{ and } 0 \le \chi_n \le 1.$$

Then for $f \in A_2$ we have $f_n = \chi_n f \in A_2$ and $\|f_n - f\|_{-1} = 0$ as $n \to \infty$. Since A_2 is dense in A_3, A_3 is dense in $\mathscr{H}_{\mathrm{E}}([a,b])$. For the case of $\nu = 0$, let $A_4 = \{f \in C_0^\infty(\mathbb{R}^{d+1}) \cap A_1 \,|\, \operatorname{supp} \hat{f} \subset \mathbb{R}^{d+1} \setminus \{0\}\}$. It is similar to show that A_4 is dense in $\mathscr{H}_{\mathrm{E}}([a,b])$.

Finally, we show that any $f \in A_3$ can be represented as a limit of finite linear sums of vectors in $\operatorname{Ran}(e_t)$, $(a < t < b)$. Let $f \in A_3$ and define

$$f_n(x_0, x) = \sum_{j=-\infty}^{\infty} \frac{1}{n} \delta\left(x_0 - \frac{j}{n}\right) f\left(\frac{j}{n}, x\right).$$

Then

$$\hat{f}_n(k_0, k) = \sum_{j=-\infty}^{\infty} \frac{1}{n} g\left(\frac{j}{n}, k\right) \frac{e^{-ijk_0/n}}{\sqrt{2\pi}},$$

where $g(X,k) = (2\pi)^{-d/2} \int f(X,x) e^{-ikx} dx$. Note that

$$|\hat{f}_n(k_0,k)| \le c_N/(|k|^2+1)^N$$

for some N and $\hat{f}_n(k_0,k) \to \hat{f}(k_0,k)$ as $n \to \infty$ for every k. Hence $\|f_n - f\|_{-1} \to 0$ as $n \to \infty$ by the Lebesgue dominated convergence theorem. Then (2) with $\nu \neq 0$ follows. In the case of $\nu = 0$, we use A_3 replaced by A_4. □

Lemma 5.16. *Let $a \le b \le t \le c \le d$. Then*

(1) $e_a e_b e_c = e_a e_c$,

(2) $e_{[a,b]} e_t e_{[c,d]} = e_{[a,b]} e_{[c,d]}$,

(3) $e_c e_b e_a = e_c e_a$,

(4) $e_{[c,d]} e_t e_{[a,b]} = e_{[c,d]} e_{[a,b]}$.

Proof. Since

$$\begin{aligned} e_a e_b e_c &= \tau_a \tau_a^* \tau_b \tau_b^* \tau_c \tau_c^* = \tau_a e^{-|a-b|\hat{\omega}_\nu} e^{-|b-c|\hat{\omega}_\nu} \tau_c^* \\ &= \tau_a e^{-|a-c|\hat{\omega}_\nu} \tau_c^* = \tau_a \tau_a^* \tau_c \tau_c^* = e_a e_c, \end{aligned}$$

(1) readily follows. Take $f, g \in \mathscr{H}_{\mathrm{E}}$. By Lemma 5.15 (2) it is clear that

$$e_{[c,d]} f = \operatorname*{s-lim}_{n\to\infty} \sum_{\alpha=1}^{N_n} f_{n_\alpha}, \quad f_{n_\alpha} \in \mathrm{Ran}(e_{t_{n_\alpha}}), \quad t_{n_\alpha} \in [c,d],$$

$$e_{[a,b]} g = \operatorname*{s-lim}_{m\to\infty} \sum_{\beta=1}^{N_m} f_{m_\beta}, \quad f_{m_\beta} \in \mathrm{Ran}(e_{t_{m_\beta}}), \quad t_{m_\beta} \in [a,b].$$

Hence by (1) we have

$$\begin{aligned}(e_{[a,b]} e_t e_{[c,d]} f, g) &= \lim_{m,n\to\infty} \sum_{\alpha,\beta=1}^{N_n, M_m} (e_t f_{n_\alpha}, g_{m_\alpha}) \\ &= \lim_{m,n\to\infty} \sum_{\alpha,\beta=1}^{N_n, M_m} (f_{n_\alpha}, g_{m_\alpha}) = (e_{[a,b]} e_{[c,d]} f, g).\end{aligned}$$

This gives (2). (3) and (4) are similarly proven. □

Define

$$\mathrm{E}_t = \mathrm{I}_t \mathrm{I}_t^* = \Gamma_{\mathrm{E}}(e_t), \quad t \in \mathbb{R}, \tag{5.5.21}$$

and

$$\mathrm{E}_{\mathscr{O}} = \Gamma_{\mathrm{E}}(e_{\mathscr{O}}), \quad \mathscr{O} \subset \mathbb{R}. \tag{5.5.22}$$

Notice that $\mathrm{E}_{\{t\}} = \mathrm{E}_t$ for $t \in \mathbb{R}$ by Lemma 5.15. The following relationship holds between the projection $\mathrm{E}_{[a,b]}$ and the set of $\Sigma_{[a,b]}$-measurable functions.

Proposition 5.17. (1) $\mathrm{Ran}(\mathrm{E}_{[a,b]}) = \mathscr{E}_{[a,b]}$,

(2) $\mathrm{E}_{[a,b]}\mathrm{E}_t\mathrm{E}_{[c,d]} = \mathrm{E}_{[a,b]}\mathrm{E}_{[c,d]}$ *and* $\mathrm{E}_{[c,d]}\mathrm{E}_t\mathrm{E}_{[a,b]} = \mathrm{E}_{[c,d]}\mathrm{E}_{[a,b]}$ *hold for* $a \le b \le t \le c \le d$,

(3) $\mathrm{E}_{[a,b]}\mathrm{E}_{[c,d]} = \mathrm{E}_{[c,d]}\mathrm{E}_{[a,b]} = \mathrm{E}_{[a,b]}$ *if* $[a,b] \subset [c,d]$.

Proof. Let $\{g_n\}_{n=1}^\infty$ be a complete orthonormal system of $\mathscr{H}_{\mathrm{E}}([a,b])$. Then

$$\begin{aligned}&\mathrm{Ran}(\mathrm{E}_{[a,b]}) \\ &= \overline{\mathrm{L.H.}\{:\phi_{\mathrm{E}}(g_1)^{n_1}\cdots\phi_{\mathrm{E}}(g_k)^{n_k}: | g_j \in \{g_n\}_{n=1}^\infty, n_j \ge 0, j = 1,\dots,k, k \ge 0\}},\end{aligned}$$

which implies that $\mathrm{Ran}(\mathrm{E}_{[a,b]}) \subset \mathscr{E}_{[a,b]}$. Noticing that $:\exp(i\phi_{\mathrm{E}}(f)): \in \mathrm{Ran}(\mathrm{E}_{[a,b]})$, we obtain $\exp(i\phi_{\mathrm{E}}(f)) \in \mathrm{Ran}(\mathrm{E}_{[a,b]})$. Thus $F(\phi_{\mathrm{E}}(f_1),\dots,\phi_{\mathrm{E}}(f_n)) \in \mathrm{Ran}(\mathrm{E}_{[a,b]})$, for $F \in \mathscr{S}(\mathbb{R}^n)$ and $f_1,\dots,f_n \in \mathscr{H}_{\mathrm{E}}([a,b])$. Since

$$\{F(\phi_{\mathrm{E}}(f_1)\cdots\phi_{\mathrm{E}}(f_n)) | F \in \mathscr{S}_{\mathrm{real}}(\mathbb{R}^d)(\mathbb{R}^n), f_1,\dots,f_n \in \mathscr{H}_{\mathrm{E}}([a,b])\}$$

is dense in $\mathscr{E}_{[a,b]}$, we conclude that $\mathscr{E}_{[a,b]} \subset \mathrm{Ran}(\mathrm{E}_{[a,b]})$. Hence (1) is obtained. (2) follows from (2) and (4) of Lemma 5.16. Since a $\Sigma_{[a,b]}$-measurable function is also $\Sigma_{[c,d]}$-measurable, (3) follows. □

Remark 5.2. (1) of Lemma 5.17 implies that $\mathrm{E}_{[a,b]}$ is the projection to the set of $\Sigma_{[a,b]}$-measurable functions in $L^2(\mathscr{Q}_\mathrm{E})$. This is a conditional expectation, and we can write $\mathrm{E}_{[a,b]}F$ as $\mathbb{E}[F|\Sigma_{[a,b]}]$. We also denote $\mathrm{E}_t F$ by $\mathbb{E}[F|\Sigma_t]$. (2) of Lemma 5.17 is called *Markov property* of the family E_s, $s \in \mathbb{R}$.

Proposition 5.18 (Markov property). *Let $F \in \mathscr{E}_{s+t}$. Then*

$$\mathbb{E}[F|\Sigma_{(-\infty,s]}] = \mathbb{E}[F|\Sigma_s]. \tag{5.5.23}$$

Proof. It follows that $\mathbb{E}[F|\Sigma_{(-\infty,s]}] = \mathrm{E}_{(-\infty,s]}\mathrm{E}_{s+t}F = \mathrm{E}_{(-\infty,s]}\mathrm{E}_s\mathrm{E}_{s+t}F$ from the Markov property. Since $\mathrm{E}_{(-\infty,s]}\mathrm{E}_s = \mathrm{E}_{(-\infty,s]}\mathrm{E}_{\{s\}} = \mathrm{E}_{\{s\}} = \mathrm{E}_s$ by Proposition 5.17, we see that $\mathrm{E}_{(-\infty,s]}\mathrm{E}_s\mathrm{E}_{s+t}F = \mathrm{E}_s\mathrm{E}_{s+t}F = \mathrm{E}_sF = \mathbb{E}[F|\Sigma_s]$. Then the proposition follows. □

5.5.3 Feynman–Kac–Nelson formula

Consider the polynomial

$$P(X) = a_{2n}X^{2n} + a_{2n-1}X^{2n-1} + \cdots + a_1X + a_0 \tag{5.5.24}$$

with $a_{2n} > 0$. For $f \in \mathscr{H}_\mathrm{M}$ define

$$H_\mathrm{I} = {:}P(\phi(f)){:}. \tag{5.5.25}$$

Note that H_I is a polynomial of $\phi(f)$ of degree $2n$ and in the Schrödinger representation there exists $\alpha > 0$ such that $H_\mathrm{I} > -\alpha$. Define the self-adjoint operator

$$H_P = \hat{H}_\mathrm{f} \dot{+} H_\mathrm{I} \tag{5.5.26}$$

acting in $L^2(\mathscr{Q})$. As was already mentioned in this section, a key ingredient in constructing a functional integral representation of $(F, e^{-tH_P}G)$ is the Trotter product formula and the Markov property of the family E_s. Before going to state the functional integral representation of e^{-tH_P} we give a corollary to Lemma 5.17.

Corollary 5.19. *Let $F \in \mathscr{E}_{[a,b]}$, $G \in \mathscr{E}_{[c,d]}$ with $a \le b \le t \le c \le d$. Then $(F, E_tG) = (F, G)$.*

Proof. By Proposition 5.17 it is immediate that

$$(F, E_tG) = (\mathrm{E}_{[a,b]}F, \mathrm{E}_t\mathrm{E}_{[c,d]}G) = (\mathrm{E}_{[a,b]}F, \mathrm{E}_{[c,d]}G) = (F, G). \qquad \square$$

Now we can prove the functional integral representation of H_P given by (5.5.26).

Theorem 5.20 (Feynman–Kac–Nelson formula). *Let* $F, G \in L^2(\mathscr{Q})$. *Then*

$$(F, e^{-tH_P}G) = \int_{\mathscr{Q}_\mathrm{E}} \overline{F_0} G_t \exp\left(-\int_0^t :P(\phi_\mathrm{E}(\delta_s \otimes f)):ds\right) d\mu_\mathrm{E}, \tag{5.5.27}$$

where $F_0 = \mathrm{I}_t F$ *and* $G_t = \mathrm{I}_t G$.

Proof. By the Trotter product formula and the fact $e^{-|t-s|H_\mathrm{f}} = \mathrm{I}_t^* \mathrm{I}_s$, we have

$$\begin{aligned}(F, e^{-tH_P}G) &= \lim_{n\to\infty} (F, (e^{-(t/n)H_\mathrm{f}} e^{-(t/n)H_\mathrm{I}})^n G) \\ &= \lim_{n\to\infty} \left(F_0, \prod_{i=1}^n (\mathrm{I}_{ti/n} e^{-(t/n)H_\mathrm{I}} \mathrm{I}_{ti/n}^*) G_t\right) \\ &= \lim_{n\to\infty} \left(F_0, \prod_{i=1}^n (\mathrm{E}_{ti/n} R_i \mathrm{E}_{ti/n}) G_t\right),\end{aligned} \tag{5.5.28}$$

where $R_i = \exp(-(t/n):P(\phi_\mathrm{E}(\delta_{it/n} f)):)$ and we used that

$$\mathrm{I}_s \exp(-tH_\mathrm{I})\mathrm{I}_s^* = \mathrm{E}_s \exp(-t:P(\phi_\mathrm{E}(\delta_s \otimes f)):)\mathrm{E}_s \tag{5.5.29}$$

as operators. In fact, we can check that

$$\mathrm{I}_t \phi(f):\prod_{i=1}^n \phi(f_i): = \mathrm{I}_t \left\{:\phi(f) \prod_{i=1}^n \phi(f_i): + \frac{1}{2} \sum_{j=1}^n (f, f_j):\prod_{i\neq j} \phi(f_i):\right\}$$

$$= :\phi_\mathrm{E}(\delta_t \otimes f) \prod_{i=1}^n \phi_\mathrm{E}(\delta_t \otimes f_i): + \frac{1}{2} \sum_{j=1}^n (\delta_t \otimes f, \delta_t \otimes f_j):\prod_{i\neq j} \phi(\delta_t \otimes f_i):$$

$$= \phi_\mathrm{E}(\delta_t \otimes f)\mathrm{I}_t:\prod_{i=1}^n \phi(f_i):.$$

Thus $\mathrm{I}_t \phi(f) \mathrm{I}_t^* = \phi_\mathrm{E}(\delta_t \otimes f)\mathrm{E}_t = \mathrm{E}_t \phi_\mathrm{E}(\delta_t \otimes f)\mathrm{E}_t$. Inductively it follows that

$$\mathrm{I}_t\Big(\prod_{j=1}^n \phi(f_j)\Big)\mathrm{I}_t^* = \mathrm{E}_t\Big(\prod_{j=1}^n \phi_\mathrm{E}(\delta_t \otimes f_j)\Big)\mathrm{E}_t.$$

A limiting argument gives then (5.5.29). Notice that

$$\begin{aligned}&\left(F_0, \prod_{i=1}^n (\mathrm{E}_{ti/n} R_i \mathrm{E}_{ti/n}) G_t\right) \\ &= \Big(\underbrace{F_0}_{\in \mathscr{E}_{\{0\}}}, \mathrm{E}_{t/n}\, \underbrace{R_1 \mathrm{E}_{t/n}(\mathrm{E}_{2t/n} R_2 \mathrm{E}_{2t/n}) \cdots (\mathrm{E}_t R_n \mathrm{E}_t) G_t}_{\in \mathscr{E}_{[t/n,t]}}\Big).\end{aligned} \tag{5.5.30}$$

Then by Corollary 5.19 we can remove $\mathrm{E}_{t/n}$ in (5.5.30). Furthermore, note that

$$\Big(F_0, \prod_{i=1}^{n}(\mathrm{E}_{ti/n}R_i\mathrm{E}_{ti/n})G_t\Big) = \Big(\underbrace{R_1F_0}_{\in\mathscr{E}_{[0,t/n]}}, \mathrm{E}_{t/n}\underbrace{(\mathrm{E}_{2t/n}R_2\mathrm{E}_{2t/n})\cdots(\mathrm{E}_tR_n\mathrm{E}_t)G_t}_{\in\mathscr{E}_{[2t/n,t]}}\Big). \tag{5.5.31}$$

Similarly, $\mathrm{E}_{t/n}$ can be removed in (5.5.31), and since

$$\Big(F_0, \prod_{i=1}^{n}(\mathrm{E}_{ti/n}R_i\mathrm{E}_{ti/n})G_t\Big) = \Big(\underbrace{R_1F_0}_{\in\mathscr{E}_{[0,t/n]}}, \mathrm{E}_{2t/n}\underbrace{R_2\mathrm{E}_{2t/n}\cdots(\mathrm{E}_tR_n\mathrm{E}_t)G_t}_{\in\mathscr{E}_{[2t/n,t]}}\Big), \tag{5.5.32}$$

also $\mathrm{E}_{2t/n}$ can be removed. Recursively, we can delete all E_s in the rightmost side of (5.5.28). Thus

$$\begin{aligned}(F, e^{-tH_P}G) &= \lim_{n\to\infty}(F_0, R_1\cdots R_nG_t) \\ &= \lim_{n\to\infty}\Big(F_0, \exp\Big(-\sum_{j=1}^{n}(t/n){:}P(\phi_\mathrm{E}(\delta_{jt/n}\otimes f)){:}\Big)G_t\Big) \\ &= \int_{\mathscr{Q}_\mathrm{E}}\overline{F_0}G_t\exp\Big(-\int_0^t{:}P(\phi_\mathrm{E}(\delta_s\otimes f)){:}ds\Big)d\mu_\mathrm{E}. \qquad \square\end{aligned}$$

5.6 Infinite dimensional Ornstein–Uhlenbeck process

5.6.1 Abstract theory of measures on Hilbert spaces

In this section we consider an infinite dimensional Ornstein–Uhlenbeck (OU) process as a Hilbert space-valued random process $(\xi_s)_{s\in\mathbb{R}}$ on the set of continuous paths $C(\mathbb{R};\mathscr{M}_{-2})$.

As explained in Remark 5.1 it is not possible to construct a Gaussian measure on the Hilbert space $\mathcal{K}$ that the covariance form defines. Nevertheless, if the covariance is given by $(A^{-1}\cdot, A^{-1}\cdot)_\mathcal{K}$ for some self-adjoint operator A with a Hilbert–Schmidt inverse, i.e., $\mathrm{Tr}(A^{-2}) < \infty$, then there exists a Gaussian measure on an enlarged Hilbert space.

Definition 5.9 (Sazanov topology). Let $\mathcal{K}$ be a real separable Hilbert space. For a non-negative trace class operator B defines the seminorm

$$p_B(f) = \sqrt{(f, Bf)}. \tag{5.6.1}$$

The *Sazanov topology* on $\mathcal{K}$ is the locally convex topology defined by the family of seminorms $\{p_B \mid B : \text{non-negative trace class operator on } \mathcal{K}\}$.

Proposition 5.21 (Minlos–Sazanov). *Let $\mathcal{K}$ be a real separable Hilbert space. Suppose that $C : \mathcal{K} \to \mathbb{C}$ satisfies $\sum_{i,j=1}^n z_i \bar{z}_j C(f_i - f_j) \geq 0$ for all $z_i, z_j \in \mathbb{C}$ and $f_i, f_j \in \mathcal{K}$. Then there exists a finite Borel measure μ on $\mathcal{K}$ such that*

$$C(f) = \int_{\mathcal{K}} e^{i(\psi,f)_{\mathcal{K}}} d\mu(\psi), \tag{5.6.2}$$

if and only if $C(\cdot)$ is continuous in the Sazanov topology.

Example 5.4. Let $B \geq 0$ be a self-adjoint bounded operator and $C_B(f) = e^{-(f,Bf)}$. It is well known that C_B is continuous in the Sazanov topology if and only if B is trace-class. Therefore, whenever B is trace-class, there is a finite Borel measure μ_B on $\mathcal{K}$ such that $\int_{\mathcal{K}} e^{i(\psi,f)_{\mathcal{K}}} d\mu_B(\psi) = e^{-(f,Bf)_{\mathcal{K}}}$.

Let $A \geq 0$ be a self-adjoint operator on $\mathcal{K}$ such that A^{-1} is a Hilbert–Schmidt operator. Define

$$C^\infty(A) = \bigcap_{n=0}^{\infty} D(A^n) \tag{5.6.3}$$

and set $\|f\|_n = \|A^{n/2} f\|_{\mathcal{K}}$. Also, by the completion

$$\mathcal{K}_n = \overline{C^\infty(A)}^{\|\cdot\|_n} \tag{5.6.4}$$

defines a Hilbert space with the scalar product $(f,g)_n = (A^{n/2} f, A^{n/2} g)_{\mathcal{K}}$ for $f, g \in D(A^{n/2})$. Then we have the sequence of Hilbert spaces

$$\cdots \subset \mathcal{K}_{+2} \subset \mathcal{K}_{+1} \subset \mathcal{K}_0 \subset \mathcal{K}_{-1} \subset \mathcal{K}_{-2} \subset \cdots. \tag{5.6.5}$$

Under the identification $\mathcal{K}_0 \cong \mathcal{K}_0^*$ we can also make

$$\mathcal{K}_n^* \cong \mathcal{K}_{-n}, \quad n \geq 1. \tag{5.6.6}$$

Denote the dual pair between $\mathcal{K}_{-2}$ and $\mathcal{K}_{+2}$ by $\langle\langle \psi, f \rangle\rangle$. An immediate corollary of the Minlos–Sazanov theorem is the following.

Lemma 5.22. *Suppose that $L \geq 0$ is a bounded self-adjoint operator on $\mathcal{K}$ and $[A^{-1}, L] = 0$. Then there exists a Borel probability measure m_L on $(\mathcal{K}_{-2}, \mathscr{B}(\mathcal{K}_{-2}))$ such that*

$$\int_{\mathcal{K}_{-2}} e^{i\langle\langle \psi, f \rangle\rangle} dm_L(\psi) = e^{-\frac{1}{2}(f,Lf)_0}. \tag{5.6.7}$$

In particular, if $L = 1$, then there exists a probability measure m_1 on $\mathcal{K}_{-2}$ such that

$$\int_{\mathcal{K}_{-2}} e^{i\langle\langle \psi, f \rangle\rangle} dm_1(\psi) = e^{-\frac{1}{2}\|f\|_0^2} \tag{5.6.8}$$

Proof. We show that A^{-1} is a Hilbert–Schmidt operator also on $\mathcal{K}_{-2}$. To see this, note that for a complete orthonormal system $\{e_i\}_{i=1}^{\infty}$ of $\mathcal{K}$ consisting of eigenvectors of A, the set $\{Ae_i\}_{i=1}^{\infty}$ is also a complete orthonormal system of $\mathcal{K}_{-2}$, and we put $\tilde{e}_i = Ae_i$. We have

$$\sum_{i=1}^{\infty}(\tilde{e}_i, A^{-2}\tilde{e}_i)_{-2} = \sum_{i=1}^{\infty}(e_i, A^{-2}e_i)_0 = \operatorname{Tr}(A^{-2}) < \infty.$$

Hence A^{-1} is Hilbert–Schmidt operator on $\mathcal{K}_{-2}$. Moreover, since

$$\|Lf\|_{-2} = \|A^{-1}Lf\|_0 = \|LA^{-1}f\|_0 \le \|L\|\|f\|_{-2},$$

L is bounded on $\mathcal{K}_{-2}$, so that LA^{-1} is a Hilbert–Schmidt operator on $\mathcal{K}_{-2}$. Now by Proposition 5.21 and Example 5.4 there exists a probability measure m_L on $\mathcal{K}_{-2}$ such that

$$\int_{\mathcal{K}_{-2}} e^{i(\psi,f)_{-2}} dm_L(\psi) = e^{-\frac{1}{2}(f,LA^{-2}f)_{-2}}, \quad f \in C^{\infty}(A).$$

Inserting $A^2 f$ into f above we have

$$\int_{\mathcal{K}_{-2}} e^{i\rho(\psi,f)} dm_L(\psi) = e^{-\frac{1}{2}(f,Lf)_0}, \tag{5.6.9}$$

where we put

$$\rho(\psi, f) = (\psi, A^2 f)_{-2}, \quad f \in C^{\infty}(A), \quad \psi \in \mathcal{K}_{-2}. \tag{5.6.10}$$

Thus we have the bound $|\rho(\psi, f)| \le \|\psi\|_{-2}\|f\|_2$. From this bound we can extend $\rho(\psi, f)$ to $f \in \mathcal{K}_{+2}$ by

$$\rho(\psi, f) = \lim_{n\to\infty} \rho(\psi, f_n), \tag{5.6.11}$$

where $f_n \to f$ in $\mathcal{K}_{+2}$. $\rho(\psi, f)$ with $f \in \mathcal{K}_{+2}$ also satisfies the bound $|\rho(\psi, f)| \le \|\psi\|_{-2}\|f\|_2$, hence $\rho(\psi, f) = \langle\langle\psi, f\rangle\rangle$. Moreover, for $f \in \mathcal{K}_2$ there exists a sequence $f_n \in C^{\infty}(A)$ such that $f_n \to f$ in both $\mathcal{K}_0$ and $\mathcal{K}_2$ since A is closed and $C^{\infty}(A)$ is a core of A. Then (5.6.9) can be extended to $f \in \mathcal{K}_{+2}$ by a limiting argument, and the lemma follows. □

We can extend Lemma 5.22 to $f \in \mathcal{K}$.

Theorem 5.23 (Existence of a Gaussian measure on a Hilbert space). *Suppose that $L \ge 0$ is a bounded self-adjoint operator on $\mathcal{K}$ and $[A^{-1}, L] = 0$. Then there exists a Borel probability measure m_L on $(\mathcal{K}_{-2}, \mathscr{B}(\mathcal{K}_{-2}))$ and a family of Gaussian random variables $\psi(f)$ indexed by $f \in \mathcal{K}$ such that*

(1) $\mathcal{K} \ni f \mapsto \psi(f) \in \mathbb{R}$ *is linear*

(2) $\psi(f) = \langle\langle \psi, f \rangle\rangle$ *when* $f \in \mathcal{K}_{+2}$

(3) *it satisfies*

$$\int_{\mathcal{K}_{-2}} e^{i\psi(f)} dm_L(\psi) = e^{-\frac{1}{2}(f,Lf)_0}. \tag{5.6.12}$$

Proof. In the case of $f \in \mathcal{K}_{+2}$, we set $\psi(f) = \langle\langle \psi, f \rangle\rangle$. Then (5.6.12) follows from Lemma 5.22. In the case of $f \in \mathcal{K} \setminus \mathcal{K}_{+2}$, $\psi(f)$ can be defined through a simple limiting procedure. By (5.6.7) it can be seen that

$$\int_{\mathcal{K}_{-2}} \psi(f)^2 dm_L(\psi) = (f, Lf)_0, \quad f \in \mathcal{K}_{+2}. \tag{5.6.13}$$

Since for $f \in \mathcal{K}$ there exists a sequence $f_n \in \mathcal{K}_{+2}$ such that $f_n \to f$ strongly in $\mathcal{K}$, we can define $\psi(f)$ by

$$\psi(f) = \operatorname*{s-lim}_{n\to\infty} \psi(f_n) \tag{5.6.14}$$

in $L^2(\mathcal{K}_{-2}, dm_L)$. Then the theorem follows by a simple limiting argument. □

Remark 5.3. Not that $\mathcal{K}_{+2} \ni f \mapsto \psi(f) \in \mathbb{R}$ is continuous with $|\psi(f)| \le \|\psi\|_{-2}\|f\|_2$, but it is not as a map $\mathcal{K} \ni f \mapsto \psi(f) \in \mathbb{R}$.

5.6.2 Fock space as a function space

In order to construct a functional integral representation of a model in quantum field theory we define boson Fock space as an L^2-space over a function space with a Gaussian measure.

From now on we will work in the following setup.

(1) *Dispersion relation.* $\omega : \mathbb{R}^d \to \mathbb{R}$ is measurable with $0 \le \omega(k) = \omega(-k)$, and $\omega(k) = 0$ only on a set of Lebesgue-measure zero

(2) *Lorentz-covariant space.*

$$\mathscr{M} = \left\{ f \in L^2_{\mathrm{real}}(\mathbb{R}^d) \,\middle|\, \|f\|^2_{\mathscr{M}} = \int \frac{|\hat{f}(k)|^2}{2\omega(k)} dk < \infty \right\}. \tag{5.6.15}$$

We also define the scalar product on $\mathscr{M}$ by

$$(f, g)_{\mathscr{M}} = \int_{\mathbb{R}^d} \overline{\hat{f}(k)} \hat{g}(k) \frac{1}{2\omega(k)} dk.$$

(3) *Hilbert–Schmidt operator.* A given positive self-adjoint operator $\mathcal{D}$ with Hilbert–Schmidt inverse on $\mathscr{M}$ and such that $\sqrt{\omega}\mathcal{D}^{-1}$ is bounded on $\mathscr{M}$. Let $\mathrm{Spec}(\mathcal{D}) = \{\lambda_i\}_{i=1}^\infty$ and $\{e_i\}_{i=1}^\infty$ be the normalized eigenvectors such that $\mathcal{D}e_i = \lambda_i e_i$. Note that $\mathrm{Tr}(\mathcal{D}^{-2}) = \sum_{i=1}^\infty \lambda_i^{-2} < \infty$. Define

$$\mathscr{M}_n = \overline{C^\infty(\mathcal{D})}^{\|\mathcal{D}^{n/2}\cdot\|_{\mathscr{M}}}. \tag{5.6.16}$$

Remark 5.4. (1) Since the Lebesgue measure of the set $\{k \in \mathbb{R}^d \,|\, \omega(k) = 0\}$ is zero, the map $1/\sqrt{\omega} : f \mapsto f/\sqrt{\omega}$ is a self-adjoint multiplication operator on $L^2(\mathbb{R}^d)$. Hence the range of $1/\sqrt{\omega}$ is dense in $L^2(\mathbb{R}^d)$; $\{f/\sqrt{\omega} | f \in D(1/\sqrt{\omega})\}$ is dense in $L^2(\mathbb{R}^d)$.

(2) Let $\omega(k) = \sqrt{|k|^2 + \nu^2}$, with $\nu \geq 0$. The positive self-adjoint operator

$$T = \left(-\frac{1}{2}\Delta + \frac{1}{2}|k|^2 + \frac{1}{2}\nu^2\right)^{(d+1)/2}$$

is a specific case of $\mathcal{D}$ above, see Example 5.2. Indeed T^{-1} is a Hilbert–Schmidt operator and $\sqrt{\omega}T^{-1}$ is bounded since the operator norm has the bound $\|\sqrt{\omega}(-\Delta + |k|^2 + \nu^2)^{-(d+1)/2}\| \leq 1$.

(3) The condition that $\sqrt{\omega}\mathcal{D}^{-1}$ is bounded on $\mathscr{M}$ will be used in Lemma 5.30 below, where we will prove path continuity of the infinite dimensional Ornstein–Uhlenbeck process. Moreover this condition yields that the embedding

$$\iota : \mathscr{M} \ni f \mapsto f \in L^2(\mathbb{R}^d)$$

is bounded. It can be indeed seen that

$$\|f\|_{L^2(\mathbb{R}^d)} = \|\sqrt{2\omega}\mathcal{D}^{-1}\mathcal{D}f\|_{L^2(\mathbb{R}^d)} \leq \|\sqrt{2\omega}\mathcal{D}\|\|f\|_{\mathscr{M}_{-2}}. \tag{5.6.17}$$

We identify $\mathscr{M}_{+2}$ with the topological dual of $\mathscr{M}_{-2}$; $\mathscr{M}_{-2}^* \cong \mathscr{M}_{+2}$, and the pairing between $\mathscr{M}_{+2}$ and $\mathscr{M}_{-2}$ is denoted by $\langle\langle\cdot,\cdot\rangle\rangle$.

Theorem 5.24 (Gaussian measure G on $\mathscr{M}_{-2}$). *Under the above set-up there exists a probability measure G on $\mathscr{M}_{-2}$ and a family of Gaussian random variables $\xi(f)$ indexed by $f \in \mathscr{M}$ such that*

(1) $\mathscr{M} \ni f \mapsto \xi(f) \in \mathbb{R}$ *is linear*

(2) $\xi(f) = \langle\langle\xi, f\rangle\rangle$ *when* $f \in \mathscr{M}_{+2}$

(3) *it follows that*

$$\int_{\mathscr{M}_{-2}} e^{i\xi(f)} d\mathsf{G} = e^{-\frac{1}{2}\|f\|_{\mathscr{M}}^2}. \tag{5.6.18}$$

Proof. This is due to Theorem 5.23 with $A = \mathscr{D}$, $L = 1$, $\mathcal{K} = \mathscr{M}$, $\psi(f) = \xi(f)$ and $m_L = \mathsf{G}$. □

Now we discuss the unitary equivalence of the boson Fock space $\mathscr{F}_{\mathrm{b}}(L^2(\mathbb{R}^d))$ and $L^2(\mathscr{M}_{-2}, d\mathsf{G})$. Let

$$\phi(f) = \int \frac{1}{\sqrt{2\omega(k)}}(a^*(k)\hat{f}(k) + a(k)\hat{f}(-k))dk, \quad f \in \mathscr{M},$$

be a scalar field in $\mathscr{F}_{\mathrm{b}}$. Recall that Ω_{b} denotes the Fock vacuum. We see that

$$(\phi(f)\Omega_{\mathrm{b}}, \phi(g)\Omega_{\mathrm{b}})_{\mathscr{F}_{\mathrm{b}}} = \int_{\mathbb{R}^d} \overline{\hat{f}(k)}\hat{g}(k)\frac{1}{2\omega(k)}dk = (f,g)_{\mathscr{M}} = \mathbb{E}_{\mathsf{G}}[\xi(f)\xi(g)]. \tag{5.6.19}$$

Denote the complexification of $\mathscr{M}$ by $\mathscr{M}_{\mathbb{C}}$. For $f \in \mathscr{M}_{\mathbb{C}}$, $f = g + ih$ with $g, h \in \mathscr{M}$, define

$$\xi(f) = \xi(g) + i\xi(h).$$

The Wick product of $\prod_{j=1}^n \xi(f_j)$ is inductively defined by

$$\begin{aligned} &{:}\xi(f){:} = \xi(f), \\ &{:}\xi(f)\prod_{j=1}^n \xi(f_j){:} = \xi(f){:}\prod_{j=1}^n \xi(f_j){:} - \frac{1}{2}\sum_{i=1}^n (f, f_i)_{\mathscr{M}_{\mathbb{C}}}{:}\prod_{j\neq i}^n \xi(f_j){:}. \end{aligned}$$

Proposition 5.25 (Unitary equivalence). *There exists a unitary operator $\theta_{\mathrm{W}} : \mathscr{F}_{\mathrm{b}} \to L^2(\mathscr{M}_{-2}, d\mathsf{G})$ such that*

(1) $\theta_{\mathrm{W}}\Omega_{\mathrm{b}} = 1$,

(2) $\theta_{\mathrm{W}}{:}\prod_{i=1}^n \phi(f_i){:}\Omega_{\mathrm{b}} = {:}\prod_{i=1}^n \xi(f_i){:}$,

(3) $\theta_{\mathrm{W}}^{-1}\xi(f)\theta_{\mathrm{W}} = \phi(f)$ *for* $f \in \mathscr{M}$ *as an operator.*

Proof. The proof is similar to that of Proposition 5.7. Define the linear operator $\theta_{\mathrm{W}} : \mathscr{F}_{\mathrm{b}} \to L^2(\mathscr{M}_{-2}, d\mathsf{G})$ by

$$\theta_{\mathrm{W}}\Omega_{\mathrm{b}} = 1, \quad \theta_{\mathrm{W}}{:}\prod_{i=1}^n \phi(f_i){:}\Omega_{\mathrm{b}} = {:}\prod_{i=1}^n \xi(f_i){:}. \tag{5.6.20}$$

By the commutation relations it follows that

$$\Big\|{:}\prod_{i=1}^n \phi(f_i){:}\Omega_{\mathrm{b}}\Big\|_{\mathscr{F}_{\mathrm{b}}} = \Big\|{:}\prod_{i=1}^n \xi(f_i){:}\Big\|_{L^2(\mathscr{M}_{-2}, d\mathsf{G})}$$

and the linear hull of ${:}\prod_{i=1}^n \phi(f_i){:}\Omega_{\mathrm{b}}$ and ${:}\prod_{i=1}^n \xi(f_i){:}$ are dense in $\mathscr{F}_{\mathrm{b}}$ and $L^2(\mathscr{M}_{-2}, d\mathsf{G})$, respectively. Then θ_{W} can be extended to a unitary operator and the proposition follows. □

Definition 5.10 (Free field Hamiltonian). The free field Hamiltonian $\tilde{H}_{\mathrm{f}}$ is defined on $L^2(\mathscr{M}_{-2}, d\mathsf{G})$ by

$$\theta_{\mathrm{W}} H_{\mathrm{f}} \theta_{\mathrm{W}}^{-1}, \tag{5.6.21}$$

where H_{f} is the free Hamiltonian of $\mathscr{F}_{\mathrm{b}}$.

In the previous section we defined the self-adjoint operator $H_P = \hat{H}_{\mathrm{f}} + :P(\phi(f)):$ in $L^2(\mathscr{Q})$ and have given the functional integral representation of the semigroup e^{-tH_P} by making use of Euclidean fields. The operator

$$\tilde{H}_P = \tilde{H}_{\mathrm{f}} + :P(\xi(f)): \tag{5.6.22}$$

is the operator associated with H_P in $L^2(\mathscr{M}_{-2}, d\mathsf{G})$, $\theta_{\mathrm{W}} H_P \theta_{\mathrm{W}}^{-1} = \tilde{H}_P$. We can also construct a functional integral representation of $e^{-t\tilde{H}_P}$ in terms of an infinite dimensional Ornstein–Uhlenbeck process which will be investigated next.

5.6.3 Infinite dimensional Ornstein–Uhlenbeck-process

The one-dimensional Ornstein–Uhlenbeck process has been discussed in Section 3.10 before, and we have seen that it is a stationary Gaussian Markov process defined on the probability space $\mathscr{X} = C(\mathbb{R}, \mathbb{R})$ with a measure characterized by having mean and covariance given by (3.10.59). The construction of an infinite dimensional Ornstein–Uhlenbeck process is similar to this.

Denote the set of $\mathscr{M}_{-2}$-valued continuous functions on the real line $\mathbb{R}$ by

$$\mathfrak{Y} = C(\mathbb{R}; \mathscr{M}_{-2}). \tag{5.6.23}$$

On $\mathfrak{Y}$ we introduce the topology derived from the metric

$$\mathrm{dist}(f, g) = \sum_{k \geq 0} \frac{(\sup_{0 \leq |x| \leq k} \|f(x) - g(x)\|_{\mathscr{M}_{-2}}) \wedge 1}{2^k}. \tag{5.6.24}$$

This metric induces the locally uniform topology, giving rise to the Borel σ-field $\mathcal{B}(\mathfrak{Y})$.

The main theorem in this section is the following.

Theorem 5.26. (1) *There exists a probability measure $\mathscr{G}$ on $(\mathfrak{Y}, \mathscr{B}(\mathfrak{Y}))$ and an $\mathscr{M}_{-2}$-valued continuous stochastic process $\mathbb{R} \ni s \mapsto \xi_s \in \mathscr{M}_{-2}$ on the probability space $(\mathfrak{Y}, \mathscr{B}(\mathfrak{Y}), \mathscr{G})$ such that*

$$\mathbb{E}_{\mathscr{G}}[(\xi_s, \xi_t)_{\mathscr{M}_{-2}}] = \frac{1}{2} \sum_{i,j=1}^{\infty} \frac{1}{\lambda_i \lambda_j} (e_i, e^{-|t-s|\hat{\omega}} e_j)_{\mathscr{M}}, \tag{5.6.25}$$

where $\mathrm{Spec}(\mathfrak{D}) = \{\lambda_j\}$ *and* $\{e_j\}$ *is the normalized eigenvector such that* $\mathfrak{D} e_j = \lambda_j e_j$.

(2) *There exists a random process* $(\xi_s(f))_{s\in\mathbb{R}}$ *indexed by* $f \in \mathscr{M}$ *on the probability space* $(\mathfrak{Y}, \mathscr{B}(\mathfrak{Y}), \mathscr{G})$ *such that*

(a) *linearity:* $\mathscr{M} \ni f \mapsto \xi_s(f) \in \mathbb{R}$ *is linear*

(b) *boundedness:* $\xi_s(f) = \langle\langle \xi_s, f\rangle\rangle$ *for* $f \in \mathscr{M}_{+2}$*, in particular,* $|\xi_s(f)| \le \|\xi_s\|_{\mathscr{M}_{-2}}\|f\|_{\mathscr{M}_{+2}}$ *for* $f \in \mathscr{M}_{+2}$

(c) *path continuity:*

(a) *if* $f \in \mathscr{M}_{+2}$*, then* $\mathbb{R} \ni s \mapsto \xi_s(f) \in \mathbb{R}$ *is continuous almost surely*

(b) *if* $f \in \mathscr{M}$*, then* $\mathbb{R} \ni s \mapsto \xi_s(f) \in L^2(\mathscr{E}_{-2}, d\mathscr{G})$ *is strongly continuous*

(d) *characteristic function:* $\xi_s(f)$ *is a Gaussian random process with respect to* $\mathscr{G}$ *with mean zero and*

$$\mathbb{E}_{\mathscr{G}}[e^{i\sum_{j=1}^n \xi_{s_j}(f_j)}] = \exp\Big(-\frac{1}{2}\sum_{i,j=1}^n (f_i, e^{-|s_i-s_j|\hat{\omega}} f_j)_{\mathscr{M}}\Big). \tag{5.6.26}$$

We will prove Theorem 5.26 below.

Definition 5.11 (Infinite dimensional Ornstein–Uhlenbeck process). The $\mathscr{M}_{-2}$-valued stochastic process $(\xi_s)_{s\in\mathbb{R}}$ on the probability space $(\mathfrak{Y}, \mathscr{B}(\mathfrak{Y}), \mathscr{G})$ given in Theorem 5.26 is called *infinite dimensional Ornstein–Uhlenbeck process*.

In order to construct $(\xi_s)_{s\in\mathbb{R}}$ we introduce the Euclidean version of $\mathscr{M}$, $\mathscr{E}$, and a self-adjoint operator D. Let

$$\mathscr{E} = \Big\{ f \in L^2_{\text{real}}(\mathbb{R}^{d+1}) \Big| \|f\|^2_{\mathscr{E}} = \int \frac{|\hat{f}(\mathbf{k})|^2}{\omega(k)^2 + \kappa^2} d\mathbf{k} < \infty \Big\}, \tag{5.6.27}$$

where $\mathbf{k} = (k, \kappa) \in \mathbb{R}^d \times \mathbb{R}$. Let D be a positive self-adjoint operator with Hilbert–Schmidt inverse on $\mathscr{E}$, and define $\mathscr{E}_n = \overline{C^\infty(D)}^{\|D^{n/2}\cdot\|_{\mathscr{E}}}$. The next lemma follows by Theorem 5.23.

Lemma 5.27. *There exists a Borel probability measure* γ *on* $(\mathscr{E}_{-2}, \mathscr{B}(\mathscr{E}_{-2}))$ *and a family of Gaussian random variables* $\tilde{\xi}(f)$ *indexed by* $f \in \mathscr{E}$ *with mean zero such that*

(1) $\mathscr{E} \ni f \mapsto \tilde{\xi}(f) \in \mathbb{R}$ *is linear*

(2) $\tilde{\xi}(f) = \langle\langle \tilde{\xi}, f\rangle\rangle_{\mathscr{E}}$ *when* $f \in \mathscr{E}_{+2}$*, where* $\langle\langle \tilde{\xi}, f\rangle\rangle_{\mathscr{E}}$ *denotes the pairing between* $\mathscr{E}_{+2}$ *and* $\mathscr{E}_{-2}$

(3) *it satisfies*

$$\int_{\mathscr{E}_{-2}} e^{i\tilde{\xi}(f)} d\gamma(\tilde{\xi}) = e^{-\frac{1}{2}\|f\|^2_{\mathscr{E}}}. \tag{5.6.28}$$

Take $g \in \mathscr{M}$. We have already checked in (5.5.8) that $\delta_t \otimes g \in \mathscr{E}$ and

$$(\delta_s \otimes f, \delta_t \otimes g)_{\mathscr{E}} = (f, e^{-|t-s|\hat{\omega}} g)_{\mathscr{M}} \tag{5.6.29}$$

for all $f, g \in K$. For every $t \in \mathbb{R}$ and $g \in \mathscr{M}$ put

$$\tilde{\xi}_t(g) = \tilde{\xi}(\delta_t \otimes g). \tag{5.6.30}$$

In particular, it follows that

$$\int_{\mathscr{E}_{-2}} \tilde{\xi}_s(f) d\gamma(\tilde{\xi}) = 0, \tag{5.6.31}$$

$$\int_{\mathscr{E}_{-2}} \tilde{\xi}_s(f)\tilde{\xi}_t(g) d\gamma(\tilde{\xi}) = (f, e^{-|t-s|\hat{\omega}} g)_{\mathscr{M}}. \tag{5.6.32}$$

Lemma 5.28. *For every fixed $t_1 < t_2 < \cdots < t_n \in \mathbb{R}$ there exists a Gaussian measure $\mathscr{G}_{t_1,\dots,t_n}$ on $\mathscr{M}_{-2}^n = \times_{k=1}^n \mathscr{M}_{-2}$ and a family of random variables $\tilde{\xi}_{t_1,\dots,t_n}(g)$ indexed by $g \in \mathscr{M}^n$ such that*

(1) *$g \ni \mathscr{M}^n \mapsto \tilde{\xi}_{t_1,\dots,t_n}(g) \in \mathbb{R}$ is linear*

(2) *$\tilde{\xi}_{t_1,\dots,t_n}(g) = \langle\langle \tilde{\xi}_{t_1,\dots,t_n}, g \rangle\rangle$ when $g \in \mathscr{M}_{+2}^n$, where $\langle\langle \cdot, \cdot \rangle\rangle$ denotes the pairing between $\mathscr{M}_{+2}^n$ and $\mathscr{M}_{-2}^n$*

(3)
$$\int_{\mathscr{M}_{-2}^n} e^{i\tilde{\xi}_{t_1,\dots,t_n}(g)} d\mathscr{G}_{t_1,\dots,t_n} = \int_{\mathscr{E}_{-2}} e^{i\sum_{j=1}^n \tilde{\xi}_{t_j}(g_j)} d\gamma(\tilde{\xi}) \tag{5.6.33}$$

for all $g = (g_1, \dots, g_n) \in \mathscr{M}_{+2}^n$.

Proof. By the definition of $\tilde{\xi}_t$ we have

$$\int_{\mathscr{E}_{-2}} e^{i\sum_{j=1}^n \tilde{\xi}_{t_j}(g_j)} d\gamma(\tilde{\xi}) = \exp\left(-\frac{1}{2} \sum_{j,l=1}^n (\delta_{t_j} \otimes g_j, \delta_{t_l} \otimes g_l)_{\mathscr{E}} \right). \tag{5.6.34}$$

The quadratic form

$$(f, g) \mapsto Q_{t_1,\dots,t_n}(f, g) = \sum_{j,l=1}^n (\delta_{t_j} \otimes f_j, \delta_{t_l} \otimes g_l)_{\mathscr{E}}$$

on $\mathscr{M}^n \times \mathscr{M}^n$ satisfies $|Q_{t_1,\dots,t_n}(f,g)| \le \|f\|_{\mathscr{M}^n} \|g\|_{\mathscr{M}^n}$. Thus there exists a bounded operator $L_{t_1,\dots,t_n} = L$ on $\mathscr{M}^n$ such that $Q_{t_1,\dots,t_n}(f,g) = (f, Lg)_{\mathscr{M}^n}$. Hence we can write

$$\int_{\mathscr{E}_{-2}} e^{i\sum_{j=1}^n \tilde{\xi}_{t_j}(g_j)} d\gamma(\tilde{\xi}) = e^{-\frac{1}{2}(g, Lg)_{\mathscr{M}^n}}. \tag{5.6.35}$$

Let $\mathscr{D}_n = \oplus^n \mathscr{D}$. Notice that

$$
\begin{aligned}
(f, L\mathscr{D}_n^{-1} g) &= \sum_{i,j=1}^{n} (\delta_{t_j} \otimes f_j, \delta_{t_l} \otimes \mathscr{D}^{-1} g_l)_{\mathscr{E}} = \sum_{i,j=1}^{n} (\delta_{t_j} \otimes \mathscr{D}^{-1} f_j, \delta_{t_l} \otimes g_l)_{\mathscr{E}} \\
&= (\mathscr{D}_n^{-1} f, Lg) = (f, \mathscr{D}_n^{-1} Lg).
\end{aligned}
$$

Thus $[L, \mathscr{D}_n^{-1}] = 0$. By Theorem 5.23 there exists a measure $\mathscr{G}_{t_1,\dots,t_n}$ and a random variable $\tilde{\xi}_{t_1,\dots,t_n}(g)$ such that

$$
\int_{\mathscr{M}_{-2}^n} e^{i\tilde{\xi}_{t_1,\dots,t_n}(g)} d\mathscr{G}_{t_1,\dots,t_n} = e^{-\frac{1}{2}(g,Lg)_{\mathscr{M}^n}},
$$

and hence (5.6.33) is obtained. □

Let $J = \mathbb{R}\times\mathbb{N}$ be the index set and $\Lambda \subset J$ be such that $|\Lambda| < \infty$, where $|\Lambda| = \#\Lambda$. Write $\mathscr{M}_{-2}^{\Lambda} = \{f : \Lambda \to \mathscr{M}_{-2}\}$, let $\pi_\Lambda : \mathscr{M}_{-2}^{J} \to \mathscr{M}_{-2}^{\Lambda}$ be the projection defined by $\pi_\Lambda f = f\lceil_\Lambda$, and define

$$
\mathscr{A} = \{\pi_\Lambda^{-1}(E) \in \mathscr{M}_{-2}^{J} \mid \Lambda \subset J, |\Lambda| < \infty, E \in \mathscr{B}(\mathscr{M}_{-2}^{|J|})\}. \tag{5.6.36}
$$

Now we construct a probability measure $\mathscr{G}$ on $(\mathscr{M}_{-2}^{J}, \sigma(\mathscr{A}))$ by using the Kolmogorov extension theorem.

Lemma 5.29. *There exists a probability measure $\mathscr{G}$ on $(\mathscr{M}_{-2}^{J}, \sigma(\mathscr{A}))$, an $\mathscr{M}_{-2}$-valued stochastic process $(\xi_t)_{t\in\mathbb{R}}$, and an $\mathbb{R}$-valued stochastic process $(\xi_s(f))_{s\in\mathbb{R}}$ with $f \in \mathscr{M}$ on $(\mathscr{M}_{-2}^{J}, \sigma(\mathscr{A}), \mathscr{G})$ such that*

(1) *its covariance is*

$$
\mathbb{E}_{\mathscr{G}}[(\xi_s, \xi_t)_{\mathscr{M}_{-2}}] = \frac{1}{2}\sum_{j=1}^{\infty} \frac{1}{\lambda_j^2}(e_j, e^{-|t-s|\hat{\omega}} e_j)_{\mathscr{M}}, \tag{5.6.37}
$$

where $\{\lambda_j\}$ and $\{e_j\}$ are given in Section 5.6.2. *In particular,*

$$
\mathbb{E}_{\mathscr{G}}[\|\xi_t\|_{\mathscr{M}_{-2}}^2] = \frac{1}{2}\mathrm{Tr}(\mathscr{D}^{-2}). \tag{5.6.38}
$$

(2) *$\mathscr{M} \ni f \mapsto \xi_s(f) \in \mathbb{R}$ is linear, and $\xi_s(f) = \langle\langle \xi_s, f\rangle\rangle$ whenever $f \in \mathscr{M}_{-2}$*

(3) *$\xi_s(f)$ is a Gaussian random process with zero mean and*

$$
\mathbb{E}_{\mathscr{G}}[e^{i\sum_{j=1}^{n}\xi_{s_j}(f_j)}] = \exp\left(-\frac{1}{2}\sum_{i,j=1}^{n}(f_i, e^{-|s_i-s_j|\hat{\omega}} f_j)_{\mathscr{M}}\right). \tag{5.6.39}
$$

Proof. Write $\tilde{e}_j = \mathscr{D}^{-1}e_j = \lambda_j^{-1}e_j$. Then $\{\tilde{e}_j\}_{j=1}^{\infty}$ forms a complete orthonormal system in $\mathscr{M}_{+2}$. For every $k_1, \ldots, k_n \in J$ with $k_j = (t_j, i_j)$, and $E_1 \times \cdots \times E_n \in \mathscr{B}(\mathbb{R}^n)$ put

$$\mu_{k_1,\ldots,k_n}(E_1 \times \cdots \times E_n) = \mathbb{E}_\gamma\Big[\prod_{j=1}^n 1_{E_j}(\tilde{\xi}_{t_j}(\tilde{e}_{i_j}))\Big]. \tag{5.6.40}$$

By Lemma 5.28 there exists a probability measure $\mathscr{G}_{t_1,\ldots,t_n}$ on $\mathscr{M}_{-2}^n$ such that

$$\mu_{k_1,\ldots,k_n}(E_1 \times \cdots \times E_n) = \int_{\mathscr{M}_{-2}^n} \prod_{j=1}^n 1_{E_j}(\tilde{\xi}_{i_j})d\mathscr{G}_{t_1,\ldots,t_n}, \tag{5.6.41}$$

where $\tilde{\xi}_{i_j} = \tilde{\xi}_{t_1,\ldots,t_n}(\underbrace{0 \oplus \cdots \overset{j}{\tilde{e}_{i_j}} \cdots \oplus 0}_{n})$. Since we can see that the family of set functions $\{\mu_\Lambda\}_{\Lambda\subset J,|\Lambda|<\infty}$ satisfies the consistency condition (5.6.40), by the Kolmogorov extension theorem there exists a probability measure $\mathscr{G}$ on $(\mathscr{M}_{-2}^J, \sigma(\mathscr{A}))$ such that

$$\mathscr{G}(\pi_\Lambda^{-1}(E_1 \times \cdots \times E_n)) = \mu_{k_1,\ldots,k_n}(E_1 \times \cdots \times E_n) \tag{5.6.42}$$

for $\Lambda = \{k_1, \ldots, k_n\} \subset J$. Let $(\xi_{s,j})_{(s,j)\in J} \in \mathbb{R}^J$. Then (5.6.42) is rewritten as

$$\mathbb{E}_{\mathscr{G}}\Big[\prod_{j=1}^n 1_{E_j}(\xi_{t_j,i_j})\Big] = \mathbb{E}_\gamma\Big[\prod_{j=1}^n 1_{E_j}(\tilde{\xi}_{t_j}(\tilde{e}_{i_j}))\Big]. \tag{5.6.43}$$

By a limiting argument we have for bounded Borel measurable functions F on $\mathbb{R}^n$ that

$$\mathbb{E}_{\mathscr{G}}[F(\xi_{s_1,i_1}, \ldots, \xi_{s_n,i_n})] = \mathbb{E}_\gamma[F(\tilde{\xi}_{s_1}(\tilde{e}_{i_1}), \ldots, \tilde{\xi}_{s_n}(\tilde{e}_{i_n}))]. \tag{5.6.44}$$

Similarly, by taking limits we arrive at

$$\mathbb{E}_{\mathscr{G}}[\xi_{s,i}] = \mathbb{E}_\gamma[\tilde{\xi}_s(\tilde{e}_i)] = 0, \tag{5.6.45}$$

$$\mathbb{E}_{\mathscr{G}}[\xi_{s,i}\xi_{t,j}] = \mathbb{E}_\gamma[\tilde{\xi}_s(\tilde{e}_i)\tilde{\xi}_t(\tilde{e}_j)] = \frac{1}{2}\frac{1}{\lambda_i\lambda_j}(e_i, e^{-|t-s|\hat{\omega}}e_j)_{\mathscr{M}}. \tag{5.6.46}$$

Let $b_j = \mathscr{D}e_j$, so that $\{b_j\}_{j=1}^{\infty}$ is a complete orthonormal system in $\mathscr{M}_{-2}$. Define the vector

$$\xi_s = \sum_{j=1}^{\infty} \xi_{s,j}b_j \in \mathscr{M}_{-2}. \tag{5.6.47}$$

We can check that $\|\xi_s\|_{\mathscr{M}_{-2}} < \infty$ almost surely with respect to $\mathscr{G}$ for every s since

$$\mathbb{E}_{\mathscr{G}}[\|\xi_s\|^2_{\mathscr{M}_{-2}}] = \mathbb{E}_{\mathscr{G}}\Big[\sum_{j=1}^{\infty} \xi_{s,j}^2\Big] = \frac{1}{2}\mathrm{Tr}(\mathscr{D}^{-2}) < \infty. \tag{5.6.48}$$

Define $\xi_s(f) = \langle\!\langle \xi_s, f\rangle\!\rangle$ for $f \in \mathscr{M}_{+2}$. Thus

$$\xi_s(f) = \sum_{j=1}^{\infty} \xi_{s,j}\langle\!\langle b_j, f\rangle\!\rangle, \quad f \in \mathscr{M}_{+2}$$

and

$$\mathbb{E}_{\mathscr{G}}[\xi_s(f)] = \lim_{n\to\infty} \sum_{j=1}^{n} \mathbb{E}_{\mathscr{G}}[\xi_{s,j}\langle\!\langle b_j, f\rangle\!\rangle] = 0. \tag{5.6.49}$$

We prove next (3). By (5.6.44) and a limiting argument we have for $f_j \in \mathscr{M}_{+2}$

$$\begin{aligned}\mathbb{E}_{\mathscr{G}}[e^{i\sum_{j=1}^n \xi_{s_j}(f_j)}] &= \mathbb{E}_{\gamma}[e^{i\sum_{j=1}^n \tilde{\xi}_{s_j}(f_j)}] \\ &= \exp\Big(-\frac{1}{2}\sum_{i,j=1}^{n} (f_i, e^{-|s_i-s_j|\hat{\omega}} f_j)_{\mathscr{M}}\Big).\end{aligned} \tag{5.6.50}$$

Now we extend $\xi_s(f)$ to $f \in \mathscr{M}$, which is a routine argument. By (5.6.50),

$$\mathbb{E}_{\mathscr{G}}[\xi_s(f)\xi_t(g)] = (f, e^{-|t-s|\hat{\omega}} g)_{\mathscr{M}}.$$

In particular, $\mathbb{E}_{\mathscr{G}}[\xi_s(f)^2] = \|f\|^2_{\mathscr{M}}$. Thus $\xi_s(f)$ for $f \in \mathscr{M}$ is defined by

$$\xi_s(f) = \operatorname*{s-lim}_{n\to\infty} \xi_s(f_n) \tag{5.6.51}$$

in $L^2(\mathscr{M}_{-2}^J; d\mathscr{G})$ for $f_n \in \mathscr{M}_{+2}$ such that $f_n \to f$ in $\mathscr{M}$. Then it is not difficult to check that (3) is satisfied for $\xi_s(f)$ with $f \in \mathscr{M}$. □

The last point is to establish continuity of sample paths ξ_s for $\mathscr{G}$.

Lemma 5.30. *There exists a continuous version of* $(\xi_t)_{t\in\mathbb{R}}$.

Proof. Using the Kolmogorov–Čentsov theorem, we only need to show that

$$\mathbb{E}_{\mathscr{G}}[\|\xi_s - \xi_t\|^4_{\mathscr{M}_{-2}}] \le C|t-s|^2 \tag{5.6.52}$$

for some $C > 0$. A direct computation gives

$$\begin{aligned}\mathbb{E}_{\mathscr{G}}[|\xi_t(g) - \xi_s(g)|^4] &= \mathbb{E}_{\gamma}[|\tilde{\xi}_t(g) - \tilde{\xi}_s(g)|^4] = 12(g, (1-e^{-|t-s|\omega})g)^2_{\mathscr{M}} \\ &= 3\Big(\int \frac{1-e^{-|t-s|\omega(k)}}{\omega(k)}|\hat{g}(k)|^2 dk\Big)^2 \le 3|t-s|^2\|g\|^4_{L^2(\mathbb{R}^d)}.\end{aligned}$$

Notice that the embedding $\iota : \mathscr{M}_{+2} \to L^2(\mathbb{R}^d)$ is bounded with the bound $\|g\|_{L^2(\mathbb{R}^d)} \leq \|\sqrt{2\omega}\mathcal{D}^{-1}\| \|g\|_{\mathscr{M}_{+2}}$. From this it follows that

$$\mathbb{E}_{\mathscr{G}}[|\xi_s(g) - \xi_t(g)|^4] \leq C\|g\|^4_{\mathscr{M}_{+2}}$$

with a constant $C > 0$. Since

$$\|\xi_s\|_{\mathscr{M}_{-2}} = \sup_{f \neq 0, f \in \mathscr{M}_{+2}} \frac{|\xi_s(f)|}{\|f\|_{\mathscr{M}_{+2}}},$$

there exists $f_\varepsilon \in \mathscr{M}_{+2}$ such that

$$\|\xi_s - \xi_t\|_{\mathscr{M}_{-2}} - \varepsilon = \frac{|\xi_s(f_\varepsilon) - \xi(f_\varepsilon)|}{\|f_\varepsilon\|_{\mathscr{M}_{+2}}},$$

we obtain that

$$\mathbb{E}_{\mathscr{G}}[\|\xi_s - \xi_t\|^4_{\mathscr{M}_{-2}}] \leq \varepsilon + C|t-s|^2 \tag{5.6.53}$$

for any $\varepsilon > 0$. Hence (5.6.52) follows. □

Proof of Theorem 5.26. We denote the continuous version of $(\xi_t)_{t \in \mathbb{R}}$ by $\bar{\xi} = (\bar{\xi}_t)_{t \in \mathbb{R}}$, and the image measure $\mathscr{G} \circ \bar{\xi}^{-1}$ on $\mathfrak{Y}$ by the same symbol $\mathscr{G}$. Thus we have constructed above the probability space $(\mathfrak{Y}, \mathscr{B}(\mathfrak{Y}), \mathscr{G})$. (1) and (2) of the theorem follow from Lemma 5.29 except for path continuity, which we show now. Whenever $f \in \mathscr{M}_{+2}$ we see that $\xi_s(f) = \langle\!\langle \xi_s, f \rangle\!\rangle$. Then $s \mapsto \xi_s(f)$ is continuous almost surely since $\mathbb{R} \ni s \mapsto \xi_s \in \mathscr{M}_{-2}$ is continuous. Next let $f \in \mathscr{M}$. In this case

$$\lim_{s \to t} \mathbb{E}_{\mathscr{G}}[|\xi_s(f) - \xi_t(f)|^2] = \lim_{s \to t} (f, (1 - e^{-|t-s|\hat{\omega}})f)_{\mathscr{M}} = 0.$$

This gives the desired result. □

5.6.4 Markov property

The stochastic process $(\xi_s)_{s \in \mathbb{R}}$ is stationary due to the form of its covariance as given by (5.6.37). Proposition 5.18 shows that the Euclidean quantum field has the Markov property. In this section we show that, moreover, $(\xi_s)_{s \in \mathbb{R}}$ is a Markov process.

Recall that $\delta_s \otimes f \in \mathscr{E}$ for $f \in \mathscr{M}$. We give the relationship of $\tilde{\xi}_s(f) = \tilde{\xi}(\delta_s \otimes f)$, $(\xi_s)_{s \in \mathbb{R}}$ and the Euclidean field $\phi_t(f)$.

Proposition 5.31. *Let F_j be Borel measurable bounded functions on $\mathbb{R}$, and $f_j \in \mathscr{M}$. Then*

$$\mathbb{E}_{\mathscr{G}}\Big[\prod_{j=1}^n F_j(\xi_{t_j}(f_j))\Big] = \mathbb{E}_\gamma\Big[\prod_{j=1}^n F_j(\tilde{\xi}_{t_j}(f_j))\Big] = \mathbb{E}_{\mu_{\mathrm{E}}}\Big[\prod_{j=1}^n F_j(\phi_{t_j}(f_j))\Big]. \tag{5.6.54}$$

Proof. Let $F \in (\mathbb{R}^n)$. From (5.6.50) it follows that

$$
\begin{aligned}
&\mathbb{E}_{\mathcal{G}}[F(\xi_{s_1}(f_1), \dots, \xi_{s_n}(f_n))] \\
&= (2\pi)^{-n/2} \int_{\mathbb{R}^d \times \dots \times \mathbb{R}^d} \check{F}(k_1, \dots, k_n) \mathbb{E}_{\mathcal{G}}[e^{i \sum_{j=1}^n k_j \xi_{s_j}(f_j)}] dk_1 \dots dk_n \\
&= (2\pi)^{-n/2} \int_{\mathbb{R}^d \times \dots \times \mathbb{R}^d} \check{F}(k_1, \dots, k_n) \mathbb{E}_{\gamma}[e^{i \sum_{j=1}^n k_j \tilde{\xi}_{s_j}(f_j)}] dk_1 \dots dk_n \\
&= \mathbb{E}_{\gamma}[F(\tilde{\xi}_{s_1}(f_1), \dots, \tilde{\xi}_{s_n}(f_n))].
\end{aligned}
$$

The first equality can be completed by a limiting argument. The second equality is proven similarly. $\square$

Lemma 5.32. *Let $\mathscr{E}' \subset \mathscr{E}$ be a closed subspace, and $P : \mathscr{E} \to \mathscr{E}'$ be the corresponding orthogonal projection. Consider $\mathscr{G}_P = \sigma(\tilde{\xi}(f) | f \in \mathscr{E}')$. Then for each $\alpha \in \mathbb{C}$ and $f \in \mathscr{E}$ it follows that*

$$\mathbb{E}_\gamma[e^{\alpha \tilde{\xi}(f)} | \mathscr{G}_P] = e^{\alpha \tilde{\xi}(Pf)} e^{\frac{\alpha^2}{2} \|f - Pf\|_{\mathscr{E}}^2}. \tag{5.6.55}$$

Proof. We have that

$$\mathbb{E}_\gamma[e^{\alpha \tilde{\xi}(f)} | \mathscr{G}_P] = \mathbb{E}_\gamma[e^{\alpha \tilde{\xi}(Pf)} e^{\alpha \tilde{\xi}(f-Pf)} | \mathscr{G}_P] = e^{\alpha \tilde{\xi}(Pf)} \mathbb{E}_\gamma[e^{\alpha \tilde{\xi}(f-Pf)} | \mathscr{G}_P], \tag{5.6.56}$$

since $\tilde{\xi}(Pf)$ is $\mathscr{G}_P$-measurable. Moreover, since $\tilde{\xi}(f - Pf)$ is independent of $\mathscr{G}_P$, we have

$$e^{\alpha \tilde{\xi}(Pf)} \mathbb{E}_\gamma[e^{\alpha \tilde{\xi}(f-Pf)}] = e^{\alpha \tilde{\xi}(Pf)} e^{\frac{\alpha^2}{2} \|f - Pf\|_{\mathscr{E}}^2}. \tag{5.6.57}$$

$\square$

For any interval $I \subset \mathbb{R}$ write

$$\mathscr{F}_I = \sigma\left(\text{L.H. } \{\xi_t(f) | t \in I, f \in \mathscr{M}\}\right).$$

Theorem 5.33 (Markov property). *Let F be a bounded $\mathscr{F}_{[s,\infty)}$-measurable function. Then it follows that*

$$\mathbb{E}_{\mathcal{G}}[F | \mathscr{F}_{(-\infty,s]}] = \mathbb{E}_{\mathcal{G}}[F | \mathscr{F}_{\{s\}}]. \tag{5.6.58}$$

Proof. Denote by $\mathscr{E}_{(-\infty,s]}$ the closed subspace of $\mathscr{E}$ generated by the linear hull of $\{\delta_r \otimes f \in \mathscr{E} | r \le s, f \in \mathscr{M}\}$, and by $\mathscr{E}_{\{s\}}$ the closed subspace generated by $\{\delta_s \otimes f \in \mathscr{E} | f \in \mathscr{M}\}$. Write $P_{(-\infty,s]}$ and $P_{\{s\}}$ for the corresponding projections. We claim that for every $t \ge s$ and every $g \in \mathscr{M}$,

$$P_{(-\infty,s]}(\delta_t \otimes g) = \delta_s \otimes e^{-(t-s)\hat{\omega}} g = P_{\{s\}}(\delta_s \otimes e^{-(t-s)\hat{\omega}} g). \tag{5.6.59}$$

Indeed, $\delta_s \otimes e^{-(t-s)\hat{\omega}} g \in \mathscr{E}_{(-\infty,s]}$, and for all $r \le s$,

$$\begin{aligned}(\delta_r \otimes f, \delta_t \otimes g)_{\mathscr{E}} &= \int_{\mathbb{R}^d} e^{-|s-r|\omega(k)} \overline{\hat{f}(k)} (e^{-(t-s)\hat{\omega}} g)\hat{\ }(k) \frac{dk}{2\omega(k)} \\ &= (\delta_r \otimes f, \delta_s \otimes e^{-(t-s)\hat{\omega}} g)_{\mathscr{E}}.\end{aligned}$$

From this we get

$$(F, \delta_t \otimes g)_{\mathscr{E}} = (F, \delta_s \otimes e^{-(t-s)\hat{\omega}} g)_{\mathscr{E}}, \quad F \in \mathscr{E}_{(-\infty,s]},$$

by linearity and approximation, and thus the first equality in (5.6.59) is shown. The second equality now follows from the fact that $\delta_s \otimes e^{-(t-s)\hat{\omega}} g$ is not only in $\mathscr{E}_{(-\infty,s]}$ but even in $\mathscr{E}_{\{s\}}$. By Lemma 5.32 we furthermore have

$$\mathbb{E}_\gamma[e^{\alpha \tilde{\xi}_t(g)} | \mathscr{G}_{P_{(-\infty,s]}}] = e^{\alpha \tilde{\xi}(P_{(-\infty,s]}\delta_t \otimes g)} \exp\left(\frac{\alpha^2}{2} \|(1 - P_{(-\infty,s]})\delta_t \otimes g\|_{\mathscr{E}}^2\right),$$

and Proposition 5.31 and (5.6.56) give

$$\mathbb{E}_{\mathscr{G}}[e^{\alpha \xi_t(g)} | \mathscr{F}_{(-\infty,s]}] = e^{\alpha \xi_s(e^{-(t-s)\hat{\omega}} g)} \exp\left(\frac{\alpha^2}{2}(g, (1 - e^{-2(t-s)\hat{\omega}})g)_{\mathscr{M}}\right) \in \mathscr{F}_{\{s\}}$$

for $t \ge 0$. Again by approximation, we find $\mathbb{E}_{\mathscr{G}}[F | \mathscr{F}_{(-\infty,s]}] \in \mathscr{F}_{\{s\}}$ for all bounded $\mathscr{F}_{[s,\infty)}$-measurable functions. Thus (5.6.58) follows. □

5.6.5 Regular conditional Gaussian probability measures

The Gaussian measure $\mathscr{G}$ is a probability measure on $(\mathfrak{Y}, \mathscr{B}(\mathfrak{Y}))$ and $\xi_0 : \mathfrak{Y} \to \mathscr{M}_{-2}$ is a random variable. Since both $\mathfrak{Y} = C(\mathbb{R}; \mathscr{M}_{-2})$ and $\mathscr{M}_{-2}$ are Polish spaces, we can define the regular conditional probability measure $\mathscr{G}^\xi$ with $\xi \in \mathscr{M}_{-2}$ by

$$\mathscr{G}^\xi(\cdot) = \mathscr{G}(\cdot | \xi_0 = \xi). \tag{5.6.60}$$

In particular we can assume that

$$\mathscr{G}^\xi(\xi_0 \in B) = 1_\xi(B), \quad B \in \mathscr{B}(\mathscr{M}_{-2}), \tag{5.6.61}$$

and

$$\mathscr{G}^\xi(\xi_0 = \xi) = 1, \quad \mathscr{G} \circ \xi_0^{-1}\text{-a.s. } \xi. \tag{5.6.62}$$

See Theorem 2.12 for (5.6.60)–(5.6.62).

We compute $\mathbb{E}^\xi_{\mathscr{G}}[\xi_t(f)]$ and $\mathbb{E}^\xi_{\mathscr{G}}[\xi_t(f)\xi_s(g)]$ for later use. Note that

$$\mathbb{E}_{\mathscr{G}}[e^{\sum_{j=1}^n \xi_{t_j}(f_j)} | \mathscr{F}_{\{0\}}] = \mathbb{E}_{\mathscr{G}}[e^{\sum_{j=1}^n \xi_{t_j}(f_j)} | \sigma(\xi_0)]. \tag{5.6.63}$$

It is seen by Lemma 2.11 that the right-hand side of (5.6.63) is a function of ξ_0, $h(\xi_0)$, and $\mathbb{E}^\xi_{\mathscr{G}}[e^{\sum_{j=1}^n \xi_{t_j}(f_j)}]$ is defined by $h(\xi_0)$ with ξ_0 replaced by ξ.

Theorem 5.34. *Suppose that $f, g \in \mathscr{M}$ and $\xi \in \mathscr{M}_{-2}$. Then it follows that*

$$\mathbb{E}^{\xi}_{\mathscr{G}}[\xi_t(f)] = \xi(e^{-|t|\hat{\omega}} f), \tag{5.6.64}$$

$$\mathbb{E}^{\xi}_{\mathscr{G}}[\xi_t(f)\xi_s(g)] = (f, e^{-|t-s|\hat{\omega}} g)_{\mathscr{M}} - (f, e^{-(|t|+|s|)\hat{\omega}} g)_{\mathscr{M}} + \xi(e^{-|t|\hat{\omega}} f)\xi(e^{-|s|\hat{\omega}} g). \tag{5.6.65}$$

Proof. In a similar way to the proof of Theorem 5.33 we see that

$$\mathbb{E}_{\mathscr{G}}[e^{\sum_{j=1}^n \xi_{t_j}(f_j)} | \mathscr{F}_{\{r\}}] = e^{\sum_{j=1}^n \xi_r(e^{-|t_j - r|\hat{\omega}} f_j)} e^{\frac{1}{2}Q}, \tag{5.6.66}$$

where

$$Q = \sum_{i,j=1}^n \{(f_i, e^{-|t_i - t_j|\hat{\omega}} f_j)_{\mathscr{M}} - (f_i, e^{-|t_i - r|\hat{\omega}} e^{-|t_j - r|\hat{\omega}} f_j)_{\mathscr{M}}\}.$$

Let $\alpha, \beta \in \mathbb{R}$. Inserting αf and βg into f_1 and f_2 in (5.6.66), respectively, and setting $n = 2$ and $r = 0$ we have

$$\mathbb{E}_{\mathscr{G}}[e^{\alpha\xi_t(f) + \beta\xi_s(g)} | \sigma(\xi_0)] = e^{\alpha\xi_0(e^{-|t|\hat{\omega}} f) + \beta\xi_0(e^{-|s|\hat{\omega}} g)} e^{\frac{1}{2}Q}, \tag{5.6.67}$$

where

$$\begin{aligned} Q = {} & \alpha^2(\|f\|^2_{\mathscr{M}} - (f, e^{-2|t|\hat{\omega}} f)_{\mathscr{M}}) + \beta^2(\|g\|^2_{\mathscr{M}} - (f, e^{-2|s|\hat{\omega}} g)_{\mathscr{M}}) \\ & + 2\alpha\beta((f, e^{-|t-s|\hat{\omega}} g)_{\mathscr{M}} - (f, e^{-(|t|+|s|)\hat{\omega}} g)_{\mathscr{M}}). \end{aligned}$$

Hence

$$\mathbb{E}^{\xi}_{\mathscr{G}}[e^{\alpha\xi_t(f) + \beta\xi_s(g)}] = e^{\alpha\xi(e^{-|t|\hat{\omega}} f) + \beta\xi(e^{-|s|\hat{\omega}} g)} e^{\frac{1}{2}Q}. \tag{5.6.68}$$

(5.6.64) is derived through $\partial_\alpha \mathbb{E}^{\xi}_{\mathscr{G}}[e^{\alpha\xi_t(f)+\beta\xi_s(g)}]_{\alpha=0}$, while formula (5.6.65) can be obtained via $\partial_\alpha\partial_\beta \mathbb{E}^{\xi}_{\mathscr{G}}[e^{\alpha\xi_t(f)+\beta\xi_t(g)}]_{\alpha=\beta=0}$. □

Using Theorem 5.34 we can also compute the covariance:

$$\begin{aligned} \mathrm{cov}(\xi_t(f); \xi_t(g)) &= \mathbb{E}^{\xi}_{\mathscr{G}}[\xi_t(f)\xi_t(g)] - \mathbb{E}^{\xi}_{\mathscr{G}}[\xi_t(f)]\mathbb{E}^{\xi}_{\mathscr{G}}[\xi_t(g)] \\ &= (f, e^{-|t-s|\hat{\omega}} g)_{\mathscr{M}} - (f, e^{-(|t|+|s|)\hat{\omega}} g)_{\mathscr{M}}. \end{aligned} \tag{5.6.69}$$

In particular, $\mathrm{cov}(\xi_t(f); \xi_t(g))$ is independent of ξ.

5.6.6 Feynman–Kac–Nelson formula by path measures

By making use of the path measure $\mathscr{G}$, an alternative functional integral representation of $(F, e^{-tH_P} G)$ in Theorem 5.20 can be given. Define

$$\tilde{H}_P = \tilde{H}_{\mathrm{f}} \dot{+} :P(\xi(f)):, \quad f \in \mathscr{M}, \tag{5.6.70}$$

where we recall that $\tilde{H}_{\mathrm{f}}$ denotes the free field Hamiltonian on $L^2(\mathscr{M}_{-2}, d\mathsf{G})$.

Theorem 5.35 (Feynman–Kac–Nelson formula). *Let $F, G \in L^2(\mathscr{M}_{-2}, d\mathsf{G})$. Then*

$$(F, e^{-t\tilde{H}_P} G) = \mathbb{E}_{\mathscr{G}}\left[\overline{F(\xi_0)} G(\xi_t) \exp\left(-\int_0^t :P(\xi_s(f)): ds\right)\right]. \tag{5.6.71}$$

Proof. Let $F(\xi) = P_1(\xi(f_1), \ldots, \xi(f_n))$ and $G(\xi) = P_2(\xi(f_1), \ldots, \xi(f_m))$ with a polynomial P_j. In the proof of Theorem 5.20 we have

$$(\theta_{\mathrm{W}}^{-1} F, e^{-tH_P} \theta_{\mathrm{W}}^{-1} G)_{\mathscr{F}_{\mathrm{b}}} = \lim_{n\to\infty} \mathbb{E}_\gamma[\overline{\theta_{\mathrm{W}}^{-1} F(\tilde{\xi}_0)} e^{-\sum_{j=1}^{n-1}(tj/n):P(\tilde{\xi}_{tj/n}(f)):} \theta_{\mathrm{W}}^{-1} G(\tilde{\xi}_t)].$$

Thus by the unitary equivalence $\theta_{\mathrm{W}} e^{-tH_P} \theta_{\mathrm{W}}^{-1} = e^{-t\tilde{H}_P}$, and Proposition 5.31 we see that

$$\begin{aligned}(F, e^{-t\tilde{H}_P} G)_{L^2(\mathscr{M}_{-2}, d\mathsf{G})} &= \lim_{n\to\infty} \mathbb{E}_{\mathscr{G}}[\overline{F(\xi_0)} e^{-\sum_{j=1}^{n-1}(tj/n):P(\xi_{tj/n}(f)):} G(\xi_t)] \\ &= \mathbb{E}_{\mathscr{G}}[\overline{F(\xi_0)} e^{-\int :P(\xi_s(f)):ds} G(\xi_t)].\end{aligned}$$

Here we used strong continuity of $s \mapsto \xi_s(f)$. By a limiting argument F, G can be extended to $F, G \in L^2(\mathscr{M}_{-2}, d\mathsf{G})$. □

Making use of the regular conditional probability measure $\mathscr{G}^\xi$ we can derive the functional integral representation of $e^{-t\tilde{H}_P}$.

Corollary 5.36. *Let $G \in L^2(\mathscr{M}_{-2}, d\mathsf{G})$. Then*

$$(e^{-t\tilde{H}_P} G)(\xi) = \mathbb{E}_{\mathscr{G}}^\xi[e^{-\int :P(\xi_s(f)):ds} G(\xi_t)], \quad \text{a.s. } \xi \in \mathscr{M}_{-2}. \tag{5.6.72}$$

Proof. The proof is straightforward. By Theorem 5.35 we have

$$(F, e^{-t\tilde{H}_P} G) = \mathbb{E}_{\mathsf{G}}\left[\overline{F(\xi)} \mathbb{E}_{\mathscr{G}}^\xi\left[G(\xi_t) \exp\left(-\int_0^t :P(\xi_s(f)): ds\right)\right]\right]. \tag{5.6.73}$$

Since F is arbitrary, the corollary follows. □

Chapter 6
The Nelson model by path measures

6.1 Preliminaries

In this section we consider a model of an electrically charged, spinless and non-relativistic particle interacting with a scalar boson field, which we call Nelson model. The coupling between particle and field is assumed to be linear. In this section we explain how the Nelson model can be derived from a Lagrangian, and we continue with a rigorous definition and analysis in the following sections.

The Lagrangian density $\mathscr{L}_{\mathrm{N}} = \mathscr{L}_{\mathrm{N}}(x,t)$, $(x,t) \in \mathbb{R}^3 \times \mathbb{R}$, of the Nelson model is described by

$$\mathscr{L}_{\mathrm{N}} = i\Psi^*\dot{\Psi} + \frac{1}{2m}\partial_j\Psi^*\partial_j\Psi + \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}\nu^2\phi^2 + \Psi^*\Psi\phi, \tag{6.1.1}$$

where $\Psi = \Psi(x,t)$ is a complex scalar field describing a non-relativistic, spinless electron, and $\phi(x,t)$ is a neutral scalar field describing scalar bosons, $\nu \geq 0$ is the mass of bosons, and $m > 0$ that of the electron. Here $\partial_\mu\phi\partial^\mu\phi = \dot{\phi}\dot{\phi} - \partial_j\phi\partial_j\phi$, $\partial_j = \partial_{x_j}$, the dots denote time derivative, and the stars denote complex conjugate. The dynamics of this system is given by the Euler–Lagrange equation leading to the system of nonlinear field equations

$$\begin{cases} (\Box + \nu^2)\phi(x,t) = \Psi^*(x,t)\Psi(x,t), \\ \left(i\partial t + \frac{1}{2m}\Delta_x\right)\Psi(x,t) = \phi(x,t)\Psi(x,t). \end{cases} \tag{6.1.2}$$

The Hamiltonian density is derived from the Legendre transform of $\mathscr{L}_{\mathrm{N}}$. The conjugate momenta are defined by

$$\Phi = \frac{\partial\mathscr{L}_{\mathrm{N}}}{\partial\dot{\Psi}} = i\Psi^*, \tag{6.1.3}$$

$$\pi = \frac{\partial\mathscr{L}_{\mathrm{N}}}{\partial\dot{\phi}} = \dot{\phi}. \tag{6.1.4}$$

Thus the Hamiltonian density $H_{\mathrm{N}} = H_{\mathrm{N}}(x,t)$ is given by

$$\begin{aligned} H_{\mathrm{N}} &= \Phi\dot{\Psi} + \pi\dot{\phi} - \mathscr{L}_{\mathrm{N}} \\ &= \frac{1}{2m}|\partial_x\Psi|^2 + \frac{1}{2}(\dot{\phi}^2 + (\partial_x\phi)^2 + \nu^2\phi^2) - \Psi^*\Psi\phi \end{aligned}$$

and the Hamiltonian

$$H_{\mathrm{N}} = \int_{\mathbb{R}^3} \left\{ \Psi^* \left(-\frac{1}{2m}\Delta_x \right) \Psi + \frac{1}{2}(\dot{\phi}^2 + (\partial_x \phi)^2 + \nu^2 \phi^2) - \Psi^* \Psi \phi \right\} dx. \quad (6.1.5)$$

Since the Nelson model describes a charged particle interacting with a scalar field, and the particle is supposed to carry low energy, there is no annihilation and creation of particles and the number of particles is conserved. In order to obtain the quantized model the kinetic part is replaced as

$$\int_{\mathbb{R}^3} \Psi^* \left(-\frac{1}{2m}\Delta_x \right) \Psi dx \to -\frac{1}{2m}\Delta$$

and the interaction as

$$-\int_{\mathbb{R}^3} \Psi^* \Psi \phi dx \to \phi(x).$$

Adding an external potential V, the formal expression of the Nelson model is

$$-\frac{1}{2m}\Delta + V + H_{\mathrm{f}} + \phi(x), \quad (6.1.6)$$

where

$$H_{\mathrm{f}} = \frac{1}{2}\int (\dot{\phi}^2 + (\partial_x \phi)^2 + \nu^2 \phi^2) dx.$$

6.2 The Nelson model in Fock space

6.2.1 Definition

We now give a rigorous definition of the Nelson model. $L^2(\mathbb{R}^d)$ describes the state space of the particle, and the scalar boson field is defined by

$$\mathscr{F}_{\mathrm{N}} = \mathscr{F}_{\mathrm{b}}(L^2(\mathbb{R}^d)), \quad (6.2.1)$$

i.e., the boson Fock space over $L^2(\mathbb{R}^d)$. The joint state space is the tensor product

$$\mathscr{H}_{\mathrm{N}} = L^2(\mathbb{R}^d) \otimes \mathscr{F}_{\mathrm{N}}. \quad (6.2.2)$$

The *free particle Hamiltonian* is described by the Schrödinger operator

$$H_{\mathrm{p}} = -\frac{1}{2}\Delta + V \quad (6.2.3)$$

acting in $L^2(\mathbb{R}^d)$. The following are standing assumptions throughout this section.

Assumption 6.1. The following conditions hold:

Dispersion relation:

$$\omega = \omega(k) = \sqrt{|k|^2 + \nu^2}, \quad \nu \geq 0. \tag{6.2.4}$$

Charge distribution: $\varphi : \mathbb{R}^d \to \mathbb{R}$, and $\overline{\hat{\varphi}(k)} = \hat{\varphi}(-k) = \hat{\varphi}(k)$, $\hat{\varphi}/\sqrt{\omega} \in L^2(\mathbb{R}^d)$, $\hat{\varphi}/\omega \in L^2(\mathbb{R}^d)$

External potential: V is Kato-decomposable and is such that H_p has a ground state Ψ_p with $H\Psi_\mathrm{p} = 0$.

Recall that by Theorem 3.54 the ground state Ψ_p of H_p is unique and it has a strictly positive version. We will choose this version of Ψ_p throughout.

The *free field Hamiltonian*

$$H_\mathrm{f} = d\Gamma(\omega) \tag{6.2.5}$$

on $\mathscr{F}_\mathrm{N}$ accounts for the energy carried by the field configuration. The *particle-field interaction Hamiltonian* H_I acting on the Hilbert space $\mathscr{H}_\mathrm{N}$ describes then the interaction energy between the boson field and the particle. To give a definition of this operator we identify $\mathscr{H}_\mathrm{N}$ as a space of $\mathscr{F}_\mathrm{N}$-valued L^2 functions on $\mathbb{R}^d$,

$$\mathscr{H}_\mathrm{N} \cong \int_{\mathbb{R}^d}^{\oplus} \mathscr{F}_\mathrm{N} dx = \left\{ F : \mathbb{R}^d \to \mathscr{F}_\mathrm{N} \,\middle|\, \int_{\mathbb{R}^d} \|F(x)\|^2_{\mathscr{F}_\mathrm{N}} dx < \infty \right\}. \tag{6.2.6}$$

$H_\mathrm{I}(x)$ is defined by the time-zero field

$$H_\mathrm{I}(x) = \frac{1}{\sqrt{2}} \{ a^*(\hat{\varphi} e^{-ikx}/\sqrt{\omega}) + a(\widetilde{\hat{\varphi}} e^{ikx}/\sqrt{\omega}) \} \tag{6.2.7}$$

for $x \in \mathbb{R}^d$, where $\widetilde{\hat{\varphi}}(k) = \hat{\varphi}(-k)$. We will use the formal notation

$$H_\mathrm{I}(x) = \int_{\mathbb{R}^d} \frac{1}{\sqrt{2\omega(k)}} (\hat{\varphi}(k) e^{-ikx} a^*(k) + \hat{\varphi}(-k) e^{ikx} a(k)) dk \tag{6.2.8}$$

for convenience. Since $\overline{\hat{\varphi}(k)} = \hat{\varphi}(-k)$, $H_\mathrm{I}(x)$ is symmetric, and it can be shown by using Nelson's analytic vector theorem that $H_\mathrm{I}(x)$ is essentially self-adjoint on the finite particle subspace of $\mathscr{F}_\mathrm{N}$. We denote the self-adjoint extension of $H_\mathrm{I}(x)$ by $\overline{H_\mathrm{I}(x)}$. The interaction H_I is then defined by the self-adjoint operator

$$H_\mathrm{I} = \int_{\mathbb{R}^d}^{\oplus} \overline{H_\mathrm{I}(x)} dx. \tag{6.2.9}$$

acting as

$$(H_\mathrm{I}\Psi)(x) = \overline{H_\mathrm{I}(x)} \Psi(x), \quad x \in \mathbb{R}^d, \tag{6.2.10}$$

with domain

$$D(H_\mathrm{I}) = \{ \Psi \in \mathscr{H}_\mathrm{N} | \Psi(x) \in D(\overline{H_\mathrm{I}(x)}), x \in \mathbb{R}^d \}. \tag{6.2.11}$$

Definition 6.1 (Nelson Hamiltonian in Fock space). Under the conditions of Assumption 6.1 the operator

$$H_\mathrm{N} = H_\mathrm{p} \otimes 1 + 1 \otimes H_\mathrm{f} + H_\mathrm{I} \tag{6.2.12}$$

in $\mathscr{H}_\mathrm{N}$ is the *Nelson Hamiltonian.*

A first natural question about the Nelson Hamiltonian is whether it is self-adjoint.

Proposition 6.1 (Self-adjointness). *Under Assumption* 6.1

(1) $H_0 = H_\mathrm{p} \otimes 1 + 1 \otimes H_\mathrm{f}$ *is self-adjoint on*

$$D(H_0) = D(H_\mathrm{p} \otimes 1) \cap D(1 \otimes H_\mathrm{f}) \tag{6.2.13}$$

and non-negative

(2) H_N *is self-adjoint on* $D(H_0)$ *and bounded from below, furthermore, it is essentially self-adjoint on any core of* H_0. *(In fact,* H_I *is infinitesimally small with respect to* H_0.*)*

Proof. H_0 is self-adjoint and non-negative as it is the sum of two commuting, non-negative self-adjoint operators. By Lemma 5.2, $D(H_\mathrm{I}) \supset D(H_0)$, Proposition 5.2 gives

$$\|\overline{H_\mathrm{I}(x)}\Psi\|_{\mathscr{F}_\mathrm{N}} \le (2\|\hat{\varphi}/\sqrt{\omega}\| + \|\hat{\varphi}\|)\|(H_\mathrm{f}+1)^{1/2}\Psi\|_{\mathscr{F}_\mathrm{N}}$$

for $\Psi \in D(H_\mathrm{f})$ and for every $x \in \mathbb{R}^d$. Thus for $\Phi \in D(1 \otimes H_\mathrm{f})$,

$$\|H_\mathrm{I}\Phi\|_{\mathscr{H}_\mathrm{N}} \le (2\|\hat{\varphi}/\sqrt{\omega}\| + \|\hat{\varphi}\|)\|1 \otimes (H_\mathrm{f}+1)^{1/2}\Phi\|_{\mathscr{H}_\mathrm{N}}.$$

Since $\|1 \otimes (H_\mathrm{f}+1)^{1/2}\Psi\| \le \|(H_0+1)^{1/2}\Psi\| \le \varepsilon\|H_0\Psi\| + (1+\frac{1}{4\varepsilon})\|\Psi\|$, the Kato–Rellich theorem yields that H_N is self-adjoint on $D(H_0)$ and bounded from below. □

6.2.2 Infrared and ultraviolet divergences

We discuss here the divergences appearing at the two ends of the spectrum. First we consider the ultraviolet divergence. Since in quantum theory the electron is regarded as a point particle, the form factor is chosen as $\varphi_\mathrm{ph}(x) = \delta(x)$. This implies

$$\int_{\mathbb{R}^d} \frac{|\hat{\varphi}_\mathrm{ph}(k)|^2}{\omega(k)} dk = \infty$$

as

$$\hat{\varphi}_\mathrm{ph}(k) = (2\pi)^{-d/2}. \tag{6.2.14}$$

This situation is *ultraviolet divergence*. In the mathematical description a more regular form factor $\hat{\varphi}$ is imposed so that

$$\int_{\mathbb{R}^d} \frac{|\hat{\varphi}(k)|^2}{\omega(k)} dk < \infty \tag{6.2.15}$$

and the field operator $\phi(\varphi(\cdot - x))$ is well-defined as an operator on $\mathscr{F}_{\mathrm{N}}$.

At the other end of the energy spectrum an infrared divergence is encountered. Suppose $\hat{\varphi}(k) = (2\pi)^{-d/2}$ for $|k| < \varepsilon$, with some $\varepsilon > 0$. In this case

$$\int_{|k|<\varepsilon} \frac{|\hat{\varphi}(k)|^2}{\omega(k)^3} dk = \infty, \quad d \le 3, \tag{6.2.16}$$

since the dispersion relation $\omega(k) = |k|$ and the dimension of the space is less than three. This *infrared divergence* is also serious. In the physical understanding of the model the ground state expectation of the boson number is finite, i.e., $(\Psi_{\mathrm{g}}, N\Psi_{\mathrm{g}}) < \infty$, with the number operator N. It will be seen in (6.5.45) below that

$$C_1 \int_{\mathbb{R}^d} \frac{|\hat{\varphi}(k)|^2}{\omega(k)^3} dk - C_2 \le (\Psi_{\mathrm{g}}, N\Psi_{\mathrm{g}}) \le C_3 \int_{\mathbb{R}^d} \frac{|\hat{\varphi}(k)|^2}{\omega(k)^3} dk$$

with constants $C_1, C_2, C_3 > 0$. Moreover, under condition (6.2.16) it can be proven that there is no ground state in $\mathscr{H}_{\mathrm{N}}$, which justifies the expectation that only a finite number of bosons couple to the particle. To cope with this difficulty we impose the *infrared regular condition*

$$\int_{\mathbb{R}^d} \frac{|\hat{\varphi}(k)|^2}{\omega(k)^3} dk < \infty. \tag{6.2.17}$$

Note that for massive bosons, $\omega(k) = \sqrt{|k|^2 + \nu^2}$ with $\nu > 0$, and thus no infrared divergence occurs. Whenever

$$\int_{\mathbb{R}^d} \frac{|\hat{\varphi}(k)|^2}{\omega(k)^3} dk = \infty \tag{6.2.18}$$

we speak of *infrared singularity*. Although the cutoffs are mathematical artifacts and not physical, we impose both the ultraviolet and infrared cutoffs above in order that the model is well defined.

Two special choices of ω and $\hat{\varphi}$ are particularly important and will be singled out. Choosing

$$\omega(k) = |k| \quad \text{and} \quad \hat{\varphi}(k) = g1_{\{\kappa<|k|<\Lambda\}} \tag{6.2.19}$$

with $0 < \kappa < \Lambda$, we arrive at the *massless Nelson model* in which the mass of bosons is zero. Here κ and Λ are respectively the infrared and ultraviolet cut-off parameters, and g is the coupling constant. Choosing

$$\omega(k) = \sqrt{|k|^2 + \nu^2} \quad \text{and} \quad \hat{\varphi}(k) = g1_{\{|k|<\Lambda\}} \tag{6.2.20}$$

with $\nu > 0$ and $\Lambda > 0$, gives the *massive Nelson model* with non-zero mass ν of bosons, ultraviolet cut-off Λ and coupling constant g. Both of these cases satisfy (6.2.17), except

$$\omega(k) = |k| \quad \text{and} \quad \hat{\varphi}(k) = g 1_{\{|k|<\Lambda\}}, \quad d \leq 3, \tag{6.2.21}$$

when the infrared singularity condition is verified.

6.2.3 Embedded eigenvalues

The Nelson Hamiltonian is of the form $H_{\mathrm{N}} = H_0 + H_{\mathrm{I}}$. The spectrum of H_0 is well known, in particular, when boson field is massless all eigenvalues of H_0 are embedded in the continuous spectrum. It is of great interest to study the behavior of the embedded eigenvalues resulting from the interaction term H_{I} in non-perturbative way. While there are few general results on the perturbation of embedded eigenvalues, the study of perturbations of the discrete spectrum can be done by Kato's theory for sufficiently weak couplings. From this theory it can be seen that under a small perturbation the discrete spectrum remains discrete. In our present context, however, the investigation of embedded eigenvalues is crucial. Since even for an arbitrarily small coupling constant the bottom of the spectrum of H_0 is an embedded eigenvalue, the existence of a ground state of H_{N} is non-trivial for any coupling constants. One of the goals of the mathematical analysis of the Nelson Hamiltonian in this chapter is to establish the behavior of embedded eigenvalues under perturbations by using functional integral methods. In particular, we will show the existence of a ground state of the Nelson Hamiltonian for all values of the coupling constant. It is, however, worth noting that the proof of existence of a ground state is not constructive. One way of studying the qualitative properties of the ground state is to apply functional integrals. Using this we get probabilistic expressions for ground state expectations $(\Psi_{\mathrm{g}}, \mathcal{O}\Psi_{\mathrm{g}})$ for a range of observables $\mathcal{O}$.

6.3 The Nelson model in function space

We will make use of the infinite dimensional Ornstein–Uhlenbeck process $(\xi_s)_{s\in\mathbb{R}}$ discussed in Section 5.6 to obtain a unitary equivalent representation of H_{N}.

From now on the real Hilbert space $\mathscr{M}$, positive self-adjoint operator $\mathcal{D}$ in $\mathscr{M}$ with a Hilbert–Schmidt inverse are assumed to be as in Section 5.6.2. Then a family of Gaussian random variables $\xi(f)$, $f \in \mathscr{M}$, on a probability space $(\mathscr{M}_{-2}, \mathscr{B}(\mathscr{M}_{-2}), \mathsf{G})$ can be constructed. It has already been seen that $L^2(\mathscr{M}_{-2}, d\mathsf{G})$ and $\mathscr{F}_{\mathrm{N}}$ are unitary equivalent under $\xi(f) \cong \phi(f)$, where

$$\phi(f) = \int_{\mathbb{R}^d} \frac{1}{\sqrt{2\omega(k)}} (\hat{f}(k)a^*(k) + \hat{f}(-k)a(k))dk$$

denotes the scalar field with $f \in \mathscr{M}$. Recall that $(\xi_s(f))_{s\in\mathbb{R}}$ is the infinite dimensional Ornstein–Uhlenbeck process on the probability space $(\mathfrak{Y}, \mathscr{B}(\mathfrak{Y}), \mathscr{G})$ with mean zero such that $\mathscr{M} \ni f \mapsto \xi_s(f) \in \mathbb{R}$ is linear, $\xi_s(f) = \langle\!\langle \xi_s, f\rangle\!\rangle$ when $f \in \mathscr{M}_{+2}$ and

$$\mathbb{E}_{\mathscr{G}}[e^{i\sum_{j=1}^n \xi_{s_j}(f_j)}] = e^{-\frac{1}{2}\sum_{i,j=1}^n (f_i, e^{-|s_i-s_j|\hat{\omega}} f_j)_{\mathscr{M}}}.$$

Furthermore, $(\xi_s(f))_{s\in\mathbb{R}}$ is a stationary Markov process. Using the regular conditional probability measure $\mathscr{G}^\xi(\cdot) = \mathscr{G}(\cdot|\xi_0 = \xi)$ we have $\mathbb{E}_{\mathscr{G}}[\cdots] = \int_{\mathscr{M}_{-2}} d\mathsf{G}\mathbb{E}_{\mathscr{G}}^\xi[\cdots]$, where $\mathbb{E}_{\mathscr{G}}^\xi = \mathbb{E}_{\mathscr{G}^\xi}$.

We now define the Nelson Hamiltonian on a function space instead of $\mathscr{H}_{\mathrm{N}}$. It is directly seen that

$$\theta_{\mathrm{W}} H_{\mathrm{I}}(x)\theta_{\mathrm{W}}^{-1} = \xi(\varphi(\cdot - x)) \tag{6.3.1}$$

for every $x \in \mathbb{R}^d$. We need one more step. Since H_{N} is an operator in $\mathscr{H}_{\mathrm{N}}$, and since up to now we only have the isomorphism $\mathscr{F}_{\mathrm{N}} \cong L^2(\mathscr{M}_{-2}, d\mathsf{G})$, we still need to transform the first factor H_{p}. Instead of using identity here, we choose the ground state transform instead since it has the benefit that the transformed Hamiltonian becomes a sum of the generator of a random process and a multiplication operator.

Recall the ground state transform $U_{\Psi_{\mathrm{p}}} : L^2(\mathbb{R}^d, d\mathsf{N}_0) \to L^2(\mathbb{R}^d, dx)$, $f \mapsto \Psi_{\mathrm{p}} f$, introduced in Section 3.10.1. Note that $d\mathsf{N}_0 = \Psi_{\mathrm{p}}^2 dx$, and write

$$\mathsf{P}_0 = \mathsf{N}_0 \otimes \mathsf{G}. \tag{6.3.2}$$

P_0 is a probability measure on $\mathbb{R}^d \otimes \mathscr{M}_{-2}$. The unitary equivalence of $L^2(\mathbb{R}^d \otimes \mathscr{M}_{-2}, d\mathsf{P}_0)$ and $\mathscr{H}_{\mathrm{N}}$ is implemented by the unitary operator

$$U_{\Psi_{\mathrm{p}}} \otimes \theta_{\mathrm{W}} : \mathscr{H}_{\mathrm{N}} \to L^2(\mathbb{R}^d \otimes \mathscr{M}_{-2}, d\mathsf{P}_0). \tag{6.3.3}$$

For convenience, we write $L^2(\mathbb{R}^d \otimes \mathscr{M}_{-2}, \mathsf{P}_0)$ simply as $L^2(\mathsf{P}_0)$, moreover $L^2(\mathsf{N}_0)$ and $L^2(\mathsf{G})$ for $L^2(\mathbb{R}^d, d\mathsf{N}_0)$ and $L^2(\mathscr{M}_{-2}, d\mathsf{G})$, respectively. Furthermore, $L^2(\mathbb{R}^d) \otimes L^2(\mathsf{G})$ is also denoted by $\mathscr{H}_{\mathrm{N}}$.

Definition 6.2 (Nelson Hamiltonian in function space). The Nelson Hamiltonian in $L^2(\mathsf{P}_0)$ is defined by

$$L_{\mathrm{N}} = L_{\mathrm{p}} \otimes 1 + 1 \otimes \tilde{H}_{\mathrm{f}} + \tilde{H}_{\mathrm{I}}, \tag{6.3.4}$$

where $L_{\mathrm{p}} = U_{\Psi_{\mathrm{p}}}(H_{\mathrm{p}} - E_{\mathrm{p}})U_{\Psi_{\mathrm{p}}}^{-1}$ with $E_{\mathrm{p}} = \inf \mathrm{Spec}(H_{\mathrm{p}})$, and $\tilde{H}_{\mathrm{I}}$ is defined by

$$\tilde{H}_{\mathrm{I}} : F(x, \xi) \mapsto \xi(\varphi(\cdot - x))F(x, \xi). \tag{6.3.5}$$

Clearly, L_{N} and $H_{\mathrm{N}} - E_{\mathrm{p}}$ are unitary equivalent under (6.3.3). We also simplify the notations H_{f} for $\tilde{H}_{\mathrm{f}}$, and H_{I} for $\tilde{H}_{\mathrm{I}}$ in what follows.

Now we turn to deriving a Feynman–Kac-type formula for the Nelson model. Recall that $d\mathscr{N}_0 = d\mathscr{N}_0^x \otimes d\mathsf{N}_0$ is the path measure of the diffusion process $(X_t)_{t\geq 0}$ corresponding to L_{p}. We write

$$\mathscr{P}_0 = \mathscr{N}_0 \otimes \mathscr{G}. \tag{6.3.6}$$

Also, we already know that

$$(f, e^{-tL_{\mathrm{p}}} g)_{L^2(\mathsf{N}_0)} = \mathbb{E}_{\mathscr{N}_0}[\overline{f(X_0)} g(X_t)] \tag{6.3.7}$$

and

$$(\Psi, e^{-t(H_{\mathrm{f}} + \xi(f))}\Phi)_{L^2(\mathsf{G})} = \mathbb{E}_{\mathscr{G}}[\overline{\Psi(\xi_0)}\Phi(\xi_t) e^{-\int_0^t \xi_s(f) ds}]. \tag{6.3.8}$$

The functional integral representation of $e^{-tL_{\mathrm{N}}}$ is a combination with (6.3.7) and (6.3.8).

Theorem 6.2 (Functional integral representation for Nelson Hamiltonian). *For $F, G \in L^2(\mathsf{P}_0)$, $t > 0$, we have*

$$(F, e^{-tL_{\mathrm{N}}} G)_{L^2(\mathsf{P}_0)} = \mathbb{E}_{\mathscr{P}_0}[\overline{F(X_0, \xi_0)} e^{-\int_0^t \xi_s(\varphi_{X_s}) ds} G(X_t, \xi_t)], \tag{6.3.9}$$

or, equivalently,

$$(\Psi_{\mathrm{p}} F, e^{-tH_{\mathrm{N}}} \Psi_{\mathrm{p}} G)_{\mathscr{H}_{\mathrm{N}}} = e^{-tE_{\mathrm{p}}} \mathbb{E}_{\mathscr{P}_0}[\overline{F(X_0, \xi_0)} e^{-\int_0^t \xi_s(\varphi_{X_s}) ds} G(X_t, \xi_t)], \tag{6.3.10}$$

where $\varphi_{X_s}(\cdot) = \varphi(\cdot - X_s)$ and $(\Psi_{\mathrm{p}} F)(x, \xi) = \Psi_{\mathrm{p}}(x) F(x, \xi)$ for every $x \in \mathbb{R}^d$.

Proof. First assume that

$$\begin{aligned} F &= F(x, \xi) = F_1(x) F_2(\xi(f_1), \dots, \xi(f_n)), \\ G &= G(x, \xi) = G_1(x) G_2(\xi(g_1), \dots, \xi(g_m)) \end{aligned}$$

with $F_1, G_1 \in \mathscr{S}(\mathbb{R}^d)$, $F_2 \in \mathscr{S}(\mathbb{R}^n)$ and $G_2 \in \mathscr{S}(\mathbb{R}^m)$. By the Trotter product formula we have

$$\begin{aligned} (F, e^{-tL_{\mathrm{N}}} G) &= \lim_{n\to\infty} (F, (e^{-(t/n)L_{\mathrm{p}}} e^{-(t/n)H_{\mathrm{I}}} e^{-(t/n)H_{\mathrm{f}}})^n G) \\ &= \lim_{n\to\infty} \int d\mathscr{N}_0 \mathbb{E}_{\mu_{\mathrm{E}}}[\overline{F(X_0, \phi_0)} G(X_t, \phi_t) e^{-\sum_{j=1}^n \phi_{jt/n}(\varphi_{X_{jt/n}})}], \end{aligned}$$

where $\phi_t(f) = \phi_{\mathrm{E}}(\delta_t \otimes f)$ is the Euclidean field, see (5.5.29). We also note that

$$\begin{aligned} &|\mathbb{E}_{\mu_{\mathrm{E}}}[\overline{F(X_0, \phi_0)} G(X_t, \phi_t) e^{-\sum_{j=1}^n \phi_{jt/n}(\varphi_{X_{jt/n}})}]| \\ &\leq \|F(X_0, \cdot)\| \|G(X_t, \cdot)\| \|e^{-\sum_{j=1}^n \phi_{jt/n}(\varphi_{X_{jt/n}})}\|_1, \end{aligned} \tag{6.3.11}$$

where $\|\cdot\| = \|\cdot\|_{L^2(\mathscr{Q})}$ and $\|\cdot\|_1 = \|\cdot\|_{L^1(\mathscr{Q}_{\mathrm{E}})}$, and we used the bound

$$\int |(\mathrm{I}_a F)(\mathrm{I}_b G)\Phi| d\mu_E \le \|\Phi\|_1 \|F\| \|G\|$$

for $a < b$, which will be proven in Corollary 7.8 later on. Moreover,

$$\|e^{-\sum_{j=1}^n \phi_{jt/n}(\varphi_{X_{jt/n}})}\|_1 \le e^{t \int dk |\hat{\varphi}(k)|^2/\omega(k)^2}. \tag{6.3.12}$$

Since $\phi_t(f)$ and $\xi_t(f)$ are identically distributed

$$(F, e^{-tL_{\mathrm{N}}} G) = \lim_{n\to\infty} \mathbb{E}_{\mathscr{P}_0}[\overline{F(X_0, \xi_0)} G(X_t, \xi_t) e^{-\sum_{j=1}^n \xi_{jt/n}(\varphi_{X_{jt/n}})}],$$

Since $s \mapsto \xi_s(\varphi_{X_s})$ is continuous almost surely, the theorem follows for F, G above. The proof is completed by a limiting argument. □

For later use we will give an alternative functional integral representation of $(F, e^{-tH_{\mathrm{N}}} G)$ in terms of the Euclidean field discussed in Chapter 5 and Brownian motion instead of diffusion $(X_t)_{t\in\mathbb{R}}$.

Theorem 6.3 (Functional integral representation for Nelson Hamiltonian). *Let $F, G \in \mathscr{H}_{\mathrm{N}}$, Then*

$$(F, e^{-tH_{\mathrm{N}}} G)_{\mathscr{H}_{\mathrm{N}}} = \int_{\mathbb{R}^d} dx \mathbb{E}^x [e^{-\int_0^t V(B_s) ds} (\mathrm{I}_0 \overline{F(B_0)}, e^{\phi_{\mathrm{E}}(\int_0^t \delta_s \otimes \varphi(\cdot - B_s) ds)} \mathrm{I}_t G(B_t))], \tag{6.3.13}$$

where I_t, $t \in \mathbb{R}$, denotes the family of isometries introduced in Definition 5.8. *Here $F, G \in \mathscr{H}_{\mathrm{N}}$ are regarded as $L^2(\mathscr{Q})$-valued L^2 functions on $\mathbb{R}^d$, and the bracket at the right-hand side of* (6.3.13) *denotes scalar product on $L^2(\mathscr{Q}_{\mathrm{E}})$.*

Proof. First we assume that $V \in C_0^\infty(\mathbb{R}^d)$. By the Trotter product formula and the factorization formula $e^{-|t-s|H_{\mathrm{f}}} = \mathrm{I}_t^* \mathrm{I}_s$, we have

$$\begin{aligned}(F, e^{-tH_{\mathrm{N}}} G) &= \lim_{n\to\infty} (F, (e^{-(t/n)H_{\mathrm{p}}} e^{-(t/n)H_{\mathrm{I}}} e^{-(t/n)H_{\mathrm{f}}})^n G) \\ &= \lim_{n\to\infty} \int_{\mathbb{R}^d} dx \mathbb{E}^x [e^{-\sum_{j=0}^n (t/n) V(B_{tj/n})} \\ &\qquad \times (\mathrm{I}_0 F(B_0), e^{\sum_{j=0}^n (t/n) \phi_{\mathrm{E}}(\delta_{tj/n} \otimes \varphi(\cdot - B_{jt/n}))} \mathrm{I}_t G(B_t))].\end{aligned}$$

Note that $s \mapsto \delta_s \otimes \varphi(\cdot - B_s)$ is strongly continuous as a map $\mathbb{R} \to \mathscr{H}_{\mathrm{E}}$, almost surely. Hence $s \mapsto \phi_{\mathrm{E}}(\delta_s \otimes \varphi(\cdot - B_s))$ is also strongly continuous as a map $\mathbb{R} \to L^2(\mathscr{Q}_E)$. Then the theorem follows for $V \in C_0^\infty(\mathbb{R}^d)$.

We extend V to Kato-decomposable class. Then $V_+ \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$ and V_- is relatively form bounded with respect to $-(1/2)\Delta$. Therefore (6.3.13) can be extended to Kato-decomposable class in a similar way to Theorem 3.31. □

As in the final dimensional setting, we will use the Feynman–Kac-type formula in Theorem 6.2 to study eigenfunctions of H_{N}. If we can suppose that H_{N} has a unique ground state Ψ_{g}, then

$$\|e^{-T(H_{\mathrm{N}}-E_{\mathrm{N}})}F\|_{\mathscr{H}_{\mathrm{N}}}^{-1}e^{-T(H_{\mathrm{N}}-E_{\mathrm{N}})}F \to (\Psi_{\mathrm{g}}, F)_{\mathscr{H}_{\mathrm{N}}}\Psi_{\mathrm{g}} \tag{6.3.14}$$

strongly as $T \to \infty$, where $E_{\mathrm{N}} = \inf \mathrm{Spec}(H_{\mathrm{N}})$. Thus the properties of Ψ_{g} can be studied by an analysis of the left-hand side of (6.3.14) via (6.3.9) and taking the limit. This is the strategy we will follow.

Corollary 6.4. *If $f, g \in L^2(\mathsf{N}_0)$, then for every $T > 0$*

$$(f \otimes 1, e^{-TL_{\mathrm{N}}}g \otimes 1)_{L^2(\mathsf{P}_0)} = \mathbb{E}_{\mathscr{N}_0}[\overline{f(X_0)}g(X_T)e^{\frac{1}{2}\int_0^T ds \int_0^T dt W(X_s-X_t,s-t)}], \tag{6.3.15}$$

where

$$W(x,t) = \int_{\mathbb{R}^d} \frac{|\hat{\varphi}(k)|^2}{2\omega(k)} e^{-ik\cdot x} e^{-\omega(k)|t|} dk. \tag{6.3.16}$$

Proof. Let $I_T = \int_0^T \xi_s(\varphi_{X_s})ds$. Then

$$\begin{aligned}(f \otimes 1, e^{-TL_{\mathrm{N}}}g \otimes 1)_{L^2(\mathsf{P}_0)} &= \mathbb{E}_{\mathscr{P}_0}[\overline{f(X_0)}g(X_t)e^{-I_T}] \\ &= \mathbb{E}_{\mathscr{N}_0}[\overline{f(X_0)}g(X_t)\mathbb{E}_{\mathscr{G}}[e^{-I_T}]].\end{aligned}$$

Note that a finite linear sum of the form $\sum_{i=j}^n \xi_{s_j}(f_j)$ is a Gaussian random variable and

$$\mathbb{E}_{\mathscr{G}}[e^{i\alpha\sum_{j=1}^n \xi_{s_j}(f_j)}] = \int_{\mathscr{M}_{-2}^n} d\gamma e^{i\alpha\phi(\sum_{j=1}^n \delta_{s_j}\otimes f_j)} = e^{-(\alpha^2/2)\|\sum_{j=1}^n \delta_{s_j}\otimes f_j\|_{\mathscr{M}_{\mathbb{C}}}^2}.$$

Then I_T is also a Gaussian random variable with covariance

$$\mathbb{E}_{\mathscr{G}}[I_T^2] = \int_0^T ds \int_0^T dt \mathbb{E}_{\mathscr{G}}[\xi_s(\varphi_{X_s})\xi_t(\varphi_{X_t})] = \int_0^T ds \int_0^T dt W(X_t - X_s, t-s).$$

Thus it follows that

$$\mathbb{E}_{\mathscr{G}}[e^{-I_T}] = \exp\left(\frac{1}{2}\int_0^T ds \int_0^T dt W(X_s - X_t, s-t)\right). \tag{6.3.17}$$

Fubini's theorem implies the claim. □

Definition 6.3 (Pair potential). We call $W(x,t)$ given by (6.3.16) *pair potential* of the Nelson model.

6.4 Existence and uniqueness of the ground state

In this section we discuss the existence and uniqueness of a ground state of H_{N} by means of the Feynman–Kac-type formula derived in the previous section.

Uniqueness of the ground state follows directly from Feynman–Kac formula of the Nelson model and the Perron–Frobenius theorem.

Corollary 6.5 (Uniqueness of ground state). *If H_{N} has a ground state, then it is unique.*

Proof. By Theorem 6.2 it is seen that $e^{-tL_{\mathrm{N}}}$ is positivity improving. The statement follows from the Perron–Frobenius theorem, in Theorem 3.54. □

Theorem 6.6 (Existence of ground state). *In addition to Assumption* 6.1 *suppose that the infrared regularity condition* (6.2.17) *and*

$$\Sigma_{\mathrm{p}} - E_{\mathrm{p}} > \int_{\mathbb{R}^d} \frac{|\hat{\varphi}(k)|^2}{2\omega(k)^2} \frac{|k|^2}{2\omega(k) + |k|^2} dk, \tag{6.4.1}$$

hold, where Σ_{p} denotes the infimum of the essential spectrum of H_{p}. Then H_{N} has a ground state.

Under (6.2.17) we have the uniform bound

$$\int_{-\infty}^{0} ds \int_{0}^{\infty} |W(X_s - X_t, s - t)| dt \leq \int_{\mathbb{R}^d} \frac{|\hat{\varphi}(k)|^2}{2\omega(k)^3} dk < \infty \tag{6.4.2}$$

on paths. In Section 6.7 we will also see that (6.2.17) is crucial for a ground state to exist, and will show that in the physically important cases the existence of a ground state fails if the infrared regularity condition does not hold. Condition (6.4.1) is a restriction on the coupling constant, which in fact only comes to effect in the case where the particle Hamiltonian H_{p} has absolutely continuous spectrum. On physical grounds (6.4.1) is not expected to be necessary. This is because the coupling to the quantum field should make the particle more heavy, thus enhancing the binding under V.

Lemma 6.7. *If* $\operatorname{ess\,inf}_{k \in K} f(k) > 0$ *for any compact set $K \subset \mathbb{R}^d$, then*

$$\lim_{T \to \infty} \frac{(f \otimes 1, e^{-(T+t)H_{\mathrm{N}}} f \otimes 1)}{(f \otimes 1, e^{-TH_{\mathrm{N}}} f \otimes 1)} = e^{-tE_{\mathrm{N}}}.$$

Proof. Recall that if μ is a measure on $\mathbb{R}$ with $\inf \operatorname{supp}(\mu) = E(\mu)$, then

$$E(\mu) = - \lim_{T \to \infty} \frac{1}{T} \ln \left(\int e^{-Tx} \mu(dx) \right), \tag{6.4.3}$$

and

$$\lim_{T\to\infty}\frac{\int e^{-(T+t)x}\mu(dx)}{\int e^{-Tx}\mu(dx)} = e^{-tE(\mu)}. \tag{6.4.4}$$

Applying (6.4.4) to the spectral measure $\mu_{f\otimes 1}$ of H_{N} associated to the vector $f\otimes 1$ yields the claim with E_{N} replaced by $\inf\operatorname{supp}(\mu_{f\otimes 1}) = E(\mu_{f\otimes 1})$. Thus it only remains to prove $E(\mu_{f\otimes 1}) = E_{\mathrm{N}}$. For this, define the class of functions

$$\mathscr{O} = \Big\{F\in L^2(\mathsf{P}_0)\Big|\, \operatorname{supp} F\subset \bigcup_{N,M>0} B_N(\mathbb{R}^d)\times B_M(\mathscr{M}_{-2})\Big\},$$

where $B_N(\mathbb{R}^d)$ and $B_M(\mathscr{M}_{-2})$ denote the balls centered in the origin in $\mathbb{R}^d$ and $\mathscr{M}_{-2}$ of radius N and M, respectively. $\mathscr{O}$ is dense in $L^2(\mathsf{P}_0)$. Pick $g\in\mathscr{O}$. Since $e^{-tH_{\mathrm{N}}}$ is positivity preserving

$$(g, e^{-TH_{\mathrm{N}}}g) \le (|g|, e^{-TH_{\mathrm{N}}}|g|) \le C^2(f\otimes 1, e^{-TH_{\mathrm{N}}}f\otimes 1) \tag{6.4.5}$$

with

$$C = \frac{\operatorname{ess\,sup}_{(x,\xi)\in\mathbb{R}^d\times\mathscr{M}_{-2}}|g(x,\xi)|}{\operatorname{ess\,inf}_{(x,\xi)\in\operatorname{supp}|g|} f(x)}. \tag{6.4.6}$$

Equality (6.4.3) shows that $E(\mu_{f\otimes 1}) \le E(\mu_g)$, for all $g\in\mathscr{O}$. However, since $\mathscr{O}$ is a dense subset in $D(H_{\mathrm{N}})$, we have

$$E(\mu_{f\otimes 1}) \ge E_{\mathrm{N}} = \inf\{E(\mu_g)|g\in\mathscr{O}\} \ge E(\mu_{f\otimes 1})$$

and thus $E_{\mathrm{N}} = E(\mu_{f\otimes 1})$. □

Recall that Ψ_{p} is the ground state of H_{p}, and for $T>0$ define

$$\Psi_{\mathrm{g}}^T = \frac{e^{-TH_{\mathrm{N}}}(\Psi_{\mathrm{p}}\otimes 1)}{\|e^{-TH_{\mathrm{N}}}(\Psi_{\mathrm{p}}\otimes 1)\|}. \tag{6.4.7}$$

Since $\|\Psi_{\mathrm{g}}^T\| = 1$, Ψ_{g}^T has a subsequence $\Psi_{\mathrm{g}}^{T'}$ such that $\Psi_{\mathrm{g}}^{T'}$ is weakly convergent to a vector Ψ_{g}^∞. We reset T' to T in what follows. Define

$$\gamma(T) = (\Psi_{\mathrm{p}}\otimes 1, \Psi_{\mathrm{g}}^T)^2. \tag{6.4.8}$$

The next criterion is useful for showing existence or non-existence of a ground state of H_{N}.

Proposition 6.8 (Criterion of existence of ground state). *Suppose that* $\lim_{T\to\infty}\gamma(T) = a$. *If* $a>0$, *then* H_{N} *has a ground state. If* $a=0$, *then* H_{N} *has no ground state.*

Proof. We may suppose that $\inf \mathrm{Spec}(H_\mathrm{N}) = 0$, so that $\lim_{T\to\infty} e^{-TH_\mathrm{N}} = 1_{\{0\}}(H_\mathrm{N})$ in strong sense. If 0 is an eigenvalue of H_N, i.e., H_N has a ground state Ψ_g, then $a = (\Psi_\mathrm{p} \otimes 1, \Psi_\mathrm{g}) > 0$, since Ψ_g is strictly positive by Corollary 6.5. Thus $a = 0$ implies that H_N has no ground state.

Suppose now that H_N has no ground state and $a > 0$. Then for some $b < a$, $\gamma(T) \geq b$ for sufficiently large T, and

$$(\Psi_\mathrm{p} \otimes 1, e^{-TH_\mathrm{N}} \Psi_\mathrm{g} \otimes 1) > b^{1/2} (\Psi_\mathrm{p} \otimes 1, e^{-2TH_\mathrm{N}} \Psi_\mathrm{g} \otimes 1)^{1/2}.$$

Letting $T \to \infty$ it follows that $\|1_{\{0\}}(H)(\Psi_\mathrm{p} \otimes 1)\| \geq b^{1/2}$. This contradicts the absence of a ground state, and thus $a = 0$. □

Corollary 6.9. *H_N has a ground state if and only if $\Psi_\mathrm{g}^\infty \neq 0$.*

Proof. Since Ψ_g^T is non-negative, the weak limit Ψ_g^∞ is also non-negative. Hence $\lim_{T\to\infty} \gamma(T) = (1, \Psi_\mathrm{g}^\infty)^2 > 0$ if $\Psi_\mathrm{g}^\infty \neq 0$ and $= 0$ if $\Psi_\mathrm{g}^\infty = 0$. Then the corollary follows from Proposition 6.8. □

Note that the choice of $\Psi_\mathrm{p} \otimes 1$ in the definition of Ψ_g^T is natural as it is the ground state of the uncoupled system, however, any other non-negative L^2-function could have been chosen.

Proof of Theorem 6.6. Write Ψ_p instead of $\Psi_\mathrm{p} \otimes 1$. By Corollary 6.9 we need to prove that Ψ_g^T is not weakly convergent to zero. Let

$$S_{[a,b]} = \frac{1}{2} \int_a^b ds \int_a^b W(X_s - X_t, s - t) dt. \tag{6.4.9}$$

Put

$$f(T,t) = (\Psi_\mathrm{g}^T, (e^{-tH_\mathrm{p}} \otimes P_0) \Psi_\mathrm{g}^T),$$

where P_0 denotes the projection onto the one dimensional subspace generated by the constant function 1 in $L^2(\mathsf{G})$. We claim that

$$\liminf_{T\to\infty} f(T,t) \geq \exp\left(-t \left(E_\mathrm{N} + \int_{\mathbb{R}^d} \frac{|\hat{\varphi}(k)|^2}{2\omega(k)^2} dk \right) - \int_{\mathbb{R}^d} \frac{|\hat{\varphi}(k)|^2 (1 + e^{-t\omega(k)})}{2\omega(k)^3} dk \right). \tag{6.4.10}$$

To prove this write

$$f(T,t) = \frac{(\Psi_\mathrm{p}, e^{-TH_\mathrm{N}} (e^{-tH_\mathrm{p}} \otimes P_0) e^{-TH_\mathrm{N}} \Psi_\mathrm{p})}{(\Psi_\mathrm{p}, e^{-(2T+t)H_\mathrm{N}} \Psi_\mathrm{p})} \, \frac{(\Psi_\mathrm{p}, e^{-(2T+t)H_\mathrm{N}} \Psi_\mathrm{p})}{(\Psi_\mathrm{p}, e^{-2TH_\mathrm{N}} \Psi_\mathrm{p})}.$$

The second ratio above converges to $e^{-E_{\rm N}t}$ as $T \to \infty$ by Lemma 6.7 since $\Psi_{\rm p} > 0$. The first ratio, call it $g(t,T)$, can be written in terms of a functional integral. The denominator is

$$\mathbb{E}_{\mathscr{N}_0}[e^{S[-T,T+t]}]e^{-(2T+t)E_{\rm p}} \tag{6.4.11}$$

due to (6.3.15) and shift invariance of $(X_t)_{t\geq 0}$. For the numerator of $g(t,T)$ notice that

$$(\Psi_{\rm p}, e^{-TH_{\rm N}}(e^{-tH_{\rm p}} \otimes P_0)e^{-TH_{\rm N}}\Psi_{\rm p}) = (h_T, e^{-tH_{\rm p}}h_T)_{L^2(\mathbb{R}^d)}, \tag{6.4.12}$$

where $h_T(x) = (1, e^{-TH_{\rm N}}\Psi_{\rm p})_{L^2(\mathsf{G})}(x)$. Also, notice that

$$\begin{aligned}\int_{\mathbb{R}^d} h_T(x)f(x)\Psi_{\rm p}(x)dx &= (f\Psi_{\rm p}\otimes 1, e^{-TH_{\rm N}}\Psi_{\rm p}\otimes 1)\\ &= \mathbb{E}_{\mathscr{P}_0}[f(X_0)e^{-\int_0^t \xi_s(\varphi_{X_s})ds}]e^{-TE_{\rm p}}\\ &= \mathbb{E}_{\mathscr{N}_0}[f(X_0)\mathbb{E}_{\mathscr{G}}[e^{-\int_0^t \xi_s(\varphi_{X_s})ds}]]e^{-TE_{\rm p}}\\ &= \int_{\mathbb{R}^d} \Psi_{\rm p}(x)^2 f(x)\mathbb{E}^x_{\mathscr{N}_0}[e^{S[0,T]}]e^{-TE_{\rm p}}dx.\end{aligned}$$

Comparing the leftmost and right-hand side expressions we have

$$h_T(x) = \Psi_{\rm p}(x)F(x)e^{-TE_{\rm p}}$$

with $F(x) = \mathbb{E}^x_{\mathscr{N}_0}[e^{S[0,T]}]$. Thus

$$\begin{aligned}(h_T, e^{-tH_{\rm p}}h_T) &= (F, e^{-tL_{\rm p}}F)e^{-(2T+t)E_{\rm p}}\\ &= \mathbb{E}_{\mathscr{N}_0}[F(X_0)F(X_t)]e^{-(2T+t)E_{\rm p}}\\ &= \int_{\mathbb{R}^d} \mathbb{E}^x_{\mathscr{N}_0}[\mathbb{E}^x_{\mathscr{N}_0}[e^{S[0,T]}]\mathbb{E}^{X_t}_{\mathscr{N}_0}[e^{S[0,T]}]]e^{-(2T+t)E_{\rm p}}d\mathsf{N}_0.\end{aligned}$$

By reflection symmetry

$$(h_T, e^{-tH_{\rm p}}h_T) = \int_{\mathbb{R}^d} \mathbb{E}^x_{\mathscr{N}_0}[\mathbb{E}^x_{\mathscr{N}_0}[e^{S[-T,0]}]\mathbb{E}^{X_t}_{\mathscr{N}_0}[e^{S[0,T]}]]e^{-(2T+t)E_{\rm p}}d\mathsf{N}_0$$

and by the Markov property

$$(h_T, e^{-tH_{\rm p}}h_T) = \int_{\mathbb{R}^d} \mathbb{E}^x_{\mathscr{N}_0}[\mathbb{E}^x_{\mathscr{N}_0}[e^{S[-T,0]}]\mathbb{E}^x_{\mathscr{N}_0}[e^{S[t,T+t]}|\sigma(X_t)]]e^{-(2T+t)E_{\rm p}}d\mathsf{N}_0.$$

Independence of X_{-t}, $t \ge 0$, and X_s, $s \ge 0$, implies

$$\begin{aligned}(h_T, e^{-tH_\mathrm{p}} h_T) &= \int_{\mathbb{R}^d} \mathbb{E}^x_{\mathcal{N}_0}[\mathbb{E}^x_{\mathcal{N}_0}[e^{S[-T,0]+S[t,T+t]}|\sigma(X_t)]] e^{-(2T+t)E_\mathrm{p}} d\mathsf{N}_0 \\ &= \int_{\mathbb{R}^d} \mathbb{E}^x_{\mathcal{N}_0}[e^{S[-T,0]+S[t,T+t]}] e^{-(2T+t)E_\mathrm{p}} d\mathsf{N}_0 \\ &= \mathbb{E}_{\mathcal{N}_0}[e^{S[-T,0]+S[t,T+t]}] e^{-(2T+t)E_\mathrm{p}}.\end{aligned}$$

Finally, using (6.4.11) it follows that

$$g(T,t) = \frac{\mathbb{E}_{\mathcal{N}_0}[e^{S[-T,0]+S[t,T+t]}]}{\mathbb{E}_{\mathcal{N}_0}[e^{S[-T,T+t]}]} = \frac{\mathbb{E}_{\mathcal{N}_0}[e^{S_\Delta + S[-T,T+t]}]}{\mathbb{E}_{\mathcal{N}_0}[e^{S[-T,T+t]}]}, \tag{6.4.13}$$

where $S_\Delta = S[-T,0] + S[t,T+t] - S[-T,T+t]$. Making use of the uniform pathwise estimate

$$\begin{aligned}|S_\Delta| &\le \frac{1}{2}\int_{\mathbb{R}^d} \frac{|\hat{\varphi}(k)|^2}{2\omega(k)} dk \left(2\int_{-\infty}^0 \int_0^\infty + 2\int_0^t \int_t^\infty + \int_0^t \int_0^t\right) e^{-\omega(k)|t-s|} ds dt \\ &\le t \int_{\mathbb{R}^d} \frac{|\hat{\varphi}(k)|^2}{2\omega(k)^2} dk + \int_{\mathbb{R}^d} \frac{|\hat{\varphi}(k)|^2 (1 + 2e^{-t\omega(k)})}{2\omega(k)^3} dk,\end{aligned}$$

we can compare the numerator and denominator of $g(t,T)$ to find that

$$g(t,T) \ge \exp\left(-t \int_{\mathbb{R}^d} \frac{|\hat{\varphi}(k)|^2}{2\omega(k)^2} dk - \int_{\mathbb{R}^d} \frac{|\hat{\varphi}(k)|^2(1 + 2e^{-t\omega(k)})}{2\omega(k)^3} dk\right).$$

This proves (6.4.10), and we see that $\liminf_{T\to\infty} \|e^{-tH_\mathrm{p}/2} \otimes P_0 \Psi^T_\mathrm{g}\|$ is non-zero. In order to show that the family Ψ^T_g does not converge to zero we replace $e^{-tH_\mathrm{p}/2} \otimes P_0$ by a compact operator, which we will choose to be a spectral projection of $e^{-tH_\mathrm{p}/2}$. Let $1_{[a,b]}(H_\mathrm{p})$ denote the projection onto the spectral subspace of H_p corresponding to $\mathrm{Spec}(H_\mathrm{p}) \cap [a,b]$. By the definition of Σ_p and by standard properties of Schrödinger operators, H_p has only finitely many eigenvalues of finite multiplicity below $\Sigma_\mathrm{p} - \delta$ for every $\delta > 0$, and thus $1_{[E_\mathrm{p}, \Sigma_\mathrm{p} - \delta]}(H_\mathrm{p}) \otimes P_0$ is a finite rank operator. Then

$$(1_{[E_\mathrm{p},\Sigma_\mathrm{p}-\delta]}(H_\mathrm{p}) \otimes P_0)\Psi^T_\mathrm{g} \to (1_{[E_\mathrm{p},\Sigma_\mathrm{p}-\delta]}(H_\mathrm{p}) \otimes P_0)\Psi^\infty_\mathrm{g}$$

in strong sense as $T \to \infty$. On the other hand, the norm of the operator $e^{-tH_\mathrm{p}} \times 1_{(\Sigma_\mathrm{p}-\delta,\infty)}$ is bounded like

$$e^{-tH_\mathrm{p}} 1_{(\Sigma_\mathrm{p}-\delta,\infty)} \le e^{-t(\Sigma_\mathrm{p}-\delta)}. \tag{6.4.14}$$

Hence

$$\begin{aligned}&(\Psi^\infty_\mathrm{g}, (e^{-tH_\mathrm{p}} 1_{[E_\mathrm{p},\Sigma_\mathrm{p}-\delta]}(H_\mathrm{p}) \otimes P_0)\Psi^\infty_\mathrm{g}) \\ &= \lim_{T\to\infty} \{(\Psi^T_\mathrm{g}, (e^{-tH_\mathrm{p}} \otimes P_0)\Psi^T_\mathrm{g}) - (\Psi^T_\mathrm{g}, (e^{-tH_\mathrm{p}} 1_{(\Sigma_\mathrm{p}-\delta,\infty)}(H_\mathrm{p}) \otimes P_0)\Psi^T_\mathrm{g})\}\end{aligned}$$

and we estimate

$$
\begin{aligned}
&(\Psi_{\rm g}^{\infty}, (e^{-tH_{\rm p}} 1_{[E_{\rm p}, \Sigma_{\rm p}-\delta]}(H_{\rm p}) \otimes P_0)\Psi_{\rm g}^{\infty}) \\
&\geq e^{-t(E_{\rm N}+C)-C(t)} - e^{-t(\Sigma_{\rm p}-\delta)} = e^{-t(E_{\rm N}+C)}(e^{-C(t)} - e^{-t(\Sigma_{\rm p}-\delta-E_{\rm N}-C)}),
\end{aligned}
$$

where we used (6.4.10) and (6.4.14), and

$$
C = \int_{\mathbb{R}^d} \frac{|\hat{\varphi}(k)|^2}{2\omega(k)^2} dk, \quad C(t) = \int_{\mathbb{R}^d} \frac{|\hat{\varphi}(k)|^2(1+e^{-t\omega(k)})}{2\omega(k)^3} dk.
$$

By choosing δ small enough and t large enough we find that $\Psi_{\rm g}^{\infty}$ is not zero provided

$$
E_{\rm N} < \Sigma_{\rm p} - \int_{\mathbb{R}^d} \frac{|\hat{\varphi}(k)|^2}{2\omega(k)^2} dk. \tag{6.4.15}
$$

The final step is to convert (6.4.15) into an equation involving the bottom $E_{\rm p}$ of the spectrum of $H_{\rm p}$ instead of that of $H_{\rm N}$. This is achieved by the inequality

$$
E_{\rm N} \leq E_{\rm p} - \int_{\mathbb{R}^d} \frac{|\hat{\varphi}(k)|^2}{2\omega(k)(\omega(k)+|k|^2/2)} dk \tag{6.4.16}
$$

which will be proved in Lemma 6.10 below. Combining (6.4.16) and (6.4.15) completes the proof. □

The following lemma provides the missing piece in the proof above.

Lemma 6.10. (6.4.16) *holds.*

Proof. Instead of $L^2(\mathsf{N}_0) \otimes L^2(\mathsf{G})$ we show (6.4.16) on $\mathscr{H}_{\rm N} = L^2(\mathbb{R}^d) \otimes \mathscr{F}_{\rm N}$. Let $P_{\rm f} = d\Gamma(k)$ be the momentum operator of $\mathscr{F}_{\rm N}$, and define

$$
\psi_f = e^{ix\otimes P_{\rm f}} \Psi_{\rm p}(x) \otimes e^{-i\Pi(f)}\Omega_{\rm N},
$$

where $\Omega_{\rm N}$ is the Fock vacuum in $\mathscr{F}_{\rm N}$, $\Pi(f)$ denotes the conjugate momentum $\Pi(f) = i(a^*(f) - a(\bar{f}))$ and f will be determined below. By a direct computation,

$$
\begin{aligned}
E_{\rm N} \leq (\psi_f, H_{\rm N}\psi_f) = E_{\rm p} &+ \frac{1}{2}\left(\int_{\mathbb{R}^d} k|f(k)|^2 dk\right)^2 \\
&+ \int_{\mathbb{R}^d} \left\{ \left(\omega(k) + \frac{1}{2}|k|^2\right)|f(k)|^2 + \frac{\hat{\varphi}(k)(\overline{f(k)} + f(k))}{\sqrt{2\omega(k)}} \right\} dk.
\end{aligned} \tag{6.4.17}
$$

Let f be such that $f(-k) = f(k)$. The second term on the right-hand side of (6.4.17) is $\left(\int k|f(k)|^2 dk\right)^2 = 0$, and the last term is

$$
\int_{\mathbb{R}^d} \left(\omega(k) + \frac{1}{2}|k|^2\right)\left(|f(k)+\Phi(k)|^2 - \Phi^2(k)\right) dk,
$$

where

$$\Phi(k) = -\frac{\hat{\varphi}(k)}{\sqrt{2\omega(k)}(\omega(k) + |k|^2/2)}.$$

The minimizer $f(k) = \Phi(k)$ yields the bound (6.4.16). $\square$

Corollary 6.11 (Existence of ground state for any coupling strength). *Suppose that* $\omega(k) = |k|$, $\hat{\varphi}(k) = g1_{\{\kappa<|k|<\Lambda\}}$ *with coupling constant* $g \in \mathbb{R}$, *and* $\int_{\mathbb{R}^d} |\hat{\varphi}(k)|^2/\omega(k)^3 dk < \infty$. *Suppose also that* $\mathrm{Spec}(H_\mathrm{p})$ *is purely discrete. Then for all* $0 < \kappa < \Lambda$ *and* $g \in \mathbb{R}$, H_N *has the unique ground state.*

Proof. Since $\Sigma_\mathrm{p} - E_\mathrm{p} = \infty$, the corollary follows from Theorem 6.6. $\square$

For confining potentials V the operator H has purely discrete spectrum (Theorem 3.20), which gives $\Sigma_\mathrm{p} - E_\mathrm{p} = \infty$.

6.5 Ground state expectations

6.5.1 General theorems

Throughout this section we suppose that the infrared regularity condition (6.2.17) holds and H_N has a ground state $\Psi_\mathrm{g} \in L^2(\mathsf{P}_0)$.

We write

$$d\mathcal{P}_T = \frac{1}{Z_T} e^{\int_{-T}^{T} \xi_s(\varphi_{X_s})ds} d\mathcal{P}_0, \tag{6.5.1}$$

with normalizing constant Z_T. Then

$$(\Psi_\mathrm{g}^T, (f \otimes 1)\Psi_\mathrm{g}^T) = \mathbb{E}_{\mathcal{P}_T}[f(X_0)].$$

Due to linearity of the coupling the field variables ξ can be integrated out in $\mathbb{E}_{\mathcal{P}_T}[f(X_0)]$ and this gives the functional integral representation

$$(\Psi_\mathrm{g}^T, (f \otimes 1)\Psi_\mathrm{g}^T) = \mathbb{E}_{\mathcal{N}_T}[f(X_0)],$$

where

$$d\mathcal{N}_T = \frac{1}{Z_T} \exp\left(\frac{1}{2}\int_{-T}^{T} ds \int_{-T}^{T} dt\, W(X_s - X_t, |s-t|)\right) d\mathcal{N}_0. \tag{6.5.2}$$

Then, formally,

$$(\Psi_\mathrm{g}, (f \otimes 1)\Psi_\mathrm{g}) = \lim_{T\to\infty} (\Psi_\mathrm{g}^T, (f \otimes 1)\Psi_\mathrm{g}^T) = \mathbb{E}_{\mathcal{N}}[f(X_0)]$$

with some measure $\mathcal{N}$. In the next theorem we establish the existence of this measure.

Theorem 6.12 (Tightness). *The family of probability measures $\{\mathcal{N}_T\}_{T\geq 0}$ is tight. In particular, there exists a subsequence T' such that $\mathcal{N}_{T'}$ is weakly convergent to a probability measure $\mathcal{N}$ on $\mathfrak{X} = C(\mathbb{R};\mathbb{R}^d)$.*

Proof. By the Prokhorov theorem it suffices to show that

(1) $\lim_{\Lambda\to\infty}\sup_T \mathcal{N}_T(|X_0|^2 > \Lambda) = 0$,

(2) $\lim_{\delta\downarrow 0}\sup_T \mathcal{N}_T\left(\max_{|t-s|<\delta, -T\leq s,t\leq T}|X_t - X_s| > \varepsilon,\right) = 0$ for any $\varepsilon > 0$.

We have $\mathcal{N}_T(|X_0|^2 > \Lambda) = (\Psi_{\rm g}^T, 1_{\{|x|^2>\Lambda\}}\Psi_{\rm g}^T)$. Let $\varepsilon > 0$ be arbitrary. Since $\Psi_{\rm g}^T \to \Psi_{\rm g}$ strongly as $T\to\infty$, there exists $T_0 > 0$ such that for all $T > T_0$,

$$(\Psi_{\rm g}^T, 1_{\{|x|^2>\Lambda\}}\Psi_{\rm g}^T) \leq (\Psi_{\rm g}, 1_{\{|x|^2>\Lambda\}}\Psi_{\rm g}) + \varepsilon. \tag{6.5.3}$$

Hence

$$\mathcal{N}_T(|X_0|^2 > \Lambda) \leq \sup_{0\leq T\leq T_0}(\Psi_{\rm g}^T, 1_{\{|x|^2>\Lambda\}}\Psi_{\rm g}^T) + (\Psi_{\rm g}, 1_{\{|x|^2>\Lambda\}}\Psi_{\rm g}) + \varepsilon. \tag{6.5.4}$$

We estimate the first factor in the right-hand side above. Note that

$$(\Psi_{\rm g}^T, 1_{\{|x|^2>\Lambda\}}\Psi_{\rm g}^T) = \frac{\mathbb{E}_{\mathcal{N}_0}[1_{\{|X_0|^2>\Lambda\}}e^{\frac{1}{2}\int_{-T}^T ds\int_{-T}^T dtW}]}{\mathbb{E}_{\mathcal{N}_0}[e^{\frac{1}{2}\int_{-T}^T ds\int_{-T}^T dtW}]}. \tag{6.5.5}$$

Since

$$\frac{1}{2}\int_{-T}^T ds\int_{-T}^T Wdt \leq \int_{\mathbb{R}^d}\frac{|\hat{\varphi}(k)|^2}{4\omega(k)^3}(e^{-2T\omega(k)} - 1 + 2T\omega(k))dk \leq aT + b$$

with some a, b, together with (6.5.5) it gives

$$(\Psi_{\rm g}^T, 1_{\{|x|^2>\Lambda\}}\Psi_{\rm g}^T) \leq \frac{\mathbb{E}_{\mathcal{N}_0}[1_{\{|X_0|^2>\Lambda\}}]e^{aT+b}}{e^{-(aT+b)}} = (\Psi_{\rm p}, 1_{\{|x|^2>\Lambda\}}\Psi_{\rm p})e^{2(aT+b)}. \tag{6.5.6}$$

Thus $\sup_{0\leq T\leq T_0}(\Psi_{\rm g}^T, 1_{\{|x|^2>\Lambda\}}\Psi_{\rm g}^T) \to 0$ as $\Lambda\to\infty$ and (1) follows.

To prove (2) it suffices to show that

$$\mathbb{E}_{\mathcal{N}_T}[|X_s - X_t|^{2n}] \leq D|t-s|^n \tag{6.5.7}$$

for some $n \geq 1$ with a constant D. We have

$$\mathbb{E}_{\mathcal{N}_T}[|X_s - X_t|^{2n}] = \sum_{\mu=1}^d\sum_{k=0}^{2n}\binom{2n}{k}(-1)^k\mathbb{E}_{\mathcal{N}_T}[(X_s^\mu)^{2n-k}(X_t^\mu)^k]$$

$$= \sum_{\mu=1}^d\sum_{k=0}^{2n}\binom{2n}{k}(-1)^k(e^{-sH_{\rm N}}\Psi_{\rm g}^T, x_\mu^{2n-k}e^{-(t-s)H_{\rm N}}x_\mu^k e^{+tH_{\rm N}}\Psi_{\rm g}^T).$$

The last term can be represented by functional integration given by Theorem 6.3 in terms of Brownian motion B_t and the Euclidean field $\phi_{\rm E}$ as

$$
\begin{aligned}
&\mathbb{E}_{\mathcal{N}_T}[|X_s - X_t|^{2n}] \\
&= \int_{\mathbb{R}^d} dx \mathbb{E}^x[|B_0 - B_{t-s}|^{2n} e^{-\int_0^{t-s} V(B_r)dr} \\
&\quad \times (\mathrm{I}_0 e^{-sH_{\rm N}} \Psi_{\rm g}^T(B_0), e^{\phi_{\rm E}(\int_0^{t-s} \delta_r \otimes \varphi(\cdot - B_r)dr)} \mathrm{I}_{t-s} e^{+tH_{\rm N}} \Psi_{\rm g}^T(B_{t-s}))] \\
&\le \int_{\mathbb{R}^d} dx \mathbb{E}^x[|B_0 - B_{t-s}|^{2n} e^{-\int_0^{t-s} V(B_r)dr} \\
&\quad \times \|e^{-sH_{\rm N}} \Psi_{\rm g}^T(B_0)\| \|e^{\phi_{\rm E}(\int_0^{t-s} \delta_r \otimes \varphi(\cdot - B_r)dr)}\|_1 \|e^{+tH_{\rm N}} \Psi_{\rm g}^T(B_{t-s})\|],
\end{aligned}
$$

where the norms with no subscript are $L^2(\mathscr{Q})$-norms, and those with subscript 1 are $L^1(\mathscr{Q}_{\rm E})$-norms. We also note that

$$
\begin{aligned}
&\|e^{\phi_{\rm E}(\int_0^{t-s} \delta_r \otimes \varphi(\cdot - B_r)dr)}\|_1 = (1, e^{\phi_{\rm E}(\int_0^{t-s} \delta_r \otimes \varphi(\cdot - B_r)dr)} 1) \\
&= e^{\frac{1}{2}\int_0^{t-s} dr \int_0^{t-s} dr' W(B_r - B_{r'}, |r-r'|)} \le e^{\frac{t-s}{2}\int_{\mathbb{R}^d} \frac{|\hat{\varphi}(k)|^2}{\omega(k)^2} dk + \int_{\mathbb{R}^d} \frac{|\hat{\varphi}(k)|^2}{\omega(k)^3} dk} = C.
\end{aligned}
\tag{6.5.8}
$$

Then by Schwarz inequality,

$$
\begin{aligned}
&\mathbb{E}_{\mathcal{N}_T}[|X_s - X_t|^{2n}] \\
&\le C \mathbb{E}\bigg[|B_0 - B_{t-s}|^{2n} \int dx e^{-\int_0^{t-s} V(B_r + x)dr} \\
&\qquad\qquad \times \|e^{-sH_{\rm N}} \Psi_{\rm g}^T(x)\| \|e^{tH_{\rm N}} \Psi_{\rm g}^T(B_{t-s} + x)\| \bigg] \\
&\le C \mathbb{E}\bigg[|B_0 - B_{t-s}|^{2n} \bigg(\int dx e^{-2\int_0^{t-s} V(B_r + x)dr} \|e^{-sH_{\rm N}} \Psi_{\rm g}^T(x)\|^2 \bigg)^{1/2} \\
&\qquad \times \bigg(\int dx \|e^{tH_{\rm N}} \Psi_{\rm g}^T(B_{t-s} + x)\|^2 \bigg)^{1/2} \bigg] \\
&\le C(\mathbb{E}[|B_0 - B_{t-s}|^{4n}])^{1/2} \bigg(\mathbb{E}\bigg[\bigg(\int dx e^{-2\int_0^{t-s} V(B_r + x)dr} \|e^{-sH_{\rm N}} \Psi_{\rm g}^T(x)\|^2 \bigg) \\
&\qquad \times \bigg(\int dx \|e^{tH_{\rm N}} \Psi_{\rm g}^T(B_{t-s} + x)\|^2 \bigg) \bigg] \bigg)^{1/2} \\
&\le D|t-s|^n \big(\sup_x \mathbb{E}^x[e^{-2\int_0^{t-s} V(B_r)dr}] \big)^{1/2} \|e^{-sH_{\rm N}} \Psi_{\rm g}^T\|_{\mathscr{H}_{\rm N}} \|e^{tH_{\rm N}} \Psi_{\rm g}^T\|_{\mathscr{H}_{\rm N}}.
\end{aligned}
$$

Finally, noticing that $\|e^{-sH_{\rm N}} \Psi_{\rm g}^T\| \|e^{tH_{\rm N}} \Psi_{\rm g}^T\| \to \|\Psi_{\rm g}\|^2$ as $T \to \infty$, we complete (2). $\square$

We now establish an explicit formula for expectations $(\Psi_{\rm g}, (g \otimes L)\Psi_{\rm g})$ of an operator $g \otimes L$ as averages with respect to the probability measure $\mathscr{N}$. In order to state our main theorem, we introduce some special elements of $\mathscr{M}_{\mathbb{C}}$. For every $T \in [0, \infty]$ and every path define

$$\varphi_{T,X}^{+}(x) = -\int_0^T e^{-|s|\hat{\omega}} \varphi_{X_s}(x) ds \quad \text{i.e. } \widehat{\varphi_{T,X}^{+}}(k) = -\int_0^T \hat{\varphi}(k) e^{-ikX_s} e^{-\omega(k)|s|} ds$$

$$\varphi_{T,X}^{-}(x) = -\int_{-T}^0 e^{-|s|\hat{\omega}} \varphi_{X_s}(x) ds \quad \text{i.e. } \widehat{\varphi_{T,X}^{-}}(k) = -\int_{-T}^0 \hat{\varphi}(k) e^{-ikX_s} e^{-\omega(k)|s|} ds,$$

where $\hat{\omega} = \omega(-i\nabla)$. We also define

$$\varphi_X^{+}(x) = -\int_0^\infty e^{-|s|\hat{\omega}} \varphi_{X_s}(x) ds,$$
$$\varphi_X^{-}(x) = -\int_{-\infty}^0 e^{-|s|\hat{\omega}} \varphi_{X_s}(x) ds.$$

It follows that $\varphi_{T,X}^{\pm} \in \mathscr{M}_{\mathbb{C}}$ and

$$(\varphi_{T,X}^{-}, \varphi_{T,X}^{+})_{\mathscr{M}_{\mathbb{C}}} = \int_{-T}^0 ds \int_0^T W(X_t - X_s, t-s) dt, \tag{6.5.9}$$

$$\|\varphi_{T,X}^{\pm}\|_{\mathscr{M}_{\mathbb{C}}}^2 \le \int_{\mathbb{R}^d} \frac{|\hat{\varphi}(k)|^2}{2\omega(k)^3} dk < \infty. \tag{6.5.10}$$

Write

$$\mathscr{P}_0^{x,\xi} = \mathscr{N}_0^x \otimes \mathscr{G}^{\xi}, \quad (x, \xi) \in \mathbb{R}^d \times \mathscr{M}_{-2}. \tag{6.5.11}$$

Theorem 6.13 (Ground state expectations for bounded operators). *Let L be a bounded operator on $L^2(\mathsf{G})$, and $g \in L^\infty(\mathbb{R}^d)$ viewed as a multiplication operator. Then*

$$\begin{aligned}
&(\Psi_{\rm g}, (g \otimes L)\Psi_{\rm g}) \\
&= \mathbb{E}_{\mathscr{N}}[(:e^{\xi(\varphi_X^{-})}:, L:e^{\xi(\varphi_X^{+})}:)_{L^2(\mathsf{G})} g(X_0) e^{-\int_{-\infty}^0 ds \int_0^\infty dt W(X_t - X_s, t-s)}].
\end{aligned} \tag{6.5.12}$$

Proof. By Theorem 6.2 we have

$$(F, \Psi_{\rm g}^T) = \frac{1}{\|e^{-TL_{\rm N}}1\|}(F, e^{-TL_{\rm N}}1) = \frac{1}{\sqrt{Z_T}} \mathbb{E}_{\mathsf{G}\times\mathsf{N}_0}[\overline{F(x,\xi)} \mathbb{E}_{\mathscr{P}_0}^{x,\xi}[e^{-\int_0^T \xi_s(\varphi_{X_s}) ds}]].$$

Hence we arrive at

$$\Psi_{\rm g}^T(x, \xi) = \frac{1}{\sqrt{Z_T}} \mathbb{E}_{\mathscr{P}_0}^{x,\xi}\left[\exp\left(-\int_0^T \xi_s(\varphi_{X_s}) ds\right)\right] \tag{6.5.13}$$

for almost every $(x,\xi) \in \mathbb{R}^d \times \mathscr{M}_{-2}$. Notice that $\mathbb{E}^{x,\xi}_{\mathscr{P}_0} = \mathbb{E}^x_{\mathscr{N}_0}\mathbb{E}^\xi_{\mathscr{G}}$. We can compute $\mathbb{E}^\xi_{\mathscr{G}}[\exp(-\int_0^T \xi_s(\varphi_{X_s})ds)]$ by using that (see Theorem 5.34)

$$\mathbb{E}^\xi_{\mathscr{G}}\left[\int_0^T \xi_s(\varphi_{X_s})ds\right] = \xi(\varphi^+_{T,X}),$$

$$\mathbb{E}^\xi_{\mathscr{G}}\left[\left(\int_0^T \xi_s(\varphi_{X_s})ds\right)^2\right] - \left(\mathbb{E}^\xi_{\mathscr{G}}\left[\int_0^T \xi_s(\varphi_{X_s})ds\right]\right)^2$$
$$= \frac{1}{2}\int_0^T ds \int_0^T dt \int_{\mathbb{R}^d} dk \frac{|\hat{\varphi}(k)|^2}{2\omega(k)} e^{-ik\cdot(X_s - X_t)}(e^{-\omega(k)|t-s|} - e^{-\omega(k)(|t|+|s|)}).$$

Now the integration with respect to $\mathscr{G}^\xi$ in (6.5.13) can be carried out with the result

$$\Psi^T_{\mathrm{g}}(x,\xi) = \frac{1}{\sqrt{Z_T}} \times \mathbb{E}^x_{\mathscr{N}_0}[e^{\xi(\varphi^+_{T,X})} e^{\frac{1}{2}\int_0^T ds \int_0^T dt \int_{\mathbb{R}^d} dk \frac{|\hat{\varphi}(k)|^2}{2\omega(k)} e^{-ik\cdot(X_s - X_t)}\left(e^{-\omega(k)|t-s|} - e^{-\omega(k)(|t|+|s|)}\right)}]. \tag{6.5.14}$$

By the Wick ordering $:e^{\xi(f)}: = e^{-(1/2)\|f\|^2_{\mathscr{M}}} e^{\xi(f)}$ from (5.2.56), and then

$$e^{\xi(\varphi^+_{T,X})} = :e^{\xi(\varphi^+_{T,X})}: e^{\frac{1}{2}\int_0^T ds \int_0^T dt \int_{\mathbb{R}^d} dk \frac{|\hat{\varphi}(k)|^2}{2\omega(k)} e^{-ik(X_s - X_t)} e^{-\omega(k)(|t|+|s|)}}. \tag{6.5.15}$$

Hence, by combining (6.5.14) and (6.5.15),

$$\Psi^T_{\mathrm{g}}(x,\xi) = \frac{1}{\sqrt{Z_T}} \mathbb{E}^x_{\mathscr{N}_0}[:e^{\xi(\varphi^+_{T,X})}: e^{\frac{1}{2}\int_0^T ds \int_0^T W(X_s - X_t, s-t)dt}]. \tag{6.5.16}$$

By the reflection symmetry of X_t, we also have

$$\Psi^T_{\mathrm{g}}(x,\xi) = \frac{1}{\sqrt{Z_T}} \mathbb{E}^x_{\mathscr{N}_0}[:e^{\xi(\varphi^-_{T,X})}: e^{\frac{1}{2}\int_{-T}^0 ds \int_{-T}^0 W(X_s - X_t, s-t)dt}]. \tag{6.5.17}$$

Now we write (6.5.17) for the left and (6.5.16) for the right entry of the scalar product $(\Psi^T_{\mathrm{g}}, (1 \otimes L)\Psi^T_{\mathrm{g}})$, use independence of X_s, $s \le 0$, and X_t, $t \ge 0$, and we add and subtract the term $-\int_{-T}^0 ds \int_0^T W(X_s - X_t, s-t)dt$ in the exponent to obtain

$$\begin{aligned}&(\Psi^T_{\mathrm{g}}, (g \otimes L)\Psi^T_{\mathrm{g}})_{L^2(\mathsf{P}_0)} \\ &= \mathbb{E}_{\mathscr{N}_T}[(:e^{\xi(\varphi^-_{T,X})}:, L:e^{\xi(\varphi^+_{T,X})}:) g(X_0) e^{-\int_{-T}^0 ds \int_0^T W(X_t - X_s, t-s)dt}].\end{aligned} \tag{6.5.18}$$

This is the finite T version of (6.5.12). It remains to take the limit $T \to \infty$. On the left-hand side of (6.5.18) this is immediate since $\Psi^T_{\mathrm{g}} \to \Psi_{\mathrm{g}}$ in $L^2(\mathsf{P}_0)$ and L

is continuous. On the right-hand side, we already know that $\mathcal{N}_{T'} \to \mathcal{N}$ weakly with some subsequence. Reset T' to T. Thus it only remains to show that the integrand of (6.5.18) is uniformly convergent in the path. For the first factor we find that $|\widehat{\varphi^{\pm}_{T,X}}(k)| \le |\hat{\varphi}(k)|/\omega(k)$ uniformly in T and paths, and $\widehat{\varphi^{\pm}_{T,X}} \to \widehat{\varphi^{\pm}_X}$ strongly in $\mathscr{M}_{\mathbb{C}}$ as $T \to \infty$ uniformly in paths. Thus $:e^{\xi(\varphi^{+}_{T,X})}: \to :e^{\xi(\varphi^{+}_{X})}:$ in $L^2(\mathsf{G})$ uniformly in path. Since the same argument applies to $\varphi^{-}_{T,X}$ and L is continuous, $(:e^{\xi(\varphi^{-}_{T,X})}:, L:e^{\xi(\varphi^{+}_{T,X})}:)$ is uniformly convergent to $(:e^{\xi(\varphi^{-}_{X})}:, L:e^{\xi(\varphi^{+}_{X})}:)$ as $T \to \infty$. Moreover, the infrared regularity condition implies

$$\left| \int_{-T}^{0} ds \int_{0}^{T} W(X_s - X_t, s-t) dt \right| \le \int_{\mathbb{R}^d} \frac{|\hat{\varphi}(k)|^2}{2\omega(k)^3} dk < \infty \tag{6.5.19}$$

and hence

$$\int_{-T}^{0} ds \int_{0}^{T} W(X_s - X_t, s-t) dt \to \int_{-\infty}^{0} ds \int_{0}^{-\infty} W(X_s - X_t, s-t) dt \tag{6.5.20}$$

uniformly in path. Hence the integrand of (6.5.18) is uniformly convergent as $T \to \infty$. □

Most operators of physical interest are, however, not bounded. Therefore we need to extend formula (6.5.12) to unbounded operators.

Theorem 6.14 (Ground state expectations for unbounded operators). *Let g be a measurable function on $\mathbb{R}^d$ and L a self-adjoint operator in $L^2(\mathsf{G})$ such that*

$$\mathbb{E}_{\mathcal{N}}[|g(X_0)|^2 \|L:e^{\xi(\varphi^{\pm}_X)}:\|^2_{L^2(\mathsf{G})}] < \infty. \tag{6.5.21}$$

Then $\Psi_{\mathrm{g}} \in D(g \otimes L)$, and (6.5.12) *holds.*

Proof. Without loss of generality we can suppose that g is real-valued, and

$$g_M(x) = \begin{cases} g(x), & |g(x)| < M, \\ M, & |g| \ge M. \end{cases}$$

Let $L_N = 1_{[-N,N]}(L)$. Then $g_M \otimes L_N$ is a bounded operator, hence (6.5.12) holds for $g_M \otimes L_N$. Let $I = \int |\hat{\varphi}(k)|^2/\omega(k)^3 dk$. Using Schwarz inequality, we have

$$\begin{aligned}
&\|(g_M \otimes L_N)\Psi_{\mathrm{g}}\|^2 \\
&= \mathbb{E}_{\mathcal{N}}[(:e^{\xi(\varphi^{-}_X)}:, L_N^2:e^{\xi(\varphi^{+}_X)}:)|g_M(X_0)|^2 e^{-\int_{-\infty}^{0} ds \int_0^{\infty} dt\, W(X_t - X_s, t-s)}] \\
&\le e^I \mathbb{E}_{\mathcal{N}}[|g_M(X_0)|^2 \|L_N:e^{\xi(\varphi^{-}_X)}:\| \|L_N:e^{\xi(\varphi^{+}_X)}:\|] \\
&\le e^I \mathbb{E}_{\mathcal{N}}[|g(X_0)|^2 \|L:e^{\xi(\varphi^{-}_X)}:\| \|L:e^{\xi(\varphi^{+}_X)}:\|],
\end{aligned}$$

which is bounded according to (6.5.21). By the monotone convergence theorem, this shows that $\Psi_g \in D(g \otimes L)$, and then we derive that $(g_M \otimes L_N)\Psi_g \to (g \otimes L)\Psi_g$ as $M, N \to \infty$. Hence

$$(\Psi_g, (g_M \otimes L_N)\Psi_g) \to (\Psi_g, (g \otimes L)\Psi_g) \tag{6.5.22}$$

as $N, M \to \infty$. On the other hand, it follows from (6.5.21) that $:e^{\xi(\varphi_X^{\pm})}: \in D(L)$ for $\mathcal{N}$-almost all paths. From this we conclude

$$(:e^{\xi(\varphi_X^-)}:, L_N:e^{\xi(\varphi_X^+)}:) \to (:e^{\xi(\varphi_X^-)}:, L:e^{\xi(\varphi_X^+)}:)$$

for $\mathcal{N}$ almost all paths as $N \to \infty$. We also have

$$(:e^{\xi(\varphi_X^-)}:, L_N:e^{\xi(\varphi_X^+)}:) \le \|:e^{\xi(\varphi_X^-)}:\| \|L_N:e^{\xi(\varphi_X^+)}:\| \le e^{I/2}\|L:e^{\xi(\varphi_X^+)}:\|$$

for all paths, and the right-hand side of the above is $\mathcal{N}$-integrable. Thus the dominated convergence theorem implies

$$\begin{aligned} &\mathbb{E}_{\mathcal{N}}[(:e^{\xi(\varphi_X^-)}:, L_N:e^{\xi(\varphi_X^+)}:) g_M(X_0) e^{-\int_{-\infty}^0 ds \int_0^\infty W(X_t - X_s, t-s)dt}] \\ &\to \mathbb{E}_{\mathcal{N}}[(:e^{\xi(\varphi_X^-)}:, L:e^{\xi(\varphi_X^+)}:) g(X_0) e^{-\int_{-\infty}^0 ds \int_0^\infty W(X_t - X_s, t-s)dt}] \end{aligned} \tag{6.5.23}$$

as $M, N \to \infty$. By (6.5.22) and (6.5.23), the proof is complete. □

6.5.2 Spatial decay of the ground state

We have already seen that in many cases the ground state of Schrödinger operators decays exponentially. Our interest is now the spatial decay of the ground state Ψ_g of H_N.

Recall the class $\mathbb{V}^{\text{upper}}$ defined in Definition 3.26. The key estimate is

Lemma 6.15 (Carmona's estimate). *Let $V \in \mathbb{V}^{\text{upper}}$. Then for any $t, a > 0$ and every $0 < \alpha < 1/2$, there exist constants $D_1, D_2, D_3 > 0$ such that*

$$\|\Psi_g(x)\|_{L^2(\mathscr{Q})} \le D_1 e^I e^{D_2 \|U\|_p t} e^{\tilde{E}_N t} (D_3 e^{-\frac{\alpha}{4}\frac{a^2}{t}} e^{-tW_\infty} + e^{-tW_a(x)}) \|\Psi_g\|_{\mathscr{H}_N}, \tag{6.5.24}$$

where $I = \int_{\mathbb{R}^d} |\hat{\varphi}(k)|^2 \omega(k)^{-3} dk$, $\widetilde{E}_N = E_N + \int_{\mathbb{R}^d} |\hat{\varphi}(k)|^2 \omega(k)^{-2} dk$ and $W_a(x) = \inf\{W(y) | |x-y| < a\}$.

Proof. Since $\Psi_g = e^{tE_N} e^{-tH_N} \Psi_g$, the functional integral representation yields that

$$\Psi_g(x) = \mathbb{E}^x[\mathrm{I}_0^* e^{-\phi_E(\int_0^t \delta_s \otimes \varphi(\cdot - B_s) ds)} \mathrm{I}_t e^{-\int_0^t V(B_s) ds} \Psi_g(B_t)]. \tag{6.5.25}$$

From this it follows directly that

$$\|\Psi_{\rm g}(x)\|_{L^2(\mathscr{Q})} \le \|e^{-\phi_{\rm E}(\int_0^t \delta_s\otimes\varphi(\cdot-B_s)ds)}\|_{L^1(\mathscr{Q})}\mathbb{E}^x[e^{-\int_0^t V(B_s)ds}\|\Psi_{\rm g}(B_t)\|_{L^2(\mathscr{Q})}]. \tag{6.5.26}$$

Since $\|e^{-\phi_{\rm E}(\int_0^t \delta_s\otimes\varphi(\cdot-B_s)ds)}\|_{L^1(\mathscr{Q})} \le e^{I+t\int_{\mathbb{R}^d}|\hat\varphi(k)|^2\omega(k)^{-2}dk}$ by (6.5.8), the lemma follows in a similar as Carmona's estimate in Lemma 3.59 for Schrödinger operators. $\square$

From the above lemma we can draw results similar to Corollaries 3.60 and 3.61 for Schrödinger operators. For $V = W - U \in \mathbb{V}^{\rm upper}$, define $\Sigma = \liminf_{|x|\to\infty} V(x)$. Notice that $\Sigma = \liminf_{|x|\to\infty} W(x)$, and $\Sigma \ge W_\infty$ holds. We only state the results.

Corollary 6.16. *Let $V = W - U \in \mathbb{V}^{\rm upper}$.*

(1) *Suppose that $W(x) \ge \gamma|x|^{2n}$ outside a compact set K, for some $n > 0$ and $\gamma > 0$. Take $0 < \alpha < 1/2$. Then there exists a constant $C_1 > 0$ such that*

$$\|\Psi_{\rm g}(x)\|_{L^2(\mathscr{Q})} \le C_1 \exp\left(-\frac{\alpha c}{16}|x|^{n+1}\right)\|\Psi_{\rm g}\|_{\mathscr{H}_{\rm N}}. \tag{6.5.27}$$

where $c = \inf_{x\in\mathbb{R}^d\setminus K} W_{|x|/2}(x)/|x|^{2n}$.

(2) *Decaying potential: Suppose that $\Sigma > \widetilde{E}_{\rm N}$, $\Sigma > W_\infty$, and let $0<\beta<1$. Then there exists a constant $C_2 > 0$ such that*

$$\|\Psi_{\rm g}(x)\|_{L^2(\mathscr{Q})} \le C_2 \exp\left(-\frac{\beta}{8\sqrt{2}}\frac{(\Sigma-\widetilde{E}_{\rm N})}{\sqrt{\Sigma - W_\infty}}|x|\right)\|\Psi_{\rm g}\|_{\mathscr{H}_{\rm N}}. \tag{6.5.28}$$

(3) *Confining potential: Suppose that $\lim_{|x|\to\infty} W(x) = \infty$. Then there exist constants $C, \delta > 0$ such that*

$$\|\Psi_{\rm g}(x)\|_{L^2(\mathscr{Q})} \le C \exp(-\delta|x|)\|\Psi_{\rm g}\|_{\mathscr{H}_{\rm N}}. \tag{6.5.29}$$

6.5.3 Ground state expectation for second quantized operators

Our next goal is to find similar expressions for the ground state expectation of second quantized operators.

Theorem 6.17. *Let L be a bounded self-adjoint operator on $L^2(\mathbb{R}^d)\cap\mathscr{M}_{\mathbb{C}}$. Then $\Psi_{\rm g} \in D(\Gamma(L)) \cap D(d\Gamma(L))$ and*

$$(\Psi_{\rm g}, \Gamma(L)\Psi_{\rm g})_{L^2(\mathsf{P}_0)} = \mathbb{E}_{\mathscr{N}}[e^{(\varphi_X^-, L\varphi_X^+)_{\mathscr{M}_{\mathbb{C}}}}e^{-\int_{-\infty}^0 ds\int_0^\infty W(X_s-X_t, s-t)dt}] \tag{6.5.30}$$

$$(\Psi_{\rm g}, d\Gamma(L)\Psi_{\rm g})_{L^2(\mathsf{P}_0)} = \mathbb{E}_{\mathscr{N}}[(\varphi_X^-, L\varphi_X^+)_{\mathscr{M}_{\mathbb{C}}}]. \tag{6.5.31}$$

Proof. Note that

$$\|\Gamma(L){:}e^{\xi(\varphi_X^{\pm})}{:}\|_{L^2(\mathsf{G})}^2 = e^{\|L\varphi_X^{+}\|_{\mathscr{M}_\mathbb{C}}^2}, \tag{6.5.32}$$

$$\|d\Gamma(L){:}e^{\xi(\varphi_X^{\pm})}{:}\|_{L^2(\mathsf{G})}^2 = (\|L\varphi_X^{\pm}\|_{\mathscr{M}_\mathbb{C}}^2 + (\varphi_X^{\pm}, L\varphi_X^{\pm})_{\mathscr{M}_\mathbb{C}}^2)e^{\|\varphi_X^{\pm}\|_{\mathscr{M}_\mathbb{C}}^2}, \tag{6.5.33}$$

and $\|\varphi_X^{\pm}\|_{\mathscr{M}_\mathbb{C}} \le \int |\hat{\varphi}|^2/2\omega^3 < \infty$. Since $\|Lf\|_{\mathscr{M}_\mathbb{C}} \le \|L\|\|f\|_{\mathscr{M}_\mathbb{C}}$ for all $f \in \mathscr{M}_\mathbb{C}$, (6.5.21) is satisfied. By Theorem 6.14, the theorem follows. □

We now discuss some particular cases.

Corollary 6.18 (Super-exponential decay of boson number). *Let $\alpha \in \mathbb{C}$. Then $\Psi_\mathrm{g} \in D(e^{\alpha N})$ and*

$$(\Psi_\mathrm{g}, e^{\alpha N}\Psi_\mathrm{g}) = \mathbb{E}_\mathscr{N}[e^{-(1-e^\alpha)\int_{-\infty}^0 ds \int_0^\infty W(X_s-X_t,s-t)dt}]. \tag{6.5.34}$$

Proof. Since $e^{\alpha N} = \Gamma(e^\alpha 1)$, Theorem 6.17 gives the result. □

By Corollary 6.18

$$(\Psi_\mathrm{g}, N^n\Psi_\mathrm{g}) = \frac{d^n}{d\alpha^n}(\Psi_\mathrm{g}, e^{\alpha N}\Psi_\mathrm{g})\lceil_{\alpha=0}. \tag{6.5.35}$$

In particular,

$$(\Psi_\mathrm{g}, N\Psi_\mathrm{g}) = \mathbb{E}_\mathscr{N}\left[\int_{-\infty}^0 ds \int_0^\infty W(X_s - X_t, s-t)dt\right]. \tag{6.5.36}$$

Corollary 6.19 (Pull-through formula). *We have*

$$(\Psi_\mathrm{g}, N\Psi_\mathrm{g}) = \int_{\mathbb{R}^d} \frac{|\hat{\varphi}(k)|^2}{2\omega(k)}\|(H_\mathrm{N} - E_\mathrm{N} + \omega(k))^{-1}e^{-ikx}\Psi_\mathrm{g}\|^2 dk. \tag{6.5.37}$$

Proof. Computing (6.5.36) we obtain

$$(\Psi_\mathrm{g}, N\Psi_\mathrm{g}) = \int_{-\infty}^0 dt \int_0^\infty ds \int_{\mathbb{R}^d} \frac{|\hat{\varphi}(k)|^2}{2\omega(k)} e^{-|t-s|\omega(k)}\mathbb{E}_\mathscr{N}[e^{-ik(X_s-X_t)}]dk.$$

Since

$$\begin{aligned}
&\int_{-\infty}^0 dt \int_0^\infty ds e^{-|t-s|\omega(k)}\mathbb{E}_\mathscr{N}[e^{-ik(X_s-X_t)}] \\
&= \int_{-\infty}^0 dt \int_0^\infty ds (e^{-ikx}\Psi_\mathrm{g}, e^{-(t-s)(H_\mathrm{N}-E_\mathrm{N}+\omega(k))}e^{-ikx}\Psi_\mathrm{g}) \\
&= \|(H_\mathrm{N} - E_\mathrm{N} + \omega(k))^{-1}e^{-ikx}\Psi_\mathrm{g}\|^2,
\end{aligned}$$

the corollary follows. □

The formula (6.5.37) is called *pull-through formula*. It can also be obtained by the formal

$$(H_{\mathrm{N}}-E_{\mathrm{N}})a(k)\Psi_{\mathrm{g}} = [(H_{\mathrm{N}}-E_{\mathrm{N}}), a(k)]\Psi_{\mathrm{g}} = (-\omega(k)+[H_{\mathrm{I}}, a(k)])\Psi_{\mathrm{g}}$$

and

$$(\Psi_{\mathrm{g}}, N\Psi_{\mathrm{g}}) = \int_{\mathbb{R}^d} \|a(k)\Psi_{\mathrm{g}}\|^2 dk = \int_{\mathbb{R}^d} \|(H_{\mathrm{N}}-E_{\mathrm{N}}+\omega(k))^{-1}[H_{\mathrm{I}}, a(k)]\Psi_{\mathrm{g}}\|^2 dk.$$

Note that $[H_{\mathrm{I}}, a(k)] = \hat{\varphi}(k)e^{-ikx}/\sqrt{2\omega(k)}$. The formula is useful to study the spectral properties of the Nelson model. The formal derivation mentioned above is justified, for instance, in Corollary 6.19.

The expectation for the number operator $N = d\Gamma(1)$ can be extended to more general operators of the form $d\Gamma(g)$.

Corollary 6.20. *Let* $0 \le g$ *be such that*

$$\int_{\mathbb{R}^d} |g(k)| \frac{|\hat{\varphi}(k)|^2}{\omega(k)^3} dk < \infty. \tag{6.5.38}$$

Then $\Psi_{\mathrm{g}} \in D(d\Gamma(g))$ *and*

$$(\Psi_{\mathrm{g}}, d\Gamma(g)\Psi_{\mathrm{g}}) = \int_{\mathbb{R}^d} dk \frac{|\hat{\varphi}(k)|^2}{2\omega(k)} g(k) \int_{-\infty}^0 ds \int_0^\infty dt e^{-\omega(k)|t-s|} \mathbb{E}_{\mathscr{N}}[e^{-ik(X_t-X_s)}]. \tag{6.5.39}$$

In particular (1)–(3) *below follow.*

(1) *Pull-through formula:*

$$\begin{aligned}(\Psi_{\mathrm{g}}, d\Gamma(g)\Psi_{\mathrm{g}}) \\ = \int_{\mathbb{R}^d} |g(k)| \frac{|\hat{\varphi}(k)|^2}{2\omega(k)} \|(H_{\mathrm{N}}-E_{\mathrm{N}}+\omega(k))^{-1} e^{-ikx}\Psi_{\mathrm{g}}\|^2 dk.\end{aligned} \tag{6.5.40}$$

(2) *Upper bound:*

$$(\Psi_{\mathrm{g}}, d\Gamma(g)\Psi_{\mathrm{g}}) \le \frac{1}{2}\int_{\mathbb{R}^d} |g(k)| \frac{|\hat{\varphi}(k)|^2}{\omega(k)^3} dk. \tag{6.5.41}$$

(3) *Lower bound: if* $\Psi_{\mathrm{g}} \in D(|x| \otimes 1)$, *then there exists* $C > 0$ *such that*

$$\frac{1}{2}\int_{\mathbb{R}^d} |g(k)| \frac{|\hat{\varphi}(k)|^2}{\omega(k)^3} (1 - C|k|^2) dk \le (\Psi_{\mathrm{g}}, d\Gamma(g)\Psi_{\mathrm{g}}). \tag{6.5.42}$$

Proof. It can be checked that with $I = \int_{\mathbb{R}^d} \frac{|\hat{\varphi}(k)|^2}{\omega(k)^3} dk$

$$\begin{aligned}\|d\Gamma(g){:}e^{\xi(\varphi_X^{\pm})}{:}\|_{L^2(\mathsf{G})}^2 &\le (\|g\varphi_X^{\pm}\|_{\mathscr{M}_{\mathbb{C}}}^2 + (\varphi_X^{\pm}, g\varphi_X^{\pm})_{\mathscr{M}_{\mathbb{C}}}^2) e^{\|\varphi_X^{\pm}\|_{\mathscr{M}_{\mathbb{C}}}^2} \\ &\le (1+I)\, e^I \int_{\mathbb{R}^d} |g(k)| \frac{|\hat{\varphi}(k)|^2}{\omega(k)^3} dk < \infty.\end{aligned}$$

Then (6.5.39) follows from Theorem 6.17. The pull-through formula can be proven in a similar way to the case involving N. The upper bound (6.5.41) follows directly from the representation (6.5.39). From $1 - (|k|^2|x|^2)/2 \le \cos k \cdot x$ we get

$$\begin{aligned}\mathbb{E}_{\mathscr{N}}[\cos(k \cdot (X_t - X_s))] &\ge 1 - \frac{|k|^2}{2} \mathbb{E}_{\mathscr{N}}[X_t^2 + X_s^2 - 2X_t X_s] \\ &\ge 1 - |k|^2 \| |x| \Psi_{\mathrm{g}} \|^2.\end{aligned}$$

The last inequality above follows from

$$\mathbb{E}_{\mathscr{N}}[X_s X_t] = \|e^{-(|t-s|/2)(H_{\mathrm{N}} - E_{\mathrm{N}})} x \Psi_{\mathrm{g}}\|^2 \ge 0.$$

Writing $C = \| |x| \Psi_{\mathrm{g}} \|^2$, we have

$$(\Psi_{\mathrm{g}}, d\Gamma(g)\Psi_{\mathrm{g}}) \ge \frac{1}{2} \int_{\mathbb{R}^d} |g(k)| \frac{|\hat{\varphi}(k)|^2}{\omega(k)^3} (1 - C|k|^2) dk.$$

This proves the lower bound. □

Corollary 6.21 (Infrared divergence). *Let $\tilde{M} \subset \mathbb{R}^d$ be a compact set and $M = \mathbb{R}^d \setminus \tilde{M}$. Suppose that $d \le 3$, $\Psi_{\mathrm{g}} \in D(|x| \otimes 1)$, $0 < \kappa < \Lambda < \infty$, and*

$$\hat{\varphi}(k) = \begin{cases} 0, & |k| < \kappa, \\ 1, & \kappa \le |k| \le \Lambda, \\ 0, & |k| > \Lambda. \end{cases}$$

Then

$$\lim_{\kappa \to 0} (\Psi_{\mathrm{g}}, d\Gamma(1_M)\Psi_{\mathrm{g}}) \begin{cases} < \infty, & 0 \notin M, \\ = \infty, & 0 \in M. \end{cases} \tag{6.5.43}$$

In particular,

$$\lim_{\kappa \to 0} (\Psi_{\mathrm{g}}, N\Psi_{\mathrm{g}}) = \infty. \tag{6.5.44}$$

Proof. By Corollary 6.20 we have

$$(\Psi_g, d\Gamma(1_M)\Psi_g) \geq \int_M \frac{|\hat{\varphi}(k)|^2}{2\omega(k)^3}(1 - C|k|^2)dk.$$

Then the corollary follows from

$$\int_M \frac{|\hat{\varphi}(k)|^2}{2\omega(k)^3}dk \begin{cases} = \infty, & 0 \in M, \\ < \infty, & 0 \notin M. \end{cases}$$

□

Remark 6.1. Since the informal description of $d\Gamma(g)$ is

$$d\Gamma(g) = \int_{\mathbb{R}^d} g(k)a^*(k)a(k)dk,$$

the above results can be compactly written as

$$\frac{|\hat{\varphi}(k)|^2}{2\omega(k)^3}(1 - C|k|^2) \leq \|a(k)\Psi_g\|^2 \leq \frac{|\hat{\varphi}(k)|^2}{2\omega(k)^3}. \tag{6.5.45}$$

The quantity in the middle of (6.5.45) is the density at momentum k of the expected number of bosons.

We can also estimate the boson number distribution of the ground state Ψ_g from both sides.

Corollary 6.22 (Boson number distribution). *Let P_n be the projection onto the nth Fock space component. Then*

$$(\Psi_g, P_n\Psi_g)_{L^2(\mathsf{P}_0)} = \frac{1}{n!}\mathbb{E}_{\mathcal{N}}[I_X^n e^{-I_X}], \tag{6.5.46}$$

where

$$I_X = (\varphi_X^+, \varphi_X^-)_{\mathscr{M}_\mathbb{C}} = \int_{-\infty}^0 ds \int_0^\infty dt\, W(X_s - X_t, s - t).$$

Proof. By the definition of Wick exponential we have

$$(:e^{\xi(\varphi_X^-)}:, P_n :e^{\xi(\varphi_X^+)}:)_{L^2(\mathsf{G})} = \frac{1}{n!}(\varphi_X^+, \varphi_X^-)^n_{\mathscr{M}_\mathbb{C}}.$$

The corollary follows from Theorem 6.13. □

Denote by $p_n = (\Psi_g, P_n\Psi_g)$ the probability of n bosons occurring in the ground state of H_N. Recall that $\omega(k) = \sqrt{|k|^2 + \nu^2}$.

Corollary 6.23. *Suppose that $\varphi \geq 0$ and $\nu > 0$. Then there exists $0 < D$ such that*

$$\frac{D^n}{n!}e^{-I} \leq p_n \leq \frac{I^n}{n!}e^{I}, \tag{6.5.47}$$

where $I = \int_{\mathbb{R}^d} \frac{|\hat{\varphi}(k)|^2}{\omega(k)^3} dk$.

Proof. The right-hand side of (6.5.47) is obvious by Corollary 6.22. Note that

$$W(X-Y,t) = \frac{1}{2}\int_{|t|}^{\infty} (\varphi_X, e^{-r\hat{\omega}}\varphi_Y) dr$$

and

$$(\varphi_X, e^{-r\hat{\omega}}\varphi_Y) = \frac{1}{2\sqrt{\pi}}\int_0^\infty \frac{re^{-r^2/(4p)}}{p^{3/2}}(\varphi_X, e^{-p\hat{\omega}^2}\varphi_Y)dp.$$

Since $e^{-r\hat{\omega}^2}$ is positivity preserving and $\varphi \geq 0$, we see that $W(X_s - X_t, s-t) \geq 0$ and hence $I_X \geq 0$. Then the left-hand side follows from

$$p_n \geq \frac{1}{n!}e^{-I}\mathbb{E}_{\mathcal{N}}\left[I_X^n\right] \geq \frac{1}{n!}e^{-I}\left(\mathbb{E}_{\mathcal{N}}\left[I_X\right]\right)^n.$$

D is the expectation of the double integral above. □

Corollary 6.24. *Let $g \in L^\infty(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$ be real-valued, and $\hat{\varphi}/\omega^{3/2} \in L^1(\mathbb{R}^d)$. Then*

$$\begin{aligned}(\Psi_{\rm g}, d\Gamma(g(-i\nabla))\Psi_{\rm g}) = \frac{1}{2(2\pi)^{d/2}}\int_{-\infty}^0 ds\int_0^\infty dt\int_{\mathbb{R}^d\times\mathbb{R}^d} dkdl\,\frac{\overline{\hat{\varphi}(k)}\hat{\varphi}(l)}{\sqrt{\omega(k)\omega(l)}}\\ \times e^{-\omega(k)|t|-\omega(l)|s|}\check{g}(k-l)\mathbb{E}_{\mathcal{N}}[e^{i(lX_t-kX_s)}].\end{aligned} \tag{6.5.48}$$

In particular, we have the upper bound

$$(\Psi_{\rm g}, d\Gamma(g(-i\nabla))\Psi_{\rm g}) \leq \frac{1}{2(2\pi)^d}\|\hat{\varphi}/\omega^{3/2}\|_{L^1}^2\|g\|_{L^1}. \tag{6.5.49}$$

Proof. In order to apply Theorem 6.17, we calculate

$$(\varphi_X^-, g(-i\nabla_k)\varphi_X^+)_{\mathscr{M}_\mathbb{C}} = \frac{1}{2}((\omega^{-1/2}\varphi_X^-)^\vee, g(\omega^{-1/2}\varphi_X^+)^\vee)_{L^2(\mathbb{R}^d)}. \tag{6.5.50}$$

Also,

$$(\omega^{-1/2}\varphi_X^+)^\vee(x) = \frac{1}{(2\pi)^{d/2}}\int_0^\infty ds\int_{\mathbb{R}^d}\frac{\hat{\varphi}(k)}{\sqrt{\omega(k)}}e^{ik(X_s+x)}e^{-\omega(k)|s|}dk,$$

and consequently

$$\begin{aligned}(6.5.50) = \frac{1}{2(2\pi)^{d/2}}\int_{-\infty}^0 ds\int_0^\infty dt\int_{\mathbb{R}^d\times\mathbb{R}^d} dkdl\,\frac{\overline{\hat{\varphi}(k)}\hat{\varphi}(l)}{\sqrt{\omega(k)\omega(l)}}\\ \times e^{i(lX_t-kX_s)}e^{-\omega(k)|t|-\omega(l)|s|}\check{g}(k-l).\end{aligned}$$

□

Let $M \subset \mathbb{R}^d$ be a measurable set. Then $(\Psi_g, d\Gamma(1_M(-i\nabla_k))\Psi_g)$ counts the expected number of bosons with position inside M. It is common to write

$$d\Gamma(1_M(-i\nabla_k)) = \int_M a^*(x)a(x)dx.$$

A special case of Corollary 6.24 is

Corollary 6.25. *Let $M \subset \mathbb{R}^d$ be a compact set and write* $\mathrm{Vol}M = \int_M dx$. *Then*

$$(\Psi_g, d\Gamma(1_M(-i\nabla_k))\Psi_g) \le \frac{1}{2(2\pi)^d}\|\hat{\varphi}/\omega^{3/2}\|_{L^1}^2 \mathrm{Vol}M. \tag{6.5.51}$$

In particular,

$$\lim_{\mathrm{Vol}\, M\to\infty} \frac{(\Psi_g, d\Gamma(1_M(-i\nabla_k))\Psi_g)}{\mathrm{Vol}\, M} \le \frac{1}{2(2\pi)^d}\|\hat{\varphi}/\omega^{3/2}\|_{L^1}^2 < \infty. \tag{6.5.52}$$

6.5.4 Ground state expectation for field operators

For $\beta \in \mathbb{R}$ and $g \in \mathscr{M}_{\mathbb{C}}$ consider $(\Psi_g, e^{\beta\xi(g)}\Psi_g)_{L^2(\mathsf{P}_0)}$, i.e., the moment generating function of the random variable $\xi \mapsto \xi_0(g)$.

Corollary 6.26 (Ground state expectations for field operators). *Let $\beta \in \mathbb{C}$. Then $(\Psi_g, e^{\beta\xi(g)}\Psi_g)_{L^2(\mathsf{P}_0)}$ is finite and*

$$(\Psi_g, e^{\beta\xi(g)}\Psi_g)_{L^2(\mathsf{P}_0)} = \mathbb{E}_{\mathscr{N}}[e^{\frac{\beta^2}{2}I_g - \beta I_X}], \tag{6.5.53}$$

where

$$I_g = \int_{\mathbb{R}^d} \frac{|\hat{g}(k)|^2}{2\omega(k)}dk, \quad I_X = \int_{\mathbb{R}^d} dk \frac{\hat{\varphi}(k)\overline{\hat{g}(k)}}{2\omega(k)} \int_{-\infty}^{\infty} ds e^{-\omega(k)|s|}e^{-ikX_s}.$$

Proof. Note that

$$\begin{aligned}
&(:e^{\xi(\varphi_X^-)}:, e^{\beta\xi_0(g)}:e^{\xi(\varphi_X^+)}:) \\
&= \exp\left(\frac{\beta^2}{2}I_g - \beta I_X\right)\exp\left(\int_{-\infty}^0 ds \int_0^\infty W(X_t - X_s, t-s)dt\right).
\end{aligned}$$

The statement is then a direct consequence of Theorem 6.14. □

By this corollary

$$(\Psi_g, \xi(g)^n\Psi_g)_{L^2(\mathsf{P}_0)} = \frac{d^n}{d\beta^n}(\Psi_g, e^{\beta\xi(g)}\Psi_g)_{L^2(\mathsf{P}_0)}\lceil_{\beta=0} \quad \text{for all } n \in \mathbb{N}. \tag{6.5.54}$$

Let H_n be the Hermite polynomial of order n defined by $H_n = (-1)^n e^{x^2/2} \frac{d^n}{dx^n} e^{-x^2/2}$. For example,

$$H_0(x) = 1, \quad H_1(x) = x, \quad H_2(x) = x^2 - 1, \quad H_3(x) = x^3 - 3x. \tag{6.5.55}$$

We see that $e^{\frac{\beta^2}{2} I_g - \beta I_X}$ can be expressed by an exponential generating function as

$$e^{\frac{\beta^2}{2} I_g - \beta I_X} = \sum_{n=0}^{\infty} H_n(i I_X/\sqrt{I_g}) \frac{(i\beta\sqrt{I_g})^n}{n!}. \tag{6.5.56}$$

Hence we derive

$$\frac{d^n}{d\beta^n} e^{\frac{\beta^2}{2} I_g - \beta I_X} \lceil_{\beta=0} = i^n H_n(i I_X/\sqrt{I_g}) I^{n/2}. \tag{6.5.57}$$

In the next corollary we will look at the mean value and variance of the random variable $\xi \mapsto \xi_0(g)$ for $g \in \mathscr{M}_{\mathbb{C}}$, using the results of Corollary 6.26.

Corollary 6.27 (Functional integral representation of $(\Psi_g, \xi(g)^n \Psi_g)$). *We have*

(1) *Average field strength: for $n = 1$*

$$(\Psi_g, \xi(g)\Psi_g) = -\int_{\mathbb{R}^d} \frac{\hat{\varphi}(k)\overline{\hat{g}(k)}}{2\omega(k)} dk \int_{-\infty}^{\infty} ds e^{-\omega(k)|s|} \mathbb{E}_{\mathscr{N}}[e^{-ikX_s}]. \tag{6.5.58}$$

(2) *Field fluctuations: for $n = 2$*

$$\begin{aligned}(\Psi_g, \xi(g)^2\Psi_g) = &\int_{\mathbb{R}^d} \frac{|\hat{g}(k)|^2}{2\omega(k)} dk \\ &+ \mathbb{E}_{\mathscr{N}}\Big[\Big(\int_{\mathbb{R}^d} \frac{\hat{\varphi}(k)\overline{\hat{g}(k)}}{2\omega(k)} dk \int_{-\infty}^{\infty} ds e^{-\omega(k)|s|} e^{ikX_s}\Big)^2\Big].\end{aligned} \tag{6.5.59}$$

Remark 6.2. (1) By using the previous result and Schwarz inequality we find that

$$(\Psi_g, \xi(g)^2\Psi_g) - (\Psi_g, \xi(g)\Psi_g)^2 \geq \int_{\mathbb{R}^d} \frac{|\hat{g}(k)|^2}{2\omega(k)} dk = (1, \xi(g)^2 1) - (1, \xi(g)1)^2.$$

The latter term represents the fluctuations of the free field. It is then seen that fluctuations increase by coupling the field to the particle.

(2) Note that

$$\mathbb{E}_{\mathscr{N}}[e^{-ikX_s}] = \int_{\mathbb{R}^d} \Psi_p(x)^2 \lambda(x)^2 e^{-ikx} dx,$$

where $\lambda(x)^2 = \int \Psi_g^2(\xi, x) d\mathsf{G}$ is the density of $\mathscr{N}$ with respect to N_0, and Ψ_p^2 is the density of N_0 with respect to Lebesgue measure and the square of the ground state of H_p.

(3) Writing $\chi = \Psi_{\mathrm{p}}^2 \lambda^2$ for the position density of the particle, and taking g to be a delta function in momentum space and in position space, respectively, we find

$$(\Psi_{\mathrm{g}}, \xi(k)\Psi_{\mathrm{g}})_{L^2(\mathsf{P}_0)} = -\frac{\hat{\varphi}(k)\hat{\chi}(k)}{(2\pi)^{d/2}\omega^2(k)} \quad k \in \mathbb{R}^d,$$

and

$$(\Psi_{\mathrm{g}}, \xi(x)\Psi_{\mathrm{g}})_{L^2(\mathsf{P}_0)} = (\chi * V_\omega * \varphi)(x) \quad x \in \mathbb{R}^d, \tag{6.5.60}$$

respectively. Here V_ω denotes the Fourier transform of $-1/\omega^2$ and is the Coulomb potential for massless bosons, i.e. for $\omega(k) = |k|$. (6.5.60) is the classical field generated by a particle with position distribution $\chi(x)dx$.

6.6 The translation invariant Nelson model

In this section we study the ground state of the translation invariant Nelson model. Translation invariant models will be revisited in Section 7.10. In the present section we only state the results without proofs.

Here we suppose that the dispersion relation is $\omega_\nu(k) = \sqrt{|k|^2 + \nu^2}$. With coupling constant $g \in \mathbb{R}$ we write the Nelson Hamiltonian as

$$H_{\mathrm{N}} = H_{\mathrm{p}} + gH_{\mathrm{I}} + H_{\mathrm{f}}.$$

Furthermore, we suppose throughout this section that $V = 0$.

Write

$$P_{\mathrm{f}\mu} = \int_{\mathbb{R}^d} k_\mu a^*(k)a(k)dk, \quad \mu = 1, \dots, d, \tag{6.6.1}$$

for the *field momentum operator*. The *total momentum operator* P_μ, $\mu = 1, \dots, d$, on $\mathscr{H}_{\mathrm{N}}$ is defined by the sum of particle and field momenta,

$$\mathrm{P}_\mu = -i\nabla_{x_\mu} \otimes 1 + 1 \otimes P_{\mathrm{f}\mu}. \tag{6.6.2}$$

It is straightforward to see that

$$[H_{\mathrm{N}}, \mathrm{P}_\mu] = 0, \quad \mu = 1, \dots, d, \tag{6.6.3}$$

as $V = 0$ is assumed. This has the implication that H_{N} is decomposable on the spectrum of the total momentum operator, $\mathrm{Spec}(\mathrm{P}_\mu) = \mathbb{R}$. We denote $\sum_{\mu=1}^d P_{\mathrm{f}\mu}^2$ by P_{f}^2.

Definition 6.4 (Nelson Hamiltonian with fixed total momentum). For each $p \in \mathbb{R}^d$, the *Nelson Hamiltonian with fixed total momentum* is defined by

$$H_{\mathrm{N}}(p) = \frac{1}{2}(p - P_{\mathrm{f}})^2 + g\phi(0) + H_{\mathrm{f}} \tag{6.6.4}$$

with domain $D(H_{\mathrm{N}}(p)) = D(H_{\mathrm{f}}) \cap D(P_{\mathrm{f}}^2)$, where

$$\phi(0) = \frac{1}{\sqrt{2}}(a^*(\hat{\varphi}/\sqrt{\omega}) + a(\tilde{\hat{\varphi}}/\sqrt{\omega})).$$

Here $p \in \mathbb{R}^d$ is the *total momentum.*

Define the unitary operator $\mathscr{T} : \mathscr{H} = L^2(\mathbb{R}_x^d) \otimes \mathscr{F}_{\mathrm{N}} \to L^2(\mathbb{R}_p^d) \otimes \mathscr{F}_{\mathrm{N}}$ by

$$\mathscr{T} = (\mathbb{F} \otimes 1) \int_{\mathbb{R}^d}^{\oplus} \exp(ix \cdot P_{\mathrm{f}}) dx, \tag{6.6.5}$$

with $\mathbb{F}$ denoting Fourier transformation from $L^2(\mathbb{R}_x^d)$ to $L^2(\mathbb{R}_p^d)$. For $\Psi \in \mathscr{H}_{\mathrm{N}}$ we see that

$$(\mathscr{T}\Psi)(p) = \frac{1}{\sqrt{(2\pi)^d}} \int_{\mathbb{R}^d} e^{-ix\cdot p} e^{ix\cdot P_{\mathrm{f}}} \Psi(x) dx.$$

Lemma 6.28. *$H_{\mathrm{N}}(p)$ is a non-negative self-adjoint operator and*

$$\mathscr{T}\left(\int_{\mathbb{R}^d}^{\oplus} \mathscr{F}_{\mathrm{N}} dp\right)\mathscr{T}^{-1} = \mathscr{H}_{\mathrm{N}}, \quad \mathscr{T}\left(\int_{\mathbb{R}^d}^{\oplus} H_{\mathrm{N}}(p) dp\right)\mathscr{T}^{-1} = H_{\mathrm{N}}. \tag{6.6.6}$$

This can be proven similarly to (7.6.11).

We can construct the functional integral representation of $e^{-tH_{\mathrm{N}}(p)}$ in a similar way as that of $e^{-tH_{\mathrm{PF}}(p)}$, which will be done in Section 7.6.

Theorem 6.29 (Functional integral representation for Nelson Hamiltonian with fixed total momentum). *If $\Psi, \Phi \in \mathscr{F}_{\mathrm{N}}$, then*

$$(\Psi, e^{-tH_{\mathrm{N}}(p)}\Phi)_{\mathscr{F}_{\mathrm{N}}} = \mathbb{E}^0[(\mathrm{I}_0\Psi, e^{-g\phi_{\mathrm{E}}(L_t)} \mathrm{I}_t e^{-id\Gamma(-i\nabla)\cdot B_t}\Phi) e^{ip\cdot B_t}], \tag{6.6.7}$$

where L_t is an $\mathscr{H}_{\mathrm{E}}$-valued integral given by

$$L_t = \int_0^t \delta_s \otimes \varphi(\cdot - B_s) ds. \tag{6.6.8}$$

The following result comes as an immediate corollary to the functional integral representation. Write $E_{\mathrm{N}}(p) = \inf \mathrm{Spec}(H_{\mathrm{N}}(p))$.

Corollary 6.30 (Uniqueness of ground state). (1) *Let* $p = 0$ *and* $\Theta = e^{-i(\pi/2)N}$. *Then* $\Theta^{-1} e^{-tH_{\mathrm{N}}(0)}\Theta$ *is positivity improving.*

(2) *The ground state of* $H_{\mathrm{N}}(0)$ *is unique whenever it exists.*

(3) $E_{\mathrm{N}}(0) \le E_{\mathrm{N}}(p)$ *holds.*

Next we review some spectral properties of $H_{\mathrm{N}}(p)$. Let

$$\Delta(p) = \inf \mathrm{Spec}_{\mathrm{ess}}(H_{\mathrm{N}}(p)) - E_{\mathrm{N}}(p). \tag{6.6.9}$$

It is known that

$$\Delta(p) = \inf_{k \in \mathbb{R}^d} (E_{\mathrm{N}}(p-k) + \omega_\nu(k)) - E_{\mathrm{N}}(p). \tag{6.6.10}$$

It follows that $\Delta(p) > 0$ for $\nu > 0$ and sufficiently small $|p|$. From this follows the important fact that since $\omega_0(0) = 0$, $\Delta(p) = 0$ for $\nu = 0$. In particular, $H_{\mathrm{N}}(p)$ has a ground state for sufficiently small $|p|$ and strictly positive $\nu > 0$. The question is the existence of a ground state of $H_{\mathrm{N}}(p)$ for $\nu = 0$.

Suppose now that $\hat{\varphi}$ is rotation invariant. Then $\Delta(p)$ is also rotation invariant, i.e., $\Delta(\mathscr{R}p) = \Delta(p)$ for all $d \times d$ orthogonal matrices $\mathscr{R}$. One of the criteria for the existence of a ground state of $H_{\mathrm{N}}(p)$ for $\nu = 0$ is

Lemma 6.31. *If*

$$\|N^{1/2}\Psi_{\mathrm{g}}(p)\| < 1 \tag{6.6.11}$$

uniformly in ν, *then* $H_{\mathrm{N}}(p)$ *for* $\nu = 0$ *has a ground state.*

To show the bound (6.6.11) a standard strategy is to apply the pull-through formula

$$\|N^{1/2}\Psi_{\mathrm{g}}(p)\|^2 = \frac{g^2}{2}\int_{\mathbb{R}^d} \left\| (H_{\mathrm{N}}(p-k) + \omega_\nu(k) - E_{\mathrm{N}}(p))^{-1} \frac{\hat{\varphi}(k)}{\sqrt{\omega_\nu(k)}} \Psi_{\mathrm{g}}(p) \right\|^2 dk, \tag{6.6.12}$$

where $\Psi_{\mathrm{g}}(p)$ denotes the ground state of $H_{\mathrm{N}}(p)$ for $\nu > 0$. A naive estimate from the pull-through formula yields

$$\|N^{1/2}\Psi_{\mathrm{g}}(p)\|^2 \le \frac{g^2}{2} \frac{\|\hat{\varphi}/\sqrt{\omega_\nu}\|^2}{\Delta_0(|p|)}. \tag{6.6.13}$$

Here $\Delta_0(|p|) = \Delta(p)$. As $\nu \to 0$, however, $\Delta_0(|p|) \to 0$ and then the right-hand side of (6.6.13) diverges. Hence the pull-through formula seems not to work. Nevertheless, functional integration gives an elegant solution for $p = 0$.

Theorem 6.32 (Existence of ground state for zero total momentum). *Let $p = 0$ and $\nu = 0$. Suppose that $\hat{\varphi}$ is rotation invariant and*

$$|g|^2 \le 2\left(\int_{\mathbb{R}^d} \frac{|\hat{\varphi}(k)|^2}{\omega_0(k)^3} dk\right)^{-1}. \tag{6.6.14}$$

Then (6.6.11) *holds. In particular, $H_{\mathrm{N}}(0)$ has a unique ground state for $\nu = 0$.*

Let $\Psi_{\mathrm{g}}^T(0) = \|e^{-TH_{\mathrm{N}}(0)}1\|^{-1} e^{-TH_{\mathrm{N}}(0)}1$. It is known that $\Psi_{\mathrm{g}}^T(0)$ is strongly convergent to the ground state $\Psi_{\mathrm{g}}(0)$ as $T \to \infty$, since $\Psi_{\mathrm{g}}(0)$ overlaps with 1. Using the Euclidean Green function associated with the Nelson model we see that

$$(\Psi_{\mathrm{g}}^T(0), e^{-\beta N}\Psi_{\mathrm{g}}^T(0)) = \mathbb{E}_{\mu_T}[e^{-g^2(1-e^{-\beta})\int_{-T}^0 ds \int_0^T W(X_s - X_t, s-t)dt}], \tag{6.6.15}$$

where

$$d\mu_T = \frac{1}{Z_T} e^{\int_{-T}^T ds \int_T^T W(X_s - X_t, s-t)dt} d\mathcal{W}^0 \tag{6.6.16}$$

is a probability measure on $\mathcal{X} = C(\mathbb{R};\mathbb{R}^d)$, Z_T denotes the normalizing constant such that $\int_{\mathcal{X}} d\mu_T = 1$, and W is given by (6.3.16). It can be seen that the right-hand side of (6.6.15) can be analytically continued from $[0,\infty)$ to the entire complex plane $\mathbb{C}$ with respect to β. We can easily see a stronger statement below.

Lemma 6.33. *It follows that $\Psi_{\mathrm{g}}^T \in D(e^{+\beta N})$ for all $\beta \in \mathbb{C}$ and* (6.6.15) *holds true for all $\beta \in \mathbb{C}$.*

Proof of Theorem 6.32. By Lemma 6.33, $(\Psi_{\mathrm{g}}^T(0), e^{-\beta N}\Psi_{\mathrm{g}}^T(0))$ is differentiable at $\beta = 0$ and then

$$\begin{aligned}(\Psi_{\mathrm{g}}^T(0), N\Psi_{\mathrm{g}}^T(0)) &= -\frac{d}{d\beta}(\Psi_{\mathrm{g}}^T(0), e^{-\beta N}\Psi_{\mathrm{g}}^T(0))|_{\beta=0} \\ &= g^2\mathbb{E}_{\mu_T}\left[\int_{-T}^0 ds \int_0^T W dt\right]. \end{aligned}\tag{6.6.17}$$

From this we can estimate

$$|(\Psi_{\mathrm{g}}^T(0), N\Psi_{\mathrm{g}}^T(0))| \le \frac{g^2}{2}\int_{\mathbb{R}^d} \frac{|\hat{\varphi}(k)|^2}{\omega_0(k)^3} dk. \tag{6.6.18}$$

The right-hand side of (6.6.18) is independent of p and ν, and $\Psi_{\mathrm{g}}^T(0)$ is strongly convergent to $\Psi_{\mathrm{g}}(0)$ as $T \to \infty$. Hence we have

$$\|N^{1/2}\Psi_{\mathrm{g}}(0)\|^2 \le \limsup_{t\to\infty} \|N^{1/2}\Psi_{\mathrm{g}}^T(0)\| \le \frac{g^2}{2}\int_{\mathbb{R}^d} \frac{|\hat{\varphi}(k)|^2}{\omega_0(k)^3} dk. \tag{6.6.19}$$

Taking $|g|^2$ as in (6.6.14), we get the desired results. □

6.7 Infrared divergence

As seen above, in order to give a rigorous mathematical definition of the Nelson model an ultraviolet cut-off was necessary to avoid ultraviolet divergences. This was done by introducing a charge distribution φ, making the coupling to the field sufficiently regular. The total charge $\int_{\mathbb{R}^d} \varphi(x)dx = g \geq 0$ measures the strength of the interaction. $\varphi(x) \to g\delta(x)$ corresponds to the point charge limit which we will consider later. First we discuss the infrared divergence problem.

Assumption 6.2. We assume $\omega(k) = |k|$, and that V is of Kato-class and satisfies the bound

$$V(x) \geq C|x|^{2\beta} \tag{6.7.1}$$

with some constant $C > 0$ and exponent $\beta > 0$.

In particular, we consider here the massless Nelson model. Under the conditions above H_{p} has a unique, strictly positive ground state Ψ_{p} at eigenvalue E_{p}, and

$$\Psi_{\mathrm{p}}(x) \leq e^{-C|x|^{\beta+1}} \tag{6.7.2}$$

holds with a constant C. Under the infrared regular condition $\int |\hat{\varphi}(k)|^2 \omega(k)^{-3} dk < \infty$ we have seen in the previous sections that H_{N} has a unique, strictly positive ground state, and we have obtained detailed information of its properties. What we discuss here is whether there is any ground state in the same space when the infrared regular condition is not assumed.

Theorem 6.34 (Absence of ground state). *Let $d = 3$, $\varphi \geq 0$ but $\varphi \not\equiv 0$, and Assumption 6.2 hold. Then H_{N} has no ground state.*

Remark 6.3. Under the assumption in Theorem 6.34, the infrared singularity

$$\int_{\mathbb{R}^3} \frac{|\hat{\varphi}(k)|^2}{\omega(k)^3} dk = \infty \tag{6.7.3}$$

holds since $\hat{\varphi}(0) > 0$.

Recall that $\gamma(T) = (\Psi_{\mathrm{p}} \otimes 1, \Psi_{\mathrm{g}}^T)^2$. In order to prove Theorem 6.34 it suffices to show that

$$\lim_{T\to\infty} \gamma(T) = 0 \tag{6.7.4}$$

by Proposition 6.8. Recall that $S[a,b] = \int_a^b ds \int_a^b dt W$ and W denotes the pair potential of the Nelson model. Since

$$\gamma(T) = \frac{(\Psi_{\mathrm{p}} \otimes 1, e^{-TL_{\mathrm{N}}} \Psi_{\mathrm{p}} \otimes 1)^2}{(\Psi_{\mathrm{p}} \otimes 1, e^{-2TL_{\mathrm{N}}} \Psi_{\mathrm{p}} \otimes 1)},$$

we can express $\gamma(T)$ by the $P(\phi)_1$-process $X = (X_t)_{t\in\mathbb{R}}$ associated with the self-adjoint operator $L_{\mathrm{p}} = \mathscr{U}^{-1}(H_{\mathrm{p}} - E(H_{\mathrm{p}}))\mathscr{U}$ as

$$\gamma(T) = \frac{(\mathbb{E}_{\mathcal{N}_0}[e^{S[0,T]}])^2}{\mathbb{E}_{\mathcal{N}_0}[e^{S[-T,T]}]}. \tag{6.7.5}$$

Here we used the shift invariance of the process, i.e.,

$$\mathbb{E}_{\mathcal{N}_0}[e^{S[-T,T]}] = \mathbb{E}_{\mathcal{N}_0}[e^{S[0,2T]}].$$

Let $\mathcal{N}_T$ be the probability measure given in (6.5.2)

Lemma 6.35. *It follows that*

$$\gamma(T) \le \mathbb{E}_{\mathcal{N}_T}[e^{-\int_{-T}^0 ds \int_0^T W(X_t - X_s, t-s)dt}]. \tag{6.7.6}$$

Proof. The numerator of (6.7.5) can be estimated by the Schwarz inequality with respect to $d\mathsf{N}_0$ and the reflection symmetry of the process as

$$\begin{aligned}(\mathbb{E}_{\mathcal{N}_0}[e^{S[0,T]}])^2 &\le \int_{\mathbb{R}^3} (\mathbb{E}^x_{\mathcal{N}_0}[e^{S[0,T]}])^2 d\mathsf{N}_0 \\ &= \int_{\mathbb{R}^3} (\mathbb{E}^x_{\mathcal{N}_0}[e^{S[0,T]}])(\mathbb{E}^x_{\mathcal{N}_0}[e^{S[-T,0]}]) d\mathsf{N}_0.\end{aligned}$$

Since X_{-s}, $s \ge 0$, and X_t, $t \ge 0$, are independent, we also see that

$$(\mathbb{E}_{\mathcal{N}_0}[e^{S[0,T]}])^2 \le \int_{\mathbb{R}^d} \mathbb{E}^x_{\mathcal{N}_0}[e^{S[0,T]+S[-T,0]}] d\mathsf{N}_0 = \mathbb{E}_{\mathcal{N}_0}[e^{S[0,T]+S[-T,0]}].$$

Since $S[0,T] + S[-T,0] = S[-T,T] - \int_{-T}^0 ds \int_0^T W(X_t - X_s, t-s)dt$, we have

$$(\mathbb{E}_{\mathcal{N}_0}[e^{S[0,T]}])^2 \le \mathbb{E}_{\mathcal{N}_0}[e^{S[-T,T] - \int_{-T}^0 ds \int_0^T W(X_t - X_s, t-s)dt}].$$

Thus

$$\begin{aligned}\gamma(T) &\le \frac{\mathbb{E}_{\mathcal{N}_0}[e^{S[-T,T] - \int_{-T}^0 ds \int_0^T W(X_t - X_s, t-s)dt}]}{\mathbb{E}_{\mathcal{N}_0}[e^{S[-T,T]}]} \\ &= \mathbb{E}_{\mathcal{N}_T}[e^{-\int_{-T}^0 ds \int_0^T W(X_t - X_s, t-s)dt}],\end{aligned}$$

and the lemma follows. $\square$

The pair potential $W(x,t)$ can be computed explicitly. The integral kernel of $e^{-|t|\sqrt{-\Delta}}$ is

$$e^{-|t|\sqrt{-\Delta}}(x,y) = \frac{1}{\pi^2}\frac{|t|}{(|x-y|^2+|t|^2)^2}.$$

Hence

$$\begin{aligned} W(x-y,t-s) &= \frac{1}{2}\int_{|t-s|}^{\infty} d|T|(e^{-ikx}\hat{\varphi}, e^{-|T|\omega}e^{-iky}\hat{\varphi}) \\ &= \frac{1}{4\pi^2}\int_{\mathbb{R}^3} du \int_{\mathbb{R}^3} dv \frac{\varphi(u)\varphi(v)}{|x-y+u-v|^2+|t-s|^2} > 0. \end{aligned} \tag{6.7.7}$$

To show that $\gamma(T) \to 0$ we first restrict to the set

$$A_T = \{\omega \in \mathfrak{X} | |X_t(\omega)| \le T^\lambda, |t| \le T\}$$

with some $\lambda < 1$. We split up the right-hand side of (6.7.6) by restricting to A_T and $\mathfrak{X} \setminus A_T$, and show that the corresponding expectations converge to zero separately.

Lemma 6.36. *It follows that*

$$\lim_{T\to\infty} \mathbb{E}_{\mathcal{N}_T}[1_{A_T} e^{-\int_{-T}^0 ds \int_0^T W(X_t-X_s,t-s)dt}] = 0. \tag{6.7.8}$$

Proof. By using the estimate

$$|X_t - X_s + x - y|^2 + |t-s|^2 \le 8T^{2\lambda} + 2|x-y|^2 + |t-s|^2,$$

on A_T, and

$$\int_{-T}^0 ds \int_0^T dt \frac{1}{a^2+|t-s|^2} \ge \log\left(\frac{a^2+T^2/2}{a^2}\right)$$

we obtain by (6.7.7) that

$$\begin{aligned} &\int_{-T}^0 ds \int_0^T W(X_t-X_s,t-s)dt \\ &= \int_{-T}^0 ds \int_0^T dt \int_{\mathbb{R}^3} dx \int_{\mathbb{R}^3} dy \frac{\varphi(x)\varphi(y)}{|X_t-X_s+x-y|^2+|t-s|^2} \\ &\ge \int_{-T}^0 ds \int_0^T dt \int_{\mathbb{R}^3} dx \int_{\mathbb{R}^3} dy \frac{\varphi(x)\varphi(y)}{8T^{2\lambda}+2|x-y|^2+|t-s|^2} \\ &= \int_{\mathbb{R}^3} dx \int_{\mathbb{R}^3} dy \varphi(x)\varphi(y) \log\left(\frac{8T^{2\lambda}+2|x-y|^2+T^2/2}{8T^{2\lambda}+2|x-y|^2}\right). \end{aligned}$$

The right-hand side in the formula above diverges as $T \to \infty$, since $\lambda < 1$. This gives (6.7.8). $\square$

Lemma 6.37. *It follows that*

$$\lim_{T\to\infty} \mathbb{E}_{\mathcal{N}_T}[1_{\mathfrak{X}\setminus A_T} e^{-\int_{-T}^0 ds \int_0^T W(X_t-X_s,t-s)dt}] = 0. \tag{6.7.9}$$

Proof. Note that

$$\int_{-T}^0 ds \int_0^T W(X_t - X_s, |t-s|)dt \le \frac{T}{2}\|\hat{\varphi}/\omega\|^2. \tag{6.7.10}$$

We have

$$\begin{aligned} \gamma(T) &\le e^{(T/4)\|\hat{\varphi}/\omega\|^2} \frac{\mathbb{E}_{\mathcal{N}_0}[1_{\mathfrak{X}\setminus A_T} e^{S[-T,T]}]}{\mathbb{E}_{\mathcal{N}_0}[e^{S[-T,T]}]} \\ &\le e^{(T/4)\|\hat{\varphi}/\omega\|^2} \frac{(\mathbb{E}_{\mathcal{N}_0}[e^{S[-T,T]}])^{1/2}}{\mathbb{E}_{\mathcal{N}_0}[e^{S[-T,T]}]} (\mathbb{E}_{\mathcal{N}_0}[1_{\mathfrak{X}\setminus A_T}])^{1/2}. \end{aligned} \tag{6.7.11}$$

Moreover, there exists a constant $\delta > 0$ such that

$$-T\delta\|\hat{\varphi}/\omega\|^2 \le |S[-T,T]| \le T\delta\|\hat{\varphi}/\omega\|^2.$$

Thus

$$\frac{(\mathbb{E}_{\mathcal{N}_0}[e^{2S[-T,T]}])^{1/2}}{\mathbb{E}_{\mathcal{N}_0}[e^{S[-T,T]}]} \le e^{2T\delta\|\hat{\varphi}/\omega\|^2}.$$

To complete the proof we argue that this exponential growth is balanced by the second factor of (6.7.11). The estimate

$$\mathbb{E}_{\mathcal{N}_0}[1_{\mathfrak{X}\setminus A_T}] \le T^{-\lambda}\sqrt{a+Tb}\exp(-cT^{\lambda(\beta+1)}) \tag{6.7.12}$$

is obtained in Lemma 6.38. Here $a, b, c > 0$. By choosing $1/(\beta+1) < \lambda < 1$, the right-hand side of (6.7.12) drops to zero as $T \to \infty$. □

Now we are in the position to prove Theorem 6.34.

Proof of Theorem 6.34. Since

$$\gamma(T) \le \mathbb{E}_{\mathcal{N}_T}[e^{-\int_{-T}^0 ds \int_0^T W(X_s,X_t,t-s)dt}] = \mathbb{E}_{\mathcal{N}_T}[1_{A_T}F] + \mathbb{E}_{\mathcal{N}_T}[1_{\mathfrak{X}\setminus A_T}F],$$

where $F = e^{-\int_{-T}^0 ds \int_0^T W(X_s,X_t,t-s)dt}$, $\lim_{T\to\infty}\mathbb{E}_{\mu_T}[1_{A_T}] = 0$ by Lemma 6.35 and $\lim_{T\to\infty}\mathbb{E}_{\mu_T}[1_{\mathfrak{X}\setminus A_T}] = 0$ by Lemma 6.36, the theorem follows. □

In order to prove (6.7.12) we make use of the path properties of the $P(\phi)_1$-process $(X_t)_{t\ge 0}$ shown in Section 3.3.10. Note that

$$\mathbb{E}_{\mathcal{N}_0}[1_{\mathfrak{X}\setminus A_T}] = \mathcal{N}_0(\sup_{t\in[-T,T]}|X_t| \ge T^\lambda).$$

Lemma 6.38. (6.7.12) *holds.*

Proof. Suppose that $f \in C^\infty(\mathbb{R}^d)$ is an even function, and

$$f(x) \begin{cases} = |x|, & |x| \geq T^\lambda, \\ \leq |x|, & T^\lambda - 1 < |x| < T^\lambda, \\ = 0, & |x| \leq T^\lambda - 1. \end{cases}$$

Then

$$\mathbb{E}_{\mathscr{N}_0}[1_{\mathfrak{X}\setminus A_T}] = \mathbb{E}_{\mathscr{N}_0}[1_{\{\sup_{|s|<T}|X_s|>T^\lambda\}}] = \mathbb{E}_{\mathscr{N}_0}[1_{\{\sup_{|s|<T}|f(X_s)|>T^\lambda\}}]. \quad (6.7.13)$$

By reflection symmetry of $(X_t)_{t\in\mathbb{R}}$

$$\mathbb{E}_{\mathscr{N}_0}[1_{\{\sup_{|s|<T}|f(X_s)|>T^\lambda\}}] = 2\mathbb{E}_{\mathscr{N}_0}[1_{\{\sup_{0\leq s\leq T}|f(X_s)|>T^\lambda\}}]$$

and by Proposition 3.111,

$$\mathbb{E}_{\mathscr{N}_0}[\sup_{|s|<T}|f(X_s)| > T^\lambda] \leq \frac{2e}{T^\lambda}((f,f)_{L^2(\mathsf{N}_0)} + T(L_\mathrm{p}^{1/2}f, L_\mathrm{p}^{1/2}f)_{L^2(\mathsf{N}_0)})^{1/2}. \quad (6.7.14)$$

We estimate the right-hand side of (6.7.14). First note that $f\Psi_\mathrm{g} \in D(H_\mathrm{p})$ and

$$H_\mathrm{p} f\Psi_\mathrm{g} = -\Delta f \cdot \Psi_\mathrm{g} - \nabla f \cdot \nabla\Psi_\mathrm{g} + E_\mathrm{p} f\Psi_\mathrm{g}.$$

By this we also see that

$$(L_\mathrm{p}^{1/2}f, L_\mathrm{p}^{1/2}f)_{L^2(\mathsf{N}_0)} = (f\Psi_\mathrm{p}, -\Delta f \cdot \Psi_\mathrm{g} - \nabla f \cdot \nabla\Psi_\mathrm{p}).$$

By the super-exponential decay $\Psi_\mathrm{p}(x) \leq e^{-C|x|^{\beta+1}}$

$$\|f\Psi_\mathrm{p}\|^2 = \int_{\mathbb{R}^3} f(x)^2\Psi_\mathrm{p}^2(x)dx \leq e^{-2CT^{\lambda(\beta+1)}}\int_{\mathbb{R}^3}|x|^2 e^{-2C|x|^{\beta+1}}dx. \quad (6.7.15)$$

Note that $\nabla f, \Delta f \in L^\infty(\mathbb{R}^3)$. Then

$$\begin{aligned}(L_\mathrm{p}^{1/2}f, L_\mathrm{p}^{1/2}f)_{L^2(\mathsf{N}_0)} &\leq C'\|f\Psi_\mathrm{p}\|(\|\nabla\Psi_\mathrm{p}\| + \|\Psi_\mathrm{p}\|) \\ &\leq C''e^{-CT^{\lambda(\beta+1)}}(\|\nabla\Psi_\mathrm{p}\| + \|\Psi_\mathrm{p}\|)\end{aligned}$$

follows. Similarly,

$$(f,f)_{L^2(\mathsf{N}_0)} \leq e^{-2CT^{\lambda(\beta+1)}}\int_{\mathbb{R}^3}|x|^2 e^{-2\gamma|x|^{\beta+1}}dx.$$

Hence

$$\mathbb{E}_{\mathscr{N}_T}[1_{\mathfrak{X}\setminus A_T}] \leq T^{-\lambda}\sqrt{a+Tb}\,e^{-cT^{\lambda(\beta+1)}}$$

with constants a, b, c. This completes the proof. □

6.8 Ultraviolet divergence

6.8.1 Energy renormalization

In this section we consider the removal of the ultraviolet cutoff in the Nelson model by using functional integration.

First we fix the version of the Nelson model discussed here. We assume $\omega(k) = |k|$, and that the external potential V is a continuous bounded function for simplicity. We introduce hard infrared and ultraviolet cutoffs. Let

$$1_\Lambda(k) = \begin{cases} 1, & |k| < \Lambda, \\ 0, & |k| \geq \Lambda. \end{cases}$$

The cutoff function of the Nelson model H_{N} is

$$\hat{\varphi}_{\kappa,\Lambda}(k) = 1_\Lambda(k)(1 - 1_\kappa(k)). \tag{6.8.1}$$

Consider the operator

$$G_\Lambda = -i\frac{g}{\sqrt{2}}\int_{\mathbb{R}^3} \frac{\hat{\varphi}_{\kappa,\Lambda}(k)}{\sqrt{|k|}}\beta(k)(e^{ik\cdot x}a(k) - e^{-ik\cdot x}a^*(k))dk, \tag{6.8.2}$$

with

$$\beta(k) = \frac{1}{|k| + |k|^2/2}. \tag{6.8.3}$$

Note that $\frac{\hat{\varphi}_{\kappa,\Lambda}(k)}{\sqrt{|k|}}\beta(k) \in L^2(\mathbb{R}^3)$ as a consequence of the infrared cutoff $\kappa > 0$, and that s-$\lim_{\Lambda\to\infty} \frac{\hat{\varphi}_{\kappa,\Lambda}(k)}{\sqrt{|k|}}\beta(k) \in L^2(\mathbb{R}^3)$ exists in L^2. The conjugation map with respect to the unitary operator e^{iG_Λ} is called *Gross transform*. A direct calculation shows that the field operators transform as

$$e^{-iG_\Lambda}a(f)e^{iG_\Lambda} = a(f) - \frac{g}{\sqrt{2}}\int_{\mathbb{R}^3} f(k)\frac{\hat{\varphi}_{\kappa,\Lambda}(k)}{\sqrt{|k|}}\beta(k)e^{-ik\cdot x}dk,$$
$$e^{-iG_\Lambda}a^*(f)e^{iG_\Lambda} = a^*(f) - \frac{g}{\sqrt{2}}\int_{\mathbb{R}^3} f(k)\frac{\hat{\varphi}_{\kappa,\Lambda}(k)}{\sqrt{|k|}}\beta(k)e^{+ik\cdot x}dk.$$

For the momentum operator we obtain

$$e^{-iG_\Lambda}pe^{iG_\Lambda} = p - A_\Lambda(x) - A_\Lambda^*(x),$$

where

$$A_\Lambda(x) = -\frac{1}{\sqrt{2}}\int_{\mathbb{R}^3} \frac{\hat{\varphi}_{\kappa,\Lambda}(k)}{\sqrt{|k|}}k\beta(k)e^{ik\cdot x}a(k)dk. \tag{6.8.4}$$

Hence, with $H_0 = H_p \otimes 1 + 1 \otimes H_f$,

$$e^{-iG_\Lambda} H_N e^{iG_\Lambda} = H_0 - g(p \cdot A_\Lambda(x) + A_\Lambda^*(x) \cdot p) + \frac{g^2}{2}(A_\Lambda(x)^2 + A_\Lambda^*(x)^2 + 2A_\Lambda(x) \cdot A_\Lambda^*(x)) + E_\Lambda. \tag{6.8.5}$$

We denote the right-hand side above by $\tilde{H}_N$. The additive constant equals

$$E_\Lambda = -\frac{g^2}{2} \int_{\mathbb{R}^3} \frac{|\hat{\varphi}_{\kappa,\Lambda}(k)|^2}{|k|} \beta(k) dk, \tag{6.8.6}$$

and diverges as $-\log \Lambda$ when $\Lambda \to \infty$.

Theorem 6.39 (Existence of Hamiltonian without ultraviolet cutoff). (1) *There exists a self-adjoint operator H_N^{ren} such that*

$$\lim_{\Lambda \to \infty} e^{-t(\tilde{H}_N - E_\Lambda)} = e^{-tH_N^{\text{ren}}}, \quad t > 0, \tag{6.8.7}$$

where the limit is in the uniform topology.

(2) *The unitary operator e^{iG_Λ} strongly converges to e^{iG_∞} as $\Lambda \to \infty$, where*

$$G_\infty = -i \frac{g}{\sqrt{2}} \int_{\mathbb{R}^3} \frac{\beta(k)}{\sqrt{|k|}} (1 - 1_\kappa(k))(e^{ik\cdot x} a(k) - e^{-ik\cdot x} a^*(k)) dk, \tag{6.8.8}$$

and then

$$\operatorname*{s-lim}_{\Lambda \to \infty} e^{-t(H_N - E_\Lambda)} = e^{iG_\infty} e^{-tH_N^{\text{ren}}} e^{-iG_\infty}, \quad t > 0. \tag{6.8.9}$$

H_N^{ren} is formally given by

$$H_N^{\text{ren}} = H_0 - g(p \cdot A + A^* \cdot p) + \frac{g^2}{2}(A^2 + A^{*2} + 2A^* \cdot A), \tag{6.8.10}$$

and

$$A = A(x) = -\frac{1}{\sqrt{2}} \int_{\mathbb{R}^3} \frac{(1 - 1_\kappa(k))}{\sqrt{|k|}} k\beta(k) e^{ik\cdot x} a(k) dk.$$

When $|k|$ is large enough, $\sqrt{|k|}\beta(k) \sim 1/|k|^{3/2}$. Notice that the test function of A is not an L^2 function:

$$\beta_\infty(k) = \frac{1}{\sqrt{|k|}} k\beta(k) e^{ik\cdot x} \notin L^2(\mathbb{R}_k^3).$$

In particular, we see that

$$[A(x), A^*(y)] = \infty, \tag{6.8.11}$$

and

$$H_{\mathrm{N}}^{\mathrm{ren}} \neq \frac{1}{2}(p - g(A + A^*))^2 + V + H_{\mathrm{f}}. \tag{6.8.12}$$

Therefore (6.8.10) is a formal description, however, we know that

$$\|A(x)\Psi\| \leq \|\beta_\infty/\sqrt{\omega_\infty}\|\|H_\infty^{1/2}\Psi\|, \tag{6.8.13}$$

where $\omega_\infty(k) = |k|$ for $|k| \geq 1$ and $= 1$ for $|k| < 1$, and

$$H_\infty = \int_{\mathbb{R}^3} \omega_\infty(k) a^*(k) a(k) dk.$$

Note that

$$\|\beta_\infty/\sqrt{\omega_\infty}\| = \int_{|k|<1} \beta(k)^2 dk + \int_{|k|\geq 1} \beta(k)^2/|k|^2 dk < \infty.$$

In spite of the fact that $\beta_\infty \notin L^2(\mathbb{R}^3)$ by (6.8.13), $A(x)$ can be defined on $D(H_\infty^{1/2})$, however, A^* cannot. By a further computation it can be seen that the quadratic form

$$\begin{aligned}(f, g) \mapsto \mathrm{q}_N(f, g) = (f, H_0 g) - g\,((pf, Ag) + (Af, pg)) \\ + \frac{g^2}{2}((A^2 f, g) + 2(Af, Ag) + (f, A^2 g))\end{aligned}$$

on $D(\bar{H}_0^{1/2}) \times D(\bar{H}_0^{1/2})$ is well defined, where $\bar{H}_0 = H_{\mathrm{p}} + H_{\mathrm{f}} - \inf\mathrm{Spec}(H_{\mathrm{p}})$. This can be checked by using the estimate

$$\|p_\mu(\bar{H}_0 + 1)^{-1/2}\| < \infty, \tag{6.8.14}$$

$$\|(\bar{H}_0 + 1)^{-1/2} A^2 (\bar{H}_0 + 1)^{-1/2}\| < \infty. \tag{6.8.15}$$

Thus there exists a self-adjoint operator associated with the form q_N, which is the definition of $H_{\mathrm{N}}^{\mathrm{ren}}$. In the next section we prove (6.8.9) by using functional integrals.

6.8.2 Regularized interaction

Instead of using the cutoff function $\hat{\varphi}_{\kappa,\Lambda}$, we define $H_{\mathrm{N}}^{\varepsilon}$ by H_{N} with cutoff function

$$\hat{\varphi}(k) = e^{-\varepsilon|k|^2/2}(1 - 1_\kappa(k)). \tag{6.8.16}$$

Then Theorem 6.39 implies that

$$e^{-t(H_{\mathrm{N}}^{\varepsilon} - E_\varepsilon)} \to e^{iG_\infty} e^{-tH_{\mathrm{N}}^{\mathrm{ren}}} e^{-iG_\infty} \tag{6.8.17}$$

as $\varepsilon \to 0$ strongly, where the additive constant is given by

$$E_\varepsilon = -\frac{g^2}{2}\int_{\mathbb{R}^3} \frac{e^{-\varepsilon|k|^2}}{|k|}\beta(k)(1-1_\kappa(k))dk. \tag{6.8.18}$$

We carry out this scheme by functional integration and show the functional integral representation of $(F, e^{iG_\infty}e^{-tH_{\mathrm{N}}^{\mathrm{ren}}}e^{-iG_\infty}G)$ through the limit

$$\lim_{\varepsilon\to 0}(F, e^{-2TH_{\mathrm{N}}^\varepsilon}G) = (F, e^{iG_\infty}e^{-tH_{\mathrm{N}}^{\mathrm{ren}}}e^{-iG_\infty}G). \tag{6.8.19}$$

In terms of two-sided Brownian motion $(B_t)_{t\in\mathbb{R}}$ we have the Feynman–Kac formula of $(F, e^{-2TH_{\mathrm{N}}^\varepsilon}G)$ in Theorem 6.3. In particular, for $F = f\otimes 1$ and $G = g\otimes 1$, we have

$$(f\otimes 1, e^{-2TH_{\mathrm{N}}^\varepsilon}g\otimes 1) = \int_{\mathbb{R}^3} dx\mathbb{E}^x[\overline{f(B_{-T})}g(B_T)e^{-\int_{-T}^T V(B_s)ds}e^{g^2S^\varepsilon/2}],$$

where

$$S^\varepsilon = \int_{-T}^T ds\int_{-T}^T W^\varepsilon(B_t - B_s, t-s)dt \tag{6.8.20}$$

and

$$W^\varepsilon(x,t) = \int_{\mathbb{R}^3}\frac{1}{2|k|}e^{-\varepsilon|k|^2}e^{-ik\cdot x}e^{-|k||t|}(1-1_\kappa(k))dk. \tag{6.8.21}$$

Definition 6.5 (Regularized pair potential and interaction). We call $W^\varepsilon(x,t)$ in (6.8.21) *regularized pair potential*, and S^ε in (6.8.20) *regularized interaction*.

We prove that

$$\mathbb{E}^x[e^{-\int_{-T}^T V(B_s)ds}e^{\frac{g^2}{2}(S^\varepsilon - 4T\varphi_\varepsilon(0,0))}] \to \mathbb{E}^x[e^{-\int_{-T}^T V(B_s)ds}e^{\frac{g^2}{2}S_{\mathrm{ren}}^0}] \tag{6.8.22}$$

as $\varepsilon \to 0$ with some interaction S_{ren}^0. Notice that $W^\varepsilon(x,t)$ is smooth, and $W^\varepsilon(x,t) \to W^0(x,t)$ for every $(x,t) \neq (0,0)$ as $\varepsilon \to 0$, where

$$W^0(x,t) = \int_{\mathbb{R}^3}\frac{1}{2|k|}e^{-ik\cdot x}e^{-|k||t|}(1-1_\kappa(k))dk. \tag{6.8.23}$$

It follows, however, that $W^\varepsilon(0,0) \to \infty$ as $\varepsilon \to 0$, thus (6.8.22) is non-trivial to obtain.

From now on we fix $T > 0$. As $\varepsilon \to 0$ only the diagonal part of the interaction has a singular term. Also, fix $0 < \tau < T$. We decompose the regularized interaction as

$$S^\varepsilon = S_1^\varepsilon + S_2^\varepsilon, \tag{6.8.24}$$

where

$$S_1^\varepsilon = 2\int_{-T}^{T} ds \int_s^{[s+\tau]} dt W^\varepsilon(B_t - B_s, t-s) \tag{6.8.25}$$

and

$$S_2^\varepsilon = 2\int_{-T}^{T} ds \int_{[s+\tau]}^{T} dt W^\varepsilon(B_t - B_s, t-s) \tag{6.8.26}$$

with the notation

$$[t] = \begin{cases} T, & t \geq T, \\ t, & |t| < T, \\ -T, & t \leq -T. \end{cases}$$

S_1^ε denotes the integral of S^ε on the diagonal $\{(t,t) \in \mathbb{R}^2 || t| \leq T\}$, and S_2^ε on its complement. Then the pathwise limit

$$S_2^\varepsilon \to S_2^0 = 2\int_{-T}^{T} ds \int_{[s+\tau]}^{T} W^0(B_t - B_s, t-s)dt \tag{6.8.27}$$

exists as $\varepsilon \to 0$. A representation in terms of a stochastic integral will help us deal with the more difficult term S_1^ε. Let

$$\varphi_\varepsilon(x,t) = \int_{\mathbb{R}^3} \frac{e^{-\varepsilon|k|^2} e^{-ik\cdot x - |k||t|}}{2|k|(|k| + |k|^2/2)}(1 - 1_\kappa(k))dk. \tag{6.8.28}$$

We give an upper bound of $\varphi_\varepsilon(x,t)$ in the lemma below.

Lemma 6.40. *There exists a constant $c > 0$ such that the following bounds on φ_ε hold uniformly in ε, x, t:*

$$\begin{aligned} |\varphi_\varepsilon(x,t)| &\leq c(1 + |\log(|t| \wedge 1)|), \\ |\nabla\varphi_\varepsilon(x,t)| &\leq c|t|^{-1}, \quad t \neq 0, \\ |\nabla\varphi_\varepsilon(x,t)| &\leq c|x|^{-1}, \quad |x| \neq 0, \end{aligned}$$

where $\nabla\varphi_\varepsilon = (\partial_1\varphi_\varepsilon, \partial_2\varphi_\varepsilon, \partial_3\varphi_\varepsilon)$ denotes the gradient of φ_ε. Furthermore, similar bounds hold for the function $\varphi_0 - \varphi_\varepsilon$ with a constant $c'(\varepsilon) > 0$ such that $\lim_{\varepsilon\to 0} c'(\varepsilon) = 0$*:*

$$\begin{aligned} |\varphi_\varepsilon(x,t) - \varphi_0(x,t)| &\leq c'(\varepsilon)(1 + |\log(|t| \wedge 1)|), \\ |\nabla\varphi_\varepsilon(x,t) - \nabla\varphi_0(x,t)| &\leq c'(\varepsilon)|t|^{-1}, \quad t \neq 0, \\ |\nabla\varphi_\varepsilon(x,t) - \nabla\varphi_0(x,t)| &\leq c'(\varepsilon)|x|^{-1}, \quad |x| \neq 0. \end{aligned}$$

Proof. After integration over the angular variables we obtain

$$\varphi_\varepsilon(x,t) = 2\pi \int_\kappa^\infty \frac{e^{-\varepsilon r^2 - r|t|}}{r(2+r)} \frac{\sin(r|x|)}{|x|} dr. \tag{6.8.29}$$

Hence

$$|\varphi_\varepsilon(x,t)| \le 2\pi C \int_0^\infty \frac{e^{-r|t|}}{2+r} dr = 2\pi C \int_0^\infty \frac{e^{-r}}{2|t|+r} dr,$$

where we used that $\sin a/a \le C$ for some $C > 0$. Thus

$$|\varphi_\varepsilon(x,t)| \le 2\pi C \int_0^1 \frac{1}{2|t|+r} dr + 2\pi C \int_1^\infty \frac{e^{-r}}{r} dr$$

which gives the bound

$$|\varphi_\varepsilon(x,t)| \le C(1 + \log(|t| \wedge 1)).$$

The first bound on the gradient follows directly by

$$\begin{aligned} |\nabla\varphi_\varepsilon(x,t)| &\le \int_{\mathbb{R}^3} \frac{1}{2(|k| + |k|^2/2)} e^{-\varepsilon|k|^2} e^{-|k||t|} dk \\ &\le 2\pi \int_0^\infty \frac{r^2}{r + r^2/2} e^{-rt} dr \le c \int_0^\infty e^{-rt} dr. \end{aligned}$$

Next consider the second bound on the gradient. Differentiation in (6.8.29) gives

$$\begin{aligned} \nabla\varphi_\varepsilon(x,t) &= 2\pi \int_\kappa^\infty \frac{e^{-\varepsilon r^2 - r|t|}}{r(2+r)} \left(\frac{\cos(r|x|) r x}{|x|^2} - \frac{\sin(r|x|) x}{|x|^3} \right) dr \\ &= \frac{2\pi x}{|x|^2} \int_\kappa^\infty \frac{e^{-\varepsilon r^2/|x|^2 - |t|r/|x|}}{r(2|x|+r)} (r\cos r - \sin r)\, dr, \end{aligned}$$

where we made the change of variables $r \mapsto r|x|$. The integral is uniformly bounded in ε and t; indeed

$$\begin{aligned} |I| &= \left| \int_0^\infty \frac{e^{-\varepsilon r^2/|x|^2 - |t|r/|x|}}{r(2|x|+r)} (r\cos r - \sin r)\, dr \right| \\ &\le \int_0^1 \frac{|r\cos r - \sin r|}{r^2} dr + \left| \int_1^\infty \frac{e^{-\varepsilon r^2/|x|^2 - |t|r/|x|}}{(2|x|+r)} \cos r\, dr \right| \\ &\quad + \left| \int_1^\infty \frac{e^{-\varepsilon r^2/|x|^2 - |t|r/|x|}}{r(2|x|+r)} \sin r dr \right| \\ &\le \int_0^1 \frac{Cr^3}{r^2} dr + \left| \int_1^\infty \frac{e^{-\varepsilon r^2/|x|^2 - |t|r/|x|}}{(2|x|+r)} \cos r dr \right| + \int_1^\infty \frac{1}{r^2} dr \end{aligned}$$

using $|r\cos r - \sin r| \le Cr^3$ for some constant C and all $r \in [0,1]$. It remains to show that the integral

$$J = \int_1^\infty \frac{e^{-\varepsilon r^2/|x|^2 - |t|r/|x|}}{(2|x|+r)} \cos r dr$$

is bounded. We have

$$J = -\frac{e^{-\varepsilon/|x|^2-|t|/|x|}}{(2|x|+1)} \sin 1 - \int_1^\infty \frac{d}{dr}\left(\frac{e^{-\varepsilon r^2/|x|^2-|t|r/|x|}}{(2|x|+r)}\right) \sin r dr.$$

Thus

$$|J| \le \sin 1 + \int_1^\infty \left|\frac{d}{dr}\left(\frac{e^{-\varepsilon r^2/|x|^2-|t|r/|x|}}{(2|x|+r)}\right)\right| dr \le \sin 1 + \int_1^\infty \frac{dr}{r^2} < \infty. \qquad \square$$

6.8.3 Removal of the ultraviolet cutoff

In this section we give the functional integral representation of the Nelson model without ultraviolet cutoff.

Lemma 6.41. *We have that*

$$\begin{aligned} S_1^\varepsilon = 4T\varphi_\varepsilon(0,0) &- 2\int_{-T}^T \varphi_\varepsilon(B_{[s+\tau]} - B_s, [s+\tau]-s) ds \\ &+ 2\int_{-T}^T ds \int_s^{[s+\tau]} \nabla\varphi_\varepsilon(B_t - B_s, t-s) dB_t. \end{aligned} \tag{6.8.30}$$

Proof. Notice that $\varphi_\varepsilon(x,t)$ solves the equation

$$\left(\partial_t + \frac{1}{2}\Delta\right)\varphi_\varepsilon(x,t) = -W^\varepsilon(x,t), \quad x \in \mathbb{R}^3, \quad t \ge 0. \tag{6.8.31}$$

Then the Itô formula yields

$$\begin{aligned} &\varphi_\varepsilon(B_{[s+\tau]} - B_s, [s+\tau]-s) - \varphi_\varepsilon(0,0) \\ &= \int_s^{[s+\tau]} \nabla\varphi_\varepsilon(B_t - B_s, t-s) dB_t + \int_s^{[s+\tau]} \left(\partial_t + \frac{1}{2}\Delta\right)\varphi_\varepsilon(B_t - B_s, t-s) dt. \end{aligned} \tag{6.8.32}$$

Hence by (6.8.31)

$$\begin{aligned} &\int_s^{[s+\tau]} W^\varepsilon(B_t - B_s, t-s) dt \\ &= \varphi_\varepsilon(0,0) - \varphi_\varepsilon(B_{[s+\tau]} - B_s, [s+\tau]-s) + \int_s^{[s+\tau]} \nabla\varphi_\varepsilon(B_t - B_s, t-s) dB_t \end{aligned} \tag{6.8.33}$$

follows. Inserting this expression into S_1^ε proves the claim. $\square$

Inspired by Lemma 6.41 we make the following

Definition 6.6 (Renormalized regularized interaction). We call

$$S^{\varepsilon}_{\mathrm{ren}} = S^{\varepsilon} - 4T\varphi_{\varepsilon}(0,0), \quad \varepsilon \geq 0, \tag{6.8.34}$$

renormalized regularized interaction.

$S^{\varepsilon}_{\mathrm{ren}} = S^{\varepsilon}_2 + S^{\varepsilon}_1 - 4T\varphi_{\varepsilon}(0,0)$ is realized as in Lemma 6.41,

$$\begin{aligned} S^{\varepsilon}_{\mathrm{ren}} = S^{\varepsilon}_2 &- 2\int_{-T}^{T} \varphi_{\varepsilon}(B_{[s+\tau]} - B_s, [s+\tau] - s)ds \\ &+ 2\int_{-T}^{T} ds \int_{s}^{[s+\tau]} \nabla\varphi_{\varepsilon}(B_t - B_s, t-s)dB_t. \end{aligned} \tag{6.8.35}$$

Lemma 6.42. *There exists a subsequence ε' such that the random variable $S^{\varepsilon'}_{\mathrm{ren}}$ converges to the random variable S^{0}_{ren} as $\varepsilon' \to 0$ almost surely with respect to $\mathcal{W}^x$, where S^{0}_{ren} denotes $S^{\varepsilon}_{\mathrm{ren}}$ with $\varepsilon = 0$.*

Proof. Write $S^{\varepsilon}_{\mathrm{ren}} = S^{\varepsilon}_2 + Y_{\varepsilon} + Z_{\varepsilon}$ with

$$Y_{\varepsilon} = 2\int_{-T}^{T} ds \int_{s}^{[s+\tau]} \nabla\varphi_{\varepsilon}(B_t - B_s, t-s)dB_t, \tag{6.8.36}$$

$$Z_{\varepsilon} = -2\int_{-T}^{T} \varphi_{\varepsilon}(B_{[s+\tau]} - B_s, [s+\tau] - s)ds. \tag{6.8.37}$$

As seen in (6.8.27), $S^{\varepsilon}_2 \to S^{0}_2$ pathwise, thus it suffices to consider the remaining two terms. Convergence of Z_{ε} follows by the estimate $|\varphi_{\varepsilon}(x,t) - \varphi_0(x,t)| \leq c'(\varepsilon)|t|^{-1}$. Next consider Y_{ε}. By the Fubini theorem the stochastic and Lebesgue integrals can be interchanged, thus Y_{ε} has the representation

$$Y_{\varepsilon} = 2\int_{-T}^{T} \left(\int_{[t-\tau]}^{t} \nabla\varphi_{\varepsilon}(B_t - B_s, t-s)ds \right) dB_t. \tag{6.8.38}$$

This is a well-defined random variable for every $\varepsilon \geq 0$. The Itô isometry gives

$$\begin{aligned} \mathbb{E}^x[|Y_{\varepsilon} - Y_0|^2] &= 4\int_{-T}^{T} \mathbb{E}^x\left[\left| \int_{[t-\tau]}^{t} (\nabla\varphi_{\varepsilon} - \nabla\varphi_0)(B_t - B_s, t-s)ds \right|^2 \right] dt \\ &\leq 4c''(\varepsilon)\int_{-T}^{T} \mathbb{E}^x\left[\left(\int_{[t-\tau]}^{t} |B_t - B_s|^{-\theta}|t-s|^{-(1-\theta)}ds \right)^2 \right] dt, \end{aligned} \tag{6.8.39}$$

where we used an interpolation to bound

$$|(\nabla\varphi_\varepsilon - \nabla\varphi_0)(x,t)| \le c''(\varepsilon)|x|^{-\theta}|t|^{-(1-\theta)} \tag{6.8.40}$$

for every $\theta \in [0,1]$ and Lemma 6.40. Here $c''(\varepsilon) \to 0$ as $\varepsilon \to 0$. With some $\theta > 1/2$, Schwarz inequality applied to the latter integral gives

$$\begin{aligned}
&\mathbb{E}^x[|Y_\varepsilon - Y_0|^2] \\
&\le 4c''(\varepsilon)\int_{-T}^{T} \mathbb{E}^x\left[\int_{[t-\tau]}^{t} |B_t - B_s|^{-2\theta} ds\right]\left(\int_{[t-\tau]}^{t} |t-s|^{-2(1-\theta)} ds\right) dt \\
&\le 4c''(\varepsilon)\tau^{2\theta-1}\int_{-T}^{T}\int_{[t-\tau]}^{t} \mathbb{E}^x[|B_t - B_s|^{-2\theta}]dsdt.
\end{aligned} \tag{6.8.41}$$

Since

$$\mathbb{E}^x[|B_t - B_s|^{-2\theta}] = \int_{\mathbb{R}^3} |x|^{-2\theta}\Pi_{t-s}(x)dx < \infty,$$

we obtain that $\mathbb{E}^x[|Y_\varepsilon - Y_0|^2] \to 0$ as $\varepsilon \to 0$. This proves that there exists a subsequence ε' such that $Y_{\varepsilon'} \to Y_0$ almost surely. □

Lemma 6.43. *For every $\alpha \in \mathbb{R}$ the random variable $\alpha S^0_{\mathrm{ren}}$ is exponentially integrable, i.e.,*

$$\mathbb{E}^x[e^{\alpha S^0_{\mathrm{ren}}}] < \infty. \tag{6.8.42}$$

Proof. Recall the decomposition $S^0_{\mathrm{ren}} = S^0_2 + Y_0 + Z_0$ and the pathwise bounds $S^0_2 \le cT^2$ and $Z_0 \le cT$. To prove exponential integrability of $\alpha S^0_{\mathrm{ren}}$ we show that αY_0 is exponentially integrable. Consider $\mathbb{E}^x[e^{2\alpha Y_0}]$. Since $2\alpha Y_0$ can be represented as the stochastic integral

$$Y_0 = \int_{-T}^{T} \Phi dB_t, \tag{6.8.43}$$

where $\Phi = 2\int_{[t-\tau]}^{t} \nabla\varphi_0(B_t - B_s, t-s)ds$, by an application of the Girsanov theorem we obtain the bound

$$\begin{aligned}
(\mathbb{E}^x[e^{2\alpha Y_0}])^2 &= (\mathbb{E}^x[e^{2\alpha\int_{-T}^{T}\Phi dB_t - 4\alpha^2\int_{-T}^{T}|\Phi|^2 dt + 4\alpha^2\int_{-T}^{T}|\Phi|^2 dt}])^2 \\
&\le \mathbb{E}^x[e^{4\alpha\int_{-T}^{T}\Phi dB_t - \frac{1}{2}(4\alpha)^2\int_{-T}^{T}|\Phi|^2 dt}]\mathbb{E}^x[e^{8\alpha^2\int_{-T}^{T}|\Phi|^2 dt}] \\
&= \mathbb{E}^x[e^{8\alpha^2\int_{-T}^{T}|\Phi|^2 dt}].
\end{aligned}$$

Here $\mathbb{E}^x[e^{4\alpha\int_{-T}^{T}\Phi dB_t - \frac{1}{2}(4\alpha)^2\int_{-T}^{T}|\Phi|^2 dt}] = 1$ is derived from the Girsanov theorem. In a similar way to (6.8.40) we have

$$\int_{-T}^{T} |\Phi|^2 dt \le 4c\tau^{2\theta-1}Q,$$

where c is the constant in Lemma 6.40,

$$Q = \int_{-T}^{T} ds \int_{s}^{[s+\tau]} |B_t - B_s|^{-2\theta} dt$$

and recall that we have chosen a suitable $\frac{1}{2} < \theta < 1$. Then we have

$$(\mathbb{E}^x[e^{2\alpha Y_0}])^2 \leq \mathbb{E}^x[e^{32c\alpha^2\tau^{2\theta-1}Q}]. \tag{6.8.44}$$

Set $\gamma = 32c\alpha^2\tau^{2\theta-1}$. By the Jensen inequality we have

$$\mathbb{E}^x[e^{\gamma Q}] \leq \int_{-T}^{T} \frac{ds}{2T} \mathbb{E}^x[e^{2T\gamma \int_s^{[s+\tau]} |B_t - B_s|^{-2\theta} dt}]. \tag{6.8.45}$$

We split up the right-hand side of (6.8.45) by integrating over $(-T, T-\tau)$ and $(T-\tau, T)$. Consider the first term. Since $[s+\tau] = s+\tau$, we have

$$\mathbb{E}^x[e^{2T\gamma \int_s^{[s+\tau]} |B_t - B_s|^{-2\theta} dt}] = \mathbb{E}^x[e^{2T\gamma \int_0^{\tau} |B_{s+t} - B_s|^{-2\theta} dt}] = \mathbb{E}^x[e^{2T\gamma \int_0^{\tau} |B_t|^{-2\theta} dt}].$$

Let $\theta < 1$. Then $1/|x|^{2\theta} \in L^p(\mathbb{R}^3)$ for $p > d/2$, thus $1/|x|^{2\theta}$ is a Kato-class potential and $\sup_x \mathbb{E}^x[e^{2T\gamma \int_0^{\tau} |B_t|^{-2\theta} dt}] < \infty$. Hence

$$\int_{-T}^{T-\tau} \frac{ds}{2T} \mathbb{E}^x[e^{2T\gamma \int_s^{[s+\tau]} |B_t - B_s|^{-2\theta} dt}] < \infty.$$

Next consider the second term. Note that $[s+\tau] = T$. In the same way as above,

$$\begin{aligned} \mathbb{E}^x[e^{2T\gamma \int_s^{T} |B_t - B_s|^{-2\theta} dt}] &= \mathbb{E}^x[e^{2T\gamma \int_0^{T-s} |B_{s+t} - B_s|^{-2\theta} dt}] \\ &= \mathbb{E}^x[e^{2T\gamma \int_0^{T-s} |B_t|^{-2\theta} dt}]. \end{aligned}$$

Hence

$$\int_{T-\tau}^{T} \frac{ds}{2T} \mathbb{E}^x[e^{2T\gamma \int_s^{[s+\tau]} |B_t - B_s|^{-2\theta} dt}] < \infty$$

and the lemma follows. □

Lemma 6.44. *It follows that*

$$\lim_{\varepsilon \downarrow 0} \mathbb{E}^x[|e^{\alpha S^{\varepsilon}_{\mathrm{ren}}} - e^{\alpha S^0_{\mathrm{ren}}}|] = 0, \quad \alpha \in \mathbb{R}. \tag{6.8.46}$$

Proof. Set $A_\varepsilon = e^{\alpha(S_2^\varepsilon + Z^\varepsilon)}$ and $B_\varepsilon = e^{\alpha Y^\varepsilon}$. Then we have

$$\begin{aligned} \mathbb{E}^x[|A_\varepsilon B_\varepsilon - A_0 B_0|] &\leq \mathbb{E}^x[|A_\varepsilon B_\varepsilon - A_0 B_\varepsilon|] + \mathbb{E}^x[|A_0 B_\varepsilon - A_0 B_0|] \\ &\leq \mathbb{E}^x[|A_\varepsilon - A_0|^2]^{1/2} \mathbb{E}^x[|B_\varepsilon|]^{1/2} + \mathbb{E}^x[|A_0||B_\varepsilon - B_0|]. \end{aligned}$$

In a similar way to the proof of Lemma 6.43 we see that $\mathbb{E}^x[|B_\varepsilon|]$ and $\mathbb{E}^x[|A_\varepsilon|]$ are uniformly bounded with respect to ε. Moreover, $\mathbb{E}^x[|A_\varepsilon - A_0|^2] \to 0$ as $\varepsilon \downarrow 0$ by the Lebesgue dominated convergence theorem. We next estimate the second term of the right-hand side above. We have

$$\mathbb{E}^x[|A_0||B_\varepsilon - B_0|] \le \mathbb{E}^x[e^{2\alpha S^0_{\mathrm{ren}}}]\mathbb{E}^x[(e^{\alpha(Y_\varepsilon - Y_0)} - 1)^2].$$

Since $\mathbb{E}^x[e^{2\alpha S^0_{\mathrm{ren}}}]$ is bounded, it suffices to show that the second factor converges to zero. We have

$$\mathbb{E}^x[(e^{\alpha(Y_\varepsilon - Y_0)} - 1)^2] = \mathbb{E}^x[e^{2\alpha(Y_\varepsilon - Y_0)}] + 1 - 2\mathbb{E}^x[e^{\alpha(Y_\varepsilon - Y_0)}]. \tag{6.8.47}$$

We will show that $\mathbb{E}^x[e^{\beta(Y_\varepsilon - Y_0)}] \to 1$ as $\varepsilon \downarrow 0$ for $\beta \in \mathbb{R}$. Then the lemma follows from (6.8.47). Let $\Phi = 2\int_{[t-T]}^t (\nabla\varphi_\varepsilon - \nabla\varphi_0)(B_t - B_s, t-s)ds$. By the Girsanov theorem

$$1 = \mathbb{E}^x[e^{\beta\int_{-T}^T \Phi dB_t - \beta^2/2\int_{-T}^T |\Phi|^2 dt}].$$

Hence it follows that

$$\begin{aligned}
&(\mathbb{E}^x[e^{\beta(Y_\varepsilon - Y_0)}] - 1)^2 \\
&= (\mathbb{E}^x[e^{\beta\int_{-T}^T \Phi dB_t - \beta^2\int_{-T}^T |\Phi|^2 dt + \beta^2\int_{-T}^T |\Phi|^2 dt} - e^{\beta\int_{-T}^T \Phi dB_t - \beta^2/2\int_{-T}^T |\Phi|^2 dt}])^2 \\
&\le \mathbb{E}^x[e^{2\beta\int_{-T}^T \Phi dB_t - 2\beta^2\int_{-T}^T |\Phi|^2 dt}]\mathbb{E}^x[|e^{\beta^2\int_{-T}^T |\Phi|^2 dt} - e^{\beta^2/2\int_{-T}^T |\Phi|^2 dt}|^2] \\
&= \mathbb{E}^x[|e^{\beta^2\int_{-T}^T |\Phi|^2 dt} - e^{\beta^2/2\int_{-T}^T |\Phi|^2 dt}|^2].
\end{aligned}$$

Here we used $\mathbb{E}^x[e^{2\beta\int_{-T}^T \Phi dB_t - 2\beta^2\int_{-T}^T |\Phi|^2 dt}] = 1$ by the Girsanov theorem again. The right-hand side above converges to zero as $\varepsilon \downarrow 0$ by the Lebesgue dominated convergence theorem. □

Theorem 6.45 (Functional integral representation of $e^{iG_\infty}e^{-tH_{\mathrm{N}}^{\mathrm{ren}}}e^{-iG_\infty}$). *It follows that*

$$\begin{aligned}
&(f \otimes 1, e^{iG_\infty}e^{-2TH_{\mathrm{N}}^{\mathrm{ren}}}e^{-iG_\infty} g \otimes 1) \\
&= \int_{\mathbb{R}^3} \mathbb{E}^x[\overline{f(B_{-T})}g(B_T)e^{-\int_{-T}^T V(B_s)ds}e^{(g^2/2)S^0_{\mathrm{ren}}}]dx,
\end{aligned} \tag{6.8.48}$$

where

$$\begin{aligned}
S^0_{\mathrm{ren}} = {} & 2\int_{-T}^T ds \int_{[s+\tau]}^T W^0(B_t - B_s, t-s)dt \\
& - 2\int_{-T}^T \varphi_0(B_{[s+\tau]} - B_s, [s+\tau] - s)ds \\
& + 2\int_{-T}^T ds \int_s^{[s+\tau]} \nabla\varphi_0(B_t - B_s, t-s)\cdot dB_t,
\end{aligned} \tag{6.8.49}$$

W^0 *is given in* (6.8.23), $0 < \tau < T$ *is arbitrary, and*

$$\varphi_0(x,t) = \int_{\mathbb{R}^3} \frac{e^{-ik\cdot x}e^{-|k||t|}}{2|k|}\beta(k)(1 - 1_\kappa(k))dk. \tag{6.8.50}$$

Proof. It suffices to prove the theorem for positive functions f, g. Let

$$f_\Lambda(x) = \begin{cases} \Lambda & f(x) > \Lambda, \\ f(x) & f(x) \le \Lambda \end{cases}$$

and g_Λ be defined similarly. Hence

$$(f_\Lambda \otimes 1, e^{-2TH_{\rm N}^\varepsilon} g_\Lambda \otimes 1) = \int_{\mathbb{R}^3} \mathbb{E}^x[\overline{f_\Lambda(B_{-T})}g_\Lambda(B_T)e^{-\int_{-T}^T V(B_s)ds}e^{g^2 S_{\rm ren}^\varepsilon/2}]dx. \tag{6.8.51}$$

The left-hand side of (6.8.51) converges to $(f_\Lambda \otimes 1, e^{iG_\infty}e^{-2TH_{\rm N}^{\rm ren}}e^{-iG_\infty}g_\Lambda \otimes 1)$ as $\varepsilon \downarrow 0$, and the right-hand side

$$\int_{\mathbb{R}^3} \mathbb{E}^x[\overline{f_\Lambda(B_{-T})}g_\Lambda(B_T)e^{-\int_{-T}^T V(B_s)ds}e^{(g^2/2)S_{\rm ren}^0}]dx \tag{6.8.52}$$

by Lemma 6.44. Thus (6.8.48) follows for f_Λ and g_Λ. Since

$$(f_\Lambda \otimes 1, e^{iG_\infty}e^{-2TH_{\rm N}^{\rm ren}}e^{-iG_\infty}g_\Lambda \otimes 1) \to (f \otimes 1, e^{iG_\infty}e^{-2TH_{\rm N}^{\rm ren}}e^{-iG_\infty}g \otimes 1)$$

as $\Lambda \to \infty$, the monotone convergence theorem yields that

$$\begin{aligned} &\lim_{\Lambda\to\infty} \int_{\mathbb{R}^3} \mathbb{E}^x[\overline{f_\Lambda(B_{-T})}g_\Lambda(B_T)e^{-\int_{-T}^T V(B_s)ds}e^{(g^2/2)S_{\rm ren}^0}]dx \\ &= \int_{\mathbb{R}^3} \mathbb{E}^x[\overline{f(B_{-T})}g(B_T)e^{-\int_{-T}^T V(B_s)ds}e^{(g^2/2)S_{\rm ren}^0}]dx. \end{aligned}$$

Thus the theorem follows. □

6.8.4 Weak coupling limit and removal of ultraviolet cutoff

In this last section we consider the weak coupling limit for the massless Nelson model with many particles. For simplicity we assume $d = 3$ and no external potential. The *N-particle Nelson Hamiltonian* is defined by

$$H_{\rm N} = \frac{1}{2}\sum_{j=1}^N(-\Delta_j) \otimes 1 + 1 \otimes H_{\rm f} + g\sum_{j=1}^N H_{\rm I}(x_j), \tag{6.8.53}$$

which acts on $L^2(\mathbb{R}^{3N}) \otimes \mathscr{F}_{\mathrm{N}}$. With given parameter $\Lambda > 0$ we define the ultraviolet cutoff function of H_{N} to be

$$\hat{\varphi}(k) = (2\pi)^{-3/2} 1_\Lambda(k). \tag{6.8.54}$$

Also, we write $H_{\mathrm{I}}(\Lambda)$ for $\sum_{j=1}^N H_{\mathrm{I}}(x_j)$ with cutoff function (6.8.54). We also introduce the scaling parameter $\kappa > 0$ such that $a(f) \to \kappa a(f)$ and $a^*(f) \to \kappa a^*(f)$. The *scaled Nelson Hamiltonian* with cutoff function (6.8.54) is then defined by

$$H_{\mathrm{N}}(\kappa, \Lambda) = H_{\mathrm{p}} + g\kappa H_{\mathrm{I}}(\Lambda) + \kappa^2 H_{\mathrm{f}}. \tag{6.8.55}$$

We call the $\lim_{\kappa\to\infty}$ in (6.8.55) *weak coupling limit*. Here we consider the asymptotic behaviour of $e^{-TH_{\mathrm{N}}(\kappa,\Lambda)}$ as $\Lambda, \kappa \to \infty$.

Before going to a rigorous computation we give a heuristic description of the asymptotic behaviour. First, note that $e^{-T\kappa^2 H_{\mathrm{f}}} \to P_{\Omega_{\mathrm{b}}}$ as $\kappa \to \infty$, with $P_{\Omega_{\mathrm{b}}}$ being the projection onto the space spanned by the Fock vacuum. Then one can expect that $e^{-TH_{\mathrm{N}}(\kappa,\Lambda)} \to e^{-TH_\infty(\Lambda)} \otimes P_{\Omega_{\mathrm{b}}}$ as $\kappa \to \infty$ with some self-adjoint operator $H_\infty(\Lambda)$ in $L^2(\mathbb{R}^{3N})$. It can indeed be seen that

$$\begin{aligned} &(f \otimes \Omega_{\mathrm{b}}, e^{-TH_{\mathrm{N}}(\kappa,\Lambda)} g \otimes \Omega_{\mathrm{b}}) \\ &= \int_{\mathbb{R}^3} dx \mathbb{E}^x[\overline{f(B_0)} g(B_T) e^{(g^2/2)\int_0^T ds \int_0^T W(X_t - X_s, t-s)dt}], \end{aligned} \tag{6.8.56}$$

where

$$W = \sum_{i,j=1}^N \frac{1}{2(2\pi)^3} \int_{|k|<\Lambda} \frac{1}{\omega(k)} e^{-ik(B_t^i - B_s^j)} \kappa^2 e^{-\kappa^2|t-s|\omega(k)} dk. \tag{6.8.57}$$

Since as $\kappa \to \infty$

$$\kappa^2 e^{-\kappa^2|t-s|\omega} \to \frac{2}{\omega}\delta(t-s), \tag{6.8.58}$$

it follows that

$$\begin{aligned} &\lim_{\Lambda\to\infty} \lim_{\kappa\to\infty} \left\{ \frac{g^2}{2} \int_0^T ds \int_0^T W(X_t - X_s, t-s)dt - g^2 TNE(\Lambda) \right\} \\ &= \frac{g^2}{4\pi} \sum_{i<j}^N \int_0^T \frac{1}{|B_t^i - B_t^j|} dt, \end{aligned} \tag{6.8.59}$$

where the renormalized term $E(\Lambda)$ is defined by the diagonal term

$$E(\Lambda) = \frac{1}{2(2\pi)^3} \int_{|k|<\Lambda} \frac{1}{\omega(k)^2} dk. \tag{6.8.60}$$

We then have

$$\lim_{\Lambda\to\infty}\lim_{\kappa\to\infty}(f\otimes\Omega_{\rm b}, e^{-T(H_{\rm N}(\kappa,\Lambda)-g^2NE(\Lambda))}g\otimes\Omega_{\rm b}) = (f, e^{-TH_\infty}g)_{L^2(\mathbb{R}^{3N})}, \tag{6.8.61}$$

where

$$H_\infty = \frac{1}{2}\sum_j^N(-\Delta_j) - \frac{g^2}{4\pi}\sum_{i<j}^N\frac{1}{|x_i-x_j|}. \tag{6.8.62}$$

We derive (6.8.62) rigorously in the theorem below.

Theorem 6.46 (Weak coupling limit and removal of ultraviolet cutoff). *We have*

$$\operatorname*{s-lim}_{\Lambda\to\infty}\operatorname*{s-lim}_{\kappa\to\infty} e^{-T(H_{\rm N}(\kappa,\Lambda)-g^2NE(\Lambda))} = e^{-TH_\infty}\otimes P_{\Omega_{\rm b}}. \tag{6.8.63}$$

Proof. Let

$$\mathscr{D} = \text{L.H.}\{f\otimes F(\phi(f_1),\dots,\phi(f_n)), f\otimes 1 | f\in\mathscr{S}(\mathbb{R}^3), F\in\mathscr{S}(\mathbb{R}^n), n\ge 1\}.$$

$\mathscr{D}$ is dense in $\mathscr{H}_{\rm N}$. It suffices to show that

$$\lim_{\Lambda\to\infty}\lim_{\kappa\to\infty}(F, e^{-T(H_{\rm N}(\kappa,\Lambda)-g^2NE(\Lambda))}G) = (F, e^{-TH_\infty}\otimes P_{\Omega_{\rm b}}G) \tag{6.8.64}$$

for $F, G\in\mathscr{D}$. Since $f\otimes F(\phi(f_1),\dots,\phi(f_n)) = (2\pi)^{n/2}\int\check{F}(k)e^{-i\sum_j k_j\phi(f_j)}dk$, we have

$$\begin{aligned}(F, e^{-TH_{\rm N}(\kappa,\Lambda)}G) &= \frac{1}{(2\pi)^{\frac{n+m}{2}}}\\ &\times\iint\check{F}(k)\check{G}(l)(f\otimes e^{-i\sum_j^n k_j\cdot\phi(f_j)}, e^{-TH_{\rm N}(\kappa,\Lambda)}g\otimes e^{-i\sum_i^m l_i\cdot\phi(g_i)})dkdl.\end{aligned}$$

Thus it suffices to consider the limit

$$\lim_{\Lambda\to\infty}\lim_{\kappa\to\infty}(f_1\otimes e^{-i\phi(g_1)}, e^{-T(H_{\rm N}(\kappa,\Lambda)-g^2NE(\Lambda))}f_2\otimes e^{-i\phi(g_2)}). \tag{6.8.65}$$

The functional integral representation of (6.8.65) in terms of the Brownian motion and the Euclidean field is given by

$$\begin{aligned}&\int_{\mathbb{R}^3}dx\mathbb{E}^x[\overline{f_1(B_0)}f_2(B_T)\\ &\times(e^{-i\phi_E(\delta_0\otimes g_1)}, e^{-i\phi_E(\delta_{\kappa^2T}\otimes g_2)}e^{-\kappa\phi_E(\sum_j\int_0^T\delta_{\kappa^2s}\otimes\varphi(\cdot-B_s^j)ds)})].\end{aligned} \tag{6.8.66}$$

Then the field variable above can be integrated out and we have

$$\begin{aligned}&(f_1 \otimes e^{-i\phi(g_1)}, e^{-T(H_{\mathrm{N}}(\kappa,\Lambda)-g^2NE(\Lambda))} f_2 \otimes e^{-i\phi(g_2)})\\ &= \int_{\mathbb{R}^3} \mathbb{E}^x[\overline{f_1(B_0)} f_2(B_T) e^{\frac{1}{4}S}]dx,\end{aligned} \tag{6.8.67}$$

where $S = -S_1 + 2iS_2 + S_3$ and

$$S_1 = \|\delta_0 \otimes g_1 - \delta_{\kappa^2 T} \otimes g_2\|^2_{\mathcal{H}_{\mathrm{E}}}, \tag{6.8.68}$$

$$S_2 = \kappa g \sum_j \left(\delta_0 \otimes g_1 - \delta_{\kappa^2 T} \otimes g_2, \int_0^T \delta_{\kappa^2 s} \otimes \varphi(\cdot - B_s^j) ds\right)_{\mathcal{H}_{\mathrm{E}}}, \tag{6.8.69}$$

$$S_3 = \kappa^2 g^2 \left\| \sum_j \int_0^T \delta_s \otimes \varphi(\cdot - B_s^j) ds \right\|^2_{\mathcal{H}_{\mathrm{E}}}. \tag{6.8.70}$$

We see that

$$\lim_{\kappa\to\infty} S_1 = \|g_1\|^2_{\mathcal{H}_{\mathrm{M}}} + \|g_2\|^2_{\mathcal{H}_{\mathrm{M}}}. \tag{6.8.71}$$

Next consider S_2. Note that for a bounded continuous function $h(s)$

$$\lim_{\kappa\to\infty} \int_0^T h(s)\kappa^2 e^{-|t-s|\kappa^2} ds = 2h(t) \tag{6.8.72}$$

for $t < T$. For every j

$$\begin{aligned}&\lim_{\kappa\to\infty} \kappa \left(\delta_0 \otimes g_1, \int_0^T \delta_{\kappa^2 s} \otimes \varphi(\cdot - B_s^j) ds\right)\\ &= \lim_{\kappa\to\infty} \frac{1}{(2\pi)^3} \int_0^T ds \int_{|k|<\Lambda} \frac{\hat{g}_1(k)}{\omega(k)} e^{-ikB_s^j} \kappa e^{-\kappa^2|s|\omega(k)} dk = 0\end{aligned}$$

and similarly

$$\lim_{\kappa\to\infty} \kappa \left(\delta_{\kappa^2 T} \otimes g_2, \int_0^T \delta_{\kappa^2 s} \otimes \varphi(\cdot - B_s^j) ds\right) = 0.$$

Hence $\lim_{\kappa\to\infty} S_2 = 0$. Finally we estimate S_3. By (6.8.72) it follows that

$$\begin{aligned}\lim_{\kappa\to\infty} S_3 &= \lim_{\kappa\to\infty} \sum_{i,j=1}^N \frac{1}{(2\pi)^3} \int_0^T dt \int_0^T ds \int_{|k|<\Lambda} \frac{1}{\omega(k)} e^{-ik(B_t^i - B_s^j)} \kappa^2 e^{-|t-s|\omega(k)\kappa^2} dk\\ &= \sum_{i,j=1}^N \frac{2}{(2\pi)^3} \int_0^T dt \int_{|k|<\Lambda} \frac{1}{\omega(k)^2} e^{-ik(B_t^i - B_t^j)} dk.\end{aligned} \tag{6.8.73}$$

Hence we have

$$\lim_{\kappa\to\infty}(f_1\otimes e^{-i\phi(g_1)}, e^{-T(H_{\rm N}(\kappa,\Lambda)-g^2NE(\Lambda))}f_2\otimes e^{-i\phi(g_2)})$$
$$=\int_{\mathbb{R}^3}\mathbb{E}^x[\overline{f_1(B_0)}f_2(B_T)e^{-\frac14(\|g_1\|^2_{\mathcal{H}_{\rm M}}+\|g_2\|^2_{\mathcal{H}_{\rm M}})}e^{\int_0^T V_\Lambda(B_s^1,\dots,B_s^N)ds}]dx,$$

where

$$V_\Lambda(x_1,\dots,x_N)=\sum_{i<j}^N\frac{g^2}{(2\pi)^3}\int_{|k|\le\Lambda}\frac{1}{\omega(k)^2}e^{-ik(x_i-x_j)}dk.$$

Note that

$$V_\Lambda(x_1,\dots,x_N)=\sum_{i<j}^N\frac{g^2}{2\pi^2}\frac{1}{|x_i-x_j|}\int_0^{\Lambda|x_i-x_j|}\frac{\sin r}{r}dr.$$

We prove that

$$|V_\Lambda(x_1,\dots,x_N)|\le\frac{g^2}{4\pi}\sum_{i<j}^N\frac{2}{|x_i-x_j|}\tag{6.8.74}$$

in Lemma 6.47 below, and thus see that

$$\lim_{\Lambda\to\infty}V_\Lambda(x_1,\dots,x_N)=\frac{g^2}{4\pi}\sum_{i<j}^N\frac{1}{|x_i-x_j|}\tag{6.8.75}$$

for every $(x_1,\dots,x_N)\in\{(x_1,\dots,x_N)\in\mathbb{R}^{3N}|x_i\ne x_j, i,j=1,\dots,N\}$. Denote the right-hand side of (6.8.75) by V. Then by dominated convergence

$$\lim_{\Lambda\to\infty}\int dx\mathbb{E}^x[\overline{f_1(B_0)}f_2(B_T)e^{-\frac14(\|g_1\|^2_{\mathcal{H}_{\rm M}}+\|g_2\|^2_{\mathcal{H}_{\rm M}})}e^{+\int_0^T V_\Lambda(B_s^1,\dots,B_s^N)ds}]$$
$$=(f_1\otimes e^{-i\phi(g_1)},(e^{-TH_\infty}\otimes P_{\Omega_{\rm b}})f_2\otimes e^{-i\phi(g_2)}).\qquad\square$$

Lemma 6.47. *We have* (6.8.74).

Proof. Let $\mu>0$ and define $V_{\Lambda,\mu}$ by V_Λ with $\omega(k)$ replaced by $\sqrt{|k|^2+\mu^2}$. Then it follows that

$$V_{\Lambda,\mu}(x_1,\dots,x_N)=\sum_{i<j}^N\frac{g^2}{2\pi^2}\frac{1}{|x_i-x_j|}\int_0^{\Lambda|x_i-x_j|}\frac{r}{r^2+\mu^2|x_i-x_j|^2}\sin r dr.$$

Define the closed curves γ_{ij}, $1\le i,j\le N$, by

$$\gamma_{ij}=\{z\in\mathbb{R}|-\Lambda|x_i-x_j|\le z\le\Lambda|x_i-x_j|\}\cup C_{ij},$$

where $C_{ij} = \{\Lambda|x_i - x_j|e^{i\theta} \in \mathbb{C} | 0 \le \theta \le \pi\}$ denotes a semi-circle. Let

$$\rho(u) = \frac{u}{u^2 + |x_i - x_j|^2 \mu^2} e^{iu}.$$

Since

$$\int_0^{\Lambda|x_i - x_j|} \frac{r}{r^2 + \mu^2 |x_i - x_j|^2} \sin r dr = \frac{1}{2} \Im \int_{-\Lambda|x_i - x_j|}^{\Lambda|x_i - x_j|} \rho(u) du$$

and

$$\int_{-\Lambda|x_i - x_j|}^{\Lambda|x_i - x_j|} \rho(u) du = \int_{\gamma_{ij}} \rho(u) du - \int_{C_{ij}} \rho(u) du,$$

we see that

$$\begin{aligned}
V_{\Lambda,\mu}(x_1, \ldots, x_N) &= \frac{g^2}{4\pi^2} \Im \sum_{i<j}^{N} \frac{1}{|x_i - x_j|} \int_{-\Lambda|x_i - x_j|}^{\Lambda|x_i - x_j|} \frac{u}{u^2 + |x_i - x_j|^2 \mu^2} e^{iu} du \\
&= \frac{2\pi g^2}{4\pi^2} \sum_{i<j}^{N} \frac{1}{|x_i - x_j|} \mathrm{Res}\left(\frac{u e^{iu}}{u^2 + |x_i - x_j|^2 \mu^2}; i\mu|x_i - x_j| \right) \\
&\quad - \frac{g^2}{4\pi^2} \Im \sum_{i<j}^{N} \frac{1}{|x_i - x_j|} \int_0^{\pi} \frac{\Lambda^2 |x_i - x_j|^2 i e^{2i\theta} e^{i\Lambda|x_i - x_j| e^{i\theta}}}{\Lambda^2 |x_i - x_j|^2 e^{2i\theta} + |x_i - x_j|^2 \mu^2} d\theta,
\end{aligned}$$

where $\mathrm{Res}(f(u); z)$ denotes the residue of $f(u)$ at $u = z$. It can be seen that

$$\mathrm{Res}\left(\frac{u e^{iu}}{u^2 + |x_i - x_j|^2 \mu^2}; i\mu|x_i - x_j| \right) = \frac{1}{2} e^{-\mu|x_i - x_j|}$$

and

$$\left| \int_0^{\pi} \frac{\Lambda^2 |x_i - x_j|^2 i e^{2i\theta} e^{i\Lambda|x_i - x_j| e^{i\theta}}}{\Lambda^2 |x_i - x_j|^2 e^{2i\theta} + |x_i - x_j|^2 \mu^2} d\theta \right| \le \pi \left| \frac{\Lambda^2}{\Lambda^2 - \mu^2} \right|.$$

Hence

$$|V_{\Lambda,\mu}(x_1, \ldots, x_N)| \le \frac{g^2}{4\pi} \sum_{i<j}^{N} \frac{1}{|x_i - x_j|} \left(e^{-\mu|x_i - x_j|} + \left| \frac{\Lambda^2}{\Lambda^2 - \mu^2} \right| \right). \tag{6.8.76}$$

Taking the limit $\mu \to 0$ on both sides above, the lemma follows. □

Remark 6.4. Let $\omega(k) = \sqrt{|k|^2 + \mu^2}$. In this case the Yukawa potential is obtained instead of the Coulomb potential, i.e.,

$$\operatorname*{s-lim}_{\Lambda\to\infty} \operatorname*{s-lim}_{\kappa\to\infty} e^{-T(H(\kappa,\Lambda)-g^2NE(\Lambda))} = e^{-TH_\infty} \otimes P_{\Omega_{\mathrm{b}}}, \tag{6.8.77}$$

where

$$H_\infty = \frac{1}{2}\sum_j^N(-\Delta_j) - \frac{g^2}{4\pi}\sum_{i<j}^N \frac{e^{-\mu|x_i-x_j|}}{|x_i - x_j|}.$$

Chapter 7
The Pauli–Fierz model by path measures

7.1 Preliminaries

7.1.1 Introduction

It has been seen in the previous section that functional integration in the framework of Euclidean quantum field theory is a powerful tool in the spectral analysis of the Nelson scalar quantum field model. In this section we consider a model of non-relativistic quantum electrodynamics (QED) by using functional integration. This involves a stochastic integral with a Euclidean quantum field version of a quantized radiation field. A significant point is that the theory is non-relativistic in the motion of the particle only, while the quantized radiation field is necessarily relativistic. In addition, it also has the feature that describes the particle "dressed" in a cloud of bosons, while its bare charge is conserved.

The Hamiltonian of non-relativistic QED is defined as a self-adjoint operator on a Hilbert space. Its spectrum is of basic interest since it involves perturbations of embedded eigenvalues in the continuous spectrum. Recently, progress in the spectral analysis of non-relativistic QED has produced an understanding of aspects of binding, spectral scattering theory, resonances, and effective mass to a fine degree. The functional integration introduced in this section offers a new point of view in the spectral analysis of non-relativistic QED. Importantly, this approach is free of perturbative elements, and it leads to a proof of existence and uniqueness of the ground state of the Hamiltonian in a surprisingly simple manner. In addition to its applications in spectral analysis, we believe that functional integration is an interesting mathematical method in its own right. As we will see, the course of developing this theory gives rise to a new class of objects in probability theory and new problems.

In this section we consider the *Pauli–Fierz Hamiltonian* H_{PF} in non-relativistic QED, in which a quantum system of a low energy electron interacting through a minimally coupling with a massless quantized radiation field is described. In this model the radiation field is quantized in the Coulomb gauge instead of the Lorentz gauge, derivative coupling is involved in its interaction by the minimal coupling, and an ultraviolet cutoff is imposed to define the model as a self-adjoint operator on a Hilbert space.

The existence of a ground state Ψ_{g} of the Pauli–Fierz Hamiltonian has been established by Bach–Fröhlich–Sigal [37] and Griesemer–Lieb–Loss [206]. For the details of the assumptions on the external potential we refer to the original paper. In particular, no infrared cutoff is assumed and no restriction on the values of the coupling constant is imposed.

Theorem 7.1. *Under given conditions on V, there exists a ground state Ψ_g of H_{PF}, and $(\Psi_g, N\Psi_g) < \infty$.*

We emphasize that in this theorem no infrared regularity condition such as (6.2.17) in the case of the Nelson model is assumed.

Conventional quantum electrodynamics uses the Lagrangian density $\mathscr{L}_{QED}(x)$. In the language of Feynman diagrams, the leading term of the effective mass is computed from the self-energy of the electron and the g-factor shift by vertex diagrams. Both diagrams include virtual photons, and give corrections to a bare mass and the g-factor. On the other hand, the effective charge e_{eff} is computed from the self-energy of photons. The photon self-energy diagram can be interpreted as the emission of the pairs of virtual electrons and positrons. The assumption that the energy of the electron is low implies that no emission of pairs of virtual electrons and positrons takes place. In the Pauli–Fierz model thus no diagram of self-energy of photons is included, and the effective charge equals the bare charge and the number of electrons stays constant.

Here is an outline of cases discussed in terms of functional integration techniques in this chapter:

(1) spinless Pauli–Fierz model

(2) relativistic Pauli–Fierz model

(3) Pauli–Fierz model with spin $1/2$

(4) translation invariant Pauli–Fierz model.

In Chapter 6 the path measure was constructed on a Hilbert space-valued path space, and the functional integrals are expressed over an infinite dimensional Ornstein–Uhlenbeck process. Intuitively, this is an analogue of the finite dimensional Ornstein–Uhlenbeck process. For the more complicated Pauli–Fierz model, however, we adopt the Euclidean quantum field method described in Chapter 5. In this case the functional integrals are constructed on the product measure space of a Lévy measure and a Gaussian measure associated with the Euclidean quantum field version of the quantized radiation field. Moreover, in this case we will have to deal with a Hilbert space-valued stochastic integral with a time dependent integrand.

7.1.2 Lagrangian QED

As in the case of Nelson's model discussed in the previous chapter, also here we start by the formulation of the model in Fock space and then proceed to an equivalent formulation in terms of path integrals. In this section we review the Lagrangian formalism of quantum electrodynamics.

We begin by some notations of the relativistically covariant theory used only in this section. Let

$$g_{\mu\nu} = \begin{cases} +1, & \mu = \nu = 0, \\ -1, & \mu = \nu = 1, 2, 3, \\ 0, & \text{otherwise} \end{cases} \tag{7.1.1}$$

and write $x = (t, \mathbf{x}) = (x^0, x^1, x^2, x^3) \in \mathbb{R}^4$, $x_\mu = g_{\mu\nu} x^\nu$. Here and in what follows in this section we understand that summation is performed over repeated indices. The index μ runs from 0 to 3 and j from 1 to 3. Also, we put $\partial x^\mu = \partial_\mu$ and $\partial x_\mu = \partial^\mu$, involving $\partial^0 = \partial_0$ and $\partial^j = -\partial_j$. The 4×4 gamma matrices are defined by

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \sigma^\mu & 0 \end{pmatrix}, \tag{7.1.2}$$

where $\sigma^0 = 1_{2\times 2}$ and σ^μ, $\mu = 1, 2, 3$, are the 2×2 Pauli matrices.

Let $\mathbf{A}^\mu = \mathbf{A}^\mu(x)$ stand for the radiation field and write $\mathbf{A}_\mu = \mathbf{A}_\mu(x) = g_{\mu\nu}\mathbf{A}^\nu(x)$, i.e., $\mathbf{A}^0 = \mathbf{A}_0$ and $\mathbf{A}^j = -\mathbf{A}_j$. The spinor field will be denoted by $\Psi = \Psi(x)$, and we write $\overline{\Psi} = \overline{\Psi(x)} = \Psi^* \gamma^0$. The Lagrangian density in QED terms is given by

$$\mathscr{L}_{\text{QED}} = \overline{\Psi}(i\gamma^\mu \partial_\mu - m)\Psi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - e\overline{\Psi}\gamma^\mu \Psi \mathbf{A}_\mu. \tag{7.1.3}$$

Here $e \in \mathbb{R}$ and $m > 0$ denote the charge and mass of the electron, respectively, $F^{\mu\nu}$ is the antisymmetric second-rank field tensor

$$F^{\mu\nu} = \partial^\mu \mathbf{A}^\nu - \partial^\nu \mathbf{A}^\mu \tag{7.1.4}$$

and $F_{\mu\nu} = g_{\mu\alpha} g_{\nu\beta} F^{\alpha\beta} = \partial_\mu \mathbf{A}_\nu - \partial_\nu \mathbf{A}_\mu$. The longitudinal component of the radiation field will be denoted by $\Phi = \mathbf{A}^0$ and the transversal by $\mathbf{A} = (\mathbf{A}^1, \mathbf{A}^2, \mathbf{A}^3)$. Let $\dot{X} = \partial_0 X$. Then the electrostatic field $\mathbb{E} = (\mathbb{E}^1, \mathbb{E}^2, \mathbb{E}^3)$ is defined by

$$\mathbb{E}^j = F^{j0} = -\partial_j \Phi - \dot{\mathbf{A}}^j, \tag{7.1.5}$$

while the magnetic field is the curl of the transversal component,

$$\mathbb{B} = (\mathbb{B}^1, \mathbb{B}^2, \mathbb{B}^3) = \nabla \times \mathbf{A}, \tag{7.1.6}$$

where $\nabla = (\partial_1, \partial_2, \partial_3)$. In particular, the field tensor is obtained as

$$F^{\mu\nu} = \begin{pmatrix} 0 & -\mathbb{E}^1 & -\mathbb{E}^2 & -\mathbb{E}^3 \\ \mathbb{E}^1 & 0 & -\mathbb{B}^3 & \mathbb{B}^2 \\ \mathbb{E}^2 & \mathbb{B}^3 & 0 & -\mathbb{B}^1 \\ \mathbb{E}^3 & -\mathbb{B}^2 & \mathbb{B}^1 & 0 \end{pmatrix}, \quad F_{\mu\nu} = \begin{pmatrix} 0 & \mathbb{E}^1 & \mathbb{E}^2 & \mathbb{E}^3 \\ -\mathbb{E}^1 & 0 & -\mathbb{B}^3 & \mathbb{B}^2 \\ -\mathbb{E}^2 & \mathbb{B}^3 & 0 & -\mathbb{B}^1 \\ -\mathbb{E}^3 & -\mathbb{B}^2 & \mathbb{B}^1 & 0 \end{pmatrix}. \tag{7.1.7}$$

By straightforward computation, using the explicit forms of $F_{\mu\nu}$ and $F^{\mu\nu}$ it follows that

$$-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} = \frac{1}{2}(\mathbb{E}^2 - \mathbb{B}^2). \tag{7.1.8}$$

From this the Lagrangian density is obtained as

$$\mathscr{L}_{\mathrm{QED}} = \overline{\Psi}(i\gamma^\mu\partial_\mu - m)\Psi + \frac{1}{2}(\mathbb{E}^2 - \mathbb{B}^2) + e\overline{\Psi}\gamma^j\Psi\mathbf{A}^j - e\Psi^*\Psi\Phi \tag{7.1.9}$$

using the fact that $\overline{\Psi}\gamma^0\Psi = \Psi^*\Psi$. From the Lagrangian we can derive the Euler-Lagrange equations through the variational principle

$$\text{(M)} \quad \partial_\mu F^{\mu\nu} = e\bar{\Psi}\gamma^\nu\Psi, \tag{7.1.10}$$

$$\text{(D)} \quad (i\gamma^\mu\partial_\mu - m)\Psi = e\gamma^\mu\mathbf{A}_\mu\Psi. \tag{7.1.11}$$

(7.1.11) is the *Dirac equation* and (7.1.10) is written as

$$\Box\mathbf{A}^\nu - \partial^\nu\partial_\mu\mathbf{A}^\mu = e\bar{\Psi}\gamma^\nu\Psi, \tag{7.1.12}$$

where $\Box = \partial_0\partial_0 - \partial_j\partial_j$, and is equivalent with

$$(\nu = 0) \quad \nabla\cdot\mathbb{E} = e\Psi^*\Psi, \tag{7.1.13}$$

$$(\nu = j) \quad \dot{\mathbb{E}}^j = (\nabla\times\mathbb{B})^j - e\bar{\Psi}\gamma^j\Psi. \tag{7.1.14}$$

Moreover, the definitions of $\mathbb{E}$ and $\mathbb{B}$ give

$$\dot{\mathbb{B}} = -\nabla\times\mathbb{E}, \tag{7.1.15}$$

$$\nabla\cdot\mathbb{B} = 0. \tag{7.1.16}$$

The system of (7.1.13)–(7.1.16) are the celebrated *Maxwell equations*.

We derive now the QED Hamiltonian by using Legendre transforms. To see this from $\mathscr{L}_{\mathrm{QED}}$ we identify the conjugate momenta

$$\Pi^0 = \frac{\partial\mathscr{L}_{\mathrm{QED}}}{\partial\dot{\mathbf{A}}_0} = 0, \tag{7.1.17}$$

$$\Pi^j = \frac{\partial\mathscr{L}_{\mathrm{QED}}}{\partial\dot{\mathbf{A}}_j} = \mathbb{E}^j \tag{7.1.18}$$

and

$$\frac{\partial\mathscr{L}_{\mathrm{QED}}}{\partial\dot{\Psi}} = i\Psi^*. \tag{7.1.19}$$

The canonical momentum Π^0 vanishes. Thus the Legendre transform gives the Hamiltonian density $\mathscr{H}_{\mathrm{QED}} = \mathscr{H}_{\mathrm{QED}}(x)$ by

$$\begin{aligned}\mathscr{H}_{\mathrm{QED}} &= \Pi^j \dot{\mathbf{A}}_j + i\Psi^*\dot{\Psi} - \mathscr{L}_{\mathrm{QED}} \\ &= \Psi^*\alpha^j(-i\partial_j + e\mathbf{A}_j)\Psi + m\beta\Psi^*\Psi + \frac{1}{2}(\mathbb{E}^2 + \mathbb{B}^2) + e\Psi^*\Psi\Phi + \nabla\Phi\cdot\mathbb{E}.\end{aligned}$$

Here $\alpha^j = \gamma^0\gamma^j$ and $\beta = \gamma^0$.

Next, in order to derive the Pauli–Fierz Hamiltonian as an analogue of full QED, instead of the Lorentz gauge the Coulomb gauge condition

$$\nabla \cdot \mathbf{A} = 0 \tag{7.1.20}$$

is used. This gauge is not covariant and (7.1.12) reduces to

$$\Box\mathbf{A}^\nu - \partial^\nu\dot{\Phi} = e\bar{\Psi}\gamma^\nu\Psi.$$

Notice that under the Lorentz gauge condition $\partial_\mu\mathbf{A}^\mu = 0$, (7.1.12) reduces to the wave equation

$$\Box\mathbf{A}^\nu = e\bar{\Psi}\gamma^\nu\Psi. \tag{7.1.21}$$

The Coulomb gauge condition (7.1.20) yields

$$\nabla\cdot\mathbb{E} = -\Delta\Phi = e\Psi^*\Psi, \tag{7.1.22}$$

where $\Delta = \partial_j\partial_j$ denotes the 3-dimensional Laplacian and the second equality is derived from the Euler–Lagrange equations (7.1.13)–(7.1.14). This is the *Poisson equation* and the longitudinal component Φ in the Coulomb gauge becomes

$$\Phi(t,\mathbf{x}) = e\int_{\mathbb{R}^3}\frac{\Psi^*(t,\mathbf{y})\Psi(t,\mathbf{y})}{4\pi|\mathbf{x}-\mathbf{y}|}d\mathbf{y}. \tag{7.1.23}$$

On decomposing the electric field $\mathbb{E} = -\nabla\Phi - \dot{\mathbf{A}}$ into a longitudinal and a transversal component, we furthermore have

$$\int_{\mathbb{R}^3}\mathbb{E}^2 d\mathbf{x} = -\int_{\mathbb{R}^3}\Phi\cdot\Delta\Phi d\mathbf{x} + \int_{\mathbb{R}^3}\dot{\mathbf{A}}^2 d\mathbf{x}$$

by the Coulomb gauge condition. Moreover,

$$-\int_{\mathbb{R}^3}\Phi\cdot\Delta\Phi d\mathbf{x} = +e^2\int_{\mathbb{R}^3}\frac{\Psi^*(t,\mathbf{y})\Psi(t,\mathbf{y})\Psi^*(t,\mathbf{x})\Psi(t,\mathbf{x})}{4\pi|\mathbf{x}-\mathbf{y}|}d\mathbf{x}d\mathbf{y}. \tag{7.1.24}$$

The term $e\Psi^*\Psi\Phi + \nabla\Phi\cdot\mathbb{E}$ in $\mathscr{H}_{\mathrm{QED}}$ vanishes since by integration,

$$\int_{\mathbb{R}^3}\nabla\Phi\cdot\mathbb{E}d\mathbf{x} = -\int_{\mathbb{R}^3}\Phi\nabla\cdot\mathbb{E}d\mathbf{x} = -e\int_{\mathbb{R}^3}\Psi^*\Psi\Phi d\mathbf{x}.$$

Finally, the Hamiltonian

$$H_{\mathrm{QED}} = \int_{\mathbb{R}^3} \mathscr{H}_{\mathrm{QED}} d\mathbf{x} \tag{7.1.25}$$

of full QED with the Coulomb gauge is given by

$$\begin{aligned} H_{\mathrm{QED}} = &\int_{\mathbb{R}^3} \Psi^* \alpha^j (-i\partial_j + e\mathbf{A}_j)\Psi d\mathbf{x} + m\beta \int_{\mathbb{R}^3} \Psi^* \Psi d\mathbf{x} \\ &+ \frac{1}{2}\int_{\mathbb{R}^3} (\dot{\mathbf{A}}^2 + \mathbb{B}^2) d\mathbf{x} + e^2 \int_{\mathbb{R}^3\times\mathbb{R}^3} \frac{\Psi^*(t,\mathbf{y})\Psi(t,\mathbf{y})\Psi^*(t,\mathbf{x})\Psi(t,\mathbf{x})}{4\pi|\mathbf{x}-\mathbf{y}|} d\mathbf{x}d\mathbf{y}. \end{aligned}$$

7.1.3 Classical variant of non-relativistic QED

In this section we derive a classical version of the non-relativistic QED Hamiltonian using the Lagrange formalism, which is an analogue of the discussion in the preceding section. Here $\overline{\Psi}(i\gamma^\mu\partial_\mu - m)\Psi$ is replaced by the N-particle classical kinetic energy $(1/2)\sum_{j=1}^N m_j \dot{q}_j^2$, $e\overline{\Psi}\gamma^j\Psi$ by the current density j, and $e\Psi^*\Psi$ by the density ρ. The Pauli–Fierz Hamiltonian in non-relativistic QED will be given by quantization of the classical version and the spinor field is changed to electrons governed by the Schrödinger operator.

Let N electrons interact with a classical electromagnetic field described by the *Maxwell equations*. Denote by $B(t,x) \in \mathbb{R}^3$ the classical magnetic field and by $E(t,x) \in \mathbb{R}^3$ the classical electric field. Consider the Maxwell equations with form factor λ,

$$\nabla\cdot E = \sum_{j=1}^N e_j \lambda(\cdot - x_j), \tag{7.1.26}$$

$$\dot{E} = \nabla\times B - \sum_{j=1}^N e_j\lambda(\cdot - x_j)\dot{q}_j, \tag{7.1.27}$$

$$\dot{B} = -\nabla\times E, \tag{7.1.28}$$

$$\nabla\cdot B = 0. \tag{7.1.29}$$

Here $x_j = x_j(t) \in \mathbb{R}^3$ denotes the position and e_j the charge of the electron labelled by j. The form factor describes the charge distribution satisfying $\lambda \geq 0$ and $\int_{\mathbb{R}^3} \lambda(x)dx = 1$. We define the current by

$$(j,\rho) = \left(\sum_{j=1}^N e_j\lambda(\cdot - x_j)\dot{q}_j,\ \ \sum_{j=1}^N e_j\lambda(\cdot - x_j)\right). \tag{7.1.30}$$

which satisfies the continuity equation $\dot{\rho} + \nabla \cdot j = 0$. The equation of motion of the electron is given by

$$m_j \ddot{q}_j = e_j(E + \dot{q}_j \times B), \tag{7.1.31}$$

where m_j is the mass of the j-th electron.

We next rewrite the Maxwell equations by introducing a vector potential A and a scalar potential ϕ. By (7.1.29) $A = (A_1(t,x), A_2(t,x), A_3(t,x))$ is required to satisfy

$$B = \nabla \times A. \tag{7.1.32}$$

Furthermore, from (7.1.28) it follows that $\nabla \times (E + \dot{A}) = 0$. The scalar potential $\phi = \phi(t,x)$ is given by

$$E = -\dot{A} - \nabla\phi. \tag{7.1.33}$$

With this the Lagrangian becomes

$$\mathscr{L}_{\mathrm{PF}} = \frac{1}{2}\sum_{j=1}^{N} m_j \dot{q}_j^2 + \int_{\mathbb{R}^3} \frac{1}{2}(E^2 - B^2)dx + \int_{\mathbb{R}^3} j \cdot A dx + \int_{\mathbb{R}^3} (-\rho\phi) dx \tag{7.1.34}$$

supplemented with the *Coulomb gauge condition*

$$\nabla \cdot A = 0. \tag{7.1.35}$$

The system (7.1.28)–(7.1.29) thus turns into

$$\square A = \nabla\dot{\phi} - j, \tag{7.1.36}$$

$$\Delta\phi = -\rho. \tag{7.1.37}$$

Inserting the definition of E (7.1.33) and taking into account the Coulomb gauge we furthermore have

$$\int_{\mathbb{R}^3} \frac{1}{2}(E^2 - B^2)dx = \int_{\mathbb{R}^3} \left\{ \frac{1}{2}\left(\dot{A}^2 - B^2\right) + \frac{1}{2}\rho\phi \right\} dx.$$

By the Poisson equation $\Delta\phi = -\rho$, we have the Coulomb potential regularized by ρ,

$$\frac{1}{2}\int_{\mathbb{R}^3} \rho\phi dx = \frac{1}{2}\sum_{i,j=1}^{N} \int_{\mathbb{R}^3} \frac{e_i e_j \lambda(x_i - y)\lambda(x_j - y')}{4\pi|y - y'|} dy dy'.$$

Now we define the Hamiltonian associated with the Lagrangian $\mathscr{L}_{\mathrm{PF}}$ through Legendre transform. We introduce the canonical momenta p_j and Π_i by

$$p_j = \frac{\partial \mathscr{L}_{\mathrm{PF}}}{\partial \dot{q}_j} = m_j \dot{q}_j + e_j \int_{\mathbb{R}^3} A(x)\lambda(x - x_j)dx$$

and

$$\Pi_i = \frac{\delta \mathscr{L}_{\mathrm{PF}}}{\delta \dot{A}_i} = \dot{A}_i, \quad i = 1, 2, 3.$$

In terms of these variables the Hamiltonian associated with $\mathscr{L}_{\mathrm{PF}}$ becomes

$$H_{\mathrm{classical}} = \sum_{j=1}^{N} p_j \dot{q}_j + \int_{\mathbb{R}^3} \Pi \cdot \dot{A} dx - \mathscr{L}_{\mathrm{PF}}.$$

A straightforward computation yields

$$\begin{aligned} H_{\mathrm{classical}} = \sum_{j=1}^{N} \frac{1}{2m_j} \left(p_j - e_j \int_{\mathbb{R}^3} A(x)\lambda(x - x_j)dx \right)^2 + \frac{1}{2} \int_{\mathbb{R}^3} (\dot{A}^2 + B^2) dx \\ + \frac{1}{2} \sum_{i,j=1}^{N} \int_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{e_i e_j \lambda(x_i - y)\lambda(x_j - y')}{4\pi |y - y'|} dy dy'. \end{aligned}$$

This Hamiltonian contains the convolution $\int_{\mathbb{R}^3} A(x)\lambda(x - x_j)dx$, the field energy $\frac{1}{2}\int_{\mathbb{R}^3}(\dot{A}^2 + B^2)dx$, the regularized Coulomb potential, and the minimal interaction. In the next section we leave the realm of classical electrodynamics and quantize $H_{\mathrm{classical}}$. We will define a self-adjoint operator H_{PF} on a suitable Hilbert space $\mathscr{H}_{\mathrm{PF}}$. In the quantum description H_{PF} will be defined by $H_{\mathrm{classical}}$ replaced by $p_j \to -i\partial_j$, $j = 1, \ldots, N$, the smeared Coulomb potential goes into a multiplication operator given by a real function V,

$$\int_{\mathbb{R}^3} A_\mu(x)\lambda(x - x_j)dx \to \mathscr{A}_\mu(\tilde{\varphi}(\cdot - x)),$$

and

$$\frac{1}{2} \int_{\mathbb{R}^3} (\dot{A}^2 + B^2)dx \to H_{\mathrm{rad}}.$$

Here $\mathscr{A}_\mu(\tilde{\varphi}(\cdot - x))$ and H_{rad} are the quantized radiation field and the free field Hamiltonian on the boson Fock space, respectively, which are defined and discussed in detail in the next section.

7.2 The Pauli–Fierz model in non-relativistic QED

7.2.1 The Pauli–Fierz model in Fock space

We now turn to defining the Pauli–Fierz Hamiltonian rigorously as a self-adjoint operator bounded from below on a Hilbert space over $\mathbb{C}$. For simplicity we assume only one electron interacting with the field.

Let

$$\mathcal{H}_{\mathrm{PF}} = L^2(\mathbb{R}^3) \otimes \mathcal{F}_{\mathrm{rad}} \tag{7.2.1}$$

be the Hilbert space describing the joint electron-photon state vectors. Here

$$\mathcal{F}_{\mathrm{rad}} = \mathcal{F}_{\mathrm{rad}}(L^2(\mathbb{R}^3 \times \{\pm\})) \tag{7.2.2}$$

is the boson Fock space over $L^2(\mathbb{R}^3 \times \{\pm\})$. The elements of the set $\{\pm\}$ account for the fact that a photon is a transversal wave perpendicular to the direction of its propagation, thus it has two components. The Fock vacuum in $\mathcal{F}_{\mathrm{rad}}$ is denoted by Ω_{ph}. We identify $\mathcal{H}_{\mathrm{PF}}$ as the set of $\mathcal{F}_{\mathrm{rad}}$-valued L^2-functions on $\mathbb{R}^3$,

$$\mathcal{H}_{\mathrm{PF}} \cong \int_{\mathbb{R}^3}^{\oplus} \mathcal{F}_{\mathrm{rad}} dx; \tag{7.2.3}$$

this will be used without further notice in what follows.

Let $a(f)$ and $a^*(f)$, $f \in L^2(\mathbb{R}^3 \times \{\pm\})$, be the annihilation operator and the creation operator, respectively. We use the identification $L^2(\mathbb{R}^3 \times \{\pm\}) \cong L^2(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3)$ and set

$$a^\sharp(f, +) = a^\sharp(f \oplus 0), \quad a^\sharp(f, -) = a^\sharp(0 \oplus f), \tag{7.2.4}$$

where $a^\sharp$ stands for either operator. Then $a^\sharp(f) = \sum_{j=\pm} a^\sharp(f_j, j)$ for $f = f_1 \oplus f_2$. In terms of (formal) kernels $a^\sharp(k, j)$, $a^\sharp(f, j)$ will be written more conveniently as

$$a^\sharp(f, j) = \int_{\mathbb{R}^3} a^\sharp(k, j) f(k) dk.$$

The finite particle subspace of $\mathcal{F}_{\mathrm{rad}}$ is given by

$$\mathcal{F}_{\mathrm{rad,fin}} = \{\Psi = \{\Psi^{(n)}\}_{n=0}^{\infty} | \Psi^{(m)} = 0 \text{ for all } m > N \text{ with some } N\}. \tag{7.2.5}$$

Next we define the quantized radiation field with a given form factor $\hat{\varphi}$. Put

$$\wp_\mu^j(x) = \frac{\hat{\varphi}(k)}{\sqrt{\omega(k)}} e_\mu^j(k) e^{-ik\cdot x} \in L^2(\mathbb{R}_k^3), \tag{7.2.6}$$

$$\tilde{\wp}_\mu^j(x) = \frac{\hat{\varphi}(-k)}{\sqrt{\omega(k)}} e_\mu^j(k) e^{ik\cdot x} \in L^2(\mathbb{R}_k^3), \tag{7.2.7}$$

for $x \in \mathbb{R}^3$, $j = \pm$ and $\mu = 1, 2, 3$, where ω is the *dispersion relation* for massless photons defined by

$$\omega(k) = \omega_{\nu=0}(k) = |k|. \tag{7.2.8}$$

Here $\hat{\varphi}$ is the Fourier transform of the charge distribution φ. The vectors $e^+(k)$ and $e^-(k)$ are called *polarization vectors*, that is, $e^+(k)$, $e^-(k)$ and $k/|k|$ form a right-hand system at $k \in \mathbb{R}^3$;

$$e^i(k) \cdot e^j(k) = \delta_{ij} 1, \quad e^j(k) \cdot k = 0, \quad e^+(k) \times e^-(k) = k/|k|. \tag{7.2.9}$$

In Proposition 7.3 we will see that the spectral analysis is independent of the choice of polarization vectors, i.e., the Hamiltonians defined through different polarizations are unitary equivalent. Thus we may fix the polarization vectors as it is most convenient.

Definition 7.1 (Quantized radiation field). The *quantized radiation field* with form factor $\hat{\varphi}$ is defined by

$$A_\mu(x) = \frac{1}{\sqrt{2}} \sum_{j=\pm} (a^*(\wp^j_\mu(x), j) + a(\tilde{\wp}^j_\mu(x), j)). \tag{7.2.10}$$

In the case of $\overline{\hat{\varphi}(k)} = \hat{\varphi}(-k)$, $A_\mu(x)$ is symmetric, and moreover essentially self-adjoint on $\mathscr{F}_{\mathrm{rad,fin}}$ by the Nelson analytic vector theorem (Proposition 5.3), (5.2.27) and (5.2.28). The condition $\overline{\hat{\varphi}(k)} = \hat{\varphi}(-k)$ is necessary and sufficient for φ being real. A physically reasonable choice of φ requires a positive function. We denote the closure of $A_\mu(x)\lceil_{\mathscr{F}_{\mathrm{rad,fin}}}$ by the same symbol. Write

$$A_\mu = \int_{\mathbb{R}^3}^{\oplus} A_\mu(x) dx, \quad A = (A_1, A_2, A_3). \tag{7.2.11}$$

A_μ is a self-adjoint operator on

$$D(A_\mu) = \left\{ F \in \mathscr{H}_{\mathrm{PF}} \,\middle|\, F(x) \in D(A_\mu(x)) \text{ and } \int_{\mathbb{R}^3} \|A_\mu(x)F(x)\|^2_{\mathscr{F}_{\mathrm{rad}}} dx < \infty \right\}$$

and acts by

$$(A_\mu F)(x) = A_\mu(x) F(x), \quad F \in D(A_\mu),$$

for almost every $x \in \mathbb{R}^3$. In terms of the kernel $a^\sharp(k, j)$, the quantized radiation field $A_\mu(x)$ can be written as

$$A_\mu(x) = \sum_{j=\pm} \frac{1}{\sqrt{2}} \int_{\mathbb{R}^3} e^j_\mu(k) \left(\frac{\hat{\varphi}(k)}{\sqrt{\omega(k)}} e^{-ik\cdot x} a^*(k, j) + \frac{\hat{\varphi}(-k)}{\sqrt{\omega(k)}} e^{ik\cdot x} a(k, j) \right) dk.$$

Since $k \cdot e^j(k) = 0$, the polarization vectors introduced above are chosen in the way that $\sum_{\mu=1}^{3} \nabla_\mu \varrho_j^\mu(x) = 0$, implying the *Coulomb gauge condition*

$$\sum_{\mu=1}^{3} \nabla_\mu \cdot A_\mu = 0. \tag{7.2.12}$$

This in turn yields $\sum_{\mu=1}^{3} [\nabla_\mu, A_\mu] = 0$.

Next we introduce the free field Hamiltonian on $\mathscr{F}_{\mathrm{rad}}$. The free field Hamiltonian H_{rad} on $\mathscr{F}_{\mathrm{rad}}$ is given in terms of the second quantization

$$H_{\mathrm{rad}} = d\Gamma(\omega \oplus \omega), \tag{7.2.13}$$

formally corresponding to

$$\sum_{j=\pm} \int_{\mathbb{R}^3} \omega(k) a^*(k,j) a(k,j) dk.$$

It leaves the n-particle subspace $\mathscr{F}_{\mathrm{rad}}^{(n)} = L^2_{\mathrm{sym}}((\mathbb{R}^3 \oplus \mathbb{R}^3)^n)$ invariant and acts as

$$\left(H_{\mathrm{rad}} \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \right)(k_1, \ldots, k_n) = \Big(\sum_{j=1}^{n} \omega(k_j) \Big) \begin{pmatrix} f_1(k_1, \ldots, k_n) \\ f_2(k_1, \ldots, k_n) \end{pmatrix}.$$

The Pauli–Fierz model describes the minimal interaction between an electron and the quantized radiation field, in which the electron is assumed to carry low energy. The electron, as in the case of the Nelson model, is described by the Schrödinger operator

$$H_{\mathrm{p}} = -\frac{1}{2}\Delta + V \tag{7.2.14}$$

in $L^2(\mathbb{R}^3)$. The assumptions on V will be specified later on below. The Hamiltonian for the non-interacting quantized radiation field and electron is thus given by

$$H_{\mathrm{PF},0} = -\frac{1}{2}\Delta \otimes 1 + 1 \otimes H_{\mathrm{rad}} \tag{7.2.15}$$

with domain

$$D_{\mathrm{PF}} = D(H_{\mathrm{PF},0}) = D\left(-\frac{1}{2}\Delta \otimes 1\right) \cap D(1 \otimes H_{\mathrm{rad}}). \tag{7.2.16}$$

The interaction is obtained by minimal coupling $-i\nabla_\mu \otimes 1 \mapsto -i\nabla_\mu \otimes 1 - eA_\mu$.

Definition 7.2 (Pauli–Fierz Hamiltonian in Fock space). The *Pauli–Fierz Hamiltonian* in non-relativistic quantum electrodynamics is defined by the operator

$$H_{\mathrm{PF}} = \frac{1}{2}(-i\nabla \otimes 1 - eA)^2 + V \otimes 1 + 1 \otimes H_{\mathrm{rad}} \tag{7.2.17}$$

in $\mathscr{H}_{\mathrm{PF}}$.

We define $H_{\mathrm{PF}}(A)$ and $H_{\mathrm{PF,I}}$ by

$$H_{\mathrm{PF}}(A) = \frac{1}{2}(-i\nabla \otimes 1 - eA)^2, \tag{7.2.18}$$

$$H_{\mathrm{PF,I}} = -e(-i\nabla \otimes 1)\cdot A + \frac{e^2}{2}A^2 + V \otimes 1. \tag{7.2.19}$$

Clearly,

$$H_{\mathrm{PF}} = H_{\mathrm{PF,0}} + H_{\mathrm{PF,I}} \tag{7.2.20}$$

by the commutation relation $\sum_{\mu=1}^{3}[\nabla_\mu, A_\mu] = 0$. In what follows we omit the tensor notation $\otimes$ for easing the notation.

Assumption 7.1. In the remainder of the chapter we use the following assumptions:

$$(1)\ \hat{\varphi}(-k) = \overline{\hat{\varphi}(k)}, \qquad (2)\ \sqrt{\omega}\hat{\varphi} \in L^2(\mathbb{R}^3),\ \ \hat{\varphi}/\omega \in L^2(\mathbb{R}^3). \tag{7.2.21}$$

We also assume that there exist $0 \le a < 1$ and $0 \le b$ such that

$$(3)\ \|Vf\| \le a\| - (1/2)\Delta f\| + b\|f\| \tag{7.2.22}$$

for $f \in D(-(1/2)\Delta)$.

Condition (1) means that $\varphi(x) = \bar{\varphi}(x)$ and hence φ is real and it ensures that H_{PF} is symmetric. By condition (2) $(-i\nabla \otimes 1)\cdot A$ and A^2 are relatively bounded with respect to $H_{\mathrm{PF,0}}$. Finally by (3) and the Kato–Rellich theorem H_{p} is self-adjoint on $D(-(1/2)\Delta)$. As a consequence, $H_{\mathrm{PF,I}}$ is relatively bounded with respect to $H_{\mathrm{PF,0}}$.

As in Section 7.4.1 before we can prove

Theorem 7.2 (Self-adjointness). *Suppose that Assumption* 7.1 *holds. Then H_{PF} is self-adjoint on D_{PF} and essentially self-adjoint on any core of $H_{\mathrm{PF,0}}$.*

In fact, for small values of $|e|$ it is not hard to see self-adjointness of H_{PF} on D_{PF} as we have

$$\|H_{\mathrm{PF,I}}\Psi\| \le (a + a_1|e| + a_2|e|^2)\|H_{\mathrm{PF,0}}\Psi\| + b(e)\|\Psi\|$$

with a given in (7.2.22) and constants a_1, a_2 and $b(e)$ whose existence is made sure by (2) of (7.2.21). Thus for e such that $(a + a_1|e| + a_2|e|^2) < 1$, H_{PF} is self-adjoint on D_{PF} by the Kato–Rellich Theorem. A more challenging problem is to show that H_{PF} is self-adjoint for all $e \in \mathbb{R}$. For the moment we assume that $|e|$ is sufficiently small.

Remarkably, Pauli–Fierz Hamiltonians with different polarization vectors are equivalent with each other. We show this next. Let $e^{\pm}$ and $\eta^{\pm}$ be polarization vectors, and $H_{\mathrm{PF}}(e^{\pm})$ and $H_{\mathrm{PF}}(\eta^{\pm})$ the corresponding Pauli–Fierz Hamiltonians, respectively.

Proposition 7.3. *$H_{\mathrm{PF}}(e^{\pm})$ and $H_{\mathrm{PF}}(\eta^{\pm})$ are unitary equivalent.*

Proof. Since for each $k \in \mathbb{R}^3$ both polarization vectors form orthogonal bases on the plane perpendicular to the vector k, there exists ϑ_k such that

$$\begin{pmatrix} e^+(k) \\ e^-(k) \end{pmatrix} = \begin{pmatrix} \cos\vartheta_k 1_3 & -\sin\vartheta_k 1_3 \\ \sin\vartheta_k 1_3 & \cos\vartheta_k 1_3 \end{pmatrix} \begin{pmatrix} \eta^+(k) \\ \eta^-(k) \end{pmatrix} \quad \text{i.e.} \quad \begin{pmatrix} e^+_\mu(k) \\ e^-_\mu(k) \end{pmatrix} = \mathscr{R}_k \begin{pmatrix} \eta^+_\mu(k) \\ \eta^-_\mu(k) \end{pmatrix},$$

where $\mathscr{R}_k = \left(\begin{smallmatrix} \cos\vartheta_k & -\sin\vartheta_k \\ \sin\vartheta_k & \cos\vartheta_k \end{smallmatrix}\right)$. Define $R : L^2(\mathbb{R}^3; \mathbb{C}^2) \to L^2(\mathbb{R}^3; \mathbb{C}^2)$ by, for almost every k,

$$R\begin{pmatrix} f \\ g \end{pmatrix}(k) = \mathscr{R}_k \begin{pmatrix} f(k) \\ g(k) \end{pmatrix}$$

and $\Gamma(R) : \mathscr{F}_{\mathrm{rad}} \to \mathscr{F}_{\mathrm{rad}}$ by second quantization of R. Then $\Gamma(R)$ is a unitary map on $\mathscr{F}_{\mathrm{rad}}$. Note that

$$R\begin{pmatrix} \eta^+_\mu f \\ \eta^-_\mu f \end{pmatrix} = \begin{pmatrix} e^+_\mu f \\ e^-_\mu f \end{pmatrix}$$

which implies that $\Gamma(R) H_{\mathrm{PF}}(\eta^{\pm}) \Gamma(R)^{-1} = H_{\mathrm{PF}}(e^{\pm})$. □

7.2.2 The Pauli–Fierz model in function space

We introduce a $\mathscr{Q}$-space associated with the quantized radiation field and reformulate the Pauli–Fierz Hamiltonian on $L^2(\mathbb{R}^3) \otimes L^2(\mathscr{Q})$ instead of $\mathscr{H}_{\mathrm{PF}}$. Furthermore, we introduce the Euclidean quantum field associated with the Pauli–Fierz Hamiltonian to derive the functional integral representation of $e^{-tH_{\mathrm{PF}}}$. A slight modification of the setup discussed in Chapter 5 is needed as the scalar field $\phi(f)$ is indexed by $f \in \mathscr{H}_{\mathrm{M}}$ while the quantized radiation field is indexed by $L^2_{\mathrm{real}}(\mathbb{R}^3)$. In particular, the family of isometries $\tau_t : \mathscr{H}_{\mathrm{M}} \to \mathscr{H}_{\mathrm{E}}$ will be modified to

$$\mathrm{j}_t : L^2_{\mathrm{real}}(\mathbb{R}^3) \to L^2_{\mathrm{real}}(\mathbb{R}^4). \tag{7.2.23}$$

For a real-valued $f \in L^2(\mathbb{R}^3)$

$$A_\mu(f) = \frac{1}{\sqrt{2}} \sum_{j=\pm} \int_{\mathbb{R}^3} e^j_\mu(k)(\hat{f}(k)a^*(k,j) + \hat{f}(-k)a(k,j))dk. \tag{7.2.24}$$

With this notation we write $A_\mu(x) = A_\mu(\tilde{\varphi}(\cdot - x))$, where $\tilde{\varphi} = (\hat{\varphi}/\sqrt{\omega})^\vee$. The relations

$$\begin{aligned}
&\text{(mean)} && (\Omega_{\mathrm{PF}}, A_\mu(f)\Omega_{\mathrm{PF}}) = 0,\\
&\text{(covariance)} && (\Omega_{\mathrm{PF}}, A_\mu(f)A_\nu(g)\Omega_{\mathrm{PF}}) = \frac{1}{2}\int_{\mathbb{R}^3} \delta^\perp_{\mu\nu}(k)\overline{\hat{f}(k)}\hat{g}(k)dk
\end{aligned} \tag{7.2.25}$$

are immediate. Here $\delta^\perp_{\mu\nu}(k)$ is the *transversal delta function* given by

$$\delta^\perp_{\mu\nu}(k) = \delta_{\mu\nu} - \frac{k_\mu k_\nu}{|k|^2}. \tag{7.2.26}$$

The identity

$$\delta^\perp_{\mu\nu}(k) = \sum_{j=\pm} e^j_\mu(k)e^j_\nu(k) \tag{7.2.27}$$

follows from the fact that the 3×3 matrix

$$\begin{pmatrix} e^+_1(k) & e^-_1(k) & k_1/|k| \\ e^+_2(k) & e^-_2(k) & k_2/|k| \\ e^+_3(k) & e^-_3(k) & k_3/|k| \end{pmatrix} \tag{7.2.28}$$

is an orthogonal matrix. Note that relation (7.2.27) is independent of the choice of polarization vectors $e^\pm(k)$. We define the 3×3 matrix $\delta^\perp(k)$ by

$$\delta^\perp(k) = (\delta^\perp_{\mu\nu}(k))_{1\le\mu,\nu\le 3}. \tag{7.2.29}$$

In order to have a functional integral representation of $(F, e^{-tH_{\mathrm{PF}}}G)$ we construct probability spaces $(\mathscr{Q}_\beta, \Sigma_\beta, \mu_\beta)$, $\beta = 0, 1$, and Gaussian random variables $\mathscr{A}_\beta(\mathbf{f})$ indexed by $\mathbf{f} \in \oplus^3 L^2_{\mathrm{real}}(\mathbb{R}^{3+\beta})$ of mean zero

$$\int_{\mathscr{Q}_\beta} \mathscr{A}_\beta(\mathbf{f})d\mu_\beta = 0 \tag{7.2.30}$$

and covariance

$$\int_{\mathscr{Q}_\beta} \mathscr{A}_\beta(\mathbf{f})\mathscr{A}_\beta(\mathbf{g})d\mu_\beta = \mathrm{q}_\beta(\mathbf{f}, \mathbf{g}). \tag{7.2.31}$$

Here the bilinear forms q_β on $(\oplus^3 L^2_{\mathrm{real}}(\mathbb{R}^{3+\beta})) \times (\oplus^3 L^2_{\mathrm{real}}(\mathbb{R}^{3+\beta}))$ are defined by

$$\mathrm{q}_0(\mathbf{f}, \mathbf{g}) = \frac{1}{2}\int_{\mathbb{R}^3} \overline{\hat{\mathbf{f}}(k)} \cdot \delta^\perp(k)\hat{\mathbf{g}}(k)dk, \tag{7.2.32}$$

$$\mathrm{q}_1(\mathbf{f}, \mathbf{g}) = \frac{1}{2}\int_{\mathbb{R}^4} \overline{\hat{\mathbf{f}}(\mathbf{k})} \cdot \delta^\perp(k)\hat{\mathbf{g}}(\mathbf{k})d\mathbf{k}, \tag{7.2.33}$$

where in what follows we set $\mathbf{k} = (k, k_0) \in \mathbb{R}^3 \times \mathbb{R}$. Note that $\delta^{\perp}(k)$ in (7.2.33) is independent of $k_0 \in \mathbb{R}$. The definitions of (7.2.32) and (7.2.33) are motivated by (7.2.25). It is, however, not straightforward to construct Gaussian random variables such as (7.2.30) and (7.2.31) since, as is seen in the definition, q_β is degenerate; q_β has non zero null space. In order to construct a $\mathscr{Q}$-space for the Pauli–Fierz Hamiltonian we have to take a quotient space with respect to the null space of q_β.

Proposition 7.4 (Gaussian process and quantized radiation field). *There exist probability spaces* $(\mathscr{Q}_\beta, \Sigma_\beta, \mu_\beta)$, $\beta = 0, 1$, *and a family of Gaussian random variables* $(\mathscr{A}_\beta(\mathbf{f}), \mathbf{f} \in \oplus^3 L_{\mathrm{real}}(\mathbb{R}^3))$ *satisfying* (7.2.30) *and* (7.2.31).

Proof. Let $\mathscr{N}_\beta = \{\mathbf{f} \in \oplus^3 L^2_{\mathrm{real}}(\mathbb{R}^3) \,|\, \mathrm{q}_\beta(\mathbf{f}, \mathbf{f}) = 0\}$. Define the Hilbert space

$$\mathscr{K}_\beta = (\oplus^3 L^2_{\mathrm{real}}(\mathbb{R}^{3+\beta}))/\mathscr{N}_\beta \tag{7.2.34}$$

as a quotient space, and take the canonical map $\pi_\beta : \oplus^3 L^2_{\mathrm{real}}(\mathbb{R}^{3+\beta}) \to \mathscr{K}_\beta$. Then

$$(\pi_\beta(\mathbf{f}), \pi_\beta(\mathbf{g}))_{\mathscr{K}_\beta} = \mathrm{q}_\beta(\mathbf{f}, \mathbf{g}).$$

By Theorem 5.9 there exists a probability space $(\mathscr{Q}_\beta, \Sigma_\beta, \mu_\beta)$ and a family of Gaussian random variables $\{\phi_\beta(\pi_\beta(\mathbf{f})) | \pi_\beta(\mathbf{f}) \in \mathscr{K}_\beta\}$ such that Σ_β is *full* and

$$\int_{\mathscr{Q}_\beta} \phi_\beta(\pi_\beta(\mathbf{f})) d\mu_\beta = 0, \quad \int_{\mathscr{Q}_\beta} \phi_\beta(\pi_\beta(\mathbf{f}))\phi_\beta(\pi_\beta(\mathbf{g})) d\mu_\beta = \mathrm{q}_\beta(\mathbf{f}, \mathbf{g}).$$

Let furthermore

$$\mathscr{A}_\beta(\mathbf{f}) = \phi_\beta(\pi_\beta(\mathbf{f})), \quad \mathbf{f} \in \oplus^3 L^2_{\mathrm{real}}(\mathbb{R}^{3+\beta}). \tag{7.2.35}$$

Then $\mathscr{A}_\beta(\mathbf{f})$ satisfies (7.2.30) and (7.2.31). □

Define the μth component of $\mathscr{A}_\beta$ by

$$\mathscr{A}_{\beta,\mu}(f) = \mathscr{A}_\beta(\oplus_{\nu=1}^3 \delta_{\mu\nu} f), \quad f \in L^2(\mathbb{R}^{3+\beta}). \tag{7.2.36}$$

In what follows we denote

$$\begin{aligned} &\text{(Minkowskian)} \quad \mathscr{A} = \mathscr{A}_0, \quad \mathrm{q} = \mathrm{q}_0, \quad \mathscr{Q} = \mathscr{Q}_0, \\ &\text{(Euclidean)} \quad \mathscr{A}_{\mathrm{E}} = \mathscr{A}_1, \quad \mathrm{q}_{\mathrm{E}} = \mathrm{q}_1, \quad \mathscr{Q}_{\mathrm{E}} = \mathscr{Q}_1 \end{aligned} \tag{7.2.37}$$

using the subscript E for Euclidean objects.

In the same way as for the scalar field we define the second quantization $\Gamma_{\beta\beta'}(T)$ for $T \in \mathscr{C}(L^2(\mathbb{R}^{3+\beta}) \to L^2(\mathbb{R}^{3+\beta'}))$ such that

$$T\mathscr{N}_\beta \subset \mathscr{N}_{\beta'}. \tag{7.2.38}$$

Here and in what follows for the operator $S : L^2(\mathbb{R}^{3+\beta}) \to L^2(\mathbb{R}^{3+\beta'})$ we use the same notation S as for the operator

$$\oplus^3 S : \oplus^3 L^2(\mathbb{R}^{3+\beta}) \to \oplus^3 L^2(\mathbb{R}^{3+\beta'}), \quad (f_1, f_2, f_3) \mapsto (Sf_1, Sf_2, Sf_3)$$

for notational convenience, and write simply $\mathscr{A}_\beta(T\mathbf{f})$ for $\mathscr{A}_\beta((\oplus^3 T)\mathbf{f})$ etc. By (7.2.38) $\pi_\beta(\mathbf{f}) \mapsto \pi_{\beta'}(T\mathbf{f})$ defines a map from $\mathscr{K}_\beta$ to $\mathscr{K}_{\beta'}$. Let thus

$$\Gamma_{\beta\beta'} : \mathscr{C}(L^2(\mathbb{R}^{3+\beta}) \to L^2(\mathbb{R}^{3+\beta'})) \to \mathscr{C}(L^2(\mathscr{Q}_\beta) \to L^2(\mathscr{Q}_{\beta'}))$$

be the functor defined by $\Gamma_{\beta\beta'}(T)1 = 1$ and

$$\Gamma_{\beta\beta'}(T) :\prod_{i=1}^n \mathscr{A}_\beta(\mathbf{f}_i): = :\prod_{i=1}^n \mathscr{A}_{\beta'}(T\mathbf{f}_i):. \tag{7.2.39}$$

We write $\Gamma_{00} = \Gamma$, $\Gamma_{11} = \Gamma_{\mathrm{E}}$ and $\Gamma_{01} = \Gamma_{\mathrm{Int}}$.

Our goal here is to discuss the Euclidean quantum field associated with the Pauli–Fierz model. We introduce a family of isometries connecting the Minkowski field and the Euclidean quantum field, as well as the scalar field discussed in Section 5.1. In contrast with the $\mathscr{Q}$-space representation of the scalar field indexed by vectors in $\mathscr{H}_{\mathrm{M}}$, the $\mathscr{Q}$-space representation $\mathscr{A}(\mathbf{f})$ for the Pauli–Fierz model is indexed by $\mathbf{f} \in \oplus^3 L^2(\mathbb{R}^3)$. So instead of the isometry $\tau_t : \mathscr{H}_{\mathrm{M}} \to \mathscr{H}_{\mathrm{E}}$ with the property $\tau_s^* \tau_t = e^{-|t-s|\omega(-i\nabla)}$, we define the isometry

$$\mathrm{j}_t : L^2_{\mathrm{real}}(\mathbb{R}^3) \to L^2_{\mathrm{real}}(\mathbb{R}^4) \tag{7.2.40}$$

through τ_t by pull-back from the operator $\mathscr{H}_{-1/2}(\mathbb{R}^3) \to \mathscr{H}_{-1}(\mathbb{R}^4)$ to the operator $L^2(\mathbb{R}^3) \to L^2(\mathbb{R}^4)$. Let $i_{-1/2} : L^2(\mathbb{R}^3) \to \mathscr{H}_{-1/2}(\mathbb{R}^3)$ and $i_{-1} : L^2(\mathbb{R}^4) \to \mathscr{H}_{-1}(\mathbb{R}^4)$ be given by

$$\widehat{i_{-1/2} f}(k) = \sqrt{\omega(k)}\, \hat{f}(k), \tag{7.2.41}$$

$$\widehat{i_{-1} f}(\mathbf{k}) = \sqrt{\omega(k)^2 + |k_0|^2}\, \hat{f}(\mathbf{k}). \tag{7.2.42}$$

Both $i_{-1/2}$ and i_{-1} are unitary operators.

Definition 7.3 (Isometry j_t). Define the family of isometries $\mathrm{j}_t : L^2_{\mathrm{real}}(\mathbb{R}^3) \to L^2_{\mathrm{real}}(\mathbb{R}^4)$ by

$$\mathrm{j}_t = (i_{-1})^{-1} \circ \tau_t \circ i_{-1/2}, \quad t \in \mathbb{R}. \tag{7.2.43}$$

By this definition, j_t can be represented as

$$\widehat{\mathrm{j}_t f}(\mathbf{k}) = \frac{e^{-itk_0}}{\sqrt{\pi}} \frac{\sqrt{\omega(k)}}{\sqrt{\omega(k)^2 + |k_0|^2}} \hat{f}(k) \tag{7.2.44}$$

or in the position representation,

$$\mathrm{j}_t f(x, x_0) = \frac{1}{\sqrt{\pi}(2\pi)^2} \int_{\mathbb{R}^4} e^{-it(k_0 - x_0) + ik\cdot x} \frac{\sqrt{\omega(k)}}{\sqrt{\omega(k)^2 + |k_0|^2}} \hat{f}(k) d\mathbf{k}. \quad (7.2.45)$$

Note that for $f \in L^2_{\mathrm{real}}(\mathbb{R}^3)$, $\overline{\mathrm{j}_t f} = \mathrm{j}_t f$, meaning that j_t preserves realness, i.e., $\mathrm{j}_t : L^2_{\mathrm{real}}(\mathbb{R}^3) \to L^2_{\mathrm{real}}(\mathbb{R}^4)$, and $\mathrm{j}_t \mathscr{N}_0 \subset \mathscr{N}_1$. In fact, it follows that $\mathrm{q}_1(\mathrm{j}_t \mathbf{f}, \mathrm{j}_t \mathbf{f}) = \mathrm{q}_0(\mathbf{f}, \mathbf{f})$. By using the definition it can also be seen that

$$\mathrm{j}_s^* \mathrm{j}_t = e^{-|t-s|\hat{\omega}}, \quad s, t \in \mathbb{R}, \quad (7.2.46)$$

where $\hat{\omega} = \omega(-i\nabla)$.

Definition 7.4 (Isometry J_t). Define the family of isometries $\mathrm{J}_t : L^2(\mathscr{Q}) \to L^2(\mathscr{Q}_{\mathrm{E}})$ by the second quantization of j_t, i.e.,

$$\mathrm{J}_t = \Gamma_{\mathrm{Int}}(\mathrm{j}_t). \quad (7.2.47)$$

By (7.2.46),

$$\mathrm{J}_t^* \mathrm{J}_s = \Gamma(e^{-|t-s|\hat{\omega}}) = e^{-|t-s|\tilde{H}_{\mathrm{rad}}}, \quad s, t \in \mathbb{R}, \quad (7.2.48)$$

holds with a self-adjoint operator $\tilde{H}_{\mathrm{rad}}$ by Proposition 3.26. As in the case of the scalar field, J_t plays an important role in the functional integral representation, which connects Minkowskian quantum fields with Euclidean quantum fields.

Note the intertwining property of J_t. Let T be a bounded multiplication operator $(Tf)(k) = T(k)f(k)$ on $L^2(\mathbb{R}^3)$, and $\hat{T}$ the bounded pseudo-differential operator $T(-i\nabla)$. It follows that

$$\mathrm{j}_t \hat{T} f = (\hat{T} \otimes 1)\mathrm{j}_t f, \quad (7.2.49)$$

where $\hat{T} \otimes 1$ is a bounded operator on $L^2(\mathbb{R}^4)$ under the identification $L^2(\mathbb{R}^4) \cong L^2(\mathbb{R}^3) \otimes L^2(\mathbb{R})$. Thus

$$\mathrm{J}_t \Gamma(\hat{T}) = \Gamma_{\mathrm{E}}(\hat{T} \otimes 1)\mathrm{J}_t. \quad (7.2.50)$$

Let h be a real-valued multiplication operator on $L^2(\mathbb{R}^3)$ and define $\hat{h} = h(-i\nabla)$. Hence

$$\mathrm{J}_t \Gamma(e^{-i\hat{h}}) = \Gamma_{\mathrm{E}}(e^{-i\hat{h}\otimes 1})\mathrm{J}_t \quad (7.2.51)$$

follows from (7.2.50). Then

$$\mathrm{J}_t d\Gamma(\hat{h}) = d\Gamma_{\mathrm{E}}(\hat{h} \otimes 1)\mathrm{J}_t \quad (7.2.52)$$

also holds. We summarize these results below.

Proposition 7.5 (Intertwining property). *Let $\hat{h} = h(-i\nabla)$ be the self-adjoint operator with real-valued symbol h. Then $\mathrm{J}_t\Gamma(e^{-i\hat{h}}) = \Gamma_{\mathrm{E}}(e^{-i\hat{h}\otimes 1})\mathrm{J}_t$ and $\mathrm{J}_t d\Gamma(\hat{h}) = d\Gamma_{\mathrm{E}}(\hat{h}\otimes 1)\mathrm{J}_t$ hold.*

The Wiener–Itô–Segal isomorphism can be extended to the isomorphism θ_{PF} between $L^2(\mathscr{Q})$ and $\mathscr{F}_{\mathrm{rad}}$. Let $\theta_{\mathrm{PF}} : \mathscr{F}_{\mathrm{rad}} \to L^2(\mathscr{Q})$ be defined by

$$\theta_{\mathrm{PF}}\Omega_{\mathrm{PF}} = 1,$$

$$\theta_{\mathrm{PF}}{:}\prod_{i=1}^{n} A(\mathbf{f}_i){:}\Omega_{\mathrm{PF}} = {:}\prod_{i=1}^{n} \mathscr{A}(\mathbf{f}_i){:}.$$

By the commutation relations

$$\left({:}\prod_{i=1}^{n} A(\mathbf{f}_i){:}\Omega_{\mathrm{PF}}, {:}\prod_{j=1}^{n} A(\mathbf{g}_j){:}\Omega_{\mathrm{PF}}\right) = \left({:}\prod_{i=1}^{n} \mathscr{A}(\mathbf{f}_i){:}, {:}\prod_{j=1}^{n} \mathscr{A}(\mathbf{g}_j){:}\right).$$

Thus θ_{PF} can be extended to the unitary operator from $\mathscr{F}_{\mathrm{rad}}$ to $L^2(\mathscr{Q})$. We denote $1\otimes\theta_{\mathrm{PF}} : \mathscr{H}_{\mathrm{PF}} \to L^2(\mathbb{R}^3)\otimes L^2(\mathscr{Q})$ by θ_{PF} for simplicity. Note that the inverse Fourier transform of $g(k,x) = e^{-ik\cdot x}\hat{\varphi}(k)/\sqrt{\omega(k)}$ equals $\check{g}(y,x) = \tilde{\varphi}(y-x)$, where

$$\tilde{\varphi} = (\hat{\varphi}/\sqrt{\omega})^{\vee} \in L^2_{\mathrm{real}}(\mathbb{R}^3). \tag{7.2.53}$$

Recall that $A_\mu(x) = A_\mu(\tilde{\varphi}(\cdot - x))$ for each x by the definition of $A_\mu(f)$. Notice that the test function of $A_\mu(f)$ is $\hat{f}$ instead of f. Hence the relations

$$\theta_{\mathrm{PF}} A_\mu(x)\theta_{\mathrm{PF}}^{-1} = \mathscr{A}_\mu(\tilde{\varphi}(\cdot - x)),$$

$$\theta_{\mathrm{PF}} H_{\mathrm{rad}}\theta_{\mathrm{PF}}^{-1} = \tilde{H}_{\mathrm{rad}},$$

follow directly. As a result

$$\theta_{\mathrm{PF}} H_{\mathrm{PF}}\theta_{\mathrm{PF}}^{-1} = \frac{1}{2}(-i\nabla - e\mathscr{A})^2 + V + \tilde{H}_{\mathrm{rad}}, \tag{7.2.54}$$

with $\mathscr{A} = (\mathscr{A}_1, \mathscr{A}_2, \mathscr{A}_3)$ and

$$\mathscr{A}_\mu = \int_{\mathbb{R}^3}^{\oplus} \mathscr{A}_\mu(\tilde{\varphi}(\cdot - x))dx, \quad \mu = 1,2,3. \tag{7.2.55}$$

In what follows we use the same notation H_{PF} for $\theta_{\mathrm{PF}} H_{\mathrm{PF}}\theta_{\mathrm{PF}}^{-1}$ and H_{rad} for $\tilde{H}_{\mathrm{rad}}$.

Definition 7.5 (Pauli–Fierz Hamiltonian in function space). The *Pauli–Fierz Hamiltonian in function space* is defined by

$$H_{\mathrm{PF}} = \frac{1}{2}(-i\nabla - e\mathscr{A})^2 + V + H_{\mathrm{rad}} \tag{7.2.56}$$

in $L^2(\mathbb{R}^3)\otimes L^2(\mathscr{Q})$.

We use the shorthand $\mathscr{H}_{\rm PF}$ for $L^2(\mathbb{R}^3) \otimes L^2(\mathscr{Q})$ and $\mathscr{H}_{\rm E} = L^2(\mathbb{R}^3) \otimes L^2(\mathscr{Q}_{\rm E})$ standing for the Euclidean space.

Remark 7.1. Note that in the $\mathscr{Q}$-space representation the test function f of $\mathscr{A}_\mu(f)$ is in the position representation but the test function f of $A_\mu(f)$ is in the momentum representation. For instance, the dispersion relation in Fock representation is $\omega(k) = |k|$, while it becomes $\omega(-i\nabla) = |-i\nabla|$ in $\mathscr{Q}$-space representation.

7.2.3 Markov property

In the scalar quantum field theory discussed in Chapter 5 the family of projections $\mathrm{E}_t = \mathrm{I}_t\mathrm{I}_t^*$, $t \in \mathbb{R}$, was defined and it was seen that the Markov property plays an important role in the the functional integral representation. Here we address a similar problem for the Pauli–Fierz model.

In the Pauli–Fierz model we will keep to the same notations e_t, E_t, etc as the similar objects for the scalar field. Let

$$e_t = \mathrm{j}_t\mathrm{j}_t^*, \quad t \in \mathbb{R}. \tag{7.2.57}$$

Then $\{e_t\}_{t\in\mathbb{R}}$ is a family of projections from $L^2_{\rm real}(\mathbb{R}^4)$ to $\operatorname{Ran}\mathrm{j}_t$. The latter is a closed subspace of $L^2_{\rm real}(\mathbb{R}^4)$. Define

$$U_{[a,b]} = \overline{\text{L.H.}\left\{f \in L^2_{\rm real}(\mathbb{R}^4) \mid f \in \operatorname{Ran}\mathrm{j}_t \text{ for some } t \in [a,b]\right\}}.$$

Let $e_{[a,b]} : L^2_{\rm real}(\mathbb{R}^4) \to U_{[a,b]}$ denote the orthogonal projection. In the same way as for the scalar quantum field theory, it follows that for $a \le b \le t \le c \le d$,

$$(1)\ e_a e_b e_c = e_a e_c, \quad (2)\ e_{[a,b]} e_t e_{[c,d]} = e_{[a,b]} e_{[c,d]}. \tag{7.2.58}$$

The projections on $L^2(\mathscr{Q}_{\rm E})$ similarly are

$$\mathrm{E}_t = \mathrm{J}_t\mathrm{J}_t^* = \Gamma_{\rm E}(e_t), \quad \mathrm{E}_{[a,b]} = \Gamma_{\rm E}(e_{[a,b]}). \tag{7.2.59}$$

Let $\Sigma_{[a,b]}$ be the minimal σ-field generated by

$$\{\mathscr{A}_{\mathrm{E}\mu}(f) \in L^2(\mathscr{Q}_{\rm E}) \mid f \in U_{[a,b]}, \mu = 1,2,3\}.$$

The set of $\Sigma_{[a,b]}$-measurable functions in $L^2(\mathscr{Q}_{\rm E})$ will be denoted by $\mathscr{E}_{[a,b]}$. The projection $\mathrm{E}_{[a,b]}$ and the set of $\Sigma_{[a,b]}$-measurable functions $\mathscr{E}_{[a,b]}$ satisfy

$$(1)\ \operatorname{Ran}\mathrm{E}_{[a,b]} = \mathscr{E}_{[a,b]}, \quad (2)\ \mathrm{E}_{[a,b]}\mathrm{E}_t\mathrm{E}_{[c,d]} = \mathrm{E}_{[a,b]}\mathrm{E}_{[c,d]}$$

for $a \le b \le t \le c \le d$. (2) above is the *Markov property* for the present model.

Next we turn to discussing the family of projections E_t, $t \in \mathbb{R}$. Let

$$\|F\|_p = \left(\int_{\mathscr{Q}_\beta} |F(\phi)|^p d\mu_\beta \right)^{1/p}$$

be L^p-norm on $(\mathscr{Q}_\beta, \mu_\beta)$ and $(\cdot,\cdot)_2$ the $L^2(\mathscr{Q}_\beta)$-scalar product. We drop the subscript β unless confusion may arise. As it was seen, $\Gamma_\beta(T)$ for $\|T\| \le 1$ is a contraction on $L^2(\mathscr{Q}_\beta)$. It also has the *hypercontractivity* property. The following sharper result is available.

Proposition 7.6 (L^p–L^q boundedness). *Let $T : L^2(\mathbb{R}^{3+\beta}) \to L^2(\mathbb{R}^{3+\beta})$ be a contraction such that*

$$\| \oplus^3 T\|^2 \le (p-1)(q-1)^{-1} \le 1$$

for some $1 \le p \le q$. Then $\Gamma_\beta(T)$ is a contraction from $L^p(\mathscr{Q}_\beta)$ to $L^q(\mathscr{Q}_\beta)$, i.e., for $\Phi \in L^p(\mathscr{Q}_\beta)$, $\Gamma_\beta(T)\Phi \in L^q(\mathscr{Q}_\beta)$ and

$$\|\Gamma_\beta(T)\Phi\|_q \le \|\Phi\|_p.$$

Note that $\Gamma(e^{-t\tilde{\omega}_\nu})$, $\nu > 0$, is a contraction from L^p to L^q if t is sufficiently large but it fails to be so for $\nu = 0$.

Lemma 7.7. *Suppose that $\nu > 0$ and let $a \le b < t < c \le d$, $F \in \mathscr{E}_{[a,b]}$ and $G \in \mathscr{E}_{[c,d]}$. Take $1 \le r < \infty$, $r < p$ and $r < q$. Suppose moreover that $e^{-2\nu(c-b)} \le (p/r-1)(q/r-1) \le 1$ and $F \in L^p(\mathscr{Q}_\mathrm{E})$ and $G \in L^q(\mathscr{Q}_\mathrm{E})$. Then $FG \in L^r(\mathscr{Q}_\mathrm{E})$ and $\|FG\|_r \le \|F\|_p\|G\|_q$. In particular, for r such that*

$$r \in \left[1, \frac{2}{1+e^{-\nu(c-b)}}\right] \cup \left[\frac{2}{1-e^{-\nu(c-b)}}, \infty\right),$$

we have $\|FG\|_r \le \|F\|_2\|G\|_2$.

Proof. Let

$$F_N = \begin{cases} F, & |F| < N \\ 0, & |F| \ge N \end{cases} \quad \text{and} \quad G_N = \begin{cases} G, & |G| < N \\ 0, & |G| \ge N. \end{cases}$$

Then $|F_N|^r \in \mathscr{E}_{[a,b]}$, $|G_N|^r \in \mathscr{E}_{[c,d]}$ and

$$\begin{aligned} \int_{\mathscr{Q}_\mathrm{E}} |F_N|^r |G_N|^r d\mu_\mathrm{E} &= (E_{[a,b]}|F_N|^r, E_{[c,d]}|G_N|^r)_2 \\ &= (|F_N|^r, \Gamma_\mathrm{E}(e_{[a,b]}e_{[c,d]})|G_N|^r)_2. \end{aligned}$$

Note that $e_{[a,b]}e_{[c,d]}$ satisfies

$$\begin{aligned}\|e_{[a,b]}e_{[c,d]}\|^2 &= \|e_{[a,b]}e_b e_c e_{[c,d]}\|^2 \le \|\mathrm{j}_b^*\mathrm{j}_c\|^2 \\ &= \|e^{-|c-b|\omega_\nu}\|^2 \le e^{-2\nu(c-b)} \le (p/r-1)(q/r-1).\end{aligned}$$

Thus by the Hölder inequality,

$$\|F_N G_N\|_r^r \le \||F_N|^r\|_{q/r}\|\Gamma_{\mathrm{E}}(e_{[a,b]}e_{[c,d]})|G_N|^r\|_s, \tag{7.2.60}$$

where $1 = 1/s + r/q$. Since

$$\|e_{[a,b]}e_{[c,d]}\|^2 \le (p/r-1)(q/r-1) = (p/r-1)(s-1)^{-1} \le 1,$$

by Proposition 7.6 it is seen that $\|\Gamma_{\mathrm{E}}(e_{[a,b]}e_{[c,d]})|G_N|^r\|_s \le \||G_N|^r\|_{p/r}$. Together with (7.2.60) this yields

$$\|F_N G_N\|_r \le \|F_N\|_q\|G_N\|_p \le \|F\|_q\|G\|_p. \tag{7.2.61}$$

Taking the limit $N \to \infty$ on both sides of (7.2.61), by monotone convergence the lemma follows. □

An immediate consequence is

Corollary 7.8. *Let $\nu = 0$, $\Phi \in L^1(\mathscr{Q}_{\mathrm{E}})$ and $F, G \in L^2(\mathscr{Q})$. Then $(\mathrm{J}_a F)(\mathrm{J}_b G)\Phi \in L^1(\mathscr{Q}_{\mathrm{E}})$ and*

$$\int_{\mathscr{Q}_{\mathrm{E}}} |(\mathrm{J}_a F)(\mathrm{J}_b G)\Phi| d\mu_{\mathrm{E}} \le \|\Phi\|_1\|F\|_2\|G\|_2 \tag{7.2.62}$$

for $a \ne b$. If $a \ne b$, then $\mathrm{J}_a^*\Phi\mathrm{J}_b$ is a bounded operator and $\|\mathrm{J}_a^*\Phi\mathrm{J}_b\| \le \|\Phi\|_1$.

Proof. First assume $\nu > 0$. Let $a < b$, and $r = \frac{2}{1-e^{-\nu(b-a)}}$ and $s > 1$ be such that $1/r + 1/s = 1$, i.e., $s = s(\nu) = r/(r-1)$. Note $s(\nu) \downarrow 1$ as $\nu \to 0$. Without loss of generality we can assume that Φ is real-valued. Truncate Φ by

$$\Phi_N = \begin{cases} N, & \Phi > N, \\ \Phi, & |\Phi| \le N, \\ -N, & \Phi < -N. \end{cases}$$

Then by Lemma 7.7

$$\begin{aligned}|(\mathrm{J}_a F, \Phi_N \mathrm{J}_b G)_2| &\le \|\Phi_N\|_s\|(\mathrm{J}_a F)(\mathrm{J}_b G)\|_r \\ &\le \|\Phi_N\|_s\|\mathrm{J}_a F\|_2\|\mathrm{J}_b G\|_2 = \|\Phi_N\|_s\|F\|_2\|G\|_2.\end{aligned}$$

Now we take the limit $\nu \to 0$ on both sides above. Rewrite J_t and j_t as $\mathrm{J}_t^{(\nu)}$ and $\mathrm{j}_t^{(\nu)}$, respectively. Then

$$(|\mathrm{J}_a^{(\nu)} F|, |\Phi_N||\mathrm{J}_b^{(\nu)} G|) \le \|\Phi_N\|_{s(\nu)}\|F\|_2\|G\|_2.$$

Notice that $\text{s-lim}_{\nu\to 0} \mathrm{J}_t^{(\nu)} = \mathrm{J}_t$ in $L^2(\mathscr{Q}_\mathrm{E})$ by $\text{s-lim}_{\nu\to 0} \mathrm{j}_t^{(\nu)} = \mathrm{j}_t$ in $L^2(\mathbb{R}^4)$ and $\lim_{\nu\to 0} \|\Phi_N\|_{s(\nu)} = \|\Phi_N\|_1$ by dominated convergence, since $|\Phi_N|_{s(\nu)} \le N^2$ for sufficiently small ν. Then we have

$$(|\mathrm{J}_a F|, |\Phi_N||\mathrm{J}_b G|)_2 \le \|\Phi_N\|_1 \|F\|_2 \|G\|_2 \le \|\Phi\|_1 \|F\|_2 \|G\|_2. \tag{7.2.63}$$

Since $|\Phi_N| \uparrow |\Phi|$ as $N \to \infty$, by monotone convergence $|\mathrm{J}_a F||\Phi||\mathrm{J}_b G| \in L^1(\mathscr{Q}_\mathrm{E})$ and (7.2.62) follows. □

7.3 Functional integral representation for the Pauli–Fierz Hamiltonian

7.3.1 Hilbert space-valued stochastic integrals

In this section we define a Hilbert space-valued stochastic integral. It will be first explained in some generality and then applied to the Pauli–Fierz model. Using the Trotter product formula and the factorization formula $e^{-|t-s|H_\mathrm{rad}} = \mathrm{J}_t^* \mathrm{J}_s$, we derive the functional integral representation of e^{-tH_PF}.

Let $\mathscr{K}$ be a Hilbert space and define

$$C^n(\mathbb{R}^3;\mathscr{K}) = \left\{ f : \mathbb{R}^3 \to \mathscr{K} \,|\, f \text{ is } n \text{ times continuously strongly differentiable} \right\}$$

and

$$C_\mathrm{b}^n(\mathbb{R}^3;\mathscr{K}) = \left\{ f \in C^n(\mathbb{R}^3;\mathscr{K}) \Big| \sup_{|z|\le n, x\in\mathbb{R}^3} \|\partial^z f(x)\|_{\mathscr{K}} < \infty \right\},$$

where $|z| = z_1 + z_2 + z_3$ for $z = (z_1, z_2, z_3)$ and $\partial^z = \partial_{x_1}^{z_1} \partial_{x_2}^{z_2} \partial_{x_3}^{z_3}$ denotes strong derivative.

The proof of the following lemma is straightforward and similar to the case of real-valued processes.

Lemma 7.9. *Let $f \in C_\mathrm{b}^1(\mathbb{R}\times\mathbb{R}^3;\mathscr{K})$. The sequence defined by*

$$\mathrm{J}_n^\mu(f) = \sum_{j=1}^{2^n} f\left(\frac{j-1}{2^n}t, B_{\frac{j-1}{2^n}t}\right)\left(B^\mu_{\frac{j}{2^n}t} - B^\mu_{\frac{j-1}{2^n}t}\right)$$

is a Cauchy sequence in $L^2(\mathscr{X}, d\mathcal{W}^x) \otimes \mathscr{K}$.

Definition 7.6 (Hilbert space-valued stochastic integral). For $f \in C_\mathrm{b}^1(\mathbb{R}\times\mathbb{R}^3;\mathscr{K})$ the limit

$$\int_0^t f(s, B_s) dB_s^\mu = \underset{n\to\infty}{\text{s-lim}}\, \mathrm{J}_n^\mu(f). \tag{7.3.1}$$

defines a *$\mathscr{K}$-valued stochastic integral.*

By the above definition

$$\mathbb{E}\left[\left(\int_0^t f(s, B_s) dB_s^\mu, \int_0^t g(s, B_s) dB_s^\nu\right)_{\mathscr{K}}\right] = \delta_{\mu\nu}\mathbb{E}\left[\int_0^t (f(s, B_s), g(s, B_s))_{\mathscr{K}} ds\right].$$

In particular, the Itô isometry

$$\mathbb{E}\left[\left\|\int_0^t f(s, B_s) dB_s^\mu\right\|_{\mathscr{K}}^2\right] = \mathbb{E}\left[\int_0^t \|f(s, B_s)\|_{\mathscr{K}}^2 ds\right]$$

holds. Similarly to the case of real-valued processes the result below follows.

Corollary 7.10. *Let $f \in C_{\mathrm{b}}^2(\mathbb{R}^3; \mathscr{K})$ and*

$$S_n^\mu(f) = \sum_{j=1}^{2^n} (f(B_{\frac{j}{2^n}t}) + f(B_{\frac{j-1}{2^n}t}))(B^\mu_{\frac{j}{2^n}t} - B^\mu_{\frac{j-1}{2^n}t}).$$

Then

$$\operatorname*{s-lim}_{n\to\infty} S_n^\mu(f) = \int_0^t f(B_s) dB_s^\mu + \frac{1}{2}\int_0^t \partial_\mu f(B_s) ds$$

holds in $L^2(\mathscr{X}, d\mathcal{W}^x) \otimes \mathscr{K}$.

Example 7.1. We will make the following specific choices for the functional integral representation of the Pauli–Fierz model below. Let $\hat{\lambda}, \omega\hat{\lambda} \in L^2(\mathbb{R}^3)$ and define the map

$$\xi : \mathbb{R}^3 \ni x \mapsto \lambda(\cdot - x) \in L^2(\mathbb{R}^3).$$

Thus $\xi \in C_{\mathrm{b}}^1(\mathbb{R}^3; L^2(\mathbb{R}^3))$ by the assumption $\omega\hat{\lambda} \in L^2(\mathbb{R}^3)$, and we can define a $L^2(\mathbb{R}^3)$-valued stochastic integral

$$\int_0^t \xi(B_s) dB_s^\mu = \int_0^t \lambda(\cdot - B_s) dB_s^\mu.$$

Later on we will construct the functional integral through the Euclidean quantum field, and will use an $L^2(\mathbb{R}^4)$-valued stochastic integral of the form

$$\int_0^t \mathrm{j}_s\tilde{\varphi}(\cdot - B_s) dB_s^\mu. \tag{7.3.2}$$

However, since

$$\frac{\|\mathrm{j}_t f - \mathrm{j}_s f\|^2}{|t-s|^2} = 2\left(\hat{f}, \frac{(1 - e^{-|t-s|\omega})}{|t-s|}\hat{f}\right)\frac{1}{|t-s|}$$

diverges as $t \to s$ for $\hat{f} \in D(\omega)$, $\mathbb{R} \times \mathbb{R}^3 \ni (s, x) \mapsto \mathrm{j}_s\tilde{\varphi}(\cdot - x) \in L^2(\mathbb{R}^4)$ is not strongly differentiable in $s \in \mathbb{R}$. Then $\mathrm{j}_s\tilde{\varphi}(\cdot - x) \notin C_{\mathrm{b}}^1(\mathbb{R} \times \mathbb{R}^3; L^2(\mathbb{R}^4))$. Therefore we need to give a proper definition of (7.3.2).

Lemma 7.11. *If $\hat{\lambda}, \omega\hat{\lambda} \in L^2(\mathbb{R}^3)$, then*

$$S_n^\mu(\lambda) = \sum_{j=1}^{2^n} \mathrm{j}_{(j-1)t/2^n}\lambda(\cdot - B_{(j-1)t/2^n})(B^\mu_{jt/2^n} - B^\mu_{(j-1)t/2^n}), \quad n \in \mathbb{N}, \tag{7.3.3}$$

is a Cauchy sequence in $L^2(\mathscr{X}, d\mathcal{W}^x) \otimes L^2(\mathbb{R}^4)$.

Proof. Write $S_n = S_n^\mu(\lambda)$ and $\eta_* = \mathrm{j}_*\lambda(\cdot - B_*)$. Then $S_{n+1} - S_n = \sum_{m=1}^{2^n} a_m$, where

$$a_m = (\eta_{(2m-1)t/2^{n+1}} - \eta_{(2m-2)t/2^{n+1}})(B^\mu_{2mt/2^{n+1}} - B^\mu_{(2m-1)t/2^{n+1}}).$$

Thus $\mathbb{E}^x[(a_i, a_j)_{L^2(\mathbb{R}^4)}] = 0$ for $i \neq j$, since $B^\mu_{2jt/2^{n+1}} - B^\mu_{(2j-1)t/2^{n+1}}$ is independent of the rest of a_i and a_j for $j > i$, and $\mathbb{E}^x[B^\mu_{2jt/2^{n+1}} - B^\mu_{(2j-1)t/2^{n+1}}] = 0$. Hence

$$\mathbb{E}^x[\|S_{n+1} - S_n\|^2] = \sum_{m=1}^{2^n} \mathbb{E}^x[\|\eta_{(2m-1)t/2^{n+1}} - \eta_{(2m-2)t/2^{n+1}}\|^2]\frac{t}{2^{n+1}}.$$

Since $\|\mathrm{j}_t f - \mathrm{j}_s g\|^2 = \|f - g\|^2 + 2(\hat{f}, (1 - e^{-|t-s|\omega})\hat{g})$ and $\|\lambda(\cdot - X) - \lambda(\cdot - Y)\| \leq |X - Y|\|\omega\hat{\lambda}\|$, we have

$$\begin{aligned}&\|\eta_{(2m-1)t/2^{n+1}} - \eta_{(2m-2)t/2^{n+1}}\|^2 \\ &\leq \|\omega\hat{\lambda}\|^2 |B^\mu_{(2m-1)t/2^{n+1}} - B^\mu_{(2m-2)t/2^{n+1}}|^2 + 2\frac{t}{2^{n+1}}\|\lambda\|\|\omega\hat{\lambda}\|.\end{aligned}$$

Finally we conclude that

$$\left(\mathbb{E}^x[\|S_m - S_n\|^2]\right)^{1/2} \leq \left(\frac{2\|\lambda\|\|\omega\hat{\lambda}\| + \|\omega\hat{\lambda}\|^2}{2}\right)^{1/2} \sum_{j=n+1}^{m} \frac{t}{2^{(j+1)/2}},$$

thus $(S_n)_{n\in\mathbb{N}}$ is a Cauchy sequence. □

Definition 7.7 ($L^2(\mathbb{R}^4)$-valued stochastic integral). Let $\hat{\lambda}, \omega\hat{\lambda} \in L^2(\mathbb{R}^3)$. Then

$$\int_0^t \mathrm{j}_s\lambda(\cdot - B_s)dB_s^\mu = \operatorname*{s-lim}_{n\to\infty} S_n^\mu, \quad \mu = 1, 2, 3, \tag{7.3.4}$$

defines an $L^2(\mathbb{R}^4)$-valued stochastic integral, where the strong limit is in the topology of $L^2(\mathscr{X}, d\mathcal{W}^x) \otimes L^2(\mathbb{R}^4)$.

By the definition it is seen that

$$\mathbb{E}^x\left[\left(\int_0^t \mathrm{j}_s\lambda(\cdot - B_s)dB_s^\mu, \int_0^t \mathrm{j}_s\rho(\cdot - B_s)dB_s^\nu\right)\right]$$
$$= \delta_{\mu\nu}\mathbb{E}^x\left[\int_0^t (\lambda(\cdot - B_s), \rho(\cdot - B_s))ds\right]. \tag{7.3.5}$$

The integral $\int_0^t \mathrm{j}_s\lambda(\cdot - B_s)dB_s^\mu$ can also be defined as a limit of a sequence of $L^2(\mathbb{R}^4)$-valued stochastic integrals, which will be used in the construction of the functional integral representation for the Pauli–Fierz Hamiltonian.

Corollary 7.12. *Suppose that* $\hat{\lambda}, \omega\hat{\lambda} \in L^2(\mathbb{R}^3)$ *and let*

$$\tilde{S}_n^\mu = \sum_{k=1}^{2^n} \int_{(k-1)t/2^n}^{kt/2^n} \mathrm{j}_{(k-1)t/2^n}\lambda(\cdot - B_s)dB_s^\mu. \tag{7.3.6}$$

Then

$$\operatorname*{s-lim}_{n\to\infty} \tilde{S}_n^\mu = \int_0^t \mathrm{j}_s\lambda(\cdot - B_s)dB_s^\mu \tag{7.3.7}$$

in $L^2(\mathscr{X}, d\mathcal{W}^x) \otimes L^2(\mathbb{R}^4)$.

Proof. Using that j_s is an isometry we obtain

$$\mathbb{E}^x\left[\left\|\tilde{S}_n^\mu - \int_0^t \mathrm{j}_s\lambda(\cdot - B_s)dB_s^\mu\right\|^2\right]$$
$$= \sum_{k=1}^{2^n} \int_{(k-1)t/2^n}^{kt/2^n} \mathbb{E}^x[\|\lambda(\cdot - B_s) - \lambda(\cdot - B_{(k-1)t/2^n})\|^2]ds \le C\frac{t}{2^n}. \qquad \square$$

7.3.2 Functional integral representation

The functional integral representation for the non-interacting Hamiltonian comes about surprisingly directly. A combination of the Feynman–Kac formula for e^{-tH_p} and the equality $e^{-tH_\mathrm{rad}} = \mathrm{J}_0^*\mathrm{J}_t$ gives

$$(F, e^{-tH_{\mathrm{PF},0}}G)_{\mathscr{H}_\mathrm{PF}} = (F, e^{-tH_\mathrm{p}}e^{-tH_\mathrm{rad}}G)_{\mathscr{H}_\mathrm{PF}} = (\mathrm{J}_0F, e^{-tH_\mathrm{p}}\mathrm{J}_tG)_{\mathscr{H}_\mathrm{E}}.$$

Proposition 7.13 (Functional integral representation for non-interacting Pauli–Fierz Hamiltonian). *Let* $F, G \in \mathscr{H}_\mathrm{PF}$. *Then*

$$(F, e^{-tH_{\mathrm{PF},0}}G)_{\mathscr{H}_\mathrm{PF}} = \int_{\mathbb{R}^3} dx\mathbb{E}^x[e^{-\int_0^t V(B_s)ds}(\mathrm{J}_0F(B_0), \mathrm{J}_tG(B_t))_{L^2(\mathscr{Q}_\mathrm{E})}]. \tag{7.3.8}$$

Our next goal is to derive a functional integral representation for $e^{-tH_{\mathrm{PF}}}$. Since

$$H_{\mathrm{PF}}(\mathscr{A}) = \frac{1}{2}(-i\nabla - e\mathscr{A})^2 \tag{7.3.9}$$

has a similar form as $H_{\mathrm{p}}(a) = \frac{1}{2}(-i\nabla - a)^2$ in the Schrödinger Hamiltonian with a vector potential a, the procedure of obtaining this functional integral will narrow down to a combination of the construction for $e^{-tH_{\mathrm{p}}(a)}$ in Section 3.5.2. We have seen that

$$(f, e^{-tH_{\mathrm{p}}(a)}g) = \int_{\mathbb{R}^3} dx \mathbb{E}^x[\overline{f(B_0)}e^{-i\int_0^t a(B_s)\circ dB_s} g(B_t)]. \tag{7.3.10}$$

On a first view, it can be expected that the functional integral representation for $(F, e^{-tH_{\mathrm{PF}}(\mathscr{A})}G)$ goes similarly with a replaced by $\mathscr{A}$ which is $L^2(\mathscr{Q})$-valued multiplication operator. There is, however, a difference. Although in the classical case $\int_0^t a(B_s)\circ dB_s$ in (7.3.10) does not depend on time s explicitly, in the Pauli–Fierz model there is dependence on s and the integral $\int_0^t \mathrm{j}_s\tilde{\varphi}(\cdot - B_s)dB_s^\mu$ appears instead of the Stratonovich integral. This time dependence originates from the semigroup of the free field Hamiltonian $e^{-tH_{\mathrm{rad}}}$. An extra difficulty is that $\mathrm{j}_s\hat{\varphi}(\cdot - x)$ is not differentiable in s.

Here is an outline of our construction of the functional integral representation of $(F, e^{-tH_{\mathrm{PF}}}G)$. Replicating the second proof of Theorem 3.65 we notice that

$$e^{-tH_{\mathrm{PF}}(\mathscr{A})} = \operatorname*{s-lim}_{n\to\infty}(\mathscr{P}_{t/n})^n$$

with a family of suitable integral operators $\mathscr{P}_s : L^2(\mathbb{R}^3; L^2(\mathscr{Q})) \to L^2(\mathbb{R}^3; L^2(\mathscr{Q}))$, $s \geq 0$. Thus in virtue of $\mathrm{J}_s^*\mathrm{J}_t = e^{-|t-s|H_{\mathrm{rad}}}$ we have

$$e^{-tH_{\mathrm{PF}}} = \operatorname*{s-lim}_{n\to\infty} \lim_{m_1\to\infty} \cdots \lim_{m_n\to\infty} \mathrm{J}_0^*\Big(\prod_{j=1}^n \mathrm{J}_{tj/n}(\mathscr{P}_{t/nm_j})^{m_j}\mathrm{J}_{tj/n}^*\Big)\mathrm{J}_t, \tag{7.3.11}$$

and by using the Markov property of $\mathrm{E}_s = \mathrm{J}_s\mathrm{J}_s^*$ we deduce

$$\begin{aligned} e^{-tH_{\mathrm{PF}}} &= \operatorname*{s-lim}_{n\to\infty} \lim_{m_1\to\infty} \cdots \lim_{m_n\to\infty} \mathrm{J}_0^*\Big(\prod_{j=1}^n \mathrm{E}_{tj/n}(\mathscr{P}_{t/nm_j,j})^{m_j}\mathrm{E}_{tj/n}\Big)\mathrm{J}_t \\ &= \operatorname*{s-lim}_{n\to\infty} \lim_{m_1\to\infty} \cdots \lim_{m_n\to\infty} \mathrm{J}_0^*\Big(\prod_{j=1}^n (\mathscr{P}_{t/nm_j,j})^{m_j}\Big)\mathrm{J}_t, \end{aligned}$$

where $\mathscr{P}_{S,T}$ denotes an operator acting on the Euclidean space $\mathscr{H}_{\mathrm{E}}$. Finally, we compute $\prod_{j=1}^n(\mathscr{P}_{t/nm_j,j})^{m_j}$ which gives the functional integral formula of $(F, e^{-tH_{\mathrm{PF}}}G)$. In what follows we will turn this argument rigorous. We first show the functional integral representation for the Pauli–Fierz Hamiltonian with sufficiently smooth and bounded V. After establishing it we extend it to more general potentials V.

Theorem 7.14 (Functional integral representation for Pauli–Fierz Hamiltonian). *Let* $V \in C_0^\infty(\mathbb{R}^3)$. *Then*

$$(F, e^{-tH_{\mathrm{PF}}} G) = \int_{\mathbb{R}^3} dx \mathbb{E}^x [e^{-\int_0^t V(B_s)ds} (\mathrm{J}_0 F(B_0), e^{-ie\mathscr{A}_{\mathrm{E}}(K_t)} \mathrm{J}_t G(B_t))_{L^2(\mathscr{Q}_{\mathrm{E}})}]. \tag{7.3.12}$$

Here K_t *denotes the* $\bigoplus^3 L^2(\mathbb{R}^4)$*-valued stochastic integral given by*

$$K_t = \bigoplus_{\mu=1}^{3} \int_0^t \mathrm{j}_s \tilde{\varphi}(\cdot - B_s) dB_s^\mu. \tag{7.3.13}$$

Proof. For the main part of the proof we may and do assume $V = 0$, and in this proof we write $\mathscr{A}_{\mathrm{E}s}(x) = \mathscr{A}_{\mathrm{E}}(\oplus^3 \mathrm{j}_s \tilde{\varphi}(\cdot - x))$. Define the family of symmetric contraction operators $\mathscr{P}_s : \mathscr{H}_{\mathrm{PF}} \to \mathscr{H}_{\mathrm{PF}}$ by

$$\mathscr{P}_s F(x) = \int_{\mathbb{R}^3} \Pi_s(x-y) e^{i\mathfrak{h}(x,y)} F(y) dy, \quad s > 0, \tag{7.3.14}$$

with $\mathscr{P}_0 F = F$, where

$$\mathfrak{h}(x,y) = -\frac{1}{2}((\mathscr{A}(x) + \mathscr{A}(y)) \cdot (x-y).$$

Here $\mathscr{P}_s$ is the quantum field version of ϱ_s defined by (3.5.7) for a Schrödinger operator with a vector potential a. By a direct computation

$$\begin{aligned}
&(F, (\mathscr{P}_{t/2^n})^{2^n} G) \\
&= \int_{\mathbb{R}^3} dx \int_{(\mathbb{R}^3)^{2^n}} \overline{F(x)} \exp\Big(i \sum_{j=1}^{2^n} \mathfrak{h}_j\Big) G(x_{2^n}) \Big(\prod_{j=1}^{2^n} \Pi_{t/2^n}(x_{j-1} - x_j)\Big) \prod_{j=1}^{2^n} dx_j
\end{aligned}$$

with $x = x_0$, $\mathfrak{h}_j = \mathfrak{h}_j(x_j, x_{j-1}) = -(1/2)(\mathscr{A}(x_j) + \mathscr{A}(x_{j-1})) \cdot (x_j - x_{j-1})$. This can be expressed by using Brownian motion as

$$(F, (\mathscr{P}_{t/2^n})^{2^n} G) = \int_{\mathbb{R}^3} dx \mathbb{E}^x [(F(B_0), e^{-ie\mathscr{A}(L_n)} G(B_t))],$$

where

$$L_n = \bigoplus_{\mu=1}^{3} \sum_{m=0}^{2^n} (\tilde{\varphi}(\cdot - B_{t\frac{m}{2^n}}) + \tilde{\varphi}(\cdot - B_{t\frac{m-1}{2^n}}))(B^\mu_{t\frac{m}{2^n}} - B^\mu_{t\frac{m-1}{2^n}}).$$

It is seen that

$$L_n \to L_t = \bigoplus_{\mu=1}^{3} \int_0^t \tilde{\varphi}(\cdot - B_s) dB_s^\mu$$

as $n \to \infty$ strongly in $\oplus^3 \left(L^2(\mathscr{X}, d\mathcal{W}^x) \otimes L^2(\mathbb{R}^3)\right)$ implying that for $F, G \in \mathscr{H}_{\mathrm{PF}}$,

$$\lim_{n\to\infty}(F, (\mathscr{P}_{t/2^n})^{2^n} G) = \int_{\mathbb{R}^3} dx \mathbb{E}^x[(F(B_0), e^{-ie\mathscr{A}(L_t)} G(B_t))]. \tag{7.3.15}$$

From (7.3.15) it follows that $|\lim_{n\to\infty}(F, (\mathscr{P}_{t/2^n})^{2^n} G)| \le \|F\|\|G\|$. Hence there exists a symmetric bounded operator S_t such that $\lim_{n\to\infty}(F, (\mathscr{P}_{t/2^n})^{2^n} G) = (F, S_t G)$. Since $(\mathscr{P}_{t/2^n})^{2^n}$ is uniformly bounded as $\|(\mathscr{P}_{t/2^n})^{2^n}\| \le 1$, the above weak convergence improves to

$$\operatorname*{s-lim}_{n\to\infty}(\mathscr{P}_{t/2^n})^{2^n} = S_t, \quad t \ge 0. \tag{7.3.16}$$

Furthermore, by (7.3.15)

$$(F, S_t G) = \int_{\mathbb{R}^3} dx \mathbb{E}^x[(F(B_0), e^{-ie\mathscr{A}(L_t)} G(B_t))]. \tag{7.3.17}$$

Thus

$$\begin{aligned}(F, S_s S_t G) &= \lim_{n\to\infty}\lim_{m\to\infty}(F, (\mathscr{P}_{s/2^n})^{2^n}(\mathscr{P}_{t/2^m})^{2^m} G)\\ &= \int_{\mathbb{R}^3} dx \mathbb{E}^x[(F(B_0), e^{-ie\mathscr{A}(L_{(s+t)})} G(B_t))] = (F, S_{s+t} G),\end{aligned}$$

giving the semigroup property $S_s S_t = S_{s+t}$, $s, t \ge 0$. It can also be seen that $w-\lim_{t\to 0} S_t = 1$ by (7.3.17), which further implies that $\operatorname{s-lim}_{t\to 0} S_t = 1$, since $\lim_{t\to 0}\|S_t F\|^2 = \lim_{t\to 0}(F, S_{2t} F) = \|F\|^2$. Finally, it is trivial to see that $S_0 = 1$. Putting these together we conclude that S_t is a symmetric C_0-semigroup, thus there exists a unique self-adjoint operator $\hat{H}_{\mathrm{PF}}(\mathscr{A})$ such that $S_t = e^{-t\hat{H}_{\mathrm{PF}}(\mathscr{A})}$, $t \ge 0$, i.e.,

$$(F, e^{-t\hat{H}_{\mathrm{PF}}(\mathscr{A})} G) = \int_{\mathbb{R}^3} dx \mathbb{E}^x[(F(B_0), e^{-ie\mathscr{A}(L_t)} G(B_t))]. \tag{7.3.18}$$

Furthermore, in a similar manner as in (3.5.8) we have

$$\lim_{t\to 0}(F, t^{-1}(1 - \mathscr{P}_t)G) = (F, H_{\mathrm{PF}}(\mathscr{A})G), \tag{7.3.19}$$

for $F, G \in C_0^\infty(\mathbb{R}^3)\hat{\otimes}\mathscr{F}_{\mathrm{rad,fin}}$. This leads to

$$\begin{aligned}(t^{-1}(e^{-t\hat{H}_{\mathrm{PF}}(\mathscr{A})} - 1)F, G) &= \lim_{n\to\infty}(t^{-1}((\mathscr{P}_{t/2^n})^{2^n} - 1)F, G)\\ &= -\int_0^1 (H_{\mathrm{PF}}(\mathscr{A})F, e^{-ts\hat{H}_{\mathrm{PF}}(\mathscr{A})} G) ds.\end{aligned} \tag{7.3.20}$$

In the second equality above we used (7.3.19). Since

$$\|H_{\mathrm{PF}}(\mathscr{A})F\| \le C(\|-\Delta F\| + \|H_{\mathrm{rad}} F\| + \|F\|) \tag{7.3.21}$$

for $F \in D((-1/2)\Delta) \cap D(H_{\mathrm{f}})$, (7.3.20) can be extended to $F \in D((-1/2)\Delta) \cap D(H_{\mathrm{f}})$ and $G \in D(\hat{H}_{\mathrm{PF}}(\mathscr{A}))$. We made use of the assumptions $\sqrt{\omega}\hat{\varphi} \in L^2(\mathbb{R}^3)$ and $\hat{\varphi}/\omega \in L^2(\mathbb{R}^3)$ in (7.3.21). Take now $t \downarrow 0$ on both sides of equality (7.3.20). Then $(F, \hat{H}_{\mathrm{PF}}(\mathscr{A})G) = (H_{\mathrm{PF}}(\mathscr{A})F, G)$ for $F \in D((-1/2)\Delta) \cap D(H_{\mathrm{f}})$ and $G \in D(\hat{H}_{\mathrm{PF}}(\mathscr{A}))$, implying that $\hat{H}_{\mathrm{PF}}(\mathscr{A}) \supset H_{\mathrm{PF}}(\mathscr{A})\lceil_{D_{\mathrm{PF}}}$ as $\hat{H}_{\mathrm{PF}}(\mathscr{A})$ is self-adjoint. Define

$$\hat{H}_{\mathrm{PF}} = \hat{H}_{\mathrm{PF}}(\mathscr{A}) \dotplus H_{\mathrm{rad}}. \tag{7.3.22}$$

We have by the Trotter product formula and the factorization $e^{-(t/n)H_{\mathrm{rad}}} = \mathrm{J}^*_{kt/n}\mathrm{J}_{(k+1)t/n}$,

$$\begin{aligned}(F, e^{-t\hat{H}_{\mathrm{PF}}}G) &= \lim_{n\to\infty}(F, (e^{-(t/n)\hat{H}_{\mathrm{PF}}(\mathscr{A})}e^{-(t/n)H_{\mathrm{rad}}})^n G)\\ &= \lim_{n\to\infty}\Big(\mathrm{J}_0 F, \Big(\prod_{i=0}^{n-1} R_i\Big)\mathrm{J}_t G\Big),\end{aligned}$$

where $R_j = \mathrm{J}_{jt/n}e^{-(t/n)\hat{H}_{\mathrm{PF}}(\mathscr{A})}\mathrm{J}^*_{jt/n}$. Using the definition of $e^{-t\hat{H}_{\mathrm{PF}}(\mathscr{A})}$ we get

$$\begin{aligned}&\mathrm{J}_s e^{-t\hat{H}_{\mathrm{PF}}(\mathscr{A})}\mathrm{J}^*_s G(x) = \operatorname*{s-lim}_{n\to\infty} \mathrm{J}_s(\mathscr{P}_{t/2^n})^{2^n}\mathrm{J}^*_s G(x)\\ &= \operatorname*{s-lim}_{n\to\infty}\int_{(\mathbb{R}^3)^{2^n}} \mathrm{J}_s \exp\Big(i\sum_{j=1}^{2^n}\mathfrak{h}_j\Big)\mathrm{J}^*_s G(x_{2^n})\Big(\prod_{j=1}^{2^n}\Pi_{t/2^n}(x_{j-1}-x_j)\Big)\prod_{j=1}^{2^n}dx_j\end{aligned}$$

with $x = x_0$. We are half way through the trying argument. Write $\delta_j = \delta_j(n,t,n_j) = t/(n2^{n_j})$ for $j = 0, 1, \ldots, n-1$ and define $\mathscr{P}_{s,j} : \mathscr{H}_{\mathrm{E}} \to \mathscr{H}_{\mathrm{E}}$ by $\mathscr{P}_s$ with $\mathfrak{h}(x,y)$ replaced by the Euclidean version $\mathfrak{h}^{tj/n}(x,y)$ given by

$$\mathfrak{h}^s(x,y) = -\frac{1}{2}\,(\mathscr{A}_{\mathrm{E}s}(x) + \mathscr{A}_{\mathrm{E}s}(y))\cdot(x-y) \tag{7.3.23}$$

as

$$\mathscr{P}_{s,j}F(x) = \int_{\mathbb{R}^3}\Pi_s(x-y)e^{i\mathfrak{h}^{tj/n}(x,y)}F(y)dy.$$

We have

$$\begin{aligned}\Big(\mathrm{J}_0 F, \Big(\prod_{i=0}^{n-1}R_i\Big)\mathrm{J}_t G\Big) &= \lim_{n_0\to\infty}\cdots\lim_{n_{n-1}\to\infty}\Big(\mathrm{J}_0 F, \prod_{i=0}^{n-1}(\mathrm{E}_i(\mathscr{P}_{\delta_i,i})^{2^{n_i}}\mathrm{E}_i)\mathrm{J}_t G\Big)\\ &= \lim_{n_0\to\infty}\cdots\lim_{n_{n-1}\to\infty}\Big(\mathrm{J}_0 F, \Big(\prod_{i=0}^{n-1}(\mathscr{P}_{\delta_i,i})^{2^{n_i}}\Big)\mathrm{J}_t G\Big).\end{aligned}$$

Here in the first equality we used the fact that

$$\mathrm{J}_s \exp\Big(i\sum_{j=1}^{2^n}\mathfrak{h}_j\Big)\mathrm{J}_s^* = \mathrm{E}_s \exp\Big(i\sum_{j=1}^{2^n}\mathfrak{h}_j^s\Big)\mathrm{E}_s, \tag{7.3.24}$$

where $\mathfrak{h}_j^s$ is defined by $\mathfrak{h}_j^s = \mathfrak{h}^s(x_j, x_{j-1})$, and in the second equality we used the Markov property of E_s. As a result we have

$$\Big(\mathrm{J}_0 F, \Big(\prod_{i=0}^{n-1} R_i\Big)\mathrm{J}_t G\Big) = \lim_{n_0\to\infty}\cdots\lim_{n_{n-1}\to\infty}\int_{\mathbb{R}^3} dx \mathbb{E}^x[(\mathrm{J}_0 F(B_0), e^{-ie\mathscr{A}_{\mathrm{E}}(K)}\mathrm{J}_t G(B_t))]$$

with $K = K(n_0, n_1, \ldots, n_{n-1}, n)$ given by

$$K = \bigoplus_{\mu=1}^{3}\sum_{j=0}^{n-1}\sum_{m=1}^{2^{n_j}} \mathrm{j}_{\frac{tj}{n}}(\tilde{\varphi}(\cdot - B_{\frac{t}{n}(\frac{m}{2^{n_j}}+j)}) + \tilde{\varphi}(\cdot - B_{\frac{t}{n}(\frac{m-1}{2^{n_j}}+j)})) \\ \times (B^\mu_{\frac{t}{n}(\frac{m}{2^{n_j}}+j)} - B^\mu_{\frac{t}{n}(\frac{m-1}{2^{n_j}}+j)})$$

and

$$K \to \bigoplus_{\mu=1}^{3}\sum_{j=0}^{n-1}\int_{tj/n}^{t(j+1)/n} \mathrm{j}_{tj/n}\tilde{\varphi}(\cdot - B_s) dB_s^\mu$$

as $n_0, n_1, \ldots, n_{n-1} \to \infty$ in $\oplus^3(L^2(\mathscr{X}, d\mathscr{W}^x) \otimes L^2(\mathbb{R}^4))$. Finally as $n \to \infty$ we have

$$(F, e^{-t\hat{H}_{\mathrm{PF}}}G) = \int_{\mathbb{R}^3} dx \mathbb{E}^x[(\mathrm{J}_0 F(B_0), e^{-ie\mathscr{A}_{\mathrm{E}}(K_t)}\mathrm{J}_t G(B_t))]. \tag{7.3.25}$$

By the construction of $\hat{H}_{\mathrm{PF}}$,

$$\hat{H}_{\mathrm{PF}} \supset \frac{1}{2}(-i\nabla - \mathscr{A})^2 + H_{\mathrm{rad}}\lceil_{D_{\mathrm{PF}}}. \tag{7.3.26}$$

In the next section we will show that

$$\hat{H}_{\mathrm{PF}} = \frac{1}{2}(-i\nabla - \mathscr{A})^2 + H_{\mathrm{rad}} \tag{7.3.27}$$

and $\hat{H}_{\mathrm{PF}}$ is self-adjoint on D_{PF}. The functional integral representation of $e^{-tH_{\mathrm{PF}}}$ including non-zero V can be obtained by the Trotter product formula

$$(F, e^{-tH_{\mathrm{PF}}}G) = \lim_{n\to\infty}(F, (e^{-(t/n)\hat{H}_{\mathrm{PF}}(\mathscr{A})}e^{-(t/n)V}e^{-(t/n)H_{\mathrm{rad}}})^n G).$$

This completes the proof. □

A rewarding application of the functional integral representation of $(F, e^{-tH_{\mathrm{PF}}}G)$ is a diamagnetic inequality as the exponent $-ie\mathscr{A}_{\mathrm{E}}(K_t)$ of $(F, e^{-tH_{\mathrm{PF}}}G)$ is purely imaginary.

Theorem 7.15 (Diamagnetic inequality). *Under the conditions of Theorem* 7.14 *it follows that*

$$|(F, e^{-tH_{\mathrm{PF}}}G)| \le (|F|, e^{-tH_{\mathrm{PF},0}}|G|). \tag{7.3.28}$$

Proof. By the functional integral representation in Theorem 7.14 we plainly have

$$\begin{aligned}|(F, e^{-tH_{\mathrm{PF}}}G)| &\le \int_{\mathbb{R}^3} dx \mathbb{E}^x[e^{-\int_0^t V(B_s)ds}(\mathrm{J}_0|F(B_0)|, \mathrm{J}_t|G(B_t)|)] \\ &= (|F|, e^{-tH_{\mathrm{PF},0}}|G|).\end{aligned}$$

Here we used that $|\mathrm{J}_s F| \le \mathrm{J}_s|F|$ since J_s is positivity preserving. □

The diamagnetic inequality shows that coupling the particle to the quantized radiation field by minimal interaction increases the ground state energy of the non-interacting system.

Corollary 7.16 (Enhanced ground state energy). *We have*

$$\inf \mathrm{Spec}(H_{\mathrm{p}}) \le \inf \mathrm{Spec}(H_{\mathrm{PF}}).$$

Proof. Theorem 7.15 implies that $\inf \mathrm{Spec}(H_{\mathrm{PF},0}) \le \inf \mathrm{Spec}(H_{\mathrm{PF}})$. Since

$$\inf \mathrm{Spec}(H_{\mathrm{PF},0}) = \inf \mathrm{Spec}(H_{\mathrm{p}}),$$

the corollary follows. □

7.3.3 Extension to general external potential

In Theorem 7.14 we assumed smoothness of the external potential. In this section we offer an extension of the functional integral representation to a wider potential class. For convenience, in this section we denote H_{PF} with $V \equiv 0$ by

$$H_0 = \frac{1}{2}(-i\nabla - e\mathscr{A})^2 + H_{\mathrm{rad}}.$$

First consider a form boundedness property of H_0. Let

$$\operatorname{sgn} F(x) = \begin{cases} F(x)/\|F(x)\|_{L^2(\mathscr{Q})} & \text{if } \|F(x)\|_{L^2(\mathscr{Q})} \ne 0, \\ 0 & \text{if } \|F(x)\|_{L^2(\mathscr{Q})} = 0. \end{cases} \tag{7.3.29}$$

Lemma 7.17. *Suppose that V is $-(1/2)\Delta$-form bounded with a relative bound a. Then $|V|$ is H_0-form bounded with a relative bound smaller than a.*

Proof. Let $\psi \in C_0^\infty(\mathbb{R}^3)$ and $\psi > 0$. Substituting $F = \operatorname{sgn}(e^{-tH_0}G) \cdot \psi$ in the diamagnetic inequality $|(F, e^{-tH_0}G)| \le (|F|, e^{-t(-(1/2)\Delta)}|G|)$, we obtain that

$$(\psi, \|e^{-tH_0}G(\cdot)\|_{L^2(\mathscr{Q})})_{L^2(\mathbb{R}^3)} \le (\psi, e^{-t(-(1/2)\Delta)}\|G\|_{L^2(\mathscr{Q})})_{L^2(\mathbb{R}^3)}.$$

Hence we have

$$\frac{\||V|^{1/2}(H_0+z)^{-1/2}G\|_{\mathscr{H}_{\rm PF}}}{\|G\|_{\mathscr{H}_{\rm PF}}} \le \frac{\||V|^{1/2}(-(1/2)\Delta+z)^{-1/2}\|G(\cdot)\|_{L^2(\mathscr{Q})}\|_{L^2(\mathbb{R}^3)}}{\|G\|_{\mathscr{H}_{\rm PF}}}.$$

Taking the supremum of both sides above with respect to $\|G\|_{\mathscr{H}_{\rm PF}} \ne 0$ we further obtain

$$\||V|^{1/2}(H_0+z)^{-1/2}\|_{\mathscr{H}_{\rm PF}} \le \||V|^{1/2}(-(1/2)\Delta+z)^{-1/2}\|_{L^2(\mathbb{R}^3)}.$$

Then the lemma follows in the similar manner as in Lemma 3.72. □

Remark 7.2. In a similar way as in Lemma 7.17 we can show that if V is relatively bounded with respect to $-(1/2)\Delta$ with a relative bound a, then V is relatively bounded with respect to H_0 with a relative bound a.

Theorem 7.18. *Take $H_0 \dotplus V_+ \dot{-} V_-$ with $V_+ \in L^1_{\rm loc}(\mathbb{R}^3)$ and V_- is $-(1/2)\Delta$-relatively form bounded with a relative bound strictly less than* 1. *The equality* (7.3.12) *holds for $H_0 \dotplus V_+ \dot{-} V_-$.*

Proof. The proof is similar to that of Theorems 3.31 and 3.73 with $h = -(1/2)\Delta$ replaced by H_0. □

7.4 Applications of functional integral representations

7.4.1 Self-adjointness of the Pauli–Fierz Hamiltonian

One of the basic problems in the spectral analysis of Hamiltonians under consideration is to show their essential self-adjointness or self-adjointness. As we have pointed out in the previous section, for sufficiently small e the Pauli–Fierz operator $H_{\rm PF}$ is self-adjoint on $D_{\rm PF}$. We will show in this section that $H_{\rm PF}$ is self-adjoint on $D_{\rm PF}$ for all $e \in \mathbb{R}$. Recall that $D_{\rm PF} = D(-(1/2)\Delta) \cap D(H_{\rm rad})$.

We have already constructed a functional integral representation for $\hat{H}_{\rm PF}$, however, it remains to be shown that $\hat{H}_{\rm PF} = H_{\rm PF}$. In the proof of Theorem 7.14 we defined the self-adjoint operator $\hat{H}_{\rm PF} = \hat{H}_{\rm PF}(\mathscr{A}) \dotplus H_{\rm rad}$ and have shown that $\hat{H}_{\rm PF} \supset H_{\rm PF}\lceil_{D_{\rm PF}}$ in (7.3.26), and

$$(F, e^{-t\hat{H}_{\rm PF}}G) = \int_{\mathbb{R}^3} dx \mathbb{E}^x[(\mathrm{J}_0 F(B_0), e^{-ie\mathscr{A}_{\rm E}(K_t)}\mathrm{J}_t G(B_t))] \tag{7.4.1}$$

in (7.3.25). Let $R_t(F,G)$ be the quadratic form defined by the right-hand side of (7.4.1) in which there is no restriction on the value of e. The right-hand side of (7.4.1) is well defined for any $e \in \mathbb{R}$. There exists a semigroup S_t such that $R_t(F,G) = (F, S_t G)$ and its generator $\hat{H}_{\rm PF}$ satisfies that $\hat{H}_{\rm PF} \supset H_{\rm PF}\lceil_{D_{\rm PF}}$. Clearly, for weak enough coupling $\hat{H}_{\rm PF} = H_{\rm PF}\lceil_{D_{\rm PF}}$. We will show that S_t leaves $D_{\rm PF}$ invariant and from that conclude that $\hat{H}_{\rm PF}$ is essentially self-adjoint on $D_{\rm PF}$. This will imply that $H_{\rm PF}$ is essentially self-adjoint on this joint domain. Finally, we will show that $H_{\rm PF}$ is closed on $D_{\rm PF}$ and therefore $H_{\rm PF}$ is self-adjoint on $D_{\rm PF}$.

We start by a general result.

Lemma 7.19. *Let K be a non-negative self-adjoint operator. Suppose that there exists a dense domain D such that $D \subset D(K)$ and $e^{-tK} D \subset D$ for all $t \geq 0$. Then $K\lceil_D$ is essentially self-adjoint.*

Proof. It suffices to show that for some $\lambda > 0$, $\mathrm{Ran}((\lambda + K)\lceil_D)$ is dense. Suppose the contrary. Then there exists nonzero f such that $(f, (\lambda + K)\psi) = 0$ for all $\psi \in D$. We have

$$\frac{d}{dt}(f, e^{-tK}\psi) = (f, -Ke^{-tK}\psi) = \lambda(f, e^{-tK}\psi).$$

Thus $(f, e^{-tK}\psi) = e^{\lambda t}(f,\psi)$. If $(f,\psi) \neq 0$, then we have $\lim_{t\to\infty} |(f, e^{-tK}\psi)| = \infty$, contradicting the fact that e^{-tK} is a contraction. Hence $(f,\psi) = 0$ for all $\psi \in D(K)$, but (f,ψ) can not equal zero for all $\psi \in D(K)$, since $D(K)$ is dense. Hence we conclude that $\mathrm{Ran}((\lambda + K)\lceil_D)$ is dense. □

The next result is an important application of Lemma 7.19 to the Pauli–Fierz Hamiltonian.

Lemma 7.20. *Suppose that*

$$|(H_{\rm PF,0}F, e^{-t\hat{H}_{\rm PF}}G)| \leq C_G \|F\| \tag{7.4.2}$$

for all $F, G \in D(H_{\rm PF,0}) = D_{\rm PF}$ with some constant C_G which may depend on G. Then $H_{\rm PF}$ is essentially self-adjoint on $D_{\rm PF}$.

Proof. Inequality (7.4.2) yields that $e^{-t\hat{H}_{\rm PF}}G \in D(H_{\rm PF,0})$ by the Riesz Theorem, i.e., $e^{-t\hat{H}_{\rm PF}}D_{\rm PF} \subset D_{\rm PF}$. Thus Lemma 7.19 implies that $\hat{H}_{\rm PF}\lceil_{D_{\rm PF}}$ is essentially self-adjoint. The corollary follows from $H_{\rm PF} = \hat{H}_{\rm PF}$ on $D_{\rm PF}$. □

In order to show (7.4.2) we need an inequality for $\int_0^t \mathrm{j}_s\tilde{\varphi}(\cdot - B_s)dB_s^\mu$ similar to the BDG inequality (Proposition 2.34). With an $\mathbb{R}^d$-valued function $a = (a_1,\dots,a_d)$, $Y_t^\mu = \int_0^t a_\mu(s, B_s)dB_s^\mu$ satisfies

$$d(Y_t^\mu)^2 = 2Y_t^\mu a_\mu(t, B_t)dB_t^\mu + a_\mu(t, B_t)^2 dt$$

and the BDG inequality

$$\mathbb{E}^x\left[\left|\int_0^t a(s,B_s)\cdot dB_s\right|^{2m}\right] \le (m(2m-1))^m t^{m-1}\mathbb{E}^x\left[\int_0^t |a(s,B_s)|^{2m} ds\right]$$

holds. We show its infinite dimensional version below. Let

$$\hat{\omega}_{\mathrm{E}} = \omega(-i\nabla)\otimes 1 \tag{7.4.3}$$

under the identification $L(\mathbb{R}^4) \cong L^2(\mathbb{R}^3)\otimes L^2(\mathbb{R})$.

Lemma 7.21 (BDG-type inequality). *If* $\omega^{(k-1)/2}\hat{\varphi} \in L^2(\mathbb{R}^3)$, *then*

$$\mathbb{E}^x\left[\left\|\hat{\omega}_{\mathrm{E}}^{k/2}\int_0^t \mathrm{j}_s\tilde{\varphi}(\cdot - B_s)dB_s^\mu\right\|^{2m}_{L^2(\mathbb{R}^4)}\right] \le \frac{(2m)!}{2^m}t^m\|\omega^{(k-1)/2}\hat{\varphi}\|^{2m}_{L^2(\mathbb{R}^3)}.$$

In particular,

$$\sup_{x\in\mathbb{R}^d}\mathbb{E}^x[\|\hat{\omega}_{\mathrm{E}}^{k/2}K_t\|^{2m}_{\oplus^3 L^2(\mathbb{R}^4)}] \le \frac{3(2m)!}{2^m}t^m\|\omega^{(k-1)/2}\hat{\varphi}\|^{2m}_{L^2(\mathbb{R}^3)},$$

where $K_t = \bigoplus_{\mu=1}^3\int_0^t \mathrm{j}_s\tilde{\varphi}(\cdot - B_s)dB_s^\mu$.

Proof. Let $(X_t)_{t\ge 0} = (X_t^1, X_t^2, X_t^3)_{t\ge 0}$ be a random process defined by

$$X_t^\mu = \left\|\int_0^t \mathrm{j}_s\lambda(\cdot - B_s)dB_s^\mu\right\|^2_{L^2(\mathbb{R}^4)}$$

with $\lambda \in L^2(\mathbb{R}^3)$. By the definition of j_s we have

$$X_t^\mu = \lim_{n\to\infty}\int_{\mathbb{R}^4}\frac{1}{\pi}\mu(\mathbf{k})^2|\lambda(k)|^2\left|\sum_{j=1}^n\int_{t(j-1)/n}^{tj/n} e^{-ik\cdot B_s}e^{-ik_0\frac{t(j-1)}{n}}dB_s^\mu\right|^2 d\mathbf{k}, \tag{7.4.4}$$

where $\mu(\mathbf{k}) = (\frac{\omega(k)}{\omega(k)^2+|k_0|^2})^{1/2}$. Since

$$\lim_{n\to\infty}\mathbb{E}\left|\sum_{j=1}^n\int_{t(j-1)/n}^{tj/n} e^{-ik\cdot B_s}e^{-ik_0\frac{t(j-1)}{n}}dB_s^\mu - \int_0^t e^{-ik\cdot B_s}e^{-ik_0 s}dB_s^\mu\right|^2 = 0,$$

there exists a subsequence n' such that

$$\sum_{j=1}^{n'}\int_{t(j-1)/n'}^{tj/n'} e^{-ik\cdot B_s}e^{-ik_0\frac{t(j-1)}{n'}}dB_s^\mu \to \int_0^t e^{-ik\cdot B_s}e^{-ik_0 s}dB_s^\mu$$

almost surely. Hence we obtain that

$$X_t^\mu = \frac{1}{\pi}\int_{\mathbb{R}^4} \mu(\mathbf{k})^2|\lambda(k)|^2|Y_t^\mu|^2 d\mathbf{k}, \tag{7.4.5}$$

where Y_t^μ is the stochastic integral defined by $Y_t^\mu = \int_0^t e^{-isk_0}e^{-ikB_s}dB_s^\mu$. The Itô formula then gives

$$d|Y_t^\mu|^2 = 2\,\mathrm{Re}(Y_t^\mu e^{-ik_0t}e^{-ikB_t})dB_t^\mu + dt. \tag{7.4.6}$$

Inserting (7.4.6) into (7.4.5) it is seen that

$$X_t^\mu = 2\,\mathrm{Re}\,\frac{1}{\pi}\int_{\mathbb{R}^4}\mu(\mathbf{k})^2|\lambda(k)|^2\left(\int_0^t Y_s^\mu e^{-ik_0s}e^{-ikB_s}dB_s^\mu\right)d\mathbf{k} + t\|\lambda\|^2,$$

where we used that $\int_{\mathbb{R}^4}\mu(\mathbf{k})^2 d\mathbf{k} = \pi$. Hence the stochastic differential equation

$$dX_t^\mu = Z_t^\mu dB_t^\mu + \|\lambda\|^2 dt \tag{7.4.7}$$

is obtained, where

$$Z_t^\mu = 2\,\mathrm{Re}\left(\mathrm{j}_t\lambda(\cdot - B_t), \int_0^t \mathrm{j}_s\lambda(\cdot - B_s)dB_s^\mu\right)_{L^2(\mathbb{R}^4)}.$$

An application of the Itô formula gives

$$\begin{aligned} d(X_t^\mu)^m &= m(X_t^\mu)^{m-1}Z_t^\mu dB_t^\mu \\ &\quad + \left(m(X_t^\mu)^{m-1}\|\lambda\|^2 + \frac{1}{2}m(m-1)(X_t^\mu)^{m-2}(Z_t^\mu)^2\right)dt. \end{aligned}$$

Taking expectation on both sides, we have

$$\mathbb{E}^x[(X_t^\mu)^m] = m\|\lambda\|^2\int_0^t \mathbb{E}^x[(X_s^\mu)^{m-1}]ds + \frac{m(m-1)}{2}\int_0^t \mathbb{E}^x[(X_s^\mu)^{m-2}(Z_s^\mu)^2]ds.$$

By the bound $|Z_t^\mu| \le 2\|\lambda\|\sqrt{X_t^\mu}$ it follows that

$$\mathbb{E}^x[(X_t^\mu)^m] \le \frac{2m(2m-1)}{2}\|\lambda\|^2\int_0^t \mathbb{E}^x[(X_s^\mu)^{m-1}]ds,$$

and by induction furthermore

$$\mathbb{E}^x[(X_t^\mu)^m] \le \frac{(2m)!}{2^m}\|\lambda\|^{2m}t^m$$

is obtained. Notice that the intertwining property $(\omega(-i\nabla)\otimes 1)\mathrm{j}_s = \mathrm{j}_s\omega(-i\nabla)$ holds. Write $\lambda = \omega(-i\nabla)^{k/2}\tilde{\varphi}$. Then $\|\lambda\| = \|\omega^{(k-1)/2}\hat{\varphi}\|$ and the proposition follows. □

Before checking (7.4.2) note that J_t intertwines the free field Hamiltonian H_{rad} and $d\Gamma_{\mathrm{E}}(\hat{\omega}_{\mathrm{E}})$, i.e.,

$$\mathrm{J}_t H_{\mathrm{rad}} = d\Gamma_{\mathrm{E}}(\hat{\omega}_{\mathrm{E}})\mathrm{J}_t. \tag{7.4.8}$$

Define

$$\Pi_{\mathrm{E}}(\mathbf{f}) = i[N, \mathscr{A}_{\mathrm{E}}(\mathbf{f})]. \tag{7.4.9}$$

Then $\Pi_{\mathrm{E}}(\mathbf{f})$ is the canonical conjugate of $\mathscr{A}_{\mathrm{E}}(\mathbf{f})$, i.e., the canonical commutation relation

$$[\mathscr{A}_{\mathrm{E}}(\mathbf{f}), \Pi_{\mathrm{E}}(\mathbf{g})] = 2i\,\mathrm{q}_{\mathrm{E}}(\mathbf{f}, \mathbf{g}) \tag{7.4.10}$$

is satisfied for all $\mathbf{f}, \mathbf{g} \in \oplus^3 L^2(\mathbb{R}^4)$. Let G be an analytic vector for $\mathscr{A}_{\mathrm{E}}(\mathbf{f})$. Then

$$e^{-i\mathscr{A}_{\mathrm{E}}(\mathbf{f})}G = \sum_{n=0}^{\infty} \frac{(-i\mathscr{A}_{\mathrm{E}}(\mathbf{f}))^n}{n!} G \tag{7.4.11}$$

holds. By the commutation relations

$$[d\Gamma_{\mathrm{E}}(\hat{\omega}_{\mathrm{E}}), \mathscr{A}_{\mathrm{E}}(\mathbf{f})] = -i\,\Pi_{\mathrm{E}}(\hat{\omega}_{\mathrm{E}}\mathbf{f}), \tag{7.4.12}$$

$$[[d\Gamma_{\mathrm{E}}(\hat{\omega}_{\mathrm{E}}), \mathscr{A}_{\mathrm{E}}(\mathbf{f})], \mathscr{A}_{\mathrm{E}}(\mathbf{f})] = -2\mathrm{q}_{\mathrm{E}}(\hat{\omega}_{\mathrm{E}}\mathbf{f}, \mathbf{f}), \tag{7.4.13}$$

together with (7.4.11), we moreover have the identity

$$d\Gamma_{\mathrm{E}}(\hat{\omega}_{\mathrm{E}})e^{-i\mathscr{A}_{\mathrm{E}}(\mathbf{f})}G = e^{-i\mathscr{A}_{\mathrm{E}}(\mathbf{f})}(d\Gamma_{\mathrm{E}}(\hat{\omega}_{\mathrm{E}}) - \Pi_{\mathrm{E}}(\hat{\omega}_{\mathrm{E}}\mathbf{f}) + \mathrm{q}_{\mathrm{E}}(\hat{\omega}_{\mathrm{E}}\mathbf{f}, \mathbf{f}))G \tag{7.4.14}$$

for $\mathbf{f} \in D(\hat{\omega}_{\mathrm{E}})$. Furthermore, by a limiting argument it can be seen that $e^{-i\mathscr{A}_{\mathrm{E}}(\mathbf{f})}$ with $\mathbf{f} \in D(\hat{\omega}_{\mathrm{E}})$ leaves $D(d\Gamma_{\mathrm{E}}(\hat{\omega}_{\mathrm{E}}))$ invariant and the identity (7.4.14) holds for $G \in D(d\Gamma_{\mathrm{E}}(\hat{\omega}_{\mathrm{E}}))$. Put

$$\mathbf{J}_t = \mathrm{J}_0^* e^{-ie\mathscr{A}_{\mathrm{E}}(K_t)}\mathrm{J}_t. \tag{7.4.15}$$

Lemma 7.22. *Let $F, G \in D(H_{\mathrm{rad}})$. Under Assumption* 7.1

$$|(H_{\mathrm{rad}}F, e^{-t\hat{H}_{\mathrm{PF}}}G)| \le C((\sqrt{t} + t)\|(H_{\mathrm{rad}} + 1)^{1/2}G\| + \|H_{\mathrm{rad}}G\|)\|F\|$$

holds. In particular, $e^{-tH_{\mathrm{PF}}}D_{\mathrm{PF}} \subset D(H_{\mathrm{rad}})$.

Proof. We see that $\mathbf{J}_t$ leaves $D(H_{\mathrm{rad}})$ invariant, and for $G \in D(H_{\mathrm{rad}})$

$$[H_{\mathrm{rad}}, \mathbf{J}_t]G = \mathrm{J}_0^* e^{-ie\mathscr{A}_{\mathrm{E}}(K_t)}\mathrm{C}_t\mathrm{J}_t G$$

by (7.4.8) and (7.4.14), where

$$\mathrm{C}_t = -e\Pi_{\mathrm{E}}(\hat{\omega}_{\mathrm{E}} K_t) + e^2 \mathrm{q}_{\mathrm{E}}(\hat{\omega}_{\mathrm{E}} K_t, K_t). \tag{7.4.16}$$

Let $F, G \in D(H_{\mathrm{rad}})$. Then the functional integral representation yields

$$(H_{\mathrm{rad}} F, e^{-t\hat{H}_{\mathrm{PF}}} G) = \int_{\mathbb{R}^3} dx \mathbb{E}^x \left[(F(B_0), [H_{\mathrm{rad}}, \mathbf{J}_t] G(B_t)) \right] + (F, e^{-t\hat{H}_{\mathrm{PF}}} H_{\mathrm{rad}} G).$$

Furthermore, the right-hand side above can be estimated as

$$|(H_{\mathrm{rad}} F, e^{-t\hat{H}_{\mathrm{PF}}} G)| \le \|F\| \left(\int_{\mathbb{R}^3} dx \left(\mathbb{E}^x [\| [H_{\mathrm{rad}}, \mathbf{J}_t] G_t \|] \right)^2 \right)^{1/2} + \|F\| \|H_{\mathrm{rad}} G\|. \tag{7.4.17}$$

Note the bound $\|[H_{\mathrm{rad}}, \mathbf{J}_t]G\| \le \|\mathrm{C}_t \mathrm{J}_t G\|$. We estimate each term in (7.4.16) separately. We have

$$\|\Pi_{\mathrm{E}}(\hat{\omega}_{\mathrm{E}} K_t) \mathrm{J}_t \Psi\| \le C(\|\hat{\omega}_{\mathrm{E}}^{1/2} K_t\| + \|\hat{\omega}_{\mathrm{E}} K_t\|) \|(H_{\mathrm{rad}} + 1)^{1/2} \Psi\|,$$
$$|\mathrm{q}_{\mathrm{E}}(\hat{\omega}_{\mathrm{E}} K_t, K_t)| \le \|\hat{\omega}_{\mathrm{E}}^{1/2} K_t\|^2$$

with a constant C. Applying our BDG-type inequality (Lemma 7.21) to $\|\hat{\omega}_{\mathrm{E}} K_t\|^{2n}$, we have

$$\int_{\mathbb{R}^3} dx (\mathbb{E}^x [\| [H_{\mathrm{rad}}, \mathbf{J}_t] G_t \|])^2 \le C'(t + t^2) \|(H_{\mathrm{rad}} + 1)^{1/2} G\|^2$$

with a constant C'. Hence the lemma follows. □

We next show that $e^{-t\hat{H}_{\mathrm{PF}}} D_{\mathrm{PF}} \subset D((-1/2)\Delta)$. Let

$$D_{\mathscr{Q}} = \left\{ F(\mathscr{A}(\mathbf{f}_1) \cdots \mathscr{A}(\mathbf{f}_n)) | \, F \in \mathscr{S}(\mathbb{R}^n), \mathbf{f}_1, \ldots, \mathbf{f}_n \in \oplus^3 L^2_{\mathrm{real}}(\mathbb{R}^3), n \ge 1 \right\}. \tag{7.4.18}$$

Lemma 7.23. *Let* $\omega^{3/2}\hat{\varphi} \in L^2(\mathbb{R}^3)$ *and take Assumption* 7.1. *Then for* $F \in D(\Delta)$ *and* $G \in D_{\mathrm{PF}}$,

$$|(\Delta F, e^{-t\hat{H}_{\mathrm{PF}}} G)| \le C(\| - \Delta G\| + (\sqrt{t} + t)\|(-\Delta + H_{\mathrm{rad}} + 1)G\|)\|F\| \tag{7.4.19}$$

with a constant $C > 0$. *In particular,* $e^{-t\hat{H}_{\mathrm{PF}}} D_{\mathrm{PF}} \subset D((-1/2)\Delta)$.

Proof. Suppose that $F = f \otimes \Phi$ and $G = g \otimes \Psi$, where $f, g \in C_0^\infty(\mathbb{R}^3)$ and $\Phi, \Psi \in D_{\mathscr{Q}}$. By functional integration we have

$$(-\Delta F, e^{-t\hat{H}_{\mathrm{PF}}} G) = -\int_{\mathbb{R}^3} dx \mathbb{E}[(\mathrm{J}_0 F(x), e^{-ie\mathscr{A}_{\mathrm{E}}(K_t)} U \mathrm{J}_t G(B_t + x))],$$

where

$$U = \Delta + \sum_{\mu=1}^{3} (-ie\mathscr{A}_{\mathrm{E}}(K_{\mu\mu}) - e^2 \mathscr{A}_{\mathrm{E}}(K_\mu)^2 - 2ie\mathscr{A}_{\mathrm{E}}(K_\mu)\cdot\nabla_\mu)$$

and

$$K_\mu = \oplus_{\nu=1}^3 \int_0^t \mathrm{j}_t \tilde{\varphi}_\mu(\cdot - B_s) dB_s^\nu, \quad \tilde{\varphi}_\mu = (-ik_\mu \hat{\varphi}/\sqrt{\omega})^\vee,$$
$$K_{\mu\mu} = \oplus_{\nu=1}^3 \int_0^t \mathrm{j}_t \tilde{\varphi}_{\mu\mu}(\cdot - B_s) dB_s^\nu, \quad \tilde{\varphi}_{\mu\mu} = (-k_\mu^2 \hat{\varphi}/\sqrt{\omega})^\vee.$$

Since we assumed that $\omega^{3/2}\hat{\varphi} \in L^2(\mathbb{R}^d)$, it follows that $k_\mu^2\hat{\varphi}/\sqrt{\omega} \in L^2(\mathbb{R}^d)$ and thus $\tilde{\varphi}_{\mu\mu}$ is well defined. Hence

$$|(-\Delta F, e^{-t\hat{H}_{\mathrm{PF}}} G)| \le \|F\| \left(\int_{\mathbb{R}^3} dx \, |\mathbb{E}\,[\|U \mathrm{J}_t G(B_t + x)\|]|^2 \right)^{1/2}. \tag{7.4.20}$$

We estimate $\mathbb{E}\,[\|U \mathrm{J}_t G(B_t + x)\|]$. By the BDG-type inequality again we have

$$\sup_{x\in\mathbb{R}^3} \mathbb{E}^x[\|\hat{\omega}_{\mathrm{E}}^{n/2} K_{\mu\mu}\|^{2m}] \le C' t^m \|\omega^{(n+3)/2}\hat{\varphi}\|^{2m} \tag{7.4.21}$$

and

$$\sup_{x\in\mathbb{R}^3} \mathbb{E}^x[\|\hat{\omega}_{\mathrm{E}}^{n/2} K_{\mu}\|^{2m}] \le C'' t^m \|\omega^{(n+1)/2}\hat{\varphi}\|^{2m} \tag{7.4.22}$$

with constants C' and C''. Let $\|f\|_\alpha = \|\omega^{\alpha/2} f\|$ and

$$A = \|f\|_{-1} + \|f\|_0, \quad B = (\|f\|_{-1} + \|f\|_0)(\|g\|_1 + \|g\|_2).$$

Combining the inequalities

$$\|a^\sharp(f)\Psi\| \le A\|(H_{\mathrm{rad}} + 1)^{1/2}\Psi\|,$$
$$\|a^\sharp(f)a^\sharp(g)\Psi\| \le C_1 B\|(H_{\mathrm{rad}} + 1)^{1/2}\Psi\| + C_2 A\|(H_{\mathrm{rad}} + 1)\Psi\|$$

with (7.4.21) and (7.4.22), we can estimate the right-hand side of (7.4.20) and show (7.4.19). By a limiting argument, the lemma follows for $F \in D(\Delta)$ and $G \in D_{\mathrm{PF}}$. □

Lemma 7.24. *Take Assumption* 7.1 *and* $\omega^{3/2}\hat{\varphi} \in L^2(\mathbb{R}^3)$. *Then* H_{PF} *is essentially self-adjoint on* D_{PF}.

Proof. By Lemmas 7.22 and 7.23 we obtain that $|(H_{\mathrm{PF},0}F, e^{-t\hat{H}_{\mathrm{PF}}}G)| \le C_G\|F\|$ for $F, G \in D(H_{\mathrm{PF},0}) = D_{\mathrm{PF}}$ with a constant C_G depending on G. Then $e^{-t\hat{H}_{\mathrm{PF}}}$ leaves D_{PF} invariant. This implies that $\hat{H}_{\mathrm{PF}}$ is essentially self-adjoint on D_{PF} and then so is H_{PF} since $H_{\mathrm{PF}} = \hat{H}_{\mathrm{PF}}$ on D_{PF}. □

Next we show self-adjointness of H_{PF} on D_{PF} without assuming $\omega^{3/2}\hat{\varphi} \in L^2(\mathbb{R}^d)$.

Lemma 7.25. *Under Assumption* 7.1 *and with* $\omega^{3/2}\hat{\varphi} \in L^2(\mathbb{R}^3)$*,* $H_{\mathrm{PF},0}(\hat{H}_{\mathrm{PF}} + z)^{-1}$ *is bounded for every* $z \in \mathbb{C}$ *with* $\operatorname{Im} z \neq 0$.

Proof. We prove that both of $\Delta(\hat{H}_{\mathrm{PF}} + z)^{-1}$ and $H_{\mathrm{rad}}(\hat{H}_{\mathrm{PF}} + z)^{-1}$ are bounded. Note that $H_{\mathrm{PF}} = \hat{H}_{\mathrm{PF}}$ on D_{PF}.

Let $p_\mu = -i\nabla_\mu$. In this proof all numbered constants are non-negative. We have

$$\|p^2\Psi\|^2 \le C_1(\|(p - e\mathscr{A})^2\Psi\|^2 + \|\mathscr{A}\cdot(p - e\mathscr{A})\Psi\|^2 + \|\mathscr{A}^2\Psi\|^2).$$

We estimate $\|\mathscr{A}\cdot(p - e\mathscr{A})\Psi\|^2$ as

$$\begin{aligned}\|\mathscr{A}\cdot(p - e\mathscr{A})\Psi\|^2 &\le C_2\sum_{\mu=1}^{3}((p - e\mathscr{A})_\mu\Psi, (H_{\mathrm{rad}} + 1)(p - e\mathscr{A})_\mu\Psi)\\ &= C_2((p - e\mathscr{A})^2\Psi, (H_{\mathrm{rad}} + 1)\Psi) + C_2\sum_{\mu=1}^{3}((p - e\mathscr{A})_\mu\Psi, -e[H_{\mathrm{rad}}, \mathscr{A}_\mu]\Psi).\end{aligned}$$

Since $\|[H_{\mathrm{rad}}, \mathscr{A}_\mu]\Psi\| \le C_2'\|(H_{\mathrm{rad}} + 1)^{1/2}\Psi\|$, we have

$$\|p^2\Psi\|^2 \le C_3(\|(p - e\mathscr{A})^2\Psi\|^2 + \|H_{\mathrm{rad}}\Psi\|^2 + \|\Psi\|^2). \tag{7.4.23}$$

We note that

$$\|\hat{H}_{\mathrm{PF}}\Psi\|^2 = \left\|\frac{1}{2}(p - e\mathscr{A})^2\Psi\right\|^2 + \|H_{\mathrm{rad}}\Psi\|^2 + \operatorname{Re}((p - e\mathscr{A})^2\Psi, H_{\mathrm{rad}}\Psi).$$

Since

$$\begin{aligned}&\operatorname{Re}((p - e\mathscr{A})^2\Psi, H_{\mathrm{rad}}\Psi)\\ &= \sum_{\mu=1}^{3}\{((p - e\mathscr{A})_\mu\Psi, H_{\mathrm{rad}}(p - e\mathscr{A})_\mu\Psi) + \operatorname{Re}((p - e\mathscr{A})_\mu\Psi, -e[H_{\mathrm{rad}}, \mathscr{A}_\mu]\Psi)\}\\ &\ge \sum_{\mu=1}^{3}\operatorname{Re}((p - e\mathscr{A})_\mu\Psi, -e[H_{\mathrm{rad}}, \mathscr{A}_\mu]\Psi)\\ &\ge -\varepsilon\|(p - e\mathscr{A})^2\Psi\|^2 - C_4\frac{1}{4\varepsilon}(\|H_{\mathrm{rad}}^{1/2}\Psi\|^2 + \|\Psi\|^2)\end{aligned}$$

for any $\varepsilon > 0$, by the trivial bound $\|H_{\mathrm{rad}}^{1/2}\Psi\|^2 \le \eta\|H_{\mathrm{rad}}\Psi\|^2 + \frac{1}{4\eta}\|\Psi\|^2$ for any $\eta > 0$, we further have

$$\|\hat{H}_{\mathrm{PF}}\Psi\|^2 \ge C_5\|(p - e\mathscr{A})^2\Psi\|^2 + C_6\|H_{\mathrm{rad}}\Psi\|^2 - C_7\|\Psi\|^2. \tag{7.4.24}$$

Together with (7.4.23) this gives

$$\|p^2\Psi\|^2 + \|H_{\mathrm{rad}}\Psi\|^2 \le C_8(\|\hat{H}_{\mathrm{PF}}\Psi\|^2 + \|\Psi\|^2). \tag{7.4.25}$$

Thus $\Delta(\hat{H}_{\mathrm{PF}} + z)^{-1}$ and $H_{\mathrm{rad}}(\hat{H}_{\mathrm{PF}} + z)^{-1}$ are bounded. □

Theorem 7.26 (Self-adjointness). *Under Assumption* 7.1

(1) *H_{PF} is self-adjoint on D_{PF} and essentially self-adjoint on any core of $H_{\mathrm{PF},0}$*

(2) *the functional integral representation*

$$(F, e^{-tH_{\mathrm{PF}}}G) = \int_{\mathbb{R}^3} dx\mathbb{E}^x[e^{-\int_0^t V(B_s)ds}(\mathrm{J}_0F(B_0), e^{-ie\mathscr{A}_{\mathrm{E}}(K_t)}\mathrm{J}_tG(B_t))] \tag{7.4.26}$$

holds.

Proof. First we assume that $\omega^{3/2}\hat{\varphi} \in L^2(\mathbb{R}^d)$. Take $V = 0$. By Lemma 7.25,

$$\|H_{\mathrm{PF},0}\Psi\| \le C'\|(\hat{H}_{\mathrm{PF}} + z)\Psi\| \tag{7.4.27}$$

holds with some C' for all $\Psi \in D(\hat{H}_{\mathrm{PF}})$. Let $\{\Psi_n\}_{n=1}^\infty \subset D(H_{\mathrm{PF},0})$ be such that $\Psi_n \to \Psi$ and $\hat{H}_{\mathrm{PF}}\Psi_n \to \Phi$ for some Φ in strong sense as $n \to \infty$. Note that $D(\hat{H}_{\mathrm{PF}}) \supset D(H_{\mathrm{PF},0}) = D_{\mathrm{PF}}$. Due to (7.4.27), $(H_{\mathrm{PF},0}\Psi_n)_n$ is a Cauchy sequence and then $\Psi \in D(H_{\mathrm{PF},0})$ since $H_{\mathrm{PF},0}$ is closed. In particular, $\hat{H}_{\mathrm{PF}}$ is thus closed on D_{PF}. This implies that $H_{\mathrm{PF}}\lceil_{D_{\mathrm{PF}}}$ is closed and H_{PF} is self-adjoint since we already proved in Lemma 7.24 that $H_{\mathrm{PF}}\lceil_{D_{\mathrm{PF}}}$ is essentially self-adjoint and $\hat{H}_{\mathrm{PF}} = H_{\mathrm{PF}}$ on D_{PF}.

Next let $V \ne 0$. As mentioned in Remark 7.2, V is relatively bounded with respect to $\hat{H}_{\mathrm{PF}}$ with a relative bound strictly smaller than 1. Self-adjointness of H_{PF} on D_{PF} follows then by the Kato–Rellich theorem.

Finally we remove the assumption $\omega^{3/2}\hat{\varphi} \in L^2(\mathbb{R}^d)$. We divide $\hat{\varphi}$ into

$$\hat{\varphi} = \hat{\varphi}_\Lambda + \hat{\varphi}_{\Lambda^\perp},$$

where $\hat{\varphi}_\Lambda = \hat{\varphi}1_\Lambda$ and $\hat{\varphi}_{\Lambda^\perp} = \hat{\varphi} - \hat{\varphi}_\Lambda$ with the indicator function 1_Λ on $|k| \le \Lambda$. Write $\mathscr{A} = \mathscr{A}_\Lambda + \mathscr{A}_{\Lambda^\perp}$, where $\mathscr{A}_\Lambda$ has the form factor $\hat{\varphi}_\Lambda$ instead of $\hat{\varphi}$, and $\mathscr{A}_{\Lambda^\perp}$ does $\hat{\varphi}_{\Lambda^\perp}$. Then H_{PF} can be split off like $H_{\mathrm{PF}} = H_{\mathrm{PF}}(\Lambda) + H_{\Lambda^\perp}$, where

$$H_{\mathrm{PF}}(\Lambda) = \frac{1}{2}(p - e\mathscr{A}_\Lambda)^2 + V + H_{\mathrm{rad}}, \quad H_{\Lambda^\perp} = -e\mathscr{A}_{\Lambda^\perp}\cdot(p - e\mathscr{A}_\Lambda) + \frac{e^2}{2}\mathscr{A}_{\Lambda^\perp}^2.$$

Since $\omega^{3/2}\hat{\varphi}_\Lambda \in L^2(\mathbb{R}^d)$, $H_{\mathrm{PF}}(\Lambda)$ is self-adjoint on D_{PF}. We shall show below that $H_{\Lambda^\perp}$ is relatively bounded with respect to $H_{\mathrm{PF}}(\Lambda)$ with a relative bound strictly smaller than one for the large Λ. The estimate is similar to the proof of Lemma 7.25. Since for any $\delta > 0$,

$$\|e\mathscr{A}_{\Lambda^\perp} \cdot (p - e\mathscr{A}_\Lambda)\Psi\| \le \delta\|(p - e\mathscr{A}_\Lambda)^2\Psi\| + \frac{e^2}{2\delta}\|\mathscr{A}^2_{\Lambda^\perp}\Psi\|,$$

we have

$$\|H_{\Lambda^\perp}\Psi\| \le \delta\|(p - e\mathscr{A}_\Lambda)^2\Psi\| + e^2\left(\frac{1}{2\delta} + 1\right)\|\mathscr{A}^2_{\Lambda^\perp}\Psi\|.$$

Similarly to the estimate in the proof of Lemma 7.25 we have for any $\varepsilon > 0$ and $\eta > 0$,

$$\begin{aligned} &\left\|\frac{1}{2}(p - e\mathscr{A}_\Lambda)^2\Psi\right\|^2 + \|H_{\text{rad}}\Psi\|^2 - \varepsilon\|(p - e\mathscr{A}_\Lambda)^2\Psi\|^2 \\ &\quad - \frac{C_1}{4\varepsilon}\left(\eta\|H_{\text{rad}}\Psi\|^2 + \frac{1}{4\eta}\|\Psi\|^2\right) \le \|H_{\text{PF}}(\Lambda)\Psi\|^2, \end{aligned} \tag{7.4.28}$$

where C_1 depends on $\|\sqrt{\omega}\hat{\varphi}_\Lambda\|$ and $\|\hat{\varphi}_\Lambda/\sqrt{\omega}\|$, but is independent of $\|\omega^{3/2}\hat{\varphi}_\Lambda\|$. In particular, $\lim_{\Lambda\to\infty} C_1 < \infty$. From this we have

$$\frac{1-4\varepsilon}{4}\|(p - e\mathscr{A}_\Lambda)^2\Psi\|^2 + \left(1 - C_1\frac{\eta}{4\varepsilon}\right)\|H_{\text{rad}}\Psi\|^2 \le \|H_{\text{PF}}(\Lambda)\Psi\|^2 + \frac{C_1}{16\varepsilon\eta}\|\Psi\|^2. \tag{7.4.29}$$

Let ε be sufficiently small and choose η such that $1 - \frac{C_1\eta}{4\varepsilon} > 0$. Thus we obtain that

$$\|(p - e\mathscr{A}_\Lambda)^2\Psi\|^2 \le \frac{4}{1-4\varepsilon}\|H_{\text{PF}}(\Lambda)\Psi\|^2 + \frac{4C_1}{16\eta\varepsilon(1-4\varepsilon)}\|\Psi\|^2.$$

Next we estimate $\|\mathscr{A}^2_{\Lambda^\perp}\Psi\|$. We have

$$\|\mathscr{A}^2_{\Lambda^\perp}\Psi\|^2 \le C_2(\|H_{\text{rad}}\Psi\|^2 + \|\Psi\|^2),$$

where C_2 depends on $\|\omega^{\alpha/2}\hat{\varphi}_{\Lambda^\perp}\|$, $\alpha = -2, -1, 0, 1$. Thus $\lim_{\Lambda\to\infty} C_2 = 0$ and by (7.4.29),

$$\|\mathscr{A}^2_{\Lambda^\perp}\Psi\|^2 \le \left(\frac{C_2}{1 - C_1\frac{\eta}{4\epsilon}}\right)\|H_{\text{PF}}(\Lambda)\Psi\|^2 + C_3\|\Psi\|^2.$$

Hence we are led to the bounds $\|(p - e\mathscr{A}_\Lambda)^2\Psi\|^2 \le C_4(\|H_{\text{PF}}(\Lambda)\Psi\|^2 + \|\Psi\|^2)$ and $\|\mathscr{A}^2_{\Lambda^\perp}\Psi\|^2 \le C_5(\|H_{\text{PF}}(\Lambda)\Psi\|^2 + \|\Psi\|^2)$, where $\lim_{\Lambda\to\infty} C_4 < \infty$ and $\lim_{\Lambda\to\infty} C_5 = 0$. From the bound

$$\|H_{\Lambda^\perp}\Psi\| \le \left(\delta C_4 + e^2\left(\frac{1}{2\delta} + 1\right)C_5\right)\|H_{\text{PF}}(\Lambda)\Psi\| + C_6\|\Psi\|$$

it follows that for sufficiently large Λ and small δ such that

$$\delta C_4 + e^2 \left(\frac{1}{2\delta} + 1 \right) C_5 < 1,$$

$H_{\Lambda^\perp}$ is relatively bounded with respect to $H_{\mathrm{PF}}(\Lambda)$ with a relative bound strictly smaller than 1. Thus H_{PF} is self-adjoint on D_{PF}. The inequality

$$\|H_{\mathrm{PF}}\Psi\| + \|\Psi\| \le C''(\|H_{\mathrm{PF},0}\Psi\| + \|\Psi\|) \tag{7.4.30}$$

can be derived directly without effort with some constant C''. Let D be a core of $H_{\mathrm{PF},0}$. For any $\Psi \in D(H_{\mathrm{PF}}) = D(H_{\mathrm{PF},0})$ there exists a sequence $\{\Psi_n\}_{n=1}^\infty \subset D$ such that $\Psi_n \to \Psi$ and $H_{\mathrm{PF},0}\Psi \to H_{\mathrm{PF},0}\Psi$. This implies that $(H_{\mathrm{PF}}\Psi_n)_n$ is a Cauchy sequence by (7.4.30). Thus $H_{\mathrm{PF}}\Psi_n$ converges to some Φ as $n \to \infty$. Since H_{PF} is closed, $\Phi = H_{\mathrm{PF}}\Psi$. Thus D is a core of H_{PF}. Since (7.4.26) with $e^{-tH_{\mathrm{PF}}}$ replaced by $e^{-t\hat{H}_{\mathrm{PF}}}$ is verified, (7.4.26) holds since $\hat{H}_{\mathrm{PF}} = H_{\mathrm{PF}}$. □

7.4.2 Positivity improving and uniqueness of the ground state

The main question addressed in this section is whether the ground state of the Pauli–Fierz Hamiltonian is unique. As seen earlier on, a way of proving this is through the Perron–Frobenius theorem, relying on the positivity improving property of the semigroup. However, in the functional integral representation of $e^{-tH_{\mathrm{PF}}}$ the phase factor $e^{-ie\mathscr{A}_{\mathrm{E}}(K_t)}$ appears, therefore intuition would say that the semigroup doe not even preserve realness. Nevertheless, we can show that this semigroup is unitary equivalent with a positivity improving operator, and that will allow us to conclude that the ground state of H_{PF} is unique whenever it exists.

Consider the multiplication operator $T_t = e^{itx}$, $t \in \mathbb{R}$, in $L^2(\mathbb{R})$ with respect to x. Although T_t is not a positivity preserving operator, the operator $\mathbb{F}T_t\mathbb{F}^{-1}$, obtained under Fourier transform $\mathbb{F}$ on $L^2(\mathbb{R})$, is a shift operator, i.e.

$$(f, \mathbb{F}T_t\mathbb{F}^{-1}g) = (f, g(\cdot + t)) \ge 0$$

for non-negative functions f and g. This implies that $\mathbb{F}T_t\mathbb{F}^{-1}$ is a positivity preserving operator but not a positivity improving operator. We extend this argument to the Pauli–Fierz model by introducing a transformation on $L^2(\mathscr{Q})$ corresponding to Fourier transform on $L^2(\mathbb{R})$.

We return to the Fock space $\mathscr{F}_{\mathrm{b}}$ in Chapter 5, in which we defined the Segal field $\Phi(f)$ and its conjugate $\Pi(f)$. Since for $\alpha \in \mathbb{R}$,

$$e^{i\alpha N}a^*(f)e^{-i\alpha N} = a^*(e^{i\alpha}f),$$
$$e^{i\alpha N}a(f)e^{-i\alpha N} = a(e^{-i\alpha}f)$$

in particular for $\alpha = \pi/2$ we have

$$e^{i(\pi/2)N} a^*(f) e^{-i(\pi/2)N} = i a^*(f),$$
$$e^{i(\pi/2)N} a(f) e^{-i(\pi/2)N} = -i a(f).$$

Thus Φ and Π are related by the unitary operator $e^{i(\pi/2)N}$ as

$$e^{i(\pi/2)N} \Phi(f) e^{-i(\pi/2)N} = \Pi(f). \tag{7.4.31}$$

(7.4.31) may suggest that $e^{i(\pi/2)N}$ can be regarded as a Fourier transform on $\mathscr{F}_{\mathrm{b}}$.

By the argument above we define the unitary operator $\mathfrak{S}_\beta$ on $L^2(\mathscr{Q}_\beta)$ by

$$\mathfrak{S}_\beta = \exp\left(-i\frac{\pi}{2}N_\beta\right), \quad \beta = 0, 1, \tag{7.4.32}$$

where N_β denotes the number operator in $L^2(\mathscr{Q}_\beta)$. We simply write $\mathfrak{S}_\beta$ for the unitary operator $1 \otimes \mathfrak{S}_\beta$ on $L^2(\mathbb{R}^3) \otimes L^2(\mathscr{Q}_\beta)$ as long as there is no risk of confusion. We have

$$\Pi_\beta(\mathbf{f}) = \mathfrak{S}_\beta^{-1} \mathscr{A}_\beta(\mathbf{f}) \mathfrak{S}_\beta \tag{7.4.33}$$

and write $\Pi_0 = \Pi$, $\Pi_1 = \Pi_{\mathrm{E}}$, $\mathfrak{S}_0 = \mathfrak{S}$ and $\mathfrak{S}_1 = \mathfrak{S}_{\mathrm{E}}$. Clearly, Π_{E} coincides with the expression under (7.4.9). It is plain that

$$[\mathscr{A}_\beta(\mathbf{f}), \Pi_\beta(\mathbf{g})] = 2i\,\mathrm{q}_\beta(\mathbf{f}, \mathbf{g})$$

and, importantly,

$$e^{i\Pi_\beta(\mathbf{g})} 1 = e^{-\mathrm{q}_\beta(\mathbf{g},\mathbf{g})} e^{-\mathscr{A}_\beta(\mathbf{g})} 1. \tag{7.4.34}$$

Moreover, the Weyl relation

$$e^{i\Pi_\beta(\mathbf{g})} e^{i\mathscr{A}_\beta(\mathbf{f})} = e^{i2\mathrm{q}_\beta(\mathbf{f},\mathbf{g})} e^{i\mathscr{A}_\beta(\mathbf{f})} e^{i\Pi_\beta(\mathbf{g})} \tag{7.4.35}$$

holds.

Lemma 7.27. *The unitary operator $e^{i\Pi(\mathbf{f})}$ is positivity preserving for any $\mathbf{f} \in \oplus^3 L^2_{\mathrm{real}}(\mathbb{R}^3)$.*

Proof. Let $F, G \in D_{\mathscr{Q}}$ (see (7.4.18)) such that

$$F = (2\pi)^{-n/2} \int_{\mathbb{R}^n} \hat{F}(k) e^{i\sum_{j=1}^n k_j \mathscr{A}(\mathbf{f}_j)} dk,$$
$$G = (2\pi)^{-m/2} \int_{\mathbb{R}^m} \hat{G}(k) e^{i\sum_{j=1}^m k_j \mathscr{A}(\mathbf{g}_j)} dk$$

with positive $F \in \mathscr{S}(\mathbb{R}^n)$ and $G \in \mathscr{S}(\mathbb{R}^m)$. Using the fact that

$$(e^{i\mathscr{A}(\mathbf{g}_1)}, e^{i\Pi(\mathbf{f})}e^{i\mathscr{A}(\mathbf{g}_2)}) = (e^{i\mathscr{A}(\mathbf{g}_1-\mathbf{g}_2)}, e^{i\Pi(\mathbf{f})})e^{2i\mathrm{q}(\mathbf{g}_2,\mathbf{f})},$$

it is directly seen that

$$(F, e^{i\Pi(\mathbf{f})}G) = \frac{1}{(2\pi)^{n+m}}\int_{\mathbb{R}^{n+m}} dkdk'\overline{\hat{F}(k)}\hat{G}(k')\exp\Big(i\sum_{j=1}^n \mathrm{q}(\mathbf{f}_j,\mathbf{f})k_j\Big) \\ \times \exp\Big(i\sum_{j=1}^m \mathrm{q}(\mathbf{g}_j,\mathbf{f})k'_j\Big)\exp\Big(-\frac{1}{2}\mathrm{q}(\mathbf{f},\mathbf{f})\Big)\mathscr{G}(k,k'),$$

where

$$\mathscr{G}(k,k') = \exp\Big(-\frac{1}{2}\sum_{i,j=1}^n\sum_{i',j'=1}^m \mathrm{q}(k_i\mathbf{f}_i - k'_{i'}\mathbf{g}_{i'}, k_j\mathbf{f}_j - k'_{j'}\mathbf{g}_{j'})\Big).$$

Hence $(F, e^{i\Pi(\mathbf{f})}G) \geq 0$. Since any nonnegative function can be approximated by functions in $D_{\mathscr{Q}}$, the lemma follows. □

Note the intertwining property $\mathrm{J}_t\mathfrak{S} = \mathfrak{S}_{\mathrm{E}}\mathrm{J}_t$ and $\mathfrak{S}_{\mathrm{E}}^{-1}e^{-ie\mathscr{A}_{\mathrm{E}}(K_t)}\mathfrak{S}_{\mathrm{E}} = e^{-ie\Pi_{\mathrm{E}}(K_t)}$. Using the functional integral representation (7.3.12), we have

$$(F, \mathfrak{S}^{-1}e^{-tH_{\mathrm{PF}}}\mathfrak{S}G) = \int_{\mathbb{R}^3} dx\mathbb{E}^x[e^{-\int_0^t V(B_s)ds}(\mathrm{J}_0F(B_0), e^{-ie\Pi_{\mathrm{E}}(K_t)}\mathrm{J}_tG(B_s))]. \tag{7.4.36}$$

Although Proposition 5.8 shows that $e^{-tH_{\mathrm{rad}}}$ is positivity preserving, we can show a stronger statement below.

Proposition 7.28 (Positivity improving for $e^{-tH_{\mathrm{rad}}}$). *The semigroup $e^{-tH_{\mathrm{rad}}}$, $t > 0$, is positivity improving.*

This proposition implies that $\mathrm{J}_t^*\mathrm{J}_0$ is positivity improving. We show furthermore that $\mathrm{J}_t^*e^{ie\Pi_{\mathrm{E}}(K_t)}\mathrm{J}_0$ is positivity improving. The key factorization identity to achieving that is

$$\mathrm{J}_t^*e^{i\Pi_{\mathrm{E}}(\mathbf{f})}\mathrm{J}_0\Psi = cAB\Psi, \tag{7.4.37}$$

for $\Psi \in D_{\mathscr{Q}}$, where $c = e^{-(\mathrm{q}_{\mathrm{E}}(\mathbf{f},\mathbf{f})+\mathrm{q}(\mathrm{j}_0^*\mathbf{f},\mathrm{j}_0^*\mathbf{f}))}$ is a constant, $A = \mathrm{J}_t^*e^{-\mathscr{A}_{\mathrm{E}}(\mathbf{f})}\mathrm{J}_0$ and $B = e^{i\Pi(\mathrm{j}_0^*\mathbf{f})}e^{\mathscr{A}(\mathrm{j}_0^*\mathbf{f})}$. We will see below that there exists A_M and B_M such that $A \geq A_M$ and $B \geq B_M$, moreover, A_M is positivity improving and B_M positivity preserving, which shows that the closure of cAB is also positivity improving, since

$$c\overline{AB\lceil_{D_{\mathscr{Q}}}} \geq cA_MB_M$$

and $B_M G \neq 0$ if $G \neq 0$. First we show (7.4.37). Note that

$$\mathrm{J}_0 G = G(\mathscr{A}_{\mathrm{E}}(\mathrm{j}_0\mathbf{g}_1), \ldots, \mathscr{A}_{\mathrm{E}}(\mathrm{j}_0\mathbf{g}_m))1.$$

By the commutation relations,

$$e^{i\Pi_{\mathrm{E}}(\mathbf{f})}\mathrm{J}_0 G1 = G(\mathscr{A}_{\mathrm{E}}(\mathrm{j}_0\mathbf{g}_1) + 2\mathrm{q}_{\mathrm{E}}(\mathrm{j}_0\mathbf{g}_1, \mathbf{f}), \ldots, \mathscr{A}_{\mathrm{E}}(\mathrm{j}_0\mathbf{g}_m) + 2\mathrm{q}_{\mathrm{E}}(\mathrm{j}_0\mathbf{g}_m, \mathbf{f}))e^{i\Pi_{\mathrm{E}}(\mathbf{f})}1.$$

From $e^{i\Pi_{\mathrm{E}}(\mathbf{f})}1 = e^{-\mathrm{q}_{\mathrm{E}}(\mathbf{f},\mathbf{f})}e^{-\mathscr{A}_{\mathrm{E}}(\mathbf{f})}1$ it then follows that

$$\begin{aligned} &e^{i\Pi_{\mathrm{E}}(\mathbf{f})}\mathrm{J}_0 G1 \\ &= e^{-\mathrm{q}_{\mathrm{E}}(\mathbf{f},\mathbf{f})}e^{-\mathscr{A}_{\mathrm{E}}(\mathbf{f})}G(\mathscr{A}_{\mathrm{E}}(\mathrm{j}_0\mathbf{g}_1) + 2\mathrm{q}_{\mathrm{E}}(\mathrm{j}_0\mathbf{g}_1, \mathbf{f}), \ldots, \mathscr{A}_{\mathrm{E}}(\mathrm{j}_0\mathbf{g}_m) + 2\mathrm{q}_{\mathrm{E}}(\mathrm{j}_0\mathbf{g}_m, \mathbf{f}))1 \end{aligned}$$

and from $\mathrm{q}_{\mathrm{E}}(\mathrm{j}_0\mathbf{g}, \mathbf{f}) = \mathrm{q}(\mathbf{g}, \mathrm{j}_0^*\mathbf{f})$, this is further

$$= e^{-\mathrm{q}_{\mathrm{E}}(\mathbf{f},\mathbf{f})}e^{-\mathscr{A}_{\mathrm{E}}(\mathbf{f})}G(\mathscr{A}_{\mathrm{E}}(\mathrm{j}_0\mathbf{g}_1) + 2\mathrm{q}(\mathbf{g}_1, \mathrm{j}_0^*\mathbf{f}), \ldots, \mathscr{A}_{\mathrm{E}}(\mathrm{j}_0\mathbf{g}_m) + 2\mathrm{q}(\mathbf{g}_m, \mathrm{j}_0^*\mathbf{f}))1. \tag{7.4.38}$$

Here

$$\widehat{\mathrm{j}_t^* h}(k) = \frac{2}{\sqrt{2\pi}}\int_{\mathbb{R}} \frac{|k|}{|k|^2 + |k_0|^2} e^{itk_0}\hat{h}(\mathbf{k})dk_0, \quad t \in \mathbb{R}.$$

The right-hand side of (7.4.38) can be written as

$$e^{-\mathrm{q}_{\mathrm{E}}(\mathbf{f},\mathbf{f})}e^{-\mathscr{A}_{\mathrm{E}}(\mathbf{f})}\mathrm{J}_0 e^{i\Pi(\mathrm{j}_0^*\mathbf{f})}Ge^{-i\Pi(\mathrm{j}_0^*\mathbf{f})}1.$$

Using again $e^{-i\Pi(\mathbf{f})}1 = e^{-\mathrm{q}_{\mathrm{E}}(\mathbf{f},\mathbf{f})}e^{\mathscr{A}(\mathbf{f})}1$ and noting that G commutes with $e^{\mathscr{A}(\mathrm{j}_0^*\mathbf{f})}$, finally we have

$$e^{i\Pi_{\mathrm{E}}(\mathbf{f})}\mathrm{J}_0 G1 = e^{-\mathrm{q}_{\mathrm{E}}(\mathbf{f},\mathbf{f}) - \mathrm{q}(\mathrm{j}_0^*\mathbf{f}, \mathrm{j}_0^*\mathbf{f})}e^{-\mathscr{A}_{\mathrm{E}}(\mathbf{f})}\mathrm{J}_0 e^{i\Pi(\mathrm{j}_0^*\mathbf{f})}e^{\mathscr{A}(\mathrm{j}_0^*\mathbf{f})}G1.$$

This gives (7.4.37). From this we have the operator identity

$$\mathrm{J}_t^* e^{i\Pi_{\mathrm{E}}(\mathbf{f})}\mathrm{J}_0 = e^{-(\mathrm{q}_{\mathrm{E}}(\mathbf{f},\mathbf{f}) + \mathrm{q}(\mathrm{j}_0^*\mathbf{f}, \mathrm{j}_0^*\mathbf{f}))}\overline{(\mathrm{J}_t^* e^{-\mathscr{A}_{\mathrm{E}}(\mathbf{f})}\mathrm{J}_0 e^{i\Pi(\mathrm{j}_0^*\mathbf{f})}e^{\mathscr{A}(\mathrm{j}_0^*\mathbf{f})})\lceil D_{\mathscr{Q}}}. \tag{7.4.39}$$

Note that $e^{-\mathscr{A}_{\mathrm{E}}(\mathbf{f})}$ and $e^{\mathscr{A}(\mathrm{j}_0^*\mathbf{f})}$ are unbounded.

Define the bounded functions obtained by the truncations

$$(e^{-\mathscr{A}_{\mathrm{E}}(\mathbf{f})})_M = \begin{cases} e^{-\mathscr{A}_{\mathrm{E}}(\mathbf{f})}, & e^{-\mathscr{A}_{\mathrm{E}}(\mathbf{f})} < M, \\ M, & e^{-\mathscr{A}_{\mathrm{E}}(\mathbf{f})} \geq M, \end{cases}$$

$$(e^{-\mathscr{A}(\mathrm{j}_0^*\mathbf{f})})_M = \begin{cases} e^{-\mathscr{A}(\mathrm{j}_0^*\mathbf{f})}, & e^{-\mathscr{A}(\mathrm{j}_0^*\mathbf{f})} < M, \\ M, & e^{-\mathscr{A}(\mathrm{j}_0^*\mathbf{f})} \geq M. \end{cases}$$

Lemma 7.29. *For $t > 0$, $\mathrm{J}_t^*(\mathrm{e}^{-\mathscr{A}_{\mathrm{E}}(\mathbf{f})})_M \mathrm{J}_0$ is positivity improving.*

Proof. It suffices to show that $(F, \mathrm{J}_t^*(\mathrm{e}^{-\mathscr{A}_{\mathrm{E}}(\mathbf{f})})_M \mathrm{J}_0 G) \neq 0$ for any non-negative F and G, but $F \not\equiv 0$ and $G \not\equiv 0$ since $\mathrm{J}_t^*(\mathrm{e}^{-\mathscr{A}_{\mathrm{E}}(\mathbf{f})})_M \mathrm{J}_0$ is positivity preserving. Suppose

$$(F, \mathrm{J}_t^*(\mathrm{e}^{-\mathscr{A}_{\mathrm{E}}(\mathbf{f})})_M \mathrm{J}_0 G) = ((e^{-\mathscr{A}_{\mathrm{E}}(\mathbf{f})})_M \mathrm{J}_t F, \mathrm{J}_0 G) = 0 \tag{7.4.40}$$

for some non-negative F and G. (7.4.40) implies that

$$\mu_E(\mathrm{supp}[(e^{-\mathscr{A}_{\mathrm{E}}(\mathbf{f})})_M \mathrm{J}_t F] \cap \mathrm{supp}[\mathrm{J}_0 G]) = 0. \tag{7.4.41}$$

As $\mathscr{A}_{\mathrm{E}}(\mathbf{f}) \in L^2(\mathscr{Q}_{\mathrm{E}})$, $\mu_E(\{\mathscr{A}_{\mathrm{E}} \in \mathscr{Q}_{\mathrm{E}} | (e^{-\mathscr{A}_{\mathrm{E}}(\mathbf{f})})_M = 0\}) = 0$, (7.4.41) implies that $\mu_E(\mathrm{supp}[\mathrm{J}_t F] \cap \mathrm{supp}[\mathrm{J}_0 G]) = 0$ and thus

$$0 = (\mathrm{J}_t F, \mathrm{J}_0 G) = (F, e^{-tH_{\mathrm{rad}}} G).$$

Since $e^{-tH_{\mathrm{rad}}}$ is positivity improving by Proposition 7.28, this is in contradiction with the assumption. □

Theorem 7.30 (Positivity improving for $e^{-tH_{\mathrm{PF}}}$). *The semigroup $\mathfrak{S}^{-1} e^{-tH_{\mathrm{PF}}} \mathfrak{S}$ is positivity improving for $t > 0$.*

Proof. Let f be real valued. Notice that if $f \geq g$, then $Tf \geq Tg$ for any positivity preserving operator T. For a non-negative $G \in D_{\mathscr{Q}}$,

$$\mathrm{J}_t^* e^{i\Pi_{\mathrm{E}}(\mathbf{f})} \mathrm{J}_0 G \geq e^{-(\mathrm{q}_{\mathrm{E}}(\mathbf{f},\mathbf{f}) + \mathrm{q}(\mathrm{j}_0^*\mathbf{f}, \mathrm{j}_0^*\mathbf{f}))} \mathrm{J}_t^*(e^{-\mathscr{A}_{\mathrm{E}}(\mathbf{f})})_M \mathrm{J}_0 e^{i\Pi(\mathrm{j}_0^*\mathbf{f})} (e^{\mathscr{A}(\mathrm{j}_0^*\mathbf{f})})_M G = S_M G.$$

Since S_M is bounded and any non-negative function can be approximated by a function in $D_{\mathscr{Q}}$, we have for an arbitrary non-negative G,

$$\mathrm{J}_t^* e^{i\Pi_{\mathrm{E}}(\mathbf{f})} \mathrm{J}_0 G \geq S_M G. \tag{7.4.42}$$

Since $\mathrm{J}_t^*(e^{-\mathscr{A}_{\mathrm{E}}(\mathbf{f})})_M \mathrm{J}_0$ is positivity improving by Lemma 7.30 and $e^{i\Pi(\mathrm{j}_0^*\mathbf{f})}(e^{\mathscr{A}(\mathrm{j}_0^*\mathbf{f})})_M$ is positivity preserving with $e^{i\Pi(\mathrm{j}_0^*\mathbf{f})}(e^{\mathscr{A}(\mathrm{j}_0^*\mathbf{f})})_M G \not\equiv 0$ for $G \geq 0$ but $G \not\equiv 0$, S_M is positivity improving and so is $\mathrm{J}_t^* e^{i\Pi_{\mathrm{E}}(\mathbf{f})} \mathrm{J}_0$. In particular, $\mathfrak{S}^{-1} \mathbf{J}_t^* \mathfrak{S}$ is positivity improving as well. Let $F, G \geq 0$, not identically vanishing. Note that

$$(F, \mathfrak{S}^{-1} e^{-tH_{\mathrm{PF}}} \mathfrak{S} G) = \int_{\mathbb{R}^3} dx \mathbb{E}^x [e^{-\int_0^t V(B_s) ds} (\mathfrak{S}^{-1} \mathbf{J}_t^* \mathfrak{S} F(B_0), G(B_t))]. \tag{7.4.43}$$

We show that (7.4.43) is strictly positive. Let $D_G = \{x \in \mathbb{R}^3 | G(x, \cdot) \not\equiv 0\}$, $D_F = \{x \in \mathbb{R}^3 | F(x, \cdot) \not\equiv 0\}$ and

$$D_{FG} = \{\omega \in \mathscr{X} | B_0(\omega) \in D_F, B_t(\omega) \in D_G\}.$$

In order to complete the proof of the theorem it suffices to see that the measure of $D_{FG} \subset \mathscr{X}$ is positive as $\mathfrak{S}^{-1}\mathbf{J}_t\mathfrak{S}$ is positivity improving. We have

$$\begin{aligned}\int dx \mathbb{E}^x[1_{D_{FG}}] &= \int_{D_F} dx \mathbb{E}^x[1_{D_G}(B_t)] \\ &= (2\pi t)^{-3/2} \int_{D_F} dx \int_{D_G} e^{-|x-y|^2/2t} dy > 0.\end{aligned}$$

Thus it is obtained that the right-hand side of (7.4.43) is strictly positive and the theorem follows. □

We proved that $\mathfrak{S}e^{-tH_{\mathrm{PF}}}\mathfrak{S}^{-1}$ is positivity improving. A direct consequence of this property is uniqueness of the ground state of H_{PF}.

Corollary 7.31 (Uniqueness of ground state). *The ground state Ψ_{g} of H_{PF} satisfies $\mathfrak{S}^{-1}\Psi_{\mathrm{g}} > 0$, and it is unique up to additive constants.*

Proof. It follows from Perron–Frobenius theorem (Theorem 3.54) and Theorem 7.30. □

By Theorem 7.30 again we have $(f \otimes 1, \mathfrak{S}\Psi_{\mathrm{g}}) \neq 0$. This allows to derive an expression of the ground state energy $\inf \mathrm{Spec}(H_{\mathrm{PF}})$ from which some of its properties follow. Write $E(e^2) = \inf \mathrm{Spec}(H_{\mathrm{PF}})$.

Theorem 7.32 (Concavity of ground state energy). *The function $e^2 \mapsto E(e^2)$ is monotonously increasing, continuous and concave.*

Proof. We know that H_{PF} has a unique ground state Ψ_{g} for any value of the coupling constant $e \in \mathbb{R}$. Since $0 \neq (f \otimes 1, \mathfrak{S}^{-1}\Psi_{\mathrm{g}}) = (f \otimes 1, \Psi_{\mathrm{g}})$ for any $e \in \mathbb{R}$ and not identically zero $f \geq 0$, we have

$$\begin{aligned}E(e^2) &= \lim_{t\to\infty}\left(-\frac{1}{t}\log(f \otimes 1, e^{-tH_{\mathrm{PF}}} f \otimes 1)\right) \\ &= \lim_{t\to\infty}\left(-\frac{1}{t}\log\int_{\mathbb{R}^3} dx \mathbb{E}^x[f(B_0)f(B_t)e^{-\int_0^t V(B_s)ds}e^{-(e^2/2)\mathrm{q_E}(K_t,K_t)}]\right).\end{aligned}$$

As $e^{-(e^2/2)\mathrm{q_E}(K_t,K_t)}$ is log-convex in e^2, $E(\cdot)$ is concave. Thus $E(e^2)$ is continuous on $(0,\infty)$. Since $E(e^2)$ is continuous at $e = 0$, $E \in C(\mathbb{R}^+)$ and $E(e^2)$ can be expressed as $E(e^2) = \int_0^{e^2} \rho(t)dt$ with a suitable positive function $\rho(t)$. This implies that $E(e^2)$ is monotonously increasing in e^2. □

7.4.3 Spatial decay of the ground state

Similarly to the Nelson model in the previous chapter we can derive the spatial exponential decay of the ground state Ψ_{g} of the Pauli–Fierz Hamiltonian. Recall the class $\mathbb{V}^{\mathrm{upper}}$ defined in Definition 3.26, and let $E_{\mathrm{PF}} = \inf \mathrm{Spec}(H_{\mathrm{PF}})$. We have the following Carmona-type estimate for the Pauli–Fierz Hamiltonian.

Lemma 7.33. *If $V \in \mathbb{V}^{\mathrm{upper}}$, then for any $t, a > 0$ and every $0 < \alpha < 1/2$, there exist constants $D_1, D_2, D_3 > 0$ such that*

$$\|\Psi_{\mathrm{g}}(x)\|_{L^2(\mathscr{Q})} \le D_1 e^{D_2 \|U\|_p t} e^{E_{\mathrm{PF}} t} (D_3 e^{-\frac{\alpha}{4}\frac{a^2}{t}} e^{-tW_\infty} + e^{-tW_a(x)}) \|\Psi_{\mathrm{g}}\|_{\mathscr{H}_{\mathrm{PF}}}, \tag{7.4.44}$$

where $W_a(x) = \inf\{W(y) | |x-y| < a\}$.

Proof. Since $\Psi_{\mathrm{g}} = e^{tE_{\mathrm{PF}}} e^{-tH_{\mathrm{PF}}} \Psi_{\mathrm{g}}$, the functional integral representation yields

$$\Psi_{\mathrm{g}}(x) = \mathbb{E}^x [\mathrm{J}_0^* e^{-ie\mathscr{A}_{\mathrm{E}}(K_t)} \mathrm{J}_t e^{-\int_0^t V(B_s) ds} \Psi_{\mathrm{g}}(B_t)]. \tag{7.4.45}$$

From this we directly obtain that

$$\|\Psi_{\mathrm{g}}(x)\|_{L^2(\mathscr{Q})} \le \mathbb{E}^x [e^{-\int_0^t V(B_s) ds} \|\Psi_{\mathrm{g}}(B_t)\|_{L^2(\mathscr{Q})}]. \tag{7.4.46}$$

Then the lemma follows in the similar way to Carmona's estimate, Lemma 3.59. □

Remark 7.3. In the Carmona estimate for the Nelson model the infrared regular condition $I = \int_{\mathbb{R}^3} |\hat{\varphi}(k)|^2/\omega(k)^3 dk < \infty$ is assumed and $e^{I/4}$ appears in the upper bound of $\|\Psi_{\mathrm{g}}(x)\|$, see Lemma 6.15. For the Pauli–Fierz model, however, $I < \infty$ is not needed and does not appear in the bound of $\|\Psi_{\mathrm{g}}(x)\|$. Furthermore, the ground state of the Pauli–Fierz Hamiltonian exists even in the case of $I = \infty$.

This lemma implies a similar result of decay of the ground state as for Schrödinger operators obtained in Corollaries 3.60 and 3.61. For $V = W - U \in \mathbb{V}^{\mathrm{upper}}$, recall that $\Sigma = \liminf_{|x|\to\infty} V(x)$. We only state the results.

Corollary 7.34 (Exponential decay of ground state). *Let $V = W - U \in \mathbb{V}^{\mathrm{upper}}$.*

(1) *Suppose that $W(x) \ge \gamma |x|^{2n}$ outside a compact set K, for some $n > 0$ and $\gamma > 0$. Take $0 < \alpha < 1/2$. Then there exists a constant $C_1 > 0$ such that*

$$\|\Psi_{\mathrm{g}}(x)\|_{L^2(\mathscr{Q})} \le C_1 \exp\left(-\frac{\alpha c}{16} |x|^{n+1}\right) \|\Psi_{\mathrm{g}}\|_{\mathscr{H}_{\mathrm{PF}}}. \tag{7.4.47}$$

where $c = \inf_{x \in \mathbb{R}^3 \setminus K} W_{|x|/2}(x)/|x|^{2n}$.

(2) *Decaying potential: Suppose that* $\Sigma > E_{\rm PF}$, $\Sigma > W_\infty$, *and Let* $0 < \beta < 1$. *Then there exists a constant* $C_2 > 0$ *such that*

$$\|\Psi_{\rm g}(x)\|_{L^2(\mathscr{Q})} \leq C_2 \exp\left(-\frac{\beta}{8\sqrt{2}} \frac{(\Sigma - E_{\rm PF})}{\sqrt{\Sigma - W_\infty}}|x|\right) \|\Psi_{\rm g}\|_{\mathscr{H}_{\rm PF}}. \tag{7.4.48}$$

(3) *Confining potential: Suppose that* $\lim_{|x|\to\infty} W(x) = \infty$. *Then there exist constants* $C, \delta > 0$ *such that*

$$\|\Psi_{\rm g}(x)\|_{L^2(\mathscr{Q})} \leq C \exp(-\delta|x|) \|\Psi_{\rm g}\|_{\mathscr{H}_{\rm PF}}. \tag{7.4.49}$$

7.5 The Pauli–Fierz model with Kato class potential

Now we consider the Pauli–Fierz model with Kato class potential V. First we are interested in defining $H_{\rm PF}$ with Kato class potential as a self-adjoint operator. This will be done through the functional integral representation established in the previous section. The expression

$$(F, e^{-tH_{\rm PF}}G) = \int_{\mathbb{R}^3} dx \mathbb{E}^x[e^{-\int_0^t V(B_s)ds}(\mathrm{J}_0 F(x), e^{-ie\mathscr{A}_{\rm E}(K_t)}\mathrm{J}_t G(B_t))].$$

implies

$$(e^{-tH_{\rm PF}}G)(x) = \mathbb{E}^x[e^{-\int_0^t V(B_s)ds}\mathrm{J}_0^* e^{-ie\mathscr{A}_{\rm E}(K_t)}\mathrm{J}_t G(B_t)]. \tag{7.5.1}$$

Conversely, we shall show that a sufficient condition to define the right-hand side of (7.5.1) is that V is of Kato class. This idea used for Schrödinger operators with Kato class potentials will be extended to the Pauli–Fierz Hamiltonian in this section.

Let V be a Kato decomposable potential and define the family of operators

$$(T_t F)(x) = \mathbb{E}^x[e^{-\int_0^t V(B_r)dr}\mathrm{J}_0^* e^{-ie\mathscr{A}_{\rm E}(K_t)}\mathrm{J}_t F(B_t)]. \tag{7.5.2}$$

Lemma 7.35. *Let* V *be a Kato decomposable potential. Then* T_t *is bounded on* $\mathscr{H}_{\rm PF}$.

Proof. Let $F \in \mathscr{H}_{\rm PF}$. We show that

$$\|T_t F\|^2_{\mathscr{H}_{\rm PF}} = \int_{\mathbb{R}^3} dx \|\mathbb{E}^x[e^{-\int_0^t V(B_r)dr}\mathrm{J}_0^* e^{-ie\mathscr{A}_{\rm E}(K_t)}\mathrm{J}_t F(B_t)]\|^2_{L^2(\mathscr{Q})} < \infty. \tag{7.5.3}$$

The left-hand side of (7.5.3) is bounded by Schwarz inequality,

$$\|T_t F\|^2_{\mathscr{H}_{\rm PF}} \leq \int_{\mathbb{R}^3} dx \mathbb{E}^0[e^{-2\int_0^t V(B_r+x)dr}]\mathbb{E}^0[\|F(B_t+x)\|^2]. \tag{7.5.4}$$

Since V is of Kato class we have $\sup_{x\in\mathbb{R}^3} \mathbb{E}^0[e^{-2\int_0^t V(B_r+x)dr}] = C < \infty$, and thus

$$\|T_t F\|^2_{\mathscr{H}_{\rm PF}} \leq C\|F\|^2_{\mathscr{H}_{\rm PF}}. \tag{7.5.5}$$

□

In what follows we show that $\{T_t : t \geq 0\}$ is a symmetric C_0-semigroup. To do that we introduce the time shift operator u_t on $L^2(\mathbb{R}^4)$ by

$$u_t f(x) = f(x_0 - t, \mathbf{x}), \quad x = (x_0, \mathbf{x}) \in \mathbb{R} \times \mathbb{R}^3. \tag{7.5.6}$$

It is straightforward that $u_t^* = u_{-t}$ and $u_t^* u_t = 1$. We denote the second quantization of u_t denoted by $U_t = \Gamma_{\mathrm{E}}(u_t)$ which acts on $L^2(\mathscr{Q}_{\mathrm{E}})$ and is a unitary map. The relationship between j_s and u_t is given next.

Lemma 7.36. *It follows that $u_t \mathrm{j}_s = \mathrm{j}_{s+t}$, for every $t, s \in \mathbb{R}$.*

Proof. In the position representation j_s is given by

$$\mathrm{j}_s f(x) = \frac{1}{\sqrt{\pi}(2\pi)^2} \int_{\mathbb{R}^4} e^{+i(k_0(x_0-s)+k\cdot\mathbf{x})} \frac{\sqrt{\omega(k)}}{\sqrt{\omega(k)^2 + |k_0|^2}} \hat{f}(k) d\mathbf{k}.$$

Hence $u_t \mathrm{j}_s f(x) = \mathrm{j}_{s+t} f(x)$. □

Lemma 7.36 implies the formula

$$U_t \mathrm{J}_s = \mathrm{J}_{s+t}. \tag{7.5.7}$$

Lemma 7.37. *Let $V \in \mathcal{K}(\mathbb{R}^3)$. Then $T_s T_t = T_{s+t}$ holds for $s, t \geq 0$.*

Proof. By the definition of T_t we have

$$T_s T_t F = \mathbb{E}^x [e^{-\int_0^s V(B_r)dr} \mathrm{J}_0^* e^{-ie\mathscr{A}_{\mathrm{E}}(K_s)} \mathrm{J}_s \mathbb{E}^{B_s} [e^{-\int_0^t V(B_r)dr} \mathrm{J}_0^* e^{-ie\mathscr{A}_{\mathrm{E}}(K_t)} \mathrm{J}_t F(B_t)]]. \tag{7.5.8}$$

By the formulae $\mathrm{J}_s \mathrm{J}_0^* = \mathrm{J}_s \mathrm{J}_s^* U_{-s}^* = \mathrm{E}_s U_{-s}^*$ and $\mathrm{J}_t = U_{-s} \mathrm{J}_{t+s}$, (7.5.8) is equal to

$$\mathbb{E}^x [e^{-\int_0^s V(B_r)dr} \mathrm{J}_0^* e^{-ie\mathscr{A}_{\mathrm{E}}(K_s)} \mathrm{E}_s \mathbb{E}^{B_s} [e^{-\int_0^t V(B_r)dr} U_{-s}^* e^{-ie\mathscr{A}_{\mathrm{E}}(K_t)} U_{-s} \mathrm{J}_{t+s} F(B_t)]]. \tag{7.5.9}$$

Now we compute $U_{-s}^* e^{-ie\mathscr{A}_{\mathrm{E}}(K_t)} U_{-s}$. Since U_s is unitary, we have

$$U_{-s}^* e^{-ie\mathscr{A}_{\mathrm{E}}(K_t)} U_{-s} = e^{-ie\mathscr{A}_{\mathrm{E}}(u_{-s}^* K_t)}$$

as an operator. The test function of the exponent $u_{-s}^* K_t$ is given by

$$u_{-s}^* K_t = \bigoplus_{\mu=1}^{3} \int_0^t \mathrm{j}_{r+s} \tilde{\varphi}(\cdot - B_r) dB_r^\mu.$$

Moreover by the Markov property of E_t, $t \in \mathbb{R}$, we may neglect E_s in (7.5.9), and by the Markov property of $(B_t)_{t\geq 0}$ we have

$$\begin{aligned}
&T_s T_t F \\
&= \mathbb{E}^x[e^{-\int_0^s V(B_r)dr} \mathrm{J}_0^* e^{-ie\mathscr{A}_{\mathrm{E}}(K_s)} \mathbb{E}^x[e^{-\int_s^{s+t} V(B_r)dr} e^{-ie\mathscr{A}_{\mathrm{E}}(K_s^{s+t})} \mathrm{J}_{s+t} F(B_{s+t})|\mathscr{F}_s]] \\
&= \mathbb{E}^x[e^{-\int_0^{s+t} V(B_r)dr} \mathrm{J}_0^* e^{-ie\mathscr{A}_{\mathrm{E}}(K_{s+t})} \mathrm{J}_{s+t} F(B_{s+t})] = T_{s+t} F,
\end{aligned}$$

where $(\mathscr{F}_t)_{t\geq 0}$ denotes the natural filtration of $(B_t)_{t\geq 0}$. □

Strong continuity of T_t on $\mathscr{H}_{\mathrm{PF}}$ can be checked similarly as previously done, while $T_0 = 1$ is trivial.

Theorem 7.38. *Let $V \in \mathcal{K}(\mathbb{R}^3)$. Under Assumption* 7.1 *$\{T_t : t \geq 0\}$ is a C_0-semigroup.*

By Theorem 7.38 there exists a self-adjoint operator $H_{\mathrm{PF}}^{\mathrm{Kato}}$ such that

$$T_t = e^{-tH_{\mathrm{PF}}^{\mathrm{Kato}}}, \quad t \geq 0. \tag{7.5.10}$$

Definition 7.8 (Pauli–Fierz Hamiltonian for Kato-class potential). We call the self-adjoint operator $H_{\mathrm{PF}}^{\mathrm{Kato}}$ *Pauli–Fierz Hamiltonian for Kato-class potential V.*

Proposition 7.39 (Diamagnetic inequality). *Let $V \in \mathcal{K}(\mathbb{R}^3)$ and take Assumption* 7.1. *Then it follows that for $F \in \mathscr{H}_{\mathrm{PF}}$,*

$$|e^{-tH_{\mathrm{PF}}^{\mathrm{Kato}}} F| \leq e^{-t(K(0)+H_{\mathrm{rad}})}|F|, \quad t \geq 0, \tag{7.5.11}$$

where $K(0)$ is the Schrödinger operator with Kato-class potential.

Proof. Since J_0^* and J_t are positivity preserving, we have $|\mathrm{J}_0^* F| \leq \mathrm{J}_0^*|F|$, $|\mathrm{J}_t F| \leq \mathrm{J}_t|F|$ and

$$|e^{-tH_{\mathrm{PF}}^{\mathrm{Kato}}} F| \leq \mathbb{E}^x[e^{-\int_0^t V(B_r)dr} \mathrm{J}_0^* \mathrm{J}_t |F(B_t)|].$$

The right-hand side above is just $e^{-t(K(0)+H_{\mathrm{rad}})}|F|$. □

7.6 Translation invariant Pauli–Fierz model

In this section we consider the translation invariant Pauli–Fierz Hamiltonian. This is obtained by setting the external potential V identically zero, resulting in the fact that H_{PF} commutes with the total momentum operator. A general argument allows then to decompose H_{PF} as $H_{\mathrm{PF}} = \int_{\mathbb{R}^3}^{\oplus} H_{\mathrm{PF}}(p)dp$. Moreover, for each $p \in \mathbb{R}^3$, the fiber Hamiltonian $H_{\mathrm{PF}}(p)$ is self-adjoint on $\mathscr{F}_{\mathrm{rad}}$. We will specify the exact form of $H_{\mathrm{PF}}(p)$

and its domain. Although H_{PF} with $V \equiv 0$ has no ground state, the fiber Hamiltonian $H_{\mathrm{PF}}(p)$ may have ground states and a functional integral representation of $e^{-tH_{\mathrm{PF}}(p)}$ can be established to study the spectrum of $H_{\mathrm{PF}}(p)$.

We start by defining the fiber Hamiltonian in the spinless case. As said, a standing assumption throughout this section is

$$V = 0. \tag{7.6.1}$$

Put

$$P_{\mathrm{f}\mu} = \sum_{j=\pm} \int k_\mu a^*(k,j) a(k,j) dk, \quad \mu = 1,2,3, \tag{7.6.2}$$

which describes the *field momentum*. The *total momentum operator* P on $\mathscr{H}_{\mathrm{PF}}$ is defined by the sum of the momentum operator for the particle and that of field:

$$\mathrm{P}_\mu = -i\nabla_{x_\mu} \otimes 1 + 1 \otimes P_{\mathrm{f}\mu}, \quad \mu = 1,2,3. \tag{7.6.3}$$

Recall that $\varrho_\mu^j(x) = \hat{\varphi}(k) e_\mu^j(k) e^{-ik\cdot x}/\sqrt{\omega(k)}$. Since

$$[-i\nabla_{x_\nu}, a^*(\varrho_\mu^j, j)] = a^*(-k_\nu \varrho_\mu^j, j),$$

$$[P_{\mathrm{f}\nu}, a^*(\varrho_\mu^j, j)] = a^*(k_\nu \varrho_\mu^j, j),$$

it follows

$$[H_{\mathrm{PF}}, \mathrm{P}_\mu] = 0, \quad \mu = 1,2,3. \tag{7.6.4}$$

This leads to a decomposition H_{PF} of the spectrum of the total momentum operator $\mathrm{Spec}(\mathrm{P}_\mu) = \mathbb{R}$, $\mu = 1,2,3$. We denote $\sum_{\mu=1}^3 P_{\mathrm{f}\mu}^2$ by P_{f}^2.

Definition 7.9 (Pauli–Fierz Hamiltonian with fixed total momentum). The *Pauli–Fierz Hamiltonian with a fixed total momentum* is

$$H_{\mathrm{PF}}(p) = \frac{1}{2}(p - P_{\mathrm{f}} - eA(0))^2 + H_{\mathrm{rad}}, \quad p \in \mathbb{R}^3,$$

with domain

$$D(H_{\mathrm{PF}}(p)) = D(H_{\mathrm{rad}}) \cap D(P_{\mathrm{f}}^2),$$

where $A_\mu(0) = A_\mu(x = 0)$ is given by

$$A_\mu(0) = \frac{1}{\sqrt{2}} \sum_{j=\pm} \int_{\mathbb{R}^3} e_\mu^j(k) \left(a^*(k,j) \frac{\hat{\varphi}(k)}{\sqrt{\omega(k)}} + a(k,j) \frac{\hat{\varphi}(-k)}{\sqrt{\omega(k)}} \right) dk.$$

Here $p \in \mathbb{R}^3$ is called *total momentum*.

It is easy to see that $H_{\rm PF}(p)$ is self-adjoint for sufficiently small coupling constants e as a consequence of the Kato–Rellich theorem. In fact, by splitting off $H_{\rm PF}(p)$ as

$$H_{\rm PF}(p) = H_{\rm PF,0}(p) + H_{\rm PF,I}(p), \tag{7.6.5}$$

with

$$H_{\rm PF,0}(p) = H_{\rm rad} + \frac{1}{2}(p - P_{\rm f})^2, \tag{7.6.6}$$

$$H_{\rm PF,I}(p) = -e(p - P_{\rm f}) \cdot A(0) + \frac{e^2}{2} A(0) \cdot A(0), \tag{7.6.7}$$

we readily see that $H_{\rm PF,I}(p)$ is relatively bounded with respect to $H_{\rm PF,0}(p)$.

We give the relationship between $H_{\rm PF}$ and $H_{\rm PF}(p)$. Define the unitary operator

$$\mathscr{T} : \mathscr{H}_{\rm PF} = L^2(\mathbb{R}^3_x) \otimes \mathscr{F}_{\rm rad} \to L^2(\mathbb{R}^3_p) \otimes \mathscr{F}_{\rm rad} \tag{7.6.8}$$

by

$$\mathscr{T} = (\mathbb{F} \otimes 1) \int_{\mathbb{R}^3}^{\oplus} \exp(ix \cdot P_{\rm f}) dx, \tag{7.6.9}$$

with $\mathbb{F}$ denoting Fourier transformation from $L^2(\mathbb{R}^3_x)$ to $L^2(\mathbb{R}^3_p)$. For $\Psi \in \mathscr{H}_{\rm PF}$,

$$(\mathscr{T}\Psi)(p) = \frac{1}{\sqrt{(2\pi)^3}} \int_{\mathbb{R}^3} e^{-ix\cdot p} e^{ix\cdot P_{\rm f}} \Psi(x) dx.$$

Let $M_\mu f(x) = x_\mu f(x)$, $\mu = 1, 2, 3$, be the multiplication by x_μ in $L^2(\mathbb{R}^3)$. Write $M = (M_1, M_2, M_3)$ and

$$\mathscr{F}_{\rm rad}^{\infty} = \text{L.H.}\left\{ \prod_{i=1}^{n} a^*(f_i, j_i)\Omega_{\rm PF} \,\middle|\, f_i \in C_0^\infty(\mathbb{R}^3),\, j_i = \pm, i = 1, \ldots, n, n \geq 1 \right\} \cup \{\Omega_{\rm PF}\}$$

and $D_\infty = C_0^\infty(\mathbb{R}^3) \hat{\otimes} \mathscr{F}_{\rm rad}^\infty$. We would like to define $H_{\rm PF}(p)$ for arbitrary values of $e \in \mathbb{R}$ and $p \in \mathbb{R}^3$ as a self-adjoint operator. As was mentioned above, for sufficiently small e, $H_{\rm PF}(p)$ is defined as a self-adjoint operator, however, for arbitrary values of e, this is not clear. Define

$$L = \overline{\frac{1}{2}(M \otimes 1 - 1 \otimes P_{\rm f} - e1 \otimes A(0))^2 + 1 \otimes H_{\rm rad} \lceil_{D_\infty}}. \tag{7.6.10}$$

We have $\mathscr{T} H_{\rm PF} \Phi = L \mathscr{T} \Phi$ for $\Phi \in D_\infty$. Since by Theorem 7.26 D_∞ is a core of $H_{\rm PF}$, and L is closed, it can be seen that $\mathscr{T}$ maps $D(H_{\rm PF})$ onto $D(L)$ with $\mathscr{T} H_{\rm PF} \mathscr{T}^{-1} = L$. Note also that

$$D(L) = \mathscr{T} D(H_{\rm PF}) = D((\overline{M \otimes 1 - 1 \otimes P_{\rm f}})^2) \cap D(1 \otimes H_{\rm rad}).$$

Theorem 7.40 (Self-adjointness). *Under Assumption* 7.1 $H_{\mathrm{PF}}(p)$ *is a non-negative self-adjoint operator, and*

$$\mathscr{T}\left(\int_{\mathbb{R}^3}^{\oplus}\mathscr{F}_{\mathrm{rad}}dp\right)\mathscr{T}^{-1}=\mathscr{H}_{\mathrm{PF}},\quad \mathscr{T}\left(\int_{\mathbb{R}^3}^{\oplus}H_{\mathrm{PF}}(p)dp\right)\mathscr{T}^{-1}=H_{\mathrm{PF}}. \tag{7.6.11}$$

Proof. Define the quadratic form

$$\mathrm{Q}_p(\Psi,\Phi)=\frac{1}{2}\sum_{\mu=1}^{3}((p-P_{\mathrm{f}}-eA(0))_\mu\Psi,(p-P_{\mathrm{f}}-eA(0))_\mu\Phi)+(H_{\mathrm{rad}}^{1/2}\Psi,H_{\mathrm{rad}}^{1/2}\Phi)$$

on $\mathscr{F}_{\mathrm{rad}}\times\mathscr{F}_{\mathrm{rad}}$, with form domain $\bigcap_{\mu=1}^{3}\left(D(P_{\mathrm{f}\mu})\cap D(A_\mu(0))\right)\cap D(H_{\mathrm{rad}}^{1/2})$. Since Q_p is densely defined and non-negative, there exists a positive self-adjoint operator $L(p)$ such that $\mathrm{Q}_p(\Psi,\Phi)=(L(p)^{1/2}\Psi,L(p)^{1/2}\Phi)$. Note that

$$L(p)=H_{\mathrm{PF}}(p) \tag{7.6.12}$$

on $D(H_{\mathrm{rad}})\cap D(P_{\mathrm{f}}^2)$. Define $\tilde{L}=\int_{\mathbb{R}^3}^{\oplus}L(p)dp$. For $\phi,\psi\in D_\infty$, we have $(\mathscr{T}\psi, L\mathscr{T}\phi)=(\mathscr{T}\psi,\tilde{L}\mathscr{T}\phi)$. Hence $\mathscr{T}^{-1}L\mathscr{T}=\mathscr{T}^{-1}\tilde{L}\mathscr{T}$ on D_∞ and then $H_{\mathrm{PF}}=\mathscr{T}^{-1}\tilde{L}\mathscr{T}$ on D_∞. Since D_∞ is a core of H_{PF} and $\tilde{L}$ is self-adjoint, it follows that $\mathscr{T}$ maps $D(H_{\mathrm{PF}})$ onto $D(\tilde{L})$ with

$$\mathscr{T}H_{\mathrm{PF}}\mathscr{T}^{-1}=\tilde{L}. \tag{7.6.13}$$

Next we prove self-adjointness of $H_{\mathrm{PF}}(p)$. The operator $H_{\mathrm{PF},0}(p)$ is self-adjoint on $D(H_{\mathrm{PF},0}(p))=D(P_{\mathrm{f}}^2)\cap D(H_{\mathrm{rad}})$ and

$$\mathscr{T}\left(\int_{\mathbb{R}^3}^{\oplus}H_{\mathrm{PF},0}(p)dp\right)\mathscr{T}^{-1}=H_{\mathrm{PF},0}.$$

Thus we have for $F\in\mathscr{H}_{\mathrm{PF}}$ such that $(\mathscr{T}F)(p)=f(p)\Phi$ with $f\in C_0^\infty(\mathbb{R}^3)$ and $\Phi\in\mathscr{F}_{\mathrm{rad}}^\infty$, by the inequality $\|H_{\mathrm{PF},0}F\|\le C\|(H_{\mathrm{PF}}+1)F\|$ derived from Theorem 7.26 and the closed graph theorem, and (7.6.13),

$$\int_{\mathbb{R}^3}|f(p)|^2\|H_{\mathrm{PF},0}(p)\Phi\|^2dp\le C^2\int_{\mathbb{R}^3}|f(p)|^2\|(L(p)+1)\Phi\|^2dp.$$

Hence

$$\|H_{\mathrm{PF},0}(p)\Phi\|\le C\|(L(p)+1)\Phi\| \tag{7.6.14}$$

follows for almost every $p\in\mathbb{R}^3$, since $f\in C_0^\infty(\mathbb{R}^d)$ is arbitrary. Since both sides of (7.6.14) are continuous in p, this inequality holds for all $p\in\mathbb{R}^3$. Thus $H_{\mathrm{PF},0}(p)(L(p)+1)^{-1}$ is bounded, and so is also $H_{\mathrm{PF},0}(p)e^{-tL(p)}$, which implies that

$e^{-tL(p)}$ leaves $D(H_{\rm rad}) \cap D(P_{\rm f}^2)$ invariant. This means that $L(p)$ is essentially self-adjoint on $D(H_{\rm rad}) \cap D(P_{\rm f}^2)$ by Lemma 7.19. Moreover, (7.6.14) implies that $L(p)$ is closed on $D(H_{\rm rad}) \cap D(P_{\rm f}^2)$, and hence $L(p)$ is self-adjoint on $D(H_{\rm rad}) \cap D(P_{\rm f}^2)$. By the basic inequality

$$\|L(p)\Phi\| \leq C'(\|H_{\rm PF,0}(p)\Phi\| + \|\Phi\|), \quad \Phi \in D(H_{\rm PF,0}(p)),$$

it is clear that $L(p)$ is essentially self-adjoint on any core of $H_{\rm PF,0}(p)$. By (7.6.12) the theorem follows. □

Self-adjointness of $H_{\rm PF}(p)$ ensures that $e^{-tH_{\rm PF}(p)}$, $t \geq 0$, is a well-defined C_0-semigroup. As in the previous section, we move to from Fock representation to Schrödinger representation in order to construct a functional integral representation. This way $H_{\rm PF}(p)$ becomes

$$H_{\rm PF}(p) = \frac{1}{2}(p - d\Gamma(-i\nabla) - \mathscr{A}(0))^2 + d\Gamma(\omega(-i\nabla)) \tag{7.6.15}$$

on $L^2(\mathscr{Q})$, where $\mathscr{A}(0) = \mathscr{A}(x = 0) = (\mathscr{A}_1(\tilde{\varphi}), \mathscr{A}_2(\tilde{\varphi}), \mathscr{A}_3(\tilde{\varphi}))$. We also use the same notation $P_{\rm f}$ and $H_{\rm rad}$ for $d\Gamma(-i\nabla)$ and $d\Gamma(\omega(-i\nabla))$, respectively, in what follows. The functional integral representation of $e^{-tH_{\rm PF}(p)}$ can be also constructed as an application of that of $e^{-tH_{\rm PF}}$.

Theorem 7.41 (Functional integral representation for Pauli–Fierz Hamiltonian with fixed total momentum). *Take Assumption* 7.1 *and let* $\Psi, \Phi \in L^2(\mathscr{Q})$. *Then*

$$(\Psi, e^{-tH_{\rm PF}(p)}\Phi) = \mathbb{E}^0[e^{ip\cdot B_t}(\mathrm{J}_0\Psi, e^{-ie\mathscr{A}_{\rm E}(K_t)}\mathrm{J}_t e^{-iP_{\rm f}\cdot B_t}\Phi)_{L^2(\mathscr{Q}_{\rm E})}]. \tag{7.6.16}$$

Proof. Write $F_s = \Pi_s \otimes \Psi$ and $G_r = \Pi_r \otimes \Phi$, where

$$\Pi_s(x) = (2\pi s)^{-3/2}\exp(-|x|^2/2s)$$

is the heat kernel, and $\Psi, \Phi \in L^2(\mathscr{Q})$. By $H_{\rm PF} = \mathscr{T}(\int_{\mathbb{R}^3}^{\oplus} H_{\rm PF}(p)dp)\mathscr{T}^{-1}$, we have

$$(F_s, e^{-tH_{\rm PF}}e^{-i\xi\cdot \mathrm{P}}G_r) = \int_{\mathbb{R}^3} dpe^{-i\xi\cdot p}((\mathscr{T}F_s)(p), e^{-tH_{\rm PF}(p)}(\mathscr{T}G_r)(p))$$

for any $\xi \in \mathbb{R}^3$. Note that $\lim_{s\to 0}(\mathscr{T}F_s)(p) = (2\pi)^{-3/2}\Psi$ strongly in $L^2(\mathscr{Q})$ for each $p \in \mathbb{R}^3$. Hence

$$\lim_{s\to 0}(F_s, e^{-tH_{\rm PF}}e^{-i\xi\cdot \mathrm{P}}G_r) = \frac{1}{\sqrt{(2\pi)^3}}\int_{\mathbb{R}^3} dp(\Psi, e^{-tH_{\rm PF}(p)}e^{-i\xi\cdot p}(\mathscr{T}G_r)(p)). \tag{7.6.17}$$

By using the fact $e^{i\xi\cdot \mathrm{P}}G_r(x) = \Pi_r(x-\xi)e^{-iP_\mathrm{f}\cdot\xi}\Phi$, we obtain by the functional integral representation in Theorem 7.14 that

$$(F_s, e^{-tH_\mathrm{PF}}e^{-i\xi\cdot\mathrm{P}}G_r) = \int_{\mathbb{R}^3} dx\mathbb{E}^x[\Pi_s(x)\Pi_r(B_t-\xi)(\mathrm{J}_0\Psi, e^{-ie\mathscr{A}_\mathrm{E}(K_t)}\mathrm{J}_t e^{-iP_\mathrm{f}\cdot\xi}\Phi)].$$

Then it follows that by $\Pi_s(x)\to\delta(x)$ as $s\to 0$,

$$\lim_{s\to 0}(F_s, e^{-tH_\mathrm{PF}}e^{-i\xi\cdot\mathrm{P}}G_r) = \mathbb{E}^0[\Pi_r(B_t-\xi)(\mathrm{J}_0\Psi, e^{-ie\mathscr{A}_\mathrm{E}(K_t)}\mathrm{J}_t e^{-i\xi\cdot P_\mathrm{f}}\Phi)]. \tag{7.6.18}$$

Combining (7.6.17) and (7.6.18) leads to

$$\begin{aligned}&\frac{1}{\sqrt{(2\pi)^3}}\int_{\mathbb{R}^3} dpe^{-i\xi\cdot p}(\Psi, e^{-tH_\mathrm{PF}(p)}(\mathscr{T}G_r)(p))\\ &= \mathbb{E}^0[\Pi_r(B_t-\xi)(\mathrm{J}_0\Psi, e^{-ie\mathscr{A}_\mathrm{E}(K_t)}\mathrm{J}_t e^{-i\xi\cdot P_\mathrm{f}}\Phi)].\end{aligned} \tag{7.6.19}$$

Since $|(\Psi, e^{-tH_\mathrm{PF}(p)}(\mathscr{T}G_r)(p))| \le \|(\mathscr{T}G_r)(p)\|$ and $\|(\mathscr{T}G_r)(\cdot)\| \in L^2(\mathbb{R}^3)$, we have $(\Psi, e^{-tH_\mathrm{PF}(p)}(\mathscr{T}G_r)(p)) \in L^2(\mathbb{R}^3_p)$. Taking the inverse Fourier transform on both sides of (7.6.19) with respect to p gives

$$\begin{aligned}&(\Psi, e^{-tH_\mathrm{PF}(p)}(\mathscr{T}G_r)(p))\\ &= \frac{1}{\sqrt{(2\pi)^3}}\mathbb{E}^0\left[\int_{\mathbb{R}^3} d\xi e^{ip\cdot\xi}\Pi_r(B_t-\xi)(\mathrm{J}_0\Psi, e^{-ie\mathscr{A}_\mathrm{E}(K_t)}\mathrm{J}_t e^{-i\xi\cdot P_\mathrm{f}}\Phi)\right]\end{aligned} \tag{7.6.20}$$

for almost every $p\in\mathbb{R}^3$. Since both sides of (7.6.20) are continuous in p, the equality stays valid for all $p\in\mathbb{R}^3$. After taking $r\to 0$ on both sides we arrive at (7.6.16). □

By Theorem 7.41 the functional integral representation of $e^{-tH_\mathrm{PF}(p)}$ can formally be written as

$$\begin{aligned}&(\Psi, e^{-tH_\mathrm{PF}(p)}\Phi)\\ &\stackrel{\text{formal}}{=} \mathbb{E}^0\left[\int_{\mathscr{Q}_\mathrm{E}} \overline{\mathrm{J}_0\Psi} e^{i\int_0^t (p-d\Gamma_\mathrm{E}(-i\nabla\otimes 1)-e\mathscr{A}_\mathrm{E}(\tilde{\varphi}(\cdot-B_s)))\cdot dB_s}\mathrm{J}_t\Phi d\mu_\mathrm{E}\right].\end{aligned}$$

In the exponent the expression $p-d\Gamma_\mathrm{E}(-i\nabla\otimes 1)-e\mathscr{A}_\mathrm{E}(\tilde{\varphi}(\cdot-B_s))$ appears suggestive of the Euclidean version of $p-d\Gamma(-i\nabla)-e\mathscr{A}(\tilde{\varphi})$.

Since intuitively $e^{-iP_\mathrm{f}\cdot B_t}$ is a shift operator on $L^2(\mathscr{Q})$, it can be expected to be a positivity preserving.

Lemma 7.42. *Let $\xi\in\mathbb{R}^3$. Then $e^{-iP_\mathrm{f}\cdot\xi}$ is positivity preserving. In particular it follows that $|e^{-iP_\mathrm{f}\cdot\xi}\Psi| \le e^{-iP_\mathrm{f}\cdot\xi}|\Psi|$.*

Proof. Let

$$F(\mathscr{A}(f_1),\ldots,\mathscr{A}(f_n)) = (2\pi)^{-n/2}\int_{\mathbb{R}^n}\hat{F}(k)e^{i\sum_{j=1}^n k_j\mathscr{A}(f_j)}dk,$$
$$G(\mathscr{A}(g_1),\ldots,\mathscr{A}(g_m)) = (2\pi)^{-m/2}\int_{\mathbb{R}^m}\hat{G}(k)e^{i\sum_{j=1}^m k_j\mathscr{A}(g_j)}dk$$

with $0 \le F \in \mathscr{S}(\mathbb{R}^n)$ and $0 \le G \in \mathscr{S}(\mathbb{R}^m)$. By $(e^{i\mathscr{A}(g_1)}, e^{iP_{\rm f}\cdot\xi}e^{i\mathscr{A}(g_2)}) = (e^{i\mathscr{A}(g_1)}, e^{i\mathscr{A}(g_2(\cdot+\xi))})$, we see that

$$(F, e^{iP_{\rm f}\cdot\xi}G) = (F(\mathscr{A}(f_1),\ldots,\mathscr{A}(f_n)), G(\mathscr{A}(g_1(\cdot+\xi)),\ldots,\mathscr{A}(g_m(\cdot+\xi)))) \ge 0.$$

Hence by a limiting argument, $(F, e^{iP_{\rm f}\cdot\xi}G) \ge 0$ for arbitrary $F, G \in L^2(\mathscr{Q})$. □

An analogue of the diamagnetic inequality for $e^{-tH_{\rm PF}}$ can be derived from the functional integral representation for $e^{-tH_{\rm PF}(p)}$.

Corollary 7.43 (Diamagnetic inequality). *It follows that*

$$|(\Psi, e^{-tH_{\rm PF}(p)}\Phi)| \le (|\Psi|, e^{-t(P_{\rm f}^2+H_{\rm f})}|\Phi|). \tag{7.6.21}$$

Proof. This follows from Lemma 7.42 and

$$|(\Psi, e^{-tH_{\rm PF}(p)}\Phi)| \le \mathbb{E}[(\mathrm{J}_0|\Psi|, \mathrm{J}_t e^{-iB_t\cdot P_{\rm f}}|\Phi|] = (|\Psi|, e^{-t(P_{\rm f}^2+H_{\rm f})}|\Phi|). \qquad \square$$

From this diamagnetic inequality, however, we can only deduce the trivial energy comparison inequality $\inf\mathrm{Spec}(P_{\rm f}^2 + H_{\rm f}) \le \inf\mathrm{Spec}(H_{\rm PF}(p)) = 0$. However, combining the unitary transformation $e^{-i(\pi/2)N}$ with the functional integral representation (7.6.16), we obtain an interesting result. Denote $E(p) = \inf\mathrm{Spec}(H_{\rm PF}(p))$.

Corollary 7.44 (Positivity improving for $e^{-tH_{\rm PF}(0)}$). *Under Assumption* 7.1 *we have*

(1) *Let* $p = 0$ *and* $\mathfrak{S} = e^{-i(\pi/2)N}$. *Then* $\mathfrak{S}^{-1}e^{-tH_{\rm PF}(0)}\mathfrak{S}$ *is positivity improving.*

(2) *The ground state of* $H_{\rm PF}(0)$ *is unique whenever it exists.*

(3) $E(0) \le E(p)$ *holds.*

Proof. In the case of $p = 0$ we remark that $e^{ip\cdot B_t} = 1$. Then we have

$$(\Psi, \mathfrak{S}^{-1}e^{tH_{\rm PF}(0)}\mathfrak{S}\Phi) = \mathbb{E}^0[(\mathrm{J}_0\Psi, e^{-ie\Pi_{\rm E}(K_t)}\mathrm{J}_t e^{-iP_{\rm f}\cdot B_t}\Phi)].$$

Since $\mathrm{J}_0^* e^{-ie\Pi_{\rm E}(K_t)}\mathrm{J}_t$ is positivity improving and $e^{-iP_{\rm f}\cdot B_t}$ is positivity preserving by Lemma 7.42, (1) follows. (2) is implied by (1) and the Perron–Frobenius theorem. We have

$$\begin{aligned}|(\Psi, \mathfrak{S}^{-1}e^{-tH_{\rm PF}(p)}\mathfrak{S}\Phi)| &\le \mathbb{E}^0[(\mathrm{J}_0|\Psi|, e^{-ie\Pi_{\rm E}(K_t)}\mathrm{J}_t e^{-iP_{\rm f}\cdot B_t}|\Phi|)]\\ &= (|\Psi|, \mathfrak{S}^{-1}e^{-tH_{\rm PF}(0)}\mathfrak{S}|\Phi|).\end{aligned}$$

This yields (3). □

7.7 Path measure associated with the ground state

7.7.1 Path measures with double stochastic integrals

The functional integral method for non-relativistic QED provides a new class of objects for probability theory. An infinite volume Gibbs measure on path space is derived from the Nelson model discussed in Chapter 6. The Pauli–Fierz model also yields a Gibbs measure with densities dependent on a double stochastic integral. Its limit can be applied to studying the ground state of the Pauli–Fierz model. As the first step we show the existence of Gibbs measures with densities dependent on a double stochastic integral.

Let Ψ_{g} be the ground state of H_{PF}, which is unique and $\mathfrak{S}^{-1}\Psi_{\mathrm{g}} > 0$ by Corollary 7.31. Although the existence of the ground state has been proven, the proof is not constructive, and therefore it does not make possible a direct study of its properties. As an alternative, the functional integral representation can be used, and ground state expectations $(\Psi_{\mathrm{g}}, \mathcal{O}\Psi_{\mathrm{g}})$ for various operators $\mathcal{O}$ can be expressed in terms of averages of the Gibbs measure.

Let M be a positive Borel measurable function on $\mathbb{R}^3$. Write

$$\mathcal{M} = d\Gamma(M(-i\nabla)) \tag{7.7.1}$$

and consider $\mathcal{O} = e^{-\beta\mathcal{M}}$ in the discussion above. The strategy is to construct a sequence $\{\Psi_{\mathrm{g}}^t\}_{t>0}$ such that $\Psi_{\mathrm{g}}^t \to \Psi_{\mathrm{g}}$ as $t \to \infty$. One candidate is

$$\Psi_{\mathrm{g}}^t = \|e^{-tH_{\mathrm{PF}}}(f \otimes 1)\|^{-1} e^{-tH_{\mathrm{PF}}}(f \otimes 1) \tag{7.7.2}$$

with $f \geq 0$, since $f \otimes 1$ overlaps with Ψ_{g} by Corollary 7.31, i.e.,

$$(\Psi_{\mathrm{g}}, f \otimes 1) = (\mathfrak{S}^{-1}\Psi_{\mathrm{g}}, f \otimes 1) > 0.$$

This gives

$$\lim_{t\to 0}(\Psi_{\mathrm{g}}^t, e^{-\beta\mathcal{M}}\Psi_{\mathrm{g}}^t) = (\Psi_{\mathrm{g}}, e^{-\beta\mathcal{M}}\Psi_{\mathrm{g}}), \tag{7.7.3}$$

i.e.,

$$(\Psi_{\mathrm{g}}, e^{-\beta\mathcal{M}}\Psi_{\mathrm{g}}) = \lim_{t\to 0} \frac{(f \otimes 1, e^{-tH_{\mathrm{PF}}} e^{-\beta\mathcal{M}} e^{-tH_{\mathrm{PF}}}(f \otimes 1))}{(e^{-tH_{\mathrm{PF}}}(f \otimes 1), e^{-tH_{\mathrm{PF}}}(f \otimes 1))}. \tag{7.7.4}$$

To have a functional integral representation of the right-hand side of (7.7.4) we need to extend the functional integration $(F, e^{-tH_{\mathrm{PF}}} e^{-\beta\mathcal{M}} e^{-sH_{\mathrm{PF}}} G)$. It was a key step before that $e^{-tH_{\mathrm{rad}}}$ could be decomposed as $e^{-|t-s|H_{\mathrm{rad}}} = \mathrm{J}_s^*\mathrm{J}_t$ leading to the functional integral representation of $(F, e^{-tH_{\mathrm{PF}}}G)$. Our goal is to extend this idea to $\mathcal{M}$.

Let $\mathcal{J}_t : L^2(\mathbb{R}^4) \to L^2(\mathbb{R}^5)$ be defined by

$$\widehat{\mathcal{J}t f}(\mathbf{k}, k_1) = \frac{e^{-ik_1 t}}{\sqrt{\pi}} \frac{\sqrt{M(k)}}{\sqrt{M(k)^2 + |k_1|^2}} \hat{f}(\mathbf{k}), \quad (\mathbf{k}, k_1) = (k, k_0, k_1) \in \mathbb{R}^3 \times \mathbb{R} \times \mathbb{R}. \tag{7.7.5}$$

Then

$$\mathcal{J}_s^* \mathcal{J}_t = e^{-|t-s|(M(-i\nabla)\otimes 1)}, \quad s, t \in \mathbb{R}, \tag{7.7.6}$$

follows. Here $M(-i\nabla) \otimes 1$ is defined under the identification $L^2(\mathbb{R}^4)) \cong L^2(\mathbb{R}^3) \otimes L^2(\mathbb{R})$. Let $(\mathcal{Q}_2, \Sigma_2, \mu_2)$ be a probability space and $\mathscr{A}_2(\mathbf{f})$ denote a Gaussian random variable indexed by $\mathbf{f} \in \bigoplus^3 L^2(\mathbb{R}^5)$ with mean zero and covariance given by

$$\int_{\mathcal{Q}_2} \mathscr{A}_2(\mathbf{f}) \mathscr{A}_2(\mathbf{g}) d\mu_2 = \frac{1}{2} \int_{\mathbb{R}^5} \overline{\hat{\mathbf{f}}(\mathbf{k}, k_1)} \cdot \delta^{\perp}(k) \hat{\mathbf{g}}(\mathbf{k}, k_1) d\mathbf{k} dk_1 = \mathrm{q}_2(\mathbf{f}, \mathbf{g}).$$

Define the contraction operator $\mathscr{I}_t : L^2(\mathcal{Q}_{\mathrm{E}}) \to L^2(\mathcal{Q}_2)$ by the second quantization of $\mathcal{J}_t$, i.e.,

$$\mathscr{I}_t = \Gamma_{12}(\mathcal{J}_t).$$

Then the factorization formula

$$e^{-|t-s| d\Gamma_{\mathrm{E}}(M(-i\nabla)\otimes 1)} = \mathscr{I}_t^* \mathscr{I}_s, \quad s, t \in \mathbb{R}, \tag{7.7.7}$$

follows.

Theorem 7.45 (Euclidean Green functions). *Let* $0 = t_0 < t_1 < \cdots < t_n = t$ *and* $0 = s_0 < s_1 < \cdots < s_n = s$. *Then*

$$\begin{aligned} &\Big(F, \Big(\prod_{j=1}^n e^{-(t_j - t_{j-1}) H_{\mathrm{PF}}} e^{-(s_j - s_{j-1}) \mathcal{M}}\Big) G\Big)_{\mathcal{H}_{\mathrm{PF}}} \\ &= \int_{\mathbb{R}^3} dx \mathbb{E}^x [e^{-\int_0^t V(B_s) ds} (\mathscr{I}_0 \mathrm{J}_0 F(B_0), e^{-ie\mathscr{A}_{\mathrm{E}}(K_t(s_0, s_1, \ldots, s_n))} \mathscr{I}_s \mathrm{J}_t G(B_t))_{L^2(\mathcal{Q}_2)}], \end{aligned} \tag{7.7.8}$$

where

$$K_t(s_0, s_1, \ldots, s_n) = \bigoplus_{\mu=1}^3 \sum_{j=1}^n \int_{t_{j-1}}^{t_j} \mathcal{J}_{s_{j-1}} \mathrm{j}_s \tilde{\varphi}(\cdot - B_s) dB_s^{\mu}.$$

Proof. We give an outline of the proof. Let $T_j = (t_j - t_{j-1})/M_j$. Substitute

$$e^{-(t_j - t_{j-1}) H_{\mathrm{PF}}} = \operatorname*{s-lim}_{M_j \to \infty} \lim_{m_1 \to \infty} \cdots \lim_{m_n \to \infty} \mathrm{J}_{t_{j-1}}^* \Big(\prod_{i=1}^{M_j} (\mathrm{J}_{iT_j} (\mathscr{P}_{T_j/2^{m_i}})^{2^{m_i}} \mathrm{J}_{iT_j}^*)\Big) \mathrm{J}_{t_j}. \tag{7.7.9}$$

The left-hand side of (7.7.8) can be expressed as the limit of

$$T(M_1, m_1^1, \ldots, m_{M_1}^1, \ldots, M_n, m_1^n, \ldots, m_{M_1}^n) = \Big(\mathrm{J}_{t_0} F, \Big(\prod_{i=1}^n \eta_i \zeta_i\Big) \mathrm{J}_{t_n} G\Big), \tag{7.7.10}$$

where

$$\eta_j = \eta_j(t_{j-1}, t_j, m_1^j, \ldots, m_{M_j}^j, M_j) = \prod_{i=1}^{M_j} \Big(\mathrm{J}_{iT_j} (\mathscr{P}_{T_j/2^{m_i^j}})^{2^{m_i^j}} \mathrm{J}_{iT_j}\Big),$$
$$\zeta_j = \mathrm{J}_{t_j} e^{-(s_j - s_{j-1})\mathcal{M}} \mathrm{J}_{t_j}^*.$$

In other words, T converges to the left-hand side of (7.7.8) as $m_i^j \to \infty$, for $i = 1, \ldots, n$, $j = 1, \ldots, M_j$, $M_j \to \infty$ for $j = 1, \ldots, n$ and $n \to \infty$. Note that

$$\mathrm{J}_{t_j} e^{-(s_j - s_{j-1})\mathcal{M}} \mathrm{J}_{t_j}^* = \mathscr{I}_{s_{j-1}}^* \mathscr{I}_{s_j} \mathrm{E}_{t_j}. \tag{7.7.11}$$

Inserting this identity on (7.7.10), and using the Markov properties of both J_t and $\mathscr{I}_s$, we obtain the theorem. □

Next we construct a functional integral representation for other types of Green functions. Define

$$L^2_\infty(\mathscr{Q}) = \{\Phi(\mathscr{A}(\mathbf{f}_1), \ldots, \mathscr{A}(\mathbf{f}_n)) | \Phi \in L^\infty(\mathbb{R}^n), \mathbf{f}_j \in \oplus^3 L^2(\mathbb{R}^3),$$
$$j = 1, \ldots, n, n \geq 0\}.$$

Theorem 7.46 (Euclidean Green functions). *Let* $0 = t_0 < t_1 < \cdots < t_n = t$. *Suppose that* $F_j = f_j \otimes \Phi_j(\mathscr{A}(\mathbf{f}_1^j), \ldots, \mathscr{A}(\mathbf{f}_{n_j}^j)) \in L^\infty(\mathbb{R}^3) \otimes L^2_\infty(\mathscr{Q})$, $j = 1, \ldots, n-1$ *and* $F_0, F_n \in \mathscr{H}_{\mathrm{PF}}$. *Then*

$$\Big(F_0, \Big(\prod_{j=1}^n e^{-(t_j - t_{j-1}) H_{\mathrm{PF}}} F_j\Big)\Big) \tag{7.7.12}$$
$$= \int_{\mathbb{R}^3} dx \mathbb{E}^x \Big[e^{-\int_0^t V(B_s) ds} \Big(\mathrm{J}_0 F_0(B_0), e^{-ie\mathscr{A}_{\mathrm{E}}(K_t)} \Big(\prod_{j=1}^{n-1} \tilde{F}_j(B_{t_j})\Big) \mathrm{J}_t F_n(B_t)\Big)_{L^2(\mathscr{Q}_2)}\Big],$$

where $\tilde{F}_j(x) = f_j(x) \Phi_j(\mathscr{A}_{\mathrm{E}}(\mathrm{j}_{t_j} \mathbf{f}_1^{\mathrm{j}}), \ldots, \mathscr{A}_{\mathrm{E}}(\mathrm{j}_{t_j} \mathbf{f}_{n_j}^{\mathrm{j}}))$.

Proof. First assume that f_j and Φ_j, $j = 1, \ldots, n$, are sufficiently smooth functions with a compact support. Note that

$$\mathrm{J}_s F_j \mathrm{J}_s^* = f_j \Phi_j(\mathscr{A}_{\mathrm{E}}(\mathrm{j}_s \mathbf{f}_1^j), \ldots, \mathscr{A}_{\mathrm{E}}(\mathrm{j}_s \mathbf{f}_{n_j}^j)) \tag{7.7.13}$$

as a bounded operator. Thus substituting (7.7.9) and (7.7.13) on the left-hand side of (7.7.12), the theorem follows. For general F_j a limiting argument can be used. □

As an application we show the path integral presentation of $(\Psi_{\rm g}, e^{-\beta N}\Psi_{\rm g})$, where N is the boson number operator and $\beta > 0$. Here and in what follows we write N for $1 \otimes N$ for notational convenience. Let

$$(1, T1)_{L^2(\mathscr{Q}_{\rm E})} = \langle T \rangle_{\rm vac}\,. \tag{7.7.14}$$

Set $M = 1$ in (7.7.1). Then $\mathcal{M} = N$ and let $\mathcal{J}_t : L^2(\mathbb{R}^4) \to L^2(\mathbb{R}^5)$ be the family of isometries defined in (7.7.5). Write

$$Y = Y(t, \beta) = \bigoplus_{\mu=1}^{3} \left(\int_{-t}^{0} \mathcal{J}_0 \mathrm{j}_s \tilde{\varphi}(\cdot - B_s) dB_s^\mu + \int_0^t \mathcal{J}_\beta \mathrm{j}_s \tilde{\varphi}(\cdot - B_s) dB_s^\mu \right),$$

$$Z = Z(t) = \bigoplus_{\mu=1}^{3} \int_{-t}^{t} \mathrm{j}_s \tilde{\varphi}(\cdot - B_s) dB_s^\mu,$$

where and in what follows in this section $(B_t)_{t\in\mathbb{R}}$ denotes two-sided 3-dimensional Brownian motion on $\mathbb{R}$. We note that B_{-s}^μ and B_t^μ are independent, for all $s, t > 0$. Let $f \geq 0$ be fixed. Using Theorem 7.46 one can compute

$$(\Psi_{\rm g}^t, e^{-\beta\mathcal{M}}\Psi_{\rm g}^t) = \frac{\int_{\mathbb{R}^3} dx \mathbb{E}^x[f(B_{-t}) f(B_t) e^{-\int_{-t}^t V(B_s)ds} \langle e^{-ie\mathscr{A}_{\rm E}(Y)}\rangle_{\rm vac}]}{\int_{\mathbb{R}^3} dx \mathbb{E}^x[f(B_{-t}) f(B_t) e^{-\int_{-t}^t V(B_s)ds} \langle e^{-ie\mathscr{A}_{\rm E}(Z)}\rangle_{\rm vac}]}. \tag{7.7.15}$$

Notice that

$$\langle e^{-ie\mathscr{A}_{\rm E}(Y)}\rangle_{\rm vac} = \exp\left(-\frac{e^2}{2} \mathrm{q}_2\,(Y, Y) \right).$$

As q_2 is a bilinear form, the identity

$$\mathrm{q}_2\,(Y, Y) = \mathrm{q}_{\rm E}(Z, Z) + 2(e^{-\beta} - 1)\mathrm{q}_{\rm E}(Z_{(-t,0)}, Z_{(0,t)}), \tag{7.7.16}$$

holds, where

$$Z_{(S,T)} = \bigoplus_{\mu=1}^{3} \int_S^T \mathrm{j}_s \tilde{\varphi}(\cdot - B_s) dB_s^\mu. \tag{7.7.17}$$

Definition 7.10. Define the family

$$d\mu_t = \frac{1}{N_t} \left(\int_{\mathbb{R}^3} f(x + B_{-t}) f(x + B_t) e^{-\int_{-t}^t V(x+B_s)ds} dx \right) e^{-(e^2/2)\mathrm{q}_{\rm E}(Z,Z)} d\mathcal{W}^0, \tag{7.7.18}$$

of probability measures on $C(\mathbb{R}; \mathbb{R}^3)$, where we recall that $d\mathcal{W}^0$ is Wiener measure on $C(\mathbb{R}; \mathbb{R}^3)$ such that $\mathcal{W}^0(B_0 = 0) = 1$. In Proposition 2.10 $\tilde{\mathcal{W}}^0$ appears for $\mathcal{W}^0$. We use, however, the notation $\mathcal{W}^0$. In (7.7.18) N_t denotes the denominator of the right-hand side of (7.7.15), regarded as the normalizing constant such that $\int_{C(\mathbb{R};\mathbb{R}^3)} d\mu_t = 1$, and $\mathrm{q}_{\rm E}(Z, Z)$ is independent of x.

Corollary 7.47. *It follows that*

$$(\Psi_{\mathrm{g}}, e^{-\beta N}\Psi_{\mathrm{g}}) = \lim_{t\to\infty}\int_{C(\mathbb{R};\mathbb{R}^3)} e^{+e^2(1-e^{-\beta})\mathrm{q}_{\mathrm{E}}(Z_{(-t,0)},Z_{(0,t)})}d\mu_t.$$

Proof. By (7.7.16) and (7.7.15) we have

$$(\Psi_{\mathrm{g}}^t, e^{-\beta N}\Psi_{\mathrm{g}}^t) = \int_{C(\mathbb{R};\mathbb{R}^3)} e^{e^2(1-e^{-\beta})\mathrm{q}_{\mathrm{E}}(Z_{(-t,0)},Z_{(0,t)})}d\mu_t. \tag{7.7.19}$$

Since $\Psi_{\mathrm{g}}^t \to \Psi_{\mathrm{g}}$ as $t\to\infty$, the corollary follows. □

The measure $d\mu_t$ includes $\mathrm{q}_{\mathrm{E}}(Z,Z)$ which has the formal expression

$$\mathrm{q}_{\mathrm{E}}(Z,Z) \stackrel{\text{formal}}{=} \sum_{\mu,\nu=1}^{3}\int_{-t}^{t} dB_s^\mu \int_{-t}^{t} dB_r^\nu W_{\mu\nu}(B_s-B_r, s-r), \tag{7.7.20}$$

where $W_{\mu\nu}$ is given by

$$W_{\mu\nu}(X,T) = \int_{\mathbb{R}^3} \delta_{\mu\nu}^{\perp}(k)\frac{|\hat{\varphi}(k)|^2}{2\omega(k)}e^{-|T|\omega(k)}e^{-ik\cdot X}dk. \tag{7.7.21}$$

Moreover

$$\mathrm{q}_{\mathrm{E}}(Z_{(-t,0)},Z_{(0,t)}) \stackrel{\text{formal}}{=} \sum_{\mu,\nu=1}^{3}\int_{0}^{t} dB_s^\mu \int_{-t}^{0} dB_r^\nu W_{\mu\nu}(B_s-B_r, s-r). \tag{7.7.22}$$

(7.7.21) expresses the effect of the quantum field. Recall that in the Nelson model instead of the double stochastic integral a double Riemann integral appears. The formal expression (7.7.20), however, is not well defined, since the integrand is not admissible. In the next section we find a suitable expression of the integral.

7.7.2 Expression in terms of iterated stochastic integrals

We define the iterated stochastic integral S_T by

$$\begin{aligned} S_T = {} & \frac{1}{2}\int_{\mathbb{R}^3}\frac{|\hat{\varphi}(k)|^2}{\omega(k)}\left(\int_{-T}^{T} e^{ik\cdot B_s}dB_s\cdot\delta^{\perp}(k)\int_{-T}^{s} e^{-\omega(k)(s-r)}e^{-ik\cdot B_r}dB_r\right)dk \\ & + T\int_{\mathbb{R}^3}\frac{|\hat{\varphi}(k)|^2}{\omega(k)}dk \end{aligned} \tag{7.7.23}$$

S_T is a well-defined expression and we use it to replace (7.7.20). We will refer to (7.7.23) as the double stochastic integral with the diagonal removed.

Theorem 7.48 (Iterated stochastic integral). *Suppose that $\hat{\varphi}$ is rotation invariant. Then we have*

$$\langle e^{-ie\mathscr{A}(Z)}\rangle_{\text{vac}} = e^{-e^2 S_T}. \tag{7.7.24}$$

Proof. We replace the time interval $[-T, T]$ with $[0, 2T]$ by shift invariance, and reset $2T$ by T for notational convenience. Using the definition of Z and dominated convergence give

$$\langle e^{-ie\mathscr{A}(Z)}\rangle_{\text{vac}} = \lim_{n\to\infty} \Big\langle \exp\Big(-ie\sum_{j=1}^{n} \mathscr{A}_{\text{E}}(\mathrm{j}_{\Delta_j}\tilde{\varphi}(\cdot - B_{\Delta_j}))\cdot \Delta B_j\Big)\Big\rangle_{\text{vac}},$$

where we set $\Delta B_j = B_{jT/n} - B_{(j-1)T/n}$ and $\Delta_j = (j-1)T/n$, $j = 1,\dots,n$. Since $\mathscr{A}_{\text{E}}(\mathbf{f})$ is Gaussian, it can be computed as

$$\begin{aligned}\langle e^{-ie\mathscr{A}(Z)}\rangle_{\text{vac}} = \lim_{n\to\infty} \exp\Big(&-\frac{e^2}{2}\int_{\mathbb{R}^3} dk\, \frac{|\hat{\varphi}(k)|^2}{2\omega(k)} \\ &\times \sum_{j=1}^{n}\sum_{l=1}^{n} e^{-|\Delta_j-\Delta_l|\omega(k)} e^{ik(B_{\Delta_j}-B_{\Delta_l})}\Delta B_j\cdot\delta^{\perp}(k)\Delta B_l\Big).\end{aligned}$$

Now the integral $\int_{\mathbb{R}^3} dk\cdots$ above can be computed as

$$\int_{\mathbb{R}^3} dk\, \frac{|\hat{\varphi}(k)|^2}{2\omega(k)} \sum_{j=1}^{n}\sum_{l=1}^{n} e^{-|\Delta_j-\Delta_l|\omega(k)} e^{ik(B_{\Delta_j}-B_{\Delta_l})}\Delta B_j\cdot\delta^{\perp}(k)\Delta B_l$$

$$= \sum_{j=1}^{n} \Delta B_j\cdot\Big(\int_{\mathbb{R}^3} dk\, \frac{|\hat{\varphi}(k)|^2}{2\omega(k)}\delta^{\perp}(k)\Big)\Delta B_j \tag{7.7.25}$$

$$+ \sum_{j=1}^{n}\int_{\mathbb{R}^3} dk\, \frac{|\hat{\varphi}(k)|^2}{\omega(k)} e^{-\Delta_j\omega(k)} e^{ikB_{\Delta_j}}\Delta B_j\cdot\delta^{\perp}(k)\Big(\sum_{l=1}^{j-1} e^{\Delta_l\omega(k)}e^{-ikB_{\Delta_l}}\Delta B_l\Big). \tag{7.7.26}$$

For the diagonal term (7.7.25) we note that

$$\int_{\mathbb{R}^3} \frac{|\hat{\varphi}(k)|^2}{2\omega(k)}\delta^{\perp}_{\mu\nu}(k)dk = \delta_{\mu\nu}\frac{2}{3}\int_{\mathbb{R}^3} \frac{|\hat{\varphi}(k)|^2}{2\omega(k)}dk$$

by rotation invariance of $\hat{\varphi}$. As $n\to\infty$, $\sum_{\mu=1}^{3}\sum_{j=1}^{n}|\Delta B_j^{\mu}|^2 \to 3T$ almost surely. Thus we find

$$\lim_{n\to\infty}\sum_{j=1}^{n}\Delta B_j\cdot\Big(\int_{\mathbb{R}^3} dk\, \frac{|\hat{\varphi}(k)|^2}{2\omega(k)}\delta^{\perp}(k)\Big)\Delta B_j = T\int_{\mathbb{R}^3}\frac{|\hat{\varphi}(k)|^2}{\omega(k)}dk$$

almost surely. For the off-diagonal term (7.7.26), we start by noting that by the definition of the Itô integral for locally bounded functions $f, g : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}$,

$$\begin{aligned}&\mathbb{E}\Big[\int_0^t ds \Big| f(s, B_s) \int_0^s g(r, B_r) dB_r^\mu \Big|^2\Big] \\ &= \int_0^t ds \mathbb{E}[|f(s, B_s)|^2] \int_0^s dr \mathbb{E}[|g(r, B_r)|^2] < \infty.\end{aligned}$$

Hence the stochastic integral of $\rho(s) = f(s, B_s) \int_0^s g(r, B_r) dB_r^\mu$ exists and for every $k \in \mathbb{R}^3$,

$$\begin{aligned}&\lim_{n\to\infty} \sum_{j=1}^n f(\Delta_j, B_{\Delta_j}) \Delta B_j \cdot \delta^\perp(k) \int_0^{\Delta_j} g(r, B_r) dB_r \\ &= \int_0^T f(s, B_s) dB_s \cdot \delta^\perp(k) \int_0^s g(r, B_r)\, dB_r \qquad (7.7.27)\end{aligned}$$

strongly in $L^2(\mathscr{X}, d\mathcal{W}^x)$. By the independence of increments of Brownian motion and the fact that $\sum_{\mu=1}^3 \mathbb{E}[(\Delta B_j^\mu)^2] = 3/n$, $\mathbb{E}[\Delta B_j^\mu] = 0$, we can estimate the difference of (7.7.27) and the off-diagonal term (7.7.26) to get

$$\begin{aligned}&\mathbb{E}\Big[\sum_{j=1}^n f(\Delta_j, B_{\Delta_j}) \Delta B_j \cdot \delta^\perp(k) \Big(\int_0^{\Delta_j} g(r, B_r)\, dB_r - \sum_{l=1}^j g(\Delta_l, B_{\Delta_l}) \Delta B_l\Big)\Big]^2 \\ &\le \|f^2\|_\infty \frac{3}{n} \sum_{\nu=1}^3 \sum_{j=1}^n \mathbb{E}\Big[\Big| \int_0^{\Delta_j} g(r, B_r)\, dB_r^\nu - \sum_{l=1}^j g(\Delta_l, B_{\Delta_l}) \Delta B_l^\nu \Big|^2\Big]. \\ &\qquad (7.7.28)\end{aligned}$$

Then the right-hand side above converges to zero as $n \to \infty$. Combining (7.7.27) and (7.7.28) we find that for every $k \in \mathbb{R}^3$,

$$\begin{aligned}&\lim_{n\to\infty} \sum_{j=1}^n f(\Delta_j, B_{\Delta_j}) \Delta B_j \cdot \delta^\perp(k) \Big(\sum_{l=1}^j g(\Delta_l, B_{\Delta_l}) \Delta B_l\Big) \\ &= \int_0^t f(s, B_s) dB_s \cdot \delta^\perp(k) \Big(\int_0^s g(r, B_r)\, dB_r\Big) \qquad (7.7.29)\end{aligned}$$

strongly in $L^2(\mathscr{X}; d\mathcal{W}^x)$. By putting $f(t, x) = e^{ik\cdot x} e^{-\omega(k)t}$, $g(t, x) = e^{-ik\cdot x} e^{\omega(k)t}$ in (7.7.29), it follows that

$$\begin{aligned}&\lim_{n\to\infty} \sum_{j=1}^n e^{-\Delta_j \omega(k)} e^{ikB_{\Delta_j}} \Delta B_j \cdot \delta^\perp(k) \Big(\sum_{l=1}^{j-1} e^{\Delta_l \omega(k)} e^{-ikB_{\Delta_l}} \Delta B_l\Big) \\ &= \int_0^T e^{ik\cdot B_s} dB_s \cdot \delta^\perp(k) \int_0^s e^{-\omega(k)(s-r)} e^{-ik\cdot B_r} dB_r. \qquad (7.7.30)\end{aligned}$$

Hence

$$\lim_{n\to\infty}(7.7.26) = \int_{\mathbb{R}^3} \frac{|\hat{\varphi}(k)|^2}{\omega(k)} dk \int_0^T e^{ik\cdot B_s} dB_s \cdot \delta^\perp(k) \int_0^s e^{-\omega(k)(s-r)} e^{-ik\cdot B_r} dB_r.$$

□

Here is a concluding summary.

Proposition 7.49. *Let μ_T be the family of probability measures on $C(\mathbb{R};\mathbb{R}^3)$ defined in Definition 7.10. Then the following identity holds:*

$$d\mu_T = \frac{1}{\hat{Z}_T}\left(\int_{\mathbb{R}^3} dx f(B_{-T}+x) f(B_T+x) e^{-\int_{-T}^T V(B_s+x)\,ds}\right) e^{-e^2\hat{S}_T} d\mathcal{W}^0,$$

where $\hat{S}_T$ is defined by S_T with the diagonal part removed

$$\hat{S}_T = \frac{1}{2}\int_{\mathbb{R}^3} dk \frac{|\hat{\varphi}(k)|^2}{\omega(k)} \int_{-T}^T e^{ik\cdot B_s} dB_s \cdot \delta^\perp(k) \int_{-T}^s e^{-\omega(k)(s-r)} e^{-ik\cdot B_r} dB_r$$

and $\hat{Z}_T$ denotes the normalizing constant such that $\int_{C(\mathbb{R};\mathbb{R}^3)} d\mu_T = 1$.

$\hat{S}_T$ may be symbolically written as

$$\hat{S}_T = \sum_{\mu,\nu=1}^{3} \int_{[-T,T]^2\setminus\{s=r\}} dB_s^\mu dB_r^\nu \int_{\mathbb{R}^3} W_{\mu\nu}(B_s - B_r, s-r) ds dr. \qquad (7.7.31)$$

7.7.3 Weak convergence of path measures

In this section we show tightness of the family of probability measures $\{\mu_t\}_{t\geq 0}$ on $C(\mathbb{R};\mathbb{R}^3)$.

Lemma 7.50. *Let $f_1,\ldots,f_{n-1} \in L^\infty(\mathbb{R}^d)$, $0 \leq f \in L^2(\mathbb{R}^3)$, and $-T = t_0 \leq t_1 \leq \cdots \leq t_n = T$, Then the n-point Green function is expressed as*

$$\frac{(f\otimes 1, e^{-(t_1-t_0)H_{\mathrm{PF}}}(f_1\otimes 1)\cdots(f_{n-1}\otimes 1)e^{-(t_n-t_{n-1})H_{\mathrm{PF}}} f\otimes 1)}{(f\otimes 1, e^{-2TH} f\otimes 1)}$$
$$= \mathbb{E}_{\mu_T}\Big[\prod_{j=1}^{n-1} f_j(B_{t_j})\Big].$$

Proof. By Theorem 7.46 we can directly see that

$$(f\otimes 1, e^{-(t_1-t_0)H_{\mathrm{PF}}}(f_1\otimes 1)\cdots(f_{n-1}\otimes 1)e^{-(t_n-t_{n-1})H_{\mathrm{PF}}} f\otimes 1)$$
$$= \int_{\mathbb{R}^3} dx \mathbb{E}^x\Big[f(B_{-T})f(B_T)\Big(\prod_{j=1}^{n-1} f_j(B_{t_j})\Big) e^{-\int_{-T}^T V(B_s)ds} \langle e^{-ie\mathscr{A}_{\mathrm{E}}(Z)}\rangle_{\mathrm{vac}}\Big].$$

By the definition of the measure μ_T the lemma follows. □

An immediate result of s-$\lim_{t\to\infty}\Psi_{\mathrm{g}}^t = \Psi_{\mathrm{g}}$ and Lemma 7.50 is as follows. Let $\rho, \rho_1, \rho_2 \in L^\infty(\mathbb{R}^3)$. Then for $t > s$,

$$\lim_{T\to\infty} \mathbb{E}_{\mu_T}[\rho(B_0)] = (\Psi_{\mathrm{g}}, (\rho \otimes 1)\Psi_{\mathrm{g}}), \tag{7.7.32}$$

$$\lim_{T\to\infty} \mathbb{E}_{\mu_T}[\rho_1(B_s)\rho_2(B_t)] = (\Psi_{\mathrm{g}}, (\rho_1 \otimes 1)e^{-(t-s)H_{\mathrm{PF}}}(\rho_2 \otimes 1)\Psi_{\mathrm{g}})e^{(t-s)E(H_{\mathrm{PF}})}, \tag{7.7.33}$$

where $E(H_{\mathrm{PF}}) = \inf \mathrm{Spec}(H_{\mathrm{PF}})$ denotes the ground state energy of H_{PF}.

Theorem 7.51 (Tightness). *Suppose that there exists the ground state of H_{PF} and*

$$\sup_{x\in\mathbb{R}^3} \mathbb{E}^x\left[\exp\left(-4\int_0^T V(B_s)ds\right)\right] < \infty \tag{7.7.34}$$

for all $T > 0$. Then there exists a subsequence T' such that $\mu_{T'}$ has a weak limit as $T' \to \infty$.

Proof. The proof is a modification of that of Theorem 6.12. By the Prokhorov theorem, it suffices to show (1) and (2) below:

(1) $\lim_{\Lambda\to\infty}\sup_T \mu_T(|B_0|^2 > \Lambda) = 0$,

(2) for any $\varepsilon > 0$, $\lim_{\delta\downarrow 0}\sup_T \mu_T(\max_{|t-s|<\delta, -T\le s,t\le T} |B_t - B_s| > \varepsilon) = 0$.

We have

$$\mu_T(|B_0|^2 > \Lambda) = (\Psi_{\mathrm{g}}^T, (1_{\{|x|^2>\Lambda\}} \otimes 1)\Psi_{\mathrm{g}}^T).$$

Note that $\Psi_{\mathrm{g}}^T \to \Psi_{\mathrm{g}}$ strongly as $T \to \infty$. Set $\alpha(T) = (\Psi_{\mathrm{g}}^T, (1_{\{|x|^2>\Lambda\}} \otimes 1)\Psi_{\mathrm{g}}^T)$, and without loss of generality we can assume that $T \ge 1$. Let $\varepsilon > 0$ be given. Since as a bounded operator $\|1_{\{|x|^2>\Lambda\}} \otimes 1\| \le 1$, there exists $T^* > 0$ independent of Λ such that,

$$\alpha(T) \le (\Psi_{\mathrm{g}}, (1_{\{|x|^2>\Lambda\}} \otimes 1)\Psi_{\mathrm{g}}) + \varepsilon \quad \forall T > T^*. \tag{7.7.35}$$

Thus

$$\sup_{1\le T}\alpha(T) \le \sup_{1\le T\le T^*}\alpha(T) + (\Psi_{\mathrm{g}}, (1_{\{|x|^2>\Lambda\}} \otimes 1)\Psi_{\mathrm{g}}) + \varepsilon. \tag{7.7.36}$$

We shall estimate $\sup_{1\le T\le T^*}\alpha(T)$. Let $\mathfrak{S} = e^{-i(\pi/2)N}$. Since $\mathfrak{S}^{-1}e^{-TH_{\mathrm{PF}}}\mathfrak{S}$ is positivity improving and $\mathfrak{S}^{-1}(f \otimes 1) = f \otimes 1$, we see that

$$\|e^{-TH_{\mathrm{PF}}}(f \otimes 1)\| = \|\mathfrak{S}^{-1}e^{-TH_{\mathrm{PF}}}\mathfrak{S}(f \otimes 1)\| > 0$$

for all $T \ge 0$. Then $\inf_{1\le T\le T^*}\|e^{-TH_{\mathrm{PF}}}(f \otimes 1)\| > 0$. Denote the left-hand side by c. Since T^* is independent of Λ, c is also independent of Λ. Thus

$$\alpha(T) \le c^{-2}(e^{-TH_{\mathrm{PF}}}(f \otimes 1), 1_{\{|x|^2>\Lambda\}}e^{-TH_{\mathrm{PF}}}(f \otimes 1)).$$

By the functional integral representation and Schwarz inequality we have

$$\begin{aligned}
&(e^{-TH_{\mathrm{PF}}}(f\otimes 1), 1_{\{|x|^2>\Lambda\}}e^{-TH_{\mathrm{PF}}}(f\otimes 1))\\
&= \int_{\mathbb{R}^3} dx \mathbb{E}^x[f(B_0)f(B_{2T})1_{\{|x|^2>\Lambda\}}(B_T)e^{-\int_0^{2T}V(B_s)ds}e^{-S_{2T}}]\\
&\le c'\int dx(\mathbb{E}^x[f(B_{2T})^2])^{1/2}f(x)(\mathbb{E}^x[1_{\{|x|^2>\Lambda\}}(B_T)])^{1/4},
\end{aligned}$$

where S_{2T} is given by (7.7.23) and $c' = \sup_{x\in\mathbb{R}^d}(\mathbb{E}^x[e^{-4\int_0^{2T}V(B_s)ds}])^{1/4}$. By Schwarz inequality again we have

$$\begin{aligned}
&(e^{-TH_{\mathrm{PF}}}(f\otimes 1), 1_{\{|x|^2>\Lambda\}}e^{-TH_{\mathrm{PF}}}(f\otimes 1))\\
&\le c'\|f\|\left(\int_{\mathbb{R}^3} dx(\mathbb{E}^x[1_{\{|x|^2>\Lambda\}}(B_T)])^{1/2}f(x)^2\right)^{1/2}.
\end{aligned}$$

Since $1\le T\le T^*$, the bound

$$\begin{aligned}
&\int_{\mathbb{R}^3} dx(\mathbb{E}^x[1_{\{|x|^2>\Lambda\}}(B_T)])^{1/2}f(x)^2\\
&\le (2\pi)^{-3/4}\int_{\mathbb{R}^3} dx f(x)^2\left(\int_{\mathbb{R}^3} dy e^{-|x-y|^2/(2T^*)}1_{\{|x|^2>\Lambda\}}(y)\right)^{1/2}
\end{aligned}$$

is obtained. Denote the right-hand side above by $C_f(\Lambda)$. Thus

$$\sup_{1\le T\le T^*}\alpha(T)\le \|f\|\frac{c'C_f(\Lambda)^{1/2}}{c^2}. \tag{7.7.37}$$

Since c and c' are independent of Λ and $C_f(\Lambda)\to 0$ as $\Lambda\to\infty$, we obtain

$$\lim_{\Lambda\to\infty}\sup_{T\ge 1}\alpha(T)\le \lim_{\Lambda\to\infty}\left(\|f\|\frac{c'C_f(\Lambda)^{1/2}}{c^2}+(\Psi_{\mathrm{g}}, 1_{\{|x|^2>\Lambda\}}\Psi_{\mathrm{g}})\right)+\varepsilon=\varepsilon.$$

Since ε is arbitrary, (1) follows. (2) can be proven in the same way as Theorem 6.12. □

Denote the weak limit of the measure $\mu_{T'}$ on $C(\mathbb{R};\mathbb{R}^3)$ by μ_∞. Using the functional integration representation of $e^{-tH_{\mathrm{PF}}}$ we show in Corollary 7.34 that if $V(x)=|x|^{2n}$, then $\|\Psi_{\mathrm{g}}(x)\|_{\mathscr{F}_{\mathrm{rad}}}\le C_1e^{-C_2|x|^{n+1}}$, while if $V(x)=-1/|x|$, then $\|\Psi_{\mathrm{g}}(x)\|_{\mathscr{F}_{\mathrm{rad}}}\le C_3e^{-C_4|x|}$ for appropriate constants C_j.

Corollary 7.52. *Assume that $\|\Psi_{\mathrm{g}}(x)\|_{\mathscr{F}_{\mathrm{rad}}}\le Ce^{-c|x|^\gamma}$ for some positive constants C, c and γ. Then*

$$\int_{C(\mathbb{R};\mathbb{R}^3)} e^{c|B_0|^\gamma}d\mu_\infty<\infty. \tag{7.7.38}$$

Proof. Let

$$\rho_m(x) = \begin{cases} e^{c|x|^\gamma}, & e^{c|x|^\gamma} \le m, \\ m, & e^{c|x|^\gamma} > m \end{cases}$$

be the truncated function of $e^{c|x|^\gamma}$. Then $(\Psi_g, (\rho_m \otimes 1)\Psi_g) = \int_{C(\mathbb{R};\mathbb{R}^3)} \rho_m(B_0) d\mu_\infty$ follows. By a limiting argument as $m \to \infty$ (7.7.38) follows. □

Using this measure μ_∞ the expectation $\langle e^{-\beta N}\rangle_{\mathrm{vac}}$ can be formally represented as

$$\langle e^{-\beta N}\rangle_{\mathrm{vac}} = \mathbb{E}_{\mu_\infty}[e^{e^2(1-e^{-\beta})\int_{-\infty}^0 dB_s^\mu \int_0^\infty dB_r^\nu W_{\mu\nu}(B_s - B_r, s-r)}].$$

It is, however, not easy to control $\int_{-t}^0 dB_s^\mu \int_0^t dB_r^\nu W_{\mu\nu}(B_s - B_r, s-r)$ as $t \to \infty$. For the Nelson model the role of the double stochastic integral is taken by a double Riemann integral and then

$$\left| \int_{-t}^0 ds \int_0^t dr W(B_s - B_r, s-r) \right| \le \frac{1}{2} \int_{\mathbb{R}^3} \frac{\hat{\varphi}(k)^2}{\omega(k)^3} dk.$$

Thus if the integral of the right-hand side is bounded, then the left-hand side is uniformly bounded with respect to the path and t.

7.8 Relativistic Pauli–Fierz model

7.8.1 Definition

In quantum mechanics the relativistic Schrödinger operator is defined by $H_{\mathrm{R}}(a) = \sqrt{(p-a)^2 + m^2} - m + V$, and the functional integral representation of $e^{-tH_{\mathrm{R}}(a)}$ is given in Chapter 3. In this section the analogue version of the Pauli–Fierz model is defined and its functional integral representation is given. The relativistic Pauli–Fierz Hamiltonian is defined by

$$H_{\mathrm{PF}}^{\mathrm{R}} = \sqrt{(-i\nabla \otimes 1 - eA)^2 + m^2} - m + V \otimes 1 + 1 \otimes H_{\mathrm{rad}} \tag{7.8.1}$$

on $\mathscr{H}_{\mathrm{PF}}$ as a self-adjoint operator.

A rigorous definition of $H_{\mathrm{PF}}^{\mathrm{R}}$ has a difficulty caused by the term containing the square root. Although a standard way to define $(-i\nabla \otimes 1 - eA)^2 + m^2$ as a self-adjoint operator is to take the self-adjoint operator associated with the quadratic form

$$F, G \mapsto \sum_{\mu=1}^3 ((-i\nabla - eA)_\mu F, (-i\nabla - eA)_\mu G) + m^2(F, G),$$

we proceed instead by finding a core of $(-i\nabla \otimes 1 - eA)^2 + m^2$ by using functional integration. In Section 7.4.1 we defined the self-adjoint operator $\hat{H}_{\mathrm{PF}}(\mathscr{A})$ by the generator of a C_0-semigroup, and have seen that

$$\hat{H}_{\rm PF}(\mathscr{A}) \supset \frac{1}{2}(-i\nabla - e\mathscr{A})^2\lceil_{D_{\rm PF}}.$$

Let $\mathscr{D} = D(\Delta) \cap C^\infty(N)$.

Lemma 7.53. *If $\omega^{3/2}\hat{\varphi} \in L^2(\mathbb{R}^3)$, then $\frac{1}{2}(-i\nabla - e\mathscr{A})^2\lceil_{\mathscr{D}}$ is essentially self-adjoint.*

Proof. By (7.3.18) we have

$$(F, e^{-t\hat{H}_{\rm PF}(\mathscr{A})}G) = \int_{\mathbb{R}^3} dx \mathbb{E}^x[(F(B_0), e^{-ie\mathscr{A}(L_t)}G(B_t))], \tag{7.8.2}$$

where we recall that $L_t = \bigoplus_{\mu=1}^3 \int_0^t \tilde{\varphi}(\cdot - B_s)dB_s^\mu$. By using (7.8.2) we show that $e^{-t\hat{H}_{\rm PF}(\mathscr{A})}$ leaves $\mathscr{D}$ invariant. Then the lemma follows.

First, it can be proven that $e^{-t\hat{H}_{\rm PF}(\mathscr{A})}\mathscr{D} \subset D(\Delta)$ in a similar manner to Lemma 7.23 with $H_{\rm rad}$ replaced by N. To see that $e^{-t\hat{H}_{\rm PF}(\mathscr{A})}\mathscr{D} \subset C^\infty(N)$, take $z \in \mathbb{N}$ and $F, G \in D(N^z)$. We have

$$(N^z F, e^{-t\hat{H}_{\rm PF}(\mathscr{A})}G) = \int_{\mathbb{R}^3} dx \mathbb{E}^x[(N^z F(B_0), e^{-ie\mathscr{A}(L_t)}G(B_t))]. \tag{7.8.3}$$

Note that

$$e^{ie\mathscr{A}(L_t)}Ne^{-ie\mathscr{A}(L_t)} = N - e\Pi(L_t) + \frac{e^2}{2}\|L_t\|^2, \tag{7.8.4}$$

where $\Pi(f) = i[N, \mathscr{A}(f)]$, and thus

$$\begin{aligned}&(N^z F, e^{-t\hat{H}_{\rm PF}(\mathscr{A})}G)\\ &= \int_{\mathbb{R}^3} dx \mathbb{E}^x\left[\left(F(B_0), e^{-ie\mathscr{A}(L_t)}\left(N - e\Pi(L_t) + \frac{e^2}{2}\|L_t\|^2\right)^z G(B_t)\right)\right].\end{aligned} \tag{7.8.5}$$

By the BDG-type inequality

$$\mathbb{E}^x\left[\left\|\int_0^t \tilde{\varphi}(\cdot - B_s)dB_s^\mu\right\|_{L^2(\mathbb{R}^3)}^{2z}\right] \le \frac{(2z)!}{2^z}t^z\|\hat{\varphi}\|_{L^2(\mathbb{R}^3)}^{2z},$$

and we obtain

$$\int_{\mathbb{R}^3} dx \mathbb{E}^x\left[\left\|\left(N - e\Pi(L_t) + \frac{e^2}{2}\|L_t\|^2\right)^z G(B_t)\right\|^2\right] \le C_z^2\|(N+1)^z G\|^2 \tag{7.8.6}$$

with a constant C_z. A combination of (7.8.5) and (7.8.6) gives

$$|(N^z F, e^{-t\hat{H}_{\rm PF}(\mathscr{A})}G)| \le C_z\|F\|\|(N+1)G\|. \tag{7.8.7}$$

This implies $e^{-t\hat{H}_{\rm PF}(\mathscr{A})}C^\infty(N) \subset C^\infty(N)$ and $e^{-t\hat{H}_{\rm PF}(\mathscr{A})}\mathscr{D} \subset \mathscr{D}$ follows. Hence $\hat{H}_{\rm PF}(\mathscr{A})$ is essentially self-adjoint on $\mathscr{D}$. □

We keep denoting the self-adjoint extension of $\hat{H}_{\mathrm{PF}}(\mathscr{A})\lceil_{\mathscr{D}}$ by the same symbol $\hat{H}_{\mathrm{PF}}(\mathscr{A})$ for simplicity, and $\sqrt{2\hat{H}_{\mathrm{PF}}(\mathscr{A})+m^2}$ by the spectral resolution of $\hat{H}_{\mathrm{PF}}(\mathscr{A})$. Recall that $(T_t)_{t\geq 0}$ is the subordinator on a probability space $(\mathscr{T}, \mathscr{B}_{\mathscr{T}}, \nu)$ with

$$\mathbb{E}_\nu^0[e^{-uT_t}] = \exp(-t(\sqrt{2u+m^2}-m)).$$

Since

$$(F, e^{-t(\sqrt{2\hat{H}_{\mathrm{PF}}(\mathscr{A})+m^2}-m)}G) = \mathbb{E}_\nu^0[(F, e^{-T_t\hat{H}_{\mathrm{PF}}(\mathscr{A})}G)],$$

we immediately have that

Lemma 7.54. *The functional integral representation*

$$(F, e^{-t(\sqrt{2\hat{H}_{\mathrm{PF}}(\mathscr{A})+m^2}-m)}G) = \int_{\mathbb{R}^3} dx\mathbb{E}^{x,0}[(F(B_0), e^{-ie\mathscr{A}(L_{T_t})}G(B_{T_t}))] \tag{7.8.8}$$

holds.

From here the diamagnetic inequality

$$|(F, e^{-t(\sqrt{2\hat{H}_{\mathrm{PF}}(\mathscr{A})+m^2}-m)}G)| \leq (|F|, e^{-t(\sqrt{-\Delta+m^2}-m)}|G|) \tag{7.8.9}$$

directly follows. In a similar way to Lemma 7.17 we also have

Lemma 7.55. (1) *If V is $\sqrt{-\Delta+m^2}-m$-form bounded with relative bound a, then $|V|$ is also $\hat{H}_{\mathrm{PF}}(\mathscr{A})$-form bounded with a relative bound smaller than a.*

(2) *If V is relatively bounded with respect to $\sqrt{-\Delta+m^2}-m$ with relative bound a, then V is also relatively bounded with respect to $\hat{H}_{\mathrm{PF}}(\mathscr{A})$ with a relative bound a.*

Definition 7.11 (Relativistic Pauli–Fierz Hamiltonian). Let $\omega^{3/2}\hat{\varphi} \in L^2(\mathbb{R}^3)$, $m > 0$ and suppose that $V = V_+ - V_-$ is such that V_- is relatively form bounded with respect to $\sqrt{-\Delta+m^2}-m$ and $D(V_+) \supset D(\Delta)$. Then we define the *relativistic Pauli–Fierz Hamiltonian* by

$$H_{\mathrm{PF}}^{\mathrm{R}} = \sqrt{2\hat{H}_{\mathrm{PF}}(\mathscr{A})+m^2} - m \dotplus V_+ \dot{-} V_- \dotplus H_{\mathrm{rad}}. \tag{7.8.10}$$

7.8.2 Functional integral representation

Now we will construct the functional integral representation of $e^{-tH_{\mathrm{PF}}^{\mathrm{R}}}$ through the Trotter product formula. Fix $t > 0$, and let $t_j = tj/2^n$, $j = 0, \ldots, 2^n$. Define an $L^2(\mathbb{R}^4)$-valued stochastic process S_n^μ on $\mathscr{X} \times \mathscr{T}$ by

$$S_n^\mu = \sum_{j=1}^{2^n} \int_{T_{t_{j-1}}}^{T_{t_j}} \mathrm{j}_{t_{j-1}} f(\cdot - B_s)dB_s^\mu, \quad \mu = 1, 2, 3, \tag{7.8.11}$$

where $f \in L^2(\mathbb{R}^3)$ and $\int_{T_{t_{j-1}}}^{T_{t_j}} \cdots dB_s^\mu = \int_T^S \cdots dB_s^\mu$ evaluated at $T = T_{t_{j-1}}$ and $S = T_{t_j}$.

Lemma 7.56. *$(S_n^\mu)_{n\in\mathbb{N}}$ is a Cauchy sequence in $L^2(\mathscr{X} \times \mathscr{T}, \mathscr{W}^x \otimes \nu) \otimes L^2(\mathbb{R}^4)$.*

Proof. For simplicity, write S_n and $T(k)$ for S_n^μ and $T_{kt/2^{n+1}}$, respectively. First we estimate $\mathbb{E}^{x,0}[\|S_{n+1} - S_n\|^2]$ for each n. It is seen directly that

$$\begin{aligned}&\mathbb{E}^{x,0}[\|S_{n+1} - S_n\|^2]\\ &= \mathbb{E}_\nu^0\Big[\sum_{j=1}^{2^n} \int_{T(2j-1)}^{T(2j)} \mathbb{E}_{\mathscr{W}}^x[\|\mathrm{j}_{T(2j-1)} f(\cdot - B_s) - \mathrm{j}_{T(2j-2)} f(\cdot - B_s)\|^2] ds\Big].\end{aligned}$$

In a similar way to the proof of Lemma 7.11, we have

$$\begin{aligned}&\mathbb{E}^{x,0}[\|S_{n+1} - S_n\|^2]\\ &\le \mathbb{E}_\nu^0\Big[\sum_{j=1}^{2^n} \int_{T(2j-1)}^{T(2j)} 2|T(2j-1) - T(2j-2)| \|\omega f\| \|f\| ds\Big]\\ &= \mathbb{E}_\nu^0\Big[\sum_{j=1}^{2^n} 2|T(2j) - T(2j-1)||T(2j-1) - T(2j-2)| \|\omega f\| \|f\|\Big].\end{aligned}$$

By stationarity and independence of increments of $(T_t)_{t\ge 0}$, we have

$$\mathbb{E}^{x,0}[\|S_{n+1} - S_n\|^2] \le \sum_{j=1}^{2^n} 2(\mathbb{E}_\nu^0[T_{t/2^{n+1}}])^2 \|\omega f\| \|f\|.$$

Note that the distribution of T_t is

$$\frac{t}{(2\pi r^3)^{1/2}} \exp\left(-\frac{1}{2}\left(\frac{t^2}{r} + m^2 r\right) + mt\right) 1_{[0,\infty)}(r)$$

and $m > 0$. Thus $\mathbb{E}_\nu^0[|T_t|] \le Ct$ with some C. Then

$$\mathbb{E}^{x,0}[\|S_{n+1} - S_n\|^2] \le \frac{t^2}{2^{n+1}} C^2 \|\omega f\| \|f\|,$$

and hence we obtain that

$$(\mathbb{E}^{x,0}[\|S_m - S_n\|^2])^{1/2} \le (\|\omega f\| \|f\|)^{1/2} \sum_{j=n+1}^{m} \frac{t}{2^{j/2}} C.$$

This implies that $(S_n)_{n\in\mathbb{N}}$ is a Cauchy sequence. □

Definition 7.12. Define the $L^2(\mathbb{R}^4)$-valued random process $\int_0^{T_t} \mathrm{j}_{(T^{-1})_s} f(\cdot - B_s) dB_s^\mu$ on the probability space $(\mathscr{X} \times \mathcal{T}, \mathscr{B}(\mathscr{X}) \times \mathscr{B}_{\mathcal{T}}, \mathcal{W}^x \otimes \nu)$ by the following strong limit of S_n^μ:

$$\int_0^{T_t} \mathrm{j}_{(T^{-1})_s} f(\cdot - B_s) dB_s^\mu = \operatorname*{s-lim}_{n\to\infty} S_n^\mu, \quad \mu = 1, 2, 3. \tag{7.8.12}$$

Remark 7.4. The subordinator $[0, \infty) \ni t \mapsto T_t \in [0, \infty)$ is monotonously increasing, but it is not injective. Thus the inverse $(T^{-1})_t$ can not be defined. (7.8.12) is a formal description of the limit of S_n^μ.

Theorem 7.57 (Functional integral representation for relativistic Pauli–Fierz Hamiltonian). *Let $\omega^{3/2}\hat{\varphi} \in L^2(\mathbb{R}^d)$. Suppose that $V = V_+ - V_-$ is such that V_- is relatively form bounded with respect to $\sqrt{-\Delta + m^2} - m$ and $D(V_+) \supset D(\Delta)$. Then*

$$(F, e^{-tH_{\mathrm{PF}}^{\mathrm{R}}} G) = \int_{\mathbb{R}^3} dx \mathbb{E}^{x,0}[e^{-\int_0^t V(B_{T_s}) ds} (\overline{\mathrm{J}_0 F(B_0)}, e^{-ie\mathscr{A}_{\mathrm{E}}(K_t^{\mathrm{rel}})} \mathrm{J}_t G(B_{T_t}))], \tag{7.8.13}$$

where

$$K_t^{\mathrm{rel}} = \bigoplus_{\mu=1}^3 \int_0^{T_t} \mathrm{j}_{(T^{-1})_s} \tilde{\varphi}(\cdot - B_s) dB_s^\mu. \tag{7.8.14}$$

Proof. We set $V = 0$ for simplicity. By the Trotter product formula,

$$(F, e^{-tH_{\mathrm{PF}}^{\mathrm{R}}} G) = \lim_{n\to\infty} (F, (e^{-(t/2^n)\hat{H}_{\mathrm{PF}}(\mathscr{A})} e^{-(t/2^n)H_{\mathrm{rad}}})^{2^n} G).$$

By the Markov property of $\mathrm{E}_t = \mathrm{J}^*\mathrm{J}_t$ the right-hand side above is equal to

$$\lim_{n\to\infty} \Big(\mathrm{J}_0 F, \prod_{j=0}^{2^n} e^{-(t/2^n)\hat{H}_{\mathrm{PF}}(\mathscr{A}_{\mathrm{E}tj/2^n})} \mathrm{J}_t G\Big),$$

where

$$\hat{H}_{\mathrm{PF}}(\mathscr{A}_{\mathrm{E}tj/2^n}) = \sqrt{(-i\nabla \otimes 1 - e\mathscr{A}_{\mathrm{E}}(\mathrm{j}_{tj/2^n}\varphi(\cdot - x))^2 + m^2} - m.$$

Thus we have

$$(F, e^{-tH_{\mathrm{PF}}^{\mathrm{R}}} G) = \lim_{n\to\infty} \int dx \mathbb{E}^{x,0}[(\overline{\mathrm{J}_0 F(B_0)}, e^{-ie\mathscr{A}_{\mathrm{E}}(K_t(n))} \mathrm{J}_t G(B_{T_t}))],$$

where

$$K_t(n) = \bigoplus_{\mu=1}^3 \sum_{j=1}^{2^n} \int_{T_{t(j-1)/2^n}}^{T_{tj/2^n}} \mathrm{j}_{t(j-1)/2^n} \tilde{\varphi}(\cdot - B_s) dB_s^\mu.$$

By Lemma 7.56 and a limiting argument the theorem follows for $V = 0$. When $H_{\mathrm{PF}}^{\mathrm{R}}$ has a bounded continuous V, the theorem can again be proven by the Trotter formula. Furthermore, it can be extended to $V = V_+ - V_-$ such that V_- is relatively form bounded with respect to $\sqrt{-\Delta + m^2} - m$ and $D(V_+) \supset D(\Delta)$, in a similar manner to Theorem 3.31. □

By using this functional integral representation we can obtain similar results to those of H_{PF}.

Corollary 7.58 (Diamagnetic inequality). *Under the assumptions of Theorem* 7.57 *and with* $E(e) = \inf \mathrm{Spec}(H_{\mathrm{PF}}^{\mathrm{R}})$

$$|(F, e^{-tH_{\mathrm{PF}}^{\mathrm{R}}} G)| \le (|F|, e^{-t(\sqrt{-\Delta+m^2}-m+H_{\mathrm{rad}})}|G|) \tag{7.8.15}$$

follows. In particular, $E(0) \le E(e)$.

Corollary 7.59 (Positivity improving for $e^{-tH_{\mathrm{PF}}^{\mathrm{R}}}$). *Under the assumptions of Theorem* 7.57 $\mathfrak{S} e^{-tH_{\mathrm{PF}}^{\mathrm{R}}} \mathfrak{S}^{-1}$ *is positivity improving, where* $\mathfrak{S} = e^{-i(\pi/2)N}$. *In particular, the ground state of* $H_{\mathrm{PF}}^{\mathrm{R}}$ *is unique.*

Corollary 7.60 (Essential self-adjointness). *Suppose that V is relatively bounded with respect to* $\sqrt{-\Delta + m^2} - m$ *with a relative bound strictly smaller than* 1, *and assume* $\omega^{3/2}\hat{\varphi} \in L^2(\mathbb{R}^d)$. *Then* $H_{\mathrm{PF}}^{\mathrm{R}}$ *is essentially self-adjoint on* $D(\sqrt{-\Delta}) \cap D(H_{\mathrm{rad}})$.

Proof. Let $V = 0$. Note that $D(H_{\mathrm{PF}}^{\mathrm{R}}) \supset D(\sqrt{-\Delta}) \cap D(H_{\mathrm{rad}})$. It has been proven above that $D(\sqrt{-\Delta}) \cap D(H_{\mathrm{rad}})$ is invariant under $e^{-tH_{\mathrm{PF}}^{\mathrm{R}}}$ in a similar way to Lemmas 7.22 and 7.23. Hence $H_{\mathrm{PF}}^{\mathrm{R}}$ is essentially self-adjoint on $D(\sqrt{-\Delta}) \cap D(H_{\mathrm{rad}})$ for $V = 0$. By the diamagnetic inequality

$$\||V|(\sqrt{(-i\nabla - e\mathscr{A})^2 + m^2} - m + H_{\mathrm{rad}} + z)^{-1}\| \le \||V|(\sqrt{-\Delta + m^2} - m + H_{\mathrm{rad}} + z)^{-1}\|$$

for z with $\mathrm{Im}\, z \neq 0$. Therefore, V is also relatively bounded with respect to the operator $\sqrt{(-i\nabla - e\mathscr{A})^2 + m^2} - m + H_{\mathrm{rad}}$ with a relative bound strictly smaller than 1. Hence the corollary follows. □

7.8.3 Translation invariant case

In the case of the relativistic Pauli–Fierz Hamiltonian with $V = 0$, it can be seen similarly to the non-relativistic translation invariant case that $[H_{\mathrm{PF}}^{\mathrm{R}}, \mathrm{P}_\mu] = 0$, where $\mathrm{P}_\mu = -i\nabla_{x_\mu} \otimes 1 + 1 \otimes P_{\mathrm{f}\mu}$. This allows then a self-adjoint operator $H_{\mathrm{PF}}^{\mathrm{R}}(p)$ such that

$$\int_{\mathbb{R}^3}^{\oplus} H_{\mathrm{PF}}^{\mathrm{R}}(p)\,dp = H_{\mathrm{PF}}^{\mathrm{R}}. \tag{7.8.16}$$

The self-adjoint operator $H^{\mathrm{R}}_{\mathrm{PF}}(p)$, $p \in \mathbb{R}^3$, is called *relativistic Pauli–Fierz Hamiltonian with a fixed total momentum* p. We can show similar results to those of $H_{\mathrm{PF}}(p)$ by using the functional integral representation of $e^{-tH^{\mathrm{R}}_{\mathrm{PF}}}$. We only state the results and leave their proof to the reader.

Theorem 7.61 (Functional integral representation for relativistic Pauli–Fierz model with fixed total momentum). *Suppose* (1) *and* (2) *of Assumption* 7.1 *hold, and assume* $\omega^{3/2}\hat{\varphi} \in L^2(\mathbb{R}^3)$. *Let* $\Psi, \Phi \in \mathscr{Q}$. *Then*

$$(\Psi, e^{-tH^{\mathrm{R}}_{\mathrm{PF}}(p)}\Phi) = \mathbb{E}^{0,0}[e^{ip\cdot B_{T_t}}(\mathrm{J}_0\Psi, e^{-ie\mathscr{A}_{\mathrm{E}}(K^{\mathrm{rel}}_t)}\mathrm{J}_t e^{-iP_{\mathrm{f}}\cdot B_{T_t}}\Phi)]. \tag{7.8.17}$$

Using this we can show that the domain $\bigcap_{\mu=1}^3 D(P_{\mathrm{f}\mu}) \cap D(H_{\mathrm{rad}})$ stays invariant under $e^{-tH^{\mathrm{R}}_{\mathrm{PF}}(p)}$. This implies

Corollary 7.62 (Essential self-adjointness). *Under the assumptions of Theorem* 7.61 $H^{\mathrm{R}}_{\mathrm{PF}}(p)$ *is essentially self-adjoint on* $\bigcap_{\mu=1}^3 D(P_{\mathrm{f}\mu}) \cap D(H_{\mathrm{rad}})$.

Denote $E(p) = \inf \mathrm{Spec}(H^{\mathrm{R}}_{\mathrm{PF}}(p))$.

Corollary 7.63. *Under the assumptions of Theorem* 7.61 *we have*

(1) *Diamagnetic inequality:*

$$|(\Psi, e^{-tH^{\mathrm{R}}_{\mathrm{PF}}(p)}\Phi)| \le (|\Psi|, e^{-t(\sqrt{P_{\mathrm{f}}^2+m^2}-m+H_{\mathrm{rad}})}|\Phi|). \tag{7.8.18}$$

(2) *Energy comparison inequality:* $E(0) \le E(p)$.

Corollary 7.64 (Positivity improving for $e^{-tH^{\mathrm{R}}_{\mathrm{PF}}(0)}$). *Under the assumptions of Theorem* 7.61 *and assuming* $p = 0$ *the operator* $\mathfrak{S}^{-1}e^{-tH^{\mathrm{R}}_{\mathrm{PF}}(0)}\mathfrak{S}$ *is positivity improving. In particular, the ground state of* $H^{\mathrm{R}}_{\mathrm{PF}}(0)$ *is unique whenever it exists.*

7.9 The Pauli–Fierz model with spin

7.9.1 Definition

In the previous sections we considered the spinless version of the Pauli–Fierz Hamiltonian ignoring the spin of the electron. In this section we define the Hamiltonian including spin $1/2$ and construct its functional integral representation with a scalar integrand dependent on a jump process.

In this section we choose a specific $\mathscr{Q}$ space instead of the abstract setup made previously. This will serve the purpose of drawing sufficient regularity of $\mathscr{A}$ with respect to $\phi \in \mathscr{Q}$. Let $\mathscr{S}(\mathbb{R}^{3+\beta})$, $\beta = 0, 1, 2, \dots$, be the set of real-valued Schwartz test functions on $\mathbb{R}^{3+\beta}$ and put $\mathscr{S}_\beta = \oplus^3 \mathscr{S}_{\mathrm{real}}(\mathbb{R}^{3+\beta})$. Put

$$\mathscr{Q}_\beta = \mathscr{S}'_\beta, \tag{7.9.1}$$

with $\mathscr{S}'_\beta$ denoting the dual space of $\mathscr{S}_\beta$. Also, denote the pairing between elements of $\mathscr{Q}_\beta$ and $\mathscr{S}_\beta$ by $\langle\langle\phi, \mathbf{f}\rangle\rangle \in \mathbb{R}$, where $\phi \in \mathscr{Q}_\beta$ and $\mathbf{f} \in \mathscr{S}_\beta$. By the Bochner–Minlos Theorem there exists a probability space $(\mathscr{Q}_\beta, \Sigma_\beta, \mu_\beta)$ such that Σ_β is the smallest σ-field generated by $\{\langle\langle\phi, \mathbf{f}\rangle\rangle \mid f \in \mathscr{S}_\beta\}$ and $\langle\langle\phi, \mathbf{f}\rangle\rangle$ is a Gaussian random variable with mean zero and covariance

$$\int_{\mathscr{Q}_\beta} \langle\langle\phi, \mathbf{f}\rangle\rangle \langle\langle\phi, \mathbf{g}\rangle\rangle d\mu_\beta = \mathrm{q}_\beta(\mathbf{f}, \mathbf{g}). \tag{7.9.2}$$

We can extend $\langle\langle\phi, \mathbf{f}\rangle\rangle$ to more general $\mathbf{f}$. For any $\mathbf{f} = \Re\mathbf{f} + i\Im\mathbf{f} \in \oplus^3 \mathscr{S}(\mathbb{R}^{3+\beta})$ put $\langle\langle\phi, \mathbf{f}\rangle\rangle = \langle\langle\phi, \Re\mathbf{f}\rangle\rangle + i\langle\langle\phi, \Im\mathbf{f}\rangle\rangle$. Since $\mathscr{S}(\mathbb{R}^{3+\beta})$ is dense in $L^2(\mathbb{R}^{3+\beta})$ and the inequality

$$\int_{\mathscr{Q}_\beta} |\langle\langle\phi, \mathbf{f}\rangle\rangle|^2 d\mu_\beta \leq \|\mathbf{f}\|^2_{\oplus^3 L^2(\mathbb{R}^{3+\beta})}$$

holds by (7.9.2), we define $\langle\langle\phi, \mathbf{f}\rangle\rangle$ for $\mathbf{f} \in \oplus^3 L^2(\mathbb{R}^{3+\beta})$ by $\langle\langle\phi, \mathbf{f}\rangle\rangle = \lim_{n\to\infty} \langle\langle\phi, \mathbf{f}_n\rangle\rangle$ in $L^2(\mathscr{Q}_\beta)$, where $\{\mathbf{f}_n\}_{n=1}^\infty \subset \oplus^3 \mathscr{S}(\mathbb{R}^{3+\beta})$ is any sequence such that $\text{s-lim}_{n\to\infty} \mathbf{f}_n = \mathbf{f}$. Thus the quantized radiation field $\mathscr{A}_\beta(\mathbf{f})$ with $\mathbf{f} \in \oplus^3 L^2(\mathbb{R}^{3+\beta})$ is realized as

$$\big(\mathscr{A}_\beta(\mathbf{f})F\big)(\phi) = \langle\langle\phi, \mathbf{f}\rangle\rangle F(\phi), \quad \phi \in \mathscr{Q}_\beta, \tag{7.9.3}$$

in $L^2(\mathscr{Q}_\beta)$, with domain

$$D(\mathscr{A}_\beta(\mathbf{f})) = \left\{ F \in L^2(\mathscr{Q}_\beta) \,\middle|\, \int_{\mathscr{Q}_\beta} |\langle\langle\phi, \mathbf{f}\rangle\rangle F(\phi)|^2 d\mu_\beta < \infty \right\}.$$

We define $\mathscr{A}_\beta(\mathbf{f}, \phi)$ for each $\phi \in \mathscr{Q}_\beta$ by

$$\mathscr{A}_\beta(\mathbf{f}, \phi) = \langle\langle\phi, \mathbf{f}\rangle\rangle, \quad \mathbf{f} \in \oplus^3 L^2(\mathbb{R}^{3+\beta}). \tag{7.9.4}$$

In the same way as in the previous section we put

$$\begin{aligned} &\text{(Minkowskian)} \quad \mathscr{A} = \mathscr{A}_0, \quad \mu = \mu_0, \quad \mathscr{Q} = \mathscr{Q}_0, \\ &\text{(Euclidean)} \quad \mathscr{A}_\mathrm{E} = \mathscr{A}_1, \quad \mu_\mathrm{E} = \mu_1, \quad \mathscr{Q}_\mathrm{E} = \mathscr{Q}_1. \end{aligned} \tag{7.9.5}$$

The Hilbert space consisting of state vectors of the Pauli–Fierz Hamiltonian with spin $1/2$ is

$$\mathscr{H}_\mathrm{S} = L^2(\mathbb{R}^3; \mathbb{C}^2) \otimes L^2(\mathscr{Q}). \tag{7.9.6}$$

The quantized radiation field $\mathscr{A} = (\mathscr{A}_1, \mathscr{A}_2, \mathscr{A}_3)$ in $\mathscr{H}_\mathrm{S}$ is defined in the same way as in the previous section by

$$\mathscr{A}_\mu = \int_{\mathbb{R}^3}^{\oplus} \mathscr{A}_\mu(\tilde{\varphi}(\cdot - x)) dx, \quad \mu = 1, 2, 3, \tag{7.9.7}$$

under the identification $\mathscr{H}_{\mathrm{S}} \cong \int_{\mathbb{R}^3}^{\oplus} \mathbb{C}^2 \otimes L^2(\mathscr{Q}) dx$. The *quantized magnetic field* is defined by the curl of $\mathscr{A}$,

$$\mathscr{B} = (\mathscr{B}_1, \mathscr{B}_2, \mathscr{B}_3) = \mathrm{rot}_x \mathscr{A}. \tag{7.9.8}$$

It is straightforward to see that

$$\mathscr{B}_\mu = \sum_{\lambda,\alpha,\nu=1}^{3} \mathscr{A}_\lambda(\delta_{\lambda\nu} \varepsilon^{\mu\alpha\nu} \partial_{x_\alpha} \tilde{\varphi}(\cdot - x)), \tag{7.9.9}$$

where $\varepsilon^{\alpha\beta\gamma}$ denotes the antisymmetric tensor. We define $\mathscr{B}_\mu(f)$ with a test function $f \in L^2(\mathbb{R}^3)$ by

$$\mathscr{B}_\mu(f) = \sum_{\lambda,\alpha,\nu=1}^{3} \mathscr{A}_\lambda(\delta_{\lambda\nu} \varepsilon^{\mu\alpha\nu} \partial_{x_\alpha} f(\cdot - x)\lceil_{x=0}). \tag{7.9.10}$$

Indeed $\mathscr{B}_\mu(f)$ corresponds to

$$B_\mu(f) = \frac{1}{\sqrt{2}} \sum_{j=\pm} (a^*(\eta_\mu^j \hat{f}, j) - a(\eta_\mu^j \tilde{\hat{f}}, j))$$

in Fock representation, where $\tilde{\hat{f}}(k) = \hat{f}(-k)$ and $\eta^j(k) = -ik \times e^j(k)$. The Euclidean version of $\mathscr{B}_\mu(g)$ with test function $g \in L^2(\mathbb{R}^4)$ is defined by

$$\mathscr{B}_{\mathrm{E}\mu}(g) = \sum_{\lambda,\alpha,\nu=1}^{3} \mathscr{A}_{\mathrm{E}\lambda}(\delta_{\lambda\nu} \varepsilon^{\mu\alpha\nu} \partial_{x_\alpha} g(\cdot - \mathbf{x})\lceil_{\mathbf{x}=0}), \quad \mathbf{x} = (x_0, x) \in \mathbb{R} \times \mathbb{R}^3. \tag{7.9.11}$$

We will use this terminology later.

Let $\sigma = (\sigma_1, \sigma_2, \sigma_3)$ be the 2×2 Pauli matrices given in (3.7.1).

Definition 7.13 (Pauli–Fierz Hamiltonian with spin 1/2). The *Pauli–Fierz Hamiltonian with spin* $1/2$ is defined by

$$H_{\mathrm{PF}}^{\mathrm{S}} = \frac{1}{2}(-i\nabla - e\mathscr{A})^2 + V + H_{\mathrm{rad}} - \frac{e}{2}\sigma \cdot \mathscr{B}. \tag{7.9.12}$$

The usual definition by minimal coupling,

$$H_{\mathrm{PF}}^{\mathrm{S}} = \frac{1}{2}(\sigma \cdot (-i\nabla - e\mathscr{A}))^2 + V + H_{\mathrm{rad}} \tag{7.9.13}$$

indeed coincides with (7.9.12), which can be seen by expanding the square above and making use of the identity $(\sigma \cdot a)(\sigma \cdot b) = a \cdot b + i\sigma \cdot (a \times b)$.

In Fock representation the Pauli–Fierz Hamiltonian with spin $1/2$ is realized as

$$\frac{1}{2}(-i\nabla - eA)^2 + V + H_{\mathrm{rad}} - \frac{e}{2}\sigma\cdot B \tag{7.9.14}$$

on $L^2(\mathbb{R}^3;\mathbb{C}^2)\otimes\mathscr{F}_{\mathrm{rad}}$, where the quantized magnetic field B is given by

$$B_\mu = \int_{\mathbb{R}^3}^{\oplus} B_\mu(x)dx, \tag{7.9.15}$$

$$B_\mu(x) = \sum_{j=\pm}\frac{-i}{\sqrt{2}}\int (k\times e^j(k))_\mu \times\left(\frac{\hat\varphi(k)}{\sqrt{\omega(k)}}e^{-ik\cdot x}a^*(k,j) - \frac{\hat\varphi(-k)}{\sqrt{\omega(k)}}e^{ik\cdot x}a(k,j)\right)dk. \tag{7.9.16}$$

Theorem 7.65 (Self-adjointness). *Under Assumption* 7.1 $H_{\mathrm{PF}}^{\mathrm{S}}$ *is self-adjoint on* D_{PF} *and bounded from below. Moreover, it is essentially self-adjoint on any core of* $H_{\mathrm{PF},0}$.

Proof. Let $H_{\mathrm{PF}}^{\mathrm{S}} = H_{\mathrm{PF}} - \frac{e}{2}\sigma\cdot\mathscr{B}$. It is shown in Theorem 7.26 that H_{PF} is self-adjoint on D_{PF}. By the bound $\|H_{\mathrm{PF}}\Psi\|^2 + C\|\Psi\|^2 \ge C'\|H_{\mathrm{rad}}\Psi\|^2$ given in (7.4.25) and $\|-\frac{e}{2}\sigma\cdot\mathscr{B}\Psi\| \le \varepsilon\|H_{\mathrm{rad}}\Psi\| + C_\varepsilon\|\Psi\|$ for arbitrary $\varepsilon > 0$ and a constant C_ε, we know that the spin interaction part $-\frac{e}{2}\sigma\cdot\mathscr{B}$ is infinitesimally small with respect to H_{PF}. Then the proposition follows. □

7.9.2 Symmetry and polarization

In this section we take the Fock representation (7.9.14). When the form factor $\hat\varphi$ and the external potential V are rotation invariant, i.e., $\hat\varphi(\mathscr{R}k) = \hat\varphi(k)$ and $V(\mathscr{R}x) = V(x)$ for any $\mathscr{R}\in\mathrm{O}(3)$, then $H_{\mathrm{PF}}^{\mathrm{S}}$ has the symmetry

$$\mathrm{SU}(2)\times\mathrm{SO}_{\mathrm{part}}(3)\times\mathrm{SO}_{\mathrm{field}}(3)\times\text{helicity}, \tag{7.9.17}$$

where SU(2) and $\mathrm{SO}_{\mathrm{part}}(3)$ come from the spin and the angular momentum of the particle, respectively, while the $\mathrm{SO}_{\mathrm{field}}(3)$ and helicity part respectively from the angular momentum and the helicity of photons.

Let $\mathscr{R}\in\mathrm{SO}(3)$ and $\hat k = k/|k|$. Note that $\mathscr{R}\hat k = \widehat{\mathscr{R}k}$. Thus the orthogonal bases $\{e^+(\mathscr{R}k), e^-(\mathscr{R}k), \widehat{\mathscr{R}k}\}$ and $\{\mathscr{R}e^+(k), \mathscr{R}e^-(k), \mathscr{R}\hat k\}$ in $\mathbb{R}^3$ at $\mathscr{R}k$ satisfy

$$\begin{pmatrix} e^+(\mathscr{R}k)\\ e^-(\mathscr{R}k)\\ \widehat{\mathscr{R}k}\end{pmatrix} = \begin{pmatrix}\cos\vartheta 1_3 & -\sin\vartheta 1_3 & 0\\ \sin\vartheta 1_3 & \cos\vartheta 1_3 & 0\\ 0 & 0 & 1_3\end{pmatrix}\begin{pmatrix}\mathscr{R}e^+(k)\\ \mathscr{R}e^-(k)\\ \mathscr{R}\hat k\end{pmatrix}, \tag{7.9.18}$$

where 1_3 is the 3×3 unit matrix and $\vartheta = \vartheta(\mathscr{R},k)$ satisfies $\cos\vartheta = \mathscr{R}e^+(k)\cdot e^+(\mathscr{R}k)$. Let $\mathscr{R} = \mathscr{R}(n,\phi)\in\mathrm{SO}(3)$ be the rotation around $n\in\mathrm{S}^2 = \{k\in\mathbb{R}^3 \,|\, |k| = 1\}$ by the angle $\phi \in \mathbb{R}$, and $\det \mathscr{R} = 1$. Also, let

$$\ell_k = k \times (-i\nabla_k) = (\ell_{k1}, \ell_{k2}, \ell_{k3}) \tag{7.9.19}$$

be the triplet of angular momentum operators in $L^2(\mathbb{R}^3_k)$. Then (7.9.18) can be rewritten as

$$e^{i\phi n\cdot \ell_k}\begin{pmatrix} e^+(k) \\ e^-(k)\end{pmatrix} = e^{-i\vartheta\Sigma_2}\begin{pmatrix} \mathscr{R} & 0 \\ 0 & \mathscr{R}\end{pmatrix}\begin{pmatrix} e^+(k) \\ e^-(k)\end{pmatrix}, \tag{7.9.20}$$

where

$$\Sigma_2 = i\begin{pmatrix} 0 & -1_3 \\ 1_3 & 0\end{pmatrix}. \tag{7.9.21}$$

In order to discuss the symmetry of $H_{\mathrm{PF}}^{\mathrm{S}}$ we introduce *coherent polarization vectors* in given directions. We make the following

Definition 7.14 (Coherent polarization vectors). The polarization vectors $e^\pm$ are *coherent polarization vectors* in direction $n \in \mathrm{S}^2$ whenever there exists $z \in \mathbb{Z}$ such that for any $\phi \in [0, 2\pi)$ and any k with $\hat{k} \neq n$,

$$\begin{pmatrix} e^+(\mathscr{R}k) \\ e^-(\mathscr{R}k)\end{pmatrix} = \begin{pmatrix} \cos(z\phi)1_3 & -\sin(z\phi)1_3 \\ \sin(z\phi)1_3 & \cos(z\phi)1_3\end{pmatrix}\begin{pmatrix} \mathscr{R} & 0 \\ 0 & \mathscr{R}\end{pmatrix}\begin{pmatrix} e^+(k) \\ e^-(k)\end{pmatrix} \tag{7.9.22}$$

or componentwise

$$\begin{pmatrix} e^+_\mu(\mathscr{R}k) \\ e^-_\mu(\mathscr{R}k)\end{pmatrix} = \begin{pmatrix} \cos(z\phi) & -\sin(z\phi) \\ \sin(z\phi) & \cos(z\phi)\end{pmatrix}\begin{pmatrix} (\mathscr{R}e^+(k))_\mu \\ (\mathscr{R}e^-(k))_\mu\end{pmatrix}, \tag{7.9.23}$$

where $\mathscr{R} = \mathscr{R}(n, \phi)$.

Assumption 7.2. $e^\pm$ are coherent polarization vectors in direction $n \in \mathrm{S}^2$.

By Assumption 7.2 we have

$$\exp\{i\phi\,(z\Sigma_2 + n\cdot \ell_k)\}\begin{pmatrix} e^+_\mu(k) \\ e^-_\mu(k)\end{pmatrix} = \begin{pmatrix} (\mathscr{R}e^+(k))_\mu \\ (\mathscr{R}e^-(k))_\mu\end{pmatrix}. \tag{7.9.24}$$

Here is an example of polarization vectors satisfying Assumption 7.2.

Example 7.2. Let $n_3 = (0,0,1)$ and $S^2_\parallel = \{(\sqrt{1-z^2}, 0, z) \in S^2 | -1 \le z \le 1\}$. Take any polarization vectors on $S^2_\parallel$: $e^\pm(k)$ for $k \in S^2_\parallel$. For $k \in S^2 \setminus S^2_\parallel$, there exists a unique $h_k \in S^2_\parallel$ and $0 < \phi < 2\pi$ such that $\mathscr{R}(n_3, \phi)h_k = k$. Then we define $e^\pm(k)$ for $k \in S^2 \setminus S^2_\parallel$ by

$$\begin{pmatrix} e^+(k) \\ e^-(k) \end{pmatrix} = \begin{pmatrix} \cos(z\phi)1_3 & -\sin(z\phi)1_3 \\ \sin(z\phi)1_3 & \cos(z\phi)1_3 \end{pmatrix} \begin{pmatrix} \mathscr{R}(n_3, \phi)e^+(h_k) \\ \mathscr{R}(n_3, \phi)e^-(h_k) \end{pmatrix}. \tag{7.9.25}$$

It is readily checked that $e^\pm$ satisfy (7.9.22) with $(n_3, z) \in S^2 \times \mathbb{Z}$.

Example 7.3. Let $n \in \mathrm{S}^2$, and

$$e^+(k) = \hat{k} \times n/\sin\vartheta, \quad e^-(k) = \hat{k} \times e^+(k), \tag{7.9.26}$$

where $\vartheta = \cos^{-1}(\hat{k} \cdot n)$. Then, since $\mathscr{R} = \mathscr{R}(n, \phi)$ satisfies that $\mathscr{R}n = n$ and $\mathscr{R}u \times \mathscr{R}v = \mathscr{R}(u \times v)$, $e^\pm(k)$ obeys (7.9.22) with $(n, 0) \in \mathrm{S}^2 \times \mathbb{Z}$, i.e., $\mathscr{R}e^\pm(k) = e^\pm(\mathscr{R}k)$. In particular, when $(n, z) = (n_3, 0) \in S^2 \times \mathbb{Z}$, $\sin\vartheta = \sqrt{k_1^2 + k_2^2}$ and

$$e^+(k) = \frac{(-k_2, k_1, 0)}{\sqrt{k_1^2 + k_2^2}}, \quad e^-(k) = \frac{(k_3k_1, -k_2k_3, k_1^2 + k_2^2)}{|k|\sqrt{k_1^2 + k_2^2}}. \tag{7.9.27}$$

Let $\mathfrak{H}_2 : L^2(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3) \to L^2(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3)$ be given by

$$\mathfrak{H}_2 = i \begin{pmatrix} 0 & -1_{L^2(\mathbb{R}^3)} \\ 1_{L^2(\mathbb{R}^3)} & 0 \end{pmatrix}. \tag{7.9.28}$$

Under Assumption 7.2 with a suitable $z \in \mathbb{Z}$ we define

$$S_\mathrm{f} = d\Gamma(z\mathfrak{H}_2) \tag{7.9.29}$$

and

$$L_\mathrm{f} = (L_{\mathrm{f},1}, L_{\mathrm{f},2}, L_{\mathrm{f},3}) = d\Gamma(\ell_k). \tag{7.9.30}$$

S_f is called the *helicity* and L_f the *angular momentum of the field*. The helicity S_f can be formally written as

$$S_\mathrm{f} = i \int z(a^*(k, -)a(k, +) - a^*(k, +)a(k, -))dk. \tag{7.9.31}$$

For $(n, z) \in \mathrm{S}^2 \times \mathbb{Z}$ define $J_\mathrm{f} = J_\mathrm{f}(n, z)$ by

$$J_\mathrm{f} = n \cdot L_\mathrm{f} + S_\mathrm{f} \tag{7.9.32}$$

and $J_\mathrm{p} = J_\mathrm{p}(n)$ by

$$J_\mathrm{p} = n \cdot \ell_x + \frac{1}{2} n \cdot \sigma. \tag{7.9.33}$$

J_{p} is the angular momentum plus spin for the particle. Write

$$\mathrm{J} = J_{\mathrm{p}} \otimes 1 + 1 \otimes J_{\mathrm{f}}. \tag{7.9.34}$$

Clearly, $\mathrm{J} = \mathrm{J}(n, z)$ is defined for each $(n, z) \in \mathrm{S}^2 \times \mathbb{Z}$.

Proposition 7.66 (J-invariance of $H_{\mathrm{PF}}^{\mathrm{S}}$). *If the polarization vectors are coherent in direction n, and $\hat{\varphi}$ and V are rotation invariant, then*

$$e^{i\phi \mathrm{J}} H_{\mathrm{PF}}^{\mathrm{S}} e^{-i\phi \mathrm{J}} = H_{\mathrm{PF}}^{\mathrm{S}}$$

for every $\phi \in \mathbb{R}$.

Proof. Write $a^\sharp(\binom{f}{g})$ for $a^\sharp(f \oplus g)$. Notice that for a rotation invariant f,

$$e^{i\phi J_{\mathrm{f}}} a^* \left(f e^{-ik\cdot x} \begin{pmatrix} e_\mu^+ \\ e_\mu^- \end{pmatrix} \right) e^{-i\phi J_{\mathrm{f}}} = a^* \left(f e^{i\phi(z\Sigma_2 + n\ell_k)} e^{-ik\cdot x} \begin{pmatrix} e_\mu^+ \\ e_\mu^- \end{pmatrix} \right).$$

Since the polarization vectors are coherent, we have by (7.9.24)

$$= a^* \left(f e^{-i\mathscr{R}k\cdot x} \begin{pmatrix} (\mathscr{R}e^+)_\mu \\ (\mathscr{R}e^-)_\mu \end{pmatrix} \right) = \sum_{\nu=1}^{3} \mathscr{R}_{\mu\nu} a^\sharp \left(f e^{-ik\cdot \mathscr{R}^{-1}x} \begin{pmatrix} e_\nu^+ \\ e_\nu^- \end{pmatrix} \right), \tag{7.9.35}$$

where $\mathscr{R} = \mathscr{R}(n, \phi) = (\mathscr{R}_{\mu\nu})_{1\le\mu,\nu\le 3}$. By (7.9.35), we see that the field part transforms as

$$\begin{aligned} e^{i\phi J_{\mathrm{f}}} H_{\mathrm{rad}} e^{-i\phi J_{\mathrm{f}}} &= H_{\mathrm{rad}}, \\ e^{i\phi J_{\mathrm{f}}} A_\mu(x) e^{-i\phi J_{\mathrm{f}}} &= (\mathscr{R}A)_\mu(\mathscr{R}^{-1}x), \end{aligned}$$

and the particle part as

$$\begin{aligned} e^{i\phi n\cdot \ell_x} x_\mu e^{-i\phi n\cdot\ell_x} &= (\mathscr{R}x)_\mu, \\ e^{i\phi n\cdot\ell_x} (-i\nabla_x)_\mu e^{-i\phi n\cdot\ell_x} &= (\mathscr{R}(-i\nabla_x))_\mu, \\ e^{i\phi n\cdot(1/2)\sigma} \sigma_\mu e^{-i\phi n\cdot(1/2)\sigma} &= (\mathscr{R}\sigma)_\mu. \end{aligned}$$

Together with all the identities above we have

$$e^{i\phi \mathrm{J}} H_{\mathrm{PF}}^{\mathrm{S}} e^{-i\phi \mathrm{J}} = \frac{1}{2}(\mathscr{R}(-i\nabla) - e\mathscr{R}\mathscr{A}(\mathscr{R}^{-1}\mathscr{R}x))^2 + H_{\mathrm{rad}} + V(\mathscr{R}x) = H_{\mathrm{PF}}^{\mathrm{S}}. \quad \square$$

From this proposition it is clear that $H_{\mathrm{PF}}^{\mathrm{S}}$ with coherent polarization vectors has the symmetry

$$\mathrm{SU}(2) \times \mathrm{SO}_{\mathrm{part}}(3) \times \mathrm{SO}_{\mathrm{field}}(3) \times \text{helicity}. \tag{7.9.36}$$

Denote the set of half integers by $\mathbb{Z}_{1/2} = \{w/2 | w \in \mathbb{Z}\}$. For each $(n, z) \in \mathrm{S}^2 \times \mathbb{Z}$, notice that

$$\operatorname{Spec}(n \cdot (\ell_x + (1/2)\sigma)) = \mathbb{Z}_{1/2}, \tag{7.9.37}$$

$$\operatorname{Spec}(n \cdot L_{\mathrm{f}}) = \mathbb{Z}, \tag{7.9.38}$$

$$\operatorname{Spec}(S_{\mathrm{f}}) = \begin{cases} \mathbb{Z}, & z \neq 0, \\ 0, & z = 0. \end{cases} \tag{7.9.39}$$

Thus for each $(n, z) \in \mathrm{S}^2 \times \mathbb{Z}$,

$$\operatorname{Spec}(\mathrm{J}) = \mathbb{Z}_{1/2} \tag{7.9.40}$$

and we have the theorem below.

Theorem 7.67. *If the polarization vectors are coherent in direction n, and $\hat{\varphi}$ and V are rotation invariant, then $\mathcal{H}_{\mathrm{S}}$ and $H_{\mathrm{PF}}^{\mathrm{S}}$ can be decomposed as*

$$\mathcal{H}_{\mathrm{S}} = \bigoplus_{w \in \mathbb{Z}_{1/2}} \mathcal{H}_{\mathrm{S}}(w), \quad H_{\mathrm{PF}}^{\mathrm{S}} = \bigoplus_{w \in \mathbb{Z}_{1/2}} H_{\mathrm{PF}}^{\mathrm{S}}(w). \tag{7.9.41}$$

Here $\mathcal{H}_{\mathrm{S}}(w)$ is the subspace spanned by eigenvectors of J *associated with eigenvalue $w \in \mathbb{Z}_{1/2}$ and $H_{\mathrm{PF}}^{\mathrm{S}}(w) = H_{\mathrm{PF}}^{\mathrm{S}} \lceil_{\mathcal{H}_{\mathrm{S}}(w)}$.*

Proof. This follows from Proposition 7.66 and the fact that $\operatorname{Spec}(\mathrm{J}) = \mathbb{Z}_{1/2}$. □

Next we consider general polarization vectors, i.e., not necessarily coherent sets. The Pauli–Fierz Hamiltonians with different polarization vectors, however, are unitary equivalent. Denote the Pauli–Fierz Hamiltonian with polarization vectors $e^{\pm}$ by $H_{\mathrm{PF}}^{\mathrm{S}}(e^{\pm})$. Combining Proposition 7.3 and Theorem 7.67, we have the corollary below.

Corollary 7.68 (Fiber decomposition of Pauli–Fierz Hamiltonian with spin 1/2; general polarization). *Suppose that $\hat{\varphi}$ and V are rotation invariant, and $e^{\pm}$ are coherent polarization vectors. Let $\eta^{\pm}$ be arbitrary polarization vectors. Then $H_{\mathrm{PF}}^{\mathrm{S}}(\eta^{\pm})$ is unitary equivalent to $\bigoplus_{w \in \mathbb{Z}_{1/2}} H_{\mathrm{PF}}^{\mathrm{S}}(e^{\pm}, w)$.*

By using the symmetries of the Pauli–Fierz Hamiltonian we can show the degeneracy of ground states of $H_{\mathrm{PF}}^{\mathrm{S}}$. Assume that V is rotation invariant and the polarization vectors $e^{\pm}$ are given by

$$e^{+}(k) = \frac{(-k_2, k_1, 0)}{\sqrt{k_1^2 + k_2^2}}, \quad e^{-}(k) = \hat{k} \times e^{+}(k). \tag{7.9.42}$$

These are coherent in direction n_3 and their helicity is zero. Let $\Lambda : \mathbb{R}^3 \to \mathbb{R}^3$ be the flip defined by

$$\Lambda \begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix} = \begin{pmatrix} k_1 \\ -k_2 \\ k_3 \end{pmatrix}.$$

Consider the unitary operators $L^2(\mathbb{R}^3;\mathbb{C}^2) \to L^2(\mathbb{R}^3;\mathbb{C}^2)$

$$\tilde{u} : \begin{pmatrix} f \\ g \end{pmatrix} \mapsto \begin{pmatrix} f \circ \Lambda \\ g \circ \Lambda \end{pmatrix}, \quad u : \begin{pmatrix} f \\ g \end{pmatrix} \mapsto \begin{pmatrix} -f \circ \Lambda \\ g \circ \Lambda \end{pmatrix}. \tag{7.9.43}$$

A computation gives

$$u^{\sharp^{-1}} k_\mu u^\sharp = \begin{cases} k_\mu, & \mu = 1,3, \\ -k_\mu, & \mu = 2, \end{cases} \qquad u^{\sharp^{-1}} \nabla_\mu u^\sharp = \begin{cases} \nabla_\mu, & \mu = 1,3, \\ -\nabla_\mu, & \mu = 2, \end{cases} \tag{7.9.44}$$

where $u^\sharp = u$ or $\tilde{u}$. From this and

$$e^-(k) = \frac{(-k_3k_1, -k_2k_3, k_1^2 + k_2^2)}{|k|\sqrt{k_1^2 + k_2^2}},$$

we have for rotation invariant f and g,

$$u^{-1} \begin{pmatrix} e_\mu^+ f \\ e_\mu^- g \end{pmatrix} = \begin{cases} \binom{e_\mu^+ f}{e_\mu^- g}, & \mu = 1,3, \\ -\binom{e_\mu^+ f}{e_\mu^- g}, & \mu = 2. \end{cases} \tag{7.9.45}$$

Then the second quantization $\Gamma(u)$ of u induces the unitary operator on $\mathscr{F}_{\mathrm{rad}}$ and for rotation invariant $\hat{\varphi}$ we obtain

$$\Gamma(u)^{-1} A_\mu(x) \Gamma(u) = \begin{cases} A_\mu(\Lambda x), & \mu = 1,3, \\ -A_\mu(\Lambda x), & \mu = 2, \end{cases} \tag{7.9.46}$$

and

$$\Gamma(u)^{-1} H_{\mathrm{rad}} \Gamma(u) = H_{\mathrm{rad}}. \tag{7.9.47}$$

Next we consider the transformation on $\sigma = (\sigma_1, \sigma_2, \sigma_3)$ given by

$$\sigma_\mu \mapsto \sigma_2 \sigma_\mu \sigma_2 = \begin{cases} -\sigma_\mu, & \mu = 1,3, \\ \sigma_\mu, & \mu = 2. \end{cases} \tag{7.9.48}$$

Under the identification $L^2(\mathbb{R}^3;\mathbb{C}^2) \cong \mathbb{C}^2 \otimes L^2(\mathbb{R}^3)$, we define $\tau = \sigma_2 \otimes \tilde{u} : L^2(\mathbb{R}^3;\mathbb{C}^2) \to L^2(\mathbb{R}^3;\mathbb{C}^2)$. This satisfies

$$\tau^{-1}(\sigma_\mu \otimes f)\tau = \begin{cases} -\sigma_\mu \otimes f \circ \Lambda^{-1}, & \mu = 1,3, \\ \sigma_\mu \otimes f \circ \Lambda^{-1}, & \mu = 2, \end{cases} \tag{7.9.49}$$

and

$$\tau^{-1}(\sigma_\mu \otimes \nabla_\mu)\tau = -\sigma_\mu \otimes \nabla_\mu, \quad \mu = 1, 2, 3. \tag{7.9.50}$$

We finally define the unitary operator $\mathfrak{J} : \mathcal{H}_{\mathrm{S}} \to \mathcal{H}_{\mathrm{S}}$ by

$$\mathfrak{J} = \tau \otimes \Gamma(u). \tag{7.9.51}$$

Combining (7.9.46)–(7.9.50), for rotation invariant $\hat{\varphi}$ and V

$$\mathfrak{J}^{-1}\sigma_\mu(-i\nabla_\mu - A_\mu)\mathfrak{J} = -\sigma_\mu(-i\nabla_\mu - A_\mu), \tag{7.9.52}$$

$$\mathfrak{J}^{-1}H_{\mathrm{rad}}\mathfrak{J} = H_{\mathrm{rad}}, \tag{7.9.53}$$

$$\mathfrak{J}^{-1}V\mathfrak{J} = V \tag{7.9.54}$$

are obtained. From these relations we can show the theorem below.

Theorem 7.69 (Reflection symmetry of $H_{\mathrm{PF}}^{\mathrm{S}}$). *If $\hat{\varphi}$ and V are rotation invariant, and the polarization vectors $e^{\pm}$ are given by* (7.9.42)*, then $H_{\mathrm{PF}}^{\mathrm{S}}(z)$ and $H_{\mathrm{PF}}^{\mathrm{S}}(-z)$ are unitary equivalent.*

Proof. Since $e^{\pm}$ is coherent in direction $(0, 0, 1)$ and its helicity is zero, J is of the form $\mathrm{J} = (\ell_{x,3} + \frac{1}{2}\sigma_3) \otimes 1 + 1 \otimes L_{\mathrm{f},3}$. It follows that

$$\mathfrak{J}^{-1}\mathrm{J}\mathfrak{J} = -\mathrm{J}. \tag{7.9.55}$$

This implies that $\mathfrak{J}$ maps $\mathcal{H}_{\mathrm{S}}(w)$ onto $\mathcal{H}_{\mathrm{S}}(-w)$. Furthermore,

$$\mathfrak{J}^{-1}H_{\mathrm{PF}}^{\mathrm{S}}\mathfrak{J} = \frac{1}{2}(-\sigma \cdot (-i\nabla - eA))^2 + V + H_{\mathrm{rad}} = H_{\mathrm{PF}}^{\mathrm{S}}.$$

Thus $\mathfrak{J}^{-1}H_{\mathrm{PF}}^{\mathrm{S}}(w)\mathfrak{J} = H_{\mathrm{PF}}^{\mathrm{S}}(-w)$ follows. □

An interesting application of Theorem 7.69 is to estimate the multiplicity of bound states of $H_{\mathrm{PF}}^{\mathrm{S}}$.

Corollary 7.70 (Degeneracy of bound states). *Suppose that V and $\hat{\varphi}$ is rotation invariant. Let M be the multiplicity of eigenvalues of $H_{\mathrm{PF}}^{\mathrm{S}}$. Then M is an even number. In particular, whenever a ground states exists, it is degenerate.*

Proof. By the unitary equivalence we may suppose that the polarization vectors of $H_{\mathrm{PF}}^{\mathrm{S}}$ are given by (7.9.42). Thus $H_{\mathrm{PF}}^{\mathrm{S}} = \bigoplus_{w \in \mathbb{Z}_{1/2}} H_{\mathrm{PF}}^{\mathrm{S}}(w)$. Let Ψ be a bound state of $H_{\mathrm{PF}}^{\mathrm{S}}$ with eigenvalue a. Thus Ψ is a bound state of $H_{\mathrm{PF}}^{\mathrm{S}}(w)$ with some $w \in \mathbb{Z}_{1/2}$, $H_{\mathrm{PF}}^{\mathrm{S}}(w)\Psi = a\Psi$. Theorem 7.69 implies $H_{\mathrm{PF}}^{\mathrm{S}}(w) \cong H_{\mathrm{PF}}^{\mathrm{S}}(-w)$, and thus there exists $\Phi \in \mathcal{H}_{\mathrm{S}}(-w)$ such that $H_{\mathrm{PF}}^{\mathrm{S}}(-w)\Phi = a\Phi$. Thus the multiplicity of a is even. □

7.9.3 Functional integral representation

As in the classical case investigated in Chapter 3, in order to construct the functional integral representation of $(F, e^{-tH_{\mathrm{PF}}^{\mathrm{S}}}G)$ with a *scalar* integrand we introduce a two-valued spin variable $\theta \in \mathbb{Z}_2 = \{-1, +1\} = \{\theta_1, \theta_2\}$ and redefine $H_{\mathrm{PF}}^{\mathrm{S}}$ in order to reduce it to a scalar operator. Since

$$H_{\mathrm{PF}}^{\mathrm{S}} = \frac{1}{2}(-i\nabla - e\mathscr{A})^2 + V + H_{\mathrm{rad}} - \frac{e}{2}\begin{pmatrix} \mathscr{B}_3 & \mathscr{B}_1 - i\mathscr{B}_2 \\ \mathscr{B}_1 + i\mathscr{B}_2 & -\mathscr{B}_3 \end{pmatrix},$$

our Hamiltonian can be regarded as an operator acting on $L^2(\mathbb{R}^3 \times \mathbb{Z}_2) \otimes L^2(\mathscr{Q})$ and is defined by

$$(H_{\mathrm{PF}}^{\mathbb{Z}_2} F)(\theta) = \left(\frac{1}{2}(-i\nabla - e\mathscr{A})^2 + V + H_{\mathrm{rad}} + \mathscr{H}_{\mathrm{d}}(\theta)\right) F(\theta) + \mathscr{H}_{\mathrm{od}}(-\theta) F(-\theta) \tag{7.9.56}$$

with $\theta \in \mathbb{Z}_2$ for $F = \binom{F(+1)}{F(-1)} \in L^2(\mathbb{R}^3 \times \mathbb{Z}_2) \otimes L^2(\mathscr{Q})$, where each component is $F(\pm) \in L^2(\mathbb{R}^3) \times L^2(\mathscr{Q})$. Here $\mathscr{H}_{\mathrm{d}}$ and $\mathscr{H}_{\mathrm{od}}$ denote the diagonal resp. off-diagonal parts of the spin interaction explicitly given by

$$\mathscr{H}_{\mathrm{d}}(\theta) = \int_{\mathbb{R}^3}^{\oplus} \mathscr{H}_{\mathrm{d}}(x, \theta) dx, \quad \mathscr{H}_{\mathrm{od}}(-\theta) = \int_{\mathbb{R}^3}^{\oplus} \mathscr{H}_{\mathrm{od}}(x, -\theta) dx, \tag{7.9.57}$$

where

$$\mathscr{H}_{\mathrm{d}}(x, \theta) = -\frac{e}{2}\theta \mathscr{B}_3(x), \quad \mathscr{H}_{\mathrm{od}}(x, -\theta) = -\frac{e}{2}\left(\mathscr{B}_1(x) - i\theta\mathscr{B}_2(x)\right). \tag{7.9.58}$$

Here $\mathscr{B}_\mu(x)$ is defined in (7.9.8). Now we define the Pauli–Fierz Hamiltonian with spin $1/2$ on $L^2(\mathbb{R}^3 \times \mathbb{Z}_2) \otimes L^2(\mathscr{Q})$ instead of $\mathscr{H}_{\mathrm{S}}$. Furthermore, we treat $\mathscr{A}$ and $\mathscr{B}$ or $\mathscr{H}_{\mathrm{d}}$ and $\mathscr{H}_{\mathrm{od}}$ as not necessarily dependent vectors. This is the same idea as the one applied to $H_{\mathbb{Z}_2}(a, b)$ discussed in Chapter 3. Write

$$\mathscr{K} = L^2(\mathbb{R}^3 \times \mathbb{Z}_2) \otimes L^2(\mathscr{Q}) \tag{7.9.59}$$

and its Euclidean version

$$\mathscr{K}_{\mathrm{E}} = L^2(\mathbb{R}^3 \times \mathbb{Z}_2) \otimes L^2(\mathscr{Q}_{\mathrm{E}}). \tag{7.9.60}$$

Definition 7.15 (Pauli–Fierz Hamiltonian with spin $1/2$ on $\mathscr{K}$). We define the *Pauli–Fierz Hamiltonian with spin* $1/2$ on $\mathscr{K}$ by $H_{\mathrm{PF}}^{\mathbb{Z}_2}$.

The key idea of constructing a functional integral representation of $e^{-tH_{\mathrm{PF}}^{\mathbb{Z}_2}}$ is to use the identity

$$\mathscr{K} \cong \int_{\mathscr{Q}}^{\oplus} L^2(\mathbb{R}^3 \times \mathbb{Z}_2) d\mu. \tag{7.9.61}$$

In other words, we regard $\mathscr{K}$ as the set of $L^2(\mathbb{R}^3 \times \mathbb{Z}_2)$-valued L^2 functions on $\mathscr{Q}$. We make the decomposition

$$H_{\mathrm{PF}}^{\mathbb{Z}_2} = \int_{\mathscr{Q}}^{\oplus} H(\phi) d\mu + H_{\mathrm{rad}},$$

where $H(\phi)$ is a self-adjoint operator on $L^2(\mathbb{R}^3 \times \mathbb{Z}_2)$. We construct a functional integral representation of $e^{-tH_{\mathrm{PF}}^{\mathbb{Z}_2}}$ through the functional integrals of $e^{-tH(\phi)}$ and $e^{-tH_{\mathrm{rad}}}$, and the Trotter product formula. In order to do that we will use the identity

$$(F, e^{-t\int_{\mathscr{Q}}^{\oplus} H(\phi) d\mu} G) = \int_{\mathscr{Q}} \left(F(\phi), e^{-tH(\phi)} G(\phi) \right) d\mu$$

while we have already done this of $(F(\phi), e^{-tH(\phi)} G(\phi))$, $\phi \in \mathscr{Q}$, in Chapter 3.

First we make the fiber decomposition of the quantized radiation field $\mathscr{A}$ and the quantized magnetic field $\mathscr{B}$ on $\mathscr{Q}$. For each $\phi \in \mathscr{Q}$ define

$$\mathscr{A}_\mu(\phi) = \int_{\mathbb{R}^3}^{\oplus} \mathscr{A}_\mu(x,\phi) dx, \quad \mathscr{B}_\mu(\phi) = \int_{\mathbb{R}^3}^{\oplus} \mathscr{B}_\mu(x,\phi) dx,$$

where

$$\mathscr{A}_\mu(x,\phi) = \mathscr{A}_\mu(\tilde{\varphi}(\cdot - x), \phi), \quad \mathscr{B}_\mu(x,\phi) = (\nabla_x \times \mathscr{A}(\tilde{\varphi}(\cdot - x), \phi))_\mu. \tag{7.9.62}$$

Recall that $\mathscr{A}_\mu(\tilde{\varphi}(\cdot - x), \phi) = \langle\!\langle \phi, \oplus_{\nu=1}^3 \delta_{\mu\nu} \tilde{\varphi}(\cdot - x) \rangle\!\rangle$. For each fiber $\phi \in \mathscr{Q}$, define the Hamiltonian $H(\phi)$ on $L^2(\mathbb{R}^3 \times \mathbb{Z}_2)$ by

$$(H(\phi)F)(x,\theta) = \left(\frac{1}{2}(-i\nabla - e\mathscr{A}(\phi))^2 + V + \mathscr{H}_{\mathrm{d}}(\phi) \right) F(x,\theta) + \mathscr{H}_{\mathrm{od}}(\phi) F(x,-\theta). \tag{7.9.63}$$

Here

$$\mathscr{H}_{\mathrm{d}}(\phi) = \int_{\mathbb{R}^3}^{\oplus} \mathscr{H}_{\mathrm{d}}(x,\theta,\phi) dx, \quad \mathscr{H}_{\mathrm{od}}(\phi) = \int_{\mathbb{R}^3}^{\oplus} \mathscr{H}_{\mathrm{od}}(x,-\theta,\phi) dx, \tag{7.9.64}$$

where

$$\mathscr{H}_{\mathrm{d}}(x,\theta,\phi) = -\frac{e}{2}\theta \mathscr{B}_3(x,\phi), \quad \mathscr{H}_{\mathrm{od}}(x,-\theta,\phi) = -\frac{e}{2}(\mathscr{B}_1(x,\phi) - i\theta\mathscr{B}_2(x,\phi)). \tag{7.9.65}$$

To prevent the off-diagonal part $\mathscr{H}_{\mathrm{od}}$ vanish we introduce $H_{\mathrm{PF},\varepsilon}^{\mathbb{Z}_2}$ in a similar manner as in $H_{\mathbb{Z}_2}^{\varepsilon}(a,b)$ in Chapter 3 by

$$\begin{aligned}(H_{\mathrm{PF},\varepsilon}^{\mathbb{Z}_2} F)(\theta) = {} & \left(\frac{1}{2}(-i\nabla - e\mathscr{A})^2 + V + H_{\mathrm{rad}} + \mathscr{H}_{\mathrm{d}}(\theta) \right) F(\theta) \\ & + \Psi_\varepsilon(\mathscr{H}_{\mathrm{od}}(-\theta)) F(-\theta),\end{aligned} \tag{7.9.66}$$

where $\Psi_\varepsilon(X) = X + \varepsilon\psi_\varepsilon(X)$ and ψ_ε is the indicator function given by (3.7.30). Also, let $H_\varepsilon(\phi)$ be the counterpart of $H(\phi)$ with $\mathscr{H}_{\mathrm{od}}(\phi)$ replaced by $\Psi_\varepsilon(\mathscr{H}_{\mathrm{od}}(\phi))$.

Recall that $\theta_\alpha = (-1)^\alpha$ and $(q_t)_{t\geq 0} = (B_t, \theta_{N_t})_{t\geq 0}$ is an $(\mathbb{R}^3 \times \mathbb{Z}_2)$-valued stochastic process on $\mathscr{X} \times \mathscr{S}$; see Section 3.7.2.

Lemma 7.71. *If $\tilde{\varphi} \in C_0^\infty(\mathbb{R}^3)$, then for every $\phi \in \mathscr{Q}$, $H_\varepsilon(\phi)$ is self-adjoint on $D(-(1/2)\Delta) \otimes \mathbb{Z}_2$ and for $g \in L^2(\mathbb{R}^3 \times \mathbb{Z}_2)$,*

$$(e^{-tH_\varepsilon(\phi)}g)(x, \theta_\alpha) = e^t \mathbb{E}^{x,\alpha}[e^{-\int_0^t V(B_s)ds} e^{Z_t(\phi,\varepsilon)} g(q_t)],$$

where

$$\begin{aligned} Z_t(\phi, \varepsilon) = &-ie\int_0^t \mathscr{A}(\tilde{\varphi}(\cdot - B_s), \phi)\cdot dB_s \\ &- \int_0^t \mathscr{H}_{\mathrm{d}}(B_s, \theta_{N_s}, \phi)ds + \int_0^{t+} W_{\varepsilon,\phi}(B_s, -\theta_{N_{s-}})dN_s \end{aligned}$$

and $W_{\varepsilon,\phi} = \log(-\Psi_\varepsilon(\mathscr{H}_{\mathrm{od}}(x, -\theta, \phi)))$.

Proof. As $\tilde{\varphi} \in C_0^\infty(\mathbb{R}^3)$, we have $\mathscr{A}_\mu(\phi) = \langle\langle \phi, \oplus_{\nu=1}^3 \delta_{\mu\nu}\tilde{\varphi}(\cdot - x)\rangle\rangle \in C_{\mathrm{b}}^\infty(\mathbb{R}_x^3)$, for every $\phi \in \mathscr{Q}$. Thus $H_\varepsilon(\phi)$ is the Pauli operator with spin $1/2$ with a smooth bounded vector potential $\mathscr{A}(\phi)$, and the off-diagonal part is perturbed by the bounded operator $\varepsilon\psi_\varepsilon(\mathscr{H}_{\mathrm{od}}(\phi))$. Hence it is self-adjoint on $D(-(1/2)\Delta)\otimes\mathbb{Z}_2$ and its functional integral representation follows by Theorem 3.86. $\square$

Next we define the operator $K_\varepsilon(\mathscr{A})$ on $\mathscr{K}$ through $H_\varepsilon(\phi)$ and the constant fiber direct integral representation (7.9.61) of $\mathscr{K}$. Take $\tilde{\varphi} \in C_0^\infty(\mathbb{R}^3)$ and define the self-adjoint operator $K_\varepsilon(\mathscr{A})$ on $\mathscr{K}$ by

$$K_\varepsilon(\mathscr{A}) = \int_{\mathscr{Q}}^{\oplus} H_\varepsilon(\phi)d\mu,$$

that is, $(K_\varepsilon(\mathscr{A})F)(\phi) = H_\varepsilon(\phi)F(\phi)$ with domain

$$D(K_\varepsilon(\mathscr{A})) = \left\{ F \in \mathscr{K} \;\middle|\; \int_{\mathscr{Q}} \|H_\varepsilon(\phi)F(\phi)\|^2_{L^2(\mathbb{R}^3\times\mathbb{Z}_2)}\, d\mu < \infty \right\}.$$

Then we are able to define the self-adjoint operator K_ε by

$$K_\varepsilon = K_\varepsilon(\mathscr{A}) \dotplus H_{\mathrm{rad}}. \tag{7.9.67}$$

In what follows we construct the functional integral representation of e^{-tK_ε} and show that

$$e^{-tK_\varepsilon} = e^{-tH_{\mathrm{PF},\varepsilon}^{\mathbb{Z}_2}}. \tag{7.9.68}$$

Let $L^2_{\rm fin}(\mathscr{Q}_\beta) = \bigcup_{m=0}^\infty \{\bigoplus_{n=0}^m L^2_n(\mathscr{Q}_\beta) \bigoplus_{n=m+1}^\infty \{0\}\}$ and define the dense subspace by

$$\mathscr{K}^\infty = C_0^\infty(\mathbb{R}^3 \times \mathbb{Z}_2) \,\hat{\otimes}\, L^2_{\rm fin}(\mathscr{Q}) \tag{7.9.69}$$

and its Euclidean version by

$$\mathscr{K}_{\rm E}^\infty = C_0^\infty(\mathbb{R}^3 \times \mathbb{Z}_2) \,\hat{\otimes}\, L^2_{\rm fin}(\mathscr{Q}_{\rm E}). \tag{7.9.70}$$

Let $\tilde{\varphi} \in C_0^\infty(\mathbb{R}^3)$. It is seen that $K_\varepsilon = H^{\mathbb{Z}_2}_{{\rm PF},\varepsilon}$ on $\mathscr{K}^\infty$, implying that $K_\varepsilon = H^{\mathbb{Z}_2}_{{\rm PF},\varepsilon}$ as a self-adjoint operator since $\mathscr{K}^\infty$ is a core of $H^{\mathbb{Z}_2}_{{\rm PF},\varepsilon}$. Moreover, $H^{\mathbb{Z}_2}_{{\rm PF},\varepsilon} \to H_{\rm PF}$ on $\mathscr{K}^\infty$ as $\varepsilon \to 0$ and $\mathscr{K}^\infty$ is a common core of the sequence $\{H^{\mathbb{Z}_2}_{{\rm PF},\varepsilon}\}_{\varepsilon\geq 0}$. Thus

$$\operatorname*{s-lim}_{\varepsilon\to 0} e^{-tH^{\mathbb{Z}_2}_{{\rm PF},\varepsilon}} = e^{-tH_{\rm PF}},$$

whence

$$(F, e^{-tH^{\mathbb{Z}_2}_{\rm PF}} G) = \lim_{\varepsilon\downarrow 0}(F, e^{-tK_\varepsilon} G) \tag{7.9.71}$$

follows. By (7.9.71) it suffices to construct a functional integral representation for the expressions at its right-hand side of (7.9.71) and then use a limiting procedure. Define the Euclidean version of $\mathscr{H}_{\rm d}(x,\theta)$ and $\mathscr{H}_{\rm od}(x,-\theta)$ by

$$\mathscr{H}_{\rm E,d}(x,\theta,s) = -\frac{e}{2}\theta \mathscr{B}_{\rm E3}\left(\mathrm{j}_s\tilde{\varphi}(\cdot - x)\right), \tag{7.9.72}$$

$$\mathscr{H}_{\rm E,od}(x,-\theta,s) = -\frac{e}{2}(\mathscr{B}_{\rm E1}(\mathrm{j}_s\tilde{\varphi}(\cdot - x)) - i\theta \mathscr{B}_{\rm E2}(\mathrm{j}_s\tilde{\varphi}(\cdot - x))), \tag{7.9.73}$$

where the Euclidean version of the quantized magnetic field is defined by

$$\mathscr{B}_{\rm E}(\mathrm{j}_s\tilde{\varphi}(\cdot - x)) = \nabla_x \times \mathscr{A}_{\rm E}(\oplus_{\nu=1}^3 \mathrm{j}_s\tilde{\varphi}(\cdot - x)). \tag{7.9.74}$$

To obtain a functional integral representation of e^{-tK_ϵ}, we apply the Trotter product formula, i.e.,

$$e^{-tK_\varepsilon} = \operatorname*{s-lim}_{n\to\infty}(e^{-(t/n)K_\epsilon(\mathscr{A})} e^{-(t/n)H_{\rm rad}})^n.$$

By the Markov property of $\mathrm{E}_s = \mathrm{J}_s\mathrm{J}_s^*$, $s \in \mathbb{R}$, it is reduced to

$$\operatorname*{s-lim}_{n\to\infty} \mathrm{J}_0^* \Big(\prod_{i=0}^{n-1} \mathrm{J}_{ti/n} e^{-(t/n)K_\varepsilon(\mathscr{A})} \mathrm{J}^*_{ti/n}\Big) \mathrm{J}_t.$$

From this we can construct a functional integral representation of e^{-tK_ε}.

Before going to construct to a functional integral representation of e^{-tK_ε}, we need a functional integral representation of $\mathrm{J}_s e^{-tK_\varepsilon(\mathscr{A})} \mathrm{J}_s^*$.

Lemma 7.72. *Let* $\tilde{\varphi} \in C_0^\infty(\mathbb{R}^3)$*,* $F, G \in \mathscr{K}_{\mathrm{E}}^\infty$*,* $F \in \mathscr{E}_{[a,b]}$ *and* $s \notin [a,b]$*. Suppose that* $V_M = \sup_{x\in\mathbb{R}^3} \mathbb{E}^x[e^{-2\int_0^t V(B_s)ds}] < \infty$*. Then*

$$(F, \mathrm{J}_s e^{-tK_\varepsilon(\mathscr{A})}\mathrm{J}_s^* G) = e^t \sum_{\alpha=1,2} \int_{\mathbb{R}^3} dx\, \mathbb{E}^{x,\alpha}[e^{-\int_0^t V(B_r)dr}(F(q_0), e^{X_t(\varepsilon,s)}\mathrm{E}_s G(q_t))]. \tag{7.9.75}$$

Here

$$X_t(\varepsilon,s) = -ie\int_0^t \mathscr{A}_{\mathrm{E}}(\mathrm{j}_s\tilde{\varphi}(\cdot - B_r))\cdot dB_r - \int_0^t \mathscr{H}_{\mathrm{E,d}}(B_r, \theta_{N_r}, s)dr + \int_0^{t+} W_{\mathrm{E},\varepsilon}(B_r, -\theta_{N_{r-}}, s)dN_r, \tag{7.9.76}$$

and

$$W_{\mathrm{E},\varepsilon}(x,-\theta,s) = \log(-\Psi_\varepsilon(\mathscr{H}_{\mathrm{E,od}}(x,-\theta,s))). \tag{7.9.77}$$

Proof. Notice that the right-hand side of (7.9.75) is bounded. Since $F(x,\theta) = \mathrm{J}_l\mathrm{J}_l^* F(x,\theta) \in \mathrm{Ran}\,\mathrm{J}_l$ for some $l \in [a,b]$ and $\mathrm{E}_s G(B_t, \theta_{N_t}) = \mathrm{J}_s\mathrm{J}_s^* G(B_t,\theta_{N_t}) \in \mathrm{Ran}\,\mathrm{J}_s$ with $l \neq s$, we obtain that $F \cdot \mathrm{E}_s G \cdot e^{X_t(\varepsilon,s)} \in L^1(\mathscr{Q}_{\mathrm{E}})$ and

$$\|F \cdot \mathrm{E}_s G \cdot e^{X_t(\varepsilon,s)}\|_1 \le \|F\|_2 \|\mathrm{E}_s G\|_2 \|e^{X_t(\varepsilon,s)}\|_1$$

by Corollary 7.8. Then it follows that

$$|\text{r.h.s. (7.9.75)}| \le e^t \mathbb{E}^{0,0}\Bigg[e^{-\int_0^t V(B_r+x)dr} \times \sum_{\alpha=1,2}\int_{\mathbb{R}^3} dx\, \|F(x,\theta_\alpha)\|_2 \|G(B_t + x, \theta_{\alpha+N_t})\|_2 \|e^{X_t(\varepsilon,s)}\|_1\Bigg]. \tag{7.9.78}$$

We will prove in Lemma 7.73 below that there exists a random variable $Y = Y(\tau)$, $\tau \in \mathscr{S}$, such that (1) $\|e^{X_t(\varepsilon,s)}\|_1^2 \le Y$, (2) Y is independent of $(x,\theta) \in \mathbb{R}^3 \times \mathbb{Z}_2$, (3) Y is independent of B_t^μ, $\mu = 1,2,3$, and (4) $\mathbb{E}^{0,0}[Y^{1/2}] < \infty$. By (7.9.78),

$$|\text{r.h.s. (7.9.75)}| \le \|G\|\|F\| V_M^{1/2}\mathbb{E}^{0,0}[Y^{1/2}] < \infty, \tag{7.9.79}$$

where we used properties (1)–(4) above.

Next we prove (7.9.75). Note that $K_\varepsilon(\mathscr{A})$ is defined by a direct integral representation and then

$$(\mathrm{J}_s^* F, e^{-tK_\varepsilon(\mathscr{A})}\mathrm{J}_s^* G) = \int_{\mathscr{Q}} d\mu((\mathrm{J}_s^* F)(\phi), e^{-tH_\varepsilon(\phi)}(\mathrm{J}_s^* G)(\phi))_{L^2(\mathbb{R}^3\times\mathbb{Z}_2)}.$$

Thus by Lemma 7.71 we have

$$
\begin{aligned}
&(\mathrm{J}_s^* F, e^{-tK_\varepsilon(\mathscr{A})}\mathrm{J}_s^* G)\\
&= e^t \sum_{\alpha=1,2} \int_{\mathbb{R}^3} dx\, \mathbb{E}^{x,\alpha}\left[e^{-\int_0^t V(B_r)dr} \int_{\mathscr{Q}} d\mu \overline{(\mathrm{J}_s^* F)(\phi, q_0)} e^{Z_t(\phi,\varepsilon)} (\mathrm{J}_s^* G)(\phi, q_t)\right].
\end{aligned}
$$

Here we used Fubini's theorem. Put

$$
\begin{aligned}
Z_t(\varepsilon) = -ie \int_0^t \mathscr{A}(\tilde{\varphi}(\cdot - B_s)) \cdot dB_s - \int_0^t \mathscr{H}_{\mathrm{d}}(B_s, \theta_{N_s}) ds \\
+ \int_0^{t+} W_\varepsilon(B_s, -\theta_{N_{s-}}) dN_s,
\end{aligned}
$$

with $W_\varepsilon(x, -\theta) = \log(-\Psi_\varepsilon(\mathscr{H}_{\mathrm{od}}(x, -\theta)))$. Pick $F, G \in \mathscr{K}_{\mathrm{E}}^\infty$. Given that $\mathrm{J}_s^* F \in L^2(\mathscr{Q})$ and $e^{Z_t(\varepsilon)}\mathrm{J}_s^* G(B_t, \theta_{N_t}) \in L^2(\mathscr{Q})$, we rewrite as

$$
\begin{aligned}
&(\mathrm{J}_s^* F, e^{-tK_\varepsilon(\mathscr{A})}\mathrm{J}_s^* G)\\
&= e^t \sum_{\alpha=1,2} \int_{\mathbb{R}^3} dx\, \mathbb{E}^{x,\alpha}[e^{-\int_0^t V(B_r)dr} (F(q_0), \mathrm{J}_s e^{Z_t(\varepsilon)} \mathrm{J}_s^* G(q_t))_{L^2(\mathscr{Q}_{\mathrm{E}})}].
\end{aligned}
$$

We can directly see that $\mathrm{J}_s e^{Z_t(\varepsilon)} \mathrm{J}_s^* = e^{X_t(\epsilon,s)} \mathrm{E}_s$, leading to (7.9.75) for $F, G \in \mathscr{K}_{\mathrm{E}}^\infty$. By a limiting argument and the bound (7.9.79) the proof can be completed. □

Lemma 7.73. *There exists a random variable $Y = Y(\tau)$ satisfying* (1)–(4) *in the proof of Lemma* 7.72.

Proof. Note that

$$
\|e^{X_t(\varepsilon,s)}\|_1^2 \le \|e^{-\int_0^t \mathscr{H}_{\mathrm{E,d}}(B_r, \theta_{N_r}, s)dr}\|_2^2 \, \|e^{\int_0^t |W_{\mathrm{E},\varepsilon}(B_r, -\theta_{N_{r-}}, s)| dN_r}\|_2^2.
$$

We estimate the right-hand side of this expression. Since $\mathscr{B}_{\mathrm{E}\mu}(f)$ defined in (7.9.11) is linear in test function f, it follows that

$$
\int_0^t \mathscr{H}_{\mathrm{E,d}}(B_r, \theta_{N_r}, s) dr = \mathscr{B}_{\mathrm{E}3}\left(-\frac{e}{2} \int_0^t \theta_{N_r} \mathrm{j}_s \tilde{\varphi}(\cdot - B_r) dr \right),
$$

$\mathscr{B}_{\mathrm{E}\mu}(f)$ is a Gaussian random variable with mean zero and covariance

$$
\int_{\mathscr{Q}_{\mathrm{E}}} \mathscr{B}_{\mathrm{E}\mu}(f) \mathscr{B}_{\mathrm{E}\nu}(g) d\mu_{\mathrm{E}} = \frac{1}{2} \int_{\mathbb{R}^4} \overline{\hat{f}(\mathbf{k})} \hat{g}(\mathbf{k}) |k|^2 \delta_{\mu\nu}(k) d\mathbf{k}, \tag{7.9.80}
$$

we have

$$
\|e^{-\int_0^t \mathscr{H}_{\mathrm{E,d}}(B_r, \theta_{N_r}, s)dr}\|_2^2 \le c_1 < \infty, \tag{7.9.81}
$$

where $c_1 = \exp(\frac{e^2}{4} t^2 \int_{\mathbb{R}^3} \frac{|\hat{\varphi}(k)|^2}{\omega(k)} |k|^2 dk)$ and c_1 is thus independent of $(x, \theta) \in \mathbb{R}^3 \times \mathbb{Z}_2$. Next consider $\| e^{\int_0^t |W_\varepsilon(B_r, -\theta_{N_{r-}}, s)| dN_r} \|_2^2$. Write $\mathscr{B}_{\mathrm{E}\mu}(t) = \mathscr{B}_{\mathrm{E}\mu}(\mathrm{j}_t \tilde{\varphi}(\cdot - B_t))$ for notational convenience. For each $\tau \in \mathscr{S}$, there exists $N = N(\tau) \in \mathbb{N}$ and points of discontinuity of $r \mapsto N_r(\tau)$, $s_1 = s_1(\tau), \ldots, s_N = s_N(\tau) \in (0, \infty)$, such that

$$\int_0^t |W_{\mathrm{E},\varepsilon}(B_r, -\theta_{N_{r-}}, s)| dN_r = \sum_{j=1}^N W_{\mathrm{E},\varepsilon}(B_{s_j}, -\theta_{N_{s_j}}, s).$$

Since $|\Psi_\varepsilon(w)| \le |w| + \varepsilon$, we have

$$\begin{aligned} &\| e^{\int_0^t |W_{\mathrm{E},\varepsilon}(B_r, -\theta_{N_{r-}}, s)| dN_r} \|_2^2 \\ &\le \left(1, \exp\left(2 \sum_{i=1}^N \log \left[\frac{|e|}{\sqrt{2}} \sqrt{\mathscr{B}_{\mathrm{E}1}(s_i)^2 + \mathscr{B}_{\mathrm{E}2}(s_i)^2 + \varepsilon^2} \right] \right) 1 \right)_2 \end{aligned}$$

where we used that $|a + ib| + \varepsilon \le \sqrt{2}\sqrt{a^2 + b^2 + \varepsilon^2}$, $a, b, \varepsilon \in \mathbb{R}$. Then the right-hand side is computed as

$$\| e^{\int_0^t |W_{\mathrm{E},\varepsilon}(B_r, -\theta_{N_{r-}}, s)| dN_r} \|_2^2 \le \left(\frac{|e|}{\sqrt{2}} \right)^{2N} \sum_{m=0}^N \varepsilon^{2(N-m)} \sum_{\mathrm{comb}_m} \| \underbrace{\mathscr{B}_{\mathrm{E}\#} \cdots \mathscr{B}_{\mathrm{E}\#}}_{m\text{-fold}} 1 \|_2^2,$$

where $\sum_{\mathrm{comb}_m}$ denotes summation over the 2^m terms in the expansion of the product $\prod_{i=1}^m (\mathscr{B}_{\mathrm{E}1}(s_i)^2 + \mathscr{B}_{\mathrm{E}2}(s_i)^2)$, $\mathscr{B}_{\mathrm{E}\#}$ denotes one of $\mathscr{B}_{\mathrm{E}\mu}(s_i)$, $\mu = 1, 2$, $i = 1, \ldots, N$. Thus we obtain

$$\| e^{\int_0^t |W_{\mathrm{E},\varepsilon}(B_r, -\theta_{N_{r-}}, s)| dN_r} \|_2^2 \le c_2(\tau), \tag{7.9.82}$$

where $c_2(\tau) = (\frac{|e|}{\sqrt{2}})^{2N} \sum_{m=0}^N \varepsilon^{2(N-m)} 2^m (\sqrt{2})^{2m} m! \| \sqrt{|k|} \hat{\varphi} \|^{2m}$. Note that $c_2(\tau)$ is independent of $(x, \theta) \in \mathbb{R}^3 \times \mathbb{Z}_2$ and B_t^μ. Write

$$Y(\tau) = c_1 c_2(\tau). \tag{7.9.83}$$

Then

$$\mathbb{E}^{0,0}[Y^{1/2}] \le e^{\frac{1}{2} \frac{e^2}{4} t^2 \| \sqrt{|k|} \hat{\varphi} \|^2} \sum_{N=0}^\infty \left(\frac{|e|}{\sqrt{2}} \right)^N \sum_{m=0}^N \frac{\varepsilon^{N-m} \sqrt{m!} 2^m \| \sqrt{|k|} \hat{\varphi} \|^m}{N!} e^{-t} < \infty. \tag{7.9.84}$$

This completes the proof of claims (1)–(4) above. □

Theorem 7.74 (Functional integral representation for Pauli–Fierz Hamiltonian with spin $1/2$). *Let* $V_M = \sup_{x\in\mathbb{R}^3} \mathbb{E}^x[e^{-2\int_0^t V(B_s)ds}] < \infty$. *Then for every* $t \geq 0$ *and all* $F, G \in \mathscr{K}$ *it follows that*

$$(F, e^{-tH_{\mathrm{PF}}^{\mathbb{Z}_2}} G) = \lim_{\varepsilon\downarrow 0}(F, e^{-tH_{\mathrm{PF},\varepsilon}^{\mathbb{Z}_2}} G) \tag{7.9.85}$$

and

$$(F, e^{-tH_{\mathrm{PF},\varepsilon}^{\mathbb{Z}_2}} G) = e^t \sum_{\alpha=1,2} \int_{\mathbb{R}^3} dx\, \mathbb{E}^{x,\alpha}[e^{-\int_0^t V(B_s)ds} (\mathrm{J}_0 F(q_0), e^{X_t(\varepsilon)} \mathrm{J}_t G(q_t))]. \tag{7.9.86}$$

Here the exponent $X_t(\varepsilon)$ *is given by*

$$\begin{aligned} X_t(\varepsilon) = {} & -ie \int_0^t \mathscr{A}_{\mathrm{E}}(\mathrm{j}_s \tilde{\varphi}(\cdot - B_s)) \cdot dB_s \\ & - \int_0^t \mathscr{H}_{\mathrm{E,d}}(B_s, \theta_{N_s}, s) ds + \int_0^{t+} \log(-\Psi_\varepsilon(\mathscr{H}_{\mathrm{E,od}}(B_s, -\theta_{N_{s-}}, s))) dN_s, \end{aligned} \tag{7.9.87}$$

or

$$X_t(\varepsilon) = -ie\mathscr{A}_{\mathrm{E}}(K_t) + \frac{e}{2}\mathscr{B}_{\mathrm{E}3}(M_t) + \int_0^{t+} \log(-\Psi_\varepsilon(\mathscr{H}_{\mathrm{E,od}}(B_s, -\theta_{N_{s-}}, s))) dN_s, \tag{7.9.88}$$

where $K_t = \bigoplus_{\mu=1}^3 \int_0^t \mathrm{j}_s \tilde{\varphi}(\cdot - B_s) dB_s^\mu$, $\mathscr{B}_{\mathrm{E}\mu}(f)$ *is defined in* (7.9.11) *and* $M_t = \int_0^t \theta_{N_s} \mathrm{j}_s \tilde{\varphi}(\cdot - B_s) ds$.

Proof. Notice that $\mathscr{B}_{\mathrm{E}\mu}(\mathrm{j}_s f)$ is a Gaussian random variable with mean zero and covariance given by

$$\int_{\mathscr{Q}_{\mathrm{E}}} \mathscr{B}_{\mathrm{E}\mu}(\mathrm{j}_s f) \mathscr{B}_{\mathrm{E}\nu}(\mathrm{j}_t g) d\mu_{\mathrm{E}} = \frac{1}{2} \int_{\mathbb{R}^3} \overline{\hat{f}(k)} \hat{g}(k) |k|^2 \delta_{\mu\nu}(k) e^{-|t-s|\omega(k)} dk.$$

Then similarly to (7.9.79) we obtain

$$|\text{r.h.s. } (7.9.86)| \leq \|F\| \|G\| e^t V_M^{1/2} \sum_{\alpha=1,2} \int_{\mathbb{R}^3} dx \mathbb{E}^{x,\alpha}[Y^{1/2}] < C,$$

where Y is a random variable given by (7.9.83) and C is a constant independent of ε. Since $e^{-tH_{\mathrm{PF},\varepsilon}^{\mathbb{Z}_2}} \to e^{-tH_{\mathrm{PF}}}$ strongly as $\varepsilon \to 0$, (7.9.85) follows. Write

$$\begin{aligned} X_{S,T}(\varepsilon, s) = {} & -ie \int_S^T \mathscr{A}_{\mathrm{E}}(\mathrm{j}_s \tilde{\varphi}(\cdot - B_r)) \cdot dB_r \\ & - \int_S^T \mathscr{H}_{\mathrm{E,d}}(B_r, \theta_{N_r}, s) dr + \int_S^{T+} W_{\mathrm{E},\varepsilon}(B_r, -\theta_{N_{r-}}, s) dN_r. \end{aligned}$$

Here $W_{\mathrm{E},\varepsilon}(x,-\theta,r)=\log(-\Psi_\varepsilon(\mathscr{H}_{\mathrm{E,od}}(x,-\theta,s)))$. Now we turn to proving (7.9.86). Take $\tilde{\varphi}\in C_0^\infty(\mathbb{R}^3)$ and define $S_{t,s}^\varepsilon:\mathscr{K}_{\mathrm{E}}^\infty\to\mathscr{K}_{\mathrm{E}}^\infty$ by

$$(S_{t,s}^\varepsilon G)(x,\theta_\alpha)=e^t\mathbb{E}^{x,\alpha}[e^{-\int_0^t V(B_r)dr}e^{X_{0,t}(\varepsilon,s)}G(q_t)].$$

Then

$$\|\mathbb{E}^{x,\alpha}[e^{-\int_0^t V(B_r)dr}e^{X_{0,t}(\epsilon,s)}G(q_t)]\|_{\mathscr{K}}\ \le\ V_M^{1/2}\,\mathbb{E}^{x,\alpha}[Y^{1/2}]\,\|G\|_{\mathscr{K}}.$$

It can be seen that $S_{t,s}^\varepsilon$ has the property:

$$S_{t,s}^\varepsilon S_{t',s'}^\varepsilon G(x,\theta_\alpha)=e^{t+t'}\mathbb{E}^{x,\alpha}[e^{-\int_0^{t+t'}V(B_r)dr}e^{X_{0,t}(\varepsilon,s)+X_{t,t+t'}(\varepsilon,s')}G(q_{t+t'})]. \tag{7.9.89}$$

Note that for $s_1\le\cdots\le s_n$,

$$e^{X_{0,t_1}(\varepsilon,s_1)+X_{t_1,t_1+t_2}(\varepsilon,s_2)+\cdots+X_{t_1+\cdots+t_{n-1},t_1+\cdots+t_n}(\varepsilon,s_n)}\in\mathscr{E}_{[s_1,s_n]} \tag{7.9.90}$$

Since $(F,e^{-tH_{\mathrm{PF},\varepsilon}^{\mathbb{Z}_2}}G)=(F,e^{-tK_\varepsilon}G)$, similarly to the proof of Theorem 5.20, by the Trotter product formula, (7.9.89), (7.9.90) and the Markov property of E_s, $s\in\mathbb{R}$, we obtain that

$$\begin{aligned}
&(F,e^{-tH_{\mathrm{PF},\varepsilon}^{\mathbb{Z}_2}}G)\\
&=\lim_{n\to\infty}\Big(\mathrm{J}_0F,\Big(\prod_{i=0}^{n-1}S_{t/n,it/n}^\varepsilon\Big)\mathrm{J}_tG\Big)\\
&=e^t\lim_{n\to\infty}\sum_{\alpha=1,2}\int_{\mathbb{R}^3}dx\,\mathbb{E}^{x,\alpha}\left[e^{-\int_0^t V(B_r)dr}\int_{\mathscr{Q}_{\mathrm{E}}}d\mu_{\mathrm{E}}\overline{\mathrm{J}_0F(x,\theta)}e^{X_t^n(\varepsilon)}\mathrm{J}_tG(q_t)\right],
\end{aligned} \tag{7.9.91}$$

where the exponent $X_t^n(\varepsilon)$ consists of three parts: $X_t^n(\varepsilon)=Y_t^n(1)+Y_t^n(2)+Y_t^n(3,\varepsilon)$ and

$$\begin{aligned}
Y_t^n(1)&=-ie\mathscr{A}_{\mathrm{E}}\left(\bigoplus_{\mu=1}^3\int_0^t \mathrm{j}_{\Delta_n(s)}\tilde{\varphi}(\cdot-B_s)dB_s^\mu\right),\\
Y_t^n(2)&=-\int_0^t\mathscr{H}_{\mathrm{E,d}}(B_s,\theta_{N_s},\Delta_n(s))ds,\\
Y_t^n(3,\varepsilon)&=\int_0^t W_{\mathrm{E},\varepsilon}(B_s,-\theta_{N_{s-}},\Delta_n(s))dN_s.
\end{aligned}$$

Here $\Delta_n(s)$ is the step function given by

$$\Delta_n(s) = \sum_{i=1}^{n} \frac{t(i-1)}{n} 1_{(t(i-1)/n, ti/n]}(s).$$

We claim that

$$\text{r.h.s. (7.9.91)} = e^t \sum_{\alpha=1,2} \int_{\mathbb{R}^3} dx\, \mathbb{E}^{x,\alpha}\left[e^{-\int_0^t V(B_s)ds} \int_{\mathscr{Q}_\mathrm{E}} d\mu_\mathrm{E} \overline{\mathrm{J}_0 F(q_0)} e^{X_t(\varepsilon)} \mathrm{J}_t G(q_t) \right] \tag{7.9.92}$$

by a limiting argument. The proof is thus complete. It only remains to show (7.9.92). Note that

$$\begin{aligned} &\sum_{\alpha=1,2} \int_{\mathbb{R}^3} dx\, \mathbb{E}^{x,\alpha}\left[e^{-\int_0^t V(B_s)ds} \int_{\mathscr{Q}_\mathrm{E}} |\mathrm{J}_0 F(q_0)|\, |\mathrm{J}_t G(q_t)|\, |e^{X_t^n(\varepsilon)} - e^{X_t(\varepsilon)}| d\mu_\mathrm{E} \right] \\ &\leq \|G\|\, \mathbb{E}^{x,\alpha}\left[\left(\sum_{\alpha=1,2} \int_{\mathbb{R}^3} dx\, e^{-2\int_0^t V(B_s)ds} \|F(x,\theta_\alpha)\|_2^2\, \|e^{X_t^n(\varepsilon)} - e^{X_t(\varepsilon)}\|_1^2 \right)^{1/2} \right] \end{aligned} \tag{7.9.93}$$

and by Schwarz inequality it follows that

$$\|e^{X_t^n(\varepsilon)}\|_1^2 \leq (1, |e^{Y_t^n(2)}|^2 1)(1, |e^{Y_t^n(3,\varepsilon)}|^2 1).$$

We estimate the right-hand side above. It readily follows that

$$(1, e^{2Y_t^n(2)} 1) \leq \exp\left(\frac{e^2}{4} t^2 \int_{\mathbb{R}^3} |\hat{\varphi}(k)|^2 |k| dk \right) = c_1, \tag{7.9.94}$$

and in the similar way as in (7.9.82), for each $\tau \in \mathscr{S}$,

$$(1, |e^{Y_t^n(3,\varepsilon)}|^2 1) \leq c_2(\tau) \tag{7.9.95}$$

is shown with $c_2(\tau)$ given in (7.9.82). Thus we conclude that

$$\|e^{X_t^n(\varepsilon)}\|_1^2 < c(\tau), \tag{7.9.96}$$

where $c(\tau) = c_1 c_2(\tau)$ and $\mathbb{E}^{x,\alpha}[c^{1/2}] < \infty$. Similarly,

$$\|e^{X_t(\varepsilon)}\|_1 < C(\tau) \tag{7.9.97}$$

and $\mathbb{E}^{x,\alpha}[C^{1/2}] < \infty$ follows for a random variable $C(\tau)$. Note that both c and C are independent of $(x,\theta) \in \mathbb{R}^3 \times \mathbb{Z}_2$, B_t^μ and n. Thus by (7.9.93) and the Lebesgue dominated convergence theorem, it suffices to show that for almost every $\tau \in \mathcal{S}$, $e^{X_t^n(\varepsilon)} \to e^{X_t(\varepsilon)}$ as $n \to \infty$ in $L^1(\mathcal{Q}_\mathrm{E})$. Put

$$Y_t(1) = -ie\mathscr{A}_\mathrm{E}\left(\bigoplus_{\mu=1}^3 \int_0^t \mathrm{j}_s\tilde{\varphi}(\cdot - B_s)dB_s^\mu\right),$$

$$Y_t(2) = -\int_0^t \mathscr{H}_{\mathrm{E,d}}(B_s, \theta_{N_s}, s)ds,$$

$$Y_t(3,\varepsilon) = \int_0^{t+} W_{\mathrm{E},\varepsilon}(B_s, -\theta_{N_{s-}}, s)dN_s.$$

Then $X_t(\varepsilon) = Y_t(1) + Y_t(2) + Y_t(3,\varepsilon)$. It follows directly

$$\begin{aligned} e^{X_t^n(\varepsilon)} - e^{X_t(\varepsilon)} &= \underbrace{e^{Y_t^n(1)}e^{Y_t^n(2)}e^{Y_t^n(3,\varepsilon)} - e^{Y_t(1)}e^{Y_t^n(2)}e^{Y_t^n(3,\varepsilon)}}_{=\mathrm{I}} \\ &\quad + \underbrace{e^{Y_t(1)}e^{Y_t^n(2)}e^{Y_t^n(3,\varepsilon)} - e^{Y_t(1)}e^{Y_t(2)}e^{Y_t^n(3,\varepsilon)}}_{=\mathrm{II}} \\ &\quad + \underbrace{e^{Y_t(1)}e^{Y_t(2)}e^{Y_t^n(3,\varepsilon)} - e^{Y_t(1)}e^{Y_t(2)}e^{Y_t(3)}}_{=\mathrm{III}}. \end{aligned} \tag{7.9.98}$$

We estimate I, II and III. It is not difficult to see that

$$\lim_{n\to\infty} \|\mathrm{I}\|_1 = 0, \quad \lim_{n\to\infty} \|\mathrm{II}\|_1 = 0. \tag{7.9.99}$$

We deal with III. Since

$$\|e^{Y_t(1)}e^{Y_t(2)}e^{Y_t^n(3,\varepsilon)} - e^{Y_t(1)}e^{Y_t(2)}e^{Y_t(3,\varepsilon)}\|_1 \le \|e^{Y_t(2)}\|_2 \,\|e^{Y_t^n(3,\varepsilon)} - e^{Y_t(3,\varepsilon)}\|_2$$

and $\|e^{Y_t(2)}\|_2^2 \le e^{4(e/2)^2 t^2 \|\sqrt{|k|}\hat{\varphi}\|^2}$, it is enough to show that $e^{Y_t^n(3,\varepsilon)} \to e^{Y_t(3,\varepsilon)}$ in $L^2(\mathcal{Q}_\mathrm{E})$. By the definition of $Y_t^n(3,\varepsilon)$ we have

$$e^{Y_t^n(3,\varepsilon)} = \prod_{i=1}^n \exp\left(\int_{t(i-1)/n}^{ti/n+} W_\varepsilon(B_s, -\theta_{N_{s-}}, t(i-1)/n)dN_s\right).$$

For every $\tau \in \mathcal{S}$ there exists $N = N(\tau) \in \mathbb{N}$ such that at the points $s_1 = s_1(\tau), \dots, s_N = s_N(\tau)$, $t \mapsto N_t(\tau)$ is not continuous. For sufficiently large n the number of s_k contained in the interval $(t(i-1)/n, ti/n]$ is at most one. Then by taking n large enough and putting $(n(s_i), n(s_i) + t/n]$ for the interval containing s_i, we get

$$e^{Y_t^n(3,\varepsilon)} = \prod_{i=1}^N (-\Psi_\varepsilon(\mathscr{H}_{\mathrm{E,od}}(B_{s_i}, -\theta_{N_{s_i-}}, n(s_i)))). \tag{7.9.100}$$

Clearly, $n(s_i) \to s_i$ as $n \to \infty$. Since $\mathscr{H}_{\mathrm{E,od}}(B_{s_i}, -\theta_{N_{s_i-}}, n(s_i))$ strongly converges to $\mathscr{H}_{\mathrm{E,od}}(B_{s_i}, -\theta_{N_{s_i-}}, s_i)$ as $n \to \infty$ in $L^2(\mathscr{Q}_{\mathrm{E}})$, we have

$$\lim_{n\to\infty} \text{r.h.s. (7.9.100)} = \prod_{i=1}^{N} (-\Psi_\varepsilon(\mathscr{H}_{\mathrm{E,od}}(B_{s_i}, -\theta_{N_{s_i-}}, s_i))). \tag{7.9.101}$$

Since the right-hand side of (7.9.101) equals $e^{Y_t(3,\varepsilon)}$, $\lim_{n\to\infty} \|e^{Y_t^n(3,\varepsilon)} - e^{Y_t(3,\varepsilon)}\|_2 = 0$ is plainly seen, and

$$\lim_{n\to\infty} \|\mathrm{III}\|_1 = 0. \tag{7.9.102}$$

A combination of (7.9.99) and (7.9.102) implies (7.9.92). $\square$

By using the functional integral representation we can estimate the ground state energy of $H_{\mathrm{PF}}^{\mathbb{Z}_2}$. Write

$$E(\mathscr{A}, \mathscr{B}_1, \mathscr{B}_2, \mathscr{B}_3) = \inf \mathrm{Spec}(H_{\mathrm{PF}}^{\mathbb{Z}_2}). \tag{7.9.103}$$

For the spinless Pauli–Fierz Hamiltonian H_{PF} we have $\inf \mathrm{Spec}(H_{\mathrm{PF}}) = E(\mathscr{A}, 0, 0, 0)$ and the diamagnetic inequality $E(0,0,0,0) \le E(\mathscr{A}, 0, 0, 0)$ was already seen. We extend this inequality to $H_{\mathrm{PF}}^{\mathbb{Z}_2}$. Define

$$(H_{\mathrm{PF}}^{\mathbb{Z}_2}(0)F)(\theta) = (H_{\mathrm{p}} + H_{\mathrm{rad}} + \mathscr{H}_{\mathrm{d}}(\theta))F(\theta) - |\mathscr{H}_{\mathrm{od}}(-\theta)|F(-\theta), \tag{7.9.104}$$

where

$$|\mathscr{H}_{\mathrm{od}}(-\theta)| = \frac{|e|}{2} \int_{\mathbb{R}^3}^{\oplus} \sqrt{\mathscr{B}_1^2(x) + \mathscr{B}_2^2(x)}\, dx \tag{7.9.105}$$

is independent of $\theta \in \mathbb{Z}_2$. $H_{\mathrm{PF}}^{\mathbb{Z}_2}(0)$ corresponds to

$$H_{\mathrm{p}} + H_{\mathrm{rad}} - \begin{pmatrix} \frac{e}{2}\mathscr{B}_3 & \frac{|e|}{2}\sqrt{\mathscr{B}_1^2 + \mathscr{B}_2^2} \\ \frac{|e|}{2}\sqrt{\mathscr{B}_1^2 + \mathscr{B}_2^2} & -\frac{e}{2}\mathscr{B}_3 \end{pmatrix}$$

acting in $L^2(\mathbb{R}^3; \mathbb{C}^2) \otimes L^2(\mathscr{Q})$. Furthermore, to avoid zeroes of the off-diagonal part to occur we also define

$$(H_{\mathrm{PF},\varepsilon}^{\mathbb{Z}_2}(0)F)(\theta) = (H_{\mathrm{p}} + H_{\mathrm{rad}} + \mathscr{H}_{\mathrm{d}}(\theta))F(\theta) - \Psi_\varepsilon(|\mathscr{H}_{\mathrm{od}}(-\theta)|)F(-\theta). \tag{7.9.106}$$

Since the spin interaction is infinitesimally small with respect to the free Hamiltonian $H_{\mathrm{p}} + H_{\mathrm{rad}}$, $H_{\mathrm{PF}}^{\mathbb{Z}_2}(0)$ and $H_{\mathrm{PF},\varepsilon}^{\mathbb{Z}_2}(0)$ are self-adjoint on D_{PF} and bounded from below. The functional integral representation of $e^{-tH_{\mathrm{PF}}^{\mathbb{Z}_2}(0)}$ is given by

$$(F, e^{-tH_{\mathrm{PF}}^{\mathbb{Z}_2}(0)}G) = \lim_{\varepsilon\downarrow 0} e^t \sum_{\alpha=1,2} \int_{\mathbb{R}^3} dx\, \mathbb{E}^{x,\alpha}[e^{-\int_0^t V(B_s)ds}(\mathrm{J}_0 F(q_0), e^{X_t^{\perp}(\varepsilon)} \mathrm{J}_t G(q_t))],$$

where

$$X_t^{\perp}(\varepsilon) = -\int_0^t \mathscr{H}_{\mathrm{E,d}}(B_s, \theta_{N_s}, s)ds + \int_0^{t+} \log(\Psi_\varepsilon(|\mathscr{H}_{\mathrm{E,od}}(B_s, s)|))dN_s. \tag{7.9.107}$$

Corollary 7.75 (Diamagnetic inequality). *It follows that*

$$|(F, e^{-tH_{\mathrm{PF}}^{\mathbb{Z}_2}} G)| \leq (|F|, e^{-tH_{\mathrm{PF}}^{\mathbb{Z}_2}(0)}|G|) \tag{7.9.108}$$

and

$$\max_{(\alpha\beta\gamma)=(123),(231),(312)} E(0, \sqrt{\mathscr{B}_\alpha^2 + \mathscr{B}_\beta^2}, 0, \mathscr{B}_\gamma) \leq E(\mathscr{A}, \mathscr{B}_1, \mathscr{B}_2, \mathscr{B}_3). \tag{7.9.109}$$

Proof. Since $|\Psi_\varepsilon(\mathscr{H}_{\mathrm{E,od}})| \leq \Psi_\varepsilon(|\mathscr{H}_{\mathrm{E,od}}|)$, $|e^{X_t(\varepsilon)}| \leq e^{X_t^{\perp}(\varepsilon)}$ and $|\mathrm{J}_t G| \leq \mathrm{J}_t|G|$ as J_t is positivity preserving, by the functional integral representation of $e^{-tH_{\mathrm{PF}}}$ we have

$$|(F, e^{-tH_{\mathrm{PF}}^{\mathbb{Z}_2}} G)| \leq \lim_{\varepsilon \downarrow 0}(|F|, e^{-tH_{\mathrm{PF},\varepsilon}^{\mathbb{Z}_2}(0)}|G|) = (|F|, e^{-tH_{\mathrm{PF}}^{\mathbb{Z}_2}(0)}|G|).$$

Thus (7.9.108) follows. From this, $E(0, \sqrt{\mathscr{B}_1^2 + \mathscr{B}_2^2}, 0, \mathscr{B}_3) \leq E(\mathscr{A}, \mathscr{B}_1, \mathscr{B}_2, \mathscr{B}_3)$ is obtained. We will show that

$$E(\mathscr{A}, \mathscr{B}_1, \mathscr{B}_2, \mathscr{B}_3) = E(\mathscr{A}, \mathscr{B}_2, \mathscr{B}_3, \mathscr{B}_1) = E(\mathscr{A}, \mathscr{B}_3, \mathscr{B}_1, \mathscr{B}_2) \tag{7.9.110}$$

by SU(2)-symmetry. Let $\mathscr{R} \in O(3)$ be such that

$$\mathscr{R}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_2 \\ x_3 \\ x_1 \end{pmatrix}.$$

Then there exists $(n, \phi) \in S^2 \times \mathbb{R}$ such that $\mathscr{R} = \mathscr{R}(n, \phi)$. Recall that $\mathscr{R}(n, \phi)$ describes the rotation around n by angle ϕ. Hence we see that $e^{i\phi n\cdot(1/2)\sigma}\sigma_\mu e^{-i\phi n\cdot(1/2)\sigma} = (\mathscr{R}\sigma)_\mu$. Now we write $H_{\mathrm{PF}}^{\mathbb{Z}_2}$ by $H_{\mathrm{PF}}^{\mathbb{Z}_2}(\mathscr{A}, \mathscr{B}_1, \mathscr{B}_2, \mathscr{B}_3)$. Thus we obtain that

$$e^{i\phi n\cdot(1/2)\sigma} H_{\mathrm{PF}}^{\mathbb{Z}_2}(\mathscr{A}, \mathscr{B}_1, \mathscr{B}_2, \mathscr{B}_3) e^{-i\phi n\cdot(1/2)\sigma} = H_{\mathrm{PF}}^{\mathbb{Z}_2}(\mathscr{A}, \mathscr{B}_2, \mathscr{B}_3, \mathscr{B}_1)$$

which implies the first equality in (7.9.110). The second one is proven in the same way. □

7.9.4 Spin-boson model

An interesting application of the functional integral representation of the Pauli–Fierz Hamiltonian with spin (Theorem 7.74) is the analysis of the *spin-boson model*. Let $\mathscr{F} = \mathscr{F}_{\rm b}(L^2(\mathbb{R}^d))$.

Definition 7.16 (Spin-boson Hamiltonian). The *spin-boson Hamiltonian* is defined by

$$H_{\rm SB} = \varepsilon\sigma_1 \otimes 1 + 1 \otimes H_{\rm f} + \sigma_3 \otimes \phi(h) \tag{7.9.111}$$

on $\mathbb{C}^2 \otimes \mathscr{F}$, where $\varepsilon \geq 0$ is a coupling constant and $\phi(h)$ is a scalar field defined by

$$\phi(h) = \int_{\mathbb{R}^d} (a^*(k)\hat{h}(-k) + a(k)\hat{h}(k))dk. \tag{7.9.112}$$

Suppose that $\hat{h}, \hat{h}/\sqrt{\omega} \in L^2(\mathbb{R}^d)$. Then $H_{\rm SB}$ is self-adjoint on $\mathbb{C}^2 \otimes D(H_{\rm f})$. $H_{\rm SB}$ is realized as

$$H_{\rm SB} = \begin{pmatrix} H_{\rm f} + \phi(h) & \varepsilon \\ \varepsilon & H_{\rm f} - \phi(h) \end{pmatrix}.$$

By introducing $\mathbb{Z}_2$, $H_{\rm SB}$ transforms into

$$H_{\rm SB}\Psi(\theta) = (H_{\rm f} + \theta\phi(h))\Psi(\theta) + \varepsilon\Psi(-\theta), \quad \theta \in \mathbb{Z}_2. \tag{7.9.113}$$

Theorem 7.76 (Functional integral representation of spin-boson model). *Let* $\Phi, \Psi \in \mathbb{C}^2 \otimes \mathscr{F}$. *Then*

$$(\Phi, e^{-tH_{\rm SB}}\Psi) = e^t \sum_{\alpha=1,2} \mathbb{E}^\alpha[(J_0\Phi(\theta_{N_0}), e^{-\phi_{\rm E}(\int_0^t \theta_{N_s} {\rm j}_s \hat{h} ds)} (-\varepsilon)^{N_t} {\rm J}_t\Psi(\theta_{N_t}))],$$
$$\varepsilon \neq 0, \tag{7.9.114}$$

$$(\Phi, e^{-tH_{\rm SB}}\Psi) = e^t \sum_{\alpha=1,2} (J_0\Phi(\theta_\alpha), e^{-\phi_{\rm E}(\theta_\alpha \int_0^t {\rm j}_s \hat{h} ds)} {\rm J}_t\Psi(\theta_\alpha)),$$
$$\varepsilon = 0. \tag{7.9.115}$$

Proof. Let $\varepsilon \neq 0$. We see that

$$H_{\rm SB}\Psi(\theta) = (H_{\rm f} + \theta\phi(h))\Psi(\theta) - e^{i\pi + \log\varepsilon}\Psi(-\theta),$$

and

$$(\Phi, e^{-tH_{\rm SB}}\Psi)$$
$$= e^t \sum_{\alpha=1,2} \mathbb{E}^\alpha[(J_0\Phi(\theta_{N_0}), e^{-\int_0^t \theta_{N_s}\phi_{\rm E}({\rm j}_s\hat{h})ds + \int_0^{t+}(i\pi + \log\varepsilon)dN_s} {\rm J}_t\Psi(\theta_{N_t}))].$$

Since $\int_0^{t+}(i\pi + \log\varepsilon)dN_s = (i\pi + \log\varepsilon)N_t$, (7.9.114) follows. Next we consider the case of $\varepsilon = 0$. As $\varepsilon \to 0$ on both sides of (7.9.114), the integrands on $\{\tau \in \mathcal{T} | N_t(\tau) \geq 1\}$ vanish and those on $K = \{\tau \in \mathcal{T} | N_t(\tau) = 0\}$ stay different from zero. Moreover, note that $N_s(\tau) = 0$, $s \leq t$, for $\tau \in K$. Hence (7.9.115) is obtained by taking the limit. □

In the case of $\varepsilon = 0$, the Hamiltonian H_{SB} is diagonal. Hence the ground state of H_{SB} with $\varepsilon = 0$ is two-fold degenerate, provided that ground states exists. Thus we estimate the dimension of $\mathrm{Ker}(H_{\mathrm{SB}} - \inf \mathrm{Spec}(H_{\mathrm{SB}}))$ for $\varepsilon \neq 0$. The transformations $\sigma_X \to -\sigma_X$, $X = 1, 2, 3$, and $a^\sharp(f) \to -a^\sharp(f)$ can be implemented by some unitary operators, and then

$$\tilde{H}_{\mathrm{SB}} = -\varepsilon\sigma_1 \otimes 1 + 1 \otimes H_{\mathrm{f}} + \sigma_3 \otimes \phi(f) \tag{7.9.116}$$

is unitary equivalent with H_{SB}. Flipping the sign $\varepsilon \to -\varepsilon$ in (7.9.114), we have

$$(\Phi, e^{-t\tilde{H}_{\mathrm{SB}}}\Psi) = e^t \sum_{\alpha=1,2} \mathbb{E}^\alpha[(J_0\Phi(\theta_\alpha), e^{-\phi_{\mathrm{E}}(\int_0^t \theta_{N_s} \mathrm{j}_s \hat{h} ds)} \varepsilon^{N_t} \mathrm{J}_t \Psi(\theta_{N_t}))]. \tag{7.9.117}$$

Corollary 7.77 (Uniqueness of ground state). *Assume that $\varepsilon \neq 0$. Then $e^{-t\tilde{H}_{\mathrm{SB}}}$, $t > 0$, is positivity improving. In particular, the ground state of H_{SB} is unique.*

7.9.5 Translation invariant case

Finally, we study the translation invariant Pauli–Fierz Hamiltonian with spin $1/2$, that is, $H^{\mathrm{S}}_{\mathrm{PF}}$ with $V \equiv 0$.

Definition 7.17 (Pauli–Fierz Hamiltonian with spin $1/2$ and fixed total momentum). The *Pauli–Fierz Hamiltonian with spin and fixed total momentum p* is defined by

$$H^{\mathrm{S}}_{\mathrm{PF}}(p) = \frac{1}{2}(p - P_{\mathrm{f}} - eA(0))^2 + H_{\mathrm{rad}} - \frac{e}{2}\sigma \cdot B(0), \quad p \in \mathbb{R}^3, \tag{7.9.118}$$

with domain $D(H^{\mathrm{S}}_{\mathrm{PF}}(p)) = D(H_{\mathrm{rad}}) \cap D(P_{\mathrm{f}}{}^2)$.

Theorem 7.78 (Self-adjointness). *If Assumption* 7.1 *holds, then $H^{\mathrm{S}}_{\mathrm{PF}}(p)$ is self-adjoint and essentially self-adjoint on any core of $H_{\mathrm{rad}} + P_{\mathrm{f}}^2$.*

Proof. The proof is similar to that of Theorem 7.65. By (7.6.14) we have

$$\|H_{\mathrm{PF},0}(p)\Phi\| \leq C\|(H_{\mathrm{PF}}(p) + 1)\Phi\|$$

for $\Phi \in D(H_{\mathrm{rad}}) \cap D(P_{\mathrm{f}}^2)$, with some constant C. From this inequality we see that, for any $\varepsilon > 0$ there exists $C_\epsilon > 0$ such that $\|H_{\mathrm{rad}}^{1/2}\Phi\| \leq \varepsilon\|H_{\mathrm{PF}}(p)\Phi\| + C_\varepsilon\|\Phi\|$ for

$\Phi \in D(H_{\rm rad}) \cap D(P_{\rm f}^2)$. Since $\|\sigma \cdot \mathscr{B}(0)\Phi\| \le \|H_{\rm rad}^{1/2}\Phi\| + b_\varepsilon\|\Phi\|$, it follows that $\sigma \cdot \mathscr{B}(0)$ is infinitesimally small with respect to $H_{\rm PF}(p)$. Thus $H_{\rm PF}^{\rm S}(p)$ is self-adjoint on $D(H_{\rm rad}) \cap D(P_{\rm f}^2)$. Essential self-adjointness follows from the trivial inequality $\|H_{\rm PF}^{\rm S}(p)\Phi\| \le C\|(H_{\rm rad} + P_{\rm f}^2)\Phi\| + \|\Phi\|$. □

As in the case of $H_{\rm PF}(p)$ we have

$$L^2(\mathbb{R}^3;\mathbb{C}^2) \otimes \mathscr{F}_{\rm rad} = \mathscr{T}\left(\int_{\mathbb{R}^3}^{\oplus} \mathbb{C}^2 \otimes \mathscr{F}_{\rm rad} dp\right)\mathscr{T}^{-1} \tag{7.9.119}$$

and

$$H_{\rm PF}^{\rm S} = \mathscr{T}\left(\int_{\mathbb{R}^3}^{\oplus} H_{\rm PF}^{\rm S}(p) dp\right)\mathscr{T}^{-1}, \tag{7.9.120}$$

where $\mathscr{T}$ is defined in (7.6.9). In $\mathscr{Q}$-representation $H_{\rm PF}^{\rm S}(p)$ is rewritten as

$$H_{\rm PF}^{\rm S}(p) = \frac{1}{2}(p - P_{\rm f} - e\mathscr{A}(0))^2 + H_{\rm rad} - \frac{e}{2}\sigma \cdot \mathscr{B}(0), \tag{7.9.121}$$

where we write $d\Gamma(-i\nabla)$ and $d\Gamma(\omega(-i\nabla))$ as $P_{\rm f}$ and $H_{\rm rad}$, respectively, and $\mathscr{B}(0) = \mathscr{B}(x = 0)$ defined in (7.9.8).

Before going to construct the functional integral representation of $e^{-tH_{\rm PF}^{\rm S}(p)}$ we look at the symmetry properties of $H_{\rm PF}^{\rm S}(p)$.

Lemma 7.79. *Let $\hat{\varphi}$ be rotation invariant. Then for any choice of the polarization vectors $H_{\rm PF}^{\rm S}(p)$ is unitary equivalent with $H_{\rm PF}^{\rm S}(\mathscr{R}^{-1}p)$, for all $\mathscr{R} \in \mathrm{SO}(3)$.*

Proof. It is sufficient to show the lemma for an arbitrary $\mathscr{R} = \mathscr{R}(m, \phi)$ with $(m, \phi) \in S^2 \times [0, 2\pi)$. For any polarization vectors $e^{\pm}$ define

$$h_{\rm f} = d\Gamma(\theta(\mathscr{R}, \cdot)\mathfrak{H}_2), \tag{7.9.122}$$

where $\theta(\mathscr{R}, k) = \cos^{-1}(e^+(\mathscr{R}k) \cdot \mathscr{R}e^+(k))$ and $\mathfrak{H}_2$ in (7.9.28). Formally,

$$h_{\rm f} = \sum_{j=\pm} \int_{\mathbb{R}^3} \theta(\mathscr{R}, k)(a^*(k, -)a(k, +) - a^*(k, +)a(k, -))dk. \tag{7.9.123}$$

Thus we obtain that

$$\begin{aligned}
e^{ih_{\rm f}}e^{i\phi m \cdot L_{\rm f}} H_{\rm rad} e^{-i\phi m \cdot L_{\rm f}} e^{-ih_{\rm f}} &= H_{\rm rad}, \\
e^{ih_{\rm f}}e^{i\phi m \cdot L_{\rm f}} P_{{\rm f}\mu} e^{-i\phi m \cdot L_{\rm f}} e^{-ih_{\rm f}} &= (\mathscr{R}P_{\rm f})_\mu, \\
e^{ih_{\rm f}}e^{i\phi m \cdot L_{\rm f}} A_\mu(0) e^{-i\phi m \cdot L_{\rm f}} e^{-ih_{\rm f}} &= (\mathscr{R}A(0))_\mu, \\
e^{ih_{\rm f}}e^{i\phi m \cdot L_{\rm f}} B_\mu(0) e^{-i\phi m \cdot L_{\rm f}} e^{-ih_{\rm f}} &= (\mathscr{R}B(0))_\mu, \\
e^{i\phi m \cdot (1/2)\sigma} \sigma_\mu e^{-i\phi m \cdot (1/2)\sigma} &= (\mathscr{R}\sigma)_\mu.
\end{aligned}$$

From these identities it follows that

$$
\begin{aligned}
& e^{ih_{\mathrm{f}}} e^{i\phi m\cdot((1/2)\sigma+L_{\mathrm{f}})} H_{\mathrm{PF}}^{\mathrm{S}}(p) e^{-i\phi m\cdot((1/2)\sigma+L_{\mathrm{f}})} e^{-ih_{\mathrm{f}}} \\
&= \frac{1}{2}(p-\mathscr{R}P_{\mathrm{f}}-e\mathscr{R}A(0))^2+H_{\mathrm{rad}}-\frac{1}{2}(\mathscr{R}\sigma)\cdot(\mathscr{R}B(0)) \\
&= H_{\mathrm{PF}}^{\mathrm{S}}(\mathscr{R}^{-1}p),
\end{aligned} \tag{7.9.124}
$$

proving the lemma. □

Let $E^{\mathrm{S}}(p,e^2)=\inf \mathrm{Spec}(H_{\mathrm{PF}}^{\mathrm{S}}(p))$. A straightforward consequence of Lemma 7.79 is as follows.

Corollary 7.80 (Rotation invariance). *Let $\hat{\varphi}$ be rotation invariant. Then for every $\mathscr{R}\in \mathrm{SO}(3)$ it follows that $E^{\mathrm{S}}(\mathscr{R}p,e^2)=E^{\mathrm{S}}(p,e^2)$.*

Theorem 7.81 (Fiber decomposition of Pauli–Fierz Hamiltonian with spin $1/2$ and fixed total momentum). *Suppose that polarization vectors are coherent in direction $n\in \mathrm{S}^2$ and $\hat{\varphi}$ is rotation invariant. Let $p/|p|=n$. Then the Hilbert space $\mathbb{C}^2\otimes\mathscr{F}_{\mathrm{rad}}$ and the self-adjoint operator $H_{\mathrm{PF}}^{\mathrm{S}}(p)=H_{\mathrm{PF}}^{\mathrm{S}}(|p|n)$ are decomposed as*

$$
\mathbb{C}^2\otimes\mathscr{F}_{\mathrm{rad}}=\bigoplus_{w\in\mathbb{Z}_{1/2}}\mathscr{F}_{\mathrm{rad}}(w),\quad H_{\mathrm{PF}}^{\mathrm{S}}(p)=\bigoplus_{w\in\mathbb{Z}_{1/2}}H_{\mathrm{PF}}^{\mathrm{S}}(p,w). \tag{7.9.125}
$$

Here $\mathscr{F}_{\mathrm{rad}}(w)$ is the subspace spanned by eigenvectors of

$$
\mathrm{J}_0=n\cdot\frac{1}{2}\sigma+n\cdot L_{\mathrm{f}}+S_{\mathrm{f}}
$$

associated with the eigenvalue $w\in\mathbb{Z}_{1/2}$ and $H_{\mathrm{PF}}^{\mathrm{S}}(|p|n,w)=H_{\mathrm{PF}}^{\mathrm{S}}(|p|n)\lceil_{\mathscr{F}_{\mathrm{rad}}(w)}$.

Proof. Let $\mathscr{R}=\mathscr{R}(n,\phi)$. Since the polarization vectors are coherent in direction n, it follows that $\theta(\mathscr{R},k)=\phi z$ with some $z\in\mathbb{Z}$, and then $h_{\mathrm{f}}=\phi S_{\mathrm{f}}$. Since $\mathscr{R}p=|p|\mathscr{R}n=|p|n=p$, we have from (7.9.124)

$$
e^{i\phi\mathrm{J}_0}H_{\mathrm{PF}}^{\mathrm{S}}(p)e^{-i\phi\mathrm{J}_0}=H_{\mathrm{PF}}^{\mathrm{S}}(\mathscr{R}^{-1}p)=H_{\mathrm{PF}}^{\mathrm{S}}(p),\quad \phi\in\mathbb{R}. \tag{7.9.126}
$$

Hence $H_{\mathrm{PF}}^{\mathrm{S}}(p)$ is reduced by J_0 and (7.9.125) follows. □

From this theorem $H_{\mathrm{PF}}^{\mathrm{S}}(p)$ with coherent polarization vectors in direction $p/|p|$ and $p\neq 0$ has the symmetry

$$
\mathrm{SU}(2)\times\mathrm{SO}_{\mathrm{field}}(3)\times\text{helicity}. \tag{7.9.127}
$$

Let $\hat{\varphi}$ be rotation invariant and $H_{\rm PF}^{\rm S}(p, e^{\pm})$ be the Hamiltonian with coherent polarization vectors given in Example 7.3 with $n = n_3 = (0, 0, 1)$, i.e.,

$$e^{+}(k) = \frac{(-k_2, k_1, 0)}{\sqrt{k_1^2 + k_2^2}}, \quad e^{-}(k) = \hat{k} \times e^{+}(k). \tag{7.9.128}$$

$H_{\rm PF}^{\rm S}(p)$ with arbitrary polarization vectors is unitary equivalent to $H_{\rm PF}^{\rm S}(p, e^{\pm})$,

$$H_{\rm PF}^{\rm S}(p) \cong H_{\rm PF}^{\rm S}(p, e^{\pm}).$$

Since there exists $\mathscr{R}$ such that $\mathscr{R}p = |p|n_3$, we have by Lemma 7.79,

$$H_{\rm PF}^{\rm S}(p, e^{\pm}) \cong H_{\rm PF}^{\rm S}(|p|n_3, e^{\pm}).$$

By (7.9.125) we obtain that

$$H_{\rm PF}^{\rm S}(|p|n_3, e^{\pm}) = \bigoplus_{w \in \mathbb{Z}_{1/2}} H_{\rm PF}^{\rm S}(|p|n_3, e^{\pm}, w).$$

Then for rotation invariant $\hat{\varphi}$, we have the unitary equivalence

$$H_{\rm PF}^{\rm S}(p) \cong \bigoplus_{w \in \mathbb{Z}_{1/2}} H_{\rm PF}^{\rm S}(|p|n_3, e^{\pm}, w). \tag{7.9.129}$$

We can show the theorem below in a similar way to Theorem 7.69.

Theorem 7.82 (Reflection symmetry). *Let $\hat{\varphi}$ be rotation invariant and the coherent polarization vectors $e^{\pm}$ be given by* (7.9.128). *Then $H_{\rm PF}^{\rm S}(p, w) \cong H_{\rm PF}^{\rm S}(p, -w)$ for $w \in \mathbb{Z}_{1/2}$ and p with* $(0, 0, 1) = p/|p|$.

Proof. The proof is similar to that of Theorem 7.69. Let $\mathfrak{J}_0 = \sigma_2 \otimes \Gamma(u)$, where u is defined in (7.9.43). Thus we have

$$\mathfrak{J}_0^{-1} L_{\mathrm{f}\mu} \mathfrak{J}_0 = \begin{cases} L_{\mathrm{f}\mu}, & \mu = 2, \\ -L_{\mathrm{f}\mu}, & \mu = 1, 3, \end{cases}$$

$$\mathfrak{J}_0^{-1} P_{\mathrm{f}\mu} \mathfrak{J}_0 = \begin{cases} P_{\mathrm{f}\mu}, & \mu = 1, 3, \\ -P_{\mathrm{f}\mu}, & \mu = 2, \end{cases}$$

$$\mathfrak{J}_0^{-1} A_\mu(0) \mathfrak{J}_0 = \begin{cases} A_\mu(0), & \mu = 1, 3, \\ -A_\mu(0), & \mu = 2, \end{cases}$$

$$\mathfrak{J}_0^{-1} \sigma_\mu \mathfrak{J}_0 = \begin{cases} -\sigma_\mu, & \mu = 1, 3, \\ \sigma_\mu, & \mu = 2. \end{cases}$$

Since $n = n_3 = (0,0,1)$ and $p = |p|n_3$, and the helicity of $e^{\pm}$ is zero, we have $\mathrm{J}_0 = \frac{1}{2}\sigma_3 + L_{\mathrm{f},3}$ and

$$\mathfrak{J}_0^{-1}\mathrm{J}_0\mathfrak{J}_0 = -\mathrm{J}_0. \tag{7.9.130}$$

We also see that $\mathfrak{J}_0^{-1}\sigma_\mu(p - P_\mathrm{f} - eA)_\mu\mathfrak{J}_0 = -\sigma_\mu(p - P_\mathrm{f} - eA)_\mu$ for $\mu = 1,2,3$. From here it follows that

$$\mathfrak{J}_0^{-1} H_{\mathrm{PF}}^{\mathrm{S}}(p)\mathfrak{J}_0 = H_{\mathrm{PF}}^{\mathrm{S}}(p). \tag{7.9.131}$$

Note also that

$$H_{\mathrm{PF}}^{\mathrm{S}}(p)\phi = H_{\mathrm{PF}}^{\mathrm{S}}(p,w)\phi, \quad \phi \in \mathscr{F}_{\mathrm{rad}}(w). \tag{7.9.132}$$

Let $\phi \in \mathscr{F}_{\mathrm{rad}}(w)$. We have $\mathfrak{J}_0\phi \in \mathscr{F}_{\mathrm{rad}}(-w)$ by (7.9.130), and $\mathfrak{J}_0^{-1} H_{\mathrm{PF}}^{\mathrm{S}}(p,-w)\mathfrak{J}_0\phi = \mathfrak{J}_0^{-1} H_{\mathrm{PF}}^{\mathrm{S}}(p)\mathfrak{J}_0\phi$ by (7.9.132). (7.9.131) implies that $\mathfrak{J}_0^{-1} H_{\mathrm{PF}}^{\mathrm{S}}(p)\mathfrak{J}_0\phi = H_{\mathrm{PF}}^{\mathrm{S}}(p)\phi = H_{\mathrm{PF}}^{\mathrm{S}}(p,w)\phi$. Thus

$$\mathfrak{J}_0^{-1} H_{\mathrm{PF}}^{\mathrm{S}}(p,-w)\mathfrak{J}_0 = H_{\mathrm{PF}}^{\mathrm{S}}(p,w)$$

follows. □

Since the operators $H_{\mathrm{PF}}^{\mathrm{S}}(p)$ with $p \neq 0$ and different polarization vectors from (7.9.128) are unitary equivalent with $H_{\mathrm{PF}}^{\mathrm{S}}(n_3|p|)$ with $e^{\pm}$ in (7.9.128), we have

Corollary 7.83 (Degeneracy of bound states). *Let $\hat{\varphi}$ be rotation invariant and M be the multiplicity of bound states of $H_{\mathrm{PF}}^{\mathrm{S}}(p)$. Then M is an even number. In particular, whenever a ground states exists, it is degenerate.*

The argument extends to the Hamiltonian without spin 1/2. With a rotation invariant $\hat{\varphi}$, similarly to (7.9.126) also $H_{\mathrm{PF}}(p)$ with $p/|p| = n$ and coherent polarization vectors in direction n satisfies

$$e^{i\phi J_\mathrm{f}} H_{\mathrm{PF}}(p) e^{-i\phi J_\mathrm{f}} = H_{\mathrm{PF}}(p), \quad \phi \in \mathbb{R}.$$

Thus $H_{\mathrm{PF}}(p)$ with coherent polarization vectors in direction $n = p/|p|$ has the symmetry

$$\mathrm{SO}_{\mathrm{field}}(3) \times \text{helicity}.$$

Since $\sigma(J_\mathrm{f}) = \mathbb{Z}$, this gives rise to the decomposition

$$\mathscr{F}_{\mathrm{rad}} = \bigoplus_{w\in\mathbb{Z}} \mathscr{F}_{\mathrm{rad}}^{\mathbf{0}}(w), \quad H_{\mathrm{PF}}(p) = \bigoplus_{w\in\mathbb{Z}} H_{\mathrm{PF}}(p,w) \tag{7.9.133}$$

where $\mathscr{F}_{\mathrm{rad}}^{\mathbf{0}}(w)$ denotes the subspace spanned by eigenvectors at eigenvalue $w \in \mathbb{Z}$ of J_f. By Corollary 7.44, as soon as $p = 0$ the ground state of $H_{\mathrm{PF}}(p = 0)$ is unique for any $e \in \mathbb{R}$. It is also known that $H_{\mathrm{PF}}(p)$ has the unique ground state for sufficiently small $|p|$ and $|e|$. We may suppose that the polarization vectors of $H_{\rm PF}(p)$ is coherent in direction $p/|p|$, since $H_{\rm PF}(p)$ with different polarization vectors are unitary equivalent. Suppose that the ground state $\Psi_{\rm g}(p)$ of $H_{\rm PF}(p)$ is unique. Then $\Psi_{\rm g}(p)$ must belong to some $\mathscr{F}^0_{\rm rad}(w)$. The only possibility is $z = 0$, since $H_{\rm PF}(|p|n, z) \cong H_{\rm PF}(|p|n, -z)$, $z \in \mathbb{Z}$, by Lemma 7.82. This implies $J_{\rm f}\Psi_{\rm g}(p) = 0$.

We now construct the functional integral representation for the translation invariant Pauli–Fierz Hamiltonian with spin $1/2$. As before, we transform $H^{\rm S}_{\rm PF}$ on $\mathbb{C}^2 \otimes \mathscr{F}_{\rm rad}$ to $H^{\mathbb{Z}_2}_{\rm PF}$ on $\ell^2(\mathbb{Z}_2) \otimes L^2(\mathscr{Q})$ which is defined by

$$(H^{\mathbb{Z}_2}_{\rm PF}(p)\Psi)(\theta) = (H_{\rm PF}(p) + \mathscr{H}_{\rm d}(0,\theta))\Psi(\theta) + \mathscr{H}_{\rm od}(0,-\theta)\Psi(-\theta), \quad (7.9.134)$$

where

$$\mathscr{H}_{\rm d}(0,\theta) = -\frac{e}{2}\theta\mathscr{B}_3(0), \quad (7.9.135)$$

$$\mathscr{H}_{\rm od}(-\theta,0) = -\frac{e}{2}(\mathscr{B}_1(0) - i\theta\mathscr{B}_2(0)) \quad (7.9.136)$$

with $\theta \in \mathbb{Z}_2$. Moreover, $H^{\mathbb{Z}_2}_{\rm PF,\varepsilon}(p)$ is defined by $H^{\mathbb{Z}_2}_{\rm PF}(p)$ with $\mathscr{H}_{\rm od}$ replaced by $\Psi_\varepsilon(\mathscr{H}_{\rm od})$, which is introduced to avoid the off-diagonal part of $H^{\mathbb{Z}_2}_{\rm PF}(p)$ becoming singular. The strategy of constructing the functional integral representation of $e^{-tH^{\rm S}_{\rm PF}(p)}$ is similar to that of the spinless case.

Theorem 7.84 (Functional integral representation for Pauli–Fierz Hamiltonian with spin 1/2 and fixed total momentum). *Let $\Phi, \Psi \in \ell^2(\mathbb{Z}_2) \otimes L^2(\mathscr{Q})$. We have*

$$\lim_{\varepsilon\downarrow 0}(\Phi, e^{-tH^{\mathbb{Z}_2}_{\rm PF,\varepsilon}(p)}\Psi) = (\Phi, e^{-tH^{\mathbb{Z}_2}_{\rm PF}(p)}\Psi) \quad (7.9.137)$$

and

$$(\Phi, e^{-tH^{\mathbb{Z}_2}_{\rm PF,\varepsilon}(p)}\Psi) = e^t \sum_{\alpha=1,2} \mathbb{E}^{0,\alpha}[e^{ip\cdot B_t}(\mathrm{J}_0\Phi(\theta_\alpha), e^{X_t(\varepsilon)}\mathrm{J}_t e^{-id P_{\rm f}\cdot B_t}\Psi(\theta_{N_t}))], \quad (7.9.138)$$

where the exponent $X_t(\varepsilon)$ is given in (7.9.87).

Proof. The proof goes along the same lines as in Theorem 7.41. □

From Theorem 7.84 we can derive energy inequalities in a similar manner to Corollary 7.43 for the spinless case. Write

$$E(p, \mathscr{A}, \mathscr{B}_1, \mathscr{B}_2, \mathscr{B}_3) = \inf \mathrm{Spec}(H^{\mathbb{Z}_2}_{\rm PF}(p)), \quad (7.9.139)$$

and define $H_{\mathrm{PF}}^{\mathbb{Z}_2}(p,0)$ by

$$(H_{\mathrm{PF}}^{\mathbb{Z}_2}(p,0)\Psi)(\theta) = (H_{\mathrm{PF}}(p) + \mathscr{H}_{\mathrm{d}}(0,\theta))\Psi(\theta) - |\mathscr{H}_{\mathrm{od}}(0,-\theta)|\Psi(-\theta), \quad (7.9.140)$$

where $|\mathscr{H}_{\mathrm{od}}(0,-\theta)| = \frac{|e|}{2}\sqrt{\mathscr{B}_1(0)^2 + \mathscr{B}_2(0)^2}$. This corresponds to

$$\frac{1}{2}(p - P_{\mathrm{f}})^2 + H_{\mathrm{rad}} - \begin{pmatrix} \frac{e}{2}\mathscr{B}_3(0) & \frac{|e|}{2}\sqrt{\mathscr{B}_1(0)^2 + \mathscr{B}_2(0)^2} \\ \frac{|e|}{2}\sqrt{\mathscr{B}_1(0)^2 + \mathscr{B}_2(0)^2} & -\frac{e}{2}\mathscr{B}_3(0) \end{pmatrix} \quad (7.9.141)$$

in $\mathbb{C}^2 \otimes L^2(\mathscr{Q})$. Note that $|\mathscr{H}_{\mathrm{od}}(0,-\theta)|$ is independent of θ. We have the energy comparison inequality

Corollary 7.85 (Diamagnetic inequality). *It follows that*

$$|(\Phi, e^{-tH_{\mathrm{PF}}^{\mathbb{Z}_2}(p)}\Psi)| \le (|\Phi|, e^{-tH_{\mathrm{PF}}^{\mathbb{Z}_2}(p,0)}|\Psi|) \quad (7.9.142)$$

and

$$\max_{(\alpha\beta\gamma)=(123),(231),(312)} E(0,0,\sqrt{\mathscr{B}_\alpha^2 + \mathscr{B}_\beta^2}, 0, \mathscr{B}_\gamma) \le E(p,\mathscr{A},\mathscr{B}_1,\mathscr{B}_2,\mathscr{B}_3). \quad (7.9.143)$$

Proof. We have

$$\begin{aligned} |(\Phi, e^{-tH_{\mathrm{PF}}^{\mathrm{S}}(p)}\Psi)| &\le e^t \lim_{\varepsilon\downarrow 0} \sum_{\theta\in\mathbb{Z}_2} \mathbb{E}^{0,\alpha}[(\mathrm{J}_0|\Phi(\theta_\alpha)|, e^{X_t^{\perp}(\varepsilon)}\mathrm{J}_t e^{-iP_{\mathrm{f}}\cdot B_t}|\Phi(\theta_{N_t})|)] \\ &= \text{r.h.s. } (7.9.142), \end{aligned}$$

where $X_t^{\perp}(\varepsilon)$ is given by (7.9.107). (7.9.143) is immediate from (7.9.142) and a similar argument to (7.9.109). □

Chapter 8
Notes and References

Notes to the Preface

Classic books of this early period include H. Poincaré: *Les méthodes nouvelles de mécanique céleste* (Gauthier-Villars, 1893, English translation: *New Methods of Celestial Mechanics*), 3 vols., see also C. Siegel and J. Moser: Vorlesungen über Himmelsmechanik, 1956 (English translation: *Lectures on Celestial Mechanics*, Springer, 1971), J. C. Maxwell: *A Treatise on Electricity and Magnetism*, Clarendon Press, Oxford, 1873 (reprinted by Dover 1954, 2003), and *Theory of Heat*, Longmans, 1872 (reprinted Dover 2001). A comprehensive account of methods of mathematical physics with a strong stress on PDE which stood as a standard work for generations is D. Hilbert and R. Courant: *Methoden der mathematischen Physik*, vols. 1–2, Springer, 1924, 1937 (with subsequent English translation).

The intellectual biographies and autobiographies of some of the main protagonists of early quantum mechanics capture the excitement of those days and convey the spirit how these ideas were born. See, for instance, W. Heisenberg: *Der Teil und das Ganze*, Piper, 1969, D. Cassidy: *Uncertainty: The Life and Science of Werner Heisenberg*, W.H. Freeman, 1993, A. Pais: *Niels Bohr's Times: In Physics, Philosophy, and Polity*, Oxford UP, 1994, G. Farmelo: *The Strangest Man: The Hidden Life of Paul Dirac, Mystic of the Atom,* Basic Books, 2009. For an account from a major figure in the development of stochastic analysis and Feynman–Kac formulae see R. Bhatia: A conversation with S.R.S. Varadhan, *Math. Intelligencer* 2008, pp. 24–42. E. Wigner's essay *The Unreasonable Effectiveness of Mathematics in the Natural Sciences* appeared in *Commun. Pure Appl. Math.*, 13 (1960). He mentions the idea attributed to Galileo that the laws of nature are written in the language of mathematics. The concept of the "book of nature" is a topos that can be traced back to antiquity and the middle ages, see the early and illuminating account given by the great German literary scholar E. R. Curtius in *Europäische Literatur und lateinisches Mittelalter*, Francke, 1954, 350ff (especially the concept of "chiffre"), and more modern discussions in *The Book of Nature*, A. Vanderjagt, K. van Berkel (eds.), vol. 1: *Antiquity and the Middle Ages*, vol. 2: *Early Modern and Modern History*, Peeters, Leuven, 2005–2006. On the quoted passage of Feynman see R. Dijkgraaf, *Rapporteur Talk: Mathematical structures*, in: *The Quantum Structure of Space and Time, Proceedings of the 23rd Solvay Conference on Physics* (D. Gross, M. Henneaux, A. Sevrin, eds.), Word Scientific, 2007; p. 91. Apparently a mathematician has replied that the time by which mathematics would be set back is "precisely the week in which God created the world"; a version of this anecdote names Mark Kac to be this mathematician.

Notes to Chapter 1

Feynman's path integral method appeared first in the papers [161, 162, 163], see also [164]. Dirac's contact transformation theory was published in [128, 129, 130]. For some historic background see [424]. Modern references on Feynman integrals include [3, 80, 86, 94, 126, 127, 407, 224, 209, 292, 313, 289, 492]. A first paper by Kac using analytic continuation of the time variable is [293], see [79, 372] for later developments. The disadvantage of Kac's approach is that it deals with the diffusion equation with dissipation instead of the time-dependent Schrödinger equation, and therefore the Feynman–Kac formula gives no direct information on the quantum dynamics. However, the time-independent solutions of both equations are the same and coincide with the eigenfunctions of the Schrödinger operator, which justifies the use of the Feynman–Kac formula in quantum mechanics.

Notes to Chapter 2

2.1

The Scottish botanist Robert Brown is credited with the observation reported in 1827 that pollen grains suspended in water perform an irregular motion, however, he was preceded by more than a century by the Dutch natural scientist van Leeuwenhoek by a similar observation. Also less known is the fact that Brown continued his observations by replacing pollen with minerals and even a piece of the Sphinx. Brownian motion has been first modelled in terms of a random process by Bachelier [38] as early as 1900 in order to describe stock prices. Einstein [146] and von Smoluchowski [452] studied in 1905/06 the fluctuations of thermal motion of particles and related them with Brownian motion, which Perrin tested in 1909 experimentally, noting the rough shape of particle paths. Langevin proposed in 1908 an equation to describe the equation of motion of a particle performing Brownian motion. In 1923 Wiener constructed Brownian motion as a random process in a mathematically rigorous way, and Lévy made important early contributions in the study of some fine details of Brownian motion. For a history see [143] as well as [316, 378].

Brownian motion in many ways stands out as a random process having a distinguished role. It is a Gaussian process, a Markov process and a martingale. It is the only Lévy process (see Section 2.4) allowing a version with continuous paths. It is the scaling limit of simple random walk, and it is intimately related with the Laplace operator in that $-\Delta$ is its infinitesimal generator, and thus also with the heat equation. There are a number of equivalent ways of defining and constructing Brownian motion, of which we have presented one and commented on others. The literature on Brownian motion is vast due to the fact that it is an object in the overlap zone of the fields and approaches mentioned above. A wealth of facts and explicit formulae can be found in [72] and the references therein, and we single out [350, 486] for further developments.

2.2

Apart from the original interest in physics, a major modern source of motivation for stochastic analysis are applications in financial mathematics. We single out [280, 284, 303, 378, 380, 406, 464] for fundamental facts on Brownian motion and general stochastic analysis. On a more specialized note, we refer to [411] for a general analysis and measure theory background, [25, 47, 141, 144, 171, 465] for probability theory discussed with an analytic point of view, [196, 319] for random processes, [137, 145, 67, 149] for Markov processes, [287] for Markov generators, [61, 288] for limit theorems, and [409] for continuous time martingales.

2.3

In 1944 Itô [282] proposed a notion of stochastic integral using the paths of Brownian motion as integrator, and he also established the chain rule for stochastic differentials, which is known today as the Itô formula. While the construction of the Itô integral departs from the classic notion of integral proceeding through approximate Riemann sums over finite divisions, it successfully copes with the difficulty that these sums do not stand a chance to converge pathwise since Brownian motion has paths of almost surely unbounded variation. However, by requiring less Itô has shown that convergence holds in L^2-sense (see Definition 3.10.36), which extends to convergence in probability (see Definition 2.23). Since the original ideas of Itô stochastic integration theory has further developed by using continuous martingales as integrators instead of Brownian motion. We refer to [226, 284, 303, 353, 354, 380, 406, 464, 409] for stochastic integration theory with Brownian or more general integrators, and to [101, 280, 380] for stochastic differential equations.

2.4

Lévy processes are random processes with stationary and independent increments, in many ways generalizing Brownian motion. In particular, they are Markov processes and semimartingales, with right continuous paths allowing left limits, which in general contain continuous pieces with jump discontinuities at random times. The generators of Lévy processes are in general non-local pseudo-differential operators. We refer [11, 52, 51, 45, 423, 426] for general results on Lévy processes, to [136, 318] for their fluctuation theory, and to the three volume set [287] which is a comprehensive study emphasizing the context of semigroups and generators. More specifically, see [11, Th. 2.3.5] for Proposition 2.45 and [11, Th. 2.4.16] for Theorem 2.47. The Itô formula for semimartingales (Proposition 2.48) is taken from [280, Chapter II, Section 5]. Subordinators are one-dimensional Lévy processes with non-decreasing paths. We refer to [51, 11, 426] for the relationship between Bernstein functions and subordinators.

Notes to Chapter 3

3.1

There is a large literature on quantum mechanics from a mathematical point of view. The series [400, 401, 402, 403] is well established and [305] is classic, all of them containing fundamental discussions on the spectral analysis of Schrödinger operators. We also refer to [99, 121, 165, 263, 330, 473, 471, 222, 120, 65] for various aspects of rigorous quantum mechanics. [443, 444] are concise but stimulating and up-to-date accounts of the spectral property of Schrödinger operators. The substantial paper [441] and the monograph [438] are seminal discussing Schrödinger semigroups and very appealing applications of the Feynman–Kac formula to the spectral analysis of Schrödinger operators.

Proposition 3.9 on the criterion for the compactness of operators is from [403, Th. XIII.64], and Proposition 3.10 from [403, XIII.4, Corollary 2]. The Kato–Rellich theorem is due to [405], and a first application to atomic Hamiltonians is [304]. Kato's inequality is due to [306]. The operator version of Kato's inequality, called abstract Kato's inequality,, is considered in [125, 240, 437, 434]. Proposition 3.16 is taken from [400, Th. VIII.5], and the KLMN theorem from [401, Th. X.17]. For Theorem 3.18 see [305, Th. VIII.3.11, Th. VIII.3.13 with Supplementary notes to Chapter VIII, 5 (p. 575)]. Rellich's criterion first appeared in [404].

The Trotter product formula (3.28) is a powerful tool in the construction of the functional integral representation of Schrödinger-type semigroups. The original paper is [475]. [307, 308] extend the Trotter product formula to the case of a quadratic form sum $A \dotplus B$ for $\overline{A + B}$ in (3.28). Proposition 3.29 is due to [309], where the Trotter formula is further extended to non-linear semigroups generated by sub-differentials of convex functions. [369, 370] use trace norm limits in the Trotter formula instead of the strong limits, assuming that $e^{-t(A \dotplus B)} P_{AB}$ is trace class (here P_{AB} denotes projection onto $D(A) \cap D(B)$). Operator norm convergence of products in the Trotter formula is obtained by [277] with an error bound $\|(e^{-(t/n)A} e^{-(t/n)B})^n - e^{-t(\overline{A+B})}\| = O(n^{-1/2})$, as $n \to \infty$. Here it is assumed that $A + B$ is essentially self-adjoint on $D(A) \cap D(B)$. That this error bound is the best possible is shown in [470]. In the case where $A + B$ is moreover self-adjoint on $D(A) \cap D(B)$, the error term can be improved to $O(n^{-1})$ [278]. Furthermore, [68] shows norm convergence in the Trotter product formula on Banach spaces.

3.2

For the classic Feynman–Kac formula see [86, 119, 292, 438]. Kac has originally considered positive potentials, however, this strong restriction has been gradually weakened (see Section 3.3 below).

There is a large and important body of research on Feynman–Kac semigroups for Schrödinger operators on bounded domains, which we cannot address in this book for lack of space. The analysis in this case involves tools of potential theory, stopping times and killed processes. The earliest papers establishing a link between Brownian motion and harmonic functions are [138, 139, 297, 298]. Further developments include the sequence of papers [272]; see [67, 117, 140, 433] for monographic treatments, and the modern [393] presenting a unifying analytic-probabilistic approach. In the context of processes generated by Schrödinger operators a seminal paper is [2], and a book length discussion of the subject is [94].

For the Schrödinger operator $H = -(1/2)\Delta + V$ the Dirichlet boundary value problem for a given bounded domain $D \subset \mathbb{R}^d$ with smooth enough boundary ∂D is

$$\begin{cases} H\psi = 0 & \text{in } D \\ \psi = \phi & \text{on } \partial D. \end{cases}$$

Let $(B_t)_{t\geq 0}$ be standard Brownian motion, and consider its first exit time $\tau_D = \inf\{t > 0 | \, B_t \notin D\}$ from D. By the definition $B_t \in D$ for $t < \tau_D$ and $B_{\tau_D} \in \partial D$ if $\tau_D < \infty$. It is possible to prove that $\tau_D < \infty$ $\mathcal{W}$-a.s. Then the Feynman–Kac formula becomes

$$\psi(x) = \mathbb{E}^x[e^{-\int_0^{\tau_D} V(B_t)dt}\phi(B_{\tau_D})]$$

and thus it offers a probabilistic representation of the solution to the Dirichlet problem. In this case the spectrum of the Laplace operator is discrete consisting of eigenvalues $0 < \lambda_0(D) < \lambda_1(D) < \cdots$.

(a) *Isoperimetric inequalities:* One line of research is addressing the question how the various properties of the spectrum depends on geometric properties of D. Let D be a convex domain and V be a convex function. In this case also the Schrödinger operator has a purely discrete spectrum $0 < \lambda_0(D, V) < \lambda_1(D, V) < \cdots$. Suppose without restricting generality that $0 \in D$, and let $\mathbb{B}$ denote a ball centered in the origin such that $|D| = |\mathbb{B}|$, where the bars denote Lebesgue measure. Then it can be shown that the first exit time of Brownian motion is maximized by the ball, i.e., $\sup_{x\in D} \mathcal{W}^x(\tau_D > t) \leq \mathcal{W}^0(\tau_{\mathbb{B}} > t)$, for all $t > 0$. This is an example of an isoperimetric inequality. Since $\lim_{t\to 0} \frac{1}{t} \log \mathcal{W}^x(\tau_D > t) = -\lambda_0(D)$, the above inequality implies the *Faber–Krahn inequality* $\lambda_0(\mathbb{B}) \leq \lambda_0(D)$, which is an isoperimetric property of the eigenvalues of the Dirichlet Laplacian [39, 89], see also [382]. The Faber–Krahn inequality has been similarly shown for Schrödinger operators for convex domains [392, 230, 231, 116], see the conditions on V in the referred papers.

Another isoperimetric inequality concerns the ratio of consecutive eigenvalues, originating from the Payne–Pólya–Weinberger conjecture [386] which originally stated that for planar domains $\lambda_1(D)/\lambda_0(D)$ is maximized when D is chosen to be a disk. The conjecture has been proven to hold in any dimension in [26, 27].

(b) *Spectral gaps:* The difference $\Lambda(D) = \lambda_1(D) - \lambda_0(D)$ is called *fundamental gap* for the Dirichlet Laplacian, and a similar object can be defined for the Dirichlet Schrödinger operator. Based on specific cases in which the eigenvalues can be computed it has been conjectured [479] that $\Lambda(D) \geq \frac{3\pi^2}{2d^2}$, where d is the diameter of D, provided both D and V are convex. An upper bound can be obtained as follows. Since by the Payne–Pólya–Weinberger conjecture $\lambda_1(D)/\lambda_0(D) \leq \lambda_1(\mathbb{B})/\lambda_0(\mathbb{B})$, it follows that $\Lambda(D) \leq \Lambda(\mathbb{B})(\lambda_0(D)/\lambda_0(\mathbb{B}))$, in which the second factor at the right-hand side can be further estimated by using the Faber–Krahn inequality. For a discussion see [28] and the references therein. For Schrödinger operators with convex V the fundamental gap conjecture was proved recently in [8].

(c) *Harnack inequalities:* In its classical form, the boundary Harnack inequality states that for any open sets $K \subset K^* \subset \mathbb{R}^d$ with $\overline{K} \subset K^*$, there exists $C > 0$ such that for all functions f harmonic and non-negative on K^* we have $f(x) \leq Cf(y)$, for all $x, y \in K$. In other words, just by knowing that a non-negative f solves $\Delta f = 0$ we can already conclude that $\inf_{x\in K} f(x) \geq \frac{1}{C} \sup_{x\in K} f(x)$, where C depends on K and K^* but not on f. A version of this inequality for Schrödinger operators was first proved by [476] using analytic methods. [2] realized that the Feynman–Kac formula can be used to obtain these results. They used the fact that when stopped at the boundary of K^*, a Feynman–Kac type formula just gives the solution of the Dirichlet problem for H and K^*. Further references include [490, 491, 233, 94].

Further interesting aspects include *heat kernel estimates* [108, 109, 42, 391, 207, 208], *functional inequalities* [421], *intrinsic ultracontractivity* [110, 111, 40], and others. For the general theory of how the Feynman–Kac formula can be used to study various classes of PDE we also refer to [144, 170, 47, 48, 466].

3.3

Kato-class has been introduced in [306]. Simon [441] studies Schrödinger semigroups with Kato-class potentials in great detail. In particular, here it is proven that e^{-tH} is a bounded operator from L^p to L^q for $1 \leq p \leq q \leq \infty$. A natural question is how the norm of this operator depends on t for large t. It is shown in [439] that $\alpha = (1/t)\ln \|e^{-tH}\|_{p,q}$ is independent of p and q; it turned out [440] that the same α is obtained by taking pointwise limits of images of non-negative functions. Since then there has been a considerable activity on the topic, see [108, 109, 316, 364, 365, 391, 488, 489], as well as [317, 469, 467]. Yet another, more recent, line of research is concerned with cases when the potential V is replaced by more singular objects such as distributions, or when Brownian motion is replaced by a Bessel process. For research in this direction see [415, 414, 416, 412, 413] and the references therein.

3.4

Theorem 3.44 can be improved by showing that $e^{-tK}(x, y)$ is even jointly continuous in x, y and t [441]. The proof of Lemma 3.47 is due to [98]. Estimates of the number of negative eigenvalues of Schrödinger operators are due to [327, 328], where the Birman–Schwinger principle and the Feynman–Kac formula are used. The proof of Lemma 3.51 is taken from [445] and [438, Ths. 8.1 and 8.2]. The Lieb–Thirring inequality [327, 332, 333] can be extended to a wide class of Schrödinger operators, including those with spin and magnetic field, we refer to [329] and references therein. Proposition 3.52 is [403, Th. XIII.52]. The Perron–Frobenius theorem first appeared in [387, 172] for matrices, and it was applied to quantum field theory by [200, 211, 212]. The Klauder phenomenon is discussed in [151, 150, 157, 435]. Proposition 3.56 is proven by [300]. Due to the Klauder phenomenon one can construct examples of Schrödinger operators with degenerate ground states. Exponential decay in L^2 sense, i.e., $\|e^{a|x|}\phi\| < \infty$ for eigenvectors is studied in [1], where also the exponential decay of eigenvectors of the divergence form $-\partial_\mu A_{\mu\nu}(x)\partial_\nu + V$ is considered. Pointwise exponential decay of eigenvectors of Schrödinger operators is investigated by [83], and by [84] for the relativistic Schrödinger operator. In the massless relativistic case, i.e., $\sqrt{-\Delta} + V$, eigenvectors decay polynomially. In the case of the divergence form $-\partial_\mu A_{\mu\nu}(x)\partial_\nu + V$ with uniform elliptic bound $c \leq |A_{\mu\nu}(x)| \leq C$ and some regularity condition on $A_{\mu\nu}$, the pointwise exponential decay of eigenvectors also can be proven in a similar way to Carmona's estimates with respect to the diffusion process associated with the divergence form. An example how far functional integration and related tools can be used to draw information on the spectral properties of a pseudo-differential operator is [342].

3.5

References on Schrödinger operators with magnetic field include [29, 30, 31]. Self-adjointness of the Schrödinger operator with magnetic field $H(a)$ is established in [325]. The alternative proof of the Feynman–Kato–Itô formula of the Schrödinger operator with magnetic field $H(a)$ is taken from [438, Section V], where $\nabla a = 0$ is assumed. The applications of functional integration with magnetic field and the diamagnetic inequality are from [147, 148, 379, 76, 169, 271].

3.6

For results on the spectrum of the relativistic Schrödinger operator see [104, 105, 239, 326]. The Feynman–Kac formula for relativistic Schrödinger operators $H_{\mathrm{R}}(a)$ with

$a = 0$ is studied [112, 84]. The case of non-zero vector potential is discussed in [115, 254]. The function $f(x) = \sqrt{x^2 + m^2} - m$ is an example of Bernstein functions. The relativistic Schrödinger operator is then realized as $f(\frac{1}{2}(-i\nabla - a)^2) + V$. A Feynman–Kac-type formula of a general Schrödinger operator of the form $\Psi(\frac{1}{2}(-i\nabla - a)^2) + V$ with Bernstein function Ψ defined on positive real line is constructed by [254].

The relativistic Schrödinger operator can be also defined through the *Weyl quantization*. This is given by $H^w = h^w(x, D) + V(x)$, where

$$(h^w u)(x) = \iint e^{i(x-y)\cdot\xi} h\left(\frac{x+y}{2}, \xi\right) u(y)(2\pi)^{-d} dy d\xi,$$

with symbol $h(x, \xi) = \sqrt{|\xi - a(x)|^2 + m^2}$. [367] shows that in general

$$(h^w(x, D))^2 \neq (-i\nabla - a)^2 + m^2.$$

The Feynman–Kac formula of e^{-tH^w} is given by [275].

3.7

In quantum mechanics the notion of spin is introduced as an internal degree of freedom of electrons. Spin plays a central role in the statistics of electrons. Another aspect of the influence of spin is in the behavior of electrons in a magnetic field. The Feynman–Kac formula for the Schrödinger operator with spin is discussed in [114, 113, 179, 180, 254]. The diamagnetic inequality (3.7.36) is due to [114, 254]. Paramagnetism of Schrödinger operators with spin $1/2$ is also an interesting subject. It has been conjectured in [266] that paramagnetism is universal, but a counterexample has been given in [32], while paramagnetism in the classical limit is discussed in [265].

For a 2-dimensional space-time Dirac operator, [273, 274, 276] proposes a path integral representation in terms of a $\mathscr{S}'(\mathbb{R}^2; M_2(\mathbb{C}))$-valued countably additive measure on $C([0, t]; \mathbb{R})$. In particular, in [276], the measure is concentrated on paths having differential coefficients equal to "$\pm 1 \times$ speed of light" in every finite time interval except at finitely many instants of time. This relates with the Zitterbewegung of Dirac particles. A similar result is also given by [113] with respect to a Poisson process. Furthermore, [270, 268, 269, 336] treat quantum probability and derive a non-commutative Feynman–Kac formula.

3.8

The Feynman–Kac type formula for relativistic Schrödinger operators with spin is due to [115, 254], where the combination of a stopping time (subordinator) and a Poisson process is applied. In [254] a Feynman–Kac formula for Schrödinger operators of the form $\Psi(\frac{1}{2}(\sigma \cdot (-i\nabla - a))^2) + V$ is given for arbitrary Bernstein functions Ψ, and an energy comparison inequality is obtained. The Schrödinger operator with spin can be transformed to the operator on $L^2(\mathbb{R}^3 \times \mathbb{Z}_2)$. Also operators on $L^2(\mathbb{R}^3 \times \{e^{2\pi i}, e^{2\pi i/2}, \ldots, e^{2\pi i/p}\})$ can be considered.

3.9

This material is taken from [443].

3.10

$P(\phi)_1$ processes have an interesting property deriving from connections with statistical mechanics and Gibbs measures. Formally such a process is the stationary solution of the SPDE

$$\dot{X}(u,t) = \frac{1}{2}\Delta_u X(u,t) - (\partial V)(X(u,t)) + \mathcal{W}, \tag{8.0.1}$$

where $\mathcal{W}$ now is *space-time white noise*. This was shown rigorously for a restricted class of potentials V by [286]. An interesting application appears in [229], where the SPDE above is used in an MCMC algorithm for sampling the conditional $P(\phi)_1$ measure in various situations. In the context of SPDE the result of [58] is also of interest: when the potential grows sufficiently rapidly, it implies uniqueness of those stationary measures of the SPDE above that are Gibbs measures, i.e. satisfy the DLR equations. It is expected, but not proven, that all stationary measures of the above SPDE satisfy the DLR equations. The $P(\phi)_1$ process associated with the divergence form

$$-\sum_{i,j=1}^{d} \partial_i a_{ij}(x)\partial_j + V(x)$$

with a confining potential can be constructed, and also that of $-\Delta + V(x)$. Refer to e.g., [192]. Moreover for the case of a non-local operator, the $P(\phi)_1$ process associated with a fractional Schrödinger operator can be constructed. We refer to [410]. Lemma 3.112 is taken from [192], which is a modification of [311].

For intrinsic ultracontractivity properties see [110, 108, 111], and for asymptotic intrinsic ultracontractivity [299].

Notes to Chapter 4

4.1

The seminal papers introducing Gibbs measures for lattice spin systems are [131, 132] and [321]. Further details on the prescribed conditional probability kernels (called specifications) can be found in [133, 168] and [396, 205, 453]. Surveys and books on Gibbs measures include the early [419], covering also classical continuous and quantum lattice systems, and [454, 183, 455, 320, 184, 395, 397, 418, 399]. In [396] a general framework is developed, while the approach of [281] is more from the perspective of the variational principle in which equilibrium states appear as tangents to the free energy functional. Reference [185] is a highly authoritative comprehensive monograph with a strong stress on DLR theory, covering a wealth of material. [348] is a systematic text on cluster expansion methods applied to Gibbs measures. Other relevant books on rigorous statistical mechanics include [217] focusing more on Ising models, [447] giving an early account of contour techniques, and [442] of lattice gases. A reappraisal of Gibbsian theory was presented in the book-length paper [478], especially in the context of renormalization group transformations. For Gibbs measures on lattice spin systems with non-compact space existence of a Gibbs measure is harder to prove: see the superstability conditions in [417], tempered Gibbs measures [322, 87], [341, 340, 337] for cases when the potential in not uniformly but pointwise or almost surely summable, and [186] for Gibbs measures indexed by subsets of an uncountable index set. For other approaches to and applications of Gibbs measures on path space(s) we refer to [5] and the references therein.

4.2

Some of the basic notions, such as local uniform domination, are adapted from discrete models, see [185]. The discussion of uniqueness of Gibbs measures follows [58]. The case $\alpha = 1$ in the uniqueness criterion suggests that a maximal configuration space on which μ is the unique Gibbs measure, a set consisting of exponentially growing paths is closer to truth than $\mathcal{X}_0$, however, establishing this is beyond the method of our proof. In [286] sets of exponentially growing paths as sets of uniqueness for the Gibbs measure were obtained, however, assuming convexity of the function $V(x) - \kappa|x|^2$ for some $\kappa > 0$, therefore the case $\alpha < 1$ is not covered by his results. The advantage of our method is that it is not limited to polynomials, nor even to continuous or semibounded potentials, in particular, local singularities and other perturbations of V in the above examples will not alter the conclusions. This corresponds to the intuition that only the behaviour of the potential at infinity should determine whether uniqueness of the Gibbs measure holds on a given set.

The existence proof for Gibbs measures with pair potential comes from [55]. For further literature on Gibbs measures with pair interaction potential we refer to [381, 343, 234, 338, 59, 55]. Densities dependent on double stochastic integrals are treated in [219, 339, 56], the first two papers using techniques of rough paths, and processes with jump discontinuities in [299].

4.3

There are several versions of cluster expansion method. A systematic monograph is [348] and useful reviews are [156, 77]. For the general combinatorial scheme see

[314, 155, 394]. We refer to [73, 358, 134, 389, 78, 49] for improving on convergence, however, these method depend on the specific models at hand.

4.4

The presentation here follows [60]. The functional central limit theorem in Theorem 4.42 follows from [237]. The above theorem still contains the possibility that the diffusion matrix is zero, which would imply subdiffusive behaviour of Brownian paths. There are two independent proofs of ruling this out, one in [60, 456] using ideas of [74], and another in [218]. Both require substantial additional techniques to the material of this book, and therefore they are omitted in our presentation.

Notes to Chapter 5

5.1

A mathematical introduction to quantum field theory is given in [17, 24, 204, 223, 436, 463]. References [285, 388, 482] treat quantum field theory from the point of view of particle physics. Useful texts on quantum field theory include [63, 70, 71, 167, 238, 368, 429, 420]. [164] is a monumental book on quantum field theory using path integrals. We also refer to [4, 86, 103, 225, 407, 427, 468] for texts in this spirit. [159] offers a concise summary of fermion functional integrals. Geometric and topological discussions on quantum field theory are [160, 182, 428, 462, 481]. Standard texts on QED are [236, 291, 301, 355, 425, 487]. Quantum field theory in curved space-time is discussed in [62, 43, 178]. [50, 349] covers spectral methods in infinite dimensional analysis.

5.2

Fock space was introduced in [166]. The second quantization $d\Gamma$ was first treated by Cook [96], where self-adjointness of the field operator $\Phi(f)$ and its conjugate momentum $\Pi(f)$ was shown. Nelson's analytic vector theorem (Proposition 5.3) appeared in [371]. Wick product was introduced in [484]. The number operator bounds (5.2.27)–(5.2.28) can be generalized to $\|(N+1)^{-m/2}T(N+1)^{-n/2}\| \leq \|W\|_{L^2}$, where $T = \int W(k_1, \ldots, k_m, p_1, \ldots, p_m) \prod_{j=1}^m a^*(k_j) \prod_{i=1}^n a(p_i) dk dp$, see [197].

5.3

Segal [430, 431] introduced $\mathscr{Q}$ space, which linked the theory of free quantum fields with probability. Using this expression the spectrum of the $P(\phi)_2$ Hamiltonian has been studied, for instance, in the series of papers [198, 200, 201, 199, 202, 203] by Glimm and Jaffe.

5.4

Theorem 5.9 is taken from [436, Section I.2]. Minlos's theorem first appeared in [356].

5.5

Nelson emphasized the functional integral approach to quantum field theory in [376], and the fundamental results concerning the Markov property in quantum field theory were established in [377, 375]. We also refer to [436] for for further aspects of Euclidean quantum field theory. Nelson's Euclidean quantum field and its lattice approximation due to [221] changed the object studied in quantum field theory before.

5.6

As explained in this section, it is impossible to construct a Gaussian measure on a real Hilbert space $\mathscr{H}$ with the given covariance. Instead of this we consider the triplet $\mathscr{M}_{+2} \subset \mathscr{M} \subset \mathscr{M}_{-2}$ under the identification $\mathscr{M}^* = \mathscr{M}$, and construct the process $t \mapsto \xi_t \in \mathscr{M}_{-2}$ such that $\xi_t(f) = \langle\!\langle \xi_t, f \rangle\!\rangle$ is a Gaussian process for $f \in \mathscr{M}_{+2}$. Similar results appear in [82], where path continuity is considered. [390] treats the Ornstein–Uhlenbeck semigroup in an infinite dimensional L^2 setting. [290] is a comprehensive study of Gaussian measures on a Hilbert space; we also refer to [135] for non-Gaussian measures. [100, 302, 398] treat infinite dimensional stochastic equations. For the Minlos–Sazanov theorem (Theorem 5.21) we refer to [302, Th. 2.3.4].

Notes to Chapter 6

6.1

In 1964 E. Nelson [373] demonstrated the mathematical existence of a meson theory with non-relativistic nucleons. He introduced a system of Schrödinger particles coupled to a quantized relativistic field, which is nowadays called the *Nelson model*. He also introduced a quantum field theory model defined by

$$H = (2m)^{-1} \int_{\mathbb{R}^d} \psi^*(x)(-\Delta)\psi(x)dx + H_{\mathrm{f}} + g \int_{\mathbb{R}^d} \psi(x) H_{\mathrm{I}}(x)\psi(x)dx,$$

where ψ is a non-relativistic complex scalar field on $\mathscr{H} = \bigoplus_{n=0}^{\infty} \mathscr{H}_n$. The restriction $H\lceil_{\mathscr{H}_n}$ is the n-nucleon Nelson model. Since this model can be reduced to each $\mathscr{H}_n$, this is essentially the same as the Nelson model considered in this chapter.

6.2

As is explained in this section, IR divergence and UV divergence are a ubiquitous difficulty not only in the Nelson model but in general the general theory of quantum fields. In 1952 Miyatake [361] and van Hove [480] found that the ground state of a Hamiltonian in Fock space weakly converges to zero when the UV cutoff removed. Shale [432] developed mathematical methods to study both IR divergence and UV divergence.

In [373] Nelson defined a self-adjoint operator with UV cutoff on the tensor product of Hilbert spaces, $L^2(\mathbb{R}^d) \otimes \mathscr{F}$. The solution of the Schrödinger equation diverges as the cutoffs tends to infinity, but the divergence amounts merely to a complex infinite phase shift due to the self-energy of the particles. A canonical transformation, which is implemented by a unitary operator when the IR cutoff is introduced, separates the divergent self-energy term. It is then shown that, after removing the UV cutoff and subtracting an infinite constant, a limit Hamiltonian exists. The infrared problem in the Nelson model was first studied by Fröhlich [173, 174]. We also refer to [81, 213, 232, 264, 310, 451, 450, 449] for earlier results on the Nelson model.

6.3

The Feynman–Kac formula of the Nelson Hamiltonian in terms of a combination of a diffusion process X_t and a $\mathscr{M}_{-2}$-valued stochastic process ξ_t is due to [57]. Here the ground state transformation of the Nelson Hamiltonian is applied. A similar Feynman–Kac formula, with no quantized field, is discussed in [334].

In [191, 192] the functional integral representation of the Nelson model defined on a static Lorentz manifold is constructed. There the diffusion process X_t is replaced by a process associated with the divergence form

$$-\sum_{\mu,\nu=1}^{3} \frac{1}{c(x)} \nabla_\mu A_{\mu\nu}(x) \nabla_\nu \frac{1}{c(x)} + V,$$

and the dispersion relation $\omega = \sqrt{-\Delta + m^2}$ expressed in the position representation is replaced by the pseudo-differential operator

$$\omega_{a,m} = \Big(-\sum_{\mu,\nu=1}^{3} \nabla_\mu a_{\mu\nu}(x) \nabla_\nu + m^2(x)\Big)^{1/2}.$$

Note that, importantly, the mass becomes position-dependent, $m(x)$. It is then not clear how to estimate the decay rate of $\widehat{\omega_{a,m}^{\alpha}\varphi}(k)$ as $|k| \to \infty$. Moreover, it is not straightforward to show existence or absence of the ground state of the Nelson model on a static Lorentz manifold by operator theory. Instead of operator theory one can apply the functional integral method introduced in Chapter 6 to show absence of ground state.

6.4

For the massive case the existence of the ground state of the translation invariant Nelson model is studied in [212, 174]. In the massless case Bach, Fröhlich and Sigal [35, 36] proved the existence and uniqueness of the ground state for sufficiently weak couplings under the infrared regular condition; they treated more general models, including the Nelson model. Spohn [460] and Gérard [188, 190] proved the existence of the ground state for arbitrary values of the coupling constant. Functional integral methods are used in [460], and operator theory in [188]. Theorem 6.6 is due to [460]. The external potential V in [460, 188] is confining. For non-confining potentials we refer to [422, 241]. In the infrared singular case, the Nelson Hamiltonian has no ground state as is proven in this chapter. However, it has a ground state when a non-Fock representation is used, see [19, 344]. The existence of a ground state without UV cutoff is due to [6] for the massive case and [245] for the massless case. In a series of papers [191, 193, 192], the Nelson model on a *static Lorenz manifold* is considered. Here the static Lorentz manifold is a 4-dimensional pseudo-Riemannian manifold with line element $ds^2 = g_{00}(x)dt \otimes dt - \sum_{i,j} \gamma_{ij}(x)dx^i \otimes dx^j$, where $g_{00} > 0$ is time-independent and $\gamma = (\gamma_{ij})$ denotes a 3-dimensional Riemannian metric. In particular, the infrared problem is studied, and criteria for existence or absence of ground states are shown. We also refer to [123, 122, 124, 187, 189, 181, 194].

6.5

This material is taken from [57]. Physically it is expected that the number of soft (i.e., low-energy) bosons in the ground state diverges when the infrared cutoff is removed, and thus a ground state cannot exist in Fock space. (6.5.44) mathematically validates the first half of this intuition. The second half, the absence of ground state, will be discussed in Section 6.6. The pull-through formula is quite useful to estimate $(\Psi_g, N\Psi_g)$, which is used in e.g. [6, 36, 37, 173, 188, 245]. Superexponential decay of the number of bosons in the Nelson model is also studied in [213] using operator theory.

6.6

The translation invariant Nelson model has been studied in e.g. [9, 81, 173, 174, 213, 362, 363, 357]. It is also related with the *polaron model*. [195] is a useful summary of various versions of polaron models. The path integral approach to this model is developed in [252, 457]. (6.6.10) is due to [174, 457]. Proposition 6.6.11 is shown in [174].

6.7

In the IR singular case, i.e., when $\int |\hat{\varphi}(k)|^2/\omega(k)^3 dk = \infty$, non-existence of the ground state of the Nelson model is proven in [22, 123, 244, 345, 344]. Theorem 6.34 is due to [345, 344].

6.8

The existence of the Nelson Hamiltonian without UV cutoff (Theorem 6.39) has been proven by Nelson [373] by using the so called Gross transform [210, 323]. Nelson [374] also showed that by using functional integration for the Hamiltonian with the potential $|x|^2$. The bounds (6.8.14)–(6.8.15) are presented in [6, 245]. The functional integral representation of the Nelson model without UV cutoff (Theorem 6.45) is due to [220, 374].

Davies considered a weak coupling limit of the form $H_\mathrm{p} + \Lambda\phi(f) + \Lambda^2 H_\mathrm{f}$ in [106, 107]. In [247] the weak coupling limit with a simultaneous removal of UV cutoff is studied by using functional integrals. Related works on the scaling limit in quantum field theory are [16, 18, 142, 384, 472]. In particular, in [16] a general criterion of convergence in the scaling limit is found.

Notes to Chapter 7

7.1

In 1938 the Pauli–Fierz model was introduced in [385], where the generation of infrared photons was studied. See also [66] for the infrared divergence. Bethe [54] and Welton [483] proposed a theoretical interpretation of the Lamb shift by a variant of the Pauli–Fierz model. We refer to [95, 355] for details, and to [461] for a summary of recent progress of the spectral analysis of the Pauli–Fierz and related models.

7.2

In the *dipole approximation* the quantized radiation field $\mathscr{A}(x)$ is replaced by $\mathscr{A}(0)$. In this case collisions between electrons and photons are ignored, and there is no derivative coupling. Under specific conditions the Pauli–Fierz model with dipole approximation and $V = 0$ can be diagonalized, i.e., $H_{\mathrm{PF}}^{\mathrm{dip}}$ can be approximated by the $3 + 3L$ dimensional harmonic oscillator

$$\frac{1}{2m}\Big(p + \sum_l^L \hat{\varphi}(l) Q_l\Big)^2 + \frac{1}{2}\sum_l^L (P_l P_l + Q_l \omega^2(l) Q_l - \omega(l)).$$

Here $[P_l, Q_{l'}] = -i\delta_{ll'}$ is satisfied. Then it can be shown that

$$H_{\mathrm{PF}}^{\mathrm{dip}} \cong \frac{1}{2(m + e^2 \frac{d-1}{d}\|\hat{\varphi}/\omega\|^2)} p^2 \otimes 1 + 1 \otimes H_\mathrm{f} + g,$$

where g is a constant. In the series of papers [12, 13, 14, 15] Arai investigated the spectrum of the Pauli–Fierz model in the dipole approximation. We also refer to [64, 459] for the scattering theory.

The spectral analysis of the Pauli–Fierz Hamiltonian without dipole approximation has seen much progress since the 1990s. The existence of a ground state has been proved by Bach, Fröhlich and Sigal [37] for weak couplings and no infrared cutoff. Griesemer, Lieb and Loss [206], Lieb and Loss [331] successively removed the restriction on the coupling constant, and have also introduced a binding condition giving a criterion for the existence of a ground state (see also [44]). We emphasize that no infrared regularity is needed to show the existence of a ground state in Fock space, in contrast to the Nelson model.

A basic assumption is the existence of a ground state for the zero coupling Hamiltonian, i.e. that H_{p} has a ground state with positive spectral gap. [259] shows that for sufficiently large coupling constants the Pauli–Fierz Hamiltonian in the dipole approximation has a ground state even when H has no ground state. This phenomenon is called *enhanced binding*. Enhanced binding for the Pauli–Fierz Hamiltonian is also studied in [23, 88, 93, 228, 258, 262].

Hypercontractivity was introduced by Nelson in [377], where also Proposition 7.6 was proved. We also refer to [446].

7.3

Functional integral representations of the Pauli–Fierz model are discussed in [158, 176, 224, 246, 461]. We emphasize that the exponent in the Feynman–Kac formula $-ie\mathscr{A}_{\mathrm{E}}(K_t) = -ie\int_0^t \mathscr{A}_{\mathrm{E}s}(\tilde{\varphi}(\cdot - B_s)) \cdot dB_s$ depends on time s explicitly, in contrast to the classical case. Gross [214] discusses a $\mathscr{Q}$-space representation of a *Proca field*, which is a model of massive quantum electrodynamics. The diamagnetic inequality (Theorem 7.15) with a quantized radiation field is obtained in [29, 246].

7.4

Essential self-adjointness of $H + V$ such that e^{-tH} is positivity preserving is studied in [154, 312]. Self-adjointness of the Pauli–Fierz Hamiltonian is established by [37] for weak couplings, and by [248, 250] for arbitrary values of coupling constants. Theorem 7.26 is taken from [250]. An alternative proof is given in [235].

An infinite dimensional version of the Perron–Frobenius theorem is due to [200, 211, 212]. Proposition 7.28 is taken from [153, 157]. The notion of a *positivity preserving* operator can be extended to *invariant cone* property. Let $\mathscr{H}$ be a real vector space. $\mathscr{K} \subset \mathscr{H}$ is a *cone* if and only if $u, v \in \mathscr{K} \to u + v \in \mathscr{K}$, $v \in \mathscr{K}$ and $a \geq 0 \to av \in \mathscr{K}$ and $\mathscr{K} \cap -\mathscr{K} = \emptyset$. The Perron–Frobenius theorem can then be extended to the invariant cone semigroup. One application is to show the uniqueness of the ground state of a fermion system, which was done by [212, 157]. Theorem 7.30 is due to [249]. We also refer to [34, 251, 347, 448, 200] for the uniqueness of the ground state of the Pauli–Fierz Hamiltonian. The spatial exponential decay of eigenvectors of the Pauli–Fierz model is discussed in [37, 206, 242].

7.5

This material is taken from [242]. $e^{-tH_{\mathrm{PF}}^{\mathrm{Kato}}}$ is also positivity improving, and the ground state of $H_{\mathrm{PF}}^{\mathrm{Kato}}$ decays exponentially. These facts are proved in the same way as for H_{PF}. Moreover, in [242] Pauli–Fierz Hamiltonians with general cutoff functions are defined through Feynman–Kac type formulae.

7.6

This material is taken from [252]. The idea to construct the functional integral representation comes from [457]; see also [7, 346]. Chen [90] shows the existence of the ground state of $H_{\mathrm{PF}}(p)$ when $p = 0$. The *effective mass* m_{eff} of the Pauli–Fierz Hamiltonian is defined by $\frac{1}{m_{\mathrm{eff}}} = \frac{1}{3}\Delta_p E(p)\lceil_{p=0}$, where $E(p)$ denotes the ground state energy of $H_{\mathrm{PF}}(p)$ [216, 215]. When a sharp infrared cutoff is introduced in H_{PF}, $E(p)$ is differentiable with respect to p_μ for sufficiently small coupling constants. It is, however, not clear that $E(\cdot)$ is twice differentiable in a neighborhood of $p = 0$, when no infrared cutoff is introduced. The infrared problem of the effective mass is studied in [33, 92, 91, 227]. The functional integral approach of the effective mass is [60, 456]. The asymptotic behaviour of effective mass with respect to the ultraviolet cutoff is studied in [261, 255].

7.7

This material is taken from [56]. The formal double stochastic integral of the type $\int dt \int ds W(B_t - B_s, t - s)$ appears in R. Feynman [163, p. 443]. The mathematical derivation is done in [249, 374, 461]. See also [97, 360].

7.8

This material is taken from [253], where also the relativistic Pauli–Fierz model *with spin* $1/2$ is studied. The spectrum of the relativistic Pauli–Fierz model is studied in [175, 257, 352, 359].

7.9

A basic reference for the material of this section is [256]. The degeneracy of ground states for weak enough coupling is proven in [251, 260]. The uniqueness of the ground state of the translation invariant spinless Pauli–Fierz Hamiltonian is also proven in [260]. Results related to Theorem 7.76 can be found in [267]. Spohn [458] treats the model by functional integration. Arai and Hirokawa [20, 21] define a generalized spin-model and prove existence and uniqueness of the ground state. The general reference for the spin-boson model is [324]. The functional integral representations of the spin-boson model follows directly from results in [256]; see also [53]. The vacuum expectation $(\sigma_\alpha \otimes \Omega, e^{-tH_{\mathrm{SB}}}\sigma_\beta \otimes \Omega)$ is given in terms of a Poisson process through the Feynman–Kac formula. This was also studied operator-theoretically in [243].

The electronic part of the Pauli–Fierz Hamiltonian is non-relativistic, given by a Schrödinger operator. Nelson's work on Euclidean field theory for spinless boson proved to be a very significant conceptual and technical stimulus in the program of constructive quantum field theory. A version for fermions has been also discussed. The Euclidean Fermi field was introduced in [383]. [485] is an axiomatic approach. See, furthermore, [46] for the Itô–Clifford integral, [10] for the fermionic Itô formula, [335] for fermion martingales, [408] for fermion path integrals, and [315] for functional integration of the Euclidean Dirac field.

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