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Approximation by Genuine Type (p, q)-Phillips Operators

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***Abstract*:** In this article, we define Genuine type (p,q)-Phillips operators. We give some basic lemmas for these operators, explore local and weighted approximation theorems and give rates of convergence of these operators. Furthermore, we provide Voronovskaja type theorem and get direct results.

***Keywords*:** (p,q)-Phillips operators, Genuine type (p,q)-Phillips operators, rate of convergence, weighted approximation, rates of approximation, Voronovskaja type theorem.

1. Introduction

q-calculus and (p,q)-calculus are one of the fastest developing notions in approximation theory [1–3]. These two notions are used effectively in many areas of sciences such as a field theory, Lie groups, quantum groups, statistics, physical sciences, etc. Today you can find a lot of articles related with the quantum and post-quantum calculus (see [4–9]).

In this work, our purpose is to define genuine type (p,q)-Phillips operators generalizations and give some approximation properties. Reader can find more information about Phillips operators and some of generalizations in [10–14]

We begin with the following notations and formulas as in [1, 12, 15–24]. Through the paper we assume $0 < q \leq p \leq 1$. For each nonnegative integer n, the (p,q)-integer $[n]_{p,q}$, (p,q)-factorial $[n]_{p,q}!$ and (p,q)-binomial are defined as follows:

$$[n]_{p,q} = p^{n-1} + p^{n-2}q + p^{n-3}q^2 + \cdots + pq^{n-2} + q^{n-1} = \begin{cases} \frac{p^n - q^n}{p-q}, \ p \neq q \neq 1; \\ np^{n-1}, \ p = q \neq 1; \\ [n]_q, \ p = 1; \\ n; \ p = q = 1 \end{cases}, \tag{1}$$

$$[n]_{p,q}! = \begin{cases} [n]_{p,q}[n-1]_{p,q}...1, \ n \geq 1 \\ 1, \ n = 0 \end{cases}$$

and

$$(a \oplus b)_{p,q}^{\infty} = (a+b)(pa+qb)\left(p^2 a + q^2 b\right)\cdots \tag{2}$$

The (p,q)-derivative $D_{p,q}f$ of a function f with respect to x is defined by [7]:

$$(D_{p,q}f)(x) = \frac{f(px) - f(qx)}{(p-q)x}, \text{ for } x \neq 0. \tag{3}$$

There are two (p,q)-analogues of the exponential functions which are defined by [21]:

$$e_{p,q}(x) = \begin{cases} \frac{1}{\left(1 \oplus \left(\frac{q}{p} - 1\right)x\right)_{p,q}^{\infty}} & , p \neq q \neq 1 \\ e_q(x) & , p = 1 \\ e^x & , p = q \end{cases} \tag{4}$$

and

$$E_{p,q}(x) = \begin{cases} \left(1 \oplus \left(1 - \frac{q}{p}\right)x\right)_{p,q}^{\infty} & , p \neq q \neq 1 \\ E_q(x) & , p = 1 \\ e^x & , p = q \end{cases}, \tag{5}$$

which satisfy

$$e_{p,q}(x)E_{p,q}(-x) = 1,$$

where $e_q(x) = \frac{1}{(1-(1-q)x)_p^{\infty}}$, $E_q(x) = (1 + (1-q)x)_q^{\infty}$.

We define (p,q)-Gamma functions as follows:

$$\Gamma_{p,q}(n) = \frac{q^{n(n-1)/2}}{p^n} \int_0^{\infty/A(1-q/p)} e_{p,q}(-t)t^{n-1} d_{p,q}t, A > 0. \tag{6}$$

Another definition of (p,q)- Gamma functions were given in [25].

2. Genuine Type of (p,q)-Phillips Operators

Suppose that f is a real valued continuous function given on the interval $[0,\infty)$. Taking into account the formulas given above, we define the genuine type (p,q)-Phillips operators as follows:

$$\begin{aligned} \mathcal{P}_{n,p,q}(f,x) = {} & [n]_{p,q} \sum_{k=1}^{\infty} s_{n,k}(p,q,x) \\ & \times \int_0^{\infty/A(1-q/p)} \frac{q^{k(k-1)}}{p^k} s_{n,k-1}(p,q,t) f\left(\frac{q}{p}t\right) d_{p,q}t \\ & + f(0)e_{p,q}(-[n]_{p,q}x). \end{aligned} \tag{7}$$

where $0 < q < p \leq 1$ and f guarantee that Eq. (6) is convergence and

$$s_{n,k}(p,q,x) = \frac{([n]_{p,q}x)^k}{[k]_{p,q}!} e_{p,q}(-[n]_{p,q}x). \tag{8}$$

Lemma 1: For every $m\in\mathbb{N}$, the following equality is provided

$$\int_0^{\infty/A(1-q/p)} s_{n,k-1}(p,q,t)t^m d_{p,q}t \tag{9}$$
$$=\frac{[k+m-1]_{p,q}!p^{k+m}}{[k-1]_{p,q}![n]_{p,q}^{m+1}q^{(k+m)(k+m-1)/2}}.$$

Proof: By using the substitution $u=[n]_{p,q}t$, we have $d_{p,q}u=[n]_{p,q}d_{p,q}t$ and taking into account definition of (p,q)-Gamma functions Eq. (6), we get the following result:

$$\int\limits_0^{\infty/A(1-q/p)} s_{n,k-1}(p,q,t)t^m d_{p,q}t$$
$$=\int\limits_0^{\infty/A(1-q/p)}\frac{u^{k-1}}{[k-1]_{p,q}!}e_{p,q}(-u)\left(\frac{u}{[n]_{p,q}}\right)^m\frac{d_{p,q}u}{[n]_{p,q}}$$
$$=\frac{[k+m-1]_{p,q}!p^{k+m}}{[k-1]_{p,q}![n]_{p,q}^{m+1}q^{(k+m)(k+m-1)/2}}.\qquad\square$$

To obtain some main results, we calculate Korovkin's monomial functions.

Lemma 2: For $e_k(t)=t^k, k=0,1,2,3,4$, we have the following equalities:

$$\begin{aligned}(i)\ \mathcal{P}_{n,p,q}(1,x)=&1,\\ (ii)\ \mathcal{P}_{n,p,q}(t,x)=&x,\\ (iii)\ \mathcal{P}_{n,p,q}(t^2,x)=&\frac{p^2}{q^2}x^2+\frac{[2]_{p,q}}{q[n]_{p,q}}x,\\ (iv)\ \mathcal{P}_{n,p,q}(t^3,x)=&\frac{p^6}{q^6}x^3+\frac{p^3q[2]_{p,q}+p^2[4]_{p,q}}{q^5[n]_{p,q}}x^2\\ &+\frac{[2]_{p,q}[3]_{p,q}}{q^3[n]_{p,q}^2}x,\end{aligned}\tag{10}$$

and

$$\begin{aligned}(v)\ \mathcal{P}_{n,p,q}(t^4,x)=&\frac{p^{12}}{q^{12}}x^4+\frac{p^8q^2[2]_{p,q}+p^7q[4]_{p,q}}{q^{11}[n]_{p,q}}x^3\\ &+\frac{p^6[6]_{p,q}}{q^{11}[n]_{p,q}}x^3+\frac{p^4q^2[2]_{p,q}[3]_{p,q}}{q^9[n]_{p,q}^2}x^2\\ &+\frac{p^3q[2]_{p,q}[5]_{p,q}+p^2[4]_{p,q}[5]_{p,q}}{q^9[n]_{p,q}^2}x^2\\ &+\frac{[2]_{p,q}[3]_{p,q}[4]_{p,q}}{q^6[n]_{p,q}^3}x.\end{aligned}$$

Proof: Using the Eq. (5) and applying it in Eq. (6) for $k=0$, we have:

$$\begin{aligned}&\mathcal{P}_{n,p,q}(1;x)\\ =&[n]_{p,q}\sum_{k=1}^{\infty}s_{n,k}(p,q,x)\frac{q^{k(k-1)}}{p^k}\frac{p^k}{[n]_{p,q}q^{k(k-1)/2}}\\ &+e_{p,q}(-[n]_{p,q}x)\\ =&\sum_{k=1}^{\infty}q^{k(k-1)/2}\frac{([n]_{p,q}x)^k}{[k]_{p,q}!}e_{p,q}(-[n]_{p,q}x)\\ &+e_{p,q}(-[n]_{p,q}x)\\ =&e_{p,q}(-[n]_{p,q}x)E_{p,q}([n]_{p,q}x)=1.\end{aligned}$$

So, (i) was proved. Applying the same computations, we get results for (ii) as below

$$\begin{aligned}&\mathcal{P}_{n,p,q}(t;x)\\ =&\frac{1}{[n]_{p,q}}\sum_{k=1}^{\infty}\frac{([n]_{p,q}x)^k}{[k-1]_{p,q}!}e_{p,q}(-[n]_{p,q}x)q^{(k^2-3k+2)/2}\\ =&\frac{1}{[n]_{p,q}}e_{p,q}(-[n]_{p,q}x)\sum_{k=0}^{\infty}q^{k(k-1)/2}\frac{([n]_{p,q}x)^{k+1}}{[k]_{p,q}!}\\ =&xe_{p,q}(-[n]_{p,q}x)E_{p,q}([n]_{p,q}x)\\ =&x.\end{aligned}$$

For (iii) using equalities

$$\begin{aligned}[k]_{p,q}=&q^{k-1}+p[k-1]_{p,q}\\ [k+1]_{p,q}=&q^k+p[k]_{p,q}\\ =&q^k+p\left(q^{k-1}+p[k-1]_{p,q}\right)\\ =&q^{k-1}(q+p)+p^2[k-1]_{p,q},\end{aligned}$$

we have the following result:

$$\begin{aligned}&\mathcal{P}_{n,p,q}(t^2,x)\\ =&\frac{1}{[n]_{p,q}^2}\sum_{k=1}^{\infty}\frac{([n]_{p,q}x)^k}{[k-1]_{p,q}!}e_{p,q}(-[n]_{p,q}x)q^{(k^2-5k+2)/2}[k+1]_{p,q}\\ =&\frac{1}{[n]_{p,q}^2}\sum_{k=1}^{\infty}\left\{\frac{([n]_{p,q}x)^k}{[k-1]_{p,q}!}e_{p,q}(-[n]_{p,q}x)q^{(k^2-5k+2)/2}\right.\\ &\times\left.\left(p^2[k-1]_{p,q}+q^{k-1}(p+q)\right)\right\}\\ =&\frac{p^2}{[n]_{p,q}^2}\sum_{k=2}^{\infty}\frac{([n]_{p,q}x)^k}{[k-2]_{p,q}!}e_{p,q}(-[n]_{p,q}x)q^{(k^2-5k+2)/2}\\ &+\frac{p+q}{[n]_{p,q}^2}\sum_{k=1}^{\infty}\frac{([n]_{p,q}x)^k}{[k-1]_{p,q}!}e_{p,q}(-[n]_{p,q}x)q^{(k^2-3k)/2}\\ =&\frac{p^2}{[n]_{p,q}^2}\sum_{k=0}^{\infty}\frac{([n]_{p,q}x)^{k+2}}{[k]_{p,q}!}e_{p,q}(-[n]_{p,q}x)q^{(k^2-k-4)/2}\\ &+\frac{[2]_{p,q}}{[n]_{p,q}^2}\sum_{k=0}^{\infty}\frac{([n]_{p,q}x)^{k+1}}{[k]_{p,q}!}e_{p,q}(-[n]_{p,q}x)q^{(k^2-k-2)/2}\\ =&\frac{p^2}{q^2}x^2+\frac{[2]_{p,q}}{q[n]_{p,q}}x.\end{aligned}$$

In (iv) using following equality

$$[k+1]_{p,q}[k+2]_{p,q} = p^6[k-1]_{p,q}[k-2]_{p,q} + q^{k-2}(p^3q[2]_{p,q}+p^2[4]_{p,q})[k-1]_{p,q} + q^{2k-2}[2]_{p,q}[3]_{p,q},$$

we get the following result

$$\begin{aligned}
&\mathcal{P}_{n,p,q}(t^3,x)\\
=&\frac{1}{[n]_{p,q}^3}\sum_{k=1}^{\infty}\frac{([n]_{p,q}x)^k}{[k-1]_{p,q}!}e_{p,q}(-[n]_{p,q}x)\\
&\times q^{(k^2-7k)/2}[k+1]_{p,q}[k+2]_{p,q}\\
=&\frac{p^6}{[n]_{p,q}^3}\sum_{k=3}^{\infty}\frac{([n]_{p,q}x)^k}{[k-3]_{p,q}!}e_{p,q}(-[n]_{p,q}x)q^{(k^2-7k)/2}\\
&+\frac{p^3q[2]_{p,q}+p^2[4]_{p,q}}{[n]_{p,q}^3}\sum_{k=2}^{\infty}\frac{([n]_{p,q}x)^k}{[k-2]_{p,q}!}\\
&\times e_{p,q}(-[n]_{p,q}x)q^{(k^2-5k-4)/2}\\
&+\frac{[2]_{p,q}[3]_{p,q}}{[n]_{p,q}^3}\sum_{k=1}^{\infty}\frac{([n]_{p,q}x)^k}{[k-1]_{p,q}!}\\
&\times e_{p,q}(-[n]_{p,q}x)q^{(k^2-3k-4)/2}\\
=&\frac{p^6}{q^6}x^3+\frac{p^3q[2]_{p,q}+p^2[4]_{p,q}}{q^5[n]_{p,q}}x^2+\frac{[2]_{p,q}[3]_{p,q}}{q^3[n]_{p,q}^2}x.
\end{aligned}$$

Considering the following equality

$$\begin{aligned}
&[k+1]_{p,q}[k+2]_{p,q}[k+3]_{p,q}\\
=&p^{12}[k-1]_{p,q}[k-2]_{p,q}[k-3]_{p,q}\\
&+q^{k-3}\left(p^6[6]_{p,q}+p^7q[4]_{p,q}\right)[k-1]_{p,q}[k-2]_{p,q}\\
&+q^{k-3}p^8q^2[2]_{p,q}[k-1]_{p,q}[k-2]_{p,q}\\
&+q^{2k-4}\left(p^4q^2[2]_{p,q}[3]_{p,q}+p^3q[2]_{p,q}[5]_{p,q}\right)[k-1]_{p,q}\\
&+q^{2k-4}p^2[4]_{p,q}[5]_{p,q}[k-1]_{p,q}\\
&+q^{3k-3}[2]_{p,q}[3]_{p,q}[4]_{p,q},
\end{aligned}$$

for (v) we have:

$$\begin{aligned}
&\mathcal{P}_{n,p,q}(t^4,x)\\
=&\frac{1}{[n]_{p,q}^4}\sum_{k=1}^{\infty}\frac{([n]_{p,q}x)^k}{[k-1]_{p,q}!}e_{p,q}(-[n]_{p,q}x)\\
&\times q^{(k^2-9k-4)/2}[k+1]_{p,q}[k+2]_{p,q}[k+3]_{p,q}\\
=&\frac{p^{12}}{[n]_{p,q}^4}\sum_{k=4}^{\infty}\frac{([n]_{p,q}x)^k}{[k-4]_{p,q}!}e_{p,q}(-[n]_{p,q}x)q^{(k^2-9k-4)/2}\\
&+\frac{p^8q^2[2]_{p,q}+p^7q[4]_{p,q}+p^6[6]_{p,q}}{[n]_{p,q}^4}\sum_{k=3}^{\infty}\frac{([n]_{p,q}x)^k}{[k-3]_{p,q}!}\\
&\times e_{p,q}(-[n]_{p,q}x)q^{(k^2-7k-10)/2}\\
&+\frac{p^4q^2[2]_{p,q}[3]_{p,q}+p^3q[2]_{p,q}[5]_{p,q}+p^2[4]_{p,q}[5]_{p,q}}{[n]_{p,q}^4}\\
&\times\sum_{k=2}^{\infty}\frac{([n]_{p,q}x)^k}{[k-2]_{p,q}!}e_{p,q}(-[n]_{p,q}x)q^{(k^2-5k-12)/2}\\
&+\frac{[2]_{p,q}[3]_{p,q}[4]_{p,q}}{[n]_{p,q}^4}\sum_{k=1}^{\infty}\frac{([n]_{p,q}x)^k}{[k-1]_{p,q}!}e_{p,q}(-[n]_{p,q}x)q^{(k^2-3k-10)/2}\\
=&\frac{p^{12}}{q^{12}}x^4+\frac{p^8q^2[2]_{p,q}+p^7q[4]_{p,q}+p^6[6]_{p,q}}{q^{11}[n]_{p,q}}x^3\\
&+\frac{p^4q^2[2]_{p,q}[3]_{p,q}+p^3q[2]_{p,q}[5]_{p,q}+p^2[4]_{p,q}[5]_{p,q}}{q^9[n]_{p,q}^2}x^2\\
&+\frac{[2]_{p,q}[3]_{p,q}[4]_{p,q}}{q^6[n]_{p,q}^3}x.
\end{aligned}$$

Proof is completed. □

Remark 1: There and further let (p_n) and (q_n) be two sequences with $0<q_n<p_n\le 1$, $p_n\to 1$, $q_n\to 1$, $p_n^n\to\alpha$ and $q_n^n\to\beta$ as $n\to\infty$, where $\alpha,\beta<1$.

Lemma 3: We have the following inequality:

$$\mathcal{P}_{n,p_n,q_n}\left((t-x)^2;x\right)\le\frac{1}{q_n^2}\left(p_n^2-q_n^2+\frac{q_n}{[n]_{p_n,q_n}}\right)\left(x^2+x\right). \tag{11}$$

Proof: By Lemma 2, we calculate second moment of function as follow:

$$\begin{aligned}
&\mathcal{P}_{n,p_n,q_n}\left((t-x)^2;x\right)\\
=&\mathcal{P}_{n,p_n,q_n}\left(t^2;x\right)-2x\mathcal{P}_{n,p_n,q_n}\left(t;x\right)+x^2\mathcal{P}_{n,p_n,q_n}\left(1;x\right)\\
=&\frac{p_n^2-q_n^2}{q_n^2}x^2+\frac{[2]_{p_n,q_n}}{q_n[n]_{p_n,q_n}}x\\
\le&\frac{1}{q_n^2}\left(p_n^2-q_n^2+\frac{2}{[n]_{p_n,q_n}}\right)\left(x^2+x\right).
\end{aligned}$$ □

3. Local Approximation of the Operator $\mathcal{P}_{n,p_n,q_n}$

Let $B[0,\infty)$ be a set of all bounded functions from $[0,\infty)$ to $\mathbb{R}$. $C_B[0,\infty)$ be a subspace of all continuous functions in $B[0,\infty)$. $C_B[0,\infty)$ is a normed space and let every $f\in C_B[0,\infty)$ be equipped with the norm $\|f\|_B=\sup|f(x)|, x\in[0,\infty)$. Let's give the determination of first modulus of continuity on finite interval $[0,a]$, $a>0$ as follows

$$\omega_{[0,a]}(f,\delta)=\sup_{0<h\le\delta, x\in[0,a]}|f(x+h)-f(x)|. \tag{12}$$

and the Peetre's K-functional is defined as

$$K_2(f,\delta)=\inf\left\{\|f-g\|_B+\delta\left\|g''\right\|_B : g\in W_\infty^2\right\},\ \delta>0$$

where $W_\infty^2=\{g\in C_B[0,\infty): g',g''\in C_B[0,\infty)\}$. By [19, p.177, Theorem 2.4], there exists a positive constant M such that

$$K_2(f,\delta)\le M\omega_2(f,\sqrt{\delta}), \tag{13}$$

where

$$\omega_2(f,\sqrt{\delta})=\sup_{0<h\le\sqrt{\delta}}\ \sup_{x\in[0,\infty)}|f(x+2h)-2f(x+h)-f(x)| \tag{14}$$

and called the second modulus of continuity. In addition to this, we have the weighted Korovkin type theorems were given by Gadzhiev [26]. Let $B_\rho[0,\infty)$ be a weighted space of all functions f, from $[0,\infty)$ to $\mathbb{R}$ with $|f(x)|\le\chi_f\rho(x)$, where $\rho(x)=1+x^2$ and χ_f is a constant depending only on f. Also $B_\rho[0,\infty)$ equipped with the norm $\|f\|_\rho=\sup\left\{\frac{|f(x)|}{\rho(x)} : 0\le x<\infty\right\}$ is a normed space. The subspace of all continuous functions in $B_\rho[0,\infty)$ and the subspace of all functions $f\in C_\rho[0,\infty)$ for which $\lim_{|x|\to\infty}\frac{|f(x)|}{\rho(x)}$ exists finitely are symbolized by $C_\rho[0,\infty)$ and $C_\rho^*[0,\infty)$, respectively.

Lemma 4: For every $x\in[0,\infty)$ and $f''\in C_B[0,\infty)$, we have the inequality

$$|\mathcal{P}_{n,p_n,q_n}(f,x)-f(x)|\le\delta_{n,p_n,q_n}(x)\left\|f''\right\|_B,$$

where $\delta_{n,p_n,q_n}(x):=\frac{1}{q_n^2}\left(p_n^2-q_n^2+\frac{2}{[n]_{p_n,q_n}}\right)(x^2+x)$.

Proof: Using Taylor's formula

$$f(t)=f(x)+(t-x)f'(x)+\int_x^t(t-u)f''(u)du,$$

and Lemma 2, we have the following equality

$$\mathcal{P}_{n,p_n,q_n}(f,x)-f(x)=\mathcal{P}_{n,p_n,q_n}\left(\int_x^t(t-u)f''(u)du;x\right).$$

Then, taking into account the inequality above and Lemma 3

$$\left|\int_x^t(t-u)f''(u)du\right|\le\|f''\|_B\frac{(t-x)^2}{2},$$

we have

$$\begin{aligned}&|\mathcal{P}_{n,p_n,q_n}(f,x)-f(x)|\\ \le&\frac{\|f''\|_B}{2}\mathcal{P}^*_{n,p_n,q_n}((t-x)^2,x)\\ \le&\delta_{n,p_n,q_n}(x)\|f''\|_B,\end{aligned}$$

where $\delta_{n,p_n,q_n}(x):=\frac{1}{q_n^2}\left(p_n^2-q_n^2+\frac{2}{[n]_{p_n,q_n}}\right)(x^2+x)$. □

Theorem 1: Let $f\in C_\rho^*[0,\infty)$, then we have the following equality

$$|\mathcal{P}_{n,p_n,q_n}(f,x)-f(x)|\le 2M\omega_2\left(f,\sqrt{\delta_{n,p_n,q_n}(x)}\right),$$

where

$M>0$ and $\delta_{n,p_n,q_n}(x)=\frac{1}{q_n^2}\left(p_n^2-q_n^2+\frac{2}{[n]_{p_n,q_n}}\right)(x^2+x)$.

Proof: From Lemma 4, for any $g\in W_\infty^2$, we get the inequality

$$\begin{aligned}&|\mathcal{P}_{n,p_n,q_n}(f,x)-f(x)|\\ \le&|\mathcal{P}_{n,p_n,q_n}(f-g,x)-(f-g)(x)+\mathcal{P}_{n,p_n,q_n}(g,x)-g(x)|.\end{aligned}$$

Using Lemma 4, we deduce

$$\begin{aligned}&|\mathcal{P}_{n,p_n,q_n}(f,x)-f(x)|\\ \le&2\|f-g\|_B+\frac{1}{2}\delta_{n,p_n,q_n}(x)\left\|g''\right\|_B.\end{aligned} \tag{15}$$

We reach the result by taking infimum over $g\in W_\infty^2$ on the right side of the above inequality and the inequality Eq. (14). □

Theorem 2: For every $f\in C_\rho^*[0,\infty)$, we have the following result

$$\lim_{n\to\infty}\|\mathcal{P}_{n,p_n,q_n}(f)-f\|_\rho=0.$$

Proof: From Lemma 3

$$\|\mathcal{P}_{n,p_n,q_n}(e_0)-e_0\|_\rho=0 \text{ and } \|\mathcal{P}_{n,p_n,q_n}(e_1)-e_1\|_\rho=0.$$

Due to Lemma 2, we get

$$\begin{aligned}\|\mathcal{P}_{n,p_n,q_n}(e_2)-e_2\|_\rho&=\sup_{x\ge0}\frac{|\mathcal{P}_{n,p_n,q_n}(t^2;x)-x^2|}{1+x^2}\\ &\le\left(\frac{p_n^2-q_n^2}{q_n^2}+\frac{[2]_{p_n,q_n}}{q_n[n]_{p_n,q_n}}\right)\sup_{x\ge0}\frac{x+x^2}{1+x^2}\\ &\le2\left(\frac{p_n^2-q_n^2}{q_n^2}+\frac{[2]_{p_n,q_n}}{q_n[n]_{p_n,q_n}}\right).\end{aligned}$$

Then we get $\lim_{n\to\infty}\left\|\mathcal{P}^*_{n,p_n,q_n}(e_2)-e_2\right\|_\rho=0$.

Thus, we obtain the proof of Theorem 2 via A. D. Gadzhiev's Theorem in [26]. □

Lemma 5: Let $f\in C_\rho^*[0,\infty)$, we have the following inequality

$$\begin{aligned}&\|\mathcal{P}_{n,p_n,q_n}(f;x)-f(x)\|_{C[0,b]}\\ \le&M\left\{(1+b)^2\eta_{p_n,q_n}(b)+\omega_{[0,b+1]}(f;\sqrt{\eta_{p_n,q_n}(b)})\right\},\end{aligned}$$

where $\eta_{p_n,q_n}(b)=\frac{1}{q_n^2}\left(p_n^2-q_n^2+\frac{2}{[n]_{p_n,q_n}}\right)(b^2+b)$.

Proof: Let $x \in [0,b]$ and $t > b+1$. Since $t-x>1$, we get

$$|f(t)-f(x)| \leq K_f(2+(t-x+x)^2+x^2) \leq 3K_f(1+b)^2(t-x)^2. \quad (16)$$

Let $x \in [0,b]$, $t < b+1$ and $\delta > 0$. Then, we give

$$|f(t)-f(x)| \leq \left(1+\frac{|t-x|}{\delta}\right)\omega_{[0,b+1]}(f,\delta). \quad (17)$$

According to Eq. (16) and Eq. (17), we have

$$|f(t)-f(x)| \leq 3K_f(1+b)^2(t-x)^2 + \left(1+\frac{|t-x|}{\delta}\right)\omega_{[0,b+1]}(f,\delta).$$

From Cauchy-Schwarz's inequality and Lemma 3, we obtain

$$\begin{aligned}
&|\mathcal{P}_{n,p_n,q_n}(f;x)-f(x)| \\
\leq & 3K_f(1+b)^2\mathcal{P}_{n,p_n,q_n}\left((t-x)^2,x\right) \\
&+\omega_{[0,b+1]}(f;\delta)\left[1+\frac{1}{\delta}\left(\mathcal{P}_{n,p_n,q_n}\left((t-x)^2,x\right)\right)^{1/2}\right] \\
\leq & 3K_f(1+b)^2\eta_{p_nq_n}(x) \\
&+\omega_{[0,b+1]}(f;\delta)\left[1+\frac{1}{\delta}\left(\eta_{p_n,q_n}(x)\right)^{1/2}\right],
\end{aligned}$$

where

$$\eta_{p_n,q_n}(x) = \frac{1}{q_n^2}\left(p_n^2-q_n^2+\frac{2}{[n]_{p_n,q_n}}\right)\left(x^2+x\right).$$

We reach the proof of Lemma 5, by choosing

$$\delta := \sqrt{\eta_{p_n,q_n}(b)} = \sqrt{\frac{1}{q_n^2}\left(p_n^2-q_n^2+\frac{2}{[n]_{p_n,q_n}}\right)(b^2+b)}$$

and $M = \max\{3K_f, 2\}$.

So, proof is completed. □

4. Voronovskaja Type Theorem

Lemma 6: From the linearity of $\mathcal{P}_{n,p_n,q_n}$ operator, we have

(i) $\lim_{n\to\infty}[n]_{p_n,q_n}\mathcal{P}_{n,p_n,q_n}((t-x)^2,x) = 2(\alpha-\beta)x^2+2x$

and

$$(ii)\ \lim_{n\to\infty}[n]_{p_n,q_n}^2\mathcal{P}_{n,p_n,q_n}((t-x)^4,x) = 12(\alpha-\beta)^2x^4+24(\alpha-\beta)x^3+12x^2.$$

Proof: (i) By Lemma 2, we can show that

$$\mathcal{P}_{n,p_n,q_n}((t-x)^2,x) = \frac{p_n^2-q_n^2}{q_n^2}x^2+\frac{[2]_{p_n,q_n}}{q_n\,[n]_{p_n,q_n}}x,$$

here

$$[n]_{p_n,q_n}\left(\frac{p_n^2-q_n^2}{q_n^2}x^2+\frac{[2]_{p_n,q_n}}{q_n\,[n]_{p_n,q_n}}x\right) = \frac{(p_n^n-q_n^n)(p_n+q_n)}{q_n^2}x^2+\frac{[2]_{p_n,q_n}}{q_n}x$$

So, we get:

$$\lim_n\left(\frac{(p_n^n-q_n^n)(p_n+q_n)}{q_n^2}\right) = 2(\alpha-\beta)$$

and

$$\lim_n\left(\frac{[2]_{p_n,q_n}}{q_n}\right) = 2.$$

For every $\varepsilon > 0$ we can define the following sets:

$$\begin{aligned}
D &:= \left\{n\in\mathbb{N} : [n]_{p_n,q_n}\mathcal{P}_{n,p_n,q_n}((t-x)^2,x) \geq \varepsilon\right\} \\
D_1 &:= \left\{n\in\mathbb{N} : \frac{(p_n^n-q_n^n)(p_n+q_n)}{q_n^2} \geq \frac{\varepsilon}{2}\right\} \\
D_2 &:= \left\{n\in\mathbb{N} : \frac{[2]_{p_n,q_n}}{q_n} \geq \frac{\varepsilon}{2}\right\}.
\end{aligned}$$

Thus $D \subseteq D_1\cup D_2$ and $\delta(D) \leq \delta(D_1)+\delta(D_2) = 0$. This implies

$$\lim_{n\to\infty}[n]_{p_n,q_n}\mathcal{P}_{n,p_n,q_n}((t-x)^2,x) = 2(\alpha-\beta)x^2+2x.$$

(ii) Similarly by Lemma 2, we obtain

$$\begin{aligned}
&\mathcal{P}_{n,p_n,q_n}((t-x)^4,x) \\
=&\left[\frac{p_n^{12}}{q_n^{12}}-\frac{4p_n^6}{q_n^6}+\frac{6p_n^2}{q_n^2}-3\right]x^4 \\
&+\frac{p_n^8q_n^2[2]_{p_n,q_n}+p_n^7q_n[4]_{p_n,q_n}+p_n^6[6]_{p_n,q_n}}{q_n^{11}[n]_{p_n,q_n}}x^3 \\
&-\frac{4p_n^3q_n[2]_{p_n,q_n}+4p_n^2[4]_{p_n,q_n}}{q_n^5[n]_{p_n,q_n}}x^3+\frac{6[2]_{p_n,q_n}}{q_n[n]_{p_n,q_n}}x^3 \\
&+\frac{p_n^4q_n^2[2]_{p_n,q_n}[3]_{p_n,q_n}+p_n^3q_n[2]_{p_n,q_n}[5]_{p_n,q_n}}{q_n^9[n]_{p_n,q_n}^2}x^2 \\
&+\frac{p_n^2[4]_{p_n,q_n}[5]_{p_n,q_n}}{q_n^9[n]_{p_n,q_n}^2}x^2 \\
&-\frac{4[2]_{p_n,q_n}[3]_{p_n,q_n}}{q_n^3[n]_{p_n,q_n}^2}x^2 \\
&+\frac{[2]_{p_n,q_n}[3]_{p_n,q_n}[4]_{p_n,q_n}}{q_n^6[n]_{p_n,q_n}^3}x.
\end{aligned}$$

Let replace the following expressions as follows

$$v_4(p_n,q_n) = \frac{p_n^{12}}{q_n^{12}} - \frac{4p_n^6}{q_n^6} + \frac{6p_n^2}{q_n^2} - 3,$$
$$v_3(p_n,q_n) = \frac{p_n^8 q_n^2 [2]_{p_n,q_n} + p_n^7 q_n [4]_{p_n,q_n} + p_n^6 [6]_{p_n,q_n}}{q_n^{11}[n]_{p_n,q_n}} - \frac{4p_n^3 q_n [2]_{p_n,q_n} + 4p_n^2 [4]_{p_n,q_n}}{q_n^5 [n]_{p_n,q_n}} + \frac{6[2]_{p_n,q_n}}{q_n [n]_{p_n,q_n}},$$
$$v_2(p_n,q_n) = \frac{p_n^4 q_n^2 [2]_{p_n,q_n}[3]_{p_n,q_n} + p_n^3 q_n [2]_{p_n,q_n}[5]_{p_n,q_n}}{q_n^9 [n]_{p_n,q_n}^2} + \frac{p_n^2 [4]_{p_n,q_n}[5]_{p_n,q_n}}{q_n^9 [n]_{p_n,q_n}^2} - \frac{4[2]_{p_n,q_n}[3]_{p_n,q_n}}{q_n^3 [n]_{p_n,q_n}^2},$$
$$v_1(p_n,q_n) = \frac{[2]_{p_n,q_n}[3]_{p_n,q_n}[4]_{p_n,q_n}}{q_n^6 [n]_{p_n,q_n}^3}.$$

So, we get the following equality:

$$\begin{aligned}
&\sup_{x\in[0,1]} [n]_{p_n,q_n}^2 \mathcal{P}_{n,p_n,q_n}((t-x)^4,x) \\
=&[n]_{p_n,q_n}^2 v_4(p_n,q_n) \sup_{x\in[0,1]} x^4 \\
&+[n]_{p_n,q_n}^2 v_3(p_n,q_n) \sup_{x\in[0,1]} x^3 \\
&+[n]_{p_n,q_n}^2 v_2(p_n,q_n) \sup_{x\in[0,1]} x^2 \\
&+[n]_{p_n,q_n}^2 v_1(p_n,q_n) \sup_{x\in[0,1]} x^1
\end{aligned}$$

From Remark 1, we have

$$\begin{aligned}
\lim_n [n]_{p_n,q_n}^2 v_4(p_n,q_n) &= 12(\alpha-\beta)^2, \\
\lim_n [n]_{p_n,q_n}^2 v_3(p_n,q_n) &= 24(\alpha-\beta), \\
\lim_n [n]_{p_n,q_n}^2 v_2(p_n,q_n) &= 12, \\
\lim_n [n]_{p_n,q_n}^2 v_1(p_n,q_n) &= 0.
\end{aligned}$$

For every $\varepsilon > 0$ we can define the following sets:

$$\begin{aligned}
N &:= \left\{n\in\mathbb{N} : [n]_{p_n,q_n}^2 \mathcal{P}_{n,p_n,q_n}((t-x)^4,x) \geq \varepsilon\right\} \\
N_1 &:= \left\{n\in\mathbb{N} : [n]_{p_n,q_n}^2 v_4 \geq \frac{\varepsilon}{4}\right\} \\
N_2 &:= \left\{n\in\mathbb{N} : [n]_{p_n,q_n}^2 v_3 \geq \frac{\varepsilon}{4}\right\} \\
N_3 &:= \left\{n\in\mathbb{N} : [n]_{p_n,q_n}^2 v_2 \geq \frac{\varepsilon}{4}\right\} \\
N_4 &:= \left\{n\in\mathbb{N} : [n]_{p_n,q_n}^2 v_1 \geq \frac{\varepsilon}{4}\right\}
\end{aligned}$$

So, $N \subseteq N_1 \cup N_2 \cup N_3 \cup N_4$ and $\delta(N) \leq \delta(N_1) + \delta(N_2) + \delta(N_3) + \delta(N_4) = 0$.

Therefore

$$\begin{aligned}
&\lim_{n\to\infty} \left\{[n]_{p_n,q_n}^2 \mathcal{P}_{n,p_n,q_n}((t-x)^4,x)\right\} \\
=&12(\alpha-\beta)^2 x^4 + 24(\alpha-\beta)x^3 + 12x^2.
\end{aligned}$$

□

Theorem 3: If $f \in W_\infty^2$ then, we have

$$\lim_{n\to\infty} [n]_{p,q} (\mathcal{P}_{n,p_n,q_n}(f,x) - f(x)) = xf''(x).$$

Proof: By Taylor's expansion of f, we have

$$f(t) = f(x) + f'(x)(t-x) + \frac{1}{2}f''(x)(t-x)^2 + \varepsilon(t,x)(t-x)^2,$$

where $\varepsilon(t,x) \to 0$ as $t \to x$. From linearity of the operators $\mathcal{P}_{n,p_n,q_n}$, using Lemma 6 and making necessary process, we obtain

$$\begin{aligned}
&\lim_{n\to\infty} [n]_{p_n,q_n} (\mathcal{P}_{n,p_n,q_n}(f,x) - f(x)) \\
=&f'(x) \lim_{n\to\infty} [n]_{p_n,q_n} \mathcal{P}_{n,p_n,q_n}(t-x,x) \\
&+\frac{1}{2}f''(x) \lim_{n\to\infty} [n]_{p_n,q_n} \mathcal{P}_{n,p_n,q_n}((t-x)^2,x) \\
&+\lim_{n\to\infty} [n]_{p_n,q_n} \mathcal{P}_{n,p_n,q_n}(\varepsilon(t,x)(t-x)^2,x).
\end{aligned}$$

Applying Cauchy-Schwarz inequality to last term, we have

$$\begin{aligned}
&\lim_{n\to\infty} [n]_{p_n,q_n} \mathcal{P}_{n,p_n,q_n}(\varepsilon(t,x)(t-x)^2,x) \\
\leq&\sqrt{\lim_{n\to\infty} \mathcal{P}_{n,p_n,q_n}(\varepsilon^2(t,x),x)} \\
&\times\sqrt{\lim_{n\to\infty} [n]_{p_n,q_n}^2 \mathcal{P}_{n,p_n,q_n}((t-x)^4,x)}.
\end{aligned}$$

Since $\lim_{n\to\infty} \mathcal{P}_{n,p_n,q_n}(\varepsilon^2(t,x),x) = 0$ and from Lemma 6 $\lim_{n\to\infty} [n]_{p_n,q_n}^2 \mathcal{P}_{n,p_n,q_n}((t-x)^4,x)$ is finite. We yield

$$\lim_{n\to\infty} [n]_{p_n,q_n} \mathcal{P}_{n,p_n,q_n}(\varepsilon(t,x)(t-x)^2,x) = 0.$$

Thus, from Lemma 6 we get

$$\lim_{n\to\infty} [n]_{p_n,q_n} (\mathcal{P}_{n,p_n,q_n}(f,x) - f(x)) = xf''(x).$$

□

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