Journal of Computer Science & Computational Mathematics, Volume 8, Issue 1, March 2018
DOI: 10.20967/jcscm.2018.01.001

On the Dual Hyperbolic Numbers and the Complex Hyperbolic Numbers

Mutlu Akar*, Salim Yüce, Serdal Şahin

Yildiz Technical University, College of Arts and Sciences, Department of Mathematics,
Davutpasa Campus, 34210 Esenler, Istanbul, Turkey
*Corresponding author email: makar@yildiz.edu.tr

Abstract: In this study, we will introduce arithmetical operations on dual hyperbolic numbers $w = x + \varepsilon y + ju + \varepsilon jv$ and hyperbolic complex numbers $w = x + iy + ju + ijv$ which forms a commutative ring. Then, we will investigate dual hyperbolic number valued functions and hyperbolic complex number valued functions. One can see that these functions have similar properties.

Keywords: hypercomplex numbers, dual hyperbolic numbers, hyperbolic complex numbers, dual numbers, hyperbolic numbers, complex numbers.

1. Introduction

It is well known that the hypercomplex numbers systems [1], are extensions of real numbers. Complex numbers

$$\left(\mathbb{C} = \left\{ z = x + iy \mid x, y \in \mathbb{R}, i^2 = -1 \right\}\right),$$

hyperbolic (double, split-complex) numbers

$$\left(\mathbb{H} = \left\{ h = x + jy \mid x, y \in \mathbb{R}, j^2 = 1, j \neq \mp 1 \right\}\right)$$

[2, 3] and dual numbers

$$\left(\mathbb{D} = \left\{ d = x + \varepsilon y \mid x, y \in \mathbb{R}, \varepsilon^2 = 0, \varepsilon \neq 0 \right\}\right)$$

are commutative instances of hypercomplex numbers systems which are defined in two-dimensions [4]. Also, the quaternion which was introduced by Hamilton [5] is an instance of hypercomplex numbers systems and defined in four dimensions, but non-commutative [4]. Hyperbolic numbers with complex coefficients are introduced by J. Cockle in 1848, [6] and called 4-dimensional tessarines, [7]. Furthermore, Rochon and Shapiro presented, in a unified manner, a variety of algebraic properties of both bicomplex numbers and hyperbolic numbers, [8]. H. H. Cheng introduced dual numbers with complex coefficients and called complex dual numbers which are also commutative, [9]. Similarly, we introduced hyperbolic numbers with dual coefficients and called dual hyperbolic numbers. We can show that this number system is commutative. Also, in analogy to complex dual numbers, we called complex hyperbolic numbers which are the hyperbolic numbers with complex coefficients (4-dimensional tessarines). It is clear that there is no differences between complex hyperbolic numbers and hyperbolic complex numbers, dual hyperbolic numbers and hyperbolic dual numbers, complex dual numbers and dual complex numbers.

In this paper, we are going to focus on dual hyperbolic numbers, hyperbolic complex numbers and their functions.

2. Dual Hyperbolic Numbers

A *dual hyperbolic number* is defined as below:

$$w = (d_1, d_2) = (x + \varepsilon y) + j(u + \varepsilon v)$$

and the set of these numbers denoted by

$$\mathbb{D}_{\mathbb{H}} = \left\{ w = d_1 + jd_2 \mid d_1, d_2 \in \mathbb{D}, j^2 = 1 \right\}.$$

A *hyperbolic dual number* is defined as below:

$$w = (h_1, h_2) = (x + ju) + \varepsilon(y + jv)$$

and the set of these numbers denoted by

$$\mathbb{H}_{\mathbb{D}} = \left\{ w = h_1 + \varepsilon h_2 \mid h_1, h_2 \in \mathbb{H}, \varepsilon^2 = 0 \right\}.$$

A dual hyperbolic number or hyperbolic dual number can be written in terms of its base elements $\{1, \varepsilon, j, \varepsilon j\}$ as

$$w = x + \varepsilon y + ju + \varepsilon jv$$

where x is called the real part, y the dual part, u the hyperbolic part, and v the dual hyperbolic part. Both dual hyperbolic numbers and hyperbolic dual numbers are identical. Clearly,

$$\begin{aligned} w &= d_1 + jd_2 \text{ for every } w \in \mathbb{D}_{\mathbb{H}} \\ &= (x + \varepsilon y) + j(u + \varepsilon v) \\ &= x + \varepsilon y + ju + \varepsilon jv \\ &= (x + ju) + \varepsilon(y + jv) \\ &= h_1 + \varepsilon h_2, \text{ for every } w \in \mathbb{H}_{\mathbb{D}}. \end{aligned}$$

For consistency, we will call w a dual hyperbolic number in the remaining part of this paper. The base elements $\{1, \varepsilon, j, \varepsilon j\}$ of dual hyperbolic numbers satisfy the following properties:

$$1.\varepsilon = \varepsilon, 1.j = j, \ \varepsilon.\varepsilon = 0, \ j.j = 1$$

$$\varepsilon.j = j.\varepsilon, \ \varepsilon.(\varepsilon j) = 0, \ j.(\varepsilon j) = \varepsilon,$$

where ε designates the pure dual unit, j designates the hyperbolic unit, and εj designates the dual hyperbolic unit. If $x \neq 0$, $u \neq 0$, and $y = v = 0$, $x + \varepsilon u$ is called a *proper* *dual number*. If all values of x, y, u and v are not zero, $x+\varepsilon y+ju+\varepsilon jv$ is called a *proper dual hyperbolic number*.

2.1 Algebraic Properties and Operations

The arithmetic operations of dual hyperbolic numbers are defined for every $w_1 = d_{11} + jd_{12}, w_2 = d_{21} + jd_{22} \in \mathbb{D}_{\mathbb{H}}$ as following table:

Table 1. Dual Hyperbolic Operations

$w_1 + w_2$	$(d_{11} + d_{21}) + j(d_{12} + d_{22})$
$w_1 - w_2$	$(d_{11} - d_{21}) + j(d_{12} - d_{22})$
$w_1 * w_2$	$(d_{11}d_{21} + d_{12}d_{22}) + j(d_{11}d_{22} + d_{12}d_{21})$
w_1 / w_2	$w_1 * conj(w_2) / \Vert w_2\Vert^2$, $\Vert w_2\Vert^2 \neq 0$ $conj(w_2)$ is anyone of defined conjugations w_2 in next subsection

The dual hyperbolic numbers form a commutative ring, a real vector space and an algebra. But it is not field. Because every element of $\mathbb{D}_{\mathbb{H}}$ doesn't have an inverse. Also, one can be confused with quaternions. Even though the dual hyperbolic numbers are commutative, quaternions are non-commutative.

2.2 Basis and Dimension

$\{1, \varepsilon, j, \varepsilon j\}$ is a basis of $\mathbb{D}_{\mathbb{H}}$. Because $\{1, \varepsilon, j, \varepsilon j\}$ is linear independent and $\mathbb{D}_{\mathbb{H}} = \mathrm{Sp}\{1, \varepsilon, j, \varepsilon j\}$. Hence, $\dim \mathbb{D}_{\mathbb{H}} = 4$.

2.3 The Conjugations and Modulus in Dual Hyperbolic Numbers

We shall define three kinds of conjugations in dual hyperbolic numbers.

2.3.1 The first kind of conjugation

We define the first kind of conjugation of dual hyperbolic number $w = d_1 + jd_2 \in \mathbb{D}_{\mathbb{H}}$ as $\overline{w} = \overline{d_1} + j\overline{d_2}$, where $\overline{d_1}, \overline{d_2}$ are dual conjugations of d_1, d_2.

Properties. For all $w_1, w_2 \in \mathbb{D}_{\mathbb{H}}$:

1. $\overline{(w_1 \pm w_2)} = \overline{w_1} \pm \overline{w_2}$
2. $\overline{(\overline{w_1})} = w_1$
3. $\overline{(w_1.w_2)} = \overline{w_1}.\overline{w_2}$
4. $\overline{(w_1 / w_2)} = \overline{w_1} / \overline{w_2}$, for $\Vert w_2\Vert^2 \neq 0$.

2.3.2 The second kind of conjugation

We define the second kind of conjugation of dual hyperbolic number $w = d_1 + jd_2 \in \mathbb{D}_{\mathbb{H}}$ as $w^* = d_1 - jd_2$.

Properties. For $\forall w_1, w_2 \in \mathbb{D}_{\mathbb{H}}$,

1. $(w_1 \pm w_2)^* = w_1^* \pm w_2^*$
2. $(w_1^*)^* = w_1$
3. $(w_1.w_2)^* = w_1^*.w_2^*$
4. $(w_1 / w_2)^* = w_1^* / w_2^*$, for $\Vert w_2\Vert_*^2 \neq 0$.

2.3.3 The third kind of conjugation

We define the third kind of conjugation of complex hyperbolic number $w = d_1 + jd_2 \in \mathbb{D}_{\mathbb{H}}$ as

$$w^{\perp} = \left(\overline{d_1 + jd_2}\right)^* = \overline{\left(d_1 + jd_2\right)^*} = \overline{d_1} - j\overline{d_2}.$$

Properties. For $\forall w_1, w_2 \in \mathbb{D}_{\mathbb{H}}$,

1. $(w_1 \pm w_2)^{\perp} = w_1^{\perp} \pm w_2^{\perp}$
2. $(w_1^{\perp})^{\perp} = w_1$
3. $(w_1.w_2)^{\perp} = w_1^{\perp}.w_2^{\perp}$
4. $(w_1 / w_2)^{\perp} = w_1^{\perp} / w_2^{\perp}$, for $\Vert w_2\Vert_{\perp}^2 \neq 0$.

2.3.4 The Modulus

We define the different moduli with respect to three kinds of conjugation as follows:

$$\begin{aligned} \Vert w\Vert^2 &= w.\overline{w} \\ &= (d_1 + jd_2)(\overline{d_1} + j\overline{d_2}) \qquad (1) \\ &= \left(\Vert d_1\Vert^2 + \Vert d_2\Vert^2\right) + 2jRe\left(d_1\overline{d_2}\right), \end{aligned}$$

$$\Vert w\Vert_*^2 = w.w^* = (d_1 + jd_2)(d_1 - jd_2) = d_1^2 - d_2^2, \qquad (2)$$

$$\begin{aligned} \Vert w\Vert_{\perp}^2 &= w.w^{\perp} \\ &= (d_1 + jd_2)(\overline{d_1} - j\overline{d_2}) \qquad (3) \\ &= \left(\Vert d_1\Vert^2 - \Vert d_2\Vert^2\right) - 2\varepsilon jDu\left(d_1\overline{d_2}\right), \end{aligned}$$

where $Re\left(d_1\overline{d_2}\right)$ is real part of dual number $d_1\overline{d_2}$ and $Du\left(d_1\overline{d_2}\right)$ is dual part of dual number $d_1\overline{d_2}$.

It is clear that, if we define a linear transformation $f : \mathbb{D}_{\mathbb{H}} \to \mathbb{D}_{\mathbb{H}}$ and obtain the matrix A, which is the corresponding matrix to linear transformation f according to basis $\{1, j, \varepsilon, \varepsilon j\}$ as,

$$A = \begin{bmatrix} x & u & 0 & 0 \\ u & x & 0 & 0 \\ y & v & x & u \\ v & y & u & x \end{bmatrix} = \begin{bmatrix} x + ju & 0 - j0 \\ y + jv & x + ju \end{bmatrix},$$

where $w = d_1 + jd_2 \in \mathbb{D}_{\mathbb{H}}$ and $d_1 = x + \varepsilon y, d_2 = u + \varepsilon v \in \mathbb{D}$. Clearly we can rewrite Eq. (1) with aid of matrix A by the formula

$$\Vert w\Vert^2 = w.\overline{w} = detA. \qquad (4)$$

In a similar way, we can obtain the corresponding matrix to linear transformation f according to basis $\{1, \varepsilon, j, \varepsilon j\}$,

$$B=\begin{bmatrix} x & 0 & u & 0 \\ y & x & v & u \\ u & 0 & x & 0 \\ v & u & y & x \end{bmatrix}=\begin{bmatrix} x+\varepsilon y & u+\varepsilon v \\ u+\varepsilon v & x+\varepsilon y \end{bmatrix},$$

Eq. (2) can be obtained again in terms of matrix B as,

$$\|w\|_*^2 = w.w^* = detB. \tag{5}$$

2.4 Algebraic Properties and Operations

$$\begin{aligned} f &: \mathbb{D}_{\mathbb{H}} \to \mathbb{D}_{\mathbb{H}} \\ d_1 + jd_2 &\to f\left(d_1 + jd_2\right) \end{aligned} \tag{6}$$

is called dual hyperbolic numbers valued function. In analogy to Sobczyk, [2] this function can be written for dual hyperbolic numbers as

$$w = w_+e_1 + w_-e_2 \tag{7}$$

$$f(w) = f(w_+)e_1 + f(w_-)e_2 \tag{8}$$

where $w_+ = d_1 + d_2$, $w_- = d_1 - d_2$, $e_1 = \frac{(1+j)}{2}$, $e_2 = \frac{(1-j)}{2}$.

Thus, with aid of Eq. (8) we can get useful results for dual hyperbolic numbers valued functions in a similar way to Cheng and Thompson [9]. Let $w = d_1 + jd_2 \in \mathbb{D}_{\mathbb{H}}$ in this case we can write the following properties,

- $\|w\|^2 = \frac{1}{2}\left(\|d_1+d_2\|^2 + \|d_1-d_2\|^2\right) + \frac{1}{2}j\left(\|d_1+d_2\|^2 - \|d_1-d_2\|^2\right)$,
- $sqrt(w) = \frac{1}{2}\left(\sqrt{|d_1+d_2|} + \sqrt{|d_1-d_2|}\right) + \frac{1}{2}j\left(\sqrt{|d_1+d_2|} - \sqrt{|d_1-d_2|}\right)$,
  $|d_1+d_2| \geq 0, |d_1-d_2| \geq 0$,
- $exp(w) = \frac{1}{2}e^{d_1}\left[\left(e^{d_2}+e^{-d_2}\right) + j\left(e^{d_2}-e^{-d_2}\right)\right]$,
- $log(w) = \frac{1}{2}\left[log\left(d_1^{\,2}-d_2^{\,2}\right) + jlog\left(\frac{d_1+d_2}{d_1-d_2}\right)\right]$,
  $d_1^2 - d_2^2 > 0$, $d_1 - d_2 \neq 0$, $\frac{d_1+d_2}{d_1-d_2} > 0$,
- $\sin(w) = \frac{1}{2}\left(sin(d_1+d_2) + sin(d_1-d_2)\right) + \frac{1}{2}j\left(sin(d_1+d_2) - sin(d_1-d_2)\right)$,
- $cos(w) = \frac{1}{2}\left(cos(d_1+d_2) + cos(d_1-d_2)\right) + \frac{1}{2}j\left(cos(d_1+d_2) - cos(d_1-d_2)\right)$,
- $tan(w) = \frac{1}{2}\left(\tan(d_1+d_2) + tan(d_1-d_2)\right) + \frac{1}{2}j\left(tan(d_1+d_2) - tan(d_1-d_2)\right)$,
- $asin(w) = \frac{1}{2}\left(asin(d_1+d_2) + asin(d_1-d_2)\right) + \frac{1}{2}j\left(asin(d_1+d_2) - asin(d_1-d_2)\right)$,
- $acos(w) = \frac{1}{2}\left(acos(d_1+d_2) + acos(d_1-d_2)\right) + \frac{1}{2}j\left(acos(d_1+d_2) - acos(d_1-d_2)\right)$,
- $atan(w) = \frac{1}{2}\left(atan(d_1+d_2) + atan(d_1-d_2)\right) + \frac{1}{2}j\left(atan(d_1+d_2) - atan(d_1-d_2)\right)$,
- $sinh(w) = \frac{1}{2}\left(sinh(d_1+d_2) + sinh(d_1-d_2)\right) + \frac{1}{2}j\left(sinh(d_1+d_2) - sinh(d_1-d_2)\right)$,
- $cosh(w) = \frac{1}{2}\left(cosh(d_1+d_2) + cosh(d_1-d_2)\right) + \frac{1}{2}j\left(cosh(d_1+d_2) - cosh(d_1-d_2)\right)$,
- $tanh(w) = \frac{1}{2}\left(tanh(d_1+d_2) + tanh(d_1-d_2)\right) + \frac{1}{2}j\left(tanh(d_1+d_2) - tanh(d_1-d_2)\right)$,
- $asinh(w) = \frac{1}{2}\left(asinh(d_1+d_2) + asinh(d_1-d_2)\right) + \frac{1}{2}j\left(asinh(d_1+d_2) - asinh(d_1-d_2)\right)$,
- $acosh(w) = \frac{1}{2}\left(acosh(d_1+d_2) + acosh(d_1-d_2)\right) + \frac{1}{2}j\left(acosh(d_1+d_2) - acosh(d_1-d_2)\right)$,
- $atanh(w) = \frac{1}{2}\left(atanh(d_1+d_2) + atanh(d_1-d_2)\right) + \frac{1}{2}j\left(atanh(d_1+d_2) - atanh(d_1-d_2)\right)$.

3. Complex Hyperbolic Numbers

The complex hyperbolic number w is defined with aid of hyperbolic numbers

$$\left(\mathbb{H} = \left\{h = h_1 + jh_2 \mid h_1, h_2 \in \mathbb{R}, \quad j^2 = 1, \quad j \neq \pm 1\right\}\right),$$

which also called split-complex or double numbers [10-12], with complex coefficients as

$$w = z_1 + jz_2,\ j^2 = 1,\ j \neq \pm 1,\ z_1, z_2 \in \mathbb{C}$$
$$w = x + iy + ju + ijv \quad x, y, u, v \in \mathbb{R},$$

where

$$i^2 = -1,\ j^2 = 1,\ (ij)^2 = -1,\ j \neq \pm 1$$
$$ij = ji,\ i(ij) = (ij)i = -j,\ j(ij) = (ij)j = i.$$

Hence, the set of complex hyperbolic numbers can be obtained as

$$\mathbb{C}_{\mathbb{H}} = \left\{w = z_1 + jz_2 \mid z_1, z_2 \in \mathbb{C},\ j^2 = 1,\ j \neq \pm 1\right\}.$$

Similarly, we can define the hyperbolic complex numbers set via complex numbers with hyperbolic coefficients as

$$\mathbb{H}_{\mathbb{C}} = \left\{ d = h_1 + ih_2 \mid h_1, h_2 \in \mathbb{H}, i^2 = -1 \right\}$$

where, $\mathbb{H}$ is the set of the hyperbolic numbers. Clearly, since these number systems $\left(\mathbb{C}_{\mathbb{H}} \text{ and } \mathbb{H}_{\mathbb{C}} \right)$ are commutative, we can see there is no differences for $\mathbb{C}_{\mathbb{H}}$ and $\mathbb{H}_{\mathbb{C}}$.

$$\begin{aligned} w &= z_1 + jz_2 \text{ for every } w \in \mathbb{C}_{\mathbb{H}} \\ &= (x+iy) + j(u+iv) \\ &= x + iy + ju + ijv \\ &= (x+ju) + i(y+jv) \\ &= h_1 + ih_2, \text{for every } w \in \mathbb{H}_{\mathbb{C}}. \end{aligned}$$

3.1 Algebraic Properties and Operations

The complex hyperbolic number system is an Abel group under addition and the multiplication has left and right distribution with respect to addition,

$w_1(w_2 + w_3) = w_1w_2 + w_1w_3$, $(w_2 + w_3)w_1 = w_2w_1 + w_3w_1$, $w_1, w_2, w_3 \in \mathbb{C}_{\mathbb{H}}$.

Also, complex hyperbolic numbers system is commutative and associative under multiplication,

$w_1w_2 = w_2w_1$, $w_1(w_2w_3) = (w_1w_2)w_3$, $w_1, w_2, w_3 \in \mathbb{C}_{\mathbb{H}}$.

Thus, we showed that this numbers system $\mathbb{C}_{\mathbb{H}}$ is a commutative ring.

We can define the arithmetic operations as following table,

Table 2. Complex Hyperbolic Operations

$w_1 + w_2$	$(z_{11} + z_{21}) + j(z_{12} + z_{22})$
$w_1 - w_2$	$(z_{11} - z_{21}) + j(z_{12} - z_{22})$
$w_1 * w_2$	$(z_{11}z_{21} + z_{12}z_{22}) + j(z_{11}z_{22} + z_{12}z_{21})$
w_1 / w_2	$w_1 * conj(w_2) / \lVert w_2 \rVert^2$, $\lVert w_2 \rVert^2 \neq 0$ $conj(w_2)$ is anyone of defined conjugations w_2 in next subsection

where $w_1 = z_{11} + jz_{12}, w_2 = z_{21} + jz_{22} \in \mathbb{C}_{\mathbb{H}}$. The complex hyperbolic numbers form a commutative ring, a real vector space and an algebra. But it is not field. Because every element of $\mathbb{C}_{\mathbb{H}}$ doesn't have an inverse. Also, one can be confused with quaternions. Even though the complex hyperbolic numbers are commutative, quaternions are non-commutative.

3.2 Basis and Dimension

$\{1, i, j, ij\}$ is a basis of $\mathbb{C}_{\mathbb{H}}$. Because $\{1, i, j, ij\}$ is linear independent and $\mathbb{C}_{\mathbb{H}} = Sp\{1, i, j, ij\}$. Hence, $\dim \mathbb{C}_{\mathbb{H}} = 4$.

3.3 The Conjugations and Modulus in Complex Hyperbolic Numbers

The conjugation is very important for algebraic properties, geometric properties of $\mathbb{C}_{\mathbb{H}}$ and for complex hyperbolic valued functions. We defined three kind of conjugations on $\mathbb{C}_{\mathbb{H}}$. Furthermore, if we multiplied a complex number with its conjugate we obtain the square of the modulus (Euclidean metric in $\mathbb{R}^2$) for the complex number. In analogy to complex numbers we can define the modulus in $\mathbb{C}_{\mathbb{H}}$. Clearly, because of three different conjugate we can define three different modulus.

3.3.1 The first kind of conjugation

We define the first kind of conjugation of complex hyperbolic number $w = z_1 + jz_2 \in \mathbb{C}_{\mathbb{H}}$ as $\overline{w} = \overline{z_1} + j\overline{z_2}$, where $\overline{z_1}, \overline{z_2}$ are complex conjugations of z_1, z_2.

Properties. For all $w_1, w_2 \in \mathbb{C}_{\mathbb{H}}$:

1. $\overline{(w_1 \pm w_2)} = \overline{w_1} \pm \overline{w_2}$
2. $\overline{\left(\overline{w_1}\right)} = w_1$
3. $\overline{(w_1.w_2)} = \overline{w_1}.\overline{w_2}$
4. $\overline{(w_1 / w_2)} = \overline{w_1} / \overline{w_2}$, for $\lVert w_2 \rVert^2 \neq 0$

3.3.2 The second kind of conjugation

We define the second kind of conjugation of complex hyperbolic number $w = z_1 + jz_2 \in \mathbb{C}_{\mathbb{H}}$ as $w^* = z_1 - jz_2$.

Properties. For $\forall w_1, w_2 \in \mathbb{C}_{\mathbb{H}}$,

1. $(w_1 \pm w_2)^* = w_1^* \pm w_2^*$
2. $\left(w_1^*\right)^* = w_1$
3. $(w_1.w_2)^* = w_1^*.w_2^*$
4. $(w_1 / w_2)^* = w_1^* / w_2^*$, for $\lVert w_2 \rVert_*^2 \neq 0$

3.3.3 The third kind of conjugation

We define the third kind of conjugation of complex hyperbolic number $w = z_1 + jz_2 \in \mathbb{C}_{\mathbb{H}}$ as $w^\perp = z_1 - jz_2$.

Properties. For $\forall w_1, w_2 \in \mathbb{C}_{\mathbb{H}}$,

1. $(w_1 \pm w_2)^\perp = w_1^\perp \pm w_2^\perp$
2. $\left(w_1^\perp\right)^\perp = w_1$
3. $(w_1.w_2)^\perp = w_1^\perp.w_2^\perp$
4. $(w_1 / w_2)^\perp = w_1^\perp / w_2^\perp$, for $\lVert w_2 \rVert_\perp^2 \neq 0$

3.3.4 The Modulus

In analogy to complex numbers, for a complex hyperbolic number $w = z_1 + jz_2 \in \mathbb{C}_{\mathbb{H}}$ where, $z_1, z_2 \in \mathbb{C}$ we can define three kind of modulus as follows:

$$\begin{aligned}\|w\|^2 &= w.\overline{w} \\ &= (z_1 + jz_2)(\overline{z_1} + j\overline{z_2}) \qquad (9) \\ &= (\|z_1\|^2 + \|z_2\|^2) + 2jRe(z_1\overline{z_2}),\end{aligned}$$

$$\|w\|_*^2 = w.w^* = (z_1 + jz_2)(z_1 - jz_2) = z_1^2 - z_2^2, \qquad (10)$$

$$\begin{aligned}\|w\|_\perp^2 &= w.w^\perp \\ &= (z_1 + jz_2)(\overline{z_1} - j\overline{z_2}) \qquad (11) \\ &= (\|z_1\|^2 - \|z_2\|^2) + 2ijIm(z_1\overline{z_2}),\end{aligned}$$

where $Re(z_1\overline{z_2})$ is real part of complex number $z_1\overline{z_2}$ and $Im(z_1\overline{z_2})$ is imaginer part of complex number $z_1\overline{z_2}$.

On the other hand, if we define a linear transformation $f:\mathbb{C}_{\mathbb{H}} \to \mathbb{C}_{\mathbb{H}}$ and obtain the matrix A, which is the corresponding matrix to linear transformation f according to basis $\{1, j, i, ij\}$ as,

$$A = \begin{bmatrix} x & u & -y & -v \\ u & x & -v & -y \\ y & v & x & u \\ v & y & u & x \end{bmatrix} = \begin{bmatrix} x+ju & -(y+jv) \\ y+jv & x+ju \end{bmatrix},$$

where $w = z_1 + jz_2 \in \mathbb{C}_{\mathbb{H}}$ and $z_1 = x+iy, z_2 = u+iv \in \mathbb{C}$. Clearly we can rewrite Eq. (9) with aid of matrix A by the formula

$$\|w\|^2 = w.\overline{w} = detA. \qquad (12)$$

In a similar way, we can obtain the corresponding matrix to linear transformation f according to basis $\{1, i, j, ij\}$,

$$B = \begin{bmatrix} x & -y & u & -v \\ y & x & v & u \\ u & -v & x & -y \\ v & u & y & x \end{bmatrix} = \begin{bmatrix} x+iy & u+iv \\ u+iv & x+iy \end{bmatrix},$$

Eq. (10) can be obtained again in terms of matrix B as,

$$\|w\|_*^2 = w.w^* = detB \qquad (13)$$

3.4 Complex hyperbolic valued function and its Properties

We can define the complex hyperbolic valued function as a function whose values are complex hyperbolic numbers,

$$\begin{aligned} f &: \mathbb{C}_{\mathbb{H}} \to \mathbb{C}_{\mathbb{H}} \\ z_1 + jz_2 &\to f(z_1 + jz_2) \end{aligned} \qquad (14)$$

Similarly, in analogy to Sobczyk, [2] this function can be written for dual hyperbolic numbers as

$$w = w_+e_1 + w_-e_2 \qquad (15)$$

$$f(w) = f(w_+)e_1 + f(w_-)e_2, \qquad (16)$$

where $w_+ = z_1 + z_2, w_- = z_1 - z_2, e_1 = \frac{(1+j)}{2}, e_2 = \frac{(1-j)}{2}$. Let $w \in \mathbb{C}_{\mathbb{H}}$ in this case we can write the following properties for complex hyperbolic valued function as dual hyperbolic valued function

- $\|w\|^2 = \frac{1}{2}(\|z_1 + z_2\|^2 + \|z_1 - z_2\|^2) + \frac{1}{2}j(\|z_1 + z_2\|^2 - \|z_1 - z_2\|^2),$
- $sqrt(w) = \frac{1}{2}(\sqrt{|z_1 + z_2|} + \sqrt{|z_1 - z_2|}) + \frac{1}{2}j(\sqrt{|z_1 + z_2|} - \sqrt{|z_1 - z_2|}),$
  $|z_1 + z_2| \geq 0, |z_1 - z_2| \geq 0,$
- $exp(w) = \frac{1}{2}e^{z_1}[(e^{z_2} + e^{-z_2}) + j(e^{z_2} - e^{-z_2})],$
- $log(w) = \frac{1}{2}\left[log(z_1{}^2 - z_2{}^2) + jlog\left(\frac{z_1 + z_2}{z_1 - z_2}\right)\right],$
  $z_1^2 - z_2^2 > 0,\ z_1 - z_2 \neq 0,\ \frac{z_1 + z_2}{z_1 - z_2} > 0,$
- $\sin(w) = \frac{1}{2}(sin(z_1 + z_2) + sin(z_1 - z_2)) + \frac{1}{2}j(sin(z_1 + z_2) - sin(z_1 - z_2)),$
- $cos(w) = \frac{1}{2}(cos(z_1 + z_2) + cos(z_1 - z_2)) + \frac{1}{2}j(cos(z_1 + z_2) - cos(z_1 - z_2)),$
- $tan(w) = \frac{1}{2}(\tan(z_1 + z_2) + tan(z_1 - z_2)) + \frac{1}{2}j(tan(z_1 + z_2) - tan(z_1 - z_2)),$
- $asin(w) = \frac{1}{2}(asin(z_1 + z_2) + asin(z_1 - z_2)) + \frac{1}{2}j(asin(z_1 + z_2) - asin(z_1 - z_2)),$
- $acos(w) = \frac{1}{2}(acos(z_1 + z_2) + acos(z_1 - z_2)) + \frac{1}{2}j(acos(z_1 + z_2) - acos(z_1 - z_2)),$
- $atan(w) = \frac{1}{2}(atan(z_1 + z_2) + atan(z_1 - z_2)) + \frac{1}{2}j(atan(z_1 + z_2) - atan(z_1 - z_2)),$
- $sinh(w) = \frac{1}{2}(sinh(z_1 + z_2) + sinh(z_1 - z_2)) + \frac{1}{2}j(sinh(z_1 + z_2) - sinh(z_1 - z_2)),$
- $cosh(w) = \frac{1}{2}(cosh(z_1 + z_2) + cosh(z_1 - z_2)) + \frac{1}{2}j(cosh(z_1 + z_2) - cosh(z_1 - z_2)),$

- $tanh(w) = \frac{1}{2}\left(tanh(z_1+z_2)+tanh(z_1-z_2)\right)$
$$+\frac{1}{2}j\left(tanh(z_1+z_2)-tanh(z_1-z_2)\right),$$
- $asinh(w) = \frac{1}{2}\left(asinh(z_1+z_2)+asinh(z_1-z_2)\right)$
$$+\frac{1}{2}j\left(asinh(z_1+z_2)-asinh(z_1-z_2)\right),$$
- $acosh(w) = \frac{1}{2}\left(acosh(z_1+z_2)+acosh(z_1-z_2)\right)$
$$+\frac{1}{2}j\left(acosh(z_1+z_2)-acosh(z_1-z_2)\right),$$
- $atanh(w) = \frac{1}{2}\left(atanh(z_1+z_2)+atanh(z_1-z_2)\right)$
$$+\frac{1}{2}j\left(atanh(z_1+z_2)-atanh(z_1-z_2)\right).$$

References

[1] I. Kantor, A. Solodovnikov, *Hypercomplex Numbers*, Springer-Verlag, New York, 1989.

[2] G. Sobczyk, "The Hyperbolic Number Plane", *The College Mathematics Journal*, vol. 26, no. 4, pp. 268-280, 1995.

[3] P. Fjelstad, "Extending Special Relativity via the Perplex Numbers", *American Journal of Physics*, vol. 54, no. 5, pp. 416-422, 1986.

[4] P. Fjelstad, G. Gal Sorin, "n-dimensional Hyperbolic Complex Numbers", *Advances in Applied Clifford Algebras*, vol. 8, no. 1, pp. 47-68, 1998.

[5] W. Hamilton, *Elements of Quaternions*, Chelsea Publishing Company, New York, 1969.

[6] J. Cockle, "On a New Imaginary in Algebra", *Philosophical magazine*, London-Dublin-Edinburgh, vol. 3, no. 34, pp. 37-47, 1849.

[7] D. Alfsmann, "On Families of 2N-dimensional Hypercomplex Algebras Suitable for Digital Signal Processing", in Proc. *European Signal Processing Conf.* (EUSIPCO 2006), Florence, Italy, 2006.

[8] D. Rochon, M. Shapiro, "On Algebraic Properties of Bicomplex and Hyperbolic Numbers", *Analele Universitatii din Oradea. Fascicola Matematica*, vol. 11, 71-110, 2004.

[9] H. H. Cheng, S. Thompson, "Dual Polynomials and Complex Dual Numbers for Analysis of Spatial Mechanisms", *Proc. of ASME 24th Biennial Mechanisms Conference*, Irvine, CA, August pp. 19-22, 1996.

[10] S. Yüce, N. Kuruoğlu, "One-Parameter Plane Hyperbolic Motions", *Advances in Applied Clifford Algebras*, vol. 18, no. 2, pp. 279-285, 2008.

[11] I.M. Yaglom, *Complex Numbers in Geometry*, Academic Press, New York, 1968.

[12] I. M. Yaglom, *A Simple Non-Euclidean Geometry and Its Physical Basis*, Springer-Verlag, New York, 1979.