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Approximation Results on Semigroup of Linear Operator

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Abstract: In this paper, $\omega - OCP_n$ (semigroup of linear operator) results on approximation theorem were established by showing that the corresponding semigroup converges as the sequence of an infinitesimal generator converge which shows that the convergence is equivalent thereby considering the continuous dependence of a semigroup $\{T(t); t \geq 0\}$ on its infinitesimal generator $A \in \omega - OCP_n$ and the continuous dependence of the infinitesimal generator $A \in \omega - OCP_n$ on the semigroup $\{T(t); t \geq 0\}$.

Keywords: ω-OCP_n, C_0-semigroup, approximation theory, convergence sequence.

1. Introduction

Interest in convergence sequence in mathematical sciences is on the increase because of its importance in some branches of science such as computational fluid, dynamics, statistical mechanics, statistics, numerical analysis, and pure mathematics. Since convergence of series is basically a test which is used to determine whether a series will sum up to a particular value, so we can say a convergence sequence describes the idea of adding up countable infinite many values rigorously. Assume X is a Banach space, $X \supseteq X_n$ be a finite set, $(T(t))_{t\geq 0}$ the C_0-semigroup, $\omega - OCP_n$ be ω-order-preserving partial contraction mapping which is an example of C_0-semigroup, $\omega - OCP_n \subseteq OCP_n$ (Order Preserving Partial Contraction Mapping). Let $Mm(\mathbb{N} \cup 0)$ be a matrix, $L(X)$ the bounded linear operator in X, P_n, the partial transformation semigroup, $\rho(A)$ a resolvent of A, where A is the infinitesimal generator of a semigroup of linear operator. This paper consist of approximation results on ω-order-preserving partial contraction mapping which is consider as a semigroup of linear operator.

Akinyele *et al.* [1], introduced some dissipative properties of ω-order-preserving partial contraction mapping in semigroup of linear operator. Balakrishnan [2], obtained an operator calculus for infinitesimal generators of semigroup. Beals [3], proved some results on abstract Cauchy problems. Butzer and Berens [4], introduced semigroup of operators and approximation. Engel and Nagel [5], obtained one-parameter semigroup for linear evolution equations. Gairola *et al.* [6], obtained rate of approximation by finite iterates of q-Durrmeyer operators, also in [7], established the q-derivatives of a certain linear positive operators. Gandhi *et al.* [8], introduced local and global results for modified S$z\backslash'\{a\}$sz - Mirakjan operators. Mishra *et al.* [9], deduced some inverse result in simultaneous approximation by Baskakov-Durrmeyer-Stancu operators and in [10], derived the integral type modification of Jain operators and its approximation properties, numerical functional analysis and optimization. Oharu [11], established some semigroups of linear operator in Banach spaces. Pazy [12], introduced semigroup of linear operators and applications to partial differential equations. Rauf and Akinyele [13], obtained ω-order-preserving partial contraction mapping and established its properties, also in [14], Rauf *et al.* established some results of stability and spectra properties on semigroup of linear operator. Seidman [15], introduced some approximations of operator semigroups. Vrabie [16], deduced some results of C_0-semigroup and its applications. Yosida [17], established and proved some results on differentiability and representation of one-parameter semigroup of linear operators.

2. Preliminaries

Definition 1 ((C_0-Semigroup) [16]): A C_0-Semigroup is a strongly continuous one parameter semigroup of bounded linear operator on Banach space.

Definition 2 ((ω-OCP_n) [13]): A transformation $\alpha \in P_n$ is called ω-order-preserving partial contraction mapping if $\forall x, y \in$Dom$\alpha : x \leq y \implies \alpha x \leq \alpha y$ and at least one of its transformation must satisfy $\alpha y = y$ such that $T(t+s) = T(t)T(s)$ whenever $t, s > 0$ and otherwise for $T(0) = I$.

Definition 3 (Approximation theory): Approximation theory is classified with how functions can best be approximated or rounded up with simpler functions such that it quantitatively characterizing the errors introduced thereby.

Definition 4 (Convergence Sequence): A sequence α_n, with $n = 0, \ldots, \infty$, is convergent when there exists a number called α, which is a complex number, that satisfies that for every $1 \leq \epsilon \leq \infty$, there exists a positive integer $\mathbb{N}$ so that

$|\alpha_n - \alpha| \leq \epsilon$ when $n \geq \mathbb{N}$.

Example 1: For any 3×3 matrix $[M_m(\mathbb{C})]$, we have for each $\lambda > 0$ such that $\lambda \in \rho(A)$ where $\rho(A)$ is a resolvent set on X.

Suppose we have

$$A = \begin{pmatrix} 3 & 2 & 1 \\ 2 & 2 & 1 \\ 3 & 2 & 2 \end{pmatrix}$$

and let $T(t) = e^{tA_\lambda}$, then

$$e^{tA_\lambda} = \begin{pmatrix} e^{3t\lambda} & e^{2t\lambda} & e^{t\lambda} \\ e^{2t\lambda} & e^{2t\lambda} & e^{t\lambda} \\ e^{3t\lambda} & e^{2t\lambda} & e^{2t\lambda} \end{pmatrix}.$$

Example 2: By the translation semigroup starting from $Af = f'$ on $C_0(\mathbb{R}_+)$ or $L^p(\mathbb{R}_+)$, $1 \le p < \infty$, the operator

$$A^2 f = f''$$

generates a bounded linear semigroup. Let us consider the slightly more involved case of several space dimensions, that is we consider the spaces $C_0(\mathbb{R}_+)$ or $L^p(\mathbb{R}_+)$, $1 \le p < \infty$. Denote by $(\cup_i(t))_{t\in\mathbb{R}_+}$ the strongly continuous semigroup given by

$$(\cup_i(t)f)(x) = f(x_1, \ldots, x_{i-1}, x_i + t, \ldots, x_n),$$

where $x \in \mathbb{R}^n$, $t \in \mathbb{R}_+$ and $1 \le i \le n$, and let Ai be its generator where $A \in \omega - OCP_n$. Obviously, these semigroups commutes as do the resolvent of A and hence of A^2.

Theorem 1 ((Hille-Yoshida) [16]): A linear operator $A : D(A) \subseteq X \to X$ is the infinitesimal generator for a C_0-semigroup of contraction if and only if

i. A is densely defined and closed; and

ii. $(0, +\infty) \subseteq \rho(A)$ and for each $\lambda > 0$, we have

$$\|R(\lambda, A)\|_{L(X)} \le \frac{1}{\lambda}. \tag{1}$$

Theorem 2 ((Lumer-Phillips) [16]): Let $A : D(A) \subseteq X \to X$ be a densely defined operator. Then A generates a C_0-semigroup of contractions on X if and only if

i. A is dissipatives; and

ii. there exists $\lambda > 0$ such that $\lambda I - A$ is surjective.

Moreover, if A generates a C_0-semigroup of contractions, then $\lambda I - A$ is surjective for any $\lambda > 0$, and we have $Re(Ax, x^*) \le 0$ for each $x \in D(A)$ and each $x^* \in F(x)$.

3. Main Results

This section presents the result of approximations on ω-OCP_n in semigroup of linear operator:

Theorem 3: Assume A and B are the infinitesimal generators of a C_0-semigroups $\{T(t); t \ge 0\}$ and $\{S(t); t \ge 0\}$ respectively where on A, $B \in \omega$-OCP_n. For every $x \in X$ and $\lambda \in \rho(A) \cap \rho(B)$, so that

$$R(\lambda; B)[T(t) - S(t)]R(\lambda; A)x$$
$$= \int_0^t S(t-s)[R(\lambda; A) - R(\lambda; B)]T(s)x ds \tag{2}$$

Proof: For every $x \in X$, $\lambda \in \rho(A) \cap \rho(B)$, and A, $B \in \omega$-OCP_n, then, the function $s \to S(t-s)R(\lambda; B)T(s)R(\lambda; A)x$ is differentiable. And a simple computation yields

$$\frac{d}{ds}[S(t-s)R(\lambda; B)T(s)R(\lambda; A)x]$$
$$= S(t-s)[-BR(\lambda; B)T(s) + R(\lambda; B)T(s)A]R(\lambda; A)x$$
$$= S(t-s)[R(\lambda; A) - R(\lambda; B)]T(s)x, \tag{3}$$

where we have used the fact that $R(\lambda; A)T(s)x = T(s)R(\lambda; A)x$. If we integrate Eq. (3) from 0 to t, then we get Eq. (2), which completes the proof. □

Theorem 4: Let $A : D(A) \subseteq X \to X$, $A \in \omega$-OCP_n and let $\{T(t); t \ge 0\}$ and $\{T_n(t); t \ge 0\}$ be semigroups generated by A and A_n respectively satisfying $\|T(t)\| \le Me^{\omega t}$, then the following are equivalent:

(i) For every $x \in X$ and $\lambda \in \rho(A)$ with $Re\lambda > \omega$, then $R(\lambda; A_n) \to R(\lambda; A)x$ as $n \to \infty$ and $\omega > 0$.

(ii) For every $x \in X$ and $t \ge 0$, $T_n(t)x \to T(t)x$ as $n \to \infty$.

More so, the convergence in (ii) is uniformly bounded on $t - intervals$.

Proof: We need to show that (i) $\Longrightarrow$ (ii). Let $x \in X$, $A \in \omega$-OCP_n, and $0 \le t \le \tau$, let consider

$$\begin{aligned}\|(T_n(t) - T(t))R(\lambda; A)x\| \le &\|T_n(t)(R(\lambda; A) - R(\lambda; A_n))x\| \\ &+ \|R(\lambda; A_n)(T_n(t) - T(t))x\| \\ &+ \|(R(\lambda; A_n) - R(\lambda; A))T(t)x\| \\ = &D_1 + D_2 + D_3.\end{aligned} \tag{4}$$

Since $\|T_n(t)\| \le Me^{\omega t}$ for $t \in [0, \tau]$,then, it means that from (i) that $D_1 \to 0$ as $n \to \infty$ uniformly on $[0, \tau]$. Also, since $t \to T(t)x$ is continuous, the set $(T(t)x : 0 \le t \le \tau)$ is compact in X and therefore $D_3 \to 0$ as $n \to \infty$ uniformly on $[0, \tau]$. Finally using Theorem 3 with $B = A_n$ we have

$$\|R(\lambda; A)(T_n(t) - T(t))R(\lambda; A)x\|$$
$$\le \int_0^t \|T_n(t-s)\|\|R(\lambda; A) - R(\lambda; A)T(s)x\|ds$$
$$\le \int_0^\tau \|T_n(t-s)\|\|R(\lambda; A) - R(\lambda; A_n)T(s)x\|ds \tag{5}$$

The integrand on the right hand side of Eq. (5) is bounded by $2M^3 e^{2\omega\tau}(Re\lambda - \omega)^{-1}\|x\|$ and its move closer to zero as $n \to \infty$. And also by Lebesgue's bounded convergence theorem, the right hand side of Eq. (5) converges to zero and therefore

$$\lim_{n\to\infty} \|R(\lambda; A_n)(T_n(t) - T(t))R(\lambda; A)x\| = 0 \tag{6}$$

and the limit in Eq. (6) is uniform at $0 \le t \le \tau$. Since every $x \in D(A)$ can be written as $x = R(\lambda; A)z$ for some $z \in X$, it follows from Eq. (6) that for all $x \in D(A)$, $D_2 \to 0$ as $n \to \infty$ uniformly at $0 \le t \le \tau$. From Eq. (4), it then follows that for all $x \in D(A^2)$ we have

$$\lim_{n\to\infty} \|T_n(t) - T(t)x\| = 0 \tag{7}$$

and the limit in Eq. (7) is uniform on $[0, \tau]$. Since $\|T_n(t) - T(t)x\|$ are uniformly bounded on $[0, \tau]$ and since $D(A^2)$ is dense in X, it follows that Eq. (7) holds for every $x \in X$ uniformly on $[0, \tau]$, and (i) $\Longrightarrow$ (ii). To show that (ii) $\Longrightarrow$ (i). Assume that (ii) holds and $Re\lambda > \omega$, then

$$\|R(\lambda; A_n)x - R(\lambda; A)x\| \le \int_0^\infty e^{-Re\lambda t}\|(T_n(t) - T(t))x\|dt. \tag{8}$$

The right-hand side of Eq. (8) converges to zero as $n \to \infty$ by (ii) and by Lebesgue's dominated convergence theorem, we have that (ii) $\implies$ (i) and this complete the proof. □

Theorem 5: Let $A_n \in \omega\text{-}OCP_n$ satisfying $\|T(t)\| \leq Me^{\omega t}$ where $M \geq 1$ and $\omega > 0$, if there exists a $\lambda_0 \in \rho(A)$ with $Re\lambda_0 > \omega$ such that:

(i) for every $x \in X$, $R(\lambda_0; A_n) \to R(\lambda_0)x$ as $n \to \infty$; and

(ii) the range of $R(\lambda_0)$ is dense in X, then there exists a unique operator $A \in \omega\text{-}OCP_n$ such that $R(\lambda_0) = R(\lambda_0; A)$.

Proof: Suppose $\omega\text{-}OCP_n = G(M,\omega)$ which implies $A \in G(M,\omega)$. Let us assume without much ambiguity that $\omega = 0$ and start by proving that $R(\lambda; A_0)x$ converges to a point as $n \to \infty$ for every $\lambda \in \rho(A)$ with $Re\lambda > 0$. Indeed, let $S = \{\lambda; Re\lambda > 0, R(\lambda; A_n)x \ converges \ as \ n \to \infty\}$, and S is open. To see this, let us expand $R(\lambda; A_n)$ in a Taylor series around a point μ at which $R(\mu; A_n)x$ converges to a particular point as $n \to \infty$. Then

$$R(\lambda; A_n) = \sum_{n=0}^{\infty} (\mu - \lambda)^{\zeta} R(\mu; A_n)^{\zeta+1}. \tag{9}$$

Assume $\|R(\mu; A_n)^{\zeta}\| \leq M(Re\mu)^{-\zeta}$, then series Eq. (9) converges in the uniform operator topology for all $\lambda \in \rho(A)$ satisfying $|\mu - \lambda|(Re\mu)^{-1} < 1$. Then, the convergence is uniform in $\lambda \in \rho(A)$ satisfying $|\mu - \lambda|(Re\mu)^{-1} \leq \vartheta < 1$ and the series of constant $\sum_{\zeta=0}^{\infty} M\vartheta^{\zeta+1}$ is a majorant to the series $\sum_{\zeta=0}^{\infty} |\mu - \lambda|^{\zeta} \|R(\mu; A_n)^{\zeta+1}\|$. This implies the convergence of $R(\lambda; A_n)x$ as $n \to \infty$ for all $\lambda \in \rho(A)$ satisfying $|\mu - \lambda|(Re\mu)^{-1} \leq \vartheta < 1$, and the set S is open as claimed. Let λ be a cluster point of S with $Re\lambda > 0$. Given $0 < \vartheta < 1$, there exists a point $\mu \in S$ such that $|\mu - \lambda|(Re\mu)^{-1} \leq \vartheta < 1$ and therefore by the first part of the proof $R(\lambda; A_n)x$ converges as $n \to \infty$, that is $\lambda \in S$. Thus S is relatively closed in $Re\lambda > 0$. Since by assumption $\lambda_0 \in S$, then we conclude that $S = \{\lambda : Re\lambda > 0\}$, and that proves (i). To prove (ii), let assume for every $\lambda \in \rho(A)$ with $Re\lambda > 0$, we define a linear operator $R(\lambda)$ by

$$R(\lambda)x = \lim_{n\to\infty} R(\lambda; A)x. \tag{10}$$

Clearly

$$R(\lambda) - R(\mu) = (\mu - \lambda)R(\lambda)R(\mu) \tag{11}$$

for $Re\lambda > 0$ and $Re\mu > 0$ and therefore $R(\lambda)$ is a pseudo resolvent on $Re\lambda > 0$ since for a pseudo resolvent, the range of $R(\lambda)$ is independent of λ, then we have by (ii) that range of $R(\lambda)$ is dense in X. Also from (i), it is clear that

$$\|R(\lambda)^{\zeta}\| \leq M(Re\lambda)^{-\zeta} \tag{12}$$

for $Re\lambda > 0$, $\zeta = 1, 2, \ldots$. In particular for real λ, $\lambda > 0$, we have

$$\|\lambda R(\lambda)\| \leq M \tag{13}$$

for $\lambda > 0$. If follows that there exists a unique closed densely defined linear operator A for which $R(\lambda) = R(\lambda; A)$ for all $A \in \omega\text{-}OCP_n$. Finally, from (ii), its follows that $A \in G(M, 0)$ which also implies $A \in \omega\text{-}OCP_n$ and the proof is complete. □

Theorem 6: Let $A_n \in \omega\text{-}OCP_n$ such that $\omega\text{-}OCP_n = G(M,\omega)$ and assume :

(i) As $n \to \infty$, $A_n x \to Ax$ for every $x \in D$, where D is a dense subset of X.

(ii) There exists a λ_0 with $Re(\lambda_0) > \omega$ for which $(\lambda_0 I - A)D$ is dense in X, then the closure $\bar{A}$ of A is in $G(M.\omega)$. If $T_n(t)$ and $T(t)$ are the semigroups of linear operator generated by $A_n \in \omega\text{-}OCP_n$ and $A \in \omega\text{-}OCP_n$ respectively, then

$$\lim_{n\to\infty} T_n(t)x = T(t)x \tag{14}$$

for all $t \geq 0$, $x \in X$ and the limit in Eq. (14) is uniform in t for t is bounded interval.

Proof: Let $y \in D$, $x = (\lambda_0 I - A)y$ and $x_n = (\lambda_0 I - A_n)y$. Since $A_n y \to Ay$ as $n \to \infty$ where $A_n \in \omega\text{-}OCP_n$. Also since $\|R(\lambda_0; A_n)\| \leq M(Re\lambda_0 - \omega)^{-1}$, it follows that

$$\lim_{n\to\infty} R(\lambda_0; A_n)x = \lim_{n\to\infty} (R(\lambda_0; A_n)(x - x_n) + y) = y, \tag{15}$$

that is $R(\lambda_0; A)$ converges on the range of $\lambda_0 I - A$. But by (ii), this range is dense in X and by our assumptions, $\|R(\lambda_0; A_n\|$ are uniformly bounded. Therefore $R(\lambda_0; A_n)x$ converges for every $x \in X$ and $A_n \in \omega\text{-}OCP_n$. Let

$$\lim_{n\to\infty} R(\lambda_0; A_n)x = R(\lambda_0)x. \tag{16}$$

From Eq. (14) it follows that the range of $R(\lambda_0)$ contains D and it therefore dense in X. Theorem 5 implies the existence of an operator $A' \in G(M,\omega)$ satisfying $R(\lambda_0) = R(\lambda_0; A')$. To conclude the proof, we need to show that $\bar{A} = A'$. So, let $x \in D$, then

$$\lim_{n\to\infty} R(\lambda_0; A_n)(\lambda_0 I - A)x = R(\lambda_0; A')(\lambda_0 I - A)x. \tag{17}$$

On the other hand, as $n \to \infty$

$$\begin{aligned} R(\lambda_0; A_n)(\lambda_0 I - A)x =& R(\lambda_0; A_n)(\lambda_0 I - A_n)x \\ &+ R(\lambda_0; A_n)(A_n - A)x \\ =& x + R(\lambda_0; A_n)(A_n - A)x \to x, \end{aligned}$$

since $\|R(\lambda_0; A_n)\|$ are uniformly bounded and for $x \in D$, $A \in \omega\text{-}OCP_n$ and $A_n x \to Ax$. Therefore

$$R(\lambda_0; A')(\lambda_0 I - A)x = x \tag{18}$$

for all $x \in D$ and $A \in \omega\text{-}OCP_n$. But Eq. (18) by implication means $A'x = Ax$ for $x \in D$ and more so $A' \supset A$. Since A' is closed, then, we have that A is closable. Next, we need to prove that $\bar{A} \supset A'$. Let $y' = A'x'$. Since $(\lambda_0 I - A)D$ is dense in X, then there exists a sequence $x_n \in D$ in the sense that

$$\begin{aligned} y_n &= (\lambda_0 I - A')x_n = (\lambda_0 I - A)x_n \to \lambda_0 x' - y' \\ &= (\lambda_0 I - A')x' \; as \; n \to \infty. \end{aligned} \tag{19}$$

Therefore

$$x_n = R(\lambda_0; A')y_n \to R(\lambda_0; A')(\lambda_0 I - A')x' = x' \; as \; n \to \infty \tag{20}$$

and

$$Ax_n = \lambda_0 x_n - y_n \to y' \; as \; n \to \infty \tag{21}$$

from Eq. (20) and Eq. (21), it follows that $y' = \bar{A}x'$ and $\bar{A} \supset A'$. Thus $\bar{A} = A'$. Hence the proof is complete. □

Theorem 7: Let $F(\rho), \rho \geq 0$ be a family of bounded linear operators satisfying

$$\|F(\rho)^k\| \leq Me^{\omega\rho k}, \; k = 1, 2, \dots \tag{22}$$

for some constant $\omega > 0$ and $M \geq 1$. Let D be a dense subset of X and let

$$\lim_{\rho \to 0} \rho^{-1}(F(\rho)x - x) = Ax \tag{23}$$

for all $x \in D$. Assume we have some λ_0 with $Re\lambda_0 > \omega$, $(\lambda_0 I - A)D$ is dense in X then A is closable and $\bar{A}$, the closable of A, satisfies $\bar{A} \in G(M, \omega)$ such that $G(M, \omega) = \omega$-OCP_n. More so, if $\{T(t); t \geq 0\}$ is the semigroup of linear operator generated by $\bar{A}$, then for every sequence of positive integers $k_n \to \infty$ satisfying $k_n \rho_n \to t$, we have

$$\lim_{n \to \infty} F(\rho_n)^{k_n} x = T(t)x \tag{24}$$

for all $x \in X$. Choosing $\rho_n k_n = t$ for every $n \in \mathbb{N}$, then the limit in Eq. (24) is uniformly bounded on t-interval.

Proof: For $\rho > 0$, consider the bounded operators $A_\rho = \rho^{-1}(F(\rho) - 1)$. These operators are the infinitesimal generators of uniformly continuous semigroups $S_\rho(t)$ satisfying

$$\|S_\rho(t)\| \leq e^{-t/\rho} \sum_{k=0}^{\infty} (\frac{t}{\rho})^k \frac{\|F(\rho)^k\|}{k!} \leq M \; exp\, \{\frac{t}{\rho}(e^{\omega\rho} - 1)\}. \tag{25}$$

Let $\epsilon > 0$ be such that $Re\lambda_0 > \omega + \epsilon$ and let $\rho_0 > 0$ be such that for $0 < \rho \leq \rho_0$, $(e^{\omega\rho} - 1)\rho^{-1} < \omega + \epsilon$. Then

$$\|S_\rho(t)\| \leq Me^{(\omega+\epsilon)t} \tag{26}$$

for $0 \leq \rho \leq \rho_0$. Suppose $n \to \infty$, $A_n x \to Ax$ for every $x \in D$ and $A \in \omega$-OCP_n, where D is dense in X, then there exists a λ_0 with $Re\lambda_0 > \omega$ for which $(\lambda_0 I - A)D$ is dense in X, then closure $\bar{A}$ of A is in $G(M, \omega)$. And suppose $T_n(t)$ and $T(t)$ are the semigroups of linear operator generated by A_n and $\bar{A}$ respectively, then it follows that A is closable and that $\bar{A} \in G(M, \omega + \epsilon)$. If $T(t)$ is the semigroup generated by $\bar{A}$, then Theorem 6 implies further that

$$\|S_\rho(t)x - T(t)x\| \to 0 \; as \; \rho \to 0 \tag{27}$$

uniformly on bounded t-intervals. On the other hand, it follows that

$$\begin{aligned} &\|S_\rho(\rho_n k_n)x - F(\rho_n)^k x\| \\ \leq& M \exp\{\omega\rho_n(k_n - 1) + (e^{\omega\rho_n} - 1)k_n\} \\ &\cdot [k_n^2(e^{\omega\rho_n} - 1)^2 + k_n e^{\omega\rho_n}]^{1/2} \cdot \rho_n \|\frac{F(\rho_n)x - x}{\rho_n}\|. \end{aligned} \tag{28}$$

Choosing $x \in D$, $\rho \to 0$, $k_n \to \infty$ such that $\rho_n k_n \to t$, it is obvious that $\rho_n k_n$, $(e^{\omega\rho_n} - 1)k_n$ and $\rho_n^{-1}\|F(\rho_n)x - x\|$ stay bounded as $n \to \infty$. Therefore, we have

$$\|S_{\rho_n}(\rho_n k_n)x - F(\rho_n)^{k_n} x\| \leq C\rho_n^{1/2} \; as \; n \to \infty. \tag{29}$$

Suppose $\rho_n = t/k_n$, then one can choose the constant C independent of t for $0 \leq t \leq \tau$. Which implies, in this case, uniform convergence on bounded intervals in Eq. (28). For all $x \in D$, we have

$$\begin{aligned} \|T(t)x - F(\rho_n)^{k_n} x\| \leq& \|T(t)x - S\rho_n(t)x\| \\ &+ \|S\rho_n(t)x - S\rho_n(k_n\rho_n)x\| \\ &+ \|S\rho_n(k_n\rho_n)x - F(\rho_n)^{k_n} x\| \\ =& I_1 + I_2 + I_3. \end{aligned}$$

From Eq. (27) and Eq. (29), it follows that $I_1 \to 0$ and $I_3 \to 0$ as $n \to \infty$. To show that $I_2 \to 0$ as $n \to \infty$, we observe that for $x \in D$, $0 \leq 1 \leq \tau$, we have for large value of n such that

$$\begin{aligned} &\|S\rho_n(t)x - S\rho_n(k_n\rho_n)x\| \\ \leq& Me^{(\omega+\epsilon)\tau}|t - \rho_n k_n| \|\frac{F(\rho_n)x - x}{\rho_n}\| \to 0 \end{aligned} \tag{30}$$

as $n \to \infty$. If $\rho_n = 1/k_n$, then $I_2 = 0$. This concludes the proof of Eq. (24) for $x \in D$. Since D is dense in X and $\|T(t) - F(\rho_n)^{k_n}\|$ are uniformly bounded, then Eq. (24) holds for every $x \in X$. Finally, the semigroup $\{T(t); t \geq 0\}$ generated by $\bar{A}$ satisfies $\|T(t)\| \leq Me^{(\omega+\epsilon)t}$ for every small enough $\epsilon > 0$ and therefore its also satisfies $\|T(t)\| \leq Me^{\omega t}$ and $\bar{A} \in G(M, \omega)$. Hence the proof is complete. □

Theorem 8: Suppose $F(\rho)$, $\rho \geq 0$ be a family of bounded linear operators satisfying

$$\|F(\rho)^k\| \leq Me^{\omega\rho k} \; where \; k = 1, 2, \dots \tag{31}$$

for some constants $\omega > 0$ and $M \geq 1$, and let A be the infinitesimal generator of a semigroup of linear operator $\{T(t); t \geq 0\}$ where $A \in \omega$-OCP_n. If

$$\rho^{-1}(F(\rho)x - x) \to Ax \; as \; \rho \to 0 \tag{32}$$

for every $x \in D(A)$, then,

$$T(t)x = \lim_{n \to \infty} F(\frac{t}{n})^n x \tag{33}$$

for all $x \in X$ and the limit is uniform on bounded t-intervals.

Proof: Since A is the infinitesimal generator of a semigroup of linear operator, where $A \in \omega\text{-}OCP_n$, then it is close and for every real λ large enough, then the range $\lambda I - A$ is all of X. Therefore, our result readily follow from Theorem 7. As a simple consequence of Theorem 8, we can prove the exponential formula

$$T(t)x = \lim_{n\to\infty}(I - \frac{t}{n}A)^{-n}x \tag{34}$$

for all $x \in X$ where $T(t)$ is a semigroup of linear operator and $A \in \omega\text{-}OCP_n$ is its infinitesimal generator. To prove Eq. (33), assume that $A \in G(M,\omega)$ and set $F(\rho) = (1 - \rho A)^{-1} = (1/\rho)(1/\rho; A)$, for $0 < \rho < 1/\omega$. It follows that

$$\|F(\rho)^n\| \le M(1 - \rho\omega)^{-n} \le Me2\omega\rho^n \tag{35}$$

for ρ small enough. Also it follows that $x \in D(A)$ then

$$\frac{1}{\rho}(F(\rho) - 1)x = A(\frac{1}{\rho}R(\frac{1}{\rho}; A)x) \to Ax. \tag{36}$$

Therefore $F(\rho) = (1 - \rho A)^{-1}$ satisfies the condition of Theorem 8 and Eq. (34) is direct consequence of this Theorem, and this achieved the proof. □

4. Conclusions

In this paper, it have been established that $\omega\text{-}OCP_n$ possesses some approximation results as a semigroup of linear operator which shows that operator defined on this special class of semigroup of linear operator can converge to a particular point.

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