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On Variance and Volatility Swaps in Oil Markets

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Abstract: This paper focuses on the pricing of variance and volatility swaps in oil markets in some steps, which we considered similarly in the work of A. Swishchuk [1]. In this study a numerical example of oil price (August 1987-October 2016) has been given.

Keywords: oil market, stochastic mean-reverting volatility, volatility swaps.

1. Introduction

Variance swaps are fairly used in the energy markets and the economic systems (see [2-8]). There has been using Ornstein-Uhlenbeck (OU) process for commodity asset with stochastic volatility continuous-time GARCH model ([9,10] one-factor model) in the study. According to Schawartz's model [11] that introduced the classical stochastic process for the spot dynamics of commodity prices like the exponential of an Ornstein-Uhlenbeck process, the model has become the standard model for energy prices processing mean-reverting features.

In this paper, we considered a risky asset in the oil market with stochastic volatility following a mean-reverting stochastic process satisfying the following SDE (continuous-time GARCH(1,1) model) :

$$d\sigma^2(t) = a\left(L - \sigma^2(t)\right)dt + \gamma\sigma^2(t)dW_t,$$

Where,

- a: speed (strength)of mean reverting;
- L: mean reverting level (or equilibrium level);
- γ: volatility of volatility σ(t);
- W_t : standard Wiener process.

Using a change of time method we find an explicit solution of this equation and by using this solution we can find the variance and volatility swaps pricing formula under the physical measure.

2. Mean-Reverting Stochastic Volatility (MRSV) Model

By introducing MRSV model some properties can be used as follows.

Let $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathrm{P})$ be a probability space with a sample space Ω, σ-algebra of Borel sets $\mathcal{F}$ and probability P, such that $\mathcal{F}_T = \mathcal{F}$, t∈[0,T].

We consider a risky asset in energy market with stochastic volatility following a mean-reverting stochastic process the following SDE

$$d\sigma^2(t) = a\left(L - \sigma^2(t)\right)dt + \gamma\sigma^2(t)dW_t, \qquad (1)$$

where a≻0, L≻0, γ ≻0 .

2.1 Explicit Solution of MRSV model

$$\text{Let} \qquad V_t := e^{at}(\sigma^2(t) - L). \qquad (2)$$

Then, from (2) and (1) we obtain

$$\begin{aligned} dV_t &= ae^{at}(\sigma^2(t) - L)dt + e^{at}d\sigma^2(t) \\ &= \sigma(V_t + e^{at}L)dW_t. \qquad (3) \end{aligned}$$

If we use the change of time approach to (3) (see [12, 13]) we find the following solution of (3)

$V_t = \sigma^2(0) - L + \breve{W}(\emptyset_t^{-1}),$

$$\sigma^2(t) = e^{-at}[\sigma^2(0) - L + \breve{W}(\emptyset_t^{-1})] + L. \qquad (4)$$

Where $\breve{W}$(t) is an $\mathcal{F}_t$-measurable standard one-dimensional Wiener process, and $\emptyset_t^{-1}$ is an inverse function to $\emptyset_t$:

$$\emptyset_t = \gamma^{-2}\int_0^t (\sigma^2(0) - L + \breve{W}(s) + e^{a\emptyset_s}L)^{-2}ds. \qquad (5)$$

.

We note that

$$\emptyset_t^{-1} = \gamma^2 \int_0^t (\sigma^2(0) - L + \breve{W}(\emptyset_t^{-1}) + e^{as}L)^2 ds.) \qquad (6)$$

This follows from (5).

2.2 Some Properties of MRSV σ²(t)

In this subsection we give the first, two moments and variance for MRSV σ²(t). From (4) we obtain:

$$E\sigma^2(t) = e^{-at}[\sigma^2(0) - L] + L. \qquad (7)$$

Remark1.

We can see that Eσ²(t) ⟶ L when t⟶+∞ .

$$E\sigma^2(t) = (e^{-at}(\sigma^2(0) - L) + L)^2 + \gamma^2 e^{-2at}\left[(\sigma^2(0) - L)^2\frac{e^{\gamma^2 t} - 1}{\gamma^2} + \frac{2L(\sigma^2(0) - L)(e^{at} - e^{\gamma^2 t})}{a - \gamma^2} + \frac{L^2((e^{2at} - e^{\gamma^2 t}}{2a - \gamma^2}\right]$$

$$\mathrm{Var}(\sigma^2(t)) = E\sigma^2(t)^2 - (E\sigma^2(t))^2$$

$$= \gamma^2 e^{-2at}\left[(\sigma^2(0) - L)^2\frac{e^{\gamma^2 t} - 1}{\gamma^2} + \frac{2L(\sigma^2(0) - L)(e^{at} - e^{\gamma^2 t})}{a - \gamma^2} + \frac{L^2(e^{2at} - e^{\gamma^2 t})}{2a - \gamma^2}\right].$$

3 Variance Swap for MRSV Model

To obtain the variance swap for σ²(t) we use Eσ²(t) , relation (7) gives:

$$E\sigma^2(t) = e^{-at}[\sigma^2(0) - L] + L.$$

Then

$E\sigma_R^2$:=EV :=
$\frac{1}{T}\int_0^T$ Eσ²(t)dt =$\frac{(\sigma^2(0)-L)}{aT}(1-e^{-aT})+L.$ (8)

Also, V:= $\frac{1}{T}\int_0^T \sigma^2(t)dt$.

4. Volatility Swap for MRSV Model

To obtain the volatility swap for σ² (t) we use the formulas E$\sqrt{V}$ =E$\sqrt{\sigma_R}$, $\sigma_R := \sqrt{V} = \sqrt{\sigma_R^2}$ u, and Brockhaus-long approximation. We find

$$E\sqrt{V} \approx \sqrt{EV} - \frac{Var\,(V)}{8(EV)^{3/2}} \tag{9}$$

We have EV calculated in (8).We need

$$\text{Var (V)} = EV^2 - (EV)^2 \tag{10}$$

From (8) it follows that (EV)² has the from:

$$(EV)^2 = \frac{(\sigma^2(0)-L)}{aT}(1-e^{-aT})^2 + 2\frac{(\sigma^2(0)-L)}{aT}(1-e^{-aT})L + L^2 \tag{11}$$

Let us calculate EV²:

$$EV^2 = \frac{1}{T^2}\int_0^T\int_0^T E\sigma^2(t)\sigma^2(s)dtds$$
$$= \frac{1}{T^2}\int_0^T\int_0^T [e^{-a(t+s)}(\sigma^2(0)-L)^2 + e^{-a(t+s)}\{\gamma^2[(\sigma^2(0)-L)^2\frac{e^{\gamma^2(t\wedge s)}-1}{\gamma^2}$$
$$+\frac{2L(\sigma^2(0)-L)(e^{a(t\wedge s)}-e^{\gamma^2(t\wedge s)})}{a-\gamma^2}+\frac{L^2(e^{2a(t\wedge s)}-e^{\gamma^2(t\wedge s)})}{2a-\gamma^2}]\}+$$
$$e^{-at}(\sigma^2(0)-L)L + e^{-as}(\sigma^2(0)-L)L + L^2]dtds \tag{12}$$

$$\text{Var (V)} = EV^2-(EV)^2$$
$$= \frac{1}{T^2}\int_0^T\int_0^T e^{-a(t+s)}\{\gamma^2[(\sigma^2(0)-L)^2\frac{e^{\gamma^2(t\wedge s)}-1}{\gamma^2} + \frac{2L(\sigma^2(0)-L)(e^{a(t\wedge s)}-e^{\gamma^2(t\wedge s)})}{a-\gamma^2}+\frac{L^2(e^{2a(t\wedge s)}-e^{\gamma^2(t\wedge s)})}{2a-\gamma^2}]\}dtds$$
$$= \frac{\sigma^2(0)-L}{T^2}\int_0^T\int_0^T e^{-a(t+s)}\left(e^{\gamma^2(t\wedge s)}-1\right)dtds + \frac{2L\gamma^2(\sigma^2(0)-L)}{(a^2-\gamma^2)T^2}\int_0^T\int_0^T e^{-a(t+s)}\left(e^{a(t\wedge s)}-e^{\gamma^2(t\wedge s)}\right)dtds$$
$$+\frac{\gamma^2L^2}{(2a-\gamma^2)T^2}\int_0^T\int_0^T e^{-a(t+s)}\left(e^{2a(t\wedge s)}-e^{\gamma^2(t\wedge s)}\right)dtds. \tag{13}$$

5. Mean-Reverting Risk-Neutral Stochastic Volatility (MRRNSV) Model

We are going to find the values of variance and volatility swaps under risk-neutral measure P^*

a→a^*:=a+λσ, L→L^*:= $\frac{aL}{a+\lambda\sigma}$, where λ: market price of risk

5.1 Risk Neutral Stochastic Volatility (RNSV) Model

According to model (1)

$$d\sigma^2(t)=a(L-\sigma^2(t))dt+\gamma\,\sigma^2(t)\,dW_t. \tag{14}$$

We want to find a probability P^*equivalent to P, under which the process e^{-rt}σ²(t) is a martingale, where r is a interest rate constant positive. The hypothesis we made on the filtration $(\mathcal{F}_t)_{t\in[0,T]}$ allows us to express the density of the probability P^* with respect to P. This density will be noted by L_T.

It is known that, there is an adopted process $(q(t))_{t\in[0,T]}$such that,for all t∈[0,T],

$L_t = e^{\int_0^t q(s)dW_s} - \frac{1}{2}\int_0^t q^2(s)ds$ a.s.

In this case,

$$\frac{dp^*}{dp} = e^{\int_0^t q(s)dW_s} - \frac{1}{2}\int_0^t q^2(s)ds = L_T.$$

Using model (11), the process q(t) is equal to

$$q(t) = -\lambda\sigma^2(t), \tag{15}$$

where λ(market price of risk) is real numbers. Now, we have

$$L_T = e^{-\lambda\int_0^T \sigma^2(u)dW_u - \frac{1}{2}\lambda\int_0^T \sigma^4(u)du}.$$

Under probability p^* and using Girasanov theorem (see [12-14]), the process (W_t^*) defined by

$$W_t^* := W_t + \lambda\int_0^t \sigma^2(u)du \tag{16}$$

Therefore, in a risk-neutral world of model (14) we obtain:

$$d\sigma^2(t)=(aL-(a+\lambda\sigma)\sigma^2(t))dt+\gamma\,\sigma^2(t)d\,W_t^*.$$

Also, we can write,

$$d\sigma^2(t)=a^*(L^*-\sigma^2(t))\,dt+\gamma\,\sigma^2(t)d\,W_t^* \tag{17}$$

where

$$a^* := a+\lambda\gamma,\quad L^* := \frac{aL}{a+\lambda\gamma} \tag{18}$$

and $W_t^*$is defined in (16)

Finally, we can see that the model in (17) is the same one as the initial model (1), and we are going to apply the change of time method to this model (17) to obtain the values of variance and volatility swaps.

5.2 Variance and Volatility Swaps for Risk-Neutral SVM

Using the same arguments as in the previous section we get the following expressions for variance and volatility. For the variance swaps, we have (see (8) and (18)):

$$E^*\sigma_R^2 := EV := \frac{1}{T}\int_0^T E\sigma^2(t)dt = \frac{(\sigma^2(0)-L^*)}{a^*T}(1-e^{-aT})+L^*, \tag{19}$$

And for the volatility swap we find (see (8) and (18))

$$E^*\sqrt{V} \approx \sqrt{E^*V} - \frac{Var^*\,(V)}{8(E^*V)^{3/2}} \tag{20}$$

5.3 Numerical Example: Oil price (August 1987-October 2016)

In first step we shall calculate the value of variance and volatility swaps prices of a monthly Oil price. To apply previous formula for calculating these values we need the values of the parameters a, L, σ_0^2 and γ (T is monthly). We can obtain the parameters values from futures prices of the crude oil for the period August 1987 to October 2016 ([15]) which we repeat the same steps as Bos *et al.* [16]. Finally, the parameters values are: a= 1.324 γ= 0.431 L= 0.7688 λ= 0.051

According to formulas (variance swaps and volatility swaps) (8), (9) and parameters values, we find the values for risk adjusted parameters $a^*$and L*

$$a^* = a+\lambda\gamma = 1.3459,$$
$$L^* = \frac{aL}{a+\lambda\gamma} = 0.7562, \text{ and } \sigma^2(0)=2,25.$$

Remark 2

For variance swap and for volatility swap with risk adjusted parameters we can use formulas (19) and (20), respectively.

6. Conclusion

This work contains the alternative method namely, change of time method, to find variance and volatility swaps using GARCH (1,1) model .Moreover, we make an application example for Oil price during the period (August 1987 to October 2016), 351 observations given. The change time method shows flexibility for oil prices data. In future prospects this Work can generalize on electricity data and energy markets. Also, we can use the price of variance and volatility swaps under Heston model (1993) for Oil energy market.

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