How to Calculate Determinants

2X2

$$\begin{aligned} ax + by &= 0 \\ cx + dy &= 0 \end{aligned} \qquad \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc \tag{1}$$

3X3

$$|A| = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} \tag{2}$$

Expand in terms of minors of any row or column (pick the row or column with the most zeros). For example, choosing the first column:

$$|A| = a \begin{vmatrix} e & f \\ h & i \end{vmatrix} - d \begin{vmatrix} b & c \\ h & i \end{vmatrix} + g \begin{vmatrix} b & c \\ e & f \end{vmatrix} \tag{3}$$

where you find the appropriate determinant in the expansion by striking out the row and column of the chosen coefficient. These smaller determinants are called minors. For example:

$$|A| = a \begin{vmatrix} \cancel{a} & \cancel{b} & \cancel{c} \\ \cancel{d} & e & f \\ \cancel{g} & h & i \end{vmatrix} - d \begin{vmatrix} \cancel{a} & b & c \\ \cancel{d} & \cancel{e} & \cancel{f} \\ \cancel{g} & h & i \end{vmatrix} + g \begin{vmatrix} \cancel{a} & b & c \\ \cancel{d} & e & f \\ \cancel{g} & \cancel{h} & \cancel{i} \end{vmatrix} \tag{4}$$

Minors have an associated sign, alternating through the matrix:

$$\begin{vmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{vmatrix} \tag{5}$$

This is why the second term in equation 3 is negative.

We could also have expanded in terms of a row. For example, choosing the second row:

$$|A| = -d \begin{vmatrix} b & c \\ h & i \end{vmatrix} + e \begin{vmatrix} a & c \\ g & i \end{vmatrix} - f \begin{vmatrix} a & b \\ g & h \end{vmatrix} \tag{6}$$

For Larger Matrices: Do the determinant in steps. For example, a 4x4 is expanded in terms of 3x3 determinants and then the 3x3 determinants are expanded in terms of 2x2 determinants.