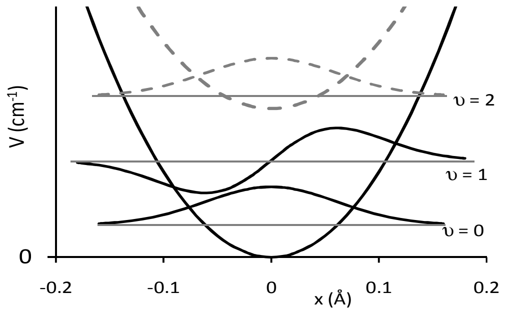
Hermite Polynomials: Harmonic Oscillator-Excited States

I. $-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\Psi + \frac{1}{2} k x^2 \Psi = E\,\Psi$

$E = h\nu_o(\upsilon + \frac{1}{2})$

$\Psi_\upsilon(x) = N_\upsilon H_\upsilon e^{-\frac{1}{2}\alpha^2 x^2}$

$\Psi_\upsilon(y) = N_\upsilon H_\upsilon e^{-y^2/2}$



$\frac{d^2}{dx^2}\Psi - \frac{2mk}{2\hbar^2}x^2\Psi = -\frac{2mE}{\hbar^2}\Psi$ multiplying I. by $-\frac{2m}{\hbar^2}$

$E = \frac{\hbar^2\alpha^2}{m}(\upsilon + \frac{1}{2})$ $\frac{2mE}{\hbar^2} = \alpha^2(2\upsilon + 1)$

$\alpha^4 = \frac{mk}{\hbar^2}$ $\frac{d^2}{dx^2}\Psi - \alpha^4 x^2\Psi = -\alpha^2(2\upsilon + 1)\Psi$

change variables:

$y = \alpha\, x$ $\frac{d}{dx} = \frac{d}{dy}\frac{dy}{dx} = \alpha\frac{d}{dy}$ $\frac{d^2}{dx^2} = \alpha^2\frac{d^2}{dy^2}$

$\alpha^2\frac{d^2}{dy^2}\Psi - \alpha^2 y^2\Psi = -\alpha^2(2\upsilon + 1)\Psi$ substituting for $\frac{d^2}{dx^2}$ and x^2

II. $\frac{d^2}{dy^2}\Psi - y^2\Psi + (2\upsilon + 1)\Psi = 0$ dividing both sides by α^2

$\frac{d}{dy}H_\upsilon e^{-y^2/2} = H_\upsilon(-y)e^{-y^2/2} + e^{-y^2/2}\frac{dH_\upsilon}{dy}$ taking the derivatives

$\frac{d^2}{dy^2} = H_\upsilon[y^2 e^{-y^2/2} - e^{-y^2/2}] + (-y)e^{-y^2/2}\frac{dH_\upsilon}{dy} + e^{-y^2/2}\frac{d^2H_\upsilon}{dy^2} + \frac{dH_\upsilon}{dy}(-y)e^{-y^2/2}$

III. $\frac{d^2}{dy^2} = e^{-y^2/2}\frac{d^2H_\upsilon}{dy^2} - 2y\,e^{-y^2/2}\frac{dH_\upsilon}{dy} + H_\upsilon(y^2 - 1)e^{-y^2/2}$ collecting terms

Substitute III. into II:

$\frac{d^2H_\upsilon}{dy^2} - 2y\frac{dH_\upsilon}{dy} + H_\upsilon(y^2 - 1) - y^2 H_\upsilon + (2\upsilon + 1)H_\upsilon = 0$

$\frac{d^2H_\upsilon}{dy^2} - 2y\frac{dH_\upsilon}{dy} + 2\upsilon H_\upsilon = 0$ Hermite Equation