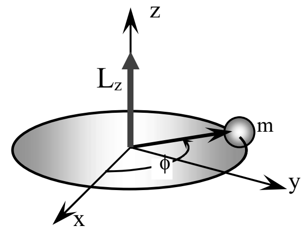
Rotation in a Plane – One Angular Dimension

Classical angular momentum: $L = I\omega = I\,\dfrac{d\phi}{dt}$



$$E_k = \frac{L^2}{2I}$$

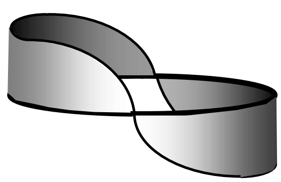
No potential energy: $$-\frac{\hbar^2}{2I}\left(\frac{\partial^2\Psi}{\partial\phi^2}\right) = E\Psi$$

$\Psi = a\,e^{im_\ell\phi}$ same form as a free particle

$\left(\dfrac{\partial\Psi}{\partial\phi}\right) = a\,im_\ell\,e^{im_\ell\phi} = im_\ell\Psi$ $\left(\dfrac{\partial^2\Psi}{\partial\phi^2}\right) = -\,a\,m_\ell^2\,e^{im_\ell\phi} = -\,m_\ell^2\,\Psi$

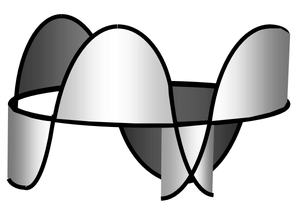
$$-\frac{\hbar^2}{2I}(-\,m_\ell^2\,\Psi) = E\,\Psi$$

$E = \dfrac{\hbar^2 m_\ell^2}{2I}$ $a^2\int_0^{2\pi} e^{-im_\ell\phi}\,e^{im_\ell\phi}\,d\phi = 1$ $\Psi = \left(\dfrac{1}{2\pi}\right)^{1/2} e^{im_\ell\phi}$



$e^{im_\ell\phi} = e^{im_\ell(\phi+2\pi)}$

$e^{im_\ell\phi} = e^{im_\ell\phi}\,e^{im_\ell 2\pi}$



$1 = e^{im_\ell 2\pi}$

$e^{im_\ell 2\pi} = \cos 2\pi m_\ell + i\sin 2\pi m_\ell = 1$

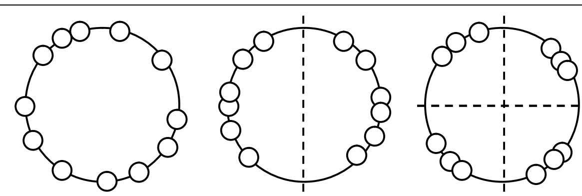
$m_\ell = 0, \pm1, \pm2, \pm3, \ldots$

$$\hat{L}_z = \frac{\hbar}{i}\frac{\partial}{\partial\phi}$$

$$\hat{L}_z\,\Psi = \frac{\hbar}{i}\frac{\partial\Psi}{\partial\phi} = \hbar m_\ell\Psi$$

↑

$L_z = \hbar m_\ell$



$m_\ell = 0$ $m_\ell = \pm1$ $m_\ell = \pm2$