Angular Momentum in 3-Dimensions

$x = r \sin\theta \cos\phi \qquad y = r \sin\theta \sin\phi \qquad z = r\cos\theta$

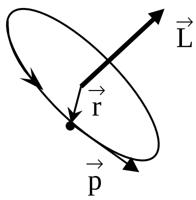
$\vec{r} = x\,\vec{i} + y\,\vec{j} + z\,\vec{k}$

$$\vec{L} = \vec{r} \times \vec{p} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ x & y & z \\ p_x & p_y & p_z \end{vmatrix} = (y\,p_z - z\,p_y)\,\vec{i} - (x\,p_z - z\,p_x)\,\vec{j} + (x\,p_y - y\,p_x)\,\vec{k}$$

$$\hat{L}_x = (\hat{y}\,\hat{p}_z - \hat{z}\,\hat{p}_y) = \frac{\hbar}{i}\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right)$$

$$\hat{L}_y = (\hat{z}\,\hat{p}_x - \hat{x}\,\hat{p}_z) = \frac{\hbar}{i}\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right)$$

$$\hat{L}_z = (\hat{x}\,\hat{p}_y - \hat{x}\,\hat{p}_x) = \frac{\hbar}{i}\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)$$



$$\hat{L}_x = \frac{\hbar}{i}\left(-\sin\phi\frac{\partial}{\partial\theta} - \cot\theta\cos\phi\frac{\partial}{\partial\phi}\right)$$

$$\hat{L}_y = \frac{\hbar}{i}\left(\cos\phi\frac{\partial}{\partial\theta} - \cot\theta\sin\phi\frac{\partial}{\partial\phi}\right)$$

$$\hat{L}_z = \frac{\hbar}{i}\left(\frac{\partial}{\partial\phi}\right)$$

$$\hat{L}^2 = \hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2 = -\hbar^2\left[\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2} + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)\right]$$

$$-\frac{\hbar^2}{2m}\nabla^2\Psi + \hat{V}(x,y,z) = E\,\Psi \qquad \text{with} \qquad \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

$$\nabla^2 = \frac{1}{r}\left(\frac{\partial^2}{\partial r^2}\right) r + \left(\frac{1}{r^2}\right)\Lambda^2$$

$\hat{L}^2$ = (angular momentum operator)$^2 = -\hbar^2\Lambda^2$

$$\Lambda^2 = \frac{1}{\sin^2\theta}\left(\frac{\partial^2}{\partial\phi^2}\right) + \left(\frac{1}{\sin\theta}\right)\left(\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta}\right)$$

Show that $\hat{L}_z = \hbar/i\ (\partial/\partial\phi)$:

Chain rule gives: $\left(\frac{\partial}{\partial\phi}\right) = \left(\frac{\partial x}{\partial\phi}\right)\left(\frac{\partial}{\partial x}\right) + \left(\frac{\partial y}{\partial\phi}\right)\left(\frac{\partial}{\partial y}\right) + \left(\frac{\partial z}{\partial\phi}\right)\left(\frac{\partial}{\partial z}\right)$

$$\left(\frac{\partial x}{\partial\phi}\right) = -r\sin\theta\sin\phi = -y \qquad \left(\frac{\partial y}{\partial\phi}\right) = r\sin\theta\cos\phi = x \qquad \left(\frac{\partial z}{\partial\phi}\right) = 0$$

$$\left(\frac{\partial}{\partial\phi}\right) = -y\left(\frac{\partial}{\partial x}\right) + x\left(\frac{\partial}{\partial y}\right) \qquad \text{and} \qquad \hat{L}_z = \frac{\hbar}{i}\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right) = \frac{\hbar}{i}\left(\frac{\partial}{\partial\phi}\right)$$