Heisenberg Uncertainty Principle: Gaussian Wave Function

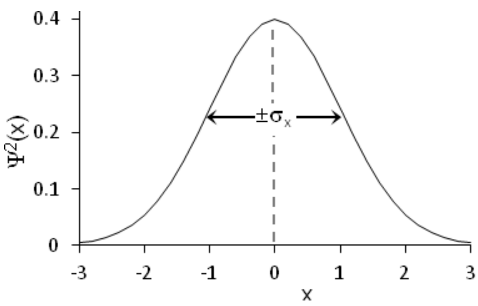
$\sigma_x \, \sigma_{px} \geq \hbar/2$ or $\delta x \, \delta p_x \geq \hbar/2$

$\sigma_x = \langle (x - \bar{x})^2 \rangle^{1/2} = (\langle x^2 \rangle - \langle x \rangle^2)^{1/2}$ $\sigma_{px} = \langle (p - \bar{p})^2 \rangle^{1/2} = (\langle p^2 \rangle - \langle p \rangle^2)^{1/2}$

$$\Psi(x) = N \, e^{-x^2/4\sigma_x^2}$$

$$N = \frac{1}{(2\pi)^{1/4} \, \sigma_x^{1/2}}$$

$\langle p \rangle = 0$ symmetrical distribution



$$\langle p^2 \rangle = \int_{-\infty}^{\infty} \Psi^* \hat{p}^2 \, \Psi \, dx$$

$$\frac{d\Psi}{dx} = N \left(\frac{-2x}{4\sigma_x^2}\right) e^{-x^2/4\sigma_x^2} = \left(\frac{-x}{2\sigma_x^2}\right) \Psi$$

(see *General Pattern* ℘5)

$$\hat{p} \, \Psi = \frac{\hbar}{i} \frac{d\Psi}{dx} = \left(\frac{-\hbar}{2i\sigma_x^2}\right) x\Psi$$

$$\hat{p}^2 \, \Psi = \hat{p}\hat{p}\Psi = \frac{\hbar}{i} \frac{d}{dx}\left(\frac{-\hbar}{2i\sigma_x^2}\right) x\Psi = \left(\frac{\hbar^2}{2\sigma_x^2}\right) \left(x \frac{d\Psi}{dx} + \Psi \frac{dx}{dx}\right)$$

$$= \left(\frac{\hbar^2}{2\sigma_x^2}\right)\left(\frac{-x^2}{2\sigma_x^2} + 1\right) \Psi = -\,\hbar^2\left(\frac{x^2}{4\sigma_x^2} - \frac{1}{2\sigma_x^2}\right) \Psi$$

$$\langle p^2 \rangle = -\hbar^2 \, N^2 \, 2 \int_0^{\infty}\left(\frac{x^2}{4\sigma_x^4} - \frac{1}{2\sigma_x^2}\right) e^{-x^2/2\sigma_x^2} \, dx$$

$$\int_0^{\infty} x^2 \, e^{-x^2/2\sigma_x^2} \, dx = \frac{2\sigma_x^2}{4} (2\pi)^{1/2} \sigma_x \qquad \int_0^{\infty} e^{-x^2/2\sigma_x^2} \, dx = \frac{1}{2} (2\pi)^{1/2} \sigma_x$$

$$\langle p^2 \rangle = -\,2 \, \hbar^2 \left(\frac{1}{(2\pi)^{1/2} \, \sigma_x}\right) \left(\frac{2\sigma_x^2}{16\sigma_x^4} (2\pi)^{1/2} \sigma_x - \frac{1}{4\sigma_x^2} (2\pi)^{1/2} \sigma_x\right) = \frac{\hbar^2}{4\sigma_x^2}$$

$$\sigma_{px} = (\langle p^2 \rangle - \langle p \rangle^2)^{1/2} = \hbar/2\sigma_x$$

$$\sigma_x \, \sigma_{px} = \sigma_x(\hbar/2\sigma_x) = \hbar/2$$

(minimum)

Harmonic oscillator: $x = r - r_o$