Particle-in-a-Box and Heisenberg Uncertainty

$\langle x\rangle = {}^{a}/_{2}$ $\qquad\qquad \langle p\rangle = 0$

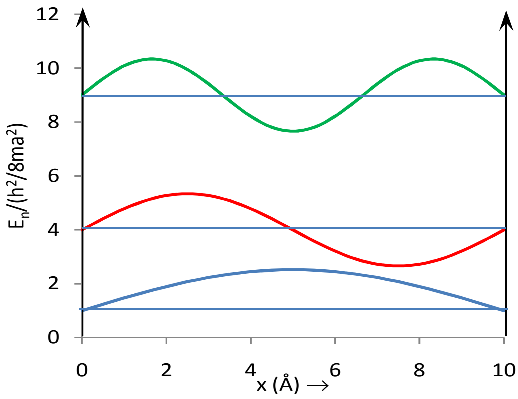
$\sigma_x\,\sigma_p \geq {}^{\hbar}/_{2}$

$\sigma_x = (\langle x^2\rangle - \langle x\rangle^2)^{1/2}$

$\sigma_p = (\langle p^2\rangle - \langle p\rangle^2)^{1/2}$

$\hat{p}^2 = \hat{p}\,\hat{p} = -\,\hbar^2\,(d^2/dx^2)$

$\Psi_1 = \left(\frac{2}{a}\right)^{1/2} \sin(\pi x/a)$



$$\langle x^2\rangle = \frac{\int_o^a \Psi_n^*\, x^2\, \Psi_n\, dx}{\int_o^a \Psi_n^*\, \Psi_n\, dx} = \int_o^a x^2\, \Psi_n^2\, dx = \left(\frac{2}{a}\right) \int_o^a x^2 \sin^2(\pi x/a)\, dx$$

Change in variables: $y = \pi x/a$, $dy/dx = \pi/a$, $dx = (a/\pi)\,dy$, $x = (a/\pi)\,y$:

$$\langle x^2\rangle = \left(\frac{2}{a}\right)\left(\frac{a}{\pi}\right)^3 \int_0^{\pi} y^2 \sin^2(y)\, dy$$

$$\int_0^{\pi} x^2 \sin^2(x)\, dx = \left[\frac{x^3}{6} - \left(\frac{x^2}{4} - \frac{1}{8}\right)\sin 2x - \frac{x\cos 2x}{4}\right]_0^{\pi} = \frac{\pi^3}{6} - \frac{\pi}{4}$$

$\sin 2x = \sin(2\pi) = 0$, $\sin 0 = 0$, $\cos 2x = \cos(2\pi) = 1$:

$$\langle x^2\rangle = \left(\frac{2}{a}\right)\left(\frac{a}{\pi}\right)^3\left(\frac{\pi^3}{6} - \frac{\pi}{4}\right) = a^2\left(\frac{1}{3} - \frac{1}{2\pi^2}\right)$$

$$\sigma_x^2 = \langle x^2\rangle - \langle x\rangle^2 = a^2\left(\frac{1}{3} - \frac{1}{2\pi^2}\right) - \frac{a^2}{4} = \frac{a^2}{12} - \frac{a^2}{2\pi^2} = \frac{a^2}{4\pi^2}\left(\frac{\pi^2}{3} - 2\right)$$

$$\sigma_x = \frac{a}{2\pi}\left(\frac{\pi^2}{3} - 2\right)^{1/2}$$

$$\langle p^2\rangle = \frac{\int_o^a \Psi_n^*\, \hat{p}^2\, \Psi_n\, dx}{\int_o^a \Psi_n^*\, \Psi_n\, dx} = -\,\hbar^2 \int_0^a \Psi_n \frac{d^2}{dx^2} \Psi_n\, dx = -\,\hbar^2\left(\frac{2}{a}\right) \int_0^a \sin(\pi x/a) \frac{d^2}{dx^2} \sin(\pi x/a)\, dx$$

$$\frac{d^2}{dx^2} \sin(\pi x/a) = \left(\frac{\pi}{a}\right)\frac{d}{dx}\cos(\pi x/a) = -\left(\frac{\pi}{a}\right)^2 \sin(\pi x/a)$$

$$\langle p^2\rangle = \hbar^2\left(\frac{\pi}{a}\right)^2\left(\frac{2}{a}\right)\int_0^a \sin^2(\pi x/a)\, dx$$

by normalization: $\left(\frac{2}{a}\right)\int_0^a \sin^2(\pi x/a)\, dx = 1$

$$\langle p^2\rangle = \hbar^2\left(\frac{\pi}{a}\right)^2$$

$$\sigma_p = (\langle p^2\rangle - \langle p\rangle^2)^{1/2} = \hbar\,({}^{\pi}/_{a})$$

$$\sigma_x\,\sigma_p = \frac{a}{2\pi}\left(\frac{\pi^2}{3} - 2\right)^{1/2} \hbar\left(\frac{\pi}{a}\right) = \frac{\hbar}{2}\left(\frac{\pi^2}{3} - 2\right)^{1/2} = 1.136\left(\frac{\hbar}{2}\right) \geq {}^{\hbar}/_{2}$$