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Hyperbolic Difference Equations: A Review of the Courant-Friedrichs-Lewy Paper in the Light of Recent Developments

Abstract: The portion of the Courant-Friedrichs-Lewy paper [*Math. Ann.* 100, 32 (1928)] that was devoted to hyperbolic difference equations is critically reviewed in terms of its basic contribution to the numerical solution of partial differential equations. Some subsequent developments are then discussed, including recent literature related to the von Neumann condition, some irreversible schemes, generalizations of the energy method, some new difference schemes, and mixed initial boundary value problems.

Introduction

At the time their paper was written, Courant, Friedrichs and Lewy[2] were interested in difference equations as a tool for proving the existence of solutions of partial differential equations; discretization was a way of constructing a sequence of finite dimensional problems whose solutions tended in the limit to the solution of the continuous problem.** For this reason the authors developed a method which was applicable to equations with variable coefficients; for the same reason they were not interested in studying all possible difference schemes but were satisfied when they found one that worked.

Today the methods of functional analysis—projection, mollification—are so well developed (thanks in large part to the pioneering work of Friedrichs) that descent to the finite dimensional case is not necessary for proving the existence of solutions of linear problems.† On the other hand, the onset of the Second World War brought an unprecedented (and unabated) pressure on the technological centers to provide numerical solutions of partial-differential equations; here the ideas introduced by C-F-L—the famous stability condition, energy inequalities, and the leap frog difference method—turned out to be basic. This is an outstanding instance of research undertaken for purely theoretical purposes§ turning out to be of immense practical importance.

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** Courant was inspired partly by Euler's use of discretization in the calculus of variations. Also, the authors were aware of the importance of difference methods for elliptic equations.

† Quite to the contrary, the difference method may be the only effective one for dealing with severely nonlinear problems, as indicated by the success of Glimm's recent work on shock waves.[6]

§ Curiously, in the late twenties the authors were interested through the work of Prandtl in hyperbolic problems of compressible gas dynamics. They considered solving these equations numerically by the method of characteristics but *not* by difference schemes.

The difference schemes of Courant-Friedrichs-Lewy

Here is a brief description of Part II of the paper, concerning hyperbolic equations: The authors start by pointing out that a scheme *cannot* be a convergent one if the ratio of the time and space mesh is so large that the domain of dependence of a point in the difference scheme does *not* contain all points of its domain of dependence in the differential equation; such a scheme ignores information which does influence the value of the solution of the differential equation. This fact is obvious to those who are used to thinking in these terms but was very much unsuspected by quite a large number of otherwise eminent scientists who, in their haste to use too large a value of Δt, fell into the trap of using difference schemes that could not converge and which in fact did diverge spectacularly.

The authors start with the centered difference scheme for the one-dimensional wave equation where $\Delta x/\Delta t$ equals the sound speed. The resulting difference equations are extremely simple and can be solved explicitly. Then the authors turn to the centered difference scheme for the wave equation on an arbitrary grid and remark that they will not bother to write down an explicit representation of the solution because "it is too complicated to yield a limiting value easily as the meshwidth tends to zero;" thereby they failed to discover the von Neumann stability criterion.*

The convergence of the difference scheme is proved by carrying over the energy method developed by Friedrichs and Lewy for the wave equation.[5] The crux of that method is a quadratic integral identity obtained as follows: multiply the wave equation by U_t, write the product as a di-

* The explicit representation was given by Lewy in a paper[16] published twenty years later.

vergence, integrate it over a lens-shaped domain, convert the volume integral into a surface integral and finally recognize that if the faces of the lens are space-like, the integrals over them are positive definite. To derive an analogous identity for their scheme, the authors multiply the difference equation by the centered approximation to U_t; they write the resulting quadratic function as a divergence sum over a tetrahedral region bounded by diagonal planes and observe that if the faces of the tetrahedron slope steeper then the characteristic cone, then the surface sums over them are positive. Although the authors do not emphasize it, the writing of the quadratic function as a divergence is much trickier for the difference scheme than for the differential operator; in the differential case it is merely integration by parts; once one has seen it he can almost certainly remember it or reconstruct it. In contrast, although I have seen the difference identity (8) [on page 63 of the original paper] a dozen times, whenever I prepare to lecture on it I waste half an hour in trying to rediscover it, then at last resort to looking it up.

In the last section of the paper the authors give an ingenious difference scheme for a certain class of second-order hyperbolic differential equations with a cross term and prove its convergence when the cross term is not too large. They observe that by a change of variables one can bring locally every hyperbolic equation into a form where the cross term is not too large. This is satisfactory if one is interested, as the authors are, in proving existence of solutions, but it may be decidedly unsatisfactory if one is interested in calculating solutions. Nevertheless, the method may be practical for equations to which it is applicable.

An interesting feature of this method (one not mentioned explicitly by the authors) is that the restriction that has to be placed on Δt to achieve convergence is more severe than merely requiring that the domain of dependence of the difference scheme contain the true domain of dependence.

Once the energy identities are derived, the authors prove convergence swiftly and elegantly. They observe that because of the positivity of the terms entering the energy identity, one obtains inequalities for the energy norm of the solution at time t in terms of the energy norm of the initial data. Analogous estimates are derived for the higher difference quotients of the solution by observing that these difference quotients satisfy an equation similar to the original one, obtained from it by differencing. With these bounds for the difference quotients, an estimate* for the maximum norm in terms of the energy norm of difference quotients is used and compactness is invoked to select a subsequence which, together with its difference quotients, converge to a limit. That limit solves the original differential equation and has the prescribed initial data; on the other hand, it follows from the energy identity for the differential equation that there can be no more than one such solution. This shows that any other convergent subsequence has the same limit and therefore the original sequence itself converges.

This indirect argument gives no error estimates; of course, using the energy inequality the authors could have estimated the difference between exact and approximate solution with the greatest of ease, had they bothered.

Some recent developments

We turn now to recent developments; these might be grouped coveniently into the following five subjects:

1) the von Neumann condition,
2) irreversible schemes,
3) generalizations of the energy method,
4) new difference schemes, and
5) mixed initial and boundary value problems.

The new developments will be illustrated rather than summarized. (For a full treatment we recommend the forthcoming second edition of Robert D. Richtmyer's book, *Initial Value Problems*).

For simplicity we shall discuss first-order systems and one-level difference operators associated with them rather than the second-order equations and second-level schemes discussed in C-F-L. In a one-level scheme the value of the approximate solution at time $t + \delta$ is obtained by applying an operator S_δ to its values at time t. S_δ is a difference operator of the form $S_\delta = \sum s_\alpha T^\alpha$, α a multi-index, T^α translation by $\delta\alpha$, and s_α a matrix-valued function. The operators S_δ act on vector-valued functions normed by the L^2 norm.

In C-F-L, the authors proved the convergence of their difference schemes by energy inequalities; in the operator language introduced here these express the uniform boundedness of powers of the operators S_δ, i.e., the existence of two constants a and b such that

$$||S_\delta^n|| \leq ae^{bt}, \quad \text{where} \quad t = n\delta, \tag{1}$$

the norm being the L^2 operator norm. A scheme satisfying (1) is called *stable*; the procedure of C-F-L demonstrates that *every stable scheme is convergent*. It turns out that something like the converse of this also is true: a scheme which converges for all square integrable data is stable in the sense of (1). Schemes which converge only for C^∞ data need be stable in a weaker sense only that these exists an integer N such that

$$||S_\delta^n|| \leq ae^{bt}c^N, \quad \text{where} \quad t = n\delta. \qquad (1)_N$$

*This is the discrete analog of the Sobelev inequality; the continuous version of that inequality, although not its sharpest form, was well known in Göttingen as an important tool in the theory of partial differential equations.

The *accuracy* of a difference scheme is measured by how closely solutions of the differential equation satisfy the difference equation. A scheme is accurate of order m if for every smooth solution u of the differential equation there is a constant k such that

$$||u(t+\delta) - S_\delta u(t)|| \leq k\delta^{m+1}. \tag{2}$$

The method of C-F-L shows that if a scheme is stable in the sense of (1), and accurate of order m in the sense of (2), then for smooth data the overall error is $O(\delta^m)$:

$$||u(t) - S_\delta^n u(0)|| \leq pe^{qt}\,\delta^m, \qquad t = n\delta. \tag{3}$$

Somewhat surprisingly, Strang proved in Ref. 20 that the *same* error estimate (3) holds also for weakly stable schemes which satisfy merely $(1)_N$, provided that the solution u is sufficiently differentiable.

Weak stability is very hard to verify for operators with variable coefficients since it is an unstable concept, in the following sense: If S_δ satisfy $(1)_N$, an operator $S'_\delta = S_\delta + M$ where the $||M|| \leq$ const. $\times\ \delta$, need not satisfy $(1)_N$. In contrast, the class of strongly stable operators satisfying (1) is stable in this sense.

In Ref. 23 Thomée makes a precise and surprisingly delicate study of the degree of stability of a class of operators in the *maximum norm.*

The symbol (amplification matrix) of the operator S_δ is

$$S(x, \xi) = \sum S_\alpha(x)e^{i\alpha\xi}.$$

For operators with coefficients independent of x it follows immediately from the isometric character of Fourier transformation that necessary and sufficient for stability is the uniform boundedness of all powers of the symbol. This reduces a stability question to a pure matrix problem; this matrix problem, however, turns out to be trickier than it seems at first glance. An obvious necessary condition, the famous one due to von Neumann, is for the spectrum of such a matrix to lie in the unit disc in the complex plane. An obvious sufficient condition is for the norm of this matrix to be ≤ 1. A non-obvious sufficient condition is that the numerical range of the matrix be in the unit disc.[1,3,4] Necessary and sufficient conditions were given by Kreiss[8] and Buchanan;[1] see also Morton[17] and Morton and Schechter.[18]

If the difference scheme is reversible, i.e., can be used to compute solutions both forward and backward in time, then both positive and negative powers of the symbol have to be bounded. In this case the von Neumann condition requires that the spectrum be located on the unit circle. One of the new developments has been the observation that although hyperbolic problems are reversible, irreversible difference schemes for them can be quite useful. Such a difference scheme was introduced by Friedrichs in Ref. 4. In some nonlinear cases the slight dissipation which makes the difference scheme irreversible is even indispensable.

It is not hard to show that the von Neumann condition is necessary for the stability of schemes with variable coefficients as well. Von Neumann conjectured that it is also sufficient; something like this has been demonstrated by a surprisingly elaborate extension of the energy method.

In the hands of Courant, Friedrichs, and Lewy, energy inequalities were derived from energy identities, obtained from difference analogs of formulas for the differentiation of products. This has been generalized and systematized by Lees[15] who has devised a large number of inequalities in this way. It was observed in Ref. 11 that all such energy inequalities can be obtained by the purely algebraic method of expressing certain non-negative symbols as sums of squares; according to Hilbert's theory of higher order positive forms, this is not always possible and so the classical energy method has inherent limitations. Nonetheless, it is possible to derive energy inequalities without energy identities as was done, for instance, in Ref. 13 with the aid of a certain amount of dissipation, and in Ref. 12 with the aid of nothing at all except sufficient differentiability of the coefficients. The theorem of Lax and Nirenberg reads: If the symbol is a sufficiently differentiable function of x, ξ and if $|S(x, \xi)| \leq 1$ for all x and ξ, then the operator S_δ satisfies the inequality.

$$||S_\delta|| \leq 1 + K\delta. \tag{4}$$

Obviously the expression of (4) implies stability in any finite time interval. The more delicate problem of proving stability when the symbol is not bounded by inequality (1) in norm but merely satisfies the von Neumann condition has been handled by Kreiss.[9] He found that if in addition to the von Neumann condition one requires dissipation in a definite sense then it is possible to introduce a new norm equivalent to the L^2 norm for which inequality (4) is true. The proof of the surprisingly delicate matrix theorems needed to do this have been simplified by Parlett.[19]

As already mentioned, earlier practical needs have called for a whole army of novel difference schemes. Among these one might single out the crude but useful method introduced by Friedrichs,[4] the more accurate methods devised by Du Fort-Frankel,[3] by Lax-Wendroff,[14] and the high-order schemes studied by Strang,[21] and many others. As more and more complicated schemes are being tried for more and more complicated systems of equations in more and more variables, the verification of the von Neumann condition, although merely algebraic, becomes quite difficult and is often done numerically on the machine.

Almost all problems of practical importance involve boundary conditions as well as initial conditions. There is a large body of practical experience with these and a modest body of theory. An analog of the von Neumann condition has been given by Godunov and Ryabenkii[7] using some observations of Gelfand. In spite of interesting beginnings, such as the work of Strang[22] who has found a relation to the theory of Wiener–Hopf equations, and some recent work of Kreiss[10] using high-order extrapolation, there is as yet no general theory for problems with variable coefficients.

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