NETWORK THEORY AND APPLICATIONS

Matrices in Combinatorics and Graph Theory

Bolian Liu and Hong-Jian Lai

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The first chapter of the book provides a brief treatment of the basics. The other chapters deal with the various decompositions of nonnegative matrices, Birkhoff type theorems, the study of the powers of nonnegative matrices, applications of matrix methods to other combinatorial problems, and applications of combinatorial methods to matrix problems and linear algebra problems.

The coverage of prerequisites has been kept to a minimum. Nevertheless, the book is basically self-contained (an Appendix provides the necessary background in linear algebra, graph theory and combinatorics). There are many exercises, all of which are accompanied by sketched solutions.

Audience

The book is suitable for a graduate course as well as being an excellent reference and a valuable resource for mathematicians working in the area of combinatorial matrix theory.

and Graph Theory
Hong-Jian Lai

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Network Theory and Applications

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Matrices in Combinatorics and Graph Theory

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FOREWORD

Combinatorics and Matrix Theory have a symbiotic, or mutually beneficial, relationship. This relationship is discussed in my paper *The symbiotic relationship of combinatorics and matrix theory*[1] where I attempted to justify this description. One could say that a more detailed justification was given in my book with H. J. Ryser entitled *Combinatorial Matrix Theory*[2] where an attempt was made to give a broad picture of the use of combinatorial ideas in matrix theory and the use of matrix theory in proving theorems which, at least on the surface, are combinatorial in nature.

In the book by Liu and Lai, this picture is enlarged and expanded to include recent developments and contributions of Chinese mathematicians, many of which have not been readily available to those of us who are unfamiliar with Chinese journals. Necessarily, there is some overlap with the book *Combinatorial Matrix Theory.* Some of the additional topics include: spectra of graphs, eulerian graph problems, Shannon capacity, generalized inverses of Boolean matrices, matrix rearrangements, and matrix completions. A topic to which many Chinese mathematicians have made substantial contributions is the combinatorial analysis of powers of nonnegative matrices, and a large chapter is devoted to this topic.

This book should be a valuable resource for mathematicians working in the area of combinatorial matrix theory.

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University of Wisconsin - Madison

[1]Linear Alg. Applics., vols. 162-4, 1992, 65-105
[2]Cambridge University Press, 1991.

PREFACE

On the last two decades or so work in combinatorics and graph theory with matrix and linear algebra techniques, and applications of graph theory and combinatorics to linear algebra have developed rapidly. In 1973, H. J. Ryser first brought in the concept "combinatorial matrix theory". In 1992, Brualdi and Ryser published "Combinatorial Matrix Theory", the first expository monograph on this subject. By now, numerous exciting results and problems, interesting new techniques and applications have emerged and been developing. Quite a few remarkable achievements in this area have been made by Chinese researchers, adding their contributions to the enrichment and development of this new theory.

The purpose of this book is to present connections among combinatorics, graph theory and matrix theory, with an emphasis on an exposition of the contributions made by Chinese scholars.

Prerequisites for an understanding of the text have been kept to a minimum. It is essential however to be familiar with elementary set notation and to have had basic knowledge in linear algebra, graph theory and combinatorics. For referential convenience, three sections on the basics of these areas are included in the Appendix, supplementing the brief introductions in the text.

The exercises which appear at the ends of chapters often supplement, extend or motivate the materials of the text. For this reason, outlines of solutions are invariably included.

We wish to make special acknowledgment to Professor $\boxed{\text{Herbert John Ryser}}$, who can be rightfully considered the father of Combinatorial Matrix Theory, and to Professor Richard Brualdi, who has made enormous contributions to the development of the theory.

There are many people to thank for their contributions to the organization and content of this book and an earlier version of it. In particular, we would like to express our sincere thanks to Professors Lizhi Hsu, Dingzhu Du, Ji Zhong, Qiao Li, Jongsheng Li, Jiayu Shao, Mingyiao Hsu, Fuji Zhang, Keming Zhang, Jingzhong Mao, and Maocheng Zhang, for their wonderful comments and suggestions. We would also like to thank Bo Zhou and Hoifung Poon for proof reading the manuscript.

Bolian Liu would like to give his special appreciation to his wife, Mo Hui, and his favorite daughters, Christy and Jolene. Hong-Jian Lai would like to give his special thanks to his wife, Ying Wu, and to his parents Jie-Ying Li and $\boxed{\text{Han-Si Lai}}$. Without our families' forbearance and support we would never have been able to complete this project.

Chapter 1

Matrices and Graphs

1.1 The Basics

Definition 1.1.1 For a digraph D with vertices $V(D) = \{v_1, v_2, \cdots, v_n\}$, let $m(v_i, v_j)$ denote the number of arcs in D oriented from v_i to v_j. The *adjacency matrix* of D is an n by n matrix $A(D) = (a_{ij})$, given by

$$a_{ij} = m(v_i, v_j).$$

We can view a graph G as a digraph by replacing each edges of G by a pair of arcs with opposite directions. Denote the resulting digraph by D_G. With this view point, we define the *adjacency matrix of* G by $A(G) = A(D_G)$, the adjacency matrix of the digraph D_G. Note that $A(G)$ is a symmetric matrix.

Note that the adjacency matrix of a simple digraph is a $(0,1)$-matrix, and so there is a one-to-one correspondence between the set of simple digraphs $D(V, E)$ with $V = \{v_1, \cdots, v_n\}$ and $\mathbf{B}_n$, the set of all (0,1) square matrices of order n: For each (0,1) square matrix $A = (a_{ij})_{n\times n}$, define an arc set E on the vertex set V by $e = (v_i, v_j) \in E$ if and only if $a_{ij} = 1$. Then we obtain a digraph $D(A)$, called *the associated digraph of* A. Proposition 1.1.1 follows from the definitions immediately.

Proposition 1.1.1 For any square (0,1) matrix A, $A(D(A)) = A$.

Definition 1.1.2 A matrix $A \in \mathbf{B}_n$ is a *permutation* matrix if each row and each column have exactly one 1-entry. Two matrices A and B are *permutation equivalent* if there exist permutation matrices P and Q such that $A = PBQ$; A and B are *permutation similar* if for some permutation matrix P, $A = PBP^{-1}$.

Let $A = (a_{ij})$ and $B = (b_{ij})$ be two matrices in $M_{m,n}$. Write $A \leq B$ if for each i and j, $a_{ij} \leq b_{ij}$.

Proposition 1.1.2 can be easily verified.

Proposition 1.1.2 Let A be a square (0,1) matrix, and let $D = D(A)$ be the associated digraph of A. Each of the following holds.

(i) Each row sum (column sum, respectively) of A is a constant r if and only if $d^+(v) = r$ ($d^-(v) = r$, respectively) $\forall v \in V(D)$.

(ii) Let $A, B \in \mathbf{B}_n$. Then $A \le B$ if and only if $D(A)$ is a spanning subgraph of $D(B)$.

(iii) There is a permutation matrix P such that $PAP^{-1} = B$ if and only if the vertices of $D(A)$ can be relabeled to obtain $D(B)$.

(iv) A is symmetric with $\mathrm{tr}(A) = 0$ if and only if $D(A) = D_G$ for some simple graph G; or equivalently, if and only if A is the adjacency matrix of a simple graph G.

(v) There is a permutation matrix P such that

$$PAP^{-1} = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix},$$

where A_1 and A_2 are square (0,1) matrices, if and only if $D(A)$ is not connected.

(vi) There is a permutation matrix P such that

$$PAP^{-1} = \begin{pmatrix} 0 & B \\ B^T & 0 \end{pmatrix},$$

where B is a (0,1) matrix, then $D(A)$ is a bipartite graph. (In this case, the matrix B is called the *reduced adjacency matrix of* $D(A)$; and $D(A)$ is called the *reduced associated bipartite graph of* B.)

(vii) The (i,j)-th entry of A^l, the l-th power of A, is positive if and only if there is a directed (v_i, v_j)-walk in $D(A)$ of length l.

(viii) A_1 is a principal square submatrix of A if and only if $A_1 = A(H)$ is the adjacency matrix of a subgraph G induced by the vertices corresponding to the columns of A_1.

Definition 1.1.3 Let $G = (V, E)$ be a graph with vertex set $V = \{v_1, v_2, \cdots, v_n\}$ and edge set $E = \{e_1, e_2, \cdots, e_q\}$, with loops and parallel edges allowed. The *incidence matrix* of G is $B(G) = (b_{ij})_{n \times q}$ whose entries are defined by

$$b_{ij} = \begin{cases} 1 & \text{it } v_i \text{ is incident with } e_j \\ 0 & \text{otherwise} \end{cases}$$

Let $\mathrm{diag}(r_1, r_2, \cdots, r_n)$ denote the diagonal matrix with diagonal entries $r_1, r_2, \cdots, r_n$.

For a digraph $D = (V, E)$ with vertex set $V = \{v_1, v_2, \cdots, v_n\}$ and arc set $E = \{e_1, e_2, \cdots, e_q\}$, with loops and parallel edges allowed, the *oriented incidence matrix* of the digraph D is $B(D) = (b_{ij})_{n\times q}$ whose entries are defined by

$$b_{ij} = \begin{cases} 1 & \text{it } v_j \text{ is an out-arc of } v_i \\ -1 & \text{it } v_j \text{ is an in-arc of } v_i \\ 0 & \text{otherwise} \end{cases}$$

Given a digraph D with oriented incidence matrix B, the matrix BB^T is called the *Laplace matrix*, (or *admittance matrix*) of D. As shown below, the Laplace matrix is independent of the orientation of the digraph D. Therefore, we can also talk about the *Laplace matrix of a graph* G, meaning the Laplace matrix of any orientation D of the graph G.

Theorems 1.1.1 and 1.1.2 below follow from these definitions and so are left as exercises.

Theorem 1.1.1 Let G be a loopless graph with $V(G) = \{v_1, v_2, \cdots, v_n\}$, and let d_i be the degree of v_i in G, for each i with $1 \le i \le n$. Let D be an orientation of G, A be the adjacency matrix of G and $C = \text{diag}(d_1, d_2, \cdots, d_n)$. Each of the following holds.

(i) If B is the incidence matrix of G, then $BB^T = C + A$.

(ii) If B is the oriented incidence matrix of D, then $BB^T = C - A$.

Theorem 1.1.2 Let G be a graph with t components and with n vertices. Let D be an orientation of G. Each of the following holds.

(i) The rank of the oriented incidence matrix B is $n - t$.

(ii) Let B_1 be obtained from B by removing t rows from B, each of these t rows corresponding to a vertex from each of the t components of G. Then the rank of B_1 is $n - t$.

Definition 1.1.4 An integral matrix $A = (a_{ij})$ is *totally unimodular* if the determinant of every square submatrix is in $\{0, 1, -1\}$.

Theorem 1.1.3 (Hoffman and Kruskal [127]) Let A be an $m \times n$ matrix such that the rows of A can be partitioned into two sets R_1 and R_2 and such that each of the following holds:

(i) each entry of A is in $\{0, 1, -1\}$;

(ii) each column of A has at most two non-zero entries;

(iii) if some column of A has two nonzero entries with the same sign, then one of the rows corresponding to these two nonzero entries must be in R_1 and the other in R_2; and

(iv) if some column of A has two nonzero entries with different signs, then either both rows corresponding to these two nonzero entries are in R_1 or both are in R_2.

Then A is totally unimodular.

Proof Note that if a matrix A satisfies Theorem 1.1.3 (i)-(iv), then so does any submatrix of A. Therefore, we may assume that A is an $n \times n$ matrix to prove $\det(A)$, the determinant of A, is in $\{0,1,-1\}$.

By induction on n, the theorem follows trivially from Theorem 1.1.3(i) when $n = 1$. Assume that $n \geq 2$ and that Theorem 1.1.3 holds for square matrices with smaller size.

If each column of A has exactly two nonzero entries, then by (iii) and (iv), the sum of the rows in R_1 is equal to that of the rows in R_2, consequently $\det(A) = 0$. If A has an all zero entry column, then $\det(A) = 0$. Therefore by (ii), we may assume that there is a column which has exactly one nonzero entry. Expand $\det(A)$ along this column and by induction, we conclude that $\det(A) \in \{0,1,-1\}$. $\square$

Corollary 1.1.3A (Poincaré [213]) The oriented incidence matrix of a graph if totally unimodular.

Corollary 1.1.3B Let G be a loopless graph (multiple edges allowed) and let B be the incidence matrix of G. Then G is bipartite if and only if B is totally unimodular.

Definition 1.1.5 For a graph G, the *characteristic polynomial* of G is the characteristic polynomial of the matrix $A(G)$, denoted by

$$\chi_G(\lambda) = \det(\lambda I - A(G)).$$

The *spectrum* of G is the set of numbers which are eigenvalues of the matrix $A(G)$, together with their multiplicities. If the distinct eigenvalues of $A(G)$ are $\lambda_1 > \lambda_2 > \cdots > \lambda_s$, and their multiplicities are $m_1, m_2, \cdots, m_s$, respectively, then we write

$$\mathrm{spec}(G) = \begin{pmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_s \\ m_1 & m_2 & \cdots & m_s \end{pmatrix}.$$

Since $A(G)$ is symmetric, all the eigenvalues λ_i's are real. When no confusion arises, we write $\det G$ or $\det(G)$ for $\det(A(G))$.

Theorem 1.1.4 Let $S(G,H)$ denote the number of subgraphs of G isomorphic to H. If $\chi_G(\lambda) = \sum_{i=0}^{n} C_i \lambda^{n-i}$, then

$$\begin{aligned} C_0 &= 1, \text{ and} \\ C_i &= (-1)^i \sum_H \det H \, S(G,H), \quad i = 1,2,\cdots,n, \end{aligned}$$

where the summation is taken over all non isomorphic subgraphs of G.

Sketch of Proof Since $\chi_G(\lambda) = \det(\lambda I - A) = \lambda^n + \cdots$, $C_0 = 1$, and for each $i = 1,2,\cdots,n$, $C_i = (-1)^i \sum_{A_i} \det A_i$, where the summation is over all the ith order principal submatrices of A. Note that A_i is the adjacency matrix of the subgraph H of G induced by the vertices corresponding to the rows of A_i (Preposition 1.1.2(vii)), and that the number of such subgraphs of G is $S(G,H)$. $\square$

Further discussion on the coefficients of $\chi_G(\lambda)$ can be found in [226].

1.2 The Spectrum of a Graph

In this section, we present certain techniques using the matrix $A(G)$ and spec(G) to study the properties of a graph G. Theorem 1.2.1 displays some facts from linear algebra and from Definition 1.1.5.

Theorem 1.2.1 Let G be a graph with n vertices, and $A = A(G)$ be the adjacency matrix of G. Each of the following holds:

(i) If $G_1, G_2, \cdots, G_k$ are components of G, then $\chi_G(\lambda) = \prod_{i=1}^{k} \chi_{G_i}(\lambda)$.

(ii) The spectrum of G is the disjoint union of the spectrums of G_i, where $i = 1, 2, \cdots, k$.

(iii) If $f(x)$ is a polynomial, and if λ is an eigenvalue of A, then $f(\lambda)$ is an eigenvalue of $f(A)$.

Theorem 1.2.2 Let G be a connected graph with n vertices and with diameter d. If G has s distinct eigenvalues, then $n \geq s \geq d + 1$.

Proof Since n is the degree of $\chi_G(\lambda)$, $n \geq s$.

Let $A = A(G)$ and suppose that $s \leq d$. Then G has two vertices v_i and v_j such that the distance between v_i and v_j is s. Let $m_A(\lambda)$ be the minimum polynomial of A. Then $m_A(A) = 0$. Since A is symmetric, the degree of $m_A(\lambda)$ is exactly s and $m_A(\lambda) = \lambda^s + \cdots$.

By Proposition 1.1.2(vii), the (i, j)-entry of A^s is positive, and the (i, j)-entry of A^l is zero, for each l with $1 \leq l \leq s - 1$. it follows that it is impossible to have $m_A(A) = 0$, a contradiction. $\square$

For integer $r, g > 1$, an r-regular graph with girth g is called an *(r, g)-graph*. Sachs [225] showed that for any $r, g > 1$, an (r, g)-graph exists. Let $f(r, g)$ denote the smallest order of an (r, g)-graph. An (r, g)-graph G with $|V(G)| = f(r, g)$ is called a *Moore graph*.

Theorem 1.2.3 (Erdös and Sachs, [84])

$$r^1 + 1 \leq f(r, 5) \leq 4(r - 1)(r^2 - r + 1).$$

Theorem 1.2.4 When $g = 5$, a Moore $(r, 5)$-graph exists only if $r = 2, 3, 7$, or 57.

Proof Let G be a Moore $(r, 5)$-graph. Then $|V(G)| = r^2+1$. Let $V(G) = \{v_1, v_2, \cdots, v_{r^2+1}\}$, and let $A = (a_{ij})$ denote the adjacency matrix of G, and denote $A^2 = (a_{ij}^{(2)})$.

Since $g = 5$, G has no 3-cycles, and so when $a_{ij} = 1$, $a_{ij}^{(2)} = 0$; and when $a_{ij} = 0$, $a_{ij}^{(2)} = 1$. It follows that

$$A^2 + A = J + (r - 1)I.$$

Add to the first row of $A^2 + A - \lambda I$ all the other rows to get

$$\det(A^2 + A - \lambda I) = (r^2 + r - \lambda)(r - 1 - \lambda)^{r^2}.$$

Therefore, $\mathrm{spec}(A^2 + A) = \begin{pmatrix} r^2+r & r-1 \\ 1 & r^2 \end{pmatrix}$.

Let λ_i, $1 \le i \le r^2+1$, be the eigenvalues of A. Then by Theorem 1.2.1(iii), $\lambda_i^2 + \lambda_i$ is an eigenvalue of $A^2 + A$, and so we may assume that

$$\lambda_1^2 + \lambda_1 = r^2 + r, \text{ and } \lambda_i^2 + \lambda_i = r - 1, \text{ for each } i = 2, 3, \cdots, r^2 + 1.$$

Hence we may assume $\lambda_1 = r$, and for some k with $2 \le k \le r^2 + 1$, $\lambda_2 = \lambda_3 = \cdots = \lambda_{k+1} = \dfrac{-1+\sqrt{4r-3}}{2}$ and $\lambda_{k+2} = \cdots = \lambda_{r^2+1} = \dfrac{-1-\sqrt{4r-3}}{2}$. Since the sum of all eigenvalues of A is zero, solving

$$r + \frac{k(-1+\sqrt{4r-3})}{2} + \frac{(r^2-k)(-1-\sqrt{4r-3})}{2} = 0,$$

we have $2k = \dfrac{(r^2-2r)}{\sqrt{4r-3}} + r^2$.

Since $k \ge 0$ is an integer and since $r \ge 2$, either $r = 2$ or for some positive integer m, $4r - 3 = (2m+1)^2$ is the square of an odd integer. Thus if $r > 2$, then $r = m^2 + m + 1$ and so

$$2k = 2m - 1 + \frac{m^2}{4}\left(2m + 3 - \frac{15}{2m+1}\right) + (m^2 + m + 1)^2.$$

As m^2 and $2m + 1$ are relatively prime, $2m + 1$ must divide 15, and so $m \in \{1, 2, 7\}$. It follows that $r \in \{2, 3, 7, 57\}$, respectively. $\square$

Moore $(r, 5)$-graphs with $r \in \{2, 3, 7\}$ has been constructed. The existence of a Moore $(57, 5)$-graph is still unknown. (See [9].)

Theorem 1.2.5 (Brown [18]) There is no r-regular graph with girth 5 and order $n = r^2+2$.

Sketch of Proof By contradiction, assume that there exists an $(r, 5)$-graph G with order $n = r^2 + 2$. For each $v \in V(G)$, let $N(v)$ denote the vertices adjacent to v in G.

Let $v_1 \in V(G)$ and let $N(v_1) = \{v_2, \cdots, v_{r+1}\}$. Since the girth of G is 5, each $N(v_i) \cap N(v_j) = \{v_1\}$ for all $2 \le i < j \le k+1$, and $N(v_1)$ is an independent set. It follows that

$$|\cup_{i=1}^{k+1} (N(v_i) - \{v_1\})| = \sum_{i=1}^{r+1} |N(v_i) - \{v_1\}| = r + r(r-1) = r^2. \tag{1.1}$$

Since $|V(G)| = r^2 + 2$, it follows by (1.1) that for any $v \in V(G)$, there is exactly one $v^* \in V(G)$ which cannot be reached form v by a path of length at most 2. Note that $(v^*)^* = v$.

Let $A = A(G)$ be the adjacency matrix of G. Since G has girth 5, it follows that $A^2 + A - (r-1)I = J - B$, where B is a permutation matrix in which each entry in the main diagonal is zero. Therefore we can relabel the vertices of G so that B is the direct sum of $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. It follows that both n and r are even. By Theorem 1.2.1(iii), if k is an eigenvalue of A, then $k^2 + k - (r-1)$ is an eigenvalue of $A^2 + A - (r-1)I$. Direct computation yields (Exercise 1.7(vi))

$$\operatorname{spec}(A^2 + A - (r-1)I) = \begin{pmatrix} n-1 & 1 & -1 \\ 1 & \frac{n}{2} & \frac{n}{2}-1 \end{pmatrix}.$$

Since A is a real symmetric matrix, A has $\frac{n}{2}$ real eigenvalues k satisfying $k^2+k-(r-1) = 1$, or $k = \frac{-1\pm s}{2}$, where $s = \sqrt{4r+1}$; and A has $\frac{n}{2} - 1$ real eigenvalues k satisfying $k^2 + k - (r-1) = 1$, or $k = \frac{-1\pm t}{2}$, where $t = \sqrt{4r-7}$. Consider the following cases.

Case 1 Both s and t are rational numbers. Then s and t must be the two odd integer 3 and 1, respectively. It follows that $r = 2$ and so G is a circle of length 6.

Case 2 Both s and t are irrationals. If there is a prime α which is a common factor of both s^2 and t^2 but α does not divide s or α does not divide t, then α divides $s^2 - t^2 = 8$, and so $\alpha = 2$. But since s^2 and t^2 are both odd numbers, none of them can have an even factor, a contradiction.

Therefore, no such α exists. It follows that if $\frac{-1+s}{2}$ is an eigenvalue of $J - B$, then $\frac{-1-s}{2}$ is also an eigenvalue of $J - B$. In other words, the number of eigenvalues of $J - B$ in the form of $\frac{-1\pm s}{2}$ is even. Similarly, the number of eigenvalues of $J - B$ in the form of $\frac{-1\pm t}{2}$ is even. But this is impossible, since one of $\frac{n}{2}$ and $\frac{n}{2} - 1$ must be odd.

Case 3 s is irrational and t is rational. Then t is an odd integer and so $-1 \pm t$ is even. Since s is irrational, the number of eigenvalues in the form $\frac{-1\pm s}{2}$ is even and the sum of all such eigenvalues is $\frac{-1}{2}\frac{n}{2} = \frac{-n}{4}$. However, since $-1 \pm t$ is even, the sum of all eigenvalues in the form of $\frac{-1\pm t}{2}$ is an integer, and so the sum of eigenvalues in the form of $\frac{-1\pm s}{2}$ is an integer. It follows that $r^2 + 2 = n \equiv 0 \pmod 4$, or $r^2 \equiv 2 \pmod 4$, which is impossible.

Case 4 s is rational and t is irrational. Since t is irrational, the eigenvalues in the form $\frac{-1\pm t}{2}$ must appear in pairs, and so the sum of all such eigenvalues is an integer. However, the sum of these eigenvalues is $\left(\frac{-1}{2}\right)\left(\frac{n}{2} - 1\right)$. Let m denote the multiplicity of $\dfrac{-1+s}{2}$. Since G is simple, the trace of A is zero. By Proposition 1.1.2(vii) and by $g = 5$,

$$0 = \operatorname{tr} A = r + \frac{m(-1+s)}{2} + \left(\frac{n}{2} - m\right)\frac{-1-s}{2} + \left(-\frac{1}{2}\right)\left(\frac{n}{2} - 1\right). \qquad (1.2)$$

Since $n = r^2 + 2$ and since $r = \dfrac{s^2-1}{4}$, substitute these into (1.2) to get

$$s^5 + 2s^4 - 2s^3 - 20s^2 + (33 - 64m)s + 50 = 0. \qquad (1.3)$$

Any positive rational solution of s in (1.3) must be a factor of 50, and so $s \in \{1, 2, 5, 10, 25, 50\}$. Among these numbers only $s = 1, 5$, and 25 will lead to these integral solutions

$$\begin{cases} s = 1 \\ m = 1 \\ r = 0 \end{cases} \quad \begin{cases} s = 5 \\ m = 12 \\ r = 6 \end{cases} \quad \begin{cases} s = 25 \\ m = 6565 \\ r = 156. \end{cases}$$

Since $n = r^2 + 2$, we can exclude the solution when $s = 1$. We outline the proof that s cannot be 5 or 25, as follows.

Since $g = 5$, the diagonal entries of A^3 must all be zero. By Theorem 1.2.1(iii), if λ is an eigenvalue of A, then λ^3 is an eigenvalue of A^3. It follows that $\mathrm{tr}(A^3)=0$. But no matter whether $s = 5$ or $s = 25$, $\mathrm{tr}(A^3) \neq 0$, and so $s = 5$ or $s = 25$ are also impossible. □

Theorem 1.2.6 Let G be a graph with n vertices such that $v \in V(G)$. If G has eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \lambda_n$, and if $G - v$ has eigenvalues $\mu_1 \geq \mu_2 \geq \cdots \mu_{n-1}$, then

$$\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \cdots \mu_{n-1} \geq \lambda_n.$$

Proof Let $A_1 = A(G - v)$, and $A = A(G)$. Then there is a row vector $\mathbf{u}$ such that

$$A = \begin{pmatrix} 0 & \mathbf{u} \\ \mathbf{u}^T & A_1 \end{pmatrix}.$$

Since A is symmetric, A has a set of n orthonormal eigenvectors $\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_n$. Let $\mathbf{e}_i$ denote the n-dimensional vector whose i-th component is one and whose other components are zeros.

Divide the eigenvalues $\lambda_1 \geq \cdots \geq \lambda_n$ into two groups: Group 1 and Group 2. A λ_i is in Group 1 if λ_i has an eigenvector $\mathbf{x}_i$ such that $\mathbf{e}_1^T\mathbf{x}_i = 0$; and a λ_i is in Group 2 if it is not in Group 1. Note that if λ_i is in Group 2, then $\mathbf{e}_1^T\mathbf{x}_i \neq 0$ (for all eigenvectors $\mathbf{x}_i$ of λ_i).

Suppose λ_i is in Group 1 and it has an eigenvector $\mathbf{x}_i$ such that $\mathbf{e}_1^T\mathbf{x}_i = 0$. Let $\mathbf{x}'_i$ denote the $(n-1)$-dimensional vector obtained from $\mathbf{x}_i$ by removing the first component of $\mathbf{x}_i$. Then as $\lambda_i A = \lambda_i \mathbf{x}_i$, $\lambda_i A_1 = \lambda_i \mathbf{x}'_i$. It follows that $\lambda_i = \mu_i$, for such λ_i's.

Rename the eigenvalues in Group 2 so that they are $\tilde{\lambda}_1 \geq \tilde{\lambda}_2 \geq \cdots \geq \tilde{\lambda}_k$. For notational convenience, let $\tilde{\mathbf{x}}_i \in \{\mathbf{x}_1, \mathbf{x}_2, \cdots .\mathbf{x}_n\}$ be the eigenvector corresponding to $\tilde{\lambda}_i$, $1 \leq i \leq k$.

Let $\mathbf{y}'$ be an eigenvector of A_1 corresponding to an eigenvalue in Group 2; and let $\mathbf{y}$ be the n-dimensional vector obtained from $\mathbf{y}'$ by adding a zero component as the first component of $\mathbf{y}$. Since the $\mathbf{x}_i$'s are orthonormal and by the definition of Group 1, both $\mathbf{y} = \sum_{i=1}^{k} b_i \tilde{\mathbf{x}}_i$ and $\mathbf{e}_1 = \sum_{i=1}^{k} c_i \tilde{\mathbf{x}}_i$, where $c_i = \mathbf{e}_1^T \tilde{\mathbf{x}}_i$.

Since $\mathbf{y}'$ is an eigenvector of A_1 corresponding to an eigenvalue μ (say), $A_1\mathbf{y}' = \mu\mathbf{y}'$. It follows that

$$\sum_{i=1}^{k} b_i\tilde{\lambda}_i\tilde{\mathbf{x}}_i = A\sum_{i=1}^{k} b_i\tilde{\mathbf{x}}_i = A\mathbf{y} = (\mathbf{u}^T\mathbf{y}')\mathbf{e}_1 + \mu\mathbf{y}' = (\mathbf{u}^T\mathbf{y}')\sum_{i=1}^{k} c_i\tilde{\mathbf{x}}_i + \mu\mathbf{y}'.$$

Therefore, for each j, the j-th component of both sides must be the same, and so

$$b_j = \frac{-(\mathbf{u}^T\mathbf{y}')c_i}{\mu - \tilde{\lambda}_j}.$$

By the definition of $\mathbf{y}$,

$$\sum_{i=1}^{k} \frac{-(\mathbf{u}^T\mathbf{y}')c_i^2}{\mu - \tilde{\lambda}_j} = \sum_{i=1}^{k} b_ic_i = \mathbf{e}_1^T\mathbf{y} = 0. \tag{1.4}$$

Equation (1.4) has k vertical asymptotes at $\mu = \tilde{\lambda}_i$, $1 \le i \le k$. It follows that (1.4) has a root $\tilde{\mu}_i \in (\tilde{\lambda}_{i+1}, \tilde{\lambda}_i)$, whence

$$\tilde{\lambda}_1 > \tilde{\mu}_1 > \tilde{\lambda}_2 > \tilde{\mu}_2 > \cdots > \tilde{\lambda}_{k-1} > \tilde{\mu}_{k-1} > \tilde{\lambda}_k.$$

This, together with the conclusion on the eigenvalues in Group 1, asserts the conclusion of the theorem. □

Corollary 1.2.6A Let G be a graph with $n > k \ge 0$ vertices. Let $V' \subset V(G)$ be a vertex subset of G with $|V'| = k$. If G has eigenvalues $\lambda_1 \ge \lambda_2 \ge \cdots \lambda_n$, then

$$\lambda_i \ge \lambda_i(G - V') \ge \lambda_{i+k}.$$

Corollary 1.2.6B Let G be a graph with $n \ge 2$ vertices. If G is not a complete graph, then $\lambda_2(G) \ge 0$.

Proof Corollary 1.2.6A follows from Theorem 1.2.6 immediately. Assume that G has two nonadjacent vertices u and v. Let $V' = V(G) - \{u, v\}$. Then by Corollary 1.2.6A, $\lambda_2(G) \ge \lambda_2(G - V') = 0$. □

Lemma 1.2.1 (J. H. Smith [71]) Let G be a connected graph with $n \ge 2$ vertices. The following are equivalent.

(i) G has exactly one positive eigenvalue.

(ii) G is a complete k-partite graph, where $2 \le k \le n - 1$.

Theorem 1.2.7 (Cao and Hong [43]) Let G be a simple graph with n vertices and without isolated vertices. Each of the following holds.

(i) $\lambda_2(G) = -1$ if and only if G is a complete graph with $n \ge 2$ vertices.

(ii) $\lambda_2(G) = 0$ if and only if $G \ne K_2$ and G is a complete k-partite graph, where

$2 \le k \le n-1$.

(iii) There exists no graph G such that $-1 < \lambda_2(G) < 0$.

Proof Direct computation yields $\lambda_2(K_n) = -1$ (Exercise 1.7(i)). Thus Theorem 1.2.7(i) follows from Corollary 1.2.6B.

Suppose that $G \neq K_2$ and that G is a complete k-partite graph with $2 \le k \le n-1$. Then by Lemma 1.2.1 and Corollary 1.2.6B, $\lambda_2(G) = 0$.

Conversely, assume that $\lambda_2(G) = 0$ (then $G \neq K_2$) and that G is not a complete k-partite graph. Since G has no isolated vertices, G must contain one of these as induced subgraphs: $2K_2$, P_4 and $K_1 \vee (K_1 \cup K_2)$. However, each of these graphs has positive second eigenvalue, and so by Corollary 1.2.6A, $\lambda_2(G) > 0$, contrary to the assumption that $\lambda_2(G) = 0$. Hence G must be a complete k-partite graph. This proves Theorem 1.2.7(ii).

Theorem 1.2.7(iii) follows from Theorem 1.2.7(i) and (ii), and from Corollary 1.2.6B. □

The proofs for the following lemmas are left as exercises.

Lemma 1.2.2 (Wolk, [277]) If G has no isolated vertices and if G^c is connected, then G has an induced subgraph isomorphic to $2K_2$ or P_4.

Lemma 1.2.3 (Smith, [71]) Let H_1, H_2, H_3 and H_4 be given in Figure 1.2.1. Then for each i with $1 \le i \le 4$, $\lambda(H_i) > \frac{1}{3}$.

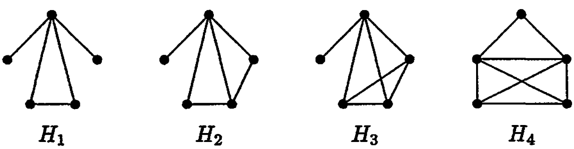

Figure 1.2.1

Lemma 1.2.4 (Cao and Hong [43]) Let $H_5 = \bar{K}_{n-3} \vee (K_1 \cup K_2)$. Then each of the following holds.

(i) $\chi_{H_5} = (\lambda^3 - \lambda^2 - 3(n-3)\lambda + n - 3)\lambda^{n-4}(\lambda + 1)$.

(ii) $\lambda_2(H_5) < \frac{1}{3}$.

Theorem 1.2.8 (Cao and Hong [43]) Let G be a graph with n vertices and without isolated vertices. Each of the following holds.

(i) $0 < \lambda_2(G) < \frac{1}{3}$ if and only if $G \cong H_5$.

(ii) If $1 > \lambda_2(G) > \frac{1}{3}$, then G contains an induced subgraph H isomorphic to a member in $\{2K_2, P_4\} \cup \{H_i : 12 \le i \le 4\}$, where the H_i's are defined in Lemma 1.2.2.

(iii) If $\lambda^{(n)} = \lambda_2(H_5)$, then $\lambda^{(n)}$ increases as n increases, and $\lim_{n\to\infty} \lambda^{(n)} = \frac{1}{3}$.

(iv) There exist no graphs G_k such that $\lambda_2(G_k) > \frac{1}{3}$ and $\lim\limits_{k\to\infty} \lambda_2(G_k) = \dfrac{1}{3}$.

Proof Part (i). By Lemma 1.2.4(ii), it suffices to prove the necessity.

Suppose $0 < \lambda_2(G) < \frac{1}{3}$. If G has an induced subgraph H isomorphic to a member in $\{2K_2, P_4\} \cup \{H_i : 12 \le i \le 4\}$, where the H_i's are defined in Lemma 1.2.3, then by Corollary 1.2.6A, $\lambda_2(G) \ge \lambda_2(H) > \frac{1}{3}$, a contradiction. Hence we have the following:

Claim 1 G does not have an induced subgraph isomorphic to a member in $\{2K_2, P_4\} \cup \{H_i : 12 \le i \le 5\}$.

By Claim 1 and Lemma 1.2.2, G^c is not connected. Hence G^c has connected components $G_1, \cdots, G_k$ with $k \ge 2$. But then $G = G_1^c \vee G_2^c \vee \cdots \vee G_k^c$. We have the following claims.

Claim 2 For each i with $1 \le i \le k$, G_i^c has an isolated vertex.

If not, then by Lemma 1.2.2, G_i^c has an induced subgraph isomorphic to $2K_2$ or P_4, contrary to Claim 1.

Claim 3 For some i, $G_i^c \cong K_1 \cup K_2$.

By $\lambda_2(G) > 0$ and by Theorem 1.2.7(ii), G is not a complete k-partite graph, and some G_i^c has at least one edge. By Claim 1, G_i^c contains $K_1 \cup K_2$ as an induced subgraph. If $|V(G_i^c)| > 3$, then G_i^c must have an induced subgraph H isomorphic to one of $2K_1 \cup K_2, K_1 \cup P_3$ and $K_1 \cup K_3$. Since $k \ge 2$, there exists a vertex $u \in V(G) - V(G_i^c)$. It follows that $V(H) \cup \{u\}$ induces a subgraph of G isomorphic to one of the H_i's in Lemma 1.2.3, contrary to Claim 1.

Claim 4 $k = 2$ and $G_2 \cong K_{n-3}$.

By Claims 2 and 3, we may assume that $G_1^c = K_1 \cup K_2$. Assume further that either $k \ge 3$ or $k = 2$ and G_2 has two nonadjacent vertices. Let $u \in V(G_2$ and $v \in V(G_3)$ if $k \ge 3$ and let $u, v \in V(G_2)$ be two nonadjacent vertices in G_2. Then $V(G_1) \cup \{u, v\}$ induces a subgraph of G isomorphic to H_4 (defined in Lemma 1.2.3), contrary to Claim 1. Thus it must be $k = 2$ and $G_2 \cong K_{n-3}$.

From Claims 3 and 4, we conclude that $G \cong H_5$.

Part (ii). It follows from the same arguments used in Part (i) and it is left as an exercise.

Part (iii). By Corollary 1.2.6A, $\lambda^{(n)}$ is increasing. Thus $\lim_{n\to\infty} \lambda^{(n)} = L$ exists. Let $f(\lambda) = \lambda^3 - \lambda^2 - 3(n-3)\lambda + n - 3$. Then by Lemma 1.2.4(i), $f(\lambda^{(n)}) = 0$, and so

$$\lambda^{(n)} = \frac{1}{3} + \frac{\lambda^{(n)} - 1}{3(n-3)}(\lambda^{(n)})^2.$$

It follows that $L = \dfrac{1}{3}$.

Part (iv). By contradiction, assume such a sequence exist. We may assume that for all k, $1 > \lambda_2(G_k) > \dfrac{1}{3}$. By Theorem 1.2.8(ii), all such G_k contains a member H in $\{2K_2, P_4\} \cup \{H_i : 12 \leq i \leq 4\}$. By Corollary 1.2.6A, $\lambda_2(G_k) \geq \lambda_2(H) > 0.334$, therefore, $\lambda_2(G_k)$ cannot have $1/3$ as a limit. $\square$

Corollary 1.2.8A Let G be a graph with n vertices and without isolated vertices. If G is not a complete k-partite graph for some k with $2 \leq k \leq n-1$, then $\lambda_2(G) \geq \lambda_2(H_5)$, where equality holds if and only if $G \cong H_5$.

In 1982, Cvetkovic [69] posed a problem of characterizing graphs with the property $0 < \lambda_2(G) \leq 1$. In 1993, P. Miroslav [200] characterized all graphs with the property $\lambda_2(G) \leq \sqrt{2}-1$. In 1995, D. Cvetkovic and S. Simic [72] obtained some important properties of the graph satisfying $\lambda_2(G) \leq \frac{\sqrt{5}-1}{2}$. Cao, Hong and Miroslav explicitly displayed all the graphs with the property $\lambda_2(G) \leq \frac{\sqrt{5}-1}{2}$. It remains difficult to solve the Cvetkovic's problem completely.

1.3 Estimating the Eigenvalues of a Graph

We start with some basics from the classical Perron-Frobenius theory on nonnegative square matrices. As the adjacency matrix of a connected graph is a special case of such matrices, some of these results can be stated in terms of connected graphs as follows.

Theorem 1.3.1 (Perron-Frobenius, [211], [93]) Let G be a connected graph with $|V(G)| \geq 2$. Each of the following holds.

(i) $\lambda_1(G)$ is positive, and is a simple root of $\chi_G(\lambda)$.

(ii) If $\mathbf{x}_1$ is an eigenvector of $\lambda_1(G)$, then $\mathbf{x}_1 > 0$.

(iii) For any $e \in E(G)$, $\lambda_1(G-e) < \lambda_1(G)$.

Definition 1.3.1 Let G be a connected graph with $|V(G)| \geq 2$. The value $\lambda_1(G)$ is called the *spectral radius* of the graph G.

Theorem 1.3.2 (Varga [265]) Let G be a connected graph with $A = A(G)$, let $\mathbf{x}_1$ denote an eigenvector of $\lambda_1(G)$. and let $\mathbf{y}$ be a positive vector. Then

$$\frac{(A\mathbf{y})^T\mathbf{y}}{|\mathbf{y}|^2} \leq \lambda_1(G) \leq \max_{1\leq j\leq n} \frac{(A\mathbf{y})^T\mathbf{e}_j}{\mathbf{y}^T\mathbf{e}_j},$$

where equality holds if and only if $\mathbf{y} = k\mathbf{x}_1$, for some constant k.

Proof Since A is a real symmetric square matrix, we can find $\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_n$ such that the $\mathbf{x}_i$'s are orthonormal and such that $\mathbf{x}_i$ is an eigenvector of $\lambda_i(G)$. By Theorem 1.3.1(ii),

$\mathbf{x}_1$ is positive. Note that $\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_n$ form a basis of the n dimensional vector space over the reals.

Let $\mathbf{y}$ be a positive vector. Then $\mathbf{y} = \sum_{i=1}^n c_i\mathbf{x}_i$. Since the $\mathbf{x}_i$'s are orthonormal, it follows by $A\mathbf{x}_i = \lambda_i\mathbf{x}_i$ that

$$\frac{(A\mathbf{y})^T\mathbf{y}}{|\mathbf{y}|^2} = \frac{\sum_{i=1}^n c_i^2\lambda_i}{\sum_{i=1}^n c_i^2} \le \frac{\sum_{i=1}^n c_i^2\lambda_1}{\sum_{i=1}^n c_i^2} = \lambda_1,$$

where equality holds if and only if either $c_2 = c_3 = \cdots = c_n = 0$ or $\lambda_1 = \lambda_2 = \cdots = \lambda_n$. By Theorem 1.3.1(i), $\lambda_1 > \lambda_2 \ge \lambda_n$, and so

$$\frac{(A\mathbf{y})^T\mathbf{y}}{|\mathbf{y}|^2} = \lambda_1$$

if and only if $\mathbf{y} = c_1\mathbf{x}_1$. This proves the lower bound.

Again by $\mathbf{y} = \sum_{i=1}^n c_i\mathbf{x}_i$ and by $A\mathbf{x}_i = \lambda_i\mathbf{x}_i$,

$$\frac{(A\mathbf{y})^T\mathbf{e}_i}{\mathbf{y}^T\mathbf{e}_i} = \frac{c_i^2\lambda_i}{c_i^2} = \lambda_i \le \lambda_1,$$

where equality holds if and only if $i = 1$, by Theorem 1.3.1(i). By Theorem 1.3.1(i), $\mathbf{x}_1 = \sum_{i=1}^n b_i\mathbf{e}_i$, where $b_i > 0$ for each i with $1 \le i \le n$. Since the $\mathbf{x}_i$'s are orthonormal and by $\mathbf{y} = \sum_{i=1}^n c_i\mathbf{x}_i$,

$$\lambda_1 = \frac{c_1\lambda_1}{c_1} = \frac{(A\mathbf{y})^T\mathbf{x}_1}{\mathbf{y}^T\mathbf{x}_1} = \frac{\sum_{i=1}^n b_i(A\mathbf{y})^T\mathbf{e}_i}{\sum_{i=1}^n b_i(\mathbf{y}^T\mathbf{x}_i)} \le \max_{1\le j\le n}\frac{b_j(A\mathbf{y})^T\mathbf{e}_j}{b_j(\mathbf{y}^T\mathbf{x}_j)} \le \lambda_1,$$

where equality holds if and only if for each j, $(A\mathbf{y})^T\mathbf{e}_j = \lambda_1(\mathbf{y}^T\mathbf{e}_j)$, and if and only if $\mathbf{y} = c_1\mathbf{x}_1$. This proves the upper bound. □

Corollary 1.3.2A Let G be a connected graph with $q = |E(G)|$ and $n = |V(G)| \ge 2$. Exactly one of the following holds.

(i) $\frac{2q}{n} < \lambda_1 < \Delta(G)$.

(ii) $\frac{2q}{n} = \lambda_1 = \Delta(G)$, G is regular and $(1, 1, \cdots, 1)^T$ is an eigenvector.

Corollary 1.3.2B(Hoffman, Walfe and Hofmeister [128], Zhou and Liu [289]) Let G be a connected graph with $q = |E(G)|$ and $n = |V(G)| \ge 2$. Let $d_1, \cdots, d_n$ be the degree sequence of G. Then

$$\frac{1}{q}\sum_{ij\in E(G)}\sqrt{d_id_j} \le \lambda_1 \le \max_{1\le i\le n}\frac{1}{d_i}\sum_{ij\in E(G)}\sqrt{d_id_j}$$

equalities hold if and only if there is a constant integer r such that $d_id_j = r^2$ for any $ij \in E(G)$.

Proof Setting $x_j = \sqrt{d_j}$ $(1 \le j \le n)$ in Theorem 1.3.2, we obtain the inequalities in Corollary 1.3.2B.

Suppose that there exists a constant integer r such that $d_i d_j = r^2$ for any $ij \in E(G)$. For $i = 1, \cdots, n$, let d be the common degree of all the vertices j adjacent to i. Then

$$\sum_{j:ij\in E(G)} \sqrt{d_j} = d_i\sqrt{d} = r\sqrt{d},$$

so that r is an eigenvalue of $A = A(G)$ with $(\sqrt{d_1}, \cdots, \sqrt{d_n})^t$ as a corresponding eigenvector. Hence $\lambda_1 = r$.

Conversely, if the equality holds in Corollary 1.3.2B, we have

$$\lambda_1\sqrt{d_i} = \sum_{j:ij\in E(G)} \sqrt{d_j}$$

for all i.

If G is regular, we are done. Otherwise let δ and Δ be respectively the minimal and maximal degrees of G. Choose u and v such that the degrees of u and v are δ and Δ. Assume that there exists a vertex w with $uw \in E(G)$ and that the degree of w is less than Δ. Then we have

$$\lambda_1 = \sum_{j:uj\in E(G)} \sqrt{\frac{d_j}{\delta}} < \sum_{j:uj\in E(G)} \sqrt{\frac{\Delta}{\delta}} = \sqrt{\delta\Delta}.$$

On the other hand,

$$\lambda_1 = \sum_{j:vj\in E(G)} \sqrt{\frac{d_j}{\Delta}} \geq \sum_{j:vj\in E(G)} \sqrt{\frac{\delta}{\Delta}} = \sqrt{\delta\Delta}.$$

a contradiction.

Assuming existence of w where $vw \in E(G)$ and the degree of w is greater than δ leads to an analogous contradiction. We have proved that whenever $ij \in E(G)$, then the degrees of i and j are δ and Δ or vice versa, and that $\lambda_1 = \sqrt{\delta\Delta}$. $\Box$

The study of the spectrum of a graph has been intensive. We present the following results with most of the proofs omitted. More results can be found in Exercises 1.11 through 1.13, and in [71]. In [131], Hong studied bounds of $\lambda_k(T_n)$ for general k (Theorem 1.3.7). Hong's results are recently refined by Shao [244] and Wu [278]. These refinements will be presented in Theorem 1.3.8.

Theorem 1.3.3 (Hong, [130]) Let $n \geq 3$ be an integer, let G be a unicyclic graph (a graph with exactly one cycle) with n vertices, and let S_n^3 denote the graph obtained from $K_{1,n-1}$ by adding a new edge joining two vertices of degree one in $K_{1,n-1}$. Then each of the following holds.

(i) $2 = \lambda_1(C_n) \leq \lambda_1(G) \leq \lambda_1(S_n^3)$, where the lower bound holds if and only if $G = C_n$; and where the upper bound holds if and only if $G = S_n^3$.

(ii) $2 \leq \lambda_1(G) \leq \sqrt{n}$, where the upper bound holds if and only if $G = S_9^3$.

Theorem 1.3.4 (Hofemeister, [121]) Let T_n denote a tree with n vertices. Then either $\lambda_2(T_n) \le \sqrt{\frac{n-3}{2}}$, or $n = 2s+2$ and T_n can be obtained from two disjoint $K_{1,s}$'s by adding a new edge joining the two vertices of degree s.

Theorem 1.3.5 (Collatz and Sinogowitz, [62]) Let G be a connected graph with $n = |V(G)|$. Then

$$2\cos\left(\frac{\pi}{n+1}\right) \le \lambda_1(G) \le n-1,$$

where the lower bound holds if and only if $G = P_n$, a path of n vertices; and where the upper bound holds if and only if G is a complete graph K_n.

Theorem 1.3.6 (Smith, [255]) Let G be a connected graph. Each of the following holds.

(i) If $\lambda_1(G) = 2$, then either $G \in \{K_{1,4}, C_n\}$, or G is one of the following graphs:

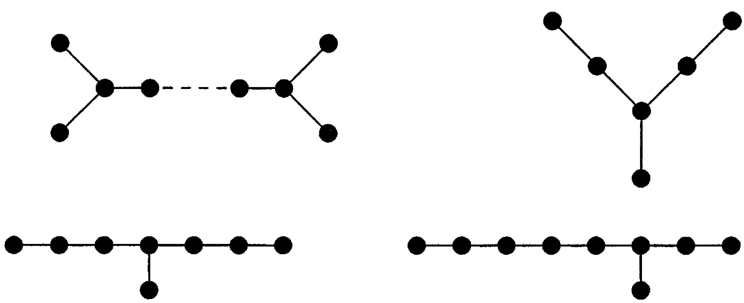

Figure 1.3.1.

(ii) If $\lambda_1(G) < 2$, then G is a subgraph of one of these graphs.

Theorem 1.3.7 (Hong, [131]) Let T_n denote a tree on $n \ge 4$ vertices. Then

$$0 \le \lambda_k(T_n) \le \sqrt{\left[\frac{n-2}{k}\right]}, \text{ for each } k \text{ with } 2 \le \left[\frac{n}{2}\right].$$

Moreover, if $n \equiv 1 \bmod k$, this upper bound of $\lambda_k(T_n)$ is best possible.

Theorem 1.3.8 Let T_n denote a tree on $n \ge 4$ vertices and F_n a forest on $n \ge 4$ vertices. Then each of the following holds.

(i) (Shao, [244]) $\lambda_k(T_n) \le \sqrt{\left[\frac{n}{k}\right] - 1}$, for each k with $2 \le \left[\frac{n}{2}\right]$. Moreover, when $n \not\equiv 1$

(mod k), this upper bound is best possible.

(ii) (Shao, [244]) When $n \equiv 1 \pmod{k}$, strict inequality holds in the upper bound in Theorem 1.3.8(i). However, there exists no $\epsilon > 0$ such that $\lambda_k(T_n) \le \sqrt{\left[\frac{n}{k}\right] - 1 - \epsilon}$, for each k with $\left[\frac{n}{2}\right]$.

(iii) (Shao, [244]) $\lambda_k(F_n) \le \sqrt{\left[\frac{n}{k}\right] - 1}$, for each k with $1 \le \left[\frac{n}{2}\right]$. This upper bound is best possible.

(iv) (Wu, [278]) Let G denote a unicyclic graph with $n \ge 4$ vertices, then $\lambda_k(G) \le \sqrt{\left[\frac{n}{k}\right] - \frac{3}{4} + \frac{1}{2}}$, for each k with $2 \le \left[\frac{n}{2}\right]$.

We shall present a proof for Theorem 1.3.7. Two lemmas are needed for this purpose. the proofs of these two lemmas are left as exercises (Exercises 1.14, 1.15 and 1.16).

Lemma 1.3.1 Let T denote a tree on $n \ge 3$ vertices. For any k with $2 \le k \le \left[\frac{n}{2}\right]$, there exists a vertex subset $V' \subseteq V(T)$ with $|V'| = k - 1$ such that each component of $T - V'$ has at most $\left[\frac{n-2}{k}\right] + 1$ vertices.

Lemma 1.3.2 Let G be a bipartite graph on $n \ge 2$ vertices. Then

$$\lambda_i(G) = -\lambda_{n+1-i}, \text{ for each } i \text{ with } 1 \le i \le \left[\frac{n}{2}\right].$$

Proof of Theorem 1.3.7 By Lemma 1.3.2, $\lambda_k(T) \ge 0$ for each i with $1 \le i \le \left[\frac{n}{2}\right]$.

By Lemma 1.3.1, $V(T)$ has a subset V' with $|V'| = k - 1$ such that each component of $T - V'$ has at most $\left[\frac{n-2}{k}\right] + 1$ vertices. By Theorem 1.2.6,

$$\lambda_k(T) \le \lambda_1(T - V') \le \sqrt{\left[\frac{n-2}{k}\right]}.$$

To prove the optimality of the inequality when $n \equiv 1 \pmod{k}$, write $n = lk + 1$, for some integer $l \ge 2$. Let $G_1, G_2, \cdots, G_k$ be disjoint graphs each isomorphic to $K_{1,l-1}$. Obtain T_n by adding a new vertex v to the disjoint union of the G_i's so that v is adjacent to the vertex of degree $l - 1$ in G_i, for all $i = 1, 2, \cdots, k$. It follows by Theorem 1.2.6 and by the fact that each component of $T_n - v$ is a $K_{1,l-1}$ and that T_n has k such components,

$$\lambda_1(T_n - v) = \lambda_1(K_{1,l-1}) = \lambda_k(T_n - v) = \lambda_k(K_{1,l-1}) = \sqrt{l-1} = \sqrt{\left[\frac{n-2}{k}\right]}.$$

By Theorem 1.2.6, $\lambda_1(T_n - v) \le \lambda_k(T_n) \le \lambda_k(T_n - v)$, and so $\lambda_k(T_n) = \sqrt{\left[\frac{n-2}{k}\right]}$. This completes the proof of Theorem 1.3.7. $\square$

As we know, an edge subset M of $E(G)$ is called a matching of G, if no two edges in M are adjacent in G. A matching with maximum number of edges is called a maximum matching and the size of it is called the edge independent number of G. The following theorem provide a sharp bound of the kth largest eigenvalue of a tree T with n vertices.

Theorem 1.3.9 (Chen [56]) Let T denote a tree on n vertices and edge independent number q. For $2 \leq k \leq q$, $\lambda_k(T) \geq \lambda_k(S^{2k-2}_{n-2k+2})$ with equality if and only if T is isomorphic to S^{2k-2}_{n-2k+2}, where S^{2k-2}_{n-2k+2} is a tree formed by making an edge from a one-degree vertex of the path P_{2k-2} to the center of the star $K_{1,n-2k+1}$.

1.4 Line Graphs and Total Graphs

Definition 1.4.1 The *line graph* $L(G)$ of a graph G is the graph whose vertices are the edges of G, where two vertices in $L(G)$ are adjacent if and only if the corresponding edges are adjacent in G.

Definition 1.4.2 The *total graph* $T(G)$ of a graph G is the graph whose vertex set is $E(G) \cup V(G)$, where two vertices in $T(G)$ are adjacent if and only if the corresponding elements are adjacent or incident in G.

Some elementary properties can be found in Exercise 1.20; and examples of line graphs and total graphs can be found in Exercise 1.21. Results in these exercises are needed in the proofs of the theorems in this section.

Theorem 1.4.1 Let G be a nontrivial graph. Each of the following holds.

(i) If λ be an eigenvalue of $L(G)$, then $\lambda \geq -2$.

(ii) If $q \geq n$, then $\lambda = -2$ is an eigenvalue of $L(G)$.

Proof Let λ be an eigenvalue of $L(G)$. By Exercise 1.20(ii), $\lambda + 2$ is an eigenvalue of B^TB. By Exercise 1.20(iv), B^TB is a semi-definite positive matrix, and so $\lambda + 2 \geq 0$.

When $q > n$, by Exercise 1.20(v), zero is an eigenvalue of B^TB, and so by Exercise 1.20(ii), $\lambda = -2$ is an eigenvalue of $L(G)$. $\square$

One of the main stream problems in this area is the relationships between $\chi_G(\lambda)$ and $\chi_{L(G)}(\lambda)$, and between $\chi_G(\lambda)$ and $\chi_{T(G)}(\lambda)$. The next two theorems by Sachs [227] and by Cvetković, Doob and Sachs [71] settled these problems for the regular graph cases.

Theorem 1.4.2 (Sachs, [227]) Let G be a k-regular graph on n vertices and q edges. Then

$$\chi_{L(G)}(\lambda) = (\lambda + 2)^{q-n}\chi_G(\lambda + 2 - k).$$

Proof Let $B = B(G)$ denote the incidence matrix of G. Define two $n + q$ by $n + q$ matrices as follows

$$U = \begin{pmatrix} \lambda I_n & -B \\ 0 & I_q \end{pmatrix}, \quad V = \begin{pmatrix} I_n & B \\ B^T & \lambda I_q \end{pmatrix}.$$

By $\det(UV) = \det(VU)$, we have

$$\lambda^q \det(\lambda I_n - BB^T) = \lambda^n \det(\lambda I_q - B^T B).$$

Note that G is k-regular, and so the matrix C in Exercise 1.28(i) is kI_n. These, together with Exercise 1.28(i) and (ii), yield

$$\begin{aligned}
\chi_{L(G)}(\lambda) &= \det(\lambda I_q - A(L(G))) \\
&= \det((\lambda+2)I_q - B^T B) \\
&= \lambda^{q-n}\det((\lambda+2)I_n - BB^T) \\
&= \lambda^{q-n}\det((\lambda+2-k)I_n - A(G)) \\
&= (\lambda+2)^{q-n}\chi_G(\lambda+2-k).
\end{aligned}$$

This proves Theorem 1.4.2. $\square$

Example 1.4.1 It follows from Theorem 1.4.2 that if

$$\operatorname{spec}(G) = \begin{pmatrix} k & \lambda_1 & \lambda_2 & \cdots & \lambda_s \\ 1 & m_1 & m_2 & \cdots & m_s \end{pmatrix},$$

then

$$\operatorname{spec}L(G) = \begin{pmatrix} 2k-1 & k-2+\lambda_1 & \cdots & k-2+\lambda_s & -2 \\ 1 & m_1 & \cdots & m_s & q-n \end{pmatrix}.$$

In particular,

$$\operatorname{spec}L(K_n) = \begin{pmatrix} 2n-4 & n-4 & -2 \\ 1 & n-11 & \frac{n(n-3)}{2} \end{pmatrix}.$$

Theorem 1.4.3 (Cretkovic, [71]) Let G be a k-regular graph with n vertices and q edges, and let the eigenvalues of G be $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. Then

$$\chi_{T(G)}(\lambda) = (\lambda+2)^{q-n}\prod_{i=1}^{n}(\lambda^2 - (2\lambda_i + k - 2)\lambda + \lambda_i^2 + (k-3)\lambda_i - k).$$

Sketch of Proof Let $A = A(G)$, $B = B(G)$ denote the adjacency matrix and the incidence matrix of G, respectively; and let $L = A(L(G))$ denote the adjacency matrix of $L(G)$. Then by the definitions and by the fact that G is k-regular,

$$BB^T = A + kI, B^T B = L + 2I, \text{ and } A(T(G)) = \begin{bmatrix} A & B \\ B^T & L \end{bmatrix}.$$

It follows that

$$\begin{aligned}
\chi_{T(G)}(\lambda) &= \begin{vmatrix} (\lambda+k)I - BB^T & -B \\ -B^T & (\lambda+2)I - B^TB \end{vmatrix} \\
&= \begin{vmatrix} (\lambda+k)I - BB^T & -B \\ -(\lambda+k+1)B^T + B^TBB^T & (\lambda+2)I \end{vmatrix} \\
&= (\lambda+2)^q \det\left(\lambda I - A + \frac{1}{\lambda+2}(A+kI)(A-(\lambda+1)I)\right) \\
&= (\lambda+2)^{q+n}\det(A^2 - (2\lambda - k + 3)A + (\lambda^2 - (k-2)\lambda - k)I) \\
&= (\lambda+2)^{q+n}\prod_{i=1}^{n}(\lambda_i^2 - (2\lambda - k+3)\lambda_i + \lambda^2 - (k-2)\lambda - k).
\end{aligned}$$

This proves the theorem.

In [235], the relationships between $\chi_G(\lambda)$ and $\chi_{L(G)}(\lambda)$, and between $\chi_G(\lambda)$ and $\chi_{T(G)}(\lambda)$ were listed as unsolved problems. These are solved by Lin and Zhang [165].

Definition 1.4.3 Let D be a digraph with n vertices and m arcs, such that D may have loops and parallel arcs. The entries of the *in-incidence matrix* $B_I = (b^i_{ij})$ of D are defined as follows:

$$b^i_{ij} = \begin{cases} 1 & \text{if vertex } v_i \text{ is the head of arc } e_j \\ 0 & \text{otherwise} \end{cases}$$

The entries of the *out-incidence matrix* $B_O = (b^o_{ij})$ of D are defined as follows:

$$b^o_{ij} = \begin{cases} 1 & \text{if vertex } v_i \text{ is the tail of arc } e_j \\ 0 & \text{otherwise} \end{cases}$$

Immediately from the definitions, we have

$$A(D) = B_O B_I^T, \text{ and } A(L(D)) = B_I^T B_O, \tag{1.5}$$

and

$$A(T(D)) = \begin{bmatrix} A(D) & B_O \\ B_I^T & A(L(D)) \end{bmatrix} \tag{1.6}$$

Theorem 1.4.4 (Lin and Zhang, [165]) Let D be a digraph with n vertices and m arcs, and let $A_L = A(L(D))$. Then

$$\chi_{L(D)}(\lambda) = \lambda^{m-n}\chi_D(\lambda). \tag{1.7}$$

Proof Let

$$U = \begin{bmatrix} \lambda I_n & -B_o \\ 0 & I_m \end{bmatrix} \text{ and } W = \begin{bmatrix} I_n & B_o \\ B_i^T & \lambda I_m \end{bmatrix}.$$

Note that $\det(UW) = \det(WU)$, and so

$$\lambda^m \det(\lambda I_n - B_o B_i^T) = \lambda^n \det(\lambda I_m - B_i^T B_i).$$

This, together with (1.5), implies that

$$\begin{aligned} \chi_{L(D)}(\lambda) &= \det(\lambda I_m - A_L) = \det(\lambda I_m - B_I^T B_O) \\ &= \lambda^{m-n} \det(\lambda I_n - B_O B_I^T) = \lambda^{m-n} \det(\lambda I_n - A(D)), \end{aligned}$$

and so (1.7) follows. $\square$

Theorem 1.4.5 Let D be a digraph with n vertices and m arcs, and let $A_L = A(L(D))$ and $A_T = A(T(D))$. Then

$$\chi_{T(D)}(\lambda) = \lambda^{m-n} \det((\lambda I_n - A)^2 - A). \tag{1.8}$$

Proof By (1.6),

$$\begin{aligned} \chi_{T(D)}(\lambda) &= \begin{vmatrix} \lambda I_n - A & -B_O \\ -B_I^T & \lambda I_m - A_L \end{vmatrix} = \begin{vmatrix} \lambda I_n - B_O B_I^T & -B_O \\ -B_I^T & \lambda I_m - B_I^T B_O \end{vmatrix} \\ &= \begin{vmatrix} \lambda I_n - B_O B_I^T & -B_O \\ -B_I^T - B_I^T(\lambda I_n - B_O B_I^T) & \lambda I_m \end{vmatrix} \\ &= \begin{vmatrix} \lambda I_n - B_O B_I^T + \frac{1}{\lambda} B_O[-(1+\lambda)B_I^T + B_I^T B_O B_I^T] & 0 \\ -(1+\lambda)B_I^T + B_I^T B_O B_I^T) & \lambda I_m \end{vmatrix} \\ &= \lambda^{m-n} \det(\lambda^2 I_n - \lambda B_O B_I^T - (1+\lambda) B_O B_I^T + B_O B_I^T B_O B_I^T) \\ &= \lambda^{m-n} \det(\lambda^2 I_n - \lambda A - n(1+\lambda)A + A^2), \end{aligned}$$

which implies (1.8). $\square$

Hoffman in [123] and [122] considered graphs each of whose eigenvalues is at least -2, and introduced the generalized line graphs. Results on generalized line graphs and related topics can be found in [123], [122] and [70].

1.5 Cospectral Graphs

Can the spectrum of a graph uniquely determine the graph? The answer is no in general as there are non isomorphic graphs which have the same spectrum. Such graphs are call *cospectral graphs*. Harary, King and Read [115] found that $K_{1,4}$ and $K_1 \cup C_4$ are the smallest pair of cospectral graphs. Hoffman [125] constructed cospectral regular bipartite graphs on 16 vertices. In fact, Hoffman (Theorem 1.5.1 below) found a construction for cospectral graphs of arbitrary size. The proof here is given by Mowshowitz [206].

Lemma 1.5.1 (Cvetković, [65]) Let G be a graph on n vertices with complement G^c, and let $H_G(t) = \sum_{k=0}^{\infty} N_k t^k$ be the generating function of the number N_k of walks of length k in G. Then

$$H_G(t) = \frac{1}{t}\left[(-1)^n \frac{\chi_{G^c}\left(-\frac{t+1}{t}\right)}{\chi_G\left(\frac{1}{t}\right)} - 1\right]. \tag{1.9}$$

Sketch of Proof For an n by n matrix $M = (m_{ij})$, let M^* denote the matrix $(\det(M)M^{-1})^T$ (if M^{-1} exists), and let $\|M\| = \sum_{i=1}^{n}\sum_{j=1}^{n} m_{ij}$. It is routine to verify that, for any real number x, $\det(M + xJ_n) = \det(M) + x\|M^*\|$

Let $A = A(G)$. By Proposition 1.1.2(vii), $N_k = \|A^k\|$. Note that when $t < \max\{\lambda_i^{-1}\}$,

$$\sum_{k=0}^{\infty} A^k t^k = (I - tA)^{-1} = (\det(I - tA))^{-1}(I - tA)^*,$$

and so

$$H_G(t) = \sum_{k=0}^{\infty} N_k t^k = \sum_{k=0}^{\infty} \|A^k\| t^k = \frac{\|(I - tA)^*\|}{\det(I - tA)}. \tag{1.10}$$

But

$$\|(I - tA)^*\| = \frac{1}{t}\left(\det(I - tA + \frac{1}{t}J) - \det(I - tA)\right). \tag{1.11}$$

By $A(G^c) = J - I - A$, (1.10) and (1.11), (1.9) obtains. $\square$

Lemma 1.5.2 Let G be an r-regular graph on n vertices and G^c its complement. Then

$$\chi_{G^c}(\lambda) = (-1)^n \left(\frac{\lambda + r + 1 - n}{\lambda + r + 1}\right) \chi_G(-\lambda - 1).$$

Sketch of Proof Since each walk of length k can start from any of the n vertices and each walk from a given vertex can have r different ways to continue, $N_k = nr^k$. It follows by Lemma 1.5.1 that (when $|t| < 1/r$),

$$\frac{1}{t}\left[(-1)^n \frac{\chi_{G^c}\left(-\frac{t+1}{t}\right)}{\chi_G\left(\frac{1}{t}\right)} - 1\right] = H_G(t) = \sum_{k=0}^{\infty} nr^k t^k = \frac{n}{1 - rt}.$$

Let $\lambda = -(t+1)/t$ to conclude.

$$\chi_{G^c}(\lambda) = (-1)^n \left(\frac{\lambda + r + 1 - n}{\lambda + r + 1}\right) \chi_G(-\lambda - 1).$$

Theorem 1.5.1 For any integer k, there exist k cospectral graphs that are connected and regular.

Proof Let H_1 and H_2 be two r-regular cospectral graphs. For each i with $0 \leq i \leq k-1$, let L_i be a graph with $k-1$ components such that i of the $k-1$ components of L_i are isomorphic to H_1, and such that the other $k-1-i$ components of L_i are isomorphic to H_2; and let $G_i = L_i^c$.

Clearly H_i's are connected, regular and mutually nonisomorphic. By Theorem 1.2.1(i), for each i with $1 \leq i \leq k-1$, $\chi_{L_i}(\lambda) = \chi_{L_0}(\lambda)$. This, together with Lemma 1.5.2, implies that $\chi_{G_i}(\lambda) = \chi_{G_0}(\lambda)$, for each i with $1 \leq i \leq k-1$. □

Schwenk [232] introduced cospectrally rooted graphs. A graph G with a distinguished vertex v is called a *rooted graph* with *root* v. Two different rooted graphs G_1 and G_2 with roots v_1 and v_2, respectively, are *cospectrally rooted* if both $\chi_{G_1}(\lambda) = \chi_{G_2}(\lambda)$ and $\chi_{G_1-v_1}(\lambda) = \chi_{G_2-v_2}(\lambda)$ hold. Graph pairs that are cospectrally rooted are characterized by Herndon and Ellzey in [120]. We now turn to the construction techniques of cospectral graphs.

Theorem 1.5.2 Let G_1 and G_2 be two graphs on n vertices with $A_i = A(G_i)$, $i = 1, 2$. The following are equivalent.

(i) $\chi_{G_1}(\lambda) = \chi_{G_2}(\lambda)$.

(ii) $\text{tr}(A_1^j) = \text{tr}(A_2^j)$, for $j = 1, 2, \cdots, n$.

(iii) There is a one-to-one correspondence between the collection of closed walks of length j in G_1 and that in G_2, for each $j = 1, 2, \cdots, n$.

Sketch of Proof Let $A_1 = A(G_1)$ and $A_2 = A(G_2)$. Note that $\text{tr}(A_1^k) = \sum_{i=1}^n \lambda_i(A_1)^k$, and so $\{\lambda_1(A_1), \cdots, \lambda_n(A_1)\}$ and $\{\text{tr}(A_1), \cdots, \text{tr}A_1^n\}$ uniquely determine each other. The same holds for A_2. This establishes the equivalence between (i) and (ii).

By Proposition 1.1.2(vii) and by Exercise 1.5, (ii) and (iii) are equivalent. □

Definition 1.5.1 (Generalized Composition, [234]) Let G_1, G_2 be two graphs with given distinct subsets $S_1 \subseteq V(G_1)$ and $S_2 \subseteq V(G_2)$, and let B be a bipartite graph with partite sets $R = \{u_1, u_2, \cdots, u_r\}$ and $S = \{w_1, w_2, \cdots, w_s\}$. The *generalized composition* $B(G_1, S_1, G_2, S_2)$ is formed by taking r copies of G_1 and s copies of G_2, every vertex in the ith G_1 is adjacent to every vertex in the jth G_2 if and only if $u_i w_j \in E(B)$.

Theorem 1.5.3 (Schwenk, Herndon and Ellzey [234]) Let $B(G_1, S_1, G_2, S_2)$ and $B(G_2, S_2, G_1, S_1)$ be defined in Definition 1.5.1. Then they have the same spectrum if one of the following holds.

(i) $r = s$.

(ii) $\chi_{G_1}(\lambda) = \chi_{G_2}(\lambda)$.

Sketch of Proof By Theorem 1.5.2, it suffices to find a one-to-one correspondence ϕ between the collection of closed walks of length j in G_1 and that in G_2, for each $j = 1, 2, \cdots, n$.

Assume first that $r = s$. Let W be a closed walk in $B(G_1, S_1, G_2, S_2)$. If W is inside an H_i, then since $r = s$, we can choose $\phi(W)$ to be the mirror image of W inside an L_i of $B(G_2, S_2, G_1, S_1)$. If W uses edges between the H_i's and the L_j's, then again by $r = s$, we can let $\phi(W)$ to be the mirror image of W in $B(G_2, S_2, G_1, S_1)$.

Now assume that $\chi_{G_1}(\lambda) = \chi_{G_2}(\lambda)$. By Theorem 1.5.2, there is a one-to-one correspondence between the closed walks in each H_i (L_j, respectively) in $B(G_1, S_1, G_2, S_2)$ and the closed walks in each H_i (L_j, respectively) in $B(G_2, S_2, G_1, S_1)$. Closed walks using edges between the H_i's and the L_j's in $B(G_1, S_1, G_2, S_2)$ can correspond to closed walks between the same H_i's and the same L_j's in $B(G_2, S_2, G_1, S_1)$. $\square$

Definition 1.5.2 (Seidel, [236], Seidel Switching) Let G be graph and let $\pi = \{T_1, T_2\}$ is a partition of $V(G)$. Construct a graph G^π as follows: $V(G^\pi) = V(G)$ and two vertices $u, v \in V(G^\pi)$ are adjacent in $G(S_1)$ if and only if either $u, v \in S_1$ and $uv \in E(G)$, or $u, v \in S_2$ and $uv \in E(G)$, or $u \in S_1$, $v \in S_2$ and $uv \notin E(G)$.

Under certain conditions, the Seifel Switching produces graphs with the same spectrum (see [236]). An extension of the Seidel Switching is the Local Switching.

Definition 1.5.3 (Godsil and McKay, [101], Local Switching) Let G be graph and let $\pi = (S_1, S_2, \cdots, S_k, S)$ is a partition of $V(G)$. Suppose that, whenever $1 \le i, j \le k$ and $v \in S$, we have

(A) any two vertices in S_i have the same number of neighbors in S_j, and

(B) v has either 0, $\frac{n_i}{2}$ or n_i neighbors in S_i, where $n_i = |S_i|$.

The graph G^π formed by *local switching in G with respect to π* is obtained from G as follows. For each $v \in S$ and $1 \le i \le k$ such that v has $\frac{n_i}{2}$ neighbors in S_i, delete these $\frac{n_i}{2}$ edges and join v instead to the other $\frac{n_i}{2}$ vertices in S_i.

Theorem 1.5.4 (Godsil and Mckay, [101]) Let G be a graph and let π be and a partition of $V(G)$ satisfying Definition 1.5.3(A) and (B). Then G and G^π are cospectral, with cospectral complements.

Sketch of Proof Let $s_0 = 0$, $s_i = |S_i|$, where $1 \le i \le k$; and let $S_{k+1} = S$ and $s_{k+1} = |S|$. Then $n = |V(G)| = \sum_{i=1}^{k+1} s_i$. Label the vertices in $V(G)$ (and in $V(G^\pi)$ also) as $V(G) = \{v_1, v_2, \cdots, v_{s_1}, v_{s_1+1}, \cdots, v_n\}$ such that for each i with $1 \le i \le k+1$, $v_j \in S_i$, for $\sum_{t=0}^{j-1} s_t + 1 \le j \le \sum_{t=0}^{j} s_t$. Let $A = A(G)$ and $A' = A(G^\pi)$ be the adjacency matrices whose columns and rows are labeled with the labeling above.

We shall find a matrix Q such that $QAQ^{-1} = A'$ to prove Theorem 1.5.4(i). The Proof for Theorem 1.5.4(ii) is similar and is left as an exercise.

Recall that J_m denotes the m by m square matrix each of whose entries is a one. Define

$$Q_m = \frac{2}{m} J_m - I_m.$$

Note that $Q_m^2 = I_m$. Define $Q = \mathrm{diag}(Q_{s_1}, Q_{s_2}, \cdots, Q_{s_k}, I_{s_{k+1}})$ to be the n by n matrix whose diagonal blocks are $Q_{s_1}, \cdots, Q_{s_k}$ and $I_{s_{k+1}}$. It is routine to verify that $QAQ^{-1} = A'$, and so this proves Theorem 1.5.4(i). □

Theorem 1.5.5 (Schwenk, Herndon, and Ellzey [234]) If G_1 is a regular graph and if $|V(G_1)| = 2|T_1|$, then $B(G_1, T_1, G_2, T_2)$ and $B(G_1, V(G_1) - T_1, G_2, T_2)$ have the same spectrum.

Sketch of Proof Use the notation in Definition 1.5.3 with S_1, S_2 in Definition 1.5.3 replaced by T_1, T_2 here in Theorem 1.5.6. Let $k = |T_1|$ and $s = |T_2|$. Then we can see that Theorem 1.5.6 follows from Theorem 1.5.5(i) by taking $\pi = \{S_1, S_2, \cdots, S_k, S\}$ in which $S_i = V(H_i)$, $1 \le i \le k$, and in which $S = \cup_{i=1}^{s} V(L_i)$. □

Codsil and Mckay also proved that graphs with the same spectrum may be constructed by taking graph products under certain conditions. We present their results below and refer the readers to [101] for proofs.

Definition 1.5.4 Let $A = (a_{ij})_{m \times n}$ and $B = (b_{ij})_{p \times q}$ be two matrices. The *Kronecker product* (also called the *tensor product*) of A and B is an $mp \times nq$ matrix $A \otimes B$, obtained from A by replacing each entry a_{ij} by a matrix $a_{ij}B$.

Theorem 1.5.6 (Godsil and Mckay, [101]) If $\chi_{G_1}(\lambda) = \chi_{G_2}(\lambda)$, and if X and H are square matrices with the same dimension, then

$$\chi_{H \otimes I + X \otimes G_1}(\lambda) = \chi_{H \otimes I + X \otimes G_2}(\lambda).$$

Theorem 1.5.7 (Godsil and Mckay, [101]) If $\chi_{G_1}(\lambda) = \chi_{G_2}(\lambda)$, if $\chi_{G_1^c}(\lambda) = \chi_{G_2^c}(\lambda)$, and if C, D, E and F are square matrices with the same dimension, then

$$\chi_{C \otimes I + D \otimes J + E \otimes G_1 + F \otimes G_1^c}(\lambda) = \chi_{C \otimes I + D \otimes J + E \otimes G_2 + F \otimes G_2^c}(\lambda)$$

It seems very difficult to determine if two given graphs have the same spectrum. Also, very little is known about what classes of graphs which can be uniquely determined by the spectrum. All these remain to be further investigated.

1.6 Spectral Radius

The spectral radius of a graph is defined in Definition 1.3.1. For a square matrix M, we can similarly define the *spectral radius* of M, $\rho(M)$, to be the maximum absolute value of an eigenvalue of M.

When $M = A(G)$ is the adjacency matrix of a graph G, $\rho(G) = \rho(A(G))$, by Theorem 1.3.1(i). Theorem 1.6.1 below summarizes some properties of non-negative matrices and the associated digraph of a matrix in $\mathbf{B}_n$.

Theorem 1.6.1 (Perron-Frobenius, [96]) Let $B \in \mathbf{B}_n$ be a matrix. Then each of the following holds.

(i) Let $\mathbf{u}$ be an eigenvector of B corresponding to $\rho(B)$. Then $\mathbf{u} \geq 0$. Moreover, if $D(B)$, the associated digraph of B, is strongly connected, then $\mathbf{u}$ is a positive vector.

(ii) Let $r_1, r_2, \cdots, r_n$ denote the row sums of B, then

$$\min\{r_1, r_2, \cdots, r_n\} \leq \rho(B) \leq \max\{r_1, r_2, \cdots, r_n\}.$$

(iii) Let $\mathbf{z}$ be a positive vector. If $B\mathbf{z} \geq r\mathbf{z}$ (or $B\mathbf{z} \leq r\mathbf{z}$, respectively) then $\rho(B) \geq r$ (or $\rho(B) \leq r$, respectively). Moreover, $\rho(H) = r$ holds if and only if both $B\mathbf{z} = r\mathbf{z}$ and $D(G)$ is strongly connected.

Definition 1.6.1 Let n and e be integers. Let $\Phi(n,e)$ denote the collection of all n by n (0,1) matrices with trace equal to zero and with exactly e entries above the main diagonal being one. Let $\Phi^*(n,e)$ denote the subcollection of $\Phi(n,e)$ such that $A = (a_{ij}) \in \Phi^*(n,e)$ if and only if $a_{kl} = 1$ for all $k < l$, $k < i$ and $l < j$ whenever $i < j$ and $a_{ij} = 1$. Define

$$\begin{aligned} f(n,e) &= \max\{\rho(A) : A \in \Phi(n,e)\} \text{ and} \\ f^*(n,e) &= \max\{\rho(A) : A \in \Phi^*(n,e)\}. \end{aligned}$$

Lemmas 1.6.1 and 1.6.2 follow immediately from Definition 1.6.1, and from linear algebra, respectively.

Lemma 1.6.1 Let $A \in \Phi(n,e)$, and let P be a permutation matrix. Then $P^TAP \in \Phi(n,e)$ and $\rho(A) = \rho(P^TAP)$.

Lemma 1.6.2 Let A and B be two square nonnegative matrices with the same dimension. If, for some nonnegative vector $\mathbf{x}$ with $|\mathbf{x}| = 1$, $B\mathbf{x} = \rho(\mathbf{x})$ and $\mathbf{x}^TA\mathbf{x} \leq \mathbf{x}^TB\mathbf{x}$, then $\rho(A) \leq \rho(B)$.

Lemma 1.6.3 Let $A' = J_s - I_s$, $\mathbf{e}_1 = (1, 0, \cdots, 0)^T$ and let

$$A'' = \begin{pmatrix} A' & \mathbf{e}_1 \\ \mathbf{e}_1^T & 0 \end{pmatrix}.$$

Then $\rho(A'') > \rho(A')$.

Proof This follows from direct computation. $\square$

Theorem 1.6.2 (Brualdi and Hoffman, [30]) For each $A \in \Phi(n,e)$, $\rho(A) \leq f^*(n,e)$, where equality holds if and only if there exists a permutation matrix P such that $PAP^{-1} \in \Phi^*(n,e)$.

Proof Choose $A \in \Phi(n,e)$ such that $\rho(A) = f(n,e)$. By Theorem 1.6.1(i), there exists a non negative vector x such that $\rho(A)\mathbf{x} = A\mathbf{x}$ and $|\mathbf{x}| = 1$. By Lemma 1.6.1, we may assume that $\mathbf{x} = (x_1, x_2, \cdots, x_n)^T$ such that $x_1 \geq x_2 \geq \cdots x_n \geq 0$. Argue by contradiction, we assume that $A \notin \Phi^*(n,e)$ and consider these two cases.

Case 1 There exist integers p and q with $p < q$, $a_{pq} = 0$ and $a_{p,q+1} = 1$.

Let B be obtained from A by permuting a_{pq} and $a_{p,q+1}$ and $a_{q+1,p}$ and a_{qp}. Note that $B - A$ has only four entries that are not equal to zero: the (p,q) and the (q,p) entries are both equal to 1, and the $(p,q+1)$ and the $(q+1,p)$ entries are both equal to -1. It follows by $x_q \geq x_{q+1}$ that

$$\begin{aligned} \mathbf{x}^T B\mathbf{x} - \mathbf{x}^T A\mathbf{x} &= \mathbf{x}^T (B - A)\mathbf{x} \\ &= (\cdots, x_q - x_{q+1}, \cdots, x_p, -x_p, \cdots)\mathbf{x} \\ &= 2x_p(x_q - x_{q+1}) \geq 0. \end{aligned}$$

Since B is a real symmetric matrix and by $|\mathbf{x}| = 1$, $\rho(B) = \mathbf{x}^T B\mathbf{x}$. By $A\mathbf{x} = \rho(A)\mathbf{x}$ and by $|\mathbf{x}| = 1$, we have $\rho(B) - \rho(A) \geq 2x_p(x_q - x_{q+1}) \geq 0$. As $\rho(A) = f(n,e)$, and as $B \in \Phi(n,e)$, it must be $x_p(x_q - x_{q+1}) = 0$, and so $\rho(B) = \mathbf{x}^T B\mathbf{x} = \rho(A)$.

If $x_p \neq 0$, then $x_q = x_{q+1}$, and so $\mathbf{x}^T B\mathbf{x} - \mathbf{x}^T A\mathbf{x} = 0$. It follows that

$$(A\mathbf{x})_q = (B\mathbf{x})_q = (A\mathbf{x})_q + (\mathbf{x})_q,$$

and so $x_q = 0$, contrary to the assumption that $x_q > 0$.

Therefore, $x_q = 0$, and so for some $s \leq p - 1$, $x_s > 0 = x_{s+1} = \cdots = x_n$. Since $\rho(A) = f(n,e)$, A has an s by s principal submatrix $A' = J - I$ at the upper left corner and $\rho(A) = \rho(A')$. But then, by Lemma 1.6.3, $f(n,e) \geq \rho(A'') > \rho(A') = \rho(A)$, contrary to the assumption that $\rho(A) = f(n,e)$.

This completes the proof for Case 1.

Case 2 Case 1 does not hold. Then there exist p and q such that $a_{pq} = 0$ and $a_{p+1,q} = 1$.

Let B be obtained from A by permuting a_{pq} and $a_{p+1,q}$ and a_{qp} and $a_{q,p+1}$. As in Case 1, we have

$$\mathbf{x}^T B\mathbf{x} - \mathbf{x}^T A\mathbf{x} = 2x_q(x_p - x_{p+1}),$$

and so it follows that $B\mathbf{x} = \rho(A)\mathbf{x}$ and $x_q(x_p - x_{p+1}) = 0$.

We may assume that $x_p = x_{p+1}$. Then $(A\mathbf{x})_p = \rho(A)x_p = (B\mathbf{x})_p = (A\mathbf{x})_p + x_q$, which forces that $x_q = \cdots = x_n = 0$. Therefore, a contradiction will be obtain as in Case 1, by applying Lemma 1.6.3. $\square$

Lemma 1.6.4 For integers $n \geq r > 0$, let F_{11} be the r by r matrix whose (1,1)-entry is $r - 1$ and each of whose other entries is zero; let $D = (d_{ij})_{n \times n}$ be a nonnegative matrix such that the left upper corner $r\times r$ principal square submatrix is F_{11}; and let $\alpha_i = \sqrt{\sum_{j=1}^n d_{ij}^2}$, where $i=r+1,\cdots,n$. Let F_{12} be the r by $n-r$ matrix whose j-th column vector has α_{r+j} in the first component and zero in all other components, $1\le j\le n-r$. Define

$$F=\begin{bmatrix} F_{11} & F_{12} \\ F_{12}^T & 0 \end{bmatrix}.$$

Then $\rho(F)\ge\rho(D)$.

Proof Let $\mathbf{x}=(x_1,\cdots,x_n)^T$ be such that $|\mathbf{x}|=1$ and such that $\rho(A)\mathbf{x}=A\mathbf{x}$. Let $\mathbf{y}=(\sqrt{x_1^2+x_2^2+\cdots+x_r^2},0,\cdots,0,x_{r+1},\cdots,x_n)^T$. Then $|\mathbf{y}|=1$, and

$$\begin{aligned}
\mathbf{x}^T D\mathbf{x} &= (r-1)x_1^2+2\sum_{j=r+1}^{n}\sum_{i=1}^{r}x_i d_{ij}x_j \\
&= (r-1)x_1^2+2\sum_{j=r+1}^{n}x_j\left(\sum_{i=1}^{r}x_i d_{ij}\right) \\
&\le (r-1)x_1^2+2\sum_{j=r+1}^{n}x_j\left(\sqrt{\sum_{i=1}^{r}x_i^2}\sqrt{\sum_{i=1}^{r}d_{ij}^2}\right) \\
&\le (r-1)x_1^2+2\sum_{j=r+1}^{n}x_j\sqrt{\sum_{i=1}^{r}x_i^2\alpha_j} \\
&\le (r-1)\sum_{i=1}^{r}x_i^2+2\sum_{j=r+1}^{n}x_j\sqrt{\sum_{i=1}^{r}x_i^2\alpha_j} \\
&= \mathbf{y}^T F\mathbf{y}
\end{aligned}$$

and so Lemma 1.6.4 follows from the Rayleigh Principle (see Theorem 6.1.3 in the Appendix). □

Lemma 1.6.5 Let F be the matrix defined in Lemma 1.6.4. Each of the following holds.

(i) $\rho(F)$ is the larger root of the equation

$$x^2-(r-1)x-\sum_{i=r+1}^{n}\alpha_i^2=0.$$

(ii) $\rho(F)\le k-1$.

(iii) $\rho(F)=k-1$ is and only if $r=k-1$.

Sketch of Proof By directly computing $\chi_F(\lambda)$, we obtain (i).

By the definition of the α_i's, we have

$$\sum_{i=r+1}^{n}\alpha_i^2=e-\binom{r}{2}.$$

It follows by (i) and by $r \le k-1$ that

$$\rho(F) = \frac{r-1+\sqrt{2k^2-r^2-2k+1}}{2} = \frac{r-1+\sqrt{(k-1)^2+k^2-r^2}}{2} \le k-1.$$

By algebraic manipulation, we have $\rho(F) = k-1$ if and only if $r = k-1$, and so (iii) follows. $\square$

Theorem 1.6.3 (Brualdi and Hoffman, [30]) Let $k > 0$ be an integer with $e = \binom{k}{2}$. Then $f(n,e) = k-1$. Moreover, a matrix $A \in \Phi(n,e)$ satisfies $\rho(A) = k-1$ if and only if A is permutation similar to

$$\begin{pmatrix} J_k^0 & 0 \\ 0 & 0 \end{pmatrix}. \tag{1.12}$$

Proof Let $A = (a_{ij}) \in \Phi^*(n,e)$. By Theorem 1.6.2, it suffices to show that $\rho(A) \le k-1$, and that $\rho(A) = k-1$ if and only if A is similar to the matrix in the theorem.

Since $A \in \Phi^*(n,e)$, we may assume that

$$A = \begin{pmatrix} J_r^0 & A_1 \\ A_1^T & 0 \end{pmatrix},$$

where $r < k-1$ and where all the entries in the first column of A_1 are 1's.

Since J_r^0 is symmetric, there is an orthonormal matrix U such that $U^T J_r^0 U$ is a diagonalized matrix. Let V be the direct sum of U and I_{n-r}, and let R be the r by r matrix whose $(1,1)$ entry is an r and whose other entries are zero. Then

$$\begin{aligned} B &= VAV^T = \begin{pmatrix} U & 0 \\ 0^T & I_{n-r} \end{pmatrix} \begin{pmatrix} J_r^0 & A_1 \\ A_1^T & 0 \end{pmatrix} \begin{pmatrix} U^T & 0 \\ 0^T & I_{n-r} \end{pmatrix} \\ &= \begin{pmatrix} R - I_r & UA_1 \\ (UA_1)^T & 0 \end{pmatrix}. \end{aligned}$$

Obtain a new n by n matrix C from B by changing all the -1's in the main diagonal of B to zeros. Note that for each nonnegative vector $\mathbf{x} = (x_1, \cdots, x_n)^T$, $\mathbf{x}^T B \mathbf{x} = \mathbf{x}^T C \mathbf{x} - \sum_{i=1}^{n-1} x_i^2 \le \mathbf{x}^T C \mathbf{x}$. By Lemma 1.6.2, $\rho(B) \le \rho(C)$.

Obtain a new matrix $D = (d_{ij})$ from C by changing every entry in UA_1 and in $(UA_1)^T$ into its absolute value. Then $\rho(C) \le \rho(D)$. Since D is nonnegative, $\rho(D)$ has a nonnegative eigenvector $\mathbf{x} = (x_1, \cdots, x_n)^T$ such that $|\mathbf{x}| = 1$. Let $\alpha_i = \sqrt{\sum_{j=1}^n d_{ij}^2}$, where $i = r+1, \cdots, n$. and let F_{11} be the r by r matrix whose (1,1)-entry is $r-1$ and each of whose other entries is zero; F_{12} the r by $n-r$ matrix whose j-th column vector has α_{r+j}

in the first component and zero in all other components, $1 \le j \le n-r$. Define

$$F = \begin{bmatrix} F_{11} & F_{12} \\ F_{12}^T & 0 \end{bmatrix}.$$

By Lemma 1.6.4, $\rho(F) \ge \rho(D) \ge \rho(A)$. By Lemma 1.6.5, $f(n,e) = \rho(A) \le \rho(F) \le k-1$. When $\rho(A) = k-1$, by Lemma 1.6.5(iii) and by $r \le k-1$, we must have $r = k-1$ and so by $e = k(k+1)/2$, A must be similar to the matrix in (1.12). □

Theorem 1.6.4 (Stanley [256]) For any $A \in \Phi(n,e)$,

$$\rho(A) \le \frac{-1+\sqrt{1+8e}}{2},$$

and equality holds if and only if $e = \binom{k}{2}$ and A is permutation similar to

$$\begin{pmatrix} J_k^0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Proof Let $A = (a_{ij})$, let r_i be the ith row sum and let $\mathbf{x} = (x_1, \cdots, x_n)^T$ be a unit eigenvector of A corresponding to $\rho(A)$. Since $Ax = \rho(A)\mathbf{x}$, we have $\rho(A)x_i = \sum_j a_{ij}x_j$. Hence, by Cauchy-Schwarz inequality,

$$\rho(A)^2 = (\sum_j a_{ij}x_j)^2 \le r_i \sum_j a_{ij}x_j^2 \le r_i(1-x_i)^2.$$

Sum up over i to get

$$\begin{aligned} \rho(A)^2 &\le 2e - \sum_i r_i x_i^2 = 2e - \sum_{i,j} a_{ij}x_i^2 \\ &= 2e - \sum_{i<j} a_{ij}(x_i^2 + x_j^2) \le 2e - \sum_{i<j} 2a_{ij}x_i x_j \\ &= 2e - \sum_{i,j} x_i a_{ij} x_j \\ &= 2e - \mathbf{x}^T A x = 2e - \rho(A), \end{aligned} \tag{1.13}$$

which implies the inequality of the theorem.

In order for the equality to hold in the theorem, all inequalities in the (1.13) must be equalities. In particular, we have

$$a_{ij}(x_i^2 + x_j^2) = 2a_{ij}x_i x_j$$

for all $i < j$. Hence either $a_{ij} = 0$ or $x_i = x_j$. Therefore, there exists a permutation matrix P such that

$$P\mathbf{x} = (y_1, y_1, \cdots, y_1, y_2, y_2, \cdots, y_2, \cdots, y_j, y_j, \cdots, y_j)$$

where $y_1, y_2, \cdots, y_j$ are distinct. It follows that

$$PAP^T = \begin{pmatrix} A_1 & & & 0 \\ & A_2 & & \\ & & \ddots & \\ 0 & & & A_j \end{pmatrix},$$

where each A_i has an eigenvector $(1, 1, \cdots, 1)^t$. Hence each A_i has equal row sums, so $\rho(A)$ is the maximum row sum of A. So $\sqrt{1+8e}$ is an integer, and $e = \binom{k}{2}$. Then $\rho(A) = k-1$, and it follows that there is one nonzero block $A_1 = J_k^0$. This completes the proof. $\square$

Let $M = (m_{ij}) \in M_n$ and let r_i and s_i be the ith row sum and the ith column sum of M, respectively, for each $i = 1, 2, \cdots, n$. A digraph D is *r-regular* if for some $M \in M_n$, $D = D(M)$ such that $r_i = s_i = r$ for $i = 1, 2, ...n$.

Theorem 1.6.5 (Gregory, Shen and Liu [100]) Let $M = (m_{ij}) \in M_n$ with trace 0 and let r_i and s_i be ith row sum and the ith column sum of M, respectively, for each $i = 1, 2, \cdots, n$. Let $r = \min_{1\le i\le n} r_i$, $S = \max_{1\le i\le n} s_i$, $m = \sum_{1\le i\le n} r_i$ and $-k = \min_{1\le i\le n}(r_i - s_i)$. If $r \ge 0$, then

$$\rho\left(\begin{bmatrix} 0 & M \\ M^t & 0 \end{bmatrix}\right) \le \sqrt{m - r(n-1) + (r-1)S + k}.$$

Moreover, if $\begin{bmatrix} 0 & M \\ M^t & 0 \end{bmatrix}$ has a positive eigenvector corresponding to its spectral radius, the equality holds if and only if the adjacency digraph of M is either an r-regular digraph or the undirected star $K_{1,n-1}$.

Proof Let ρ denote $\rho\left(\begin{bmatrix} 0 & M \\ M^t & 0 \end{bmatrix}\right)$. Let $\begin{bmatrix} 0 & M \\ M^t & 0 \end{bmatrix}\begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix} = \rho\begin{bmatrix} x \\ y \end{bmatrix}$ for some non-zero vector $\begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix}$. Then

$$M\mathbf{y} = \rho\mathbf{x}. \tag{1.14}$$

Let $\mathbf{x} = (x_1 \cdots x_i \cdots x_n)^T$ and $\mathbf{y} = (y_1 \cdots y_i \cdots y_n)^T$. Since $\rho > 0$, by Exercise 1.21, we may assume that $\sum_{1\le i\le n} x_i^2 = \sum_{1\le i\le n} y_i^2 = 1$. by (1.14) and Cauchy-Schwarz inequality, for each $i = 1, 2, \cdots, n$,

$$\begin{aligned} \rho^2 x_i^2 &= (\sum_i m_{ij} y_j)^2 = (\sum_{j:m_{ij} \neq 0} m_{ij} y_j)^2 \\ &\leq \sum_{j:m_{ij} \neq 0} m_{ij}^2 \sum_{j:m_{ij} \neq 0} y_j^2 \\ &= r_i (1 - \sum_{j:m_{ij}=0} y_j^2). \end{aligned}$$

Sum up the inequalities above over i to obtain

$$\rho^2 = \sum_i \rho^2 x_i^2 \leq \sum_i r_i (1 - \sum_{j:m_{ij}=0} y_j^2) = m - \sum_i r_i \sum_{j:m_{ij}=0} y_j^2 \tag{1.15}$$

Now

$$\begin{aligned} \sum_i \sum_{j:m_{ij}=0} y_j^2 &= \sum_i r_i y_i^2 + \sum_i r_i \sum_{j:j \neq i, m_{ij}=0} y_j^2 \\ &\geq \sum_i r_i y_i^2 + r \sum_i \sum_{j:j \neq i, m_{ij}=0} y_j^2 \\ &= \sum_i r_i y_i^2 + r \sum_i (n - 1 - s_i) y_i^2 \\ &= \sum_i (r_i - s - i) y_i^2 - (r-1) \sum_i s_i y_i^2 + r(n-1) \\ &\geq -k \sum_i y_i^2 - (r-1) S \sum_i y_i^2 + r(n-1) \\ &= -k - (r-1)S + r(n-1). \end{aligned}$$

Therefore, by (1.15), $\rho \leq \sqrt{m - r(n-1) + (r-1)S + k}$.

In order for equality to hold , all equalities in the above argument must be equalities. In particular, for each $i = 1, 2, \cdots, n$,

$$r_i \sum_{j:j \neq i, m_{ij}=0} y_j^2 = r \sum_{j:j \neq i, m_{ij}=0} y_j^2,$$

$$(r_i - s_i) y_i^2 = -k y_i^2,$$

and

$$(r-1) s_i y_i^2 = (r-1) S y_i^2.$$

Suppose $y_i > 0$. Then $r_i - s_i = -k$, $(r-1)s_i = (r-1)S$, and r_i equals either r or $n-1$. Since $\sum_i r_i = \sum_i s_i$, we have $r_i = s_i$ for all i. These implies that the adjacency digraph of M is either an r-regular digraph or the undirected star $K_{1,n-1}$.

On the other hand, it is easy to verify that

$$\rho = \sqrt{m - r(n-1) + (r-1)S + k},$$

if the adjacency digraph of M is either an r-regular digraph or the undirected star $K_{1,n-1}$.
$\square$

Corollary 1.6.5A (Liu [172]) Suppose M is an $n \times n$ matrix with trace 0. Let $r = \min_{1\le i\le n} r_i$, $S = \max_{1\le i\le n} s_i$, $m = \sum_{1\le i\le n} r_i$ and $-k = \min_{1\le i\le n}(r_i - s_i)$. If $r \geq 1$, then

$$\rho(M) \le \sqrt{m - r(n-1) + (r-1)S + k}.$$

Moreover, if M is irreducible then equality holds if and only if the $D(M)$, the associate digraph of M, is either an r-regular digraph or the undirected star $K_{1,n-1}$.

Proof The inequality follows from Exercise 1.20 and Theorem 1.6.5 immediately. Now suppose that equality holds and that M is irreducible. Let $\mathbf{x}$ be an eigenvector of M corresponding to $\rho(M)$. Then $\mathbf{x}$ is a positive vector by Theorem 1.6.1. By Exercise 1.20 $\rho(M) = \rho\left(\begin{bmatrix} 0 & M \\ M^t & 0 \end{bmatrix}\right)$ and $\begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix}$ is a positive eigenvector of $\begin{bmatrix} 0 & M \\ M^t & 0 \end{bmatrix}$ corresponding to $\rho(M)$. By Theorem 1.6.4, the adjacency digraph of M is either an r-regular digraph or the undirected star $K_{1,n-1}$.

On the other hand, if $D(M)$ is either an r-regular digraph or the undirected star $K_{1,n-1}$, then it is routine to verify that $\rho(M) = \sqrt{m - r(n-1) + (r-1)S + k}$. $\square$

Corollary 1.6.5B (Hong, [132]) Suppose M is an $n \times n$ irreducible matrix with trace 0 and $r_i = s_i$ for $i = 1, 2, \cdots, n$. Let $m = \sum_{1\le i\le n} r_i$. Then

$$\rho(M) \le \sqrt{m - n + 1}$$

with equality if and only if M is either the complete graph K_n or the star $K_{1,n-1}$.

Corollary 1.6.5C (Cao, [42], Liu [172]) Let G be a simple connected graph with n vertices and e edges. Then

$$\rho(G) \le \sqrt{2e - r(n-1) + (r-1)}$$

with equality if and only G is either the star $K_{1,n-1}$ or a regular graph.

Corollary 1.6.5D (Hong, [132]) Let G be a connected graph with n vertices and e edges. Then

$$\rho(G) \le \sqrt{2e - n + 1}$$

with equality if and only if G is either the star $K_{1,n-1}$ or the complete graph K_n.

The next theorem consider the spectral radius of a bipartite graph. Thus we do not assume that M is square or that $\mathrm{tr}(M) = 0$.

Theorem 1.6.6 (Gregory, Shen and Liu [100]) Suppose M is an $a \times b$ matrix. Let $r = \min_{1\le i\le a} r_i$, $R = \max_{1\le i\le a} r_i$, $s = \min_{1\le i\le b} s_i$, $S = \max_{1\le i\le b} s_i$, $m = \sum_{1\le i\le a} r_i$.

Then

$$\rho\left(\begin{bmatrix} 0 & M \\ M^T & 0 \end{bmatrix}\right) \le \begin{cases} \sqrt{m - ar + Sr}, \\ \sqrt{m - bs + Rs}, \\ \sqrt{m - \frac{ar+bs}{2} + \frac{Sr+Rs}{2}}. \end{cases}$$

Moreover , if $\begin{bmatrix} 0 & M \\ M^T & 0 \end{bmatrix}$ has a positive eigenvector, then the spectral radius attains any of the above upper bounds if and only if $r = R$, $s = S$ (and so $ar = bs$).

Proof We only prove the first inequality. The second can be proved by replacing M^t for M, and the third follows from the first and the second easily. We shall adopt the notations in the proof of Theorem 1.6.5. By (1.15),

$$\begin{aligned} \rho^2 &\le m \quad - \sum_i r_i \sum_{j:m_{ij}=0} y_j^2 \le m - r \sum_i \sum_{j:m_{ij}=0} y_j^2 \\ &= \quad m - r \sum_i (a - S) \sum_i y_i^2 = m - ar + Sr. \end{aligned}$$

Now suppose equality holds and $y_i > 0$ for all i. Then $r > 1$. These imply $r = R$ and $s = S$. On the other hand, if $r = R$, $s = S$ and $ar = bs$, then it is easy to verify that

$$\begin{bmatrix} 0 & M \\ M^t & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \sqrt{rs} \begin{bmatrix} x \\ y \end{bmatrix},$$

where $x_i = \sqrt{b}$ and $y_j = \sqrt{a}$ for all $1 \le i \le a$, $1 \le j \le b$. By Theorem 1.6.1, $\rho = \sqrt{rs} = \sqrt{m - ar + Sr}$. □

Recently, Ellingham and Zha obtained a new result on $\rho(G)$ for a simple planar graph of order n.

Theorem 1.6.7 (Ellingham and Zha, [81]) let G be a simple planar graph with n vertices. Then

$$\rho(G) \le 2 + \sqrt{2n - 6}.$$

An analogue study has also been conducted for nonsymmetric matrices in $\mathbf{B}_n$.

Definition 1.6.2 Let $\mathcal{M}(n, d)$ be the collection of n by n (0,1) matrices each of which has exactly d entries being one; and let $\mathcal{M}^*(n, d)$ be the subset of $\mathcal{M}(n, d)$ such that $A = (a_{ij}) \in \Phi^*$ if and only if $A \in \mathcal{M}(n, d)$ and for each i with $1 \le i \le n$, $a_{1i} \ge a_{2i} \ge \cdots \ge a_{ni}$ and $a_{i1} \ge a_{i2} \ge \cdots \ge a_{in}$. Denote

$$g(n, e) = \max\{\rho(A) : A \in \mathcal{M}(n, e)\} \text{ and } g^*(n, e) = \max\{\rho(A) : A \in \mathcal{M}^*(n, e)\}.$$

Example 1.6.2 The following matrices are members in $\mathcal{M}(3,7)$ achieving the maximum spectral radius $g(3,7) = 1+\sqrt{2}$. Only the first one is in $\mathcal{M}^*(3,7)$.

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

The value of $g(n,d)$ has yet to be completely determined. Most of the results on $g(n,d)$ and $g^*(n,d)$ are done by Brualdi and Hoffman.

Theorem 1.6.8 (Schwarz, [231]) $g(n,d) = g^*(n,d)$.

Theorem 1.6.9 (Brualdi and Hoffman [30]) Let $k > 0$ be an integer. Then $g(n,k^2) = k$. Moreover, for $A \in \mathcal{M}(n,k^2)$, $\rho(A) = k$ if and only if there exists a permutation matrix P such that

$$P^{-1}AP = \begin{bmatrix} J_k & 0 \\ 0 & 0 \end{bmatrix}.$$

Let $k > 2$ be an integer. Define

$$Z_k = \begin{bmatrix} J_k & * \\ * & * \end{bmatrix},\ Z_2 = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},\ Z_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

where there is a single 1-entry somewhere in the asterisked region of Z_k.

Theorem 1.6.10 (Brualdi and Hoffman [30]) Let $k > 0$ be an integer. The $g(n,k^2+1) = k$. Moreover, for $A \in \mathcal{M}(n,k^2+1)$, $\rho(A) = k$ if and only if there is a permutation matrix P such that $P^{-1}AP = Z_k$.

Friedland [92] and Rowlinson [221] also proved different upper bounds $\rho(A)$, under various conditions. Theorem 1.6.11 below extends Theorem 1.6.9 when $s = 0$; and Theorem 1.6.13 below was conjectured by Brualdi and Hoffman in [30].

Theorem 1.6.11 (Friedland [92]) Let $e = \binom{k}{2} + s$, where $s < k$. Then for any $A \in \Phi(n,e)$,

$$\rho(A) \le \frac{k-1+\sqrt{(k-1)^2+4s}}{2}.$$

Theorem 1.6.12 (Friedland [92]) Let $e = \binom{k}{2} + k - 1$, where $k \ge 2$. Then for any

$A \in \Phi(n,e)$,

$$\rho(A) \leq \frac{k-2+\sqrt{k^2+4k-4}}{2},$$

where equality holds if and only if there exists a permutation matrix A such that $P^{-1}AP = H^0_{k+1} + 0$, where

$$H^0_{k+1} = \begin{bmatrix} & J^0_{k-1} & & \begin{matrix} 1 & \cdots & 1 \\ \vdots & & \vdots \\ 1 & \cdots & 1 \end{matrix} \\ \begin{matrix} 1 & \cdots & 1 \\ \vdots & & \vdots \\ 1 & \cdots & 1 \end{matrix} & & & 0 \end{bmatrix}$$

Theorem 1.6.13 (Rowlinson [221]) Let $e = \begin{pmatrix} k \\ 2 \end{pmatrix}$ with $k > s \geq 0$. If $A \in \Phi(n,e)$ such that $\rho(A) = \phi(n,e)$, then $G(A)$ can be obtained from a K_k by adding $n-k$ additional vertices $v_1, \cdots, v_{n-k}$ such that v_1 is adjacent to exactly s vertices in K_k, and such that $v_2, \cdots, v_{n-k}$ are isolated vertices in $G(A)$.

Theorem 1.6.14 (Friedland [92]) For $d = k^2 + t$ where $1 \leq t \leq 2k$, and for each $A \in \mathcal{M}(n,e)$,

$$\rho(A) \leq \frac{k+\sqrt{k^2+2t}}{2}.$$

Moreover, equality holds if and only if $t = 2k$ and there is a permutation matrix P such that

$$P^{-1}AP = \begin{pmatrix} E_d & 0 \\ 0 & 0 \end{pmatrix},$$

where $d = k^2 + 2k$, and

$$E_d = \begin{pmatrix} J_k & \mathbf{1} \\ \mathbf{1}^T & 0 \end{pmatrix}.$$

Theorem 1.6.15 (Friedland [92]) For $d = k^2 + 2k - 3$ where $k > 1$, and for each $A \in \mathcal{M}(n,e)$,

$$\rho(A) \leq \frac{k-1+\sqrt{k^2+6k-7}}{2}.$$

Moreover, when $k > 2$, equality holds if and only if there is a permutation matrix P such that

$$P^{-1}AP = \begin{pmatrix} K_{k+1} & 0 \\ 0 & 0 \end{pmatrix},$$

where for a $k-1$ by 2 matrix $L = J_{(k-1)\times 2}$,

$$H_{k+1} = \begin{pmatrix} J_{k-1} & L \\ L^T & 0 \end{pmatrix}.$$

To study the lower bound of $\rho(A)$, it is more convenient to classify the (0,1) matrices by the number of entries equal to zero instead of by the number of entries equal to one.

Definition 1.6.4 Let n and r be integers. Let $\overline{\mathcal{M}}(n,r)$ be the collection of n by n (0,1) matrices each of which has exactly r entries being zero; and let $\overline{\mathcal{M}}^*(n,r)$ be the subset of $\overline{\mathcal{M}}(n,r)$ such that $A = (a_{ij}) \in \overline{\Phi}^*(n,r)$ if and only if $A \in \overline{\mathcal{M}}(n,r)$ and for each i with $1 \le i \le n$, $a_{1i} \ge a_{2i} \ge \cdots \ge a_{ni}$ and $a_{i1} \ge a_{i2} \ge \cdots \ge a_{in}$. Denote

$$\begin{aligned} \bar{g}(n,e) &= \max\{\rho(A) : A \in \overline{\mathcal{M}}(n,r)\} \text{ and} \\ \bar{g}^*(n,e) &= \max\{\rho(A) : A \in \overline{\mathcal{M}}^*(n,r)\}. \end{aligned}$$

Once again, a result of Schwarz in [231] indicates that $\bar{g}(n,r) = \bar{g}^*(n,r)$. Also, the following are straight forward.

$$\bar{g}(n,r) = \begin{cases} 0 & \text{if } r \ge \begin{pmatrix} n+1 \\ 2 \end{pmatrix} \\ 1 & \text{if } \begin{pmatrix} n \\ 2 \end{pmatrix} \le r < \begin{pmatrix} n+1 \\ 2 \end{pmatrix}. \end{cases}$$

Therefore, it remains to study the value of $\bar{g}(n,r)$ when $r < \begin{pmatrix} n \\ 2 \end{pmatrix}$.

Theorem 1.6.16 (Schwarz, [231]) Let $A \in \overline{\mathcal{M}}(n,r)$ Then for some permutation matrix Q, there exists a sequence of matrices $A_0 = Q^T AQ$, $A_1, \cdots, A_s = B$ such that

(i) $B \in \overline{\mathcal{M}}^*, r)$

(ii) A_{i+1} is obtained from A_i by switching a 0 and a 1 in A_i where the 0 immediately precedes the 1 in some row or immediately follows the 1 in some column $(i = 0, 2, \cdots, s-1)$,

(iii) $\rho(A) \ge \rho(A_i) \ge \rho(B)$, $(i = 1, \cdots, s-1)$.

The following Theorem 1.6.17 is due to Brualdi and Solheid. The proof here is given by Li [160].

Theorem 1.6.17 (Brualdi and Solheid, [38]) Let r be an integer with $0 \le r \le \lfloor \frac{n^2}{4} \rfloor$, then

$$g(n,r) = \frac{n + \sqrt{n^2 - 4r}}{2}.$$

Moreover, for $A \in \overline{\mathcal{M}}(n,r)$, $\rho(A) = \bar{g}(n,r)$ if and only if there is a permutation matrix P such that $P^{-1}AP$ is equal to

$$\begin{pmatrix} J_k & C \\ J_{lk} & J_l \end{pmatrix}, \tag{1.16}$$

where k, l are non negative integers such that $k + l = n$.

Proof Let $A = (a_{ij}) \in \overline{\Phi}^*(n,r)$. Let $r_i = n - \sum_{j=1}^n a_{ij}$, $1 \le i \le n$. Since $A \in \overline{\mathcal{M}}^*(n,r)$, $r_1 \ge r_2 \ge \cdots \ge r_n$ and $\sum_{i=1}^n r_n = r$. Let $\mathbf{x} = (x_1, x_2, \cdots, x_n)^T$ be a eigenvector of A corresponding to the eigenvalue $\rho(A)$ such that $|\mathbf{x}| = 1$. Then by $\rho(A)\mathbf{x} = A\mathbf{x}$, for each i with $1 \le i \le n$,

$$\rho(A)x_i = \sum_{j=1}^{n-r_i} x_j = 1 - \sum_{j=n-r_i+1}^{n} x_j.$$

It follows by $\rho(A) > 0$ that

$$x_n \ge x_{n-1} \ge \cdots \ge x_1, \quad \sum_{j=n-r_i+1}^{n} x_j \le r_i x_n, (1 \le i \le n) \tag{1.17}$$

and

$$\rho(A)x_n \le 1. \tag{1.18}$$

This implies that for each i,

$$\rho(A)x_i = 1 - \sum_{j=n-r_i+1}^{n} x_j \ge 1 - r_i x_n,$$

and so summing up for i yields

$$\rho(A) \ge n - \sum_{i=1}^{n} r_i x_n = n - r x_n.$$

By $\rho(A) > 0$ and by $\rho(A)x_n \le 1$,

$$\rho(A)^2 \ge \rho(A)(n - r x_n) \ge n\rho(A) - r.$$

Solving this inequality for $\rho(A)$ gives

$$\rho(A) \ge \frac{n + \sqrt{n^2 - 4r}}{2}. \tag{1.19}$$

Note that equality in (1.19) holds if and only if both (1.17) and (1.18) are equalities. In other words, for each j with $n - r_1 + 1 \le j \le n$, both $\rho(A)x_j = \sum_{i=1}^{n} x_i = 1$ and $r_j = 0$, and so there exists a permutation matrix P such that $P^{-1}AP = B$ with $k = n - r_1$ and $l = r_1$.

It remains to show that if $A \in \overline{\mathcal{M}}(n,r) - \overline{\mathcal{M}}^*(n,r)$ and if A is not permutation similar to a matrix of the form in (1.16).

Consider a matrix $A \in \overline{\mathcal{M}}(n,r)$ and let $A_0 = Q^TAQ$, $A_1, \cdots, A_s = B$ be matrices satisfying Theorem 1.6.16. Since $B \in \overline{\mathcal{M}}^*(n,r)$, if B is not permutation similar to the form in (1.16), then $\rho(A) \geq \rho(B) > \bar{g}(n,r)$. Thus we may assume that there is a number j such that the matrix $F = A_{j+1}$ has the form (1.16) and $k = r$, but the matrix $E = A_j$ does not have the form (1.16) for any k. Without loss of generality, we assume, by Theorem 1.6.16, that

$$E = \begin{bmatrix} J_r & C' \\ J_{n-r,r} & D \end{bmatrix}, \; F = \begin{bmatrix} J_r & C \\ J_{n-r,r} & J_{n-r} \end{bmatrix},$$

where C' is obtained from C by replacing a 0 at the (r,t) position of C by a 1, for some t with $1 \leq t \leq n-r$, and where D is obtained from J_{n-r} by replacing a 1 at the $(1,t)$ position by a 0.

Denote $E = (e_{ij})$ and let $\mathbf{x} = (x_1, \cdots, x_n)^T$ with unit length be an eigenvector of E corresponding to the eigenvalue $\rho(E)$. By the choice of E, E does not take the form of (1.16) for any k. In fact, for $k = r+1$, there is 0 in the first column of C'. Let $e_i = 1 - e_{i,r+1}$, then $\sum_{i=1}^n e_i > 0$. Since the $(r+2)$ row of E does not have a 0, we can deduce, from the $r+1$ and the $r+2$ rows of $E\mathbf{x} = \rho(E)\mathbf{x}$, that $0 \leq x_{r+1} < x_{r+2} = x_{r+3} = \cdots = x_n$, and

$$\rho(E)x_i = 1 - [(r_i - e_i)x_n + e_i x_{r+1}].$$

Summing up for i yields

$$\rho(E) = n - \left(\sum_{i=1}^n (r_i - e_i)x_n + \sum_{i=1}^n e_i x_{r+1}\right) > n - rx_n,$$

and

$$|\rho(E)|^2 - n\rho(E) + r > 0.$$

It follows that

$$\rho(A) \geq \rho(E) > \frac{1}{2}(n + \sqrt{n^2 - 4r}) = \bar{g}(n,r),$$

which completes the proof. □

1.7 Exercises

Exercise 1.1 Prove Proposition 1.1.2.

Exercise 1.2 Prove Theorem 1.1.1 and Theorem 1.1.2.

Exercise 1.3 Let $A = A(K_n)$ be the adjacency matrix of the complete graph of order n. Show that

$$A^k = \left(\frac{(n-1)^k - (-1)^k}{n}\right) J_n + (-1)^k I_n.$$

Exercise 1.4 Prove Corollary 1.1.3A and Corollary 1.1.3B.

Exercise 1.5 The number $\mathrm{tr}A^k$ is the number of closed walks of length k in G.

Exercise 1.6 Let J_n denote the $n \times n$ matrix whose entries are all ones and let s, t be two numbers. Show that the matrix $sJ_n - tI_n$ has eigenvalues t (with multiplicity $n-1$) and $ns + t$.

Exercise 1.7 Let K_n, C_n and P_n denote the complete graph, the cycle, the path with n vertices, respectively, and let $K_{r,s}$ denote the complete bipartite graph with r vertices on one side and s vertices on the other. Let $m = 2k$ be an even number, J be the $m \times m$ matrix in which each entry is a 1, and B is the direct sum of k matrices each of which is $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Verify each of the following:

(i) $\mathrm{spec}(K_n) = \begin{pmatrix} -1 & n-1 \\ n-1 & 1 \end{pmatrix}$.

(ii) $\mathrm{spec}(K_{r,s}) = \begin{pmatrix} 0 & \sqrt{rs} & -\sqrt{rs} \\ r+s-2 & 1 & 1 \end{pmatrix}$.

(iii) $\mathrm{spec}(C_n) = \left\{ 2\cos\left(\frac{2\pi j}{n}\right) : j = 0, 1, \cdots, n-1 \right\}$.

(iv) $\mathrm{spec}(P_n) = \left\{ 2\cos\left(\frac{\pi j}{n+1}\right) : j = 1, \cdots, n \right\}$.

(v) $\mathrm{spec}(J - B - I) = \begin{pmatrix} -2 & 0 & m-2 \\ k-1 & k & 1 \end{pmatrix}$.

(vi) $\mathrm{spec}(J - B) = \begin{pmatrix} -1 & 1 & m-1 \\ k-1 & k & 1 \end{pmatrix}$.

Exercise 1.8 Let G be a simple graph with n vertices and q edges, and let $m(\Delta)$ be the number of 3-cycles of G. Show that

$$\chi_G(\lambda) = \lambda^n - q\lambda^{n-2} - 2m(\Delta)\lambda^{n-3} + \cdots.$$

Exercise 1.9 Prove Lemmas 1.2.2, 1.2.3, and 1.2.4.

Exercise 1.10 Prove Corollary 1.3.2A and Corollary 1.3.2B, using Theorem 1.3.2.

Exercise 1.11 Let G be a graph with $n \geq 3$ vertices and let $v_1, v_2 \in V(G)$ such that v_1 has degree 1 in G and such that $v_1v_2 \in E(G)$. Show that

$$\chi_G(\lambda) = \lambda\chi_{G-v_1}(\lambda) - \chi_{G-\{v_1,v_2\}}(\lambda).$$

Exercise 1.12 Show that is G is a connected graph with $n = |V(G)| \geq 2$ vertices and with a degree sequence $d_1 \leq d_2 \leq \cdots \leq d_n$. Then each of the following holds.

(i) $\lambda_1(G) \leq \left(\max\limits_{1 \leq i \leq n} \sum\limits_{ij \in E(G)} d_j \right)^{\frac{1}{2}}$.

(ii) (H. S. Wilf, [275]) Let $q = |E(G)|$. Then $\lambda_1(G) \leq \sqrt{\dfrac{2q(n-1)}{n}}$.

Exercise 1.13 Let T_n denote a tree with $n \geq 2$. Then

$$2\cos\left(\frac{\pi}{n+1}\right) \leq \lambda_1(G) \leq \sqrt{n-1},$$

where the lower bound holds if and only if $T_n = P_n$, the path with n vertices; and where the upper bound holds if and only $T_n = K_{1,n-1}$.

Exercise 1.14 Let T denote a tree on $n \geq 3$ vertices. For any k with $2 \leq k \leq n-1$, there exists a vertex $v \in V(T)$ such that the components of $T - v$ can be labeled as $G_1, G_2, \cdots, G_c$ so that either

(i) $|V(G_i)| \leq \left\lceil \dfrac{n-2}{k} \right\rceil + 1$, for all i with $1 \leq i \leq$ c; or

(ii) $|V(G_i)| \leq \left\lceil \dfrac{n-2}{k} \right\rceil + 1$, for all i with $1 \leq i \leq c-1$, and $|V(G_c)| \leq n - 2 - \left\lceil \dfrac{n-2}{k} \right\rceil$.

Exercise 1.15 Prove Lemma 1.3.1.

Exercise 1.16 Let G be a graph on $n \geq 2$ vertices. Prove each of the following.

(i) (Lemma 1.3.2) If G is bipartite, then its spectrum is symmetric with respect to the origin. In other words,

$$\lambda_i(G) = -\lambda_{n+1-i}(G), \text{ for each } i \text{ with } 1 \leq i \leq \lfloor \frac{n}{2} \rfloor.$$

(ii) If G is connected, then G is bipartite if and only if $-\lambda_1(G)$ is an eigenvalue of G.

(iii) If the spectrum of G is symmetric with respect to the origin, then G is bipartite.

Exercise 1.17 Let $\rho(A)$ be the maximum absolute value of an eigenvalue of matrix A. Show that for any matrix $M \in \mathbf{B}_n$,

$$\rho(M) \leq \rho\left(\begin{bmatrix} 0 & M \\ M^T & 0 \end{bmatrix}\right).$$

Exercise 1.18 If $\begin{bmatrix} 0 & M \\ M^T & 0 \end{bmatrix}\begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix} = \lambda\begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix}$, then $||\mathbf{x}|| = ||\mathbf{y}||$ whenever $\lambda \neq 0$.

Exercise 1.19 Let G be a graph on n vertices and G^c be the complement of G. Then

$$\rho(G) + \rho(G^c) \leq -\frac{1}{2} + \sqrt{2(n-1)^2 + \frac{1}{2}},$$

$$\rho(G)\rho(G^c) < \frac{(n-1)^2}{2} - \frac{1}{4}\sqrt{(n-1)^2 + \frac{1}{2}} + \frac{3}{16}.$$

Exercise 1.20 Let G be a loopless (n, q)-graph with a degree sequence $d_1, d_2, \cdots, d_n$. Let $B(G)$ denote the incidence matrix of G (Definition 1.1.5), and let $C = \mathrm{diag}(d_1, d_2, \cdots, d_n)$. Show each of the following.

(i) $A(L(G)) = B^T B - C$.

(ii) $A(L(G)) = B^T B - 2I_q$.

(iii) If $q \geq n$, then $\chi_{B^T B}(\lambda) = \lambda^{q-n}\chi_{BB^T}(\lambda)$.

(iv) The matrix $B^T B$ is *semi-definite positive* (that is, a real symmetric matrix each of whose eigenvalues is bigger than or equal to zero).

(v) If $q > n$, then zero is an eigenvalue of $B^T B$.

Exercise 1.21 Let G be given below. Find $L(G)$ and $T(G)$.

(i) $G = K_3$.

(ii) $G = K_4$.

(iii) G is the graph obtained by deleting an edge from a K_4.

(iv) What is the relationship between $L(K_4)$ and $T(K_3)$? Are they isomorphic?

(v) What is the relationship between $L(K_{n+1})$ and $T(K_n)$? Are they isomorphic? (See [5]).

Exercise 1.22 Let $A \in \Phi(n, e)$ such that $G(A)$ is a connected simple planar graph. Then $\rho(A) \leq \sqrt{5n - 11}$.

Exercise 1.23 Let $A \in \Phi(n, e)$ such that $G(A)$ is a connected graph, and let $\lambda_1, \cdots, \lambda_n$ be the eigenvalues of A. Then $\sum_{i=2}^{n} \lambda_i^2 \geq n - 1$, where equality holds if and only if $G(A) \cong K_{1,n-1}$ or $G(A) \cong K_n$.

Exercise 1.24 Let A be an n by n square matrix, and let B be an r by r submatrix of A with $r \geq 3$. Then

$$\rho(A) \leq \sqrt{\mathrm{tr}(AA^T) - \lambda_2(BB^T) - \lambda_3(BB^T)}.$$

1.8 Hints for Exercises

Exercise 1.1 Apply the definitions.

Exercise 1.2 To prove Theorem 1.1.1(i), note that when $i \neq j$, the dot product of the ith row and jth column of B is the number of edges joining v_i and v_j in G; and when $i = j$, this dot product is the degree of v_i. The proof for (ii) is similar, taking the orientation into account.

To prove Theorem 1.1.2, we can label the vertices of G so that the oriented incidence matrix B of G is the direct sum of the oriented incidence matrix B_i of G_i, $1 \le i \le t$, where $G_1, \cdots, G_t$ are the components of G. Thus it suffices to prove Theorem 1.1.3 when $t = 1$. Assume that $t = 1$.

Let $\mathbf{a}_i$ denote the ith row of B. Since each column of B has exactly one 1-entry and one (-1)-entry, $\sum_{i=1}^{n} \mathbf{a}_i = \mathbf{0}$, and so the rank of B is at most $n-1$. Assume then that there exist scalars c_i, not all $c_i = 0$, such that $\sum_{i=1}^{n} c_i \mathbf{a}_i = \mathbf{0}$, then use the fact that each column of B has exactly one 1-entry and one (-1)-entry to argue that all c_i must be the same, and so $\sum_{i=1}^{n} \mathbf{a}_i = \mathbf{0}$. Therefore, the rank of B is exactly $n-1$.

Exercise 1.3 Argue by induction on n. Note that $A(K_n) = J_n - I_n$.

Exercise 1.4 Apply Theorem 1.1.3 to the oriented incidence matrix of a graph with $R_2 = \emptyset$.

For Corollary 1.1.3B, note that G has a bipartition X and Y of $V(G)$, which yields a partition of the rows of the incidence matrix B into R_1 and R_2, and so by Theorem 1.1.3, B is unimodular.

Conversely, assume that G has an odd cycle or length $r > 1$, which corresponds to an r by r submatrix B' of B with $|\det(B')| = 2$, and so B cannot be unimodular.

Exercise 1.5 - 1.7 Proceed induction on k in Exercise 1.5, and compute the corresponding determinants in the other two exercises. Note that Exercise 1.7(v) and (vi) are equivalent.

Exercise 1.8 Apply Theorem 1.1.4. $C_0 = 1$. Since $\det(A(K_1))=0$, $C_1 = 0$. Note that $\det(A(K_2)) = -1$, and so $C_2 = \det(A(K_2))S(G, K_2) = -q$. Since $\det(A(K_3)) = 2$ and $S(G, K_3) = m(\Delta)$, $C_3 = (-1)^3 \det(A(K_3))S(G, K_3) = -2m(\Delta)$.

Exercise 1.9 For Lemma 1.2.2, if G has two components, then G has an induced $2K_2$. Hence we may assume that G has only one component. If G has an induced cycle of length at least 5, then done. Since G^c is connected, $|V(G)| \ge 6$. If G has a vertex of degree one, the we can easily find an induced $2K_2$ or a P_4, as long as G^c is connected. Assume that G has no vertices of degree one. Then argue by induction to $G - v$. Note that $G - v$ has no isolated vertices. If $\overline{G - v}$ is not connected, then v is a cut vertex of G^c, and so $G - v$ contains a spanning complete bipartite subgraph H with bipartition $X \cup Y$. It is now easy to find either an induced $2K_2$ or an induced P_4. If $\overline{G - v}$ is connected, then apply induction.

Lemma 1.2.3 can be done by straight forward computations.

For Lemma 1.2.4, it suffices to show that $\chi_{H_5}(\lambda)$ is equal to (by adding (-1)Column

1 to Column 2 and Column 3)

$$\begin{vmatrix} \lambda & 0 & 0 & -1 & \cdots & -1 \\ 0 & \lambda & -1 & -1 & \cdots & -1 \\ 0 & -1 & \lambda & -1 & \cdots & -1 \\ -1 & -1 & -1 & \lambda & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \\ -1 & -1 & -1 & 0 & \cdots & \lambda \end{vmatrix} = \begin{vmatrix} \lambda & -\lambda & -\lambda & -1 & \cdots & -1 \\ 0 & \lambda & -1 & -1 & \cdots & -1 \\ 0 & -1 & \lambda & -1 & \cdots & -1 \\ -1 & 0 & 0 & \lambda & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \\ -1 & 0 & 0 & 0 & \cdots & \lambda \end{vmatrix}$$

(then expand along the Column 1)

$$= \lambda^{n-2}\begin{vmatrix} \lambda & -1 \\ -1 & \lambda \end{vmatrix} + \sum_{i=4}^{n}(-1)^{i+1}(-1)(-1)^i\lambda^{n-4}\begin{vmatrix} -\lambda & -\lambda & -1 \\ \lambda & -1 & -1 \\ -1 & \lambda & -1 \end{vmatrix}.$$

Exercise 1.10 For Corollary 1.3.2A, let $\mathbf{y} = J_{1,n}$. For Corollary 1.3.2B, let $\mathbf{y} = (\sqrt{d_1}, \sqrt{d_2}, \cdots,$

Exercise 1.11 Let G be a graph with $n \geq 3$ vertices and let $v_1, v_2 \in V(G)$ such that v_1 has degree 1 in G and such that $v_1v_2 \in E(G)$. Show that

$$\chi_G(\lambda) = \lambda\chi_{G-v_1}(\lambda) - \chi_{G-\{v_1,v_2\}}(\lambda).$$

Exercise 1.12 Apply Corollary 1.3.2B to get $\lambda_1 \leq \max_{1\leq i\leq n} \frac{1}{d_i} \sum_{ij\in E(G)} \sqrt{d_id_j}$. By Cauchy-Schwarz inequality, $\frac{d_i}{\sum_{ij\in E(G)}\sqrt{d_id_j}} \leq \sqrt{\sum_{ij\in E(G)} d_j}$, which gives Part (i).

Apply Part (i) to show Part (ii). It suffices to show that for each i, $\sum_{ij\in E(G)} d_j \leq \frac{2q(n-1)}{n}$. This is true if $nd_i \geq 2q$, since $\sum_{ij\in E(G)} d_j \leq 2q - d_i \leq \frac{2q(n-1)}{n}$.

For i with $nd_i < 2q$, then $\sum_{ij\in E(G)} d_j \leq \Delta(G)\frac{2q}{n} \leq \frac{2q(n-1)}{n}$.

Exercise 1.13 Apply the reduction formula in Exercise 1.11.

Exercise 1.14 Let T denote a tree on $n \geq 3$ vertices. Pick a vertex $v_1 \in V(T)$. If v_1 is not the desired vertex, then $T - v_1$ has either a component with more than $n-2-\lfloor (n-2)/k \rfloor$ vertices, or at least two components each of which has more than $\lfloor (n-2)/k \rfloor + 1$ vertices.

Assume that T_1 is a component of $T - v_1$ with more than $\lfloor (n-2)/k \rfloor + 1$ vertices. Let $v_2 \in V(T_1)$ be a vertex adjacent to v_1 in T. Then in $T - v_2$, the component containing v_1 has at most $n-2-\lfloor (n-2)/k \rfloor$ vertices, and the other components of $T - v_2$ are subgraphs of $T_1 - v_2$.

If again v_2 is not the desired vertex, then $T_1 - v_2$ has a component T_2 with more than $\lfloor (n-2)/k \rfloor + 1$ vertices. Let $v_3 \in V(T_2)$ be a vertex adjacent to v_2 in T. Then in $T - v_3$, the component containing v_2 has at most $n - 2 - \lfloor (n-2)/k \rfloor$ vertices, and the other components of $T - v_3$ are subgraphs of $T_2 - v_3$.

Repeat this process a finite number of times to get the desired v.

Exercise 1.15 Apply Exercise 1.14 when $k = 2$. Assume $k \geq 3$. By Exercise 1.14, we assume that T has a vertex v_1 such that Exercise 1.14(ii) holds. Let $T_1 = G_c$ in Exercise 1.14(ii) and let $n_1 = |V(T_1)|$.

If $n_1 \leq k$, then V' can be chosen to include v_1, and as many vertices in $V(T_1)$ as possible. Thus assume that $n_1 > k$.

By Exercise 1.14, T has a vertex v_2 such that $T - v_2$ has a component T_2 with n_2 vertices, and such that each other component of $T - v_2$ has at most $\lfloor (n_1 - 2)/(k-1) \rfloor + 1$ vertices, and

$$\lfloor \frac{n_1 - 1}{k-2} \rfloor + 1 \leq \lfloor \frac{n - 2\lfloor (n-2)/k \rfloor - 2}{k-1} \rfloor + 1 \leq \lfloor \frac{n-2}{k} \rfloor.$$

Note that

$$n_2 \leq n_1 - 2 - \lfloor \frac{n_1 - 2}{k-1} \rfloor.$$

Such a process may be repeated at most $k-1$ times before we obtain a desirable subset V'.

Exercise 1.16 Let $A = A(G) = (a_{ij})$ and $\lambda_i = \lambda_i(G)$. Note that $\Lambda_1 > 0$.
(i) Assume that G is bipartite with a bipartition $\{S^+, S^-\}$. Let $\mathbf{x} = (x_1, \cdots, x_n)^T$ be such that $\lambda_i \mathbf{x} = A\mathbf{x}$. Define, for each $j = 1, 2, \cdots, n$,

$$w_j = \begin{cases} x_j & \text{if } v_j \in S^+ \\ -x_j & \text{if } v_j \in S^- \end{cases}$$

and let $\mathbf{w} = (w_1, \cdots, w_n)^T$. Then we can routinely verify that $A\mathbf{w} = -\lambda_i \mathbf{w}$.
(ii) Suppose $-\lambda_1$ is an eigenvalue. Assume $A\mathbf{x} = -\lambda_1 \mathbf{x}$ where $\mathbf{x} = (x_1, \cdots, x_n)^T$. Let $\mathbf{w} = (|x_1|, \cdots, |x_n|)^T$. Then

$$|\mathbf{x}^T A \mathbf{x}| = |-\lambda_1 \mathbf{x}^T \mathbf{x}| = \lambda_1 \mathbf{x}^T \mathbf{x} = \lambda_1 \mathbf{w}^T \mathbf{w}.$$

Thus

$$\begin{aligned} \lambda_1 \mathbf{w}^T \mathbf{w} &= -\mathbf{x}^T A \mathbf{x} \\ &\leq |-\mathbf{x}^T A \mathbf{x}| = |\sum_i \sum_j a_{ij} x_i x_j| \\ &\leq \sum_i \sum_j a_{ij} |x_i||x_j| = \mathbf{w}^T A \mathbf{w} \\ &\leq \lambda_1 \mathbf{w}^T \mathbf{w}. \end{aligned}$$

(Here we used the fact that $\lambda_1(G) = \max_{|\mathbf{u}|\neq 0}\{\frac{\mathbf{u}^T A\mathbf{u}}{|\mathbf{u}|^2}\}$. See Theorem 6.1.3 in the Appendix) It follows that $\lambda_1 \mathbf{w} = A\mathbf{w}$, By Theorem 1.3.1(ii), and by the assumption that G is connected, $|x_i| > 0$, for all i with $1 \le i \le n$. It also follows that all inequalities above must be equalities, and so all nonzero terms in

$$\sum_i \sum_j a_{ij} x_i x_j = -\lambda_1 |\mathbf{x}|^2$$

must have the same sign. Therefore, all summands must be negative, which implies that if $a_{ij} \neq 0$, then $x_i x_j < 0$. Let $S^+ = \{v_i : x_i > 0\}$ and $S^- = \{v_i : x_i < 0\}$. Then $\{S^+, S^-\}$ is a bipartition of $V(G)$, and so G is bipartite.

The other direction follows from (i).

(iii) Apply (ii) and argue componentwise.

Exercise 1.17 Suppose $Mx = \rho(M)\mathbf{x}$ for some non-zero vector $\mathbf{x}$. Then $\mathbf{x}^T M^T = \rho(M)\mathbf{x}^T$ and

$$\begin{aligned}\rho\left(\begin{bmatrix} 0 & M \\ M^T & 0 \end{bmatrix}\right) &\ge \frac{[\mathbf{x}^T\ \mathbf{x}^T]\begin{bmatrix} 0 & M \\ M^T & 0 \end{bmatrix}\begin{bmatrix} \mathbf{x} \\ \mathbf{x} \end{bmatrix}}{[\mathbf{x}^T\ \mathbf{x}^T]\begin{bmatrix} \mathbf{x} \\ \mathbf{x} \end{bmatrix}} \\ &= \frac{\mathbf{x}^T M\mathbf{x} + \mathbf{x}^T M^T \mathbf{x}}{2\|x\|^2} = \rho(M).\end{aligned}$$

Exercise 1.18 Since $M\mathbf{y} = \lambda\mathbf{x}$ and $M^T\mathbf{x} = \lambda\mathbf{y}$, we have $\mathbf{x}^T M = \lambda\mathbf{y}^T$ and $\lambda\mathbf{x}^T\mathbf{x} = \mathbf{x}^T M\mathbf{y} = \lambda\mathbf{y}^T\mathbf{y}$.

Exercise 1.19 Note that G or G^c must be connected, and so we may assume that G is connected. By Theorem 1.6.5 and its corollaries,

$$\begin{aligned}\rho(G) + \rho(G^c) &\le \sqrt{2e - n + 1} + [-\frac{1}{2} + \sqrt{\frac{1}{4} + n(n-1) - 2e}], \\ (\rho(G)\rho(\bar{G}) + \frac{1}{2})^2 &\le (n-1)^2 + \frac{1}{4} + 2\sqrt{(2e - n + 1))\frac{1}{4} + n(n-1) - 2e)} \\ &< 2(n-1)^2 + \frac{1}{2}.\end{aligned}$$

Exercise 1.20 Apply definitions and Theorem 1.1.1.

Exercise 1.21 A formal proof for $L(K_{n+1}) \equiv T(K_n)$ can be found in [5], but we can get the main idea by working on $L(K_4) \equiv T(K_3)$.

Exercise 1.22 For a simple planar graph G, $|E(G)| \le 3|V(G)| - 6$.

Exercise 1.23 Note that $\sum_{i=1}^n \lambda_i^2 = 2e$. Then it follows from Corollary 1.6.5D.

Exercise 1.24 Note that if P and Q are permutation matrices, then

$$(PAQ)(PAQ)^T = PAA^TP^T.$$

Hence we may assume that

$$A = \begin{bmatrix} B & C \\ D & E \end{bmatrix} = \begin{bmatrix} X \\ Y \end{bmatrix}.$$

Then $A^TA = X^TX + Y^TY$, and

$$AA^T = \begin{bmatrix} XX^T & XY^T \\ YX^T & YY^T \end{bmatrix}, \; XX^T = BB^T + CC^T.$$

Since YY^T abd CC^T are semidefinite positive,

$$\begin{aligned} \lambda_2(A^TA) + \lambda_3(A^TA) &\geq \lambda_2(X^TX) + \lambda_3(X^TX) = \lambda_2(XX^T) + \lambda_3(XX^T) \\ &= \lambda_2(BB^T) + \lambda_3(BB^T). \end{aligned}$$

It follows by $\rho(A)^2 \leq \lambda_1(A^TA) = \text{tr}(A^TA) - \sum_{i=2}^{n} \lambda_i(A^TA)$ that

$$\rho(A)^2 \leq \text{tr}(A^TA) - \lambda_2(BB^T) - \lambda_3(BB^T).$$

Chapter 2

Combinatorial Properties of Matrices

Let F be a subset of a number field, and let $M_{m,n}(F)$ denote the set of all $m \times n$ matrices with entries in F, and let $M_n(F) = M_{n,n}(F)$. Note that $\mathbf{B}_n = M_n(\{0,1\})$. We write $M_n^+ = M_n(\{r \geq 0 |\ r \text{ is real}\})$ and $M_n^* = M_n(\{r > 0 |\ r \text{ is real}\})$. When the set F is not specified, we write $M_{m,n}$ and M_n for $M_{m,n}(F)$ and $M_n(F)$, respectively.

Let $A \in M_{m,n}$, and let k and r be integers with $1 \leq k \leq m$ and $1 \leq l \leq n$. For $1 \leq i_1 < i_2 < \cdots < i_k \leq m$ and $1 \leq j_1 < j_2 < \cdots < j_l \leq n$, write $\alpha = (i_1, i_2, \cdots, i_k)$ and $\beta = (j_1, j_2, \cdots, j_l)$. Let $A[i_1, i_2, \cdots, i_k | j_1, j_2, \cdots, j_l] = A[\alpha|\beta]$ denote the *submatrix* of A whose (p,q)th entry is $a_{i_p j_q}$. We also say that $A[\alpha|\beta]$ is obtained from A by deleting the rows not indexed in α and columns not indexed in β.

Similarly, $A[i_1, i_2, \cdots, i_k | j_1, j_2, \cdots, j_l) = A[\alpha|\beta)$ denotes the matrix obtained from A by deleting the rows not indexed in α and columns indexed in β; $A(\alpha|\beta]$ denotes the matrix obtained from A by deleting the rows indexed in α and columns not indexed in β; and $A(\alpha|\beta)$ the matrix obtained from A by deleting the rows indexed in α and columns indexed in β.

Nonnegative matrices can be classified as reducible matrices and irreducible matrices by the relation $\simeq_p$, or as partly decomposable matrices and fully indecomposable matrices by the relation $\sim_p$.

2.1 Irreducible and Fully Indecomposable Matrices

Definition 2.1.1 A matrix $A \in M_n$ is *reducible* if $A \simeq_p A_1$, where

$$A_1 = \begin{bmatrix} B & 0 \\ C & D \end{bmatrix},$$

and where B is an l by l matrix and D is an $(n-l)$ by $(n-l)$ matrix, for some $1 \le l \le n-1$. The matrix A is *irreducible* if it is not reducible. The next theorem follows from the definitions.

Theorem 2.1.1 Let $A = (a_{ij}) \in M_n^+$ for some $n > 1$, and let m denote the degree of the minimal polynomial of A. The following are equivalent.

(i) A is irreducible.

(ii) There exist no indices $1 \le i_1 < i_2 < \cdots < i_l \le n$ with $1 \le l < n$ such that $A[i_1, \cdots i_l | i_1, \cdots, i_l) = 0$.

(iii) A^T is irreducible.

(iv) $(I + A)^{n-1} > 0$.

(v) There is a polynomial $f(x)$ over the complex numbers such that $f(A) > 0$.

(vi) $1 + A + \cdots + A^{m-1} > 0$.

(vii) $(I + A)^{m-1} > 0$.

(viii) For each cell (i, j), there is an integer $k > 0$ such that $a_{ij}^{(k)}$, the (i, j)th entry of A^k, is positive.

(ix) $D(A)$ is strongly connected.

Sketch of Proof By definitions, (i) $\Longleftrightarrow$ (ii) $\Longleftrightarrow$ (iii), (iv) $\Longrightarrow$ (v), (vi) $\Longrightarrow$ (vii) $\Longrightarrow$ (viii) $\Longrightarrow$ (i).

To see that (v) $\Longrightarrow$ (vi), let $f(x)$ be a polynomial such that $f(A) > 0$. Let $m_A(x)$ denote the minimum polynomial of A. Then $f(x) = g(x)m_A(x) + r(x)$, where the degree of $r(x)$ is less than m, the degree of $m_A(x)$. Since $m_A(A) = 0$, $r(A) = f(A) > 0$, and this proves (vi).

To see that (i) $\Longrightarrow$ (ix), suppose that $D(A)$ has strongly connected components $D_1, D_2, \cdots, D$ for some $k > 1$. Then we may assume that D has not arcs from $V(D_1)$ to a vertex not in $V(D_1)$. Let $i_1, i_2, \cdots, i_l$, where $1 \le i_1 < i_2 \cdots < i_j \le n$, be integers representing the vertices of $V(D_1)$. Then $A[i_1 \cdots i_j | i_1 \cdots i_j) = 0$, contrary to (i).

It remains to show that (ix) $\Longrightarrow$ (vi), which follows from Proposition 1.1.1 (vii). $\square$

Definition 2.1.2 A matrix $A \in M_n$ is *partly decomposable* if $A \sim_p A_1$, where

$$A_1 = \begin{bmatrix} B & 0 \\ C & D \end{bmatrix},$$

and where B is an l by l matrix and D is an $(n-l)$ by $(n-l)$ matrix, for some $1 \le l \le n-1$. The matrix A is *fully indecomposable* if it is not partly decomposable.

Theorem 2.1.2 Let $A = (a_{ij}) \in M_n^+$ for some $n > 1$. The following are equivalent.

(i) A is fully indecomposable.

(ii) For any r with $1 \le r < n$, A does not have an $r \times (n-r)$ submatrix which equals $0_{r\times(n-r)}$.

(iii) For any $\emptyset \ne X \subset V(D(A))$, $|N(X)| > |X|$, where $N(X) = \{v \in V(D(A))$: such that $D(A)$ has an arc (uv) from a vertex $u \in X$ to $v\}$.

Proof By definition, (i) $\iff$ (ii). The graph interpretation of (ii) is (iii). □

Theorem 2.1.3 Let $A = (a_{ij}) \in M_n^+$. Each of the following holds.

(i) If for any r with $1 \le r < n$, A does not have an $r \times (n-r)$ submatrix which equals $0_{r\times(n-r)}$, then there exist no indices $1 \le i_1 < i_2 < \cdots < i_l \le n$ with $1 \le l < n$, $A[i_1, \cdots i_l | i_1, \cdots, i_l) = 0$.

(ii) If A is fully indecomposable, then A is irreducible.

(iii) Suppose that $a_{ii} > 0$, for each i with $1 \le i \le n$. Then A is fully indecomposable if and only if A is irreducible.

(iv) A is irreducible if and only if $A + I$ is fully indecomposable.

(v) (Brualdi, Parter and Schneider, [35]) A is fully indecomposable if and only if $A \sim_p B$ for some irreducible matrix $B = (b_{ij})$ such that $b_{ii} > 0$, $1 \le i \le n$.

Sketch of Proof (i) is straightforward. (ii) follows from Theorem 2.1.1(ii) and Theorem 2.1.2(ii).

Argue by induction on n to prove (iii). If A is not fully decomposable, then by Theorem 2.1.2(ii), and by the assumption that $a_{ii} > 0$, $A[i_1 \cdots i_l | i_1 \cdots i_l) = 0$ for some $i_1, i_2, \cdots, i_l$, and so A is not irreducible. The other direction follows from (ii).

(iv) follows from (iii) and (ii). □

Definition 2.1.3 A *diagonal* of a matrix $A = (a_{ij}) \in \mathbf{B}_n$ is a collection T of n entries $a_{1i_1}, a_{2i_2}, \cdots, a_{ni_n}$ of A such that $\{i_1, i_2, \cdots, i_n\} = \{1, 2, \cdots, n\}$. A diagonal T of A is a *nonzero diagonal* if no member of T is zero. An entry a_{ij} of A is a 1-entry (0-entry, respectively) if $a_{ij} = 1$ ($a_{ij} = 0$, respectively).

From this view point and by Theorems 2.1.2 and 2.1.3, the following can be concluded.

Theorem 2.1.4 (Brualdi, Parter and Schneider, [35]) Let $A \in \mathbf{B}_n$. Then A is fully indecomposable if and only if every one entry of A lies in a nonzero diagonal, and every zero entry of A lies in a diagonal with exactly one zero member.

Corollary 2.1.3 Let $A \in M_n^+$ with a nonzero diagonal. Then A is fully indecomposable if and only if there is a permutation matrix P, such that PA is irreducible.

Proof Since A has a nonzero diagonal, there is a permutation matrix P such that $PA = (a'_{ij})$ with the property $a'_{ii} > 0$, $1 \le i \le n$. By Theorem 2.1.3(iii), PA is irreducible if and only if PA is fully indecomposable. □

2.2 Standard Forms

Certain canonical forms of a matrix $A \in M_n^+$ can be obtained in terms of irreducible matrices and fully indecomposable matrices.

Theorem 2.2.1 Let $A \in M_n^+$. Each of the following holds.

(i) $A \simeq_p B$ for some $B \in M_n^+$ with the following form (called the *Frobenius normal form* of A):

$$B = \begin{pmatrix} A_1 & 0 & 0 & \cdots & 0 & \cdots & 0 \\ 0 & A_2 & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \\ 0 & 0 & A_g & 0 & 0 & \cdots & 0 \\ A_{g+1,1} & \cdots & A_{g+1,g} & A_{g+1} & 0 & \cdots & 0 \\ A_{g+2,1} & \cdots & A_{g+2,g} & A_{g+2,g+1} & A_{g+2} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\ A_{k,1} & \cdots & A_{k,g} & A_{k,g+1} & \cdots & \cdots & A_k \end{pmatrix}, \tag{2.1}$$

where $A_1, \cdots, A_g, A_{g+1}, \cdots, A_k$ are irreducible matrices, and where for $g < q \le k$, some matrix among $A_{q,1}, \cdots, A_{q,q-1}$ is not a zero matrix.

(ii) If $A_i \in M_{n_i}$, then the parameters k, g and $n_1, \cdots, n_k$ are uniquely determined by A.

Proof Let $D = D(A)$. Note that the strong components of D can be labeled as $D_1, D_2, \cdots, D_k$ such that D has an arc from a vertex in $V(D_i)$ to a vertex in $V(D_j)$ only if $i < j$. We may assume that $D_1, D_2, \cdots, D_g$ are the source components, the strong components with out degree zero.

For each i with $1 \le i \le k$ let $\alpha_i = \{j : v_j \in V(D_i)\}$. Then $A[\alpha_i|\alpha_i]$ is irreducible by Corollary 2.1.1.

By the labeling of the strong components of D, for each i with $1 \le i \le g$, $A[\alpha_i|\alpha_i) = 0$; for $g < q \le r \le k$, $A[\alpha_q|\alpha_r) = 0$. Since $D_{g+1}, \cdots, D_k$ are not source components, some of $A[\alpha_q|\alpha_1], \cdots, A[\alpha_q|\alpha_{q-1}]$ is not a zero matrix. This proves (i).

Since g represents the number of source components of $D(A)$, k the number of strong components of $D(A)$, and n_i's are the number of vertices in the ith strong component of $D(A)$, all these quantities are uniquely determined by $D(A)$. This proves (ii). □

Example 2.2.1 Let $X \in M_3$, and let $0_3 \in M_3$ be the zero matrix. Let

$$A_1 = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, A_2 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}, A = \begin{bmatrix} A_1 & 0_3 \\ X & A_2 \end{bmatrix}.$$

Then A is in a Frobenius normal form. If $X = 0_3$, then we can permute A_1 and A_2 to get another Frobenius normal form. If $X \neq 0_3$, then the Frobenius normal form of A is unique.

Example 2.2.2 If $A \simeq_p B$ and if A is irreducible, then B is also irreducible, since $\simeq_p$ is an equivalence relation. However, if A is irreducible and P, Q are permutation matrices, then PA or AQ may not be irreducible. For example, the matrix obtained by permuting the first two rows of the irreducible matrix A_1 in Example 2.2.1 is reducible, by Theorem 2.1.1.

Example 2.2.2 indicates that certain reducible matrices may become irreducible by the relation $\sim_p$. Therefore it is natural to ask what matrix A has the property that for some permutation matrices P and Q, PAQ is irreducible? Note that $PAQ = (PAP^T)PQ$, and so we may only consider such matrix A that AQ is irreducible. This is answered by Brualdi in Theorem 2.2.2 below.

The proofs for the following two lemmas are left as exercises.

Lemma 2.2.1 Let $A \in M_n^+$ and let $D = D(A)$ with $V(D) = \{v_1, v_2, \cdots, v_n\}$. Let $P_{s,t}$ be the matrix obtained from $I_n \in M_n$ by permuting the sth row with the tth row, and let $A' = AP_{s,t}$. Then $D(A')$ can be obtained from $D(A)$ by changing all the in-arcs of v_s in $D(A)$ into in-arcs of v_t in $D(A')$; and by changing all the in-arcs of v_t in $D(A)$ into in-arcs of v_s in $D(A')$.

Lemma 2.2.2 Let D be a digraph with strong components $D_1, \cdots, D_k$ such that D does not have a source (a vertex with in degree zero) nor a sink (a vertex with out degree zero), and such that

(*) each arc of D goes from a vertex in a $V(D_i)$ to a vertex in $\cup_{j=1}^{i} V(D_j)$, $(1 \le i \le k)$.
Pick a vertex $u_i \in V(D_i)$, where $1 \le i \le k$, and denote $u_0 = u_k$. Obtain a new digraph D' from D by the following *Procedure Redirecting*:

For each $i = 1, 2, \cdots, k$,

redirect all the in-arcs of u_i *in* $D(A)$ *to in-arcs of* u_{i-1}.

Then D' is strongly connected.

Theorem 2.2.2 (Brualdi, [25]) Let $A \in M_n^+$. There exists a permutation matrices Q such that AQ is irreducible if and only if A does not have a zero row or a zero column.

Proof If A has a zero row or a zero column, so does AQ, and so AQ is reducible.

Conversely, assume that A has neither a zero row nor a zero column. By Theorem 2.2.1, we may assume that A has the following Frobenius normal form

$$\begin{bmatrix} A_1 & 0 & 0 & \cdots & 0 \\ A_{21} & A_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ A_{k1} & A_{k2} & A_{k3} & \cdots & A_k \end{bmatrix}.$$

If $k = 1$, then A is irreducible. Hence we assume that $k > 1$. Therefore, $D = D(A)$ has k strong components $D_1, \cdots, D_k$ such that $A_i = A(D_i)$, for each i with $1 \le i \le k$. Moreover, each arc of D goes from a vertex in a $V(D_i)$ to a vertex in $\cup_{j=1}^{i} V(D_j)$.

Pick a vertex $u_i \in V(D_i)$, where $1 \le i \le k$, and denote $u_0 = u_k$. Obtain a new digraph D' from D by Procedure Redirecting as in Lemma 2.2.2.

By Lemma 2.2.1, this procedure amounts to multiplying a permutation matrix Q to A from the right, such that $D' = D(AQ)$. By Corollary 2.1.1, it suffices to show that D' is strongly connected. Therefore, the theorem follows from Lemma 2.2.2. $\square$

Example 2.2.3 By Theorem 2.1.4, if $A \in M_n^+$ is fully indecomposable, then A must have a nonzero diagonal. Also by Theorem 2.1.4, the identity matrix $I_n \in M_n^+$ has a nonzero diagonal but I_n is not fully indecomposable.

The Frobenius form (2.1) can also be applied to obtain standard forms for matrices $A \in M_n^+$ with $\rho_A = n$ under permutation equivalence relation, where ρ_A stands for the term rank of a matrix A (see Appendix, Section 6.2). The analogous result for non square matrices can be similarly obtained, as stated in Theorem 2.2.4.

Theorem 2.2.3 Let $A \in M_n^+$ be such that $\rho_A = n$. Then each of the following holds.

(i) $A \sim_p B$ for some $B \in M_n^+$ with the following form (called the *equivalence normal form* of A),

$$B = \begin{bmatrix} A_1 & 0 & 0 & \cdots & 0 & \cdots & 0 \\ 0 & A_2 & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \\ 0 & 0 & A_g & 0 & 0 & \cdots & 0 \\ A_{g+1,1} & \cdots & A_{g+1,g} & A_{g+1} & 0 & \cdots & 0 \\ A_{g+2,1} & \cdots & A_{g+2,g} & A_{g+2,g+1} & A_{g+2} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\ A_{k,1} & \cdots & A_{k,g} & A_{k,g+1} & \cdots & \cdots & A_k \end{bmatrix} \tag{2.2}$$

where $A_1, \cdots, A_g, A_{g+1}, \cdots, A_k$ are fully indecomposable matrices, and where for $g < q \le k$, some matrix among $A_{q,1}, \cdots, A_{q,q-1}$ is not a zero matrix.

(ii) If $A_i \in M_{n_i}$, then the parameters k, g and $n_1, \cdots, n_k$ are uniquely determined by A.

Proof Since $\rho_A = n$, there is a permutation matrix Q such that $AQ = (a'_{ij})$ with $a'_{ii} > 0$, $1 \le i \le n$. By Theorem 2.2.1, there is a permutation matrix P such that $B = P^T(AQ)P$ is a Frobenius normal form in (2.1). Let $B = (b_{ij})$. Since P is a permutation matrix, and since $a'_{ii} > 0$, we have $b_{ii} > 0$, where $1 \le i \le n$. By Theorem 2.2.1, each A_i is irreducible. By Theorem 2.1.3(iii), each A_i is fully indecomposable. $\square$

In [163], a stronger conclusion on the uniqueness of (2.2) is proved. That is, if the matrix A has an equivalence normal form B in (2.2), and is also permutation equivalent to

$$B' = \begin{bmatrix} A'_1 & 0 & 0 & \cdots & 0 & \cdots & 0 \\ 0 & A'_2 & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \\ 0 & 0 & A'_h & 0 & 0 & \cdots & 0 \\ A'_{h+1,1} & \cdots & A'_{h+1,h} & A'_{h+1} & 0 & \cdots & 0 \\ A'_{h+2,1} & \cdots & A'_{h+2,h} & A'_{h+2,h+1} & A'_{h+2} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\ A'_{l,1} & \cdots & A'_{l,g} & A'_{l,h+1} & \cdots & \cdots & A'_l \end{bmatrix},$$

then $k = l$, $g = h$, and $A'_1, \cdots, A'_h$ can be obtained by permuting $A_1, \cdots, A_g$, and $A'_{h+1}, \cdots, A'_l$ can be obtained by permuting $A_{g+1}, \cdots, A_k$.

Theorem 2.2.4 Let $A \in M^+_{m,n}$ such that A has no zero rows nor zero columns. Then A is permutation equivalent to the following matrix

$$B = \begin{bmatrix} A_L & 0 & 0 & \cdots & 0 & 0 \\ & A_1 & 0 & \cdots & 0 & 0 \\ & & A_2 & \cdots & 0 & 0 \\ & & & \vdots & & \\ & * & & & A_k & 0 \\ & & & & & A_H \end{bmatrix}$$

where $A_L \in M_{r_l, s_l}$, for some $r_l > s_l$ with $\rho_{A_L} = s_l$, where $A_H \in M_{r_h, s_h}$, for some $r_h < s_h$ with $\rho_{A_H} = r_h$, and where each A_i, $1 \le i \le k$, is a fully indecomposable square matrix.

Definition 2.2.1 Let $A \in \mathbf{B}_n$ such that $A \ne 0$. A has a *total support* if every 1 entry of A lies in a nonzero diagonal.

Example 2.2.4 Let $A = (a_{ij}) \in \mathbf{B}_n$ with the following form

$$A = \begin{bmatrix} X & Z \\ 0 & Y \end{bmatrix},$$

where $X \in M_k$ and $Y \in M_l$. If there is an entry $a_{ij} = 1$ in Z, then A does not have a nonzero diagonal that contains this entry. Assume such a diagonal T exists. Then $a_{ij} \in T$ and $|T| = n$. Delete the ith row and jth column from A to get a matrix $B \in M_{n-1}$. Then $T - \{a_{ij}\}$ must be a nonzero diagonal of B. However, the elements in $T - \{a_{ij}\}$ must be in either the first $k-1$ rows or the last $l-1$ columns, and so $|T-\{a_{ij}\}| \le (k-1)+(l-1) = n-2$, contrary to $|T| = n$.

Example 2.2.4 indicates that if A has a total support, by Theorem 2.2.3, A is permutation equivalent to a direct sum of fully indecomposable matrices. The converse holds also, again using Theorem 2.2.3.

Corollary 2.2.3 Let $A \in \mathbf{B}_n$. Then A has a total support if and only if $A \sim_p B$, where B is a direct sum of fully indecomposable matrices.

Theorem 2.2.5 Let $A \in \mathbf{B}_n$ and $A \neq 0$ and let G be the reduced associated bipartite graph of A (see Proposition 1.1.2(vii)). Suppose that A has a total support. Then A is fully indecomposable if and only if G is connected.

Proof If A is not fully indecomposable, then by Corollary 2.2.3, $A \sim_p B$, where B is the direct sum of at least two matrices. Therefore, G is not connected.

Conversely, if G is not connected, then A is permutation equivalent to

$$A' = \begin{bmatrix} X & 0 \\ 0 & Y \end{bmatrix},$$

where $X \in M_{p,q}$ with $1 \le p+q \le 2n-1$. Without loss of generality, assume $p \le q$. Then $\lambda_A \le p+(n-q) = n-(q-p)$, as the first p rows and the last $n-q$ column of A' contain all positive entries of A'. Since A has total support, $n = \rho_A = \lambda_A$, and so $p = q$. It follows that A' has $0_{p,n-q}$ as a submatrix, and so A is not fully indecomposable.

2.3 Nearly Reducible Matrices

By Theorem 2.1.1, a matrix $A \in M_n^+$ is irreducible if and only if $D(A)$ is strong. A minimally strong digraph corresponds to a nearly reducible matrix.

Definition 2.3.1 A digraph D is a *minimally strong digraph* if D is strongly connected, but for any arc $e \in E(D)$, $D - e$ is not strongly connected. For convenience, the graph K_1 is regarded as a minimally strong digraph. A matrix $A \in M_n^+$ is *nearly reducible* if $D(A)$ is a minimally strong digraph.

As an example, a directed cycle is a minimally strong digraph. Some of the properties of minimally strong digraphs are listed in Proposition 2.3.1 and Proposition 2.3.2 below, which follow from the definition immediately.

Proposition 2.3.1 Let D be a minimally strong digraph. Each of the following holds.

(i) D has no loops nor parallel arcs.

(ii) Any directed cycle of D with length at least 4 has no chord. In other words, if $C = v_1v_2\cdots v_kv_1$ is a directed cycle of D, then $(v_i, v_j) \notin E(D) - E(C)$, for any $v_i, v_j \in V(C)$.

(iii) If $|V(D)| \geq 2$, then D contains at least one vertex v such that $d^+(v) = d^-(v) = 1$. (Such a vertex is called a *cyclic vertex* of D.)

(iv) If D has a cut vertex v, then each v-component of D is minimally strong.

Definition 2.3.2 Let D be a digraph, and H a subgraph of D. The *contraction* D/H is the digraph obtained from D by identifying all the vertices in $V(H)$ into a new single vertex, and by deleting all the arcs in $E(H)$. If $W \subseteq V(D)$ is a vertex subset, then write $D/W = D/D[W]$.

Proposition 2.3.2 Let D be a minimally strong digraph and let $W \subseteq V(D)$. If $D[W]$ is strong, then both $D[W]$ and D/W are minimally strong.

Proposition 2.3.3 If D is a minimally strong digraph with $n = |V(D)| \geq 2$, then D must have at least two cyclic vertices.

Proof Since D is strong with $n \geq 2$, D must have a directed cycle, and so the proposition holds when $n = 2$. By Proposition 2.3.1(iii) and by induction, we may assume that D has no cut vertices and that $n \geq 3$.

If every directed cycle of D has length two, then since D is minimally directed and by $n \geq 3$, at least one of the two vertices in a directed cycle of length two is a cut vertex of D. Therefore, D must have a directed cycle C of length $m \geq 3$. We may assume that $D \neq C$, and so $n - m \geq 1$. Thus D/C has $n - m + 1 \geq 2$ vertices, and so by induction, D/C has at least two cyclic vertices, v_1 and v_2 (say). If both $v_1, v_2 \in V(D) - V(C)$, then they are both cyclic vertices of D. Thus we assume that $v_1 \in V(D) - V(C)$ and v_2 is the contraction image of C in D/C. Then D has exactly two arcs between $V(D) - V(C)$ and $V(C)$. Since $|V(C)| = m \geq 3$, C must contain a cyclic vertex of D. $\square$

Definition 2.3.3 Let $D = (V, E)$ be a digraph. A directed path $P = v_0v_1\cdots v_m$ is a *branch* of D if each of the following holds.

(B1) Neither v_0 nor v_m is a cyclic vertex of D, and

(B2) Each vertex in $P^o = \{v_1, v_2, \cdots, v_{m-1}\}$ is a cyclic vertex of D (vertices in P^o are called the *internal vertices* of the branch P), and

(B3) $D[V - P^o]$ is strong.

The number m is the *length* of the branch. Note that $W = \emptyset$ or $v_0 = v_m$ is possible.

Proposition 2.3.4 Let D be a minimally strong digraph with $n = |V(D)| \geq 3$. Then either D is a directed cycle, or D has a branch with length at least 2.

Proof Assume that D is not a directed cycle. Let $U = \{u \in V(D) | u \text{ is not cyclic}\}$. Then $U \neq \emptyset$. Define a new digraph $D' = (U, E')$ such that for any $u, u' \in U$, $(u, u') \in E'$ if and only if D has a directed (u, u')-path. Since D is strong, D' is also strong and has no cyclic vertices.

By Proposition 2.3.3, D' is not minimally strong and so there must be an arc $e' \in E'$ such that $D' - e'$ is also strong. Since D is minimally strong, and since $D' - e'$ is strong, e' must correspond to a branch in D, which completes the proof. $\square$

Theorem 2.3.5 (Hartfiel, [118]) For integers $n > m \geq 1$, let $F_1 \in M^+_{n-m,m}$ such that $F_1 = E_{1,s}$, for some s with $1 \leq s \leq m$, $F_2 \in M^+_{m,n-m}$ such that $F_2 = E_{t,n-m}$, for some t with $1 \leq t \leq m$, and let $A_0 \in M^+_{n-m}$ be the following matrix.

$$A_0 = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix}.$$

Then every nearly reducible matrix $A \in \mathbf{B}_n$ is permutation similar to a matrix $B \in M^+_n$ with the form

$$B = \begin{bmatrix} A_0 & F_1 \\ F_2 & A_1 \end{bmatrix}, \tag{2.3}$$

where $A_1 = (a'_{ij}) \in M^+_m$ is nearly reducible with $a'_{ts} = 0$.

Proof This is trivial if D is a directed cycle ($m = 1$ and $A_1 = A(K_1)$). Hence assume $n \geq 3$ and by Proposition 2.3.4, $D(A)$ has a branch $v_0 v_1 \cdots v_m$, and so there is a permutation matrix P such that PAP^T has the form of B in (2.3). $\square$

Theorem 2.3.6 (Brualdi and Hedrick [29]) Let D be a minimally strong digraph with $n = |V(D)| \geq 2$ vertices. Then

$$n \leq |E(D)| \leq 2(n-1).$$

Moreover, $|E(D)| = n$ is and only if D is a directed cycle; and $|E(D)| = 2(n-1)$ if and only if D is obtained from tree T by replacing each edge of T by a pair of arcs with opposite directions.

Proof Since D is strong, $d^+(v) \geq 1$ for each $v \in V(D)$. Thus $|E(D)| \geq |V(D)| = n$, with equality holds if and only if every vertex of D is cyclic, and so D must be a directed cycle. It remains to prove the upper bound.

The upper bound holds trivially for $n = 1$ and $n = 2$. Assume $n \geq 3$. By Proposition 2.3.4, D has a branch $P = v_0v_1 \cdots v_t$ with $t \geq 2$ such that $D' = D - P^o$ is strong. By induction, $|E(D')| \leq 2(|V(D')| - 1)$. Since $|E(D)| = |E(D')| + t$ and $|V(D)| = |V(D')| + t - 1$,

$$|E(D)| = |E(D')| + t \leq 2(|V(D')| - 1) + t = 2(n-1) - (t-2) \leq 2(n-1).$$

Assume $|E(D)| = 2(n-1)$. Then $t = 2$ and by induction, D' is obtained from a tree T' with $n-1$ vertices by replacing each edge of T' by a pair of oppositely oriented arcs. If $v_0 \neq v_t$, then there is a directed (v_0, v_t)-path P'in D' with length at least one. Since P is a (v_0, v_t)-path, all the arcs in P' may be deleted from D, and the resulting digraph is still strong, contrary to the assumption that D is minimally strong. Hence $v_0 = v_t$ and so the theorem follows by induction. $\square$

Corollary 2.3.6 Let $n \geq 2$ and let $k > 0$ be integers. The there exists a minimally strong digraph D with $|V(D)| = n$ and $|E(D)| = k$ if and only if $n \leq k \leq 2(n-1)$.

Proof The only if part following from Theorem 2.3.6.

Assume $n \leq k \leq 2(n-1)$. Construct a digraph D_k on the vertex set $V = \{v_1, v_2, \cdots, v_n\}$ with these arcs:

$$\begin{aligned} E \cdot &= \{(v_iv_{i+1}), (v_{i+1}v_i)|1 \leq i \leq k-n\} \\ &\cup\{(v_{k-n+j}v_{k-n+j+1})|1 \leq j \leq n-1\} \cup \{(v_nv_{k-n+1})\}. \end{aligned}$$

It is routine to check that D is a minimally strong digraph with n vertices and k arcs. $\square$

Definition 2.3.4 For a matrix $A \in M_n$, the *density* of A, denoted by $\|A\|$, is the sum of all entries of A. Note that when $A \in \mathbf{B}_n$, $\|A\|$ is also the number of positive entries of A.

Theorem 2.3.6 and Corollary 2.3.6 can be stated in terms of (0,1) matrices. The proof is straight forward and is omitted.

Theorem 2.3.7 Let $A \in \mathbf{B}_n$ be a nearly reducible matrix with $n \geq 2$. Then each of the following holds.

(i) $n \leq \|A\| \leq 2(n-1)$.

(ii) $\|A\| = n$ if and only if $A = A(D)$ for a directed cycle D.

(iii) $\|A\| = 2(n-1)$ if and only if $A = A(G)$ for a tree G.

2.4 Nearly Decomposable Matrices

In order to better understand the behavior of fully indecomposable matrices, we investigate the properties of those matrices that are fully indecomposable matrices but the replacing any positive entry by a zero entry in such a matrix will result in a partly decomposable matrix. For notational convenience, we let E_{ij} denote the matrix whose (i, j)-entry is one and whose other entries are zero.

Definition 2.4.1 A matrix $A = (a_{ij}) \in \mathbf{B}_n$ is *nearly decomposable* if A is fully indecomposable and for each $a_{pq} > 0$, $A - a_{pq}E_{pq}$ is partly decomposable.

Theorem 2.4.1 Let $A \in \mathbf{B}_n$ be a nearly decomposable matrix. Each of the following holds.

(i) There exist permutation matrices P and Q such that $PAQ \geq I$.

(ii) For each pair of permutation matrices P and Q such that $PAQ \geq I$, $PAQ - I$ is nearly reducible.

Proof By Theorem 2.1.3(v), there exist permutation matrices P and Q such that $PAQ \geq I$. For the sake of simplicity, we may assume that $A \geq I$. By Theorem 2.1.3(iv), $A - I$ is irreducible. If for some $B \in \mathbf{B}_n$ such that C is irreducible and $C \leq A - I$, then by Theorem 2.1.3(iv), $C + I \leq A$ is fully indecomposable. Since A is nearly decomposable, it must be $C = A - I$, and so $A - I$ is nearly reducible. $\square$

Example 2.4.1 If B is a nearly reducible matrix, by Theorem 2.1.3(iv), $B + I$ is fully indecomposable. But $B + I$ may not be nearly decomposable. Let

$$B = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \text{ and } C = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}.$$

Then B is nearly reducible and C is fully indecomposable. Note that $C \leq B + I$ and $C \neq B + I$. Hence $B + I$ is not near decomposable.

Theorem 2.3.5 can be applied to obtain a recurrence standard form for nearly decomposable matrices. The following notation for A_0 will be used throughout this section. For integers $n > m \geq 1$, let $A_0 \in M_{n-m}^+$ be the following matrix.

$$A_0 = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 1 \end{bmatrix}.$$

Lemma 2.4.1 follows from Theorem 2.2.5, and so its proof is left as an exercise.

Lemma 2.4.1 Let B be a matrix of the form

$$B = \begin{bmatrix} A_0 & F_1' \\ F_2' & B_1 \end{bmatrix}. \tag{2.4}$$

If $B_1 = (b_{ij}) \in M_{n-m}^+$ is a fully indecomposable matrix, if F_1' has a 1-entry in the first row, and if F_2' has a 1-entry in the last column, then B is fully indecomposable.

Lemma 2.4.2 Let $B_1 = (b_{ij}) \in M_m^+$ denote the submatrix in (2.4). If B in (2.7) is nearly indecomposable, then each of the following holds.

(i) B_1 is nearly indecomposable.

(ii) $F_1' = E_{1,s}$, for some s with $1 \le s \le m$ and $F_2' = E_{t,n-m}$, for some t with $1 \le t \le m$.

(iii) If $m \ge 2$, then $b_{ts} = 0$.

Proof If some 1-entry of B_1 can be replaced by a 0-entry to result in a fully indecomposable matrix, then by Lemma 2.4.1, B is not nearly indecomposable, contrary to the assumption of Lemma 2.4.2. Therefore, B_1 must be nearly indecomposable. Similarly, by the assumption that B is nearly indecomposable, Lemma 2.4.2(ii) must hold.

It remains to show Lemma 2.4.2(iii). Suppose $m \ge 2$. Since $m \ge 2$ and since B_1 is fully indecomposable, $B_1 \ne 0$. Given B in the form of (2.4), every nonzero diagonal of B_1 can be extended to a nonzero diagonal of B.

If $b_{ts} = 1$, then let B' be the matrix obtained from B by replacing the 1-entry b_{ts} by a 0-entry. Since B_1 is fully indecomposable, b_{ts} lies in a nonzero diagonal L of B_1. Removing b_{ts} from L, adding the only 1-entry in F_1 and F_2, and utilizing the non main diagonal 1-entries of A_0, we have a nonzero diagonal of B'. Therefore, B' has total support. It is easy to check that the reduced associated bipartite graph of B' is connected, and so by Theorem 2.2.5, B' is fully indecomposable, contrary to the assumption that B is nearly indecomposable. Thus $b_{ts} = 0$. $\square$

Theorem 2.4.2 (Hartfiel, [118]) For integers $n > m \ge 1$, let $F_1 \in \mathbf{B}_{n-m,m}$ such that $F_1 = E_{1,s}$, for some s with $1 \le s \le m$, $F_2 \in \mathbf{B}_{m,n-m}$ such that $F_2 = E_{t,n-m}$, for some t with $1 \le t \le m$. Then every nearly decomposable matrix $A \in \mathbf{B}_n$ is permutation equivalent to a matrix $B \in M_n^+$ with the form

$$B = \begin{bmatrix} A_0 & F_1 \\ F_2 & A_1 \end{bmatrix}, \tag{2.5}$$

where either $m = 1$ and $A_1 = 0$ or $m \ge 3$ and $A_1 = (a'_{ij}) \in M_m(0,1)$ is nearly decomposable with $a'_{ts} = 0$.

Proof By Theorem 2.1.3(v), $A \sim_p A'$ with $A' \ge I$. By Theorem 2.4.1(ii), $A' - I$ is nearly irreducible. By Theorem 2.3.5, there is a permutation matrix P such that $P(A'-I)P^{-1} =$

$PA'P^{-1} - I$ has the form in (2.5), Assume that $m \geq 2$, then by Lemma 2.4.2, A_1 is nearly indecomposable with at least one 0-entry. Since nearly indecomposable matrices in M_2^+ has no 0-entries, $m \geq 3$. $\square$

Example 2.4.2 A matrix $A \in \mathbf{B}_n$ with the form in Theorem 2.4.2 may not be nearly decomposable. Consider

$$A_1 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix}.$$

Then A_1 is nearly decomposable. However, $B - E_{3,2}$ is fully indecomposable and so B, having the form in Theorem 2.4.2, is not nearly decomposable.

Theorem 2.4.3 (Minc, [194]) Let $A \in \mathbf{B}_n$ be a nearly decomposable matrix. Then

$$\left.\begin{array}{ll} n & \text{if } n = 1 \\ 2n & \text{if } n \geq 2 \end{array}\right\} \leq \|A\| \leq \left\{\begin{array}{ll} 3n-2 & \text{if } n \leq 2 \\ 3(n-1) & \text{if } n \geq 3 \end{array}\right. \tag{2.6}$$

Proof The upper bound is trivial if $n \leq 2$, and so we assume that $n \geq 3$. By Theorem 2.1.3(iv) and by Theorem 2.3.7, $\|A - I\| \leq 2(n-1)$ with equality if and only if there is a tree T on n vertices such that $A - I = A(T)$. Note that if $n = 2$, then T has two pendant vertices (vertices of degree one); and if $n \geq 3$, then T has at most $n - 1$ pendant vertices. It follows that

$$\|A - I\| \leq \left\{\begin{array}{ll} 2(n-1) & \text{if } n = 2 \\ 2(n-1) - 1 & \text{if } n \geq 3, \end{array}\right.$$

which implies the upper bound in (2.6).

The lower bound in (2.6) is trivial if $n = 1$. When $n \geq 2$, note that each row of a fully indecomposable matrix has at least two 1-entries, and so the lower bound in (2.6) follows. $\square$

Theorem 2.4.4 Let $n \geq 3$ and $k > 0$ be integers such that $2n \leq k \leq 3(n-1)$. Then there exists $A \in \mathbf{B}_n$ such that A is nearly decomposable and $\|A\| = k$.

Sketch of Proof Write $k = 2(n-1) + s$ for some s with $2 \leq s \leq n-1$. Let T^s denote a tree on n vertices with exactly s vertices of degree one. (For example, T^s can be obtained by subdividing edges in a $K_{1,s}$.) Let T^s_o denote the graph obtained from T^s by attaching a loop at each vertex of degree one of T^s. Then $A(T^s_o)$ is nearly decomposable and $\|A(T^s_o)\| = k$. $\square$

2.5 Permanent

Definition 2.5.1 Let $n \geq m \geq 1$ be integers and let $A = (a_{ij}) \in M_{m,n}$. The *permanent* of A is

$$\mathrm{Per}(A) = \sum_{(i_1, i_2, \cdots, i_m) \in P_m^n} a_{1i_1} a_{2i_2} \cdots a_{mi_m},$$

where P_m^n is the set of all m-permutations of the integers $1, 2, \cdots, n$.

Both Proposition 2.5.1 and Proposition 2.5.2 follow directly from the related definitions.

Proposition 2.5.1 Let D be a directed graph with $|V(D)| = n \geq 1$ vertices and without parallel arcs, and let G be a simple graph on $n = 2m \geq 2$ vertices.

(i) Let $A = A(D)$. Each term in $\mathrm{Per}(A)$ is a one, which corresponds to a spanning subgraph of D consisting with disjoint directed cycles. Thus Per (A) counts the number of such subgraphs of D.

(ii) Let $A = A(G)$. If G is a bipartite graph with $A = \begin{bmatrix} 0 & B \\ B^T & 0 \end{bmatrix}$, for some $B \in M_m$, then $\mathrm{Per}(B)$ counts the number of perfect matchings of D.

Proposition 2.5.2 Let $A = (a_{ij}) \in M_{m,n}$ with $n \geq m \geq 1$. Each of the following holds.

(i) Let c be a scalar and let A' be obtained from A by multiplying the ith row of A by c. Then $\mathrm{Per}(A') = c\,\mathrm{Per}(A)$.

(ii) Fix i with $1 \leq i \leq m$. Suppose for each j with $1 \leq j \leq n$, $a_{ij} = a'_{ij} + a''_{ij}$. Let A' and A'' be obtained from A by replacing the (i,j)th entry of A by a'_{ij} and a''_{ij}, respectively. Then $\mathrm{Per}(A) = \mathrm{Per}(A') + \mathrm{Per}(A'')$.

(iii) If $A \sim_p B$, then $\mathrm{Per}(A) = \mathrm{Per}(B)$.

(iv) If $m = n$, then $\mathrm{Per}(A) = \mathrm{Per}(A^T)$.

(v) If $D_1 \in M_m$ and $D_2 \in M_n$ are diagonal matrices, then $\mathrm{Per}(D_1 A D_2) = \mathrm{Per}(D_1)$ $\mathrm{Per}(A)$ $\mathrm{Per}(D_2)$.

The following examples indicate that permanent and determinant behave quite differently.

Example 2.5.1 The permanent as a scalar function is not multiplicative. We can routinely check that

$$\mathrm{Per}\left(\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}\right) \neq \mathrm{Per}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \mathrm{Per}\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}.$$

Example 2.5.2 We can also verify the following:

$$\mathrm{Per}\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \det\begin{bmatrix} a_{11} & -a_{12} \\ a_{21} & a_{22} \end{bmatrix}.$$

However, Polyá in 1913 indicated that one cannot compute Per(A) by computing det(A'), where A' is obtained from A by changing the signs of some entries of A. Consider $A_3 = (a_{ij}) \in M_3$. Then

$$\begin{aligned} \det A_3 &= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} \\ &\quad - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31}. \end{aligned}$$

Assume that the signs of some entries can be changed so that the last three terms in det(A_3) can be positive. Then an odd number of sign changes is needed to make the last three term positive. On the other hand, to keep the first three term remaining the same sign, an even number of sign changes would be needed. A contradiction obtains.

For examples in M_n with $n > 3$, we can consider the direct sum of A_3 and I_{n-3} and repeat the argument above.

Theorem 2.5.1 (Laplace Expansion) Let $A = (a_{ij}) \in M_{m,n}$ with $n \geq m \geq 1$. Let $\alpha = (i_1, i_2, \cdots, i_k)$ be a fixed k-tuple of indices with $1 \leq i_1 < \cdots < i_k \leq m$, and $\beta = (j_1, j_2, \cdots, j_k)$ denote an general k-tuple of indices with $1 \leq j_1 < \cdots < j_k \leq n$. Then

$$\mathrm{Per}(A) = \sum_{\text{all possible } \beta} \mathrm{Per}A[\alpha|\beta] \, \mathrm{Per}A(\alpha|\beta).$$

In particular, we have the formula of the expansion of Per(A) by the ith row:

$$\mathrm{Per}(A) = \sum_{j=1}^{n} a_{ij} \, \mathrm{Per}A(i|j).$$

Sketch of Proof Each term in Per$A[\alpha|\beta]$ multiplying each term in Per$A(\alpha|\beta)$ is one term in Per(A). For a fixed β, there are $k!$ terms in a fixed Per$A[\alpha|\beta]$ and $\binom{n-k}{m-k}(m-k)!$ terms in Per$A(\alpha|\beta)$. There are $\binom{n}{k}$ different choices of β. Therefore, there are

$$\binom{n}{k} k! \binom{n-k}{m-k} (m-k)! = \binom{n}{m} m!$$

terms in the right hand side expression, which equals the number of all terms in Per(A). $\square$

To introduce Ryser's formula for computing Per(A), we need a weighted version of the Principle of Inclusion and Exclusion (Theorem 2.5.2 below). Let S be a set with $|S| = n \geq 1$, let F be a field and let $W : S \mapsto F$ be a function (called the *weight* function). Let $P_1, P_2, \cdots, P_N$ be N properties involving elements in S and denote $P = \{P_1, \cdots, P_N\}$.

For any subset $\{P_{i_1}, P_{i_2}, \cdots, P_{i_r}\} \subseteq P$ write

$$W(P_{i_1}, P_{i_2}, \cdots, P_{i_r}) = \sum_{s \in \cap_{j=1}^{r} P_{i_j}} W(s) \tag{2.7}$$

and

$$W(r) = \sum_{\text{all possible } \{P_{i_1}, P_{i_2}, \cdots, P_{i_r}\} \subseteq P} W(P_{i_1}, P_{i_2}, \cdots, P_{i_r}).$$

The following identities are needed in the proof of the next theorem.

$$\binom{k}{m}\binom{t}{k} = \binom{t}{m}\binom{t-m}{t-k}. \tag{2.8}$$

$$\sum_{j=0}^{t-m} (-1)^j \binom{t-m}{t-(m+j)} = (1-1)^{t-m} = 0. \tag{2.9}$$

Theorem 2.5.2 For an integer m with $1 \le m \le N$, let

$$E(m) = \sum \{W(s) \mid s \in S \text{ and } s \text{ has exactly } m \text{ properties out of } P\}.$$

Then

$$E(m) = \sum_{j=0}^{N-m} (-1)^j \binom{m+j}{m} W(m+j).$$

Sketch of Proof Assume that an $s \in S$ satisfies exactly t properties out of P. Consider the contribution of s to the right hand side of the equality. If $t < m$, then s makes no contribution; if $t = m$, then the contribution of s is $W(s)$; and if $t > m$, then the contribution of s is

$$\begin{aligned}
& \left[\sum_{j=0}^{t-m} (-1)^j \binom{t}{m+j}\binom{m+j}{m}\right] W(s) \\
= & \binom{t}{m}\left[\sum_{j=0}^{t-m} (-1)^j \binom{t-m}{t-(m+j)}\right] W(s) \\
= & \binom{t}{m}(0) W(s) = 0,
\end{aligned}$$

Then the theorem follows by (2.8) and (2.9). □

Corollary 2.5.2 With the same notation in Theorem 2.5.2, we can write

$$E(0) = W(0) - W(1) + W(2) - \cdots + (-1)^N W(N).$$

Theorem 2.5.3 (Ryser, [197]) Let $A \in M_{m,n}$ with $n \geq m \geq 1$. For each r with $1 \leq r \leq n$, let A_r denote a matrix obtained from A by replacing some r columns of A by all zero columns, $S(A_r)$ the product of the m row sums of A_r, and $\sum S(A_r)$ the sum of the $S(A_r)$'s with the summation taken over all possible A_r's. Then

$$\operatorname{Per}(A) = \sum_{j=0}^{m-1} \binom{n-m+j}{j} \sum S(A_{n-m+j}).$$

Proof Let S be the set of all m-combinations of the column labels $1, 2, \cdots, n$. Then each $s \in S$ has the form $(j_1, j_2, \cdots, j_m)$, where $1 \leq j_1 < j_2 < \cdots < j_m \leq n$. Define

$$W(s) = W(j_1, j_2, \cdots, j_m) = a_{1j_1} a_{2j_2} \cdots a_{mj_m},$$

and $P_i = \{(j_1, j_2, \cdots, j_m) \in S | i \in \{j_1, j_2, \cdots, j_m\}\}$, $1 \leq i \leq n$. Then $W(r) = \sum S(A_r)$ and so Theorem 2.5.3 follows from Theorem 2.5.2. $\square$

Corollary 2.5.3 With the same notation in Theorem 2.5.3, we can write

$$\operatorname{Per}(A) = S(A) - \sum S(A_1) + \sum S(A_2) - \cdots + (-1)^{n-1} \sum S(A_{n-1}).$$

Both Theorem 2.5.1 and Theorem 2.5.3 are not easy to apply in actual computation. Therefore, estimating the upper and lower bounds of $\operatorname{Per}(A)$ becomes important. The proofs of the following lemmas are left as exercises.

Lemma 2.5.1 Let $A = (a_{ij}) \in \mathbf{B}_{m,n}$ with $n \geq m \geq 1$. If for each i, $\sum_{j=1}^{n} a_{ij} \geq m$, then $\operatorname{Per}(A) > 0$.

Lemma 2.5.2 Let $A = (a_{ij}) \in \mathbf{B}_{m,n}$ with $n \leq m \leq 1$. If for each k with $1 \leq k \leq m-1$, every $k \times n$ submatrix of A has at least $k+1$ nonzero columns, then every $(m-1) \times (n-1)$ submatrix A' of A satisfies $\operatorname{Per}(A') > 0$.

Theorem 2.5.4 (Hall-Mann-Ryser, [116]) Let $A \in \mathbf{B}_{m,n}$ with $n \leq m \geq 1$ such that each row of A has at leat t 1-entries. Then each of the following holds.

(i) If $t \geq m$, then $\operatorname{Per}(A) \geq t!/(t-m)!$.

(ii) If $t \leq m$ and if $\operatorname{Per}(A) > 0$, then $\operatorname{Per}(A) \geq t!$.

Sketch of Proof By Lemma 2.5.1, we may assume that $\operatorname{Per}(A) > 0$ for each t with $1 \leq t \leq n$.

Argue by induction on m. The theorem holds trivially when $m = 1$. Assume $m > 1$.

Case 1 For some $1 \leq h \leq m-1$ and $B \in M_h$, $A \sim_p A_1$ where

$$A_1 = \begin{bmatrix} B & 0 \\ C & D \end{bmatrix}.$$

Then each row of B must have all the t positive entries, and so $t \le h \le m-1$. Moreover, $\text{Per}(A) = \text{Per}(B)\ \text{Per}(D) > 0$. By induction, $\text{Per}(B) \ge t!$ and so $\text{Per}(A) \ge t!$.

Case 2 Case 1 does not occur.

Then for each k with $1 \le k \le m-1$, every $k \times n$ submatrix of A must have at least $k+1$ nonzero columns. By Lemma 2.5.2, for each submatrix $A(i|j)$, obtained from A by deleting the ith row and the jth column, satisfies $\text{Per}(A(i|j)) > 0$. Note that each row of $A(i|j)$ has at least $(t-1)$ positive entries. By induction,

$$\text{Per}(A(i|j)) = \begin{cases} (t-1)! & \text{if } t-1 \le m-1 \\ \frac{(t-1)!}{(t-m)!} & \text{if } t-1 \ge m-1. \end{cases}$$

It follows by $\sum_{j=1}^{m} a_{ij} \ge t$ that when $t \le m$,

$$\begin{aligned} \text{Per}(A) &= \sum_{j=1}^{n} a_{ij}\ \text{Per}(A(1|j)) \\ &\ge \sum_{j=1}^{n} a_{ij}(t-1)! \ge t!; \end{aligned}$$

and when $t \ge m$,

$$\begin{aligned} \text{Per}(A) &= \sum_{j=1}^{n} a_{ij}\ \text{Per}(A(1|j)) \\ &\ge \sum_{j=1}^{n} a_{ij}\frac{(t-1)!}{(t-m)!} \ge \frac{t!}{(t-m)!}. \end{aligned}$$

The theorem now follows by induction. $\square$

Theorem 2.5.5 (Minc, [195]) Let $A \in \mathbf{B}_n$ be fully indecomposable. Then

$$\text{Per}\ (A) \ge \|A\| - 2n + 2. \tag{2.10}$$

An improvement of Theorem 2.5.5 can be found in Exercise 2.14. Gibson [99] gave another improvement.

Theorem 2.5.6 (Gibson, [99]) Let $A \in \mathbf{B}_n$ be fully indecomposable such that each row of A has at least t positive entries. Then

$$\text{Par}(A) \ge \|A\| - 2n + 2 + \sum_{i=1}^{t-1}(i! - 1).$$

An important upper bound of $\text{Per}(A)$ was conjectured Minc in [196] and proved by Bregman [16]. The proof needs two lemmas and is due to Schrijver [230]).

Lemma 2.5.3 Let $A \in \mathbf{B}_n$ with $\mathrm{Per}(A) > 0$. Let S denote the set of permutations on n elements such that $\sigma \in S$ if and only if $\prod_{i=1}^{n} a_{i\sigma(i)} = 1$. Then each of the following holds.

(i) $\prod_{i=1}^{n} \prod_{a_{ik}=1} (\mathrm{Per}(A(i|k))^{\mathrm{Per}(A(i|k))} = \prod_{\sigma \in S} \prod_{i=1}^{n} \mathrm{Per}(A(i|\sigma(i)))$.

(ii) If $r_1, \cdots, r_n$ are the row sums of A, then $\prod_{i=1}^{n} r_i^{\mathrm{Per}A} = \prod_{\sigma \in S} \prod_{i=1}^{n} r_i$.

Sketch of Proof For fixed i and k, the number of $\mathrm{Per}(A(i|k))$ factors on the left hand side of (i) is $\mathrm{Per}(A(i|k))$ when $a_{ik} = 1$ and 0 otherwise; and the number of $\mathrm{Per}(A(i|k))$ factors on the right hand side of (i) is the number of permutations $\sigma \in S$ such that $\sigma(i) = k$, which is $\mathrm{Per}(A(i|k))$ when $a_{ik} = 1$ and 0 otherwise.

For (ii), the number of r_i factors on either side equals $\mathrm{Per}(A)$. $\square$

Lemma 2.5.4 Assume that $0^0 = 1$. If $t_1, t_2, \cdots, t_n$ are non negative real numbers, then

$$\left(\frac{1}{n}\sum_{i=1}^{n} t_k\right)^{\sum_{k=1}^{n} t_k} \le \prod_{k=1}^{n} t_k^{t_k}.$$

Sketch of Proof By the convexity of the function $x \log x$,

$$\left(\frac{1}{n}\sum_{k=1}^{n} t_k\right) \log \left(\frac{1}{n}\sum_{k=1}^{n} t_k\right) \le \frac{1}{n}\sum_{k=1}^{n} t_k \log t_k,$$

and so the lemma follows. $\square$

Theorem 2.5.7 (Minc-Bregman,[16]) Let $A \in \mathbf{B}_n$ be a matrix with row sums $r_1, r_2, \cdots, r_n$. Then

$$\mathrm{Per}(A) \le \prod_{i=1}^{n} (r_i!)^{\frac{1}{r_i}}.$$

Proof Argue by induction on n. By Lemma 2.5.3,

$$\begin{aligned}
(\mathrm{Per}A)^{n\mathrm{Per}(A)} &= \prod_{i=1}^{n} \left(\sum_{k=1}^{n} a_{ik}\ \mathrm{Per}A(i|k)\right)^{\sum_{k=1}^{n} a_{ik}\mathrm{Per}(A(i|k))} \\
&\le \prod_{i=1}^{n} \left(r_i^{\mathrm{Per}(A)} \prod_{k,\ a_{ik}=1} (\mathrm{Per}A(i|k))^{\mathrm{Per}(A(i|k))}\right).
\end{aligned}$$

By Lemma 2.5.4,

$$(\mathrm{Per}(A))^{n\mathrm{Per}(A)} \le \prod_{\sigma \in S} \left[\left(\prod_{i=1}^{n} r_i\right) \prod_{i=1}^{n} \mathrm{Per}(A(i|\sigma(i)))\right].$$

Apply induction to each $(A(i|\sigma(i)))$ to get

$$\begin{aligned}
\prod_{i=1}^{n} \operatorname{Per}(A(i|\sigma(i))) &\leq \prod_{i=1}^{n}\left[\left(\prod_{j\neq i,\ a_{j\sigma(i)}=0} (r_j!)^{\frac{1}{r_j}}\right)\left(\prod_{j\neq i,\ a_{j\sigma(i)}=1} ((r_j-1)!)^{\frac{1}{r_j-1}}\right)\right] \\
&= \prod_{j=1}^{n}\left[\left(\prod_{i\neq j,\ a_{j\sigma(i)}=0} (r_j!)^{\frac{1}{r_j}}\right)\left(\prod_{i\neq j,\ a_{j\sigma(i)}=1} ((r_j-1)!)^{\frac{1}{r_j-1}}\right)\right] \\
&= \prod_{j=1}^{n}\left[(r_j!)^{\frac{n-r_j}{r_j}}((r_j-1)!)^{\frac{r_j-1}{r_j-1}}\right].
\end{aligned}$$

where the last equality above is obtained by counting the number of $(r_j!)^{\frac{1}{r_j}}$ factors and the number of $((r_j-1)!)^{\frac{1}{r_j-1}}$ factors: for fixed j and $\sigma \in S$, there are $n-r_j$ of i's satisfying $i \neq j$ and $a_{j\sigma(i)} = 0$ and there are $r_j - 1$ of i's satisfying $i \neq j$ and $a_{j\sigma(i)} = 1$. It follows that

$$\begin{aligned}
\prod_{i=1}^{n} \operatorname{Per}(A(i|\sigma(i))) &\leq \prod_{\sigma\in S}\left[\left(\prod_{i=1}^{n} r_i\right)\left(\prod_{j=1}^{n}(r_j!)^{\frac{n-r_j}{r_j}}(r_j-1)!\right)\right] \\
&= \prod_{\sigma\in S}\left(\prod_{i=1}^{n}(r_i!)^{\frac{n}{r_j}}\right) \\
&= \left(\prod_{i=1}^{n}(r_i!)^{\frac{1}{r_i}}\right)^{n\operatorname{Per}(A)}.
\end{aligned}$$

This proves the theorem. □

An upper bound of permanents of fully indecomposable matrices in M_n^+ was obtained by Foregger [90]. A few lemmas are needed for the proof.

Lemma 2.5.5 Let $A = (a_{ij}) \in M_n^+$ with $n \geq 2$ be a fully indecomposable matrix with an entry $a_{rs} \geq 2$. Then

$$\operatorname{Per}(A(r|s)) \leq \frac{\operatorname{Per}(A) - 1}{2}.$$

Lemma 2.5.6 Let $n \geq 3$ be an integer. Then $n! < 2^{n(n-2)}$.

Lemma 2.5.7 (Foregger, [90]) Let $A \in M_n^+$ be a fully indecomposable matrix. If each row sum of A is at least 3, Then

$$\operatorname{per}(A) < 2^{\|A\|-2n}.$$

Lemma 2.5.8 Let $n \geq 2$ and let $A = (a_{ij}) \in M_n^+$ be a fully indecomposable matrix. Then there exists a fully indecomposable $B \in \mathbf{B}_n$ and an integer $j \geq 0$ such that $B \leq A$, $\|A\| - \|B\| = j$ and $\operatorname{Per}(A) \leq 2^j \operatorname{Per}(B) - (2^j - 1)$.

Sketch of Proof If $A \in \mathbf{B}_n$, then $j = 0$ and $B = A$. Therefore, we assume that A has an entry $a_{rs} \geq 2$. Let $A' = A - E_{rs}$. By Lemma 2.5.5 and since $\mathrm{Per}(A) = \mathrm{Per}(A') + \mathrm{Per}(A(r|s))$, it follows that

$$\mathrm{Per}(A) \leq 2\,\mathrm{Per}(A') - 1.$$

Argue by induction on the number of entries bigger than one to complete the proof of the lemma. $\square$

Theorem 2.5.8 (Foregger, [90]) Let $A \in M_n^+$ be a fully indecomposable matrix. Then

$$\mathrm{per}(A) \leq 2^{\|A\|-2n} + 1.$$

Sketch of Proof Argue by induction on n and the theorem is trivial if $n = 1$. Assume $n \geq 2$. By Lemma 2.5.7, we may assume that $r_1 \in \{1, 2\}$. Since A is fully indecomposable and by Theorem 2.1.2, $r_1 = 2$ and we may assume that the first row of A is $1, 1, 0, \cdots, 0$. Let A' denote the matrix obtained from A by adding Column 1 to Column 2 of A, and then deleting both Row 1 and Column 1. By Theorem 2.1.2, A' is also fully indecomposable and $\|A'\| = \|A\| - 2$.

Case 1 $A \in \mathbf{B}_n$. Then

$$\mathrm{Per}(A) = \mathrm{Per}(A(1|1)) + \mathrm{Per}(A(1|2)) = \mathrm{Per}(A').$$

By induction, $\mathrm{Per}(A) = \mathrm{Per}(A') \leq 2^{\|A'\|-2(n-1)} + 1 = 2^{\|A\|-2n} + 1$.

Case 2 A has an entry at least 2. Then by Lemma 2.5.8, there exist a matrix $B \in \mathbf{B}_n$ and an integer j satisfying Lemma 2.5.8. Applying Case 1 to B to get

$$\begin{aligned} \mathrm{Per}(A) &= 2^j\,\mathrm{Per}(B) - (2^j - 1) \\ &\leq 2^j(2^{\|B\|-2n} + 1) - (2^j - 1) = 2^{\|A\|-2n} + 1. \end{aligned}$$

This completes the proof. $\square$

Foregger also characterized matrices which have equality in Theorem 2.5.8. Interested reader are referred to [90]. Brualdi and Gibson noted that a matrix with total support can be viewed as the direct sum of totally indecomposable matrices under the $\sim_p$ relation, Theorem 2.5.8 can be extended.

Theorem 2.5.9 (Brualdi and Gibson, [27]) Let $A \in M_n^+$ be a matrix with a total support with t fully indecomposable blocks. Then

$$\mathrm{per}(A) < 2^{\|A\|-2n+t}.$$

The upper bound of Per(A) for fully indecomposable matrices was further improved by Donald *et al* [75].

Theorem 2.5.10 (Donald, Elwin, Hager, and Salomon, [75]) Let $A = (a_{ij}) \in M_n^+$ be a fully indecomposable matrix with row sums $r_i = \sum_{j=1}^n a_{ij}$, $1 \le i \le n$, and column sums $s_j = \sum_{i=1}^n a_{ij}$, $1 \le j \le n$. Then

$$\mathrm{Per}(A) \le 1 + \min\left\{ \prod_{i=1}^n (r_i - 1),\ \prod_{i=1}^n (s_i - 1) \right\}.$$

Brualdi *et al* considered upper bounds of Per(A) when $A \in \mathbf{B}_n$ with restrictions on $\|A\|$. If $\sigma = \|A\|$ for some $A \in \mathbf{B}_n$, then $\tau = n^2 - \sigma$ is the number of zero entries of A. For τ with $0 \le \tau \le n^2$, Let

$$\mathcal{U}(n,\tau) = \{A \in \mathbf{B}_n : \|A\| = n^2 - \tau\}.$$

and

$$\mu(n,\tau) = \max\{\ \mathrm{Per}(A) : A \in \mathcal{U}(n,\tau)\}.$$

Note that if $\tau > n^2 - n$, then $\mu(n,\tau) = 0$.

Theorem 2.5.11 (Brualdi, Goldwasser and Michael, [28]) Let $A \in \mathcal{U}(n,\tau)$ with $\tau \le n^2 - n$. Let $\sigma = n^2 - \tau$ and $r = \lfloor \sigma/n \rfloor$. Then

$$\mathrm{Per}(A) \le (r!)^{\frac{nr+n-\sigma}{r}} ((r+1)!)^{\frac{\sigma-nr}{r+1}}. \tag{2.11}$$

Sketch of Proof By Theorem 2.5.7, we need to estimate $\prod_{i=1}^n (r_i!)^{\frac{1}{r_i}}$, under the conditions that r_i's are positive integers and that $\sum_{i=1}^n r_i = \sigma$.

To do that, we first we establish the following inequality. For integers m, t with $m \ge 2$ and $t \ge 1$,

$$((m+t-1)!)^{\frac{1}{m+t-1}} (m!)^{\frac{1}{m}} > ((m+t)!)^{\frac{1}{m+t}} ((m-1)!)^{\frac{1}{m-1}}. \tag{2.12}$$

For any integer $k \ge 2$, we have $k^2 > (k-1)(k+1)$ and so

$$k^{2k(k-1)} > [(k-1)(k+1)]^{k(k-1)}.$$

It follows that

$$\frac{k^{(k+2)(k-1)}}{(k-1)^{(k+1)(k-2)}} > \frac{(k+1)^{k(k-1)}(k-1)^2}{k^{(k-1)(k-2)}},$$

and so

$$\prod_{k=2}^{s} \frac{k^{(k+2)(k-1)}}{(k-1)^{(k+1)(k-2)}} > \prod_{k=2}^{s} \frac{(k+1)^{k(k-1)}(k-1)^2}{k^{(k-1)(k-2)}}.$$

Cancel the common factors from both the numerators and the denominators to get

$$s^{(s+2)(s-1)} > (s+1)^{s(s-1)}((s-1)!)^2 \tag{2.13}$$

Multiplying $s^{s^2-s}((s-1)!)^{2s^2-2}$ to both sides of (2.13) and then raise both sides of (2.13) to the power of $1/(s(s-1)(s+1))$ yields

$$(s!)^{\frac{2}{s}} > ((s-1)!)^{\frac{1}{s-1}}((s+1)!)^{\frac{1}{s+1}}.$$

It follows that

$$\prod_{s=m}^{m+t-1}(s!)^{\frac{2}{s}} > \prod_{s=m}^{m+t-1}((s-1)!)^{\frac{1}{s-1}}((s+1)!)^{\frac{1}{s+1}}, \tag{2.14}$$

and so (2.12) follows from (2.14).

By (2.12),

$$\max\left\{\prod_{i=1}^{n}(r_i!)^{\frac{1}{r_i}} \ : \ r_i\text{'s are positive integers and } \sum_{i=1}^{n} r_i = \sigma\right\}$$

can be reached if there exists some integers a, b and r satisfying

$$\begin{cases} a+b & = n \\ ar+b(r+1) & = \sigma, \end{cases} \tag{2.15}$$

such that a of the r_i's are equal to $r = \lfloor \sigma/n \rfloor$, b of the r_i's are equal to $r+1$.

Note that the unique solution of (2.15) is

$$\begin{cases} a & = nr+n-\sigma \\ b & = \sigma - nr. \end{cases}$$

Therefore, by Theorem 2.5.7

$$\begin{aligned} \mu(n,\tau) &= \max\{\operatorname{Per}A \ : \ A \in \mathcal{U}(n,\tau)\} \\ &\le \max\left\{\prod_{i=1}^{n}(r_i!)^{\frac{1}{r_i}} \ : \ r_i\text{'s are positive integers and } \sum_{i=1}^{n} r_i = \sigma\right\} \\ &= (r!)^{\frac{a}{r}}((r+1)!)^{\frac{b}{r+1}}. \end{aligned}$$

This completes the proof. □

The bound in (2.11) is reachable, for certain integer values. For example, choose integers a and b such that

$$a = \frac{nr+n-\sigma}{r} \text{ and } b = \frac{\sigma - nr}{r+1}.$$

Let A be the direct sum of a J_r's and b J_{r+1}'s. Then $A \in \mathcal{U}(n,\tau)$ and $\mathrm{Per}(A) = (r!)^a((r+1)!)^b$.

Brualdi, Goldwasser and Michael also proved the following in [28]. The extremal matrices are characterized in [28].

Theorem 2.5.12 (Brualdi, Goldwasser and Michael, [28]) Let n, τ be integers such that $n > 3$, $n^2 - 2n \le \tau \le n^2 - n$. Then

$$\mu(n,\tau) = 2^{\lfloor (n^2-\tau-n)/2 \rfloor}.$$

2.6 The Class $\mathcal{U}(\mathbf{r},\mathbf{s})$

The study of the class $\mathcal{U}(\mathbf{r},\mathbf{s})$ was started by Ryser and Fulkerson, among others, in the early 1950's. The central problems have been the existence of matrices in $\mathcal{U}(\mathbf{r},\mathbf{s})$, the structure of $\mathcal{U}(\mathbf{r},\mathbf{s})$, and counting and estimating $|\mathcal{U}(\mathbf{r},\mathbf{s})|$.

Definition 2.6.1 Let $\mathbf{r} = (r_1, r_2, \cdots, r_m)^T$ and $\mathbf{s} = (s_1, s_2, \cdots, s_n)^T$ be two non negative vectors. We say $\mathbf{r}$ is increasing if $r_1 \ge r_2 \ge \cdots \ge r_m$. Similar definition holds for $\mathbf{s}$. Let $\mathcal{U}(\mathbf{r},\mathbf{s})$ denote the set of all matrices $A = (a_{ij}) \in \mathbf{B}_{m,n}$ such that $\sum_{j=1}^{n} a_{ij} = r_i, 1 \le i \le m$ and $\sum_{i=1}^{m} a_{ij} = s_j, 1 \le j \le n$.

When $A \in \mathcal{U}(\mathbf{r},\mathbf{s})$, $\mathbf{r},\mathbf{s}$ are called the *row sum vector* and *column sum vector* of A.

The examples below present different views of the matrices in $\mathcal{U}(\mathbf{r},\mathbf{s})$.

Example 2.6.1 Given $\mathbf{r} = (r_1, r_2, \cdots, r_m)^T$ and $\mathbf{s} = (s_1, s_2, \cdots, s_n)^T$, the set $\mathcal{U}(\mathbf{r},\mathbf{s}) \ne \emptyset$ if and only if there exists a bipartite graph with bipartition (X, Y) with $X = \{x_1, \cdots, x_m\}$ and $Y = \{y_1, \cdots, y_n\}$ such that the degree of each x_i is r_i and the degree of each y_j is s_j.

Example 2.6.2 Suppose $\mathbf{r} = (r_1, r_2, \cdots, r_m)^T$ and $\mathbf{s} = (s_1, s_2, \cdots, s_n)^T$ satisfy both $r_i > 0$ and $s_j > 0$ for all i and j, and $\sum_{i=1}^m r_i = \sum_{j=1}^n s_j = t$. Let T be a set with t elements. Then $\mathcal{U}(\mathbf{r},\mathbf{s}) \ne \emptyset$ if and only if there exist two partitions of T

$$T = F_1 \cup F_2 \cup \cdots \cup F_m = G_1 \cup G_2 \cup \cdots \cup G_n,$$

such that $|F_i| = r_i, |G_j| = s_j$ and $|F_i \cap G_j| \in \{0, 1\}$, where $1 \le i \le m$ and $1 \le j \le n$.

Definition 2.6.2 Given a non negative vector $\mathbf{r} = (r_1, r_2, \cdots, r_m)^T$, let

$$\delta_i = (1, 1, \cdots, 1, 0, \cdots, 0)^T, \ (1 \le i \le m),$$

be an n dimensional vector with 1's in the first r_i components and 0's elsewhere. A matrix of the form

$$\overline{A} = [\delta_1, \delta_2, \cdots, \delta_m]^T \tag{2.16}$$

is a *maximal matrix* with row sum vector $\mathbf{r}$.

For two n-dimensional non negative vectors $\mathrm{s} = (s_1, s_2, \cdots, s_n)^T$ and $\mathrm{s}' = (s_1', s_2', \cdots, s_n')$ The vector s is *majorized by* s', written

$$\mathrm{s} \prec \mathrm{s}',$$

if each of the following holds after necessary subscripts renumbering:
(2.6.2A) $s_1 \geq s_2 \geq \cdots \geq s_n$ and $s_1' \geq s_2' \geq \cdots \geq s_n'$,
(2.6.2B) For each i with $1 \leq i \leq n-1$, $\sum_{j=1}^{i} s_j \leq \sum_{j=1}^{i} s_j'$, and
(2.6.2C) $\sum_{j=1}^{n} s_j = \sum_{j=1}^{n} s_j'$.

Proposition 2.6.1 summerizes two observations that follow from the definition.

Proposition 2.6.1 Let $\overline{A}$ be the matrix defined in (2.16). Then
(i) the column sum vector $\overline{\mathrm{s}} = (\overline{s}_1, \overline{s}_2, \cdots, \overline{s}_n)^T$ of $\overline{A}$ is monotone, and
(ii) $\sum_{i=1}^{m} r_i = \sum_{j=1}^{n} \overline{s}_j$, and
(iii) $\mathcal{U}(\mathbf{r}, \overline{\mathrm{s}}) = \{\overline{A}\}$.

Gale and Ryser independently proved the necessary and sufficient condition (Theorem 2.6.1 below) for $\mathcal{U}(\mathbf{r}, \mathrm{s})$ to be non empty. Their constructive proofs can be found in [95] and [223]. Gale and Ryser's proofs cannot be used to estimate $|\mathcal{U}(\mathbf{r}, \mathrm{s})|$.

The necessity of Theorem 2.6.1 is straightforward and is left as an exercise. The sufficiency of Theorem 2.6.1 follows from Theorem 2.6.2, due to Wei [273], which also gives a nontrivial lower bound for $|\mathcal{U}(\mathbf{r}, \mathrm{s})|$.

Theorem 2.6.1 (Gale [95] and Ryser [223]) Given non negative vectors $\mathbf{r} = (r_1, r_2, \cdots, r_m)^T$ and $\mathrm{s} = (s_1, s_2, \cdots, s_n)^T$ with $r_i \leq n$ for each i with $1 \leq i \leq m$. Let $\overline{\mathrm{s}}$ be defined as in Definition 2.6.2. Then $\mathcal{U}(\mathbf{r}, \mathrm{s}) \neq \emptyset$ if and only if $\mathrm{s} \prec \overline{\mathrm{s}}$.

Definition 2.6.3 Suppose that $\mathrm{s}' = (s_1', \cdots, s_n')^T$ and $\mathrm{s}'' = (s_1'', \cdots, s_n'')^T$ are two non negative monotone vectors with $\mathrm{s}'' \prec \mathrm{s}'$. Let ν be the smallest subscript such that $s_\nu'' - s_\nu' > 0$ and μ be the largest subscript such that $s_\mu' - s_\mu'' > 0$. Define

$$W(\mathrm{s}'', \mathrm{s}') = \begin{pmatrix} s_\mu' - s_\nu' \\ \min\{s_\mu' - s_\mu'', s_\nu'' - s_\nu'\} \end{pmatrix},$$

and define a new vector $\mathrm{s}^{(1)} = (s_1^{(1)}, \cdots, s_n^{(1)})^T$ according to the two different cases as follows:
(2.6.3A) When $s_\mu' - s_\mu'' \geq s_\nu'' - s_\nu' > 0$,

$$\mathrm{s}^{(1)} = (s_1', \cdots, s_{\mu-1}' s_\mu' - (s_\nu'' - s_\nu'), s_{\mu+1}', \cdots, s_{\nu-1}', s_\nu'', \cdots, s_n')^T.$$

(2.6.3B) When $s_\nu'' - s_\nu' \geq s_\mu' - s_\mu'' > 0$,

$$\mathrm{s}^{(1)} = (s_1', \cdots, s_{\mu-1}' s_\mu'', s_{\mu+1}', \cdots, s_{\nu-1}', s_\nu' + (s_\nu' - s_\nu''), s_{\nu+1}', \cdots, s_n')^T.$$

We can routinely verify that $s'' \prec s^{(1)} \prec s'$ (Exercise 2.19). Moreover, there exist an integer $k \geq 1$ and non negative n-dimensional vectors $s^{(0)}, s^{(1)}, \cdots, s^{(k)}$ such that

$$s'' = s^{(k)} \prec s^{(k-1)} \prec \cdots \prec s^{(1)} \prec s^{(0)} = s',$$

which is called a *whole chain* from s' to s''.

Theorem 2.6.2 (Wei, [273]) Given non negative vectors $\mathbf{r} = (r_1, r_2, \cdots, r_m)^T$ and $s = (s_1, s_2, \cdots, s_n)^T$ with $r_i \leq n$ for each i with $1 \leq i \leq m$. Let $\bar{s}$ be defined as in Definition 2.6.2. If s is monotone and if $s \prec \bar{s}$, then

$$|\mathcal{U}(\mathbf{r}, s)| \geq \prod_{i=0}^{k-1} W(s, s^{(i)}) \geq 1,$$

where $s^{(0)}, s^{(1)}, \cdots, s^{(k-1)}$ are the elements in the whole chain from $\bar{s}$ to s.

Proof Denote the whole chain from s to $\bar{s}$ by

$$s = s^{(k)} \prec s^{(k-1)} \prec \cdots \prec s^{(1)} \prec s^{(0)} = \bar{s}, \tag{2.17}$$

where $\bar{s}$ is the vector defined in Proposition 2.6.1(i).

Let s'' be a one of the vectors in the whole chain (2.17), let s' be a vector satisfying $s'' \prec s'$ and let $A' = (a'_{ij}) \in \mathcal{U}(\mathbf{r}, s')$. Assume that $s'_\mu > s'_\nu$ and let B' the $m \times 2$ submatrix A':

$$B' = \begin{bmatrix} a'_{1\mu} & a'_{1\nu} \\ a'_{2\mu} & a'_{2\nu} \\ \vdots & \vdots \\ a'_{n\mu} & a'_{n\nu} \end{bmatrix}.$$

Note that B' has at least $s'_\mu - s'_\nu \geq 2$ rows of the form $(1, 0)$. By Definition 2.6.3, to construct the whole chain, we should exchange the elements in the $s''_\nu - s'_\nu$ rows of the form $(1, 0)$, if (2.6.3A) holds, or exchange the elements in the $s'_\mu - s''_\mu$ rows of the form $(1, 0)$, if (2.6.3B) holds. In either case, elements in $\min\{s'_\mu - s''_\mu, s''_\nu - s'_\mu\}$ rows in B' that are of the form $(1, 0)$ should be exchanged. The number of different ways to perform these exchanges is $W(s'', s')$.

It follows by (2.17), by Proposition 2.6.1(iii) and by the argument above that we can generate at least $\prod_{i=0}^{k-1} W(s, s^{(i)})$ different matrices that are all in $\mathcal{U}(\mathbf{r}, s)$. $\square$

Example 2.6.3 (Wei, [273]) Let

$$\mathbf{r} = (7, 5, 5, 4, 4, 3, 2, 2, 1)^T \text{ and } s = (7, 6, 4, 4, 4, 4, 4)^T.$$

Then $\bar{s} = (9,8,6,5,3,1,1)^T$ and $s \prec \bar{s}$. A whole chain from $\bar{s}$ to s can be found as follows:

$$\begin{aligned}
s^{(0)} = \quad \bar{s} \quad &= (9,8,6,5,3,1,1)^T, \quad W(s, s^{(0)}) = \binom{2}{1} \\
s^{(1)} \quad &= (9,8,6,4,4,1,1)^T, \quad W(s, s^{(1)}) = \binom{5}{2} \\
s^{(2)} \quad &= (9,8,4,4,4,3,1)^T, \quad W(s, s^{(2)}) = \binom{5}{1} \\
s^{(3)} \quad &= (9,7,4,4,4,4,1)^T, \quad W(s, s^{(3)}) = \binom{6}{1} \\
s^{(4)} \quad &= (9,6,4,4,4,4,2)^T, \quad W(s, s^{(0)}) = \binom{7}{2} \\
s^{(5)} \quad &= s \ = \ (7,6,4,4,4,4,4)^T.
\end{aligned}$$

Therefore,

$$|\mathcal{U}(\mathbf{r}, s)| \leq \binom{2}{1}\binom{5}{2}\binom{5}{1}\binom{6}{1}\binom{7}{2} = 12600.$$

In [268], Wan refined the concept of totals chains and improved the lower bound in Theorem 2.6.2. An application of Theorem 2.6.2 can be found in Exercise 2.20.

Example 2.6.4 (Ryser, [223]) Let

$$A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ and } A_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Suppose $A \in \mathcal{U}(\mathbf{r}, s)$ which contains a submatrix A_1. Let A' be obtained from A by changing this submatrix A_1 into A_2, while keeping all other entries of A unchanged. Then $A' \in \mathcal{U}(\mathbf{r}, s)$. The operation of replacing the submatrix A_1 by A_2 to get A' from A is called an *interchange*, and we say that A' is obtained from A by performing an interchange on A_1.

Theorem 2.6.3 (Ryser, [223]) For $A, A' \in \mathcal{U}(\mathbf{r}, s)$, A' can be obtained from A by performing a finite sequence of interchanges.

Proof Without loss of generality, we assume that both $\mathbf{r} = (r_1, \cdots, r_m)^T$ and $s = (s_1, \cdots, s_n)^T$ are monotone. Argue by induction on n. Note that the theorem holds trivially if $n = 1$, and so we assume that $n \geq 2$, and that the theorem holds for smaller values of n.

Denote $A = (a_{ij})$ and $A' = (a'_{ij})$. Consider the $m \times 2$ matrix

$$M = \begin{bmatrix} a_{1n} & a'_{1n} \\ a_{2n} & a'_{2n} \\ \vdots & \vdots \\ a_{mn} & a'_{mn} \end{bmatrix}.$$

The rows of M have 4 types: (1,1), (1,0), (0,1) and (0,0). Since $A, A' \in \mathcal{U}(\mathbf{r}, \mathbf{s})$, A and A' have the same column sums and so rows of Type (1,0) and (0,1) must occur in pairs.

If M does not have a row of Type (1,0), then M can only have rows of Type (1,1) or (0,0), which implies that the two columns of M are identical. The theorem obtains by applying induction to the two submatrices consisting of the first $n-1$ columns of A and A.

Therefore we assume that M has some rows of Type (1,0) and (0,1). Let $j = j(A, A')$ be the smallest row label such that Row j of M is either (1,0) or (0,1). Without loss of generality, assume that $(a_{jn}, a'_{jn}) = (0, 1)$.

By the minimality of j, there exists an integer k with $j+1 \le k \le n\}$ such that $a_{kn} = 1$. Since $a'_{jn} = 1$ and since A and A' has the same row sums, A has at least a 1-entry in Row j. Let $a_{ji_1}, \cdots, a_{ji_l}$ be the 1-entries of A in the jth row. Then $1 \le l \le r_j$.

Since $\mathbf{r}$ is monotone, $r_j \ge r_k$. As $a_{jn} = 0$ and $a_{kn} = 1$, we may assume that $a_{kt} = 0$ for some $t \in \{i_1, i_2, \cdots, i_l\}$. Thus M has a submatrix

$$B = \begin{bmatrix} a_{jt} & a_{jn} \\ a_{kt} & a_{kn} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Let A_1 be the matrix obtained from A by performing an interchange on B. Note that $j(A_1, A') \ge j(A, A')+1$, which means we can perform at most m interchanges to transform A to a matrix $A'' \in \mathcal{U}(\mathbf{r}, \mathbf{s})$ such that the last column of A'' is identical with the last column of A', and so the theorem can be proved by induction. $\square$

Definition 2.6.4 If both $\mathbf{r}$ and $\mathbf{s}$ are monotone, then $\mathcal{U}(\mathbf{r}, \mathbf{s})$ is *normalized class*. For a matrix $A = (a_{ij})$ in a normalized class $\mathcal{U}(\mathbf{r}, \mathbf{s})$, a 1-entry a_{ij} is called an *invariant* 1 if no sequence of interchanges applied to A makes $a_{ij} = 0$. Thus by Theorem 2.6.3, for a normalized class $\mathcal{U}$, either every matrix in $\mathcal{U}$ has an invariant 1, in which case we say that $\mathcal{U}$ is with invariant 1; or none matrix in $\mathcal{U}$ has an invariant 1, in which case we say that $\mathcal{U}$ is without invariant 1.

Theorem 2.6.4 (Ryser, [222]) The normalized class $\mathcal{U}(\mathbf{r}, \mathbf{s})$ if with invariant 1's if and only if every matrix $A \in \mathcal{U}(\mathbf{r}, \mathbf{s})$ can be interchanged into the form

$$\begin{bmatrix} J_{ef} & * \\ * & 0 \end{bmatrix}.$$

Here the integers e and f may not be unique, but they are determined by $\mathbf{r}$ and $\mathbf{s}$, and are independent of the choice of $A \in \mathcal{U}(\mathbf{r}, \mathbf{s})$.

Proof The sufficiency is trivial since every entry in J is an invariant 1. To prove the necessity, assume that $\mathcal{U} = \mathcal{U}(\mathbf{r}, \mathbf{s})$ is with invariant 1's and that every matrix of $\mathcal{U}$ has an invariant 1 at the (e, f)th entry with $e + f$ maximized.

Then each $A = (a_{ij}) \in \mathcal{U}$ can be viewed as

$$\begin{bmatrix} W & X \\ Y & Z \end{bmatrix},$$

where $W \in M_{e,f}(0, 1)$. We must show that $W = J$ and $Z = 0$.

If $W \neq J$, then by the monotonity of $\mathbf{r}$ and $\mathbf{s}$, at most two interchanges applied to A would change a_{ef} into a 0-entry, contrary to the assumption that $a_{ef} = 1$ is invariant. Therefore, $W = J$ and every entry in W is an invariant 1.

Note that since $e + f$ is maximized, we may assume that $a_{l,f+1} = 0$, for some t with $1 \le l \le e$; for otherwise $a_{e,f+1}$ would be an invariant 1, contrary to the maximality of $e + f$.

Now if Z has a 1-entry in Row t (say), then we may assume that $a_{t,f+1} = 1$; for otherwise by the monotonity of $\mathbf{r}$ and $\mathbf{s}$, an interchange can be applied to get $a_{t,f+1} = 1$, where $e + 1 \le t \le m$. If for some i with $1 \le j \le e$, $a_{t,j} = 0$, then an interchange on the entries $a_{l,j}, a_{l,f+1}, a_{t,j}$ and $a_{t,f+1}$ would make $a_{l,j} = 0$, contrary to the fact that every entry in W is an invariant 1. Therefore, every $a_{t,j}$ is also an invariant 1, $1 \le j \le e$, contrary to the maximality of $e + f$. It follows that $Z = 0$, as desired. □

In [31], Theorem 2.6.4 was applied to study the lattice structure of the subspaces linearly spanned by matrices in $\mathcal{U}(\mathbf{r}, \mathbf{s})$:

$$\mathcal{L}(\mathbf{r}, \mathbf{s}) = \left\{ \sum_{i=1}^{t} c_i A_i \ : \ A_i \in \mathcal{U}(\mathbf{r}, \mathbf{s}) \text{ and } c_i \in \mathbf{Z},\ 1 \le i \le t \right\}.$$

In [268], Wan studied the distribution and the counting problems of invariant 1's in a class $\mathcal{U}(\mathbf{r}, \mathbf{s})$.

Using binary numbers, Wang [269] gave a formula of $|\mathcal{U}(\mathbf{r}, \mathbf{s})|$. For an integer $k > 0$, let $I(k)$ denote the number of 1's in the binary expression of the integer $k - 1$; let p_i be number of components of $\mathbf{r}$ that are equal to i, $1 \le i \le n$; and for $\mathbf{s} = (s_1, \cdots, s_n)^T$, let $q_j = s_j - p_n$, $1 \le j \le m$.

Theorem 2.6.5 (Wang, [269])

$$|\mathcal{U}(\mathbf{r}, \mathbf{s})| = \sum_{t_{ijk} \ge 0} \prod_{i,j=1}^{n-1} \prod_{k=1}^{2j-1} \binom{n_{ijk}}{m_{ijk}},$$

where n_{ijk} and m_{ijk} are functions of p_i, q_j and t_{ijk}, given by the recurrence definition: for $1 \le i \le n-1$, $1 \le j \le n-1$ and $1 \le k \le 2^{j-1}$,

$$
\begin{aligned}
n_{i11} &= p_i \\
m_{ijk} &= \begin{cases} t_{ijk} & \text{if } i+j-n \le I(k) < i, \ (i \text{ and } k \text{ cannot be both equal to 1}), \\ 0 & \text{if } I(k) \ge i, \\ n_{ijk} & \text{if } I(k) < i+j-n. \end{cases} \\
n_{ijk} &= \begin{cases} n_{i(j-1)\frac{k+1}{2}} - m_{i(j-1)\frac{k+1}{2}} & \text{if } k \text{ is odd and } j > 1, \\ m_{i(j-1)\frac{k}{2}} & \text{if } k \text{ is even}, \end{cases} \\
m_{1j1} &= q_j - \sum_{i=2}^{n-1} \sum_{k=1}^{2^{j-1}} m_{ijk}.
\end{aligned}
$$

2.7 Stochastic Matrices and Doubly Stochastic Matrices

Definition 2.7.1 Given a matrix $A = (a_{ij}) \in M_n^+(\mathbf{R})$, A is a *stochastic* matrix if for each i with $1 \le i \le n$, $\sum_{j=1}^n a_{ij} = 1$; and A is a *doubly stochastic* matrix if for each i with $1 \le i \le n$, both $\sum_{j=1}^n a_{ij} = 1$ and $\sum_{j=1}^n a_{ji} = 1$. Define

$$\Omega_n = \{P \in M_n^+(\mathbf{R}) \mid P \text{ is doubly stochastic}\}.$$

Theorem 2.7.1 Let $A \in M_n^+(\mathbf{R})$. Each of the following holds.

(i) A is stochastic if and only if $AJ = J$, where $J = J_n$.

(ii) A is stochastic if and only if 1 is an eigenvalue of A and the vector $\mathbf{e} = J_{n\times 1}$ is an eigenvector corresponding to the eigenvalue 1 of A.

(iii) $A \in \Omega_n$ if and only if $AJ = JA = J$, where $J = J_n$.

(iv) If A is stochastic and if $A \sim_p B$, then B is also stochastic.

(v) If $A \in \Omega_n$ and if $A \sim_p B$, then B is also doubly stochastic.

(vi) Suppose A be a direct sum of two matrices A_1 and A_2. Then A is stochastic if and only if both A_1 and A_2 are stochastic.

(vii) Suppose A be a direct sum of two matrices A_1 and A_2. Then $A \in \Omega_n$ if and only if both A_1 and A_2 are doubly stochastic.

Proof (i)-(iii) follow directly from Definition 2.7.1; (iv) and (vi) follow from (i); (v) and (vii) follow from (iii), respectively. $\square$

Theorem 2.7.2 If $A \in \Omega_n$ is doubly stochastic, then $\mathrm{Per}(A) > 0$.

Proof Suppose that $A \in \Omega_n$. If $\mathrm{Per}(A) = 0$, then by Theorem 6.2.1 in the Appendix, A is permutation similar to

$$B = \begin{bmatrix} X & Y \\ 0_{p\times q} & Z \end{bmatrix},$$

where $p + q = n + 1$. It follows that

$$n = \|B\| \geq \|X\| + \|Z\| = p + q = n + 1,$$

a contradiction. $\square$

The following Theorem 2.7.3 was conjectured by Van der Waerden [266], and was independently proved in 1980 by Falikman [85] and Egorysev[79]. A simpler proof of Theorem 2.7.3 can be found in [199] and [163].

Theorem 2.7.3 (Van der Waerden-Falikman-Egorysev, [266], [85], [79]) If $A \in \Omega_n$, then $\mathrm{Per}(A) \geq \dfrac{n!}{n^n}$.

Example 2.7.1 Let $A = \dfrac{1}{n} J_n$. Then $\mathrm{Per}(A) = \dfrac{n!}{n^n}$.

Theorem 2.7.4 If $A \in \Omega_n$, then $A \simeq_p B$, where B is a direct sum of doubly stochastic irreducible matrices.

Proof Arguing by induction on n. It suffices to show that if A is a doubly stochastic matrix, then $A \simeq_p B$, where B is a direct sum of doubly stochastic matrices.

Nothing needs a proof if A is irreducible. Assume that a doubly stochastic matrix A is reducible. By Definition 2.1.1, $A \sim_p B$ for some $B \in M_n$ of the form

$$B = \begin{bmatrix} X & Y \\ 0 & Z \end{bmatrix},$$

where $X \in M_k$ and $Z \in M_{n-k}$ for some integer $k > 0$. By Theorem 2.7.1(v), B is doubly stochastic, and so both $\|X\| = k$ and $\|Z\| = n - k$. It follows that $n = \|B\| = \|X\| + \|Y\| + \|Z\| = k + \|Y\| + (n - k)$, and so $\|Y\| = 0$, which implies that $Y = 0$. By Theorem 2.7.1(vii), both X and Z are doubly stochastic. $\square$

Theorem 2.7.5 (Birkhoff, [8]) Let $A \in M_n^+$. Then $A \in \Omega_n$ if and only if there exist an integer $t > 0$ and positive numbers $c_1, c_2, \cdots, c_t$ with $c_1 + c_2 + \cdots + c_t = 1$, and permutation matrices $P_1, \cdots, P_t$ such that

$$A = \sum_{i=1}^{t} c_i P_i.$$

Proof If $A = \sum_{i=1}^t c_i P_i$, then $AJ = \sum_{i=1}^t c_i P_i J = \sum_{i=1}^t c_i J = J$. Similarly, $JA = J$. Therefore $A \in \Omega_n$ by Theorem 2.7.1(iii).

Assume now that $A \in \Omega_n$. The necessity will be proved by induction on $p(A)$, the number of positive entries of A. By Theorem 2.7.2, $p(A) \geq n$.

If $p(A) = n$, then A is itself a permutation matrix, and so $t = 1$, $c_1 = 1$ and $P_1 = A$. The theorem holds.

Assume $p(A) > n$. By Theorem 2.7.2, $\text{Per}(A) > 0$, and so there must be a permutation π on n elements such that the product $a_{1\pi(1)}a_{2\pi(2)}\cdots a_{n\pi(n)} > 0$. Let $c_1 = \min\{a_{1\pi(1)}, a_{2\pi(2)}, \cdots, a_{n\pi(n)}\}$ and let $P_1 = (p_{ij})$ denote the permutation matrix such that $p_{ij} = 1$ if and only if $(i,j) = (i, \pi(i))$, for $1 \leq i \leq n$. Since A is doubly stochastic, $0 \leq c_1 \leq 1$. If $c_1 = 1$, then $a_{i\pi(i)} = 1$ for all $1 \leq i \leq n$, and so by Theorem 2.7.1(iii), $A = P_1$. Therefore, $p(A) = n$, contrary to the assumption that $p(A) > n$. Therefore $c_1 < 1$. Let

$$A_1 = \frac{1}{1-c_1}(A - c_1 P_1),$$

Then $A_1 J = JA_1 = J$ and so A_1 is also double stochastic with $p(A_1) = p(A) - 1$. By induction, there exist positive integers $t, c_2', \cdots, c_t'$ with $c_2' + \cdots + c_t' = 1$, and permutation matrices $P_2, \cdots, P_t$ such that

$$A_1 = \sum_{i=2}^{t} c_i' P_i.$$

It follows that

$$A = (1-c_1)A_1 + c_1 P_1 = c_1 P_1 + (1-c_1)(c_2' P_2 + \cdots + c_t' P_t).$$

Since $c_1 + (1-c_1)(c_2' + \cdots + c_t') = 1$, the theorem follows by induction. $\square$

Definition 2.7.2 A matrix $A \in \mathbf{B}_n$ is of *doubly stochastic type* if A can be obtained from a doubly stochastic matrix B by changing every positive entry of B into a 1. Let Ω_n denote the set of all $n \times n$ doubly stochastic matrices.

A digraph D is *k-cyclic* if D is a disjoint union of k directed cycles. Let D be a digraph on n vertices $\{v_1, \cdots, v_n\}$, and let $l_1, l_2, \cdots, l_n$ be non negative integers. Then $D(l_1, l_2, \cdots, l_n)$ denote the digraph obtained from D by attaching l_i loops at vertex v_i, $1 \leq i \leq n$.

Example 2.7.2 The matrix

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix},$$

is not of doubly stochastic type (Exercise 2.25).

Theorem 2.7.6 Let $A \in \mathbf{B}_n$. Then A is a matrix of doubly stochastic type if and only if for some integers $t, k_1, \cdots, k_t > 0$, $D(A)$ is a disjoint union of t spanning subgraphs $D_1, D_2, \cdots, D_t$ of $D(A)$, where D_i is k_i-cyclic, $1 \leq i \leq t$.

Sketch of Proof Note that $P \in M_n$ is a permutation matrix if and only if $D(P)$ is a k-cyclic digraph on n vertices, for some integer k depending only on P. Thus Theorem 2.7.6 follows from Theorem 2.7.5. $\square$

Definition 2.7.3 Let D be a digraph on n vertices $\{v_1, \cdots, v_n\}$, and let $l_1, l_2, \cdots, l_n$ be non negative integers. Then $D(l_1, l_2, \cdots, l_n)$ denote the digraph obtained from D by attaching l_i loops at vertex v_i, $1 \le i \le n$.

A digraph D is *Eulerian* if for every vertex $v \in V(D)$, $d^+(v) = d^-(v)$.

Corollary 2.7.6 A digraph D is Eulerian if and only if for some integers $l_1, \cdots, l_n \ge 0$, the adjacency matrix $A(D(l_1, \cdots, l_n))$ is of doubly stochastic type.

Sketch of Proof Note that D is Eulerian if and only if D is the disjoint union of directed cycles $C_1, C_2, \cdots, C_t$ of D. Since each C_i can be made a spanning subgraph by adding a loop at each vertex not in C_i, and so the corollary follows from Theorem 2.7.6. $\square$

Definition 2.7.4 Let $r, s > 0$ be real numbers and $n \ge m > 0$ be integers satisfying $rm = sn$. Let $\mathbf{r}_0$ denote an m-dimensional vector each of whose component is r, and let $\mathbf{s}_0$ denote an n-dimensional vector each of whose component is s.

In general, for an integer $k > 0$, let $\mathbf{s} = \mathbf{k}_0$ denote a vector each of whose component is k; and define $\mathcal{U}_n(k) = \mathcal{U}(\mathbf{k}_0, \mathbf{k}_0) \cap \mathbf{B}_n$. Thus every matrix $A \in \mathcal{U}_n(k)$ has row sum k and column sum k, for every row and column of A. In particular, $\mathcal{U}_n(1) = \Omega_n$.

Theorem 2.7.7 Let $A \in \mathbf{B}_{m,n}$ for some $n \ge m > 0$. Then $A \in \mathcal{U}(\mathbf{r}_0, \mathbf{s}_0)$ if and only if there exist integers $t, c_1, \cdots, c_t > 0$ and permutation matrices $P_1, \cdots, P_t$, such that

$$A = c_1 P_1 + c_2 P_2 + \cdots + c_t P_t.$$

Sketch of Proof If $n = m$, then $r = s$ and so $\frac{1}{r}A$ is doubly stochastic. Therefore, Theorem 2.7.7 follows from Theorem 2.7.5.

Assume that $m < n$. Then consider

$$A' = \begin{bmatrix} \frac{1}{r}A \\ \frac{1}{n}J_{(n-m)\times n} \end{bmatrix}.$$

Then $A' \in \mathcal{U}_n(\mathbf{j})$, and so Theorem 2.7.7 follows by applying Theorem 2.7.5 to A'. $\square$

Theorem 2.7.8 Let $A \in \mathbf{B}_n$. Then $A \in \mathcal{U}_n(k)$ if and only if there exist permutation matrices $P_1, \cdots, P_k$ such that

$$A = \sum_{i=1}^{k} P_k.$$

Proof It follows from Theorem 2.7.5. $\square$

Example 2.7.3 Let G be a k-regular bipartite graph with bipartite sets X and Y. If $|X| = |Y| = n$, then the reduced adjacency matrix of A is in $\mathcal{U}(k)$, and so by Theorem 2.7.8, $E(G)$ can be partitioned into k perfect matchings.

Theorem 2.7.9 If $A \in \mathcal{U}_n(k)$, then

$$\mathrm{Per}(A) \geq \frac{n!k^n}{n^n} \sim \left(\frac{k}{e}\right)^n \sqrt{2\pi n}.$$

Proof This follows from Theorem 2.7.3. $\square$

Another lower bound of $\mathrm{Per}(A)$ for $A \in \mathcal{U}_n(k)$ can be found in Exercise 2.26. When $k = 3$, the lower bound in Theorem 2.7.9 has been improved to ([198])

$$\mathrm{Per}(A) \geq 6\left(\frac{4}{3}\right)^{n-3}.$$

However, the problem of determining the exact lower bound of $\mathrm{Per}(A)$ for $A \in \mathcal{U}_n(k)$ remains open.

2.8 Birkhoff Type Theorems

Definition 2.8.1 Recall that $\mathcal{P}_n$ denotes the set of all n by n permutation matrices, and that Ω_n denotes the set of all n by n doubly stochastic matrices.

For two matrices $A, B \in M_n$, define

$$\begin{aligned} \mathcal{P}(A,B) &= \{P \in \mathcal{P}_n \;:\; AP = PB\} \\ \overline{\mathcal{P}(A,B)} &= \left\{\sum_i c_i P_i \;:\; \sum_i c_i = 1 \text{ and } P_i \in \mathcal{P}(A,B)\right\}. \end{aligned}$$

Also define

$$\Omega_n(A,B) = \{P \in \Omega_n \;:\; AP = PB\}.$$

When $A = B$, write $\mathcal{P}(A)$, $\overline{\mathcal{P}(A)}$ and $\Omega_n(A)$ for $\mathcal{P}(A,A)$, $\overline{\mathcal{P}(A,A)}$ and $\Omega_n(A,A)$, respectively. Note that when $A = A(G)$ is the adjacency matrix of a graph G, $\mathcal{P}(A)$ is the set of all automorphisms of G.

Example 2.8.1 Let G be the vertex disjoint union of a 3-cycle and a 4-cycle, let $A = A(G)$ and $B = \frac{1}{7}J_7$. Then $AB = BA$ and so $B \in \Omega_7(A)$. However, as there is no automorphism of G that maps a vertex in the 3-cycle to a vertex in the 4-cycle, $B \notin \overline{\mathcal{P}(A)}$.

Definition 2.8.2 A graph G is *compact* if $\overline{\mathcal{P}(A(G))} = \Omega_n(A(G))$.

Example 2.8.2 Note that $G(I_n)$ is the graph with n vertices and n loop edges such that a loop is attached at each vertex of $G(I_n)$. Note that $\mathcal{P}(I_n) = \mathcal{P}_n$.

Thus Birkhoff Theorem (Theorem 2.7.5) may be restated as

$$\Omega_n(I_n) = \Omega_n = \overline{\mathcal{P}_n} = \overline{\mathcal{P}(I_n)},$$

which is equivalent to saying that $G(I_n)$ is compact. Tinhofer [260] indicated that theorems on compact graph families may be viewed as Birkhoff type theorems.

Definition 2.8.3 Let G be a graph with n vertices and without multiple edges. Let G^* be the graph obtained from G by attaching a loop at each vertex of G. Thus $K_n^* = G(J_n)$ is the graph obtained from the complete graph K_n by attaching a loop at each vertex of K_n.

The graph G can be viewed as a subgraph of K_n^*. Moreover, if G is loopless, then G can be viewed as a subgraph of K_n. The *full completement* of G is $G^{fc} = K_n^* - E(G)$. If G is loopless, then the *completement* of G is $G^c = K_n - E(G)$.

The proof of the following Theorem 2.8.1 is straightforward.

Theorem 2.8.1 Each of the following holds.

(i) If G is compact, then G^{fc} is also compact.

(ii) If a loopless graph G is compact, then G^* is also compact.

(iii) If a loopless graph G is compact, then G^c is also compact.

(iv) K_n, K_n^*, K_n^c are compact graphs.

Theorem 2.8.2 (Tinhofer, [260]) A tree is compact.

Theorem 2.8.3 (Tinhofer, [260]) For $n \geq 3$, the n-cycle C_n is compact.

Proof Let $V(C_n) = \mathbf{Z}_n$, the integers modulo n, and denote $A = A(C_n) = (a_{ij})$, where

$$a_{ij} = \begin{cases} 1 & \text{if } j \equiv i \pm 1 \pmod{n} \\ 0 & \text{otherwise.} \end{cases}$$

It suffices to show $\Omega_n(A) \subseteq \overline{\mathcal{P}(A)}$.

Argue by contradiction, assume that there is an $X = (x_{i,j}) \in \Omega_n(A) \setminus \overline{\mathcal{P}(C_n)}$ such that $p(X)$, the number of positive entries of X, is minimized. To find a contradiction, we shall show the existence of a real number ϵ with $1 > \epsilon > 0$, matrices $Y \in \Omega_n$ and $P \in \mathcal{P}(A)$ such that $X = (1-\epsilon)Y + \epsilon P$ and such that $p(Y) < p(X)$.

Since $XA = AX$,

$$x_{i+1,j} + x_{i-1,j} = x_{i,j-1} + x_{i,j+1}, \quad \text{for all } i,j \text{ with } i,j \in \mathbf{Z}_n.$$

It follows that for all i,j with $1 \leq i,j \leq n$,

$$x_{i+1,j-i} - x_{i,j-i-1} = x_{1,j} - x_{n,j-1} \text{ and } x_{i+1,j+i} - x_{i,j+i+1} = x_{1,j} - x_{n,j+1}.$$

Note that the right hand sides are independent of i.

In each of the cases below, a matrix $P = (p_{ik}) \in \mathcal{P}(A)$ is found with the property that $x_{ik} > 0$ whenever $p_{ik} > 0$. Let $\epsilon = \min\{x_{ik} | p_{ik} = 1\}$. If $\epsilon = 1$, the $X = P \in \mathcal{P}(A)$, contrary to that assumption that X is a counterexample. Hence $0 < \epsilon < 1$. Let $Y = \frac{1}{1-\epsilon}(X - \epsilon P)$. Then $p(Y) < p(X)$ and by $XA = AX, PA = AP$ and by Theorem 2.7.1(iii), $Y \in \Omega_n(A)$, and so a contradiction obtains.

Case 1 $x_{1,j} - x_{n,j-1} > 0$, for some fixed j.

Define $P = (p_{ik})$ as follows:

$$p_{ik} = \begin{cases} 1 & \text{if } k \equiv j + 1 - i \pmod n \\ 0 & \text{otherwise.} \end{cases}$$

As $x_{i+1,j-1} - x_{i,j-i-1} = x_{1,j} - x_{n,j-1}$, $x_{ik} > 0$ whenever $p_{ik} > 0$. Note that P is the reflection about the axis through the positions $\frac{j+1}{2}$ and $\frac{n+j+1}{2}$, and so $P \in \mathcal{P}(A)$.

Case 2 $x_{1,j} - x_{n,j-1} < 0$, for some fixed j.

In this case, define P to be the reflection about the axis through the positions $\frac{j-1}{2}$ and $\frac{n+j-1}{2}$. The proof is similar to that for Case 1.

Case 3 $x_{1,j} - x_{n,j+1} > 0$, for some fixed j.

Define $P = (p_{ik})$ as follows:

$$p_{ik} = \begin{cases} 1 & \text{if } k \equiv j + i - 1 \pmod n \\ 0 & \text{otherwise.} \end{cases}$$

As $x_{i+1,j+i} - x_{i,j+i+1} = x_{1,j} - x_{n,j+1} > 0$, $x_{ik} > 0$ whenever $p_{ik} > 0$. Note that P is a clockwise rotation of C_n, and so $P \in \mathcal{P}(A)$.

Case 4 $x_{1,j} - x_{n,j+1} < 0$, for some fixed j.

In this case, define P to be an anticlockwise rotation, similar to that in Case 3.

Case 5 $x_{1,j} = x_{n,j+1} = x_{n,j-1}$, for all j with $1 \le j \le n$.

Then, $x_{i+1,j} = x_{i,j+1} = x_{i-1,j} = x_{i,j-1}$, for all i, j with $1 \le i, j \le n$. It follows that there exist matrices U, V and numbers $\alpha, \beta \ge 0$, such that $X = U + V$ and such that

$$U_{ij} = \begin{cases} \alpha & \text{if } i - j \equiv 0 (\bmod 2) \\ 0 & \text{otherwise,} \end{cases} \quad \text{and } V_{ij} = \begin{cases} \beta & \text{if } i - j \equiv 1 (\bmod 2) \\ 0 & \text{otherwise.} \end{cases}$$

Note that if $\alpha > 0$, then $U \ge$ the sum of some reflections; if $\beta > 0$, then $V \ge$ the sum of some reflections. Thus we can proceed as in Case 1-4 to express

$$X = (1 - \epsilon)Y + \epsilon P,$$

for some $P \in \mathcal{P}(A)$ and $Y \in \Omega_n(A)$. If $\alpha = \beta = 0$, then $X = 0$.

Therefore, in any case, if $X \neq 0$, a desired P can be found and so this completes the proof. $\square$

Definition 2.8.4 Let $A \in M_n$ be a matrix. Define

$$\begin{aligned} \text{Cone}(A) &= \{B \in M_n^+ : BA = AB\} \\ \hat{\mathcal{P}}(A) &= \left\{ \sum_{P \in \mathcal{P}(A)} c_P P : c_P \geq 0 \right\}. \end{aligned}$$

It follows immediately from definitions that

$$\hat{\mathcal{P}}(A) \subseteq \text{Cone}(A).$$

Let G be a graph with $A = A(G)$. If $\hat{\mathcal{P}}(A) = \text{Cone}(A)$, then G is a *supercompact* graph.

Theorem 2.8.4 Let $\hat{\mathcal{P}}_n = \left\{ \sum_{P \in \mathcal{P}(A)} c_P P : c_P \geq 0 \right\}$ be a set of $n \times n$ matrices. Let G be a graph on n vertices with $A = A(G)$. Each of the following holds.

(i) If $Y \in \hat{\mathcal{P}}_n$, then $G(Y)$ is a regular graph. (In other words, $Y \in \mathcal{U}_n(q)$ for some number q.)

(ii) If $Y \in \hat{\mathcal{P}}(A)$ and if $Y \neq 0$, then there exists a number $q > 0$ such that $\frac{1}{q}Y \in \overline{\mathcal{P}(A)}$. Moreover, $q = 1$ if and only if $Y \in \Omega_n$.

(iii) (Brualdi, [19]) If G is supercompact, then G is compact and regular.

Sketch of Proof (i) follows from Theorem 2.7.7. For (ii), note that $Y = \sum_P c_P P$. Therefore let $q = \sum_P c_P$ and apply Birkhoff Theorem (Theorem 2.7.5) to conclude (ii).

For (iii), by definitions, $\overline{\mathcal{P}(A)} \subseteq \Omega_n(A)$. It suffices to show the other direction of containment when G is supercompact. Let $A = A(G)$ and assume that $\hat{\mathcal{P}}(A) = \text{Cone}(A)$. By (i), $G = G(A)$ is regular. By definitions and by (ii), $\Omega_n(A) \subseteq \text{Cone}(A) = \hat{\mathcal{P}}(A) \subseteq \overline{\mathcal{P}(A)}$. $\square$

Example 2.8.3 There exist compact graphs that are not supercompact. Let G be a tree with $n \geq 3$ vertices. By Theorem 2.8.3, G is compact. Since $n \geq 3$, G is not regular, and so G is not supercompact, by Theorem 2.8.4(i).

Example 2.8.4 A compact regular graph may not be supercompact. Let G be the disjoint union of two K_2's. With appropriate labeling,

$$A = A(G) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

Let

$$X = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

We can easily check that $XA = XA$ and so $X \in \text{Cone}(A)$. But by Theorem 2.8.4(i), $X \notin \hat{\mathcal{P}}(A)$, and so G is not supercompact.

Theorem 2.8.5 (Brualdi, [19]) For $n \geq 1$, the n-cycle C_n is supercompact.

Theorem 2.8.6 (Brualdi, [19]) Let $n, k > 0$ be integers such that $n = kl$, let H be a supercompact graph on k vertices, and let G be the disjoint union of l copies of H. Then G is compact.

Proof Let $B = A(H)$ and $A = A(G)$. Then A is the direct sum of l matrices each of which equals B.

By contradiction, assume that G is not compact, and so there exists a matrix $X \in \Omega_n(A) - \overline{\mathcal{P}(A)}$ with $p(X)$, the number of nonzero entries of X, is as small as possible. Write

$$X = \begin{bmatrix} X_{11} & X_{12} & \cdots & X_{1l} \\ X_{21} & X_{22} & \cdots & X_{2l} \\ \vdots & \vdots & \ddots & \vdots \\ X_{l1} & X_{l2} & \cdots & X_{ll} \end{bmatrix},$$

where each $X_{ij} \in M_k^+$. Since $XA = AX$, for all $1 \leq i, j \leq l$, $X_{ij}B = BX_{ij}$, and so $X_{ij} \in \hat{\mathcal{P}}(B)$, by the assumption that H is supercompact. By Theorem 2.8.4(i), there is a number $q_{ij} \geq 0$ such that $X_{ij} \in \mathcal{U}_k(q_{ij})$. Let $Q = (q_{ij})$ denote the $l \times l$ matrix. Since $X \in \Omega_n$, $Q \in \Omega_l$. By Theorem 2.7.5, $Q = \sum_{P \in \mathcal{P}_l} c_P P$, where $\sum c_P = 1$. Therefore, there exists a $P = (p_{ij}) \in \mathcal{P}_l$, which corresponds to a permutation σ on $\{1, 2, \cdots, l\}$, such that $q_{s,\sigma(s)} > 0$ for all $1 \leq s \leq l$.

Fix an $s = 1, 2, \cdots, l$. Since $X_{s,\sigma(s)} \in \hat{\mathcal{P}}(B)$, $X_{s,\sigma(s)} = \sum_{P \in \mathcal{P}_l(B)} c_P P$, where $c_P \geq 0$. Hence there exists a $P_s \in \mathcal{P}_k(B)$, which corresponds to an automorphism σ_s of H, such that for all $1 \leq u \leq k$, the $(u, \sigma_s(u))$-entry of $X_{s,\sigma(s)}$ is positive.

Construct a new matrix $R = (r_{ij}) \in M_n$ as follows. Write

$$R = \begin{bmatrix} R_{11} & R_{12} & \cdots & R_{1l} \\ R_{21} & R_{22} & \cdots & R_{2l} \\ \vdots & \vdots & \ddots & \vdots \\ R_{l1} & R_{l2} & \cdots & R_{ll} \end{bmatrix}, \text{ where } R_{i,j} = \begin{cases} P_s & \text{if } i = s \text{ and } j = \sigma(s) \\ 0 & \text{otherswise.} \end{cases}$$

Since $P_sB = BP_s$, $RA = AR$. Since $P_s \in \mathcal{P}_k$, $R \in \mathcal{P}_n$. Thus R is an automorphism of G and so $R \in \mathcal{P}(A)$. Moreover, $x_{ij} > 0$ whenever $r_{ij} = 1$, for all $1 \le i,j \le n$. Let $\epsilon = \min\{x_{ij} : r_{ij} = 1\}$. If $\epsilon = 1$, then since $X, R \in \Omega_n$, $X = R \in \mathcal{P}(A) \subseteq \overline{\mathcal{P}(A)}$, contrary to the choice of X. Therefore, $0 < \epsilon < 1$. Let $Y = \frac{1}{1-\epsilon}(X - \epsilon R)$. Then by Theorem 2.7.1(iii), and by $X, R \in \Omega_n(A)$, $Y \in \Omega_n(A)$ with $p(Y) < p(X)$. By the minimality of X, $Y \in \overline{\mathcal{P}(A)}$, and so $X = (1-\epsilon)Y + \epsilon R \in \overline{\mathcal{P}(A)}$, contrary to the choice of X. $\square$

Corollary 2.8.6 If G is the disjoint union of C_k (cycles of length k), then G is compact.

When $k = 1$, Corollary 2.8.6 yields Birkhoff Theorem. Therefore, Corollary 2.8.6 is an extension of Theorem 2.7.5. See Exercise 2.30 for other applications of Theorem 2.8.6.

Definition 2.8.5 Let $k, m, n > 0$ be integers with $n = km$. A graph G on n vertices is a *complete k-equipartite* graph, (a $C.k - e$ graph for short), if $V(G) = \cup_{i=1}^{m}$ is a disjoint union with $|V_i| = k$, for all $1 \le i \le m$, such that two vertices in $V(G)$ are joined by an edge $e \in E(G)$ if and only if these two vertices belong to different V_i's.

Let G be a $C.k - e$ graph on n vertices. If $k = 1$, then $G = K_n$; if $2k = n$, then $G = K_{\frac{n}{2},\frac{n}{2}}$. Theorem 2.8.6 can be applied to show that $C.k - e$ graphs are also compact graphs (Exercise 2.30).

Theorem 2.8.7 (Brualdi, [19]) Let $n = 2m \ge 2$ and let $M \subseteq E(K_{m,m})$ be a perfect matching of $K_{m,m}$. Then $K_{m,m} - M$ is compact.

Proof We may assume that $m \ge 3$. Write $V(K_{m,m} - M) = V_1 \cup V_2$, where

$$V_1 = \{v_1, v_2, \cdots, v_m\} \text{ and } V_2 = \{v_{m+1}, v_{m+2}, \cdots, v_{2m}\}.$$

and $M = \{e_i \mid e_i \text{ joins } v_i \text{ to } v_{m+i}, \text{ where } 1 \le i \le m\}$. Hence we may assume that

$$A = A(K_{m,m} - M) = \begin{bmatrix} 0 & J_m - I_m \\ J_m - I_m & 0 \end{bmatrix}.$$

By contradiction, there exists an $X \in \Omega_n(A) - \overline{\mathcal{P}(A)}$ with $p(X)$, the number of positive entries of X, minimized. Write

$$X = \begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix}.$$

Then by $AX = XA$,

$$\begin{aligned} X_1(J_m - I_m) &= (J_m - I_m)X_4 \\ X_4(J_m - I_m) &= (J_m - I_m)X_1 \end{aligned}$$

and so $(X_1 + X_4)J_m = J_m(X_1 + X_4)$. It follows that there exists a number $a \ge 0$ such that a is the common value of each row sum and each column sum of $X_1 + X_4$. Similarly, there exists a number $b \geq 0$ such that b is the common value of each row sum and each column sum of $X_2 + X_3$.

Let $r_1, \cdots, r_m$ denote the row sums of X_1 and let $s_1, \cdots, s_m$ denote the column sums of X_1. Then the row sums and column sums of X_4 are respectively $a - r_1, \cdots, a - r_m$ and $b - s_1, \cdots, b - s_m$.

Let z_{ij} denote the (i,j)-entry of X_1. Since $X_1 - X_4 = X_1 J_m - J_m X_4$, then (i,j)-entry of X_4 is $z_{ij} + a - t_i - s_j$. By the definition of a, $J_m(X_1 + X_4) = aJ_m$, and so for fixed j,

$$\begin{aligned} a &= \sum_{i=1}^{m}(z_{ij} + z_{ij} + a - r_i - s_j) \\ &= 2s_j + ma - (r_1 + r_2 + \cdots + r_m) - ms_j. \end{aligned}$$

Thus by $m > 2$ that, for each j with $1 \leq j \leq m$,

$$s_j = \frac{1}{m-2}\left[(m-1)a - \sum_{i=1}^{m} r_i\right].$$

It follows that $s_1 = s_2 = \cdots = s_m$. Similarly, $r_1 = r_2 = \cdots = r_m$. Hence there exist $a_1, a_4 \geq 0$ such that $a = a_1 + a_4$, $X_1 \in \mathcal{U}(a_1)$, $X_4 \in \mathcal{U}(a_4)$, and

$$X_4 = X_1 + (a - 2a_1)J_m.$$

Compare the row sums and column sums of the matrices both sides to see $a - a_1 = a_4 = a_1 + m(a - 2a_1)$, and so

$$(m-1)(a - 2a_1) = 0.$$

Since $m > 2$, $a = 2a_1$ and so $X_1 = X_4$. Similarly, $X_2 = X_3$, and so

$$X = \begin{bmatrix} X_1 & X_2 \\ X_2 & X_1 \end{bmatrix},$$

where $X_1 \in \mathcal{U}(a_1)$, $X_2 \in \mathcal{U}(a_2)$, and $a_1 + a_2 = 1$.

Suppose first that $a_1 \neq 0$. Then $\frac{1}{a_1}X_1 \in \Omega_m$ and so by Theorem 2.7.5, there is a $Q = (q_{ij}) \in \mathcal{P}_m$ such that $z_{ij} > 0$ whenever $q_{ij} = 1$. Let τ denote the permutation on $\{1, 2, \cdots, m\}$ corresponding to Q, and let $P \in \Omega_{2m}$ be the direct sum of Q and Q. Then P corresponds to the automorphism of $K_{m,m} - M$ which maps v_i to $v_{\tau(i)}$ and v_{m+i} to $v_{m+\tau(i)}$, and so $P \in \mathcal{P}(A)$.

Let $\epsilon = \min\{z_{ij} \ : \ q_{ij} = 1\}$. If $\epsilon = 1$, then $X = P$, contrary to the assumption that $X \notin \overline{\mathcal{P}(A)}$. Therefore, $0 < \epsilon < 1$. Let $Y = \frac{1}{1-\epsilon}(X - \epsilon P)$. Then by Theorem 2.71.(iii) and by $X, P \in \Omega_n(A)$, $Y \in \Omega_n(A)$ with $p(Y) \leq p(X) - 2$. By the choice of X, $Y \in \overline{\mathcal{P}(A)}$. But then $X = (1-\epsilon)Y + \epsilon P \in \overline{\mathcal{P}(A)}$, contrary to the assumption that $X \notin \overline{\mathcal{P}(A)}$.

The proof for the case when $a_2 \neq 0$ is similar. This completes the proof. $\square$

It is not difficult to see that $K_{mm} - M$ $(m \geq 3)$ is also supercompact.

Theorem 2.8.8 (Liu and Zhou, [187]) Let G be the 1-regular graph on $n = 2m \geq 4$ vertices. Then G is not supercompact.

Proof Let $A = A(G)$ and consider these two cases.

Case 1 $n \equiv 0 \pmod 4$. Then A can be written as

$$A = \begin{bmatrix} 0 & \cdots & 0 & I_2 \\ 0 & \cdots & I_2 & 0 \\ \vdots & & \vdots & \vdots \\ I_2 & \cdots & 0 & 0 \end{bmatrix}.$$

Define

$$X = \begin{bmatrix} Y & 0 & \cdots & 0 \\ 0 & Y & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & Y \end{bmatrix}, \text{ where } Y = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

Then $X \in \mathbf{B}_n$ and $AX = XA$, and so $X \in \mathrm{Cone}(A)$. However, as the row sums of X are not a constant, $X \notin \hat{\mathcal{P}}(A)$ by Theorem 2.8.4(i), and so G is not compact.

Case 2 $n \equiv 2 \pmod 4$. Let

$$P_{(1,1)} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \text{ and } Y = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

Then A can be written as

$$A = \begin{bmatrix} 0 & 0 & \cdots & 0 & \cdots & 0 & I_2 \\ 0 & 0 & \cdots & 0 & \cdots & I_2 & 0 \\ \vdots & \vdots & & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & P_{(1,1)} & \cdots & 0 & 0 \\ 0 & I_2 & \cdots & 0 & \cdots & 0 & 0 \\ I_2 & 0 & \cdots & 0 & \cdots & 0 & 0 \end{bmatrix}.$$

Define

$$X = \begin{bmatrix} Y & 0 & \cdots & 0 & \cdots & 0 & 0 \\ 0 & Y & \cdots & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & I_2 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & \cdots & Y & 0 \\ 0 & 0 & \cdots & 0 & \cdots & 0 & Y \end{bmatrix}.$$

Then $X \in \text{Cone}(A) \setminus \hat{\mathcal{P}}(A)$ as shown in Case 1, and so G is not compact. □

Example 2.8.5 The complement of a supercompact graph may not be supercompact. Let $G = C_4$ denote the 4-cycle. Then G is compact. But G^c is a 1-regular graph on 4 vertices, which is not supercompact, by Theorem 2.8.8. Since G^c is not supercompact, it follows by Definition 2.8.4 that G^{fc} is not supercompact either.

Definition 2.8.6 Let $k, m > 0$ be integers. A graph G is called an (m,k)-*cycle* if $V(G)$ can be partitioned into $V_1 \cup V_2 \cup \cdots \cup V_m$ with $|V_i| = k, (1 \le i \le m)$ such that an edge $e \in E(G)$ if and only if for some i, one end of e is in V_i and the other in V_{i+1}, where $i \equiv 1, 2, \cdots, m \pmod{m}$.

Example 2.8.6 An $(m,1)$-cycle is an m-cycle. When $m = 2$ or $m = 4$, an (m,k)-cycle is a complete bipartite graph $K_{\frac{mk}{2},\frac{mk}{2}}$. If $A(C_m) = B$, then an (m,k)-cycle has adjacency matrix $B \otimes J_m$. For example, the adjacency matrix of the (4,2)-cycle is

$$\begin{bmatrix} 0 & J_2 & 0 & J_2 \\ J_2 & 0 & J_2 & 0 \\ 0 & J_2 & 0 & J_2 \\ J_2 & 0 & J_2 & 0 \end{bmatrix}.$$

Theorem 2.8.9 (Brualdi, [19]) Let $m, k > 0$ be integers. An (m,k)-cycle is supercompact if either $k = 1$, or $k \ge 2$ and $m = 4$, or $k \ge 2$ and $m \not\equiv 0 \pmod 4$.

Example 2.8.7 The $(8,2)$-cycle is not compact, and so it is not supercompact. To see this, let $A = A(C_8) \otimes J_2$ be the adjacency matrix of the $(8,2)$-cycle, and let

$$X_{ij} = \begin{cases} \begin{bmatrix} \frac{1}{4} & 0 \\ 0 & 0 \end{bmatrix} & \text{if } j - i \equiv 0, 1 \pmod 4 \\ \begin{bmatrix} 0 & 0 \\ 0 & \frac{1}{4} \end{bmatrix} & \text{if } j - i \equiv 2, 3 \pmod 4 \end{cases}$$

Let $X = (X_{ij})$ denote the matrix in M_{16} which is formed by putting each of the blocks X_{ij}, $1, \le i, j \le 2$ in the ijth position of a 2×2 matrix. Then $X \in \Omega_n(A) \setminus \overline{\mathcal{P}(A)}$. (Exercise 2.31).

Open Problems

(i) Find new compact graph families.

(ii) Find new techniques to construct compact graphs from supercompact graphs.

(iii) Construct new supercompact graphs. It is known that when $k = 1$, a $C.k - e$ graph is supercompact; and when $k = 2$, a $C.k - e$ graph is compact. What can we say for $k \geq 3$?

(iv) Is there another kind of graphs whose relationship with supercompact graphs is similar to that between supercompact graphs and compact graphs?

2.9 Exercise

Exercise 2.1 (This is needed in the proof of Theorem 2.2.1) Let D be a directed graph. Prove each of the following.

(i) If D has no directed cycles, then G has a vertex v with out degree zero.

(ii) D has no directed cycles if and only if the vertices of G can be so labeled $v_1, v_2, \cdots, v_n$ that $(v_i v_j) \in E(D)$ only if $i < j$. (Such a labeling is called a *topological ordering*.)

(iii) If $D_1, D_2, \cdots, D_k$ are the strongly connected components of D, then there is some D_i such that G has no arc from a vertex in $V(D_i)$ to a vertex $V(D) - V(D_i)$. (In this case we say that D_i is a *source component*, and write $\delta^+(D_i) = 0$.)

(iv) The strong components of D can be labeled as $D_1, D_2, \cdots, D_k$ such that D has an arc from a vertex in $V(D_i)$ to a vertex in $V(D_j)$ only if $i < j$.

Exercise 2.2 Prove Lemma 2.2.1.

Exercise 2.3 Prove Lemma 2.2.2.

Exercise 2.4 Let $A \in M_n$ be a matrix with the form

$$A = \begin{bmatrix} A_1 & 0 & 0 & \cdots & 0 \\ * & A_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ * & * & \cdots & A_{k-1} & 0 \\ * & * & \cdots & * & A_k \end{bmatrix},$$

where each A_i is a square matrix, $1 \leq i \leq k$. Show that A has a nonzero diagonal if and only if each A_i has a nonzero diagonal.

Exercise 2.5 Let $A = (a_{ij}) \in M_n^+$. Then A is nearly reducible if and only if A is irreducible and for each $a_{pq} > 0$, $A - a_{pq}E_{pq}$ is reducible.

Exercise 2.6 Prove Proposition 2.3.1.

Exercise 2.7 Let D be a digraph, let $W \subseteq V(D)$ be a vertex subset and let H be a subgraph of D. Prove each of the following.

(i) If D is minimally strong and if $D[W]$ is strong, then $D[W]$ is minimally strong.
(ii) If D is strong, then D/H is also strong.
(iii) If both H and D/H are strong, then D is strong.

Exercise 2.8 A permutation $(a_1, a_2, \cdots, a_n)$ of $1, 2, \cdots, n$ is an *n-derangement* if $a_i \neq i$, for each i with $1 \le i \le n$. Show each of the following.

(i) $\mathrm{Per}(J_n) = n!$.

(ii) $\mathrm{Per}(J_n - I_n) = n! \sum_{k=0}^{n} \frac{(-1)^k}{k!}$.

Exercise 2.9 For $n \ge 3$, let $A = (a_{ij}) \in M_n$ be a matrix with $a_{ij} = 0$ for each i and j with $1 \le i \le n-1$ and $i+2 \le j \le n$, (such a matrix is called a Hessenberg matrix). Show that the signs of some entries of A can be changed so that $\mathrm{Per}(A) = \det(A)$.

(Hint: change the sign of each $a_{i,i+1}$, $1 \le i \le n-1$.)

Exercise 2.10 A matrix $A = (a_{ij}) \in M_{r,n}$ with $n \ge r \ge 1$ is a $r \times n$ *normalized Latin rectangle* if $a_{1i} = i$, $1 \le i \le n$, if each row of A is a permutation of $1, 2, \cdots, n$ and if each column of A is an r-permutation of $1, 2, \cdots, n$. Let $K(r, n)$ denote the number of $r \times n$ normalized Latin rectangles. Show that $K(2, n) = \mathrm{Per}(J_n - I_n)$.

Exercise 2.11 Prove Lemma 2.5.1.

Exercise 2.12 Prove Lemma 2.5.2.

Exercise 2.13 Prove Theorem 2.5.5.

Exercise 2.14 Let $A \in M_n^+$ be fully indecomposable. Then Par $(A) \ge \|A\| - 2n + 2$.

Exercise 2.15 Prove Lemma 2.5.5.

Exercise 2.16 Prove Lemma 2.5.6.

Exercise 2.17 Prove Lemma 2.5.7.

Exercise 2.18 Prove Theorem 2.5.9.

Exercise 2.19 Using the notation in Definition 2.6.3, Show that each of the following holds.

(i) $\mathbf{s}'' \prec \mathbf{s}^{(1)} \prec \mathbf{s}'$.

(ii) there exist an integer $k \ge 1$ and k non negative n-dimensional vectors $\mathbf{s}^{(1)}, \mathbf{s}^{(2)}, \cdots, \mathbf{s}^{(k)}$ such that

$$\mathbf{s}'' = \mathbf{s}^{(k)} \prec \mathbf{s}^{(k-1)} \prec \cdots \prec \mathbf{s}^{(1)} \prec \mathbf{s}'.$$

Exercise 2.20 Let $\mathbf{r} = \mathbf{s} = (k, k, \cdots, k)^T$ be two $n0$-dimensional vectors. If $k|n$, show that

$$|\mathcal{U}(\mathbf{r},\mathbf{s})| \geq \frac{(n!)^k}{(k!)^n}.$$

Exercise 2.21 Let $A = (a_{ij}) \in M_n$, let $\mathbf{x} = (x_1, x_2, \cdots, x_n)^T > 0$ and let $(A\mathbf{x})_i = \sum_{j=1}^{n} a_{ij}x_j$. Each of the following holds.

(i) If A is nonnegative and irreducible, then $\min\limits_{1\leq i\leq n} \frac{(A\mathbf{x})_i}{x_i} = \rho(A) \leq \max\limits_{1\leq i\leq n} \frac{(A\mathbf{x})_i}{x_i}$.

(ii) If A is nonnegative, then $\min\limits_{x_i>0} \frac{(A\mathbf{x})_i}{x_i} = \rho(A) \leq \max\limits_{x_i>0} \frac{(A\mathbf{x})_i}{x_i}$.

Exercise 2.22 Show that if A is a stochastic matrix, then $\rho(A) = 1$.

Exercise 2.23 Let $A, B \in M_n^+$. Prove each o the following:

(i) If both A and B are stochastic, show that AB is also stochastic.

(ii) If both A and B are doubly stochastic, show that AB is also doubly stochastic.

Exercise 2.24 If $A \in M_n^+$ is doubly stochastic, then $A \sim B$, where B is a direct sum of doubly stochastic fully indecomposable matrices.

Exercise 2.25 Show that the matrix A in Example 2.7.2 is not of doubly stochastic type.

Exercise 2.26 Show that if $A \in \mathcal{U}_n(k)$, then $\mathrm{Per}(A) \geq k!$.

Exercise 2.27 Prove Theorem 2.8.1.

Exercise 2.28 Without turning to Theorem 2.8.2, prove the star $K_{1,n-1}$ is compact.

Exercise 2.29 Prove Theorem 2.8.5 by imitate the proof of Theorem 2.8.3.

Exercise 2.30 Show that G is compact if

(i) G is a disjoint union of K_m's.

(ii) G is a disjoint union of K_m^*'s.

(iii) G is a disjoint union of T_m's, where T_m is a tree on m vertices.

(iv) G is a $C.k - e$ graph.

Exercise 2.31 Show that $X \in \Omega(A) \setminus \overline{\mathcal{P}(A)}$ in Example 2.8.7.

2.10 Hints for Exercises

Exercise 2.1 For (i), if no such v (a source) exists, then walking randomly through the arcs, we can find a directed cycle.

For (ii), use (i) to find a source. Label the source with the largest available number, then delete the source and go on by induction.

For (iii), we can contract each strong component into a vertex to apply the result in (i).

Exercise 2.2 Suppose that $A = (a_{ij})$ and $A' = (a'_{ij})$. Then $a_{is} = a'_{it}$ and $a_{it} = a'_{is}$, for each i with $1 \le i \le n$. For each $a_{is} > 0$, there are a_{is} arcs from v_i to v_s in $D(A)$. Since $a'_{it} = a_{is}$, there are a_{is} ares from v_i to v_t in $D(A')$. Thus all the arcs getting into v_s in $D(A)$ will be redirected to v_t in $D(A')$. Similarly, all the arcs getting into v_t in $D(A)$ will be redirected to v_s in $D(A')$.

Exercise 2.3 We only show that case when $k = 2$. The general case can be proved similarly.

Let $u \in V(D') = V(D)$. We shall show that D' has a directed (u_2, u)-path and a (u, u_2)-path. If $u \in V(D_2)$, then since D_2 is a strong component, D_2 has a (u_2, u)-path which is still in D'. Also, D_2 has a (u, u')-path for some vertex u' with $(u'u_2) \in E(D_2)$. Note that in D', $(u'u_2) \in E(D')$. Since u_1 is not a source and by (*), there is a vertex $v \in V(D_1)$ such that $(vu_1) \in E(D)$. It follows that D' has a (u, u_2)-path that goes from u, through u' and v with the last arc (v, u_2).

If $u \in V(D_1)$, since u_1 is not a sink and by (*), there is a $u'' \in V(D_1)$ such that $(u''u_1) \in E(D)$, whence D' has a (u, u_2)-path that contains a (u, u'')-path in D_1 with the last arc $(u''u_2)$. Also, since u_2 is no a source, there is a vertex $v \in V(D_2)$ such that $(vu_2) \in E(D)$. Hence D' has a (u_2, u)-path that contains a (u_2, v)-path in D_2, the arc (v, u_1), and a (u_1, u)-path in D_1.

Exercise 2.4 Induction on k. By the definition of a diagonal, we can see that A_k must have a nonzero diagonal. Argue by induction to the submatrix by deleting the rows and columns containing entries in A_k.

Exercise 2.5 Apply Definition 2.3.1.

Exercise 2.6 Apply definitions only.

Exercise 2.7 (i) If $D[W]$ has an arc a such that $D[W] - a$ is strong, then $D - a$ is also strong, and so D is not minimal. Hence $D[W]$ is minimally strong.

(ii) and (iii): definitions.

Exercise 2.8

(i) Show that $\text{Per}(J_n)$ counts the number of permutations on n elements.

(ii) Show that $\text{Per}(J_n - I_n)$ counts the number of n derangements.

Exercise 2.9 If $\text{Per}(A) = 0$, then A has a zero submatrix $H \in M_{s,n-s+1}(0)$, which has at least $n - m + 1$ columns.

Exercise 2.11 If $A \in \mathbf{B}_{m,n}$ and if $\text{Per}(A) = 0$, then A has $0_{s\times(n-s+1)}$ as a submatrix, for some $s > 0$ (Theorem 6.2.1 in the Appendix); and this submatrix has at least $n - m + 1$

columns.

Exercise 2.12 In this case, each $k\times(n-1)$ submatrix of A has at least k nonzero columns, and so by Theorem 6.2.1 in the Appendix, $\mathrm{Par}(A')>0$ for each $(m-1)\times(n-1)$ submatrix A'.

Exercise 2.13 It suffices to prove Theorem 2.5.5 for nearly indecomposable matrices. Argue by induction on n. When $n=1$, (2.10) is trivial. By Theorem 2.4.2, we may assume that

$$A=\begin{bmatrix} A_0 & F_1 \\ F_2 & B \end{bmatrix},$$

where B is nearly indecomposable. By induction,

$$\mathrm{Per}(A)\le \ \mathrm{Per}(B)+1\ge \|B\|-2m+2+1.$$

Exercise 2.14 Let $A=(a_{ij})$. If $a_{ij}\le 1$, then this is Theorem 2.5.5. Assume that some $a_{rs}>0$. Let $B=A-a_{rs}E_{rs}$. By induction on $\|A\|$, we have

$$\begin{aligned}\mathrm{Per}(A) &= \mathrm{Per}(B)+\ \mathrm{per}A(r|s)\ge\|B\|-2n+1\\ &= \|A\|-2n+2.\end{aligned}$$

Exercise 2.15 Since $n\ge 2$ and since A is fully indecomposable, there exists a $t\ne s$ such that $a_{rt}>0$. Hence

$$\begin{aligned}\mathrm{Per}(A) &= \sum_{k=1}^{n} a_{rk}\ \mathrm{Per}(A(r|k))\\ &\ge a_{rs}\ \mathrm{Per}(A(r|s))+a_{rt}\ \mathrm{Per}(A(r|t))\\ &\ge 2\ \mathrm{Per}(A(r|s))+1\end{aligned}$$

Exercise 2.16 Argue by induction on $n\ge 3$. Assume that $(n-1)!<2^{(n-1)(n-3)}$. Then the theorem will follow if $n\le 2^{2(n-2)}$.

Consider the function $f(x)=2(x-2)-\log_2 x$. Note that $f(3)=2-\log_2 3>2-\log_2 4=0$, and $f'(x)>0$. Hence $f(x)>0$ and so $3\le n\le 2^{2(n-2)}$.

Exercise 2.17 Let $r_1,r_2,\cdots,r_n$ denote the row sums of A. If $r_i\ge 3$ for each i, then by Theorem 2.5.7 and by Lemma 2.5.6,

$$\mathrm{Per}(A)\le\prod_{i=1}^{n}(r_i!)^{\frac{1}{r_i}}<\prod_{i=1}^{n}2^{r_i-2}=2^{\|A\|-2n}.$$

Exercise 2.18 Argue by induction on $t\ge 1$ and apply Theorem 2.5.8.

Exercise 2.19

(i). By Definition 2.6.2, the first $\mu - 1$ components of $\mathbf{s}^{(1)}$ and $\mathbf{s}'$ are identical, and $s_\mu^{(1)} \le s'_\mu$. Thus comparing the sum of the first k components and considering the cases when $k \le \lambda - 1$ and $k \ge \lambda$, we can conclude that $\mathbf{s}^{(1)} \prec \mathbf{s}'$.

Note that $s_\mu^{(1)} \ge s''_\mu$ and $s_\lambda^{(1)} > s'_\lambda$. comparing the sum of the first k components and considering the cases when $k \le \mu - 1$ and $k \ge \mu$, we also conclude that $\mathbf{s}'' \prec \mathbf{s}^{(1)}$.

(ii). The number of identical components between $\mathbf{s}''$ and $\mathbf{s}^{(1)}$ is more than that between $\mathbf{s}''$ and $\mathbf{s}'$. Apply (i) to $\mathbf{s}''$ and $\mathbf{s}^{(1)}$ to find $\mathbf{s}^{(2)}$ such that $\mathbf{s}'' \prec \mathbf{s}^{(2)} \prec \mathbf{s}^{(1)}$. Repeat this process to get (ii).

Exercise 2.20 Apply Theorem 2.6.2.

Exercise 2.21 Let $Y = \operatorname{diag}(x_1, \cdots, x_n)$. Then $Y^{-1}AY = (a_{ij}x_j/x_i)$, and A is irreducible, $Y^{-1}AY$ is also irreducible, and $\rho(Y^{-1}AY) = \rho(A)$. It follows that

$$\min_{1\le i\le n} \frac{(A\mathbf{x})_i}{x_i} = \rho(Y^{-1}AY) = \rho(A) \le \max_{1\le i\le n} \frac{(A\mathbf{x})_i}{x_i}.$$

If A is nonnegative, then let $\epsilon > 0$ be a real number and let $A_\epsilon = A + \epsilon J_n$. Then by the above,

$$\min_{1\le i\le n} \frac{(A_\epsilon\mathbf{x})_i}{x_i} = \rho(A_\epsilon) \le \max_{1\le i\le n} \frac{(A_\epsilon\mathbf{x})_i}{x_i}.$$

Thus (ii) follows by letting $\epsilon \to 0$.

Exercise 2.22 Since A is stochastic, $A\mathbf{j} = A$. Therefore apply Exercise 2.21 with $\mathbf{x} = \mathbf{j}$ to get

$$\rho(A) \le \max_{1\le i\le n} \frac{(A\mathbf{j})_i}{1} = 1.$$

Show that if A is a stochastic matrix, then $\rho(A) = 1$.

Exercise 2.23 Apply Theorem 2.7.1(ii) and (iii).

Exercise 2.24 Imitate the proof for Theorem 2.7.4.

Exercise 2.25 By contradiction, assume that there is a doubly stochastic matrix $B = (b_{ij})_{2\times 2}$ such that only $b_{21} = 0$. Then apply Theorem 2.7.1(iii) to see that $a_{12} = 0$.

Exercise 2.26 Apply Hall-Mann-Ryser Theorem.

Exercise 2.27 Let $A = A(G)$. Then the adjacency matrices of G^{fc}, G^* and G^c are $J_n - A, A + I_n$ and $J_n - I_n - A$, respectively. Apply these to prove (i)-(iii). For (iv), it suffices to show K_n is compact, or $\Omega(J_n - I_n) = \Omega_n = \overline{\mathcal{P}(J_n - I_n)}$.

Exercise 2.28 Let $G = K_{1,n-1}$ and $A = A(G)$. Then

$$A = \begin{bmatrix} 0 & 1 & 1 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots & \\ 1 & 0 & 0 & \cdots & 0 \end{bmatrix} \text{ and so } \overline{\mathcal{P}(A)} = \begin{bmatrix} 1 & 0 \\ 0 & P \end{bmatrix} \in \overline{\mathcal{P}}_n,$$

where $P \in \mathbf{B}_{n-1}$ is a permutation matrix. It suffices to show $\Omega_n(A) \subseteq \overline{\mathcal{P}(A)}$. Let $Q \in \Omega_n(A)$. Then since $A = Q^{-1}AQ$,

$$Q = \begin{bmatrix} 1 & 0 \\ 0 & Q_1 \end{bmatrix},$$

for some $Q_1 \in \Omega_{n-1}$. It follows that $Q_1 = \sum_i c_i Q_i'$, where Q_i' are permutation matrices in $\mathbf{B}_{n-1}$. It follows that

$$Q = \sum_i c_i P_i \in \overline{\mathcal{P}(A)}, \text{ where } P_i = \begin{bmatrix} 1 & 0 \\ 0 & Q_i' \end{bmatrix}.$$

Exercise 2.29 Let $A = A(C_n)$ and $0 \neq X = (x_{ij}) \in \text{Cone}(A)$, $X \geq 0$ and $XA = AX$. In the proof of Theorem 2.8.5, we know that there is a permutation σ (corresponding to the permutation matrix $P = p_{ij}) \in \mathcal{P}(A))$ such that $x_{i\sigma(i)} > 0$ for all i. Let $\epsilon = \min\{x_{i\sigma(i)}, i = 1, 2, \cdots, n\}$. As $P \in \mathcal{P}(A)$, $(X - \epsilon P) \in \text{Cone}(A)$, and $X - \epsilon P$ has one more 0-entry than X.

Then argue by induction to show that $X \in \hat{\mathcal{P}}(A)$.

Exercise 2.30 (i)-(iii) follows directly from Theorem 2.8.6. For (iv), note that G^{fc} is the disjoint union of K_k^*'s.

Chapter 3

Powers of Nonnegative Matrices

Powers of nonnegative matrices are of great interests since many combinatorial properties of nonnegative matrices have been discovered in the study of the powers of nonnegative matrices and the indices associated with these powers.

A standard technique in this area is to study the associated (0,1)-matrix of a nonnegative matrix. Given a nonnegative matrix A, we associate A with a matrix $A' \in \mathbf{B}_n$ obtained by replacing any nonzero entry of A with a 1-entry of A'. Many of the combinatorial properties of a nonnegative matrix can be obtained by investigating the associated (0,1)-matrix and by treating this associated (0,1)-matrix as a Boolean matrix (to be defined in Section 3.2).

Quite a few of the problems in the study of powers will lead to the Frobenius Diophantine Problem. Thus we start with a brief introductory section on this problem.

3.1 The Frobenius Diophantine Problem

Certain Diophantine equations will be encountered in the study of powers of nonnegative matrices. This section presents some results and methods on this topic. As usual, for integers $a_1, a_2, \cdots, a_s$, let $\gcd(a_1, \cdots, a_s)$ denote that greatest common divisor of $a_1, \cdots, a_s$, and let $\mathrm{lcm}(a_1, \cdots, a_s)$ denotes that least common multiple of $a_1, \cdots, a_s$.

Theorem 3.1.1 Let $a_1, a_2 > 0$ be integers with $\gcd(a_1, a_2) = 1$. Define $\phi(a_1, a_2) = (a_1 - 1)(a_2 - 1)$. Each of the following holds.

(i) For any integer $n \geq \phi(a_1, a_2)$, the equation $a_1x_1 + a_2x_2 = n$ has a nonnegative integral solution $x_1 \geq 0$ and $x_2 \geq 0$.

(ii) The equation $a_1x_1 + a_2x_2 = \phi(a_1, a_2) - 1$ does not have any nonnegative integral solution.

Proof Let $n \geq (a_1 - 1)(a_2 - 1)$. Note from number theory that any solution x_1 and x_2 of $a_1x_1 + a_2x_2 = n$ can be presented by

$$\begin{cases} x_1 & = x_1' + a_2 t \\ x_2 & = x_2' - a_1 t \end{cases}$$

where x_1', x_2' are a pair of integers satisfying the equation, and where t can be any integer.

Since $a_1 \geq 1$ and since x_2' is an integer, we can choose t so that $a_1 t \leq x_2' \leq a_1 t + a_1 - 1$. Therefore $x_2 = x_2' - a_1 t \geq 0$. Since $n > a_1a_2 - a_1 - a_2$ and by this choice of t,

$$\begin{aligned} x_1 a_1 &= (x_1' + a_2 t)a_1 = n - (x_2' - a_1 t)a_2 \\ &> a_1a_2 - a_1 - a_2(a_1 - 1)a_2 = -a_1 \end{aligned}$$

and so $x_1 = x_1' + a_2 t \geq 0$. This proves (i).

Argue by contradiction to prove (ii). Assume that there exist nonnegative integers x_1, x_2 so that $a_1x_1 + a_2x_2 = a_1a_2 - a_1 - a_2$, which can be written as $a_1a_2 = (x_1 + 1)a_1 + (x_2 + 1)a_2$. It follows by $\gcd(a_1, a_2) = 1$ that $a_1|(x_2 + 1)$ and $a_2|(x_1 + 1)$. Therefore $x_2 + 1 \geq a_1$ and $x_1 + 1 \geq a_2$, and so $a_1a_2 = (x_1+1)a_1 + (x_2+1)a_2 \geq 2a_1a_2$, a contradiction obtains. $\square$

Theorem 3.1.2 Let $s, n, a_1, \cdots, a_s$ be positive integers with $s > 2$ such that $\gcd(a_1, \cdots, a_s) = 1$. There exists an integer $N = N(a_1, \cdots, a_s)$, such that the equation

$$a_1x_1 + \cdots + a_sx_s = n,$$

has nonnegative integral solution $x_1 \geq 0, x_2 \geq 0, \cdots, x_s \geq 0$ whenever $n > N(a_1, \cdots, a_n)$.

Proof When $s = 2$, Theorem 3.1.2 follows from Theorem 3.1.1 with $N(a_1, a_2) = (a_1 - 1)(a_2 - 1) - 1$.

Assume that $s \geq 3$ and argue by induction on s. Let d denote $\gcd(a_1, \cdots, a_{s-1})$. Then $\gcd(d, a_s) = 1$, and so there is an integer d_s with $0 \leq b_s \leq d - 1$ such that $a_s b_s \equiv n \pmod d$. Write $a_i = a_i' d$, $1 \leq i \leq s - 1$. Then the equation $a_1x_1 + \cdots + a_sx_s = n$ becomes

$$a_1'x_1 + \cdots + a_{s-1}'x_{s-1} = \frac{n - a_s b_s}{d}. \tag{3.1}$$

By induction, there exists an integer $N(a_1', a_2', \cdots, a_{s-1}')$ such that the equation (3.1) has nonnegative integral solution $x_1 = b_1, \cdots, x_{s-1} = b_{s-1}$, whenever

$$\frac{n - a_s b_s}{d} \geq \frac{n - a_s(d-1)}{d} > N(a_1', a_2', \cdots, a_{s-1}').$$

Let $N(a_1, \cdots, a_s) = a_s(d-1) + dN(a_1', \cdots, a_{s-1}')$. Then $a_1x_1 + \cdots a_sx_s = n$ has nonnegative integral solution $x_1 = b_1, \cdots x_s = b_s$ whenever $n > N(a_1, \cdots, a_s)$, and so the theorem is proved by induction. $\square$

Definition 3.1.1 Let $s, a_1, \cdots, a_s$ be positive integers with $s > 2$ such that $\gcd(a_1, \cdots, a_s) = 1$. By Theorem 3.1.2, there exists a smallest positive integer $\phi(a_1, \cdots, a_s)$ such that any integer $n \geq \phi(a_1, \cdots, a_s)$ can be expressed as $n = a_1x_1 + \cdots a_sx_s$ for some nonnegative integers $x_1, \cdots, x_s$. This number $\phi(a_1, \cdots, a_s)$ is called the *Frobenius number*. The *Frobenius problem* is to determine the exact value of $\phi(a_1, \cdots, a_s)$.

Theorem 3.1.1 solves the Frobenius problem when $s = 2$. This problem remains open when $s \geq 3$.

Theorem 3.1.3 (Ke, [143]) Let $a_1, a_2, a_3 > 0$ be integers with $\gcd(a_1, a_2, a_3) = 1$. Then

$$\phi(a_1, a_2, a_3) \leq \frac{a_1a_2}{\gcd(a_1, a_2)} + a_3 \gcd(a_1, a_2) - \sum_{i=1}^{3} a_i + 1. \tag{3.2}$$

Moreover, equality holds when

$$a_3 > \frac{a_1a_2}{(\gcd(a_1, a_2))^2} - \frac{a_1}{\gcd(a_1, a_2)} - \frac{a_2}{\gcd(a_1, a_2)}. \tag{3.3}$$

Note that a_1, a_2, a_3 can be permuted in both (3.2) and (3.3).

Proof Let $d = \gcd(a_1, a_2)$ and write $a_1 = a_1'd$ and $a_2 = a_2'd$. Let u_1, u_2, x_0, y_0, z_0 be integers satisfying $a_1'u_1 + a_2'u_2 = 1$ and $a_1x_0 + a_2y_0 + a_3z_0 = n$ respectively. We can easily see (Exercise 3.1) that any integral solution of $a_1x + a_2y + a_3z = n$ can be expressed as

$$\begin{cases} x = x_0 + a_1't_1 - u_1a_3t_2 \\ y = y_0 - a_1t_1 - u_2a_3t_2 \\ z = z_0 + dt_2, \end{cases}$$

where t_1, t_2 can be any integers. Choose t_2 so that $-dt_2 \leq z_0 \leq -dt_2 + d - 1$, and then choose t_1 so that $-a_2't_1 \leq x_0 - u_1a_3t_2 \leq -a_2't_1 + a_2' - 1$. Note that these choices of t_1 and t_2 make $x \geq 0$ and $z \geq 0$. Let $n \geq \frac{a_1a_2}{\gcd(a_1,a_2)} + a_3 \gcd(a_1, a_2) - \sum_{i=1}^{3} a_i + 1$. By the choices of t_1, t_2 ad n,

$$\begin{aligned} a_2y &= a_2(y_0 - a_1't_1 - u_2a_3t_2) \\ &= n - a_1x - a_3z \geq n - a_1(a_2' - 1) - a_3(d-1) \\ &= n - \frac{a_1a_2}{d} - a_3d + a_1 + a_3 > -a_2. \end{aligned}$$

Thus $y \geq 0$. This proves (3.2).

Now assume (3.3). We shall show that

$$a_1x + a_2y + a_3z = \frac{a_1a_2}{\gcd(a_1, a_2)} + a_3 \gcd(a_1, a_2) - \sum_{i=1}^{3} a_i, \tag{3.4}$$

has no nonnegative integral solutions, and so equality must hold in (3.2). By contradiction, assume that there exist nonnegative integers x, y, z satisfying (3.4). Note that $\frac{a_1 a_2}{\gcd(a_1, a_3)} = da_1' a_2'$ and that $a_3 \gcd(a_1, a_2) = a_3 d$, and so We have

$$da_1' a_2' + a_3 d - \sum_{i=1}^{3} a_i = a_1 x + a_2 y + a_3 z.$$

It follows that

$$d(a_1' a_2' + a_3) = da_1'(x+1) + da_2'(y+1) + a_3(z+1).$$

Since $\gcd(d, a_3) = 1$, we must have $d|(z+1)$, and so $z + 1 = dk$ for some integer $k > 0$. Cancel d both sides to get

$$a_1' a_2' = a_1'(x+1) + a_2'(y+1) + a_3(k-1).$$

If $k > 1$, then $a_1' a_2' \geq a_1' + a_2' + a_3$, contrary to (3.3). Thus $k = 1$. Then by $\gcd(a_1', a_2') = 1$, $a_1' | (y+1)$ and $a_2' | (x+1)$, which lead to a contradiction $a_1' a_2' \geq 2a_1' a_2'$. $\square$

Some of the major results in this area are listed below. In each of these theorems, it is assumed that $s \geq 2$ and that $a_1, \cdots, a_s$ are integers with $a_1 > a_2 > \cdots > a_s > 0$ and with $\gcd(a_1, \cdots, a_s) = 1$. Exercises 3.2 and 3.3 are parts of the proof for Theorem 3.1.7.

Theorem 3.1.4 (Schur, [158])

$$\phi(a_1, \cdots, a_s) \leq (a_1 - 1)(a_s - 1).$$

Theorem 3.1.5 (Brauer and Seflbinder, [15]) Let $d_i = \gcd(a_1, a_2, \cdots, a_i)$, $1 \leq i \leq s-1$. Then

$$\phi(a_1, \cdots, a_s) \leq \frac{a_1 a_2}{d_1} + \sum_{i=1}^{s-3} \frac{a_{i+2} d_i}{d_{i+1}} + a_s d_{s-2} - \sum_{i=1}^{s} a_i + 1.$$

When $s = 3$, Theorem 3.1.5 gives inequality (3.2).

Theorem 3.1.6 (Roberts, [217]) Let $a \geq 2$ and $d > 0$ be integers with $d \not| a$. Let $a_j = a + jd$, $(0 \leq j \leq s)$, then

$$\phi(a_0, a_1, \cdots, a_s) \leq \left(\lfloor \frac{a-2}{s} \rfloor + 1\right) a + (d-1)(a-1).$$

Theorem 3.1.7 (Lewin, [158])

$$\phi(a_1, \cdots, a_s) \leq \lfloor \frac{(a_1 - 2)^2}{2} \rfloor.$$

Theorem 3.1.8 (Lewin, [156]) If $s \geq 3$, then

$$\phi(a_1, \cdots, a_s) \leq \lfloor \frac{(a_1 - 2)(a_2 - 1)}{2} \rfloor.$$

Theorem 3.1.9 (Vitek, [267]) Let i be the largest integer such that $\frac{a_i}{a_s}$ is not an integer. One of the following holds.

(i) If there is an a_j, such that there exist for all choices of nonnegative integers μ and γ, $a_j \neq \mu a_s + \gamma a_i$, then

$$\phi(a_1, \cdots, a_s) \leq \lfloor \frac{a_s}{2} \rfloor (a_1 - 2).$$

(ii) If no such a_j exists, then

$$\phi(a_1, \cdots, a_s) \leq (a_s - 1)(a_i - 1).$$

Theorem 3.1.10 (Vitek, [267]) If $s \geq 3$, then

$$\phi(a_1, \cdots, a_s) \leq \lfloor \frac{(a_{s-1} - 1)(a_1 - 2)}{2} \rfloor.$$

3.2 The Period and The Index of a Boolean Matrix

Definition 3.2.1 A matrix $A \in \mathbf{B}_n$ can be viewed as a *Boolean matrix*. The *Boolean matrix multiplication and addition* of (0,1) matrices can be done as they were real matrices except that the addition of entries in Boolean matrices follows the Boolean way:

$$a + b = \max\{a, b\}, \text{ where } a, b \in \{0, 1\}.$$

Unless otherwise stated, the addition and multiplication of all $(0, 1)$ matrices in this section will be Boolean.

Theorem 3.2.1 Let $A \in \mathbf{B}_n$. There exist integers $p > 0$ and $k > 0$ such that each of the following holds.

(i) If $n \geq k$ and $n - k = sp + r$, where $0 \leq r \leq p - 1$, then $A^n = A^{k+r}$.

(ii) $\{I, A, A^2, \cdots, A^k, \cdots, A^{k+p-1}\}$ with the Boolean matrix multiplication forms a semigroup.

(iii) $\{A^k, \cdots, A^{k+p-1}\}$ with the Boolean matrix multiplication forms a cyclic group with identity A^e and generator A^{e+1}, for some $e \in \{k, k + 1, \cdots, k + p - 1\}$.

Proof It suffices to show (i) since (ii) and (iii) follow from (i) immediately.

Since $|\mathbf{B}_n| = 2^{n^2}$ is a finite number, the infinite sequence $I, A, A^2, A^3, \cdots$ must have repeated members. Let k be the smallest positive integers such that there exist a smallest integer $p > 0$ satisfying $A^k = A^{k+p}$.

Then for any integer $s > 0$, $A^{k+sp} = A^{k+p+(s-1)p} = A^{k+p}A^{(s-1)p} = A^{k+(s-1)p} = \cdots = A^k$, and so (i) obtains. □

Definition 3.2.2 For a matrix $A \in \mathbf{B}_n$, the smallest positive integers p and k satisfying Theorem 3.2.1 are called the *period of convergence* of A and the *index of convergence* of A, denoted by $p = p(A)$ and $k = k(A)$, respectively. Very often $p(A)$ and $k(A)$ are just called the period and the index of A, respectively.

Definition 3.2.3 A irreducible matrix $A \in M_n$ is *primitive* if there exists an integer $k > 0$ such that $A^k > 0$. An irreducible matrix A is *imprimitive* if A is not primitive.

Example 3.2.1 Let D be a directed n-cycle for some integer $n \geq 2$ and let $A = A(D)$. Then since D is strong, A is irreducible. However, A is not primitive.

Definition 3.2.4 Let D be a strong digraph. Let $l(D) = \{l > 0 : D$ has a directed cycle of length $l\} = \{l_1, l_2, \cdots, l_s\}$. The *index of imprimitivity* of D is $d(D) = \gcd(l_1, l_2, \cdots, l_s)$.

Theorem 3.2.2 Let $A \in M_n^+$ and let $D = D(A)$. Then A is primitive if and only if both of the following holds:

(i) D is strong, and

(ii) $d(D) = 1$.

Proof Suppose that A is primitive. Then A is irreducible and so (i) holds. Moreover, there is an integer $k > 0$ such that $A^k > 0$. Note that if $A^k > 0$, then $A^{k+1} > 0$ also. It follows by Proposition 1.1.2(vii) and by Exercise 3.4, that $d(D)$ must be a divisor for both k and $k+1$, and so $d(D) = 1$.

Conversely, assume that A satisfies both (i) and (ii). Then A is irreducible. Let $l(D) = \{l_1, l_2, \cdots, l_s\}$. By (ii), $\gcd(l_1, l_2, \cdots, l_s) = 1$. By Theorem 3.1.2, $\phi(l_1, \cdots, l_s)$ exists.

Denote $V(D) = \{1, 2, \cdots, n\}$. For each pair $i, j \in V(D)$, by (i), D has a spanning directed trail $T(i,j)$ from i to j. Let $d(i,j) = |E(T(i,j))|$. Define

$$k = \max_{i,j \in V(D)} d(i,j) + \phi(l_1, \cdots, l_s).$$

Then for each fixed pair $i, j \in V(D)$, $k = d(i,j) + a$, where $a \geq \phi(l_1, \cdots, l_s)$. By the definition of $\phi(l_1, \cdots, l_s)$ and by Proposition 1.1.2(viii), D has a closed trail L of length a. Since $T(i,j)$ is spanning, $T(i,j)$ and L together form a directed (i,j)-trial of D. By Proposition 1.1.2(vii), $A^k > 0$ and so A is primitive. □

Definition 3.2.5 Let $d \geq 2$ be an integer. A digraph D is *cyclically d-partite* if $V(D)$ can be partitioned into d sets $V_1, V_2, \cdots, V_d$ such that D has an arc $(u,v) \in E(D)$ only if $u \in V_p$ and $v \in V_q$ such that $q - p = 1$ or $q - p = 1 - d$.

Lemma 3.2.1 Let D be a strong graph with $V(D) = \{v_1, v_2, \cdots, v_n\}$. For each i with

$1 \le i \le n$, let d_i be the greatest common divisor of the lengths of all closed trails of D containing v_i, and let $d = d(D)$. The each of the following holds.

(i) $d_1 = d_2 = \cdots = d_n = d$.

(ii) For each pair of vertices $v_i, v_j \in V(D)$, if P_1 and P_2 are two (v_i, v_j)-walks of D, then $|E(P_1)| \equiv |E(P_2)| \pmod{d}$.

(iii) $V(D)$ can be partitioned into $V_1 \cup V_2 \cup \cdots \cup V_d$ such that any (v_i, v_j)-trial $T_{i,j}$ with $v_i \in V_i$ and $v_j \in V_j$ has length $|E(T_{i,j})| \equiv j - i \pmod{d}$.

(iv) If $d \ge 2$, then D is cyclically d-partite.

Sketch of Proof (i). Fix $v_i, v_j \in V(D)$. Assume that D has a (v_i, v_j)-trail $T_{i,j}$ of length s, a (v_j, v_i)-trail $T_{j,i}$ of length t, and a (v_j, v_j)-trail of length t_j. Then both $s + t$ and $s + t + t_j$ are lengths of closed trails containing v_i, and so $d_i | (s + t)$ and $d_i | (s + t + t_j)$. It follows that $d_i | h_j$, and so $d_i | d_j$. Since i, j are arbitrary, $d_1 = d_2 = \cdots = d_n = d'$, and $d | d'$.

Let l be a length of a directed cycle C of D. Then C contains a vertex v_i (say), and so $d' | l$. It follows that $d' | d$.

(ii). Let Q be a (v_j, v_i)-trail. Then each $P_i \cup Q$ is a closed trail containing v_i. By (i), $|E(P_1)| + |E(Q)| \equiv |E(P_2)| + |E(Q)| \pmod{d}$.

(iii). Fix v_1. For $1 \le i \le d$, let $V_i = \{v_i \in V :$ any directed (v_1, v_i)-path has length $i \bmod d\}$. If $v_i \in V_i$ and $v_j \in V_j$, then D has a directed (v_1, v_i)-path of length l', and a directed (v_i, v_j)-path of length l. Thus $l' \equiv i \bmod d$ and $l + l' \equiv j \bmod d$, and so $l \equiv j - i \bmod d$.

(iv) follows from (iii). $\square$

By direct matrix computation, we obtain Lemma 3.2.2 below.

Lemma 3.2.2 Let $A \in M_n^+$ be a matrix such that for some positive integers $n_1, n_2, \cdots, n_d$ matrices $A_i \in M_{n_i, n_{i+1}}$ with each A_i having no zero row nor zero column,

$$A = \begin{bmatrix} 0 & A_1 & & \\ & 0 & A_2 & \\ & & \ddots & A_{d-1} \\ A_d & & & 0 \end{bmatrix} \tag{3.5}$$

Then each of the following holds.

(i) $A^d = \text{diag}(B_1, \cdots, B_d)$, where $B_j = \prod_{i=j}^{d+j-1} A_i$ and where the subscripts are counted modulo d.

(ii) If for some integer $m > 0$, $A^m = \text{diag}(J_1, J_2, \cdots, J_d)$, such that each J_i is a non-zero square matrix, then $d | m$.

Theorem 3.2.3 Let $A \in M_n^+$ be an irreducible matrix with $d = d(D(A)) > 1$. Each of the following holds.

(i) There exist positive integers $n_1, n_2, \cdots, n_d$ and matrices $A_i \in M_{n_i, n_{i+1}}$, and a permutation matrix P such that PAP^{-1} has the form of (3.5).

(ii) Each A_i in (i) has no zero row nor zero column, $1 \le i \le d$.

(iii) $\prod_{i=1}^{d} A_i$ is primitive.

Proof (i). Let $D = D(A)$. By Lemma 3.2.1(iii), $V(D)$ has a partition $V_1 \cup V_2 \cup \cdots \cup V_d$ satisfying Lemma 3.2.1(iii). Let $n_i = |V_i|$, $(1 \le i \le d)$. By Lemma 3.2.1(iii), any arc of D is directed from a vertex in V_i to a vertex in V_{i+1}, $i \equiv 1, 2, \cdots, d \pmod d$. With such a labeling, D has an adjacency matrix of the form in (i).

(ii) follows from the assumption that D is irreducible.

(iii). Let $l(D) = \{l_1, l_2, \cdots, l_s\}$. Then $\gcd\left(\frac{l_1}{d}, \frac{l_2}{d}, \cdots, \frac{l_s}{d}\right) = 1$, and so by Theorem 3.1.2, $k_0 = \phi(\frac{l_1}{d}, \cdots, \frac{l_s}{d})$ exists.

Choose $u, v \in V_1$. Since D is strong, there exists a directed closed walk $W(u,v)$ from u to v. By Lemma 3.2.1(iii), d divides $|E(W(u,v))|$. Let

$$t = \max_{u,v \in V_1} \left\{ \frac{|E(W(u,v))|}{d} + k_0 \right\}.$$

Then $t \ge k_0$, and so for any $u, v \in V_1$, $td = |E(W(u,v))| + kd$ for some integer $k \ge k_0$. By Theorem 3.1.2, and since $V(W(u,v)) = V(D)$, for any pair of vertices $u, v \in V_1$, D has a directed (u,v)-walk of length td. It follows by Proposition 1.1.2(vii) that every entry of the $n_1 \times n_1$ submatrix in the upper left conner of $(PAP^{-1})^{td}$ is positive, and so by Lemma 3.2.2(i), $B_1^t > 0$. This proves (iii). $\square$

The corollaries below follow from Theorem 3.2.3.

Corollary 3.2.3A Let $A \in M_n^+$ be irreducible with $d = d(D(A))$. Then d is the largest positive integer such that $D(A)$ is cyclically d-partite.

Corollary 3.2.3B Let $A \in M_n^+$ be an irreducible matrix with $d = d(D(A))$. Then each of the following holds.

(i) If A has the form (3.5) and satisfies Theorem 3.2.3(ii), then for each j with $1 \le j \le d$, $B_j = \prod_{i=j}^{d+j-1} A_i$ is primitive, where the subscripts are counted modulo d.

(ii) (Dulmage and Mendelsohn, [77]) There is a permutation matrix Q (called a *canonical transformer* of A) such that

$$Q^{-1}A^dQ = \operatorname{diag}(B_1, B_2, \cdots, B_d),$$

where each B_i is primitive.

(iii) (Dulmage and Mendelsohn, [77]) Let Q be a canonical transformer of A. The number $d = d(D(A))$ is the smallest power of $Q^{-1}AQ$ which has the form of $\operatorname{diag}(B_1, B_2, \cdots, B_d)$, where each B_i is primitive.

Corollary 3.2.3C Let $A \in M_n^+$ be a irreducible matrix with $d(D(A)) > 1$. Then $p(A) = d(D(A))$.

Corollary 3.2.3D Let $A \in \mathbf{B}_n$. Each of the following holds.

(i) $p(A) = 1$ if and only if A is primitive.

(ii) If $p = p(A) > 1$, then A is similar to

$$\begin{bmatrix} 0 & A_1 & & & \\ & 0 & A_2 & & \\ & & \ddots & A_{p-1} & \\ A_p & & & 0 & \end{bmatrix}, \tag{3.6}$$

such that each A_i is primitive. (The form (3.6) is called the *imprimitive standard form* of the matrix A, and the integer p is the *index of imprimitivity* of A.)

Lemma 3.2.3 Let $A \in \mathbf{B}_n$ be a matrix having the form (3.5) and satisfy Theorem 3.2.3(ii). If $\prod_{i=1}^{d} A_i$ is irreducible, then A is also irreducible and $d|p(A)$.

Sketch of Proof By Lemma 3.2.2(i),

$$A^d = \operatorname{diag}(B_1, B_2, \cdots, B_d),$$

where $B_1 = \prod_{i=1}^{d} A_i$ is irreducible. By Theorem 2.1.1(v), there is a polynomial $f(x)$ such that $f(B_1) > 0$.

Let $g(x) = xf(x)$. Then for each $i = 1, 2, \cdots, d$,

$$g(B_i) = B_i f(B_i) = (A_i A_{i+1} \cdots A_d) f(B_1)(A_1 A_2 \cdots A_{i-1}).$$

Since A satisfies Theorem 3.2.3(ii), each A_i has no zero rows nor zero columns. It follows by $f(B_1) > 0$ and by the operations of Boolean matrices that $g(B_i) > 0$. Direct computation leads to

$$(I + A + \cdots + A^{d-1})g(A^d) > 0,$$

and so A is irreducible by Theorem 2.1.1(vi).

Let $p = p(A)$. By Corollary 3.2.3C, $p = d(D(A))$. Let $m > 0$ be a length of a closed trail of $D(A)$. Then A^m has a diagonal 1-entry. By Lemma 3.2.2(ii), $d|m$, and so $d|p(A)$. □

Theorem 3.2.4 Let $B \in \mathbf{B}_n$ such that $B \simeq_p A$ for some A such that A has the form in (3.5) with $d > 1$ and satisfies Theorem 3.2.3(ii) and Theorem 3.2.3(iii). Then B is imprimitive and $d = p(B)$.

Proof Since $B \simeq_p A$, $p(B) = p(A)$ and B is primitive exactly when A is primitive. Thus it suffices to show that A is imprimitive and $p(A) = d$.

By Lemma 3.2.3 and since A satisfies Theorem 3.2.3(ii) and Theorem 3.2.3(iii), A is irreducible and $d|p(A)$. Thus $p(A) > 1$ and so by Corollary 3.2.3D, A is imprimitive. It remains to show that $p(A)|d$.

By Corollary 3.2.3D(ii) and Lemma 3.2.2, $A^d = \mathrm{diag}(B_1, B_2, \cdots, B_d)$, where the B_i's are defined in Lemma 3.2.2(ii). Since A satisfies Theorem 3.2.3(iii), B_1 is primitive, and so for some integer $k > 0$, both $B_1^k > 0$ and $B_1^{k+1} > 0$. It follows by Proposition 1.1.2(vii) that $D(A)$ has closed walks of length kd and $(k+1)d$, and so $d(D(A))|d$. By Corollary 3.2.3C, $p(A) = d(D(A))$, and so $p(A)|d$, as desired. □

Example 3.2.1 If $A \simeq_p B$, then $D(A)$ and $D(B)$ are isomorphic, and so by Corollary 3.2.3C and Corollary 3.2.3D that A is primitive if and only if B is primitive. However, if $A \sim_p B$, then that A is primitive may not imply that B is primitive. Consider

$$A = \begin{bmatrix} 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

We can show (Exercise 3.11) that A is primitive, B is imprimitive and $A \sim_p B$.

Theorem 3.2.5 (Shao, [246]) Let $A \in \mathbf{B}_n$ The there exists a $Q \in \mathcal{P}_n$ such that AQ is primitive if and only if each of the following holds:

(i) Each row and each column of A has at least one 1-entry;

(ii) $A \notin \mathcal{P}_n$; and

(iii)

$$A \not\sim_p \begin{bmatrix} 0 & 1 & \cdots & 1 \\ 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & \cdots & 0 \end{bmatrix}.$$

Theorem 3.2.6 (Moon and Moser, [204]) Almost all (0,1)-matrices are primitive. In other words, if P_n denote the set of all primitive matrices in $\mathbf{B}_n$, then

$$\lim_{n\to\infty} \frac{|P_n|}{|\mathbf{B}_n|} = 1.$$

3.3 The Primitive Exponent

Definition 3.3.1 A digraph D is *primitive digraph* if $A(D)$ is primitive. Let D be a primitive digraph with $V = V(D)$. For $u, v \in V$, let

$$\begin{aligned} \gamma(u,v) &= \min\{k \,:\, D \text{ has an } (u,v)\text{-walk of length } k\} \text{ and} \\ \gamma(D) &= \max_{u,v \in V(D)} \gamma(u,v). \end{aligned}$$

Let A be a primitive matrix. The *primitive exponent* of A is $\gamma(A) = \gamma(D(A))$.

Propositions 3.3.1 and 3.3.2 below provide an equivalent definition and some elementary properties of $\gamma(A)$.

Proposition 3.3.1 Let A be a primitive matrix and let $D = D(A)$. Each of the following holds.

(i) $\gamma(A) = \min\{k \,:\, A^k > 0\}$.

(ii) If $\gamma(A) = k$, then for any $u, v \in V(D)$, D has an (u,v)-walk of length at least k.

(iii) For $u, v \in V(D)$, let $d_D(u,v)$ be the shortest length of an (u,v)-walk in D, and let $l(D) = \{l_1, l_2, \cdots, l_s\}$. Then

$$\gamma(u,v) \le d_D(u,v) + \phi(l_1, \cdots, l_s).$$

Sketch of Proof Apply Proposition 1.1.2(vii) and Definition 3.3.1 to get (i). (ii) follows from (i) and (iii) follows from (ii) and Definition 3.1.1. □

Proposition 3.3.2 (Dulmage and Mendelsohn, [77]) Let D be a primitive digraph and let $A = A(D)$. Each of the following holds.

(i) For any integer $m > 0$, A^m is primitive.

(ii) Given any $u \in V(D)$, there exists a smallest integer $h_u > 0$ such that for every $v \in V(D)$, D has an (u,v)-walk of length h_u. (This number h_u is called the *reach* of the vertex u.)

(iii) Fix an $u \in V(D)$. For any integer $h \ge h_u$, D has an (u,v)-walk of length h for every $v \in V(D)$.

(iv) If $V(D) = \{1, 2, \cdots, n\}$, then $\gamma(D) = \max\{h_1, h_2, \cdots, h_n\}$.

Sketch of Proof (i) and (ii) follow from Definitions 3.2.3 and 3.3.1, and (iv) follows from (iii). (iii) can be proved by induction on $h' = h - h_u$. When $h' > 0$, since D is strong, there exists $w \in V(D)$ such that $(w,v) \in E(D)$. By induction D has an (u,w)-walk of length $h_u + h' - 1 = h - 1$.

Theorem 3.3.1 (Dulmage and Mendelsohn, [77]) Let D be a primitive digraph with $|V(D)| = n$, and let s be the length of a directed cycle of D. Then

$$\gamma(D) \le n + s(n-2).$$

Proof Let C_s be a directed cycle of D with length s. Note that $D(A^s)$ has a loop at each vertex in $V(C_s)$.

Let $u, v \in V(D) = V(D(A^s))$ be two vertices. If $u \in V(C_s)$, then since $D(A^s)$ is primitive (Proposition 3.3.2), and since $D(A^s)$ has a loop at u, $D(A^s)$ has an (u,v)-walk of length $n-1$. By Proposition 1.1.2(vii), the (u,v)-entry of $A^{s(n-1)}$ is positive. Hence D has a (u,v)-walk of length $s(n-1)$.

If $u \notin V(C_s)$, then since D is strong, D has an (u,w)-walk of length at most $n-s$ for some $w \in V(C_s)$, and so D has an (u,v)-walk of length at most $n-s+s(n-1) = n+s(n-2)$. It follows by Proposition 3.3.2 that $\gamma(D) \le n + s(n-2)$. □

Corollary 3.3.1A (Wielandt, [274]) Let D be a primitive digraph with n vertices, then

$$\gamma(D) \le (n-1)^2 - 1.$$

Sketch of Proof By induction on n. When $n \ge 2$, D has a cycle of length $s > 1$, since D is strong. Since D is primitive, $s \le n-1$, and so Corollary 3.3.1A follows from Theorem 3.3.1. □

Proposition 3.3.3 Fix $i \in \{1, 2\}$ and let D be a primitive digraph with $n \ge 3+i$ vertices. Each of the following holds.

(i) If $\gamma(D) = (n-1)^2 + 2 - i$, then D is isomorphic to D_i.

(ii) There is no primitive digraph on n vertices such that

$$n^2 - 3n + 4 < \gamma(D) < (n-1)^2.$$

Proof (i). Let s be the length of a shortest directed cycle of D. By Theorem 3.3.1, $(n-1)^2 + 2 - i = \gamma(D) \le n + s(n-2)$. It follows that $s = n-1$. Build D from this $n-1$ directed cycle to see that D must be D_1 or D_2.

(ii). By (i), $s \le n-2$, and so by Theorem 3.3.1, $\gamma(D) \le n^2 - 3n + 4$. □

Example 3.3.1 Let $C_n = v_1 v_2 \cdots, v_n v_1$ be a directed cycle with $n \ge 4$ vertices. Let D_1 be the digraph obtained from C_n by adding an arc (v_{n-1}, v_1). Then by Proposition 3.3.2 and Theorem 3.1.2,

$$\gamma(D_1) = \max\{h_{v_i}\} = h_{v_n} = n + \phi(n, n-1) = (n-1)^2 + 1.$$

Thus the bound in Corollary 3.3.1A is best possible.

Example 3.3.2 (Continuation of Example 3.3.1) Assume that $n \ge 5$. Let D_2 be obtained from D_1 by adding an arc (v_n, v_2). Note that $\gamma(D_2) = (n-1)^2$.

Proposition 3.3.3 indicates that D_i is the only digraph, up to isomorphism, that have $\gamma(D_i) = (n-1)^2 + 2 - i$, $(1 \le i \le 2)$. Moreover, there will some integers k such that $1 \le k \le (n-1)^2 + 1$ but no primitive digraph D satisfies $\gamma(D) = k$.

Example 3.3.3 (Holladay and Varga, [129]) Let $n \ge d > 0$ be integers. If $A \in M_n^+$ is irreducible and has d positive diagonal entries, then A is primitive and $\gamma(A) \le 2n-d-1$ (Proposition 3.3.4(ii) below). Let $A(n,d) \in M_n$ be the following matrix

$$A(n,d) = \begin{bmatrix} 1 & 1 & & & & \\ 0 & \ddots & \ddots & & & \\ \vdots & & 1 & \ddots & & \\ \vdots & & & 0 & \ddots & \\ 0 & & & & \ddots & 1 \\ 1 & & & & 0 & \end{bmatrix},$$

where $A(n,d)$ has exactly d positive diagonal entries. Then $\gamma(A(n,d)) = 2n-d-1$.

Proposition 3.3.4 Let D be a strong digraph with $n = |V(D)|$ and let $d > 0$ be an integer. Then each of the following holds.

(i) If D has a loop, then D is primitive.

(ii) If D has loops at d distinct vertices, then $\gamma(D) \le 2n-d-1$.

(iii) The bound in (ii) is best possible.

Sketch of Proof (i) follows from Theorem 3.2.1.

(ii). By (i), D is primitive. For any $u, v \in V(D)$, D has an (u,w)-walk of length $n-d$ for some vertex w with a loop, and a (w,v)-walk of length at most $n-1$. Thus $h_u \le 2n-d-1$, and (ii) follows from Proposition 3.3.2(iv).

(iii). Compute $\gamma(A(n,d))$ for the graph $A(n,d)$ in Example 3.3.3 to see $\gamma(A(n,d)) = 2n-d-1$. $\square$

Definition 3.3.2 For integers $b > a > 0$ and $n > 0$, let

$$\begin{aligned} [a,b]^o &= \{k : k \text{ is an integer and } a \le k \le b\} \\ E_n &= \{k : \text{there exists a primitive matrix } A \in M_n^+ \text{ such that } \gamma(A) = k\}. \end{aligned}$$

By Theorem 3.3.1 and by Proposition 3.3.3, $E_n \subset [1,(n-1)^2+1]^o$.

Theorem 3.3.2 (Liu, [168]) Let $n-1 \ge d \ge 1$ be integers let $P_n(d)$ be the set of primitives matrix in M_n^+ with $d > 0$ positive diagonal entries. If $k \in \{2,3,\cdots,2n-d-1\}$, then there exists a matrix $A \in P_n(d)$ such that $\gamma(A) = k$.

Sketch of Proof For any integer $k \in \{2,3,\cdots,n\}$ we construct a digraph D whose adjacency matrix A satisfying the requirements.

If $1 \le d < k \le n$, the consider the adjacency matrix of the digraph D in Figure 3.3.1.

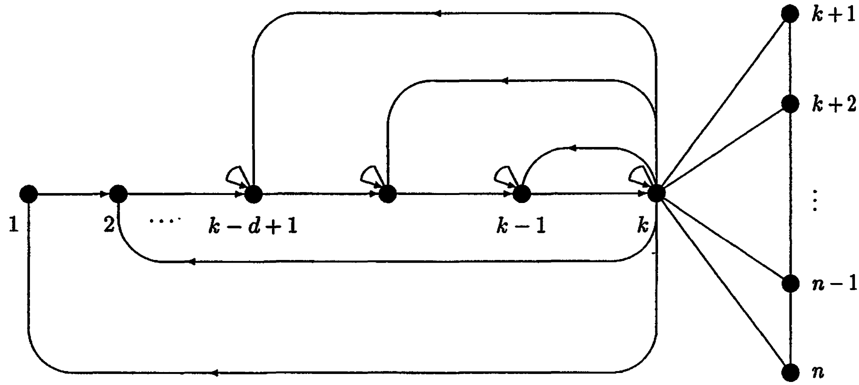

Figure 3.3.1

Note that

$$\gamma(i,j)\begin{cases} = k & \text{if } i=j=1, \\ \le k & \text{otherwise.}\end{cases}$$

Thus $\gamma(A) = k$ in this case.

Digraphs needed to prove the other cases can be constructed similarly, and their constructions are left as an exercise. $\square$

Theorem 3.3.3 (Shao, [245]) Let $A \in \mathbf{B}_n$ be symmetric and irreducible.

(i) A is primitive if and only if $D(A)$ has a directed cycle of odd length.

(ii) If A is primitive, then $\gamma(A) \le 2n-2$, where equality holds if and only if

$$A \simeq_p B = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & \cdots & 1 & 1 \end{bmatrix}.$$

Proof Let $D = D(A)$. Since A is reduced and symmetric, D is strong and every arc of D lies in a directed 2-cycle. Thus by Theorem 3.2.2, D is primitive if and only if D has a directed odd cycle.

Assume that A is primitive. Then A^2 is also primitive by Proposition 3.3.2. Since A is symmetric, a loop is attached at each vertex of $V(D(A^2)) = V(D)$. Thus by Proposition 3.3.4(ii), $\gamma(A^2) \le n-1$, and so $A^{2(n-1)} > 0$. Hence $\gamma(A) \le 2n-2$.

Assume further that $\gamma(A) = 2n-2$. Then in $D(A^2)$, there exist a pair of vertices u, v such that the shortest length of a (u,v)-path in $D(A^2)$ is $n-1$. It follows that $D(A^2)$ must be an (u,v)-path with a loop at every vertex and with each arc in a 2-cycle. If D has a vertex adjacent to three distinct vertices $u', v', w' \in V(D)$, then $u'v'w'u'$ is a directed 3-cycle in $D(A^2)$, contrary to the fact that $D(A^2)$ is an (u,v)-path with a loop attaching at each vertex. It follows that $D(A)$ is a path of n vertices and has at least one loop. By $\gamma(A) = 2n-2$ again, D has exactly one loop which is attached at one end of the path. $\square$

Theorem 3.3.4 (Shao, [245]) For all $n \ge 1$, $E_n \subseteq E_{n+1}$. Moreover, if $n \ge 4$, then $E_n \subset E_{n+1}$.

Proof Let $t \in E_n$, and let $A = (a_{ij}) \in \mathbf{B}_n$ be a primitive matrix with $\gamma(A) = t$. Construct a matrix $B = (b_{ij}) \in \mathbf{B}_{n+1}$ as follows. The $n \times n$ upper left corner submatrix of B is A, for $1 \le i \le n$, $b_{i,n+1} = a_{i,n}$, for $1 \le j \le n$, $b_{n+1,j} = a_{n,j}$, and $b_{n+1,n+1} = a_{n,n}$. Let $V(D(A)) = \{1, 2, \cdots, n\}$. Then $D(B)$ is the digraph obtained from $D(A)$ by adding a new vertex $n+1$ such that $(i, n+1) \in E(D(B))$ if and only if $(i,n) \in E(D(A))$, such that $(n+1, j) \in E(D(B))$ if and only if $(n,j) \in E(D(A))$, and such that $(n+1, n+1) \in E(D(B))$ if and only if $(n,n) \in E(D(A))$. By Theorem 3.2.2, B is also primitive. By Definition 3.3.1 (or by Exercise 3.13(iv)), $\gamma(B) = \gamma(A)$, and so $t \in E_{n+1}$.

If $n \ge 4$, then by Example 3.3.1, $n^2+1 \in E_{n+1} - E_n$, and so the containment must be proper. $\square$

Ever since 1950, when Wielandt published his paper [274] giving a best possible upper bound of $\gamma(A)$, the study of $\gamma(A)$ has been focusing on the problems described below. Let $\mathcal{A}$ denote a class of primitive matrices.

(MI) The *Maximum Index* problem: estimate upper bounds of $\gamma(A)$ for $A \in \mathcal{A}$.
(IS) The *Set of Indices* problem: determine the exponent set

$$E_n(\mathcal{A}) = \{m \,:\, m > 0 \text{ is an integer such that for some } A \in \mathcal{A},\ \gamma(A) = m\}.$$

(EM) The *Extremal Matrix* problem: determine the matrices with maximum exponent in a given class $\mathcal{A}$. That is, the set

$$EM(\mathcal{A}) = \{A \in \mathcal{A} \,:\, \gamma(A) = \max\{\gamma(A') \,:\, A' \in \mathcal{A}\}\}.$$

(MS) The *Set of Matrices* problem: for a $\gamma_0 \in \cup_n E_n(\mathcal{A})$, determine the set of matrices

$$MS(\mathcal{A}, \gamma_0) = \{A \in \mathcal{A} \,:\, \gamma(A) = \gamma_0\}.$$

In fact, Problem EM is a special case of Problem MS.

We are to present a brief survey on these problems, which indicates the progresses made in each of these problems by far. First, let us recall and name some classes of matrices.

Some Classes of Primitive Matrices

Notation	Definition
P_n	$n \times n$ primitive matrices in $\mathbf{B}_n$
$P_n(d)$	matrices in P_n with d positive diagonal entries
T_n	matrices $A \in P_n$ such that $D(A)$ is a tournament
F_n	fully indecomposable matrices in P_n
DS_n	$P_n \cap \Omega_n$
CP_n	circulant matrices in P_n
NR_n	nearly reducible matrices in P_n
S_n	symmetric matrices in P_n
S_n^0	matrices in S_n with zero trace

Problem MI

This area seems to be the one that has been studied most thoroughly.

Notation	Authors and References	Results
P_n	Wielandt, [274]	$\gamma(A) \le (n-1)^2 + 1$
	Dulmage and Mendelsohn, [77]	$\gamma(A) \le n + s(n-2)$
$P_n(d)$	Holladay and Varga, [129]	$\gamma(A) \le 2n - d - 1$
T_n	Moon and Pullman, [205]	$\gamma(A) \le n + 2$
F_n	Schwarz, [232]	$\gamma(A) \le n - 1$
DS_n	Lewin, [159]	$\gamma(A) \le \begin{cases} \lfloor \frac{n^2}{4} + 1 \rfloor & \text{if } n = 5, 6, \text{ or } n \equiv 0 \pmod 4 \\ \lfloor \frac{n^2}{4} \rfloor & \text{otherwise} \end{cases}$
CP_n	Kim-Butler and Krabill [144]	$\gamma(A) \le n - 1$
NR_n	Brualdi and Ross, [36]	$\gamma(A) \le n^2 - 4n + 6$
S_n	Shao, [245]	$\gamma(A) \le 2n - 2$
S_n^0	Liu et al, [177]	$\gamma(A) \le 2n - 4$

Problem IS

Let $w_n = (n-1)^2 + 1$. Wielandt (1950, [274]) showed that $E_n \subseteq [1, w_n]^o$; Dulmage and Mendelsohn (1964, [77]) showed that $E_n \subset [1, w_n]^o$.

In 1981, Lewin and Vitek [157] found all gaps (numbers in $[1, w_n]^o$ but not in E_n) in $\left[\lfloor \frac{w_n}{2} \rfloor + 1, w_n\right]^o$ and conjectured that $\left[1, \lfloor \frac{w_n}{2} \rfloor\right]^o$ has no gaps.

Shao (1985, [247]) proved that this Levin-Vitek Conjecture is valid for sufficiently large n and that $\left[1, \lfloor \frac{w_n}{4} \rfloor + 1\right]^o$ has no gaps. However, when $n = 11$, $48 \notin E_{11}$ and so the conjecture has one counterexample.

Zhang continued and complete the work. He proved (1987, [282]) that the Levin-Vitek Conjecture holds for all n except $n = 11$. Thus the set E_n for the class P_n is completely determined.

Results concerning the exponent set in special classes of matrices are listed below.

Notation	Authors and References	Results
$P_n(n)$	Guo, [110]	$[1, n-1]^o$
$P_n(d)$	Liu, [168]	$[2, 2n-d-1]^o$ $1 \leq d < n$
T_n	Moon and Pullman, [205]	$[3, n+2]^o$
F_n	Pan, [209]	$[1, n-1]^o$
DS_n		Unsolved
CP_n		Unsolved
NR_n	Shao and Hu, [249]	Characterized
S_n	Shao, [245]	$[1, 2n-2]^o \setminus S$ $S = \{m$ is an odd integer and $n \leq m \leq 2n-3\}$
S_n^0	Liu et al, [177]	$[2, 2n-4]^o \setminus S_1$ $S_1 = \{m$ is an odd integer and $n-2 \leq m \leq 2n-5\}$

Problem EM

In Proposition 3.3.3, it is indicated that

$$EM(P_n) = \{PD_1P^{-1} \ : \ P \text{ is a permutation matrix}\},$$

where D_1 is defined in Example 3.3.1. However, for certain classes of primitive matrices, the extremal matrices may not be unique under the $\simeq_p$ equivalence and are hard to characterize, even when under the condition that the number of positive entries is minimized ([168]). Several such problems remain unsolved.

Results concerning Problem EM in special classes of matrices are summarized below.

Notation	Status	Authors and References
$P_n(d)$	Solved	Liu and Shao, [182]
T_n	Solved	Moon and Pullman, [205]
F_n	Partially Solved	Pan, [209]
DS_n	Solved	Zhou and Liu [290]
CP_n	Solved	Huang, [135]
NR_n	Solved	Brualdi and Ross, [36]
S_n	Solved	Shao, [245]
S_n^0	Solved	Liu et al, [177]

Problem MS

This problem appears to be harder. From the view point of matrix equations, this is to find all primitive matrices solution A to the equation $A^k = J$, for any $k \in E_n$. The study of this problem is just the beginning. See [209], [292] and [270], among others.

3.4 The Index of Convergence of Irreducible Nonprimitive Matrices

Throughout this section, the matrices and the matrix operations will also be Boolean.

Definition 3.4.1 Recall that $p(A)$ denotes the period of a matrix $A \in \mathbf{B}_n$ and $k(A)$ the index of convergence of A (Definition 3.2.2). For integers $n \geq p \geq 1$ with $n > 1$, define

$$\mathbf{IB}_{n,p} = \{A \in \mathbf{B}_n \ : \ p(A) = p\}.$$

Note that $\mathbf{IB}_{n,1} = P_n$, the set of all n by n primitive matrices. Denote

$$\bar{k}(n,p) = \max\{k(A) \ : \ A \in \mathbf{IB}_{n,p}\}.$$

Theorem 3.4.1 (Heap and Lynn, [119]) Write $n = pr + s$ for integers r and s such that $0 \leq s < p$. Then

$$\bar{k}(n,p) \leq p(r^2 - 2r + 2) + 2s.$$

Equality holds when $s = 0$.

Note that Wielandt's theorem (Corollary 3.3.1A) is the special case of Theorem 3.4.1 when $p = 1$. The next theorem, due to Schwarz (Theorem 3.4.2), implies Theorem 3.4.1. The proof here is also given by Schwarz.

Theorem 3.4.2 (Schwarz, [233]) Write $n = pr+s$ for integers r and s such that $0 \leq s < p$

and let

$$\omega_n = \begin{cases} r^2 - 2r + 2 & \text{if } r > 1 \\ 0 & \text{if } r = 1. \end{cases}$$

Then $\bar{k}(n,p) \le p\omega_n + s$.

Some notations and lemmas are needed in the proof of Theorem 3.4.2.

Definition 3.4.2 Let $A \in \mathbf{B}_n$ with $p(A) = p$. If A is not primitive, then by Corollary 3.2.3D, A is permutation similar to

$$\begin{bmatrix} 0 & A_1 & & & \\ & 0 & A_2 & & \\ & & 0 & \ddots & \\ & & & & A_{p-1} \\ A_p & & & & 0 \end{bmatrix}, \tag{3.7}$$

where each $A_i \in M_{n_i,n_{i+1}}$, $(i = 1, 2, \cdots, p \pmod p)$ is primitive. Denote the matrix of the form (3.7) by $(n_1, A_1, n_2, A_2, \cdots, n_p, A_p, n_1)$, or simply $(A_1, A_2, \cdots, A_p)$, when no confusion should arise.

Fix an integer $p \ge 1$. For any integers $m \ge 0$ and $n_1, n_2, \cdots, n_p > 0$, and any $Z_i \in M_{n_i,n_{i+1}}$, $(i = 1, 2, \cdots, p \pmod p)$, define $(Z_1, Z_2, \cdots, Z_p)_m$ to be the block matrix $[A_{ij}]$ such that

$$A_{ij} = \begin{cases} Z_i & \text{if } j - i \equiv m \pmod p \\ 0 & \text{otherwise.} \end{cases}$$

Thus $(A_1, A_2, \cdots, A_p) = (A_1, A_2, \cdots, A_p)_1$.

For any $m, j \ge 1$, define

$$A_{j+p} = A_j,\ A_j(0) = I_{n_j},\ \text{and } A_j(m) = A_j A_{j+1} \cdots A_{j+m-1}.$$

The following three lemmas are obtained form the definitions and by straightforward computations.

Lemma 3.4.1 If $A = (A_1, A_2, \cdots, A_p)$, then

(i) $A^m = (A_1(m), A_2(m), \cdots, A_p(m))_m$.

(ii) $A_i(ph) = (A_i(p))^h$, for each $i = 1, 2, \cdots, p \pmod p$, and for each integer $h \ge 1$.

(iii) For integers k, j, l, q with $k, j, l > 0$, $q \ge 0$, $j + q \le p$, and $1 \le l \le p$, $A_l(k) = A_l(q)A_{l+q}(k - q)$.

Lemma 3.4.2 Let $A, B \in \mathbf{B}_n$. Each of the following holds.

(i) If AB is primitive, then A has no zero rows and B has no zero columns.

(ii) If A has no zero columns and B has no zero rows, and if AB is primitive, then BA is also primitive. Moreover,

$$|\gamma(BA)-\gamma(AB)|\le 1.$$

(iii) If $A=(A_1,A_2,\cdots,A_p)\in \mathbf{B}_n$ is an irreducible matrix with $p(A)=p$, then each $A_i(p)$ $(1\le i\le p)$, is primitive, and $|\gamma(A_i(p))-\gamma(A_j(p))|\le 1$, $(1\le i,j\le p)$.

Lemma 3.4.3 Let $A=(n_1,A_1,n_2,A_2,\cdots,n_p,A_p,n_1)=(A_1,A_2,\cdots,A_p)\in \mathbf{B}_n$ be an irreducible matrix with $p(A)=p$. Then $k(A)$ is the smallest integer $k>0$ such that $A_i(k)=J$, for all $i=1,2,\cdots,p$.

Lemma 3.4.4 Let $A=(A_1,A_2,\cdots,A_p)\in \mathbf{B}_n$ be an irreducible matrix with $p(A)=p$. Let t be an integer with $1\le t\le p$. If for $1\le i_1<i_2<\cdots<i_t\le p$, $\gamma_{i_j}=\gamma(A_{i_j}(p))$, then

$$k(A)\le p\max\{\gamma_{i_1},\gamma_{i_2},\cdots,\gamma_{i_t}\}+p-t.$$

Proof Let $h=\max\{\gamma_{i_1},\gamma_{i_2},\cdots,\gamma_{i_t}\}$ and $k=ph+p-t$. Since $h\ge\gamma(A_{i_j}(p))$,

$$A_{i_j}(ph)=(A_{i_j}(p))^h=J^h=J.$$

For any $1\le l\le p$, since

$$|\{i_1,\cdots,i_t\}|+|\{l,l+1,l+2,\cdots,l+p-t\}|=p+1>p,$$

there exist j and q $(1\le j\le t,\ 0\le q\le p-t)$ such that some $i_j\equiv l+q \pmod p$, and so $A_{ij}=A_{l+q}$. It follows that for each $l=1,2,\cdots,p$,

$$\begin{aligned}
A_l(k) &= A_l(q)A_{l+q}(k-q)\\
&= A_l(q)A_{l+q}(ph)A_{l+q+ph}(k-ph-q)\\
&= A_l(q)A_{i_j}(ph)A_{l+q+ph}(k-ph-q)\\
&= A_l(q)JA_{l+q+ph}(p-t-q)=J.
\end{aligned}$$

Therefore $k(A)\le k$ follows by Lemma 3.4.3. □

Lemma 3.4.5 Let $A=(n_1,A_1,n_2,A_2,\cdots,A_p,n_1)\in \mathbf{B}_n$ be an irreducible matrix with $p(A)=p$, and let $m=\min\{n_1,n_2,\cdots,n_p\}$. Then

$$k(A)\le p(m^2-2m+3)-1.$$

Proof It follows from Lemma 3.4.4 and Corollary 3.3.1A. □

Proof of Theorem 3.4.2 Since $k(A)=k(PAP^{-1})$ for any permutation matrix P, we may assume that $A=(n_1,A_1,n_2,A_2,\cdots,n_p,A_p,n_1)$, where $n_1+n_2+\cdots+n_p=n$. Let $m=\min\{n_1,\cdots,n_p\}$. Since $n=rp+s$ where $0\le s<p$, $m\le r$.

Case 1 $m \le r-1$. Then $r \ge m+1 \ge 2$, and so by Lemma 3.4.5,

$$\begin{aligned} k(A) &\le p(m^2-2m+3)-1 \le p(r^2-4r+6)-1 \\ &< p(r^2-2r+2)+s. \end{aligned}$$

Case 2 $m = r$. Since $n_1+n_2+\cdots+n_p = n = pr+s = pm+s$, there must be $1 \le i_1 < i_2 < \cdots < i_{p-s} \le p$ such that $n_{i_1} = \cdots = n_{i_{p-s}} = r$. For each $j = 1, 2, \cdots, p-s$, $A_{i_j}(p) \in \mathbf{B}_r$ is primitive, and $\gamma_{i_j} = \gamma(A_{i_j}(p)) \le r^2 - 2r + 2$ (by Corollary 3.3.1A). It follows by Lemma 3.4.4 with $t = p-s$ that

$$k(A) \le p\max\{\gamma_{i_1}, \cdots, \gamma_{i_{p-s}}\} + p - (p-s) \le p(r^2-2r+2)+s.$$

When $r = 1$, $n = p+s$. Thus among any $s+1$ members in $\{n_1, \cdots, n_p\}$, at least one of them is 1. Accordingly, one of the matrices of $A_i, A_{i+1}, \cdots, A_{i+s-1}$ is a J, since these matrices has no zero rows nor zero columns. It follows that $A_i(s) = J$, $i = 1, 2, \cdots, p$, and so $k(A) \le s$, by Lemma 3.4.3. □

Schwarz indicated in [233] that in Theorem 3.4.2, the upper bound can be reached when $n = 7$ and $p = 2$. Can the upper bound be reached for general n and p? Shao and Li [252] completely answered this question. Two of their results are presented below. Further details can be found in [252].

Theorem 3.4.3 (Shao and Li, [252]) Let $A \in \mathbf{IB}_{n,p}$ with $p = 2$ and $n = 2r+1$, $r > 1$. Then $k(A) = \bar{k}(n,p)$ if and only if there is a permutation matrix P such that $PAP^{-1} \in \{M_1, M_2, M_3\}$, where

$$M_1 = \begin{bmatrix} 0 & H_1 \\ Y_1 & 0 \end{bmatrix}, \; M_2 = \begin{bmatrix} 0 & H_1 \\ Y_2 & 0 \end{bmatrix}, \; M_3 = \begin{bmatrix} 0 & H_2 \\ Y_1 & 0 \end{bmatrix},$$

and where

$$H_1 = \begin{bmatrix} 0 & 1 & & & \\ & 0 & \ddots & & \\ & \ddots & \ddots & & 1 \\ & \cdots & & \cdots & \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \end{bmatrix}_{(r+1)\times r}; \; H_2 = \begin{bmatrix} 0 & 1 & & & \\ & 0 & \ddots & & \\ & \ddots & \ddots & & 1 \\ & \cdots & & \cdots & \\ 1 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 0 & \cdots & 0 \end{bmatrix}_{(r+1)\times r}$$

and

$$Y_1 = \begin{bmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & 1 \end{bmatrix}_{r\times(r+1)}, \; Y_2 = \begin{bmatrix} 1 & & & 1 \\ & \ddots & & \\ & & 1 & 0 \end{bmatrix}_{r\times(r+1)}.$$

Theorem 3.4.4 (Shao and Li, [252]) When $r > 1$, $r = 1$ and $s > 0$, or $r = 1$ and $s = 0$, the matrices $A \in \mathrm{IB}_{n,p}$ with $k(A) = \bar{k}(n,p)$ can be partitioned into $2^s + s \cdot 2^{s-1}$, 2^{s-1}, and 1 equivalence classes, respectively, under the relation $\simeq_p$.

Theorem 3.4.2 can be viewed as a Wielandt type upper bound of the index of convergence of irreducible nonprimitive matrices. In order to obtain a Dulmage-Mendelsohn type upper bound, we need a few more notions.

Definition 3.4.3 Let $A \in \mathrm{IB}_{n,p}$. For $1 \le i,j \le n$, let $k_A(i,j)$ be the smallest integer $k \ge 0$ such that $(A^{l+p})_{i,j} = (A^l)_{i.j}$, for all $l \ge k$; and define $m_A(i,j)$ be the smallest integer $m \ge 0$ such that $(A^{m+ap})_{i,j} = 1$ for all $a \ge 0$.

Example 3.4.1 Let $A \in \mathrm{IB}_{n,p}$ and let $D = D(A)$ with $V(D) = \{v_1, v_2, \cdots, v_n\}$. By Proposition 1.1.2(vii), $k_A(i,j)$ is the smallest integer $k \ge 0$ such that for each $l \ge k$, D has a directed (v_i, v_j)-walk of length $l+p$ if and only if D has a directed (v_i, v_j)-walk of length l; and $m_A(i,j)$ is the smallest integer $m \ge 0$ such that for any $a \ge 0$, D has a directed (v_i, v_j)-walk of length $m + ap$.

Definition 3.4.4 Let $A \in \mathrm{IB}_{n,p}$ and $D = D(A)$ with $V(D) = \{v_1, \cdots, v_n\}$, $l(A) = \{l_1, l_2, \cdots, l_s\}$ denote the set of lengths of directed cycles in $D(A)$, and let $d_{l(A)}(i,j)$ denote the shortest length of a directed (v_i, v_j)-walk in D that intersects directed cycles in D of lengths in $l(D)$.

Also, let $p = \gcd(l_1, l_2, \cdots, l_s)$, and define

$$\tilde{\phi}(l_1, l_2, \cdots, l_s) = p \cdot \phi(\frac{l_1}{p}, \frac{l_2}{p}, \cdots, \frac{l_s}{p}).$$

The following two results, due to Shao and Li [251], indicate that to estimate $k(A)$, we can study $d_{l(D)}$ and $\tilde{\phi}(l_1, \cdots, l_s)$, therefore the problem can be approached by graph theory techniques and by number theory techniques.

Theorem 3.4.5 (Shao and Li [251]) Let $A \in \mathrm{IB}_{n,p}$. For $1 \le i,j \le n$, each of the following holds.

(i) $k(A) = \max_{1 \le i,j \le n}\{k_A(i,j)\}$, and

(ii) $k_A(i,j) = \begin{cases} m_A(i,j) - p + 1 & \text{if } m_A(i,j) \ge p-1 \\ 0 & \text{if } m_A(i,j) < p-1. \end{cases}$

Proof It suffices to prove (ii). Let $m = m_A(i,j)$ and $k = k_A(i,j)$. Let $D = D(A)$ with $V(D) = \{v_1, \cdots, v_n\}$. Suppose that $l \ge m - p + 1$ is an integer.

If $l \equiv m \pmod p$, then $l = m + ap$ for some integer $a \ge 0$. By Definition 3.4.3, $(A^l)_{i,j} = (A^{l+p})_{i,j} = 1$. Assume that $l \not\equiv m \pmod p$. By Proposition 1.1.2(vii), D has a directed (v_i, v_j)-walk of length exactly m. It follows by Lemma 3.2.1(ii) that D does not have directed (v_i, v_j)-walk of length exactly l and $l+p$, and so $(A^l)_{ij} = (A_{ij}^{l+p}) = 0$.

Thus for each $l \geq m-p+1$, $(A^l)_{ij} = (A^{l+p})_{ij}$, and so by Definition 3.4.3, $k \leq m-p+1$. On the other hand, by Definition 3.4.3, $(A^{m-p})_{ij} = 0 \neq 1 = (A_m)_{ij}$, and so $k \geq m-p+1$. $\square$

Theorem 3.4.6 (Shao and Li [251]) Let $A \in \mathbf{IB}_{n,p}$ and $D = D(A)$. Let $l(D) = \{l_1, l_2, \cdots, l_s\}$. For $1 \leq i, j \leq n$,

$$m_A(i.j) \leq d_{l(A)}(i,j) + \tilde{\phi}(l_1, l_2, \cdots, l_s).$$

Proof Let W be a shortest directed (v_i, v_j)-walk which intersects directed cycles of each length in $l(D)$. Then by the definition of the Frobenius number, for an integer $a \geq 0$, D has a directed (v_i, v_j)-walk of length exactly $d_{L(A)}(i,j) + \tilde{\phi}(l_1, l_2, \cdots l_s\} + ap$, and so the theorem follows by Definition 3.4.3. $\square$

By applying the techniques used in Theorems 3.4.5 and 3.4.6, Shao, Shao and Wu obtained the Dulmage-Mendelsohn type upper bound.

Theorem 3.4.7 (Shao, [242], [243], Wu and Shao [279]) Let $A \in \mathbf{IB}_{n,p}$, $D = D(A)$ and $l = \min\{l_i \in l(D)\}$. Then

$$k(A) \leq n + l\left(\lfloor \frac{n}{p} \rfloor - 2\right).$$

Theorem 3.4.8 (Shao, [242]) Let $A \in \mathbf{IB}_{n,p}$ and $D = D(A)$. Write $n = pr + s$, where $0 \leq s < p$. If $l(D)$ contains three distinct cycle lengths different from p, then

$$k(A) \leq p\lfloor \frac{r^2 - 2r + 4}{2} \rfloor + s.$$

Definition 3.4.5 Let $A \in \mathbf{IB}_{n,p}$. Define

$$\begin{aligned} m(A) &= \max_{1 \leq i,j \leq n} m_A(i,j) \\ IM_{n,p} &= \{m(A) \; : \; A \in \mathbf{IB}_{n,p}\} \\ I_{n,p} &= \{k(A) \; : \; A \in \mathbf{IB}_{n,p}\} \end{aligned}$$

Shao and Li [251] investigated the structure of $I_{n,p}$ and found that $I_{n,p}$ may have gaps. The complete characterization of $I_{n,p}$ is done by Shao and Li [251] and Wu and Shao [279]. Lemma 3.4.6 below can be obtained by quoting Lemmas 3.4.3 and 3.4.4 (with $t = 1$).

Lemma 3.4.6 (Shao and Li, [251]) Let $A = (A_1, A_2, \cdots, A_p) \in \mathbf{IB}_{n,p}$. Then for $i = 1, 2, \cdots, p$,

$$p\gamma(A_i(p)) - p + 1 \leq k(A) \leq p\gamma(A_i(p)) + p - 1.$$

Theorem 3.4.9 (Shao and Li, [251]) Let n, p, k_1, k_2 be positive integers. such that $n = pr + s$, where $0 \le s < p$. If for all k with $k_1 \le k \le k_2$, $k \notin E_r$, then for all m with $k_1 p \le m \le k_2 p$, $m \notin I_{n,p}$. In particular (when $k_1 = k_2 = k$), if $k \notin E_r$, then $kp \notin I_{n,p}$.

Proof Suppose that there exists an m with $k_1 p \le m \le k_2 p$ and $m \in I_{n,p}$. Then there exists an $A \in \mathbf{IB}_{n,p}$ with $m = k(A)$.

By Corollary 3.2.3D, we may assume that there exist matrices $A_1, A_2, \cdots, A_p$ such that $A = (n_1, A_1, n_2, A_2, \cdots, n_p, A_p, n_1)$ satisfying Theorem 3.2.3.

Since $n_1 + n_2 + \cdots + n_p = n = pr + s < p(r+1)$, there must be some $n_j \le r$. By Corollary 3.2.3B, $A_j(p)$ is an $n_j \times n_j$ primitive matrix, and so by Theorem 3.3.4, $\gamma_j = \gamma(A_j(p)) \in E_{n_j} \subseteq E_r$.

On the other hand, by Lemma 3.4.6,

$$p\gamma_i - p + 1 \le m \le p\gamma_i + p - 1,$$

and so $k_1 \le \gamma_i \le k_2$, contrary to the assumption that any k with $k_1 \le k \le k_2$ is not in E_r. $\square$

Example 3.4.2 Dulmage and Mendelsohn [77] showed that if $r \ge 4$ is even, then for any k with $r^2 - 4r + 7 \le k \le r^2 - 2r$, we have $k \notin E_r$. By Theorem 3.4.9, for any $p > 0$, $s \ge 0$ and $n = pr + s$, we have $pk \notin I_{n,p}$, and so $I_{n,p}$ may have gaps.

Theorem 3.4.10 (Shao and Li, [251], Wu and Shao [279]) Let $n = pr + s$ with $0 \le s < p$. Then

$$I_{n,p} = I_1(n,p) \cup I_2(n,p),$$

where

$$\begin{aligned} I_1(n,p) &= \left\{1, 2, \cdots, p\lfloor \frac{r^2 - 2r + 4}{2} \rfloor + s\right\} \text{ and} \\ I_2(n,p) &= \bigcup_{\substack{n \ge r_1 \ge r_2 \ge 4p \\ r_1 + r_2 \ge (r+3)p \\ \gcd(r_1, r_2) = p}} \left\{r_2\left(\frac{r_2}{p} - 1\right), \cdots, r_2\left(\frac{r_2}{p} - 2\right) + n\right\} \end{aligned}$$

3.5 The Index of Convergence of Reducible Nonprimitive Matrices

In 1970, Schwarz presented the first upper bound of $k(A)$ for reducible matrices in $\mathbf{B}_n$. Little had been done on this subject until Shao obtained a Dulmage-Mendelsohn type upper bound in 1990.

Theorem 3.5.1 (Schwarz, [233]) For each $A \in \mathbf{B}_n$, $k(A) \le (n-1)^2 + 1$. Moreover, if A is reducible, then $k(A) \le (n-1)^2$.

Theorem 3.5.2 (Shao, [243]) Let $A \in \mathbf{B}_n$, $D = D(A)$, and $D_1, D_2, \cdots, D_c$ the strong components of D. Let $n_0 = \max_{1 \le i \le c}\{|V(D_i)|\}$, and let s_0 be the maximum of shortest lengths of directed cycles in each D_i, $1 \le i \le c$, if D has a directed cycle, or 0 if D is acyclic. Then

$$k(A) \le n + s_0 \left(\frac{n_0}{d(D)} - 2 \right).$$

Theorem 3.5.2 implies Theorem 3.5.1 and Theorem 3.3.1 (Exercise 3.17). In Theorem 3.5.3 below, Shao [243] applied Theorem 3.5.2 to estimate the upperbound of $k(A)$ for reducible matrices A.

Lemma 3.5.1 Let $X \in \mathbf{B}_n$ have the following form:

$$X = \begin{bmatrix} B & 0 \\ \mathbf{x}^T & a \end{bmatrix} \tag{3.8}$$

where $B \in \mathbf{B}_{n-1}$ and $a \in \{0, 1\}$. Each of the following holds.

(i) If $a = 0$, then $k(B) \le k(X) \le k(B) + 1$.

(ii) If $a = 1$, then $k(B) \le k(X) \le \max\{k(B), n-1\}$.

Proof The relationship between X^{k+1} and B^{k+1} can be seen in Exercise 3.19. Thus by Definition 3.2.2, $k(B) \le k(X)$. When $a = 0$, Lemma 3.5.1(i) follows immediately from direct matrix computation (Exercise 3.19(i)).

Assume that $a = 1$. Since $B \in \mathbf{B}_{n-1}$, for any $k \ge n-2$, $I + B + \cdots B^k = I + B + \cdots + B^{n-2}$, by Proposition 1.1.2(vii). By matrix computation (Exercise 3.19(ii)), if $m \ge \max\{k(B), n-1\}$, then $X^m = X^{m+p(B)}$, and so $k(X) \le \max\{k(B), n-1\}$. □

Lemma 3.5.2 Let $n \ge 3$ be an integer and let $A \in \mathbf{B}_n$ be a reducible matrix. If $k(A) > n^2 - 5n + 9$, then there exists a matrix $X \in \mathbf{B}_n$ with the form in (3.8) such that $B \in \mathbf{B}_{n-1}$ is primitive, such that $D(B)$ has a shortest directed cycle with length $n-1$, and such that either $A \simeq_p X$ or $A^T \simeq_p X$.

Proof Let $D = D(A)$. If every strong component of D is a single vertex, then $A^n = A^{n+1} = 0$, and so $k(A) \le n < n^2 - 5n + 9$, a contradiction. Hence D must have a strong component D_1 with $|V(D_1)| > 1$. By Theorem 3.5.2,

$$k(A) \le n + s_0 \left(\frac{n_0}{d(D)} + 2 \right), \tag{3.9}$$

where s_0 and n_0 are defined in Theorem 3.5.2. Since A is reducible,

$$s_0 \le n_0 \le n - 1. \tag{3.10}$$

If $s_0 \le n-3$, then by (3.9) and (3.10), $k(A) \le n+(n-3)(n-1-2) = n^2-5n+9$, a contradiction.

If $s_0 = n-2$ and $n_0 = n-2$, then by (3.9), $k(A) \le n+(n-2)(n-4) < n^2-5n+9$, another contradiction.

If $s_0 = n-1$, then by (3.10), $n_0 = n-1$. Then D has a strong component D_1 which is a directed cycle of $n-1$ vertices. Therefore we may assume that A or A^T has the form (3.8), and that the submatrix B in (3.8) must be a permutation matrix. It follows that $B^0 = I_{n-1} = B^{n-1}$, and so $k(B) = 0$. By Lemma 3.5.1, $k(A) \le \max\{k(B)+1, n-1\} \le n-1 < n^2-5n+9$, a contradiction.

Therefore it must be the case that $s_0 = n-2$ and $n_0 = n-1$, and so we may assume that A or A^T has the form (3.8) and that the associate directed graph $D(B)$ of the submatrix B in (3.8) has shortest directed cycle length $n-2$.

It remains to show that B is primitive. If not, then by Theorem 3.2.2, every directed cycle of $D(B)$ has length exactly $n-2$. Since $D(B)$ is strong, $B = B^{n-1}$, and so $k(B) = 0$, which implies by Lemma 3.5.1 that $k(A) \le n^2-5n+9$, a contradiction. Therefore, B must be primitive. $\square$

Theorem 3.5.3 (Shao, [243]) Let $A \in \mathbf{B}_n$ be a reducible matrix. Then

$$k(A) \le (n-2)^2 + 2. \tag{3.11}$$

Moreover, when $n \ge 4$, equality holds in (3.11) if and only if $A \simeq_p R_n$ or $A^T \simeq_p R_n$, where

$$R_n = \begin{bmatrix} 0 & 1 & & & & 0 \\ 0 & 0 & 1 & & & 0 \\ & \ddots & \ddots & & & \vdots \\ & & \ddots & 0 & 1 & 0 \\ 1 & 1 & & 0 & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 & 0 \end{bmatrix}.$$

Proof Note that (3.11) holds trivially when $n \in \{1,2\}$. Assume that $n \ge 3$. Since $n^2-5n+9 \le (n-2)^2+2$, we may assume that $k(A) > n^2-5n+9$. By Lemma 3.5.2, we may assume that A or A^T has the form (3.8). By Theorem 3.5.1 or Theorem 3.5.2, $k(B) \le (n-1)^2+1$ and so (3.11) follows by Lemma 3.5.1.

Now assume that $n \ge 4$ and that $A \simeq_p R_n$ or $A^T \simeq_p R_n$. Then direct computation yields

$$R_n^{(n-2)^2+1} = \begin{bmatrix} & & & & 0 \\ & & J_{n-1} & & 1 \\ & & & & 0 \\ 0 & 1 & \cdots & 1 & 0 \end{bmatrix},$$

and so $k(R_n) = (n-2)^2 + 2$. Thus $k(A) = k(R_n) = (n-2)^2 + 2$.

Conversely, assume that $n \geq 4$ and $k(A) = (n-2)^2 + 2 > n^2 - 5n + 9$. By Lemma 3.5.2, we may assume that $A \simeq_p X$ or $A^T \simeq_p X$, where X is of the form (3.8). Note that $k(B) \leq (n-1)^2 + 1$. If $a = 1$ in (3.8), then by Lemma 3.5.1,

$$k(A) = k(X) \leq \max\{k(B), n-1\} \leq (n-2)^2 + 1,$$

contrary to the assumption that $k(A) = (n-2)^2 + 2$. Therefore we must have $a = 0$, and so by

$$(n-2)^2 + 2 = k(A) = k(X) \leq k(B) + 1 \leq (n-2)^2 + 2,$$

$k(B) = (n-2)^2 + 1$. By Proposition 3.3.1 and Proposition 3.3.3, $D(B)$ must be the digraph in Example 3.3.1 with $n-1$ vertices, and so we may assume that B is the $(n-1) \times (n-1)$ upper left corner submatrix of R_n. It remains to show that the vector in the last row of A in (3.8) is $\mathbf{x}^T = (1, 0, 0, \cdots, 0)$.

By direct computation,

$$B^{(n-2)^2} = \begin{bmatrix} 0 & J_{(n-2)\times 1} \\ J_{1\times(n-2)} & J_{n-2} \end{bmatrix}.$$

Since $k(X) = k(A) = (n-2)^2+2$, $X^{(n-2)^2+1} \neq X^{(n-2)^2+2}$, and so $\mathbf{x}B^{(n-2)^2} \neq \mathbf{x}^T B^{(n-2)^2+1}$. Therefore $\mathbf{x}^T = (1, 0, 0, \cdots, 0)$, and so $A \simeq_p R_n$ or $A^T \simeq_p R_n$. □

Theorem 3.5.4 Let $n \geq 1$ be an integer. There is no reducible matrix $A \in \mathbf{B}_n$ such that

$$n^2 - 5n + 9 < k(A) < (n-2)^2. \tag{3.12}$$

Proof By contradiction, assume that there exists a reducible matrix $A \in \mathbf{B}_n$ satisfying (3.12). Then $n \geq 7$. By Lemma 3.5.2, $A \simeq_p X$ or $A^T \simeq_p X$ where X is the matrix in (3.8) such that B is primitive and such that the shortest length of a directed cycle in $D(B)$ is $n-2$. By Proposition 3.3.3 and Proposition 3.3.1, there are exactly two such digraphs (as described in Proposition 3.3.3), and $k(B) \geq (n-2)^2$. By Lemma 3.5.1, $(n-2)^2 > k(A) = k(X) \geq k(B) \geq (n-2)^2$, a contradiction. □

Theorem 3.5.5 Let $n \geq 13$ be an integer. If n is odd, then there is no matrix $A \in \mathbf{B}_n$ such that

$$n^2 - 4n + 6 < k(A) < n^2 - 3n + 2, \text{ or } n^2 - 3n + 4 < k(A) < (n-1)^2;$$

if n is even, then there is no matrix $A \in \mathbf{B}_n$ such that

$$n^2 - 4n + 6 < k(A) < (n-1)^2.$$

Sketch of Proof Assume such an A exists. By Proposition 3.3.1 and Proposition 3.3.3(ii), A is not primitive. By Theorem 3.5.3, A is not reducible. Therefore, A must be irreducible and imprimitive, and so $p(A) = p \geq 2$. Write $n = pr + s$ with $0 \leq s < p$. By Theorem 3.4.2,

$$\begin{aligned} k(A) &\leq p(r^2 - 2r + 2) + s \\ &= p(r^2 + 5) - 2(pr + s) - 3(p - s) \\ &\leq pr^2 + 5p - 2n - 3 \leq \frac{n^2}{p} + 3n - 3 \\ &\leq \frac{n^2}{2} + 3n - 3 \leq n^2 - 4n + 6, \end{aligned}$$

and so Theorem 3.5.5 obtains. □

For integers $j, m, n \geq 0$, let

$$\begin{aligned} E_m + j &= \{a + j \,:\, a \in E_m\}; \\ RI_n &= \{k \,:\, k(A) = k \text{ for some reducible } A \in \mathbf{B}_n\}; \\ BI_n &= \{k \,:\, k(A) = k \text{ for some } A \in \mathbf{B}_n\} \end{aligned}$$

Jiang and Shao completely determined RI_n and BI_n. For Problem IS in other classes of matrices, the study has just begun.

Theorem 3.5.6 (Jiang and Shao, [139])

$$\begin{aligned} RI_n &= \bigcup_{i=1}^{n-1} \bigcup_{j=0}^{i} (E_{n-i} + j), \\ BI_n &= \bigcup_{i=0}^{n-1} \bigcup_{j=0}^{i} (E_{n-i} + j). \end{aligned}$$

Given a matrix $A \in \mathbf{B}_n$, if $D(A)$ has a directed cycle of length s and if one of the length s directed cycle in $D(A)$ has exactly t vertices, we say that A has s-cycle positive elements on exactly t rows.

Theorem 3.5.7 (Zhou and Liu, [288]) Let $n \geq t \geq 1$ be integers. Let $A \in \mathbf{B}_n$ be a matrix such that with s-cycle positive elements on exactly t rows. If $s = 1$ or if s is a prime number, then

$$k(A) \leq \begin{cases} (n - t - 1)^2 + 1 & \text{if } t \leq n - \sqrt{s(n-1) + \frac{1}{4}} - \frac{3}{2}, \\ (s+1)n - t - s & \text{if } t > n - \sqrt{s(n-1) + \frac{1}{4}} - \frac{3}{2}. \end{cases}$$

The upper bound is best possible except when $t > n-\sqrt{s(n-1)+\frac{1}{4}}-\frac{3}{2}$ and $\gcd(s,n) > 1$.

Theorem 3.5.7 has been improved by Zhou [285]. An important case of Theorem 3.5.7 is when $s = 1$.

Theorem 3.5.8 (Liu and Shao, [183], Liu and Li, [179]) Let $n \geq d \geq 1$ be integers. Suppose that $A \in \mathbf{B}_n$ has d positive diagonal entries. Then

$$k(A) \leq \begin{cases} (n-d-1)^2+1 & \text{if } 1 \leq d \leq \lfloor \frac{2n-3-\sqrt{4n-3}}{2} \rfloor \\ 2n-d-1 & \text{if } d \geq \lceil \frac{2n-3-\sqrt{4n-3}}{2} \rceil \end{cases}$$

Let $I_n(d) = \{k \ : \ k = k(A)$ for some $A \in \mathbf{B}_n$ with d diagonal elements $\}$. Liu *et al* completely determined $I_n(d)$ as follows.

Theorem 3.5.9 (Liu, Li and Zhou, [181])

$$I_n(d) = \begin{cases} \{1,2,\cdots,2n-d-1\} \cup \left(\bigcup_{i=0}^{n-d} \bigcup_{j=0}^{i} E_{n-d-i+j} \right) & \text{if } 1 \leq d \leq \frac{2n-3-\sqrt{4n-3}}{2} \\ \{1,2,\cdots,2n-d-1\} & \text{if } d \geq \frac{2n-3-\sqrt{4n-3}}{2}. \end{cases}$$

Theorem 3.5.10 (Liu, Shao and Wu, [184]) If $A \in \Omega_n$, then

$$k(A) \leq \begin{cases} \lceil \frac{n^2}{4} + 1 \rceil & \text{if } n = 5,6 \text{ or } n \equiv 0 \pmod 4 \\ \lceil \frac{n^2}{4} \rceil & \text{otherwise.} \end{cases}$$

Moreover, these bounds are best possible.

The extremal matrices of $k(A)$ in Theorem 3.5.8 and Theorem 3.5.10 have been characterized by Zhou and Liu ([288] and [287]).

3.6 Index of Density

Definition 3.6.1 For a matrix $A \in \mathbf{B}_n$, the *maximum density* of A is

$$\mu(A) = \max_{m \geq 1}\{\|A^m\|\},$$

and the *index of maximum density* of A is

$$h(A) = \min\{m > 0 \ : \ \|A^m\| = \mu(A)\}.$$

For matrices in $\mathbf{IB}_{n,p}$, define

$$\bar{h}(n,p) = \max\{h(A) \ : \ A \in \mathbf{IB}_{n,p}\}.$$

Example 3.6.1 Let $A \in \mathbf{B}_n$ be a primitive matrix. Then $\mu(A) = n^2$ and $h(A) = \gamma(A)$. Thus the study of the index of density will be mainly on imprimitive matrices. For a generic matrix $A \in \mathbf{B}_n$ with $p(A) > 1$, $\mu(A) < n^2$ and $h(A) \le k(A)+p-1$ (Exercise 3.23).

Theorem 3.6.1 Let $A = (n_1, A_1, n_2, A_2, \cdots, n_p, A_p, n_1) \in \mathrm{IB}_{n,p}$ and denote

$$B_0 = \begin{bmatrix} J_{n_1\times n_1} & 0 & \cdots & 0 \\ 0 & J_{n_2\times n_2} & \cdots & 0 \\ & \cdots & \cdots & \\ 0 & 0 & \cdots & J_{n_p\times n_p} \end{bmatrix},$$

$$B_1 = \begin{bmatrix} 0 & J_{n_1\times n_2} & 0 & \cdots & 0 \\ 0 & 0 & J_{n_2\times n_3} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & J_{n_{p-1}\times n_p} \\ J_{n_p\times n_1} & 0 & 0 & \cdots & 0 \end{bmatrix},$$

and $B_i = B_1^i$, $1 \le i \le p-1$. Then

$$k(A) = \min\{m > 0 \ : \ A^m = B_j, j \equiv m \pmod p, 0 \le j \le p-1\}.$$

Proof Let

$$m_0 = \min\{m > 0 \ : \ A^m = B_j, j \equiv m \pmod p, 0 \le j \le p-1\}.$$

Let $k = k(A)$ and write $k = rp + j$, where $0 \le j < p$. Since each $A_i(p)$ is primitive, $\gamma(A_i(p))$ exists. Let $e = \max_{1\le i\le p}\{\gamma(A_i(p))\}$. Then $A^{ep} = B_0$, and so

$$A^k = A^{k+ep} = A^{(r+e)p+j}.$$

However, as $A^{ep} = B_0$, $A^{(r+e)p} = B_0$ also, and so $A^k == A^{k+ep} = A^{(r+e)p+j} = B_j$. Thus $m_0 \le k$.

On the other hand, write $m_0 = lp + j$ with $0 \le j < p$. Then $A^{m_0} = B_j$, and so $A^{m_0+p} = B_j A^p = B_j$, and so $k \le m_0$. □

Corollary 3.6.1 Let $A = (n_1, A_1, n_2, A_2, \cdots, n_p, A_p, n_1) \in \mathrm{IB}_{n,p}$ and let $\gamma_i = \gamma(A_i(p))$. Then for each $i = 1, 2, \cdots, p$,

$$p(\gamma_i - 1) < k(A) < p(\gamma_i + 1).$$

Proof Note that $A^0 = I$, for each i, by the definition of γ_i,

$$(A_i(p))^{\gamma_i - 1} < J \Longrightarrow A^{p(\gamma_i-1)} < B_0 \Longrightarrow k(A) > p(\gamma_i - 1).$$

To show $k(A) < p(\gamma_i + 1)$, for each j with $1 \le j \le p$, it suffices to show that $A_j(p(\gamma_i) - 1) = J$. Write $i \equiv j + t \pmod p$, where $0 \le t < p$. Then $A_i = A_{j+t}$ and so $(A_{j+t}(p))^{\gamma_i} = J$. It follows

$$\begin{aligned} A_j(p(\gamma_i + 1) - 1) &= A_j(p\gamma_i + p - 1) \\ &= (A_j \cdots A_{j+i} \cdots A_{j+p-1})^{\gamma_i} A_j \cdots A_{j+p-2} \\ &= A_j(t)(A_{j+t}(p))^{\gamma_i} A_{j+t}(p - 1 - t) = J. \end{aligned}$$

□

Definition 3.6.2 Let $\mathbf{a}^T = (a_1, a_2, \cdots, a_p)$ be a vector. The *circular period* of $\mathbf{a}$, denoted by $\tau(\mathbf{a})$ or $\tau(a_1, a_2, \cdots, a_p)$, is the smallest positive integer m such that

$$(a_1, a_2, \cdots, a_p) = (a_{m+1}, \cdots, a_p, a_1, \cdots, a_m).$$

With this definition, $\tau(a_1, a_2, \cdots, a_p) | p$.

Heap and Lynn [119] started that investigation of $h(A)$ and (n, p). Shao and Li [251] gave an explicit expression of $h(A)$ in terms of the circular period of a vector, and completely determined $\bar{h}(n, p)$.

Theorem 3.6.2 (Heap and Lynn, [119], Shao and Li, [251]) Let $A \in \mathrm{IB}_{n,p}$ with the form $A = (n_1, A_1, \cdots, n_p, A_p, n_1)$, and let $\tau = \tau(n_1, n_2, \cdots, n_p)$. Each of the following holds.

(i) $\mu(A) = \sum_{i=1}^{p} n_i^2$,

(ii) $h(A) = \min\{m \; : \; m \ge k(A), \; \tau | m\} = \tau \lfloor \frac{k(A)}{\tau} \rfloor$.

Sketch of Proof Let $m \ge 0$ be an integer with $m \equiv j \pmod p$, such that $0 \ge j < p$. Define $n_j = n_{j'}$ whenever $j \equiv j' \pmod p$. By Theorem 3.6.1,

$$m \ge k(A) \Longrightarrow A^m = B_j \Longrightarrow \|A^m\| = \sum_{i=1}^{p} n_i n_{i+j},$$

and

$$m < k(A) \Longrightarrow A^m < B_j \Longrightarrow \|A^m\| < \sum_{i=1}^{p} n_i n_{i+j}.$$

As $\sum_{i=1}^{p} n^2 - \sum_{i=1}^{p} n_i n_{i+j} = \frac{1}{2} \sum_{i=1}^{p} (n_i - n_{i+j})^2 \ge 0$, for each integer $m \ge 0$, $\|A^m\| \le \sum_{i=1}^{p} n_i^2$. Moreover, $\|A^m\| = \sum_{i=1}^{p} n_i^2$ if and only if $m \ge k(A)$ and $n_i = n_{i+j}$, for each $i = 1, 2, \cdots, p$; if and only if $m \ge k(A)$ and $\tau | j$. Thus (i) and (ii) follow from the definitions of $\mu(A)$ and $h(A)$. □

Theorem 3.6.3 (Shao and Li, [251]) For integers $n \geq p \geq 1$, write $n = rp + s$, where $0 \leq s < p$. Let

$$\bar{k}(m,p) = \max\{k(A) \ : \ A \in \mathbf{IB}_{n,p}\}.$$

Then

$$\begin{aligned}
\bar{h}(n,p) &= p\lfloor\frac{\bar{k}(n,p)}{p}\rfloor \\
&= \begin{cases} p(r^2 - 2r + 2) & \text{if } r > 1, s = 0 \\ p(r^2 - 2r + 3) & \text{if } r > 1, 0 < s < p \\ p & \text{if } r = 1, 0 < s < p\,. \\ 1 & \text{if } r = 1, s = 0. \end{cases}
\end{aligned}$$

Moreover, if $k(A) = \bar{k}(n,p)$, then $h(A) = \bar{h}(n,p)$.

Sketch of Proof For each $A \in \mathbf{IB}_{n,p}$, $A \simeq_p (n_1, A_1, \cdots, n_p, A_p, n_1)$.
Let $\tau = tau(n_1, n_2, \cdots, n_p)$.

By Theorem 3.6.2,

$$k(A) = \tau\lfloor\frac{k(A)}{\tau}\rfloor \leq p\lfloor\frac{k(A)}{p}\rfloor \leq p\lfloor\frac{\bar{k}(n,p)}{p}\rfloor.$$

Assume that for some $A \simeq_p (n_1, A_1, \cdots, n_p, A_p, n_1) \in \mathbf{IB}_{n,p}$, $k(A) = \bar{k}(n,p)$. By Theorem 3.4.2 and Theorem 3.4.3, we may assume that $(n_1, n_2, \cdots, n_p) = (r+1, \cdots, r+1, r, \cdots, r)$, and so $h(A) = p\lfloor\frac{\bar{k}(n,p)}{p}\rfloor$. $\square$

Definition 3.6.3 The index set for $h(A)$ is

$$H(n,p) = \{h(A) \ : \ A \in \mathbf{IB}_{n,p}\}.$$

Thus $H(n,1) = E_n$.

Theorem 3.6.4 (Shao and Li, [251]) For integers $n \geq p \geq 1$, write $n = rp + s$, where $0 \leq s < p$. Each of the following holds.

(i) If $k \notin E_r$ and if $k_1 \leq k \leq k_2$, then for each integer m with $pk_1 < m \leq pk_2$, $m \notin H(n,p)$.

(ii) If r is odd and if $r \geq 5$, then $[p(r^2 - 3r + 5) + 1, p(r^2 - 2r)]^o \cap H(n,p) = \emptyset$.

(iii) If r is even and if $r \geq 4$, then $[p(r^2 - 4r + 7) + 1, p(r^2 - 2r)]^o \cap H(n,p) = \emptyset$.

Definition 3.6.4 For integers $n \geq p \geq 1$, let $\mathbf{SIB}_{n,p}$ denote the set of all symmetric imprimitive irreducible matrices.

Example 3.6.2 Let $A \in \mathbf{SIB}_{n,2}$ and let $D = D(A)$. Then $D(A)$ is a bipartite graph and the diameter of D is $k(A) + 1$.

Theorem 3.6.5 Let $A = (n_1, A_1, n_2, A_2, n_1) \in \mathbf{SIB}_{n,2}$. Then

$$h(A) = \begin{cases} k(A) \text{ if } n_1 = n_2 \\ 2\lfloor \frac{k(A)}{2} \rfloor \text{ if } n_1 \neq n_2. \end{cases}$$

Proof This follows from Theorem 3.6.2. $\square$

Example 3.6.3 Define

$$SK_{n,2} = \{k(A) \ : \ A \in \mathbf{SIB}_{n.2}\} \text{ and } SH_{n,2} = \{h(A) \ : \ A \in \mathbf{SIB}_{n.2}\}.$$

For integers $n \geq k+2 \geq 3$, let $G(n,k)$ to be the graph obtained from $K_{n-k,1}$ by replacing an edge of $K_{n-k,1}$ by a path k edges. Then we can show that $k(A(G)) = k$, and so

$$SK_{n,2} = \{1, 2, \cdots, n-2\}.$$

The same technique can be used to show the following Theorem 3.6.6.

Theorem 3.6.6 (Shao and Li, [251]) Let $n \geq 2$ be an integer. Each of the following holds.
(i) If n is even, then $SH_{n,2} = [1, n-2]^o$.
(ii) If n is odd, then $SH_{n,2}$ consists of all even integers in $[2, n-1]^o$.

For tournaments, Zhang et al completely determined the index set of maximum density.

Theorem 3.6.7 (Zhang, Wang and Hong, [284]) Let $ST_n = \{h(A) \ : \ A \in T_n\}$. Then

$$ST_n = \begin{cases} \{1\} & \text{if } n = 1,2,3 \\ \{1,9\} & \text{if } n = 4 \\ \{1,4,6,7,9\} & \text{if } n = 5 \\ \{1,2,\cdots,8,9\} \setminus \{2\} & \text{if } n = 6 \\ \{1,2,\cdots,n+2\} \setminus \{2\} & \text{if } n = 7,8,\cdots,15 \\ \{1,2,\cdots,n+2\} & \text{if } n \geq 16. \end{cases}$$

3.7 Generalized Exponents of Primitive Matrices

The main purpose of this section is to study that generalized exponents $exp(n,k)$, $f(n,k)$ and $F(n,k)$, to be defined in Definitions 3.7.1 and 3.7.2, and to estimate their bounds.

Definition 3.7.1 For a primitive digraph D with $V(D) = \{v_1, v_2, \cdots, v_n\}$, and for $v_i, v_j \in V(D)$, define $exp_D(v_i, v_j)$ to be the smallest positive integer p such that for each integer $t \geq p$, D has a directed (v_i, v_j)-walk of length t. By Proposition 3.3.1, this integer exists. For each $i = 1, 2, \cdots, n$, define

$$exp_D(v_i) = \max_{1 \leq j \leq n} \{exp_D(v_i, v_j)\}.$$

For convenience, we assume that the vertices of D are so labeled that

$$exp_D(v_1) \le exp_D(v_2) \le \cdots \le exp_D(v_n).$$

With this convention, we define, for integers $n \ge k \ge 1$,

$$exp(n,k) = \max_{\substack{D \text{ is primitive and} \\ |V(D)| = n}} \{exp_D(v_k)\}.$$

Let D be a primitive digraph with $|V(D)| = n$ and let $X \subseteq V(D)$ with $|X| = k$. Define $exp_D(X)$ to be the smallest positive integer p such that for each $u \in V(D)$, there exists a $v \in X$ such that D has a directed (u,v)-walk of length at least p; and define the *kth lower multi-exponent of D* and the *kth upper multi-exponent of D* as

$$f(D,k) = \min_{X \subseteq V(D)} \{exp_D(X)\} \text{ and } F(D,k) = \max_{X \subseteq V(D)} \{exp_D(X)\},$$

respectively. We further define, for integers $n \ge k \ge 1$,

$$f(n,k) = \max_{\substack{D \text{ is primitive and} \\ |V(D)| = n}} \{f(D,k)\} \text{ and}$$

$$F(n,k) = \max_{\substack{D \text{ is primitive and} \\ |V(D)| = n}} \{F(D,k)\}.$$

These parameters $exp(n,k)$, $f(n,k)$ and $F(n,k)$ can be viewed as generalized exponents of primitive matrices (Exercise 3.25).

Example 3.7.1 Denote $exp(n) = exp(n,n)$. By Corollary 3.3.1A and Example 3.3.1, $exp(n) = (n-1)^2 + 1$.

Definition 3.7.2 Let D_n denote the digraph obtained from reversing every arc in the digraph D_1 in Example 3.3.1, and write $V(D_n) = \{v_1, v_2, \cdots, v_n\}$. For convenience, for $j > n$, define $v_j = v_i$ if and only if $j \equiv i \pmod{n}$. For each $v_i \in V(D_n)$ and integer $t \ge 0$, let $R_t(i)$ be the set of vertices in D_n that can be reached by a directed walk in D of length t.

Lemma 3.7.1 Let $v_m \in V(D_n)$, let $t \ge 0$ be an integer. Write $t = p(n-1) + r$, where $p, r \ge 0$ are integers such that $0 \le r \le n-1$. Each of the following holds.

(i) If $t \ge (n-2)(n-1) + 1$, then $R_t(1) = V(D_n)$.

(ii) If $t \le (n-2)(n-1) + 1$, then $R_t(1) = \{v_{-r}, v_{1-r}, \cdots, v_{p-r}, v_{p-r+1}\}$.

(iii) If $t \ge (n-2)(n-1) + m$, then $R_t(m) = V(D_n)$.

(iv) If $0 \le t \le m-1$, then $R_t(m) = \{v_{m-t}\}$.
(v) If $m-1 \le t \le (n-2)(n-1)+m$, then $R_t(m) = R_{t-m+1}(1)$.

Proof (i) and (ii) follows directly from the structure of D_n. (iii), (iv) and (v) follows from (i) and (ii), and the fact that in D_n, there is exactly one arc from v_k to v_{k-1}, $2 \le k \le n$.

Theorem 3.7.1 Let $n \ge k \ge 1$ be integers. Each of the following holds.
(i) $exp_{D_n}(k) = n^2 - 3n + k + 2$.
(ii) $f(D_n, k) = 1 + (2n-k-2)\lfloor \frac{n-1}{k} \rfloor - k\lfloor \frac{n-1}{k} \rfloor^2$.

Proof By Lemma 3.7.1,

$$exp_{D_n}(k) = (n-2)(n-1) + k = n^2 - 3n + k + 2.$$

Note that when $k = 1$, (ii) follows by Example 3.3.1. Assume that $k < n$. Write $n - 1 = qk+s$, where $0 \le s < k$. Then the right hand side of (ii) becomes $(q-1)(n-1)+1+s(q+1)$.

We construct two subsets X and Y in $V(D_n)$ as follows.

Let $X = \{v_{i_1}, v_{i_2}, \cdots, v_{i_k}\}$ such that $i_1 = 1$, and such that, for $j \ge 2$,

$$i_j = \begin{cases} i_{j-1} + q + 1 & \text{if } 2 \le j \le s+1 \\ i_{j-1} + q & \text{if } s+2 \le j \le k. \end{cases}$$

Let $Y = \{v_{i_j+q-1} \ : \ 1 \le j \le s\}$ and $\overline{Y} = V(D_n) \setminus Y$. We make these claims.

Claim 1 If $X^* = \{u_1, u_2, \cdots, u_k\} \subseteq V(D_n)$, then

$$exp_{D_n}(X^*) \ge (q-1)(n-1) + 1 + s(q+1).$$

Note that from any vertex $v \in X^*$, v can reach at most $n-s$ vertices by using directed walks of length $(n-1)(q-1)+1$. If a vertex v_l, where $1 \le l < s(q+1)-1$, cannot be reached from X^* by directed walks of length $(n-1)(q-1)+1$, then adding a directed walk of length $s(q+1)-1$ cannot reach the vertex v_n. Thus Claim 1 follows.

Claim 2 Every vertex in $\overline{Y}$ can be reached from a vertex in X by a directed walk of length $1 + (n-1)(q-1)$ in D_n.

In fact, by Lemma 3.7.1,

$$\begin{aligned} R_{1+(n-1)(q-1)}(v_{i_1}) &= \{v_{n-1}, v_n, v_1, v_2, \cdots, v_{i_1+q-2}\} \\ R_{1+(n-1)(q-1)}(v_{i_2}) &= \{v_{i_2-1}, v_{i_2}, \cdots, v_{i_2+q-2}\} \\ \cdots & \quad \cdots \\ R_{1+(n-1)(q-1)}(v_{i_k}) &= \{v_{i_k-1}, v_{i_k}, \cdots, v_{i_k+q-2}\}. \end{aligned}$$

Thus Claim 2 becomes clear since

$$(i_j - 1) - (i_{j-1} + q - 2) = i_j - i_{j-1} - q + 1 = \begin{cases} 2 & \text{if } 2 \le j \le s+1 \\ 1 & \text{if } s+2 \le j \le k. \end{cases}$$

Claim 3 Every vertex $v_i \in V(D_n)$ can be reached by a directed walk from a vertex in $\overline{Y}$ of length $s(q+1)$; but not every vertex can be reached by a directed walk from a vertex in $\overline{Y}$ of length $s(q+1)-1$.

It suffices to indicate that v_n cannot be reached by a directed walk from a vertex in $\overline{Y}$ of length $s(q+1)-1$. In fact, since $s(q+1)-1 = i_s + q - 1$, if v_n can be reached by a directed walk in D_n of length $s(q+1)-1$, then the initial vertex of the walk must be $v_{s(q+1)-1} \in Y$.

By Claims 2 and 3, $f(D_n, k) \le exp_{D_n}(X) \le (q-1)(n-1)+1+s(q+1)$. This, together with Claim 1, implies (ii). □

Theorem 3.7.2 Let $n \ge k \ge 1$ be integers. Then

$$F(D_n, k) = (n-1)(n-k)+1.$$

Proof Let $X' = \{v_1, v_2, \cdots, v_{k-1}, v_n\}$. By Lemma 3.7.1, D_n has no directed walk of length $(n-1)(n-k)$ from a vertex in X' to v_n. Thus $F(D_n, k) \ge (n-1)(n-k)+1$.

On the other hand, by Lemma 3.7.1 again, for any vertex $v_i \in V(D_n)$, the end vertices of directed walks from v_i of length $(n-1)(n-k)+1$ consists of $n-k+1$ consecutive vertices in a section of the directed cycle $v_1 v_2 \cdots v_n v_1$. Since any k distinct such sections must cover all vertices of D_n, for any $X \subseteq V(D_n)$ with $|X| = k$, $exp_{D_n}(X) \le (n-1)(n-k)+1$. This proves the theorem. □

Lemma 3.7.2 Let $n \ge k \ge 1$ be integers and let D be a primitive digraph with $V(D_n) = \{v_1, v_2, \cdots, v_n\}$. If D has a loop at v_i, $1 \le i \le r$, then

$$exp_D(k) \le \begin{cases} n-1 & \text{if } k \le r \\ n-1+k-r & \text{if } k \ge r. \end{cases}$$

Proof Assume that D has a loop at $v_1, v_2, \cdots, v_r$. Then $exp_D(v_i) \le n-1$, $1 \le i \le r$. Thus if $k \le r$, then $exp_D(k) \le n-1$.

Assume $k > r$ and $L = \{v_1, \cdots, v_r\}$. Since D is strong, $V(D)$ has a subset X with $|X| = k-r$ such that any vertex in X can reach a vertex in L with a directed walk of length at most $k-r$, and any vertex in L can reach a vertex in X with a directed walk of length at most $k-r$. Thus $exp_D(v) \le (n-1)+(k-r), \forall v \in X \cup L$. □

Lemma 3.7.3 Let $n \ge k \ge 2$ be integers and let D be a primitive digraph with $|V(D)| = n$. Then

$$exp_D(k) \le exp_D(k-1)+1.$$

Proof Assume that $exp_D(v_i) = exp_D(i)$, $1 \le i \le n$. Since D is strong, D has a vertex v such that $(v_i, v) \in E(D)$. □

Theorem 3.7.3 (Brualdi and Liu, [33]) Let $n \geq k \geq 1$ be integers and let D be a primitive digraph with $|V(D)| = n$. If s is the shortest length of a directed cycle of D, then

$$exp_D(k) \leq \begin{cases} s(n-1) & \text{if } k \leq s \\ s(n-1+k-s) & \text{if } k > s \end{cases}$$

Sketch of Proof Given D, construct a new digraph $D^{(s)}$ such that $V(D^{(s)}) = V(D)$, where $(x, y) \in E(D^{(s)})$ if and only if D has a directed (x, y)-walk of length s. Then D' has at least s vertices attached with loops, and so Theorem 3.7.3 follows from Lemma 3.7.2. □

Theorem 3.7.4 (Brualdi and Liu, [33]) Let $n \geq k \geq 1$ be integers. Then

$$exp(n, k) = n^2 - 3n + k + 2.$$

Proof Let D be a primitive digraph with $|V(D)| = n$. By Lemma 3.7.3, $exp_D(k) \leq exp_D(1) + (k-1)$. Let s denote the shortest length of directed cycles in D. If $s \leq n-2$, then by Theorem 3.7.3, $exp_D(1) \leq n^2 - 3n + 2$ and so the theorem obtains.

Since D is primitive, by Theorem 3.2.2, $n \leq n-1$. Assume now $s = n-1$. Since D is strong, D must have a directed cycle of length n and so D has D_n (see Definition 3.7.2) as a spanning subgraph. Theorem 3.7.4 follows from Theorem 3.7.1(i) and Theorem 3.7.3. □

Shao et al [253] proved that the extremal matrix of $exp(n, k)$ is the adjacency matrix of D_n. In [185] and [241], the exponent set for $exp_D(k)$ was partially determined.

Lemma 3.7.4 Let $n \geq k > s > 0$ be integers and let D be a primitive digraph with $|V(D)| = n$ and with s the shortest length of a directed cycle of D. Then $f(D, k) \leq n - k$.

Proof Let $Y \subset V(D)$ be the set of vertices of a directed cycle of D with $|Y| = s$. Since D is strong, $V(D)$ has a subset X such that $Y \subset X \subseteq V(D)$ and such that every vertex in $X \setminus Y$ can be reached from a vertex in Y by a directed walk with all vertices in X. Thus any vertex in $V(D)$ can be reached from a vertex in X by a directed walk of length exactly $n - k$. □

Lemma 3.7.5 Let $n \geq s > k \geq 1$ be integers. If D has a shortest directed cycle of length s, then

$$f(D, k) \leq 1 + s(n-k-1).$$

Proof Let $C_s = x_1 x_2 \cdots x_s x_1$ be a directed cycle in D. Since D is strong, we may assume that there exists $z \in V(D) \setminus V(C_s)$ such that $(x_1, z) \in E(D)$.

Let $X = \{x_1, x_2, \cdots, x_k\}$, and let Y be the set of vertices in D that can be reached from vertices in X by a directed path of length 1. Then $\{z, x_2, \cdots, x_{k+1}\} \subseteq Y$.

Construct a new digraph $D^{(s)}$ with $V(D^{(s)}) = V(D)$, where $(u,v) \in E(D^{(s)})$ if and only if D has a directed (u,v)-walk of length s. Then D' has a loop at each of the vertices $x_2, \cdots x_{k+1}$, and $(x_2, z) \in E(D^{(s)})$. Thus, each vertex in $D^{(s)}$ can be reached from a vertex in Y by a directed walk of length exactly $n-k-1$, and so every vertex in D can be reached from a vertex in X by a directed walk of length exactly $1 + s(n-k-1)$. □

Lemma 3.7.6 can be proved in a way similar to the proof for Lemma 3.7.5, and so its proof is left as an exercise.

Lemma 3.7.6 Let $n > s \geq k \geq 1$ be integers such that $k|s$. Let D be a primitive digraph with $|V(D)| = n$ and with a directed cycle of length s. Then

$$f(D,k) \leq 1 + \frac{s(n-k-1)}{k}.$$

Theorem 3.7.5 (Brualdi and Liu, [33]) Let $n > k \geq 1$ be integers. Then

$$f(n,k) \leq n^2 - (k+2)n + k + 2.$$

Sketch of Proof Any primitive digraph on n vertices must have a directed cycle of length $s \leq n-1$, by Theorem 3.2.2. Thus Theorem 3.7.5 follows from Lemmas 3.7.4 and 3.7.5. □

Theorem 3.7.6 Let $n > k \geq 1$ be integers such that $k|(n-1)$. Let $f^*(n,k) = \max\{f(D,k) : D$ is a primitive digraph on n vertices with a directed cycle of length s and $k|s\}$. Then

$$f^*(n,k) = \frac{n^2 - (k-2)n + 2k + 1}{k}.$$

Proof This follows by combining Lemma 3.7.6 and Theorem 3.7.1(ii). □

Lemma 3.7.7 Let D be a primitive digraph with $|V(D)| = n$, and let s and t denote the shortest length and longest length of directed cycles in D, respectively. Then

$$F(D, n-1) \leq \max\{n-s, t\}.$$

Proof Pick $X \subset V(D)$ with $|X| = n-1$. If $V(C) \subseteq X$ for some directed cycle C of length p, where $s \leq p \leq t$, then any vertex in D can be reached by a directed walk from a vertex in $V(C)$ of length $n-p$. Hence we assume that no directed cycle of D is contained in X. Let u denote the only vertex in $V(D) \setminus X$. Then every directed cycle of D contains u.

Let C_1 be a directed cycle of length t in D. Then $u \in V(C_1)$. Since D is strong, every vertex lies in a directed cycle of length at most t, and so every vertex in X can be reached from a vertex in X by a directed walk of length exactly t.

Since D is primitive, and by Theorem 3.2.2, D has a directed cycle C_2 of length q with $0 < q < t$. Let $t = mq + r$ with $0 < r \leq q$. let $v \in V(C_1)$ be the $(t-r)$th vertex from u. Then C_1 has a directed (v,u)-path. By repeating C_2 m times, D has a directed (v,u)-walk of length t. Hence $exp_D(X) \leq \max\{n-s, t\}$. □

Theorem 3.7.7 $F(n, n-1) = n$.

Proof By Theorem 3.7.2, $F(n, n-1) \geq F(D_n, n-1) = n$. By Lemma 3.7.7, for any primitive digraph D with $|V(D)| = n$, $F(D, n-1) \leq \max\{n-s, t\} \leq \max\{n-1, n\} = n$. □

Lemma 3.7.8 Let $n \geq m \geq 1$ be integers and let D be a primitive digraph with $|V(D)| = n$ such that D has loops at m vertices. Then for any integer k with $n \geq k \geq 1$,

$$F(D,k) \leq \begin{cases} n-1 & \text{if } k > n-m \\ 2n-m-k & \text{if } k \leq n-m. \end{cases}$$

Proof Let $X \subseteq V(D)$ with $|X| = k$. Assume first that D has a loop at a vertex $v \in X$. Then every vertex of D can be reached from v by a directed walk of length exactly $n-1$, and so $F(D,k) \leq n-1$. Note that when $k > n-m$, X must have such a vertex v.

Assume then $k \leq n-m$ and no loops is attached to any vertex of X. Then X has a vertex x such that D has a directed (x, w)-path of length at most $n-m-k+1$, for some vertex $w \in V(D)$ at which a loop of D is attached. Thus any vertex in D can be reached from a vertex in X by a directed walk of length exactly $2n-m-k$. □

Theorem 3.7.8 (Brualdi and Liu, [33]) Let $n \geq k \geq 1$ and $s > 0$ be integers. If a primitive digraph D with $|V(D)| = n$ has a directed cycle of length s, then

$$F(D,k) \leq \begin{cases} s(n-1) & \text{if } k > n-s \\ s(2n-s-k) & \text{if } k \leq n-s. \end{cases}$$

Sketch of Proof Apply Lemma 3.7.8 to $D^{(s)}$. □

Theorem 3.7.9 (Liu and Li, [179]) Let $n \geq k \geq 1$ be integers, and let D be a primitive digraph with $|V(D)| = n$ and with shortest directed cycle length s. Then

$$F(D,k) \leq (n-k)s + (n-s).$$

Proof It suffices to prove the theorem when $n > k \geq 1$. Let C_s be a directed cycle of length s and let $X \subseteq V(D)$ be a subset with $|X| = k < n$. Let $v \in V(D)$ and let

$t \geq (n-k)s + (n-s)$. We want to find a vertex $x \in X$ such that D has a directed (x, v)-walk of length exactly t.

Fix $v \in V(D)$. Then there is a vertex $x' \in X$ such that D has a directed (x', v)-walk of length $d \leq n - s$. Since C_s is a directed cycle, then for any $h \geq d$, there exists a vertex $x'' \in V(C_s)$ such that D has a directed (x'', v)-walk of length h.

Note that in $D^{(s)}$, x'' is a vertex at which a loop is attached. Since $|X| = k$ and since $D^{(s)}$ has a loop atx'', we can find $x \in X$ such that $D^{(s)}$ has a directed (x, x'')-walk of length $n - k$. Thus D has a directed (x, x'')-walk of length $s(n-k)$, and so D has a directed (x, v)-walk of length $t \geq s(n-k) + (n-s)$, consisting of a directed (x, x'')-walk and a directed (x', v)-walk. $\square$

Theorem 3.7.10 (Liu and Li, [179]) Let $n > k \geq 1$ be integers. Then

$$F(n, k) = (n-1)(n-k) + 1.$$

Proof Let D be a primitive digraph with $|V(D)| = n$ and let s denote the shortest length of a directed cycle of D. Since D is primitive, $s \geq n - 1$. Thus by Theorem 3.7.9,

$$\begin{aligned} F(D, k) &= s(n-k) + n - s = s(n-k-1) + n \\ &= (n-1)(n-k-1) + n = (n-1)(n-k) + 1. \end{aligned}$$

Theorem 3.7.10 proves a conjecture in [33]. By Theorem 3.7.2, the bound in Theorem 3.7.10 is best possible. The extremal matrices for $F(D, k)$ have been completely determined by Liu and Zhou [186]. The determination of $f(n, k)$ for general values of n and k remains unsolved.

Conjecture Let $n \geq k + 2 \geq 4$ be integers. Show that

$$f(n, k) = 1 + (2n - k - 2)\left\lfloor \frac{n-1}{k} \right\rfloor - \left\lfloor \frac{n-1}{k} \right\rfloor^2 k.$$

3.8 Fully indecomposable exponents and Hall exponents

Definition 3.8.1 For integer $n > 0$, let $\mathbf{F}_n$ denote the collection of fully indecomposable matrices in $\mathbf{B}_n$, and $\mathbf{P}_n$ the collection of primitive matrices in $\mathbf{B}_n$. For a matrix $A \in \mathbf{P}_n$, define $f(A)$, the *fully indecomposable exponent* of A, to be the smallest integer $k > 0$ such that $A^k \in \mathbf{F}_n$. For an integer $n > 0$, define

$$f_n = \max\{f(A) \ : \ A \in \mathbf{P}_n\}.$$

The Proposition 3.8.1 follows from the definitions.

Proposition 3.8.1 Let $n > 0$ be an integer. Then

$$\mathbf{P}_n = \{A \,:\, A \in \mathbf{B}_n \text{ and for some integer } k > 0, A^k \in \mathbf{F}_n\}.$$

Schwarz [232] posed the problem to determine f_n, and he conjectured that $f_n \leq n$. However, Chao [53] presented a counterexample.

Example 3.8.1 Let

$$M_5 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Then we can compute M_5^i, for $i = 2, 3, 4, 5$ to see that $f(M_5) \geq 6$. In fact, Chao in [53] showed that for every integer $n \geq 5$, there exists an $A \in \mathbf{P}_n$ such that $f(A) > n$. However, Chao and Zhang [54] showed that if $\mathrm{tr}A > 0$, then $f_n \leq n$.

Example 3.8.2 For a matrix $A \in \mathbf{P}_n$ and an integer $k > 1$, that $A^k \in \mathbf{F}_n$ does not imply $A^{k+1} \in \mathbf{F}_n$. Let

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Then we can verify that $A^8, A^9 \in \mathbf{F}_7$, $A^{10}, A^{11} \notin \mathbf{F}_7$, and $A^i \in \mathbf{F}_7$ for all $i \geq 12$.

Definition 3.8.2 For a matrix $A \in \mathbf{P}_n$, define $f^*(A)$, the ***strict fully indecomposable exponent*** of A, to be the smallest integer $k > 0$ such that for every $i \geq k$, $A^i \in \mathbf{F}_n$. Define

$$f_n^* = \max\{f^*(A) \,:\, A \in \mathbf{P}_n\}.$$

Thus in Example 3.8.2, $f(A) = 8$, $f^*(A) = 12$.

Proposition 3.8.2 Let $A \in \mathbf{P}_n$. Then

$$f(A) \leq f^*(A) \leq \gamma(A).$$

Proof Let $k = f(A)$. Then $A^k \in \mathbf{F}_n$, and so $f(A) \le f^*(A)$. By Proposition 3.3.1, for any and $k' \ge \gamma(A)$, we have $A^{k'} > 0$, and so $A^{k'} \in \mathbf{F}_n$. Therefore, $f^*(A) \le \gamma(A)$. $\square$

Lemma 3.8.1 Let $A \in \mathbf{B}_n$ and $D = D(A)$ with $V(D) = \{v_1, v_2, \cdots, v_n\}$. For a subset $X \subseteq V(D)$, and for an integer $t > 0$, let $R_t(X)$ denote the set of vertices in D which can be reached from a vertex in X by a directed walk of length t, and let $R_0(X) = X$. Then for an integer $k > 0$, the following are equivalent.

(i) $A^k \in \mathbf{F}_n$.

(ii) For every non empty subset $X \subseteq V(D)$, $|R_k(X)| > |X|$.

Proof This is a restatement of Theorem 2.1.2. $\square$

Lemma 3.8.2 Let D be a strong digraph with $V(D) = \{v_1, v_2, \cdots, v_n\}$ and let $W = \{v_{i_1}, v_{i_2}, \cdots, v_{i_s}\} \subseteq V(D)$ be vertices of D at which D has a loop. Then for each integer $t > 0$,

$$|R_t(W)| \ge \min\{s + t, n\}.$$

Proof Suppose that $R_t(W) \ne V(D)$. Since D is strong, there exist $u \in R_t(W)$ and $v \in V(D) \setminus R_t(W)$ such that $(u, v) \in E(D)$. Since $v \notin R_t(W)$, we may assume that the distance in D from v_{i_1} to u is t, and the distance in D from v_{i_j} to u is at least t, for any j with $1 \le j \le s$. As the directed (v_{i_1}, u) in $R_t(W)$ contains no vertices in $W - \{v_{i_1}\}$, $|R_t(W)| \ge (s-1) + (t+1) = s + t$. $\square$

Theorem 3.8.1 (Brualdi and Liu, [31]) let $n \ge s \ge 1$ be integers. Suppose that $A \in \mathbf{P}_n$ has s positive diagonal entries. Then

$$f^*(A) \le n - s + 1.$$

Proof Let $D = D(A)$ and let W denote the set of vertices of D at which a loop is attached. Let $X \subset V(D)$ be a subset with $n > |X| = k > 0$. By Lemma 3.8.1, it suffices to show that

$$|R_t(X)| \ge |X| + 1, \text{ for each } t \ge n - s + 1. \tag{3.13}$$

Since $n > k$, we may assume that $|R_t(X)| < n$. If $X \cap W \ne \emptyset$, then by Lemma 3.8.2 and since $t \ge n - s + 1$

$$|R_t(X)| \ge |R_t(X \cap W)| \ge |X \cap W| + t \ge |X \cap W| + n - s + 1 \ge |X| + 1.$$

Thus we assume that $X \cap W = \emptyset$. Let $x^* \in X$ and $w^* \in W$ such that the distance d from x^* to w^* is minimized among all $x \in X$ and $w \in W$. By the minimality of d,

$$d \le n + 1 - |X| - |W| = n - s + 1 - k < t.$$

Since $w^* \in W$, x^* can reach every vertex in $R_k(w^*)$ by a directed walk in D of length exactly t. By Lemma 3.8.2, $|R_t(X)| \geq |R_t(\{w^*\})| \geq |R_k(w^*)| \geq k+1$, and so (3.13) holds also. □

Corollary 3.8.1A (Chao and Zhang, [54]) Suppose $A \in \mathbf{P}_n$ with $\mathrm{tr}(A) > 0$. Then

$$f(A) \leq f^*(A) \leq n.$$

Corollary 3.8.1B Let $A \in \mathbf{P}_n$ such that $D(A)$ has a directed cycle of length r and such that $D(A)$ has s vertices lying in directed cycles of length r. Then $f(A) \leq r(n-s+1)$. In particular, if D has a Hamilton directed cycle, then $f(A) \leq n$.

Corollary 3.8.1C Let $A \in \mathbf{P}_n$ such that the diameter of $D(A)$ is d. Then

$$f(A) \leq 2d(n-d).$$

Corollary 3.8.1D Let $A \in \mathbf{P}_n$ be a symmetric matrix with $\mathrm{tr}(A) = 0$. Then $f(A) \leq 2$.

Proof Corollary 3.8.1A follows from Theorem 3.8.1. For Corollary 3.8.1B, argue by Theorem 3.8.1 that $(A^r)^{n-s-1} \in \mathbf{F}_n$. If the diameter of D is d, then D has a directed cycle of length $r \leq 2d$, and D has at least $d+1$ vertices lying in directed cycles of length r. Thus the other corollaries follow from Corollary 3.8.1B. □

Theorem 3.8.2 (Brualdi and Liu, [31]) For $n \geq 1$,

$$f_n \leq \lceil \frac{(n-1)(n+3)}{4} \rceil.$$

Proof Let $A \in \mathbf{P}_n$ and $D = D(A)$. Since D is strong, D has a directed cycle of length $r > 0$. Let s be the number of vertices in D lying in directed cycles of length r. By Corollary 3.8.1B,

$$f(A) \leq r(n-s+1) \leq r(n-r+1).$$

When n is odd, since D is primitive, D must have a directed closed walk of length different from $(n+1)/2$. Since $r(n-r+1)$ is a quadratic function in r, we have

$$f(A) \leq \begin{cases} \frac{n^2+2n}{4} & \text{if } n \text{ is even} \\ \frac{n^2+2n-3}{4} & \text{if } n \text{ is odd,} \end{cases}$$

which completes the proof. □

Conjecture 3.8.1 (Brualdi and Liu, [31]) For $n \geq 5$, $f_n = 2n-4$. Example 3.8.1 can be extended for large values of n and so we can conclude that $f_n \geq 2n-4$.

Liu [170] proved Conjecture 3.8.1 for primitive matrices with symmetric 1-entries.

Example 3.8.3 Let $n \geq 5$ and $k \geq 2$ be integers with $n \geq k+3$. Let D be the digraph with $V(D) = \{v_1, v_2, \cdots, v_n\}$ and with $E(D) = \{(v_i, v_{i+1}) : 1 \leq i \leq n-k\} \cup \{(v_{n-k+1}, v_1)\} \cup \{(v_{n-k-1}, v_j), (v_j, v_1) : n-k+2 \leq j \leq n\}$. Let $A = A(D)$ and let $X_k = \{v_{k-k+1}, \cdots, v_n\}$. Then we can see that for each $i = 1, 2, \cdots, k$, $|R_{i(n-k)-1}(X_k)| = i$, and so $f^*(A) \geq k(n-k)$. (See Exercise 3.25 for more discussion of this example.)

Lemma 3.8.3 Let D be a strong digraph with $|V(D)| = n$, and let C_r be a directed cycle of D of length $r > 0$.

(i) If $X \subseteq V(C_r)$, then

$$R_{ir+j}(X) \subseteq R_{(i+1)r+j}(X), \quad (i \geq 0,\ 0 \leq j \leq r-1).$$

(ii) If $X = V(C_r)$, then $R_i(X) \subseteq R_{(i+1)}(X)$, for each $i \geq 0$.

Proof (i). Let $z \in R_{ir+j}(X)$ and $x \in X$. Since $x \in V(C_r)$, any direct (x,z)-walk of length $ir+j$ can be extended to a direct (x,z)-walk of length $(i+1)r+j$ by taking an additional tour of C_r.

(ii). Let $z \in R_i(X)$ and $x \in X = V(C_r)$. Let $x' \in V(C_r)$ be the vertex such that $(x', x) \in E(C_r)$. Then D has a directed (x', z)-walk of length $i+1$. □

Lemma 3.8.4 Let $r > s > 0$ be two coprime integers, and let D be a digraph consists of exactly two directed cycles C_r and C_s, of length r and s, respectively, such that $V(C_r) \cap V(C_s) \neq \emptyset$. If $\emptyset \neq X \subseteq V(C_r)$, then

$$|R_i(X)| \geq \min\{n, |X| + l\},\ i \geq lr \text{ and } l > 1. \tag{3.14}$$

Proof Let $X^{(i)}$ denote the vertices in $V(C_r)$ that can be reached from vertices in X by a directed walk in C_r of length i. Thus if $i \equiv j \pmod r$, $X^{(i)} = X^{(j)}$.

Assume first that $r \leq i < 2r$. If $X = V(C_r)$, then since $i \geq r$, $|R_i(X)| \geq \min\{n, |X| + i\}$, and so (3.14) holds.

Thus we assume $X \neq V(C_r)$ and $l = 1$. If $R_i(X) \not\subseteq V(C_r)$, then $|R_i(X)| \geq |X| + 1$. Assume then $R_i(X) \subseteq V(C_r)$. If (3.14) does not hold when $l = 1$, then $|R_i(X)| = |X|$, and so $R_i(X) = X^{(i)}$. Since $R_i(X) \subseteq V(C_r)$, we have $R_{i-s}(X) \subseteq R_i(X)$, which implies that $X^{(i-s)} = X^{(i)}$. Therefore if $Y = X^{(i-s)} \subseteq V(C_r)$, then $Y^{(s)} = Y$, contrary to the assumption that r and s are coprime.

Now assume that $l \geq 2$ and argue by induction on l. Note that (3.14) holds if $|R_{(l-1)r+j}(X)| = n$, and so assume that $|R_{(l-1)r+j}(X)| < n$.

If $R_{lr+j}(X) = R_{(l-1)r+j}(X)$, then for each $t > l-1$, $R_{tr+j}(X) = R_{(l-1)r+j}(X)$. Since D is primitive, for t large enough, we have $|R_{(l-1)r+j}(X)| = |R_{tr+j}(X)| = n$, a contradiction. Hence by Lemma 3.8.3, $R_{(l-1)r+j}(X) = R_{l)r+j}(X)$, for each j with

$0 \le j \le r-1$. It follows

$$\begin{aligned} |R_{lr+j}(X)| &\ge |R_{(l-1)r+j}(X)| + 1 \\ &\ge \min\{n, |X| + (l-1)\} + 1 \\ &\ge \min\{n, |X| + l\} \end{aligned}$$

which completes the proof. $\square$

Theorem 3.8.3 (Brualdi and Liu, [31]) Let $A \in \mathbf{P}_n$. If $D(A)$ has exactly 2 different lengths of directed cycles,then

$$f^*(A) \le \lfloor \frac{(n+1)^2}{4} \rfloor.$$

Proof Let $D(A)$ has directed cycles C_r and C_r, of lengths r and s, respectively, such that r and s are coprime and such that $V(C_r) \cap V(C_s) \ne \emptyset$. Let D^* denote the subgraph of D induced by $E(C_r) \cup E(C_s)$.

Let $Y \subseteq V(D)$ be a subset, where $1 \le k = |Y| \le n-1$. First assume that $|Y \cap V(C_r)| \ge p \ge 1$. By Lemma 3.8.4, $|R_i(Y)| \ge k+1$, $(i \ge (k-p+1)r)$, and so by $r \le n-(k-p)$, it follows that

$$(k-p+1)r \le \lfloor \frac{1}{4}(n+1)^2 \rfloor.$$

Now assume that $Y \cap V(C_r) = \emptyset$. Then $r \le n-k$ and D has a directed (y,x)-walk from a vertex $y \in Y$ to a vertex $x \in V(C_r)$ of length t, where $t \le n-r-k+1$. By lemma 3.8.4,

$$|R_i(\{x\})| \ge k+1, \ i \ge kr.$$

Therefore $|R_i(Y)| \ge k+1$, for $i \ge kr+n-r-k+1$. It follows that

$$kr+n-r-k+1 \le \lfloor \frac{n^2}{4} \rfloor + 1.$$

Hence for all Y $\emptyset \ne Y \subseteq X$,

$$|R_i(Y)| \ge |Y| + 1, \ i \ge \lfloor \frac{(n+1)^2}{4} \rfloor,$$

which completes the proof. $\square$

From the discussions above on f_n^* (see also Exercise 3.25), we can see that the order of f_n^* will fall between $O(n^2/4)$ and $O(n^2/2)$. It was conjectured that

$$f_n^* \le \lfloor (n+1)^2/4 \rfloor$$

and this conjecture has been proved by Liu and Li [178].

Definition 3.8.3 A matrix $A \in \mathbf{B}_n$ is called a *Hall matrix* if there exists a permutation matrix Q such that $Q \le A$. Let $\mathbf{H}_n$ denote the collection of all Hall matrices in $\mathbf{P}_n$.

$$\tilde{\mathbf{H}}_n = \{A \in \mathbf{B}_n \ : \ A^k \in \mathbf{H}_n \text{ for some integer } k\}.$$

For an matrix $A \in \tilde{\mathbf{H}}_n$, $h(A)$, the *Hall exponent* of A, is the smallest integer $k > 0$ such that $A^k \in \mathbf{H}_n$. Define

$$h_n = \max\{h(A) \ : \ A \in \tilde{\mathbf{H}}_n \cap \mathbf{IB}_n\},$$

where $\mathbf{IB}_n$ is the collection of irreducible matrices in $\mathbf{B}_n$.

Similarly, for an matrix $A \in \tilde{\mathbf{H}}_n$, $h^*(A)$, the *strict Hall exponent* of A, is the smallest integer $k > 0$ such that $A^i \in \mathbf{H}_n$, for all integer $i \ge k$. Define $\mathbf{H}_n^* = \{A \in \mathbf{B}_n \ : \ h^*(A)$ exists as a finite number$\}$, and

$$h_n^* = \max\{h^*(A) \ : \ A \in \tilde{\mathbf{H}}_n \cap \mathbf{IB}_n\},$$

Example 3.8.4 In general, $\mathbf{P}_n \subset \tilde{\mathbf{H}}_n$. When $n > 1$, if P is a permutation matrix with $\mathrm{tr}(P) = 0$, then $P \in \tilde{\mathbf{H}}_n \setminus \mathbf{P}_n$.

Example 3.8.5 Let

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 \end{bmatrix}.$$

Then we can verify that $A \in \mathbf{P}_7 \setminus \mathbf{H}_7$. $A^2 \in \mathbf{H}_7$ but $A^3 \not\in \mathbf{H}_7$, and $A^i \in \mathbf{H}_7$, for all $i \ge 4$.

Proposition 3.8.3 follows from Hall's Theorem for the existence of a system of distinct representatives (Theorem 1.1 in [222]); and the other proposition is obtained from the definitions and Corollary 3.3.1A.

Proposition 3.8.3 Let $A \in \mathbf{B}_n$ and let $D = D(A)$. Each of the following holds.

(i) A is Hall if and only if for any integers $r > 0$ and $s > 0$ with $r + s > n$, A does not have an $0_{r\times s}$ as a submatrix.

(ii) For some integer $k > 0$, $A^k \in \mathbf{H}_n$ if and only if for each nonempty subset $X \subseteq V(D)$, $|R_k(X)| \ge |X|$.

Proposition 3.8.4 Each of the following holds:

(i) If $A \in \mathbf{H}_n^*$, then

$$h(A) \le h^*(A) < \gamma(A) \le n^2 - 2n + 2.$$

(ii) If $A \in \mathbf{P}_n$, then $h(A) \le f(A)$ and $h^*(A) \le f^*(A)$.

(iii) For each $n > 1$, $\mathbf{F}_n \subseteq \mathbf{H}_n$.

Example 3.8.7 Let

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix}.$$

Then $A^k \in \mathbf{H}_4$ if and only if $4|k$, and so $A \in \tilde{\mathbf{H}}_n \setminus \mathbf{H}_n^*$.

Example 3.8.8 It is possible that $h^*(A) > f(A)$. Let

$$A = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \end{bmatrix}.$$

Then $A \notin \mathbf{H}_{10}$, $A^2 \in \mathbf{F}_{10}$, (and so $A \in \mathbf{P}_{10}$ and $A^2 \in \mathbf{H}_{10}$), $A^3 \notin \mathbf{H}_{10}$ but $A^k \in \mathbf{F}_{10} \subseteq \mathbf{H}_{10}$, for any $k \ge 4$. Therefore,

$$h^*(A) = f^*(A) = 4 > 2 = f(A) = h(A).$$

Definition 3.8.4 Let $A \in \mathbf{B}_n$ and let

$$\begin{bmatrix} A_{11} & 0 & \cdots & 0 \\ A_{12} & A_{22} & \cdots & 0 \\ & \cdots & \cdots & \\ A_{p1} & A_{p2} & \cdots & A_{pp} \end{bmatrix}$$

be the Frobenius normal form (Theorem 2.2.1) of A. By Theorem 2.2.1, each A_{ii} is irreducible, and will be called an *irreducible block* of A, $i = 1, 2, \cdots, p$. A block A_{ii} is a *trivial block* of A if $A_{ii} = 0_{1\times 1}$.

By definition, we can see that if A has a trivial block, then $A \notin \mathbf{H}_n$, and so Lemma 3.8.5 obtains.

Lemma 3.8.5 Let $A \in \mathbf{B}_n$. Then $A \in \mathbf{H}_n$ if and only if every irreducible block of A is a Hall matrix.

Theorem 3.8.4 (Brualdi and Liu, [33]) Let $A \in \mathbf{B}_n$. Then $A \in \tilde{\mathbf{H}}_n$ if and only if the Frobenius standard form of A does not have a trivial irreducible block.

Proof We may assume that A is in the standard form. If A has a trivial irreducible block. Then for any k, A^k also has a trivial irreducible block, and so $A^k \notin \mathbf{H}_n$.

Assume then that A has no trivial irreducible block. Then each vertex $v_i \in V(D(A))$ lies in a directed cycle of length m_i, $(1 \le i \le n)$. Let $p = \operatorname{lcm}(m_1, m_2, \cdots, m_n)$. Then each diagonal entry of A^p is positive, and so $A \in \tilde{\mathbf{H}}_n$. $\square$

Definition 3.8.5 Recall that if $A \in \mathbf{B}_n$ is irreducible, then A is permutation similar to

$$\begin{bmatrix} 0 & B_1 & 0 & \cdots & 0 \\ 0 & 0 & B_2 & \cdots & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & 0 & \cdots & B_{h-1} \\ B_h & 0 & 0 & \cdots & 0 \end{bmatrix} \tag{3.15}$$

where $B_i \in M_{k_i \times k_{i+1}}$, $(1 \le i \le h)$ and $k_{h+1} = k_1$. These integers k_i's are the *imprimitive parameters* of A.

Let $P \in \mathbf{B}_h$ be a permutation matrix and let $Y_1, Y_2, \cdots, Y_h \in \mathbf{B}_k$ be h matrices. Then $P(Y_1, Y_2, \cdots, Y_h)$ denotes a matrix in $\mathbf{B}_{kh}$ obtained by replacing the only 1-entry of the ith row of P by Y_i, and every 0-entry of P by a $0_{k\times k}$, $(1 \le i \le h)$.

Theorem 3.8.5 (Brualdi and Liu, [33]) Let $A \in \mathbf{IB}_n$. Then $A \in \mathbf{H}_n^*$ if and only if all the imprimitive parameters are identical.

Proof We may assume that A has the form in (3.15). Define $X_1 = B_1 B_2 \cdots B_h$, $X_2 = B_2 B_3 \cdots B_h B_1$, $\cdots$, $X_h = B_h B_1 \cdots B_{h-1}$.

Suppose first that $k = k_1 = k_2 = \cdots = k_h$. Then the matrices $X_1, X_2, \cdots, X_h$ are in $\mathbf{P}_k$, and so there exists an integer $e > 0$ such that $X_i^p = J_k$, for any integer $p \ge e$ and $1 \le i \le h$.

Let $q \ge eh$ be an integer and write $q = fh + r$, where $f \ge e$ and where $0 \le r < h$. Then A^q has the form $P(Y_1, Y_2, \cdots, Y_h)$ for some permutation matrix P, where $Y_i = X_i^f A_i \cdots A_{i+r-1}$, $(1 \le i \le h$, and the subscripts are counted modulo $h)$. Since $f \ge e$, $X_i^f = J$; since A is irreducible, each A_i has no zero columns. It follows that $Y_i = J$, $(1 \le i \le h)$, and so $A^q \in \mathbf{H}_n$. Hence $A \in \mathbf{H}_n^*$.

Conversely, assume that A does not have identical imprimitive parameters. Without lost of generality, we assume $k_1 < k_2$. Note that for each integer $f > 0$, A^{fh+1} takes the same form as in (3.15) with $X_i^f B_i$ replacing B_i, $1 \le i \le h$. It follows that A^{fh} has

$0_{(n-k_1)\times k_2}$ as a submatrix. Since $(n-k_1)+k_2 > n$, $A^{fh} \notin \mathbf{H}_n$, by Proposition 3.8.3. $\square$

The following results concerning the Hall exponent, analogous to those concerning the fully indecomposable exponent, are obtained by Brualdi and Liu [33] with similar techniques.

Theorem 3.8.6 (Brualdi and Liu, [33]) Let $A \in \mathbf{IB}_n$. If $\mathrm{tr}(A) = s > 0$, then $h^*(A) \le n-s$.

Theorem 3.8.7 (Brualdi and Liu, [33]) Let $n \ge 3$. Then

$$h_n \le \lfloor \frac{n^2-1}{4} \rfloor.$$

Theorem 3.8.9 (Brualdi and Liu, [33], Zhou and Liu, [286]) Let $A \in \mathbf{P}_n$. Then $h^*(A) \le \lfloor n^2/4 \rfloor$.

Shen et al [240] introduced the exponent of r-indecomposability as a generalization of fully indecomposable exponents and Hall exponents. Several useful results have been obtained in [240] and [176].

Definition 3.8.7 The exponents can be defined in a weak sense. For a matrix $A \in \mathbf{IB}_n$, the weak exponents are defined as follows.

(i) The *weak primitive exponents* of A, denoted by $e_w(A)$, is the smallest integer $p > 0$ such that $A + A^2 + \cdots + A^p \in \mathbf{P}_n$.

(ii) The *weak fully indecomposable exponent* of A, denoted by $f_w(A)$, is the smallest integer $p > 0$ such that $A + A^2 + \cdots + A^p \in \mathbf{F}_n$.

(iii) The *weak Hall exponent* of A, denoted by $h_w(A)$, is the smallest integer $p > 0$ such that $A + A^2 + \cdots + A^p \in \mathbf{H}_n$.

Theorem 3.8.10 (Liu, [171]) For $A \in \mathbf{IB}_n$,

$$e_w(A) \le 2, \;\; f_w(A) \le \lfloor \frac{n}{2} \rfloor + 1, \;\; \text{and } f_w(A) \le \lceil \frac{n}{2} \rceil.$$

Moreover, each of these bounds is best possible.

Theorem 3.8.11 (Liu, [171]) For $A \in \mathbf{IB}_n$, let $WE(n), FE(n)$ and $HE(n)$ denote the set of integers that can be the value of $e_w(A)$, $f_w(A)$ and $h_w(A)$, respectively. Then

$$\begin{aligned} WE(n) &= \{1,2\} \\ FE(n) &= \{1,2,\cdots,\lfloor \frac{n}{2} \rfloor + 1\} \\ HE(n) &= \{1,2,\cdots,\lceil \frac{n}{2} \rceil\}. \end{aligned}$$

3.9 Primitive exponent and other parameters

The investigation of the relationship between $\gamma(A)$ and the diameter of $D(A)$, and between $\gamma(A)$ and the eigenvalues of A has recently begun and is on the rise.

We use these notations in this section: For a matrix $A \in \mathbf{B}_n$ and $D = D(A)$, $m = m(A)$ denotes the degree of the minimal polynomial of A; $d = d(D)$ denotes the diameter of D, and $d(v_i, v_j)$ the distance from v_i to v_j in D.

Given a matrix A, let $(A)_{ij}$ denote the (i, j)-entry of A. Therefore, if $A = (a_{ij})$, then $(A)_{i,j} = a_{ij}$. Similarly, given an n-dimensional vector $\mathbf{a}$, $(\mathbf{a})_j$ denotes the jth component of $\mathbf{a}$. Finally, for an integer k with $1 \le k \le n$, let $\mathbf{e}_k$ denote the n-dimensional vector whose kth component is a 1 and whose other components are 0.

Proposition 3.9.1 Let $A \in \mathbf{P}_n$, $D = D(A)$, $d = d(D)$ and $m = m(A)$. Each of the following holds.

(i) $\gamma(A) \ge d$. Moreover, if each diagonal entry of A is positive, then $\gamma(A) = d$.

(ii) If the length s of a shortest directed cycle of D is not bigger than d, then $\gamma(A) \le dn$.

(iii) (See [96])

$$\begin{cases} I + A + \cdots + A^{m-1} > 0 \\ A + A^2 + \cdots A^m > 0 \end{cases}$$

(iv) If $V(D) = \{v_1, v_2, \cdots, v_n\}$, then

$$d(v_i, v_j) \le m - 1 + \delta_{ij}, \text{ where } \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \ne j. \end{cases}$$

Proof (i) follows from the graphical meaning of $\gamma(A)$. To prove (ii), consider the graph $D^{(s)}$. Each vertex of a shortest directed cycle C_s is a loop vertex in $D^{(s)}$, from which any vertex can be reached by a directed walk of length at most $n-1$. Thus in D, any vertex u can reach a vertex in C_s by a directed walk of length at most d, and any vertex in $V(C_s)$ to any other vertex v by a directed walk of length at most $s(n-1)$. It follows that $\gamma(A) \le d + s(n-1) \le d + d(n-1)$. □

Problem 3.9.1 By examine the graph in Example 3.3.1, it may be natural to conjecture that if $A \in \mathbf{P}_n$, and if d is the diameter of $D(A)$, then

$$\gamma(A) \le d^2 + 1. \tag{3.16}$$

Note that the degree m of the minimal polynomial of A and d are related by $m \ge d + 1$. A weaker conjecture will be

$$\gamma(A) \le (m-1)^2 + 1. \tag{3.17}$$

Hartwig and Neumann proved (3.17) conditionally. Lemma 3.9.1 below follows from Proposition 3.9.1 and Proposition 1.1.2(vii).

Lemma 3.9.1 (Hartwig and Neumann, [117]) Let $A \in \mathbf{P}_n$, $D = D(A)$ and $m = m(A)$. Suppose $V(D) = \{v_1, v_2, \cdots, v_n\}$.

(i) If v_k is a loop vertex of D, then $A^{m-1}\mathbf{e}_k > 0$.

(ii) If each vertex of D is a loop vertex, then $A^{m-1} > 0$.

Theorem 3.9.1 (Hartwig and Neumann, [117]) Let $A \in \mathbf{P}_n$, $D = D(A)$ and $m = m(A)$. Then $\gamma(A) \le (m-1)^2 + 1$, if one of the following holds for each vertex $v \in V(D)$:

(i) v lies in a directed cycle of length at most $m-1$,

(ii) v can be reached from a vertex lying in a directed cycle of length at most $m-1$ by a directed walk of length one, or

(iii) v can reach a vertex lying in a directed cycle of length at most $m-1$ by a directed walk of length one.

Sketch of Proof Let $V(D) = \{v_1, v_2, \cdots, v_n\}$ and assume that v_k lies in a directed cycle of length $j_k < m$. Then v_k is a loop vertex in $D(A^{j_k})$. Since $A^{j_k} \in \mathbf{P}_n$ with $m(A^{j_k}) \le m(A) = m$, it follows by Lemma 3.9.1 that $(A^{j_k})^{m-1}\mathbf{e}_k > 0$, and so

$$A^{(m-1)^2}\mathbf{e}_k = A^{[(m-1)-j_k](m-1)}\left[(A^{j_k})^{m-1}\mathbf{e}_k\right] > 0.$$

Thus (i) implies the conclusion by Lemma 3.9.1.

Assume then v_k can be reached from a vertex lying in a directed cycle of length at most $m-1$ by a directed walk of length one, then argue similarly to see that

$$A^{(m-1)^2+1}\mathbf{e}_k = A^{(m-1)^2}(A\mathbf{e}_k) > 0,$$

and so (ii) implies the conclusion by Lemma 3.9.1 also.

That (iii) implies the conclusion can be proved similarly by considering A^T instead of A, and so the proof is left as an exercise. $\square$

Theorem 3.9.2 (Hartwig and Neumann, [117]) Let $A \in \mathbf{P}_n$ with $m = m(A)$. Then

$$\gamma(A) \le m(m-1).$$

Proof Let $D = D(A)$ with $V(D) = \{v_1, v_2, \cdots, v_n\}$. By Proposition 3.9.1(iii), for each $v_k \in V(D)$, there is an integer j_k with $1 \le j_k \le m$ such that v_k is a loop vertex of $D(A^{j_k})$. By Lemma 3.9.1, $(A^{j_k})^{m-1}\mathbf{e}_k > 0$. It follows that

$$A^{m(m-1)}\mathbf{e}^{(k)} = A^{(m-j_k)(m-1)}\left[(A^{j_k})^{m-1}\mathbf{e}_k\right] > 0,$$

and so $A^{m(m-1)} > 0$, by Lemma 3.9.1(ii). $\square$

Theorem 3.9.3 (Hartwig and Neumann, [117]) Let $A \in \mathbf{P}_n$ be symmetric with m the degree of minimal polynomial of A. Then $\gamma(A) \le 2(m-1)$.

Sketch of Proof As A is symmetric, every vertex of $D(A^2)$ is a loop vertex. Then apply Lemma 3.9.1 to see $(A^2)^{m-1}\mathbf{e}_k > 0$. □

Theorem 3.9.4 (Hartwig and Neumann, [117]) Let $A \in \mathbf{P}_n$ such that $D(A)$ has a directed cycle of length $k > 0$, and let m and m_{A^k} be the degree of the minimal polynomial of A and that of A^k, respectively. Then $\gamma(A) \le (m-1) + k(m_{A^k} - 1)$.

Proof Let C_k denote a directed cycle of length k. Then $V(C_k)$ are loop vertices in $D(A^k)$. Any vertex in $D(A^k)$ can be reached from a vertex in $V(C_k)$ by a directed walk of length at most $m_{A^k} - 1$, and so in $D(A)$, any vertex can reach another by a directed walk (via vertices in $V(C_k)$) of length at most $k(m_{A^k} - 1) + (m-1)$. □

Theorem 3.9.5 (Hartwig and Neumann, [117]) Let $A \in \mathbf{P}_n$ such that A has r distinct eigenvalues. Then $D(A)$ contains a directed cycle of length at most r.

Proof If $\rho(A)$, the spectrum radius of A, is zero, then $A^2 = 0$. Thus $r = 1$ and, by Proposition 1.1.2(vii), $D(A)$ has no directed cycles. Assume that $\rho(A) > 0$ and that

$$\operatorname{Spec}(A) = \begin{pmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_r \\ l_1 & l_2 & \cdots & l_r \end{pmatrix}.$$

Argue by contradiction, assume that every directed cycle of $D(A)$ has length longer than r. Then for each k with $1 \le k \le r$, $\operatorname{tr}(A^k) = 0$, by Proposition 1.1.2(vii). Thus

$$\begin{bmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_r \\ \lambda_1^2 & \lambda_2^2 & \cdots & \lambda_r^2 \\ \ddots & & & \ddots \\ \lambda_1^r & \lambda_2^r & \cdots & \lambda_r^r \end{bmatrix} \begin{bmatrix} l_1 \\ l_2 \\ \ddots \\ l_r \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \ddots \\ 0 \end{bmatrix}. \tag{3.18}$$

Note that (3.18) is equivalent to the homogeneous system

$$\begin{bmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_r \\ \ddots & & & \ddots \\ \lambda_1^{r-1} & \lambda_2^{r-1} & \cdots & \lambda_r^{r-1} \end{bmatrix} \begin{bmatrix} \lambda_1 l_1 \\ \lambda_2 l_2 \\ \ddots \\ \lambda_r l_r \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \ddots \\ 0 \end{bmatrix}. \tag{3.19}$$

The determinant of the coefficient matrix in (3.19) is a Vandermonde determinant with $\lambda_i \ne \lambda_j$, whenever $i \ne j$. Thus the system in (3.19) can only have a zero solution $\lambda_1 l_1 = \lambda_2 l_2 = \cdots \lambda_r l_r = 0$, a contradiction. □

Corollary 3.9.5A ([117]) Let $A \in \mathbf{P}_n$ with $m = m(A)$. If A has at most $m-2$ distinct eigenvalues, then $\gamma(A) \le (m-1)^2$.

The conjectured (3.17) remains unsolved in [117]. In 1996, Shen proved the stronger form of (3.16), therefore also proved (3.17).

For a simple graph G, Delorme and Sole [73] proved $\gamma(G)$ can have a much smaller upper bound.

Theorem 3.9.6 (Delorme and Sole, [73]) Let G be a connected simple graph with diameter d. If every vertex of G lies in a closed walk of an odd length at most $2g+1$, then $\gamma(G) \le d+g$. In particular, if G is not bipartite, then $\gamma(G) \le 2d$.

Example 3.9.1 The equality $\gamma(G) = 2d$ may be reached. Consider these examples: G is the cycle of length $2k+1$ ($d=k$ and $\gamma = 2k$); $G = K_n$ with $n > 2$ ($d = 1$ and $\gamma = 2$); and G is the Petersen graph ($d = 2$ and $\gamma = 4$).

The relationship between $\gamma(A)$ and the eigenvalues of A is not clear yet. Chung obtained some upper bounds of $\gamma(A)$ in terms of eigenvalues of A. For convenience, we extend the definition of $\gamma(A)$ and define $\gamma(A) = \infty$ when A is imprimitive.

Theorem 3.9.7 (Chung, [58]) Let G be a k-regular graph and with eigenvalues λ_i so labeled that $|\lambda_1| \le |\lambda_2| \le \cdots \le |\lambda_n|$. Then

$$\gamma(A) \le \lceil \frac{\log(n-1)}{\log k - \log|\lambda_2|} \rceil.$$

Proof Let $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ are orthonormal eigenvectors corresponding to $\lambda_1, \lambda_2, \cdots, \lambda_n$, respectively, such that

$$\mathbf{u}_1 = \frac{1}{\sqrt{n}} J_{n\times 1} \text{ and } \lambda_1 = k.$$

Thus if $\left(\frac{k}{|\lambda_2|}\right)^m > n-1$, then

$$\begin{aligned}
(A^m)_{r,s} &= \sum_i \lambda_i^m (\mathbf{u}_i \mathbf{u}_i^T)_{r,s} \\
&\ge \frac{k^m}{n} - \left| \sum_{i>1} \lambda_i^m (\mathbf{u}_i)_r (\mathbf{u}_i)_s \right| \\
&\ge \frac{k^m}{n} - |\lambda_2|^m \left\{ \sum_{i>1} |(\mathbf{u}_i)_r (\mathbf{u}_i)_s| \right\} \\
&\ge \frac{k^m}{n} - |\lambda_2|^m \left\{ \sum_{i>1} |(\mathbf{u}_i)_r|^2 \right\}^{\frac{1}{2}} \left\{ \sum_{i>1} |(\mathbf{u}_i)_s|^2 \right\}^{\frac{1}{2}} \\
&= \frac{k^m}{n} - |\lambda_2|^m \{1 - (\mathbf{u}_i)_r^2\}^{\frac{1}{2}} \{1 - (\mathbf{u}_i)_s^2\}^{\frac{1}{2}} \\
&> 0.
\end{aligned}$$

Therefore, if $m > \lfloor \frac{\log(n-1)}{\log k - \log|\lambda_2|} \rfloor$, then $A^m > 0$. □

With similar techniques, Chung also obtained analogous bounds for non regular graphs and digraphs.

Theorem 3.9.8 (Chung, [58]) Let G be a simple graph with eigenvalues λ_i so labeled that $|\lambda_1| \le |\lambda_2| \le \cdots \le |\lambda_n|$. Let $\mathbf{u}_1$ be an eigenvector corresponding to λ_1, let $w = \min_i\{|(\mathbf{u}_1)_i|\}$ and let $d(G)$ denote the diameter of G. Then

$$d(G) \le \gamma(A) \le \lceil \frac{\log(1-w^2) - \log w^2)}{\log|\lambda_1| - \log|\lambda_2|} \rceil.$$

Theorem 3.9.9 (Chung, [58]) Let $A \in \mathbf{B}_n$ be such that each row sum of A is k, and A has n eigenvectors which form an orthonormal basis. Then

$$\gamma(A) \le \lceil \frac{\log(n-1)}{\log k - \log|\lambda_2|} \rceil.$$

For further improvement along this line, readers are referred to Delorme and Sole [73]. For a matrix $A \in \mathbf{B}_{m,n}$, define its *Boolean rank* $b(A)$ to be the smallest positive integer k such that some $F \in \mathbf{B}_{m,k}$ and $G \in \mathbf{B}_{k,n}$, $A = FG$. With Lemma 3.4.2, Gregory et al obtained the following.

Theorem 3.9.10 (Gregory, Kirkland and Pullman, [106]) Let $A \in P_n$. Then $\gamma(A) \le (b(A)-1)^2 + 2$.

The next theorem can be obtained by applying Lemma 3.7.3 and Exercise 3.23.

Theorem 3.9.11 (Liu and Zhou, [185], Neufeld and Shen, [207]) Let $A \in P_n$ and let r denote the largest outdegree of vertices of a shortest cycle length with length s in $D(A)$. Then

$$\gamma(A) \le s(n-r) + n \le (n-r+1)^2 + r - 1.$$

Open Problem From Proposition 3.9.1(i), one would ask the question what the matrices A with $\gamma(A) = d$ are? In other words, the problem is to determine the set

$$\{A \in \mathbf{P}_n \ : \ \gamma(A) = d, \text{ where } d \text{ is the diameter of } D(A)\}.$$

3.10 Exercises

Exercise 3.1 Let $a_1, a_2, a_3 > 0$ be integers with $\gcd(a_1, a_2, a_3) = 1$. Let $d = \gcd(a_1, a_2)$ and write $a_1 = a_1'd$ and $a_2 = a_2'd$. Let u_1, u_2, x_0, y_0, z_0 be integers satisfying $a_1'u_1 +$

$a_2'u_2 = 1$ and $a_1x_0 + a_2y_0 + a_3z_0 = n$ respectively. Show that all integral solutions of $a_1x + a_2y + a_3z = n$ can be presented as

$$\begin{cases} x = x_0 + a_1't_1 - u_1a_3t_2 \\ y = y_0 - a_1t_1 - u_2a_3t_2 \\ z = z_0 + dt_2, \end{cases}$$

where t_1, t_2 can be any integers.

Exercise 3.2 Let $s \geq 2$ be an integer and suppose $r_1, r_2, \cdots, r_s$ are real numbers such that $r_1 \geq r_2 \geq \cdots \geq r_s \geq 1$. Show that

$$\sum_{i=1}^{s-1}\left(\frac{r_i}{r_{i+1}} - 1\right) + r_s \leq r_1.$$

Exercise 3.3 Assume that Theorem 3.1.7 holds for $s = 3$. Prove Theorem 3.1.7 by induction on s.

Exercise 3.4 Let D be a strong digraph. Let $d'(D)$ denote the g.c.d. of directed closed trail lengths of D. Show that $d'(D) = d(D)$.

Exercise 3.5 Let D be a cyclically k partite directed graph. Show each of the following.

(i) If D has a directed cycle of length m, then $k|m$.

(ii) If $h|k$, then D is also cyclically h-partite.

Exercise 3.6 Show that if $A \in M_n^+$ is irreducible with $d = d(D(A)) > 1$, and if $h|d$, then, there exists a permutation matrix P such that $PA^hP^{-1} = \text{diag}(A_1, A_2, \cdots, A_h)$.

Exercise 3.7 Prove Corollary 3.2.3A.

Exercise 3.8 Prove Corollary 3.2.3B. For (i), imitate the proof for Theorem 3.2.3(iii).

Exercise 3.9 Prove Corollary 3.2.3C.

Exercise 3.10 Prove Corollary 3.2.3D.

Exercise 3.11 Show that in Example 3.2.1, A is primitive, B is imprimitive and $A \sim_p B$.

Exercise 3.12 Let D_1, D_2 be the graphs in Examples 3.3.1 and 3.3.2. Show that $\gamma(D_i) = (n-1)^2 + 2 - i$.

Exercise 3.13 Complete the proof of Theorem 3.3.2.

Exercise 3.14 Prove Lemma 3.4.1.

Exercise 3.15 Prove Lemma 3.4.2.

Exercise 3.16 Prove Lemma 3.4.3.

Exercise 3.17 Prove Lemma 3.4.5.

Exercise 3.18 Let $A \in \mathbf{B}_p$ and let n_0, s_0 be defined as in Theorem 3.5.2. Apply Theorem 3.5.2 to prove each of the following.

(i) If $A \in \mathbf{IB}_{n,p}$, then $k(A) \le n + s_0 \left(\frac{n}{p} - 2\right)$.

(ii) Wielandt's Theorem (Corollary 3.3.1A).

(iii) Theorem 3.5.1.

Exercise 3.19 Let X be a matrix with the form in Lemma 3.5.1. Show each of the following.

(i) If $a = 0$, then

$$X^{k+1} = \left[\begin{array}{c|c} B^{k+1} & 0 \\ \hline \mathbf{x}^T B^k & 0 \end{array}\right]$$

(ii) If $a = 1$, then

$$X^{k+1} = \left[\begin{array}{c|c} B^{k+1} & 0 \\ \hline \mathbf{x}^T (I + B + \cdots + B^k) & 1 \end{array}\right]$$

Exercise 3.20 Suppose that $A \in \mathbf{B}_n$ with $p(A) > 1$. Show that $\mu(A) < n^2$ and $h(A) \le k(A) + p - 1$.

Exercise 3.21 Let $n > 0$ denote an integer, and let D be a primitive digraph with $V(D) = \{v_1, \cdots, v_n\}$ such that $exp_D(v_1) \le exp_D(v_2) \le \cdots \le exp_D(v_n)$. Show that

(i) $F(D, 1) = exp_D(v_n) = \gamma(D)$ and $f(D, 1) = exp_D(v_1)$.

(ii) $f(n, n) = 0$, $f(n, 1) = exp(n, 1)$, and $F(n, 1) = exp(n)$.

Exercise 3.22 Suppose that r is the largest outdegree of vertices of a shortest cycle with length s in D. Show that $exp_D(1) \le s(n - r) + 1$.

Exercise 3.23 Let $A \in \mathbf{B}_n$ be a primitive matrix and let $D = D(A)$. For each positive $k \le n$, show each of the following.

(i) $exp_D(k)$ is the smallest integer $p > 0$ such that A^p has k all one rows. (That is $J_{k \times n}$ is a submatrix of A^p.)

(ii) $f(D, k)$ is the smallest integer $p > 0$ such that A^p has a $k \times n$ submatrix which does not have a zero column.

(iii) $F(D, k)$ is the smallest integer $p > 0$ such that A^p does not have a $k \times n$ submatrix which had a zero column.

Exercise 3.24 Let $n \ge k \ge 1$. Then

$$f(D_n, k) = \begin{cases} 1 + \dfrac{(n-k-1)(n-1)}{k} & \text{if } n - 1 \equiv 0 (\mathrm{mod}\ k) \\ 2(n-k) - 1 & \text{if } \frac{n}{2} \le k < n - 1 \end{cases}$$

Exercise 3.25 Show that $f(n, n-1) = 1$ and $f(n, 1) = n^2 - 3n + 3$.

Exercise 3.26 Prove Lemma 3.7.6.

Exercise 3.27 Let D be the digraph

(i) Show that $f^*(A) = k(n-k)$.

(ii) Show that for $n \geq 5$,

$$\lceil \frac{n}{2} \rceil \lfloor \frac{n}{2} \rfloor \leq f_n^* \leq n^2 - 2n + 2.$$

Exercise 3.28 Let $A \in \mathbf{P}_n$ with $m = m(A)$. If $D(A)$ has a directed cycle of length at most $m-2$, then $\gamma(A) \leq (m-1)^2$.

Exercise 3.29 Let $A \in \mathbf{P}_n$ with $m = m(A) \geq 4$. If every eigenvalue of A is real, then

$$\gamma(A) \leq 3(m-1) \leq (m-1)^2.$$

Exercise 3.30 Prove Corollary 3.9.5A.

Exercise 3.31 Let $A \in \mathbf{P}_n$ with $m = m(A)$. If A has a real eigenvalue with multiplicity at least 3, or if A has a non real eigenvalue of multiplicity at least 2, then $\gamma(A) \leq (m-1)^2$.

Exercise 3.32 Prove Theorem 3.9.10.

Exercise 3.33 Prove Theorem 3.9.11.

3.11 Hint for Exercises

Exercise 3.1 First, we can routinely verify that for integers t_1, t_2,

$$\begin{cases} x = x_0 + a_1' t_1 - u_1 a_3 t_2 \\ y = y_0 - a_1 t_1 - u_2 a_3 t_2 \\ z = z_0 + d t_2, \end{cases}$$

satisfy the equation $a_1 x + a_2 y + a_3 z = n$.

Conversely, let x, y, z be an integral solution of the equation $a_1 x + a_2 y + a_3 z = n$. Since

$$a_1 x + a_2 y + a_3 z = n, \text{ and } a_1 x_0 + a_2 y_0 + a_3 z_0 = n,$$

we derive that

$$d(a_1(x - x_0) + b_1(y - y_0)) = -c(z - z_0).$$

Since $\gcd(c, d) = 1$, there exists an integer t_2 such that $z = z_0 + dt_2$. It follows that

$$a_1(x - x_0) + b_1(y - y_0) = -ct_2.$$

Since $a_1u_1 + b_1u_2 = 1$, we have $a_1(-u_1ct_2) + b_1(-u_2ct_2) = -ct_2$, and so

$$a_1(x - x_0 + u_1ct_2) + b_1(y - y_0 + u_2ct_2) = 0.$$

It follows that there exists an integer t_1 such that

$$x - x_0 + a_1't_1 - u_1a_3t_2, \text{ and } y = y_0 - a_1t_1 - u_2a_3t_2.$$

Exercise 3.2 Argue by induction on $s \geq 2$.

Exercise 3.3 First prove a fact on real number sequence.

Let u_1, u_2 be real numbers such that $u_1 \geq u_2 \geq 1$. Then

$$\frac{u_1}{u_2} - 1 + u_2 \leq u_1.$$

This can be applied to show that if $u_1 \geq u_2 \geq \cdots \geq u_k \geq 1$, then

$$\sum_{i=1}^{k-1}\left(\frac{u_i}{u_{i+1}} - 1\right) + u_k \leq u_1.$$

For number $a_1, a_2, \cdots, a_s$ satisfying Theorem 3.1.7, let $\gcd(a_1, d_2) = d_1$, $\gcd(a_1, a_2, a_3) = d_2$, $\gcd(a_1, \cdots, a_{s-1}) = d_{s-2}$, $s - 1 > 2$. Then

$$\begin{aligned}\phi(a_1, \cdots, a_s) &\leq \frac{a_1a_2}{d_1} + \frac{a_3d_1}{d_2} + \cdots + \frac{a_{s-1}d_{s-3}}{a_{s-2}} \\ &\quad + a_sa_{s-2} - \sum_{i=1}^{s} a_i + 1 \\ &< \frac{a_1a_2}{d_1} + a_3\left[\sum_{i=1}^{s-3}\left(\frac{d_i}{d_{i+1}} - 1\right) + d_{s-2}\right].\end{aligned}$$

This, together with the fact above on real number sequence, implies that

$$\phi(a_1, \cdots, a_s) < \frac{a_1a_2}{d_1} + a_3d_1.$$

If, among $a_1, a_2, \cdots, a_s$, there are $s - 1$ numbers which are relatively prime, then by induction, Theorem 3.1.7 holds. Therefore, $d_1 > 1$.

If $d_1 = p^a$ is a prime power, for some prime number p and integer $a > 0$, then $d_{s-2} = p^b$ for some integer b with $0 < b < a$. Hence we have $\gcd(a_1, a_2, \cdots, a_s) = p^\delta$ for some integer $\delta > 0$, a contradiction. Therefore, d_1 must have at least two prime factors. Thus $d_1 \geq 6$, $a_2 \leq a_1 - d_1$, $a_3 \leq a_2 - 2 \leq n - 8$. As $d_1|a_2$, we have $d_1 \leq n/2$, and so

$$\phi(a_1, \cdots, a_s) < \frac{a_1a_2}{d_1} + a_3d_1 \leq \frac{n(n - d_1)}{d_1} + (n - d_1 - 2)d_1.$$

As $\frac{n(n-d_1)}{d_1}$ is a decreasing function of d_1, ad as $d_1 \geq 6$, we have both $\frac{n(n-d_1)}{d_1} \leq (n-2)^2/4$ and $(n-d_1-2)d_1 \leq (n-2)^2/4$.

Exercise 3.4 Note that $d'|d$ since cycles are closed trails. By Euler, a closed trail is an edge-disjoint union of cycles, and so $d|d'$.

Exercise 3.5 Apply Definition 3.2.5 and combine the partite sets.

Exercise 3.6 By Corollary 3.2.3A and argue similarly to the proof of Lemma 3.2.2(i).

Exercise 3.7 By the definition of $d(D)$ (Definition 3.2.4).

Exercise 3.8 For (i), imitate the proof for Theorem 3.2.3(iii). (ii) is obtained by direct computation.

Exercise 3.9 By Lemma 3.2.2(i), $PA^dP^{-1} = \text{diag}(B_1, \cdots, B_d)$. By Corollary 3.2.3B, B_i is primitive. Therefore, $B_i^{m_i} > 0$ for some smallest integer $m_i > 0$. Let $m = \max_i\{m_i\}$. Then $B_i^m > 0$ and $B_i^{m+1} > 0$, and so $p(A) = d$, by definition.

Exercise 3.10 (i) follows from definition immediately. Assume that $p > 1$. Then by Corollary 3.2.3C, $p = d = d(D(A))$. Then argue by Theorem 3.2.3.

Exercise 3.11 It can be determined if A is primitive by directly computing the sequence $A, A^2, \cdots$. An alternative way is to apply Theorem 3.2.2. The digraph $D(A)$ has a 3-cycle and a 4-cycle, and so $d(D(A)) = 1$, and A is primitive. Do the same for $D(B)$ to see $d(D(B)) = 3$. Move Column 1 of A to the place between Column 3 and Column 4 of A to get B, and so $A \sim_p B$.

Exercise 3.12 Direct computation gives $\gamma(D_1) = \gamma(v_n, v_n)$ and $\gamma(D_2) = \gamma(v_1, v_n)$,

Exercise 3.13 Complete the proof of Theorem 3.3.2. The following take care of the unfinished cases.

If $k \in \{2, 3, \cdots, n\}$ and $k \leq d \leq n-1 \leq n$, then write $d = k + l$ for some integer l with $0 \leq l \leq n-k-1$. Consider the adjacency matrix of the digraph D in figure below.

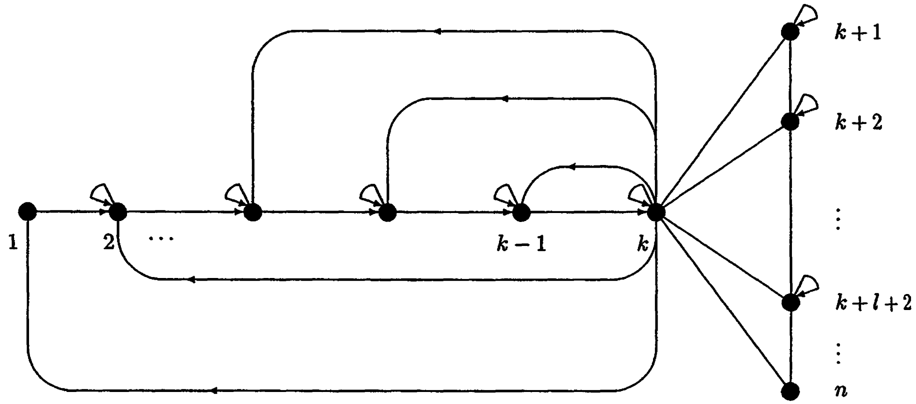

Figure: when $k \in \{2, 3, \cdots, n\}$ and $k \le d \le n-1 \le n$

Again, we have

$$\gamma(i,j) \begin{cases} = k & \text{if } i = j = 1, \\ \le k & \text{otherwise,} \end{cases}$$

and so $\gamma(A) = k$ in this case also.

Now assume that $k \in \{n+1, n+2, \cdots, 2n-d-1\}$. Note that we must have $d < n-1$ in this case. Write $k = 2n - l$ for some integer l with $d+1 \le l \le n-1$. Consider the adjacency matrix of the digraph D in the figure below

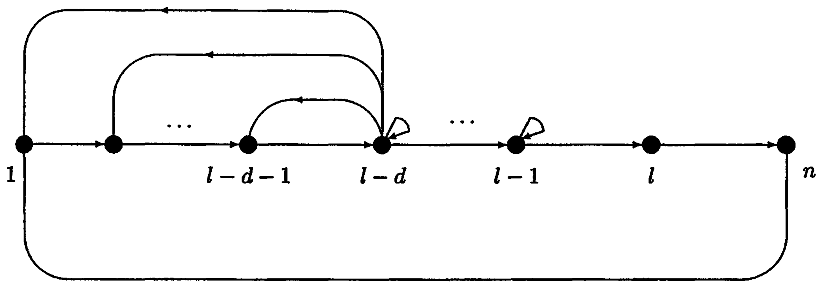

Figure: when $k \in \{n+1, n+2, \cdots, 2n-d-1\}$.

Note that in this case, we must have $d < n-1$. Thus

$$\gamma(i,j) \begin{cases} = 2n-l = k & \text{if } i = j \text{ and } j = n, \\ \le 2n-l = k & \text{otherwise,} \end{cases}$$

and so $\gamma(A) = k$, as desired.

Exercise 3.14 Definition 3.4.2.

Exercise 3.15 (i) follows from Definition 3.2.3.
(ii). Use $(BA)^{k+1} = B(AB)^k B$ and $UJV = J$.
(iii). Apply (ii).

Exercise 3.16 Let k be an integer such that $A_i(k) = J$, $\forall i = 1, 2, \cdots, p$. Then by Lemma 3.4.1, $A^k = (A_1(k), \cdots, A_p(k))_k$ and $A^{k+p} = (A_1(k+p), \cdots, A_p(k+p))_{k+p}$. Thus $A_i(k) = A_i(k+p)$, $\forall i$, and $k+p \equiv p \pmod p$, and so $A^k = A^{k+p}$. It follows that $k(A) \le k$.

Conversely, assume that for some j, $A_j(k-1) \ne J$. Note that $A^{k-1} = (A_1(k-1), \cdots, A_p(k-1))_{k-1}$ and $A^{k-1+p} = (A_1(k-1+p), \cdots, A_p(k-1+p))_{k-1+p}$. Since $A_j(k-1+p) = J \ne A_j(k-1)$, $A^{k-1} \ne A^{k-1+p}$, and so $k(A) > k-1$.

Exercise 3.17 Let $m = n_i$ for some i. Then $A_i(p) \in M_{m,m}$ is primitive, and so by Corollary 3.3.1A, $\gamma(A_i(p)) \le m^2 - 2m + 2$. Apply Lemma 3.4.4 with $t = 1$ and $i_1 = i$ to get the answer.

Exercise 3.18 (i). When A is irreducible, $n = n_0$ and $p = d(D)$. (ii). Theorem 3.3.1 follows from (i) with $p = 1$. (iii). When A is reducible, apply a decomposition.

Exercise 3.19 Argue by induction on k.

Exercise 3.20 By Definition (of primitive matrix), $\mu(A) = n^2$ if and only if A is primitive, and if and only if $p(A) = 1$. The inequality of $h(A)$ follows from the definitions of $p(A)$ and $k(A)$.

Exercise 3.21 Let $n > 0$ denote an integer, and let D be a primitive digraph with $V(D) = \{v_1, \cdots, v_n\}$ such that $exp_D(v_1) \le exp_D(v_2) \le \cdots \le exp_D(v_n)$. Show that

(i) $F(D,1) = exp_D(v_n) = \gamma(D)$ and $f(D,1) = exp_D(v_1)$.

(ii) $f(n,n) = 0$, $f(n,1) = exp(n,1)$, and $F(n,1) = exp(n)$.

Exercise 3.22 Let $w \in V(C_s)$ and $d^+(w) = r$. Let $V_1 = \{v \mid (w,v) \in E(D)\}$. Then $|V_1| = r$. Denote $V(C_s) \cap V_1 = \{w_1\}$. Then D has a directed path of length s from w_1 to a vertex in V_1. In D^s, there is at most one vertex, say x, which cannot be reached from loop w_1 by a walk of length $n-r$. Thus a path of length $n-r+1$ from w_1 to x must pass through some vertex z (say) of V_1, and so there is a path of length $n-r$ from z to x. It follows that there is a walk of length $s(n-r)+1$ from w to x in $D(A)$.

Exercise 3.23 Apply definitions.

Exercise 3.24 Apply Theorem 3.7.1.

Exercise 3.25 By Theorem 3.7.5, $f(n,n-1) \le 1$ and $f(n,1) \le n^2-3n+3$. By Theorem 3.7.1(ii), $f(D_n,1) = n^2-3n+3$.

Exercise 3.26 Let $t = \dfrac{s}{k}$ and let C_s denote a directed cycle of length s. Pick $X = \{x_1, x_2, \cdots, x_k\} \subseteq V(C_s)$ such that C_s has a directed (x_i, x_{i+1})-path of length t, where $x_j = x'_j$ whenever $j \equiv j' \pmod{k}$. Since D is primitive and since $n > s$, we may assume that $(x_1, z) \in E(D)$ for some $z \in V(D) \setminus V(C_s)$. Let Y be the set of vertices that can be reached from vertices in X by a directed path of length 1. Then $\{x_{i_1}, \cdots, x_{i_k}, z\} \subseteq Y$.

Construct $D^{(t)}$ as in the proof of Lemma 3.7.5. Note that $x_{i_1} \cdots x_{i_k} x_{i_1}$ is a directed cycle of $D^{(t)}$ of length k and $(x_1, z) \in E(D^{(t)})$. Thus any vertex in $D^{(t)}$ can be reached from a vertex in Y by a directed walk of length exactly $n-k-1$.

Exercise 3.27 First use Example 3.8.3 to show that $f_n^* \le k(n-k)$. As a quadratic function in k, $k(n-k)$ has a maximum when $k = n/2$. The other inequality of (ii) comes from Proposition 3.8.2 and Wielandt' Theorem (Corollary 3.3.1A).

Exercise 3.28 Apply Theorem 3.9.4 with $m_{A^k} \le m$ and $k \le m-2$.

Exercise 3.29 Since $\rho(A) > 0$ and $\mathrm{tr}(A) > 0$, $D(A)$ must have a directed cycle of length $2 \le m-2$.

Exercise 3.30 Apply Theorem 3.9.5 and then Exercise 3.42.

Exercise 3.31 In either case, A has at most $m-2$ distinct eigenvalues. Apply Corollary 3.9.5A.

Exercise 3.32 Suppose that $A = X_{n\times b}Y_{b\times n}$. By Lemma 3.4.2, $\gamma(A) \leq \gamma(XY) + 1 \leq (b-1)^2 + 2$.

Exercise 3.33 Apply Lemma 3.7.3 and Exercise 3.22.

Chapter 4

Matrices in Combinatorial Problems

4.1 Matrix Solutions for Difference Equations

Consider the *difference equation* (also called *recurrence relation*) with given boundary conditions

$$u_{n+k} = a_1u_{n+k-1} + a_2u_{n+k-2} + \cdots + a_ku_n + b_n \tag{4.1}$$

$$u_i = c_l,\ 0 \le l \le k-1, \tag{4.2}$$

where the constants $a_1, \cdots, a_k, c_0, \cdots, c_{k-1}$ and the sequence $\langle b_n \rangle$ are given. A solution to this equation is a sequence $\langle u_n \rangle$ satisfying (4.1) and (4.2). If $b_n = 0$, for all n, then the resulting equation is the *corresponding homogeneous equation* to (4.1).

Definition 4.1.1 The equation

$$\lambda^k - a_1\lambda^{k-1} - a_2\lambda^{k-2} - \cdots - a_k = 0 \tag{4.3}$$

is called the *characteristic equation* of the difference equation in (4.1), and the matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ & \vdots & \vdots & & \vdots & \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ a_k & a_{k-1} & a_{k-2} & \cdots & a_2 & a_1 \end{bmatrix} \tag{4.4}$$

is called the *companian matrix* of equation (4.3). Note that by Hamilton-Cayley Theorem,

$$A^k - a_1A^{k-1} - a_2A^{k-2} - \cdots - a_kI = 0$$

A usual way to solve (4.1) and (4.2) is to solve the characteristic equation of the difference equation, to obtained the *homogeneous solution*, which satisfies the difference equation (4.1) when the constant b_n on the right hand side of the equation is set to 0, and the *particular solution*, which satisfies the difference equation with b_n on the right hand side.

The homogeneous solution is usually obtained by solving the characteristic equation (4.3). However, when k is large, (4.3) is difficult to solve. The purpose of this section is to introduce an alternative way of solving (4.1), via matrix techniques.

Theorem 4.1.1 (Liu, [169]) Let A be the companian matrix in (4.4), and let

$$\begin{aligned} C &= (c_0, c_1, \cdots, c_{k-1})^T \\ B_j &= (0, 0, \cdots, 0, b_j)^T,\ j = 0, 1, 2, \cdots \end{aligned}$$

and let

$$A^m C + A^{m-1}B_0 + A^{m-2}B_1 + \cdots + A^{k-1}B_{m-k} = \left(a^{(m)}, \cdots\right)^T. \tag{4.5}$$

Then $\langle a^{(m)}\rangle$ is a solution to (4.1).

Proof By (4.4),

$$\begin{aligned} A^m C &= \sum_{i=1}^{k} a_i A^{m-i} C, \text{ and} \\ A^{m-j-1}B_j &= \sum_{i=1}^{k} a_i A^{m-j-1} B_j,\ j = 1, 2, \cdots. \end{aligned}$$

Thus by (4.5),

$$\begin{aligned} & \left(a^{(n+k)}, \cdots\right)^T \\ = \ & A^{n+k}C + A^{m+k-1}B_0 + A^{m+k-2}B_1 + \cdots + A^{k-1}B_n \\ = \ & \sum_{i=1}^{k} a_i A^{n+k-i}C + \sum_{i=1}^{k} a_i A^{n+k-1-i}B_0 + \cdots + \sum_{i=1}^{k} a_i A^{k-i}B_{n-1} + A^{k-1}B_n \\ = \ & a_1\left(A^{n+k-1} + \sum_{i=1}^{n} A^{n+k-1-i}B_{i-1}\right) + a_2\left(A^{n+k-2}C + \sum_{i=1}^{n-1} A^{n+k-2-i}B_{i-1}\right) \\ & + \sum_{i=2}^{k} a_i A^{k-i}B_{n-1} + a_3\left(A^{n+k-3}C + \sum_{i=1}^{n-2} A^{n+k-3-i}B_{i-1}\right) \\ & + \sum_{i=3}^{k} a_i A^{k+1-i}B_{n-2} + \cdots + a_k\left(A^n C + \sum_{i=1}^{n-k+1} A^{n-i}B_{i-1}\right) \\ & + a_k A^{k-2}B_{n-k+1} + A^{k-1}B_n. \end{aligned}$$

Note that

$$\begin{aligned}
A^i &= \begin{bmatrix} 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ & \vdots & & & & \vdots & \end{bmatrix}, \text{ for } i = 0, 1, 2, \cdots, k-1. \\
A^i B_j &= (0, \cdots)^T, \ 0 \le i \le k-2, \text{ and any } j, \\
A^{k-1} B_n &= (b_n, \cdots)^T.
\end{aligned}$$

It follows that

$$\begin{aligned}
\left(a^{(n+k)}, \cdots\right)^T &= a_1 \left(a^{(n+k-1)}, \cdots\right)^T + a_2 \left(a^{(n+k-2)}, \cdots\right)^T + (0, \cdots)^T \\
&\quad + a_3 \left(a^{(n+k-3)}, \cdots\right)^T + (0, \cdots)^T \\
&\quad + \cdots \\
&\quad + a_k \left(a^{(n)}, \cdots\right)^T + (0, \cdots)^T + (b_n, \cdots)^T.
\end{aligned}$$

Therefore, $a^{(n+k)} = \sum_{i=1}^{k} a_i a^{(n+k-i)} + b_n$, for each $n \ge 0$, and so (4.1) is satisfied.

When $0 \le n \le k-1$, by (4.5) again, we have

$$\begin{aligned}
\left(a^{(0)}, \cdots\right)^T &= A^0 C = (c_0, \cdots)^T \\
\left(a^{(1)}, \cdots\right)^T &= AC = (c_1, \cdots)^T \\
&\cdots \\
\left(a^{(k-1)}, \cdots\right)^T &= A^{k-1} C = (c_{k-1}, \cdots)^T
\end{aligned}$$

and so (4.2) is also satisfied. □

We shall find a combinatorial expression of the $a^{(m)}$'s. Let $A^m = (a_{i,j}^{(m)})$. Then by (4.5), we have, for $m \ge k$, that

$$a^{(m)} = \sum_{i=0}^{k-1} c_i a_{1,i+1}^{(m)} + \sum_{i=0}^{m-k} b_i a_{1,k}^{(m-i-1)}. \tag{4.6}$$

Definition 4.1.2 For a matrix $A = (a_{ij}) \in M_n$, define a weighted digraph D (called the *weighted associate digraph of* A) as follows. Let $V(D) = \{1, 2, \cdots, n\}$. For $i, j \in V(D)$, an arc $(i, j) \in E(D)$ with weight $w(i, j) = a_{ij}$ exists if and only if $a_{ij} \ne 0$. If $T = v_1 \to v_2 \to \cdots \to v_k$ is a directed walk of D, then the weight of T is $w(T) = \prod_{i=1}^{k-1} w(v_i, v_{i+1})$. Therefore if $A^m = (a_{i,j}^{(m)})$, then $a_{ij}^{(m)}$ is the sum of the weights of all directed (i, j)-walks of length m (the weighted version of Proposition 1.1.2(vii)).

Lemma 4.1.1 $a_{ij}^{(m)} = a_{jj}^{(m+1-j)}$.

Proof Note that for $1 \le m \le k-1$,

$$a_{ij}^{(m)} = a_{jj}^{(m+1-j)} = \begin{cases} 1 & \text{if } m = j-1 \\ 0 & \text{if } j \le m \le k-1. \end{cases}$$

Now assume $m \ge k$. Note that any directed $(1, j)$-walk of length m must be the form

$$1 \to 2 \to \cdots \to j \to \cdots \to k \to \cdots \to j,$$

which yields a one-to-one correspondence between all directed (i, j)-walks of length m and the directed (j, j)-walks of length $m - j + 1$.

Since $w(i, i+1) = a_{i,i+1} = 1$, for all $1 \le i \le j-1$, we have $a_{ij}^{(m)} = a_{jj}^{(m+1-j)}$. □

Lemma 4.1.2 Define $f^{(t)} = 0$ for each $t < 0$, $f^{(0)} = 1$ and

$$f^{(m)} = \sum_{\substack{s_1 + 2s_2 + \cdots + ks_k = n \\ s_i \ge 0,\ (i = 1, 2, \cdots, k)}} \binom{s_1 + s_2 + \cdots + s_k}{s_1, s_2, \cdots, s_k} a_1^{s_1} a_2^{s_2} \cdots a_k^{s_k}.$$

Then for $j = 1, 2, \cdots, k$,

$$a_{jj}^{(m)} = \sum_{i=1}^{j} a_{k-i+1} f^{(m-k+i-1)} \tag{4.7}$$

Proof By Definition 4.1.2, D has these directed (k, k)-walks:

Type	Walk	Length	Weight
C_1	$k \to k$	1	a_1
C_2	$k \to k-1 \to k$	2	a_2
C_3	$k \to k-2 \to k-1 \to k$	3	a_3
$\vdots$	$\vdots$	$\vdots$	$\vdots$
C_k	$k \to 1 \to 2 \to \cdots \to k$	k	a_k

Therefore, any directed (k, k)-walk of length m must have s_1 of Type C_1, s_2 of Type $C_2, \cdots, s_k$ of Type C_k.

For any j with $1 \le j \le k-1$, D has these directed (j, j)-walks:

Type	Walk
C_1'	$j \to \cdots \to k \cdots \to k \to 1 \to 2 \to \cdots \to j$
C_2'	$j \to \cdots \to k \cdots \to k \to 2 \to 3 \to \cdots \to j$
C_3'	$j \to \cdots \to k \cdots \to k \to 3 \to 4 \to \cdots \to j$
$\vdots$	$\vdots$
C_j'	$j \to \cdots \to k \cdots \to k \to j$

For each i with $1 \le i \le j$, the first directed (j,k)-walk of length $k-j$ and the last directed (k,j)-walk of length $j-i+1$ of C_i' form a directed closed walk of length $k-i+1$. Thus, for each j with $1 \le j \le k$,

$$a_{jj}^{(m)}$$
$$= \sum_{j=1}^{j} \sum_{\substack{s_1+2s_2+\cdots+ks_k=m \\ s_i \ge 0,\, t \ne k-i+1 \\ s_t > 0,\, t=k-i+1}} \begin{pmatrix} s_1+s_2+\cdots+(s_{k-i+1}-1)+\cdots+s_k \\ s_1,s_2,\cdots,(s_{k-i+1}-1),\cdots,s_k \end{pmatrix} a_1^{s_1}a_2^{s_2}\cdots a_k^{s_k}$$
$$= \sum_{j=1}^{j} a_{k-i+1} \sum_{\substack{s_1+2s_2+\cdots+ks_k=m-k+i-1 \\ s_i \ge 0,\,(t=1,2,\cdots,k)}} \begin{pmatrix} s_1+s_2+\cdots+s_k \\ s_1,s_2,\cdots,s_k \end{pmatrix} a_1^{s_1}a_2^{s_2}\cdots a_k^{s_k}$$

Therefore the lemma follows by the definition of $f^{(m)}$. □

Theorem 4.1.2 (Liu, [169]) The solution for (4.1) and (4.2) is

$$u_m = a_{k-1}f^{(m-k+1)} + \sum_{j=1}^{k-1} c_{j-1}\sum_{i=1}^{j} a_{k-i+1}f^{(m-k-j+i)} + \sum_{j=1}^{m-k+1} b_{j-1}f^{(m-k-j+1)}$$

Proof This follows from Theorem 4.1.1, (4.6) and (4.7). □

Corollary 4.1.2A Another way to express u_m is

$$u_m = \sum_{j=1}^{k} c_{j-1}\sum_{i=1}^{j} a_{k-i+1}f^{(m-k-j+i)} + \sum_{j=1}^{m-k+1} b_{j-1}f^{(m-k-j+1)}$$

Corollary 4.1.2B (Tu, [261]) Let k and r be integers with $1 \le r \le k-1$. The difference equation

$$\begin{cases} u_{n+k} = au_{n+r} + bu_n + b_n \\ u_0 = c_0, u_1 = c_1, \cdots, u_{k-1} = c_{k-1} \end{cases}$$

has solutions

$$u_m = \sum_{j=0}^{r-1} c_j b f^{(m-k-j)} + \sum_{j=r}^{k-1} c_j f^{(m-j)} + \sum_{j=1}^{m-k+1} b_{j-1}f^{(m-k-j+1)}.$$

where

$$f^{(m)} = \sum_{\substack{kx+(k-r)y=m \\ x,y\ge 0}} \begin{pmatrix} x+y \\ y \end{pmatrix} b^x a^y,\ (m \ge 0)$$

Proof Let $a_k = b$, $a_{k-r} = a$ and all other $a_i = 0$. Then Corollary 4.1.2B follows from Theorem 4.1.2. □

Corollary 4.1.2C Letting $a = b = 1$, $b_n = 0$ $r = 1$, and $c_0 = c_1 = 1$ in Corollary 4.1.2B, we obtain the Fibonacci sequence

$$\begin{aligned} u_m &= c_0 f^{(m-2)} + c_1 f^{(m-1)} = f^{(m-2)} + f^{(m-1)} \\ &= f^{(m)} = \sum_{\substack{2x+y=m \\ x,y \geq 0}} \binom{x+y}{y} = \sum_{k=0}^{\lfloor m/2 \rfloor} \binom{m-k}{k}. \end{aligned}$$

Example 4.1.1 Solve the difference equation

$$\begin{cases} F_{n+5} = 2F_{n+4} + 3F_n + (2n-1) \\ F_0 = 1, F_1 = 0, F_2 = 1, F_3 = 2, F_4 = 3. \end{cases}$$

In this case,

$$\begin{aligned} & k = 5, r = 4, a = 2, b = 3, b_n = 2n - 1 \\ & c_0 = 1, c_1 = 0, c_2 = 1, c_3 = 2, c_4 = 3. \end{aligned}$$

and so

$$\begin{aligned} F_n &= 3 \sum_{x=0}^{\lfloor (n-5)/5 \rfloor} \binom{n-4x-5}{x} 3^x 2^{n-5x-5} \\ &+3 \sum_{x=0}^{\lfloor (n-7)/5 \rfloor} \binom{n-4x-7}{x} 3^x 2^{n-5x-7} \\ &+6 \sum_{x=0}^{\lfloor (n-8)/5 \rfloor} \binom{n-4x-8}{x} 3^x 2^{n-5x-8} \\ &+3 \sum_{x=0}^{\lfloor (n-4)/5 \rfloor} \binom{n-4x-4}{x} 3^x 2^{n-5x-4} \\ &+ \sum_{j=1}^{n-4} (2j-3) \sum_{x=0}^{\lfloor (n-4-j)/5 \rfloor} \binom{n-4x-4-j}{x} 3^x 2^{n-5x-4-j}. \end{aligned}$$

4.2 Matrices in Some Combinatorial Configurations

Incidence matrix is a very useful tool in the study of some combinatorial configurations. In this section, we describe how incidence matrices can be applied to investigate the properties of system of distinct representatives, of bipartite graph coverings, and of certain incomplete block designs.

Definition 4.2.1 Let $X = \{x_1, x_2, \cdots, x_n\}$ be a set and let $\mathcal{A} = \{X_1, X_2, \cdots, X_m\}$ denote a family of subsets of X. (Members in a family may not be distinct.)

The *incidence matrix* of $\mathcal{A}$ is an matrix $A = (a_{ij}) \in \mathbf{B}_{m,n}$ satisfying:

$$a_{ij} = \begin{cases} 1 & \text{if } x_j \in X_i \\ 0 & \text{if } x_j \notin X_i, \end{cases}$$

where $1 \le i \le m$ and $1 \le j \le n$.

Example 4.2.1 The incidence of elements of X in members of $\mathcal{A}$ can also be represented by a bipartite graph G with vertex partite sets $X = \{x_1, x_2, \cdots, x_n\}$ and $Y = \{y_1, y_2, \cdots, y_m\}$ such that $x_i y_j \in E(G)$ if and only if $x_i \in X_j$, for each $1 \le i \le n$ and $1 \le j \le m$. Let A be the incidence matrix of $\mathcal{A}$. Note that a set of k mutually independent entries (entries that are not lying in the same row or same column, see Section 6.2 in the Appendix) of A corresponds to k edges in $E(G)$ that are mutually disjoint (called a *matching* in graph theory),

In a graph H, a vertex and an edge are said to *cover* each other if they are incident. A set of vertices cover all the edges of H is called a *vertex cover* of H. Since a line in the incidence matrix A of $\mathcal{A}$ corresponds to either an element in X or a member in $\mathcal{A}$, either of which is a vertex in G. Therefore, Theorem 6.2.2 in the Appendix says that in a bipartite graph, the number of edges in a maximum matching is equal to the number of vertices in a minimum vertex cover.

Definition 4.2.2 A family of elements $(x_i \ : \ i \in I)$ in S is a *system of representatives* (SR) of $\mathcal{A}$ if $x_i \in A_i$, for each $i \in I$. An SR $(x_i \ : \ i \in I)$ is a *system of distinct representatives* (SDR) of $\mathcal{A}$ if for each $i, j \in I$, if $i \ne j$, then $x_i \ne x_j$.

Example 4.2.2 Let $X = \{1, 2, 3, 4, 5\}$, $X_1 = X_2 = \{1, 2, 4\}$, $X_3 = \{2, 3, 5\}$ and $X_4 = \{1, 2, 4, 5\}$. Then both $D_1 = \{1, 2, 3, 4\}$ and $D_2 = \{4, 2, 5, 1\}$ are SDRs for the same family X.

However, for the same ground set X, if we redefine $X_1 = \{1, 2\}, X_2 = \{2, 4\}, X_3 = \{1, 2, 4\}$ and $X_4 = \{1, 4\}$. Then this family $\{X_1, X_2, X_3, X_4\}$ does not have an SDR.

Example 4.2.3 Let A be the incidence matrix of $\mathcal{A}$. By Definitions 4.2.1 and 4.2.2, A set of k mutually independent entries of A corresponds to a subset of k distinct elements in X such that for some k members $X_{i_1}, X_{i_2}, \cdots, X_{i_k}$ of $\mathcal{A}$, we have $x_j \in X_{i_j}$, for each $1 \le j \le k$ (called a *partial transversal* of $\mathcal{A}$). Thus a partial transversal of $|\mathcal{A}|$ elements is just an SDR of $\mathcal{A}$.

Several major results concerning transversals are given below. Proposition 4.2.1 is straightforward, while the proofs for Theorems 4.2.1, 4.2.2 and 4.2.3 can be found in [113].

Proposition 4.2.1 Let $X = \{x_1, x_2, \cdots, x_n\}$ be a set and let $\mathcal{A} = \{X_1, X_2, \cdots, X_m\}$ denote a family of subsets of X. Let $A \in \mathbf{B}_{m,n}$ denote the incidence matrix of $\mathcal{A}$. Each of the following holds.

(i) The family $\mathcal{A}$ has an SDR if and only if ρ_A, the term rank of A, is equal to m.
(ii) The number of SDRs of $\mathcal{A}$ is equal to per(A).

Theorem 4.2.1 (P. Hall) A family $A = \{X_i \mid i \in I\}$ has an SDR if and only if for each subset $J \subseteq I$, $|\cup_{j\in J} X_j| \geq |J|$.

Given a family $X = \{X_1, X_2, \cdots, X_m\}$, $N(X) = N(X_1, \cdots, X_m)$ denotes the number of SDRs of the family.

Theorem 4.2.2 (M. Hall) Suppose the family $X = \{X_1, X_2, \cdots, X_m\}$ has an SDR. If for each i, $|X_i| \geq k$, then

$$N(X) \geq \begin{cases} k! & \text{if } k \leq m \\ \frac{k!}{(k-m)!} & \text{if } k > m. \end{cases}$$

Van Lint obtains a better lower bound in [162]. For integers $m > 0$ and $n_1, \cdots, n_m$, let

$$F_m(n_1, n_2, \cdots, n_m) = \Pi_{i=0}^{m-1} \max\{n_{i+1} - i, 1\}.$$

Theorem 4.2.3 (Van Lint, [162]) Suppose the family $X = \{X_1, X_2, \cdots, X_m\}$ has an SDR. If for each i, $|X_i| \geq n_i$, then

$$N(X) \geq F_m(n_1, n_2, \cdots, n_m).$$

Definition 4.2.3 Let S be a set. A partition of S is a collection of subsets $\{A_1, A_2, \cdots, A_m\}$ such that
(i) $S = \cup_{i=1}^m A_i$, and
(ii) $A_i \cap A_j = \emptyset$, whenever $i \neq j$.
Suppose that S has two partitions $\{A_1, A_2, \cdots, A_m\}$ and $\{B_1, B_2, \cdots, B_m\}$. A subset $E \subseteq S$ is a *system of common representatives* (abbreviated as SCR) if for each $i, j \in \{1, 2, \cdots, m\}$,

$$E \cap A_i \neq \emptyset \text{ and } E \cap \cap B_j \neq \emptyset.$$

Theorem 4.2.4 Suppose that S has two partitions $\{A_1, A_2, \cdots, A_m\}$ and $\{B_1, B_2, \cdots, B_m\}$. Then these two partitions have an SCR if and only if for any integer k with $1 \leq k \leq m$, and for any k subsets $A_{i_1}, A_{i_2}, \cdots, A_{i_k}$, the union $\cup_{j=1}^k A_{i_j}$ contains at most k distinct members in $\{B_1, B_2, \cdots, B_m\}$.

Interested readers can find the proofs for Theorems 4.2.1-4.2.4 in [222] and [162].

Definition 4.2.4 For integers $k, t, \lambda \geq 0$, a family $\{X_1, X_2, \cdots, X_b\}$ of subsets (called the *blocks* of a ground set $X = \{x_1, x_2, \cdots, x_v\}$ is a *t-design*, denoted by $S_\lambda(t, k, v)$, if

(i) $|X_i| = k$, and

(ii) for each t element subset T of X, there are exactly λ blocks that contain T.

An $S_\lambda(2, k, v)$ is also called a *balanced incomplete block design* (abbreviated BIBD, or a (v, k, λ)-BIBD) . A BIBD with $b = v$ is a *symmetric balanced incomplete block design* (abbreviated SBIBD, or a (v, k, λ)-SBIBD). An $S_1(2, k, v)$ is a *Steiner* system.

Example 4.2.4 The incidence matrix of an $S_1(2, 3, 7)$ (a Steiner triple system):

$$A = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Proposition 4.2.2 The five parameters of a BIBD are not independent. They are related by these equalities:

(i) $rv = bk$.

(ii) $\lambda(v-1) = r(k-1)$.

Theorem 4.2.5 Let $A = (a_{ij}) \in \mathbf{B}_{b,v}$ denote the incidence matrix of a BIBD $S_\lambda(2, k, v)$. If $v > k$, then

(i) $A^T A$ is nonsingular, and

$$(A^T A)^{-1} = ((r-\lambda)I_v + \lambda J_v)^{-1} = \frac{1}{r-\lambda}\left(I_v - \frac{\lambda}{rk} J_v\right).$$

(ii) (Fisher inequality) $b \geq v$.

Proof By the definition of an $S_\lambda(2, k, v)$, we have the following:

$$\begin{aligned} r(k-1) &= \lambda(v-1) \\ AJ_{v,b} &= kJ_b \\ J_{v,b}A &= rJ_v \\ A^T A &= (r-\lambda)I_v + \lambda J_v. \end{aligned}$$

If $v > k$, then $r > \lambda$, and so the matrix $(r-\lambda)I_v + \lambda J_v$ has eigenvalues $r + (v+1)\lambda$ and $\lambda - r$ (with multiplicity $v - 1$). Since all v eigenvalues of $A^T A$ are nonzero, $A^T A$ is nonsingular, and the rank of A is v. By direct computation,

$$((r-\lambda)I_v + \lambda J_v)^{-1} = \frac{1}{r-\lambda}\left(I_v - \frac{\lambda}{rk} J_v\right).$$

Note that $A \in \mathbf{B}_{b,v}$ and the rank of A is v. Fisher inequality $b \geq v$ now follows. $\square$

Theorem 4.2.6 (Bruck-Ryser-Chowla, [222]) If a (v, k, λ)-SBIBD exists, and if v is odd, then the equation

$$x^2 = (k - \lambda)y^2 + (-1)^{\frac{v-1}{2}} \lambda z^2$$

has a solution in x, y, and z, not all zero.

We omit the proof of Theorem 4.2.6, which can be found in [222]. In the following, we shall apply linear algebra and matrix techniques to derive the Cannor inequalities.

From linear algebra, it is well known that the matrices $P = A(A^TA)^{-1}A^T$ and $Q = I_b - P$, representing the projections to the column space of A and to is orthogonal complement, respectively, are symmetric and semipositive definite. Therefore, by Theorem 4.2.5 and by the definition of an $S_\lambda(2, k, v)$,

$$\begin{aligned}
P &= A(A^TA)^{-1}A^T = A((r-\lambda)I_v + \lambda J_v)^{-1}A^T \\
&= \frac{1}{r-\lambda}\left(AA^T - \frac{\lambda}{rk}AJ_vA^T\right) \\
&= \frac{1}{r-\lambda}\left(AA^T - \frac{\lambda}{rk}AkJ_{v,b}\right) \\
&= \frac{1}{r-\lambda}\left(AA^T - \frac{\lambda}{rk}k^2J_b\right) \\
&= \frac{1}{r-\lambda}\left(AA^T - \frac{\lambda k}{r}J_b\right)
\end{aligned}$$

and

$$Q = I_b - P = I_b - \frac{1}{r-\lambda}\left(AA^T - \frac{\lambda k}{r}J_b\right).$$

By the definition of an incidence matrix, any $m \times m$ principal submatrix of Q can be obtained as follows: Pick a subfamily $\mathcal{A}' = \{X_1', X_2', \cdots, X_m'\}$ of $\mathcal{A}$. Let μ_{ij} denote the common elements in both X_i' and in X_j', $(1 \leq i \leq m,\ 1 \leq j \leq m)$, and let $U = (\mu_{ij})$. Then

$$Q_m = I_m - \frac{1}{r-\lambda}\left(U - \frac{\lambda k}{r}J_b\right)$$

is an $m \times m$ principal submatrix.

Since Q is semidefinite positive, $\det(Q_m) \geq 0$. Hence we have the following.

Theorem 4.2.7 (Cannor inequalities) With the notations above, if Q_m is a principal submatrix of $Q = I_b - P$, then $\det(Q_m) \geq 0$.

Note that there are $2^b - 1$ such inequalities. When $m = 1$, this yields Fisher inequality. When $m = 2$, Cannor inequalities say that the number μ of common elements in two blocks must satisfy

$$(r-k)(r-\lambda) \geq |\lambda k - r\mu|.$$

Readers interested in design theory are referred to [76].

4.3 Decomposition of Graphs

Definition 4.3.1 Let G be a loopless graph. A *bipartite decomposition* of G is a collection edge-disjoint complete bipartite subgraphs $\{G_1, G_2, \cdots, G_r\}$ of G such that $E(G) = \cup_{i+1}^{r} E(G_i)$.

Since a graph with a loop cannot have a bipartite decomposition, we assume all graphs in this section are loopless.

Example 4.3.1 Let H be a graph and let $v \in V(H)$. Denote $E_H(v)$ the edges in H incident with v. For a complete graph $G = K_n$ with $V(G) = \{v_1, v_2, \cdots, v_n\}$, for $1 \leq i \leq n-1$, let

$$G_i = (G - \{v_1, \cdots, v_{i-1}\})[E_{G-\{v_1,\cdots,v_{i-1}\}}(v_i)].$$

(Thus G_i is the subgraph of $G - \{v_1, \cdots, v_{i-1}\}$ induced by the edges incident with v in $G - \{v_1, \cdots, v_{i-1}\}$.) Then $\{G_1, G_2, \cdots, G_{n-1}\}$ is a bipartite decomposition of K_n.

What is the smallest number r such that K_n has a bipartite decomposition of r subgraphs? This was first answered by Graham and Pollak. Different proofs were later given by Tverberg and by Peck.

Theorem 4.3.1 (Graham and Pollak, [105], Tverberg, [263], and Peck, [210]) If $\{G_1, G_2, \cdots, G_r\}$ is a bipartite decomposition of K_n, then $r \geq n-1$.

Theorem 4.3.2 (Graham and Pollak, [105]) Let G be a multigraph with n vertices with a bipartite decomposition $\{G_1, G_2, \cdots, G_r\}$. Let $A = A(G) = (a_{ij})$ be the adjacency matrix of G, and let n_+, n_- denote the number of positive eigenvalues and the number of negative eigenvalues of A, respectively. Then $r \geq \max\{n_+, n_-\}$.

Proof A complete bipartite subgraph G_i of G with vertex partite sets X_i and Y_i can be obtained by selecting two disjoint nonempty subsets X_i and Y_i from $V(G)$. Therefore we write (X_i, Y_i) for G_i, $1 \leq i \leq r$.

Let $z_1, \cdots, z_n$ be n unknowns, let $\mathbf{z} = (z_1, z_2, \cdots, z_n)^T$, and let

$$q(\mathbf{z}) = \mathbf{z}^T A \mathbf{z} = 2 \sum_{1 \leq i < j \leq n} a_{ij} z_i z_j.$$

For each (X_i, Y_i),

$$q_i(\mathbf{z}) = \left(\sum_{a_k \in X_i} z_k - \sum_{a_l \in Y_i} z_l \right).$$

Since $\{G_1, G_2, \cdots, G_r\}$ is a bipartite decomposition of G,

$$q(\mathbf{z}) = \mathbf{z}^T A\mathbf{z} = 2\sum_{i=1}^{r} q_i(\mathbf{z}),$$

By the identity $4ab = (a+b)^2 - (a-b)^2$,

$$q(\mathbf{z}) = \frac{1}{2}\left(\sum_{i=1}^{r} l_i'(\mathbf{z})^2\right)\left(\sum_{i=1}^{r} l_i''(\mathbf{z})^2\right),$$

where $l_i'(\mathbf{z})$ and $l_i''(\mathbf{z})$ are linear combinations of $z_1, z_2, \cdots, z_n$.

Note that $l_1', l_2', \cdots, l_r'$ will take zero values over a vector subspace W of dimension at least $n - r$. Therefore, $q(\mathbf{z})$ is semidefinite negative over W.

Let E^+ denote the vector subspace spanned by the eigenvectors of A corresponding to the positive eigenvectors. Then E^+ has dimension n_+ and $q(\mathbf{z})$ is positive definite over E^+. It follows that $(n-r) + n_+ \le n$, and so $r \ge n_+$. Similarly, $r \ge n_-$. □

Note that $A(K_n) = J_n - I_n$ has $n-1$ negative eigenvalues. Thus Theorem 4.3.2 implies Theorem 4.3.1.

Definition 4.3.2 A bipartite decomposition $\{G_1, G_2, \cdots, G_r\}$ of a graph G may be viewed as an edge coloring of $E(G)$ with r colors, such that the edges colored with color i induce the complete bipartite subgraph G_i. A subgraph H of G is *multi-colored* if no two edges of H received the same color.

Theorem 4.3.2 has been extended by Alon, Brualdi and Shader in [3]. We refer the interested readers to [3] for a proof.

Theorem 4.3.3 (Alon, Brualdi and Shader, [3]) Let G be a graph with n vertices such that $A = A(G)$ has n_+ positive eigenvalues and n_- negative eigenvalues. Then in any bipartite decomposition $\{G_1, G_2, \cdots, G_r\}$, there is a multi-colored forest with at least $\max\{n_+, n_-\}$ edges. In particular, any bipartite decomposition of K_n contains a multi-colored tree.

Theorem 4.3.4 (Caen and Gregory, [44]) Let $n \ge 2$ and let $K_{n,n}^*$ denote the graph obtained from $K_{n,n}$ by deleting edges in a perfect matching.

(i) If $K_{n,n}^*$ has a bipartite decomposition $\{G_1, \cdots, G_r\}$, then $r \ge n$.

(ii) If $r = n$, then there exist integers $p > 0$ and $q > 0$ such that $pq = n-1$ and each G_i is isomorphic to $K_{p,q}$.

Proof Let $\{X, Y\}$ denote the bipartition of $V(G_{n,n}^*)$, where $X = \{x_1, x_2, \cdots, x_n\}$ and $Y = \{y_1, y_2, \cdots, y_n\}$. For each i with $1 \le i \le r$, G_i has bipartition $\{X_i, Y_i\}$. Let G_i' be the subgraph induced by the edge set $E(G_i)$, and let $A_i = A(G_i')$ be the adjacency matrix of G_i'. Note that

$$J - I = A_1 + A_2, + \cdots + A_r. \tag{4.8}$$

For each i with $1 \le i \le r$, let

$$\mathbf{x}_i = (x_1^{(i)}, x_2^{(i)}, \cdots, x_n^{(i)})^T \text{ and } \mathbf{y}_i = (y_1^{(i)}, y_2^{(i)}, \cdots, y_n^{(i)})^T$$

such that for $k = 1, 2, \cdots, n$,

$$x_k^{(i)} = \begin{cases} 1 & \text{if } x_k \in X_i \\ 0 & \text{otherwise} \end{cases} \text{ and } y_k^{(i)} = \begin{cases} 1 & \text{if } y_k \in Y_i \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, $A_i = \mathbf{x}_i\mathbf{y}_i^T$, $1 \le i \le r$. Let $A_X = (\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_n) \in \mathbf{B}_{n,r}$ and $A_Y = (\mathbf{y}_1, \mathbf{y}_2, \cdots, \mathbf{y}_n)^T \in \mathbf{B}_{r,n}$. Then by (4.8),

$$J - I = A_X A_Y. \tag{4.9}$$

By (4.9), $n = \operatorname{rank}(J - I) \le r$, and so Theorem 4.3.4(i) follows.

By (4.9), for each i with $1 \le i \le r$, $\mathbf{y}_i^T\mathbf{x}_i = 0$. For integers i, j with $1 \le i, j \le n$, define $U \in \mathbf{B}_{n,n-1}$ to be the matrix obtained from A_X by deleting Column i and Column j from A_X and by adding an all 1 column as the first column of U; and define $V \in \mathbf{B}_{n-1,n}$ to be the matrix obtained from A_Y by deleting Row i and Row j from A_Y and by adding an all 1 row as the first row of V. Then

$$UV = I_n + \mathbf{x}_i\mathbf{y}_i^T + \mathbf{x}_j\mathbf{y}_j^T$$

is a singular matrix. Let

$$U_1 = (\mathbf{x}_i, \mathbf{x}_j) \text{ and } V_1 = \begin{pmatrix} \mathbf{y}_i^T \\ \mathbf{y}_j^T \end{pmatrix}.$$

It follows that

$$\begin{aligned} 0 &= \det(UV) = \det(I_n + U_1V_1) = \det(I_2 + V_1U_1) \\ &= 1 - (\mathbf{y}_i^T\mathbf{x}_j)(\mathbf{y}_j^T\mathbf{x}_i), \end{aligned}$$

and so for each i, j,

$$\mathbf{y}_i^T\mathbf{x}_j = \mathbf{y}_j^T\mathbf{x}_i = 1.$$

If $r = n$, then by (4.9), we must have

$$A_Y A_X = J - I, \tag{4.10}$$

and so by (4.9) and (4.10),

$$A_X J = J A_X, \text{ and } A_Y J = J A_Y,$$

and so all the rows of A_X have the same row sum and all the columns of A_X have the same column sum and so there exists an integer $p \geq 0$ such that $A_X J = JA_X = pJ$. Similarly, there exists an integer $q \geq 0$ such that $A_Y J = JA_Y = qJ$. It follows that

$$(n-1)J = (J-I)J = (A_X A_Y)J = A_X(A_Y J) = A_X(qJ) = (pq)J,$$

and so Theorem 4.3.3(ii) obtains. $\square$

Definition 4.3.3 Let $m_1, m_2, \cdots, m_t$ be positive integers, and let G be a graph. A *complete* $(m_1, m_2, \cdots, m_t)$*-decomposition* of G is a collection of edge-disjoint subgraphs $\{G_1, \cdots, G_t\}$ such that $E(G) = \cup_{i=1}^t E(G_i)$ and such that each G_i is a complete m_i-partite graph.

Another extension of Theorem 4.3.1 is the next theorem, which is first proposed by Hsu [134].

Theorem 4.3.5 (Liu, [173]) If K_n has a complete $(m_1, m_2, \cdots, m_t)$-decomposition, then

$$n \leq \sum_{i=1}^{t}(m_i - 1) + 1.$$

Proof Suppose that $\{G_1, \cdots, G_t\}$ is a complete $(m_1, \cdots, m_t)$-decomposition of G, where G_i is a complete m_i partite graph with partite sets $A_{i,1}, A_{i,2}, \cdots, A_{i,m_i}$, $(1 \leq i \leq t)$.

Let $x_1, x_2, \cdots, x_n$ be real variables and for $i, j = 1, 2, \cdots, t$, let

$$L_{i,j} = \sum_{k \in A_{i,j}} x_k.$$

Note that

$$\sum_{i=1}^{t} \sum_{1 \leq j < k \leq m_i} L_{i,j} L_{i,k} = \sum_{1 \leq i < j \leq m_i} x_i x_j.$$

Consider, for each i with $1 \leq i \leq t$, the system of equations

$$\begin{cases} L_{i,1} = 0 \\ L_{i,2} = 0 \\ \cdots \\ L_{i,m_i - 1} = 0 \\ \sum_{k=1}^{n} x_k = 0 \end{cases}$$

By contradiction, assume that $n > \sum_{i=1}^{t}(m_i - 1) + 1$. Then in the system there are more variables than equations, and so the system must have a nonzero solution $(x_1, x_2, \cdots, x_n)$.

For such a solution,

$$\begin{aligned} 0 &= \left(\sum_{i=1}^{n} x_i\right)^2 = 2\sum_{1\le i<j\le n} x_i x_j + \sum_{i=1}^{n} x_i^2 \\ &= 2\sum_{i=1}^{t}\sum_{1\le j<k\le n} L_{i,j}L_{i,k} + \sum_{i=1}^{n} x_i^2 \\ &= \sum_{i=1}^{n} x_i^2 > 0 \end{aligned}$$

a contradiction. □

Corollary 4.3.5 (Huang, [136]) If K_n has a complete $(m_1, m_2, \cdots, m_t)$-decomposition with $m_1 = m_2 = \cdots = m_t = m$, then

$$t \le \frac{n-1}{m-1}.$$

Corollary 4.3.5 follows immediately from Theorem 4.3.5. When $m = 2$, Corollary 4.3.5 yields Theorem 4.3.1.

Definition 4.3.4 Let G be a simple graph and let $x, y \in V(G)$. Let $P_k(x,y)$ denote a collection of k internally disjoint (x,y)-paths:

$$P_k(x,y) = \{P_1, P_2, \cdots, P_k\},$$

where the paths are so labeled that $|E(P_1)| \ge |E(P_2)| \ge \cdots \ge |E(P_k)|$. The k-distance between x and y is

$$d_k(x,y) = \min_{\text{all possible } P_k(x,y)} \{|E(P_k)|\};$$

and the *k-diameter* of G is

$$d_k(G) = \max_{x,y\in V(G)} d_k(x,y).$$

The following problem was posed by Hsu [134]: Given two sequences of natural numbers $L_t = \{l_1, l_2, \cdots, l_t\}$ and $D_t = \{d_1, d_2, \cdots, d_t\}$, is it possible to decompose a K_n ($n > 2$) into subgraphs $F_1, F_2, \cdots, F_t$ such that each F_i is k_i-connected and such that $d_{l_i}(F_i) \le d_i$, $1 \le i \le t$. Such a decomposition, if exists, is called an *(L_t, D_t)-decomposition* of K_n.

Let $f(L_t, D_t)$ denote the smallest integer n such that K_n has an (L_t, D_t)-decomposition. When $l_1 = l_2 = \cdots = l_t = l$ and $d_1 = d_2 \cdots = d_t = d$, we write $f(l,d,t)$ for $f(L_t, D_t)$.

Theorem 4.3.6 Let $L_t = \{l_1, l_2, \cdots, l_t\}$ and $D_t = \{d_1, d_2, \cdots, d_t\}$, and let $M = \sum_{i=1}^{t}(l_i + 1)l_i$. Then

$$f(L_t, D_t) \ge \frac{1}{2}(1 + \sqrt{1+4M}). \tag{4.11}$$

Proof Suppose that K_n has an (L_t, D_t)-decomposition $F_1, F_2, \cdots, F_t$. Then

$$\binom{n}{2} = \sum_{i=1}^{t} |E(F_i)| \geq \frac{1}{2}\sum_{i=1}^{t} |V(F_i)| l_i \geq \frac{1}{2}\sum_{i=1}^{t}(l_i+1)l_i = \frac{M}{2}.$$

Therefore, $n(n-1) \geq M$, and so (4.11) obtains. □

Corollary 4.3.6 $f(l,d,t) \geq \dfrac{1+\sqrt{1+4tl(l+1)}}{2}$.

Proof When $l_1 = l_2 = \cdots = l_t = l$,

$$M = \sum_{i=1}^{t}(l_i+1)l_i = tl(l+1),$$

which implies Corollary 4.3.6. □

Theorem 4.3.7 Let $L_t = \{l_1, l_2, \cdots, l_t\}$ and $D_t = \{d_1, d_2, \cdots, d_t\}$ with each $d_i > 1$. Then

$$f(L_t, D_t) \leq \sum_{i=1}^{t} l_i + 1. \tag{4.12}$$

Proof Since a complete (l_i+1)-partite graph is l_i-connected, it suffices to decompose K_n into t subgraphs $F_1, F_2, \cdots, F_t$ such that each F_i is a complete (l_i+1)-partite graph. Thus (4.12) follows from Theorem 4.3.5. □

By Theorem 4.3.6 and Theorem 4.3.7, we obtain the corollary below.

Corollary 4.3.6 Let $M = \sum_{i=1}^{t} l_i(l_i+1)$. Then

(i) $(1+\sqrt{1+4M})/2 \leq f(L_t, D_t) \leq \sum_{i=1}^{t} l_i + 1$.

(ii) $(1+\sqrt{1+4tl(l+1)})/2 \leq f(l,d,t) \leq tl+1$.

The upper bound in Corollary 4.3.6(ii) can be improved by applying a result of Huang [136].

Theorem 4.3.7 (Huang, [136]) Let b, m, n be integers such that $n = b(m-1)+m-(b-1)$ and $2 \leq b \leq m-1$. Then K_n can be decomposed into $b+1$ complete m-partite subgraphs.

Theorem 4.3.8 For $3 \leq t \leq l+1$ and $d \geq 2$,

$$f(l,d,t) \leq tl - t + 3. \tag{4.13}$$

4.4 Matrix-Tree Theorems

Let D be a digraph. Aside from the adjacency matrix $A(D)$, there are other matrices associated with D. The study of these matrices can reveal combinatorial properties of the digraph D.

Definition 4.4.1 Let D be a digraph with n vertices and without multiple arcs and loops. Let $A(D) \in \mathbf{B}_n$ be the adjacency matrix of D. The *path matrix* of D is

$$P(D) = \sum_{k=1}^{n} A^k(D),$$

where the multiplication and addition are Boolean.

Theorem 4.4.1 Let D be a digraph with $V(D) = \{v_1, v_2, \cdots, v_n\}$ and denote $P(D) = (p_{ij})$.

(i) D is strongly connected if and only if $P(D) = J$.

(ii) Define

$$P(D) \odot P^T(D) = \begin{bmatrix} p_{11}^2 & p_{12}p_{21} & \cdots & p_{1n}p_{n1} \\ p_{21}p_{12} & p_{22}^2 & \cdots & p_{2n}p_{n2} \\ & \cdots & \cdots & \\ p_{n1}p_{1n} & p_{n2}p_{2n} & \cdots & p_{nn}^2 \end{bmatrix}.$$

If the nonzero entries in the ith row of $P(D) \odot P^T(D)$ is

$$p_{ij_1}p_{j_1i}, p_{ij_2}p_{j_2i}, \cdots, p_{ij_k}p_{j_ki},$$

then $D_1 = D[\{v_i, v_{j_1}, v_{j_2}, \cdots, v_{j_k}\}]$, the subgraph of D induced by the vertices $\{v_i, v_{j_1}, v_{j_2}, \cdots, v_{j_k}$ is a maximal strong subgraph of D containing v_i (that is, for any $u \in V(D) - V(D_1)$, $D[V(D_1) \cup \{u\}]$ is not strong).

Proof We only need to prove Part (ii).

Note that for each $t = 1, 2, \cdots, k$, $p_{ij_t} = p_{j_ti} = 1$. Therefore D has a directed (v_i, v_{j_t})-path and a directed (v_{j_t}, v_i)-path, and so D has a closed walk W_0 such that W_0 contains all vertices $v_i, v_{j_1}, \cdots, v_{j_k}$.

If $v_j \in V(W_0) - \{v_i\}$, then D has a directed closed walk containing v_i and v_j, and so $p_{ij} = p_{ji} = 1$. It follows that $j \in \{j_1, j_2, \cdots, j_k\}$, and so $W_0 = D_1$.

Suppose that there exists a $v_j \in V(D) - V(D_1)$ such that $D[V(D_1) \cup \{v_j\}]$ is also a strong subgraph of D. Then once again we have $p_{ij} = p_{ji} = 1$, and so $j \in \{j_1, j_2, \cdots, j_k\}$, a contradiction again. $\square$

Definition 4.4.2 Let D be a digraph. For a vertex $v \in V(D)$ and an arc $e \in E(D)$, write $v = \text{tail}(e)$ if in D e is an out-arc of v; and write $v = \text{head}(e)$ if e is an in-arc of v.

The incidence matrix of D is $B(D) = (b_{ij}) \in M_{n,m}$, defined in Definition 1.1.3. Let $B_f(D)$ denote a matrix obtained from $B(D)$ by deleting a row of $B(D)$.

We can imitate the proof of Theorem 4.4.1 to obtain the following.

Proposition 4.4.1 Let $c \geq 1$ be an integer and let G denote the underlying graph of D (that is, G is obtained from D by replacing each directed edge of D by an undirected edge). Then G has c components if and only if

$$r(B(D)) = r(B_f(D)) = n - c,$$

where $r(B(D))$ and $r(B_f(D))$ are the rank of $B(D)$ and the rank of $B_f(D)$, respectively.

Theorem 4.4.2 Let D be a digraph with n vertices and let G denote the underlying graph of D. The arcs $e_{j_1}, e_{j_2}, \cdots, e_{j_{n-1}}$ form a spanning tree of G if and only if the submatrix of $B_f(D)$ consists of the columns corresponding to these arcs has determinant 1 or -1.

Proof Let B_1 denote the submatrix of $B_f(D)$ consists of the columns corresponding to the arcs $e_{j_1}, e_{j_2}, \cdots, e_{j_{n-1}}$. The subgraph D_1 induced by these arcs has n vertices and $n-1$ arcs, and $B_f(D_1) = B_1$. If $\det(B_1) \in \{1, -1\}$, then $r(B_1) = n-1$, and so by Proposition 4.4.1, D_1 is a spanning tree in G.

Conversely, if D_1 is a spanning tree of G, then by Proposition 4.4.1, $r(B_1) = n-1$, and so by Theorem 1.1.4, B_1 is a nonsingular unimodular square matrix. It follows that $\det(B_1) \in \{1, -1\}$. $\square$

Definition 4.4.3 Let G be a connected labeled graph. Define $\tau(G)$ to be the number of distinct labeled spanning trees of G. If D is a weakly connected digraph, and if G is the underlying graph of D, then define $\tau(D) = \tau(G)$.

Theorem 4.4.3 (Matrix-Tree Theorem) If D is weakly connected, the number of spanning trees of the underlying graph of D is

$$\tau(D) = \det(B_f(D) \cdot D_f^T(D)).$$

Proof By Binet-Cauchy formula and by Theorem 4.4.2, $\det(B_f(D) \cdot D_f^T(D)) = \tau(D) \cdot (\pm 1)^2 = \tau(D)$. $\square$

Theorem 4.4.4 (Cayley) Let $n \geq 1$ be an integer. Then $\tau(K_n) = n^{n-2}$. Here K_n denotes a labeled complete graph on n vertices.

Proof Let $B_f(K_n) = (b_{ij})$. Then $B_f(K_n)B_f^T(K_n) = (b'_{ij}) \in M_{n-1,n-1}$, where for $i, j = 1, 2, \cdots, n-1$,

$$b'_{ij} = \sum_{k=1}^{n(n-1)/2} b_{ik}b_{jk} = \begin{cases} \sum (b_{ik})^2 = n-1 & \text{if } i = j \\ -1 & \text{if } i \neq j. \end{cases}$$

It follows that

$$B_f(K_n)B_f^T(K_n) = \begin{bmatrix} n-1 & -1 & \cdots & -1 \\ -1 & n-1 & \cdots & -1 \\ & \cdots & \cdots & \\ -1 & -1 & \cdots & n-1 \end{bmatrix},$$

and so by Theorem 4.4.3, $\tau(K_n) = \det(B_f(K_n)B_f^T(K_n)) = n^{n-2}$. $\square$

Definition 4.4.4 Let D be a digraph with $V(D) = \{v_1, v_2, \cdots, v_n\}$. Let d_i^+, d_i^- denote the outdegree and the indegree of v_i in D, respectively, where $1 \le i \le n$. Define $M_{\text{out}} = \text{diag}(d_1^+, d_2^+, \cdots, d_n^+)$ and $M_{\text{in}} = \text{diag}(d_1^-, d_2^-, \cdots, d_n^-)$, and define

$$C_{\text{out}} = M_{\text{out}} - A(D), \text{ and } C_{\text{in}} = M_{\text{in}} - A(D).$$

Definition 4.4.5 Let D be a digraph and let $v \in V(D)$. A *v-source tree* is a weakly connected subgraph in which v has indegree zero and every other vertex has indegree exactly one. A *v-sink tree* is a weakly connected subgraph in which v has outdegree zero and every other vertex has outdegree exactly one.

The Propositions below follow from Definitions 4.4.4 and 4.4.5.

Proposition 4.4.2 Each of the following holds.

(i) For each row of C_{out}, the row sum is zero.

(ii) Each column sum of C_{out} is zero if and only if D is eulerian. (That is, D has a directed trail that uses each arc exactly once).

(iii) Each column sum of C_{in} is zero.

(iv) Each row sum of C_{out} is zero if and only if D is eulerian.

(v) Write $C_{\text{out}} = (c_{ij})$. For each i with $1 \le i \le n$, the cofactor of the element c_{ij} in $\det(C_{\text{out}})$ is equal to the cofactor of the element c_{i1} in $\det(C_{\text{out}})$.

(vi) Write $C_{\text{in}} = (c'_{ij})$. For each j with $1 \le j \le n$, the cofactor of the element c_{ij} in $\det(C_{\text{in}})$ is equal to the cofactor of the element c_{1j} in $\det(C_{\text{in}})$.

Proposition 4.4.3 The underlying graph of a v-source tree or a v-sink tree is a tree.

Theorem 4.4.5 Let D be a digraph with $V(D) = \{v_1, v_2, \cdots, v_n\}$. Write $C_{\text{out}} = (c_{ij})$ and $C_{\text{in}} = (c'_{ij})$. Each of the following holds.

(i) For each j with $j = 1, 2, \cdots, n$, the cofactor C_{ij} of the element c_{ij} in $\det(C_{\text{out}})$ is the number of spanning v_1-sink trees of D.

(ii) For each j with $j = 1, 2, \cdots, n$, the cofactor C'_{ji} of the element c'_{ji} in $\det(C_{\text{in}})$ is the number of spanning v_1-source trees of D.

Sketch of Proof By duality, only Part(i) needs a proof.

Without loss of generality, assume that $(v_i, v_1) \in E(D)$. (The case when $(v_1, v_i) \in E(D)$ is similar). Let D' and D'' denote the digraphs obtained from D by deleting and by contracting (v_i, v_1), respectively. Then

$$C_{\text{out}}(D') = \begin{bmatrix} c_{11} & \cdots & c_{1i} & \cdots & c_{1n} \\ & \cdots & & \cdots & \\ 0 & \cdots & c_{ii}-1 & \cdots & c_{in} \\ & \cdots & & \cdots & \\ c_{n1} & \cdots & c_{ni} & \cdots & c_{nn} \end{bmatrix}.$$

Since (v_i, v_1) is contracted, $C_{\text{out}}(D'')$ can be obtained from $C_{\text{out}}(D)$ by deleting the ith row and the ith column, and by replacing c_{11} by $c_{11} + c_{ii} - 1$, c_{1j} by $c_{1j} + c_{ij}$, and c_{j1} by $c_{j1} + c_{ji}$, $2 \le j \le n$.

Let $C(D_1)$ and $C(D_1')$ denote the cofactors of the $(i,1)$-element in $C_{\text{out}}(D)$ and $C_{\text{out}}(D')$, respectively; and let $C(D_1'')$ denote the cofactor of the $(1,1)$-element in $C_{\text{out}}(D'')$. Then

$$\begin{aligned} C(D_1) &= \begin{vmatrix} c_{22} & \cdots & c_{2i} & \cdots & c_{2n} \\ & \cdots & & \cdots & \\ c_{i2} & \cdots & c_{ii}-1 & \cdots & c_{in} \\ & \cdots & & \cdots & \\ c_{n2} & \cdots & c_{ni} & \cdots & c_{nn} \end{vmatrix} + \begin{vmatrix} c_{22} & \cdots & c_{2i} & \cdots & c_{2n} \\ & \cdots & & \cdots & \\ 0 & \cdots & 1 & \cdots & 0 \\ & \cdots & & \cdots & \\ c_{n2} & \cdots & c_{ni} & \cdots & c_{nn} \end{vmatrix} \\ &= C(D_1') + C(D_1''). \end{aligned}$$

Let $T(D_1)$ denote the number of spanning v_1-sink trees in D. Given a spanning v_1-sink tree T, either T is a spanning v_1-sink tree in D' if $(v_i, v_1) \notin E(T)$ or T', obtained from T by contracting (v_i, v_1), is a spanning v_1-sink tree in D'' if $(v_i, v_1) \in E(T)$. It follows that

$$T(D_1) = T(D_1') + T(D_1'').$$

With this reduction formula, Part(i) of Theorem 4.4.5 can now be proved by induction on $|E(D)|$ to show that $T(D_1) = C(D_1)$. $\square$

4.5 Shannon Capacity

The Shannon capacity $\theta(G)$ of a graph G, first introduced by Shannon [238], originates from the theory of error correcting codes. The determination of $\theta(G)$ is very difficult even for a graph with very few vertices. In 1979, Lovász introduced a matrix method to study $\theta(G)$ and successfully solved the long lasting problem for determining $\theta(C_5)$. We shall introduce this method of Lovász in this section.

Throughout this section, the vertices of a graph G will be letters in an given alphabet, where two letters (vertices of G) are adjacent if and only if these letters can be confused. Thus the maximum number of single letter message that can be sent without the danger of confusion is $\alpha(G)$, called the *independence number* of G, which is the maximum cardinality of a vertex subset $V' \subseteq V(G)$ such that the vertices in V' are mutually nonadjacent (vertex subset with this property will be called an *independent* vertex set).

Definition 4.5.1 Let H_1 and H_2 be two graphs with $V_1 = V(H_1)$ and $V_2 = V(H_2)$. The *strong product* of H_1 and H_2, is the graph $H_1 * H_2$, whose vertex set if the product $V_1 \times V_2$, where two vertices (u_1, u_2) and (v_1, v_2) are adjacent if and only if one of the following occurs:

(i) $u_1 = v_1$ and $u_2v_2 \in E(H_2)$; or

(ii) $u_2 = v_2$ and $u_1v_1 \in E(H_1)$; or

(iii) both $u_1v_1 \in E(H_1)$ and $u_2v_2 \in E(H_2)$.

Let $k > 0$ be an integer. For a graph G, define $G^{(1)} = G$, and $G^{(k)} = G^{(k-1)} * G$.

Example 4.5.1 Call two k-letter words are *confundable* if for each $1 \le i \le k$, their ith letters are equal or can be confused. Thus in $G^{(k)}$, vertices are k-letters, and two k-letter words are adjacent in $G^{(k)}$ if and only if they are confundable.

Definition 4.5.2 The *Shannon capacity* of G, is the number

$$\Theta(G) = \sup_k \sqrt[k]{\alpha(G^{(k)})} = \lim_{k \to \infty} \sqrt[k]{\alpha(G^{(k)})}.$$

By Definitions 4.5.1 and 4.5.2, it follows that

$$\Theta(G) \ge \alpha(G). \tag{4.14}$$

Shannon in [238] showed that if a graph G is *perfect* (that is, $\chi(G) = \omega(G)$), then (4.14) holds. For other graphs, like the 5-cycle C_5, Shannon only showed

$$\sqrt{5} \le \Theta(C_5) \le \frac{5}{2}. \tag{4.15}$$

Recall that if $\mathbf{u} = (u_1, u_2, \cdots, u_n)^T$ and $\mathbf{v} = (v_1, v_2, \cdots, v_m)^T$ are two vectors, then their *tensor product* is

$$\mathbf{u} \otimes \mathbf{v} = (u_1v_1, u_1v_m, u_2v_1, \cdots, u_2v_m, \cdots u_nv_1, \cdots, u_nv_m)^T.$$

Definition 4.5.3 Let G be a simple graph with vertex set $\{1, 2, \cdots, n\}$. An *orthonormal representation* of G is a system of unit vectors $\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\}$ such that $\mathbf{v}_i$ and $\mathbf{v}_j$ are orthogonal if and only if the vertices i and j are not adjacent.

The *value* of an orthonormal representation $\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\}$ is

$$\min_{\mathbf{c}} \max_{1 \le i \le n} \frac{1}{(\mathbf{c}^T \mathbf{u}_i)^2},$$

where $\mathbf{c}$ runs over all unit vectors. If $\mathbf{c}$ yields the minimum, then $\mathbf{c}$ is called the *handle* of the representation $\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\}$. Let

$$\theta(G) = \min\{ \text{ value of } \{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\}\},$$

where the minimum runs over all orthonormal representation of G. The minimum yielding representation $\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\}$ is an *optimal representation.*

Proposition 4.5.1 Let G and H be graphs. Each of the following holds.

(i) Every graph G has an orthonormal representation.

(ii) If $\{\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n\}$ and $\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_m\}$ are orthonormal representations of G and H, respectively, then the vectors $\{\mathbf{u}_i \otimes \mathbf{v}_j \ : \ 1 \le i \le n, 1 \le j \le m\}$ is an orthonormal representation of $G * H$.

(iii) $\theta(G * H) \le \theta(G)\theta(H)$.

(iv) $\alpha(G) \le \theta(G)$.

Proof For (i), we can take an orthonormal basis of an n-dimensional real vector space. Proposition 4.5.1(ii) follows from the following fact in linear algebra: If $\mathbf{u}, \mathbf{v}, \mathbf{x}, \mathbf{y}$ are vectors, then

$$(\mathbf{u} \otimes \mathbf{v})^T(\mathbf{x} \otimes \mathbf{y}) = (\mathbf{u}^T\mathbf{x})(\mathbf{v}^T\mathbf{y}). \tag{4.16}$$

Let $\{\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n\}$ and $\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_m\}$ be optimal representations of G and H, with handles $\mathbf{c}$ and $\mathbf{d}$, respectively.

By (4.16), $\mathbf{c} \otimes \mathbf{d}$ is a unit vector, and so by (4.16) again,

$$\begin{aligned} \theta(G * H) &\le \max_{i,j} \frac{1}{((\mathbf{c} \otimes \mathbf{d})^T(\mathbf{u}_i \otimes \mathbf{v}_j))^2} \\ &= \max_{i,j} \frac{1}{(\mathbf{c}^T\mathbf{u}_i)^2} \frac{1}{(\mathbf{d}^T\mathbf{v}_j)^2} = \theta(G)\theta(H). \end{aligned}$$

This proves (iii).

Without loss of generality, assume that $\{1, 2, \cdots, k\}$ is a maximum independent vertex set in G. Thus $\mathbf{u}_1, \cdots, \mathbf{u}_k$ are mutually orthogonal, and so

$$1 = |\mathbf{c}|^2 \ge \sum_{i=1}^{k}(\mathbf{c}^T\mathbf{u}_i)^2 \ge \frac{\alpha(G)}{\theta(G)}.$$

Hence (iv) follows. $\square$

By Proposition 4.5.1, for each $k > 0$, $\alpha(G^{(k)}) \le \theta(G^{(k)}) \le (\theta(G))^k$, and so by Definition 4.5.2, Theorem 4.5.1 below obtains.

Theorem 4.5.1 (Lovász, [189])

$$\Theta(G) \le \theta(G).$$

Proposition 4.5.2 Let $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ is an orthonormal representation of G and $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n$ is an orthonormal representation of G^c, the complement of G. Each of the following holds.

(i) If $\mathbf{c}$ and $\mathbf{d}$ are two vectors, then

$$\sum_{i=1}^{n} (\mathbf{u}_i \otimes \mathbf{v}_i)^T (\mathbf{c} \otimes \mathbf{d}) = \sum_{i=1}^{n} (\mathbf{u}_i^T \mathbf{c})(\mathbf{v}_i^T \mathbf{d}) \le |\mathbf{c}|^2 |\mathbf{d}|^2.$$

(ii) If $\mathbf{d}$ is a unit vector, then

$$\theta(G) \ge \sum_{i=1}^{n} (\mathbf{v}_i^T \mathbf{d})^2.$$

(iii) $\theta(G)\theta(G^c) \ge n$.

Proof By (4.16), the vectors $\mathbf{u}_i \otimes \mathbf{v}_i$, $(1 \le i \le n)$ form an orthonormal basis and so Proposition 4.5.2(i) follows from

$$|\mathbf{c} \otimes \mathbf{d}|^2 \ge \sum_{i=1}^{n} (\mathbf{u}_i \otimes \mathbf{v}_i)^T (\mathbf{c} \otimes \mathbf{d})$$

and from (4.16).

Let $\mathbf{c}$ be the handle for an optimal representation $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ of G. Then Proposition 4.5.2(ii) follows from Proposition 4.5.2(i).

Let $\mathbf{d}$ be the handle for an optimal representation $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n$ of G^c. Then Proposition 4.5.2(iii) follows from Proposition 4.5.2(ii). $\square$

Let G be a graph on vertices $\{1, 2, \cdots, n\}$. $\mathcal{A}$ denote the set of $n \times n$ real symmetric matrix $A = (a_{ij})$ such that

$$a_{ij} = 1 \text{ if } i = j \text{ or if } i \text{ and } j \text{ are nonadjacent},$$

and let $\mathcal{B}$ denote the set of all $n \times n$ positive semidefinite matrices $B = (b_{ij})$ such that

$$\begin{aligned} b_{ij} &= 0 \text{ for all pairs of nonadjacent vertices } (i, j) \\ \mathrm{tr}(B) &= 1 \end{aligned}$$

Note that $\mathrm{tr}(BJ)$ is the sum of all the entries in B.

Theorem 4.5.2 (Lovász, [189]) Let G be a graph with vertices $\{1, 2, \cdots, n\}$. Then each of the following holds.

(i) $\theta(G) = \min_{A \in \mathcal{A}} \{\lambda_{\max}(A)\}$.

(ii) $\theta(G) = \max_{B \in \mathcal{B}} \{\mathrm{tr}(BJ)\}$.

Proof Let $\{\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n\}$ be an optimal representation of G with handle $\mathbf{c}$. Define for $1 \le i, j \le n$,

$$\begin{aligned} a_{ij} &= 1 - \frac{\mathbf{u}_i^T \mathbf{u}_j}{(\mathbf{c}^T \mathbf{u}_i)(\mathbf{c}^T \mathbf{u}_j)}, \quad i \ne j, \\ a_{ii} &= 1 \end{aligned}$$

and $A = (a_{ij})_{n \times n}$. Then $A \in \mathcal{A}$. Note that the elements in the matrix $\theta(G)I - A$ are

$$\begin{aligned} -a_{ij} &= \left(\mathbf{c} - \frac{\mathbf{u}_i}{\mathbf{c}^T \mathbf{u}_i}\right)^T \left(\mathbf{c} - \frac{\mathbf{u}_j}{\mathbf{c}^T \mathbf{u}_j}\right) \quad i \ne j \\ \theta(G) - a_{ii} &= \left|\mathbf{c} - \frac{\mathbf{u}_i}{\mathbf{c}^T \mathbf{u}_i}\right|^2 + \left(\theta(G) - \frac{1}{(\mathbf{c}^T \mathbf{u}_i)^2}\right). \end{aligned}$$

It follows that $\theta(G)I - A$ is positive semidefinite and so $\lambda_{\max}(A) \le \theta(G)$.

Conversely, let $A = (a_{ij}) \in \mathcal{A}$ and let $\lambda = \lambda_{\max}(A)$. Then $\lambda I - A$ is positive semidefinite, and so there exist vectors $\mathbf{x}_1, \cdots, \mathbf{x}_n$ and a matrix $X = [\mathbf{x}_1, \cdots, \mathbf{x}_n]$ such that $X^T X = \lambda I - A$. Since λ is an eigenvalue of A, $\operatorname{rank}(X) < n$, and so there exists a unit vector $\mathbf{c}$ which is perpendicular to all the $\mathbf{x}_i$'s. Let

$$\mathbf{u}_i = \frac{1}{\sqrt{\lambda}}(\mathbf{c} + \mathbf{x}_i).$$

Then for $1 \le i, j \le n$

$$\begin{aligned} |\mathbf{u}_i|^2 &= \frac{1}{\lambda}(1 + |\mathbf{x}_i|^2) = 1 \\ \mathbf{u}_i^T \mathbf{u}_j &= \frac{1}{\lambda}(1 + \mathbf{x}_i^T \mathbf{x}_j) = 0, \text{ if } i \ne j \text{ are not adjacent.} \end{aligned}$$

Therefore, $\{\mathbf{u}_1, \cdots, \mathbf{u}_n\}$ is a orthonormal representation of G with

$$\frac{1}{(\mathbf{c}^T \mathbf{u}_i)^2}, \quad 1 \le i \le n.$$

Hence $\theta(G) \le \lambda$. This completes that proof for Theorem 4.5.2(i).

By Theorem 4.5.2(i), there exists an $n \times n$ matrix $A = (a_{ij}) \in \mathcal{A}$ such that $\lambda_{\max}(A) = \theta(G)$. Let $B \in \mathcal{B}$. Then by the choice of A and by the definition of $\mathcal{B}$,

$$\operatorname{tr}(BJ) = \sum_{i=1}^{n} \sum_{j=1}^{n} b_{ij} = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} b_{ij} = \operatorname{tr}(AB),$$

and so

$$\theta(G) - \operatorname{tr}(BJ) = \operatorname{tr}((\theta(G)I - A)B). \tag{4.17}$$

It follows that both $\theta(G)I - A$ and B are positive semidefinite. Let $\lambda_1, \lambda_2, \cdots, \lambda_n$ be the eigenvalues of B, and let $\mathbf{w}_1, \cdots, \mathbf{w}_n$ be mutually orthogonal eigenvectors corresponding to $\lambda_1, \cdots, \lambda_n$, respectively. Then

$$\operatorname{tr}((\theta(G)I - A)B) = \sum_{i=1}^{n} \mathbf{e}_i^T(\theta I - A)B\mathbf{w}_i = \sum_{i=1}^{n} \lambda_i \mathbf{w}_i^T(\theta I - A)\mathbf{w}_i \geq 0.$$

This, together with (4.17), implies that $\theta(G) \geq \max_{B\in\mathcal{B}} \operatorname{tr}(BJ)$.

Conversely, let $E(G) = \{i_t j_t \ : \ 1 \leq t \leq m\}$. Consider the $(m+1)$-dimensional vectors

$$\hat{\mathbf{h}} = (h_{i_1}h_{j_1}, \cdots, h_{i_m}h_{j_m}, (\sum h_i)^2)^T,$$

where $\mathbf{h} = (h_1, h_2, \cdots, h_n)^T$ ranges though all unit vectors, and let $\hat{\mathcal{H}}$ denote the set of all such vectors $\hat{\mathbf{h}}$. Note that $\hat{\mathcal{H}}$ is a compact subset in the $(m+1)$-dimensional real space.

Claim Let $\mathbf{z} = (0, 0, \cdots, 0, \theta(G))^T$. Then there exist vectors $\hat{\mathbf{h}}_1, \hat{\mathbf{h}}_2, \cdots, \hat{\mathbf{h}}_N \in \hat{\mathcal{H}}$, and nonnegative real numbers $c_1, c_2, \cdots, c_N$ such that

$$\sum_{i=1}^{N} c_i = 1, \tag{4.18}$$

$$\sum_{i=1}^{N} c_i \mathbf{h}_i = \mathbf{z}. \tag{4.19}$$

If not, then there exists a vector $\mathbf{a} = (a_1, a_2, \cdots, a_m, y)^T$ and a constant a such that $\mathbf{a}^T\hat{\mathbf{h}} \leq a$, for all $\hat{\mathbf{h}} \in \hat{\mathcal{H}}$ but $\mathbf{a}^T\mathbf{z} > a$.

Note that for $\mathbf{h} = (1, 0, \cdots, 0)^T$, the corresponding $\hat{\mathbf{h}}$ satisfies $\mathbf{a}^T\hat{\mathbf{h}} \leq a$, and so $y \leq a$. On the other hand, that $\mathbf{a}^T\mathbf{z} > a$ implies $\theta(G)y > a$, and so $\theta(G)y > a \geq y$. By Proposition 4.5.1(iv), $\theta(G) \geq 1$, and so $a \geq y > 0$. We may assume that $y = 1$, and so $\theta(G) > a$.

Now define $A = (a_{ij})$ with

$$a_{ij} = \begin{cases} \frac{1}{2}a_k + 1, & \text{if } \{i, j\} = \{i_k, j_k\} \\ 1 & \text{otherwise} \end{cases}$$

then that $\mathbf{a}^T\hat{\mathbf{h}} \leq a$ can be written as

$$\hat{\mathbf{h}}^T A \hat{\mathbf{h}} = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}h_ih_j \leq a.$$

Since $\lambda_{\max}(A) = \max\{\mathbf{x}^T A\mathbf{x} \ : \ |\mathbf{x}| = 1\}$, this implies that $\lambda_{\max}(A) \leq a$. However, $A \in \mathcal{A}$, and so by Theorem 4.5.2(i), $\theta(G) \leq a$, a contradiction. This proves the claim.

Therefore (4.18) and (4.19) hold. Set

$$\begin{aligned} \mathbf{h}_p &= (h_{p,1}, h_{p,2}, \cdots, h_{p,n})^T \\ b_{ij} &= \sum_{p=1}^{N} c_p h_{p,i} h_{p,j} \\ B &= (b_{ij}) \end{aligned}$$

Then B is symmetric and positive semidefinite. By (4.18) $\mathrm{tr}(B) = 1$. By (4.19),

$$\begin{aligned} b_{i_k, j_k} &= 0, (1 \le k \le m) \\ \mathrm{tr}(BJ) &= \theta(G). \end{aligned}$$

Therefore $B \in \mathcal{B}$, and so $\theta(G) \le \max_{B \in \mathcal{B}} \mathrm{tr}(BJ)$. This complete the proof of Theorem 4.5.2(ii). $\square$

Corollary 4.5.2 There exists an optimal representation $\{\mathbf{u}_1, \cdots, \mathbf{u}_n\}$ such that

$$\theta(G) = \frac{1}{(\mathbf{c}^T \mathbf{u}_i)^2}, \ \ 1 \le i \le n.$$

Theorem 4.5.3 (Lovász, [189]) Let $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n$ range over all orthonormal representation of G^c, and $\mathbf{d}$ over all unit vectors. Then

$$\theta(G) = \max \sum_{i=1}^{n} (\mathbf{d}^T \mathbf{v}_i)^2.$$

Proof By Proposition 4.5.2(ii), it suffices to show that for some $\mathbf{v}_i$'s and some $\mathbf{d}$, $\theta(G) \le \sum_{i=1}^{n} (\mathbf{d}^T \mathbf{v}_i)^2$.

Pick a matrix $B = (b_{ij}) \in \mathcal{B}$ such that $\mathrm{tr}(BJ) = \theta(G)$. Since B is positive semidefinite, there exists vectors $\mathbf{w}_1, \mathbf{w}_2, \cdots, \mathbf{w}_n$ such that

$$b_{ij} = \mathbf{w}_i^T \mathbf{w}_j, \ \ 1 \le i, j \le n.$$

Since $B \in \mathcal{B}$,

$$\sum_{i=1}^{n} |\mathbf{w}_i|^2 = 1, \text{ and } \left| \sum_{i=1}^{n} \mathbf{w}_i \right|^2 = \theta(G).$$

Set

$$\mathbf{v}_i = \frac{\mathbf{w}_i}{|\mathbf{w}_i|}, \ (1 \le i \le n) \text{ and } \mathbf{d} = \left(\sum_{i=1}^{n} \mathbf{w}_i \right) / \left| \sum_{i=1}^{n} \mathbf{w}_i \right|.$$

Then the $\mathbf{v}_i$'s form a orthonormal representation of G^c, and so by Cauchy-Schwarz inequality,

$$\begin{aligned} \sum_{i=1}^{n} (\mathbf{d}^T \mathbf{v}_i)^2 &= \left(\sum_{i=1}^{n} |\mathbf{w}_i|^2 \right) \left(\sum_{i=1}^{n} (\mathbf{d}^T \mathbf{v}_i)^2 \right) \\ &\ge \left(\sum_{i=1}^{n} |\mathbf{w}_i| (\mathbf{d}^T \mathbf{v}_i) \right)^2 = \left(\sum_{i=1}^{n} (\mathbf{d}^T \mathbf{w}_i) \right)^2 \\ &= \left| \mathbf{d}^T \sum_{i=1}^{n} \mathbf{w}_i \right|^2 = \left| \sum_{i=1}^{n} \mathbf{w}_i \right|^2 = \theta(G). \end{aligned}$$

This proves the theorem. $\square$

Corollary 4.5.3 $\chi(G) \geq \theta(G^c)$.

Proof Let $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n$ be an orthonormal representation of G, $\mathbf{d}$ a unit vector. Suppose that $\chi(G) = k$ and $V_1, V_2, \cdots, V_k$ be a partition of $V(G)$ corresponding to a proper k-coloring. Then by Theorem 4.5.3,

$$\sum_{i=1}^{n}(\mathbf{d}^T\mathbf{v}_i)^2 = \sum_{m=1}^{k}\sum_{i\in V_m}^{n}(\mathbf{d}^T\mathbf{v}_i)^2 \leq \sum_{i=1}^{k} 1 = k.$$

Thus the corollary follows. $\square$

Theorem 4.5.4 (Lovász, [189]) Let G be a graph on vertices $\{1, 2, \cdots, n\}$. $\mathcal{A}^*$ denote the set of $n \times n$ real symmetric matrix $A^* = (a_{ij})$ such that

$$a_{ij} = 0 \text{ whenever } i \text{ and } j \text{ are adjacent.}$$

Let $\lambda_1(A^*) \geq \lambda_2(A^*) \geq \cdots \geq \lambda_n(A^*)$ be eigenvalues of an $A^*\mathcal{A}^*$. Then

$$\theta(G) = \max_{A^*\in\mathcal{A}^*}\left\{1 - \frac{\lambda_1(A^*)}{\lambda_n(A^*)}\right\}.$$

Sketch of Proof Let $A^* = (a_{ij}) \in \mathcal{A}^*$ and let $\mathbf{f} = (f_1, f_2, \cdots, f_n)^T$ be an eigenvector of $\lambda_1(A^*)$ such that $|\mathbf{f}|^2 = -1/\lambda_n(A^*)$ (note that since $\mathrm{tr}(A^*) = 0$, $\lambda_n(A^*) < 0$).

Define the matrices $F = \mathrm{diag}(f_1, f_2, \cdots, f_n)$ and $B = (b_{ij}) = F(A^* - \lambda_n(A^*)I)F$. Then $B \in \mathcal{B}$, and so by Theorem 4.5.2,

$$\begin{aligned}\theta(G) &\geq \mathrm{tr}(BJ) = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}f_if_j - \lambda_n(A^*)\sum_{i=1}^{n} f_i^2 \\ &= \sum_{i=1}^{n}[\lambda_1(A^*)f_i^2 - \lambda_n(A^*)f_i^2] = 1 - \frac{\lambda_1(A^*)}{\lambda_n(A^*)}.\end{aligned}$$

Conversely, by Theorem 4.5.2, choose $B \in \mathcal{B}$ so that $\theta(G) = \mathrm{tr}(BJ)$, and then follow basically an inversion of the argument above. $\square$

Corollary 4.5.4 (Hoffman, [123]) Let G be a graph with $A = A(G)$, and let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ be eigenvalues of A. Then

$$\chi(G) \geq 1 - \frac{\lambda_1}{\lambda_n}.$$

Proof This follows from Corollary 4.5.3 and Theorem 4.5.4. $\square$

Theorem 4.5.5 (Lovász, [189]) Let G be a regular graph on n vertices and let $\Lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ be the eigenvalues of $A = A(G)$. Then

$$\theta(G) \leq \frac{-n\lambda_n}{\lambda_1 - \lambda_n}. \tag{4.20}$$

Equality holds in (4.20) if the automorphism group of G is edge-transitive.

Proof Let $\mathbf{v}_i$, $(1 \le i \le n)$ be eigenvectors corresponding to λ_i, respectively. Since G is regular, $\mathbf{v}_1 = \mathbf{j} = (1, 1, \cdots, 1)^T$, and each $\mathbf{v}_i$ is also an eigenvector of J.

Note that real number x, the matrix $J - xA \in \mathcal{A}$, and so by Theorem 4.5.2, $\lambda_{\max}(J - xA) \ge \theta(G)$. However, the eigenvalues of $J - xA$ are $n - x\lambda_1, -x\lambda_2, \cdots, -x\lambda_n$, and so $\lambda_{\max}(J - xA)$ is either the first or the last. Hence the optimal choice of $x = n/(\lambda_1 - \lambda_n)$, and so (4.20) follows from Theorem 4.5.2.

Assume now that the automorphism group Γ of G is edge-transitive. Let $C = (c_{ij}) \in \mathcal{A}$ such that $\lambda_{\max}(C) = \theta(G)$. Note that each element in Γ corresponds to a permutation matrix. Define

$$\overline{C} = \frac{1}{|\Gamma|} \sum_{P \in \Gamma} P^{-1} C P.$$

Then we can verify that $\overline{C} \in \mathcal{A}$ and has the form $J - xA$. By Theorem 4.5.2, $\theta(G) = \lambda_{\max}(\overline{C})$ and so equality holds in (4.20). $\square$

Corollary 4.5.5A For an odd integer $n > 0$,

$$\theta(C_n) = \frac{n \cos(\pi/n)}{1 + \cos(\pi/n)}.$$

Corollary 4.5.5B $\Theta(C_5) = \sqrt{5}$.

Proof Applying Theorem 4.5.5 to $G = C_n$, we have Corollary 4.5.5A. In particular, $\theta(C_5) = \sqrt{5}$, and so by Proposition 4.5.1, $\Theta(C_5) \le \sqrt{5}$. Thus Corollary 4.5.5B follows from (4.15). $\square$.

We conclude this section with an application of Shannon capacity, due to Lovász [189]. For integers n, r with $n \ge 2r > 0$, let $K(n, r)$ denote the graph whose vertices are r-subsets of an n-element set S, where two subsets are adjacent in $K(n, r)$ if and only if they are disjoint.

Theorem 4.5.6 (Lovász, [189])

$$\Theta(K(n, r)) = \binom{n-1}{r-1}.$$

Proof For each $s \in S$, all the r-subsets containing s form an independent set in $K(n, r)$, and so

$$\Theta(K(n, r)) \ge \alpha(K(n, r)) \ge \binom{n-1}{r-1}.$$

On the other hand, as the automorphism group of $K(n,r)$ is edge-transitive, (4.20) may be used to derive the desired inequality. Since $K(n,r)$ is regular, $\mathbf{j}$ is an eigenvector for the eigenvalue $\binom{n-r}{r}$ of $K(n,r)$.

Let $1 \le t \le r$. For each $T \subset S$ with $|T| = t$, let x_T be a real number such that

$$\text{for every } U \subset S \text{ with } |U| = t-1, \sum_{U \subset T} x_T = 0. \tag{4.21}$$

There are $\binom{n}{t} - \binom{n}{t-1}$ linearly independent $\binom{n}{t}$-dimensional vectors of the type $(\cdots, x_T, \cdots)$. For each such vector, and for each $A \subset S$ with $|A| = r$, define

$$\overline{x}_A = \sum_{T \subset A, |T|=t} x_T.$$

Then there are $\binom{n}{t} - \binom{n}{t-1}$ linearly independent $\binom{n}{r}$-dimensional vectors of the type

$$\overline{\mathbf{x}} = (\cdots, \overline{x}_A, \cdots). \tag{4.22}$$

We shall show that every vector $\overline{\mathbf{x}}$ of the form (4.22) is an eigenvector of the adjacency matrix of $K(n,r)$, with eigenvalue $(-1)^t \binom{n-r-t}{r-t}$.

In fact, for any $A_0 \subset S$ with $|A_0| = r$, and for each $0 \le i \le t$, define

$$\beta_i = \sum_{|T \cap A_0| = i} x_T.$$

Then we have

$$\sum_{A \cap A_0 = \emptyset} \overline{x}_A = \sum_{\substack{T \cap A_0 = \emptyset \\ |T| = t}} \binom{n-r-t}{r-t} x_T = \binom{n-r-t}{r-t} \beta_0.$$

For each $0 \le i \le t$, and for each $U \subset S$ with $|U| = t-1$ and with $|U \cap A_0| = i$, by (4.21), we obtain an recurrence relation for β_i:

$$(i+1)\beta_{i+1} + (t-i)\beta_i = 0,$$

which yields

$$\beta_i = (-1)^i \binom{t}{i} \beta_0.$$

Therefore,

$$\beta_0 = (-1)^t \beta_t = (-1)^t \overline{x}_{A_0},$$

as desired. Hence $\overline{x}_{A_0}$ is an eigenvector of the adjacency matrix of $K(n,r)$.

By this construction we have found

$$1 + \sum_{t=1}^{r} \left(\binom{n}{t} - \binom{n}{t-1} \right) = \binom{n}{r}$$

linearly independent eigenvectors of $K(n,r)$, as eigenvectors belonging to different eigenvalues are linearly independent. It follows that all the eigenvalues of $K(n,r)$ are

$$(-1)^t \binom{n-r-t}{r-t}, \quad t = 0, 1, 2, \cdots, r,$$

and so the largest and the smallest are the ones with $t = 0$ and $t = 1$, respectively. Now the proof for Theorem 4.5.6 is complete by applying (4.20). $\square$

Corollary 4.5.6A The Petersen graph, which is isomorphic to $K(5,2)$, has capacity 4.

Corollary 4.5.6B (Erdös, Ko, Rado [82])

$$\alpha(K(n,r)) = \Theta(K(n,r)) = \binom{n-1}{r-1}.$$

4.6 Strongly Regular Graphs

In 1966, Erdös, Renyi and Sòs proved the so called friendship theorem [83]: In a society with a finite population, if every two people have exactly one common friend, then there must be a person who is a friend of every other member in the society. The friendship theorem can be stated in graph theory as follows, whose proof will be postponed.

Theorem 4.6.1 (Erdös, Renyi and Sòs, [83]) Let G be a graph on n vertices. If for each pair of distinct vertices x and y, there is exactly one vertex w adjacent to both x and y, then n is odd and G has a vertex $u \in V(G)$ such that u is adjacent to every vertex in $V(G-u)$ and such that $G-u$ is a 1-regular graph.

Definition 4.6.1 For a graph G with $v \in V(G)$, let $N(v) = N_G(v)$ denote the vertices that are adjacent to v in G.

Let n, k, λ, μ be non negative integers with $n \geq 3$. A k-regular graph G on n vertices is an (n, k, λ, μ)-*strongly regular graph* if each of the following holds:

(4.6.1A) If $u, v \in V(G)$ such that u and v are adjacent in G, then $|N(u) \cap N(v)| = \lambda$.

(4.6.1B) If $u, v \in V(G)$ such that u and v are not adjacent in G, then $|N(u) \cap N(v)| = \mu$.

Example 4.6.1

(i) The 4-cycle is a (4,2,0,2)-strongly regular graph.

(ii) The 5-cycle is a (5,2,0,1)-strongly regular graph.

(iii) If $n \geq 6$, the n-cycle is not a strongly regular graph.

(iv) The Petersen graph is a (10,3,0,1)-strongly regular graph.

(v) The disjoint union of two copies of K_3 is a (6,2,1,0)-strongly regular graph.

(vi) For $m \geq 2$, the complete bipartite graph $K_{m,m}$ is a $(2m, m, 0, m)$-strongly regular graph.

Proposition 4.6.1 Let G be a k-regular graph with n vertices and let $A = A(G)$. Then G is an (n, k, λ, μ)-strongly regular graph if and only if one of the following holds.

(i) $A^2 = kI + \lambda A + \mu(J - I - A)$.

(ii) $A^2 - (\lambda - \mu)A - (k - \mu)I = \mu J$.

Proof By Proposition 1.1.2(vii), and by Definition 4.6.1, that G is an (n, k, λ, μ)-strongly regular graph is equivalent to Proposition 4.6.1(i), and it is straightforward to see that Proposition 4.6.1(i) and Proposition 4.6.1(ii) are equivalent. $\square$

Proposition 4.6.2 Let G be an (n, k, λ, μ)-strongly regular graph. Then each of the following holds.

(i) $k(k - \lambda - 1) = \mu(n - k - 1)$

(ii) The complement of G is an $(n, n-k-1, n-2k+\mu-2, n-2k+\lambda)$-strongly regular graph.

(iii) $\mu = 0$ if and only if G is the disjoint union of some K_{k+1}'s.

Proof This follows from Proposition 4.6.1(i) and (ii). $\square$

Theorem 4.6.2 Let G be a connected (n, k, λ, μ)-strongly regular graph, let $l = n - k - 1$ and let

$$\begin{aligned} d &= (\lambda - \mu)^2 + 4(k - \mu), \\ \delta &= (k + l)(\lambda - \mu) + 2k. \end{aligned} \tag{4.23}$$

Then the spectrum of $A = A(G)$ is

$$\mathrm{Spec}(A) = \begin{pmatrix} k & \rho & \sigma \\ 1 & r & s \end{pmatrix},$$

where

$$\begin{aligned} \rho &= \tfrac{1}{2}\left(\lambda - \mu + \sqrt{d}\right) \geq 0, \\ \sigma &= \tfrac{1}{2}\left(\lambda - \mu - \sqrt{d}\right) \leq -1, \end{aligned} \tag{4.24}$$

and where

$$r = \frac{1}{2}\left(k + l - \frac{\delta}{\sqrt{d}}\right),$$
$$s = \frac{1}{2}\left(k + l + \frac{\delta}{\sqrt{d}}\right). \tag{4.25}$$

Proof Let $\lambda_1(A) \geq \cdots \geq \lambda_n(A)$ denote the eigenvalues of A. By Corollary 1.3.2A, $\lambda_1(A) = k$. Since G is connected, A is irreducible, and so the multiplicity of $\lambda_1(A) = k$ is 1.

By Proposition 4.6.1(i),

$$(A - kI)(A^2 - (\lambda - \mu)A - (k - \mu)I) = 0,$$

and so ρ and σ in (4.24) are the other eigenvalues of A.

If $d = 0$, then by $k \geq mu$, $\lambda = \mu = k$. However, since G is a (n, k, λ, μ)-strongly regular graph, $\lambda \leq k - 1$, a contradiction. Hence $d > 0$, and so $k > \rho > \sigma$. By (4.24), write

$$\lambda = k + \rho + \sigma + \rho\sigma, \ \mu = k + \rho\sigma.$$

Since $k \geq \mu$, we have $\rho \geq 0$ and $\mu \leq 0$. If $\sigma = 0$, then $\lambda = k + \rho \geq k$, contrary to $\lambda \leq k - 1$. Hence $\sigma < 0$.

Consider $\overline{G}$, the complement of G. By algebraic manipulation, we obtain the corresponding parameters for G^c as follows:

$$\overline{d} = d, \overline{\rho} = -\sigma - 1, \overline{\sigma} = -\rho - 1.$$

As for G^c, $\overline{\rho} \geq 0$, we must have $\rho \geq 0$ and $\sigma \leq -1$.

Finally, we note that if r and s are multiplicities of ρ and σ, respectively, then

$$\begin{cases} r + s = n - 1 \\ k + r\rho + s\sigma = 0 \end{cases}$$

and by algebraic manipulation again, (4.25) obtains. $\square$

Theorem 4.6.3 Let G be an (n, k, λ, μ)-strongly regular graph, let $l = n - k - 1$, and adopt the notations in (4.23), (4.24) and (4.25).

(i) If $\delta = 0$, then

$$\lambda = \mu - 1, k = l = 2\mu = r = s = \frac{n-1}{2}.$$

(ii) If $\delta \neq 0$, then $\sqrt{d}$, ρ, σ are integers. Moreover, if n is even, then $\sqrt{d}|\delta$ but $2\sqrt{d} \nmid \delta$, and if n is odd, then $2\sqrt{d}|\delta$.

Sketch of Proof If $\delta = 0$, then by (4.23), $2k/(\mu - \lambda) = k + 1 > k$, and so $0 < \mu - \lambda < 2$, which yields $\lambda = \mu - 1$. The other equalities follow from Proposition 4.6.2(ii) and (4.25).

The conclusion when $\delta \neq 0$ follow by (4.24) and (4.25). $\square$

A strongly regular graph satisfying Theorem 4.6.3(i) is also called a *conference graph.*

By Theorem 4.6.2, if G is a connected strongly regular graph, then G has exactly 3 distinct eigenvalues.

Theorem 4.6.4 Let G be a connected regular graph. Then G is strongly regular if and only if G has exactly 3 distinct eigenvalues.

We are now ready to apply Theorem 4.6.2 to present a proof, due to Cameron [45], for Theorem 4.6.1.

Proof for Theorem 4.6.1 Let G be a graph satisfying the conditions of Theorem 4.6.1.
Claim 1 If u and v are not adjacent in G, then u and v must have the same degree.

Let $u, v \in V(G)$ be distinct non adjacent vertices. Then there exists a unique vertex w such that w is adjacent to both u and v. Also, G has vertices $x \neq v$ and $y \neq u$ such that x is the only vertex adjacent to both u and w and y is the only vertex adjacent to v and w.

If $z \notin \{w, x\}$ is a vertex adjacent to u, then there exists a unique $z' \notin \{w, y\}$ such that z' is adjacent to both z and v. Note that we can exchange u and v to get the same conclusion on v. Therefore, u and v must have the same degree.
Claim 2 If G is a k-regular graph, then $G = K_1$, or $G = K_3$.

If G is k-regular, then G is an $(n, k, 1, 1)$-strongly regular graph. By Theorem 4.6.2, $s - r = \delta/\sqrt{d} = k/\sqrt{k-1}$ is an integer, and so $(k-1)|k^2$. It follows that either $k = 0$ or $k = 2$, and so G is either K_1 or K_3.

By Claim 2, we may assume that G is not a regular graph. Let u and v be two vertices with different degrees. Then by Claim 1, u and v must be adjacent. Let w be the only vertex in G adjacent to both u and v, and renaming u and v if necessary, we may assume that w and u have different degrees. Let $x \in V(G) - \{u, v, w\}$. Then by Claim 1 and by the assumption that u and v have different degrees, x must be adjacent to either u or v. Similarly, x must be adjacent to either u or w.

However, since u is the only vertex adjacent to both v and w, x must be adjacent to u. Therefore, u is a vertex in G that is adjacent to every other vertex in G, and each component of $G - u$ is a K_2. $\square$

C. W. H. Lam and J. H. van Lint [149] considered a generalization of friendship theorem to loopless digraphs, first proposed by A. J. Hoffman.

Definition 4.6.2 A loopless digraph D is a *k-friendship graph* if for any pair of distinct vertices $u, v \in V(D)$, D has a unique directed (u, v)-walk of length k, and for every vertex $u \in V(D)$, D has no (u, u)-walk of length k.

Note that if $A \in \mathbf{B}_n$ satisfying $A^k = J - I$, then by Proposition 1.1.2(vii), $D(A)$ is a k-friendship graph. The converse of this also follows from Proposition 1.1.2(vii). Therefore, to determine the existence of a k-friendship graph, it is equivalent to showing that the equation $A^k = J - I$ has a solution.

Example 4.6.2 Let F_2 denote the directed cycle of length 2. Then for any odd integer $k > 0$, F_2 is a k-friendship, (called a *fish* in [149]).

Proposition 4.6.3 Let D be a k-friendship graph with n vertices. and let $A = A(D)$. Then each of the following holds.

(i) $\mathrm{tr}(A) = 0$.

(ii) $A^k = J - I$.

(iii) For some integer $c \geq 0$, $AJ = JA = cJ$. (Therefore, A has constant row and column sums c; and the digraph D has indegree and outdegree c in each vertex.)

(iv) The integer c in (ii) satisfies

$$n = c^k + 1. \tag{4.26}$$

Proof Since D is loopless, $\mathrm{tr}(A) = 0$. By Proposition 1.1.2(vii) and Definition 4.6.2, $A^k = J - I$. Multiply both sides of $A^k = J - I$ by A to get

$$A^{k+1} = JA - A = AJ - A,$$

which implies that $AJ = JA = cJ$, for some integer c. Multiply both sides of $A^k = J - I$ by J, and apply Proposition 4.6.3(ii) to get

$$c^k J = A^k J = J^2 - J = nJ - J = (n-1)J,$$

and so $c^k = n - 1$. $\square$

Theorem 4.6.5 (Lam and Van Lint, [149]) If k is even, no k-friendship graph exists.

Proof Let $k = 2l$. Assume that there exists a k-friendship graph D with n vertices. Let $A = A(D)$ and let $A_1 = A^l$. By Proposition 4.6.3(i), $A_1^2 = J - I$. Therefore by Proposition 4.6.3, n must satisfy $n = c^2 + 1$ for some integer c. The eigenvalues of A must then be c with multiplicity 1, and i and $-i$ (where i is the complex number satisfying $i^2 = -1$) with equal multiplicities. This implies $\mathrm{tr}(A) = c$, contrary to Proposition 4.6.3(i). $\square$

For k odd, a solution for $A^k = J - I$ is obtained for each $n = c^k + 1$, where $c > 0$ is an integer. Consider the integers mod n. For each integer v with $1 \leq v \leq c$, define the permutation matrix $P_v = (p_{ij}^{(v)})$ by

$$p_{ij}^{(v)} = \begin{cases} 1 & \text{if } j \equiv v - ci \pmod{n} \\ 0 & \text{otherwise} \end{cases} \quad (0 \leq i, j \leq n-1).$$

and define

$$A = \sum_{v=1}^{c} P_v. \tag{4.27}$$

In fact, the matrix A of order n has as its first row $(0,1,1,\cdots,1,0,\cdots,0,0)$ where there are c ones after the initial 0. Subsequent rows of A are obtained by shifting c positions to the left at each step. The matrix A^k is the sum of all the matrix products of the form

$$P_{\alpha_1} P_{\alpha_2} \cdots P_{\alpha_k},$$

where $(\alpha_1, \alpha_2, \cdots, \alpha_k)$ runs through $\{1,2,\cdots,c\}^k$, the set of all k element subsets of $\{1,2,\cdots,c\}$.

The matrix $P_{\alpha_1} P_{\alpha_2} \cdots P_{\alpha_k}$ is a permutation matrix corresponding to the permutation

$$x \mapsto (-c)^k x + \sum_{i=1}^{k} \alpha_i (-c)^{k-i} \pmod{n}. \tag{4.28}$$

Theorem 4.6.6 (Lam and Van Lint, [149]) The matrix A defined in (4.27) satisfies $A^k = J - I$ for odd integer $k \geq 1$.

Proof Note that if $(\alpha_1, \alpha_2, \cdots, \alpha_k) \in \{1,2,\cdots,c\}^k$, then

$$1 \leq \left| \sum_{i=1}^{k} \alpha_i (-c)^{k-i} \right| \leq n - 1$$

(which is obtained by letting the α_i's alternate between 1 and c). Similarly, if $(\beta_1, \beta_2, \cdots, \beta_k) \in \{1,2,\cdots,c\}^k$ also, then by (4.26), $c^k = n - 1$, and so

$$\left| \sum_{i=1}^{k} \alpha_i (-c)^{k-i} - \sum_{i=1}^{k} \beta_i (-c)^{k-i} \right| \leq (c-1) \sum_{i=1}^{k} c^{k-i} = n - 2.$$

It follows that

$$\sum_{i=1}^{k} \alpha_i (-c)^{k-i} \equiv \sum_{i=1}^{k} \beta_i (-c)^{k-i} \pmod{n}$$

implies that the two sums are equal which is possible only if $(\alpha_1, \alpha_2, \cdots, \alpha_k) = (\beta_1, \beta_2, \cdots, \beta_k)$. Therefore if $(\alpha_1, \alpha_2, \cdots, \alpha_k)$ runs through $2, \cdots, c\}^k$, the permutations in (4.28) form the set of permutations of the form $x \mapsto x + \gamma$ $(1 \leq \gamma < n)$. This proves the theorem. □

Theorem 4.6.7 (Lam and Van Lint, [149]) Let A be defined in (4.27) and let $D = D(A)$. Then the dihedral group of order $2(c+1)$ is a group of automorphisms of the graph D.

Proof In the permutation $x \mapsto (-c)x + v$ which defines P_v, substitute $x = y + \lambda(c^k + 1)/(c+1)$. The result is the permutation $y \mapsto (-c)y + v$. Hence for $\lambda = 0,1,2,\cdots,c$, we find a permutation which leaves A invariant. These substitutions form a cyclic group of order c.

In the same way it can be found that the substitution

$$x = 1 - y + \lambda(c^k + 1)/(c + 1)$$

maps P_v to the permutation $y \mapsto (-c)y + (c + 1 - v)$ and therefore leaves A invariant. This, together with the cyclic group of order c above, yields a dihedral group acting on D. $\square$

It is not known whether the solution of $A^k = J - I$ is unique or not. In [149], it was shown that when $k = 3$ and $n = 9$, the dihedral group of Theorem 4.6.7 is the full automorphism group of the friendship graph D. However, whether the dihedral group in Theorem 4.6.7 is the full automorphism group of D in general remains to be determined.

We conclude this section by presenting two important theorems in the field. Let $T(m) = L(K_m)$ denote the line graph of the complete graph K_m, and let $L_2(m) = L(K_{m,m})$ denote the line graph of the complete bipartite graph $K_{m,m}$. Note that $T(m)$ is an $(m(m-1)/2, 2(m-2), m-2, 4)$-strongly regular graph, and $L_2(m)$ is an $(m^2, 2(m-2), m-2, 2)$-strongly regular graph.

Theorem 4.6.8 (Chang, [51] and [52], and Hoffman, [126]) Let $m \geq 4$ be an integer. Let G be an $(m(m-1)/2, 2(m-2), m-2, 4)$-strongly regular graph. If $m \neq 8$, then G is isomorphic to $T(m)$; and if $m = 8$, then G is isomorphic to one of the four graphs, one of which is $T(8)$.

Theorem 4.6.9 (Shrikhande, [254]) Let $m \geq 2$ be an integer and let G be an $(m^2, 2(m-2), m-2, 2)$-strongly regular graph. If $m \neq 4$, then G is isomorphic to $L_2(m)$; and if $m = 4$, then G is isomorphic to one of the two graphs, one of which is $L_2(4)$.

4.7 Eulerian Problems

In this section, linear algebra and systems of linear equations will be applied to the study of certain graph theory problems. Most of the discussions in this section will be over $GF(2)$, the field of 2 elements. Let $V(m, 2)$ denote the m-dimensional vector space over $GF(2)$. Let $\mathbf{B}^e_{n,m}$ denote the matrices $B \in \mathbf{B}_{n,m}$ such that all the column sums of B are positive and even. For subspaces V and W of $V(m, 2)$, $V + W$ is the subspace spanned by the vectors in $V \cup W$. Let $E = \{e_1, e_2, \cdots, e_m\}$ be a set. For each vector $\mathbf{x} = (x_1, x_2, \cdots, x_m)^T \in V(m, 2)$, the map

$$\mathbf{x} = (x_1, x_2, \cdots, x_m)^T \leftrightarrow E_{\mathbf{x}} = \{e_i \ : \ \text{where } x_i = 1, \ (1 \leq i \leq m)\} \tag{4.29}$$

yields a bijection between the subsets of E and the vectors in $V(m,2)$. Thus we also use $V(E,2)$ for $V(m,2)$, especially when we want to indicate the vectors in the vector space are indexed by the elements in E. Therefore, for a subset $E' \subseteq E$, it makes sense to use $V(E',2)$ to denote a subspace of $V(E,2)$ which consists of all the vectors whose ith component is always 0 whenever $e_i \in E - E'$, $1 \le i \le m$.

For two matrices B_1, B_2, write $B_1 \subseteq B_2$ to mean that B_1 is submatrix of B_2.

Throughout this section, $\mathbf{j} = (1,1,\cdots,1)^T$ denotes the m-dimensional vector each of whose component is a 1.

Definition 4.7.1 A matrix $B \in \mathbf{B}^e_{n,m}$ is *separable* if B is permutation similar to

$$\begin{bmatrix} B_{11} & 0 \\ 0 & B_{22} \end{bmatrix},$$

where $B_{11} \in \mathbf{B}_{n_1,m_1}$ and $B_{22} \in \mathbf{B}_{n_2,m_2}$ such that $n = n_1 + n_2$ and $m = m_1 + m_2$, for some positive integers n_1, n_2, m_1, m_2. A matrix B is *nonseparable* if it is not separable.

For a matrix $B \in \mathbf{B}^e_{n,m}$ with $\operatorname{rank}(B) \ge n-1$, a submatrix B' of B is *spanning in B* if $\operatorname{rank}(B') \ge n-1$. Note that for every $B \in \mathbf{B}^e_{n,m}$, each column sum of B is equal to zero modulo 2. Thus if $B \in \mathbf{B}^e_{n,m}$ is nonseparable, then over $GF(2)$, $\operatorname{rank}(B) = n-1$.

A matrix $B \in \mathbf{B}^e_{n,m}$ is *even* if $B\mathbf{j} = \mathbf{0} \pmod 2$; and B is *Eulerian* if B is both nonseparable and even.

A matrix $B \in \mathbf{B}_{n,m}$ is *simple* if it has no repeated columns and does not contain a zero column. In other words, in a simple matrix, the columns are mutually distinct, and no column is a zero column.

Example 4.7.1 When G is a graph and $B = B(G)$ is the incidence matrix of G, G is connected if and only if B is nonseparable; every vertex of G has even degree if and only if B is even; G is a simple graph if and only if B is simple; and G is eulerian (that is, both even and connected) if and only if B is eulerian.

Proposition 4.7.1 By Definition 4.7.1, Each of the following holds.

(i) If $B_1, B_2 \in \mathbf{B}_{n,m}$ are two permutation similar matrices, then B_1 is nonseparable if and only if B_2 is nonseparable.

(ii) Suppose that $B_1 \in \mathbf{B}_{n,m}$ and $B_2 \in \mathbf{B}_{n,m'}$ are matrices such that $B_1 \subseteq B_2$. If B_1 is nonseparable, then B_2 is nonseparable.

(iii) Suppose that $B_1 \in \mathbf{B}_{n,m}$ and $B_2 \in \mathbf{B}_{n,m'}$ are matrices such that $B_1 \subseteq B_2$. If B_1 has a submatrix B which is spanning in B_1, then B is also spanning in B_2.

(iv) Let $B = B(G) \in \mathbf{B}_{n,m}$ be an incident matrix of a graph G. If B is nonseparable, then $\operatorname{rank}(B) = n-1$.

Proof The first three claims follow from Definition 4.7.1. If $B = B(G)$ is nonseparable, then G is connected with n vertices, and so G has a spanning tree T with $n-1$ edges.

The submatrix of B consisting of the $n-1$ columns corresponding to the edges of T will be a matrix of rank $n-1$. $\square$

Definition 4.7.2 Let $A, B \in \mathbf{B}^e_{n,m}$ be matrices such that $A \subseteq B$. We say that A is *cyclable* in B if there exists an even matrix A' such that $A \subseteq A' \subseteq B$; and that A is *subeulerian* in B if there exists an eulerian matrix A' such that $A \subseteq A' \subseteq B$.

A matrix $B \in \mathbf{B}^e_{n,m}$ is *supereulerian* if there exists a matrix $B'' \in \mathbf{B}^e_{n,m''}$ for some integer $m'' \leq m$ such that $B'' \subseteq B$ and such that B'' is eulerian.

Let G be a graph, and let $B = B(G)$ be the incidence matrix of G. Then G is subeulerian (supereulerian, respectively) if and only if $B(G)$ is subeulerian (supereulerian, respectively).

Example 4.7.2 Let G be a graph and $B = B(G)$ be the incidence matrix of G. Then G is subeulerian if and only if G is a spanning subgraph of an eulerian graph; and G is supereulerian if and only if G contains a spanning eulerian subgraph.

What graphs are subeulerian? what graphs are supereulerian? These are questions proposed by Boesh, Suffey and Tindell in [12]. The same question can also be asked for matrices. It has been noted that the subeulerian problem should be restricted to simple matrices, for otherwise we can always construct an eulerian matrix B' with $B \in B'$ by adding additional columns, including a copy of each column of B.

The subeulerian problem is completely solved in [12] and in [138]. Jaeger's elegant proof will be presented later. However, as pointed out in [12], the supereulerian problem seems very difficult, even just for graphs. In fact, Pulleyblank [214] showed that the problem to determine if a graph is supereulerian is NP complete. Catlin's survey [48] and its update [57] will be good sources of the literature on supereulerian graphs and related problems.

Definition 4.7.3 Let $B \in \mathbf{B}_{n,m}$, let $E(B)$ denote the set of the labeled columns of B. We shall use $V(B,2)$ to denote $V(E(B),2)$. For a vector $\mathbf{x} \in V(E(B),2)$, let $E_{\mathbf{x}}$ denote the subset of $E(B)$ defined in (4.29). We shall write $V(B-\mathbf{x},2)$ for $V(E(B)-E_{\mathbf{x}},2)$, and write $V(\mathbf{x},2)$ for $V(E_{\mathbf{x}},2)$.

Therefore, $V(B-\mathbf{x},2)$ is a subspace of $V(B,2)$ consisting of all the vectors whose ith component is always 0 whenever the ith component of $\mathbf{x}$ is 1, $(1 \leq i \leq m)$, while $V(\mathbf{x},2)$ is the subspace consisting of all the vectors whose ith component is always 0 whenever the ith component of $\mathbf{x}$ is 0, $(1 \leq i \leq m)$.

For a matrix $B \in \mathbf{B}_{n,m}$ and a vector $\mathbf{x} \in V(B,2)$, we say that Column i of B is chosen by $\mathbf{x}$ if and only if the ith component of $\mathbf{x}$ is 1. Let $B_{\mathbf{x}}$ denote the submatrix of B consisting of all the columns chosen by $\mathbf{x}$. If $E' \subseteq E(B)$, then by the bijection (4.29), there is a vector $\mathbf{x} \in V(B,2)$ such that $E_{\mathbf{x}} = E'$. Define $B_{E'} = B_{\mathbf{x}}$. Conversely, for each submatrix $A \subseteq B$ with $A \in \mathbf{B}_{n,m}$, there exists a unique vector $\mathbf{x} \in V(B,2)$ such that $B_{\mathbf{x}} = A$. Then denote this vector $\mathbf{x}$ as $\mathbf{x}_A$.

A vector $\mathbf{x} \in V(B,2)$ is a *cycle* (of B) if $B_{\mathbf{x}}$ is even, and is *eulerian* (with respect to B) if $B_{\mathbf{x}}$ is eulerian. Note that the set of all cycles, together with the zero vector $\mathbf{0}$, form a vector subspace $\mathbf{C}$, called the *cycle space* of B; $\mathbf{C}^{\perp}$, the maximal subspace in $V(B,2)$ orthogonal to $\mathbf{C}$, is the *cocycle space* of B.

For $\mathbf{x}, \mathbf{y} \in V(m,2)$, write $\mathbf{x} \le \mathbf{y}$ if $\mathbf{y} - \mathbf{x} \ge 0$, and in this case we say that $\mathbf{y}$ contains $\mathbf{x}$. Let $B \in \mathbf{B}_{n,m}$ be a matrix and let $\mathbf{x} \in V(B,2)$ be a vector. The vector $\mathbf{x}$ is *cyclable* in B if there exists an cycle $\mathbf{y} \in V(B,2)$ such that $\mathbf{x} \le \mathbf{y}$. Denote the number of non zero components of a vector $\mathbf{x} \in V(n,2)$ by $\|\mathbf{x}\|_0$.

Theorem 4.7.1 Let $B \in \mathbf{B}^e_{n,m}$. A vector $\mathbf{x}$ is cyclable in B if and only if $\mathbf{x}$ does not contain a vector $\mathbf{z}$ in the cocycle space of B such that $\|\mathbf{z}\|_0$ is odd.

Proof Let $\mathbf{C}$ and $\mathbf{C}^{\perp}$ denote the cycle space and the cocycle space of B, respectively. Let $\mathbf{x} \in V(B,2)$. Then, by the definitions, the following are equivalent.

(A) $\mathbf{x}$ is cyclable in B.

(B) there exists a $\mathbf{y} \in \mathbf{C}$ such that $\mathbf{x} \le \mathbf{y}$.

(C) there exists a $\mathbf{y} \in \mathbf{C}$ such that $\mathbf{x} = \mathbf{y} + (\mathbf{y} + \mathbf{x}) \in \mathbf{C} + V(B - \mathbf{x}, 2)$.

(D) $\mathbf{x} \in \mathbf{C} + V(B - \mathbf{x}, 2)$.

Therefore, $\mathbf{x}$ is cyclable if and only if $\mathbf{x} \in \mathbf{C} + V(B - \mathbf{x}, 2)$. Note that

$$[\mathbf{C} + V(B - \mathbf{x}, 2)]^{\perp} = \mathbf{C}^{\perp} \cap V(B - \mathbf{x}, 2)^{\perp} = \mathbf{C}^{\perp} \cap V(\mathbf{x}, 2).$$

It follows that $\mathbf{x}$ is cyclable if and only if $\mathbf{x}$ is orthogonal to every vector in the subspace $\mathbf{C}^{\perp} \cap V(\mathbf{x}, 2)$. Since $\mathbf{x}$ contains every vector in $V(\mathbf{x}, 2)$, and since for every nonzero vector $\mathbf{z} \in \mathbf{C}^{\perp}$, $\|\mathbf{z}\|_0$ is odd, we conclude that $\mathbf{x}$ is cyclable if and only if $\mathbf{x}$ does not contain a vector $\mathbf{z}$ in the cocycle space of B such that $\|\mathbf{z}\|_0$ is odd. $\square$

Theorem 4.7.2 (Jaeger, [138]) Let $A, B \in \mathbf{B}^e_{n,m}$ be matrices such that $A \subseteq B$. Each of the following holds.

(i) A is cyclable in B if and only if there exists no vector $\mathbf{z}$ in the cocycle space of B such that $\|\mathbf{z}\|_0$ is odd.

(ii) If, in addition, A is a nonseparable. Then A is subeulerian in B if and only if there exists no vector $\mathbf{z}$ in the cocycle space of B such that $\|\mathbf{z}\|_0$ is odd.

Proof By Definition 4.7.2 and since A is nonseparable, A is subeulerian in B if and only if the vector $\mathbf{x}_A$ is cyclable in B. Therefore, Theorem 4.7.2(ii) follows from Theorem 4.7.2(i).

By Theorem 4.7.1, $\mathbf{x}_A$ is cyclable in B if and only if $\mathbf{x}_A$ does not contain a vector $\mathbf{z}$ in the cocycle space of B such that $\|\mathbf{z}\|_0$ is odd. This proves Theorem 4.7.1(i). $\square$

Theorem 4.7.3 (Boesch, Suffey and Tindell, [12], and Jaeger, [138]) A connected simple graph on n vertices is subeulerian if and only if G is not spanned by a complete bipartite with an odd number of edges.

Proof Let G be a connected simple graph on n vertices. Theorem 4.7.3 obtains by applying Theorem 4.7.2 with $A = B(G)$ and $B = A(K_n)$ (Exercise 4.12). $\square$

Definition 4.7.4 A vector $\mathbf{b} \in V(n, 2)$ is an even vector if $||\mathbf{b}|| \equiv 0 \pmod 2$. A matrix $H \in \mathbf{B}_{n,m}$ is *collapsible* if for any even vector $\mathbf{b} \in V(n, 2)$, the system

$$H\mathbf{x} = \mathbf{b},$$

has a solution $\mathbf{x}$ such that $H_{\mathbf{x}}$ is nonseparable and is spanning in H, (such a solution $\mathbf{x}$ is called a $\mathbf{b}$-*solution*).

Let n, m, n_1, m_1, m_2 be integers with $n \geq n_1 \geq 0$, and $m \geq m_1, m_2 \geq 0$. Let $B_{11} \in \mathbf{B}_{n_1,m_1}, B_{12} \in \mathbf{B}_{n_1,m_2}, B_{22} \in \mathbf{B}_{n-n_1,m_2}$ and $H \in \mathbf{B}_{n-n_1,m-(m_1+m_2)}$. Let s_i denote the column sum of the ith column of B_{22}, $1 \leq i \leq m_2$, and let $B \in \mathbf{B}_{n,m}$ with the following form

$$B = \begin{bmatrix} B_{11} & B_{12} & 0 \\ 0 & B_{22} & H \end{bmatrix}. \tag{4.30}$$

Define $B/H \in \mathbf{B}_{n-n_2+1,m-m_2}$ to be the matrix of the following form

$$B/H = \begin{bmatrix} B_{11} & B_{12} \\ 0 & \mathbf{v}_H^T \end{bmatrix},$$

where $\mathbf{v}_H^T \in V(m, 2)$ such that $\mathbf{v}_H^T = (v_{m_1+1}, v_{m_1+2}, \cdots, v_{m_1+m_2})$ and such that $v_{m_1+i} \equiv s_i \pmod 2$, $(1 \leq i \leq m_2)$.

Proposition 4.7.2 Let $B_1, B_2 \in \mathbf{B}_{n,m}$ such that B_1 is permutation similar to B_2. Then each of the following holds.

(i) B_1 is collapsible if and only if B_2 is collapsible.

(ii) B_1 is supereulerian if and only if B_2 is supereulerian.

Proof Suppose that $B_1 = PB_2Q$. Let $\mathbf{b} \in V(n, 2)$ be an even vector. Then since $\mathbf{b}$ is even, $\mathbf{b}' = P^{-1}\mathbf{b} \in V(n, 2)$ is also even. Since B_1 is collapsible, $B_1\mathbf{y} = \mathbf{b}'$ has solution $\mathbf{y} \in V(m, 2)$ such that $(B_1)_{\mathbf{y}}$ is nonseparable.

Let $\mathbf{x} = Q^{-1}\mathbf{y}$. Then

$$P\mathbf{b} = \mathbf{b}' = B_1\mathbf{y} = PB_2Q\mathbf{y} = PB_2\mathbf{x},$$

and so $B\mathbf{x} = \mathbf{b}$ has a solution $\mathbf{x} = Q\mathbf{y}$.

Note that $P^{-1}(B_1)_y = (B_2Q)_y = (B_2)_x$. Therefore, B_x is nonseparable, by Proposition 4.7.1 and by the fact that $(B_1)_y$ is nonseparable. This proves Proposition 4.7.2(i).

Proposition 4.7.2(ii) follows from Proposition 4.7.2(i) by letting $\mathbf{b} = \mathbf{0}$ in the proof above. $\square$

Proposition 4.7.3 If $H \in \mathbf{B}^e_{n,m}$ is collapsible, then H is nonseparable, and $\mathrm{rank}(H) = n - 1$.

Proof By Definition 4.7.4, the system $H\mathbf{x} = \mathbf{0}$ has a $\mathbf{0}$-solution $\mathbf{x}$, and so $H_\mathbf{x}$ is nonseparable and spanning in H, and so Proposition 4.7.3 follows from Proposition 4.7.1. $\square$

Theorem 4.7.4 Let H be a collapsible matrix and let B be a nonseparable matrix of the form in (4.30). Each of the following holds.

(i) If B/H is collapsible, then B is collapsible.

(ii) If B/H is supereulerian, then B is supereulerian.

Proof We adopt the notation in Definition 4.7.4. Let $\mathbf{b}$ be an even vector, and consider the system of linear equations

$$\begin{bmatrix} B_{11} & B_{12} & 0 \\ 0 & B_{22} & H \end{bmatrix} \begin{pmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \\ \mathbf{x}_3 \end{pmatrix} = \begin{pmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \end{pmatrix} = \mathbf{b}, \tag{4.31}$$

where $\mathbf{b}_1 \in V(n_1, 2), \mathbf{b}_2 \in V(n - n_1, 2)$, $\mathbf{x}_1 \in V(m_1, 2), \mathbf{x}_2 \in V(m_2, 2)$ and $\mathbf{x}_3 \in V(m - m_1 - m_2, 2)$. Let

$$\delta = \begin{cases} 0 & \text{if } \mathbf{b}_1 \text{ is even} \\ 1 & \text{otherwise} \end{cases}, \quad \text{and } \mathbf{b}' = \begin{pmatrix} \mathbf{b}_1 \\ \delta \end{pmatrix}.$$

Then $\mathbf{b}'$ is an even vector. Since B/H is collapsible,

$$(B/N)\mathbf{x}_{12} = \begin{bmatrix} B_{11} & B_{12} \\ 0 & \mathbf{v}_H^T \end{bmatrix} \begin{pmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{pmatrix} = \begin{pmatrix} \mathbf{b}_1 \\ \delta \end{pmatrix},$$

has a $\mathbf{b}'$-solution $\mathbf{x}_{12}$. Therefore

$$(G/H)_{\mathbf{x}_{12}} \text{ is nonseparable and spanning in } G/H. \tag{4.32}$$

Since $\mathbf{b}$ is even, by the definition of δ, $\mathbf{b}_2 - B_{22}\mathbf{x}_2$ is also even. Since H is collapsible, the system

$$H\mathbf{x}_3 = \mathbf{b}_2 - B_{22}\mathbf{x}_2$$

has a $(\mathbf{b}_2 - B_{22}\mathbf{x}_2)$-solution $\mathbf{x}_3$. Therefore,

$$H_{\mathbf{x}_3} \text{ is nonseparable and spanning in } H. \tag{4.33}$$

Now let

$$\mathbf{x} = \begin{pmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \\ \mathbf{x}_3 \end{pmatrix}.$$

We have

$$B\mathbf{x} = \begin{bmatrix} B_{11} & B_{12} & 0 \\ 0 & B_{22} & H \end{bmatrix} \begin{pmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \\ \mathbf{x}_3 \end{pmatrix} = \begin{bmatrix} B_{11}\mathbf{x}_1 + B_{12}\mathbf{x}_2 \\ B_{22}\mathbf{x}_2 + H\mathbf{x}_3 \end{bmatrix} = \begin{pmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \end{pmatrix}.$$

Thus, to see that $\mathbf{x}$ is a $\mathbf{b}$-solution for equation (4.31), it remains to show that $B_{\mathbf{x}}$ is nonseparable and is spanning in B.

Claim 1 $B_{\mathbf{x}}$ is nonseparable.

Suppose that $B_{\mathbf{x}}$ is separable. We may assume that

$$B_{\mathbf{x}} = \begin{bmatrix} X & 0 \\ 0 & Y \end{bmatrix}, \text{ where } X \neq 0 \text{ and } Y \neq 0. \tag{4.34}$$

If $\begin{bmatrix} X \\ 0 \end{bmatrix}$ has a submatrix $\begin{bmatrix} X_H \\ 0 \end{bmatrix}$ that is a submatrix of $\begin{bmatrix} 0 \\ H \end{bmatrix}$, and if $\begin{bmatrix} 0 \\ Y \end{bmatrix}$ has a submatrix $\begin{bmatrix} 0 \\ Y_H \end{bmatrix}$ that is a submatrix of $\begin{bmatrix} 0 \\ H \end{bmatrix}$, then

$$\begin{bmatrix} 0 \\ H_{\mathbf{x}_3} \end{bmatrix} = \begin{bmatrix} X_H & 0 \\ 0 & Y_H \end{bmatrix}.$$

By (4.33), $H_{\mathbf{x}_3}$ is nonseparable, and so we must have $X_H = 0$. By the definition of $\mathbf{B}^{e}_{n,m}$, B has no zero columns, and so

$$\text{the columns chosen by } \mathbf{x}_3 \text{ are in the last } m - (m_1 + m_2) \text{ columns of } B. \tag{4.35}$$

By (4.34) and (4.35), $\begin{bmatrix} X \\ 0 \end{bmatrix}$ has a submatrix $\begin{bmatrix} X_{B/H} \\ 0 \end{bmatrix}$ that is a submatrix of $\begin{bmatrix} B_{11} & B_{12} \\ 0 & B_{22} \end{bmatrix}$, and $\begin{bmatrix} 0 \\ Y \end{bmatrix}$ has a submatrix $\begin{bmatrix} 0 \\ Y_{B/H} \end{bmatrix}$ that is a submatrix of $\begin{bmatrix} B_{11} & B_{12} \\ 0 & B_{22} \end{bmatrix}$. By (4.31) and (4.33), $X_{B/H} \neq 0$. If $Y_{B/H} \neq 0$, then $(B/H)_{\mathbf{x}_{12}}$ is separable, contrary to (4.32). Therefore, $Y_{B/H} = 0$, and so

$$\text{the columns chosen by } \mathbf{x}_{12} \text{ are in the first } (m_1 + m_2) \text{ columns of } B. \tag{4.36}$$

By (4.34), (4.35) and (4.36), $Y = H_{\mathbf{x}_3}$ and X is the first n_1 rows of $B/H_{\mathbf{x}_{12}}$. This implies that the last row of $B/H_{\mathbf{x}_{12}}$ is a zero row, and so $B/H_{\mathbf{x}_{12}}$ is separable, contrary to (4.31). This proves Claim 1.

Claim 2 B_x is spanning in B.

It suffices to show that $\text{rank}(B_x) = n - 1$. Since $(B/H)_{x_{12}}$ spans in B/H, there exist l_1 column vectors $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_{l_1}$ in the first m_1 columns of B, and l_2 column vectors $\mathbf{w}_1, \mathbf{w}_2, \cdots, \mathbf{w}_{l_2}$ in the middle m_2 columns of B such that $l_1 + l_2 = n_1$ and such that $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_{l_1}, \mathbf{w}_1, \mathbf{w}_2, \cdots, \mathbf{w}_{l_2}$ are linearly independent over $GF(2)$. Since H_{x_3} spans in H, there exist column vectors $\mathbf{u}_1, \cdots, \mathbf{u}_{n_2-1}$ in the last $m - (m_1 + m_2)$ columns of B such that $\mathbf{u}_1, \cdots, \mathbf{u}_{n-n_1-1}$ are linearly independent over $GF(2)$. It remains to show that $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_{l_1}, \mathbf{w}_1, \mathbf{w}_2, \cdots, \mathbf{w}_{l_2}$ and $\mathbf{u}_1, \cdots, \mathbf{u}_{n-n_1-1}$ are linearly independent. If not, there exist constants $c_1, \cdots, c_{l_1}, c'_1 \cdots, c'_{l_2}, c''_1, \cdots, c''_{n_2-1}$ such that

$$\sum_{i=1}^{l_1} c_i \mathbf{v}_i + \sum_{i=1}^{l_2} c'_i \mathbf{w}_i + \sum_{i=1}^{n-n_1-1} c''_i \mathbf{u}_i = \mathbf{0}. \tag{4.37}$$

Consider the first n_1 equations in (4.37), and since $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_{l_1}, \mathbf{w}_1, \mathbf{w}_2, \cdots, \mathbf{w}_{l_2}$ are linearly independent over $GF(2)$, we must have $c_1 = \cdots = c_{l_1} = 0$ and $c'_1 = \cdots = c'_{l_2} = 0$. This , together with the fact that $\mathbf{u}_1, \cdots, \mathbf{u}_{n-n_1-1}$ are linearly independent, implies that $c''_1 = \cdots = c''_{n-n_1-1} = 0$. Therefore $\text{rank}(B_x) = n - 1$, as expected. $\square$

Definition 4.7.5 Let $B \in \mathbf{B}^e_{n,m}$. Let $\tau(B)$ denote the largest possible number k such that $E(B)$ can be partitioned into k subsets $E_1, E_2, \cdots, E_k$ such that each B_{E_i} is both nonseparating and spanning in B, $1 \le i \le k$.

Example 4.7.3 Let G be a graph and let $B = B(G)$. Then $\tau(B)$ is the spanning tree packing number of G, which is the maximum number of edge-disjoint spanning trees in G.

Proposition 4.7.4 Let $B \in \mathbf{B}^e_{n,m}$ be a matrix with $\tau(B) \ge 1$. Then for any even vector $\mathbf{b} \in V(n, 2)$, the system $B\mathbf{x} = \mathbf{b}$ has a solution.

Proof We may assume that $B \in \mathbf{B}^e_{n,n-1}$ and $\text{rank}(B) = n-1$, for otherwise by $tau(B) \ge 1$, we can pick a submatrix B' of B such that B' is nonseparating and spanning in B to replace B. Since $\mathbf{b}$ is even and since every columns sum of A is even, it follows that the $\text{rank}([B, \mathbf{b}]) = n - 1 = \text{rank}(B)$ also. Therefore, $B\mathbf{x} = \mathbf{b}$ has a solution. $\square$

Theorem 4.7.5 Let $B \in \mathbf{B}^e_{n,m}$ be a matrix with $\tau(B) \ge 2$. Then B is collapsible.

Proof We need to show that for every even vector $\mathbf{b} \in V(n, 2)$, $B\mathbf{x} = \mathbf{b}$ has a $\mathbf{b}$-solution.

Since $\tau(B) \ge 2$, we may assume that for some $B_1 \in \mathbf{B}^e_{n_1,m}$ and $B_2 \in \mathbf{B}^e_{n-n_1,m}$, $B = [B_1, B_2]$ such that each B_i is nonseparable and spanning in B. Let $\mathbf{x}_1 = (1, 1, \cdots, 1, 0, \cdots, 0)^T \in V(n, 2)$ such that the first n_1 components of $\mathbf{x}_1$ are 1, and all the other components of $\mathbf{x}_1$ are 0.

Write $B = [B_1, B_2] = [B_1, 0] + [0, B_2]$. Since $B_1 \in \mathbf{B}^e_{n_1,m}$, $[B_1, 0]\mathbf{x}_1$ is even, and so the vector $\mathbf{b} - [B_1, 0]\mathbf{x}_1 \in V(n, 2)$ is also even.

Since $\tau([0, B_2]) \geq 1$, by Proposition 4.7.4, the system $[0, B_2]\mathbf{x}_2 = \mathbf{b} - [B_1, 0]\mathbf{x}_1$ has a solution $\mathbf{x}_2$, such that the first n_1 components of $\mathbf{x}_2$ are 0.

Let $\mathbf{x} = \mathbf{x}_1 + \mathbf{x}_2$. Then since the last $n - n_1$ components of $\mathbf{x}_1$ are 0 and the first n_1 components of $\mathbf{x}_2$ are 0, we have

$$B\mathbf{x} = [B_1, B_2](\mathbf{x}_1 + \mathbf{x}_2) = \mathbf{b}.$$

By the definition of $\mathbf{x}_1$, $B_{\mathbf{x}}$ contains B_1 as a submatrix, and so $B_{\mathbf{x}}$ is both spanning in B and nonseparable, by Proposition 4.7.1. $\square$

Theorem 4.7.6 (Catlin, [47] and Jaeger, [138]) If a graph G has two edge-disjoint spanning trees, then G is collapsible, and supereulerian.

We close this section by mentioning a completely different definition of E ulerian matrix in the literature. For a square matrix A whose entries are in $\{0, 1, -1\}$, Camion [46] called the matrix A *Eulerian* if the row sums and the column sums of A are even integers, and he proved the following result.

Theorem 4.7.7 (Camion [46]) An $m \times n$ $(0, 1, -1)$ matrix A is totally unimodular if and only if the sum of all the elements of each Eulerian submatrix of A is a multiple of 4.

4.8 The Chromatic Number

Graphs considered in this section are simple, and groups considered in this section are all finite Abelian groups. The focus of this section is certain linear algebra approach in the study of graph coloring problems.

Let Γ denote a group and let $p > 0$ be an integer. Denote by $V(p, \Gamma)$ the set of p-tuples $(g_1, g_2, \cdots, g_p)^T$ such that each $g_i \in \Gamma$, $(1 \leq i \leq p)$. Given $\mathbf{g} = (g_1, g_2, \cdots, g_p)^T$ and $\mathbf{h} = (h_1, h_2, \cdots, h_p)^T$, we write $\mathbf{g} \not\equiv \mathbf{h}$ to mean that $g_i \neq h_i$ for every i with $1 \leq i \leq p$. For notational convenience, we assume that the binary operation of Γ is addition and that 0 denotes the additive identity of Γ. We also adopt the convention that for integers $1, -1, 0$, and for an element $g \in \Gamma$, the multiplication $(1)(g) = g, (0)(g) = 0$, the additive identity of Γ, and $(-1)g = -g$, the additive inverse of g in Γ.

Let G be a graph, $k \geq 1$ be an integer, and let $C(k) = \{1, 2, \cdots, k\}$ be a set of k distinct elements (referred as colors in this section). A function $c : V(G) \mapsto C(k)$ is a *proper vertex k-coloring* if $f(u) \neq f(v)$ whenever $uv \in E(G)$. Elements in the set $C(k)$ are referred as colors. A graph G is *k-colorable* if G has a proper k-coloring. Note that a graph G has a proper k-coloring if and only if $V(G)$ can be partitioned into k independent sets, each of which is called a *color class.* The smallest integer k such that G is k-colorable is $\chi(G)$, the *chromatic number* of G. If for every vertex $v \in V(G)$, $\chi(G-v) < \chi(G) = k$, then G is *k-critical* or just critical.

Let G be a graph with n vertices and m edges. We can use elements in Γ as colors. Arbitrarily assign orientations to the edges of G to get a digraph $D = D(G)$, and let $B = B(D)$ be the incidence matrix of D. Then a proper $|\Gamma|$-coloring is an element $\mathbf{c} \in V(n,\Gamma)$ such that

$$B^T\mathbf{c} \not\equiv \mathbf{0},$$

where $\mathbf{0} \in V(m,\Gamma)$.

Viewing the problem in the nonhomogeneous way, for an element $\mathbf{b} \in V(m,\Gamma)$, an element $\mathbf{c} \in V(n,\Gamma)$ is a *$(\Gamma,\mathbf{b})$-coloring* if

$$B^T\mathbf{c} \not\equiv \mathbf{b}. \tag{4.38}$$

Definition 4.8.1 Let Γ denote a group. A graph G is *Γ-colorable* if for any $\mathbf{b} \in V(m,\Gamma)$, there is always a $(\Gamma,\mathbf{b})$-coloring $\mathbf{c}$ satisfying (4.38).

Vectors in $V(|E(G)|,\Gamma)$ can be viewed as functions from $E(G)$ into Γ, and vectors in $V(|V(G)|,\Gamma)$ can be viewed as functions from $V(G)$ into Γ. With this in mind, for any $\mathbf{b} \in V(|E(G)|,\Gamma)$ and $e \in E(G)$, $\mathbf{b}(e)$ denotes the component in $\mathbf{b}$ labeled with element e. Similarly, for any $\mathbf{c} \in V(|V(G)|,\Gamma)$ and $v \in V(G)$, $\mathbf{c}(v)$ denotes the component in $\mathbf{c}$ labeled with element v.

Therefore, we can equivalently state that for a function $\mathbf{b} \in V(m,\Gamma)$, a $(\Gamma,\mathbf{b})$-coloring is a function $\mathbf{c} \in V(n,\Gamma)$ such that for each oriented edge $e = (x,y) \in E(G)$, $\mathbf{c}(x) - \mathbf{c}(y) \neq \mathbf{b}(e)$; and that a graph G is Γ-colorable if, under some fixed orientation of G, for ant function $\mathbf{b} \in V(m,\Gamma)$, G always has a $(\Gamma,\mathbf{b})$-coloring.

Proposition 4.8.1 If for one orientation $D = D(G)$, that G is Γ-colorable, then for any orientation of G, G is also Γ-colorable.

Proof Let D_1 and D_2 be two orientations of G, and assume that G is Γ-colorable under D_1. It suffices to show that when D_2 is obtained from D_1 by reversing the direction of exactly one edge, G is Γ-colorable under D_2.

Let $B_1 = B(D_1)$ and $B_2 = B(D_2)$. We may assume that B_1 and B_2 differ only in Row 1, where the first row of B_2 equals the first row of B_1 multiplied by (-1).

Let $\mathbf{b} = (b_1, b_2, \cdots, b_m)^T \in V(m,\Gamma)$. Then $\mathbf{b}' = (-b_1, b_2, \cdots, b_m)^T \in V(m,\Gamma)$ also. Since G is Γ-colorable under D_1, there exists a $(\Gamma,\mathbf{b}')$-coloring $\mathbf{c}' = (c_1, c_2, \cdots, c_n)^T \in V(n,\Gamma)$. Note that $\mathbf{c} = (-c_1, c_2, \cdots, c_n)^T \in V(n,\Gamma)$ is a $(\Gamma,\mathbf{b})$-coloring, and so G is also Γ-colorable under D_2. $\square$

Definition 4.8.2 Let G be a simple graph. The define the *group chromatic number* $\chi_1(G)$ to be the smallest integer k such that whenever Γ is a group with $|\Gamma| \geq k$, G is Γ-colorable.

Example 4.8.1 (Lai and Zhang, [152]) For any positive integers m and k, let G be a graph with $(2m+k)+(m+k)^{m+k}-1$ vertices formed from a complete subgraph K_m on m vertices and a complete bipartite subgraph K_{r_1,r_2} with $r_1 = m+k$ and $r_2 = (m+k)^{m+k}$ such that

$$|V(K_m) \cap V(K_{r_1,r_2})| = 1.$$

We can routinely verify that $\chi(G) = m$ and $\chi_1(G) = m+k$ (Exercises 4.19).

Immediately from the definition of $\chi_1(G)$, we have

$$\chi(G) \leq \chi_1(G). \tag{4.39}$$

and

$$\chi_1(G') \leq \chi_1(G), \text{ if } G' \text{ is a subgraph of } G. \tag{4.40}$$

More properties on the group chromatic number $\chi_1(G)$ can be found in the exercises. We now present the Brooks coloring theorem for the group chromatic number.

Theorem 4.8.1 (Lai and Zhang, [152]) Let G be a connected graph with maximum degree $\Delta(G)$. Then

$$\chi_1(G) \leq \Delta(G) + 1,$$

where equality holds if and only if $G = C_n$ is the cycle on n vertices, or $G = K_n$ is the complete graph on n vertices.

The proof of Theorem 4.8.1 has been divided into several exercises at the end of this chapter.

Modifying a method of Wilf [275], we can applied Theorem 4.8.1 to prove an improvement of Theorem 4.8.1, yielding a better upper bound of $\chi_1(G)$ in terms of $\lambda_1(G)$, the largest eigenvalue of G.

Lemma 4.8.1 Let G be a graph with $\chi_1(G) = k$. Then G contains a connected subgraph G' such that $\chi_1(G) = k$ and $\delta(G') \geq k-1$.

Proof By Definition 4.8.1, G is Γ-colorable if and only if each component of G is Γ-colorable. Therefore, we may assume that G is connected.

By (4.40), G contains a connected subgraph G' such that $\chi_1(G) = k$ but for any proper subgraph G'' of G', $\chi_1(G'') < k$. Let $n = |V(G')|$ and $m = |E(G')|$.

If $\delta(G') < k-1$, then G' has a vertex v of degree at most $d \le k-2$ in G'. Note that by the choice of G', $\chi_1(G'-v) \le k-1$. By Proposition 4.8.1, we assume that G' is a digraph such that all the edges incident with v are directed from v, and such that v corresponds to the last row of $B = B(G')$. Let $v_1, v_2, \cdots, v_d$ be the vertices adjacent to v in G', and correspond to the first d rows of B, respectively; and let $e_i = (v, v_i)$, $(1 \le i \le d)$ denote the edges incident with v in G', and correspond to the first d columns of B, respectively.

Let Γ be a group with $|\Gamma| = k-1$. For any $\mathbf{b} = (b_1, b_2, \cdots, b_d, b_{d+1}, \cdots, b_m)^T \in V(m, \Gamma)$. Let $\mathbf{b}' = (b_{d+1}, \cdots, b_m)^T = \in V(m-d, \Gamma)$ Since $\chi_1(G'-v) \le k-1 = |\Gamma|$, there exists a $(\Gamma, \mathbf{b}')$-coloring $\mathbf{c}' = (c_1, c_2, \cdots, c_{n-1})^T \in V(n-1, \Gamma)$.

Note that $|\Gamma - \{b_1+c_1, b_2+c_2, \cdots, b_d+c_d\}| \ge (k-1)-d > 0$, there exists a $c_n \in \Gamma - \{b_1+c_1, b_2+c_2, \cdots, b_d+c_d\}$. Let $\mathbf{c} = (c_1, \cdots, c_{n-1}, c_n)^T \in V(n, \Gamma)$. Since $c_n - c_i \ne b_i$, $1 \le i \le d$, $\mathbf{c}$ is a $(\Gamma, \mathbf{b})$-coloring of G'. As Γ and $\mathbf{b}$ are arbitrary, it follows that $\chi_1(G') \le k-1$, contrary to the assumption that $\chi_1(G') = k$. $\square$

Theorem 4.8.2 Let G be a connected graph and let λ_1 be the largest eigenvalue of $A(G)$. Each of the following holds.

(i)

$$\chi_1(G) \le 1 + \lambda_1, \tag{4.41}$$

where equality holds if and only if G is a compete graph or a cycle.

(ii) (Wilf, [275])

$$\chi(G) \le 1 + \lambda_1,$$

where equality holds if and only if G is a compete graph or an odd cycle.

Proof By (4.39), and by the well know fact that $\chi(C_n) \le 3$ for any cycle C_n on $n \ge 3$ vertices, with equality holds if and only n is odd, it is straightforward to see that Theorem 4.8.2(ii) follows from Theorem 4.8.2(i).

By Lemma 4.8.1, G has a connected subgraph G' satisfying Lemma 4.8.1. Thus

$$\lambda_1(G) + 1 \ge \lambda_1(G') + 1 \ge \chi_1(G). \tag{4.42}$$

By Theorem 1.6.1, $\lambda_1(G') \ge \delta(G') \ge k-1$, and so (4.41) obtains. Assume that equality holds in (4.41). Then

$$\lambda_1(G) = k - 1 = \chi_1(G) - 1 \tag{4.43}$$

Hence equalities hold everywhere in (4.42). By Corollary 1.3.2A, G' is $(k-1)$ regular.

If $k > 3$, then by Theorem 4.8.1, $G' = K_k$. Denote

$$V(G) = \{v_1, \cdots, v_k, v_{k+1}, \cdots, v_n\}$$

such that $V(G') = \{v_1, \cdots, v_k\}$. Then

$$A = (a_{i,j}) = A(G) = \begin{bmatrix} J_k - I_k & A_{12} \\ A_{21} & A_{22} \end{bmatrix}.$$

We want to show that $k = n$ and so $G = G'$. If $k < n$, then let $\mathbf{x} = (1, 1, \cdots, 1, \epsilon, 0, \cdots, 0)^T$ be a n-dimensional vector such that the first k components of $\mathbf{x}$ are 1, and the $(k+1)$th component is ϵ and all the other components are 0. By Theorem 1.3.2,

$$\lambda_1(G) \geq \frac{(A\mathbf{x})^T\mathbf{x}}{\|\mathbf{x}\|^2} = \frac{k(k-1) + 2\epsilon \sum_{j=1}^{k} a_{j,k+1}}{k + \epsilon^2}.$$

Note that $\sum_{j=1}^{k} a_{j,k+1} \geq 0$. If $\sum_{j=1}^{k} a_{j,k+1} > 0$, then choose ϵ so that $2\sum_{j=1}^{k} a_{j,k+1} > \epsilon(k-1)$, which results in $\lambda_1(G) > k-1$, contrary to (4.43). Therefore, $\sum_{j=1}^{k} a_{j,k+1} = 0$, and so $a_{j,k+1} = 0$ for each j with $1 \leq j \leq k$. Repeating this process yields that $A_{12} = 0$, and so $A_{21} = A_{12}^T = 0$, contrary to the assumption that G is connected. Therefore, we must have $n = k$, and so $G = G' = K_n$.

With a similar argument, we can also show that when $k = 2$, $G = G' = C_n$ is a cycle. □

Corollary 4.8.2 Let G be a connected graph with n vertices and m edges. Then

$$\chi_1(G) \leq 1 + \sqrt{\frac{2m(n-1)}{n}}. \tag{4.44}$$

Equality holds if and only if G is a complete graph.

Proof Let $\lambda_1, \lambda_2, \cdots, \lambda_n$ denote the eigenvalues of G. By Schwarz inequality, by $\sum_{i=1}^{n} \lambda_i = 0$ and $\sum_{i=1}^{n} \lambda_i^2 = 2m$,

$$\lambda_1^2 = (-\lambda_1)^2 = \left(\sum_{i=2}^{n} \lambda_i\right)^2 \leq (n-1)\sum_{i=2}^{n} \lambda_i^2 = (n-1)(2m - \lambda_1^2).$$

Therefore, (4.44) follows from that (4.41).

Suppose equality holds in (4.44). Then by Theorem 4.8.2(i), G must be a complete graph or a cycle. But in this case, we must also have $\lambda_2 = \lambda_3 = \cdots = \lambda_n$, and so G must be a K_n. (see Exercise 1.7 for the spectrums of K_n and C_n.) □

The following result unifies Brooks coloring theorem and Wilf coloring theorem.

Theorem 4.8.3 (Szekers and Wilf, [258], Cao, [42]) Let $f(G)$ be a real function on a graph G satisfying the properties (P1) and (P2) below:

(P1) If H is an induced subgraph of G, then $f(H) \leq f(G)$.

(P2) $f(G) \geq \delta(G)$ with equality if and only if G is regular.

Then $\chi(G) \leq f(G) + 1$ with equality if and only if G is an odd cycle or a complete graph.

We now turn to the lower bounds of $\chi(G)$. For a graph G, $\omega(G)$, the ***clique number*** of G, is the maximum k such that G has K_k as a subgraph. We immediately have a trivial lower bound for $\chi(G)$:

$$\chi_1(G) \geq \chi(G) \geq \omega(G).$$

Theorem 4.8.4 below present a lower bound obtained by investigating $A(G)$, the adjacency matrix of G. A better bound (Theorem 4.8.5 below) was obtained by Hoffman [123] by working on the eigenvalues of G.

Theorem 4.8.4 If G has n vertices and m edges, then

$$\chi(G) \geq \lfloor \frac{n^2}{n^2 - 2m} \rfloor.$$

Proof Note that G has a proper k-coloring if and only if $V(G)$ can be partitioned into k independent subsets $V_1, V_2, \cdots, V_k$. Let $n_i = |V_i|$, $(1 \leq i \leq k)$. we may assume that the adjacency matrix $A(G)$ has the form

$$A(G) = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1k} \\ A_{21} & A_{22} & \cdots & A_{2k} \\ & \cdots & \cdots & \\ A_{k1} & A_{k2} & \cdots & A_{kk} \end{bmatrix}, \tag{4.45}$$

where the rows in $[A_{i1}, A_{i2}, \cdots, A_{ik}]$ correspond to the vertices in V_i, $1 \leq i \leq k$. Since V_i is independent, $A_{ii} = 0 \in \mathbf{B}_{n_i}$, $(1 \leq i \leq k)$, and so

$$2m = \|A(G)\| \leq n^2 - \sum_{i=1}^{k} n_i^2. \tag{4.46}$$

It follows by Schwarz inequality that

$$k \sum_{i=1}^{k} n_i^2 \geq \left(\sum_{i=1}^{k} n_i \right)^2 = n^2,$$

and so Theorem 4.8.4 follows by (4.46). $\square$

Lemma 4.8.2 (Hoffman, [123]) Let A be a real symmetric matrix with the form (4.45) such that each A_{ii} are square matrices. Then

$$\lambda_{\max}(A) + (k-1)\lambda_{\min}(A) \leq \sum_{i=1}^{k} \lambda_{\max}(A_{ii}).$$

Theorem 4.8.5 (Hoffman, [123]) Let G be a graph with n vertices and $m > 0$ edges, and with eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. Then

$$\chi(G) \geq 1 - \frac{\lambda_1}{\lambda_n}. \tag{4.47}$$

Proof Let $k = \chi(G)$. Then $V(G)$ can be partitioned into k independent subsets $V_1, \cdots, V_k$, and so we may assume that $A(G)$ has the form in (4.45), where $A_{ii} = 0$, $1 \le i \le k$. By Lemma 4.8.2, we have

$$\lambda_1 + (k-1)\lambda_n \le 0.$$

However, since $m > 0$, we have $\lambda_n < 0$, and so (4.47) follows. $\square$

4.9 Exercises

Exercise 4.1 Solve the difference equation

$$\begin{cases} F_{n+3} = 2F_{n+1} + F_n + n, \\ F_0 = 1, F_1 = 0, F_3 = 1. \end{cases}$$

Exercise 4.2 Let $S_\lambda(t,k,v)$ be a t-design. Show that

$$b = \lambda \begin{pmatrix} v \\ t \end{pmatrix} / \begin{pmatrix} k \\ t \end{pmatrix}.$$

Exercise 4.3 Show that for $i = 0, 1, \cdots, t$, a t-design $S_\lambda(t,k,v)$ is also an i-design $S_{\lambda_i}(i,k,v)$, where

$$\lambda_i = \lambda \begin{pmatrix} v-i \\ t-i \end{pmatrix} / \begin{pmatrix} k-i \\ t-i \end{pmatrix}.$$

Exercise 4.4 Prove that $bk = vr$ in BIBD.

Exercise 4.5 Prove Theorem 4.3.8.

Exercise 4.6 Let $A = (a_{ij})$ and $B = (b_{ij})$ be the adjacency matrix and the incidence matrix of digraph $D(V,E)$ respectively. Show that

$$\sum_{i=1}^{|V|}\sum_{j=1}^{|E|} b_{ij} = 0 \text{ and } \sum_{i=1}^{|V|}\sum_{j=1}^{|V|} a_{ij} = |E|.$$

Exercise 4.7 For graphs G and H, show that $\theta(G * H) = \theta(G)\theta(H)$.

Exercise 4.8 If G has an orthonormal representation in dimension d, then $\theta(G) \le d$.

Exercise 4.9 Let G be a graph on n vertices.

(i) If the automorphism group Γ of G is vertex-transitive, then both $\theta(G)\theta(G^c) = n$ and $\Theta(G)\Theta(G^c) \le n$.

(ii) Find an example to show that it is necessary for Γ to be vertex-transitive.

Exercise 4.10 If the automorphism group Γ of G is vertex-transitive, each of the following holds.

(i) $\Theta(G * G^c) = |V(G)|$.

(ii) if, in addition, that G is self-complementary, then $\Theta(G) = \sqrt{|V(G)|}$.

(iii) $\Theta(C_5) = \sqrt{5}$.

Exercise 4.11 Prove Proposition 4.6.2.

Exercise 4.12 Prove Theorem 4.7.3.

Exercise 4.13 (Cao, [42]) Let v be a vertex of a graph G. The k-degree of v is defined to be the number of walks of length k from v. Let $\Delta_k(G)$ be the maximum k-degree of vertices in G. Show that

(i) $\Delta_k(G)$ is the maximum row sum of $A^k(G)$.

(ii) For a connected graph G,

$$\chi(G) \leq [\Delta_k(G)]^{1/k} + 1,$$

where equality holds if and only if G is an odd cycle or a complete graph.

Exercise 4.14 Let Γ be an Abelian group. Then a graph G is Γ-colorable if and only if every block of G is Γ-colorable.

Exercise 4.15 (Lai and Zhang, [152]) Let H be a subgraph of a graph G, and Γ be an Abelian group. Then (G, H) is said to be Γ-*extendible* if for any $\mathbf{b} \in V(|E(G)|, \Gamma)$, and for any $(\Gamma, \mathbf{b}_1)$-coloring $\mathbf{c}_1$ of H, where $\mathbf{b}_1$ is the restriction of $\mathbf{b}$ in $E(H)$ (as a function), there is a $(\Gamma, \mathbf{b})$-coloring $\mathbf{c}$ of G such that the restriction of $\mathbf{c}$ in $V(H)$ is $\mathbf{c}_1$ (as a function).

Show that if (G, H) is Γ-extendible and H is Γ-colorable, then G is Γ-colorable.

Exercise 4.16 (Lai and Zhang, [152]) Let G be a graph and suppose that $V(G)$ can be linearly ordered as $v_1, v_2, \cdots, v_n$ such that $d_{G_i}(v_i) \leq k$ $(i = 1, 2, \cdots, n)$, where $G_i = G[\{v_1, v_2, \cdots, v_i\}]$ is the subgraph of G induced by $\{v_1, v_2, \cdots, v_i\}$. Then for any Abelian group Γ with $|\Gamma| \geq k+1$, (G_{i+1}, G_i) $(i = 1, 2, \cdots, n-1)$ is Γ-extendible. In particular, G is Γ-colorable.

Exercise 4.17 (Lai and Zhang, [152]) Let G be a graph. Then

$$\chi_1(G) \leq \max_{H \subseteq G}\{\delta(H)\} + 1.$$

Exercise 4.18 (Lai and Zhang, [152]) For any complete bipartite graph $K_{m,n}$ with $n \geq m^m$, $\chi_1(K_{m,n}) = m + 1$.

Exercise 4.19 (Lai and Zhang, [152]) For any positive integers m and k, there exists a graph G such that $\chi(G) = m$ and $\chi_1(G) = m + k$.

Exercise 4.20 Let G be a graph. Show that

(i) If G is a cycle on $n \geq 3$ vertices, then $\chi_1(G) = 3$.

(ii) $\chi_1(G) \leq 2$ if and only if G is a forest.

Exercise 4.21 Prove Theorem 4.8.1.

4.10 Hints for Exercises

Exercise 4.1 Apply Corollary 4.1.2B to obtain $k = 3, r = 1, \alpha = 2, \beta = 1, b_n = n$, $c_0 = 1, c_1 = 0, c_2 = 1$.

Exercise 4.2 Count the number of t-subsets in $S_\lambda(t,k,v)$ in two different ways.

Exercise 4.3 For any subset $S \subseteq X$ with $|S| = i$, the number of = ways of taking t-subsets containing S from X is $\begin{pmatrix} v-i \\ t-i \end{pmatrix}$, while each t-subset belongs to λ of the X_i's in an $S_\lambda(t,k,v)$. Thus the number of X_i's containing S is $\lambda \begin{pmatrix} v-i \\ t-i \end{pmatrix}$.

On the other hand, the number of ways of taking t-subset containing S from X is $\begin{pmatrix} k-i \\ t-i \end{pmatrix}$ where S belongs to λ_i of the X_i's. Hence the number of X_i's containing S is $\lambda_i \begin{pmatrix} k-i \\ t-i \end{pmatrix}$.

Exercise 4.4 Count the repeated number of v elements in two different ways.

Exercise 4.5 Let $m = l+1$ and $b = t-1$ and apply Theorem 4.3.7. Then there exists a complete graph K_n with $n = b(m-1)+m-(b-1) = (t-1)l+(l+1)-(t-2) = tl-t+3$, which can be decomposed into t complete $(l+1)$-partite subgraphs $F_1, F_2, \cdots, F_t$. Clearly $d_k(F_i) \leq 2 \leq d$, and so (4.13) follows.

Exercise 4.6 In the incidence matrix, every row has exactly a $+1$ and a -1, and so the first double sum is 0. The second double sum is the sum of the in-degrees of all vertices, and so the sum is $|E(D)|$.

Exercise 4.7 By Proposition 4.5.1, it suffices to show that $\theta(G * H) \leq \theta(G)\theta(H)$.

Let $\mathbf{v}_1, \cdots, \mathbf{v}_n$, and $\mathbf{w}_1, \cdots, \mathbf{w}_m$ be orthonormal representations of G^c and $\overline{H}$, respectively, and let $\mathbf{c}$ and $\mathbf{d}$ be unit vectors such that

$$\sum_{i=1}^{n}(\mathbf{v}_i^T\mathbf{c})^2 = \theta(G), \; \sum_{j=1}^{m}(\mathbf{w}_j^T\mathbf{d})^2 = \theta(H).$$

Then the $\mathbf{v}_i \otimes \mathbf{w}_j$'s form an orthonormal representation of $G^c * \overline{H}$. Since $G^c * \overline{H} \subseteq \overline{G * H}$, the $\mathbf{v}_i \otimes \mathbf{w}_j$'s form an orthonormal representation of $\overline{G * H}$. Note that $\mathbf{c} \otimes \mathbf{d}$ is a unit vector. Hence

$$\theta(G * H) \geq \sum_{i=1}^{n}\sum_{j=1}^{m}((\mathbf{v}_i \otimes \mathbf{w}_j)^T(\mathbf{c} \otimes \mathbf{d}))^2$$

$$= \sum_{i=1}^{n}(\mathbf{v}_i^T\mathbf{c})^2 \sum_{j=1}^{m}(\mathbf{w}_j^T\mathbf{d})^2.$$

Exercise 4.8

Let $\mathbf{u}_1, \cdots, \mathbf{u}_n$ be an orthonormal representation of G in dimension d. Then $\mathbf{u}_1 \otimes \mathbf{u}_1, \cdots, \mathbf{u}_n \otimes \mathbf{u}_n$ also form an orthonormal representation of G. Let $\mathbf{e}_1, \cdots, \mathbf{e}_d$ be an orthonormal basis, and let

$$\mathbf{b} = \frac{1}{\sqrt{d}}(\mathbf{e}_1 \otimes \mathbf{e}_1 + \mathbf{e}_2 \otimes \mathbf{e}_2 + \cdots + \mathbf{e}_d \otimes \mathbf{e}_d).$$

Then $|\mathbf{b}| = 1$ and

$$(\mathbf{u}_i \otimes \mathbf{u}_i)^T\mathbf{b} = \frac{1}{\sqrt{d}}.$$

Hence $\theta(G) \leq d$ follows by the definition of θ.

Exercise 4.9 (i). View elements in Γ as $n \times n$ permutation matrices. By Theorem 4.5.2, there exists a $B \in \mathcal{B}$ such that $\mathrm{tr}(BJ) = \theta(G)$, and consider the matrix

$$\overline{B} = (\overline{b}_{ij}) = \frac{1}{|\Gamma|}\left(\sum_{P \in \Gamma} P^{-1}BP\right).$$

Then as $PJ = JP = J$, we have $\overline{B} \in \mathcal{B}$ and $\mathrm{tr}(\overline{B}J) = \theta(G)$. Since Γ is vertex transitive, $\overline{b}_{ii} = 1/n$. Imitate the proof of Theorem 4.5.3 to construct orthonormal representation $\mathbf{v}_1, \cdots, \mathbf{v}_n$ and the unit vector $\mathbf{d}$.

Note that in the proof of Theorem 4.5.3, equality in the Cauchy-Schwarz inequality held, and so we have in fact (using the notation in the proof of Theorem 4.5.3)

$$(\mathbf{d}^T\mathbf{v}_i)^2 = \theta(G)|\mathbf{w}_i|^2 = \theta(G)b_{ii}.$$

Hence back to the proof of this exercise, we have

$$(\mathbf{d}^T\mathbf{v}_i)^2 = \frac{\theta(G)}{n}.$$

It follows by the definition of $\theta(G^c)$ that

$$\theta(G^c) \leq \max_{1 \leq i \leq n} \frac{1}{(\mathbf{d}^T\mathbf{v}_i)^2} = \frac{n}{\theta(G)}.$$

The other assertion of (i) follows from Theorem 4.5.1.

(ii). Take $G = K_{1,n-1}$, for $n \geq 3$.

Exercise 4.10 It suffices to show (i). Note that the "diagonal" of $G*G^c$ is an independent vertex set in $G * G^c$.Hence

$$\Theta(G * G^c) \geq \alpha(G * G^c) \geq |V(G)|.$$

On the other hand, by Theorems 4.5.1 and 4.5.4 and by the previous exercise,

$$\Theta(G * G^c) \leq \theta(G * G^c) = \theta(G)\theta(G^c)).$$

Exercise 4.11 Let $\mathbf{j} = (1, 1, \cdots, 1)^T \in \mathbf{B}_{n,1}$. Multiply both sides of the equation in Proposition 4.6.1(i) from the right by $\mathbf{j}$ to get $k^2 = k + \lambda k + \mu(n-1-k)$, and so Proposition 4.6.2(i) obtains.

Let $l = n - k - 1$. Then G^c, the complement of G, is l-regular. By Proposition 4.6.1,

$$(J - I - A)^2 = lI + (l - k + \mu - 1)(J - I - A) + (l - k + \lambda + 1)A.$$

This implies Proposition 4.6.2(ii), by Proposition 4.6.1.

If G is the disjoint union of some K_{k+1}'s, then it is routine to check that G is an $(n, k, k-1, 0)$-strongly regular graph. Conversely, assume that G is an $(n, k, \lambda, 0)$-strongly regular graph. By Proposition 4.6.2(ii) and since $\mu = 0$, $\lambda = k - 1$, which implies that each component of G is a complete graph.

Exercise 4.12 By Theorem 4.7.2 with $B' = B(K_n)$, a simple nonseparable matrix $A \subseteq B(K_n)$ is subeulerian if and only if there exists no vector $\mathbf{z}$ in the cocycle space of $B(K_n)$ such that $\|\mathbf{z}\|_0$ is odd.

Exercise 4.13 (i) follows from Proposition 1.1.2(vii). For (ii), note that $[\Delta_k(G)]^{1/k}$ satisfies properties (P1) and (P2) in Theorem 4.8.3.

Exercise 4.14 Let Γ be an Abelian group. If a graph G is Γ-colorable, then every subgraph of G is also γ-colorable. In particular, every block of G is Γ-colorable.

It suffices to prove the converse for connected graphs with two blocks. Let G be a connected graph with two blocks G_1 and G_2 and assume that G_1 and G_2 are Γ-colorable. Let v_0 be the cut vertex of G. Then $v_0 \in V(G_1) \cap V(G_2)$.

Let $m = |E(G)|$, $m_i = |E(G_i)|$ with $1 \leq i \leq 2$, and let $V(m_i, \Gamma)$ denote the set of m_i-tuples whose components are in Γ and are labeled with the edges in G_i.

For any $\mathbf{b} \in V(m, \Gamma)$, we can get two vectors $\mathbf{b}_1 \in V(m_1, \Gamma)$ and $\mathbf{b}_2 \in V(m_2, \Gamma)$. Since G_1 and G_2 are Γ-colorable, there exist a $(\Gamma, \mathbf{b}_1)$-coloring $\mathbf{c}_1 \in V(|V(G_1)|, \Gamma)$ and an $(\Gamma, \mathbf{b}_2)$-coloring $\mathbf{c}_2 :\in V(|V(G_2)|, \Gamma)$. Let $\mathbf{c}_i(v_0) = g_i$, where $1 \leq i \leq 2$, and for each $v \in V(G_2)$, let $\mathbf{c}_2'(v) = \mathbf{c}_2(v) + g_1 - g_2$. Then

$$\mathbf{c}(v) = \begin{cases} \mathbf{c}_1(v), & \text{if } v \in V(G_1) \\ \mathbf{c}_2'(v), & \text{if } v \in V(G_2). \end{cases}$$

It is routine to verify that $\mathbf{c}$ is a $(\Gamma, \mathbf{b})$-coloring of G.

Exercise 4.15 For any $\mathbf{b} \in V(|E(G)|, \Gamma)$, since H is Γ-colorable, H has a $(\Gamma, \mathbf{b}_H)$-coloring $\mathbf{c}_H : V(H) \mapsto \Gamma$, where $\mathbf{b}_H$ is the restriction of $\mathbf{B}$ in $E(H)$. Since (G, H) is Γ-extendible, $\mathbf{c}_H$ can be extended to a (Γ, G)-coloring.

Exercise 4.16 Let D be an orientation of $E(G_{i+1})$ such that every $e = v_{j_1}v_{j_2} \in E(G_{i+1})$ is directed from v_{j_1} to v_{j_2} if $j_1 > j_2$ and from v_{j_2} to v_{j_1} otherwise. For any $\mathbf{b} \in V(|E(G_{i+1})|, \Gamma)$ and any $(\Gamma, {}_1)$-coloring $\mathbf{c}_1$ of G_i, where $\mathbf{b}_1$ is the restriction of $\mathbf{b}$ in $E(G_i)$, we define a function $\mathbf{c} : V(G_{i+1}) \mapsto \Gamma$ as follows: Assuming that $v_{i_1}v_{i+1}, v_{i_2}v_{i+1}, \cdots, v_{i_r}v_{i+1}$ are all the edges joining v_{i+1} $(0 \le r \le k)$ in G_{i+1}, we let $\mathbf{c}(v) = \mathbf{c}_1(v)$ if $v \in V(G_i)$, and let $\mathbf{c}(v_{i+1}) = g'$ such that $g' \in \Gamma' = \Gamma - \{\mathbf{c}(v_{i_p}) + \mathbf{b}(v_{i_p}v_{i+1}) | p = 1, 2, \cdots, r\}$. Since $|\Gamma| \ge k+1$, $\Gamma' \ne \emptyset$, and so $\mathbf{c}$ can be defined. It is routine to verify that $\mathbf{c}$ is a $(\Gamma, \mathbf{b})$-coloring, and so (G_{i+1}, G_i) is Γ-extendible, where $i = 1, 2, \cdots, n-1$.

By Exercise 4.15 and since G_1 is Γ-colorable, it follows that G is Γ-colorable.

Exercise 4.17 Let $|V(G)| = n$, $k = \max_{H \subseteq G}\{\delta(H)\} + 1$, and v_n be a vertex of degree at most k. Put $H_{n-1} = G - \{v_n\}$. By assumption H_{n-1} has a vertex, say v_{n-1}, of degree at most k. Put $H_{n-2} = G - \{v_n, v_{n-1}\}$. Repeating this process we obtain a sequence $v_1, v_2, \cdots, v_n$ such that each v_j is joined to at most k vertices preceding it. Now Exercise 4.17 follows from Exercise 4.16.

Exercise 4.18 Assume that $K_{m,n}$ has the vertex bipartition (X, Y) with $X = \{x_1, x_2, \cdots, x_m\}$ and $Y = \{y_1, y_2, \cdots, y_n\}$. Let Γ be an Abelian group with $\Gamma| \le m$ and D be an orientation of $E(K_{m,n})$ such that every $e = x_iy_j \in E(K_{m,n})$ is directed from y_j to x_i.

Denote the set of all functions $\mathbf{c} : V(K_{m,n}) \mapsto \Gamma$ by $C(K_{m,n}, \Gamma)$. For every function $\mathbf{c} \in C(K_{m,n}, \Gamma)$, we can get a function $\mathbf{c}_X : X \mapsto \Gamma$. Let $C(X, \Gamma) = \{\mathbf{c}_X \ : \ \mathbf{c} \in C(K_{m,n}, \Gamma)\}$. Since $|\Gamma| \le m$, $|C(X, \Gamma)| = m^{|\Gamma|} \le m^m$. Assume that $C(X, \Gamma) = \{\mathbf{c}_1, \mathbf{c}_2, \cdots, \mathbf{c}_r\}$, where $r = m^{|\Gamma|}$. Now we define $\mathbf{f}_l : E(K_{m,n}) \mapsto \Gamma)$ $(l = 1, 2, \cdots, r)$ as follows: If $l \ne j$, let $\mathbf{f}_l(x_iy_j) = 0$ for every i, and otherwise let $\mathbf{f}_l(x_iy_l) = a_{l_i} \in \Gamma$ such that $\{\mathbf{c}_l(x_i) + a_{l_i} : i = 1, 2, \cdots, m\} = \Gamma$. Let $\mathbf{f} = \sum_{l=1}^{r} \mathbf{f}_l$. Then for any function $c : V(K_{m,n}) \mapsto \Gamma$, there exists at least one arc $e = y_jx_i \in E(K_{m,n})$ such that $\mathbf{c}(y_i) - \mathbf{c}(x_i) = \mathbf{f}(e)$. Hence $\chi_1(K_{m,n}) \ge m+1$.

On the other hand, by Exercise 4.17, $\chi_1(K_{m,n}) \le m+1$.

Exercise 4.19 Let G be the graph in Example 4.8.1 Since G contains a K_m, $\chi(G) = m$. Apply Exercise 4.18 to show that $\chi_1(G) = m + k$.

Exercise 4.20 To see that if G contains a cycle, than $\chi_1(G) \ge 3$, it suffices to show that $\chi_1(C_n) > 2$ for any $n \ge 3$, where C_n denote a cycle on n vertices. Let Z_2 denote the group of 2 element. If $\chi_1(C_n) = 2$, then C_n is Z_2-colorable. Denote $V(C_n) = \{v_1, v_2, \cdots, v_n\}$, such that the orientation of G is $\{(v_i, v_{i+1}) \ : \ i = 1, 2, \cdots, n \pmod{n}\}$. Assume first that n is even. Let $\mathbf{b} = (0, 0, \cdots, 0, 1)^T : E(C_n) \mapsto Z_2$. (Note that $\mathbf{b}(v_n, v_1) = 1$.) By the assumption that G is Z_2-colorable, G has a $(\Gamma, \mathbf{b})$-coloring $\mathbf{c} : V(G) \mapsto Z_2$ such that $\mathbf{c}(v_i) - \mathbf{c}(v_{i+1}) \neq \mathbf{b}(v_i, v_{i+1})$, where $i = 1, 2, \cdots, n \pmod n$. If $\mathbf{c}(v_n) = 1$, then $\mathbf{c}(v_{2i+1}) = 0$ and $\mathbf{c}(v_{2i}) = 1$, and so $\mathbf{c}(v_n) - \mathbf{c}(v_0) = 1 = \mathbf{b}(v_n, v_1)$, a contradiction. (This shows that $\chi_1(C_n) > 2$. Together with Exercise 4.17, the above implies that $\chi_1(G) = 3$.)

If n is odd, then choose $\mathbf{b} = \mathbf{0}$, and a similar argument works also.

On the other hand, we can routinely argue by induction on $|V(G)|$ to show that if G is a tree, then $\chi_1(G) \leq 2$.

Exercise 4.21 If G is connected and not regular of degree $\Delta(G)$, then $max_{H\subset G}\delta(H) \leq \Delta(G) - 1$ and so $\chi_1(G) \leq \Delta(G)$. Without loss of generality, let G be 2-connected and $\Delta(G)$-regular. If G is a complete graph, then $\chi_1(G) = |V(G)| = \Delta(G) + 1$.

If $\Delta(G) = 2$, then G is a cycle and so by Exercise 4.20(i), $\chi_1(G) = 3 = \Delta(G) + 1$. If G is 3-connected and G is not complete, then there are three vertices v_1, v_2 and v_n ($n = |V(G)|$) in G such that $v_1v_n, v_2v_n \in E(G)$ and $v_1v_2 \notin E(G)$. If G is 2-connected, let $\{v_n, v'\}$ be a cut set of G. Then there are two vertices v_1 and v_2 belonging to different endblocks of $G - v_n$. Now, we arrange the vertices of $G - \{v_1, v_2\}$ in nonincreasing order of their distance from v_n, say $v_3, v_4, \cdots, v_n$. Then the sequence $v_1, v_2, \cdots, v_n$ is such that each vertex other than v_n is adjacent to at least one vertex following it, namely each vertex other than v_n is joined to at most $\Delta(G) - 1$ vertices preceding it.

Let D be an orientation of $E(G)$ such that every $e = v_iv_j \in E(G)$ is directed from v_i to v_j if $i > j$ and from v_j to v_i otherwise. For any $\mathbf{b} : E(G) \mapsto \Gamma$, where $|\Gamma| \geq \Delta(G)$, we define $\mathbf{c} : V(G) \mapsto \Gamma$ as follows: Assign $a_1 \in \Gamma$ to $\mathbf{c}(v_1)$ and $a_2 \in \Gamma$ to $\mathbf{c}(v_2)$ such that $a_1 + \mathbf{b}(v_1v_n) = a_2 + \mathbf{b}(v_2v_n)$; for v_j $(3 \leq j \leq n)$, let $v_{i_1}v_j, v_{i_2}v_j, \cdots, v_{i_r}v_j \in E(G)$ $(r \leq \Delta(G) - 1$ if $j < n)$ be the edges joining v_j and having $i_p < j$ $(p = 1, 2, \cdots, r)$, and assign a_j to $c(v_j)$ such that $a_j \in \Gamma_j = \Gamma - \{\mathbf{c}(v_{i_p}) + \mathbf{b}(v_{i_p}v_j) | p = 1, 2, \cdots, r\}$. If $j < n$, then $r \leq \Delta(G) - 1$ and so $\Gamma_j \neq \emptyset$; if $j = n$, then $\Gamma_n \neq \emptyset$; since $a_1 + \mathbf{b}(v_1v_n) = a_2 + \mathbf{b}(v_2v_n)$.

It is now routine to verify that $\mathbf{c}$ is a $(\Gamma, \mathbf{b})$-coloring.

Chapter 5

Combinatorial Analysis in Matrices

5.1 Combinatorial Representation of Matrices and Determinants

Definition 5.1.1 For a matrix $A = (a_{ij}) \in M_{m,n}$, the *weighted bipartite graph* of A, denoted by $\overline{K}A$, has vertex set $V = V_1 \cup V_2$ and edge set E, where $V_1 = \{u_1, u_2, \cdots, u_m\}$ and $V_2 = \{v_1, v_2, \cdots, v_n\}$ such that $u_i v_j \in E$ with weight a_{ij} if and only if $a_{ij} \neq 0$.

To represent a determinant by graphs, we adopt the convention to view $\overline{K}A$ as a weighted complete bipartite graph $K_{m,n}$ with partite sets V_1 and V_2, such that $u_i v_j \in E$ with weight a_{ij}, for all $u_i \in V_1$ and $v_j \in V_2$.

If $A = (a_{ij}) \in M_n$, then we can extend the definition of $D(A)$ by defining $D(A)$ as the *weighted digraph of* A such that $V(D(A)) = \{v_1, v_2, \cdots, v_n\}$, where $(v_i, v_j) \in E(D(A))$ with weight a_{ij} if and only if $a_{ij} \neq 0$.

For the convenience to interpret a determinant by digraphs, we can view $D(A)$ as a complete digraph on n vertices with a loop at every vertex by assigning a_{ij} to the arc (v_i, v_j) for every $i, j = 1, 2, \cdots, n$.

Example 5.1.1 For the matrix

$$A = \begin{bmatrix} 2 & -3 & 0 & 0 \\ 0 & 1.2 & 0 & -1 \\ 5 & 0 & 3 & -6 \end{bmatrix},$$

the weighted bipartite graph is

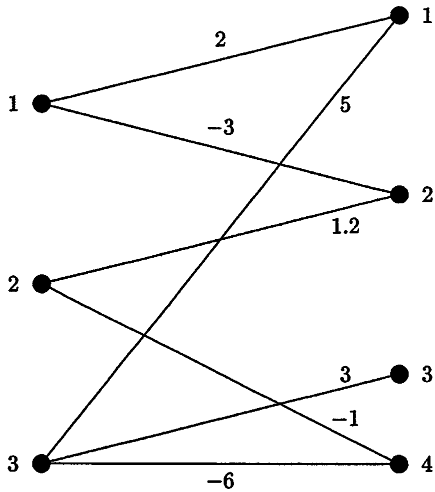

Figure 5.1.1

Example 5.1.2 Let $A = (a_{ij}) \in M_n$ be a square matrix. Let $K_{n,n}$ denote the weighted bipartite graph of A with partite sets $V_1 = \{u_1, u_2, \cdots, u_n\}$ and $V_2 = \{v_1, v_2, \cdots, v_n\}$. For any permutation π in the symmetric group on n letters, the edge subset

$$F_\pi = \{(u_1 v_{\pi(1)}), (u_2 v_{\pi(2)}), \cdots, (u_n v_{\pi(n)})\}$$

is a perfect matching in $K_{n,n}$. Therefore, this yields a one to one correspondence between $\mathcal{S}_n$, the symmetric group on n letters, and $\mathcal{M}_n$, the set of all perfect matchings of $K_{n,n}$.

For each $\pi \in \mathcal{S}_n$, define

$$\operatorname{sign}(\pi) = \begin{cases} 1 & \text{if } \pi \text{ is an even permutation} \\ -1 & \text{if } \pi \text{ is an odd permutation.} \end{cases}$$

Let $W_A(F_\pi) = \operatorname{sign}(\pi) \prod_{(u_i v_j) \in F_\pi} a_{ij}$. Then the determinant of A can be written as

$$\det(A) = \sum_{F_\pi \in \mathcal{M}_n} W_A(F_\pi).$$

Example 5.1.3 Let $A = (a_{ij}) \in M_n$ be a square matrix. Let $D(A)$ denote the weighted digraph of A with $V(D(A)) = \{v_1, v_2, \cdots, v_n\}$. For any permutation π in the symmetric group on n letters, the arcs

$$F_\pi = \{(v_1, v_{\pi(1)}), (v_{\pi(1)}, v_{\pi(\pi(1))}), \cdots ,\}$$

form a subdigraph called a *1-factor* in $D(A)$. Therefore, this yields a one to one correspondence between $\mathcal{S}_n$, the symmetric group on n letters, and $\mathcal{D}_n$, the set of all 1-factors of $D(A)$.

Note that each 1-factor F_π is a disjoint union of directed cycles in $D(A)$. For a directed cycle C, define

$$W_A(C) = -\prod_{(v_i,v_j)\in E(C)} a_{ij}, \text{ and } W_A(F_\pi) = \prod_{C\in F_\pi} W_A(C).$$

Note that if k denotes the number of disjoint directed cycles in F_π, then

$$(-1)^k = (-1)^n(-1)^{n-k} = (-1)^n \operatorname{sign}(\pi).$$

Therefore, the determinant of A can be written as

$$\det(A) = (-1)^n \sum_{F_\pi\in\mathcal{D}_n} W_A(F_\pi).$$

Definition 5.1.2 Two matrices $A, B \in M_n$ are ***diagonally similar*** if there is a non singular diagonal matrix $D = \operatorname{diag}(d_1, d_2, \cdots, d_n)$ such that $DAD^{-1} = B$.

Example 5.1.4 If A and B are diagonally similar, then $D(A) = D(B)$, when weights are ignored and zero weighted arcs dropped. However, it is possible that $D(A) = D(B)$, but A and B are not diagonally similar. Consider the example:

$$A = \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}, \text{ and } B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Clearly $D(A) = D(B)$. If there were a D=diag(d_1, d_2) such that $DAD^{-1} = B$, then

$$\begin{cases} 2d_1d_2^{-1} = 1 \\ d_2d_1^{-1} = 1. \end{cases}$$

A contradiction obtains.

Theorem 5.1.1 (Fiedler and Ptak, [88]) Let $A, B \in M_n$ be two square matrices such that A is irreducible. Then A and B are diagonally similar if and only if each of the following holds:

(i) $D(A) = D(B)$, and

(ii) For each directed cycle C in $D(A)$, $W_A(C) = W_B(C)$.

Proof Assume first that A and B are diagonally similar. Then $a_{ij} = 0$ if and only if $b_{ij} = 0$, for any $i, j = 1, 2, \cdots, n$, and so $D(A) = D(B)$. Let $D = D(A) = D(B)$. By assumption, there is a D=diag$(d_1, d_2, \cdots, d_n)$ with $d_i \neq 0$ such that $DAD^{-1} = B$. Thus,

$$d_i a_{ij} d_j^{-1} = b_{ij}, \text{ for any } i, j = 1, 2, \cdots, n.$$

If follows that for each directed cycle $C = v_{i_1} v_{i_2} \cdots v_{i_k} v_{i_1}$,

$$\begin{aligned} W_B(C) &= -b_{i_1,i_2} b_{i_2,i_3} \cdots b_{i_k,i_1} \\ &= -d_{i_1} a_{i_1,i_2} d_{i_2}^{-1} d_{i_2} a_{i_2,i_3} d_{i_3}^{-1} \cdots d_{i_k} a_{i_k,i_1} d_{i_1}^{-1} \\ &= -a_{i_1,i_2} a_{i_2,i_3} \cdots a_{i_k,i_1} = W_A(C). \end{aligned}$$

Conversely, assume that both (i) and (ii) hold. Since A is irreducible, $D(A)$ is strongly connected, and so by $D(A) = D(B)$, B is also irreducible. We argue by induction on $|E(D(A))|$ (or equivalently, the number of non zero entries of A) to show that A is diagonally similar to B.

Suppose first that $D = v_1 v_2 \cdots v_n v_1$ is a directed cycle. By assumption,

$$a_{1,2} a_{2,3} \cdots a_{n,1} = -W_A(C) = -W_B(C) = b_{1,2} b_{2,3} \cdots b_{n,1}.$$

Define

$$\begin{aligned} d_1 &= 1 \\ d_{i+1} &= \frac{d_i a_{i,i+1}}{b_{i,i+1}}, \text{ for } i = 1, 2, \cdots, n-1. \end{aligned}$$

Let $D = \text{diag}(d_1, d_2, \cdots, d_n)$. Then $DAD^{-1} = B$.

Now assume that D is obtained from a strong digraph D' by adding a directed path $v_k v_{k-1} \cdots v_2 v_1 v_n$. (Thus D' may be viewed as $D(A')$, where A' is obtained from A by changing the nonzero entries $a_{k,k-1}, \cdots, a_{2,1}, a_{1,n}$ to zeros. This can be done since D is strong, and so every arc of D lies in a directed cycle of D). Therefore,

$$A = \left[\begin{array}{c|c} & a_{1,n} \\ a_{i,i+1} & \\ \hline a_{k,k-1} & \\ & A' \end{array}\right], \text{ and } B = \left[\begin{array}{c|c} & b_{1,n} \\ a_{i,i+1} & \\ \hline b_{k,k-1} & \\ & B' \end{array}\right],$$

where $A', B' \in M_{n-k+1}$ are irreducible, and, in A, only $a_{k,k-1}, \cdots, a_{2,1}, a_{1,n}$ are nonzero entries outside A'; and in B, only $b_{k,k-1}, \cdots, b_{2,1}, b_{1,n}$ are nonzero entries outside B'. By induction, there is a nonsingular $D' = \text{diag}(d_k, \cdots, d_n)$ such that $D'A'(D')^{-1} = B'$. Define

$$d_i = \frac{d_{i+1} a_{i+1,i}}{b_{i+1,i}}, \text{ for each } i = 1, 2, \cdots, k-1,$$

and let $D = \text{diag}(d_1, \cdots, d_n)$. Then $DAD^{-1} = B$, and so A and B are diagonally similar. $\square$

Shao and Cheng [248] modified the conditions in Theorem 5.1.1(ii) to obtain necessary and sufficient conditions for matrices A and B which are not necessarily irreducible to be diagonally similar. Interested readers are referred to [248] for further details.

5.2 Combinatorial Proofs in Linear Algebra

Combinatorial methods have been applied to prove results in matrices. For example, the Jacobi identity was studied combinatorially by Jackson and Foato ([89]). Brualdi gave combinatorial proofs of the Jordon canonical form of a matrix ([20]) and he also showed that the elementary divisors of a matrix can be determined combinatorially ([21] and [22]). In this section, we present the combinatorial proofs of two matrix identities given by Zeilberger ([280]).

Once again we adopt the following notational convention: For a matrix A, $(A)_{ij}$ denotes the (i,j)-entry of A.

Theorem 5.2.1 (Cayley-Hamilton) Let $A \in M_n$. Then $\chi_A(A) = 0$.

Proof Fix an $A \in M_n$. Let

$$\chi_A(\lambda) = \det(\lambda I - A) = \lambda^n + a_1\lambda^{n-1} + \cdots + a_k\lambda^{n-k} + \cdots + a_{n-1}\lambda + a_n.$$

Therefore, it suffices to prove this matrix identity:

$$A^n + a_1A^{n-1} + \cdots + a_kA^{n-k} + \cdots + a_{n-1}A + a_nI_n = 0. \tag{5.1}$$

As in Example 5.1.3, let $D(A)$ denote the weighted digraph of A with $V(D(A)) = \{v_1, v_2, \cdots, v_n\}$ (viewed as a weighted complete digraph on n vertices, where an arc (v_i, v_j) has weight a_{ij}), $\mathcal{D}_n$ the set of all 1-factors of $D(A)$ and let $\mathcal{D}_n^*$ denote the set of all 1-factors of all subdigraphs of $D(A)$. Then

$$\det(-A) = \sum_{F_\pi \in \mathcal{D}_n} W_A(F_\pi) \text{ and } \det(I_n - A) = \sum_{F_\pi \in \mathcal{D}_n^*} W_A(F_\pi).$$

Note that in (5.1), a_k is the sum of the weights of all subgraphs of $D(A)$ induced by k element subsets of $V(D(A))$, and that the (i,j)-entry of A^{n-k} is the total weight of all directed (v_i, v_j)-walks of length $n-k$.

Fix i and j, let $\mathcal{A} = \mathcal{A}(i,j)$ denote the collection of such subgraph pairs (P, C) in $D(A)$:

(A1) P is a directed (v_i, v_j)-walk,

(A2) C is an arc disjoint union of directed cycles, (A3) $|E(P)| + |E(C)| = n$.

Let $o(C)$ denote the number of cycles in C. Then the weight of (P, C) can be defined as follows.

$$W(P,C) = (-1)^{o(C)} \prod_{(v_k,v_l)\in E(P)\cup E(C)} a_{kl}.$$

To verify (5.1), it suffices to verify both of the following claims.

Claim 1 The (i, j)-entry of the left hand side of (5.1) is

$$W(\mathcal{A}(i,j)) = \sum_{(P,C)\in\mathcal{A}(i,j)} W(P,C).$$

Fix a k with $0 \le k \le n$. Suppose that $|E(P)| = n - k$. Then $\prod_{(v_m,v_l)\in E(P)} a_{ml}$ is the total weight of the directed (v_i, v_j)-walk P, and the (i, j)-entry of A^{n-k} is the sum of the total weight of all directed (v_i, v_j)-walk of length $n - k$. By (A3), $|E(C)| = k$, and so $E(C)$ corresponds to a k element subset of $V(D(A))$. The total weight of these k-element subsets is equal to the sum of all $k \times k$ principal submatrices of $(-A)$, which is a_k. In other words,

$$a_k(A^{n-k})_{i,j} = \left(\sum_{\substack{(P,C)\,\in\,\mathcal{A}(i,j)\\ \text{with } |E(P)|\,=\,n-k}} \prod_{(v_m,v_l)\in E(P)} a_{ml} \right)\left(\sum_{\substack{(P,C)\,\in\,\mathcal{A}(i,j)\\ \text{with } |E(C)|\,=\,k}} \prod_{(v_m,v_l)\in E(C)} a_{ml} \right).$$

Thus Claim 1 follows by summing up from $k = 0, 1, \cdots, n$.

Claim 2 For each (i, j),

$$\sum_{(P,C)\in\mathcal{A}(i,j)} W(P,C) = 0.$$

Fix a $(P, C) \in \mathcal{A}(i, j)$. Starting from v_i, P may revisit a vertex that is already in P, in which case P contains a directed cycle C_1; or P will visit a vertex that is in C, in which case P and a directed cycle C_2 in C share a common vertex. Obtain a new pair $(P', C') \in \mathcal{A}(i, j)$ as follows: In the former case, move C_1 from P to C, and in the latter case, move C_2 from C to P. Note that $W(P, C) = -W(P', C')$, and that the correspondence between (P, C) and (P', C') is a bijection. Therefore, Claim 2 follows. $\square$

Theorem 5.2.2 Let $A = (a_{ij}), B = (b_{ij}) \in M_n$. Then $\det(AB) = \det(A)\det(B)$.

Proof Let $\mathcal{S}_n$ denote the symmetric group on n letters. For each $\pi \in \mathcal{S}_n$, define

$$\begin{aligned} w_A(\pi) &= \operatorname{sign}(\pi) a_{1,\pi(1)} a_{2,\pi(2)} \cdots a_{n,\pi(n)}, \\ w_B(\pi) &= \operatorname{sign}(\pi) b_{1,\pi(1)} b_{2,\pi(2)} \cdots b_{n,\pi(n)}. \end{aligned}$$

Note the difference between the definitions of w_A here and that of W_A previously. With these notations,

$$\det(A) = w_A(\mathcal{S}_n) = \sum_{\pi \in \mathcal{S}_n} w_A(\pi), \text{ and } \det(B) = w_B(\mathcal{S}_n) = \sum_{\pi \in \mathcal{S}_n} w_B(\pi).$$

Note that

$$(AB)_{i,j} = \sum_{k=0}^{n} a_{ik} b_{kj}.$$

Thus we can introduce a digraph $D = D(A, B)$ with $V(D) = \{v_1, v_2, \cdots, v_n\}$, where there are two parallel arcs from each v_i to each v_j, one with weight a_{ij} (denoted by $v_i \to_A v_j$) and the other with weight b_{ij} (denoted by $v_i \to_B v_j$). With this model, we represent $a_{ik}b_{kj}$ by a path $v_i v_k v_j$. This also motivates the following notation.

Let $Z(n)$ denote the set of pairs (f, π), where f is a function from $\{1, 2, \cdots, n\}$ into itself and where $\pi \in \mathcal{S}_n$. Define

$$w(f, \pi) = \operatorname{sign}(\pi) \prod_{i=1}^{n} (a_{if(i)} b_{f(i)\pi(i)}).$$

Note that

$$\det(AB) = w(Z(n)) = \sum_{(f,\pi) \in Z(n)} w(f, \pi). \tag{5.2}$$

If $f \in \mathcal{S}_n$, then $f^{-1}\pi \in \mathcal{S}_n$, and so $w(f, \pi) = w_A(f) w_B(f^{-1}\pi)$. If follows that

$$\begin{aligned} \sum_{f,\pi \in \mathcal{S}_n} w(f, \pi) &= \sum_{f,\pi \in \mathcal{S}_n} w_A(f) w_B(f^{-1}\pi) \qquad (5.3) \\ &= \sum_{f \in \mathcal{S}_n} w_A(f) \sum_{\pi \in \mathcal{S}_n} w_B(\pi) = \det(A)\det(B). \end{aligned}$$

By (5.2) and (5.3), it suffices to show that

$$\sum_{f \notin \mathcal{S}_n, \pi \in \mathcal{S}_n} w(f, \pi) = 0. \tag{5.4}$$

Since $f \notin \mathcal{S}_n$, there exist $b, i, i' \in \{1, 2, \cdots, n\}$ such that $f(i) = f(i') = b$, where $i \neq i'$. Then, D has arcs $v_i \to_A v_b$ and $v_{i'} \to_A v_b$. Choose a smallest b such that $|f^{-1}(b)| \geq 2$, and after b is chosen, choose $i, i' \in f^{-1}(b)$ such that $i + i'$ is smallest.

If v_i and $v_{i'}$ are in the same cycle of π, that is, π has a cycle

$$v_i \to_A v_b \to_B v_{\pi(i)} \to_A \cdots \to_B v_{i'} \to_A v_b \to_B v_{\pi(i')} \to_A \cdots v_i,$$

then there is a $\pi' \in \mathcal{S}_n$, such that x is not in the cycle above, $\pi'(x) = \pi(x)$, and such that the cycle above is broken into two cycles:

$$v_i \to_A v_b \to_B v_{\pi(i')} \to_A \cdots \to_B v_i \text{ and } v_{\pi(i)} \to_A \cdots \to_B v_{i'} \to_A v_b \to_B v_{\pi(i)}.$$

If v_i and $v_{i'}$ are not in the same cycle of π, that is, π has cycles

$$v_i \to_A v_b \to_B v_{\pi(i)} \to_A \cdots \to_B v_i \text{ and } v_{i'} \to_A v_b \to_B v_{\pi(i')} \to_A \cdots \to_B v_{i'},$$

then there is a $\pi' \in S_n$, such that x is not in these two cycles, $\pi'(x) = \pi(x)$, and such that these two cycles are combined into a cycle:

$$v_i \to_A v_b \to_B v_{\pi(i'} \to_A \cdots \to_B v_{i'} \to_A v_b \to_B v_{\pi(i')} \to_A \cdots v_i.$$

Thus, $(f, \pi) \longleftrightarrow (f, \pi')$ is a one to one correspondence from the set $\{(f, \pi) : f \notin S_n$ and $\pi \in S_n\}$ onto itself such that

$$w(f, \pi) = -w(f, \pi').$$

Therefore, (5.4) must hold. □

5.3 Generalized Inverse of a Boolean Matrix

Definition 5.3.1 Let $\mathbf{B}_{m,n}$ denote the set of all $m \times n$ Boolean matrices. A *generalized inverse* (or just a *g-inverse*) of a matrix $A \in \mathbf{B}_{n,m}$ is a matrix $B \in \mathbf{B}_{n,m}$ such that $ABA = A$.

A matrix $A = (a_{ij}) \in \mathbf{B}_{m,n}$ can be represented by a bipartite digraph $B(R_m, S_n)$, called the *bipartite digraph representation* of A, where $R_m = \{u_1, u_2, \cdots, u_m\}$ and $S_n = \{v_1, v_2, \cdots, v_n\}$ such that (u_i, v_j) is an arc if and only if $a_{ij} > 0$. Conversely, given a bipartite digraph $B(R_m, S_n)$ whose arcs are all directed from a vertex in R_m to a vertex in S_n, there is a matrix $A \in \mathbf{B}_{m,n}$, denoted by $M(B(R_m, S_n))$, whose bipartite digraph representation is $B(R_m, S_n)$.

Note that in our notation, the arcs in a bipartite digraph $B(V_1, V_2)$ are always directed from V_1 to V_2. Thus $B(R_m, S_n)$ and $B(S_n, R_m)$ have identical vertex sets but the arcs are directed oppositely.

If $A \in \mathbf{B}_{m,n}$ has a bipartite digraph representation $B(R_m, S_n)$ and if $M(B(S_n, R_m))$ is the bipartite digraph representation of a generalized inverse of A, then $B(S_n, R_m)$ is called the *g-inverse graph* of A (or of $B(R_m, S_n)$).

If $G = B(R_m, S_n) \cup B(S_n, R_m)$, then G is called the *combined graph* of $B(R_m, S_n)$ and $B(S_n, R_m)$.

Example 5.3.1 The bipartite digraph representation of the matrix

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

is the bipartite graph in Figure 5.3.1.

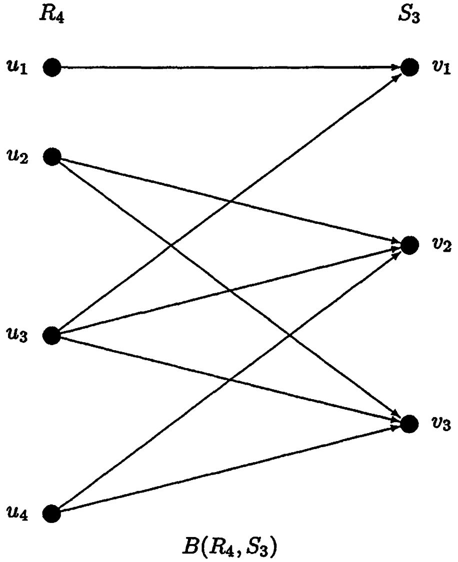

Figure 5.3.1

Example 5.3.2 (Combined graph) Let A be the matrix in Example 5.3.1 with representation $B(R_4, S_3)$ and $G = B(R_4, S_3) \cup B(S_3, R_4)$. (See Figure 5.3.2). We can verify that $B_1 = M(S_3, R_4)$ is a g-inverse of A, where

$$B_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

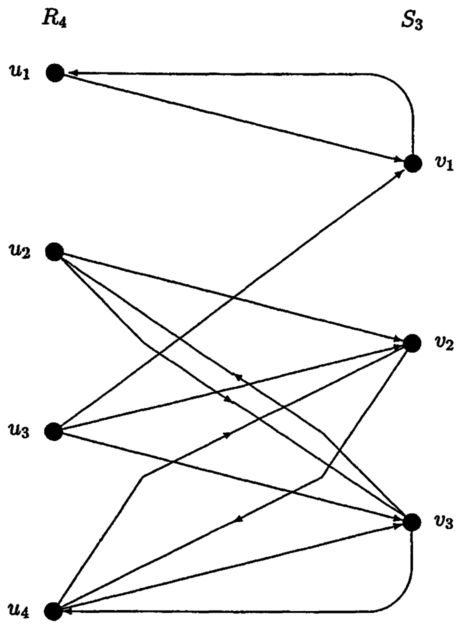

Figure 5.3.2

Example 5.3.3 (Nonuniqueness of g-inverses) Each of the following is a g-inverse of the matrix A in Example 5.3.1.

$$B_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad B_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}, \quad B_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Definition 5.3.2 The set of all g-inverses of A is denoted by A^-. A matrix $B \in A^-$ is a *maximum g-inverse* of A and is denoted by $\max A^-$, if B has the maximum number of nonzero entries among all the matrices in $A-$; a matrix $B \in A^-$ is a *minimum g-inverse* of A and is denoted by $\min A^-$, if B has the minimum number of nonzero entries among all the matrices in A^-.

Note that both $\max A^-$ and $\min A^-$ are sets of matrices. For notational convenience, we also use $\max A^-$ to denote a matrix in $\max A^-$, and use $\min A^-$ to denote a matrix in $\min A^-$.

Example 5.3.4 Let A be the matrix in Example 5.3.1. Then $\max A^- = B_1$ and both B_2 and B_4 are $\min A^-$. In fact, Zhou [291] and Liu [167] proved that while $\min A^-$ may not be unique, $\max A^-$ is unique as long as $A^- \neq \emptyset$.

Theorem 5.3.1 Let $A \in \mathbf{B}_{m,n}$ with a bipartite digraph representation $B(R_m, S_n)$. For a graph $B(S_n, R_m)$, the following are equivalent.

(i) $B(S_n, R_m)$ is a g-inverse graph of A.

(ii) In the combined graph $G = B(R_m, S_n) \cup B(S_n, R_m)$, for each pair of vertices (u_i, v_j) with $u_i \in R_m$ and $v_j \in S_n$, (u_i, v_j) is an arc of G if and only if G has a directed (u_i, v_j)-walk of length 3.

Proof Denote $M = (g_{ij}) = M(B(S_n, R_m))$ and $A = (a_{ij})$.

Suppose first that $B(S_n, R_m)$ is a g-inverse graph of A. Then $AMA = A$, and so for each i, j,

$$\sum_{p,q} a_{ip} g_{pq} a_{qj} = a_{ij}. \tag{5.5}$$

If (u_i, v_j) is an arc in G with $u_i \in R_m$ and $v_j \in S_n$, then $a_{ij} = 1$, and so by (5.5), there must be a pair (p, q) such that $a_{ip} g_{pq} a_{qj} = a_{ij} = 1$. Hence $a_{ip} = g_{pq} = a_{qj} = 1$, which implies that G has a directed (u_i, v_j)-walk of length 3.

If (u_i, v_j) is not an arc in G, then by (5.5), $a_{ip} g_{pq} a_{qj} = 0$ for all choices of (p, q), and so G does not have any directed (u_i, v_j)-walk of length 3, and so (ii) must hold.

Conversely, assume that (ii) holds. Then $AMA = A$ follows immediately by (5.5), and so (i) follows. $\square$

Definition 5.3.3 Let $G = B(R_m, S_n) \cup B(S_n, R_m)$ be a combined graph. For each pair of vertices (u, v), if $(u, v) \in E(G)$ only if G has a directed (u, v)-walk of length 3, then we say that the pair (u, v) has the *(1-3) property*; similarly, if G has a directed (u, v)-walk of length 3 only if $(u, v) \in E(G)$, then we say that the pair (u, v) has the *(3-1) property.*

For a vertex $u \in R_m$, if for each $v \in S_n$, (u, v) has the (1-3) property (or (3-1) property, respectively), then we say that u is a vertex with the (1-3) property (or (3-1) property, respectively).

If each vertex in R_m has the (1-3) property (or (3-1) property, respectively), then we say that G has the (1-3) property (or (3-1) property, respectively).

With these terminology, Theorem 5.3.1 can be restated as follows.

Theorem 5.3.1' Let $A \in \mathbf{B}_{m,n}$ and let $B(R_m, S_n)$ be the bipartite digraph representation of A. Then a bipartite digraph $B(S_n, R_m)$ is a g-inverse graph of A if and only if the combined graph $G = B(R_m, S_n) \cup B(S_n, R_m)$ has both the (1-3) property and the (3-1) property.

Corollary 5.3.1A Suppose that $B(S_n, R_m)$ is a g-inverse graph of $B(R_m, S_n)$. Then for every pair of vertices $u \in R_m$ and $v \in S_n$ in the combined graph $G = B(R_m, S_n) \cup B(S_n, R_m)$, either $d(u, v) = 1$ or $d(u, v) = \infty$.

Proof Note that if $d(u, v) = k < \infty$, then k must be an odd integer. By Theorem 5.3.1, if $k > 1$, then $k > 3$. Take a shortest directed (u, v)-path $P = v_0 v_1 \cdots v_k$ in G, where $v_0 = u$ and $v_k = v$. Then by Theorem 5.3.1, $d(v_0, v_3) = 1$, contrary to the assumption that P is a shortest path. $\square$

Corollary 5.3.1B Suppose that $B(S_n, R_m)$ is a g-inverse graph of $B(R_m, S_n)$. Then for vertices $u_1, u_2 \in R_m$ and $v_1, v_2 \in S_n$ in the combined graph $G = B(R_m, S_n) \cup B(S_n, R_m)$, if $(u_1, v_1), (u_2, v_2) \in E(G)$, and if $(u_1, v_2) \notin E(G)$. then $(v_1, u_2) \notin E(G)$.

Lemma 5.3.1 In the digraph $G = B(R_m, S_n) \cup B(S_n, R_m)$, if for some $u \in R_m$ and $v \in S_n$, $d^+(u) = 0$ or $d^-(v) = 0$, then the pair $\{u, v\}$ has (1-3) property and (3-1) property.

Proof G does not have a directed (u, v)-path of length 1 or 3. $\square$

Lemma 5.3.2 In the digraph $G = B(R_m, S_n) \cup B(S_n, R_m)$, For any $u \in R_m$, if for any $v \in S_n$, at least one vertex in $\{u, v\}$ lies in a directed cycle of length 2, then u has (1-3) property.

Proof By Definition 5.3.3, we may assume that $(u, v) \in E(G)$. If one of u or v lies in a directed 2-cycle, then G has a directed (u, v)-walk of length 3. $\square$

Given a combined graph $G = B(R_m, S_n) \cup B(S_n, R_m)$, when (u_1, v_1), $(u_2, v_2) \in E(G)$ and $(u_1, v_2) \notin E(G)$, we say that the pair $\{v_1, u_2\}$ is a *forbidden pair*. An arc $(u, v) \in E(G)$ with $u \in R_m$ and $v \in S_n$ is called a *single arc* if neither u not v lies in a directed cycle of length 2.

We now present an algorithm to determine if a matrix $A \in \mathbf{B}_{m,n}$ has a g-inverse, and construct one if it does. The validity of this algorithm will be proved in Theorem 5.3.2, which follows the algorithm.

Algorithm 5.3.1

(Step 1) For a given $A \in \mathbf{B}_{m,n}$, construct the bipartite digraph representation of A, denote it by $B(R_m, S_n)$.

(Step 2) Construct a bipartite digraph $B_0 = B(S_n, R_m)$ as follows:

For each pair of vertices $u \in R_m$ and $v \in S_n$, an arc $(v, u) \in E(B_0)$ if and only if $\{v, u\}$ is a not a forbidden pair.

(Step 3) Let $G = B(R_m, S_n) \cup B_0$.

If for each single arc (u, v) of G, the pair $\{u, v\}$ has (1-3) property, then B_0 is a g-inverse graph of $B(R_m, S_n)$, and $M(B_0)$ is the maximum g-inverse of A; otherwise A does not have a g-inverse.

Example 5.3.5

$$A = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad M(B_0) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

Since G has no single arcs, $M(B_0)$ is the maximum g-inverse of A.

Lemma 5.3.3 The graph $G = B(R_m, S_n) \cup B_0$ produced in Step 3 of Algorithm 5.3.1 has (3-1) property.

Proof Pick $u \in R_m$ and $v \in S_n$. Suppose that G has a directed (u, v)-walk $uu'v'v$ of length 3, but $(u, v) \notin E(G)$. Then $\{u', v'\}$ is a forbidden pair, and so by Step 2, $(v', u') \notin E(G)$, a contradiction.

Lemma 5.3.4 Let $B'(S_n, R_m)$ be a g-inverse graph of A, then $B'(S_n, R_m)$ is a subgraph of B_0.

Proof Let

$$G = B(R_m, S_n) \cup B_0(S_n, R_m), \text{ and } G' = B(R_m, S_n) \cup B'(S_n, R_m).$$

Let $u \in R_m$ and $v \in S_n$. If $(u, v) \notin E(G)$, then $\{u, v\}$ is a forbidden pair, by Step 2 of Algorithm 5.3.1. By Corollary 5.3.1B, and since $B'(S_n, R_m)$ is a g-inverse of $B(R_m, S_n)$, we have $(u, v) \notin E(G')$. $\square$

Theorem 5.3.2 Let $A \in \mathbf{B}_{m,n}$ with a bipartite digraph representation $B(R_m, S_n)$, and let $B_0 = B_0(S_n, R_m)$ be the bipartite digraph produced by Algorithm 5.3.1. The following are equivalent.

(i) A has a g-inverse.

(ii) In the combined graph $G = B(R_m, S_n) \cup B_0$, if for some $u \in R_m, v \in S_n$, $(u, v) \in E(G)$ is a single arc, then the pair $\{u, v\}$ has (1-3) property.

Proof Assume first that $B' = B'(S_n, R_m)$ is a g-inverse of A, and let $G' = B(R_m, S_n) \cup B'$. If Part(ii) is violated, then there exists an arc $(u, v) \in E(G')$ with $u \in R_m$ and $v \in S_n$ such that G' has not (u, v)-trail of length 3. By Lemma 5.3.2, (u, v) must be a single arc. It follows that G' does not have a (1-3) property, and so by Theorem 5.3.1', B' is not a g-inverse of A, a contradiction.

Conversely, by Theorem 5.3.2(ii) and by Lemma 5.3.2, B_0 has (1-3) property. By Theorem 5.3.1', by Lemma 5.3.3 and by the fact that B_0 has (1-3) property, B_0 is a g-inverse of A. □

Now we consider properties of a minimum g-inverse of A. We have the following observation, stated as Proposition 5.3.1. The straightforward proof for this proposition is left as an exercise.

Proposition 5.3.1 Let $A \in \mathbf{B}_{m,n}$. Each of the following holds.

(i) Let $B \in \mathbf{B}_{n,m}$ be a matrix. If for some $\min A^-$, we have $\min A^- \leq B \leq \max A^+$, then B is also a g-inverse of A.

(ii) If $B_0(S_n, R_m)$ is a $\max A^+$, then a minium g-inverse $B^*(S_n, R_m)$ of A can be obtained from $B_0(S_n, R_m)$ by deleting arcs such that $G = B(R_m, S_n) \cup B^*(S_n, R_m)$ has (1-3) property and such that $B^*(S_n, R_m)$ has the minimum number of arcs among all g-inverses of A.

(iii) Let (v, u) with $v \in S_n$ and $u \in R_m$ be an arc in $B_0(S_n, R_m)$. If $d^+(u) = 0$ or $d^-(v) = 0$ in $G = B(R_m, S_n) \cup B_0(S_n, R_m)$, then the arc (v, u) can be deleted and the resulting graph $B_0 - (v, u)$ is also a g-inverse of A.

Theorem 5.3.3 Let $B(R_m, S_n)$ be the bipartite representation of A, let $B^*(S_n, R_m)$ be a minimum g-inverse of $B(R_m, S_n)$, and let $G^* = B(R_m, S_n) \cup B^*(S_n, R_m)$. If (u, v) is a single arc of G^*, then G^* must have a directed $K_{2,2}$ as a subgraph whose arcs are all directed from R_m to S_n, and such that (u, v) is an arc of this directed $K_{2,2}$.

Proof Since $B^*(S_n, R_m)$ is a g-inverse, (u, v) has (1-3) property. Since (u, v) is a single arc, G^* has a directed (u, v)-path uv_1u_1v with $u, u_1 \in R_m$ and $v, v_1 \in S_n$.

If (u_1, v_1) is in G^*, then G^* has a desirable $K_{2,2}$. If (u_1, v_1) is not an arc in G^*, then as $(u, v-1)$ has (1-3) property, G^* has either a directed (u, v_1)-path $uv_2u_2v_1$ with $u_2 \neq u$, or a directed 2-cycle $v_1u_2v_1$. In either case, since (u_2, v) has (1-3) property, G^* contains (u_2, v), and so a desirable $K_{2,2}$ must exist. □

By Theorem 5.3.3, we can first apply Algorithm 5.3.1 to construct a g-inverse of $B_0(S_n, R_m)$, and then obtain a minimum g-inverse by deleting arcs from B_0. Interested readers are referred to [174] for details.

5.4 Maximum Determinant of a (0,1) Matrix

What is the maximum value of $|\det(A)|$, if A ranges over all matrices in $\mathbf{B}_n$? What is the least upper bound of $|\det(A)|$, if A ranges over all matrices in M_n? The Hadamard inequality (Theorem 5.4.1) gives an upper bound of $|\det(A)|$, but determining the least upper bound seems very difficult. This section will be devoted to the discussion of this problem.

Theorem 5.4.1 (Hadamard, [111]) Let $A = (a_{ij}) \in M_n$. Then

$$|\det (A)|^2 \leq \prod_{j=1}^{n} \sum_{i=1} a_{ij}^2.$$

Moreover, if each $a_{ij} \in \{1, -1\}$, then

$$|\det (A)| \leq \prod_{j=1}^{n} \sqrt{\sum_{i=1} a_{ij}^2} \leq n^{\frac{n}{2}}, \tag{5.6}$$

where equality in (5.6) holds if and only if $AA^T = nI_n$.

Definition 5.4.1 Let $M_n(1, -1)$ denote the collection of all $n \times n$ matrices whose entries are 1 or -1. A matrix $H \in M_n(1, -1)$ is a *Hadamard matrix* if $HH^T = nI_n$.

Proposition 5.4.1 Suppose that there exists an $n \times n$ Hadamard matrix. Each of the following holds.

(i) $\alpha_n = \sqrt{n^n}$.

(ii) $n = 1, 2$ or $n \equiv 0 \pmod 4$.

Proof (i) follows by Theorem 5.4.1.

Suppose that $H = (h_{ij})$ is a Hadamard matrix. By the definition of a Hadamard matrix, $HH^T = H^T H = nI_n$. Hence, for $n > 2$,

$$\sum_{j=1}^{n} (h_{1j} + h_{2j})(h_{1j} + h_{3j}) = \sum_{j=1}^{n} h_{1j}^2 = n.$$

Since $h_{1j} + h_{2j} = \pm 2, 0$ and $h_{1j} + h_{3j} = \pm 2, 0$, the left hand side of the equality above is divisible by 4, and so (ii) holds. $\square$

It has been conjectured that a Hadamard matrix exists if and only if $n = 1, 2$, or $n \equiv 0 \pmod 4$. See [237].

For each integer $n \geq 1$, define

$$\begin{aligned} \alpha_n &= \max\{ \det(H) \ : \ H \in M_n(1, -1)\}, \\ \beta_n &= \max\{ \det(B) \ : \ B \in \mathbf{B}_n\}. \end{aligned}$$

When $n \not\equiv 0 \pmod 4$, the value of α_n is determined by Ehlich [80]. Williamson [276] showed that for $n \geq 2$, $\alpha_n = 2^{n-1}\beta_n$. Therefore, we can study β_n in order to determine α_n.

When A belongs to some special classes of (0,1) matrices, the studies of the least upper bound of $|\det(A)|$ were conducted by Ryser with an algebraic approach and by Brualdi and Solheid with a graphical approach.

Example 5.4.1 Let $A \in \mathbf{B}_n$ be the incidence matrix of a symmetric 2-design $S_\lambda(2,k,n)$. Then

$$AA^T = A^TA = (k-\lambda)I_n + \lambda J_n.$$

It follows that

$$|\det(A)| = k(k-\lambda)^{\frac{n-1}{2}}.$$

Note that the parameters satisfy $\lambda(n-1) = k(k-1)$. Ryser [224] found that the incidence matrix of symmetric 2-design $S_\lambda(2,k,n)$ yields in fact the extremal value of $|\det(A)|$, among a class of matrices in $\mathbf{B}_n$ with interesting properties.

Theorem 5.4.2 (Ryser, [224]) Let $Q = (q_{ij}) \in \mathbf{B}_n$ with $||Q|| = t$. Let $k \geq \lambda \geq 0$ be integers such that $\lambda(n-1) = k(k-1)$. If

$$t \leq kn \text{ and } \lambda \leq k-\lambda, \text{ or if } t \geq kn \text{ and } k-\lambda \leq \lambda, \tag{5.7}$$

then $|\det\ (Q)| \leq k(k-\lambda)^{\frac{n-1}{2}}$.

Proof For a matrix $E \in \mathbf{B}_n$, let $E(x,y)$ denote the matrix obtained from E by replacing each 1-entry of E by an x, and each 0-entry of E by a y. With this notation, set

$$p = \frac{k-\lambda}{\lambda},\ \ Q_1 = Q(-p,1) \ \text{ and } \overline{Q} = \begin{bmatrix} p & \mathbf{z} \\ \mathbf{z}^T & Q_1 \end{bmatrix}, \tag{5.8}$$

where $\mathbf{z}^T = (\sqrt{p}, \sqrt{p}, \cdots, \sqrt{p})$. Let $S_i = \sum_{j=1}^n q_{ij}^2$, for each i with $1 \leq i \leq n$. By Theorem 5.4.1,

$$|\det(\overline{Q})| \leq \sqrt{p^2+np}\prod_{i=1}^n \sqrt{p+S_i}. \tag{5.9}$$

Note that $\sum_{i=1}^n S_i = tp^2 + (n^2-t) = t(p^2-1)+n^2$, and that

$$p^2 + np = p\left(\frac{k-\lambda+\lambda n}{\lambda}\right) = \frac{k^2(k-\lambda)}{\lambda^2}.$$

It follows by (5.7) that $\sum_{i=1}^n S_i \leq kn(p^2-1)+n^2$. For each i, let $\overline{S}_i$ be a quantity such that

$$\overline{S}_i \geq S_i, (1 \leq i \leq n), \text{ and } \sum_{i=1}^n \overline{S}_i \leq kn(p^2-1)+n^2.$$

Thus

$$\begin{aligned}\sum_{i=1}^n (p+\overline{S}_i) &= kn(p^2-1)+n^2+np = n\left(kp^2 + \frac{\lambda n - \lambda k + k - \lambda}{\lambda}\right) \\ &= nkp(p+1) = \frac{n(k-\lambda)k^2}{\lambda^2}.\end{aligned}$$

It follows

$$\prod_{i=1}^{n}(p+\overline{S}_i) \le \left(\frac{1}{n}\sum_{i=1}^{n}(p+\overline{S}_i)\right)^n \le \left(\frac{(k-\lambda)k^2}{\lambda^2}\right)^n. \tag{5.10}$$

Combine (5.9) and (5.10) to get

$$\begin{aligned} |\det(\overline{Q})| &\le \frac{k\sqrt{k-\lambda}}{\lambda}\prod_{i=1}^{n}\sqrt{p+\overline{S}_i} \\ &\le \frac{k\sqrt{k-\lambda}}{\lambda}\left(\frac{k\sqrt{k-\lambda}}{\lambda}\right)^n = \left(\frac{k\sqrt{k-\lambda}}{\lambda}\right)^{n+1}. \end{aligned}$$

By (5.8), we can multiply the first row of $\overline{Q}$ by $-1/\sqrt{p}$ and add to the other rows of $\overline{Q}$ to get

$$|\det(\overline{Q})| = p|\det(Q(-k/\lambda,0))| \le \left(\frac{k\sqrt{k-\lambda}}{\lambda}\right)^{n+1}. \tag{5.11}$$

Note that $|\det(Q(-k/\lambda,0))| = (k/\lambda)^n|\det(Q)|$. It follows that

$$p|\det(Q)| \le \frac{k}{\lambda}\left(\sqrt{k-\lambda}\right)^{n+1}.$$

Therefore the theorem obtains. □

Theorem 5.4.3 (Ryser, [224]) Let $Q=(q_{ij})\in \mathbf{B}_n$ be a matrix. If $|\det(Q)| = k(k-\lambda)^{\frac{n-1}{2}}$, then Q is the incidence matrix of a symmetric 2-design $S_\lambda(2,k,n)$.

Proof If $|\det(Q)| = k(k-\lambda)^{\frac{n-1}{2}}$, then

$$p\left|\det Q\left(-\frac{k}{\lambda},0\right)\right| = \left(\frac{k\sqrt{k-\lambda}}{\lambda}\right)^{n+1}.$$

Define $\overline{Q}$ as in (5.8) and employ the notations in Theorem 5.4.2. By (5.11),

$$|\det(\overline{Q})| = \left(\frac{k\sqrt{k-\lambda}}{\lambda}\right)^{n+1},$$

and so equality must hold in (5.10), which implies

$$p+\overline{S}_i = \frac{(k-\lambda)k^2}{\lambda^2}, \quad 1\le i\le n.$$

It follows that $\overline{QQ}^T = k^2(k-\lambda)/\lambda^2 I_{n+1}$, and so

$$Q_1Q_1^T = \frac{k^2}{\lambda^2}(k-\lambda)I_n - pJ_n. \tag{5.12}$$

For each i, let $r_i = \sum_{j=1}^{n} q_{ij}$. By (5.12),

$$\begin{aligned} p^2 r_i + (n - r_i) &= \frac{k^2}{\lambda^2}(k-\lambda) - p \\ (p^2-1)r_i &= \frac{k^2}{\lambda^2}(k-\lambda) - p - n. \end{aligned}$$

Hence $r_i = k$, for each i with $1 \le i \le n$.

For $i \ne j$ with $1 \le i, j \le n$, let f denote the dot product of the ith row and the jth row of Q. By (5.12),

$$fp^2 - 2(k-f)p + n - 2k + f = -p$$
$$f(p^2 + 2p + 1) = 2kp - p + 2k - n.$$

It follows that $fk^2/\lambda^2 = k^2/\lambda$ and so $f = \lambda$.

Therefore, Q is the incidence matrix of a symmetric 2-design $S_\lambda(2,k,n)$. $\square$

The following theorem of Ryser follows by combining Theorems 5.4.2 and 5.4.3.

Theorem 5.4.4 (Ryser, [224]) Let $n > k > \lambda > 0$ be integers such that $\lambda(n-1) = k(k-1)$. Let $Q \in \mathbf{B}_n$ with $\|Q\| = t = kn$. Then

$$|\det(Q)| \le k(k-\lambda)^{\frac{n-1}{2}},$$

where equality holds if and only if A is the incidence matrix of a 2-design $S_\lambda(2,k,n)$.

Definition 5.4.2 Let $A \in \mathbf{B}_n$. Define two bipartite graphs $G_0(A)$ and $G_1(A)$ as follows. Both $G_0(A)$ and $G_1(A)$ have vertex partite sets $U = \{u_1, u_2, \cdots, u_n\}$ and $V = \{v_1, v_2, \cdots, v_n\}$. An edge $u_iv_j \in E(G_0(A))$ ($u_iv_j \in E(G_1(A))$, respectively) if and only if $a_{ij} = 0$ ($a_{ij} = 1$, respectively). Note that $G_0(A) = G_1(J_n - A)$.

A matrix A is *acyclic* if $G_1(A)$ is acyclic, and A is *complementary acyclic* if $G_0(A)$ is acyclic.

A matrix $A \in \mathbf{B}_n$ is *complementary triangular* if A has only 1's above the its main diagonal. For example, $J_n - A$ is a triangular matrix.

For each integer $n \ge 1$, define

$$f_n = \max\{|\det(A)| \,:\, A \in \mathbf{B}_n \text{ and } A \text{ is complementary acyclic}\}.$$

Example 5.4.2 Since an acyclic graph has at most one perfect matching, if A is acyclic, then $\det(A) \in \{0, -1, 1\}$.

Example 5.4.3 Suppose A is complementary acyclic, and let B be the matrix obtained from A by permuting two rows of A. Then B is also complementary acyclic with $\det(B) = -\det(A)$. Therefore,

$$f_n = \max\{\det(A) \,:\, A \in \mathbf{B}_n \text{ and } A \text{ is complementary acyclic}\}.$$

Brualdi and Solheid successfully obtained the least upper bound of $|\det(A)|$, where A ranges in some subsets of $\mathbf{B}_n$ with the complementary acyclic property. Their results are presented below. Interested readers are referred to [39] for proofs.

Theorem 5.4.5 (Brualdi and Solheid, [40]) Let $n \geq 3$ be an integer and let $A \in \mathbf{B}_n$ be a complementary acyclic matrix such that A has a row or column of all ones. Then

$$|\det(A)| \leq n - 2. \tag{5.13}$$

For $n \geq 4$, equality in (5.13) holds if and only if A or A^T is permutation equivalent to

$$L_n = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 0 & & & \\ \vdots & & J_{n-1} - I_{n-1} & \\ 0 & & & \end{bmatrix}. \tag{5.14}$$

For $n = 3$, equality in (5.13) holds if and only if A or A^T is permutation equivalent to one of these matrices

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

Definition 5.4.3 For a matrix $A \in \mathbf{B}_n$, the *complementary term rank* of A, ρ_{J-A}, is the term rank of $J_n - A$.

Theorem 5.4.6 (Brualdi and Solheid, [40]) Let $n \geq 3$ be an integer and let $A \in \mathbf{B}_n$ be a complementary acyclic matrix with $\rho_{J-A} = n - 1$. Then

$$|\det(A)| \leq \begin{cases} n - 2 & \text{if } 3 \leq n \leq 8 \\ \lfloor \frac{n-3}{2} \rfloor \lceil \frac{n-3}{2} \rceil & \text{if } n \geq 8. \end{cases} \tag{5.15}$$

For $n \geq 4$, equality holds in (5.15) if and only if A or A^T is permutation equivalent to L_n as defined in (5.14) (when $4 \leq n \leq 8$), or

$$\begin{bmatrix} J_{\lfloor \frac{n-1}{2} \rfloor} & \mathbf{j} & J \\ 0^T & 1 & \mathbf{j}^T \\ Z & 0 & J_{\lceil \frac{n-1}{2} \rceil} - I_{\lceil \frac{n-1}{2} \rceil} \end{bmatrix} \quad (n \geq 8),$$

where Z has at most one 0.

Theorem 5.4.7 (Brualdi and Solheid, [40]) Let $n \geq 2$ be an integer and let $A \in \mathbf{B}_n$ be a complementary acyclic matrix with $\rho_{J-A} = n$. Then

$$|\det(A)| \leq \begin{cases} n - 2 & \text{if } n \leq 5 \\ \lfloor \frac{n-1}{2} \rfloor \lceil \frac{n-1}{2} \rceil & \text{if } n \geq 5. \end{cases} \tag{5.16}$$

Equality holds in (5.16) if and only if A or A^T is permutation equivalent to $J_n - I_n$, or

$$\begin{bmatrix} J_{\lfloor \frac{n-1}{2} \rfloor} - I_{\lfloor \frac{n-1}{2} \rfloor} & \mathbf{j} & J \\ \mathbf{0}^T & 0 & \mathbf{j}^T \\ J & \mathbf{0} & J_{\lceil \frac{n-1}{2} \rceil} - I_{\lceil \frac{n-1}{2} \rceil} \end{bmatrix} \quad (n \geq 5).$$

More details can be found in [39]. For most of the matrix classes, the determination of the maximum determinant of matrices in a given class is still open.

5.5 Rearrangement of (0,1) Matrices

The rearrangement problem of an n-tuple was first studied by Hardy, Littlewood and Pòlya [114]. Schwarz [231] extended the concept to square matrices.

Definition 5.5.1 Let $(a) = (a_1, a_2, \cdots, a_n)$ be an n-tuple and let π be a permutation on the set $\{1, 2, \cdots, n\}$. Then $(a_\pi) = (a_{\pi(1)}, a_{\pi(2)}, \cdots a_{\pi(n)})$ is a *rearrangement* of (a).

A matrix $A \in M_n$ can be viewed as an n^2-tuple, and so we can define the rearrangement of a square matrix in a similar way.

Let π be a permutation on $\{1, 2, \cdots, n\}$, and let $A = (a_{ij}) \in M_n$. The matrix $A_\pi = (a'_{ij})$ is called a *permutation* of A if $a'_{ij} = a_{\pi(i),\pi(j)}$, for all $1 \leq i, j \leq n$. Clearly a permutation or a transposition of a matrix A is a rearrangement of A. We call a rearrangement *trivial* if it is a permutation, or a transposition, or a combination of permutations and transpositions. Two matrices $A_1, A_2 \in M_n$ are *essentially different* if A_2 is a nontrivial rearrangement of A_1.

For each $A = (a_{ij}) \in M_n$, define $\|A\| = \sum_{i=1}^n \sum_{j=1}^n a_{ij}$. Proposition 5.5.1 below follows immediately from the definitions.

Proposition 5.5.1 For matrices $A_1, A_2 \in M_n$, each of the following holds.

(i) If A_1 is a rearrangement of A_2, then $\|A_1\| = \|A_2\|$.

(ii) If A_1 is a trivial rearrangement of A_2, then $\|A_1^2\| = \|A_2^2\|$.

Definition 5.5.2 Let

$$\overline{N} = \max\{\|A^2\| \,:\, A \in M_n\}, \text{ and } \tilde{N} = \min\{\|A^2\| \,:\, A \in M_n\}.$$

Let $\overline{\mathcal{U}} = \{A \in M_n \,:\, \|A^2\| = \overline{N}\}$ and $\tilde{\mathcal{U}} = \{A \in M_n \,:\, \|A^2\| = \tilde{N}\}$, let $\overline{\mathcal{C}}$ denote the set of matrices $A = (a_{ij}) \in M_n$ such that for each i, $a_{ij} \geq a_{ij'}$ whenever $j < j'$, and such that for each j, $a_{ij} \geq a_{i'j}$ whenever $i < i'$; and let $\tilde{\mathcal{C}}$ denote the set of matrices $A = (a_{ij}) \in M_n$ such that for each i, $a_{ij} \geq a_{ij'}$ whenever $j < j'$, and such that for each j, $a_{ij} \leq a_{i'j}$ whenever $i < i'$.

Theorem 5.5.1 (Schwarz, [231])

$$\overline{\mathcal{U}} \cap \overline{\mathcal{C}} \neq \emptyset, \text{ and } \tilde{\mathcal{U}} \cap \tilde{\mathcal{C}} \neq \emptyset.$$

Definition 5.5.3 For a matrix $A \in M_n$, let $\lambda_1(A)$ and $\lambda_n(A)$ denote the maximum and the minimum eigenvalues of A, respectively. Let

$$\begin{aligned} \bar{\lambda} &= \max\{\lambda_1(A) \ : \ A \in M_n\}, \text{ and} \\ \tilde{\lambda} &= \min\{\lambda_n(A) \ : \ A \in M_n\}; \end{aligned}$$

and let

$$\begin{aligned} \overline{\mathcal{B}} &= \{A \in M_n \ : \ \lambda_1(A) = \bar{\lambda}\} \text{ and} \\ \tilde{\mathcal{B}} &= \{A \in M_n \ : \ \lambda_n(A) = \tilde{\lambda}\}. \end{aligned}$$

Theorem 5.5.2 (Schwarz, [231])

$$\overline{\mathcal{B}} \cap \overline{\mathcal{C}} \neq \emptyset, \text{ and } \tilde{\mathcal{B}} \cap \tilde{\mathcal{C}} \neq \emptyset.$$

Definition 5.5.4 For integer $n \geq 1$ and σ with $1 \leq \sigma \leq n^2$, let $\mathcal{U}_n(\sigma) = \{A \in \mathbf{B}_n \ : \ \|A\| = \sigma\}$. Let

$$\begin{aligned} \overline{N}_n(\sigma) &= \max\{\|A^2\| \ : \ A \in \mathcal{U}_n(\sigma)\}, \text{ and} \\ \tilde{N}_n(\sigma) &= \min\{\|A^2\| \ : \ A \in \mathcal{U}_n(\sigma)\}. \end{aligned}$$

Let $\overline{\mathcal{U}}_n(\sigma)$ denote the set of matrices $A = (a_{ij}) \in \mathcal{U}_n(\sigma)$ such that for each i, $a_{ij} \geq a_{ij'}$ whenever $j < j'$, and such that for each j, $a_{ij} \geq a_{i'j}$ whenever $i < i'$; and let $\tilde{\mathcal{U}}_n(\sigma)$ denote the set of matrices $A = (a_{ij}) \in \mathcal{U}_n(\sigma)$ such that for each i, $a_{ij} \geq a_{ij'}$ whenever $j < j'$, and such that for each j, $a_{ij} \leq a_{i'j}$ whenever $i < i'$.

Proposition 5.5.2 follows from Theorem 5.5.1 and the definitions.

Proposition 5.5.2 For integers $n \geq 1$ and σ with $1 \leq \sigma \leq n^2$, each of the following holds.

(i) $\overline{N}_n(\sigma) = \max\{\|A^2\| \ : \ A \in \overline{\mathcal{U}}_n(\sigma)\}$, and $\tilde{N}_n(\sigma) = \min\{\|A^2\| \ : \ A \in \tilde{\mathcal{U}}_n(\sigma)\}$.

(ii) Let $A = (a_{ij}) \in \overline{\mathcal{U}}_n(\sigma)$, let $s_i = \sum_{j=1}^n a_{ij}$ and $r_i = \sum_{i=1}^n a_{ji}$ denote the ith row sum and the ith column sum of A, respectively. Then

$$\overline{N}_n(\sigma) \geq \|A^2\| = \sum_{i=1}^n r_i s_i \geq \frac{1}{n}\left(\sum_{i=1}^n r_i\right)\left(\sum_{i=1}^n s_i\right) = \frac{\sigma^2}{n}.$$

Example 5.5.1

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \in \overline{\mathcal{U}}_3(6) \text{ and } \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \in \tilde{\mathcal{U}}_3(6).$$

Aharoni discovered the following relationship between $\overline{N}_n(\sigma)$ and $\overline{N}_n(n^2-\sigma)$, and between $\tilde{N}_n(\sigma)$ and $\tilde{N}_n(n^2-\sigma)$.

Theorem 5.5.3 (Aharoni, [1]) Let n and σ be integers with $n \geq 1$ and $1 \leq \sigma \leq n^2$. If $A \in \mathcal{U}_n(\sigma)$, then

(i) $\|A^2\| = 2\sigma n - n^3 + \|(J_n - A)^2\|$.

(ii) $\overline{N}_n(\sigma) = 2\sigma n - n^3 + \overline{N}_n(n^2-\sigma)$.

(iii) $\tilde{N}_n(\sigma) = 2\sigma n - n^3 + \tilde{N}_n(n^2-\sigma)$.

Parts (ii) and (iii) of Theorem 5.5.3 follow from Part (i) of Theorem 5.5.3 and the observation that if $A \in \mathcal{U}_n(\sigma)$, then $J_n - A \in \mathcal{U}_n(n^2-\sigma)$. In [1], Aharoni constructed four types of matrices for any $1 \leq \sigma \leq n^2$, and proved that among these four matrices, there must be one A such that $\overline{N}_n(\sigma) = \|A^2\|$.

Theorem 5.5.3(ii) and (iii) indicate that to study $\overline{N}_n(\sigma)$ and $\tilde{N}_n(\sigma)$, it suffices to consider the case when $\sigma \geq n^2/2$. The next result of Katz, points out that an extremal matrix reaching $\overline{N}_n(\sigma)$ would have all its 1-entries in the upper left corner principal submatrix.

Theorem 5.5.4 (Katz, [142]) Let n, k be integers with $n^2 \geq k^2 \geq n^2/2 > 0$. Then

$$\overline{N}_n(\sigma) = k^3.$$

Corollary 5.5.4 Let n, k be integers with $n^2 \geq k^2 \geq n^2/2 > 0$. Then

$$\overline{N}_n(n^2-k^2) = k^3 - 2k^2 n + n^3.$$

To study $\tilde{N}_n(\sigma)$, we introduce the square bipartite digraph of a matrix, which plays a useful role in the study of $\|A^2\|$.

Definition 5.5.5 For a $A = (A_{ij}) \in \mathbf{B}_n$, let $K(A)$ be a directed bipartite graph with vertex partite sets (V_1, V_2), where $V_1 = \{u_1, u_2, \cdots, u_n\}$ and $V_2 = \{v_1, v_2, \cdots, v_n\}$, representing the row labels and the column labels of A, respectively. An arc (u_i, v_j) is in $E(K)$ if and only if $a_{ij} = 1$.

Let K_1 and K_2 be two copies of $K(A)$ with vertex partite sets (V_1, V_2) and (V_1', V_2'), respectively, where $V_1' = \{u_1', u_2', \cdots, u_n'\}$ and $V_2' = \{v_1', v_2', \cdots, v_n'\}$, and where $(u_i', v_j') \in E(K_2)$ if and only if $a_{ij} = 1$. The ***square bipartite digraph*** of A, denoted by $SB(A)$, is the digraph obtained from K_1 and K_2 by identifying v_i with u_i', for each $i = 1, 2, \cdots, n$.

The next proposition follows from the definitions.

Proposition 5.5.3 Let $A \in \mathbf{B}_n$. Each of the following holds.

(i) $\|A^2\|$ is the total number of directed paths of length 2 from a vertex in V_1 to a vertex in V_2'.

(ii) For each $v_i \in V_2$ in $SB(A)$, $d^-(v_i) = s_i$ is the ith row sum of A, and $d^+(v_i) = r_i$ is the ith column sum of A.

(iii) $\|A^2\| = \sum_{i=1}^{n} d^-(v_i)d^+(v_i) = \sum_{i=1}^{n} r_i s_i$.

Example 5.5.2 The square bipartite graph of the matrix A_1 in Example 5.5.1 is as follows:

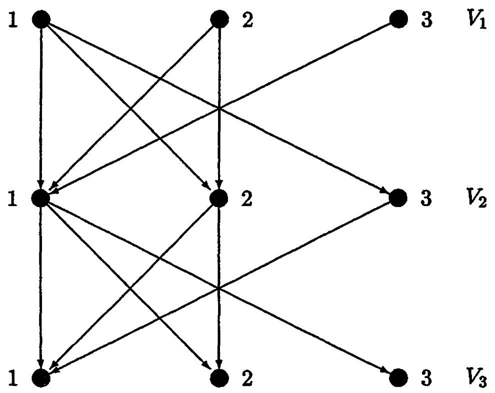

Figure 5.5.1 The graph in Example 5.5.1

Example 5.5.3 The value $\|M^2\|$ may not be preserved under taking rearrangements. Consider the matrices

$$A_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ and } A_2 = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Then A_1 and A_2 are essentially different and $\|A_1^2\| \neq \|A_2^2\|$.

Theorem 5.5.5 (Brualdi and Solheid, [38]) If $\sigma \geq n^2 - \lfloor \frac{n}{2} \rfloor \lceil \frac{n}{2} \rceil$, then

$$\tilde{N}_n = 2\sigma n - n^3.$$

Moreover, for $A \in \mathcal{U}_n(\sigma)$, $\|A^2\| = \tilde{N}_n(\sigma)$ if and only if A is a permutation similar to

$$\begin{bmatrix} J_k & X \\ J_{l,k} & J_l \end{bmatrix}, \tag{5.17}$$

where $X \in M_{k,l}$ is an arbitrary matrix, and where $k \geq 0$ and $l \geq 0$ are integers such that $k + l = n$.

Sketch of Proof Construct a square bipartite digraph $D = SB(A_1)$ as follows: every vertex in $\{u_1, \cdots, u_l\}$ is directed to every vertex in $\{v_{l+1}, \cdots, v_n\}$, where $l > 0$ is an integer at most n. By Proposition 5.5.3(i), $\|A_1^2\| = 0$. Let $A = J_n - A_1$. By Theorem 5.5.3(i) and by $\|A_1^2\| = 0$,

$$\|A^2\| = 2\sigma n - n^3 = \tilde{N}_n(\sigma),$$

where $\sigma = \|A\| = \|J_n - A_1\| \geq n^2 - \lfloor \frac{n}{2} \rfloor \lceil \frac{n}{2} \rceil$.

By Theorem 5.5.3(i), if $A \in \mathcal{U}_n(\sigma)$ satisfies $\|A^2\| = 2\sigma n - n^3$, then $\|(J_n - A)^2\| = 0$, and so by Proposition 5.5.3(i), $SB(J_n - A)$ must be some digraph as a subgraph of the one constructed above, (renaming the vertices if needed). Therefore, A must be permutation similar to a matrix of the form in (5.17). □

Proposition 5.5.4 Let $A \in \tilde{\mathcal{U}}_n(\sigma)$ and let $D = SB(A)$ with vertex set $V_1 \cup V_2 \cup V_2'$, using notations in Definition 5.5.5.

(i) If for some $i < n$ and $j > 1$, $(u_i, v_j) \in E(D)$, then both $(u_i, v_{j-1}) \in E(D)$ and $(u_{i+1}, v_j) \in E(D)$.

(ii) If $\sigma \geq \begin{pmatrix} n \\ 2 \end{pmatrix}$, and if $\|A^2\| = \tilde{N}_n(\sigma)$, then in D, that $i > j$ implies that $(u_i, v_j) \in E(D)$.

(iii) If $\sigma \geq \begin{pmatrix} n \\ 2 \end{pmatrix}$, and if $A \in \tilde{\mathcal{U}}_n(\sigma)$ and $\|A^2\| = \tilde{N}_n(\sigma)$, then every entry under the main diagonal in A is a 1-entry.

Proof Part (i) follows from the Definition 5.5.4 and Definition 5.5.5. Part (iii) follows from Part (ii) immediately.

To prove Part (ii), we argue by contradiction. Assume that there is a pair p and q such that $p > q$ but $(u_p, v_q) \notin E(D)$. Since $\sigma \geq n(n-1)/2$ and by Proposition 5.5.4(i), there must be an i such that $(u_i, v_i) \in E(D)$. Obtain a new bipartite digraph $D_1 = SB(A_0)$ from D by deleting $(u_i, v_i), (v_i, v_i')$ and then by adding (u_p, v_q) and (v_p, v_q'). Note that

$$\|A^2\| - \|A_0^2\| = (d^-(v_i) - d^-(v_p)) + (d^+(v_i) - d^+(v_q)) - 1, \tag{5.18}$$

where the degrees are counted in D. By Proposition 5.5.4(i) again,

$$d^-(v_i) \geq n - (i-1),\ d^+(v_i) \geq i,\ d^-(v_p) \leq n - p, \text{ and } d^+(v_q) \geq q - 1. \tag{5.19}$$

It follows by (5.18) and (5.19) that

$$\|A^2\| - \|A_0^2\| \geq p - q + 1 \geq 2,$$

contrary to the assumption that $\|A^2\| = \tilde{N}_n(\sigma)$. □

Theorem 5.5.6 (Brualdi and Solheid, [38]) If $\sigma = \binom{n}{2}$, then

$$\tilde{N}_n = \binom{n}{3}.$$

Moreover, if $A \in \mathcal{U}_n(\sigma)$ and $\|A^2\} = \tilde{N}_n(\sigma)$, then A is permutation similar to L_n, the matrix in $\mathbf{B}_n$ each of whose 1-entry is below the main diagonal.

Proof This follows from Proposition 5.5.4(iii). □

To investigate the case when $\binom{n}{2} < \sigma < n^2 - \lfloor\frac{n}{2}\rfloor\lceil\frac{n}{2}\rceil$, we establish a few lemmas.

Lemma 5.5.1 (Liu, [175]) Let $A = (a_{ij}) \in \mathcal{U}_n(\sigma)$, let $A(\sigma + e_{pq})$ denote the matrix obtained from A by replacing a 0-entry a_{pq} by a 1-entry, and let s_p and r_q denote the pth column sum and the qth row sum, respectively. Then

$$\|A^2(\sigma + e_{pq})\| = \begin{cases} \|A^2\| + s_p + r_q & \text{if } p \neq q \\ \|A^2\| + s_p + r_q + 1 & \text{if } p = q \end{cases}$$

Proof Note that $SB(A(\sigma+e_{pq})$ is obtained from $SB(A)$ by adding the arcs $(u_p, v_q), (v_p, v'_q)$. If $p \neq q$, the number of the newly created directed paths of length 2 from V_1 to V'_2 in $SB(A(\sigma + e_{pq})$ is $d^+(v_q) + d^-(v_p) = r_q + s_p$; if $p = q$, an additional path $u_p v_p v'_p$ is also created, and so Lemma 5.5.1 follows from Proposition 5.5.3(i). □

Lemma 5.5.2 (Liu, [175]) Let L_n be the matrix in $\mathbf{B}_n$ each of whose 1-entry is below the main diagonal. An (i,j)-entry in L_n is called an *upper entry* if $i \leq j$. Let A denote the matrix obtained from L_n by changing the upper entries at (i_t, j_t), where $1 \leq t \leq r$, from 0 to 1. If all the i_t's are distinct and all the j_t's are distinct, then

$$\|A^2\| = \binom{n}{3} + \sum_{t=1}^{r} \Delta(i_t, j_t),$$

where

$$\Delta(i_t, j_t) = \begin{cases} (n-1) - i_t + j_t & \text{if } i_t < j_t \\ n - i_t + j_t & \text{if } i_t = j_t. \end{cases}$$

Proof This follows immediately from Lemma 5.5.1 and Theorem 5.5.6. □

With Proposition 5.5.4(iii) and Lemma 5.5.2, it is not difficult to prove the following theorem.

Theorem 5.5.7 (Liu, [175]) Let $\sigma = \binom{n}{2} + k$, where $1 \le k \le n$, then

$$\tilde{N}_n(\sigma) = \binom{n}{3} + kn,$$

Theorem 5.5.8 (Brualdi and Solheid, [38]) If $\sigma = \binom{n+1}{2}$, then

$$\tilde{N}_n = \binom{n}{3} + n^2 = \binom{n+2}{3}.$$

Moreover, if $A \in \mathcal{U}_n(\sigma)$ and $\|A^2\} = \tilde{N}_n(\sigma)$, then A is permutation similar to L_n^*, the matrix in $\mathbf{B}_n$ each of whose 1-entry is on and below the main diagonal.

Proof This follows from Theorem 5.5.7 with $k = n$. $\square$

The case when $\sigma = \binom{n+1}{2} + k$ with $1 \le k \le n-1$ can be studied similarly. With the same arguments as in the proofs for Lemmas 5.5.1 and 5.5.2, we can derive the lemmas below, which provide the needed arguments in the proof of Theorem 5.5.9 (Exercise 5.10).

Lemma 5.5.3 If $\sigma \ge \binom{n+1}{2}$, and if $A \in \tilde{\mathcal{U}}_n(\sigma)$ and $\|A^2\| = \tilde{N}_n(\sigma)$, then every entry on or under the main diagonal in A is a 1-entry.

Lemma 5.5.4 Let L_n^* be the matrix in $\mathbf{B}_n$ each of whose 1-entry is on or below the main diagonal. Let A denote the matrix obtained from L_n^* by changing the upper entries at (i_t, j_t), where $1 \le t \le r$, from 0 to 1. If all the i_t's are distinct and all the j_t's are distinct, then

$$\|A^2\| = \binom{n+2}{3} + \sum_{t=1}^{r}(n + 1 - i_t + j_t).$$

Theorem 5.5.9 (Liu, [175]) Let $\sigma = \binom{n+1}{2} + k$, where $1 \le k \le \lfloor \frac{n}{2} \rfloor$, then

$$\tilde{N}_n(\sigma) = \binom{n+2}{3} + k(n+2),$$

Theorem 5.5.10 (Brualdi and Solheid, [38]) If $n \ge 2$ is even and if $\sigma = \binom{n+1}{2} + \frac{n}{2}$, then

$$\tilde{N}_n = 8\binom{\frac{n}{2}+2}{3}.$$

Proof Apply Theorem 5.5.9 with $k = n/2$. $\square$

Conjecture 5.5.1 (Brualdi and Solheid, [38]) Let k, l, n and q be integers such that $1 \le k \le n$, $n = qk + l$ and $0 \le l < k$. Let

$$\sigma_{n,k} = \binom{q+1}{2} k^2 + nl,$$

and let

$$A_{n,k} = \begin{bmatrix} J_k & & & & 0 \\ & \ddots & & & \\ & & J_k & & \\ & J & & J_k & \\ & & & & J_k \end{bmatrix},$$

where every entry of $A_{n,k}$ below the mail diagonal is a 1-entry. Then

$$\tilde{N}_n(\sigma_{n,k}) = \|A_{n,k}^2\|.$$

Theorem 5.5.10 proved Conjecture 5.5.1 for the special case when $k = 2$ and n is even. To further approach this conjecture, we introduce some notations.

Through the end of this section, let $L_n^*(S_r)$ be the matrix obtained from L_n^* by changing the 0-entries into 1-entries at the positions $(i, i+1)$, with $i_0 \le i \le i_0 + r - 1$, for some $1 \le i_0 \le n-1$ (and call these newly added 1-entries an S_r); let $L_n^*(T_r)$ be the matrix obtained from L_n^* by changing the 0-entries into 1-entries at the positions $(i, i+j)$, with $i_0 \le i \le i_0 + r - 1$ and $1 \le j \le r - (i - i_0)$, for some $1 \le i_0 \le n-1$ (and call these newly added 1-entries a T_r).

Let $\Delta(S_r) = \|(L_n^*(S_r))^2\| - \|(L_n^*)^2\|$ and $\Delta(T_r) = \|(L_n^*(T_r))^2\| - \|(L_n^*)^2\|$. Let $\|T_r\| = r(r+1)/2$ denote the number of 1-entries of a T_r and let $\|S_r\| = r$ denote the number of 1-entries of a S_r.

Lemma 5.5.5

$$\Delta(T_r) = \binom{r+1}{2}\left(n + \frac{2(r+2)}{3}\right).$$

Proof By Lemma 5.5.1,

$$\begin{aligned} \Delta(T_r) &= n\sum_{j=1}^{r} j + \sum_{j=1}^{r} j(j+1) \\ &= n\binom{r+1}{2} + \binom{r+1}{2} + \frac{2r+1}{3}\binom{r+1}{2} \\ &= \binom{r+1}{2}\left(n + \frac{2(r+2)}{3}\right). \end{aligned}$$

This proves Lemma 5.5.5. □

Lemma 5.5.6 If $\|T_r\| = \sum_{i=1}^{t} \|T_{r_i}\|$ for some $t > 1$, then $\Delta(T_r) > \sum_{i=1}^{t} \Delta(T_{r_i})$.

Proof By assumption and by $r > r_i$,

$$\begin{aligned}\binom{r+1}{2}(r+2) &= \sum_{i=1}^{t}\binom{r_i+1}{2}(r+2)\\ &> \binom{r_i+1}{2}(r_i+2).\end{aligned}$$

It follows by Lemma 5.5.5 that $\Delta(T_r) > \sum_{i=1}^{t} \Delta(T_{r_i})$. □

The next two lemmas can be proved similarly.

Lemma 5.5.7

$$\Delta(S_r) = r(n+3) - 1.$$

Lemma 5.5.8 If $\|S_r\| = \sum_{i=1}^{t} \|S_{r_i}\|$ for some $t > 1$, then $\Delta(S_r) > \sum_{i=1}^{t} \Delta(S_{r_i})$.

Theorem 5.11 Suppose that

$$\sigma = \binom{n+1}{2} + k,\ \lfloor \frac{n}{2} \rfloor < k \le \lceil \frac{n+1}{2} \rceil.$$

Then

$$\tilde{N}_n(\sigma) = \binom{n+2}{3} + k(n+3) - \lfloor \frac{n-1}{2} \rfloor.$$

Proof Assume that $A \in \mathcal{U}_n(\sigma)$ with $\|A^2\| = \tilde{N}_n(\sigma)$. By Proposition 5.5.2, we may assume that $A \in \tilde{\mathcal{U}}_n(\sigma)$, and so L_n^* is a submatrix of A. By Lemma 5.5.8, the minimum of $\|A^2\|$ can be obtained by putting $k - \lfloor \frac{n-1}{2} \rfloor$ of S_2 and $2\lfloor \frac{n-1}{2} \rfloor - k$ of S_1 above the main diagonal of L_n^*. Therefore,

$$\begin{aligned}\tilde{N}_n(\sigma) &= \|(L_n^*)^2\| + \sum_{i=1}^{k-\lfloor \frac{n-1}{2} \rfloor} \Delta(S_2) + \sum_{i=1}^{2\lfloor \frac{n-1}{2} \rfloor - k} \Delta(S_1)\\ &= \binom{n+2}{3} + k(n+3) - \lfloor \frac{n-1}{2} \rfloor.\end{aligned}$$

This completes the proof. □

5.6 Perfect Elimination Scheme

This section is devoted to the discussion of perfect elimination schemes of matrices and graph theory techniques can be applied in the study.

Definition 5.6.1 Let $A = (a_{ij}) \in M_n$ be a nonsingular matrix. The following process converting A into I is called the *Gauss elimination process.*

For each $t = 1, 2, \cdots, n$,

(1) select a nonzero entry at some (i_t, j_t)-cell (called a pivot),

(2) apply row and column operations to convert this entry into 1, and to convert the other entries in Row i_t and Column i_t into zero.

The resulted matrix can then be converted to the identity matrix I by row permutations only. The sequence $(i_1, j_1), (i_2, j_2), \cdots, (i_n, j_n)$ is called a pivot sequence.

A *perfect elimination scheme* is a pivot sequence such that no zero entry of A will become a nonzero entry in the Gauss elimination process.

Example 5.6.1 For the matrix

$$\begin{bmatrix} 4 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix},$$

The pivot sequence $(1,1), (2,2), (3,3), (4,4)$ is not a perfect elimination scheme since the 0-entry at (3,2) will become a nonzero entry in the process. On the other hand, the pivot sequence $(4,4), (3,3), (2,2), (1,1)$ is a perfect elimination scheme.

Example 5.6.2 There exists matrices that does not have a perfect elimination scheme. Consider

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 \end{bmatrix}.$$

Note that if for a given (i, j), there exist s, t such that $a_{sj} \neq 0$, $a_{it} \neq 0$ but $a_{st} = 0$, then, a_{ij} cannot be a pivot in a perfect elimination scheme. With this in mind, we can easily check that the matrix A in this example does not have a perfect elimination scheme.

Example 5.6.3 Matrices of the following type always have a perfect elimination scheme,

$$A = \begin{bmatrix} * & * & & & & \\ * & * & * & & 0 & \\ & * & * & * & & \\ & & * & * & \ddots & \\ & 0 & & \ddots & \ddots & * \\ & & & & * & * \end{bmatrix},$$

where $*$ denotes a nonzero entry.

Definition 5.6.2 Let G be a simple graph, let $\omega(G), \chi(G), \alpha(G)$ and $k(G)$ denote the clique number (maximum order of a clique), the chromatic number, the independence number (maximum cardinality of an independence set), and the clique covering number (minimum number of cliques that covers $V(G)$), respectively. For a subset $A \subseteq V(G)$, $G[A]$ denotes the subgraph of G induced by A. A graph G is *perfect* if G satisfies any one of the following:

(P1) $\omega(G[A]) = \chi(G[A]), \forall A \subset V(G)$.
(P2) $\alpha(G[A]) = k(G[A]), \forall A \subset V(G)$.
(P3) $\omega(G)\alpha(G) \geq |A|, \forall A \subset V(G)$.

(The equivalence of that (P1), (P2) and (P3) is shown by Lovász [190] in 1972.)

Definition 5.6.3 A graph G is *chordal* if G does not have an induced cycle of length longer than three.

Given a graph G, a vertex $v \in V(G)$ is a *simplicial* vertex if $G[N(v)]$ is a clique. If $[v_1, v_2, \cdots, v_n]$ is an ordering of $V(G)$ such that v_i is simplicial in $G[\{v_i, \cdots, v_n\}]$, for each $i = 1, 2, \cdots, n-1$, then $[v_1, \cdots, v_n]$ is a *perfect vertex elimination scheme* of just a *perfect scheme.*

An edge $e = xy$ in a bipartite graph H is *bisimplicial* if the induced subgraph $H[N(x) \cup N(y)]$ is a complete bipartite graph. Given an ordering $[e_1, e_2, \cdots, e_m]$ of pairwise nonadjacent edges in H, let S_i denote the vertices in H that are incident with $\{e_1, e_2, \cdots, e_i\}$, where $1 \leq i \leq m$, and let $S_0 = \emptyset$. An ordering $[e_1, e_2, \cdots, e_m]$ of edges in H is a *partial scheme* if the e_i's are pairwise nonadjacent edges in H and if for each $i \geq 1$, e_i is bisimplicial in $H - S_{i-1}$. An ordering $[e_1, e_2, \cdots, e_m]$ of edges in H is a *perfect edge elimination scheme* (or just a scheme) if it is a partial scheme and if $H - S_m$ is edgeless. A bipartite graph H possessing a scheme is a *perfect elimination bipartite graph.*

Theorem 5.6.1 (Dirac, [74]) Let G be a chordal graph. Then G has a simplicial vertex. If G is not a complete graph, then G has two nonadjacent simplicial vertices.

Sketch of Proof This is trivial if G is a complete graph. Argue by induction on $|V(G)|$

and assume that G has two nonadjacent vertices u and v and a minimum vertex set S separating u and v. Let G_u and G_v denote the connected components of $G - S$ containing u and v, respectively.

Let $G[V(G_u) \cup S]$ and $G[V(G_v) \cup S]$ denote the subgraphs of G induced by $V(G_u) \cup S$ and by $V(G_v) \cup S$, respectively.

By induction, either since $G[V(G_u) \cup S]$ contains two nonadjacent simplicial vertices, whence one of these two vertices must be in G_v, as $G[V(G_u) \cup S]$ is induced; or $G[V(G_u) \cup S]$ is complete. Apply induction to $G[V(G_v) \cup S]$ as well, and so the theorem follows by induction. $\square$

Theorem 5.6.2 (Fulkerson and Gross, [94]) Let G be a simple graph. The following are equivalent.

(i) G is chordal.

(ii) G has a perfect scheme such that any simplicial vertex of G can be the first vertex in a perfect scheme.

(iii) Every minimal separation set of G induces a complete subgraph.

Proof (iii) $\Longrightarrow$ (i). Let $u, x, v, y_1, y_2, \cdots, y_k, a$ be a cycle of G with length $k+3 \geq 4$. Note that any vertex subset S separating u and v must contain $x, y_1, \cdots, y_k$, and so $xy_i \in E(G)$ is a chord of this cycle.

(i) $\Longrightarrow$ (iii). Let S be a minimal vertex set separating u and v in G. Let G_u and G_v be the components of $G \setminus S$ containing u and v, respectively. Since S is minimal, each $s \in S$ is adjacent to some vertex in G_u and some vertex in G_v. For each pair of distinct vertices $x, y \in S$, there exists an (x, y)-path P whose internal vertices are all in G_u, and an (x, y)-path Q whose internal vertices are all in G_v. It follows by (i) that $xy \in E(G)$, and so S induces a complete subgraph of G.

(i) $\Longrightarrow$ (ii). Argue by induction. By Theorem 5.6.1, G has a simplicial vertex v, and so $G - v$ is also a chordal graph, which has a perfect scheme. The add v to this scheme to get a perfect scheme of G with v being the first vertex.

(ii) $\Longrightarrow$ (i). Let C be a cycle of G and let $v \in V(C)$ such that in a perfect scheme of G, v is the first among all vertices in $V(C)$. As $|N(v) \cap V(C)| \geq 2$, that v is simplicial implies that C has a chord. $\square$

Theorem 5.6.3 below gives an inductive structural description of chordal graphs. A proof of Theorem 5.6.3 can be found in [155].

Theorem 5.6.3 (Leukev, Rose and Tarjan, [155]) If G is a chordal graph, then there exists a sequence of chordal graphs G_i, $0 \leq i \leq s$, such that $G_0 = G$, G_s is a complete graph and for each $1 \leq i \leq s-1$, G_i is obtained by adding a new edge from G_{i-1}.

Definition 5.6.4 The notions of associated digraph and the bipartite representation of a

(0,1) matrix will be extended. Let $A = (a_{ij}) \in M_n$.

The associate directed digraph $D(A)$ of A has vertex set $\{v_1, v_2, \cdots, v_n\}$ such that $(v_i, v_j) \in E(D(G))$ if and only if both $a_{ij} \neq 0$ and $i \neq j$. When A is symmetric, $D(A)$ can be viewed as a graph, and in this case, we write $G(A)$ for $D(A)$.

The bipartite representation $B(A)$ of A has vertex partite sets $X = \{x_1, x_2, \cdots, x_n\}$ and $Y = \{y_1, y_2, \cdots, y_n\}$, representing the rows and the columns of A, respectively, such that $(x_i y_j) \in E(B(A))$ if and only if $a_{ij} \neq 0$. For each i, x_i and y_i are *partners* corresponding to the vertex v_i in $D(A)$.

Some observations from the definitions and from the remarks in Example 5.6.2 are listed in Proposition 5.6.1.

Proposition 5.6.1 Let $A = (a_{ij}) \in M_n$.

(i) If A is symmetric, then $D(A)$ can be viewed as a graph. An entry a_{ii} is a pivot if and only if v_i is simplicial in $D(A)$.

(ii) Suppose that A is symmetric such that each $a_{ii} \neq 0$. A perfect elimination scheme of A with each a_{ii} as a pivot corresponds to a perfect vertex elimination scheme of $G(A)$.

(iii) A bisimplicial edge of $B(A)$ corresponds to a pivot in A; and a perfect elimination scheme of A corresponds to a perfect edge elimination scheme of $B(A)$.

Theorem 5.6.4 (Columbic [60]) and Rose [218]) Let $A = (a_{ij}) \in M_n$ be a symmetric matrix with $a_{ii} \neq 0$, $1 \leq i \leq n$. Each of the following is equivalent.

(i) A has a perfect elimination scheme.

(ii) A has a perfect elimination scheme of A with each a_{ii} as a pivot.

(iii) $G(A)$ is a chordal graph.

Proof Part (ii) and Part (iii) are equivalent by Theorem 5.6.2. As (ii) trivially implies (i), it suffices to show that (i) implies (iii).

By Proposition 5.6.1(iii), we may assume that $B(A)$ has a perfect edge elimination scheme $[e_1, \cdots, e_m]$.

Suppose $G(A)$ has a chordless cycle $v_{a_1} v_{a_2} \cdots v_{a_p} v_{a_1}$ for some $p \geq 4$. Since each $a_{ii} \neq 0$, the induced subgraph $H = B(A)[\{x_{a_1}, x_{a_2}, \cdots, x_{a_p}, y_{a_1}, y_{a_2}, \cdots, y_{a_p}\}]$ is a 3-regular graph with edge set

$$\begin{aligned} &\{(x_{a_i} y_{a_{i+1}}) \ : \ 1 \leq i \leq p-1\} \cup \{(x_{a_p} y_{a_1}), (y_{a_p} y_{a_1})\} \cup \\ \cup \ &\{(y_{a_i} x_{a_{i+1}}) \ : \ 1 \leq i \leq p-1\} \cup \{x_{a_i} y_{a_i} \ : \ 1 \leq i \leq p\}. \end{aligned}$$

Since $[e_1, \cdots, e_m]$ is a perfect edge elimination scheme, there is a smallest j such that e_j is incident with a vertex in $V(H)$. The edge e_j cannot be in $E(H)$ since no edge in $E(H)$ is bisimplicial. Without loss of generality, we assume that $e_j = x_{a_1} y_s$, for some $y_s \notin V(H)$. Denote x_s the partner of y_s.

Since $p \geq 4$, in H, $N(x_{a_1}) = \{y_{a_1}, y_{a_2}, y_{a_p}\}$ and $N(y_{a_1}) \cap N(y_{a_2}) \cap N(y_{a_p}) = \{x_{a_1}\}$. Since $e_j = x_{a_1} y_s$ is bisimplicial, $V(H) \cap N(y_s) = \{x_{a_1}\}$. By symmetry, $V(H) \cap N(x_s) = \{y_{a_1}\}$. Note that $x_{a_1} y_{a_1}, x_s y_s \in E(B(A))$ but $x_s y_{a_2} \notin E(B(A))$, contrary to the fact that $x_{a_1} y_s$ is bisimplicial. Therefore, $p = 3$ and $G(A)$ must be chordal. $\Box$

We conclude this section with two more results by Columbic and Goss [61] on the properties of perfect elimination bipartite graphs. Interested readers are referred to [61] for proofs.

Theorem 5.6.5 (Columbic and Goss, [61]) If H is a perfect elimination bipartite graph and if $e = xy$ is bisimplicial in H, then $H - \{x, y\}$ is also a perfect elimination bipartite graph.

Theorem 5.6.5 (Columbic and Goss, [61]) If a bipartite graph H has no pair of disjoint edges, the it is a perfect elimination bipartite graph.

5.7 Completions of Partial Hermitian Matrices

All the matrices considered in this section will be over the complex number field. Denote by $\overline{z}$ the conjugate of a complex number z, and by $\overline{A}$ the conjugate of a matrix A with complex entries. An $n \times n$ matrix H is *Hermitian* if $H = H^* = \overline{H^T}$.

An $n \times n$ matrix A is positive *definite Hermitian* (positive *semidefinite Hermitian*, respectively) if for any n-dimensional complex vector $\mathbf{x} \neq \mathbf{0}$, $\overline{\mathbf{x}^T} A\mathbf{x} > 0$ (≥ 0, respectively). While all positive definite Hermitian matrices are positive semidefinite Hermitian, the matrix J_n, when $n \geq 2$, is positive semidefinite Hermitian but not positive semidefinite Hermitian.

A Hermitian matrix $A = (a_{ij})$ is a *partial Hermitian* matrix if some of the a_{ij}'s are not specified. These unspecified entries may be viewed as blank slots. Given a partial Hermitian matrix A, is it possible to file the blank slots with certain complex number so that the resulting matrix is positive semidefinite Hermitian? If the answer is affirmative, then the resulting Hermitian matrix if a *positive semidefinite completion* of the partial Hermitian matrix A. Results in this section are mostly from the work of Grone, Johnson, Sá and Wolkowwicz [107]. Some of the facts and properties of positive semidefinite Hermitian matrix are listed in the proposition below.

Proposition 5.7.1 Each of the following holds.

(i) Let $A_1, A_2, \cdots, A_k$ be positive semidefinite Hermitian matrices. If $c_1, c_2, \cdots, c_k$ are nonnegative real numbers, then $\sum_{i=1}^{k} c_i A_i$ is also positive semidefinite Hermitian.

(ii) Let $A = (a_{ij})$ be an $n \times n$ Hermitian matrix. Then each a_{ii} is a real number, for $i = 1, 2, \cdots, n$.

(iii) A is positive semidefinite Hermitian if and only if every eigenvalue of A is nonnegative.

(iv) Let A be a Hermitian matrix. Then A is positive semidefinite if and only if for each $i = 1, 2, \cdots, n$, $\det(A_i) \geq 0$, where A_i is an $i \times i$ principal submatrix of A.

(v) Let $A = (a_{ij})$ be a Hermitian matrix. Then

$$\det(A) \leq \prod_{i=1}^{n} a_{ii}. \tag{5.20}$$

Moreover, if $a_{ii} > 0$ for each $i = 1, 2, \cdots, n$, then equality holds in (5.20) if and only if A is diagonal.

Definition 5.7.1 Let G be a graph vertex set $\{v_1, v_2, \cdots, v_n\}$, and with loops permitted. A *G-partial matrix* is a set of complex numbers, denoted $(a_{ij})_G$, where a_{ij} is defined if and only if $(v_i v_j \in E(G)$. (As G is undirected, a_{ij} is defined if and only if a_{ji} is defined).

A *completion* of $(a_{ij})_G$ is an $n \times n$ matrix $M = (m_{ij})$ such that $m_{ij} = a_{ij}$ whenever $v_i v_j \in E(G)$. We say that M is a *positive completion* (a *nonnegative completion*, respectively) if and only if M is a completion of $(a_{ij})_G$ and M is positive definite (positive semidefinite, respectively).

Given a graph G with $V(G) = \{v_1, c_2, \cdots, v_n\}$, the G-partial matrix $(a_{ij})_G$ is *G-partial positive* (*G-partial nonnegative*, respectively), if both of the following hold.

(5.7.1A) For all $1 \leq i, j \leq n$, $a_{ij} = \overline{a_{ji}}$, and

(5.7.1B) For any complete subgraph K of G, the corresponding principal submatrix $(a_{ij})_{v_i, v_j \in V(K)}$ of $(a_{ij})_G$ is positive definite (positive semidefinite, respectively).

Let G be a subgraph of J. A J-partial matrix $(b_{ij})_J$ *extends* a G-partial matrix (a_{ij}) if $b_{ij} = a_{ij}$ for every $v_i v_j \in E(G) \subseteq E(J)$.

A graph G is *completable* , *(nonnegative-completable* respectively) if and only if any G-partial positive (G-partial nonnegative, respectively) matrix has a positive (nonnegative, respectively) completion.

Let $LP(G)$ denote the set of vertices of G at which G has loops. Without loss of generality, we assume that either $LP(G) = \emptyset$ or $LP(G) = \{v_1, v_2, \cdots, v_k\}$, for some $1 \leq k \leq n$. For notational convenience, we also use $LP(G)$ to denote the subgraph of G induced by $LP(G)$.

Proposition 5.7.3 below indicates that the terms "nonnegative-completable" and "completable" coincide.

Proposition 5.7.2 (Grone, Johnson, Sá and Wolkowicz, [107]) G is completable (nonnegative-completable respectively) if and only if $LP(G)$ is completable (nonnegative-completable respectively).

Proof It suffices to show that completable case. If G is completable, then $LP(G)$ is also completable, since any complete subgraph must be contained in $LP(G)$. For the converse, denote

$$A_G(x) = \begin{bmatrix} A_{11} & A_{12} \\ A_{12}^* & xI + H \end{bmatrix},$$

where A_{11} is positive definite $k \times k$ (representing a positive completion of an $LP(G)$-partial positive matrix), and H is $(n-k)\times(n-k)$ Hermitian. Note that $A_G(x)$ is positive definite for sufficient large positive x. □

Proposition 5.7.3 (Grone, Johnson, Sá and Wolkowicz, [107]) Let G be a graph with $G = LP(G)$. Then G is completable if and only if G is nonnegative-completable.

Proof Assume first that G is completable. For a G-partial nonnegative matrix $(a_{ij})_G$, define, for each integer $n > 0$, $A_k = (a_{ij})_G + \frac{1}{n}I$. Then each A_n is a G-partial positive matrix, and so A_n has a positive completion M_n. Since $G = LP(G)$, the sequence A_n is bounded and so has a convergent subsequence with limit A, which will be a nonnegative completion of $(a_{ij})_G$.

Assume then G is nonnegative-completable, and let $(a_{ij})_G$ be a G-partial positive matrix. Choose $\epsilon > 0$ so that $(a_{ij})_G - \epsilon I$ is still a G-partial positive matrix. Then $(a_{ij})_G - \epsilon I$ has a nonnegative completion A, which yields a positive completion $A + \epsilon I$ of $(a_{ij})_G$. □

Definition 5.7.2 An *ordering* of a graph G is a bijection $\alpha : V(G) \mapsto \{1, 2, \cdots, n\}$, and for vertices $u, v \in V(G)$, we say that u *follows* v (with respect to the ordering α) if $\alpha(v) < \alpha(u)$. A graph G is a *band graph* if there exists an ordering α of G and an integer m with $2 \le m \le n$ such that

$$uv \in E(G) \text{ if and only if } |\alpha(u) - \alpha(v)| \le m - 1.$$

Example 5.7.1 Let G denote the graph with

$$V(G) = \{v_1, v_2, v_3, v_4\} \text{ and } E(G) = \{v_1v_2, v_2v_3, v_3v_4, v_1v_3, v_2v_4\}.$$

Let α be the map defined by $\alpha(v_i) = i$, and let $m = 3$. Then we can check that G is a band graph.

Note that every band graph is a chordal graph. But a chordal graph may not be a band graph. Let H denote the graph obtained from the 3-cycle $u_1u_2u_3u_1$ by adding two new vertices u_4 and u_5 such that u_4 is only adjacent to u_2 and u_5 is only adjacent to u_3. Then H has only one 3-cycle and so it is chordal. If H were a band graph, then there would exist a bijection α and an integer $2 \le m \le 5$ satisfying the definition of a band graph. Since H has a 3-cycle, $m > 2$. If $m \ge 4$, the vertex u_i with $\alpha(u_i) = 3$ will have degree 4, absurd. Now assume that $m = 3$. Then the vertices u_i with $\alpha(u_i) = 1$ or $\alpha(u_i) = 5$ will have degree 2, whereas H has only one vertex of degree 2, absurd again.

Theorem 5.7.1 (Dym and Gohberg, [78]) Every band graph is completable.

Theorem 5.7.2 (Grone, Johnson, Sá and Wolkowicz, [107]) A graph G is completable if and only if G is chordal.

With the remarks before Theorem 5.7.1, we can see that Theorem 5.7.2 generalizes Theorem 5.7.1. The proof of Theorem 5.7.2 requires several lemmas and a new notion. A cycle C in a graph G is *minimal* if C is an induced cycle in G. Note that any minimal cycle is chordless. For an edge $e = uv \in E(G)$, $G + e$ denotes the graph obtained from G by joining u and v by the edge e.

Lemma 5.7.1 Let G be a graph. The following are equivalent.

(i) G has no minimal cycle of length exactly 4.

(ii) For any pair of distinct nonadjacent vertices $u, v \in V(G)$, the graph $G + uv$ has a unique maximal complete subgraph H with $u, v \in V(H)$.

Proof (i) $\Longrightarrow$ (ii). Let $u, v \in V(G)$ be distinct nonadjacent vertices and let H_1 and H_2 denote two maximal complete subgraphs in $G + uv$ with $u, v \in V(H_1) \cap V(H_2)$, but H_i does not contain H_{3-i} as a subgraph. Then there exist $z_i \in V(H_i) - \{u, v\}$ such that $z_1z_2 \notin E(G + uv)$. But then, $G[\{z_1, z_2, u, v\}]$ is a minimal 4-cycle of G, a contradiction.

(ii) $\Longrightarrow$ (i). Suppose that $G[\{z_1, z_2, u, v\}]$ is a minimal 4-cycle of G. Then $uv \notin E(G)$ and $u \neq v$. Hence $G + uv$ has a unique maximal complete subgraph H containing uv. Note that $z_1, z_2 \in V(H)$, and so $z_1z_2 \in E(G)$, absurd. $\Box$

Lemma 5.7.2 Let G' be a subgraph of G induced by $V(G')$. If G is completable, then G' is also completable.

Proof Let $(a'_{ij})_{G'}$ be aG'-partial nonnegative matrix. Define a G-partial nonnegative matrix $(a_{ij})_G$ as follows.

$$a_{ij} = \begin{cases} a'_{ij} & \text{if } v_iv_j \in E(G') \\ 0 & \text{otherwise.} \end{cases}$$

Then $(a_{ij})_G$ has a nonnegative completion M. The principal submatrix M' corresponding to rows and columns indexed by $V(G')$ is then a nonnegative completion of $(a'_{ij})_{G'}$. $\Box$

Lemma 5.7.3 If a $k \times k$ positive semidefinite matrix $A = (a_{ij})$ satisfies $a_{ij} = 1$ whenever $|i - j| \leq 1$, then $A = J_k$.

Proof We may assume that $k \geq 4$ since the case for $k = 2, 3$ are easy to verify. If there exists an $a_{ij} =\neq 1$ for some $1 \leq i < j \leq k$, then $j - i \geq 2$. Let A' denote the principal submatrix of A corresponding to the rows and columns i, $i+1$ and j. Then A' satisfies the hypothesis of the lemma and so $a_{ij} = 1$, absurd. This completes the proof. $\square$

Proof of Theorem 5.7.2 Assume that G is completable but G is not chordal. Then G has an induced cycle C of length at least 4. Without loss of generality, assume that $C = v_1 v_2 \cdots v_k v_1$. Note that C is an induced subgraph of G. Define a C-partial matrix $(a'_{ij})_C$ as follows.

$$a'_{ij} = \begin{cases} 1 & \text{if } |i-j| \le 1, \text{ and } \{i,j\} \ne \{1,k\} \\ -1 & \text{if } (i,j) = (1,k) \text{ or } (i,j) = (k,1). \end{cases}$$

Then $(a'_{ij})_C$ is a C-partial nonnegative matrix (we need to check only the principal minors of order 2). By Lemma 5.7.3, $(a'_{ij})_C$ is not completable to a positive semidefinite matrix, and so by Lemma 5.7.2, G is not completable either, a contradiction.

Conversely, assume that G is chordal but not a complete graph. Let $G = G_0, G_1, \cdots, G_s$ be a sequence of chordal graphs satisfying the conditions of Theorem 5.6.3. Let $A = (a_{ij})_G$ be a G-partial positive matrix. We shall show that there exists a G_1-partial positive matrix A_1 which extends A, and so the sufficiency of Theorem 5.7.2 will follow by induction on s.

Let $uv \in E(G_1) - E(G)$. By Lemma 5.7.1, there exists a unique maximal complete subgraph H of G_1 with $u, v \in V(H)$. Without loss of generality, assume that $V(H) = \{v_1, v_2, \cdots, v_p\}$ with $u = v_1$ and $v = v_p$. For any complex z, let $A_1(z)$ denote the G_1-partial matrix extending A with the $(1,p)$-entry being z and the $(p,1)$-entry $\bar{z}$. Let $M(z)$ denote the principal submatrix of $A_1(z)$ corresponding to the vertices in $V(H)$. Then $A_1(z)$ is a G_1 partial positive matrix if and only if $M(z)$ is a positive matrix. Thus it suffices to show that there exists an z_0 such that $M(z_0)$ is positive definite.

However, with $\alpha : V(H) \mapsto \{1, 2, \cdots, p\}$ defined by $\alpha(v_i) = i$, the edges in $E(H)$ are exactly the edges

$$\{v_i v_j \ : \ |i-j| \le p-2, 1 \le i \le j \le p\}.$$

and so such a z_0 exists, by Theorem 5.7.1. $\square$

5.8 Estimation of the Eigenvalues of a Matrix

Throughout this section $\mathbf{C}$ denotes the field on complex numbers. All the matrices considered in this section will be over $\mathbf{C}$. Let $A = (a_{ij})$ be an $n \times n$ matrix, $D = \text{diag}(a_{11}, a_{22}, \cdots, a_{nn})$ and $B - A - D$. For a $z \in \mathbf{C}$, define $A_z = D + zB$. Then $A_0 = D$ and $A_1 = A$. Note that the eigenvalues of A_0 are $a_{11}, a_{22}, \cdots, a_{nn}$. As the zeros of a polynomial vary continuously with the coefficients of the polynomial, it is natural to guess intuitively that when $|z|$ is small, the eigenvalues of A_z will be close to $a_{11}, a_{22}, \cdots$, or a_{nn} on the complex plane. This is confirmed by Gersgorin.

Definition 5.8.1 Let $A = (a_{ij})$ be an $n \times n$ matrix, and let

$$R_i(A) = \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}|, \ 1 \leq i \leq n.$$

The matrix A is *diagonal dominant* if

$$|a_{ii}| \geq R_i(A), \ 1 \leq i \leq n. \tag{5.21}$$

If (5.21) is strict, then A is *strict diagonal dominant.*

Theorem 5.8.1 (Gersgorin, [98]) Let $A = (a_{ij})$ be an $n \times n$ matrix. Then all eigenvalues of A lie in the region $G(A)$, which is the union of the closed discs

$$G(A) = \bigcup_{i=1}^{n} \{z \in \mathbf{C} \ : \ |z - a_{ii}| \leq R_i(A)\} \tag{5.22}$$

Moreover, if the union of k of these n discs form a connected region on the complex plane and if this connected region is disjoint from the other $n - k$ discs, then this connected region has exactly k eigenvalues of A.

Proof Let λ be an eigenvalue of A, and let $\mathbf{x} = (x_1, x_2, \cdots, x_n)^T$ be an eigenvector belonging to λ. Pick a component x_p such that

$$|x_p| \geq |x_i|, 1 \leq i \leq n.$$

Since $\mathbf{x} \neq \mathbf{0}$, $|x_p| > 0$. Thus

$$\lambda x_p = \sum_{j=1}^{n} a_{pj} x_j,$$

This is equivalent to

$$x_p(\lambda - a_{pp}) = \sum_{\substack{j=1 \\ j \neq i}}^{n} a_{pj} x_j.$$

It follows that

$$\begin{aligned} |x_p(\lambda - a_{pp})| &= \left| \sum_{\substack{j=1 \\ j \neq p}}^{n} a_{pj} x_j \right| \leq \sum_{\substack{j=1 \\ j \neq p}}^{n} |a_{pj} x_j| \\ &\leq |x_p| \sum_{\substack{j=1 \\ j \neq p}}^{n} |a_{pj}| = |x_p| R_p. \end{aligned}$$

This proves the first assertion of Theorem 5.8.1.

Let $A = D+B$, where $D = \text{diag}(a_{11}, a_{22}, \cdots, a_{nn})$ and let $A_\epsilon = D+\epsilon B$, for some $\epsilon > 0$. Note that $R_i(A_\epsilon) = R_i(\epsilon B) = \epsilon R_i(B) = \epsilon R_i(A)$. Without loss of generality, assume that the first k discs

$$\bigcup_{i=1}^{k}\{z \in \mathbf{C} : |z - a_{ii}| \le R_i(A)\}$$

form a connected region $G_k(A)$ on the complex plane, which is disjoint from other $n-k$ discs. Note that for any ϵ with $0 \le \epsilon \le 1$,

$$G_k(A_\epsilon) = \bigcup_{i=1}^{k}\{z \in \mathbf{C} : |z - a_{ii}| \le \epsilon R_i(A)\}$$

lies in the region $G_k(A)$, and so $G_k(A_\epsilon)$ is also disjoint from the other discs $\{z \in \mathbf{C} : |z - a_{ii}| \le R_i(A)\}$, $k+1 \le i \le n$.

For each i, the continuous curve $\lambda_i(A_\epsilon)$ with $(0 \le \epsilon \le 1)$ lies in the disc $\{z \in \mathbf{C} : |z - a_{ii}| \le R_i(A)\}$. Hence $G_k(A)$ has at least k eigenvalues of A. By the same reason, $\bigcup_{i=k+1}^{n}\{z \in \mathbf{C} : |z - a_{ii}| \le \epsilon R_i(A)\}$ also contains at least $n-k$ eigenvalues of A.

Since the region $G_k(A)$ and $\bigcup_{i=k+1}^{n}\{z \in \mathbf{C} : |z - a_{ii}| \le \epsilon R_i(A)\}$ are disjoint, and since A has at most n eigenvalues, the second assertion of Theorem 5.8.1 now obtains. □

Each of the closed discs in (5.22) will be called a *Gersgorin disc.* Applying Theorem 5.8.1, we can obtain the following theorem.

Theorem 5.8.2 Let $A = (a_{ij})$ be an $n \times n$ matrix such that A is strict diagonal dominant. Then

(i) A is nonsingular.

(ii) If $a_{ii} > 0$, $1 \le i \le n$, then every eigenvalue of A is has a positive real part.

(iii) If A is Hermitian, and if $a_{ii} > 0$, where $1 \le i \le n$, then all eigenvalues of A are positive.

Example 5.8.1 A strict diagonal dominant matrix is nonsingular but a diagonal dominant matrix may be singular. The following is an example.

$$\begin{bmatrix} 4 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

Theorem 5.8.3 Let $A = (a_{ij})$ be an $n \times n$ matrix, and let λ be an eigenvalue of A lying on the boundary of $G(A)$, with an eigenvector $\mathbf{x} = (x_1, x_2, \cdots, x_n)^T$. Let x_p be the component of $\mathbf{x}$ such that $|x_p| = \max_{1 \le i \le n} |x_i|$. Each of the following holds.

(i) If for some k, $|x_k| = |x_p|$, then $|\lambda - a_{kk}| = R_k(A)$. (In other words, λ also lies in the kth Gersgorin disc.)

(ii) Suppose that for some $k = 1, 2, \cdots, n$, $|x_k| = |x_p|$. If for some $j \neq k$, $a_{kj} \neq 0$, then $|x_j| = |x_k|$.

Proof Since λ lies in the boundary of $G(A)$, for each $i = 1, 2, \cdots, n$, $|\lambda - a_{ii}| \geq R_i(A)$. It follows that when $|x_k| = |x_p|$,

$$\begin{aligned}
|x_k||\lambda - a_{kk}| &= \left| \sum_{\substack{j=1 \\ j \neq k}}^{n} a_{kj} x_j \right| \leq \sum_{\substack{j=1 \\ j \neq k}}^{n} |a_{kj} x_j| \\
&\leq |x_p| \sum_{\substack{j=1 \\ j \neq k}}^{n} |a_{kj}| = |x_k| \sum_{\substack{j=1 \\ j \neq k}}^{n} |a_{kj}| = |x_k| R_k,
\end{aligned}$$

and so equalities must hold everywhere. Thus both (i) holds and

$$\sum_{\substack{j=1 \\ j \neq k}}^{n} |a_{kj}|(|x_k| - |x_j|) = 0$$

holds, and so by the fact that each $|a_{kj}|(|x_k| - |x_j|) \geq 0$, (ii) must follow as well. $\square$

Corollary 5.8.3 Let $A = (a_{ij})$ be an $n \times n$ matrix such that $a_{ij} \neq 0$, $1 \leq i, j \leq n$, and let λ be an eigenvalue of A lying on the boundary of $G(A)$, with an eigenvector $\mathbf{x} = (x_1, x_2, \cdots, x_n)^T$.

(i) Every Gersgorin disc contains λ.

(ii) For $1 \leq i, j \leq n$, $|x_i| = |x_j|$.

Brauer considered taking two rows when determining the radii of the Gersgorin discs instead of just one row, and successfully extended Theorem 5.8.1.

Theorem 5.8.4 (Brauer, [14]) Let $A = (a_{ij})$ be an $n \times n$ matrix. Then all eigenvalues of A lie in the region

$$\bigcup_{\substack{i,j=1 \\ i \neq j}} \{z \in \mathbf{C} \,:\, |z - a_{ii}||z - a_{jj}| \leq R_i(A) R_j(A)\}. \tag{5.23}$$

Proof Let λ be an eigenvalue of A with an eigenvector $\mathbf{x} = (x_1, x_2, \cdots, x_n)^T$. Let $|x_p| = \max_{1 \leq i \leq n} |x_i| > 0$. Since each a_{ii} lies in the region (5.23), λ must be there also when $\mathbf{x}$ has only one nonzero component.

Now assume that $\mathbf{x}$ has two nonzero components. Let $|x_p| \geq |x_q| \geq |x_i|$ for all i with

$i \in \{1, 2, \cdots, n\} - \{p, q\}$. As before, we have

$$\begin{aligned}|x_p||\lambda - a_{pp}| &= \left|\sum_{\substack{j=1\\ j\neq p}}^{n} a_{pj}x_j\right| \le \sum_{\substack{j=1\\ j\neq p}}^{n} |a_{pj}||x_j| \\ &\le |x_q| \sum_{\substack{j=1\\ j\neq p}}^{n} |a_{pj}| = |x_q| \sum_{\substack{j=1\\ j\neq p}}^{n} |a_{pj}| = |x_q|R_p,\end{aligned}$$

which yields

$$|\lambda - a_{pp}| \le R_p \frac{|x_q|}{|x_p|}. \tag{5.24}$$

However, we also have

$$\begin{aligned}|x_q||\lambda - a_{qq}| &= \left|\sum_{\substack{j=1\\ j\neq q}}^{n} a_{qj}x_j\right| \le \sum_{\substack{j=1\\ j\neq q}}^{n} |a_{qj}||x_j| \\ &\le |x_p| \sum_{\substack{j=1\\ j\neq q}}^{n} |a_{qj}| = |x_p| \sum_{\substack{j=1\\ j\neq p}}^{n} |a_{pj}| = |x_p|R_q,\end{aligned}$$

which yields

$$|\lambda - a_{qq}| \le R_q \frac{|x_p|}{|x_q|}. \tag{5.25}$$

Therefore the theorem follows by combining (5.24) and (5.25). $\square$

Motivated by the success of Brauer, it is natural to consider more than two rows in a time. Attempts were made by Brauer and by Marcus and Minc. Unfortunately, this does not lead to a successful generalization of Theorem 5.8.1. (See comments in [23]). Brualdi ([23]) discovered new connection between the distributions of eigenvalues and closed walks and connectivity in digraphs, and thereby finding a new extension.

Definition 5.8.2 A digraph D is *cyclically connected* if for any vertex v, there exists a vertex w such that D has a directed (v, w)-walk and a directed (w, v)-walk. Thus in a cyclic connected digraph D, every vertex lies in a nontrivial directed closed walk (a trivial directed closed walk is just a loop). We define a matrix A to be *weakly irreducible* if $D(A)$ is cyclically connected.

For each vertex $v \in V(D)$, let $N^+(v)$ to be the set of vertices $u \in V(D) - \{v\}$ such that $(v, u) \in E(D)$. Note that if D is cyclic connected, then $N^+(v) \neq \emptyset$, $\forall v \in V(D)$.

Throughout this section, the collection of all nontrivial directed closed walks of $D(A)$ is denoted by $C(A)$. For a $W = v_{i_1}v_{i_2}\cdots v_{i_k}v_{i_{k+1}} \in C(A)$ (where $v_{i_{k+1}} = v_{i_1}$), write

$$\prod_{v_i \in W} |z - a_{ii}| = \prod_{j=1}^{k} |z - a_{i_j i_j}| \text{ and } \prod_{v_i \in W} R_i(A) = \prod_{j=1}^{k} R_{i_j}(A).$$

A *pre-order* on a set V is a relation $\preceq$ satisfying
(PO1) $x \preceq x$, $\forall x \in V$, and
(PO2) $x \preceq y$ and $y \preceq z$ imply that $x \preceq z$, $\forall x, y, z \in V$.

Theorem 5.8.5 (Brualdi, [23]) Let $A = (a_{ij})$ be an $n \times n$ matrix. If A is weakly irreducible, then every eigenvalue of A lies in the region

$$\bigcup_{W \in C(A)} \left\{ z \in \mathbf{C} \, : \, \prod_{v_i \in W} |z - a_{ii}| \le \prod_{v_i \in W} R_i(A) \right\}. \tag{5.26}$$

Proof Let λ be an eigenvalue of A. If for some i, $\lambda = a_{ii}$, then clearly λ lies in the region (5.26). Thus we assume that $\lambda \ne a_{ii}$, $\forall i$, and let $\mathbf{x} = (x_1, x_2, \cdots, x_n)^T$ be an eigenvector belonging to λ. Then define a pre-order on $V(D(A))$ a follows:

$$v_i \preceq v_j \longleftrightarrow |x_i| \le |x_j|.$$

Claim 1 λ lies in the region (5.26) if there exists a $W \in C(A)$ with these properties
(A) $W = v_{i_1}v_{i_2}\cdots v_{i_k}v_{i_{k+1}}$ with $v_{i_1} = v_{i_{k+1}}$ and $k \ge 2$.
(B) For each $j = 1, 2, \cdots, k$, $v_m \preceq v_{i_{i+1}}$ for all $v_m \in N^+(x_{i_j})$
(C) $|x_{i_j}| > 0$, $1 \le j \le k$.
If such a W exists, then by $A\mathbf{x} = \lambda\mathbf{x}$, for $j = 1, 2, \cdots, k$,

$$(\lambda - a_{i_j i_j})x_{i_j} = \sum_{\substack{m=1 \\ m \ne i_j}}^{n} a_{i j_m} x_m = \sum_{v_m \in N^+(v_{i_j})} a_{i j_m} x_m.$$

Therefore,

$$\begin{aligned} |\lambda - a_{i_j i_j}||x_{i_j}| &= \left| \sum_{v_n \in N^+(v_{i_j})} a_{i j_m} x_m \right| \le \sum_{v_n \in N^+(v_{i_j})} |a_{i j_m}||x_m| \\ &\le \sum_{v_n \in N^+(v_{i_j})} |a_{i j_m}||x_{i_{j+1}}| = R_{i_j}|x_{i_{j+1}}|. \end{aligned} \tag{5.27}$$

It follows

$$\prod_{j=1}^{k} |\lambda - a_{i_j i_j}||x_{i_j}| \le \prod_{j=1}^{k} R_{i_j}|x_{i_{j+1}}|. \tag{5.28}$$

Note that

$$\prod_{j=1}^{k} |\lambda - a_{i_j i_j}| = \prod_{v_i \in V(W)} |\lambda - a_{ii}|, \text{ and } \prod_{j=1}^{k} R_{i_j} = \prod_{v_i \in W} R_i.$$

As $v_{i_{k+1}} = v_{i_1}$ and so $x_{i_{k+1}} = x_{i_1}$. Hence

$$\prod_{j=1}^{k} |x_{i_j}| = \prod_{j=1}^{n} |x_{i_{j+1}}|. \tag{5.29}$$

Combine (5.28) and (5.29) to get

$$\prod_{v_i \in W} |\lambda - a_{ii}| \le \prod_{v_i \in W} R_i, \tag{5.30}$$

and so λ lies in the region (5.26).

Claim 2 A closed walk satisfying (A), (B) and (C) in Claim 1 exists.

Since $\mathbf{x} \neq \mathbf{0}$, there exists $x_i \neq 0$. Note that

$$(\lambda - a_{ii})x_i = \sum_{\substack{j=1 \\ j \neq i}}^{n} a_{ij}x_j = \sum_{v_j \in N^+(v_i)} a_{ij}x_j.$$

This, together with $|x_i| > 0$ and $\lambda - a_{ii} \neq 0$, implies that among the vertices in $N^+(v_i)$, there must be at least one $v_j \in N^+(v_i)$ such that $x_j \neq 0$.

Let $v_{i_1} = v_i$ and let $v_{i_2} \in N(v_{i_1})$ such that for any $v \in N^+(v_{i_1})$, $v \preceq v_{i_2}$. Note that $x_{i_2} \neq 0$.

Inductively, assume that a walk $v_{i_1}v_{i_2}\cdots v_{i_{j-1}}v_{i_j}$, satisfying (B) and (C) in Claim 1, has been constructed. Since $x_{i_j} \neq 0$, we can repeat the above to find $v_{i_{j+1}} \in N^+(v_j)$ such that for any $v \in N^+(v_j)$ $v \preceq v_{i_{j+1}}$.

Since $D(A)$ has only finitely many vertices, a closed walk satisfying (A), (B) and (C) of Claim 1 must exists. This proves Claim 2, as well as the theorem. □

Theorem 5.8.6 (Brualdi, [23]) Let $A = (a_{ij})$ be an $n \times n$ irreducible matrix and let λ be a complex number.

If λ lies in the boundary of the region (5.26), then for any $W \in C(A)$, λ also lies in the boundary of each region

$$\left\{ z \in \mathbf{C} \; : \; \prod_{v_i \in W} |z - a_{ii}| \le \prod_{v_i \in W} R_i(A) \right\}. \tag{5.31}$$

Proof Note that an irreducible matrix is also weakly irreducible. All the argument in the proof of the previous theorem remains valid here. We shall use the same notation as in the proof of the previous theorem. Note that Claim 2 in the proof of Theorem 5.8.5 remains valid here.

Since $R_i > 0$, for each i. If $\lambda = a_{ii}$ for some i, then λ cannot be in the boundary of (5.26). Hence $\lambda \neq a_{ii}$, $1 \leq i \leq n$.

Fix a $W \in C(A)$ that satisfies (A),(B) and (C) of Claim 1 in the proof of Theorem 5.8.5. Since λ lies in the boundary the region (5.26), for each $v_i \in V(W)$, we have $|\lambda - a_{ii}| \geq R_i$, and so

$$\prod_{v_i \in W} |z - a_{ii}| \geq \prod_{v_i \in W} R_i(A). \tag{5.32}$$

By (5.30), we must have equality in (5.32), and so λ lies in the boundary of (5.31) for this W.

Note that when equality holds in (5.32), we must have, for any $j = 1, 2, \cdots, k$, that equalities hold everywhere in (5.27). Therefore, for any closed walk in $C(A)$ satisfying (A), (B) and (C) of Claim 1 in the proof of Theorem 5.8.5, for any $v_{i_j} \in V(W)$ and for any $v_m \in N^+(v_{i_j})$,

$$|x_m| = |x_{i_{j+1}}| = c_{i_{j+1}} \text{ is a constant.} \tag{5.33}$$

Define

$$K = \{v_i \in V(D(A)) \ : \ |x_m| = c_i = \text{ constant, for any } v_m \in N^+(v_i)\}.$$

By Claim 2 in the proof for Theorem 5.8.5, $K \neq \emptyset$. If we can show that $K = V(D(A))$, then for any closed walk $W \in C(A)$, W will satisfy (A), (B) and (C) of Claim 1 in the proof of Theorem 5.8.5, and so λ will be on the boundary of the region (5.31) for this W.

Suppose, by contradiction, then some $v_q \in V(D(A)) - K$. Since $D(A)$ is strongly connected, $D(A)$ has a shortest directed walk from a vertex in K to v_q. Since it is shortest, the first arc of this walk is from a vertex in K to a vertex v_f not in K. Adopting the same pre-order of $D(A)$ as in the proof for Claim 2 of Theorem 5.8.5, we can similarly construct a walk by letting $v_{i_1} = v_f$, v_{i_2} is chosen from $N^+(v_{i_1})$ so that for any $v \in N^+(v_{i_1})$, $v \preceq v_{i_2}$. Since $D(A)$ is strong, $N^+(v_i) \neq \emptyset$, for every $v_i \in V(D(A))$. Once again, such a walk satisfies (B) and (C) in Claim 1 in the proof of Theorem 5.8.5.

In each step to find the next v_{i_j}, we choose $v_{i_j} \notin K$ whenever possible, and if we have to choose $v_{i_j} \in K$, then choose $v_{i_j} \in K$ so that v_{i_j} is in a shortest directed walk from a vertex in K to a vertex not in K. Since $|V(D(A)) - K|$ is finite, a vertex v not in K will appear more than once in this walk, and so a closed walk $W' \in C(A)$ is found, satisfying (A), (B) and (C) of Claim 1 in the proof of Theorem 5.8.5, and containing v. But then, by (5.33), every vertex in W' must be K, contrary to the assumption that $v \not{i}nK$. Hence $V(D(A)) = K$. This completes the proof. □

Corollary 5.8.6 Let $A = (a_{ij})$ be an $n \times n$ matrix. Then A is nonsingular if one of the following holds.

(i) A is weakly irreducible and

$$\prod_{v_i \in W} |a_{ii}| > \prod_{v_i \in W} R_i, \text{ for any } W \in C(A).$$

(ii) A is irreducible and

$$\prod_{v_i \in W} |a_{ii}| \geq \prod_{v_i \in W} R_i, \text{ for any } W \in C(A),$$

and strict inequality holds for at least one $W \in C(A)$.

Proof In either case, by Theorems 5.8.5 or 5.8.6, the region (5.26) does not contain 0; and when A is irreducible, the boundary of the region (5.26) does not contain 0, either. $\square$

5.9 M-matrices

In this section, we will once again restrict our discussion to matrices over the real numbers, and in particular, to nonnegative matrices. Throughout this section, for an integer $p > 0$, denote $\langle p \rangle = \{1, 2, \cdots, p\}$. For the convenience of discussion, a matrix $A \in M_n$ is often written in the form

$$A = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1q} \\ A_{21} & A_{22} & \cdots & A_{2q} \\ & \cdots & \cdots & \\ A_{p1} & A_{p2} & \cdots & A_{pq} \end{bmatrix}, \tag{5.34}$$

where each *block* $A_{ij} \in M_{m_i, n_j}$ and where $m_1 + m_2 + \cdots + m_p = n = n_1 + n_2 + \cdots + n_q$. In this case, we write $A = (A_{ij}), i = 1, 2, \cdots, p$ and $j = 1, 2, \cdots, q$. A vector $\mathbf{x} = (\mathbf{x}_1^T, \mathbf{x}_2^T, \cdots, \mathbf{x}_p^T)^T$ is said to *agree with the blocks of the matrix* (5.34) if $\mathbf{x}_i$ is an m_i-dimensional vector, $1 \leq i \leq p$. When $\mathbf{x} = (\mathbf{x}_1^T, \mathbf{x}_2^T, \cdots, \mathbf{x}_p^T)^T$, $\mathbf{x}_i^T$ is also called the ith component of $\mathbf{x}$, for convenience.

Definition 5.9.1 Recall that if $A \geq 0$ and $A \in M_n$, then A is permutation similar to its

Frobenius Standard from (Theorem 2.2.1)

$$B = \begin{pmatrix} A_{11} & 0 & 0 & \cdots & 0 & \cdots & 0 \\ 0 & A_{22} & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \\ 0 & 0 & A_{gg} & 0 & 0 & \cdots & 0 \\ A_{g+1,1} & \cdots & A_{g+1,g} & A_{g+1,g+1} & 0 & \cdots & 0 \\ A_{g+2,1} & \cdots & A_{g+2,g} & A_{g+2,g+1} & A_{g+2,g+2} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\ A_{k,1} & \cdots & A_{k,g} & A_{k,g+1} & \cdots & \cdots & A_{kk} \end{pmatrix}, \tag{5.35}$$

By Theorem 2.1.1, each irreducible diagonal block A_{ii}, $1 \le i \le k$, corresponds to a strong component of $D(A)$. Throughout this section, let $\rho_i = \rho(A_{ii})$, the spectral radius of A_{ii}, for each i with $1 \le i \le k$.

Label the strong components of $D(A)$ (diagonal blocks in (5.35)) with elements in $\langle k \rangle$, and define a partial order $\preceq$ on $\langle k \rangle$ as follows: for $i, j \in \langle k \rangle$, $i \preceq j$ if and only if in $D(A)$, there is a directed path from a vertex in the jth strong component to a vertex in the ith strong component; and $i \prec j$ means $i \preceq j$ but $i \ne j$.

The partial order $\preceq$ yields a digraph, called the *reduced graph* $R(A)$ of A, which has vertex set $\langle k \rangle$, where $(i, j) \in E(R(A))$ if and only if $i \prec j$. Note that by the definition of strong components, $R(A)$ is has no directed cycles.

If a matrix A has the form (5.35), then denote $\rho_i = \rho(A_{ii})$, $1 \le i \le k$. Let $M = \lambda I - A$ be an M-matrix. Then the ith vertex in $R(M)$ (that is, the vertex corresponding to A_{ii} in $R(A)$) is a *singular vertex* if $\lambda = \rho_i$. The singular vertices of reduced graph $R(M)$ is also called the singular vertices of the matrix M.

For matrix of the form (5.34), define

$$\gamma_{ij} = \begin{cases} 1 & \text{if } i = j \text{ or } A_{ij} \ne 0 \\ 0 & \text{otherwise.} \end{cases}$$

Also, let

$$R_{ij} = \max \gamma_{ih}\gamma_{hl} \cdots \gamma_{qj},$$

where the maximum is taken over all possible sequences $\{i, h, \cdots, q, j\}$.

Proposition 5.9.1 With the notation above, each of the following holds.

(i) If A is a Frobenius Standard form (5.35), then with the partial order $\preceq$, we can equivalently write,

$$R_{ij} = \begin{cases} 1 & \text{if } j \preceq i \\ 0 & \text{otherwise.} \end{cases}$$

(ii)

$$\sum_{h=1}^{p} R_{ih}R_{hj} \geq R_{ij} \geq R_{il}R_{lj}, \quad 1 \leq l \leq p$$

(iii)

$$\sum_{h=1,h\neq i}^{p} \gamma_{ih}R_{hj} \geq R_{ij} \geq \max_{h\neq i} \gamma_{ih}R_{hj}, \quad \text{if } i \neq j. \tag{5.36}$$

Example 5.9.1 Let

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ -a & -1 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ -b & -c & -1 & -1 & 1 \end{bmatrix}$$

where a, b, c are nonnegative real numbers. Then $R(A)$ is the graph in Figure 5.9.1.

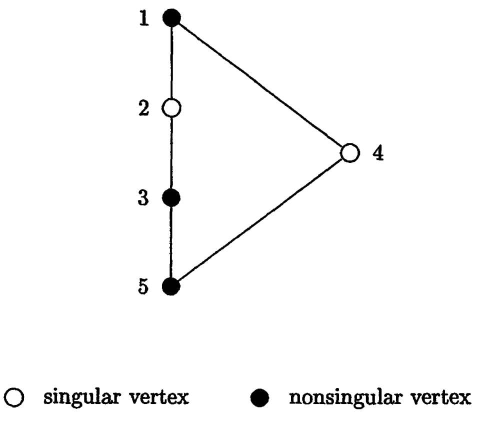

Figure 5.9.1

Definition 5.9.2 A matrix $B = (b_{ij}) \in M_n$ is an *M-matrix* if each of the following holds.
(5.9.2A) $a_{ii} \geq 0$, $1 \leq i \leq n$.

(5.9.2B) $b_{ij} \le 0$, for $i \ne j$ and $1 \le i, j \le n$.

(5.9.2C) If $\lambda \ne 0$ is an eigenvalue of B, then λ has a positive real part.

Proposition 5.9.2 summerizes some observations made in [229] and [216].

Proposition 5.9.2 (Schneider, [229] and Richman and Schneider, [216]) Each of the following holds.

(i) A is an M-matrix if and only if there exists a nonnegative matrix $P \ge 0$ and a number $\rho \ge \rho(P)$ such that $A = \rho I - P$, and A is a singular M-matrix if and only if $A = \rho(P)I - P$.

(ii) If $A = (A_{ij})$ is an M-matrix in the Frobenius standard form (5.35), then the diagonal blocks A_{ii}, $1 \le i \le k$, are irreducible M-matrices.

(iii) The blocks below the main diagonal, A_{ij}, $1 \le j < i \le k$, are non-negative. In other words, $-A_{ij} \ge 0$.

Lemma 5.9.1 Let $A = (A_{ij})$, $i, j = 1, 2, \cdots k$ be an M-matrix in a Frobenius Standard form (5.35), and let $\mathbf{x} = (\mathbf{x}_1^T, \mathbf{x}_2^T, \cdots, \mathbf{x}_k^T)^T$ agreeing with the blocks of A. For an $a \in \langle k \rangle$ and for an $h > a$, let

$$\begin{cases} \mathbf{x}_i = \mathbf{0} & \text{if } R_{ia} = 0 \\ \mathbf{x}_i >> 0 & \text{if } R_{ia} = 1. \end{cases} \quad i = 1, 2, \cdots, h-1. \tag{5.37}$$

If

$$\begin{cases} \mathbf{y}_1 = \mathbf{0} & \text{and} \\ \mathbf{y}_i = -\sum_{j=1}^{i-1} A_{ij}\mathbf{x}_j, & i = 1, 2, \cdots, k, \end{cases} \tag{5.38}$$

then

$$\begin{cases} \mathbf{y}_h = \mathbf{0} & \text{if } R_{ha} = 0 \\ \mathbf{y}_h > 0 & \text{if } R_{ha} = 1. \end{cases} \tag{5.39}$$

Proof Since $A_{hj}\mathbf{x}_j \ge 0$, we have $\mathbf{y}_h \ge 0$, and $\mathbf{y}_h = \mathbf{0}$ if and only if $A_{hj}\mathbf{x}_j = 0$, $j = 1, 2, \cdots, h-1$. By (5.37), $\mathbf{y}_h = \mathbf{0}$ if and only if

$$\gamma_{hj}R_{ja} = 0, \ \ j = 1, 2, \cdots, h-1. \tag{5.40}$$

Since $\gamma_{hj} = 0$ whenever $h < j$, we also have

$$\sum_{j=1}^{h-1} \gamma_{hj}R_{ja} = \sum_{j=1, j\ne i}^{k} \gamma_{hj}R_{ja}, \text{ and } \max_{j<h} \gamma_{hj}R_{ja} = \max_{j \ne h} \gamma_{hj}R_{ja}.$$

As $h \ne a$, it follows by (5.36) that (5.40) holds if and only if $R_{ha} = 0$. □

Theorem 5.9.1 (Schneider, [229]) Let $A = (A_{ij})$ be an M-matrix in Frobenius standard form (5.35), and let a be a singular vertex of $R(A)$. Then there exists a vector $\mathbf{x} =$

$(\mathbf{x}_1^T, \mathbf{x}_2^T, \cdots, \mathbf{x}_k^T)^T$ such that $A\mathbf{x} = \mathbf{0}$ and

$$\begin{cases} \mathbf{x}_i >> 0 & \text{if } R_{ia} = 1 \text{ (that is, } a \preceq i) \\ \mathbf{x}_i = 0 & \text{otherwise.} \end{cases}$$

Proof Let $\mathbf{x} = (\mathbf{x}_1^T, \mathbf{x}_2^T, \cdots, \mathbf{x}_k^T)^T$ be given and let $\mathbf{y} = (\mathbf{y}_1^T, \mathbf{y}_2^T, \cdots, \mathbf{y}_k^T)^T$ be defined by (5.38). Then $A\mathbf{x} = \mathbf{0}$ if and only if

$$A_{ii}\mathbf{x}_i = \mathbf{y}_i, \quad i = 1, 2, \cdots, k. \tag{5.41}$$

Now $\mathbf{x}_i = \mathbf{0}$, for all $i < a$. Note that since A is an M-matrix, the block matrix A_{aa} has an eigenvector $\mathbf{x}_a >> 0$ belonging to the eigenvector 0. Since A is in Frobenius standard form, we have $R_{ia} = 0$, when $i < a$, and $R_{aa} = 1$. Thus $\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_a$ satisfy (5.37). By induction, assume that $\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_{h-1}$ satisfy (5.37), for some $h > a$. If $\mathbf{y}_1, \mathbf{y}_2, \cdots, \mathbf{y}_{h-1}$ satisfy (5.38), then $\mathbf{y}_h$ also satisfies (5.39), by Lemma 5.9.1. Thus if $R_{ha} = 0$, then $\mathbf{y}_h = \mathbf{0}$; and if $\mathbf{x}_h = \mathbf{0}$, then $\mathbf{x}_h$ satisfies (5.41) with $i = h$.

If $R_{ha} = 1$, then $\mathbf{y}_h > 0$, and so as a is a singular vertex, we conclude that A_{hh} is nonsingular. Note that as the inverse of a nonsingular M-matrix, $A_{hh}^{-1} >> 0$. Therefore, $\mathbf{x}_h = A_{hh}^{-1}\mathbf{y}_h >> 0$. Thus $\mathbf{x}_h$ satisfies (5.41) with $i = h$, and so the theorem follows by induction. □

Corollary 5.9.1A A singular M matrix must have an eigenvector $\mathbf{x} > 0$, belonging to the eigenvalue 0.

Corollary 5.9.1B Let A be an M-matrix, and let $\gamma_1, \gamma_2, \cdots, \gamma_s$ are singular vertices of $R(A)$. If $R(A)$ has no singular vertex that can reach to another singular vertex, then A has s linearly independent eigenvectors $\mathbf{x}^1, \mathbf{x}^2, \cdots, \mathbf{x}^s$ which belong to the eigenvalue 0 and satisfy (5.37) (with $a = \gamma_j$).

Example 5.9.2 Let A be the M-matrix

$$A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Then $R(A)$ is edgeless and has two vertices, one singular and one nonsingular. The vector $\mathbf{x} = (1, 0)^T$ is the unique unit eigenvector with $\mathbf{x} > 0$.

Definition 5.9.3 Let A be an M-matrix and let the Jordan blocks of A corresponding to the eigenvalue 0 be $J_{n_1}, J_{n_2}, \cdots, J_{n_s}$, where $n_1 \geq n_2 \geq \cdots \geq n_s > 0$. Then the *Segré characteristic* of A is the sequence $(n_1, n_2, \cdots, n_s)$. The *Jordan graph* of A (with respect to the eigenvalue 0) is an array consisting with s columns of $*$'s, where the jth column (counted from the left to right) has n_j $*$'s. The *Wyre characteristic* of A is a sequence $(w_1, w_2, \cdots, w_u)$, where w_i is the number of $*$'s in the ith row (counted from the bottom and up) of $J(A)$. Note that $w_1 \geq w_2 \geq \cdots \geq w_s > 0$.

The first element in the Segré characteristic of A is the *index* of A, denoted $\operatorname{ind}(A)$. Thus $\operatorname{ind}(A) = n_1$. We have these observations.

Proposition 5.9.3 Let A be a matrix with Segré characteristic $(n_1, n_2, \cdots, n_s)$ and Wyre characteristic $(w_1, w_2, \cdots, w_u)$. Then each of the following holds.

(i) $\operatorname{ind}(A) = u = n_1$.

(ii) The number 0 is an eigenvalue of A if and only if $\operatorname{ind}(A) > 0$.

(iii) If $\operatorname{ind}(A) > 0$, then $w_1 \ge w_2 \ge \cdots \ge w_u > 0$.

(iv) For an integer k with $1 \le k \le u$, $w_1 + w_2 + \cdots + w_k$ is the dimension of $\operatorname{Ker}(A^k)$, the null space of A^k.

Example 5.9.3 If A has four Jordan blocks corresponding to eigenvalue 0 which are of order 3, 2, 2, and 1, respectively. Then $\operatorname{ind}(A) = 3$, the Segré characteristic of A is $(3, 2, 2, 1)$, and the Wyre characteristic of A is $(4, 3, 1)$. The Jordan graph $J(A)$ is

$$\begin{array}{cccc} * & & & \\ * & * & * & \\ * & * & * & * \end{array}$$

Definition 5.9.4 Let A be an M-matrix with $u = \operatorname{ind}(A)$. The null space $\operatorname{Ker}(A^u)$ is called the *generalized eigenspace* of A, and is denoted by $E(A)$. For a vector $\mathbf{x}$, if none of $\mathbf{x}$, $(-A)\mathbf{x}, \cdots, (-A)^{k-1}\mathbf{x}$ is $\mathbf{0}$, but $(-A)^k\mathbf{x} = \mathbf{0}$, then the sequence $\mathbf{x}, (-A)\mathbf{x}, \cdots, (-A)^{k-1}\mathbf{x}$ is a *Jordan chain* of length k. A *Jordan basis* of $E(A)$ is a basis of $E(A)$, which is a union of Jordan chains. A Jordan chain (or a Jordan basis) is *nonnegative* if each vector in the chain (basis) is nonnegative.

Let $R(A)$ denote the reduced graph of an M-matrix A in standard form (5.35). A vertex i in $R(A)$ is a *distinguished vertex* if in $R(A)$, $\rho_i > \rho_j$ whenever $i \prec j$. Thus a singular vertex i is distinguished if and only if j is a nonsingular vertex whenever $i \prec j$.

Let the singular vertices of $R(A)$ be $w_1, w_2, \cdots, w_q$. The *singular graph* of A is a graph $S(A)$ whose vertices are $w_1, w_2, \cdots, w_q$, where $w_i \preceq w_j$ if and only if $w_i \preceq w_j$ in $R(A)$. If for each w_i in $S(A)$, the vertices $\{w : w \preceq w_i\}$ with the order $\preceq$ is a linear order, then $S(A)$ is a *rooted forest.*

Let Λ_1 denote the set of maximal elements (with respect to the order $\preceq$) in $S(A)$. Note that elements in Λ_1 are distinguished vertices of $S(A)$. For $j = 2, 3, \cdots$, let

$$\Lambda_j = \{ \text{ the maximal elements in } S(A) \setminus \bigcup_{i=1}^{j} \Lambda_i \}.$$

Put the vertices in Λ_1 on the lowest level, and for $j \ge 2$, put vertices in Λ_j one level higher than those in Λ_{j-1}. Then remove the directions by $\preceq$. The resulting graph, denoted $S_*(A)$, is the *level diagram* of A. Label the levels of $S_*(A)$ from bottom up. The highest level label is the *high* of $S_*(A)$. Let $\gamma_j = |\Lambda_j|$, $j = 1, 2, \cdots, h$, where h is the high of $S_*(A)$. Then $(\gamma_1, \gamma_2, \cdots, \gamma_h)$ is the *level characteristic* of A.

For a subset $Q \subseteq \Lambda_j$, where $2 \le j \le h$, $\Delta(Q) \subseteq \Lambda_{j-1}$ is the subset consists of all the vertices j such that for some $i \in Q$, $i \preceq j$.

Example 5.9.4 Let $S_*(A)$ be a diagram given as follows:

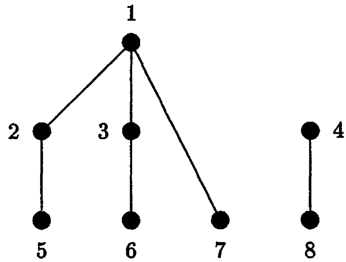

Figure 5.9.2

Then $\Lambda_1 = \{5, 6, 7, 8\}$, $\Lambda_2 = \{2, 3, 4\}$, $\Lambda_1 = \{1\}$; $\Delta(1) = \{2, 3\}$ but $7 \notin \Delta(1)$. $\Delta(\{2, 3\}) = \{5, 6\}$. The height of $S_*(A)$ is $h = 3$, and the level characteristic of A is $(4, 3, 1)$.

Lemma 5.9.2 Let $A = (A_{ij}), 1 \le i, j \le k$ be a singular M-matrix in standard form (5.35), and let $\gamma_1, \gamma_2, \cdots, \gamma_s$ be the singular vertices of A such that $\gamma_1 < \gamma_2 < \cdots < \gamma_s$. If A has m linearly independent eigenvectors belonging to the eigenvalue 0, then for each integer $n \le m$, there exists an eigenvector $\mathbf{x}$, belonging to the eigenvalue 0, such that for some $i \le \gamma_{n+s-m}$, $(\mathbf{x}_j)_i$, the ith component of $\mathbf{x}_j$, is nonzero.

Proof The conclusion is obvious if $\gamma_{n+s-m} = k$. Assume that $\gamma_{n+s-m} < k$, and let $\mathbf{x}^1, \cdots, \mathbf{x}^m$ be linearly independent eigenvectors of A belonging to 0, such that for $i = 1, 2, \cdots, \gamma_{n+s-m}$ and for $j = n, n+1, \cdots, m$, $\mathbf{x}_i^j = 0$, where $\mathbf{x}_i^j$ is the ith component of $\mathbf{x}^j$.

Let $\mu = \gamma_{n+s-m} + 1$. The vectors $((\mathbf{x}_\mu^j)^T, \cdots, (\mathbf{x}_k^j)^T)^T$, $j = n, n+1, \cdots, m$ are $m - n + 1$ linearly independent eigenvectors of the matrix $B = (A_{ij})$, $i, j = \mu, \mu+1, \cdots, m$, belonging to the eigenvalue 0. However, the multiplicity of 0 in B is the same as the number of singular vertices in $R(B)$, which is $m - n$, a contradiction. $\square$

Lemma 5.9.3 Let $A = (A_{ij}), 1 \le i, j \le k$ be a singular M-matrix in standard form (5.35). Let $\gamma_1 < \gamma_2 < \cdots < \gamma_s$ be the singular vertices of A. If A has s linearly independent eigenvectors belonging to the eigenvalue 0, then there exist s eigenvectors $\mathbf{x}^j = (x_1^j, x_2^j, \cdots)^T$, $1 \le j \le s$, belonging to the eigenvalue 0, such that

$$\begin{cases} x_i^j = 0 & \text{if } i < \gamma_j \\ x_i^j \neq 0 & \text{if } i = \gamma_j, \end{cases} \quad j = 1, 2, \cdots, s. \tag{5.42}$$

Proof Let $\mathbf{z}^j = (z_1^j, z_2^j, \cdots)^T$, $1 \le j \le s$ be linearly independent eigenvectors belonging to the eigenvalue 0.

Note that $z_i^j = 0$ for each $i < \gamma_1$ and each $1 \le j \le s$. By Lemma 5.9.2 with $m = s$ and $n = 1$, we may assume that for some j, we have $z_i^j \neq 0$, if $i = \gamma_1$. This $\mathbf{z}^j$ can be chosen as $\mathbf{x}^1$.

Inductively, assume that eigenvectors $\mathbf{x}^1, \mathbf{x}^2, \cdots, \mathbf{x}^n, \mathbf{z}^{n+1}, \mathbf{z}^{n+2}, \cdots, \mathbf{z}^s$ have been found, all belonging to the eigenvalue 0, such that

(A) (5.42) holds for $j = 1, 2, \cdots, n$, and

(B) For $j = n+1, n+2, \cdots, s$, $z_{ij} = 0$ whenever $i < \gamma_n$.

Set $a = \gamma_n$ to get

$$A_{aa}\mathbf{x}_a^n = A_{aa}\mathbf{z}_a^j = 0, \quad j = n+1, n+2, \cdots, s.$$

As the null space of an irreducible singular M-matrix is one dimensional, it follows that

$$z_a^j = \lambda_j x_a^n, \quad j = n+1, \cdots, s.$$

Let $\mathbf{y}^j = \mathbf{z}^j - \lambda_j \mathbf{x}^n$, $j = n+1, \cdots, s$. Then $\mathbf{x}^1, \cdots, \mathbf{x}^n, \mathbf{y}^{n+1}, \cdots, \mathbf{y}^s$ are linearly independent eigenvectors belonging to the eigenvalue 0. Moreover, if $i \le \gamma_n$, then $y_i^j = 0$ for each $j = n+1, n+2, \cdots, s$. By Lemma 5.9.2, there exists a $j \ge n+1$ such that when $i = \gamma_{n+1}$, $y_i^j \neq 0$. Choose this $\mathbf{y}^j$ to be $\mathbf{x}^{n+1}$. Therefore,

(A) (5.42) holds for $j = 1, 2, \cdots, n, n+1$, and

(B) For $j = n+2, \cdots, s$, $y_i^j = 0$ whenever $i < \gamma_n$.

Thus the lemma follows by induction. $\square$

Lemma 5.9.4 below can be easily proved.

Lemma 5.9.4 Let A be an irreducible singular M-matrix and let $\mathbf{x}$ be a vector. If either $A\mathbf{x} \ge 0$, or if $A\mathbf{x} \le 0$, then $A\mathbf{x} = 0$.

Theorem 5.9.2 (Schneider, [229]) Let $A = (A_{ij})$, $1 \le i, j \le k$ be a singular M-matrix in standard form (5.35). The following are equivalent.

(i) The Segré characteristic of A is $(1, 1, \cdots, 1)$.

(ii) In $R(A)$, no singular vertex of A can be reached from another singular vertex.

Proof Let $S = \{\gamma_1, \gamma_2, \cdots, \gamma_s\}$ be the set of all singular vertices of $R(A)$ such that $\gamma_1 < \gamma_2 < \cdots < \gamma_s$.

Note that the eigenvalue 0 has s linearly independent eigenvectors if and only if the Segré characteristic of A is $(1, 1, \cdots, 1)$. Therefore, by Corollary 5.9.1B, that (ii) implies that the eigenvalue 0 has s linearly independent eigenvectors, and so (i) must hold.

Conversely, assume that (i) holds, and so the eigenvalue 0 has s linearly independent eigenvectors $\mathbf{x}^1, \mathbf{x}^2, \cdots, \mathbf{x}^s$. By Lemma 5.9.3, we may assume that these s eigenvectors satisfy (5.42).

By contradiction, we assume that for some distinct $\alpha, \beta \in S$, $\alpha \prec \beta$. Thus $R_{\beta\alpha} = 1$. Choose such a pair of $\alpha, \beta \in S$ such that

$$R_{\beta\alpha} = 1, \beta > \alpha, \text{ and } \beta - \alpha \le \beta' - \alpha', \forall \alpha'\beta' \in S \text{ with } \alpha' \ne \beta' \text{ and } R_{\beta'\alpha'} = 1. \tag{5.43}$$

By (5.43), if $\alpha \le \sigma \le \delta \le \beta$, $\sigma, \delta \in S$, then $R_{\delta\sigma} = 0$ except both $\sigma = \alpha$ and $\delta = \beta$. Let $B = (A_{ij})$, $i, j = \alpha, \cdots, \beta - 1$. Let $\delta_1 < \delta_2 < \cdots < \delta_\gamma$ be singular vertices of $R(B)$. Then $\delta_1 = \alpha = \gamma_j$, for some j. By Corollary 5.9.1B, B has γ linearly independent eigenvectors $\mathbf{z}^h$, $h = 1, 2, \cdots, \gamma$, where each $\mathbf{z}^h = ((\mathbf{z}^h_\alpha)^T, \cdots, (\mathbf{z}^h_{\beta-1})^T)^T$ satisfies (5.37), $h = 1, 2, \cdots, \gamma$. Note that the multiplicity of the eigenvalue 0 of B is γ, every eigenvector of B belonging to 0 is a linear combination of $\mathbf{z}^h$, $h = 1, 2, \cdots, \gamma$.

As $\alpha = \gamma_j$, $((\mathbf{x}^j_\alpha)^T, \cdots, (\mathbf{x}^j_{\beta-1})^T)^T$ is an eigenvector of B belonging to the eigenvalue 0. Therefore, $(\mathbf{x}^j_i) = \sum_{h=1}^{\gamma} \lambda_h \mathbf{z}^h_i$, $i = 1, 2, \cdots, \beta - 1$. Since $\mathbf{x}^j_\alpha \ne \mathbf{0}$, and since $\mathbf{z}^j_\alpha = \mathbf{0}$ when $h = 2, 3, \cdots, \gamma$, we have $\lambda_1 \ne 0$.It follows that

$$A_{\beta\beta}\mathbf{x}^j_\beta = \sum_{h=1}^{\gamma} \lambda_h \mathbf{y}^h_\beta, \text{ where } \mathbf{y}^h_\beta = -\sum_{h=1}^{\gamma} A_{\beta i}\mathbf{z}^h_i, \quad h = 1, 2, \cdots, \gamma.$$

When $i = 1, 2, \cdots, \alpha - 1$, set $\mathbf{z}^h_i = \mathbf{0}$, and apply Lemma 5.9.1 to get

$$\begin{aligned} \mathbf{y}^h_\beta > 0 &\quad \text{if } R_{\beta\gamma} = 1 \\ \mathbf{y}^h_\beta = 0 &\quad \text{if } R_{\beta\gamma} = 0 \end{aligned}$$

Thus

$$\mathbf{y}^1_\beta > 0, \text{ and } \mathbf{y}^h_\beta = 0, \text{ for each } h = 2, 3, \cdots, \gamma.$$

It follows that $A_{\beta\beta}\mathbf{x}^j_\beta = \lambda_1 \mathbf{y}^1_\beta$, and so either

$$A_{\beta\beta}\mathbf{x}^j_\beta > 0 \text{ or } 0 > A_{\beta\beta}\mathbf{x}^j_\beta,$$

contrary to Lemma 5.9.4. Therefore, if $\alpha, \beta \in S$, $\alpha \prec \beta$, we must have $R_{\beta\alpha} = 0$. □

Corollary 5.9.2 Let $A = (A_{ij})$, $1 \le i, j \le k$ be a singular M-matrix in standard form (5.35). Then $S_*(A)$ has the horizontal level form

$$* \;\; * \;\; * \cdots *$$

if and only if the Jordan graph $J(A)$ has the same form.

Example 5.9.5 Let A be the matrix in Example 5.9.1. The both $S_*(A)$ and $J(A)$ are

$$* \quad *$$

Schneider also discovered a relationship between Wyre characteristic of a matrix A and the singular graph $S(A)$ of A, as follows. Interested readers are referred to [229] for further details.

Theorem 5.9.3 (Schneider, [229]) Let $A = (A_{ij})$, $1 \le i, j \le k$ be a singular M-matrix in standard form (5.35). The following are equivalent.

(i) The Wyre characteristic of A is $(1, 1, \cdots, 1)$.

(ii) The singular graph $S(A)$ is linearly ordered.

Example 5.9.6 Let A be the M-matrix

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -a & 0 & 0 & 0 & 0 \\ -b & -u & 1 & 0 & 0 \\ -c & -v & -w & 0 & 0 \\ -d & -e & -f & -g & 1 \end{bmatrix}$$

where all the elements below the main diagonal are nonpositive, and where either $uw > 0$ or $v > 0$. Then $S(A)$ is

and so both $S_*(A)$ and $J(A)$ are

$$\begin{array}{c} * \\ | \\ * \end{array}$$

Therefore, the eigenvalue 0 has exactly one 2×2 Jordan block.

A generalization of Theorem 5.9.2 and Theorem 5.9.3, called Rothblum index theorem, was obtained by Rothblum.

Theorem 5.9.4 (Rothblum, [220]) Let $A = (A_{ij})$, $1 \le i, j \le k$ be a singular M-matrix in standard form (5.35). Then $\text{ind}(A)$ equals the height of the diagram $S_*(A)$, that i, the length of a longest singular vertex chain in $R(A)$.

We conclude this section with several open problems in this area, proposed by Schneider, [228].

(1) What is the relationship between $S(A)$ and $J(A)$?

(2) When does $S_*(A) = J(A)$ hold?

(3) Given $S(A)$, for an M-matrix B, what conditions on $J(B)$ will assure that $S(A) = S(B)$?

(4) Given $J(A)$, for an M-matrix B, what conditions on $S(B)$ will assure $J(B) = J(A)$?

Some progresses towards these problems have been made. Interested readers are referred to [13].

5.10 Exercises

Exercise 5.1 Show that $A = (a_{ij})$ and $B = (b_{ij})$, where a_{ij}'s are complex numbers and where each $b_{ij} \in \{0, 1\}$, are diagonally similar if and only if there exist complex numbers $d_1, d_2, \cdots, d_n$, where $d_i \neq 0$, $i = 1, 2, \cdots, n$, such that $a_{ij} = d_i d_j^{-1}$ for every $a_{ij} \neq 0$.

Exercise 5.2 Given $A \in M_n$ such that A is irreducible. What is the sufficient and necessary condition for A to be diagonally similar to a $(0, 1)$ matrix?

Exercise 5.3 Give a counterexample to show that, in Exercise 5.2, the condition that A is irreducible is necessary.

Exercise 5.4 Let M and A be the matrices defined in the proof of Theorem 5.3.1. Express g_{ij}, the entries of the matrix M, in terms of the entries of A by verifying each of the following.

(i)

$$g_{ij} = \begin{cases} 0 & \text{if there exist } x \text{ and } y \text{ such that } a_{xi} = a_{jy} = 1 \text{ and } a_{xy} = 0, \\ 1 & \text{otherwise.} \end{cases}$$

(ii)

$$g_{ij} = \prod_{x,y} [1 - a_{xi} a_{jy} (1 - a_{xy})].$$

Exercise 5.5 Let $A = (a_{ij}), A_0 = (g_{ij}), \alpha_k = (a_{k1}, a_{k2}, \cdots, a_{kn}), \beta_k = (g_{1k}, g_{2k}, \cdots, g_{nk})^T$ (as in the proof of Theorem 5.3.1). Let $(c_{ij}) = (a_{ij}) \oplus (b_{ij})$ denote the matrix in which each $c_{ij} = a_{ij} \cdot b_{ij}$. Show that

(1) if $\alpha_t = \alpha - i_1 + \alpha_{i_2} + \cdots + \alpha_{i_k}$ in A, then $\beta_t = \beta_{i_1} \oplus \beta_{i_2} \oplus \cdots \oplus \beta_{i_k}$ in A_0.

(2) If $a_k = a_i$, then $\beta_k = \beta_i$.

(3) If $a_t = 0$ then $\beta_t = (1, 1, \cdots, 1)^T$.

Exercise 5.6 Let $A \in \mathbf{B}_n$ be a nonsingular matrix. Show that $\rho_{J-A} \geq n - 1$.

Exercise 5.7 Let $A \in \mathbf{B}_n$ be a complementary acyclic matrix with $\rho_{J-A} = n-1$. If A contains an all 1 row and an all 1 column. Then $\det A = \pm 1$.

Exercise 5.8 Let $M \in M_n$, and for $i = 1, 2, \cdots, n$ let r_i and s_i denote the ith row sum and the ith column sum, respectively. Show that $\|M^2\| = \sum_{v=1}^{n} r_v s_v$.

Exercise 5.9 Prove Theorem 5.5.7.

Exercise 5.10 Prove Theorem 5.5.9.

Exercise 5.11 Let H be a bipartite graph with vertex bipartition sets $V(H) = X \cup Y$. If H does not have a pair of disjoint edges, then each vertex z, where z is not isolated, is an end point of a bisimplicial edge in H.

Exercise 5.12 Prove Theorem 5.6.5.

Exercise 5.13 Prove Proposition 5.7.2 for the case when G is nonnegative completable.

Exercise 5.14 Prove Theorem 5.8.2.

Exercise 5.15 Prove Corollary 5.8.3.

Exercise 5.16 Let $A = (a_{ij})$ be an $n \times n$ diagonal dominant matrix such that $A > 0$. Show that if there exists an $i \in \{1, 2, \cdots, n\}$ such that $|a_{ii}| > R_i'$. Then A is nonsingular.

Exercise 5.17 Prove Corollary 5.9.1A.

Exercise 5.18 Prove Corollary 5.9.1B.

Exercise 5.19 Prove Lemma 5.9.4.

Exercise 5.20 For a stochastic matrix A, the order of each Jordon = block corresponds to eigenvalue 1.

Exercise 5.21 Find an example to show that there exist matrices A and B with $S(A) = S(B)$ but $J(A) \neq J(B)$.

5.11 Hints for Exercises

Exercise 5.1 Apply $a_{ij} = d_i b_{ij} d_j^{-1}$.

Exercise 5.2 By Theorem 5.1.1 and Exercise 5.1, the condition is $W_A(C) = 1$ for each directed cycle C in $D(A)$.

Exercise 5.3 Consider weighted digraph D, with $V(D) = \{1, 2, 3, 4\}$ and $E = \{e_1, e_2, e_3, e_4\}$, where $e_1 = (1,2), e_2 = (1,4), e_3 = (2,3), e_4 = (3,1), e_5 = (3,4)$. Assign the weights as follows: $W(e_1) = W(e_3) = W(e_4) = 1$, $W(e_2) = 2$ and $W(e_5) = 3$. Then D satisfies $W_D(C) = 1$. But $A(D)$ is not diagonally similar to any $(0,1)$ matrix.

Exercise 5.4 Apply the definitions of M and A in the proof of Theorem 5.3.1.

Exercise 5.5 Without loss of generality, suppose that $\alpha_3 = \alpha_1 + \alpha_2$. We show that $\beta_3 = \beta_1 \oplus \beta_2$. If $a_{i3} = 0$, then by Exercise 5.4(i), there exist x, y such that $a_{xi} = a_{3y} = 1$, $a_{xy} = 0$. Let $a_{iy} = 1$. Since $a_{3y} = a_{1y} + a_{2y}$, by Exercise 5.4(ii) $g_{1i} = 0$. Hence $g_{i3} = g_{i1} \cdot g_{i2}$. Conversely let $g_{i1} = 0$. We have $g_{i3} = 0$. Thus $g_{i3} = g_{i1} \cdot g_{i2}$. If $a_t = a_k$, then $\beta_t = \beta_k$. If $a_t = 0, a_{ty} = 0, y = 1 - n$, then by Exercise 5.4(ii) $g_{it} = 1, i = 1 - n$.

Exercise 5.6 Consider a set of lines including e rows and $f =$ columns. Let

$$A = \begin{pmatrix} B & C \\ D & J_{n-e,n-f} \end{pmatrix}.$$

Since A is nonsingular, columns $C_{f+1}, \cdots, C_n$ of A are linearly independent. But $\begin{pmatrix} C \\ J \end{pmatrix}$ has at most $e+1$ independent rows. Thus $e+1 \geq n-f$, $e+f \geq n-1$. By Kőnig Theorem the inequality follows.

Exercise 5.7 Suppose $A = \begin{pmatrix} 1 & 1\cdots1 \\ 1 & \\ \vdots & B \\ 1 & \end{pmatrix}$. Since B is complementary acyclic, $J_{n-1} - B$ is permutation equivalent to a triangular matrix. Thus B is permutation equivalent to a complementary triangular T. Since $\rho_{J-B} = n-1$, T contains a main diagonal with all zero and all entries above the main diagonal of T are 1. Thus $\det A = \pm 1$.

Exercise 5.8 Let $M = (m_{ij})$. Then

$$\begin{aligned} \|M^2\| &= \sum_{i,j=1}^{n} \sum_{v=1}^{n} m_{iv} m_{vj} \\ &= \sum_{v=1}^{n} \sum_{i,j=1}^{n} m_{vj} m_{iv} = \sum_{v=1}^{n} r_v s_v. \end{aligned}$$

Exercise 5.9 This follows from Proposition 5.5.4(iii) and Lemma 5.5.2. To construct a matrix $A \in \mathbf{B}_n$ reaching this bound, we can add k 1-entries in the main diagonal of the matrix L_n.

Exercise 5.10 This follows from Lemmas 5.5.3 and 5.5.4. To construct a matrix $A \in \mathbf{B}_n$ reaching this bound, we can add k 1-entries above the main diagonal of the matrix L_n^*.

Exercise 5.11 If not, let $z \in Y$ and $x_0 z \in E$. We can construct an infinite chain of subsets of X as follows, $X_0 \subset X_1 \subset \cdots \subset \cdots$, contrary to the finiteness of X. Suppose $X_k = \{x_0, x_1, \cdots, x_k\} \subseteq X$ and $Y_k = \{z, y_1, y_2, \cdots, y_k\} \subseteq Y$ such that $x_i y_j \in E$ if and only if $i < j$ for all $0 \leq i, j \leq k$, and such that $x_i z \in E$ for $0 \leq i \leq k$.

Since $x_k z$ is not bisimplicial, there exist vertices x and $y(\neq z)$ such that $x_k y, xz \in E$ but $xy \notin E$. Thus $y \notin Y_k$. Therefore $x_i y_{i+1}$ and $x_k y$ are not pair of disjoint edges, which implies $x_i y \in E$. But $xy \notin E$. Thus $x \notin X_k$. Let $X_{k+1} = x_k \cup \{y_{k+1}\}$. The algorithm can continue infinitely, and so a contradiction obtains.

Exercise 5.12 Apply Exercise 5.8 and Theorem 5.6.5.

Exercise 5.13 Imitate the proof for the completable graph case.

Exercise 5.14 If A is strict diagonal dominant, then does not lie in any of the Gersgorin discs of A, and so 0 is not an eigenvector of A. Therefore, A is nonsingular.

When all $a_{ii} > 0$, since A is diagonal dominant, each Gersgorin disc is lying on the right half complex plane and so (ii) must hold.

By Proposition 5.7.1(ii), all eigenvalues of a hermitian matrix are real, and so (iii) follows from (i) and (ii).

Exercise 5.15 Apply Theorem 5.8.3.

Exercise 5.16 If not, then 0 is an eigenvalue of A. Since A is diagonal dominant, 0 must lie on the boundary of $G(A)$. By Theorem 5.8.3 every Gersgorin disc contains 0. But $|a_{ii}| > R_i'$, the i-th disc does not contain 0, a contradiction.

Exercise 5.17 and 5.18 Corollary 5.9.1A follows immediately from Theorem 5.9.1. For Corollary 5.9.1B, let $a = \gamma_j$ in Theorem 5.9.1. Then by Theorem 5.9.1, there exist eigenvectors $\mathbf{x}^1, \mathbf{x}^2, \cdots, \mathbf{x}^s$ which belong to the eigenvalue 0 and satisfy (5.37). Assume that

$$\lambda_1 \mathbf{x}^1 + \lambda_2 \mathbf{x}^2 + \cdots + \lambda_h \mathbf{x}^h = \mathbf{0}$$

Then for $i = 1, 2, \cdots, k$, $\sum_{h=1}^{s} \lambda_h \mathbf{x}_i^h = \mathbf{0}$. Note that since $a = \gamma_j$ is a singular vertex, if $\beta = \gamma_h$, $(h \neq j)$, then $\mathbf{x}_a^h = \mathbf{0}$, and so $\lambda_j \mathbf{x}_a^j = \mathbf{0}$. As $\mathbf{x}_a^j \neq \mathbf{0}$, we must have $\lambda_j = 0$, and so $\mathbf{x}^1, \mathbf{x}^2, \cdots, \mathbf{x}^s$ are linearly independent.

Exercise 5.19 Since A is an irreducible M-matrix, there exists a vector $\mathbf{u} >> 0$ such that $\mathbf{u}^T A = 0$. Therefore, we have $\mathbf{u}^T A\mathbf{x} = 0$. Since $A\mathbf{x} \geq 0$ (or $A\mathbf{x} \leq 0$), and since $\mathbf{u} >> 0$, we must have $A\mathbf{x} = 0$.

Exercise 5.20 Note that all singular vertices in $R(A)$ are endpoint (or start point). Apply Theorem 5.9.2.

Exercise 5.21 Example: $A(a) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 \\ -a & -1 & 0 & 0 \end{pmatrix}$, $a > 0$. $S(A(1)) = S(A(2))$. But $J(A(1)) \neq J(A(2))$.

Chapter 6

Appendix

6.1 Linear Algebra and Matrices

Let F be a field and let $n, m \geq 1$ be integers. Then $M_{m,n}(F)$ denote the set of all $m \times n$ matrices over F, and $M_n(F) = M_{n,n}(F)$. When F is not specified, or is the real numbers field or the complex number field, we omit the field and simply write $M_{m,n}$ and M_n instead. For a matrix $A = (a_{ij}) \in M_{m,n}$, and for $1 \leq i \leq m$ and $1 \leq j \leq n$, the symbol $(A)_{ij}$ denotes the (i,j)-entry of A, whereas A_{ij} denote a block submatrix of A. If $\mathbf{v} = (x_1, x_2, \cdots, x_n)^T$ denotes an n-dimensional vector, then $(\mathbf{v})_i$ denotes the ith component x_i of $\mathbf{v}$.

For a matrix $A = (a_{ij}) \in M_{m,n}$, we say that A is a *positive matrix* and write $A > 0$ if $a_{ij} \geq 0$ for all i and j with $1 \leq i \leq m$ and $1 \leq j \leq n$, but $A \neq 0$. If $A = 0$ is possible, then A is a *nonnegative matrix*, and we write $A \geq 0$ instead. For an n dimensional vector $\mathbf{x} = (x_1, x_2, \cdots, x_n)^T$, we similarly define $\mathbf{x} > 0$ and $\mathbf{x} \geq 0$ and call $\mathbf{x}$ a *positive vector* or a *nonnegative vector*, respectively. If, $a_{ij} > 0$ for all i and j with $1 \leq i \leq m$ and $1 \leq j \leq n$, then A is *strictly positive* and we write $A >> 0$. Similarly we define a strictly positive vector $\mathbf{x}$ and write $\mathbf{x} >> 0$.

Let $A \in M_n$. The *characteristic polynomial* of A is $\chi_A(\lambda) = \det(A - \lambda I_n)$; and the *minimum polynomial* of A, denoted $m_A(\lambda)$, is the unique monic polynomial with the minimum degree among all monic polynomials that annihilate A.

For $A_i \in M_{n_i}$, for each i with $1 \leq i \leq c$, we write

$$\operatorname{diag}(A_1, A_2, \cdots, A_c) = \begin{bmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & A_c \end{bmatrix} .(A.1) \qquad (6.1)$$

When $A_i = (a_i) \in M_1$, A in (A.1) becomes $\mathrm{diag}(a_1, a_2, \cdots, a_c)$, and is a diagonal matrix. The *trace* of a matrix $A = (a_{ij}) \in M_n$, is $\mathrm{tr}(A) = \sum_{i=1}^n a_{ii}$.

Certain facts about trace and determinant of a matrix are summarized in Theorem 6.1.1, while some of the useful facts in linear algebra and in the theory of non negative matrices are listed in the theorems that follow.

Theorem 6.1.1 Let $A, B \in M_n$ with eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n$. Then

(i) A is nonsingular if and only if $\lambda_i \neq 0$ for each $i = 1, 2, \cdots, n$.

(ii) Let A_{11} be a square matrix, and suppose that A has the form

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}.$$

If $A_{11}, A_{12}, A_{21}, A_{22}$ all have the same dimension and satisfy $A_{11}A_{21} = A_{21}A_{11}$, then $\det(A) = \det(A_{11}A_{22} - A_{21}A_{12})$. Furthermore, if A_{11} is nonsingular, then $\det(A) = \det(A_{11}) \det(A_{22} - A_{21}A_{11}^{-1}A_{12})$.

Theorem 6.1.2 (Hamilton-Cayley Theorem) Let $A \in M_n$. Then $\chi_A(A) = 0$.

Theorem 6.1.3 (The Rayleigh Principle, see Chapter 6 of [91]) Let $A \in M_n$ be a real symmetric matrix, let $\lambda_1 = \lambda_1(A)$ be the largest eigenvalue of A, and let $\lambda_n = \lambda_n(A)$ be the smallest eigenvalue of A. Each of the following holds.

(i) $\lambda_1 = \max\limits_{|\mathbf{u}|=1} \mathbf{u}^T A\mathbf{u} = \max\limits_{|\mathbf{u}|\neq 0} \dfrac{\mathbf{u}^T A\mathbf{u}}{|\mathbf{u}|^2}$.

(ii) $\lambda_n = \min\limits_{|\mathbf{u}|=1} \mathbf{u}^T A\mathbf{u} = \min\limits_{|\mathbf{u}|\neq 0} \dfrac{\mathbf{u}^T A\mathbf{u}}{|\mathbf{u}|^2}$.

Theorem 6.1.4 (Chapter 6 of [91]) Let $A = (a_{ij}) \in M_n$ be a positive, and let $\lambda_1 \geq \lambda_2 \leq \cdots \leq \lambda_n$ be the eigenvalues of A. Then each of the following holds.

(i) λ_1 has multiplicity one.

(ii) If $n \geq 2$, then $\lambda_1 > |\lambda_i|$ for all i with $2 \leq i \leq n$.

Theorem 6.1.5 (Chapter 6 of [91]) Let $A = (a_{ij})$ be an n by n real matrix and let $A^+ = (|a_{ij}|)$. Then $\max_{1\leq i\leq n}\{|\lambda_i(A)|\} \leq \max_{1\leq i\leq n}\{|\lambda_i(A^+)|\}$.

Theorem 6.1.6 Let $A = (a_{ij}) \in M_n^+$ be such that if for any $i \neq j$, either $a_{ij} > 0$, or there exist $t_1, t_2, \cdots, t_r$ such that $a_{it_1} > 0, a_{t_1t_2} > 0, \cdots, a_{t_rj} > 0$, then $(I + A)^{n-1}$ is positive.

Proof If A is a (0,1) matrix, then the assumption says that $D(A)$, the associated digraph of A (see Definition 1.1.1), is a strongly connected digraph, and so for any pair of vertices, there is a path from one to the other with length at most $n-1$, which proves the conclusion.

In general, for each $r \leq n-2$, the (i, j) entry of A^{r+1} is $\sum_{t_1, t_2, \cdots, t_r} a_{it_1}a_{t_1t_2}\cdots a_{t_rj}$, which is positive if and only if one of its terms is positive. Therefore,

$$(I+A)^{n-1} = \sum_{r=0}^{n-1} \binom{n-1}{r} A^r > 0.$$

This proves the theorem. $\square$

In Section 4.4, Cauchy-Binet formula is used to prove the Matrix-Tree Theorem. We state this useful formula here. For a reference of this formula, see [133].

Theorem 6.1.7 (Cauchy-Binet Formula) For integers $m > 0, n > 0$ and r with $1 \le r \le \min\{m, n\}$, let $A \in M_{m,k}$ and $B \in M_{k,n}$, $C = AB \in M_{m,n}$, and let $\alpha = (i_1, i_2, \cdots, i_r)$ and $\beta = (j_1, j_2, \cdots, j_r)$ be two r-tuples of indices such that $1 \le i_1 < i_2 < \cdots < i_r \le m$ and $1 \le j_1 < j_2 < \cdots < j_r \le n$. Then

$$\det(A[\alpha|\beta]) = \sum_{\gamma} \det(A[\alpha|\gamma])\det(B[\gamma|\beta]).$$

6.2 The Term Rank and the Line Rank of a Matrix

Definition 6.2.1 For a matrix $A \in \mathbf{B}_{m,n}$, a *line* of A is either a row or a column of A. The *line rank* of A, denoted λ_A, is the minimum number of lines that contain all positive entries of A. Two positive entries of A are *independent* if they are not in the same line of A. The *term rank* of A, denoted ρ_A, is the maximum number of independent positive entries in A.

The proposition below follows immediately from the definition.

Proposition 6.2.1 Let $A, B \in \mathbf{B}_n$. Each of the following holds.

(i) If there are permutation matrices $P, Q \in \mathbf{B}_n$ such that $A = PBQ$, then $\lambda_A = \lambda_B$ and $\rho_A = \rho_B$.

(ii) If $A = B^T$, then $\lambda_A = \lambda_B$ and $\rho_A = \rho_B$.

Theorem 6.2.1 Let $A \in \mathbf{B}_{m,n}$. The following are equivalent.

(i) $\lambda_A < m$.

(ii) $\rho_A < m$.

(iii) For some integers p and q with $1 \le p \le m$ and $1 \le q \le n$ and with $p + q = n + 1$, A contains a $0_{p\times q}$ as a submatrix.

Proof By Definition 6.2.1, (i) and (iii) are equivalent. To see that (ii) and (iii) are equivalent, first observe that if $m < n$, then let

$$\bar{A} = \begin{bmatrix} A \\ J_{(n-m)\times n} \end{bmatrix}.$$

It follows that $\rho_A < m \Longleftrightarrow \rho_{\tilde{A}<n}$ and that A has a $0_{p\times(n-p+1)}$ submatrix $\Longleftrightarrow$ $\tilde{A}$ has a $0_{p\times(n-p+1)}$ submatrix. Therefore it suffices to prove that (ii) and (iii) are equivalent when $m = n$.

Note that (ii) and (iii) are equivalent if $m = n = 1$. Assume that $m = n > 1$.

If A has a $0_{p\times q}$ as a submatrix with $p + q = m + 1$, then the p rows containing this $0_{p\times q}$ submatrix has at most $m - q = p - 1$ non zero columns, and so it is impossible for A to have p positive entries among these p rows such that no two of these p positive entries are in the same line. Hence $\rho_A < m$.

Now assume that $\rho_A < m$, that (ii) and (iii) are equivalent for smaller values of m, and that $A \neq 0$.

Since $A \neq 0$, we may assume that $a_{ij} > 0$ for some $1 \le i, j \le m$. Let $A(i|j)$ denote the matrix obtained from A by deleting the row and column of A containing a_{ij}. Then $\rho_{A(i|j)} < m - 1$. By induction, $A(i|j)$ has a $0_{p_1\times q_1}$ as a submatrix, for some p_1, q_1 with $p_1 + q_1 = m$ and $1 \le p_1, q_1 \le m - 1$. By Proposition 6.2.1(i), we may assume that

$$A = \begin{bmatrix} X & 0 \\ Z & Y \end{bmatrix},$$

where $X \in \mathbf{B}_{p_1,p_1}$, $Y \in \mathbf{B}_{m-p_1,m-p_1}$ and $Z \in \mathbf{B}_{m-p_1,p_1}$. Since $\rho_A < m$, either $\rho_X < p_1$ or $\rho_Y < m - p_1$. Without loss of generality, we assume that $\rho_X < p_1$. By induction, X contains a $0_{p_2\times q_2}$ as a submatrix for some $1 \le p_2, q_2 \le p_1$ with $p_2 + q_2 = p_1 + 1$. It follows that A has a $0_{p_2\times(q_2+m-p_1)}$ as a submatrix, where $p_2 + (q_2 + m - p_1) = m + 1$. Thus the equivalence between (ii) and (iii) now follows by induction. $\square$

Theorem 6.2.2 (König, [148]) For any $A \in \mathbf{B}_{m,n}$, $\lambda_A = \rho_A$.

Proof By Proposition 6.2.1(ii), we may assume that $m \le n$.

If all positive entries of A are contained in λ_A lines, then each of these lines can contain at most one independent positive entry of A, and so $\lambda_A \ge \rho_A$. Therefore, it suffices to show $\lambda_A \le \rho_A$.

By Theorem 6.2.1, we may assume that $\lambda_A < m$. Suppose that A has e rows and f columns, with $e + f = \lambda_A$, which contain all the nonzero entries of A. By Proposition 6.2.1, we may assume that the first e rows and the first f columns of A contain all nonzero entries of A, and so

$$A = \begin{bmatrix} X & Y \\ Z & W \end{bmatrix},$$

where $X \in \mathbf{B}_{e,f}, Y \in \mathbf{B}_{e,n-f}, Z \in \mathbf{B}_{m-e,f}$ and $W = 0_{m-e,n-f}$.

If we can show $\rho_Y = e$, then by Proposition 6.2.1(ii), the same augment yields $\rho_Z = f$, and so $\rho_A \ge \rho_Y + \rho_Z = e + f = \lambda_A$, as desired. Thus we only need to show $\rho_Y = e$.

Suppose $e > 0$. Then $e = \lambda_A - f < n - f$. If $\rho_Y < e$, then by Theorem 6.2.1, $\rho_Y < e$. As $W = 0_{m-e,n-f}$ and by the definition of λ_A, $\lambda_A \le f + \lambda_Y < f + e$, a contradiction. $\square$

6.3 Graph Theory

A *graph* G consists of a nonempty set $V(G)$ of elements called *vertices*, and a set $E(G)$ of elements called *edges*, and a relation of *incidence* that associates with each edge e two vertices u and v, called the *ends* of e. Sometimes a graph G with vertex set $V = V(G)$ and edge set $E = E(G)$ is denoted $G(V, E)$, to indicate the vertex and edge sets. If an edge e has ends u, v, then e is *incident* with u and v, and u and v are *adjacent* to each other. In this case we write $e = (uv)$ or $e = uv$. An edge with identical ends is a *loop*, and one with distinct ends is a *link*. Two or more edges with the same pair of vertices as their common ends are *multiple* edges. A graph G is *simple* if G does not have loops nor multiple edges.

Given graphs G and H, if $V(H) \subseteq V(G)$, $E(H) \subseteq E(G)$ and for each edge $e \in E(H)$, the ends of e in H are the same ends of e in G, then H is a *subgraph* of G, and G a *supergraph* of H. If $V(G) = V(H)$ and $H \subseteq G$, then H is a *spanning* subgraph of G.

Let $G = G(V, E)$ be a graph, and $S \subseteq E(G)$ be an edge subset. The subgraph $H = H(V', S)$ of G with V' being the set of vertices in G that are incident with an edge in S is the subgraph *induced* by the edge set S, and will be denoted by $G[S]$.

Let $G = G(V, E)$ be a graph, and $R \subseteq V(G)$ be a vertex subset. The subgraph $H = H(R, E')$ of G with E' be the set of edges in G whose ends are in R is the subgraph *induced* by the vertex set R, and will be denoted by $G[R]$.

Let x and y be vertices of a graph G. An (x, y)-*walk* in G is a sequence

$$W = \{v_0, e_1, v_1, e_2, v_2, \cdots, v_{k-1}, e_k, v_k\},$$

where $x = v_0, v_1, \cdots, v_k = y$ are in $V(G)$, $e_1, \cdots, e_k$ are in $E(G)$, and v_{i-1} and v_i are the ends of e_i, $1 \le i \le k$. The vertices $v_1, \cdots, v_{k-1}$ are *internal vertices* of W. The *length* of W is the number of edges in W, namely, $|E(W)| = k$. We allow $k = 0$ and so v_0 itself is a walk of length zero.

A walk W is a *trail* if the edges $e_1, \cdots, e_k$ are all distinct, and a *path* if all vertices $v_0, \cdots, v_k$ are distinct. We then define (x, y)-*trails* and (x, y)-*paths* similarly. A trail with $v_0 = v_k$ is a *closed trail*. A closed trail with the vertices $v_0 = v_k, v_1, \cdots, v_{k-1}$ being distinct is a *cycle*. The *girth* of a graph G is the shortest length of a cycle of G.

The *degree* of a vertex v in a graph G denoted by $d_G(v)$, is the number of edges in G that are incident with v, where a loop counts as two edges. The *maximum* and the *minimum* degree of the vertices of a graph G are denoted by $\Delta(G)$ and $\delta(G)$, respectively. A graph G is r-*regular* if $\Delta(G) = \delta(G) = r$. Theorem 6.3.1 below follows by counting the incidences of a graph in different ways.

Theorem 6.3.1 ([115]) Let G be a graph, then

$$\sum_{v \in V(G)} d_G(v) = 2|E(G)|.$$

A graph G is *planar* if G can be drawn on the plane so that no two edges intersect. The next theorem can be derived from the Euler Polyhedron Formula and Theorem 6.3.1.

Theorem 6.3.2 ([115]) Let G be a simple planar graph. Then

$$|E(G)| \leq 3|V(G)| - 6.$$

A simple graph on n vertices is the *complete graph on n vertices*, denoted K_n, if every pair of distinct vertices are adjacent. A maximal complete subgraph H of a graph G is called a *clique* if H is isomorphic to a complete graph. The *clique number* of a graph G, denoted $\omega(G)$, is the maximum k such that G has K_k as a subgraph.

The *chromatic number* of a graph G, denoted $\chi(G)$, is the minimum number k such that the vertices of G can be colored with k distinct colors so that no two vertices with the same color are adjacent in G. A subset $X \subset V(G)$ is called an *independent* set of G if no two vertices in X are adjacent in G. The *independence number* of G, denoted $\alpha(G)$, is the maximum cardinality of an independent subset of G. Immediately following these definitions, we have the following result.

Theorem 6.3.3 ([115]) Let G be a graph. Then

$$\chi(G)\alpha(G) \geq |V(G)|, \text{ and } \alpha(G) = \omega(G^c).$$

Let G and H be two vertex disjoint simple graphs. The (*disjoint*) *union* of G and H, denoted by $G \cup H$, is the graph with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$. The *join* of G and H, denoted by $G \vee H$, is the simple graph obtained from $G \cup H$ by adding all possible additional edges that join a vertex in G with a vertex in H.

A graph G is *connected* if $V(G)$ has only one equivalence class. Note that G is connected if and only if for every pair of vertices $x, y \in V(G)$, G has an (x, y)-path.

A *digraph* (directed graph) D is obtained from a graph G by assigning each edge $e \in E(G)$ a direction. We call D an *orientation* of G. If e has u and v as ends and if e is oriented from u (*tail*) to v (*head*), then we denote the oriented e by (u, v). The oriented edges in a digraph D are called *arcs*. We sometimes denote a directed graph D by $D(V, E)$ to indicate the vertex set V and the arc set E of D. *Loops* in a digraph are defined similar to those in a graph. But two or more arcs with the same heads and the same tails are *multiple arcs*. For a vertex $v \in V(D)$, we denote the number of arcs in D with v as a head by $d_D^-(v)$, called the *indegree* of v in D. Similarly, the number of arcs in D with v as a tail is denoted by $d_D^+(v)$, called the *outdegree* of v in D. The sum of $d_D^-(v)$ and $d_D^+(v)$ is the degree of v in D, denoted by $d_D(v)$. The subscript D may be omitted when the digraph D is understood in the context. Similar to the undirected graph case, the following can be obtained by counting $|E(D)|$ in different ways.

Theorem 6.3.4 Let D be a digraph, then

$$\sum_{v \in V(D)} d_G^-(v) = \sum_{v \in V(D)} d_G^-(v) = |E(D)|.$$

For $x, y \in V(D)$, a ***directed*** (x, y)***-walk*** (or (x, y)-diwalk) in D is a walk as defined above, with an additional requirement that each e_i be directed from v_{i-1} to v_i. Sometimes we use the notation $T = v_1 \to v_2 \to \cdots \to v_k$ to denote a directed walk T which starts from v_1 and ends at v_k, where $v_i \to v_{i+1}$ indicates that the arc (v_i, v_{i+1}) is in the walk T. Directed trails, paths, cycles in a digraph D are defined similarly.

A digraph D is ***strong*** if for any pair of vertices $u, v \in V(D)$, D has a directed (u, v)-walk. A maximal strong subdigraph of D is called a ***strong component*** of D. We also say that a digraph D is ***weakly connected*** if the underlying graph of D, obtained from D by erasing all the directions on the arcs of D, is a connected graph.

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