Chapter 10

Nuclear Properties

Note to students and other readers: This Chapter is intended to supplement Chapter 3 of Krane's excellent book, "Introductory Nuclear Physics". Kindly read the relevant sections in Krane's book first. This reading is supplementary to that, and the subsection ordering will mirror that of Krane's, at least until further notice.

A nucleus, discovered by Ernest Rutherford in 1911, is made up of nucleons, a collective name encompassing both neutrons (n) and protons (p).

Name	**symbol**	**mass (MeV/c^2**	**charge**	**lifetime**	**magnetic moment**
neutron	n	939.565378(21)	0 e	881.5(15) s	-1.91304272(45) $\mu_{\rm N}$
proton	p	938.272046(21)	1 e	stable	2.792847356(23) $\mu_{\rm N}$

The neutron was theorized by Rutherford in 1920, and discovered by James Chadwick in 1932, while the proton was theorized by William Prout in 1815, and was discovered by Rutherford between 1917 and 1919m and named by him, in 1920.

Neutrons and protons are subject to all the four forces in nature, (strong, electromagnetic, weak, and gravity), but the strong force that binds nucleons is an intermediate-range force that extends for a range of about the nucleon diameter (about 1 fm) and then dies off very quickly, in the form of a decaying exponential. The force that keeps the nucleons in a nucleus from collapsing, is a short-range repulsive force that begins to get very large and repulsive for separations less than a nucleon radius, about $\frac{1}{2}$ fm. See Fig. 10.1 (yet to be created).

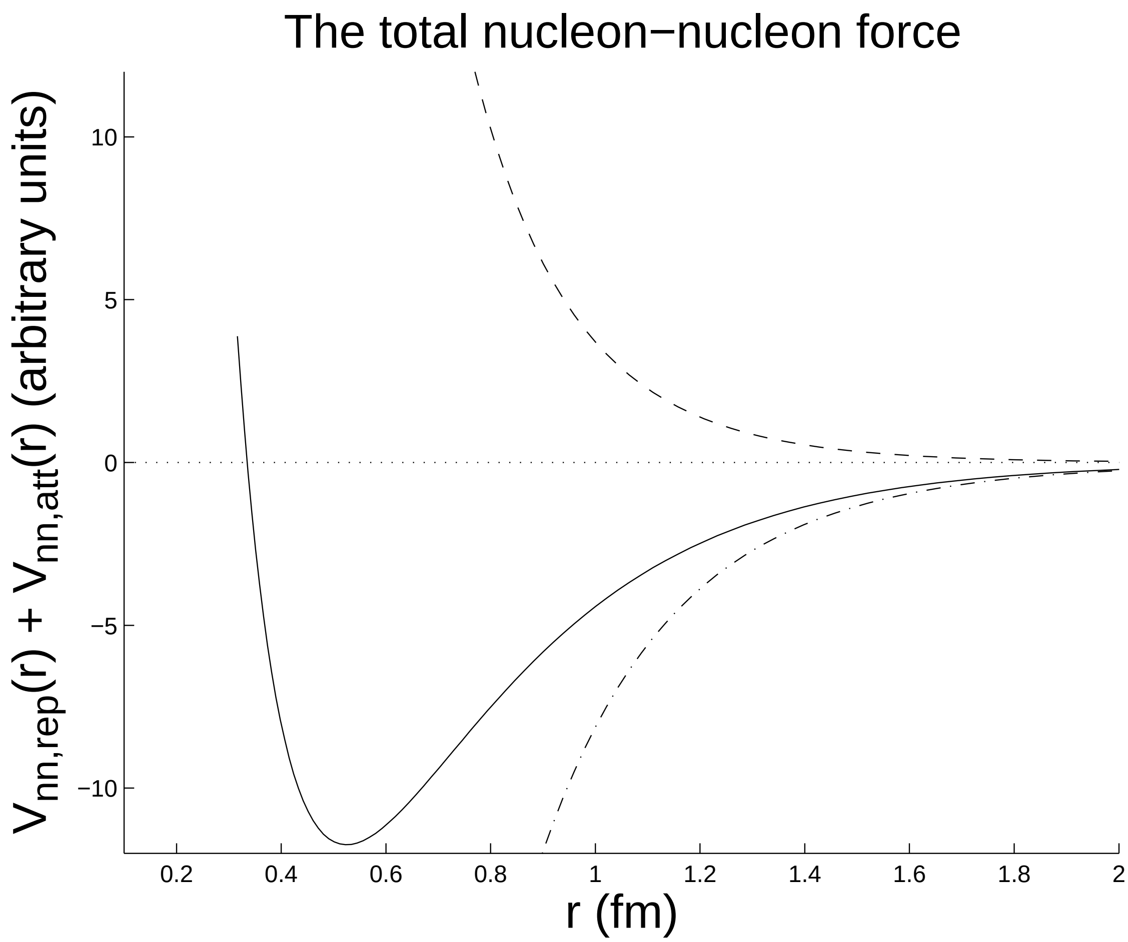


Figure 10.1: A sketch of the nucleon-nucleon potential.

The n-n, n-p, and p-p nuclear forces are all almost identical. (There are some important differences.) Of course, there is an additional p-p Coulombic repulsive potential, but that is separate from the nuclear force.

Owing to these nuclear forces between individual nucleons, a nucleus is tightly bound. The consequence is, from the attractive/repulsive form of the nuclear force, that the nucleons are in very close proximity. One can almost imagine a nucleus being made up of incompressible nucleonic spheres, sticking to one other, with a "contact" potential, like ping-pong balls smeared with petroleum jelly. A further consequence of the nuclear force is that nucleons in the nuclear core move, in what seems to be, a constant potential formed by the attraction of its nearby neighbors, only those that are in contact with it. A nucleon at the surface of a nucleus has fewer neighbors, and thus, is less tightly bound.

Nucleons are spin-$\frac{1}{2}$ particles (*i.e.* fermions). Hence the Pauli Exclusion Principle applies. That is, no two identical nucleons may possess the same set of quantum numbers. Consequently, we can "build" a nucleus, much as we built up an atom (in NERS311), by placing individual electrons into different quantum "orbitals", with orbitals being filled according to energy hierarchy, with a maximum of two electrons (spin up and spin down) to an orbital. Nucleons are formed in much the same way, except that all the force is provided by the other constituent nucleons, and there are two different "flavors" of nucleon, the neutron and the proton.

So, it seems that we could build a nucleus of almost any size, were it not for two physical facts that prevent this. The Pauli Exclusion Principle prevents the di-nucleon from being bound. Thus, uniform neutron matter does not exist in nature, except in neutron stars, where gravity, a long-range force, provides the additional binding energy to enable neutron matter to be formed. Thus, to build nuclei, we need to add in approximately an equal proportion of protons. However, this also breaks down because of Coulomb repulsion, for A (the total number of nucleons) greater than about 200 or so.

Moderate to large size nuclei also have more neutrons in the mix, thereby pushing the protons farther apart. It is all a matter of balance, between the Pauli Exclusion Principle and the Coulomb repulsion. And, that balance is remarkably delicate. The di-neutron is *not* bound, but *just* not bound. The deuteron *is* bound, but only *just* so. The alpha particle is tightly bound, but there are no stable $A = 5$ nuclei. ^{5}He $(2p + 3n)$ has a half-life of only 7.9×10^{-22} seconds, while ^{5}Li $(3p + 2n)$ has a half-life of only $\approx 3 \times 10^{-22}$ seconds. Those lifetimes are so short, that the unbalanced nucleon can only make a few orbits of the nucleus before it breaks away. Nature is delicately balanced, indeed.

Since we have argued that nuclei are held together by a "contact" potential, it follows that nuclei would tend to be spherical in "shape", and hence[1] it is reasonable to make mention of ...

[1] Admittedly, these are classical concepts. However, classical concepts tend to be very useful when discussing nuclei as these objects seem to straddle both the classical and quantum descriptions of its nature, with one foot set solidly in both.

10.1 The Nuclear Radius

Like the atom, the radius of a quantum object is not a precisely defined quantity; it depends on how that characteristic is measured. We can, with the proper tools, ask some very interesting things about the nucleus. Let us assume that the charge-independence of the nucleus means that the proton charge density and the neutron charge density are the same. Thus, a measure of the proton charge distribution yields direct knowledge of the neutron charge distribution. (In actual fact, the proton charge density distribution is forced to greater radius by Coulomb repulsion, but this effect is almost negligible.)

How may we measure the proton charge distribution?

In Nuclear and Particle Physics, the answer to this question usually takes some form of "Bang things together and see what happens!" In this case, we'll use electrons as the projectile and the nucleus as the target. The *scattering amplitude* is given by a proportionality (describing the constants necessary to convert the $\propto$ to an $=$ would be an unnecessary distraction):

$$F(\vec{k}_i, \vec{k}_f) \propto \langle e^{i\vec{k}_f\cdot\vec{x}}|V(\vec{x})|e^{i\vec{k}_i\cdot\vec{x}}\rangle \ , \tag{10.1}$$

where $e^{i\vec{k}_i\cdot\vec{x}}$ is the initial unscattered electron wavefunction, $e^{i\vec{k}_f\cdot\vec{x}}$ is the final scattered electron wavefunction, and $\vec{k}_i/\vec{k}_f$ are the initial/final wavenumbers.

Evaluating ...

$$\begin{aligned}
F(\vec{k}_i, \vec{k}_f) &\propto \int \mathrm{d}\vec{x}\ e^{-i\vec{k}_f\cdot\vec{x}}V(\vec{x})e^{i\vec{k}_i\cdot\vec{x}} \\
&\propto \int \mathrm{d}\vec{x}\ V(\vec{x})e^{i(\vec{k}_i-\vec{k}_f)\cdot\vec{x}} \\
&\propto \int \mathrm{d}\vec{x}\ V(\vec{x})e^{i\vec{q}\cdot\vec{x}} \ ,
\end{aligned}$$

where $\vec{q} \equiv \vec{k}_i - \vec{k}_f$ is called the *momentum transfer*.

Thus, we see that scattering amplitude is proportional to the 3D Fourier Transform of the potential.

$$F(\vec{k}_i, \vec{k}_f) \equiv F(\vec{q}) \propto \int \mathrm{d}\vec{x}\ V(\vec{x})e^{i\vec{q}\cdot\vec{x}} \ , \tag{10.2}$$

For the present case, we apply the scattering amplitude to the case where the incident electron scatters from a much heavier nucleus that provides a scattering potential of the form:

$$V(\vec{x}) = -\frac{Ze^2}{4\pi\epsilon_0} \int \mathrm{d}\vec{x}' \; \frac{\rho_p(\vec{x}')}{|\vec{x} - \vec{x}'|} \; , \tag{10.3}$$

where $\rho_p(\vec{x}')$ is the number density of protons in the nucleus, normalized so that:

$$\int \mathrm{d}\vec{x}' \; \rho_p(\vec{x}') \equiv 1 \; . \tag{10.4}$$

That is, the potential at $\vec{x}$ arises from the electrostatic attraction of the elemental charges in $\mathrm{d}\vec{x}$, integrated over all space. In order to probe the shape of the charge distribution, the reduced wavelength of the electron, $\lambda/2\pi$, must be less than the radius of the nucleus. Evaluating ...

$$\frac{\lambda}{2\pi} = \frac{\hbar}{p_e} = \frac{\hbar c}{p_e c} \approx \frac{\hbar c}{E_e} = \frac{197 \text{ [MeV.fm]}}{E_e} < R_N \; ,$$

where R_N is the radius of the nucleus. The above is a relativistic approximation. (That is why the $\approx$ appears; $p_e c \approx E_e$.) The calculation is justified, however, since the inequality implies that the energy of the electron-projectile must be many 10s or 100s of MeV for the condition to hold. As we raise the electron energy even more, and it approaches 1 GeV or more, we can even begin to detect the individual charges of the constituent particles of the protons (and neutrons), the constituent quarks.

Proceeding with the calculation, taking the potential in (10.3) and putting it in (10.2), results in:

$$F(\vec{q}) \propto \left(-\frac{Ze^2}{4\pi\epsilon_0}\right) \int \mathrm{d}\vec{x} \; \int \mathrm{d}\vec{x}' \; \frac{\rho_p(\vec{x}')}{|\vec{x} - \vec{x}'|} e^{i\vec{q}\cdot\vec{x}} \; . \tag{10.5}$$

We choose the constant of proportionality in $F(\vec{q})$, to require that $F(0) \equiv 1$. The motivation for this choice is that, when $\vec{q} = 0$, the charge distribution is known to have no effect on the projectile. If a potential has no effect on the projectile, then we can rewrite (10.5) as

$$F(0) = 1 \propto \left(-\frac{Ze^2}{4\pi\epsilon_0}\right) \int \mathrm{d}\vec{x} \; \int \mathrm{d}\vec{x}' \; \frac{\rho_p(\vec{x}')}{|\vec{x} - \vec{x}'|} \; , \tag{10.6}$$

thereby determining the constant of proportionality. The details of this calculation will be left to enthusiastic students to discover for themselves. The final result is:

$$F(\vec{q}) = \int \mathrm{d}\vec{x} \; \rho_p(\vec{x}) e^{i\vec{q}\cdot\vec{x}} \; . \tag{10.7}$$

Thus, we have determined, at least for charge distributions scattering other charges, that the scattering amplitude is the Fourier Transform of the charge distribution.

This realization is one of the most important discoveries of nuclear structure physics: namely, that a measurement of the scattering of electrons (or other charged particles) from charge distributions, yields a direct measure of the shape of that charge distribution. One merely has to invert the Fourier Transform.

We also note, from (10.4) that $F(0) = 1$.

10.1.1 Application to spherical charge distributions

Most nuclei are spherical in shape, so it behooves us to examine closely, the special case of spherical charge distributions. In this case, $\rho_p(\vec{x}) = \rho_p(r)$, and we write (10.7) more explicitly in spherical polar coordinates:

$$F(\vec{q}) = \int_0^{2\pi} \mathrm{d}\phi \int_0^{\infty} r^2 \mathrm{d}r \ \rho_p(r) \int_0^{\pi} \sin\theta \mathrm{d}\theta \ e^{iqr\cos\theta} \ . \tag{10.8}$$

The only "trick" we have used is to align our coordinate system so that $\vec{q} = q\hat{z}$. This is permissible since the charge distribution is spherically symmetric and there is no preferred direction. Hence, we choose a direction that makes the arithmetic easy. The remaining integrals are elementary, and one can easily show that:

$$F(q) = \frac{4\pi}{q} \int_0^{\infty} r \mathrm{d}r \ \rho_p(r) \sin qr \ . \tag{10.9}$$

Figure 10.2: From “Introductory Nuclear Physics”, by Kenneth Krane

Figure 10.3: From “Introductory Nuclear Physics”, by Kenneth Krane

Figure 10.4: From "Introductory Nuclear Physics", by Kenneth Krane

Conclusions from the data shown?

1. The central density, is (roughly) constant, almost independent of atomic number, and has a value about $0.13/\text{fm}^3$. This is very close to the density nuclear in the infinite radius approximation,
$$\rho_0 = 3/(4\pi R_0^3)\ .$$

2. The "skin depth", s, is (roughly) constant as well, almost independent of atomic number, with a value of about 2.9 fm, typically. The skin depth is usually defined as the difference in radii of the nuclear densities at 90% and 10% of maximum value.

3. Measurements suggest a best fit to the radius of nuclei:
$$R_N = R_0 A^{1/3} \quad ; \quad R_0 \approx 1.22\ [\text{fm}], 1.20 \longrightarrow 1.25 \ \text{ is common.} \tag{10.10}$$
however, values from 1.20⟶1.25 are commonly found

   A convenient parametric form of the nuclear density was psoposed by Woods and Saxon (*ca.* 1954).
$$\rho_N(r) = \frac{\rho_0}{1 + \exp\left(\frac{r - R_N}{t}\right)}$$
where t is a surface thickness parameter, related to s, by $s = 4t\log(3)$.

An example of this distribution is shown in Figure 10.5

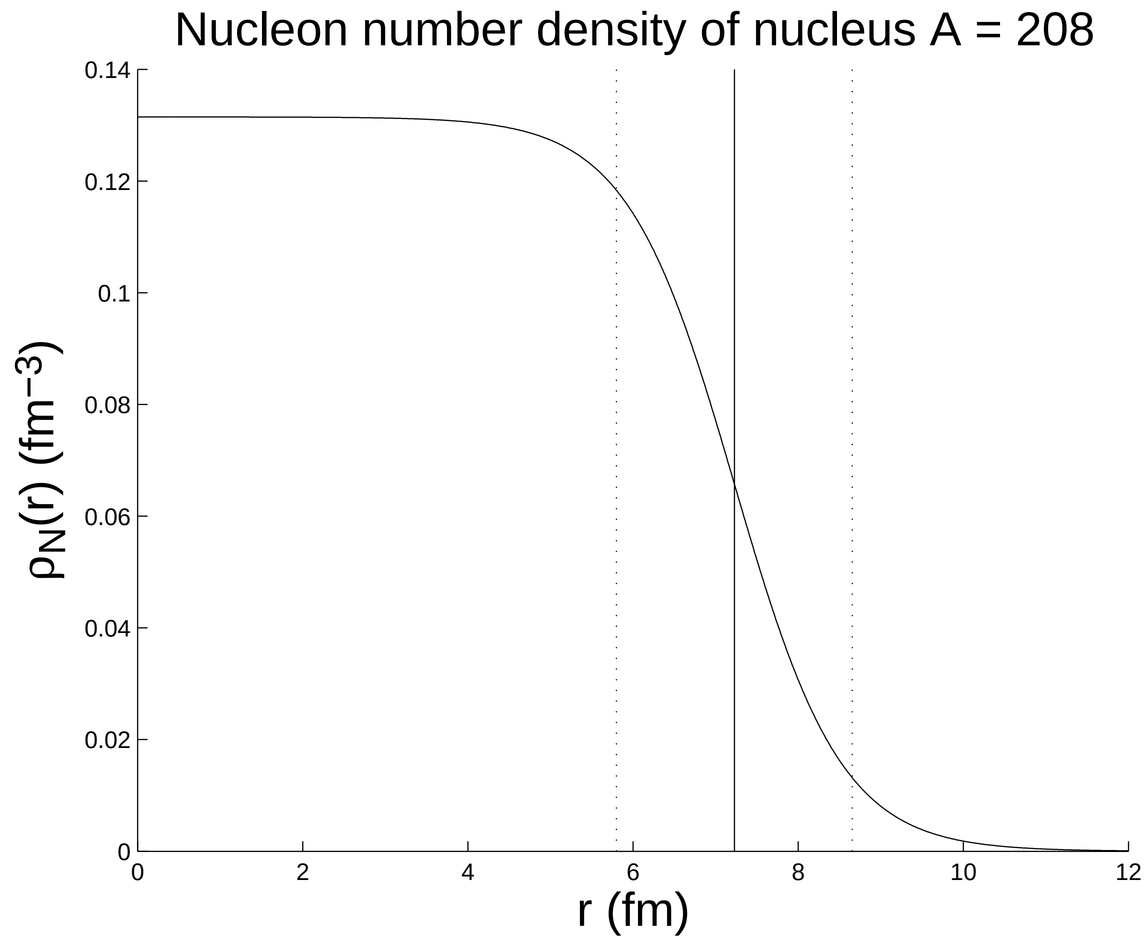


Figure 10.5: The Woods-Saxon model of the nucleon number density. In this figure, $A = 208$, $R_0 = 1.22$ (fm), and $t = 0.65$ (fm). The skin depth is shown, delimited by vertical dotted lines.

Let's work out a specific, but important realization of a charge distribution, namely, a uniform proton distribution, up to some radius R_N, the radius of the nucleus.

Example: Uniform nucleon charge density

In this case, the normalized proton density takes the form:

$$\rho_p(r) = \frac{3}{4\pi R_N^3}\Theta(R-r) \ . \tag{10.11}$$

Thus, combining (10.9) and (10.11), gives, after some reorganization:

$$F(q) = \frac{3}{(qR_N)^3}\int_0^{(qR_N)} \mathrm{d}z \ z\sin z \ , \tag{10.12}$$

which is easily evaluated to be,

$$F(q) = \frac{3[\sin(qR_N) - qR_N\cos(qR_N)]}{(qR_N)^3} \ , \tag{10.13}$$

for which $F(0) = 1$, as expected.

<u>**Technical side note:**</u>
The following `Mathematica` code was useful in deriving the above relations.

```
(* Here Z == q*R_N: *)

(3/Z^3)*Integrate[z Sin[z], {z,0,Z}]

Series[3*(Sin[Z] - Z*Cos[Z])/Z^3,{Z,0,2}]
```

Graphical output of (10.13) is given in Figure 10.6. We note, in particular, the zero minima when $\tan(qR_N) = qR_N$. The shape of the lobes is determined by the nuclear shape, while the minima are characteristic of the sharp edge. Measurements do not have such deep minima, since the nuclear edge is blurred, and the projectile energies are are not exact, but slightly distributed, and the detectors have imperfect resolution. However, the measurements do, unambiguously, reveal important details of the nuclear shape.

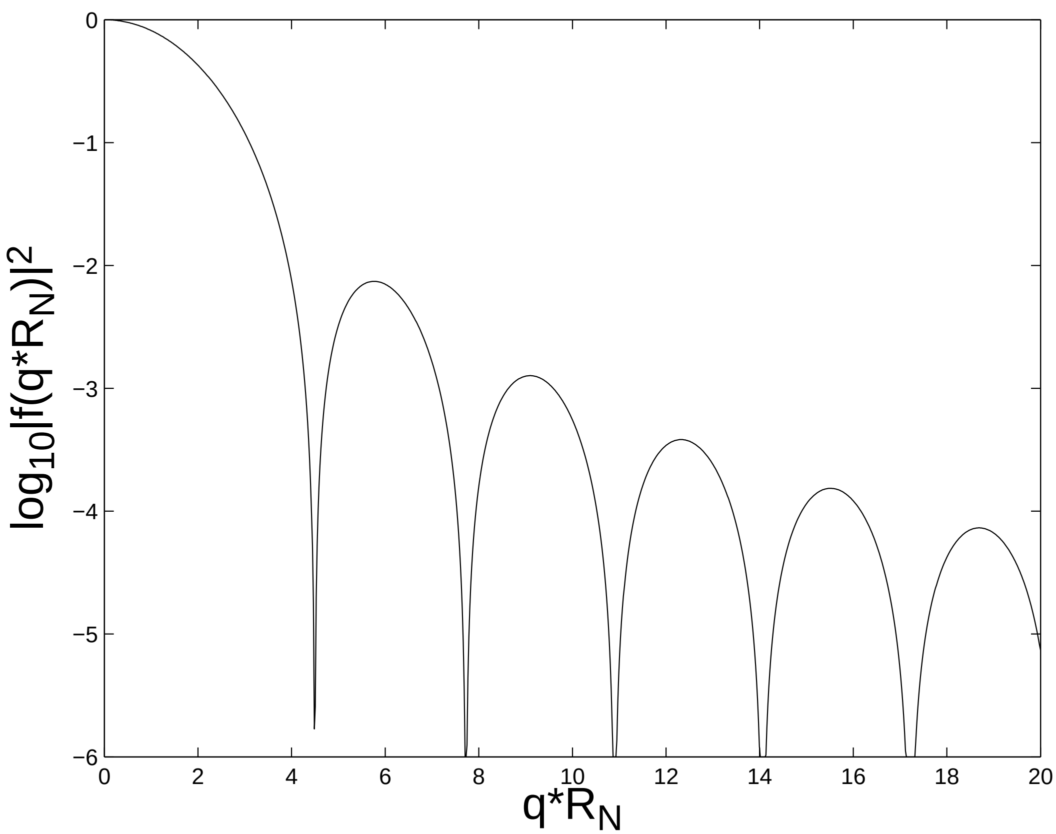


Figure 10.6: Graphical output corresponding to (10.13).

Technical side note:

The following `Matlab` code was useful in producing the above graph.

```
N = 1000; fMin = 1e-6; zMax = 20; % Graph data
z = linspace(0,zMax,N); f = 3*(sin(z) - z.*cos(z))./z.^3;
f(1) = 1; % Overcome the singularity at 0
f2 = f.*f;
for i = 1:N
   f2(i) = max(fMin,f2(i));
end
plot(z,log10(f2),'-k')
xlabel('\fontsize{20}q*R_N')
ylabel('\fontsize{20}log_{10}|f(q*R_N)|^2')
```

10.1.2 Nuclear shape data from electron scattering experiments

---

Technical side note:
The mathematical details can be found in the supplemental notes.

Most of the mathematical detail is given in the supplementary notes to this lecture. Those notes obtain the following, very significant result.

What is measured in a scattering experiment is the relative intensity of deflected projectiles (e), scattered into different angles, by the nucleus (N). This is also known as the scattering cross section, differential in scattering angle. The result is that:

$$\frac{\mathrm{d}\sigma_{eN}}{\mathrm{d}\Omega} = \frac{\mathrm{d}\sigma_{eN}^{\mathrm{Ruth}}}{\mathrm{d}\Omega}|F(q)|^2 \ , \tag{10.14}$$

where $\mathrm{d}\sigma_{eN}^{\mathrm{Ruth}}/\mathrm{d}\Omega$ is the classical Rutherford cross section discussed in NERS311 (but re-derived in the supplemental notes to include relativistic kinematics, and $F(q)$ is the scattering amplitude we have been discussing so far. $|F(q)|^2$ is the scattering amplitude, modulus squared. (It can, in general, be complex.)

Hence, we have a direct experimental determination of the form factor, as a ratio of measurement data (the measured cross section), and a theoretical function, the Rutherford cross section.

$$|F(q)|^2 = \left(\frac{\mathrm{d}\sigma_{eN}^{\mathrm{meas}}}{\mathrm{d}\Omega}\right) \bigg/ \left(\frac{\mathrm{d}\sigma_{eN}^{\mathrm{Ruth}}}{\mathrm{d}\Omega}\right) \ . \tag{10.15}$$

All that remains is to take the square root, and invert the Fourier Transform, to get $\rho(r)$. This is always done via a relatively simple numerical process.

Although the form factor $|F(q)|^2$ is given in terms of q, we may cast it into more recognizable kinematic quantities as follows. Recall,

$$q = \sqrt{q^2} = \sqrt{|\vec{k}_i - \vec{k}_f|^2} = \sqrt{2k^2(1-\cos\theta)} \ , \tag{10.16}$$

the final step above being obtained since this is an elastic scattering process, where $k = |\vec{k}_i| = |\vec{k}_f|$ and $\vec{k}_i \cdot \vec{k}_f = k^2 \cos\theta$.

Thus, electron scattering experiments yield exquisitely detailed data on the shape of nuclei. Figure 3.11 in Krane depicts some very detailed data that shows the departure from the classical Rutherford scattering cross section, as the projectile's energy, α-particles in this case, is increased. The classical interpretation is that the projectile is penetrating the nucleus. The Quantum Mechanical picture is that the projectile's wave function has a wave number small enough to start resolving the finite size of the nucleus. We now examine another way that experiments can yield information about the nuclear shape.

10.1.3 Nuclear size from spectroscopy measurements

Nuclear and atomic spectroscopy, the technique of measuring the energies of nuclear and atomic transitions, is one of the most precise measurements in nuclear science. If that is the case, then spectroscopy ought to be able to measure differences in transition energies that arise from the finite nuclear size.

Assume, for the sake of argument, that the nucleus is a sphere of radius R_N. An ideal probe of the effect of a finite-sized nucleus *vs.* a point-nucleus (as in the Schrödinger atomic model), would be a $1s$ atomic state, since, of all the atomic electron wavefunctions, the $1s$ state has the most probability density in the vicinity of the nucleus.

The shift of energy of the $1s$ can be estimated as follows:

$$\Delta E_{1s} = \langle \psi_{1s} | V_\circ(r) - V_\cdot(r) | \psi_{1s} \rangle \ , \tag{10.17}$$

where the ψ_{1s} is the $1s$ wavefunction for the point-like nucleus, $V_\circ(r)$ is the Coulomb potential for the finite nucleus, and $V_\cdot(r)$ is the point-like Coulomb potential. This way of estimating energy shifts comes formally from "1st-order perturbation theory", where it is assumed that the difference in potential has only a small effect on the wavefunctions. For a uniform sphere of charge, we know from Classical Electrostatics, that $V_\circ(r) = V_\cdot(r)$ for $r \geq R_N$.

$$\begin{aligned} V_\circ(r \leq R_N) &= -\frac{Ze^2}{4\pi\epsilon_0 R_N}\left[\frac{3}{2} - \frac{1}{2}\left(\frac{r}{R_N}\right)^2\right] \\ V_\circ(r \geq R_N) &\equiv V_\cdot(r) = -\frac{Ze^2}{4\pi\epsilon_0 r} \end{aligned} \tag{10.18}$$

We evaluate this by combining (10.18) with (10.17) and using the hydrogenic wavefunctions given in NERS311 and also in Krane II (Tables 2.2 and 2.5), and obtain:

$$\Delta E_{1s} = \frac{Ze^2}{4\pi\epsilon_0 R_N}\frac{4Z^3}{a_0^3}\int_0^{R_N} \mathrm{d}r r^2 e^{-2Zr/a_0}\left[\frac{R_N}{r} - \frac{3}{2} + \frac{1}{2}\left(\frac{r}{R_N}\right)^2\right] \ . \tag{10.19}$$

In unitless quantities, we may rewrite the above as:

$$\Delta E_{1s} = Z^2\alpha^2(m_ec^2)\left(\frac{2ZR_N}{a_0}\right)^2 \int_0^1 \mathrm{d}z e^{-(2ZR_N/a_0)z}\left[z - \frac{3}{2}z^2 + \frac{z^4}{2}\right] . \tag{10.20}$$

Across all the elements, the dimensionless parameter $(2ZR_N/a_0)$ spans the range $2\times10^{-5} \longrightarrow\approx 10^{-2}$. Hence, the contribution to the exponential, in the integral, is inconsequential. The remaining integral is a pure number and evaluates to 1/10. Thus, we may write:

$$\Delta E_{1s} \approx \frac{1}{10} Z^2\alpha^2(m_ec^2)\left(\frac{2ZR_N}{a_0}\right)^2 . \tag{10.21}$$

This correction is about 1 eV for $Z = 100$ and much smaller for lighter nuclei.

Nuclear size determination from an isotope shift measurement

Let us imagine how we are to determine the nuclear size, by measuring the energy of the photon that is given off, from a $2p \rightarrow 1s$ transition.

The Schrödinger equation predicts that the energy of the photon will be given by:

$$(E_{2p\rightarrow 1s})_\circ = (E_{2p\rightarrow 1s})_\cdot + \langle\psi_{2p}|V_\circ(r) - V_\cdot(r)|\psi_{2p}\rangle - \langle\psi_{1s}|V_\circ(r) - V_\cdot(r)|\psi_{1s}\rangle , \tag{10.22}$$

or,

$$(\Delta E_{2p\rightarrow 1s})_\circ = \langle\psi_{2p}|V_\circ(r) - V_\cdot(r)|\psi_{2p}\rangle - \langle\psi_{1s}|V_\circ(r) - V_\cdot(r)|\psi_{1s}\rangle , \tag{10.23}$$

expressing the change in the energy of the photon, due to the effect of finite nuclear size.

The latter term, $\langle\psi_{1s}|V_\circ(r) - V_\cdot(r)|\psi_{1s}\rangle$, has been calculated in (10.21). We now consider the former term, $\langle\psi_{2p}|V_\circ(r) - V_\cdot(r)|\psi_{2p}\rangle$. Figure 10.7 shows the $1s$ and $2p$ hydrogenic radial probabilities for the $1s$ and $2p$ states, each divided by their respective maxima. (This corresponds to having divided the $2p$ function by a factor of about 89.) The vertical line near the origin is the radius of an $A = 208$ nucleus, assuming $R_N = 1.22A^{1/3}$. That radius has been multiplied by a factor of 10 for display purposes. The actual value is $ZR_N/a_0 = 0.0112$, assuming further, that $Z = 82$.

As can be seen from this figure, the overlap of the $2p$ state is many orders of magnitude smaller than that of the $1s$ state. Hence, the term $\langle\psi_{2p}|V_\circ(r) - V_\cdot(r)|\psi_{2p}\rangle$ may be safely ignored in (10.23). Therefore, we can conclude, from (10.21), that the photon's energy is reduced by,

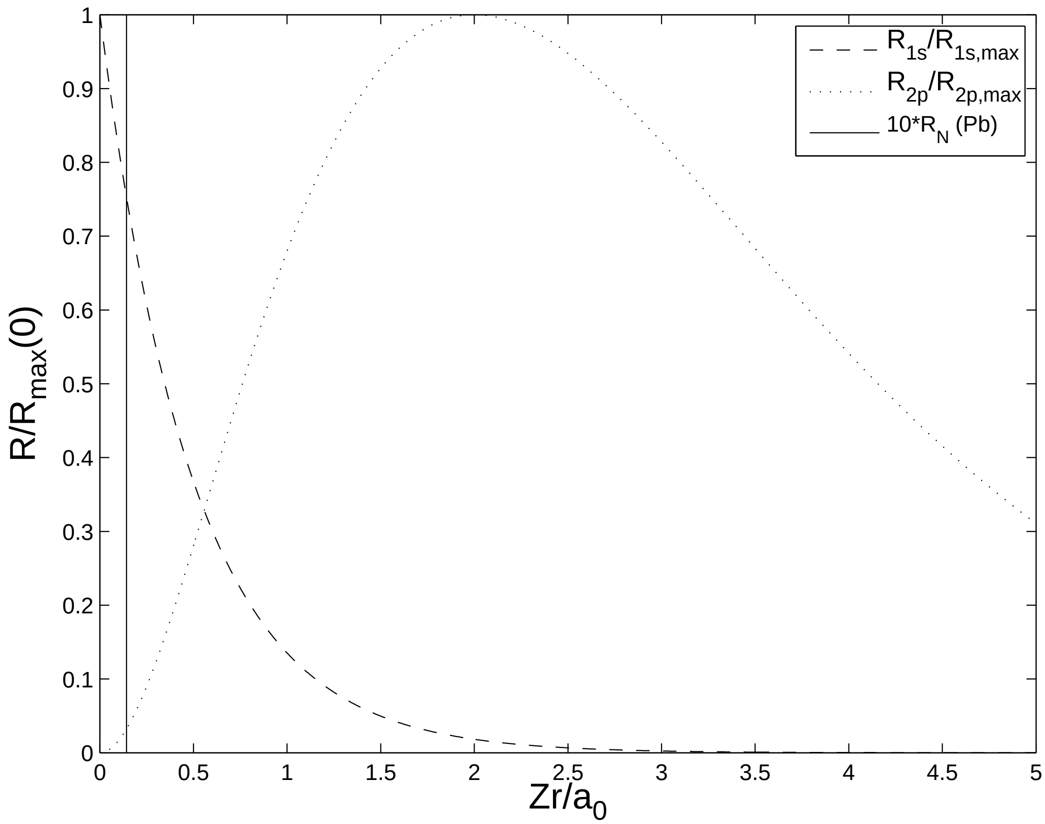


Figure 10.7: Overlap of $1s$ and $2p$ electronic orbitals with the nuclear radius. The nuclear radius depicted is for $A = 208$ and has been scaled upward by 10 for display purposes.

$$\Delta E_{2p\to 1s} \approx -\frac{1}{10} Z^2\alpha^2(m_e c^2)A^{2/3}\left(\frac{2ZR_0}{a_0}\right)^2 \ , \tag{10.24}$$

for a uniformly charged nucleus with radius $R_N = R_0 A^{1/3}$.

However, we have yet to make the connection to a measurement, because the measurement of a photon's energy from a realistically shaped nucleus can not be compared with that of an identical atom with a point nucleus. That does not exist in nature. Instead, consider the following: the transition energy for two isotopes of the same element, A and A'. The difference in this transition energy may be determined experimentally, and we obtain:

$$\Delta E_{2p\to 1s}(A) - \Delta E_{2p\to 1s}(A') = \frac{1}{10} Z^2\alpha^2(m_e c^2)\left(\frac{2ZR_0}{a_0}\right)^2 (A'^{2/3} - A^{2/3}) \ . \tag{10.25}$$

The measured quantity is called the K X-ray *isotope shift*. The following few pages show measurements of isotope shifts, for K X-Rays and optical photon isotope shifts.

Figure 10.8: Fig 3.6 from Krane, K X-ray shifts for Hg.

Figure 10.9: Fig 3.7 from Krane, optical shifts for Hg.

A better probe of nuclear shape can be done by forming muonic atoms, formed from muons (usually from cosmic rays), that replace an inner K-shell electron, and has significant overlap of its wavefunction with the nucleus.

Figure 10.10: Fig 3.8 from Krane, K X-ray shifts for muonic Fe.

All these data are consistent with a nuclear size with a radius, $R_N = R_0 A^{/3}$, and a value for $R_0 \approx 1.2$ fm.

Charge radius from Coulomb energy in mirror nuclei

A mirror-pair of nuclei are two nuclei that have the same atomic mass, but the number of protons in one, is the number of neutrons in the other, and the number of protons and neutrons in one of the nuclei differs by only 1. So, if Z is the atomic number of the higher atomic number mirror nucleus, it has $Z-1$ neutrons. Its mirror pair has $Z-1$ protons and Z neutrons. The atomic mass of both is $2Z-1$. Examples of mirror pairs are: ^{3}H/^{3}He, and ^{39}Ca/^{39}K.

These mirror-pairs are excellent laboratories for investigating nuclear radius since the nuclear component of the binding energy of these nuclei ought to be the same, if the strong force does not distinguish between nucleons. The only remaining difference is the Coulomb self-energy. For a charge distribution with Z protons, the Coulomb self-energy is:

$$E_{\rm C} = \frac{1}{2}\frac{Z^2 e^2}{4\pi\epsilon_0}\int {\rm d}\vec{x}_1 \rho_p(\vec{x}_1)\int {\rm d}\vec{x}_2 \rho_p(\vec{x}_2)\frac{1}{|\vec{x}_1 - \vec{x}_2|} \ . \tag{10.26}$$

The factor of 1/2 in front of (10.26) accounts for the double counting of repulsion that takes place when one integrates over the nucleus twice, as implied in (10.26).

For a uniform, spherical charge distribution of the form,

$$\rho_p(\vec{x}) = \frac{3}{4\pi R_N^3}\Theta(R_N - r) \ . \tag{10.27}$$

Figure 10.11: Fig 3.9 from Krane, composite K X-ray shift data.

As shown below:

$$E_{\rm C} = \frac{3}{5}\frac{Z^2e^2}{4\pi\epsilon_0 R_N} \ . \tag{10.28}$$

For a uniform, spherical charge distribution, given by (10.27):

$$\begin{aligned}
E_{\rm C} &= \frac{1}{2}\frac{Z^2e^2}{4\pi\epsilon_0}\left(\frac{3}{4\pi R_N^3}\right)^2 \int_{|\vec{x}_1|\le R_N} {\rm d}\vec{x}_1 \int_{|\vec{x}_2|\le R_N} {\rm d}\vec{x}_2 \frac{1}{|\vec{x}_1-\vec{x}_2|} \\
&= \frac{1}{2}\frac{Z^2e^2}{4\pi\epsilon_0 R_N}\left(\frac{3}{4\pi}\right)^2 \int_{|\vec{u}_1|\le 1} {\rm d}\vec{u}_1 \int_{|\vec{u}_2|\le 1} {\rm d}\vec{u}_2 \frac{1}{|\vec{u}_1-\vec{u}_2|} \\
&= \frac{1}{2}\frac{Z^2e^2}{4\pi\epsilon_0 R_N} I \ ,
\end{aligned} \tag{10.29}$$

where

$$I = \left(\frac{3}{4\pi}\right)^2 \int_{|\vec{u}_1|\le 1} {\rm d}\vec{u}_1 \int_{|\vec{u}_2|\le 1} {\rm d}\vec{u}_2 \frac{1}{|\vec{u}_1-\vec{u}_2|} \ . \tag{10.30}$$

From (10.30), one sees that I has the interpretation as a pure number representing the average of $|\vec{u}_1-\vec{u}_2|^{-1}$, for two vectors, $\vec{u}_1$ and $\vec{u}_2$, integrated uniformly over the interior of a unit sphere. So, now it just remains, to calculate I. We'll work this out explicitly because the calculation is quite delicate. Features of this derivation are seen in several areas of Nuclear and Radiological Science.

Expanding the 3-dimensional integrals in (10.30) results in:

$$I = \left(\frac{3}{4\pi}\right)^2 \int_0^{2\pi} {\rm d}\phi_1 \int_0^{\pi} {\rm d}\theta_1 \sin\theta_1 \int_0^1 {\rm d}u_1 u_1^2 \int_0^{2\pi} {\rm d}\phi_2 \int_0^{\pi} {\rm d}\theta_2 \sin\theta_2 \int_0^1 {\rm d}u_2 u_2^2 \frac{1}{|\vec{u}_1-\vec{u}_2|} \ .$$

The following expression results from having done both azimuthal integrals, once having aligned the z-axis of the coordinate system with $\vec{u}_1$, when performing the 3 inner integrals. Then with the transformation $\cos\theta_1 \rightarrow \mu_1$ and $\cos\theta_2 \rightarrow \mu_2$, we obtain:

$$\begin{aligned}
I &= \left(\frac{9}{2}\right) \int_0^1 {\rm d}u_1 u_1^2 \int_0^1 {\rm d}u_2 u_2^2 \int_{-1}^1 {\rm d}\mu_2 \frac{1}{\sqrt{u_1^2+u_2^2-2u_1u_2\mu_2}} \\
&= \left(\frac{9}{2}\right) \int_0^1 {\rm d}u_1 u_1 \int_0^1 {\rm d}u_2 u_2 [(u_1+u_2) - |u_1-u_2|]
\end{aligned}$$

$$
\begin{aligned}
&= 9\int_0^1 \mathrm{d}u_1 u_1 \left[\int_0^{u_1} \mathrm{d}u_2 u_2^2 + u_1 \int_{u_1}^1 \mathrm{d}u_2 u_2\right] \\
&= 9\int_0^1 \mathrm{d}u_1 \left[\frac{u_1^2}{2} - \frac{u_1^4}{6}\right] \\
&= 9\left[\frac{1}{6} - \frac{1}{30}\right] \\
&= \frac{6}{5}\ . \qquad (10.31)
\end{aligned}
$$

A common error in performing the above integral results from ignoring the absolute value in the 2nd step. Recall that $\sqrt{a^2} = |a|$, not a.

Finally, combining (10.29) and (10.31) gives us the final result expressed in (10.28).

The Coulomb energy differences are measured through β-decay endpoint energies (more on this later in the course), which yield very good information on the nuclear radius. The difference in Coulomb energies is given by:

$$
\begin{aligned}
\Delta E_{\mathrm{C}} &= \frac{3}{5}\frac{e^2}{4\pi\epsilon_0 R_N}[Z^2 - (Z-1)^2] \\
&= \frac{3}{5}\frac{e^2}{4\pi\epsilon_0 R_N}(2Z-1) \\
&= \frac{3}{5}\frac{e^2}{4\pi\epsilon_0 R_0}A^{2/3}\ , \qquad (10.32)
\end{aligned}
$$

where, in the last step, we let $R_N = R_0 A^{1/3}$. (Recall, $A = 2Z - 1$ for mirror nuclei.)

Figure 10.12: Fig 3.10 from Krane, Coulomb energy differences.

10.2 Mass and Abundance of Nuclei

Note to students: Read 3.2 in Krane on your own. You are responsible for this material, but it will not be covered in class.

10.3 Nuclear Binding Energy

In this section , we discuss several ways that the binding energy of the nucleus is tabulated in nuclear data tables. Nuclear binding energy is always related to the atomic mass, an experimentally derived quantity, one that is obtained with great precision through spectroscopy measurements, at least for nuclei that are stable enough. We start by discussing the binding energy of an atom, and then draw the analogy with the binding energy of the nucleus.

The rest mass energy of a neutral atom, $m_{_A}c^2$, and the rest mass energy of its nucleus, $m_{_N}c^2$, are related by:

$$m_{_A}c^2 = m_{_N}c^2 + Zm_ec^2 - B_e(Z, A) \ , \tag{10.33}$$

where $B_e(Z, A)$ is the *electronic* binding, the sum of the binding energies of all the electrons in the atomic cloud. The total electronic binding energy can be as large as 1 MeV in the heavier atoms in the periodic table. However, this energy is swamped by factors of 10^5–10^6 by the rest mass energy of the nucleus, approximately $A \times 1000$ MeV. Hence, the contribution of the electronic binding is often ignored, particularly when mass *differences* are discussed, as the electronic binding component largely cancels out. We shall keep this in mind, however.

One may estimate the total electronic binding as done in the following example.

Technical aside: Estimating the electronic binding in Pb:
Lead has the following electronic configuration:
$1s^22s^22p^63s^23p^63d^{10}4s^24p^64d^{10}5s^25p^64f^{14}5d^{10}6s^26p^2$,
or, occupancies of 2, 8, 18, 32, 18, 4 in the $n = 1, 2, 3, 4, 5, 6$ atomic shells. Thus,

$$B_e(82, 208) \approx (82)^2(13.6 \text{ eV})\left(2 + \frac{8}{2^2} + \frac{18}{3^2} + \frac{32}{4^2} + \frac{18}{5^2} + \frac{4}{6^2}\right) = 0.8076 \text{ MeV} \ .$$

This is certainly an overestimate, since electron repulsion in the atomic shells has not been accounted for. However, the above calculation gives us some idea of the magnitude of the total electronic binding. (A more refined calculculation gives 0.2074 MeV, indicating that the overestimate is as much as a factor of 4.) So, for the time being, we shall ignore the total electronic binding but keep it in mind, should the need arise.

By analogy, and more apropos for our purposes, we state the formula for the *nuclear binding energy*, $B_N(Z, A)$, for atom X, with atomic mass $m(^AX)$:

$$B_N(Z, A) = \left\{Zm_p + Nm_n - \left[m(^AX) - Zm_e\right]\right\}c^2 \ . \tag{10.34}$$

Since

$$m_p + m_e \approx m(^1H) \ ,$$

we may rewrite (10.34) as

$$B_N(Z,A) = [Zm(^1H) + Nm_n - m(^AX)]c^2 \ . \tag{10.35}$$

We emphasize, however, that electron binding energy is being ignored, henceforth[2].

Thus, we have obtained the binding energy of the nucleus in terms of the atomic mass of its neutral atom, $m(^AX)$. Conventionally, atomic masses are quoted in terms of the *atomic mass unit*, u. The conversion factor is $uc^2 = 931.494028(23)$ MeV.

Occasionally, it is the nuclear binding energy that is tabulated (it may be listed as *mass defect* or *mass excess*), in which case that data may be used to infer the atomic mass. A word of warning, however. Don't assume that the uses of *mass defect* or *mass excess* are consistent in the literature. One must always consult with the detailed descriptions of the data tables, to see the exact definition employed in that document.

Separation energies

Other measured data of interest that shine some light on the binding energy, as well as the nuclear structure of a given nucleus, is the neutron separation energy, S_n. That is the energy required to liberate a neutron from the nucleus, overcoming the strong attractive force. From the binding energy expressed in (10.35), we see that S_n takes the form:

$$\begin{aligned} S_n &= B_N\left({}^A_ZX_N\right) - B_N\left({}^{A-1}_{\ \ Z}X_{N-1}\right) \\ &= \left[m\left({}^{A-1}_{\ \ Z}X_{N-1}\right) - m\left({}^A_ZX_N\right) + m_n\right]c^2 \ . \end{aligned} \tag{10.36}$$

The proton separation energy is a similar quantity, except that it also accounts for the repulsion by the other protons in the nucleus.

$$\begin{aligned} S_p &= B_N\left({}^A_ZX_N\right) - B_N\left({}^{A-1}_{Z-1}X_N\right) \\ &= \left[m\left({}^{A-1}_{Z-1}X_N\right) - m\left({}^A_ZX_N\right) + m\left({}^1H\right)\right]c^2 \ . \end{aligned} \tag{10.37}$$

Thus we see from (10.35), that measurement of atomic mass yields direct information on the binding energy. We also see from (10.36) and (10.37), that measurements of neutron and proton separation energies yield direct information on the difference in nuclear binding energy between two nuclei that differ in A by one neutron or proton.

[2] To adapt these equations to account for electronic binding, (10.34) would take the form:

$$B_N(Z,A) = [Z(m_p + m_e) + Nm_{_N} - m(^AX)]c^2 - B_N(Z,A) \ .$$

There are 82 stable[3] elements. $^{209}_{83}$Bi, the most stable isotope of Bi, has a measured half-life of $(19 \pm 2) \times 10^{18}$ years (α-decay). Those 82 stable elements have 256 stable isotopes. Tin[4] has 10 stable isotopes ranging from ^{112}Sn–^{126}Sn. These stable isotopes, plus the more than 1000 unstable but usable nuclei (from the standpoint of living long enough to provide a direct measurement of mass), can have their binding energy characterized by a universal fitting function, the semiempirical formula for $B(Z,A) \equiv B_N(Z,A)$, a five-parameter empirical fit to the 1000+ set of data points. (The subscript N is dropped to distinguish B as the formula derived from data fitting.

Semiempirical Mass Formula – Binding Energy per Nucleon

The formula for $B(Z,A)$ is given conventionally as:

$$B(Z,A) = a_{\rm V} A - a_{\rm S} A^{2/3} - a_{\rm C} Z(Z-1)A^{-1/3} - a_{\rm sym}\frac{(A-2Z)^2}{A} + a_{\rm p}\frac{(-1)^Z[1+(-1)^A]}{2}A^{-3/4} \ . \tag{10.38}$$

The numerical values of the filling constants and the meaning of each term are given in the following table:

a_i	[MeV]	Description	Source
$a_{\rm V}$	15.5	Volume attraction	Liquid Drop Model
$a_{\rm S}$	16.8	Surface repulsion	Liquid Drop Model
$a_{\rm C}$	0.72	Coulomb repulsion	Liquid Drop Model + Electrostatics
$a_{\rm sym}$	23	n/p symmetry	Shell model
$a_{\rm p}$	34	n/n, p/p pairing	Shell model

Table 10.1: Fitting parameters for the nuclear binding energy

The explanation of each term follows:

Volume attraction: This term represents the attraction of a core nucleon to its surrounding neighbors. The nuclear force is short-medium range, therefore, beyond the immediate neighbors, there is no further attraction. Thus we expect this term to be attractive, and proportional to the number of nucleons. Add one nucleon to the core, and the binding energy

[3]Let us use, as a working definition, that "stability" means "no measurable decay rate".

[4]Tin's remarkable properties arise from the fact that it has a "magic" number of protons (50). This "magic" number represents a major closed proton shell, in the "shell model" of the nucleus, that we shall study soon. Tin's remarkable properties don't stop there! Tin has 28 known additional unstable isotopes, ranging from ^{99}Sn–^{137}Sn! It even has a "doubly-magic" isotope, ^{100}Sn, with a half-life of about 1 s, discovered in 1994. Tin is <u>the</u> superstar of the "Chart of the Nuclides". And you thought tin was just for canning soup!

goes up by the same amount, regardless of what A is. Another way to see this is: the "bulk term" is proportional to the volume of material, thus it is proportional to R_N^3, or A, since $R_N \propto A^{1/3}$. This comes from considering the nucleus to be formed of an incompressible fluid of mutually-attracting nucleons, *i.e.* the Liquid Drop Model of the nucleus.

Surface attraction: The volume term overestimates the attraction, because the nucleons at the surface lack some of the neighbors that attract the core nucleons. Since the surface is proportional to R_N^2, this term is proportional to $A^{2/3}$, and is repulsive.

Coulomb repulsion: The Coulomb repulsion is estimated from (10.28). This term is proportional to $1/R_N$, or $A^{-1/3}$. The Z^2 is replaced by $Z(Z-1)$ since a proton does not repulse itself. As discussed previously, this term is derived from Electrostatics, but within the Liquid Drop Model, in which the electrostatic charge is considered to be spread continuously through the drop.

n/p symmetry: The Nuclear Shell Model predicts that nuclei like to form with equal numbers of protons and neutrons. This is reflected by the per nucleon factor of $[(A-2Z)/A]^2$. This "repulsion" minimizes (vanishes) when $Z = N$.

n/n, p/p pairing: The Nuclear Shell Model also predicts that nuclei prefer when protons or neutrons are paired up in $n-n$, $p-p$ pairs. This factor is attractive, for an even-even nucleus (both Z and N are even), repulsive for an odd-odd nucleus, and zero otherwise. The $A^{-3/4}$ term is not easy to explain, and different factors are seen in the literature.

For a graphical representation of B/A, see figure 3.17 in Krane.

Using the expression (10.38) and adapting (10.35), we obtain the *semiempirical mass formula:*

$$m(^AX) = Zm(^1H) + Nm_n - B(Z,A)/c^2 \ , \tag{10.39}$$

that one may use to estimate $m(^AX)$ from measured values of the binding energy, or vice-versa.

Application to β-decay

β-decay occurs when a proton or a neutron in a nucleus converts to the other form of nucleon, $n \to p$, or $p \to n$. (An unbound neutron will also β-decay.) This process preserves A. Therefore, one may characterize β-decay as an isobaric (*i.e.* same A) transition. For fixed A, (10.39) represents a parabola in Z, with the minimum occuring at (Note: There is a small error in Krane's formula below.):

$$Z_{\rm min} = \frac{[m_n - m(^1{\rm H})]c^2 + a_{\rm C}A^{-1/3} + 4a_{\rm sym}}{2a_{\rm C}A^{-1/3} + 8a_{\rm sym}A^{-1}} \ . \tag{10.40}$$

We have to use some caution when using this formula. When A is odd, there is no ambiguity. However, when the decaying nucleus is odd-odd, the transition picks up an additional loss in mass of $2a_{\rm p}A^{-3/4}$, because an odd-odd nucleus becomes an even-even one. Similarly, when an even-even nucleus decays to an odd-odd nucleus, it picks up a gain of $2a_{\rm p}A^{-3/4}$ in mass, that must be more than compensated for, by the energetics of the β-decay.

Figure 3.18 in Krane illustrates this for two different decays chains.

(10.40) can very nearly be approximated by:

$$Z_{\rm min} \approx \frac{A}{2}\frac{1}{1+(1/4)(a_{\rm C}/a_{\rm sym})A^{2/3}} \ . \tag{10.41}$$

This shows clearly the tendency for $Z \approx N$ for lighter nuclei. For heavier nuclei, $A \approx 0.41$.

Binding Energy per Nucleon

The binding energy per nucleon data is shown in Krane's Figure 3.16 and the parametric fit shown in Krane's Figure 3.17. There are interesting things to note. $B(Z,A)/A$ ...

- peaks at about $A = 56$ (Fe). Iron and nickel (the iron core of the earth) are natural endpoints of the fusion process.
- is about 8 MeV $\pm$ 10 % for $A > 10$.

10.4 Angular Momentum and Parity

The total angular momentum of a nucleus is formed from the sum of the individual constituents angular momentum, $\vec{l}$, and spin, $\vec{s}$, angular momentum. The symbol given to the nuclear angular momentum is I. Thus,

$$\vec{I} = \sum_{i=1}^{A}(\vec{l}_i + \vec{s}_i) \ . \tag{10.42}$$

These angular momenta add in the Quantum Mechanical sense. That is:

$$\begin{aligned} \langle \vec{I}^2 \rangle &= \hbar^2 I(I+1) \\ I &= 0, \frac{1}{2}, 1, \frac{3}{2} \cdots \end{aligned}$$

$$
\begin{aligned}
\langle I_z \rangle &= \hbar m_{_I} \\
m_{_I} &: -I \leq m_{_I} \leq I \\
\Delta m_{_I} &: \text{integral}
\end{aligned}
\tag{10.43}
$$

Since neutron and proton spins are half-integral, and orbital angular momentum is integral, it follows that I is half-integral for odd-A nuclei, and integral for even-A nuclei.

Recall that parity is associated with a quantum number of ± 1, that is associated with the inversion of space. That is, if Π is the parity operator, acting on the composite nuclear wave function, $\Psi(\vec{x}; A, Z)$,

$$
\Pi\Psi(\vec{x}; A, Z) = \pm\Psi(-\vec{x}; A, Z) \ . \tag{10.44}
$$

The plus sign is associated with "even parity" and the minus sign with "odd parity".

Total spin and parity are measurable, and a nucleus is said to be in an I^π configuration. For example, ^{235}U has $I^\pi = \frac{7}{2}^-$, while ^{238}U has $I^\pi = 0^+$.

10.5 Nuclear Magnetic and Electric Moments

10.5.1 Magnetic Dipole Moments of Nucleons

We have learned from atomic physics, that the magnetic fields generated by moving charges, has a small but measurable effect on the energy levels of bound electrons in an atom. For example, the apparent motion of the nucleus about the electron (in the frame where the electron is at rest), leads to "fine structure" changes in atomic spectra. This arises because the nucleus can be thought of as a closed current loop, generating its own magnetic field, and that magnetic field exerts a torque on the spinning electron. Although the electron is a "point particle", that point charge is spinning, generating its own magnetic field. We know that two magnets exert torques on each other, attempting to anti-align the magnetic poles.

The nucleus itself, is made up of protons and neutrons that have intrinsic spin as well, generating their own "spin" magnetic fields, in addition to the orbital one. That provides an additional torque on the electron spins, resulting in the "hyperfine structure" of atomic energy levels.

"Superhyperfine structure" results from additional torques on the electron resulting from neighboring atoms in condensed materials, yet another set of forces on the electron.

These energy differences are small, but, nonetheless important, for interpreting atomic spectra. However, we are now concerned with nucleons, in a tightly-bound nucleus, all in close proximity to each other, all moving with velocities of about $0.001 \rightarrow 0.1$c. This is a radical departure from the leisurely orbit of an electron about a nucleus. This is a "mosh pit" of thrashing, slamming nucleons. The forces between them are considerable, and play a vital role in the determination of nuclear structure.

The orbital angular momentum can be characterized in classical electrodynamics in terms of a magnetic moment, $\vec{\mu}$:

$$\vec{\mu} = \frac{1}{2}\int \mathrm{d}\vec{x}\; \vec{x} \times J(\vec{x}) \;, \tag{10.45}$$

where $J(\vec{x})$ is the *current density*. For the purpose of determining the orbital angular momentum's contribution to the magnetic moment, the nucleons can be considered to be point-like particles. For point-like particles,

$$\mu = |\vec{\mu}| = g_l l \mu_N \;, \tag{10.46}$$

where l is the orbital angular momentum quantum number, g_l is the *g-factor or gyromagnetic ratio* ($g_l = 1$ for protons, $g_l = 0$ for neutrons, since the neutrons are neutral), and the nuclear magnetron, μ_N is:

$$\mu_N = \frac{e\hbar}{2m_p} \;, \tag{10.47}$$

defined in terms of the single charge of the proton, e, and its mass, m_p. Its current measured value is $\mu_N = 5.05078324(13) \times 10^{-27}$ J/T.

Intrinsic spins of the nucleons also result in magnetic moments. These are given by:

$$\mu = g_s s \mu_N \;, \tag{10.48}$$

where the *spin g factors* are known to be, for the electron, proton, neutron and muon:

Type	g_s (measured)	g_s (theory)
e	-2.002319043622(15)	agree!
p	5.585694713(90)	?
n	-3.82608545(46)	?
μ	-2.0023318414(12)	2.0023318361(10)

A simple(!) application of Dirac's *Relativistic Quantum Mechanics* and *Quantum Electrodynamics* (*aka* QED) leads to the prediction, $g_s = 2$ for the electron. The extra part comes from the *zitterbewegung* of the electron[5]. The fantastic agreement of g_s for the electron, between measurement and theory, 12 decimal places, is considered to be the most remarkable achievement of theoretical physics, and makes QED the most verified theory in existence.

I'm not aware of any theory for the determination of the nucleon g-factors. However, the measured values allow us to reach an important conclusion: The proton must be something very different from a point charge (else its g_s would be close to 2), and the neutron must be made up of internal charged constituents (else its g_s would be 0). These observations laid the groundwork for further investigation that ultimately led to the discovery (albeit indirectly), that neutrons and protons are made up of quarks. (Free quarks have never been observed.) This led to the development of *Quantum Chromodynamics* (*aka* QCD), that describes the the strong force in fundamental, theoretical terms. The unification of QCD, QED, and the weak force (responsible for β-decay) is called *The Standard Model* of particle physics.

Measurement and Theory differ, however, for the muon's g_s. It has been suggested that there is physics beyond The Standard Model that accounts for this.

Measurements of magnetic moments of nuclei abound in the literature. These magnetic moments are composites of intrinsic spin as well as the orbital component of the protons. Nuclear models provide estimates of these moments, and measured moments yield important information on nuclear structure. Table 3.2 in Krane provides some examples. Further exploration awaits our later discussions on nuclear models.

10.5.2 Quadrupole Moments of Nuclei

The electric quadrupole moment is derived from the following considerations.

The electrostatic potential of the nucleus is given by:

$$V(\vec{x}) = \frac{Ze}{4\pi\epsilon_0} \int \mathrm{d}\vec{x}' \; \frac{\rho_p(\vec{x}')}{|\vec{x} - \vec{x}'|} \; . \tag{10.49}$$

Now, imagine that we are probing the nucleus from a considerable distance, so far away from it, that we can only just discern the merest details of its shape. Given that $\rho_p(\vec{x}')$ is highly localized in the vicinity of the nucleus and our probe is far removed from it, we may expand (10.49) in a Taylor expansion in $|\vec{x}'|/|\vec{x}|$. Thus we obtain:

$$V(\vec{x}) = \frac{Ze}{4\pi\epsilon_0} \left[\frac{1}{|\vec{x}|} \int \mathrm{d}\vec{x}' \; \rho_p(\vec{x}') + \frac{\vec{x}}{|\vec{x}|^3} \cdot \int \mathrm{d}\vec{x}' \; \vec{x}' \rho_p(\vec{x}') + \frac{1}{2|\vec{x}|^5} \int \mathrm{d}\vec{x}' \; \left(3(\vec{x}\cdot\vec{x}')^2 - |\vec{x}|^2|\vec{x}'|^2\right) \rho_p(\vec{x}') \cdots \right] . \tag{10.50}$$

[5]According to Wikipedia, the term *zitterbewegung* is derived from German, meaning "trembling motion". According to Zack Ford (NERS312-W10 student), the word is derived from "cittern movements", a "cittern" (or "citter") being an old (Renaissance-era) instrument very similar to a guitar. I like Zack's definition better.

This simplifies to:

$$V(\vec{x}) = \frac{Ze}{4\pi\epsilon_0}\left[\frac{1}{|\vec{x}|} + \frac{Q}{2|\vec{x}|^3}\cdots\right] , \tag{10.51}$$

where

$$Q = \int \mathrm{d}\vec{x}\ \left(3z^2 - r^2\right)\rho_p(\vec{x}) . \tag{10.52}$$

We have used $\int \mathrm{d}\vec{x}\ \rho_p(\vec{x}) \equiv 1$ for the first integral in (10.50). This is simply a statement of our conventional normalization of $\rho_p(\vec{x})$. We also used $\int \mathrm{d}\vec{x}\ \vec{x}\rho_p(\vec{x}) \equiv 0$ in the second integral in (10.50). This is made possible by choosing the "center of charge" as the origin of the coordinate system for the integral. Finally, the third integral resulting in (10.52), arises from the conventional choice, when there is no preferred direction in a problem, and set the direction of $\vec{x}'$ to align with the z'-axis, for mathematical convenience.

Technical note: *The second integral can be made to vanish through the choice of a center of charge. This definition is made possible because the charge is of one sign. Generally, when charges of both signs are involved in an electrostatic configuration, and their respective centers of charge are different, the result is a non-vanishing term known as the electric dipole moment. In this case, the dipole moment is given by:*

$$\vec{d} = \int \mathrm{d}\vec{x}\ \vec{x}\rho(\vec{x}) .$$

Finally, when it is not possible to choose the z-axis to be defined by the direction of $\vec{x}$, but instead, by other considerations, the quadrupole becomes a tensor, with the form:

$$Q_{ij} = \int \mathrm{d}\vec{x}\ \left(3x_ix_j - |\vec{x}|^2\right)\rho_p(\vec{x}) .$$

The quantum mechanics analog to (10.52) is:

$$Q = \int \mathrm{d}\vec{x}\ \psi_N^*(\vec{x})(3z^2 - r^2)\psi_N(\vec{x}) , \tag{10.53}$$

where $\psi_N(\vec{x})$ is the composite nuclear wave function. The electric quadrupole moment of the nucleus is also a physical quantity that can be measured, and predicted by nuclear model theories. See Krane's Table 3.3.

Closed book "exam-type" problems

1. **Nuclear Form Factor**

   The nuclear form factor, $F(\vec{q})$, is defined as follows:

$$F(\vec{q}) = \int \mathrm{d}\vec{x}\, \rho_p(\vec{x}) e^{i\vec{q}\cdot\vec{x}} \ .$$

   If $\rho_p(\vec{x})$ is the proton density, normalized so that:

$$\int \mathrm{d}\vec{x}\, \rho_p(\vec{x}) \equiv 1 \ ,$$

   show:

   (a)

$$F(0) = 1$$

   (b)

$$F(\vec{q}) = F(q) = \frac{4\pi}{q} \int r\mathrm{d}r\, \rho_p(r) \sin(qr) \ ,$$

   for spherically symmetric nuclei ($\rho_p(\vec{x}) = \rho_p(r)$).

   (c) Given that:

$$\rho_p(\vec{x}) = N\Theta(R_N - r) \ ,$$

   where R_N is the radius of the nucleus, find an expression for the normalization constant, N above, in terms of R_N.

   (d) For the proton distribution implied by (c), show:

$$F(q) = \frac{3}{(qR_N)^3}[\sin(qR_N) - qR_N \cos(qR_N)] \ .$$

   Hint: $\int \mathrm{d}x\, x \sin x = \sin x - x \cos x$.

   (e) From the expression for $F(q)$ in part (d) above, show that,

$$\lim_{q\to 0} F(q) = 1 \ ,$$

   using either a Taylor expansion, or l'Hôpital's Rule.

2. **Nuclear Coulomb Repulsion Energy**

   (a) The potential felt by a proton due to a charge distribution made up of the $Z-1$ other protons in the nucleus, is given by

   $$V(\vec{x}) = \frac{(Z-1)e^2}{4\pi\epsilon_0} \int \mathrm{d}\vec{x}' \, \frac{\rho_p(\vec{x}')}{|\vec{x}-\vec{x}'|} \ ,$$

   where

   $$\int \mathrm{d}\vec{x}' \, \rho(\vec{x}') \equiv 1 \ .$$

   With the assumption that the protons are uniformly distributed through the nucleus up to radius R_N, that is, $\rho(\vec{x}') = \rho_0$, for $0 \le |\vec{x}'| \le R_N$, show,

   $$V(r) = \frac{(Z-1)e^2}{4\pi\epsilon_0 R_N} \left[\frac{3}{2} - \frac{1}{2}\left(\frac{r}{R_N}\right)^2 \right] \ .$$

   From this expression, what can you tell about the force on a proton inside the nucleus?

   (b) The Coulomb self-energy of a charge distribution is given by

   $$E_c = \frac{e}{4\pi\epsilon_0}\left(\frac{1}{2}\right) \int \mathrm{d}\vec{x} \int \mathrm{d}\vec{x}' \, \frac{\rho(\vec{x})\rho(\vec{x}')}{|\vec{x}-\vec{x}'|} \ ,$$

   where

   $$Ze = \int \mathrm{d}\vec{x}' \, \rho(\vec{x}') \ .$$

   Justify the factor of (1/2) in the above expression?

   (c) Starting with the result of part a) or part b) of this problem, show, for a uniform charge distribution $\rho(\vec{x}') = \rho_0$, for $0 \le |\vec{x}'| \le R_N$, that,

   $$E_c = \frac{3}{5}\frac{Z(Z-1)e^2}{4\pi\epsilon_0}\frac{1}{R_N} \ .$$

3. **Nuclear Binding Energy**

   The semi-empirical formula for the total binding energy of the nucleus is:

   $$B(Z,A) = a_{\mathrm{v}}A - a_{\mathrm{s}}A^{2/3} - a_{\mathrm{c}}Z(Z-1)A^{-1/3} - a_{\mathrm{sym}}\frac{(A-2Z)^2}{A} + p(N,Z)a_{\mathrm{p}}A^{-3/4} \ ,$$

   where $p(N,Z)$ is 1 for even-N/even-Z, -1 for odd-N/odd-Z, and zero otherwise.

   (a) Identify the physical meaning of the 5 terms in $B(A,Z)$. Explain which terms would go up, or go down with the addition of one additional neutron or one additional proton.

(b) With A fixed, $B(Z,A)$ is a quadratic expression in Z. Find its minimum, and discuss.

(c) Show that $p_{\mathrm{N,Z}}$ can be written:

$$p(Z,N) = \frac{1}{2}[(-1)^Z + (-1)^N] \ ,$$

or equivalently,

$$p(Z,A) = \frac{1}{2}(-1)^Z[(-1)^A + 1] \ .$$

(d) The neutron separation energy is defined by:

$$S_{\mathrm{n}} = B(Z,A) - B(Z,A-1) \ ,$$

and the proton separation energy is defined by:

$$S_{\mathrm{p}} = B(Z,A) - B(Z-1,A-1) \ .$$

Using the large A approximation, namely:

$$(A-1)^n \approx A^n - nA^{n-1} \ ,$$

develop approximate expressions for S_{n} and S_{p}.

4. **Quadrupole Moment**

The quadrupole moment, Q, in the liquid drop model of the nucleus is defined by:

$$Q = \int \mathrm{d}\vec{x}\rho_p(\vec{x})(3z^2 - r^2) \ ,$$

where $\rho_p(\vec{x})$ is the charge density per unit volume, normalized in the following way:

$$\int \mathrm{d}\vec{x}\rho_(\vec{x}) \equiv 1 \ .$$

We consider an ellipsoidal nucleus with a sharp nuclear edge, with its surface being given by:

$$\frac{x^2+y^2}{a^2} + \frac{z^2}{b^2} = 1 \ ,$$

where the larger of a or b is the semimajor axis, and the smaller of the two, the semiminor axis.

(a) Sketch the shape of this nucleus, for both the prolate and oblate cases.

(b) Show that the volume of this nucleus is $V = (4\pi/3)a^2b$.

(c) Find an expression for Q, that involves only Z, a, and b.

(d) Discuss the cases $a > b$, $a < b$, $a = b$. Even if you have not found an expression for Q, you should be able to discuss this effectively, given its integral form above, and say something about the sign of Q depending on the relative size of a and b.

5. **Nuclear Structure and Binding Energy**

(a) **Theoretical foundations**

i. What is the liquid drop model of the nucleus?

ii. What is the shell model of the nucleus?

iii. How is Classical Electrostatics employed in describing the structure of the nucleus? Cite two examples.

(b) **Binding energy of the nucleus**

The semi-empirical formula for the total binding energy of the nucleus is:

$$B(Z,A) = a_{\rm V}A - a_{\rm S}A^{2/3} - a_{\rm C}Z(Z-1)A^{-1/3} - a_{\rm sym}\frac{(A-2Z)^2}{A} + a_{\rm p}\frac{(-1)^Z[1+(-1)^A]}{2}A^{-3/4} \ .$$

i. In the table below...

A. In the 3rd column, identify the nuclear model that gives rise to this term. The answer is started for you, in the 2nd row. Complete the remaining rows.

B. In the 4th column, identify how this term arises (qualitative explanation) from the nuclear model identified in the 3rd column. The answer is started for you, in the 2nd row. Complete the remaining rows.

C. In the 5thth column, indicate with a "yes", "no", or "maybe", if this term would cause $B(Z, A)$ to go up with the addition of one more proton.

D. In the 6thth column, indicate with a "yes", "no", or "maybe", if this term would cause $B(Z, A)$ to go up with the addition of one more neutron.

ii. Justify the exact dependence on Z and A for the $a_{\rm V}$, $a_{\rm S}$ and $a_{\rm C}$ terms. Qualitatively explain the dependence on Z and A for the $a_{\rm sym}$ term.

a_i	[MeV]	Theoretical origin	Description	$+p$?	$+n$?
$a_{\mathrm{V}}\cdots$	15.5	The theoretical model that suggests the term starting with a_{V} is the __________ model of the nucleus.	The term involving a_{V} comes from the idea that...		
$a_{\mathrm{S}}\cdots$	16.8				
$a_{\mathrm{C}}\cdots$	0.72				
$a_{\mathrm{sym}}\cdots$	23				
$a_{\mathrm{p}}\cdots$	34				

iii. With A fixed, $B(Z, A)$ is a quadratic expression in Z. Find its extremum, and discuss the relationship between the quadratic expression and β-decay.

6. **The Modeling of Protons in the Nucleus**

One approach to accounting for the effect of the electric charge of the protons, on the structure of the nucleus, is to combine the classical ideas of electrostatics with the liquid drop model of the nucleus. The approach starts with a calculation of the electrostatic potential of the nucleus, $V_{\mathrm{es}}(\vec{x})$, due to a distribution of protons, $\rho_p(\vec{x}')$.

$$V_{\mathrm{es}}(\vec{x}) = \frac{Ze}{4\pi\epsilon_0}\int \mathrm{d}\vec{x}' \; \frac{\rho_p(\vec{x}')}{|\vec{x}-\vec{x}'|} \quad ; \quad \int \mathrm{d}\,\vec{x}'\rho_p(\vec{x}') \equiv 1 \; .$$

$\vec{x}$ is a vector that can be positioned anywhere. $\vec{x}'$, the vector over which we integrate, is positioned only within the confines of the nucleus. The origin of the coordinate system for $\vec{x}'$, is located at the *center of charge*. In the following, $\hat{n}$ is a unit vector pointing in the direction of $\vec{x}$.

(a) Show, for $|\vec{x}| >> |\vec{x}'|$, that;

$$\begin{aligned} V_{\rm es}(\vec{x}) &= \frac{Ze}{4\pi\epsilon_0|\vec{x}|}\left[1 + \frac{\hat{n}\cdot\vec{P}}{|\vec{x}|} + \frac{Q}{2|\vec{x}|^2} + \mathcal{O}\left(\frac{1}{|\vec{x}'|^3}\right)\right] \quad \text{where} \\ \vec{P} &\equiv \hat{n}\cdot\int {\rm d}\vec{x}'\ \vec{x}'\rho_p(\vec{x}') \\ Q &\equiv \int {\rm d}\vec{x}'\ \left[3(\hat{n}\cdot\vec{x}')^2 - |\vec{x}'|^2\right]\rho_p(\vec{x}')\ . \end{aligned}$$

(b) Whether or not you derived the above expression, interpret the meaning of the 3 terms, $(1, \vec{P}, Q)$ in

$$\left[1 + \frac{\hat{n}\cdot\vec{P}}{|\vec{x}|} + \frac{Q}{2|\vec{x}|^2}\right]\ .$$

(c) Why does $\vec{P} = 0$ for the nucleus.

(d) If $\hat{n}$ is aligned with $\hat{z}$, show that:

$$Q = \int {\rm d}\vec{x}\ \rho_p(\vec{x})(3z^2 - r^2)\ .$$

(e) Show that $Q = 0$ if $\rho_p(\vec{x})$ is spherically symmetric, but otherwise arbitrary. That is, show:

$$Q = \int {\rm d}\vec{x}\ \rho_p(r)(3z^2 - r^2) = 0\ .$$

(f) The connection between the nuclear wavefunction, $\psi_N(\vec{x})$ and the proton density, in the liquid drop model of the nucleus is given by:

$$|\psi_N(\vec{x})|^2 = \rho_p(\vec{x})\ .$$

Justify this assumption.

(g) If the nucleus is a uniformly charged ellipsoid of the form:

$$\frac{x^2+y^2}{a^2} + \frac{z^2}{b^2} = 1\ ,$$

Show that

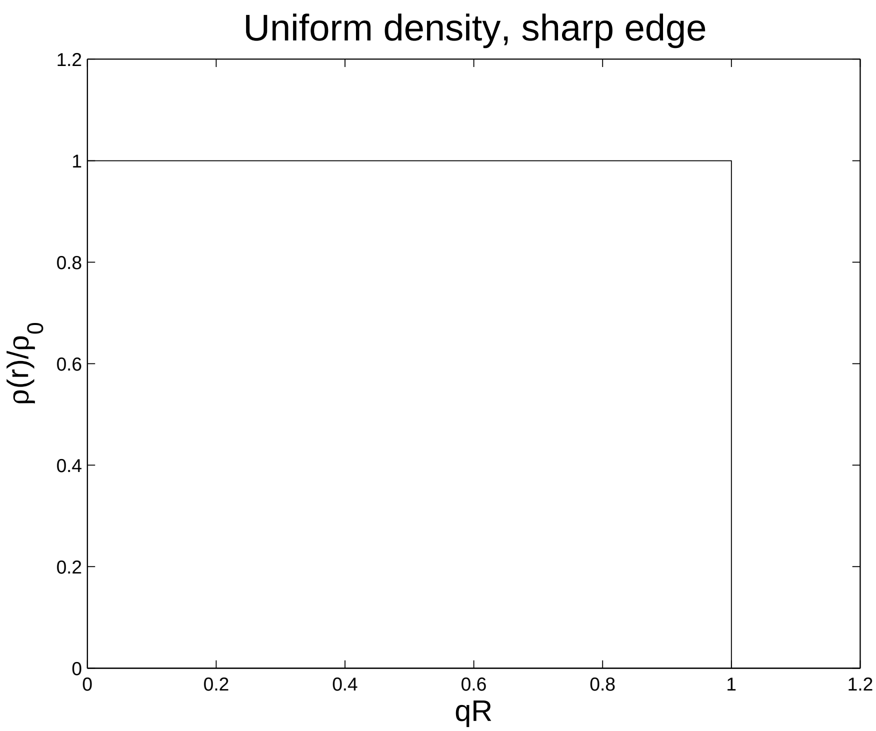
$$Q = \frac{2}{5}(b^2 - a^2)\ .$$

Assignment-type problems

1. If all the matter on earth collapsed to a sphere with the same density as the interior of a nucleus, what would the radius of the earth be? Cite all sources of data you used.

2. **The effect of the nuclear edge on $F(\vec{q})$**

   (a) Find $F(\vec{q})$ for a uniformly charged sphere. For a uniform sphere, $\rho_{\rm p}(r) = \rho_0\Theta(R-r)$, where R is the radius of the nucleus.[6]



   Plot[7] $\log(|F(\vec{q})|^2)$ *vs.* qR.

[6]Mathematical note:

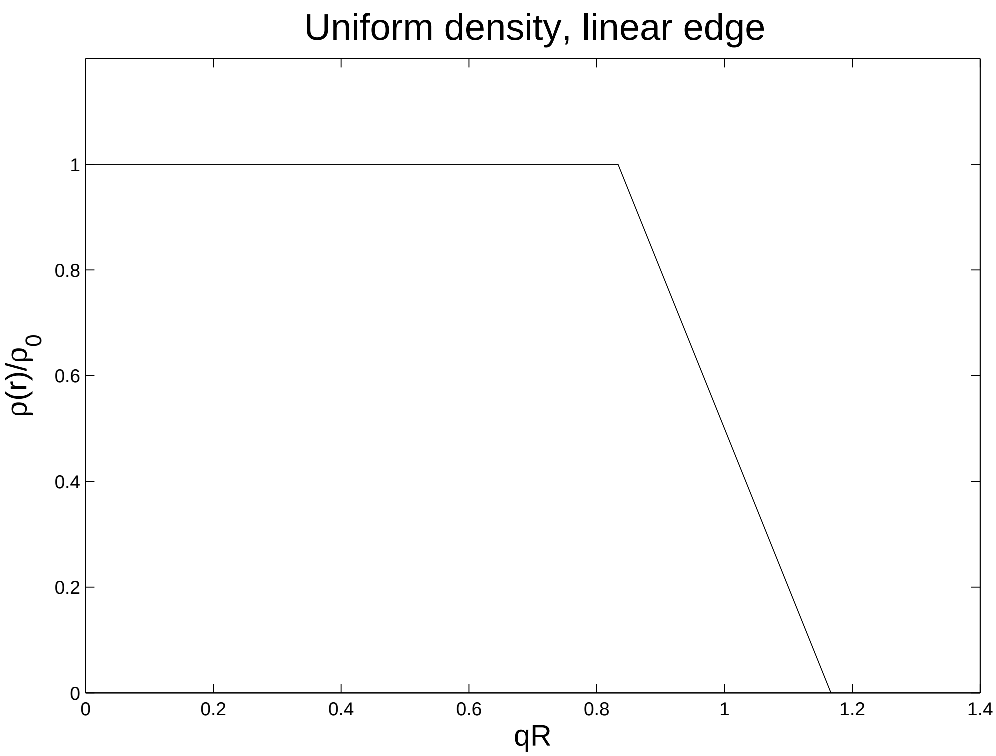
$$\begin{aligned}\Theta(z) &= 1\ ;\ z > 0\\ &= 0\ ;\ z < 0\end{aligned}$$

[7]Technical note: $|F(\vec{q})|)^2$ can be zero, so its logarithm would be $-\infty$, causing the plots to look strange. We need the logarithm to see the full structure of $|F(\vec{q})|^2$. So, in order to make the plots look reasonable, you will probably have to adjust the scale on the y-axis in an appropriate fashion.

(b) Do the same as in (a) but include a linear edge in the model of the nucleus. That is,

$$\begin{aligned}\rho_{\mathrm{p}}(r) &= \rho_0\Theta([R-t/2]-r) \\ &= \rho_0[1/2-(r-R)/t]\ ;\ R-t/2\leq r\leq R+t/2\ ,\end{aligned}$$

where t is the nuclear "skin depth", and, mathematically, can take any value between 0 and $2R$. Note that $\rho_{\mathrm{p}}(R-t/2)=\rho_0$ and $\rho_{\mathrm{p}}(R+t/2)=0$. Note also, that when $t=0$, the nuclear shape is the same as in part (a).



Plot $\log(|F(\vec{q})|^2)$ *vs.* qR as in part (a) showing several values of t over the entire domain of t, $0\leq t\leq 2R$. Compare with the result of part (a).

(c) Compare and discuss your results. Does the elimination of the sharp edge eliminate the sharp minima in $\log(|F(\vec{q})|^2)$? What else could be contributing to the reduction of the sharpness of these minima?

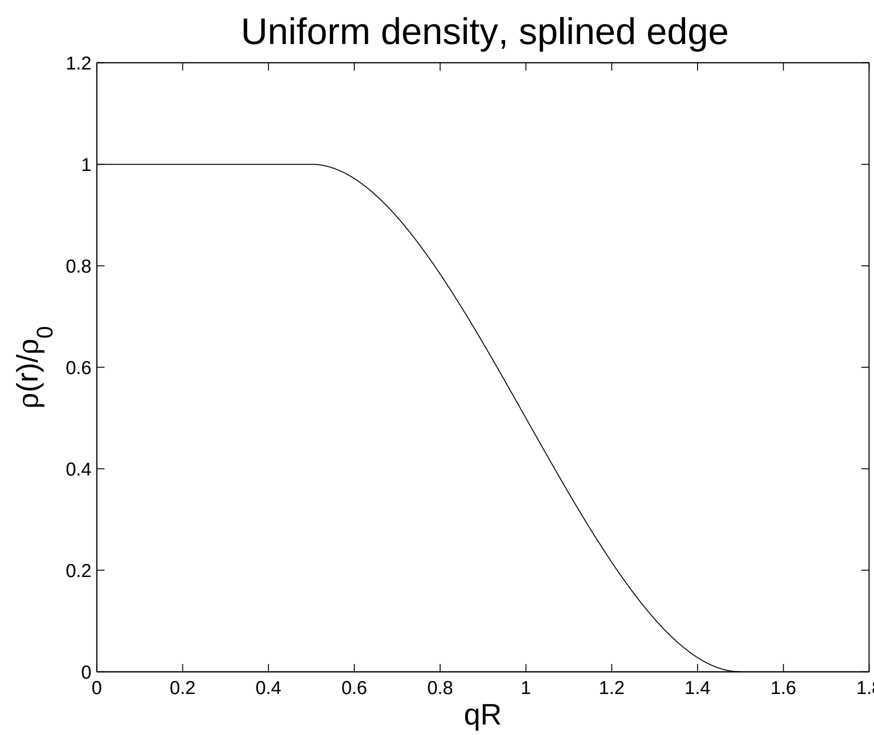
(d) Do the same as in (b) and (c) but include a cubic edge in the model of the nucleus. That is,

$$\begin{aligned}\rho_{\rm p}(r) &= \rho_0\Theta([R-t/2]-r) \\ &= \rho_0(A+Br+Cr^2+Dr^3)\ ;\ R-t/2\le r\le R+t/2\ .\end{aligned}$$

Arrange the four constants A, B, C, D such that:

$$\begin{aligned}\rho_{\rm p}(R-t/2) &= \rho_0 \\ \rho'_{\rm p}(R-t/2) &= 0 \\ \rho_{\rm p}(R+t/2) &= 0 \\ \rho'_{\rm p}(R+t/2) &= 0\end{aligned}$$

.



Plot $\log(|F(\vec{q})|^2)$ *vs.* qR as in part (b) showing several values of t over the entire domain of t, $0\le t\le 2R$. Compare with the result of parts (a) and (b).

3. **Perturbation of atomic energy levels**

   (a) Read carefully and understand the text on pages 49–55 in Krane on this topic. Krane makes the assertion that ΔE_{2p} can be ignored for atomic electron transitions. Verify this assertion by repeating the calculation for ΔE_{2p} and obtain a relationship for $\Delta E_{2p}/\Delta E_{1s}$ for atomic electrons and muons. Evaluate numerically for ^{12}C and ^{208}Pb.

   (b) The results in (a), for muons, is suspect, because the muon's wavefunctions have significant overlap with the physical location of the nucleus, and the shape of the edge of the nucleus may play a significant role. Repeat the analysis of part (a), but introduce a skin depth using one of the models in Question 2) or some other model of your choosing. What do you conclude?

4. Show:

$$\frac{6}{5} = \left(\frac{3}{4\pi}\right)^2 \int_{|\vec{u}|\le 1} \mathrm{d}\vec{u} \int_{|\vec{u}'|\le 1} \mathrm{d}\vec{u}' \left(\frac{1}{|\vec{u}-\vec{u}'|}\right) ,$$

   which is used to find the energy of assembly of the protons in a nucleus.

   Then, consider:

$$I(n) = \left(\frac{3}{4\pi}\right)^2 \int_{|\vec{u}|\le 1} \mathrm{d}\vec{u} \int_{|\vec{u}'|\le 1} \mathrm{d}\vec{u}' \left(\frac{1}{|\vec{u}-\vec{u}'|^n}\right) ,$$

   where $n \geq 0$, and is an integer. From this, make conclusions as to the fundamental forces in nature, and their forms as applied to classical and quantum physics.

5. **"Pop Quiz" Review questions on Quantum Mechanics**

   (a) In words, describe "The Compton interaction".

   (b) Derive the relationship between the scattered γ energy and its scattering angle in the Compton interaction.

   (c) What is the time dependent Schrödinger equation in 1D? 3D? What is the time independent Schrödinger equation in 1D? 3D? What is the main application of solutions to the time independent Schrödinger equation?

   (d) Describe and state the expressions (1D and 3D) for the i) probability density, and ii) the probability current density.

   (e) What is the Heisenberg Uncertainty Principle and what does it mean?

   (f) What is the Pauli Exclusion Principle and what does it mean?

   (g) For a particle of mass m with velocity $\vec{v}$, what is its i) momentum, ii) total energy, iii) kinetic energy in both non-relativistic and relativistic formalisms.

   (h) For a massless particle with momentum $\vec{p}$, what is its i) total energy, ii) kinetic energy.

(i) State the Conservation of Energy and the Conservation of Momentum equations for a 2-body interaction involving two masses, m_1 and m_2 with initial velocities $\vec{v}_1$ and $\vec{v}_2$. Perform this in both non-relativistic and relativistic formalisms.

6. We have seen that the calculation of the scattering rate due to electron scattering from a bare nucleus approaches the Rutherford scattering law in the limit of small q. Discuss the onset of the departure of this change, via the nuclear form factor, as q gets larger. Adopt a more realistic model for the distribution of charge in the nucleus and see if you can match the data for the experiments shown in Figures 3.1 and 3.2 in Krane.

7. For low q, the electrons impinging on a nucleus have the positive charge of the nucleus screened by the orbital electrons. Develop an approximate model for the screening of the nuclei by the orbital electrons and demonstrate the effect on the Rutherford scattering law. Show that the forward scattering amplitude is finite and calculate its numerical value using sensible numbers for the parameters of your model.

8. The numerical data of the average shift given in Figure 3.8 of Krane can be matched very closely by introducing a realistic positive charge distribution in the nucleus. Introduce such a model and demonstrate that you obtain the correct numerical answer.

9. Calculate the muonic K X-ray shift for Fe using one of the following nuclear shapes:

$$
\begin{aligned}
\frac{\rho(r)}{\rho_0} &= 1 \text{ for } r \leq R - t_{\mathrm{min}}/2 \\
&= \frac{R + t_{\mathrm{min}}/2 - r)}{t_{\mathrm{min}}} \text{ for } R - t_{\mathrm{min}}/2 \leq r \leq R + t_{\mathrm{min}}/2 \\
&= 0 \text{ for } r \geq R + t_{\mathrm{min}}/2
\end{aligned}
$$

or, the Fermi distribution,

$$
\frac{\rho(r)}{\rho_0} = \frac{1}{1 + \exp[(r - R)/a]} \ .
$$

Compare with data. You may use $R = R_0 A^{1/3}$ with any value between 1.20 and 1.25 fm to got the K X-rays in the right place. Use $t_{\mathrm{min}} = 2.3$ fm, and make sure to interpret a in the right way with respect to t_{min}, if you use the Fermi distribution.