POISSON-NEWTON FORMULAS AND DIRICHLET SERIES

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ABSTRACT. We prove that a Poisson-Newton formula, in a broad sense, is associated to each Dirichlet series with a meromorphic extension to the whole complex plane. These formulas simultaneously generalize the classical Poisson formula and Newton formulas for Newton sums. Classical Poisson formulas in Fourier analysis, classical summation formulas as Euler-McLaurin or Abel-Plana formulas, explicit formulas in number theory and Selberg trace formulas in Riemannian geometry appear as special cases of our general Poisson-Newton formula. We also associate to finite order meromorphic functions general Poisson-Newton formulas that yield many classical integral formulas.

We dedicate this article to Daniel Barsky and Pierre Cartier for their interest and constant support

1. INTRODUCTION

All classical Poisson formulas for functions in Fourier analysis result from the general distributional Poisson formula

$$\sum_{n \in \mathbb{Z}} e^{i \frac{2\pi}{\lambda} nt} = \lambda \sum_{k \in \mathbb{Z}} \delta_{\lambda k} \,, \tag{1}$$

which is an identity of distributions identifying an infinite sum of exponentials, converging in the sense of distributions, and a purely atomic distribution. This distributional formula is related to the simplest finite Dirichlet series

$$f(s) = 1 - e^{-\lambda s} \,.$$

It is interesting to observe that on the left hand side of (1) we have an exponential sum

$$W(f) = \sum_{\rho} e^{\rho t} \,,$$

where the sum runs over the zeros $\rho_n = \frac{2\pi i}{\lambda} n$, $n \in \mathbb{Z}$ of f, and on the right hand side of (1) we have a sum of atomic masses at the multiples of the fundamental frequency λ. One can say that the frequencies associated to the zeros are resonant at the fundamental frequencies. Taking the Fourier transform we obtain the dual Poisson formula that is of the same form where we exchange zeros and fundamental frequencies. Thus the fundamental frequencies are also resonant at the zeros.

The main purpose of this article is to show that this is general and to each meromorphic Dirichlet series f we can associate a distributional Poisson formula

$$(2) \qquad W(f) = \sum_{\rho} n_\rho e^{\rho t} = \sum_{\mathbf{k}} \langle \boldsymbol{\lambda}, \mathbf{k} \rangle b_{\mathbf{k}} \, \delta_{\langle \boldsymbol{\lambda}, \mathbf{k} \rangle} \,,$$

where the first sum of exponentials runs over the divisor of f, i.e., zeros and poles ρ with multiplicities n_ρ, and the second sum runs over non-zero sequences $\mathbf{k} = (k_1, k_2, \ldots) \in \mathbb{N}^\infty$ of non-negative integers, all of them zero but finitely many, and $\langle \boldsymbol{\lambda}, \mathbf{k} \rangle = \sum \lambda_j k_j$. The equality holds in $\mathbb{R}_+^*$. Conversely, we prove that any such Poisson formula comes from a Dirichlet series.

The distribution

$$W(f) = \sum_{\rho} n_\rho e^{\rho t}$$

is well defined in $\mathbb{R}_+^*$ and is called the *Newton-Cramer distribution* of f. We name it after Newton because it appears as a distributional interpolation of the Newton sums to exponents $t \in \mathbb{R}$, since in the complex variable[1] $z = e^s$ the zeros are the $\alpha = e^\rho$ so

$$W(f)(t) = \sum_{\alpha} \alpha^t \,,$$

and for integer values $t = m \in \mathbb{Z}$ we get (in case of convergence) the Newton sums

$$W(f)(m) = S_m = \sum_{\alpha} \alpha^m \,.$$

There is a precise theorem behind this observation. We show that our Poisson-Newton formula for a finite Dirichlet series f with a single fundamental frequency is strictly equivalent to the classical Newton relations. This is the reason why we name also after Newton our general Poisson formulas.

Writing $\rho = i\gamma$ we see that the sum $W(f)$ of the left hand side of (2) is the Fourier transform of the atomic Dirac distributions δ_γ and we can formally write

$$\sum_{\gamma} n_\rho \hat{\delta}_\gamma = \sum_{\mathbf{k}} \langle \boldsymbol{\lambda}, \mathbf{k} \rangle b_{\mathbf{k}} \, \delta_{\langle \boldsymbol{\lambda}, \mathbf{k} \rangle} \,.$$

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[1] The variable $z = e^s$ or better $z = e^{-s}$ is the proper variable when dealing with Dirichlet series.

The form of this formula, relating zeros to fundamental frequencies, strongly reminds other distributional formulas in other contexts. In number theory, more precisely in the theory of zeta and L-functions, the same type of identities do appear as "explicit formulas" associated to non-trivial zeros of the zeta and other L-functions. These *explicit formulas*, when written in distributional form, reduce to a single distributional relation that identifies a sum of exponentials associated to the divisor of the zeta or L-function and an atomic distribution associated to the location of prime numbers. Usually the sum runs over non-trivial zeros, and the sum over trivial zeros appears hidden in other forms as a *Weil functional*, which is classically interpreted as corresponding to the "infinite prime"[2]. For that reason, Delsarte labeled this formula as "Poisson formula with rest" (see [11]), the "rest" refers to the sum over the trivial part of the divisor. More precisely, for the Riemann zeta function, we have in $\mathbb{R}_+^*$

$$\sum_\rho n_\rho e^{\rho t} + W_0(f) = \sum_p \sum_{k \geq 1} \log p \, \delta_{k \log p} \ ,$$

where the sum on the left runs over the non-trivial (i.e., non-real) zeros ρ, and the sum over p runs over prime numbers. Conjecturally, the non-trivial zeros are simple, i.e., $n_\rho = 1$. The term $W_0(f)$ is the sum over the trivial (real) divisor and is computable

$$W_0(f)(t) = -e^t + \sum_{n \geq 1} e^{-2nt} = -e^t + \frac{1}{e^{2t} - 1} \ ,$$

and corresponds to Delsarte "rest", or to the Weil functional of the infinite prime. Also we have in this case

$$\sum_\rho n_\rho e^{\rho t} = e^{t/2} V(t) + e^{t/2} V(-t) \ ,$$

where

$$V(t) = \sum_{\Re \gamma > 0} e^{i\gamma t} \ ,$$

is the classical Cramer function, studied by H. Cramer [9], where $\rho = \frac{1}{2} + i\gamma$. This motivates that we name our distribution $W(f)$ also after Cramer.

In Riemannian geometry, we have the same structure for the Selberg trace formula for compact surfaces with constant negative curvature. With the relevant difference that Selberg zeta function is of order 2, which gives a "rest" of order 2 also. Selberg formula relates the length of primitive geodesics, which play the role of prime numbers, and the eigenvalues of the Laplacian, which give the zeros of the Selberg zeta function. For non-negative constant curvature, the formulas are of a different nature and the distribution on the right side are no longer simple atomic Diracs, but also higher order derivatives appear. This will be discussed elsewhere. In the context of

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[2]It may be more appropriate to talk of the prime $p = 1$.

dynamical systems and semiclassical quantization, we have Gutzwiller trace formula, which relates the structure of the periodic orbits of a classical mechanical system to the energy levels of the associated quantum system.

The interpretation and analogy of these formulas with "Poisson formulas" was noticed long time ago. We should mention in particular the classical work of A.P. Guinand [13], J. Delsarte [11], A. Weil [28], and results related to Hamburger theorem [15], [19]. Already the title of Delsarte's article points to the Poisson flavor of these formulas *"Formules de Poisson avec reste"*. More recently this analogy between Poisson and explicit formulas and its relation with zeta-regularization is studied for the Selberg trace formula for surfaces with constant negative curvature by P. Cartier and A. Voros in [7]. General Poisson formulas for Riemannian manifolds relating the spectrum of positive elliptic operators and the length spectrum were developed by J. Chazarain [10] and J.J. Duistermaat and V.W. Guillemin [12], and the Dirichlet series associated to the spectrum of the heat equation by S. Minakshisundaram and Å. Pleijel [21], after foundational work by T. Carleman [8].

Our goal is to put in the proper context, generalize and make precise the analogy of Poisson and trace formulas, and derive a general class of Poisson formulas that contain all such instances. More precisely, to each meromorphic Dirichlet series of finite order we associate a Poisson-Newton formula. All relevant known formulas can be generated in this way. On the other hand the fact that *explicit formulas* in number theory and Selberg trace formula can be seen as a generalization of Newton formulas, seems to be a new interpretation.

It is important to remark that in our general setting the Poisson-Newton formulas are independent from a possible functional equation for the Dirichlet series f, contrary to what happens in classical formulas. As a matter of fact, we do associate a Poisson-Newton formula to any Dirichlet series with no functional equation. This is sometimes hidden in the classical theory where explicit formulas and functional equations come hand to hand. For instance, most proofs in the literature derive the explicit formula for the zeta function using its functional equation, and the "rest" term, borrowing Delsarte terminology, is computed from the fudge factor from the functional equation. Nevertheless our approach shows that the functional equation is not related to the existence of an explicit formula. Moreover, for an f having a functional equation, the "rest" term in our Poisson-Newton formula emerges from the non-symmetric part of the divisor of f.

Although independent, these questions are interrelated. As is well known, it is a classical and basic procedure since Riemann foundational memoir [24] to derive functional equations for the Dirichlet series of f from Poisson formulas for other Dirichlet series (for the θ-function in the case of the Riemann zeta function).

We also derive a general Poisson-Newton formula associated to finite order mero-morphic functions $f(s)$ which are not necessarily Dirichlet series, but have their divisor contained in a left half plane. In this general situation, the Newton-Cramer distribution is still defined by the exponential series (converging in $\mathbb{R}_+$ as distribution)

$$W(f)(t) = \sum_\rho e^{\rho t}.$$

This time, the distribution is no longer a sum of purely atomic measures in $\mathbb{R}_+^*$. A particularly important case is when the divisor of f if left-oriented, i.e., contained in a left cone. Then the Newton-Cramer distribution is a θ-distribution and it is a distribution given by an analytic function in $\mathbb{R}_+^*$. An application of our general Poisson-Newton formula gives a collection of classical formulas: Gauss formula for the logarithmic derivative of the Γ-function, Binet formula for the logarithm of the Γ-function, general Gauss and Binet formulas for higher order Barnes Γ-functions, etc.

The structure at 0 of the distributions appearing in the Poisson-Newton formula is interesting. In the construction of Newton-Cramer distribution we have some param-eter freedom that is irrelevant for the structure of the distribution in $\mathbb{R}^*$, but not at 0. But precisely the variation of this parameter gives a distributional form of differ-ent infinite Euler-MacLaurin type formulas associated to each Dirichlet series. More precisely, the classical Euler-MacLaurin formula (as well as Abel-Plana summation formula) can be derived from the simplest case of the Dirichlet series $f(s) = 1 - e^{-s}$. The distributional infinite Euler-MacLaurin formula sheds some light on Ramanujan's theory of the "constant" of a diverging series. Most of the other summation formulas over the semi-group generated by the frequencies of the Dirichlet series seem new.

2. DIRICHLET SERIES

We consider a non-constant Dirichlet series

(3)
$$f(s) = 1 + \sum_{n \geq 1} a_n \, e^{-\lambda_n s},$$

with $a_n \in \mathbb{C}$ and

$$0 < \lambda_1 < \lambda_2 < \ldots$$

with (λ_n) a finite set (equivalently, take the sequence (a_n) with all but finitely many elements being zero) or $\lambda_n \to +\infty$, such that we have a half plane of absolute con-vergence (see [18] for background on Dirichlet series), i.e., for some $\bar\sigma \in \mathbb{R}$ we have

$$\sum_{n \geq 1} |a_n| \, e^{-\lambda_n \bar\sigma} < +\infty.$$

It is classical ([18], p.8) that

$$\bar{\sigma} = \limsup \frac{\log(|a_1| + |a_2| + \ldots + |a_n|)}{\lambda_n} \ .$$

The Dirichlet series (3) is therefore absolutely and uniformly convergent on right half-planes $\Re s \geq \sigma$, for any $\sigma > \bar{\sigma}$.

We assume that f has a meromorphic extension of finite order to all the complex plane $s \in \mathbb{C}$. We denote by (ρ) the set of zeros and poles of f, and the integer n_ρ is the multiplicity of ρ (positive for zeros and negative for poles, with the convention $n_\rho = 0$ if ρ is neither a zero nor pole). The convergence exponent of f is the minimum integer $d \geq 0$ such that

$$\sum_{\rho \neq 0} |n_\rho| \, |\rho|^{-d} < +\infty \ .$$

We have $d = 0$ if and only if f is a rational function, which cannot be a Dirichlet series, thus $d \geq 1$. Indeed we always have $d \geq 2$ (see Corollary 3.8). The order o of f satisfies $d \leq [o] + 1$.

Since f has finite order, we have the Hadamard factorization of f (see [2], p.208)

$$f(s) = s^{n_0} e^{Q_f(s)} \prod_{\rho \neq 0} E_m(s/\rho)^{n_\rho} \ ,$$

where $m = d - 1 \geq 0$ is minimal for the convergence of the product with

$$E_m(z) = (1 - z) e^{z + \frac{1}{2} z^2 + \ldots + \frac{1}{m} z^m} \ ,$$

and Q_f is a polynomial uniquely defined up to the addition of an integer multiple of $2\pi i$. The genus of f is defined as the integer

$$g = \min(\deg Q_f, m) \ ,$$

and in general we have $d \leq g+1$ and $g \leq o \leq g+1$ (see [2], p.209). For a meromorphic Dirichlet series we prove that in fact $d = g + 1$ (see Corollary 3.7).

The origin plays no particular role, thus we may prefer to use Hadamard product with origin at some $\sigma \in \mathbb{C}$,

$$f(s) = (s - \sigma)^{n_\sigma} e^{Q_{f,\sigma}(s)} \prod_{\rho \neq \sigma} E_m \left(\frac{s - \sigma}{\rho - \sigma} \right)^{n_\rho} \ .$$

We have, uniformly on $\Re s$,

$$\lim_{\Re s \to +\infty} f(s) = 1 \ ,$$

thus

$$\sigma_1 = \sup_\rho \Re \rho < +\infty \ ,$$

so $f(s)$ has neither zeros nor poles on the half plane $\Re s > \sigma_1$. Sometimes in the applications σ_1 is a pole of f because when the coefficients (a_n) are real and positive then f contains a singularity at $\bar{\sigma}$ by a classical theorem of Landau (see [18], Theorem 10, p.10). The singularity is necessarily a pole by our assumptions, and in general $\sigma_1 = \bar{\sigma}$.

Associated to the divisor $\mathrm{div}(f) = \sum n_\rho \rho$, we define a distribution $W(f) = \sum n_\rho e^{\rho t}$ on $\mathbb{R}_+^*$. We do this as follows.

Consider the space $\mathcal{S}$ of C^∞-functions of rapid decay on $\mathbb{R}$ (i.e., $\varphi \in \mathcal{S}$ if and only if for any $n, m > 0$, $|t|^n D^m \varphi \to 0$, as $t \to \pm\infty$). The dual space $\mathcal{S}'$ is the Schwartz space of tempered distributions. As $\mathcal{D} = C_0^\infty \subset \mathcal{S}$, we have that $\mathcal{S}' \subset \mathcal{D}'$, where $\mathcal{D}'$ is the space of distributions.

Lemma 2.1. *For finite sets A, consider the family of locally integrable functions*

$$\tilde{W}_A(f) = \left(\sum_{\rho \in A} e^{\rho t}\right) \mathbf{1}_{\mathbb{R}_+} .$$

There is a family of distributions $W_A(f)$ which coincides with $\tilde{W}_A(f)$ in $\mathbb{R}^*$, and which converges in $\mathbb{R}$ (over the filter of finite sets A), to a distribution $W(f)$ in $\mathcal{D}'$.

This distribution has support contained in $\mathbb{R}_+$, is Laplace transformable, and $e^{-\sigma_1 t} W(f) \in \mathcal{S}'$. More precisely, if σ_1 is not a zero nor pole (resp. it is a zero or pole), $e^{-\sigma_1 t} W(f)$ is the (distributional) d-th derivative of a uniformly bounded continuous function (resp. continuous function) on $\mathbb{R}$ with support in $\mathbb{R}_+$.

More precisely, we have

$$W(f) = e^{\sigma_1 t} \frac{D^d}{Dt^d} \left((K_d(t) - K_d(0)) \mathbf{1}_{\mathbb{R}_+} \right) ,$$

where

$$K_d(t) = \left(n_{\sigma_1} \frac{t^d}{d!} \right) \mathbf{1}_{\mathbb{R}_+} + \sum_{\rho \neq \sigma_1} \left(\frac{n_\rho}{(\rho - \sigma_1)^d} e^{(\rho - \sigma_1)t} \right) \mathbf{1}_{\mathbb{R}_+}.$$

Proof. We prove first the lemma when σ_1 is not a zero nor pole. We define

$$(4) \qquad K_\ell(t) = \sum_\rho \left(\frac{n_\rho}{(\rho - \sigma_1)^\ell} e^{(\rho - \sigma_1)t} \right) \mathbf{1}_{\mathbb{R}_+}.$$

Then for $\ell \geq d$, K_ℓ is absolutely convergent for $t \in \mathbb{R}_+$ since

$$\left| e^{(\rho - \sigma_1)t} \right| = e^{\Re(\rho - \sigma_1)t} \leq 1 ,$$

and K_ℓ is a uniformly bounded function in $\mathbb{R}$, continuous in $\mathbb{R}^*$, since

$$|K_\ell| \leq \sum_\rho \frac{|n_\rho|}{|\rho - \sigma_1|^\ell} < \infty \ .$$

The function

$$F_\ell(t) = (K_\ell(t) - K_\ell(0))\mathbf{1}_{\mathbb{R}_+}$$

is a uniformly bounded continuous function on $\mathbb{R}$, for $\ell \geq d$.

For a finite set A, denote by

$$K_{\ell,A}(t) = \sum_{\rho \in A} \left(\frac{n_\rho}{(\rho - \sigma_1)^\ell} e^{(\rho-\sigma_1)t} \right) \mathbf{1}_{\mathbb{R}_+}$$

the corresponding sum over $\rho \in A$, and $F_{\ell,A}(t) = (K_{\ell,A}(t) - K_{\ell,A}(0))\mathbf{1}_{\mathbb{R}_+}$. On $\mathbb{R}^*$,

$$\tilde{W}_A(f) = \left(\sum_{\rho \in A} n_\rho\, e^{\rho t} \right) \mathbf{1}_{\mathbb{R}_+} = e^{\sigma_1 t} \frac{d^d}{dt^d} F_{d,A}(t).$$

We consider

$$W_A(f) = e^{\sigma_1 t} \frac{D^d}{Dt^d} F_{d,A},$$

taking the distributional derivative.

For a smooth function (resp. function with polynomial growth) K on $\mathbb{R}$ and a test function φ with compact support (resp. in the Schwarz class) we have

$$\left\langle \frac{D}{Dt}(K\mathbf{1}_{\mathbb{R}_+}), \varphi \right\rangle = -\langle K\mathbf{1}_{\mathbb{R}_+}, \varphi' \rangle$$

$$= -\int_0^{+\infty} K(t)\varphi'(t)\, dt$$

$$= -[K(t)\varphi(t)]_0^{+\infty} + \int_0^{+\infty} K'(t)\varphi(t)\, dt$$

$$= K(0)\varphi(0) + \langle K'\mathbf{1}_{\mathbb{R}_+}, \varphi \rangle \ ,$$

thus

$$\frac{D}{Dt}(K\mathbf{1}_{\mathbb{R}_+}) = K'\mathbf{1}_{\mathbb{R}_+} + K(0)\delta_0 \ .$$

Then since $K'_{\ell,A} = K_{\ell-1,A}$ we get

$$\frac{D^d}{Dt^d} F_{d,A} = K_{0,A}(t) + K_{1,A}(0)\delta_0 + K_{2,A}(0)\delta'_0 + \ldots + K_{d-1,A}(0)\delta_0^{(d-2)}$$

$$= K_{0,A}(t) + \left(\sum_{\rho \in A} \frac{n_\rho}{\rho - \sigma_1} \right) \delta_0 + \left(\sum_{\rho \in A} \frac{n_\rho}{(\rho - \sigma_1)^2} \right) \delta'_0 + \ldots + \left(\sum_{\rho \in A} \frac{n_\rho}{(\rho - \sigma_1)^{d-1}} \right) \delta_0^{(d-2)} \ .$$

Thus the difference between $\tilde{W}_A(f)$ and $W_A(f)$ is a distribution supported at $\{0\}$.

We have the convergence $F_{d,A} \to F_d$, uniformly as continuous functions on $\mathbb{R}$. Thus we have the same limit $F_{d,A} \to F_d$ in the distributional sense. Then taking the limit as distributions, $W_A(f) \to W(f)$, where

$$W(f) = e^{\sigma_1 t} \frac{D^d}{Dt^d} F_d \ ,$$

which is the d-th derivative of a uniformly bounded continuous function on $\mathbb{R}$ with support on $\mathbb{R}_+$, as stated.

When σ_1 is part of the divisor, then we do the same proof with

$$K_\ell(t) = \left(n_{\sigma_1} \frac{t^\ell}{\ell!} \right) \mathbf{1}_{\mathbb{R}_+} + \sum_{\rho \neq \sigma_1} \left(\frac{n_\rho}{(\rho - \sigma_1)^\ell} e^{(\rho - \sigma_1)t} \right) \mathbf{1}_{\mathbb{R}_+},$$

which adds to $W(f)$ a term $n_{\sigma_1} e^{\sigma_1 t}$. $\qquad \square$

Note that we can write

$$W(f)|_{\mathbb{R}_+^*} = \lim_A \tilde{W}_A(f)|_{\mathbb{R}_+^*} = \sum_\rho n_\rho \, e^{\rho t} \ ,$$

as a distribution on $\mathbb{R}_+^*$. But if $d \geq 2$, the family of distributions $\tilde{W}_A(f)$ is not converging to a distribution in $\mathbb{R}$ because the sums

$$\sum_\rho \frac{n_\rho}{(\rho - \sigma_1)^\ell} \ ,$$

are not absolutely convergent for $\ell = 1, \ldots, d-1$ (by the definition of d). On the other hand, the same argument shows that for $\ell \geq d$, $K_{\ell,A}$ has a limit K_ℓ in the sense of distributions, and for any $k \geq 0$

$$\frac{D^k}{Dt^k} F_{\ell,A} \to \frac{D^k}{Dt^k} F_\ell \ .$$

It is important to note that $W(f)|_{\mathbb{R}_+^*}$ is independent of the choices of σ_1 and of taking d larger than the exponent of convergence. The only dependence on σ_1 and d is located at the structure of the distribution at 0.

Proposition 2.2. *Let d be the exponent of convergence of f as before, and let $d' \geq d$. We define for $\sigma \in \mathbb{C} - \{\rho\}$, $\ell \geq 0$, and a finite subset $A \subset \{\rho\}$*

$$K_{\ell,A}(t, \sigma) = \sum_{\rho \in A} \left(\frac{n_\rho}{(\rho - \sigma)^\ell} e^{(\rho - \sigma)t} \right) \mathbf{1}_{\mathbb{R}_+} \ ,$$

$$F_{\ell,A}(t, \sigma) = (K_{\ell,A}(t, \sigma) - K_{\ell,A}(0, \sigma)) \mathbf{1}_{\mathbb{R}_+} \ ,$$

then the following limits exist in the sense of distributions

$$\tilde{W}(f, d', \sigma) = \lim_A e^{\sigma t} \frac{D^{d'}}{Dt^{d'}} K_{d',A}(t, \sigma),$$

$$W(f, d', \sigma) = \lim_A e^{\sigma t} \frac{D^{d'}}{Dt^{d'}} F_{d',A}(t, \sigma),$$

and $W(f, d', \sigma)|_{\mathbb{R}^*} = \tilde{W}(f, d', \sigma)|_{\mathbb{R}^*} = W(f)|_{\mathbb{R}^*}$ *is independent of* σ *and* $d' \geq d$.

Proof. We shall deal with the first case, the second one is similar. The existence of the limit is proved as before. Take

$$\tilde{W}_A(f, d', \sigma) = e^{\sigma t} \frac{D^{d'}}{Dt^{d'}} K_{d',A}(t, \sigma) = e^{\sigma t} \frac{D^{d'}}{Dt^{d'}} \left(e^{(\sigma_1 - \sigma)t} K_{d',A}(t, \sigma_1) \right),$$

where σ_1 is the one considered before. Now, since $d' \geq d$, $K_{d',A}(t, \sigma_1)$ converges to a distribution $K_{d'}(t, \sigma_1)$. So $\tilde{W}_A(f, d', \sigma)$ converges as distribution to

$$\tilde{W}(f, d', \sigma) = e^{\sigma t} \frac{D^{d'}}{Dt^{d'}} \left(e^{(\sigma_1 - \sigma)t} K_{d'}(t, \sigma_1) \right) = e^{\sigma t} \frac{D^{d'}}{Dt^{d'}} K_{d'}(t, \sigma).$$

In $\mathbb{R}_-^*$ the independence on σ is clear since the distributions vanish. In $\mathbb{R}_+^*$, first we note that for $\ell \geq 0$,

$$\frac{\partial}{\partial t} K_{\ell, A} = K_{\ell-1, A}$$

and

$$\frac{D}{Dt}(K \mathbf{1}_{\mathbb{R}_+}) = K' \mathbf{1}_{\mathbb{R}_+} + K(0) \delta_0 .$$

So we have, in the sense of distributions:

$$\frac{D}{Dt} K_{\ell, A} = K_{\ell-1, A} + K_{\ell, A}(0, \sigma) \, \delta_0$$

$$\frac{D^\ell}{Dt^\ell}(K_{\ell, A}) = K_{0, A} + K_{1, A}(0, \sigma) \, \delta_0 + K_{2, A}(0, \sigma) \, \delta_0' + \ldots + K_{\ell, A}(0, \sigma) \, \delta_0^{(\ell-1)}$$

Using this, we get

$$(5) \qquad e^{\sigma t} \frac{D^{d'}}{Dt^{d'}} K_{d',A}(t, \sigma) = e^{\sigma t} K_{0, A}(t, \sigma) + e^{\sigma t} \sum_{\ell=1}^{d'} K_{\ell, A}(0, \sigma) \, \delta_0^{(\ell-1)} .$$

From this last expression we see that away from 0 the distributional limit, that we know to exist, is independent of σ and d', since $e^{\sigma t} K_{0, A}(t, \sigma) = \sum_{\rho \in A} e^{\rho t}$ is independent of σ, and the other summand is supported at zero. $\qquad \square$

In section 4 we study in more detail the parameter dependence at 0.

3. Poisson-Newton formula

On the half plane $\Re s > \sigma_1$, $\log f(s)$ is well defined taking the principal branch of the logarithm. Then we can define the coefficients $(b_\mathbf{k})$ by

$$(6) \qquad -\log f(s) = -\log\left(1 + \sum_{n \geq 1} a_n\, e^{-\lambda_n s}\right) = \sum_{\mathbf{k} \in \Lambda} b_\mathbf{k}\, e^{-\langle \boldsymbol{\lambda}, \mathbf{k} \rangle s}\;,$$

where $\Lambda = \{\mathbf{k} = (k_n)_{n \geq 1}\,|\,k_n \in \mathbb{N}, ||\mathbf{k}|| = \sum |k_n| < \infty, ||\mathbf{k}|| \geq 1\}$, and $\langle \boldsymbol{\lambda}, \mathbf{k} \rangle = \lambda_1 k_1 + \ldots + \lambda_l k_l$, where $k_n = 0$ for $n > l$. Note that the coefficients $(b_\mathbf{k})$ are polynomials on the (a_n). More precisely, we have

$$(7) \qquad b_\mathbf{k} = \frac{(-1)^{||\mathbf{k}||}}{||\mathbf{k}||}\, \frac{||\mathbf{k}||!}{\prod_j k_j!}\, \prod_j a_j^{k_j}\;.$$

Note that if the λ_n are $\mathbb{Q}$-dependent then there are repetitions in the exponents of (6).

3.1. Hadamard interpolation.

Lemma 3.1. *Consider a discrete set* $\{\rho\} \subset \mathbb{C}$ *with the property that*

$$\sum_{\rho \neq 0} |n_\rho|\,|\rho|^{-d} < +\infty\,.$$

Let $\sigma_1 \in \mathbb{C}$. *We have that*

$$G(s) = \frac{n_{\sigma_1}}{s - \sigma_1} - \sum_{\rho \neq \sigma_1} n_\rho \left(\frac{1}{\rho - s} - \sum_{l=0}^{d-2} \frac{(s - \sigma_1)^l}{(\rho - \sigma_1)^{l+1}}\right)$$

$$= \frac{n_{\sigma_1}}{s - \sigma_1} + \sum_{\rho \neq \sigma_1} n_\rho \frac{(s - \sigma_1)^{d-1}}{(\rho - \sigma_1)^{d-1}} \frac{1}{s - \rho}\;.$$

is a meromorphic function in $\mathbb{C}$, *and has a simple pole with residue* n_ρ *at each* ρ.

Proof. We start by noting that

$$\frac{1}{\rho - s} - \sum_{l=0}^{d-2} \frac{(s - \sigma_1)^l}{(\rho - \sigma_1)^{l+1}} = \frac{(s - \sigma_1)^{d-1}}{(\rho - \sigma_1)^{d-1}} \frac{1}{\rho - s}\;.$$

Consider a disk $D(0, R) \subset \mathbb{C}$, and split $G(s) = G_1(s) + G_2(s)$, where $G_1(s)$ corresponds to the sum of those $\rho \in D(0, R)$, and $G_2(s)$ to the sum over the remaining ρ's.

Now, for $s \in D(0, R/2)$, we have

$$(8) \qquad |G_2(s)| \leq C|s - \sigma_1|^d \sum_\rho |n_\rho| \, |\rho|^{-d} < \infty \ ,$$

thus we get the absolute and uniform convergence of the series in $D(0, R/2)$. As $G_1(s)$ has simple poles at ρ with residues n_ρ, in $D(0, R/2)$, we get the required properties for $G(s)$ in $D(0, R/2)$. This happens for every $R > 0$, thereby the result.

$\square$

Now we can define the Hadamard interpolation associated to the divisor $\sum n_\rho \, \rho$. We define it up to a multiplicative constant which is irrelevant when we consider its logarithmic derivative as we will do.

Definition 3.2. *We define the Hadamard interpolation as*

$$f_H(s) = \exp\left(\int G(s) \, ds\right).$$

The divisor of f_H is $\operatorname{Div} f_H = \sum n_\rho \, \rho$.

Definition 3.3. *Consider a meromorphic function f with divisor $\sum n_\rho \, \rho$. Fix σ_1 as above. The discrepancy of f is defined as the difference of the logarithmic derivatives*

$$P_f = \frac{f_H'}{f_H} - \frac{f'}{f} = G - \frac{f'}{f}$$

Lemma 3.4. *The discrepancy P_f is a polynomial of degree $\leq g - 1$.*

Proof. We recall the Hadamard factorization

$$f(s) = (s - \sigma_1)^{n_{\sigma_1}} e^{Q_{f,\sigma_1}(s)} \prod_{\rho \neq \sigma_1} \left[E_{d-1}\left(\frac{s - \sigma_1}{\rho - \sigma_1}\right)\right]^{n_\rho} \ ,$$

where Q_{f,σ_1} is a polynomial of degree $\leq g$ which is uniquely defined up to the addition of an integer multiple of $2\pi i$. The product is absolutely convergent because of the definition of the convergence exponent d. Taking the logarithmic derivative, we have that

$$\frac{f'}{f} = Q_{f,\sigma_1}' + G(s) \ ,$$

so $P_f = -Q_{f,\sigma_1}'$ and the result follows. $\square$

3.2. **Poisson-Newton formula.** The main result is the following Poisson-Newton formula associated to the Dirichlet series f. Consider its polynomial discrepancy P_f of degree $\leq g - 1$,

$$P_f(s) = c_0 + c_1 s + \ldots + c_{g-1} s^{g-1} .$$

The inverse Laplace transform of P_f is the distribution supported at $\{0\}$

$$\mathcal{L}^{-1}(P_f) = c_0 \delta_0 + c_1 \delta_0' + \ldots + c_{g-1} \delta_0^{(g-1)} .$$

Theorem 3.5. *As distributions in $\mathbb{R}$ we have*

$$W(f) = \sum_{k=0}^{g-1} c_k \delta_0^{(k)} + \sum_{\mathbf{k} \in \Lambda} \langle \boldsymbol{\lambda}, \mathbf{k} \rangle \, b_{\mathbf{k}} \; \delta_{\langle \boldsymbol{\lambda}, \mathbf{k} \rangle} .$$

The structure at 0 of $W(f)$ depends on the function f and its comparison with the Hadamard interpolation. The structure out of 0 only depends on the divisor and is independent of parameter choices. In some sense it is the most canonical part. Sometimes we refer to the "full Poisson-Newton formula" the one of the main theorem with the structure at 0.

Corollary 3.6. *As distributions on $\mathbb{R}_+^*$ we have*

$$W(f)|_{\mathbb{R}_+^*} = \sum_{\mathbf{k} \in \Lambda} \langle \boldsymbol{\lambda}, \mathbf{k} \rangle \, b_{\mathbf{k}} \; \delta_{\langle \boldsymbol{\lambda}, \mathbf{k} \rangle} .$$

Proof. We prove the theorem by taking the right-sided Laplace transform of $W(f)$ (we use the interval $[-1, \infty)$):

$$\langle W(f), e^{-st} \rangle_{[-1,\infty)} = \left\langle \frac{D^d}{Dt^d} F_d(t), e^{(\sigma_1 - s)t} \right\rangle_{[-1,\infty)}$$

$$= \int_0^\infty (-1)^d (K_d(t) - K_d(0)) \frac{d^d}{dt^d} e^{(\sigma_1 - s)t} dt$$

$$= n_{\sigma_1} (-1)^d \frac{(\sigma_1 - s)^d}{d!} \int_0^{+\infty} t^d e^{(\sigma_1 - s)t} \, dt +$$

$$+ \sum_\rho \frac{n_\rho}{(\rho - \sigma_1)^d} (-1)^d (\sigma_1 - s)^d \left(\int_0^{+\infty} e^{(\rho - \sigma_1)t} e^{(\sigma_1 - s)t} dt - \int_0^{+\infty} e^{(\sigma_1 - s)t} dt \right)$$

$$= \frac{n_{\sigma_1}}{s - \sigma_1} - \sum_\rho n_\rho \frac{(s - \sigma_1)^d}{(\rho - \sigma_1)^d} \left(\frac{1}{\rho - s} - \frac{1}{\sigma_1 - s} \right)$$

$$= \frac{n_{\sigma_1}}{s - \sigma_1} - \sum_\rho n_\rho \frac{(s - \sigma_1)^{d-1}}{(\rho - \sigma_1)^{d-1}} \frac{1}{\rho - s}$$

$$= G(s).$$

On the other hand, consider the distribution

$$V = \sum_{\mathbf{k}} \langle \boldsymbol{\lambda}, \mathbf{k} \rangle \, b_{\mathbf{k}} \, \delta_{\langle \boldsymbol{\lambda}, \mathbf{k} \rangle}.$$

Its Laplace transform is

$$
\begin{aligned}
\langle V, e^{-ts} \rangle_{[-1,\infty)} &= \left\langle \sum_{\mathbf{k}} \langle \boldsymbol{\lambda}, \mathbf{k} \rangle \, b_{\mathbf{k}} \, \delta_{\langle \boldsymbol{\lambda}, \mathbf{k} \rangle}, e^{-ts} \right\rangle_{[-1,\infty)} \\
&= \sum_{\mathbf{k}} \langle \boldsymbol{\lambda}, \mathbf{k} \rangle \, b_{\mathbf{k}} e^{-\langle \boldsymbol{\lambda}, \mathbf{k} \rangle s} \\
&= -(-\log f(s))' \\
&= \frac{f'(s)}{f(s)} \\
&= G(s) - P_f(s) \\
&= \langle W(f), e^{-st} \rangle_{[-1,\infty)} - P_f(s) \, .
\end{aligned}
$$

By uniqueness of the Laplace transform for distributions (see [30], Theorem 8.3-1, p.225), we have

$$W(f) = V + \mathcal{L}^{-1}(P_f),$$

where $\mathcal{L}^{-1}(P_f)$ is the inverse Laplace transform of the polynomial P_f. This is a distribution supported at $\{0\}$. Hence we get the theorem and the corollary

$$W(f)|_{\mathbb{R}_+^*} = V \, .$$

$\square$

Just inspecting the order of the distributions appearing in both sides of the Poisson-Newton formula, we get an interesting corollary for Dirichlet series. We know that $d \leq g + 1$. In fact we do have equality $d = g + 1$.

Corollary 3.7. *For a meromorphic Dirichlet series we have*

$$d = g + 1 = \deg Q_f + 1 = \deg P_f + 2 \, .$$

Proof. We inspect the order of the distributions in the Poisson-Newton-formula. We recall that $W(f)$ is, as distribution, the d-th derivative of a continuous function. But $\delta_0^{(l)}$ is not the d-th derivative of a continuous function for $l \geq d-1$. Thus $\deg P_f \leq d-2$ so $\deg Q_f \leq d-1$ and $g \leq d-1$, hence $g + 1 = d$. $\square$

It is clear that $d \geq 1$ for a meromorphic Dirichlet series, but we have in fact $d \geq 2$.

Corollary 3.8. *For a meromorphic Dirichlet series we have a convergence exponent at least* 2 *and order at least* 1*:*

$$d \geq 2$$

and

$$o \geq 1 \ .$$

Proof. As before we inspect the order of the distributions in the Poisson-Newton-formula. The right hand side contains Dirac distributions at the frequencies, hence it is at least a second derivative of a continuous function. In the left hand side we have $W(f)$ that is the d-th derivative of a continuous function. This gives $d \geq 2$.

Also we know that $d \leq o + 1$, hence $o \geq 1$. $\qquad\square$

3.3. **Symmetric Poisson-Newton formula.** Let $f(s)$ be a Dirichlet series with exponent of convergence d, and fix σ as before. We have defined a distribution $W(f, \sigma)(t) = (\sum n_\rho e^{\rho t}) \mathbf{1}_{\mathbb{R}_+}$ supported on $\mathbb{R}_+$. If we make the change of variables $t \mapsto -t$, we have the distribution $W(f, \sigma)(-t) = (\sum n_\rho e^{-\rho t}) \mathbf{1}_{\mathbb{R}_-}$, which is formally defined as

$$(-1)^d e^{-\sigma t} \frac{D^d}{Dt^d} \left((K_d(-t) - K_d(0)) \mathbf{1}_{\mathbb{R}_-} \right) .$$

This is independent of σ on $\mathbb{R}_-^*$ and has a contribution at zero dependent on the parameter.

The sum

$$\widehat{W}(f, \sigma) = W(f, \sigma)(t) + W(f, \sigma)(-t)$$

is a distribution on $\mathbb{R}$, whose only dependence on σ is at zero, and which formally it is equal to $\sum_\rho n_\rho e^{\rho|t|}$, $t \in \mathbb{R}$.

Theorem 3.9. *For a Dirichlet series f, we have in $\mathbb{R}$,*

$$\widehat{W}(f, \sigma)(t) = 2 \sum_{l=0}^{\frac{g-1}{2}} c_{2l} \, \delta_0^{(2l)} + \sum_{\mathbf{k} \in \Lambda \cup (-\Lambda)} \langle \boldsymbol{\lambda}, |\mathbf{k}| \rangle \, b_{|\mathbf{k}|} \, \delta_{\langle \boldsymbol{\lambda}, \mathbf{k} \rangle} \ .$$

Proof. The Poisson-Newton formula for f is

$$W(f, \sigma)(t) = \sum_{\mathbf{k} \in \Lambda} \langle \boldsymbol{\lambda}, \mathbf{k} \rangle b_{\mathbf{k}} \, \delta_{\langle \boldsymbol{\lambda}, \mathbf{k} \rangle} + \sum_{l=0}^{g-1} c_l \, \delta_0^{(l)} \ ,$$

Making the change of variables $t \mapsto -t$, we have

$$W(f, \sigma)(-t) = \sum_{\mathbf{k} \in -\Lambda} \langle \boldsymbol{\lambda}, |\mathbf{k}| \rangle b_{|\mathbf{k}|} \, \delta_{\langle \boldsymbol{\lambda}, \mathbf{k} \rangle} + \sum_{l=0}^{g-1} (-1)^l c_l \, \delta_0^{(l)} \ ,$$

where $|\mathbf{k}| = -\mathbf{k}$, for $\mathbf{k} \in -\Lambda$.

Adding the two formulas, we get the symmetric formula stated in the theorem for

$$\widehat{W}(f,\sigma)(t) = W(f,\sigma)(t) + W(f,\sigma)(-t) = \sum_\rho n_\rho e^{\rho|t|} \ .$$

$\square$

Consider a Dirichlet series $f(s) = 1 + \sum a_n e^{\lambda_n s}$ and let

$$\bar{f}(s) = \overline{f(\bar{s})} = 1 + \sum \bar{a}_n e^{\lambda_n s}$$

be its conjugate. Then $\bar{f}$ is a Dirichlet series whose zeros are the $\{\bar{\rho}\}$ and $n_{\bar{\rho}} = n_\rho$. Also $b_{\mathbf{k}}(\bar{f}) = \overline{b_{\mathbf{k}}(f)}$. The Poisson-Newton formula for $\bar{f}$ is

$$W(\bar{f},\bar{\sigma})(t) = \sum_{\mathbf{k}\in\Lambda} \langle \boldsymbol{\lambda}, \mathbf{k}\rangle \overline{b_{\mathbf{k}}}\, \delta_{\langle \boldsymbol{\lambda},\mathbf{k}\rangle} + \sum_{l=0}^{g-1} \bar{c}_l\, \delta_0^{(l)} \ ,$$

Corollary 3.10. *For a real analytic Dirichlet series f, that is $\bar{f}(s) = f(s)$, we have that for $\sigma \in \mathbb{R}$, the numbers c_l and $b_{\mathbf{k}}$ are real.*

The converse also holds.

The last point is due to the fact that the association $f \mapsto W(f)$ is one-to-one, as its inverse is the Laplace transform.

3.4. **Poisson-Newton formula with parameters.** Observing that the space of Dirichlet series is invariant by the change of variables $s \mapsto \alpha s + \beta$, with $\alpha > 0$ and $\beta \in \mathbb{C}$, we get a parameter version of the main theorem.

Corollary 3.11. *Let $\alpha > 0$ and $\beta \in \mathbb{C}$. As distributions on $\mathbb{R}_+^*$ we have*

$$e^{-\frac{\beta}{\alpha}t}\, W(f)(t/\alpha)|_{\mathbb{R}_+^*} = \sum_{\mathbf{k}\in\Lambda} \alpha\langle \boldsymbol{\lambda}, \mathbf{k}\rangle\, e^{-\langle \boldsymbol{\lambda},\mathbf{k}\rangle\beta}\, b_{\mathbf{k}}\, \delta_{\alpha\langle \boldsymbol{\lambda},\mathbf{k}\rangle} \ .$$

Proof. This results by applying Corollary 3.6 to $g(s) = f(\alpha s + \beta)$, which is a Dirichlet series for $\alpha > 0$,

$$g(s) = 1 + \sum_{n\geq 1} a_n e^{-\lambda_n \beta} e^{-(\lambda_n \alpha)s} \ .$$

The zeros of g are the numbers $\left(\frac{\rho-\beta}{\alpha}\right)$, and $b_{\mathbf{k}}$ is changed to $e^{-\langle \boldsymbol{\lambda},\mathbf{k}\rangle\beta}b_{\mathbf{k}}$.

$\square$

We can also give a parameter version of the full Poisson-Newton formula.

Corollary 3.12. *Let $\alpha > 0$ and $\beta \in \mathbb{C}$. As distributions in $\mathbb{R}$ we have*

$$e^{-\frac{\beta}{\alpha}t} \, W(f)(t/\alpha) = \sum_{k=0}^{g-1} c_k(\sigma_1') \delta_0^{(k)} + \sum_{\mathbf{k} \in \Lambda} \alpha \langle \boldsymbol{\lambda}, \mathbf{k} \rangle \, e^{-\langle \boldsymbol{\lambda}, \mathbf{k} \rangle \beta} \, b_{\mathbf{k}} \; \delta_{\alpha \langle \boldsymbol{\lambda}, \mathbf{k} \rangle} \, ,$$

where the $c_k(\sigma_1')$ are the coefficients of the discrepancy polynomial for $\sigma = \sigma_1' = \frac{\sigma_1 - \Re \beta}{\alpha}$.

Proof. Just observe that σ_1 becomes $\sigma_1' = \frac{\sigma_1 - \Re \beta}{\alpha}$ for $g(s) = f(\alpha s + \beta)$. $\qquad\qquad\square$

3.5. **Symmetric Poisson-Newton formula with parameters.** The space of real analytic Dirichlet series is invariant by the change of variables $s \mapsto \alpha s + \beta$, with $\alpha > 0$ and $\beta \in \mathbb{R}$, then we get a parameter version of the symmetric Poisson-Newton formula of the previous section.

Theorem 3.13. *Let $\alpha > 0$ and $\beta \in \mathbb{R}$. For a real analytic Dirichlet series f, that is $\bar{f}(s) = \overline{f(\bar{s})} = f(s)$, we have in $\mathbb{R}$*

$$e^{-\frac{\beta}{\alpha}|t|}(W(f)(t) + W(f)(-t)) = \sum_{\rho} n_\rho e^{(\rho - \beta)|t|/\alpha} =$$

$$= 2 \sum_{l=0}^{\frac{g-1}{2}} c_{2l}(\sigma_1') \, \delta_0^{(2l)} + \sum_{\mathbf{k} \in \Lambda \cup (-\Lambda)} \alpha \, \langle \boldsymbol{\lambda}, |\mathbf{k}| \rangle e^{-\langle \boldsymbol{\lambda}, |\mathbf{k}| \rangle \beta} \, b_{|\mathbf{k}|} \; \delta_{\alpha \langle \boldsymbol{\lambda}, \mathbf{k} \rangle} \, ,$$

with $\sigma_1' = \frac{\sigma_1 - \beta}{\alpha}$.

In particular, for $\alpha = 1$ and $g = 1$ that we use in the applications, we get

Corollary 3.14. *Let $\beta \in \mathbb{R}$. For a real analytic Dirichlet series f, that is $\bar{f}(s) = \overline{f(\bar{s})} = f(s)$, of genus $g = 1$ we have in $\mathbb{R}$*

$$\sum_{\rho} n_\rho e^{(\rho - \beta)|t|} = 2c_0(\sigma_1 - \beta) \, \delta_0 + \sum_{\mathbf{k} \in \Lambda \cup (-\Lambda)} \langle \boldsymbol{\lambda}, |\mathbf{k}| \rangle e^{-\langle \boldsymbol{\lambda}, |\mathbf{k}| \rangle \beta} \, b_{|\mathbf{k}|} \; \delta_{\langle \boldsymbol{\lambda}, \mathbf{k} \rangle} \, .$$

3.6. **General Poisson-Newton formula.** We can observe that in the proof of the Poisson-Newton formula we barely used the Dirichlet character of f. The construction of G, the discrepancy polynomial P_f, and the Hadamard interpolation f_H can be carried out in general for any meromorphic function of finite order with its divisor contained in a left half plane. The existence of the Newton-Cramer distribution $W(f)$ also holds in this generality. The choice of σ_1 is only relevant for the structure at 0 of the Newton-Cramer distribution and for determining the exponential decay of the test functions to which we can apply the distribution formula. Only in order to compute the logarithmic derivative f'/f we have used the Dirichlet series expansion. If we take over the proof for a meromorphic funciton f of finite order and with divisor contained in a left half plane, we get the following general theorem:

Theorem 3.15. *Let f be a meromorphic function of finite order with convergence exponent d and its divisor contained in a left half plane. Let $W(f)$ be its Newton-Cramer distribution and $P_f(s) = c_0 + c_1 s + \ldots + c_{g-1} s^{g-1}$ be the discrepancy polynomial. We have in $\mathbb{R}$,*

$$W(f) = \sum_{j=0}^{g-1} c_j \delta_0^{(j)} + \mathcal{L}^{-1}(f'/f) \ ,$$

or

$$\mathcal{L}(W(f)) = \langle W(f), e^{-st} \rangle_{\mathbb{R}_+} = \frac{f'(s)}{f(s)} + P_f(s) \ .$$

The inverse Laplace transform $\mathcal{L}^{-1}(F)$ ia a well defined distribution of finite order when F has polynomial growth on a half plane, which is the case of $F = f'/f$ when f is of finite order and has divisor contained on a left half plane. It is defined as follows. Take m which is two units more than the order of growth of F, and define

$$L(t) = \int_{-\infty}^{+\infty} \frac{F(c+iu)}{(c+iu)^m} e^{(c+iu)t} \frac{du}{2\pi} \ .$$

This is a continuous function, which vanishes on $\mathbb{R}_-$. It is independent of the choice of c (subject to $\Re c > \sigma_1$). Then

$$\mathcal{L}^{-1}(F)(t) := \frac{D^m}{Dt^m} L(t),$$

which is a distribution of order at most $m - 2$.

More explicitly, for an appropriate test function φ, letting $\psi(t) = \varphi(t)e^{ct}$, we have

$$
\begin{aligned}
\langle \mathcal{L}^{-1}(F), \varphi \rangle &= \langle L(t), (-1)^m \varphi^{(m)}(t) \rangle \\
&= \int_{\mathbb{R}} \int_{-\infty}^{+\infty} \frac{F(c+iu)}{(c+iu)^m} (-1)^m \varphi^{(m)}(t) e^{(c+iu)t} \frac{du}{2\pi} dt \\
&= \int_{-\infty}^{+\infty} (-1)^m \frac{F(c+iu)}{(c+iu)^m} \left(\int_{\mathbb{R}} \varphi^{(m)}(t) e^{(c+iu)t} dt \right) \frac{du}{2\pi} \\
&= \int_{-\infty}^{+\infty} \frac{F(c+iu)}{(c+iu)^m} \left(\int_{\mathbb{R}} (c+iu)^m \varphi(t) e^{(c+iu)t} dt \right) \frac{du}{2\pi} \\
&= \int_{-\infty}^{+\infty} F(c+iu) \hat{\psi}(-u) \frac{du}{2\pi} \ ,
\end{aligned}
$$

doing m integrations by parts in the penultimate line.

We can give a symmetric version of the general Poisson-Newton formula: We denote

$$\widehat{W}(f, \sigma) = W(f, \sigma)(t) + W(f, \sigma)(-t),$$

which is a distribution on $\mathbb{R}$.

Theorem 3.16. *We have in* $\mathbb{R}$

$$\widehat{W}(f,\sigma)(t) = 2\sum_{l=0}^{\frac{g-1}{2}} c_{2l}\,\delta_0^{(2l)} + \left(\mathcal{L}^{-1}(f'/f)(t) + \mathcal{L}^{-1}(f'/f)(-t)\right).$$

3.7. General symmetric Poisson-Newton formula with parameters. Let $\alpha > 0$, $\beta \in \mathbb{C}$. Take $\sigma_1' = \frac{\sigma-\beta}{\alpha}$. We denote, as a slight abuse of notation,

$$e^{-\frac{\beta}{\alpha}|t|}\widehat{W}(f,\sigma)(t) = e^{-\frac{\beta}{\alpha}t}W(f,\sigma)(t) + e^{\frac{\beta}{\alpha}t}W(f,\sigma)(-t)\,.$$

Theorem 3.17. *We have in* $\mathbb{R}$

$$e^{-\frac{\beta}{\alpha}|t|}\widehat{W}(f,\sigma)(t/\alpha) = 2\sum_{l=0}^{\frac{g-1}{2}} c_{2l}(\sigma_1')\,\delta_0^{(2l)} + \left(e^{-\frac{\beta}{\alpha}t}\mathcal{L}^{-1}(f'/f)(t/\alpha) + e^{\frac{\beta}{\alpha}t}\mathcal{L}^{-1}(f'/f)(-t/\alpha)\right).$$

Corollary 3.18. *For a real analytic function f and $\beta \in \mathbb{R}$ to the right of all zeroes of f, we have in* $\mathbb{R}$

$$e^{-\frac{\beta}{\alpha}|t|}\widehat{W}(f,\sigma)(t) = 2\sum_{l=0}^{\frac{g-1}{2}} c_{2l}(\sigma_1')\,\delta_0^{(2l)} + e^{-\frac{\beta}{\alpha}t}\mathcal{L}_{\beta}^{-1}\left(2\Re(f'/f)\right)(t/\alpha).$$

We prove this using the following result

Lemma 3.19. *For a real analytic function F, and γ to the right of the zeroes, we have*

$$e^{-\gamma t}\mathcal{L}^{-1}(F)(t) + e^{\gamma t}\mathcal{L}^{-1}(F)(-t) = e^{-\gamma t}\mathcal{L}_{\gamma}^{-1}\left(2\Re F\right)(t)$$

Proof. We have

$$e^{-ct}\mathcal{L}^{-1}(F)(t) = \int_{-\infty}^{+\infty} F(c+iu)e^{iut}\frac{du}{2\pi}$$

Analogously,

$$\begin{aligned}
e^{ct}\mathcal{L}^{-1}(F)(-t) &= \int_{-\infty}^{+\infty} F(c+iu)e^{-iut}\frac{du}{2\pi}\\
&= \int_{-\infty}^{+\infty} F(c-iv)e^{ivt}\frac{dv}{2\pi}\\
&= \int_{-\infty}^{+\infty} \overline{F(c+iv)}e^{ivt}\frac{dv}{2\pi}
\end{aligned}$$

Adding both, we get

$$\int_{-\infty}^{+\infty} 2\left(\Re F(c+iv)\right)e^{ivt}\frac{dv}{2\pi}$$

as required. $\qquad\qquad\square$

3.8. **Converse theorem.** Conversely we show that any Poisson-Newton formula is associated to a Dirichlet series.

Proposition 3.20. *Let $\sum_\rho n_\rho \, \rho$ be a divisor with convergence exponent equal to d and contained in a left half plane $\Re \rho \leq \sigma_1$. Suppose that a Poisson-Newton formula*

$$\left(\sum_\rho n_\rho e^{-\rho t} \right) \mathbf{1}_{\mathbb{R}_+} = \sum_{n \geq 1} b_n \delta_{\mu_n} + \sum_{j=0}^{d-2} c_j \delta_0^{(j)}$$

holds in $\mathbb{R}$, in the sense of an equality of distributions, acting on functions with fast enough exponential decay. We assume $0 < \mu_1 < \mu_2 < \ldots$, where (μ_n) are finitely many or $\mu_n \to +\infty$, and $\sum_{n \geq 1} |b_n| e^{-\mu_n \bar\sigma} < \infty$. for some $\bar\sigma < \infty$. Then there is a Dirichlet series f meromorphic on $\mathbb{C}$, whose Poisson-Newton formula is the one given.

Proof. The condition $\sum |n_\rho| \, |\rho|^{-d} < \infty$ allows us to define the function $G(s, \sigma)$ associated to the divisor, and the corresponding Hadamard interpolation $f_H(s, \sigma) = \exp\left(\int G(s, \sigma) \, ds \right)$. The function $f_H(s, \sigma)$ is meromorphic on $\mathbb{C}$ and has divisor of zeros and poles equal to $\sum n_\rho \, \rho$. For the function f_H, we have defined a distribution $W(f_H)$ supported in $\mathbb{R}_+$, which equals with $\sum n_\rho e^{-\rho t}$ in $\mathbb{R}_+^*$ (the precise meaning of the later is our definition of the distributional sense given in Section 2). Therefore,

$$W(f_H) = \left(\sum_\rho n_\rho e^{-\rho t} \right) \mathbf{1}_{\mathbb{R}_+} + \sum_{j=0}^{d-2} c_j' \delta_0^{(j)} \ .$$

By our hypothesis, we have the equality

$$W(f_H) = \sum_{n \geq 1} b_n \delta_{\mu_n} + \sum_{j=0}^{d-2} c_j'' \delta_0^{(j)}$$

as distributions paired against functions with fast enough exponential decay. So for $\Re s \geq \bar\sigma$ (enlarging $\bar\sigma$ if necessary), we have

$$\frac{f_H'}{f_H} = G(s) = \langle W(f_H), e^{-ts} \rangle$$

$$= \left\langle \sum_{n \geq 1} b_n \delta_{\mu_n} + \sum_{j=0}^{d-2} c_j'' \delta_0^{(j)}, e^{-ts} \right\rangle$$

$$= F(s) + P(s),$$

where $P(s)$ is a polynomial of degree at most $d-2$, and

$$F(s) = \left\langle \sum_{n \geq 1} b_n \delta_{\mu_n}, e^{-ts} \right\rangle = \sum_{n \geq 1} b_n e^{-\mu_n s} \ .$$

Note that the assumptions mean that this can be paired against e^{-st} for $\Re s \geq \bar{\sigma}$ and the last sum is uniformly convergent. So for $\Re s \geq \bar{\sigma}$, we have

$$f(s) := e^{\int F(s)} = e^{-\int P(s)} f_H(s) \,.$$

Then $f(s)$ is a meromorphic function on $\mathbb{C}$, and in a right half-plane, we have

$$f(s) = \exp \left(\sum b_n e^{-\mu_n s} \right) = 1 + \sum_{n \geq 1} a_n e^{-\lambda_n s} \,,$$

which is a Dirichlet series. $\qquad \square$

4. Parameter dependence

We analyze the dependence of $W(f, d', \sigma)$ on d' and σ, which is concentrated at 0. We write

$$R_A(\sigma, d') = \sum_{\ell=1}^{d'} e^{\sigma t} K_{\ell,A}(0, \sigma) \, \delta_0^{(\ell-1)}$$

for the remainder in (5). Note that this is not convergent, as the first summand in the right hand side (5) is also not convergent. But

$$R_A(\sigma_1, d_1') - R_A(\sigma_2, d_2') = W_A(f, \sigma_1, d_1') - W_A(f, \sigma_2, d_2')$$

is convergent. The dependence on $d' \geq d$ is easy to describe, since the $K_{\ell,A}(0, \sigma)$ do converge for $\ell \geq d$.

Lemma 4.1. *Let $d' = d + a$, $a \geq 1$. Then*

$$W(f, \sigma, d') = W(f, \sigma, d) + \sum_{j=1}^{a} K_{d+j}(0, \sigma) e^{\sigma t} \, \delta_0^{(d-1+j)}$$

$$= W(f, \sigma, d) + \sum_{k=0}^{d-1+a} \left(\sum_{\max(1, k-d+1) \leq j \leq a} \binom{d-1+j}{k} (-\sigma)^{d-1+j-k} K_{d+j}(0, \sigma) \right) \delta_0^{(k)}$$

In particular $W(f, \sigma, d)$ is obtained from $W(f, \sigma, d')$ by removing the higher order derivatives of δ_0 strictly larger than $d-1$ and a convergent part in lower order derivatives.

Proof. Just note that for $n \geq 0$ and $\sigma \in \mathbb{C}$,

$$e^{\sigma t} \, \delta_0^{(n)} = \sum_{k=0}^{n} \binom{n}{k} (-\sigma)^{n-k} \delta_0^{(k)} \,.$$

$\qquad \square$

So from now on, we shall restrict to $d' = d$ and write $W(f, \sigma) = W(f, \sigma, d)$. We write

$$R_A(\sigma) = W_A(f, \sigma) - W_A(f, \sigma_1) = \sum_{l=0}^{d-1} (K_{l+1,A}(0, \sigma)e^{\sigma t} - K_{l+1,A}(0, \sigma_1)e^{\sigma_1 t})\delta_0^{(l)}$$

$$= \sum_{k=0}^{d-1} r_{k,A}(\sigma)\delta_0^{(k)}$$

with for $k = 0, 1, \ldots, d-1$

$$c_{k,A}(\sigma) = \sum_{l=k}^{d-1} \binom{l}{k}(-\sigma)^{l-k}K_{l+1,A}(0, \sigma)$$

$$r_{k,A}(\sigma) = c_{k,A}(\sigma) - c_{k,A}(\sigma_1) \ ,$$

and

$$R(\sigma) = \lim_A R_A(\sigma) = \sum_{k=0}^{d-1} r_k(\sigma)\delta_0^{(k)} \ ,$$

with

$$r_k = \lim_A r_{k,A} \ .$$

Note that the last coefficient for $k = d-1$ is clearly convergent and the following holds:

Proposition 4.2. *We have that* $r_{d-1}(\sigma) = K_d(0, \sigma) - K_d(0, \sigma_1) = \sum_\rho \frac{n_\rho}{(\rho-\sigma)^d} - K_d(0, \sigma_1)$ *is a meromorphic function on* $\mathbb{C}$ *with poles of order d at the ρ's.*

For $k \le d-2$, we do not have such an explicit description, but still a similar result holds.

Proposition 4.3. *For $k = 0, 1, \ldots, d-2$, the coefficient $r_k(\sigma)$ is a meromorphic function on $\sigma \in \mathbb{C}$ with poles of order d at the ρ's.*

Proof. This results from the parameter dependence of the Hadamard interpolation. We have defined earlier

$$G(s, \sigma) = -\sum_\rho n_\rho \left(\frac{1}{\rho - s} - \sum_{l=0}^{d-2} \frac{(s-\sigma)^l}{(\rho-\sigma)^{l+1}} \right) = \sum_\rho n_\rho \frac{(s-\sigma)^{d-1}}{(\rho-\sigma)^{d-1}} \frac{1}{s - \rho} \ .$$

Then we write

$$f_H(s, \sigma) = \exp\left(\int G(s, \sigma)\, ds \right),$$

and

$$P_f(s, \sigma) = \frac{f'_H(s, \sigma)}{f_H(s, \sigma)} - \frac{f'(s)}{f(s)} = G(s, \sigma) - \frac{f'(s)}{f(s)}$$

and

$$P_f(s, \sigma) = c_0(\sigma) + c_1(\sigma)s + \ldots + c_{d-2}(\sigma)s^{d-2} \ .$$

Assume that 0 is not in the support of the divisor. Then observe that

$$c_j(\sigma) = \frac{1}{j!} \, P_f^{(j)}(0, \sigma),$$

taking the derivative with respect to s, $j = 0, 1, \ldots, d - 2$. As $G(s, \sigma)$ has poles of order $d - 1$ at ρ (as a function on σ), the same happens for $c_j(\sigma)$.

$\square$

4.1. Newton-Cramer functions associated to divisors.

Consider a divisor $D = \sum n_\rho \, \rho$ supported in a left half-plane and of convergence exponent d. This means that $\Re\rho \leq \sigma_1$ for some $\sigma_1 \in \mathbb{R}$ and $\sum_{\rho \neq 0} |n_\rho| \, |\rho|^{-d} < \infty$. Then, for any $\sigma \in \mathbb{C} - \{\rho\}$ and $d' \geq d$, we have an associated distribution

$$W(D, \sigma, d')$$

constructed as in Section 2. All these distributions are supported in $\mathbb{R}_+$, and they are independent of σ and d' in $\mathbb{R}_*^+$. Moreover, we denote $W(D, \sigma) = W(D, \sigma, d)$.

The distributions $W(D, \sigma)$ are Laplace transformable, i.e., they can be paired against e^{-st}, for $\Re s > \sigma_0$, for some σ_0 depending on D and σ. Define

$$g_D(s, \sigma) = \langle W(D, \sigma), e^{-st} \rangle_{\mathbb{R}}$$

and

$$f_D(s, \sigma) = \exp\left(-\int g_D(s, \sigma) \, ds \right).$$

Proposition 4.4. *All $g_D(s, \sigma)$, for different σ, are equal up to the addition of a polynomial of degree $d - 2$ in s. They have simple poles exactly at the ρ, with residues n_ρ.*

The functions $f_D(s, \sigma)$ are meromorphic on $\mathbb{C}$. They have the same divisor and they differ by a Weierstrass exponential factor of order at most $d - 1$.

5. Applications.

5.1. **Classical Poisson formula.** Consider for $\lambda = \lambda_1 > 0$ the entire function
$$f(s) = 1 - e^{-\lambda s}.$$
This function has order 1, exponent of convergence $d = 2$, genus $g = 1$, and its zeros are
$$\rho_k = i\frac{2\pi}{\lambda}k,$$
for $k \in \mathbb{Z}$. Also we have
$$-\log f(s) = -\log\left(1 - e^{-\lambda s}\right) = \sum_{k=1}^{+\infty} \frac{1}{k}e^{-\lambda k s},$$
i.e., with our notation $\Lambda = \mathbb{N}^*$, $b_k = 1/k$.

Therefore the Poisson-Newton formula in $\mathbb{R}_+^*$ gives
$$\left(\sum_{k\in\mathbb{Z}} e^{i\frac{2\pi}{\lambda}kt}\right)_{\big|\mathbb{R}_+^*} = \lambda \sum_{k\geq 1} \delta_{\lambda k}.$$
The distribution on the left side, when considered without restricting to $\mathbb{R}_+^*$, is even. It follows then
$$\left(\sum_{k\in\mathbb{Z}} e^{i\frac{2\pi}{\lambda}kt}\right)_{\big|\mathbb{R}^*} = \lambda \sum_{k\in\mathbb{Z}^*} \delta_{\lambda k}.$$
Now the distribution on the left side, without restricting to $\mathbb{R}^*$, is λ-periodic on $\mathbb{R}$. So we get the full classical Poisson formula, identifying what we get at 0,
$$\sum_{k\in\mathbb{Z}} e^{i\frac{2\pi}{\lambda}kt} = \lambda \sum_{k\in\mathbb{Z}} \delta_{\lambda k}.$$

If we start with
$$f(s) = 1 - ae^{-\lambda s},$$
for some $a \in \mathbb{C}^*$, we get also the classical Poisson formula. We have for $k \in \mathbb{Z}$,
$$\rho_k = \frac{1}{\lambda}\log a + i\frac{2\pi}{\lambda}k,$$
and
$$b_k = \frac{1}{k}a^k.$$
Therefore
$$\left(a^{t/\lambda}\sum_{k\in\mathbb{Z}} e^{i\frac{2\pi}{\lambda}kt}\right)_{\big|\mathbb{R}_+^*} = \lambda \sum_{k\geq 1} a^k\delta_{\lambda k} = \lambda a^{t/\lambda} \sum_{k\geq 1} \delta_{\lambda k},$$

which gives the same formula as before.

Application of the symmetric Poisson-Newton formula. As expected from the symmetric form of the classical Poisson formula, one can recover the formula in a more direct way from the symmetric Poisson-Newton formula.

It is also interesting to clarify the structure of the Newton-Cramer distribution at 0. It helps to understand why the Dirac δ_0 appearing in the right side of the classical Poisson formula is of a different nature than the other $\delta_{\lambda k}$ for $k \neq 0$, something that was intuitively suspected from the analogy with trace formulas (see a comment on this in [7], p.2).

In order to use the symmetric Poisson-Newton formula we compute the discrepancy polynomial P_f for $f(s) = 1 - e^{-\lambda s}$. We have that $\sigma_1 = 0$ is a zero. From the classical Hadamard factorization

$$\sinh(\pi s) = \pi s \prod_{k \in \mathbb{Z}^*} \left(1 - \frac{s}{ik}\right) e^{\frac{s}{ik}} \ ,$$

we get the Hadamard factorization for f,

$$f(s) = 2e^{-\lambda s/2} \sinh(\lambda s/2) = s\lambda e^{-\lambda s/2} \prod_{k \in \mathbb{Z}^*} \left(1 - \frac{s}{\rho_k}\right) e^{\frac{s}{\rho_k}} \ .$$

Note that this is equivalent to

$$G(s) = \frac{1}{s} - \sum_{k \in \mathbb{Z}^*} \left(\frac{1}{\rho_k - s} - \frac{1}{\rho_k}\right)$$

$$= \frac{\lambda/2}{\tanh(\lambda s/2)} \ .$$

Thus $Q_f(s) = (\log \lambda + 2\pi i n) - \frac{\lambda}{2}s$, with $n \in \mathbb{Z}$, and

$$P_f(s) = -Q'_f(s) = c_0 = \frac{\lambda}{2} \ .$$

Therefore we can apply the symmetric Poisson-Newton formula (Theorem 3.9) and we get

$$\sum_{k \in \mathbb{Z}} e^{i\frac{2\pi}{\lambda}k|t|} = 2c_0\delta_0 + \lambda \sum_{k \in \mathbb{Z}^*} \delta_{\lambda k}$$

$$= \lambda\delta_0 + \lambda \sum_{k \in \mathbb{Z}^*} \delta_{\lambda k}$$

$$= \lambda \sum_{k \in \mathbb{Z}} \delta_{\lambda k} \ .$$

We finally observe that

$$\sum_{k \in \mathbb{Z}} e^{i\frac{2\pi}{\lambda}k|t|} = \sum_{k \in \mathbb{Z}} e^{i\frac{2\pi}{\lambda}kt} \ ,$$

because we can reorder freely a converging (in the distribution sense) infinite series of distributions

$$\sum_{k \in \mathbb{Z}} e^{i\frac{2\pi}{\lambda}k|t|} = 1 + 2\sum_{k=1}^{+\infty} \cos\left(\frac{2\pi}{\lambda}k|t|\right)$$

$$= 1 + 2\sum_{k=1}^{+\infty} \cos\left(\frac{2\pi}{\lambda}kt\right)$$

$$= \sum_{k \in \mathbb{Z}} e^{i\frac{2\pi}{\lambda}kt} \ .$$

5.2. **Newton formulas.** We show in this section how the Poisson-Newton formula is a generalization to Dirichlet series of Newton formulas which express Newton sums of roots of a polynomial equation in terms of its coefficients (or elementary symmetric functions).

Let $P(z) = z^n + a_1 z^{n-1} + \ldots + a_n$ be a polynomial of degree $n \geq 1$ with zeros $\alpha_1, \ldots, \alpha_n$ repeated according to their multiplicity. For each integer $m \geq 1$, the Newton sums of the roots are the symmetric functions

$$S_m = \sum_{j=1}^{n} \alpha_j^m \ .$$

From the fundamental theorem on symmetric functions, these Newton sums can be expressed polynomially with integer coefficients in terms of elementary symmetric functions, i.e., in terms of the coefficients of P. These are the Newton formulas. For instance, if for $k \geq 1$

$$\Sigma_k = \sum_{1 \leq i_1 < \ldots < i_k \leq n} \alpha_{i_1} \ldots \alpha_{i_k} = (-1)^k a_k \ ,$$

then we have

$$S_1 = \Sigma_1$$
$$S_2 = \Sigma_1^2 - 2\Sigma_2$$
$$S_3 = \Sigma_1^3 - 3\Sigma_2\Sigma_1 + 3\Sigma_3$$
$$S_4 = \Sigma_1^4 - 4\Sigma_2\Sigma_1^2 + 4\Sigma_3\Sigma_1 + 2\Sigma_2^2 - 4\Sigma_4$$

$$\vdots$$

We recover them applying the Poisson-Newton formula to the finite Dirichlet series

$$f(s) = e^{-\lambda ns} P(e^{\lambda s}) = 1 + a_1 e^{-\lambda s} + \ldots + a_n e^{-\lambda ns} .$$

The zeros of f are the $(\rho_{j,k})$ with $j = 1, \ldots, n$, $k \in \mathbb{Z}$, and

$$e^{\rho_{j,k}} = \alpha_j^{1/\lambda} e^{\frac{2\pi i}{\lambda} k} .$$

Thus, using the classical Poisson formula, its Newton-Cramer distribution can be computed in $\mathbb{R}$ as

$$\sum_\rho e^{\rho t} = \sum_{j=1}^n \alpha_j^{(1/\lambda)t} \sum_{k \in \mathbb{Z}} e^{\frac{2\pi i}{\lambda} kt}$$

$$= \sum_{j=1}^n \alpha_j^{(1/\lambda)t} \lambda \sum_{m \in \mathbb{Z}} \delta_{m\lambda}$$

$$= \lambda \sum_{m \in \mathbb{Z}} \left(\sum_{j=1}^n \alpha_j^m \right) \delta_{m\lambda}$$

$$= \lambda \sum_{m \in \mathbb{Z}} S_m \, \delta_{m\lambda}$$

Now, using the Poisson-Newton formula in $\mathbb{R}_+^*$

$$\sum_\rho e^{\rho t} = \sum_{\mathbf{k} \in \Lambda} \langle \boldsymbol{\lambda}, \mathbf{k} \rangle \, b_{\mathbf{k}} \, \delta_{\langle \boldsymbol{\lambda}, \mathbf{k} \rangle} ,$$

taking into account the repetitions in the right side, and that $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_n) = (\lambda, 2\lambda, \ldots, n\lambda)$, we have using the formula (7) for the $b_{\mathbf{k}}$

$$S_m = m \sum_{k_1 + 2k_2 + \ldots + nk_n = m} b_{\mathbf{k}} = m \sum_{k_1 + 2k_2 + \ldots + nk_n = m} \frac{(||k|| - 1)!}{\prod_j k_j} \prod_j \Sigma_j^{k_j} ,$$

which gives the explicit Newton relations. Moreover, Newton relations are equivalent to the Poisson-Newton formula in $\mathbb{R}_+^*$ in this case.

For example, for $m = 4$,

$$S_4 = 4 \left(b_{(4,0,0,0)} + b_{(2,1,0,0)} + b_{(1,0,1,0)} + b_{(0,2,0,0)} + b_{(0,0,0,1)} \right) ,$$

and from

$$b_{(4,0,0,0)} = \frac{1}{4}\Sigma_1^4$$

$$b_{(2,1,0,0)} = -\Sigma_1^2\Sigma_2$$

$$b_{(1,0,1,0)} = \Sigma_1\Sigma_3$$

$$b_{(0,2,0,0)} = \frac{1}{2}\Sigma_2^2$$

$$b_{(0,0,0,1)} = -\Sigma_4$$

we get

$$S_4 = \Sigma_1^4 - 4\Sigma_2\Sigma_1^2 + 4\Sigma_3\Sigma_1 + 2\Sigma_2^2 - 4\Sigma_4 \ .$$

5.3. **Abel-Plana summation formula.** The full Poisson-Newton formula for $f(s) = 1 - e^{-\lambda s}$ is a "half" classical Poisson formula and can be written for $\lambda = 1$ as

(9)
$$\sum_{n \geq 0} \delta_n = \frac{1}{2}\delta_0 + \left(\sum_{k \in \mathbb{Z}} e^{2\pi ikt}\right)\mathbf{1}_{\mathbb{R}_+} \ .$$

We check now that this is essentially the Abel-Plana summation formula (see [1], [22] and [6]). The Abel-Plana summation formula compares an infinite sum with the corresponding integral.

Theorem 5.1. (Abel-Plana summation formula) *Let f be a holomorphic function in a domain containing the right half plane $\mathbb{H}_+ = \{\Re z > 0\}$ and continuous in $\overline{\mathbb{H}}_+$ with*

$$\lim_{y \to +\infty} |f(x \pm iy)|e^{-2\pi y} = 0$$

uniformly on compact sets of x, and such that

$$\int_0^{+\infty} |f(x+iy) - f(x-iy)|e^{-2\pi y}dy$$

exists for $x \in \mathbb{R}_+$ and tends to 0 when $x \to +\infty$.

We have

$$\lim_{N \to +\infty}\left(\sum_{n=0}^{N} f(n) - \int_0^{N+1/2} f(t)\,dt\right) = \frac{1}{2}f(0) + i\int_0^{+\infty} \frac{f(it) - f(-it)}{e^{2\pi t} - 1}\,dt \ ,$$

and when the sum converges,

$$\sum_{n=0}^{+\infty} f(n) - \int_0^{+\infty} f(t)\,dt = \frac{1}{2}f(0) + i\int_0^{+\infty} \frac{f(it) - f(-it)}{e^{2\pi t} - 1}\,dt \ .$$

Proof. We apply the full Poisson-Newton formula (9) to a smooth truncation f_N of f with compact support, so in particular f_N is in the Schwarz class. We consider $f_N(x + iy) = \chi_N(x)f(x + iy)$ with $\chi_N(x)$ equal to 1 in a neighborhood of $[0, N]$ and vanishing outside $[-\epsilon, N + 1/2 + \epsilon]$ with $0 < \epsilon < 1/2$. We also require that $\|f\mathbf{1}_{[0,N]} - f_N\|_{L^1(\mathbb{R})} \to 0$ when $N \to +\infty$. The function f_N coincides with f and is holomorphic in a neighborhood of the strip $0 \leq \Re z \leq N + 1/2$. Applying to f_N the half Poisson formula, we have

$$\sum_{n=0}^{+\infty} f_N(n) = \sum_{n=0}^{N} f(n)$$

$$= \frac{1}{2}f(0) + \sum_{k\in\mathbb{Z}} \int_0^{+\infty} f_N(t)e^{2\pi ikt}dt$$

$$= \frac{1}{2}f(0) + \int_0^{+\infty} f_N(t)\,dt + \sum_{k=1}^{+\infty} \int_0^{+\infty} f_N(t)e^{2\pi ikt}dt - \sum_{k=1}^{+\infty} \int_0^{-\infty} f_N(-t)e^{2\pi ikt}dt$$

$$= \frac{1}{2}f(0) + \int_0^{N+1/2} f(t)\,dt + \sum_{k=1}^{+\infty} \left(\int_0^N f(t)e^{2\pi ikt}dt - \int_0^{-N} f(-t)e^{2\pi ikt}dt \right) + o(1) \ ,$$

where $o(1) \to 0$ when $N \to +\infty$.

For $R > 0$, in the domain of holomorphy of f, and using Cauchy theorem, we decompose each integral into three line integrals over a rectangular contour:

$$\int_0^N f(t)e^{2\pi ikt}dt = \int_0^{iR} f(z)e^{2\pi ikz}dz + \int_{iR}^{N+iR} f(z)e^{2\pi ikz}dz + \int_{N+iR}^N f(z)e^{2\pi ikz}dz$$

$$\int_0^{-N} f(-t)e^{2\pi ikt}dt = \int_0^{iR} f(-z)e^{2\pi ikz}dz + \int_{iR}^{-N+iR} f(-z)e^{2\pi ikz}dz + \int_{-N+iR}^{-N} f(-z)e^{2\pi ikz}dz$$

When $R \to +\infty$ the second integral of each line tends to 0 because of the first hypothesis on f. The substraction of the third integrals gives

$$\int_{N+iR}^N f(z)e^{2\pi ikz}dz - \int_{-N+iR}^{-N} f(-z)e^{2\pi ikz}dz = i\int_R^0 f(N+iu)e^{-2\pi ku}du - i\int_R^0 f(N-iu)e^{-2\pi ku}du$$

$$= -i\int_0^R (f(N+iu) - f(N-iu))e^{-2\pi ku}du \ .$$

And the second hypothesis on f shows that this last expression tends to 0 when $N \to +\infty$. So in the limit we are left with

$$\sum_{n=0}^{N} f(n) - \int_{0}^{N+1/2} f(t)\, dt = \frac{1}{2}f(0) + i \int_{0}^{+\infty} \frac{f(it) - f(-it)}{e^{2\pi t} - 1}\, dt + o(1) \ ,$$

and the result follows. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ □

5.4. Euler-MacLaurin formula and generalizations. We have from equation (5)

Proposition 5.2. *For $m \geq d$ we have*

$$\frac{D^m}{Dt^m} K_{m,A}(t,\sigma) = K_{0,A}(t,\sigma) + \sum_{j=1}^{m} K_{j,A}(0,\sigma)\delta_0^{(j-1)} \ .$$

And passing to the limit in A we have

$$e^{-\sigma t} W(f) = - \sum_{j=d+1}^{m} K_j(0,\sigma)\delta_0^{(j-1)} + \frac{D^m}{Dt^m} K_m(t,\sigma) \ .$$

The half Poisson formula can be written as

$$\sum_{n=0}^{+\infty} \delta_n = \frac{1}{2}\delta_0 + W(1 - e^{-s}) \ .$$

We prefer to work with $g(s) = \frac{1-e^{-s}}{s}$ in order to make the simplest choice $\sigma = 0$. Observing that

$$W(1 - e^{-s}) = \mathbf{1}_{\mathbb{R}_+} + W\left(\frac{1 - e^{-s}}{s}\right) \ ,$$

we have

$$\sum_{n=0}^{+\infty} \delta_n = \frac{1}{2}\delta_0 + \mathbf{1}_{\mathbb{R}_+} + W\left(\frac{1 - e^{-s}}{s}\right) \ .$$

Now we apply the formula of Proposition 5.2 with $\sigma = 0$ and $d = 1$ to $g(s) = \frac{1-e^{-s}}{s}$. Then we have for $j \geq d + 1$

$$K_j(0,0) = \sum_{k \in \mathbb{Z}^*} \frac{1}{(2\pi i k)^j} \ ,$$

thus $K_j(0,0) = 0$ when j is odd. Recalling that

$$\zeta(2l) = (-1)^{l+1} \frac{B_{2l}(2\pi)^{2l}}{2(2l)!} \ ,$$

where the (B_n) are the Bernouilli numbers, we have for j even

$$K_{2l}(0,0) = 2(-1)^l (2\pi)^{-2l} \sum_{k \geq 1} k^{-2l} = 2(-1)^l (2\pi)^{-2l} \zeta(2l) = -\frac{B_{2l}}{(2l)!} \ .$$

Thus for m even, or by replacing m by $2m$

Theorem 5.3. (Infinite distributional Euler-MacLaurin formula) *We have*

$$\sum_{n=0}^{+\infty} \delta_n = \frac{1}{2}\delta_0 + \mathbf{1}_{\mathbb{R}_+} + \sum_{l=1}^{m} \frac{B_{2l}}{(2l)!}\delta_0^{(2l-1)} + \frac{D^{2m}}{Dt^{2m}}K_{2m}(t,0) \ .$$

This strongly reminds Euler-MacLaurin formula in distributional form. We apply this to a test function f. Observe that

$$\langle \delta_0^{(2l-1)}, f \rangle = -f^{(2l-1)}(0) \ ,$$

and

$$\left\langle \frac{D^{2m}}{Dt^{2m}}K_{2m}, f \right\rangle = \langle K_{2m}, f^{(2m)} \rangle$$

$$= \left\langle \left(\sum_{k\in\mathbb{Z}^*} \frac{e^{2\pi ikt}}{(2\pi ik)^{2m}} \right) \mathbf{1}_{\mathbb{R}_+}, f^{(2m)} \right\rangle$$

$$= (-1)^m \sum_{k\geq 1} \int_0^{+\infty} \frac{e^{2\pi ikt} + e^{-2\pi ikt}}{(2\pi k)^{2m}} f^{(2m)}(t) \, dt \ .$$

Theorem 5.4. (Infinite Euler-MacLaurin formula) *For a test function f in the Schwarz class*

$$\sum_{n=0}^{+\infty} f(n) = \frac{1}{2}f(0) + \int_0^{+\infty} f(t)dt - \sum_{l=1}^{m} \frac{B_{2l}}{(2l)!}f^{(2l-1)}(0) + (-1)^m \sum_{k\geq 1} \int_0^{+\infty} \frac{e^{2\pi ikt} + e^{-2\pi ikt}}{(2\pi k)^{2m}} f^{(2m)}(t) \, dt \ .$$

In order to recover the standard finite Euler-Maclaurin formula, we substract the infinite formula from itself after a translation by an integer $N \geq 0$.

Theorem 5.5. (Distributional Euler-MacLaurin formula) *We have*

$$\sum_{n=0}^{N} \delta_n = \frac{1}{2}(\delta_0 + \delta_N) + \mathbf{1}_{[0,N]} + \sum_{l=1}^{m} \frac{B_{2l}}{(2l)!}(\delta_0^{(2l-1)} - \delta_N^{(2l-1)}) + \frac{D^{2m}}{Dt^{2m}}\left(\sum_{k\in\mathbb{Z}^*} \frac{e^{2\pi ikt}}{(2\pi k)^{2m}} \right) \mathbf{1}_{[0,N]} \ .$$

A particular case is a finite version of the "half" classical Poisson formula that we used in the previous section for deriving Abel-Plana formula.

Theorem 5.6. (Finite half Poisson formula) *We have*

$$(10) \qquad \sum_{n=0}^{N} \delta_n = \frac{1}{2}(\delta_0 + \delta_N) + \mathbf{1}_{[0,N]} + \left(\sum_{k\in\mathbb{Z}^*} e^{2\pi ikt} \right) \mathbf{1}_{[0,N]} \ .$$

Now, we check that this finite distributional Euler-MacLaurin formula gives the usual formula.

Theorem 5.7. *For $N, m \geq 0$ and f a C^{2m} function in $[0, N]$ we have*

$$\sum_{n=0}^{N} f(n) = \int_0^N f(t)dt + \frac{1}{2}(f(0) + f(N)) + \sum_{k=1}^{m} \frac{B_{2k}}{(2k)!} \left(f^{(2k-1)}(N) - f^{(2k-1)}(0) \right)$$

$$+ (-1)^m \sum_{k=1}^{+\infty} \int_0^N \frac{e^{2\pi ikt} + e^{-2\pi ikt}}{(2\pi k)^{2m}} f^{(2m)}(t)dt$$

Proof. By density of the Schwarz class in the C^{2m} topology of C^{2m} functions in $[0, N]$, it is enough to prove the formula for a function f in the Schwarz class. Then it is a direct application of the previous finite distributional formula. $\square$

With our approach we can generalize Euler-MacLaurin formula by choosing the parameter $\sigma \neq 0$.

Theorem 5.8. *For $\sigma \neq 0$ we have:*

$$\sum_{n=0}^{+\infty} \delta_n = \frac{1}{2}\delta_0 + \mathbf{1}_{\mathbb{R}_+} - \sum_{j=d+1}^{m} e^{-\sigma t} K_j(0, \sigma)\delta_0^{(j-1)} + e^{-\sigma t}\frac{D^m}{Dt^m}K_m(t, -\sigma)$$

$$= \frac{1}{2}\delta_0 + \mathbf{1}_{\mathbb{R}_+} - \sum_{k=0}^{m-1} L_k(\sigma)\,\delta_0^{(k)} + e^{-\sigma t}\frac{D^m}{Dt^m}K_m$$

where

$$L_k(\sigma) = \sum_{l=\max(d+1,k+1)}^{m} \binom{l-1}{k} \sigma^{l-1-k} K_l(0, -\sigma) \ ,$$

and

$$K_l(0, -\sigma) = \sum_{k\in\mathbb{Z}^*} \frac{1}{(2\pi ik + \sigma)^l} = (2\pi i)^{-l} \left(\zeta\left(l, \frac{\sigma}{2\pi i}\right) + (-1)^l \zeta\left(l, -\frac{\sigma}{2\pi i}\right) \right) \ ,$$

where we use Hurwitz zeta function

$$\zeta(s, q) = \sum_{n=0}^{+\infty} \frac{1}{(n+q)^s} \ .$$

Remark 5.9. The approach presented to the classical Euler-MacLaurin summation formula consists in applying our Poisson-Newton formula to the simplest Dirichlet series $f(s) = 1 - e^{-s}$. It is clear that this admits an infinite number of generalizations applying it to arbitrary Dirichlet series. We will develop these questions elsewhere, but we want to notice that in general we get summation formulas over the set of

frequencies of well behaved Dirichlet series. In particular, we can obtain summation formulas for quasi-peroiodic sequences, etc

5.5. **Ramanujan theory of the constant of a series.** Ramanujan developed an heuristic theory of summation of series from Euler-MacLaurin formula. He associated to some divergent series its "constant" that in his words "It is like the center of gravity of a body" (see [23] p. 40 of the first notebook). We refer to chapter 6 of [5] for an overview of the theory as exposed in Ramanujan's first notebook. As explained by Berndt, Hardy attempted to formalize Ramanujan's theory in [17] in order to make proper sense of Ramanujan's constant for summable series. This does not seem quite in the spirit of Ramanujan. Its theory makes sense for sums of the type

$$\sum_{n=1}^{+\infty} f(n) \ ,$$

where the general term $f(n)$ is given by an analytic function f with global properties.

Writing in the following form our infinite distributional form of Euler-McLaurin, theorem 5.3,

$$\sum_{n=1}^{+\infty} \delta_n - \mathbf{1}_{\mathbb{R}_+} = -\frac{1}{2}\delta_0 + \sum_{l=1}^{m} \frac{B_{2l}}{(2l)!}\delta_0^{(2l-1)} + \frac{D^{2m}}{Dt^{2m}}K_{2m}(t,0) \ .$$

we see that it is natural to define the Ramanujan class of tests functions f with global properties for which

$$\lim_{m \to +\infty} \left\langle \frac{D^{2m}}{Dt^{2m}}K_{2m}(t,0), f \right\rangle = 0 \ .$$

For these it is natural to define the Ramanujan distribution RC of infinite order and the Ramanujan constant of the series as Ramanujan does by

$$RC(f) = \langle RC, f \rangle = -\frac{1}{2}f(0) - \sum_{l=1}^{+\infty} \frac{B_{2l}}{(2l)!}f^{(2l-1)}(0) \ .$$

In some sense we can write

$$RC(f) = \sum_{n=1}^{+\infty} f(n) - \int_0^{+\infty} f(t) \ dt \ ,$$

and $RC(f)$ weights the "equilibrium" between the divergence of the series and the divergence of the integral.

In this way we can develop rigorously Ramanujan theory, and this will be done elsewhere.

5.6. **Location of the divisor of a Dirichlet series.** From the Poisson formula, we get that the distribution $W(f)$ is an atomic distribution on $\mathbb{R}_+^*$. Thus the sum of exponentials associated to the zeros cannot be a convergent series for $t \in \mathbb{R}_+^*$. But the following lemma gives a simple condition which would imply the analytic convergence of the sum. For $\theta_1 < \theta_2$, $\theta_2 - \theta_1 < \pi$, denote by $C(\theta_1, \theta_2)$ the cone of values of $s \in \mathbb{C}$ with

$$\theta_1 < \operatorname{Arg} s < \theta_2 \,.$$

Lemma 5.10. *If $\{\rho\} \subset C(\theta_1, \theta_2)$, then*

$$W(t) = \sum_\rho n_\rho e^{\rho t} \,,$$

is a holomorphic function in $C(\pi/2 - \theta_1, 3\pi/2 - \theta_2)$.

Proof. For $t \in C(\pi/2 - \theta_1, 3\pi/2 - \theta_2)$ we have $\Re(\rho t) < 0$, whence

$$\left| e^{\rho t} \right| < 1 \,,$$

and the series $K_\ell(t)$ defined in (4) is holomorphic in that region, so the results follows. $\qquad\square$

From this we obtain the following straightforward corollary:

Corollary 5.11. *The divisor of any Dirichlet series cannot be contained in a cone $C(\theta_1, \theta_2)$ for $\pi/2 < \theta_1 < \theta_2 < 3\pi/2$.*

In particular, once established the functional equation for the Riemann zeta function and its unique pole at $s = 1$, we get that it must have an infinite number of non-real zeros in the critical strip $0 < \Re s < 1$ (this is not a hard result in any case).

The above motivates the definition of θ-distribution:

Definition 5.12. *A θ-distribution is a Newton-Cramer distribution W in $\mathbb{R}$, associated to a finite order divisor $\sum_\rho n_\rho \rho$ such that the series*

$$W(t) = \sum_\rho n_\rho e^{\rho t}$$

is absolutely convergent and defines an analytic function in compact sets of $\mathbb{R}_+^*$.

We also talk about germs of θ-distributions for the germ to the right of 0 of θ-distributions. Also we define left-directed divisors:

Definition 5.13. *A divisor $D = \sum_\rho n_\rho \rho$ is left-directed if it is contained in a left cone $C(\theta_1, \theta_2)$ with $\pi/2 < \theta_1 < \theta_2 < 3\pi/2$.*

Thus we proved above that no non-constant Dirichlet series has a left-directed divisor and:

Proposition 5.14. *If f is a meromorphic function of finite order with a left-directed divisor, then its Newton-Cramer distribution $W(f)$ is a θ-distribution.*

5.7. **Hadamard regularization of θ-distributions.** For a θ-distribution, one may ask naturally what is the relation of the Hadamard regularization at 0 of the germ of analytic function $W(t)$ on $]0, \epsilon[$ and the distribution W. We show that by the local Hadamard regularization we recover the local structure of the germ defined by W at 0. But the theorem we present below has also a global meaning in $\mathbb{R}_+$. It shows that the Hadamard formula gives a global integral formula for the full distribution W.

Theorem 5.15. *Let $W = W(\sigma_1, d)$ be a θ-distribution in $\mathbb{R}$ associated to a left-directed divisor with convergence exponent $d \geq 2$. If $\sigma_1 \in \mathbb{R}$ is the vertex of a left cone containing the divisor, then there are constants $C > 0$ and $\epsilon > 0$ such that for $t > 0$*

$$|W(t)| \leq Ct^{-d}e^{(\sigma_1 - \epsilon)t} .$$

The distribution W can be computed by Hadamard regularization at 0 for a test function φ such that $\psi = e^{\sigma_1 t}\varphi \in \mathcal{S}$ is in the Schwartz class,

$$\langle W, \varphi \rangle = \int_0^{+\infty} W(t)e^{-\sigma_1 t} \left(\psi(t) - \psi(0) - \psi'(0)t - \ldots - \frac{\psi^{(d-2)}(0)}{(d-2)!}t^{d-2} \right) dt$$

$$= \int_0^{+\infty} W(t) \left(\varphi(t) - e^{-\sigma_1 t} \sum_{l=0}^{d-2} \frac{1}{l!} \left(\sum_{k=0}^{d-l-2} \frac{(\sigma_1 t)^k}{k!} \right) \varphi^{(l)}(0) \right) dt$$

$$= \int_0^{+\infty} W(t) \left(\varphi(t) - \sum_{l=0}^{d-2} \frac{1}{l!} \varphi^{(l)}(0)t^l \right) dt + \sum_{l=0}^{d-2} c_l(\sigma_1) \, \varphi^{(l)}(0) ,$$

where

$$c_l(\sigma_1) = \frac{1}{l!} \int_0^{+\infty} W(t)t^l e^{-\sigma_1 t} R_{d-1-l}(\sigma_1 t) \, dt ,$$

and

$$R_n(x) = e^x - \sum_{k=0}^{n} \frac{x^k}{k!} = \sum_{k=n+1}^{+\infty} \frac{x^k}{k!} .$$

Therefore the distribution W only differs by a contribution at 0 from the classical Hadamard regularization H,

$$W = H + \sum_{l=0}^{d-2} c_l(\sigma_1) \, \delta_0^{(l)} .$$

Note that we can extend the integrals to the whole of $\mathbb{R}$ since $W = 0$ in $\mathbb{R}_-^*$.

Proof. From definition 6.1 the divisor is contained in a left strict cone, so by proposition 5.10 we have that W is holomorphic in a t-cone containing the positive real axis, in particular it is analytic in $\mathbb{R}_+^*$. Moreover $e^{\sigma_1 t} W$ is the d-th derivative of the uniformly bounded holomorphic function K_d in that cone. Since σ_1 is not part of the divisor, we can consider $\sigma_1 - \epsilon$ instead of σ_1 and open slightly the cone in order to have a left cone with vertex at $\sigma_1 - \epsilon$ still containing the divisor. Thus we do have the better estimate

$$|K_d(t)| \leq C e^{-\epsilon \Re t} ,$$

because we have in the new cone in the t-plane

$$\Re((\rho - \sigma_1)t) \leq -\epsilon \Re t < 0 .$$

For $t_0 \in \mathbb{R}_+$, we have that the distance $d(t_0)$ to the boundary of the t-cone is $\geq C|t_0|$, and the estimate of the theorem results from Cauchy estimate,

$$|W(t_0)| = \frac{1}{d!} e^{\sigma_1 t_0} \left| \int_{C(t_0, d(t_0))} \frac{K_d(z)}{(z - t_0)^{d+1}} \frac{dz}{2\pi i} \right| \leq C|t_0|^{-d} e^{(\sigma_1 - \epsilon)t_0} .$$

Note that in the same way we obtain the control of the derivatives, for $l = 0, 1, \ldots, d$ and $t > 0$

$$\left| \frac{d^l}{dt^l}(K_d(t) - K_d(0)) \right| = \left| \frac{d^{l+1}}{dt^{l+1}} K_{d-1}(t) \right| \leq C t^{-l-1} e^{-\epsilon t} .$$

Observing that $K_d(t) - K_d(0) \to 0$ for $t \to 0$ in the cone, we obtain that for $t \to 0$, and $l = 1, 2, \ldots, d-1$

$$\frac{d^l}{dt^l}(K_d(t) - K_d(0)) = o(t^{-l}) .$$

Define

$$\tilde{\psi}(t) = \psi(t) - \psi(0) - \psi'(0)t - \ldots - \frac{\psi^{(d-2)}(0)}{(d-2)!} t^{d-2} ,$$

and observe that near 0

$$\frac{d^l}{dt^l}\tilde{\psi}(t) = \mathcal{O}(t^{d-1-l}) .$$

Therefore, for $l = 1, 2, \ldots, d-1$ we have

$$\lim_{t \to 0} \frac{d^l}{dt^l}(K_d(t) - K_d(0)) \frac{d^{d-l-1}}{dt^{d-l-1}} \tilde{\psi}(t) = 0 .$$

Using that for $t > 0$

$$W(t) = e^{\sigma_1 t} \frac{d^d}{dt^d}(K_d(t) - K_d(0)),$$

we have performing d integrations by parts

$$\langle W, \varphi \rangle = \langle K_d(t) - K_d(0), (-1)^d \frac{d^d}{dt^d}(e^{\sigma_1 t}\varphi) \rangle$$

$$= \int_0^{+\infty} (-1)^d (K_d(t) - K_d(0)) \frac{d^d}{dt^d}(\psi)\, dt$$

$$= \int_0^{+\infty} (-1)^d (K_d(t) - K_d(0)) \frac{d^d}{dt^d} \left(\psi(t) - \psi(0) - \psi'(0)t - \ldots - \frac{\psi^{(d-2)}(0)}{(d-2)!} t^{d-2} \right) dt$$

$$= \int_0^{+\infty} \left(\frac{d^d}{dt^d} K_d(t) \right) \left(\psi(t) - \psi(0) - \psi'(0)t - \ldots - \frac{\psi^{(d-2)}(0)}{(d-2)!} t^{d-2} \right) dt$$

where we used the estimates for the vanishing at the evaluations at 0 and $+\infty$. This proves the Hadamard regularization formula stated in the theorem. $\qquad\square$

6. FUNCTIONAL EQUATIONS.

We give in this section a precise definition of the property of "having a functional equation". We know no reference in the classical literature.

We say that the divisor D_1 is contained in the divisor D_2, and denote this by

$$D_1 \subset D_2\ ,$$

if any zero, resp. pole, of D_1 is a zero, resp. pole, of D_2, and $|n_\rho(D_1)| \leq |n_\rho(D_2)|$ for all $\rho \in \mathbb{C}$.

Definition 6.1. *The meromorphic function f has a functional equation if there exists $\sigma^* \in \mathbb{R}$ and a divisor $D \subset \mathrm{Div}\, f$ contained in a left cone $\sigma_1 + C(\theta_1, \theta_2)$, with $\pi/2 < \theta_1 < \theta_2 < 3\pi/2$, such that $\mathrm{Div}\, f - D$ is infinite and symmetric with respect to the vertical line $\{\Re s = \sigma^*\}$.*

Proposition 6.2. *If f has a functional equation and the divisor of f is contained in a left half plane then $\sigma^* \in \mathbb{R}$ is unique.*

Proof. Otherwise, if they were two distinct values σ^*, then $\mathrm{Div}\, f$ would have an infinite subdivisor invariant by a real translation and this contradicts the hypothesis that the divisor of f is contained in a left half plane. $\qquad\square$

Proposition 6.3. *If f has a functional equation and the divisor of f is contained in a left half plane then $\mathrm{Div}\, f - D$ is contained in a vertical strip. The minimal strip $\{\sigma_- < \Re s < \sigma_+\}$, $\sigma_+ < \sigma_1$, with this property is the critical strip and $\sigma^* = \frac{\sigma_- + \sigma_+}{2}$ is its center.*

Proof. Since $\mathrm{Div}(f)$ has no zeros nor poles for $\Re s > \sigma_1$, the divisor of $\mathrm{Div}\, f - D$ is contained in a vertical strip due to the symmetry. The minimal vertical strip has to be compatible with the functional equation, hence $\sigma^* = \frac{\sigma_- + \sigma_+}{2}$. $\qquad\square$

Proposition 6.4. *If f has a functional equation and the divisor of f is contained in a left half plane then there is a unique minimal divisor D (i.e., with $|n_\rho(D)|$ minimal for all $\rho \in \mathbb{C}$), and a unique decomposition $D = D_0 + D_1$, D_0 and D_1 with disjoint supports, with D_0 contained in a left cone $\sigma^* + C(\theta_1, \theta_2)$, with $\pi/2 < \theta_1 < \theta_2 < 3\pi/2$, and D_1 a finite divisor contained in the half plane $\{\Re s > \sigma^*\}$, such that $\mathrm{Div}\, f - D$ is infinite and symmetric with respect to the vertical line $\{\Re s = \sigma^*\}$.*

Proof. We start with D minimal as in the definition, and we define D_0 to be the part of D to the left of $\{\Re s = \sigma^*\}$ and D_1 the remaining part. It is easy to see that D_0 is contained in a left cone with vertex at σ^*. $\qquad\square$

Theorem 6.5. *If f has a functional equation and the divisor of f is contained in a left half plane then there exists a meromorphic function χ with $\mathrm{Div}\, \chi = D \subset \mathrm{Div}\, f$, such that the function $g(s) = \chi(s) f(s)$ satisfies the functional equation*

$$g(2\sigma^* - s) = g(s) \ .$$

Moreover, we can write $\chi = \chi_0 \cdot R$ with $\mathrm{Div}\, \chi_0 = D_0 - \tau^* D_1$ and $\mathrm{Div}\, R = D_1 + \tau^* D_1$, where τ is the reflexion along $\Re s = \sigma^*$, and R is a unique rational function up to a non-zero multiplicative constant.

The meromorphic function χ (or χ_0) is uniquely determined up to a factor $\exp h(s - \sigma^*)$ where h is an even entire function. If f has convergence exponent $d < +\infty$, then we can take χ of convergence exponent d, and then χ is uniquely determined up to a factor $\exp P(s - \sigma^*)$ where P is an even polynomial of degree less than d. In particular, when f is of order 1 then χ and χ_0 are uniquely determined up to a non zero multiplicative constant.

Proof. From Proposition 6.3 we know that σ^* is uniquely determined as the center of the critical strip (which is defined only in terms of the divisor of f). Translating everything by σ^* we can assume that $\sigma^* = 0$. By minimality the divisor of χ is uniquely determined. Then χ is uniquely determined up to a factor $\exp h(s)$ where h is an entire function. If $\hat\chi(s) = (\exp h(s))\chi(s)$ gives also a functional equation for f, then we have

$$f(s) = \frac{\chi(-s)}{\chi(s)} f(-s) = \frac{\chi(-s)}{\chi(s)} \frac{\hat\chi(s)}{\hat\chi(-s)} f(s).$$

Therefore

$$\exp(h(s) - h(-s)) = 1,$$

so for some $k \in \mathbb{Z}$,

$$h(s) - h(-s) = 2\pi i k \ .$$

Specializing for $s = 0$ we get $k = 0$ and h is even.

When f is of convergence exponent $d < +\infty$, and since the divisor of χ is contained in the divisor of f, then we can take χ of convergence exponent at most d. $\qquad\square$

If f is real analytic, then it is easy to see that χ must be real analytic up to the Weierstrass factor. We will always choose χ to be real analytic. Then $g = \chi f$ is real analytic, and we have a four-fold symmetry and g is symmetric with respect to the vertical line $\Re s = \sigma^*$.

It is of interest to dissociate the contribution to the distribution $W(f)$ of the part of the divisor of f that comes from the divisor of χ_0. This part is the θ-distribution $W(\chi_0)$ for which we have Hadamard regularization formula. For a test function φ such that $\psi = e^{\sigma_1 t}\varphi \in \mathcal{S}$ is in the Schwartz class,

$$\langle W(\chi_0), \varphi \rangle = \int_0^{+\infty} W(\chi_0)(t)e^{-\sigma_1 t}\left(\psi(t) - \psi(0) - \psi'(0)t - \ldots - \frac{\psi^{(d-2)}(0)}{(d-2)!}t^{d-2}\right) dt$$

$$= \int_0^{+\infty} W(\chi_0)(t)\left(\varphi(t) - \sum_{l=0}^{d-2}\frac{1}{l!}\varphi^{(l)}(0)t^l\right) dt + \sum_{l=0}^{d-2} c_l(\sigma_1)\,\varphi^{(l)}(0) ,$$

where

$$c_l(\sigma_1) = \frac{1}{l!}\int_0^{+\infty} W(\chi_0)(t)t^l e^{-\sigma_1 t} R_{d-1-l}(\sigma_1 t) \, dt ,$$

where

$$R_n(x) = e^x - \sum_{k=0}^{n}\frac{x^k}{k!} = \sum_{k=n+1}^{+\infty}\frac{x^k}{k!} .$$

Example. For the Riemann zeta function $f(s) = \zeta(s)$ we have $\sigma^* = 1/2$, $\sigma_- = 0$, $\sigma_+ = 1$, $D_0 = -2\mathbb{N}^*$, $D_1 = \{1\}$, and

$$\chi(s) = \pi^{-s/2}\Gamma(s/2)s(s-1),$$
$$\chi_0(s) = \pi^{-s/2}\Gamma(s/2),$$
$$P(s) = s(s-1).$$

Note that

$$g(s) = \chi(s)\zeta(s) = \pi^{-s/2}\Gamma(s/2)s(s-1)\zeta(s) = 2\xi(s) .$$

(using Riemann's classical notation for ξ).

Next, we determine when a finite Dirichlet series satisfies a functional equation.

Proposition 6.6. *A finite Dirichlet series*

$$f(s) = 1 + \sum_{n=1}^{N} a_n e^{-\lambda_n s} \ ,$$

satisfies a functional equation if and only if it is of the form

$$f(s) = e^{\mu s} \sum_{i=0}^{[(N-1)/2]} a_i \big(e^{(-\lambda_i + \mu)s} + c\, e^{(\lambda_i - \mu)s} \big),$$

where $c = 1$ if N is even, $c = \pm 1$ if N is odd.

Proof. The Dirichlet series $f(s)$ is of order 1. Suppose that there is some $\chi(s)$ of order 1 with zeros and poles in a left cone such that $g(s) = \chi(s)f(s)$ is symmetric with respect to $\Re s = \sigma^*$. By translating, we can assume $\sigma^* = 0$.

The zeros of $f(s)$ lie in a strip, since $e^{-\lambda_n s} f(-s)$ is also a Dirichlet series. Therefore $\chi(s)$ has finitely many zeros and poles, and hence $\chi(s) = \frac{Q_1(s)}{Q_2(s)} e^{\mu s}$, for some polynomials $Q_1(s), Q_2(s)$. The functional equation $g(s) = g(-s)$ reads

$$Q_1(s)Q_2(-s) \sum_{n=0}^{N} a_n e^{(\mu - \lambda_n)s} = Q_2(s)Q_1(-s) \sum_{n=0}^{N} a_n e^{(\lambda_n - \mu)s} \ ,$$

where we have set $a_0 = 1$, $\lambda_0 = 0$.

From this it follows that $Q_1(s)Q_2(-s) = c\, Q_2(s)Q_1(-s)$, $c \in \mathbb{C}^*$. It follows easily that $c = \pm 1$. Also $0, \lambda_1, \ldots, \lambda_N$ is a sequence symmetric with respect to $\mu = \lambda_N/2$. So $\lambda_{N-i} = 2\mu - \lambda_i$ and $a_{N-i} = a_i$.

If N even, then $\lambda_{N/2} = \mu$, $c = 1$, and

$$\sum_{n=0}^{N} a_n e^{-\lambda_n s} = e^{\mu s} \sum_{i=0}^{N/2-1} a_i \big(e^{(-\lambda_i + \mu)s} + e^{(\lambda_i - \mu)s} \big) + a_{N/2} e^{\mu s} \ .$$

If N is odd, then

$$\sum_{n=0}^{N} a_n e^{-\lambda_n s} = e^{\mu s} \sum_{i=0}^{(N-1)/2} a_i \big(e^{(-\lambda_i + \mu)s} + c\, e^{(\lambda_i - \mu)s} \big),$$

where if $c = -1$, we have $\chi(s) = s\, e^{\mu s}$. $\square$

An example without functional equation. Consider the elementary Dirichlet series

(11) $$f(s) = 1 + a_1 e^{-\lambda_1 s} + a_2 e^{-\lambda_2 s}$$

with $0 < \lambda_1 < \lambda_2$ and $a_j \neq 0$. It is an entire function on $\mathbb{C}$ of order 1.

If λ_1, λ_2 are rationally dependent, then we may write $f(s) = 1 + a_1 \left(e^{\lambda s}\right)^{k_1} + a_2 \left(e^{\lambda s}\right)^{k_2}$, for $\lambda_1 = k_1 \lambda$, $\lambda_2 = k_2 \lambda$, $k_1, k_2 > 0$ and coprime. We can compute the zeros solving the algebraic equation $1 + a_1 X^{k_1} + a_2 X^{k_2} = 0$. Therefore, the zeros of $f(s)$ lie in at most k_2 vertical lines, and they form k_2 arithmetic sequences of the same purely imaginary step.

If λ_1, λ_2 are rationally independent, then we cannot compute the zeros in general. We know that they lie in a half-plane $\Re s < \sigma_1$. Also $a_2^{-1} e^{\lambda_2 s} f(s)$ converges to 1 for $\Re s \to -\infty$. So the zeros of $f(s)$ are located in a half-plane $\Re s > \sigma_2$. Hence in a strip. By Corollary 5.11, there are infinitely many zeros in that strip.

Now, let (ρ) be the set of zeros. Then $\Lambda = \{\mathbf{k} = (k_1, k_2) \in \mathbb{N}^2 \,|\, (k_1, k_2) \neq (0, 0)\}$, and

$$b_{\mathbf{k}} = \frac{(-1)^{k_1 + k_2}}{k_1 + k_2} \binom{k_1 + k_2}{k_1} a_1^{k_1} a_2^{k_2}$$

and

$$\sum n_\rho e^{\rho t} = \sum_{\mathbf{k}} (\lambda_1 k_1 + \lambda_2 k_2) b_{\mathbf{k}} \delta_{\lambda_1 k_1 + \lambda_2 k_2} ,$$

on $\mathbb{R}_+^*$.

By Proposition 6.6, the Dirichlet series (11) does not have a functional equation unless $\lambda_2 = 2\lambda_1$.

6.1. Functional equation and Hadamard factorization.

Let f be a meromorphic function of finite order which has its divisor contained in a left half plane and which has a functional equation. In order to simplify we assume that σ^* is not part of the divisor. For $g(s) = \chi(s) f(s)$ we have

$$g(2\sigma^* - s) = g(s) ,$$

and when we express this symmetry in the Hadamard factorization, we get

$$Q_{g, \sigma^*}(2\sigma^* - s) = Q_{g, \sigma^*}(s) ,$$

hence if we write

$$Q_{g, \sigma^*}(s) = \sum_k a_k (s - \sigma^*)^k ,$$

the symmetry implies that all odd coefficients are zero $a_1 = a_3 = \ldots = 0$.

We observe also that

$$Q_g = Q_\chi + Q_f ,$$

and for the discrepancy polynomials

$$P_g = P_\chi + P_f .$$

Also if the exponent of convergence is $d = 2$, pairing the Weierstrass factors for symmetric zeros ρ and $2\sigma^* - \rho$, gives

$$E_{d-1}\left(\frac{s - \sigma^*}{\rho - \sigma^*}\right) \cdot E_{d-1}\left(\frac{s - \sigma^*}{(2\sigma^* - \rho) - \sigma^*}\right) = -\frac{1}{\rho - \sigma^*}(s - \rho)(s - (2\sigma^* - \rho)) \ .$$

Therefore for $d = 2$, the discrepancy polynomial is constant and we have

$$c_0(\chi) = P_\chi = P_{\chi_0} = c_0(\chi_0) \ .$$

Note that this also holds if we know that $g = 2$, in particular when $d = 3$ and f is a Dirichlet series by Corollary 3.7. On the other hand the functional equation implies that

$$c_0(g) = 0 \ ,$$

therefore we obtain:

Proposition 6.7. *Let f be a meromorphic function of exponent of convergence $d = 2$, or $g = 2$, which has its divisor contained in a left half plane and has a functional equation. We assume that σ^* is not part of the divisor.*

We have

$$c_0(\chi_0, \sigma^*) + c_0(f, \sigma^*) = 0 \ .$$

6.2. **Gauss formula for the logarithmic derivative of the Γ-function.** From our general Poisson-Newton formula we get in one stroke Gauss and related formulas for the logarithmic derivative of the Γ-function.

The Γ-function was defined by Euler by the limit

$$\Gamma(s) = \lim_{n \to +\infty} \frac{(n-1)!}{s(s+1)\dots(s+n-1)} n^s \ ,$$

which shows that $f(s) = 1/\Gamma(s)$ is an entire function with simple zeros at $\rho_n = -n$ for $n = 0, 1, 2, \dots$ Hence the exponent of convergence is $d = 2$. The divisor of f is left-directed and the associated Newton-Cramer distribution is the θ-distribution

$$W(1/\Gamma)(t) = \sum_{n=0}^{+\infty} e^{(-n)t} = \frac{1}{1 - e^{-t}} \ .$$

From Euler's definition we can derive directly the Hadamard factorization

$$f(s) = \frac{1}{\Gamma(s)} = se^{\gamma s} \prod_{n=1}^{+\infty} \left(1 + \frac{s}{n}\right) e^{-s/n} \ ,$$

where γ is Euler's constant

$$\gamma = \lim_{n \to +\infty} \sum_{k=1}^{n} \frac{1}{k} - \log n = \int_0^{+\infty} \left(\frac{1}{1 - e^{-t}} - \frac{1}{t}\right) e^{-t} \, dt$$

(see [29], p.246 for the integral expression). This gives the Hadamard interpolation for $\sigma_1 = 0$. Unfortunately $\sigma_1 = 0$ is a zero of f and the Hadamard regularization is divergent at $+\infty$. So we choose $\sigma_1 = 1$ (for example) and derive the associated Hadamard factorization

$$f(s) = \frac{1}{\Gamma(s)} = \frac{1}{s-1}\frac{1}{\Gamma(s-1)} = e^{\gamma(s-1)} \prod_{n=0}^{+\infty} \left(1 - \frac{s-1}{-n-1}\right) e^{\frac{s-1}{-n-1}} .$$

Thus we have for some $n \in \mathbb{Z}$,

$$Q(s) = 2\pi i n - \gamma + \gamma s ,$$

and the discrepancy is a constant polynomial

$$P_f(s) = -Q'(s) = -\gamma = c_0 .$$

Applying the general Poisson-Newton formula to f we have

$$\mathcal{L}(W(f)) = \mathcal{L}\left(\frac{e^{-t}}{1 - e^{-t}}\right) = -\gamma - \frac{\Gamma'(s)}{\Gamma(s)} .$$

And by the Hadamard regularization formula applied with $\sigma_1 = 1$, $d = 2$ and the test function $\varphi(t) = e^{-st}$, $\psi(t) = e^{(1-s)t}$,

$$\mathcal{L}(W(f)) = \langle W(f), e^{-st}\rangle = \int_0^{+\infty} \frac{1}{1 - e^{-t}} e^{-t} \left(e^t e^{-st} - 1\right) dt$$

$$= \int_0^{+\infty} \frac{1}{1 - e^{-t}} \left(e^{-st} - e^{-t}\right) dt$$

Therefore we get

$$\frac{\Gamma'(s)}{\Gamma(s)} = -\gamma - \int_0^{+\infty} \frac{1}{1 - e^{-t}} \left(e^{-st} - e^{-t}\right) dt ,$$

and plugging the integral expression for Euler's constant we finally prove Gauss formula for the logarithmic derivative of the Γ-function [29], p.246).

Theorem 6.8. (C.F. Gauss)

$$\frac{\Gamma'(s)}{\Gamma(s)} = \int_0^{+\infty} \left(\frac{e^{-t}}{t} - \frac{e^{-st}}{1 - e^{-t}}\right) dt .$$

So, according to our interpretation, Gauss integral formula is equivalent to all Newton relations for the roots e^{-n} for all exponents $t \in \mathbb{R}_+^*$. This interpretations seems new.

In a similar way we can obtain Binet formula for the logarithm of the Γ-function (see the section on the Γ-function in [29]), as well as general formulas for higher

Γ-functions, the first example being Barnes Γ-function. This will be developed in future versions of this article.

6.3. **Explicit formulas for Riemann zeros.** In this section we apply our Poisson-Newton formula to the Riemann zeta function. We obtain a non-classical form of the Explicit Formula in analytic number theory. The classical forms can be derived from our distributional formula.

Explicit formulas in analytic number theory go back to the original memoir of Riemann [24] on the analytic properties of Riemann zeta function where it is the central point of the derivation of Riemann's asymptotic formula for the growth of the number of primes. It relates prime numbers with non-trivial zeros of Riemann zeta function. Despite the mystery about the precise location of the non-trivial zeros, many of such formulas were developped at the end of the XIXth century and the beginning of the XXth century (see [20]). Later, general explicit formulas were developped by A.P. Guinand [13], J. Delsarte [11], A. Weil [28] and K. Barner [4], these last ones in general distributional form. A classical form of this Explicit Formula is the following by K. Barner [4] :

Theorem 6.9. *For an appropriate test function φ with Fourier transform $\hat{\varphi}$ analytic in a large enough strip, we have*

$$\sum_{\gamma} \hat{\varphi}(\gamma) = \hat{\varphi}(i/2) + \hat{\varphi}(-i/2) + \frac{1}{2\pi} \int_{\mathbb{R}} \Psi(t)\hat{\varphi}(t) \ dt$$

$$- \sum_{p,k\geq 1} (\log p)p^{-k/2} \left(\varphi(k\log p) + \varphi(-k\log p) \right) ,$$

where the $\gamma's$ run over the non-trivial zeros of ζ, the p'sover prim numbers, and

$$\Psi(t) = -\log \pi + \Re \left(\frac{\Gamma'}{\Gamma}(1/4 + it/2) \right) .$$

Let's see how one can recover this classical formula from our Poisson-Newton formula.

We consider the Riemann zeta function defined for $\Re s > 1$ by

$$\zeta(s) = \sum_{n\geq 1} n^{-s} = \sum_{n\geq 1} e^{-s\log n} ,$$

which is a Dirichlet series with $\lambda_n = \log(n+1)$ and $\sigma_1 = 1$ in our notation. It has a meromorphic extension to the complex plane $s \in \mathbb{C}$ with a single simple pole at $s = 1$. It has order $o = 1$ and convergence exponent $d = 2$.

For $\Re s > 1$ we have the Euler product which gives the relation of the zeta function with prime numbers,

$$\zeta(s) = \prod_p (1 - p^{-s})^{-1},$$

where the product is running over the prime numbers p. Thus

$$-\log \zeta(s) = -\sum_{p,\,k \geq 1} \frac{p^{-ks}}{k} = -\sum_{p,\,k \geq 1} \frac{1}{k} e^{-k(\log p)s}.$$

The vector of fundamental frequencies is $\boldsymbol{\lambda} = (\log 2, \log 3, \log 5, \ldots)$. We have $b_{\mathbf{k}} = 0$ when $\mathbf{k}$ has more than one non-zero entry, and $b_{\mathbf{k}} = -1/k$ for $\langle \boldsymbol{\lambda}, \mathbf{k} \rangle = k \log p$.

The Riemann zeta function has a functional equation with $\sigma^* = 1/2$, $\sigma_- = 0$ and $\sigma_+ = 1$. We have, using the notations of section 6,

$$g(s) = g(1 - s)$$
$$g(s) = \chi(s)\zeta(s)$$
$$\chi(s) = \pi^{-s/2}\Gamma(s/2)s(s-1),$$
$$\chi_0(s) = \pi^{-s/2}\Gamma(s/2),$$
$$R(s) = s(s-1).$$

The Riemann zeta function has a single simple pole at $\rho = 1$, and simple real zeros at $\rho = -2n$, for $n = 1, 2, \ldots$, and non-real zeros in the critical strip $\sigma_- = 0 < \Re s < 1 = \sigma_+$, $\rho = 1/2 + i\gamma$. The Riemann Hypothesis conjectures that $\gamma \in \mathbb{R}$, i.e., that all non-real zeros have real part $1/2$. These non-real zeros are conjectured to be simple. Following the tradition we will repeat them according to their multiplicity, so we may skip the multiplicities $n_\rho = 1$ in our subsequent formulas.

The Riemann zeta function is real analytic, and we can apply the symmetric Poisson-Newton formula (theorem 3.9) and the Poisson-Newton formula with parameters (theorem 3.13). More precisely, in order to get the classical formulas and exploit the functional equation, we apply the Poisson-Newton formula with parameters, Corollary 3.14, for $\beta = \sigma^* = 1/2$, so $\sigma_1' = 1 - 1/2 = 1/2$ and we get

Theorem 6.10.
$$\sum_\rho n_\rho e^{(\rho - 1/2)|t|} = 2c_0(\zeta, 1/2)\,\delta_0 - \sum_{p,k \geq 1} (\log p) p^{-k/2} \left(\delta_{k \log p} + \delta_{-k \log p} \right).$$

The contribution at 0, $c_0(\zeta, 1/2)$ is computed in the Appendix, and we have

Theorem 6.11.
$$c_0(\zeta, 1/2) = -\frac{\log \pi}{2} - \frac{\gamma}{2} - 2 \log 2.$$

We can compute explicitly the contribution of the real divisor to the distribution on the left handside:

$$W_0(t) = -e^{|t|/2} + e^{-|t|/2} \sum_{n \geq 1} e^{-2n|t|} = -e^{|t|/2} + e^{-5|t|/2} \frac{1}{1 - e^{-2|t|}} = -e^{|t|/2} + e^{-\frac{3}{2}|t|} \frac{1}{2 \sinh |t|}.$$

Note that $-e^{|t|/2} W_0 = W(\chi)(t) + W(\chi)(-t) = W(\chi_0)(t) + W(\chi_0)(-t) + W(R)(t) + W(R)(-t)$.

So the associated Poisson-Newton formula on $\mathbb{R}$ is

$$\sum_\gamma e^{i\gamma|t|} + W_0(t) = 2c_0(\zeta, 1/2)\, \delta_0 - \sum_{p,k \geq 1} (\log p) p^{-k/2} \left(\delta_{k \log p} + \delta_{-k \log p} \right)$$

$$= 2c_0(\zeta, 1/2)\, \delta_0 - \sum_{n \geq 1} \frac{\Lambda(n)}{\sqrt{n}} \left(\delta_{\log n} + \delta_{-\log n} \right),$$

where Λ is the Von Mangoldt function, $\Lambda(p^k) = \log p$ and $\Lambda(n) = 0$ if n is not the power of a prime number.

For a test function $\varphi \in \mathcal{S}$ in the Schwartz class, consider its Fourier transform

$$\hat{\varphi}(x) = \int_{\mathbb{R}} \varphi(t) e^{-ixt}\, dt.$$

Observe that

$$\hat{\varphi}(\gamma) = \int_{\mathbb{R}} \varphi(t) e^{-i\gamma t}\, dt = \int_{\mathbb{R}_+} \left(\varphi(t) e^{-i\gamma t} + \varphi(-t) e^{-i(-\gamma)t} \right) dt.$$

By the real analyticity of $\zeta(s)$, the set of non-trivial zeros is real symmetric, $(\gamma) = (-\gamma)$, hence

$$\sum_\gamma \hat{\varphi}(\gamma) = \int_{\mathbb{R}_+} (\varphi(t) + \varphi(-t)) \left(\sum_\gamma e^{i\gamma t} \right) dt.$$

Thus applying now our Poisson-Newton formula to the test function φ we get

$$\sum_\gamma \hat{\varphi}(\gamma) + W_0[\varphi] = 2c_0(\zeta, 1/2)\varphi(0) - \sum_{p,k \geq 1} (\log p) p^{-k/2}(\varphi(k \log p) + \varphi(-k \log p)),$$

where $W_0[\varphi]$ is the functional

$$W[\varphi] = \int_{\mathbb{R}} W_0(t) \varphi(t)\, dt.$$

We compute more precisely this functional. We have

$$W_0 = -e^{-|t|/2}(W(\chi)(t) + W(\chi)(-t))$$

$$= -e^{-|t|/2}(W(\chi_0)(t) + W(\chi_0)(-t) + W(R)(t) + W(\bar{R})(-t)).$$

We assume that $\hat{\varphi}$ is holomorphic in a neighborhood of the strip $|\Im t| \leq 1/2$, then we have by the general symmetric Poisson-Newton formula (or by direct computation)

$$
\begin{aligned}
\langle -e^{-|t|/2}(W(R)(t) + W(R)(-t)), \varphi \rangle &= -\int_{\mathbb{R}} e^{-|t|/2}(e^{|t|} + 1)\varphi(t)\, dt \\
&= -\int_{\mathbb{R}} 2\cosh(|t|/2)\varphi(t)\, dt \\
&= -\int_{\mathbb{R}} 2\cosh(t/2)\varphi(t)\, dt \\
&= -\int_{\mathbb{R}} \varphi(t)e^{t/2}\, dt - \int_{\mathbb{R}} \varphi(t)e^{-t/2}\, dt \\
&= -\hat{\varphi}(i/2) - \hat{\varphi}(-i/2)\ .
\end{aligned}
$$

Now, again using the general symmetric Poisson-Newton formula, more precisely, Corollary 3.18 with $\alpha = 1$ and $\beta = 1/2$ applied to χ_0 that is real analytic, we have

$$
-e^{-|t|/2}(W(\chi_0)(t) + W(\chi_0)(-t)) = 2c_0(\chi_0, 1/2)\delta_0 + \mathcal{L}_{1/2}^{-1}\left(2\Re\left(\frac{\chi_0'}{\chi_0}\right)\right)\ .
$$

And using Proposition 6.7,

(12)
$$
c_0(\chi_0, 1/2) + c_0(\zeta, 1/2) = 0\ ,
$$

thus the Poisson-Newton formula applied to an appropriate test function φ is

$$
\begin{aligned}
\sum_{\gamma} \hat{\varphi}(\gamma) =\ &\hat{\varphi}(i/2) + \hat{\varphi}(-i/2) + \left\langle \mathcal{L}_{1/2}^{-1}\left(2\Re\left(\frac{\chi_0'}{\chi_0}\right)\right), \varphi \right\rangle \\
&- \sum_{p,k \geq 1} (\log p)p^{-k/2}(\varphi(k\log p) + \varphi(-k\log p))\,,
\end{aligned}
$$

Now, we have

$$
\frac{\chi_0'(s)}{\chi_0(s)} = -\frac{1}{2}\log\pi + \frac{1}{2}\frac{\Gamma'(s/2)}{\Gamma(s/2)}\ ,
$$

so

$$
\left\langle \mathcal{L}_{1/2}^{-1}\left(2\Re\left(\frac{\chi_0'}{\chi_0}\right)\right), \varphi \right\rangle = \frac{1}{2\pi}\int_{\mathbb{R}} \Psi(t)\hat{\varphi}(t)\, dt\ ,
$$

where

$$
\Psi(t) = -\log\pi + \Re\left(\frac{\Gamma'}{\Gamma}(1/4 + it/2)\right)\ .
$$

Thus we recover the classical form of the Explicit formula stated at the beginning of the section. Historically this form is due to Barner that gave a new form of the Weil functional. Barner's derivation is based on an integral formula, Barner formula, that can be directly derived from our general Poisson-Newton formula.

Note that our "explicit formula" appears more concise that the classical formulation, and even more if we use Corollary 3.14 with $\beta = 0$

Theorem 6.12. *We have*

$$\sum_{\rho} n_\rho e^{\rho|t|} = 2c_0(\zeta, 0)\,\delta_0 - \sum_{p,k\geq 1} (\log p)\,(\delta_{k\log p} + \delta_{-k\log p})\ ,$$

and

$$c_0(\zeta, 0) = -\log(2\pi)\ .$$

We can compute $c_0(\zeta, 0)$ from the known Hadamard factorization of Riemann zeta function. We have (see [26] p.31):

$$\zeta(s) = \frac{e^{bs}}{2(s-1)\Gamma(s/2+1)}\prod_{\rho} E_1(s/\rho) = \frac{e^{bs}}{s(s-1)\Gamma(s/2)}\prod_{\rho} E_1(s/\rho)\ ,$$

where the product is over the non-trivial zeros and

$$b = \log(2\pi) - 1 - \gamma/2\ .$$

Now, we have

$$\frac{1}{s-1} = -e^s\,(E_1(s/1))^{-1}\ ,$$

thus

$$c_0(\zeta, 0) = -Q_\zeta = -b - 1 + c_0\,(1/\Gamma(s/2), 0)\ .$$

But from the Hadamard factorization of the Γ-function we have

$$\frac{1}{\Gamma(s/2)} = \frac{s}{2}e^{\frac{\gamma}{2}s}\prod_{n\geq 1} E_1(s/(-2n))\ ,$$

thus

(13) $$c_0\,(1/\Gamma(s/2), 0) = -\frac{\gamma}{2}\ ,$$

and

(14) $$c_0(\zeta, 0) = -\log(2\pi)\ .$$

We have the final formula:

Theorem 6.13.

$$\sum_{\rho} n_\rho e^{\rho|t|} = -2\log(2\pi)\,\delta_0 - \sum_{p,k\geq 1} (\log p)\,(\delta_{k\log p} + \delta_{-k\log p})\ .$$

Remark 6.14. **(Newton relations interpretation)** Again, by our general interpretation, the Explicit formula appears as the "Newton relations" which links the non-trivial zeros with the primes, the primes playing a similar role than coefficients in Newton formulas.

Remark 6.15. (**General Explicit Formulas**) The derivation given of the classical distributional Explicit Formula is general and applies to any Dirichlet series of order 1 with the required conditions. In this sense the Poisson-Newton formula can be seen as the general Explicit Formula associated to a Dirichlet series. The structure at 0 needs to be computed in general. But when we have a functional equation, one can apply the Poisson-Newton formula with the parameter well chosen so that the structure at 0 vanishes from the formula (as we have done in the previous section for the Riemann zeta function). The divisor on the left cone gives the general "Weil functional" and again, by application of the general Poisson-Newton formula with parameters and using Hadamard regularization for this θ-distribution we get a general Barner integral formula for the functional. Thus we get a general Explicit Formula with the same structure as for the classical one for Riemann zeta function.

6.4. General Guinand equation.

The Newton-Cramer distribution $W(f)$ can be naturally be decomposed in the form of an hiperfunction $W(f) = W_+(f) + W_-(f)$ by separating zeros with positive and negative imaginary parts. Both $W_+(f)$ and $W_-(f)$ are analytic functions on cones sharing $\mathbb{R}_+$ in its boundary. If f is real analytic, then its zeros are symmetric with respect to the real axes, giving a relation between $W_+(f)$ and $W_-(f)$. This relation plugged into

$$W(f) = W_+(f) + W_-(f) \ ,$$

givesa functional equation for $W_+(f)$ which generalizes Guinand functional equation for the Cramer function associated to Riemann zeta-function.

Therefore this proves that we have general Guinand equations for the generalization of the Cramer function for general real analytic Dirichlet series.

6.5. Selberg Trace formula.

It is well known that Selberg trace formula was developed by analogy with the Explicit Formulas in analytic number theory and that this was the original motivation by Selberg (see [25], [7]). In this section we explain this folklore analogy by showing that Selberg Trace Formula results from the Poisson-Newton formula applied to Selberg zeta function. The approach is very similar to that of the previous section and we have a unified treatment of both formulas. The only relevant difference is that Selberg zeta function is of order 2.

We consider a compact Riemannian surface X of genus $h \geq 2$ with a metric of constant negative curvature. Let $\mathcal{P}$ be the set of primitive geodesics. The Selberg zeta function is defined in the half plane $\Re s > 1$ by the Euler product

$$\zeta_X(s) = \prod_{p \in \mathcal{P}} \prod_{k \geq 0} \left(1 - e^{\tau(p)(s+k)}\right) \ ,$$

where $\tau(p)$ is the length of the geodesic p.

We have

$$-\log \zeta_X(s) = \sum_p \sum_{k \geq 0} \sum_{l \geq 1} \frac{1}{l} \, e^{-\tau(p)(s+k)l}$$

$$= \sum_{p,l \geq 1} \frac{1}{l} \, e^{-\tau(p)ls} \, \frac{1}{1 - e^{\tau(p)l}}$$

$$= \sum_{p,l \geq 1} \frac{1}{l} \, e^{-\tau(p)l/2} \, \frac{1}{2\sinh(\tau(p)l/2)} \, e^{-\tau(p)ls}$$

Thus we compute the coefficients

$$b_{p,l} = \frac{1}{l} \, e^{-\tau(p)l/2} \, \frac{1}{2\sinh(\tau(p)l/2)} \ ,$$

and the frequencies

$$\langle \boldsymbol{\lambda}, (p, l) \rangle = \lambda_{p,l} = \tau(p)l \ .$$

One of the fundamental results of the theory is that ζ_X has a meromorphic extension to the complex plane of order 2, exponent of convergence $d = 3$, thus genus $g = 2$ by Corollary 3.7, has a functional equation with $\sigma^* = 1/2$, and its zeros are the following (see [27], p.129):

- Trivial zeros at $s = -k$ with $k = 0, 1, 2, \ldots$ with multiplicity $2(h-1)(2k+1)$.
- Non-trivial zeros $s = 1/2 \pm i\gamma_n$, $n = 0, 1, 2, \ldots$, where $1/4 + \gamma_n^2$ are the eigenvalues of the positive Laplacian $-\Delta_X$ on X counted with multiplicity. The lowest eigenvalue 0 yields two zeros, $s = 1$ that is simple, and the trivial zero $s = 0$ with multiplicity $2(h-1)$ (we exclude the case of $1/4$ as eigenvalue).

For $n < 0$ write $\gamma_n = -\gamma_{-n}$. Therefore the Newton-Cramer distribution decomposes as

$$W(\zeta_X) = V(\zeta_X) + W_0(\zeta_X) \ ,$$

where W_0 is the contribution of the trivial zeros and V the contribution of the non-trivial ones. We compute on $\mathbb{R}^*$ with $\beta = 1/2$

$$\widehat{W_0}(\zeta_X, 1/2)(t) = \sum_{n \in \mathbb{Z}} 2(h-1)(2n+1)e^{(-n-1/2)|t|}$$

$$= 4(h-1) \sum_{n \geq 0} (n+1/2)e^{-(n+1/2)|t|}$$

$$= -4(h-1)\frac{d}{d|t|} \left(\frac{1}{2\sinh(|t|/2)} \right)$$

$$= (h-1)\frac{\cosh(t/2)}{(\sinh(t/2))^2} \ .$$

And we have

$$\widehat{V}(\zeta_X, 1/2)(t) = \sum_{n \in \mathbb{Z}} e^{i\gamma_n |t|} = 2 \sum_{n \geq 0} \cos(\gamma_n t) \ .$$

Now we apply the symmetric Poisson-Newton formula with parameter (Corollary 3.14) with $\beta = 1/2$, and we get

$$\widehat{W}_0(\zeta_X, 1/2) + \widehat{V}(\zeta_X, 1/2) = 2c_0(\zeta_X, 1/2) + \sum_{p,l \in \mathbb{Z}^*} |\langle \boldsymbol{\lambda}, (p,l) \rangle| e^{-|\langle \boldsymbol{\lambda},(p,l) \rangle|/2} \, b_{p,|l|} \, \delta_{\langle \boldsymbol{\lambda},(p,l) \rangle}$$

$$2 \sum_{\gamma} e^{i\gamma t} + (h-1) \frac{\cosh(t/2)}{(\sinh(t/2))^2} = 2 \sum_{p,l \in \mathbb{Z}^*} \frac{\tau(p)}{4 \sinh(\tau(p)|l|/2)} \, \delta_{\tau(p)l} \ ,$$

where we used $c_0(\zeta_X, 1/2) = 0$ by Proposition 6.7.

This yields the classical Selberg Trace Formula as stated in [7]:

Theorem 6.16. *(Selberg Trace Formula) We have*

$$\sum_{\gamma} e^{i\gamma t} = -\frac{1}{2}(g-1) \frac{\cosh(t/2)}{(\sinh(t/2))^2} + \sum_{p,l \in \mathbb{Z}^*} \frac{\tau(p)}{4 \sinh(\tau(p)|l|/2)} \, \delta_{\tau(p)l} \ .$$

We can now manipulate the integral expression for the "Weil functional" à la Barner, using the general Poisson-Newton formula as we have done in the previous section, etc. These computations will be done elsewhere.

Remark 6.17. **(Gutzwiller Trace formula)** The Selberg trace formula is just a particular case of the Gutzwiller Trace formula in Quantum Chaos (see [14]). We see that in general Gutzwiller Trace Formula, that is the central formula in quantum chaos, results from the application of the Poisson-Newton formula to the dynanmical zeta function of the Dynamical System when this zeta function has an analytic extension to the whole complex plane. Thus non-trivial zeros are related to the quantum energy levels and the frequencies to the classical periodic orbits.

6.6. **Lifting formulas.** The "lifting formulas" developed in this section are examples of Poisson-Newton formulas for Dirichlet series of infinite order. They have a transalgebraic meaning that will be developped elsewhere.

We have normalized our Dirichlet series by $a_0 = 1$, but we can carry out the same analysis in general for

$$f(s) = a_0 + \sum_{n \geq 1} a_n \, e^{-\lambda_n s} \ ,$$

with $a_0 \neq 0$.

We can write

$$f(s) = a_0 \left(1 + \sum_{n \geq 1} \frac{a_n}{a_0} e^{-\lambda_n s} \right) ,$$

and we have the associated Poisson-Newton formula in $\mathbb{R}_+^*$

$$\sum_{\rho} n_\rho e^{\rho t} = \sum_{\mathbf{k}} \langle \boldsymbol{\lambda}, \mathbf{k} \rangle \frac{b_{\mathbf{k}}}{a_0^{||\mathbf{k}||}} \delta_{\langle \boldsymbol{\lambda}, \mathbf{k} \rangle} ,$$

where the first sum is over the zeros (ρ) of f. But we can also write

$$f(s) = (a_0 - 1) + \left(1 + \sum_{n \geq 1} a_n e^{-\lambda_n s} \right) = (a_0 - 1) + g(s) .$$

Note that the zeros $\{\eta\}$ of g are the preimages by f of $a_0 - 1$. Hence we have proved

Proposition 6.18. *We have in $\mathbb{R}_+^*$, where the first sum is taken with multiplicity*

$$\sum_{\eta; f(\eta)=a_0-1} e^{\eta t} = \sum_{\mathbf{k}} \langle \boldsymbol{\lambda}, \mathbf{k} \rangle \frac{b_{\mathbf{k}}}{a_0^{||\mathbf{k}||}} \delta_{\langle \boldsymbol{\lambda}, \mathbf{k} \rangle} .$$

Observe that when $||\mathbf{k}|| = 1$, say $\mathbf{k} = (0, \ldots, 0, 1, 0, \ldots)$ with 1 at the j-th place, then

$$b_{\mathbf{k}} = -a_j .$$

Now adding these Poisson-Newton formulas for $a_0 = 1, 2, 3, \ldots$ we get

Corollary 6.19. *In $\mathbb{R}_+^*$ we have*

$$\sum_{m=0}^{+\infty} \left(\sum_{\rho; f(\rho)=m} e^{\rho t} + \sum_{n=1}^{+\infty} \lambda_n a_n \delta_{\lambda_n} \right) = \sum_{\mathbf{k} \in \Lambda; ||\mathbf{k}|| \geq 2} \langle \boldsymbol{\lambda}, \mathbf{k} \rangle b_{\mathbf{k}} \, \zeta(||\mathbf{k}||) \, \delta_{\langle \boldsymbol{\lambda}, \mathbf{k} \rangle} .$$

Or also

Corollary 6.20. *In $\mathbb{R}_+^* - \{\lambda_n\}$, we have*

$$\sum_{\rho \in f^{-1}(\mathbb{Z})} e^{\rho t} = \sum_{\mathbf{k} \in \Lambda; ||\mathbf{k}|| \ even} \langle \boldsymbol{\lambda}, \mathbf{k} \rangle b_{\mathbf{k}} \, (2 \, \zeta(||\mathbf{k}||) - 1) \, \delta_{\langle \boldsymbol{\lambda}, \mathbf{k} \rangle} .$$

7. Appendix

In this appendix we determine the relation between $Q_{f,\sigma}$ and $Q_{f,0}$. In particular, this the variation of the coefficient $c_0(f, \sigma)$ from $c_0(f, 0)$ and we apply this to compute $c_0(\zeta, 1/2)$.

Let f be of finite order and consider the Hadamard factorization of f (see [2] p.208)

$$f(s) = s^{n_0} e^{Q_f(s)} \prod_{\rho \neq 0} E_m(s/\rho)^{n_\rho} \ ,$$

where $m = d - 1 \geq 0$ is minimal for the convergence of the product with

$$E_m(z) = (1 - z)e^{z + \frac{1}{2}z^2 + \ldots + \frac{1}{m}z^m} \ ,$$

and Q_f is a polynomial uniquely defined up to the addition of an integer multiple of $2\pi i$. Consider now $\sigma \in \mathbb{C}$ and the corresponding Hadamard factorization centered at σ,

$$f(s) = (s - \sigma)^{n_\sigma} e^{Q_{f,\sigma}(s)} \prod_{\rho \neq \sigma} E_m \left(\frac{s - \sigma}{\rho - \sigma} \right)^{n_\rho} \ .$$

We want to understand the difference between these two factorizations. We take logarithmic derivatives to get

$$\frac{n_\sigma}{s - \sigma} + Q'_{f,\sigma} + \sum_{\rho \neq 0, \sigma} n_\rho \frac{(s - \sigma)^m}{(\rho - \sigma)^m} \frac{1}{s - \rho} + n_0 \frac{(s - \sigma)^m}{(-\sigma)^m} \frac{1}{s} =$$

$$= \frac{n_0}{s} + Q'_f + \sum_{\rho \neq 0, \sigma} n_\rho \frac{s^m}{\rho^m} \frac{1}{s - \rho} + n_\sigma \frac{s^m}{\sigma^m} \frac{1}{s - \sigma}$$

Therefore

$$Q'_{f,\sigma} - Q'_f = n_0 \frac{(-\sigma)^m - (s - \sigma)^m}{(-\sigma)^m s} + n_\sigma \frac{s^m - \sigma^m}{\sigma^m (s - \sigma)}$$

$$+ \sum_{\rho \neq 0, \sigma} n_\rho \frac{s^m (\rho - \sigma)^m - (s - \sigma)^m \rho^m}{\rho^m (\rho - \sigma)^m} \frac{1}{s - \rho}$$

For $m = 1$ this reduces to

$$(15) \qquad Q'_{f,\sigma} - Q'_f = \frac{n_0}{\sigma} + \frac{n_\sigma}{\sigma} + \sum_{\rho \neq 0, \sigma} n_\rho \frac{-\sigma}{\rho(\rho - \sigma)}$$

For $m = 2$, it becomes

$$Q'_{f,\sigma} - Q'_f = n_0 \frac{2\sigma - s}{\sigma^2} + n_\sigma \frac{s + \sigma}{\sigma^2} + \sum_{\rho \neq 0, \sigma} n_\rho \frac{(-2\rho\sigma + \sigma^2)s + \rho\sigma^2}{\rho^2 (\rho - \sigma)^2} \ .$$

We also can calculate the discrepancy when we change from σ to σ' by considering

$$Q_{f.\sigma'} - Q_{f.\sigma} = (Q_{f.\sigma'} - Q_f) - (Q_{f.\sigma} - Q_f).$$

For $m = 1$, it is of the form $A s + B$, where

$$A = \frac{n_{\sigma'}}{(\sigma' - \sigma)^2} - \frac{n_\sigma}{(\sigma' - \sigma)^2} + \sum_{\rho \neq \sigma, \sigma'} n_\rho \frac{(\sigma + \sigma' - 2\rho)(\sigma' - \sigma)}{(\rho - \sigma)^2 (\rho - \sigma')^2}$$

$$B = n_{\sigma'} \frac{\sigma' - 2\sigma}{(\sigma' - \sigma)^2} - n_\sigma \frac{\sigma - 2\sigma'}{(\sigma' - \sigma)^2} + \sum_{\rho \neq \sigma, \sigma'} n_\rho \frac{(\rho\sigma + \rho\sigma' - 2\sigma\sigma')(\sigma' - \sigma)}{(\rho - \sigma)^2 (\rho - \sigma')^2} .$$

To compute $c_0(\zeta, 1/2)$, note that $c_0(\zeta, 1/2) = -c_0(\chi_0, 1/2)$ by (12). The value of

$$c_0(\chi_0, 0) = \frac{\log \pi}{2} + \frac{\gamma}{2}$$

follows from (13). The zeros of χ_0 are the negative integers $-n$, $n \geq 0$, and are simple. Hence the formula (15) reads (for σ not a pole of χ_0)

$$-c_0(\chi_0, \sigma) + c_0(\chi_0, 0) = -\frac{1}{\sigma} + \sum_{n=1}^{\infty} (-1) \frac{-\sigma}{(-n)(-n - \sigma)}$$

$$= -\frac{1}{\sigma} + \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n + \sigma} \right)$$

$$= \frac{\Gamma'(\sigma)}{\Gamma(\sigma)} + \gamma,$$

where the last formula follows from the expression for the logarithmic derivative of the the Γ-function, the digamma function ψ,

$$\psi(s) = \frac{\Gamma'(s)}{\Gamma(s)} = -\frac{1}{s} - \gamma + \sum_{n=1}^{+\infty} \left(\frac{1}{n} - \frac{1}{n + s} \right) ,$$

which results from its Hadamard factorization.

Finally, We get

Theorem 7.1. *We have, for $\sigma \notin -\mathbb{Z}$*

$$c_0(\chi_0, \sigma) = \frac{\log \pi}{2} - \frac{\gamma}{2} - \psi(\sigma) .$$

In particular, for $\sigma = 1/2$, we have (see [3] entry 6.3.3 p. 258)

$$\psi(1/2) = -2\log 2 - \gamma .$$

Theorem 7.2. *We have,*

$$c_0(\zeta, 1/2) = -c_0(\chi_0, 1/2) = -\frac{\log \pi}{2} - \frac{\gamma}{2} - 2\log 2 .$$

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