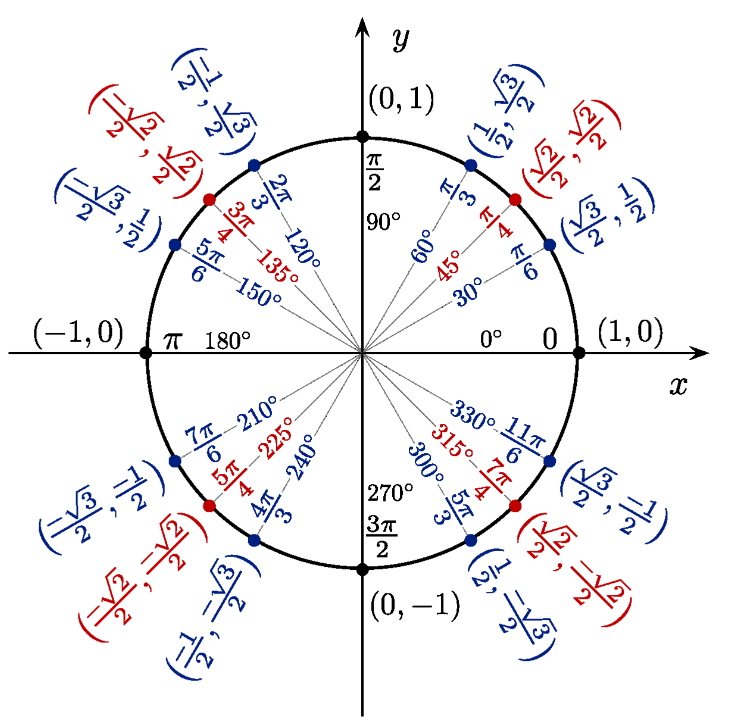
# List of trigonometric identities

In mathematics, **trigonometric identities** are equalities that involve trigonometric functions and are true for every single value of the occurring variables. Geometrically, these are identities involving certain functions of one or more angles. They are distinct from triangle identities, which are identities involving both angles and side lengths of a triangle. Only the former are covered in this article.

These identities are useful whenever expressions involving trigonometric functions need to be simplified. An important application is the integration of non-trigonometric functions: a common technique involves first using the substitution rule with a trigonometric function, and then simplifying the resulting integral with a trigonometric identity.



Cosines and sines around the unit circle

## Notation

### Angles

This article uses Greek letters such as alpha ($\alpha$), beta ($\beta$), gamma ($\gamma$), and theta ($\theta$) to represent angles. Several different units of angle measure are widely used, including degrees, radians, and grads:

$$1 \text{ full circle } = 360 \text{ degrees} = 2\pi \text{ radians } = 400 \text{ grads}.$$

The following table shows the conversions for some common angles:

| **Degrees** | 30° | 60° | 120° | 150° | 210° | 240° | 300° | 330° |
|---|---|---|---|---|---|---|---|---|
| **Radians** | $\frac{\pi}{6}$ | $\frac{\pi}{3}$ | $\frac{2\pi}{3}$ | $\frac{5\pi}{6}$ | $\frac{7\pi}{6}$ | $\frac{4\pi}{3}$ | $\frac{5\pi}{3}$ | $\frac{11\pi}{6}$ |
| **Grads** | 33⅓ grad | 66⅔ grad | 133⅓ grad | 166⅔ grad | 233⅓ grad | 266⅔ grad | 333⅓ grad | 366⅔ grad |

| **Degrees** | 45° | 90° | 135° | 180° | 225° | 270° | 315° | 360° |
|---|---|---|---|---|---|---|---|---|
| **Radians** | $\frac{\pi}{4}$ | $\frac{\pi}{2}$ | $\frac{3\pi}{4}$ | $\pi$ | $\frac{5\pi}{4}$ | $\frac{3\pi}{2}$ | $\frac{7\pi}{4}$ | $2\pi$ |
| **Grads** | 50 grad | 100 grad | 150 grad | 200 grad | 250 grad | 300 grad | 350 grad | 400 grad |

Unless otherwise specified, all angles in this article are assumed to be in radians, though angles ending in a degree symbol (°) are in degrees.

### Trigonometric functions

The primary trigonometric functions are the sine and cosine of an angle. These are sometimes abbreviated sin($\theta$) and cos($\theta$), respectively, where $\theta$ is the angle, but the parentheses around the angle are often omitted, e.g., sin $\theta$ and cos $\theta$.

The tangent (tan) of an angle is the ratio of the sine to the cosine:

$$\tan\theta = \frac{\sin\theta}{\cos\theta}.$$

Finally, the reciprocal functions secant (sec), cosecant (csc), and cotangent (cot) are the reciprocals of the cosine, sine, and tangent:

$$\sec\theta = \frac{1}{\cos\theta}, \quad \csc\theta = \frac{1}{\sin\theta}, \quad \cot\theta = \frac{1}{\tan\theta} = \frac{\cos\theta}{\sin\theta}.$$

These definitions are sometimes referred to as ratio identities.

## Inverse functions

The inverse trigonometric functions are partial inverse functions for the trigonometric functions. For example, the inverse function for the sine, known as the **inverse sine** ($\sin^{-1}$) or **arcsine** (arcsin or asin), satisfies

$$\sin(\arcsin x) = x \quad \text{for} \quad |x| \le 1$$

and

$$\arcsin(\sin x) = x \quad \text{for} \quad |x| \le \pi/2.$$

This article uses the notation below for inverse trigonometric functions:

| **Function** | sin | cos | tan | sec | csc | cot |
|---|---|---|---|---|---|---|
| **Inverse** | arcsin | arccos | arctan | arcsec | arccsc | arccot |

## Pythagorean identity

The basic relationship between the sine and the cosine is the Pythagorean trigonometric identity:

$$\cos^2\theta + \sin^2\theta = 1$$

where $\cos^2\theta$ means $(\cos(\theta))^2$ and $\sin^2\theta$ means $(\sin(\theta))^2$.

This can be viewed as a version of the Pythagorean theorem, and follows from the equation $x^2 + y^2 = 1$ for the unit circle. This equation can be solved for either the sine or the cosine:

$$\sin\theta = \pm\sqrt{1-\cos^2\theta} \quad \text{and} \quad \cos\theta = \pm\sqrt{1-\sin^2\theta}.$$

### Related identities

Dividing the Pythagorean identity through by either $\cos^2\theta$ or $\sin^2\theta$ yields two other identities:

$$1+\tan^2\theta=\sec^2\theta \quad \text{and} \quad 1+\cot^2\theta=\csc^2\theta.$$

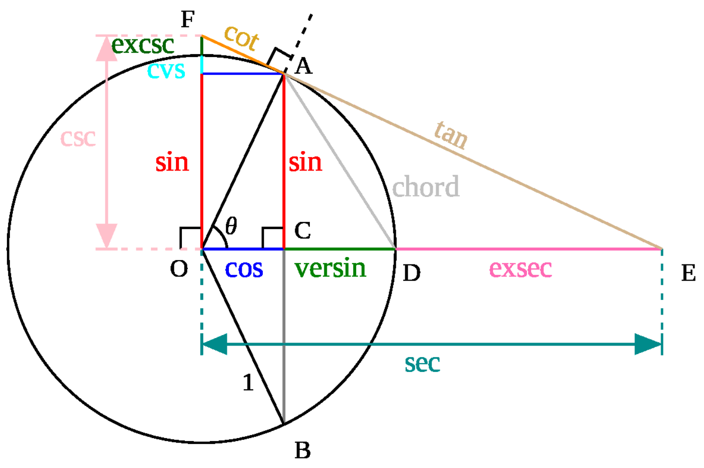
Using these identities together with the ratio identities, it is possible to express any trigonometric function in terms of any other (up to a plus or minus sign):

**Each trigonometric function in terms of the other five.[1]**

| in terms of | $\sin\theta$ | $\cos\theta$ | $\tan\theta$ | $\csc\theta$ | $\sec\theta$ | $\cot\theta$ |
|---|---|---|---|---|---|---|
| $\sin\theta =$ | $\sin\theta$ | $\pm\sqrt{1-\cos^2\theta}$ | $\pm\frac{\tan\theta}{\sqrt{1+\tan^2\theta}}$ | $\frac{1}{\csc\theta}$ | $\pm\frac{\sqrt{\sec^2\theta-1}}{\sec\theta}$ | $\pm\frac{1}{\sqrt{1+\cot^2\theta}}$ |
| $\cos\theta =$ | $\pm\sqrt{1-\sin^2\theta}$ | $\cos\theta$ | $\pm\frac{1}{\sqrt{1+\tan^2\theta}}$ | $\pm\frac{\sqrt{\csc^2\theta-1}}{\csc\theta}$ | $\frac{1}{\cos\theta}$ | $\pm\frac{\cot\theta}{\sqrt{1+\cot^2\theta}}$ |
| $\tan\theta =$ | $\pm\frac{\sin\theta}{\sqrt{1-\sin^2\theta}}$ | $\pm\frac{\sqrt{1-\cos^2\theta}}{\cos\theta}$ | $\tan\theta$ | $\pm\frac{1}{\sqrt{\csc^2\theta-1}}$ | $\pm\sqrt{\sec^2\theta-1}$ | $\frac{1}{\cot\theta}$ |
| $\csc\theta =$ | $\frac{1}{\sin\theta}$ | $\pm\frac{1}{\sqrt{1-\cos^2\theta}}$ | $\pm\frac{\sqrt{1+\tan^2\theta}}{\tan\theta}$ | $\csc\theta$ | $\pm\frac{\sec\theta}{\sqrt{\sec^2\theta-1}}$ | $\pm\sqrt{1+\cot^2\theta}$ |
| $\sec\theta =$ | $\pm\frac{1}{\sqrt{1-\sin^2\theta}}$ | $\frac{1}{\cos\theta}$ | $\pm\sqrt{1+\tan^2\theta}$ | $\pm\frac{\csc\theta}{\sqrt{\csc^2\theta-1}}$ | $\sec\theta$ | $\pm\frac{\sqrt{1+\cot^2\theta}}{\cot\theta}$ |
| $\cot\theta =$ | $\pm\frac{\sqrt{1-\sin^2\theta}}{\sin\theta}$ | $\pm\frac{\cos\theta}{\sqrt{1-\cos^2\theta}}$ | $\frac{1}{\tan\theta}$ | $\pm\sqrt{\csc^2\theta-1}$ | $\pm\frac{1}{\sqrt{\sec^2\theta-1}}$ | $\cot\theta$ |

## Historic shorthands

The versine, coversine, haversine, and exsecant were used in navigation. For example the haversine formula was used to calculate the distance between two points on a sphere. They are rarely used today.




All of the trigonometric functions of an angle $\theta$ can be constructed geometrically in terms of a unit circle centered at $O$. Many of these terms are no longer in common use.

| Name(s) | Abbreviation(s) | Value[2] |
|---|---|---|
| versed sine, versine | $\operatorname{versin}(\theta)$<br>$\operatorname{vers}(\theta)$<br>$\operatorname{ver}(\theta)$ | $1-\cos(\theta)$ |
| versed cosine, vercosine | $\operatorname{vercosin}(\theta)$ | $1+\cos(\theta)$ |
| coversed sine, coversine | $\operatorname{coversin}(\theta)$<br>$\operatorname{cvs}(\theta)$ | $1-\sin(\theta)$ |
| coversed cosine, covercosine | $\operatorname{covercosin}(\theta)$ | $1+\sin(\theta)$ |
| half versed sine, haversine | $\operatorname{haversin}(\theta)$ | $\frac{1-\cos(\theta)}{2}$ |
| half versed cosine, havercosine | $\operatorname{havercosin}(\theta)$ | $\frac{1+\cos(\theta)}{2}$ |
| half coversed sine, hacoversine<br>cohaversine | $\operatorname{hacoversin}(\theta)$ | $\frac{1-\sin(\theta)}{2}$ |
| half coversed cosine,<br>hacovercosine<br>cohavercosine | $\operatorname{hacovercosin}(\theta)$ | $\frac{1+\sin(\theta)}{2}$ |
| exterior secant, exsecant | $\operatorname{exsec}(\theta)$ | $\sec(\theta)-1$ |
| exterior cosecant, excosecant | $\operatorname{excsc}(\theta)$ | $\csc(\theta)-1$ |
| chord | $\operatorname{crd}(\theta)$ | $2\sin\left(\frac{\theta}{2}\right)$ |

## Symmetry, shifts, and periodicity

By examining the unit circle, the following properties of the trigonometric functions can be established.

### Symmetry

When the trigonometric functions are reflected from certain angles, the result is often one of the other trigonometric functions. This leads to the following identities:

| Reflected in $\theta = 0$[3] | Reflected in $\theta = \pi/2$ (co-function identities)[4] | Reflected in $\theta = \pi$ |
|---|---|---|
| $\sin(-\theta) = -\sin\theta$ | $\sin(\frac{\pi}{2}-\theta) = +\cos\theta$ | $\sin(\pi-\theta) = +\sin\theta$ |
| $\cos(-\theta) = +\cos\theta$ | $\cos(\frac{\pi}{2}-\theta) = +\sin\theta$ | $\cos(\pi-\theta) = -\cos\theta$ |
| $\tan(-\theta) = -\tan\theta$ | $\tan(\frac{\pi}{2}-\theta) = +\cot\theta$ | $\tan(\pi-\theta) = -\tan\theta$ |
| $\csc(-\theta) = -\csc\theta$ | $\csc(\frac{\pi}{2}-\theta) = +\sec\theta$ | $\csc(\pi-\theta) = +\csc\theta$ |
| $\sec(-\theta) = +\sec\theta$ | $\sec(\frac{\pi}{2}-\theta) = +\csc\theta$ | $\sec(\pi-\theta) = -\sec\theta$ |
| $\cot(-\theta) = -\cot\theta$ | $\cot(\frac{\pi}{2}-\theta) = +\tan\theta$ | $\cot(\pi-\theta) = -\cot\theta$ |

### Shifts and periodicity

By shifting the function round by certain angles, it is often possible to find different trigonometric functions that express the result more simply. Some examples of this are shown by shifting functions round by π/2, π and 2π radians. Because the periods of these functions are either π or 2π, there are cases where the new function is exactly the same as the old function without the shift.

| **Shift by π/2** | **Shift by π**<br>**Period for tan and cot**[5] | **Shift by 2π**<br>**Period for sin, cos, csc and sec**[6] |
|---|---|---|
| $\sin(\theta + \frac{\pi}{2}) = +\cos\theta$<br>$\cos(\theta + \frac{\pi}{2}) = -\sin\theta$<br>$\tan(\theta + \frac{\pi}{2}) = -\cot\theta$<br>$\csc(\theta + \frac{\pi}{2}) = +\sec\theta$<br>$\sec(\theta + \frac{\pi}{2}) = -\csc\theta$<br>$\cot(\theta + \frac{\pi}{2}) = -\tan\theta$ | $\sin(\theta + \pi) = -\sin\theta$<br>$\cos(\theta + \pi) = -\cos\theta$<br>$\tan(\theta + \pi) = +\tan\theta$<br>$\csc(\theta + \pi) = -\csc\theta$<br>$\sec(\theta + \pi) = -\sec\theta$<br>$\cot(\theta + \pi) = +\cot\theta$ | $\sin(\theta + 2\pi) = +\sin\theta$<br>$\cos(\theta + 2\pi) = +\cos\theta$<br>$\tan(\theta + 2\pi) = +\tan\theta$<br>$\csc(\theta + 2\pi) = +\csc\theta$<br>$\sec(\theta + 2\pi) = +\sec\theta$<br>$\cot(\theta + 2\pi) = +\cot\theta$ |

## Angle sum and difference identities

These are also known as the *addition and subtraction theorems* or *formulæ*. They were originally established by the 10th century Persian mathematician Abū al-Wafā' Būzjānī. One method of proving these identities is to apply Euler's formula. The use of the symbols $\pm$ and $\mp$ is described in the article plus-minus sign.

| | |
|---|---|
| **Sine** | $\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$ [7][8] |
| **Cosine** | $\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$ [8][9] |
| **Tangent** | $\tan(\alpha \pm \beta) = \frac{\tan\alpha \pm \tan\beta}{1 \mp \tan\alpha\tan\beta}$ [8][10] |
| **Arcsine** | $\arcsin\alpha \pm \arcsin\beta = \arcsin\left(\alpha\sqrt{1-\beta^2} \pm \beta\sqrt{1-\alpha^2}\right)$ [11] |
| **Arccosine** | $\arccos\alpha \pm \arccos\beta = \arccos\left(\alpha\beta \mp \sqrt{(1-\alpha^2)(1-\beta^2)}\right)$ [12] |
| **Arctangent** | $\arctan\alpha \pm \arctan\beta = \arctan\left(\frac{\alpha \pm \beta}{1 \mp \alpha\beta}\right)$ [13] |

### Matrix form

The sum and difference formulae for sine and cosine can be written in matrix form as:

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}$$

$$= \begin{pmatrix} \cos\theta\cos\phi - \sin\theta\sin\phi & -\cos\theta\sin\phi - \sin\theta\cos\phi \\ \sin\theta\cos\phi + \cos\theta\sin\phi & -\sin\theta\sin\phi + \cos\theta\cos\phi \end{pmatrix}$$

$$= \begin{pmatrix} \cos(\theta+\phi) & -\sin(\theta+\phi) \\ \sin(\theta+\phi) & \cos(\theta+\phi) \end{pmatrix}$$

This shows that these matrices form a representation of the rotation group in the plane (technically, the special orthogonal group *SO*(2)), since the composition law is fulfilled: subsequent multiplications of a vector with these two matrices yields the same result as the rotation by the sum of the angles.

### Sines and cosines of sums of infinitely many terms

$$\sin\left(\sum_{i=1}^{\infty}\theta_i\right) = \sum_{\text{odd } k\geq 1}(-1)^{(k-1)/2}\sum_{\substack{A\subseteq\{\,1,2,3,\dots\}\\|A|=k}}\left(\prod_{i\in A}\sin\theta_i\prod_{i\not\in A}\cos\theta_i\right)$$

$$\cos\left(\sum_{i=1}^{\infty}\theta_i\right) = \sum_{\text{even } k\geq 0}(-1)^{k/2}\sum_{\substack{A\subseteq\{\,1,2,3,\dots\}\\|A|=k}}\left(\prod_{i\in A}\sin\theta_i\prod_{i\not\in A}\cos\theta_i\right)$$

In these two identities an asymmetry appears that is not seen in the case of sums of finitely many terms: in each product, there are only finitely many sine factors and cofinitely many cosine factors.

If only finitely many of the terms $\theta_i$ are nonzero, then only finitely many of the terms on the right side will be nonzero because sine factors will vanish, and in each term, all but finitely many of the cosine factors will be unity.

### Tangents of sums

Let $e_k$ (for $k$ = 0, 1, 2, 3, ...) be the $k$th-degree elementary symmetric polynomial in the variables

$$x_i = \tan\theta_i$$

for $i$ = 0, 1, 2, 3, ..., i.e.,

$$\begin{aligned}
e_0 &= 1\\
e_1 &= \sum_i x_i &&= \sum_i \tan\theta_i\\
e_2 &= \sum_{i<j} x_i x_j &&= \sum_{i<j}\tan\theta_i\tan\theta_j\\
e_3 &= \sum_{i<j<k} x_i x_j x_k &&= \sum_{i<j<k}\tan\theta_i\tan\theta_j\tan\theta_k\\
&\ \ \vdots &&\ \ \vdots
\end{aligned}$$

Then

$$\tan\left(\sum_i\theta_i\right) = \frac{e_1 - e_3 + e_5 - \cdots}{e_0 - e_2 + e_4 - \cdots}.$$

The number of terms on the right side depends on the number of terms on the right side.

For example:

$$\begin{aligned}
\tan(\theta_1+\theta_2) &= \frac{e_1}{e_0 - e_2} = \frac{x_1 + x_2}{1\ -\ x_1x_2} = \frac{\tan\theta_1 + \tan\theta_2}{1\ -\ \tan\theta_1\tan\theta_2},\\
\tan(\theta_1+\theta_2+\theta_3) &= \frac{e_1 - e_3}{e_0 - e_2} = \frac{(x_1+x_2+x_3)\ -\ (x_1x_2x_3)}{1\ -\ (x_1x_2+x_1x_3+x_2x_3)},\\
\tan(\theta_1+\theta_2+\theta_3+\theta_4) &= \frac{e_1 - e_3}{e_0 - e_2 + e_4}\\
&= \frac{(x_1+x_2+x_3+x_4)\ -\ (x_1x_2x_3+x_1x_2x_4+x_1x_3x_4+x_2x_3x_4)}{1\ -\ (x_1x_2+x_1x_3+x_1x_4+x_2x_3+x_2x_4+x_3x_4)\ +\ (x_1x_2x_3x_4)},
\end{aligned}$$

and so on. The case of only finitely many terms can be proved by mathematical induction.[14]

### Secants and cosecants of sums

$$\sec\left(\sum_i \theta_i\right) = \frac{\prod_i \sec\theta_i}{e_0 - e_2 + e_4 - \cdots}$$

$$\csc\left(\sum_i \theta_i\right) = \frac{\prod_i \sec\theta_i}{e_1 - e_3 + e_5 - \cdots}$$

where $e_k$ is the $k$th-degree elementary symmetric polynomial in the $n$ variables $x_i = \tan\theta_i$, $i = 1, ..., n$, and the number of terms in the denominator and the number of factors in the product in the numerator depend on the number of terms in the sum on the left. The case of only finitely many terms can be proved by mathematical induction on the number of such terms. The convergence of the series in the denominators can be shown by writing the secant identity in the form

$$e_0 - e_2 + e_4 - \cdots = \frac{\prod_i \sec\theta_i}{\sec\left(\sum_i \theta_i\right)}$$

and then observing that the left side converges if the right side converges, and similarly for the cosecant identity.

For example,

$$\sec(\alpha+\beta+\gamma) = \frac{\sec\alpha\sec\beta\sec\gamma}{1 - \tan\alpha\tan\beta - \tan\alpha\tan\gamma - \tan\beta\tan\gamma}$$

$$\csc(\alpha+\beta+\gamma) = \frac{\sec\alpha\sec\beta\sec\gamma}{\tan\alpha + \tan\beta + \tan\gamma - \tan\alpha\tan\beta\tan\gamma}$$

## Multiple-angle formulae

| | |
|---|---|
| **$T_n$ is the $n$th Chebyshev polynomial** | $\cos n\theta = T_n(\cos\theta)$ [15] |
| **$S_n$ is the $n$th spread polynomial** | $\sin^2 n\theta = S_n(\sin^2\theta)$ |
| **de Moivre's formula, $i$ is the imaginary unit** | $\cos n\theta + i\sin n\theta = (\cos(\theta) + i\sin(\theta))^n$ [16] |

### Double-, triple-, and half-angle formulae

These can be shown by using either the sum and difference identities or the multiple-angle formulae.

| | | | |
|---|---|---|---|
| **Double-angle formulae**[17][18] | | | |
| $\sin 2\theta = 2\sin\theta\cos\theta$ $= \frac{2\tan\theta}{1+\tan^2\theta}$ | $\cos 2\theta = \cos^2\theta - \sin^2\theta$ $= 2\cos^2\theta - 1$ $= 1 - 2\sin^2\theta$ $= \frac{1-\tan^2\theta}{1+\tan^2\theta}$ | $\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$ | $\cot 2\theta = \frac{\cot^2\theta - 1}{2\cot\theta}$ |
| **Triple-angle formulae**[15][19] | | | |
| $\sin 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta$ $= 3\sin\theta - 4\sin^3\theta$ | $\cos 3\theta = \cos^3\theta - 3\sin^2\theta\cos\theta$ $= 4\cos^3\theta - 3\cos\theta$ | $\tan 3\theta = \frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$ | $\cot 3\theta = \frac{3\cot\theta - \cot^3\theta}{1 - 3\cot^2\theta}$ |
| **Half-angle formulae**[20][21] | | | |

| | | | |
|---|---|---|---|
| $\sin\frac{\theta}{2} = \operatorname{sgn}\left(2\pi-\theta+4\pi\left\lfloor\frac{\theta}{4\pi}\right\rfloor\right)\sqrt{\frac{1-\cos\theta}{2}}$ <br> $\left(\text{or }\ \sin^2\frac{\theta}{2} = \frac{1-\cos\theta}{2}\right)$ | $\cos\frac{\theta}{2} = \operatorname{sgn}\left(\pi+\theta+4\pi\left\lfloor\frac{\pi-\theta}{4\pi}\right\rfloor\right)\sqrt{\frac{1+\cos\theta}{2}}$ <br> $\left(\text{or }\ \cos^2\frac{\theta}{2} = \frac{1+\cos\theta}{2}\right)$ | $\tan\frac{\theta}{2} = \csc\theta - \cot\theta = \pm\sqrt{\frac{1-\cos\theta}{1+\cos\theta}} = \frac{\sin\theta}{1+\cos\theta} = \frac{1-\cos\theta}{\sin\theta}$ <br> $\tan\frac{\eta+\theta}{2} = \frac{\sin\eta+\sin\theta}{\cos\eta+\cos\theta}$ <br> $\tan\left(\frac{\theta}{2}+\frac{\pi}{4}\right) = \sec\theta+\tan\theta$ <br> $\sqrt{\frac{1-\sin\theta}{1+\sin\theta}} = \frac{1-\tan(\theta/2)}{1+\tan(\theta/2)}$ <br> $\tan\tfrac{1}{2}\theta = \frac{\tan\theta}{1+\sqrt{1+\tan^2\theta}}$ <br> $\text{for}\quad \theta\in\left(-\tfrac{\pi}{2},\tfrac{\pi}{2}\right)$ | $\cot\frac{\theta}{2} = \csc\theta + \cot\theta = \pm\sqrt{\frac{1+\cos\theta}{1-\cos\theta}} = \frac{\sin\theta}{1-\cos\theta} = \frac{1+\cos\theta}{\sin\theta}$ |

The fact that the triple-angle formula for sine and cosine only involves powers of a single function allows one to relate the geometric problem of a compass and straightedge construction of angle trisection to the algebraic problem of solving a cubic equation, which allows one to prove that this is in general impossible using the given tools, by field theory.

A formula for computing the trigonometric identities for the third-angle exists, but it requires finding the zeroes of the cubic equation $x^3 - \frac{3x+d}{4} = 0$, where $x$ is the value of the sine function at some angle and $d$ is the known value of the sine function at the triple angle. However, the discriminant of this equation is negative, so this equation has three real roots (of which only one is the solution within the correct third-circle) but none of these solutions is reducible to a real algebraic expression, as they use intermediate complex numbers under the cube roots, (which may be expressed in terms of real-only functions only if using hyperbolic functions).

## Sine, cosine, and tangent of multiple angles

For specific multiples, these follow from the angle addition formulas, while the general formula was given by 16th century French mathematician Vieta.

$$\sin n\theta = \sum_{k=0}^{n} \binom{n}{k} \cos^k\theta\, \sin^{n-k}\theta\, \sin\left(\frac{1}{2}(n-k)\pi\right)$$
$$\cos n\theta = \sum_{k=0}^{n} \binom{n}{k} \cos^k\theta\, \sin^{n-k}\theta\, \cos\left(\frac{1}{2}(n-k)\pi\right)$$

In each of these two equations, the first parenthesized term is a binomial coefficient, and the final trigonometric function equals one or minus one or zero so that half the entries in each of the sums are removed. tan $n\theta$ can be written in terms of tan $\theta$ using the recurrence relation:

$$\tan\,(n{+}1)\theta = \frac{\tan n\theta + \tan\theta}{1 - \tan n\theta\ \tan\theta}.$$

cot $n\theta$ can be written in terms of cot $\theta$ using the recurrence relation:

$$\cot\,(n{+}1)\theta = \frac{\cot n\theta\ \cot\theta - 1}{\cot n\theta + \cot\theta}.$$

### Chebyshev method

The Chebyshev method is a recursive algorithm for finding the $n^{\text{th}}$ multiple angle formula knowing the $(n-1)^{\text{th}}$ and $(n-2)^{\text{th}}$ formulae.[22]

The cosine for $nx$ can be computed from the cosine of $(n-1)x$ and $(n-2)x$ as follows:

$$\cos nx = 2\cdot\cos x\cdot\cos(n-1)x - \cos(n-2)x$$

Similarly $\sin(nx)$ can be computed from the sines of $(n-1)x$ and $(n-2)x$

$$\sin nx = 2\cdot\cos x\cdot\sin(n-1)x - \sin(n-2)x$$

For the tangent, we have:

$$\tan nx = \frac{H + K\tan x}{K - H\tan x}$$

where $H/K = \tan(n-1)x$.

### Tangent of an average

$$\tan\left(\frac{\alpha+\beta}{2}\right) = \frac{\sin\alpha+\sin\beta}{\cos\alpha+\cos\beta} = -\frac{\cos\alpha-\cos\beta}{\sin\alpha-\sin\beta}$$

Setting either $\alpha$ or $\beta$ to 0 gives the usual tangent half-angle formulæ.

### Viète's infinite product

$$\cos\left(\frac{\theta}{2}\right)\cdot\cos\left(\frac{\theta}{4}\right)\cdot\cos\left(\frac{\theta}{8}\right)\cdots = \prod_{n=1}^{\infty}\cos\left(\frac{\theta}{2^n}\right) = \frac{\sin(\theta)}{\theta} = \operatorname{sinc}\theta.$$

## Power-reduction formula

Obtained by solving the second and third versions of the cosine double-angle formula.

| Sine | Cosine | Other |
|---|---|---|
| $\sin^2\theta = \frac{1-\cos 2\theta}{2}$ | $\cos^2\theta = \frac{1+\cos 2\theta}{2}$ | $\sin^2\theta\cos^2\theta = \frac{1-\cos 4\theta}{8}$ |
| $\sin^3\theta = \frac{3\sin\theta-\sin 3\theta}{4}$ | $\cos^3\theta = \frac{3\cos\theta+\cos 3\theta}{4}$ | $\sin^3\theta\cos^3\theta = \frac{3\sin 2\theta-\sin 6\theta}{32}$ |
| $\sin^4\theta = \frac{3-4\cos 2\theta+\cos 4\theta}{8}$ | $\cos^4\theta = \frac{3+4\cos 2\theta+\cos 4\theta}{8}$ | $\sin^4\theta\cos^4\theta = \frac{3-4\cos 4\theta+\cos 8\theta}{128}$ |
| $\sin^5\theta = \frac{10\sin\theta-5\sin 3\theta+\sin 5\theta}{16}$ | $\cos^5\theta = \frac{10\cos\theta+5\cos 3\theta+\cos 5\theta}{16}$ | $\sin^5\theta\cos^5\theta = \frac{10\sin 2\theta-5\sin 6\theta+\sin 10\theta}{512}$ |

and in general terms of powers of $\sin\theta$ or $\cos\theta$ the following is true, and can be deduced using De Moivre's formula, Euler's formula and binomial theorem.

| | Cosine | Sine |
|---|---|---|
| if $n$ is odd | $\cos^n\theta = \frac{2}{2^n}\sum_{k=0}^{\frac{n-1}{2}}\binom{n}{k}\cos\left((n-2k)\theta\right)$ | $\sin^n\theta = \frac{2}{2^n}\sum_{k=0}^{\frac{n-1}{2}}(-1)^{\left(\frac{n-1}{2}-k\right)}\binom{n}{k}\sin\left((n-2k)\theta\right)$ |
| if $n$ is even | $\cos^n\theta = \frac{1}{2^n}\binom{n}{\frac{n}{2}} + \frac{2}{2^n}\sum_{k=0}^{\frac{n}{2}-1}\binom{n}{k}\cos\left((n-2k)\theta\right)$ | $\sin^n\theta = \frac{1}{2^n}\binom{n}{\frac{n}{2}} + \frac{2}{2^n}\sum_{k=0}^{\frac{n}{2}-1}(-1)^{\left(\frac{n}{2}-k\right)}\binom{n}{k}\cos\left((n-2k)\theta\right)$ |

## Product-to-sum and sum-to-product identities

The product-to-sum identities or prosthaphaeresis formulas can be proven by expanding their right-hand sides using the angle addition theorems. See beat (acoustics) and phase detector for applications of the sum-to-product formulæ.

| Product-to-sum[23] |
|---|
| $\cos\theta\cos\varphi = \frac{\cos(\theta-\varphi)+\cos(\theta+\varphi)}{2}$ |
| $\sin\theta\sin\varphi = \frac{\cos(\theta-\varphi)-\cos(\theta+\varphi)}{2}$ |
| $\sin\theta\cos\varphi = \frac{\sin(\theta+\varphi)+\sin(\theta-\varphi)}{2}$ |
| $\cos\theta\sin\varphi = \frac{\sin(\theta+\varphi)-\sin(\theta-\varphi)}{2}$ |
| $\tan(\theta)\tan(\varphi) = \frac{cos(\theta-\varphi)-cos(\theta+\varphi)}{cos(\theta-\varphi)+cos(\theta+\varphi)}$ |
| $\prod_{k=1}^{n}\cos\theta_k = \frac{1}{2^n}\sum_{e\in S}\cos(e_1\theta_1+\cdots+e_n\theta_n)$ where $S=\{1,-1\}^n$ |

| Sum-to-product[24] |
|---|
| $\sin\theta\pm\sin\varphi = 2\sin\left(\frac{\theta\pm\varphi}{2}\right)\cos\left(\frac{\theta\mp\varphi}{2}\right)$ |
| $\cos\theta+\cos\varphi = 2\cos\left(\frac{\theta+\varphi}{2}\right)\cos\left(\frac{\theta-\varphi}{2}\right)$ |
| $\cos\theta-\cos\varphi = -2\sin\left(\frac{\theta+\varphi}{2}\right)\sin\left(\frac{\theta-\varphi}{2}\right)$ |

### Other related identities

If $x$, $y$, and $z$ are the three angles of any triangle, or in other words

$$\text{if } x+y+z=\pi=\text{half circle,}$$
$$\text{then } \tan(x)+\tan(y)+\tan(z)=\tan(x)\tan(y)\tan(z).$$

(If any of $x$, $y$, $z$ is a right angle, one should take both sides to be ∞. This is neither +∞ nor −∞; for present purposes it makes sense to add just one point at infinity to the real line, that is approached by tan(θ) as tan(θ) either increases through positive values or decreases through negative values. This is a one-point compactification of the real line.)

$$\text{If } x+y+z=\pi=\text{half circle,}$$
$$\text{then } \sin(2x)+\sin(2y)+\sin(2z)=4\sin(x)\sin(y)\sin(z).$$

### Hermite's cotangent identity

Charles Hermite demonstrated the following identity.[25] Suppose $a_1, \ldots, a_n$ are complex numbers, no two of which differ by an integer multiple of $\pi$. Let

$$A_{n,k} = \prod_{\substack{1\le j\le n \\ j\neq k}} \cot(a_k - a_j)$$

(in particular, $A_{1,1}$, being an empty product, is 1). Then

$$\cot(z-a_1)\cdots\cot(z-a_n) = \cos\frac{n\pi}{2} + \sum_{k=1}^n A_{n,k}\cot(z-a_k).$$

The simplest non-trivial example is the case $n = 2$:

$$\cot(z-a_1)\cot(z-a_2) = -1+\cot(a_1-a_2)\cot(z-a_1)+\cot(a_2-a_1)\cot(z-a_2).$$

### Ptolemy's theorem

$$\begin{aligned}
&\text{If } w+x+y+z=\pi=\text{half circle,}\\
&\qquad \text{then } \sin(w+x)\sin(x+y)\\
&\qquad\qquad = \sin(x+y)\sin(y+z)\\
&\qquad\qquad = \sin(y+z)\sin(z+w)\\
&\qquad\qquad = \sin(z+w)\sin(w+x) = \sin(w)\sin(y)+\sin(x)\sin(z).
\end{aligned}$$

(The first three equalities are trivial; the fourth is the substance of this identity.) Essentially this is Ptolemy's theorem adapted to the language of modern trigonometry.

## Linear combinations

For some purposes it is important to know that any linear combination of sine waves of the same period or frequency but different phase shifts is also a sine wave with the same period or frequency, but a different phase shift. In the case of a non-zero linear combination of a sine and cosine wave[26] (which is just a sine wave with a phase shift of $\pi/2$), we have

$$a\sin x + b\cos x = \sqrt{a^2+b^2}\cdot\sin(x+\varphi)$$

where

$$\varphi = \begin{cases} \arcsin\left(\frac{b}{\sqrt{a^2+b^2}}\right) & \text{if } a \ge 0,\\ \pi - \arcsin\left(\frac{b}{\sqrt{a^2+b^2}}\right) & \text{if } a < 0,\end{cases}$$

or equivalently

$$\varphi = \operatorname{sgn}(b)\arccos\left(\frac{a}{\sqrt{a^2+b^2}}\right)$$

or even

$$\varphi = \arctan\left(\frac{b}{a}\right) + \begin{cases} 0 & \text{if } a\ge 0,\\ \pi & \text{if } a<0,\end{cases}$$

or using the atan2 function

$$\varphi = \operatorname{atan2}(b, a).$$

More generally, for an arbitrary phase shift, we have

$$a\sin x + b\sin(x+\alpha) = c\sin(x+\beta)$$

where

$$c = \sqrt{a^2 + b^2 + 2ab\cos\alpha},$$

and

$$\beta = \arctan\left(\frac{b\sin\alpha}{a + b\cos\alpha}\right) + \begin{cases} 0 & \text{if } a + b\cos\alpha \ge 0, \\ \pi & \text{if } a + b\cos\alpha < 0. \end{cases}$$

For the most general case, see Phasor addition.

## Lagrange's trigonometric identities

These identities, named after Joseph Louis Lagrange, are:[27][28]

$$\sum_{n=1}^{N} \sin n\theta = \frac{1}{2}\cot\frac{\theta}{2} - \frac{\cos(N+\frac{1}{2})\theta}{2\sin\frac{1}{2}\theta}$$

$$\sum_{n=1}^{N} \cos n\theta = -\frac{1}{2} + \frac{\sin(N+\frac{1}{2})\theta}{2\sin\frac{1}{2}\theta}$$

A related function is the following function of $x$, called the Dirichlet kernel.

$$1 + 2\cos(x) + 2\cos(2x) + 2\cos(3x) + \cdots + 2\cos(nx) = \frac{\sin\left(\left(n+\frac{1}{2}\right)x\right)}{\sin(x/2)}.$$

## Other sums of trigonometric functions

Sum of sines and cosines with arguments in arithmetic progression[29]:

$$\sin\varphi + \sin(\varphi+\alpha) + \sin(\varphi+2\alpha) + \cdots$$
$$\cdots + \sin(\varphi+n\alpha) = \frac{\sin\left(\frac{(n+1)\alpha}{2}\right)\cdot\sin\left(\varphi+\frac{n\alpha}{2}\right)}{\sin\frac{\alpha}{2}}.$$

$$\cos\varphi + \cos(\varphi+\alpha) + \cos(\varphi+2\alpha) + \cdots$$
$$\cdots + \cos(\varphi+n\alpha) = \frac{\sin\left(\frac{(n+1)\alpha}{2}\right)\cdot\cos\left(\varphi+\frac{n\alpha}{2}\right)}{\sin\frac{\alpha}{2}}.$$

For any $a$ and $b$:

$$a\cos(x) + b\sin(x) = \sqrt{a^2+b^2}\cos(x - \operatorname{atan2}(b,a))$$

where atan2($y$, $x$) is the generalization of arctan($y$/$x$) that covers the entire circular range.

$$\tan(x) + \sec(x) = \tan\left(\frac{x}{2} + \frac{\pi}{4}\right).$$

The above identity is sometimes convenient to know when thinking about the Gudermannian function, which relates the circular and hyperbolic trigonometric functions without resorting to complex numbers.

If $x$, $y$, and $z$ are the three angles of any triangle, i.e. if $x + y + z = \pi$, then

$$\cot(x)\cot(y) + \cot(y)\cot(z) + \cot(z)\cot(x) = 1.$$

# Certain linear fractional transformations

If $f(x)$ is given by the linear fractional transformation

$$f(x) = \frac{(\cos\alpha)x - \sin\alpha}{(\sin\alpha)x + \cos\alpha},$$

and similarly

$$g(x) = \frac{(\cos\beta)x - \sin\beta}{(\cos\beta)x + \sin\beta},$$

then

$$f(g(x)) = g(f(x)) = \frac{(\cos(\alpha+\beta))x - \sin(\alpha+\beta)}{(\sin(\alpha+\beta))x + \cos(\alpha+\beta)}.$$

More tersely stated, if for all $\alpha$ we let $f_\alpha$ be what we called $f$ above, then

$$f_\alpha \circ f_\beta = f_{\alpha+\beta}.$$

If $x$ is the slope of a line, then $f(x)$ is the slope of its rotation through an angle of $-\alpha$.

# Inverse trigonometric functions

$$\arcsin(x) + \arccos(x) = \pi/2$$
$$\arctan(x) + \operatorname{arccot}(x) = \pi/2.$$
$$\arctan(x) + \arctan(1/x) = \begin{cases} \pi/2, & \text{if } x > 0 \\ -\pi/2, & \text{if } x < 0 \end{cases}$$

## Compositions of trig and inverse trig functions

| | |
|---|---|
| $\sin[\arccos(x)] = \sqrt{1-x^2}$ | $\tan[\arcsin(x)] = \frac{x}{\sqrt{1-x^2}}$ |
| $\sin[\arctan(x)] = \frac{x}{\sqrt{1+x^2}}$ | $\tan[\arccos(x)] = \frac{\sqrt{1-x^2}}{x}$ |
| $\cos[\arctan(x)] = \frac{1}{\sqrt{1+x^2}}$ | $\cot[\arcsin(x)] = \frac{\sqrt{1-x^2}}{x}$ |
| $\cos[\arcsin(x)] = \sqrt{1-x^2}$ | $\cot[\arccos(x)] = \frac{x}{\sqrt{1-x^2}}$ |

# Relation to the complex exponential function

$e^{ix} = \cos(x) + i\sin(x)$ [30] (Euler's formula),

$$e^{-ix} = \cos(-x) + i\sin(-x) = \cos(x) - i\sin(x)$$

$e^{i\pi} = -1$ (Euler's identity),

$$\cos(x) = \frac{e^{ix} + e^{-ix}}{2}$$ [31]

$$\sin(x) = \frac{e^{ix} - e^{-ix}}{2i}$$ [32]

and hence the corollary:

$$\tan(x) = \frac{e^{ix} - e^{-ix}}{i(e^{ix} + e^{-ix})} = \frac{\sin(x)}{\cos(x)}$$

where $i^2 = -1$.

## Infinite product formulae

For applications to special functions, the following infinite product formulae for trigonometric functions are useful:[33][34]

$$\sin x = x \prod_{n=1}^{\infty}\left(1 - \frac{x^2}{\pi^2 n^2}\right) \qquad \cos x = \prod_{n=1}^{\infty}\left(1 - \frac{x^2}{\pi^2(n-\frac{1}{2})^2}\right)$$

$$\sinh x = x \prod_{n=1}^{\infty}\left(1 + \frac{x^2}{\pi^2 n^2}\right) \qquad \cosh x = \prod_{n=1}^{\infty}\left(1 + \frac{x^2}{\pi^2(n-\frac{1}{2})^2}\right)$$

$$\frac{\sin x}{x} = \prod_{n=1}^{\infty}\cos\left(\frac{x}{2^n}\right) \qquad |\sin x| = \frac{1}{2}\prod_{n=0}^{\infty}\sqrt[2^{n+1}]{|\tan(2^n x)|}$$

## Identities without variables

The curious identity

$$\cos 20^\circ \cdot \cos 40^\circ \cdot \cos 80^\circ = \frac{1}{8}$$

is a special case of an identity that contains one variable:

$$\prod_{j=0}^{k-1}\cos(2^j x) = \frac{\sin(2^k x)}{2^k \sin(x)}.$$

Similarly:

$$\sin 20^\circ \cdot \sin 40^\circ \cdot \sin 80^\circ = \frac{\sqrt{3}}{8}.$$

The same cosine identity in radians is

$$\cos\frac{\pi}{7}\cos\frac{2\pi}{7}\cos\frac{3\pi}{7} = \frac{1}{8},$$

Similarly:

$$\tan 50^\circ \cdot \tan 60^\circ \cdot \tan 70^\circ = \tan 80^\circ.$$

$$\tan 40^\circ \cdot \tan 30^\circ \cdot \tan 20^\circ = \tan 10^\circ.$$

The following is perhaps not as readily generalized to an identity containing variables (but see explanation below):

$$\cos 24^\circ + \cos 48^\circ + \cos 96^\circ + \cos 168^\circ = \frac{1}{2}.$$

Degree measure ceases to be more felicitous than radian measure when we consider this identity with 21 in the denominators:

$$\cos\left(\frac{2\pi}{21}\right) + \cos\left(2\cdot\frac{2\pi}{21}\right) + \cos\left(4\cdot\frac{2\pi}{21}\right) + \cos\left(5\cdot\frac{2\pi}{21}\right) + \cos\left(8\cdot\frac{2\pi}{21}\right) + \cos\left(10\cdot\frac{2\pi}{21}\right) = \frac{1}{2}.$$

The factors 1, 2, 4, 5, 8, 10 may start to make the pattern clear: they are those integers less than 21/2 that are relatively prime to (or have no prime factors in common with) 21. The last several examples are corollaries of a basic fact about the irreducible cyclotomic polynomials: the cosines are the real parts of the zeroes of those polynomials; the sum of the zeroes is the Möbius function evaluated at (in the very last case above) 21; only half of the zeroes are present above. The two identities preceding this last one arise in the same fashion with 21 replaced by 10 and 15, respectively.

Many of those curious identities stem from more general facts like the following[35]:

$$\prod_{k=1}^{n-1} \sin\left(\frac{k\pi}{n}\right) = \frac{n}{2^{n-1}}$$

and

$$\prod_{k=1}^{n-1} \cos\left(\frac{k\pi}{n}\right) = \frac{\sin(\pi n/2)}{2^{n-1}}$$

Combining these gives us

$$\prod_{k=1}^{n-1} \tan\left(\frac{k\pi}{n}\right) = \frac{n}{\sin(\pi n/2)}$$

If $n$ is an odd number ($n = 2m + 1$) we can make use of the symmetries to get

$$\prod_{k=1}^{m} \tan\left(\frac{k\pi}{2m+1}\right) = \sqrt{2m+1}$$

## Computing π

An efficient way to compute π is based on the following identity without variables, due to Machin:

$$\frac{\pi}{4} = 4\arctan\frac{1}{5} - \arctan\frac{1}{239}$$

or, alternatively, by using an identity of Leonhard Euler:

$$\frac{\pi}{4} = 5\arctan\frac{1}{7} + 2\arctan\frac{3}{79}.$$

## A useful mnemonic for certain values of sines and cosines

For certain simple angles, the sines and cosines take the form $\sqrt{n}/2$ for $0 \le n \le 4$, which makes them easy to remember.

$$\begin{matrix}
\sin 0 & = & \sin 0^\circ & = & \sqrt{0}/2 & = & \cos 90^\circ & = & \cos\left(\frac{\pi}{2}\right) \\
\sin\left(\frac{\pi}{6}\right) & = & \sin 30^\circ & = & \sqrt{1}/2 & = & \cos 60^\circ & = & \cos\left(\frac{\pi}{3}\right) \\
\sin\left(\frac{\pi}{4}\right) & = & \sin 45^\circ & = & \sqrt{2}/2 & = & \cos 45^\circ & = & \cos\left(\frac{\pi}{4}\right) \\
\sin\left(\frac{\pi}{3}\right) & = & \sin 60^\circ & = & \sqrt{3}/2 & = & \cos 30^\circ & = & \cos\left(\frac{\pi}{6}\right) \\
\sin\left(\frac{\pi}{2}\right) & = & \sin 90^\circ & = & \sqrt{4}/2 & = & \cos 0^\circ & = & \cos 0
\end{matrix}$$

## Miscellany

With the golden ratio φ:

$$\cos\left(\frac{\pi}{5}\right) = \cos 36^\circ = \frac{\sqrt{5}+1}{4} = \frac{\varphi}{2}$$

$$\sin\left(\frac{\pi}{10}\right) = \sin 18^\circ = \frac{\sqrt{5}-1}{4} = \frac{\varphi-1}{2} = \frac{1}{2\varphi}$$

Also see exact trigonometric constants.

### An identity of Euclid

Euclid showed in Book XIII, Proposition 10 of his *Elements* that the area of the square on the side of a regular pentagon inscribed in a circle is equal to the sum of the areas of the squares on the sides of the regular hexagon and the regular decagon inscribed in the same circle. In the language of modern trigonometry, this says:

$$\sin^2(18^\circ) + \sin^2(30^\circ) = \sin^2(36^\circ).$$

Ptolemy used this proposition to compute some angles in his table of chords.

## Calculus

In calculus the relations stated below require angles to be measured in radians; the relations would become more complicated if angles were measured in another unit such as degrees. If the trigonometric functions are defined in terms of geometry, their derivatives can be found by verifying two limits. The first is:

$$\lim_{x\to 0} \frac{\sin x}{x} = 1,$$

verified using the unit circle and squeeze theorem. The second limit is:

$$\lim_{x\to 0} \frac{1-\cos x}{x} = 0,$$

verified using the identity $\tan(x/2) = (1 - \cos x)/\sin x$. Having established these two limits, one can use the limit definition of the derivative and the addition theorems to show that $(\sin x)' = \cos x$ and $(\cos x)' = -\sin x$. If the sine and cosine functions are defined by their Taylor series, then the derivatives can be found by differentiating the power series term-by-term.

$$\frac{d}{dx}\sin x = \cos x$$

The rest of the trigonometric functions can be differentiated using the above identities and the rules of differentiation:[36][37][38]

$$\frac{d}{dx}\sin x = \cos x, \qquad \frac{d}{dx}\arcsin x = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}\cos x = -\sin x, \qquad \frac{d}{dx}\arccos x = \frac{-1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}\tan x = \sec^2 x, \qquad \frac{d}{dx}\arctan x = \frac{1}{1+x^2}$$

$$\frac{d}{dx}\cot x = -\csc^2 x, \qquad \frac{d}{dx}\operatorname{arccot} x = \frac{-1}{1+x^2}$$

$$\frac{d}{dx}\sec x = \tan x \sec x, \qquad \frac{d}{dx}\operatorname{arcsec} x = \frac{1}{|x|\sqrt{x^2-1}}$$

$$\frac{d}{dx}\csc x = -\csc x \cot x, \qquad \frac{d}{dx}\operatorname{arccsc} x = \frac{-1}{|x|\sqrt{x^2-1}}$$

The integral identities can be found in "list of integrals of trigonometric functions". Some generic forms are listed below.

$$\int \frac{du}{\sqrt{a^2 - u^2}} = \sin^{-1}\left(\frac{u}{a}\right) + C$$
$$\int \frac{du}{a^2 + u^2} = \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right) + C$$
$$\int \frac{du}{u\sqrt{u^2 - a^2}} = \frac{1}{a}\sec^{-1}\left|\frac{u}{a}\right| + C$$

### Implications

The fact that the differentiation of trigonometric functions (sine and cosine) results in linear combinations of the same two functions is of fundamental importance to many fields of mathematics, including differential equations and Fourier transforms.

## Exponential definitions

| Function | Inverse function[39] |
|---|---|
| $\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$ | $\arcsin x = -i\ln\left(ix + \sqrt{1 - x^2}\right)$ |
| $\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$ | $\arccos x = -i\ln\left(x + \sqrt{x^2 - 1}\right)$ |
| $\tan\theta = \frac{e^{i\theta} - e^{-i\theta}}{i(e^{i\theta} + e^{-i\theta})}$ | $\arctan x = \frac{i}{2}\ln\left(\frac{i + x}{i - x}\right)$ |
| $\csc\theta = \frac{2i}{e^{i\theta} - e^{-i\theta}}$ | $\operatorname{arccsc} x = -i\ln\left(\frac{i}{x} + \sqrt{1 - \frac{1}{x^2}}\right)$ |
| $\sec\theta = \frac{2}{e^{i\theta} + e^{-i\theta}}$ | $\operatorname{arcsec} x = -i\ln\left(\frac{1}{x} + \sqrt{1 - \frac{i}{x^2}}\right)$ |
| $\cot\theta = \frac{i(e^{i\theta} + e^{-i\theta})}{e^{i\theta} - e^{-i\theta}}$ | $\operatorname{arccot} x = \frac{i}{2}\ln\left(\frac{x - i}{x + i}\right)$ |
| | |
| $\operatorname{cis}\theta = e^{i\theta}$ | $\operatorname{arccis} x = \frac{\ln x}{i} = -i\ln x = \arg x$ |

## Miscellaneous

### Dirichlet kernel

The **Dirichlet kernel** $D_n(x)$ is the function occurring on both sides of the next identity:

$$1 + 2\cos(x) + 2\cos(2x) + 2\cos(3x) + \cdots + 2\cos(nx) = \frac{\sin\left[\left(n + \frac{1}{2}\right)x\right]}{\sin\left(\frac{x}{2}\right)}.$$

The convolution of any integrable function of period $2\pi$ with the Dirichlet kernel coincides with the function's $n$th-degree Fourier approximation. The same holds for any measure or generalized function.

## Weierstrass substitution

If we set

$$t = \tan\left(\frac{x}{2}\right),$$

then[40]

$$\sin(x) = \frac{2t}{1+t^2} \text{ and } \cos(x) = \frac{1-t^2}{1+t^2} \text{ and } e^{ix} = \frac{1+it}{1-it}$$

where $e^{ix} = \cos(x) + i\sin(x)$, sometimes abbreviated to cis($x$).

When this substitution of $t$ for $\tan(x/2)$ is used in calculus, it follows that $\sin(x)$ is replaced by $2t/(1 + t^2)$, $\cos(x)$ is replaced by $(1 - t^2)/(1 + t^2)$ and the differential $dx$ is replaced by $(2\,dt)/(1 + t^2)$. Thereby one converts rational functions of $\sin(x)$ and $\cos(x)$ to rational functions of $t$ in order to find their antiderivatives.

# Notes


[1] Abramowitz and Stegun, p. 73, 4.3.45

[2] Abramowitz and Stegun, p. 78, 4.3.147

[3] Abramowitz and Stegun, p. 72, 4.3.13–15

[4] The Elementary Identities (http://jwbales.home.mindspring.com/precal/part5/part5.1.html)

[5] Abramowitz and Stegun, p. 72, 4.3.9

[6] Abramowitz and Stegun, p. 72, 4.3.7–8

[7] Abramowitz and Stegun, p. 72, 4.3.16

[8] Weisstein, Eric W., " Trigonometric Addition Formulas (http://mathworld.wolfram.com/TrigonometricAdditionFormulas.html)" from MathWorld.

[9] Abramowitz and Stegun, p. 72, 4.3.17

[10] Abramowitz and Stegun, p. 72, 4.3.18

[11] Abramowitz and Stegun, p. 80, 4.4.42

[12] Abramowitz and Stegun, p. 80, 4.4.43

[13] Abramowitz and Stegun, p. 80, 4.4.36

[14] Bronstein, Manuel (1989). "Simplification of real elementary functions". In G. H. Gonnet (ed.). *Proceedings of the ACM-SIGSAM 1989 International Symposium on Symbolic and Algebraic Computation*. ISSAC'89 (Portland US-OR, 1989-07). New York: ACM. pp. 207–211. doi:10.1145/74540.74566. ISBN 0-89791-325-6.

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[16] Abramowitz and Stegun, p. 74, 4.3.48

[17] Abramowitz and Stegun, p. 72, 4.3.24–26

[18] Weisstein, Eric W., " Double-Angle Formulas (http://mathworld.wolfram.com/Double-AngleFormulas.html)" from MathWorld.

[19] Abramowitz and Stegun, p. 72, 4.3.27–28

[20] Abramowitz and Stegun, p. 72, 4.3.20–22

[21] Weisstein, Eric W., " Half-Angle Formulas (http://mathworld.wolfram.com/Half-AngleFormulas.html)" from MathWorld.

[22] Ken Ward's Mathematics Pages, http://www.trans4mind.com/personal_development/mathematics/trigonometry/multipleAnglesRecursiveFormula.htm

[23] Abramowitz and Stegun, p. 72, 4.3.31–33

[24] Abramowitz and Stegun, p. 72, 4.3.34–39

[25] Warren P. Johnson, "Trigonometric Identities à la Hermite", *American Mathematical Monthly*, volume 117, number 4, April 2010, pages 311–327

[26] Proof at http://pages.pacificcoast.net/~cazelais/252/lc-trig.pdf

[27] Eddie Ortiz Muñiz (February 1953). "A Method for Deriving Various Formulas in Electrostatics and Electromagnetism Using Lagrange's Trigonometric Identities". *American Journal of Physics* **21** (2): 140. doi:10.1119/1.1933371.

[28] Alan Jeffrey and Hui-hui Dai (2008). "Section 2.4.1.6". *Handbook of Mathematical Formulas and Integrals* (4th ed.). Academic Press. ISBN 978-0-12-374288-9.

[29] Michael P. Knapp, Sines and Cosines of Angles in Arithmetic Progression (http://evergreen.loyola.edu/mpknapp/www/papers/knapp-sv.pdf)

[30] Abramowitz and Stegun, p. 74, 4.3.47

[31] Abramowitz and Stegun, p. 71, 4.3.2

[32] Abramowitz and Stegun, p. 71, 4.3.1

[33] Abramowitz and Stegun, p. 75, 4.3.89–90

[34] Abramowitz and Stegun, p. 85, 4.5.68–69
[35] Weisstein, Eric W., " Sine (http://mathworld.wolfram.com/Sine.html)" from MathWorld
[36] Abramowitz and Stegun, p. 77, 4.3.105–110
[37] Abramowitz and Stegun, p. 82, 4.4.52–57
[38] Finney, Ross (2003). *Calculus : Graphical, Numerical, Algebraic*. Glenview, Illinois: Prentice Hall. pp. 159–161. ISBN 0-13-063131-0.
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[40] Abramowitz and Stegun, p. 72, 4.3.23

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- Abramowitz, Milton; Stegun, Irene A., eds. (1972). *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*. New York: Dover Publications. ISBN 978-0-486-61272-0

## External links

- Values of Sin and Cos, expressed in surds, for integer multiples of 3° and of 5⅝° (http://www.jdawiseman.com/papers/easymath/surds_sin_cos.html), and for the same angles Csc and Sec (http://www.jdawiseman.com/papers/easymath/surds_csc_sec.html) and Tan (http://www.jdawiseman.com/papers/easymath/surds_tan.html).

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# TRIGONOMETRY FORMULAS

$$\cos^2(x)+\sin^2(x)=1 \qquad 1+\tan^2(x)=\sec^2(x) \qquad \cot^2(x)+1=\csc^2(x)$$

$$\cos(x\pm y)=\cos(x)\cos(y)\mp\sin(x)\sin(y)$$
$$\sin(x\pm y)=\sin(x)\cos(y)\pm\cos(x)\sin(y)$$

$$\tan(x\pm y)=\frac{\tan(x)\pm\tan(y)}{1\mp\tan(x)\tan(y)}$$

$$\sin(2x)=2\sin(x)\cos(x)$$
$$\cos(2x)=\begin{cases}\cos^2(x)-\sin^2(x)\\ 2\cos^2(x)-1\\ 1-2\sin^2(x)\end{cases}$$
$$\tan(2x)=\frac{2\tan(x)}{1-\tan^2(x)}$$

$$c^2=a^2+b^2-2ab\cos(C)$$
$$\frac{\sin(A)}{a}=\frac{\sin(B)}{b}=\frac{\sin(C)}{c}$$

$$\sin^2(x)=\frac{1-\cos(2x)}{2}$$
$$\cos^2(x)=\frac{1+\cos(2x)}{2}$$
$$\tan^2(x)=\frac{1-\cos(2x)}{1+\cos(2x)}$$

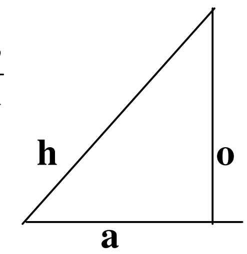
$$\cos\left(\frac{x}{2}\right)=\pm\sqrt{\frac{1+\cos(x)}{2}}$$
$$\sin\left(\frac{x}{2}\right)=\pm\sqrt{\frac{1-\cos(x)}{2}}$$
$$\tan\left(\frac{x}{2}\right)=\pm\sqrt{\frac{1-\cos(x)}{1+\cos(x)}}$$

$$\sin(x)\sin(y)=\tfrac{1}{2}[\cos(x-y)-\cos(x+y)]$$
$$\cos(x)\cos(y)=\tfrac{1}{2}[\cos(x-y)+\cos(x+y)]$$
$$\sin(x)\cos(y)=\tfrac{1}{2}[\sin(x+y)+\sin(x-y)]$$
$$\cos(x)\sin(y)=\tfrac{1}{2}[\sin(x+y)-\sin(x-y)]$$

$$\sin(x)+\sin(y)=2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$$
$$\sin(x)-\sin(y)=2\sin\left(\frac{x-y}{2}\right)\cos\left(\frac{x+y}{2}\right)$$
$$\cos(x)+\cos(y)=2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$$
$$\cos(x)-\cos(y)=-2\sin\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)$$

For two vectors **A** and **B, A·B** = ||**A**||||**B**||cos(θ)

The well known results: soh, cah, toa




soh: s stands for sine, o stands for opposite and h stands for hypotenuse, $\sin x = \dfrac{o}{h}$

cah: c stands for cosine, a stands for adjacent h stands for hypotenuse, $\cos x = \dfrac{a}{h}$

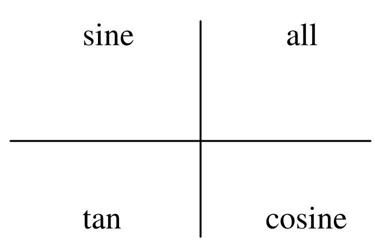
toa: t stands for tan, o stands for opposite and a stands for adjacent, $\tan x = \dfrac{o}{a}$

Where $x$ is the angle between the hypotenuse and the adjacent.
Other three trigonometric functions have the following relations:
$\csc x = \dfrac{1}{\sin x} = \dfrac{h}{o}$, $\sec x = \dfrac{1}{\cos x} = \dfrac{h}{a}$ and $\cot x = \dfrac{1}{\tan x} = \dfrac{a}{o}$

## Important values:

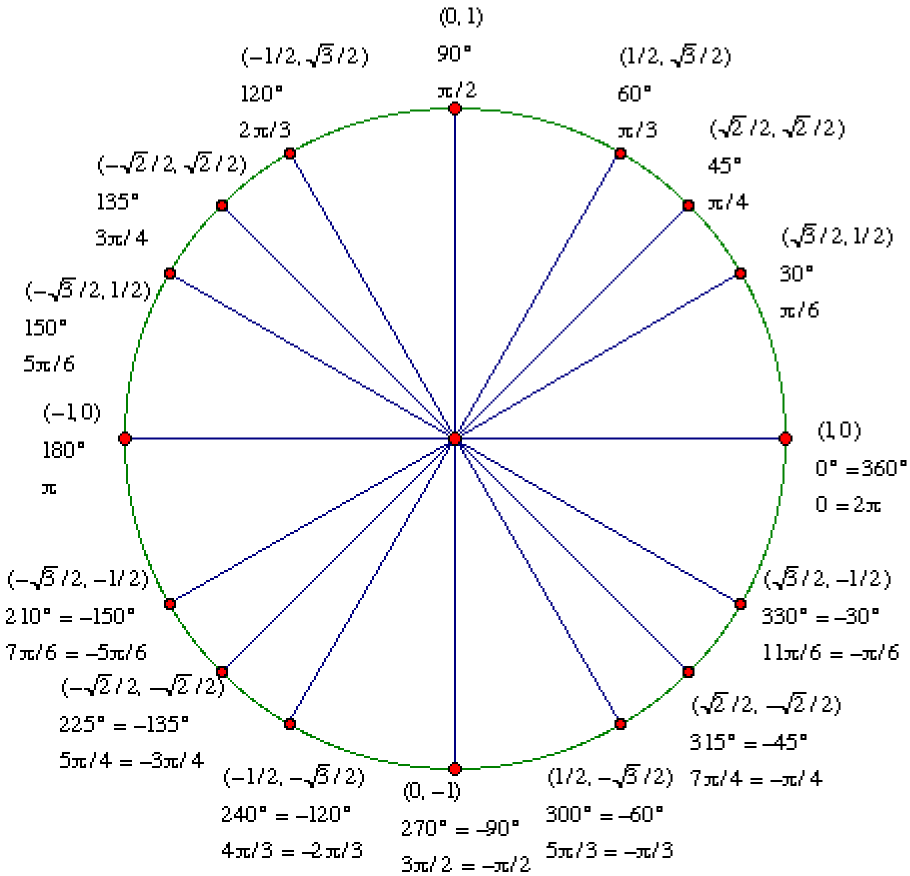
| | $0$ | $30^0 = \frac{\pi}{6}$ | $45^0 = \frac{\pi}{4}$ | $60^0 = \frac{\pi}{3}$ | $90^0 = \frac{\pi}{2}$ |
|---|---|---|---|---|---|
| sin | $0$ | $\frac{1}{2}$ | $\frac{\sqrt{2}}{2}$ | $\frac{\sqrt{3}}{2}$ | $1$ |
| cos | $1$ | $\frac{\sqrt{3}}{2}$ | $\frac{\sqrt{2}}{2}$ | $\frac{1}{2}$ | $0$ |
| tan | $0$ | $\frac{1}{\sqrt{3}}$ | $1$ | $\sqrt{3}$ | undefined |
| csc | undefined | $2$ | $\sqrt{2}$ | $\frac{2}{\sqrt{3}}$ | $1$ |
| sec | $1$ | $\frac{2}{\sqrt{3}}$ | $\sqrt{2}$ | $2$ | undefined |
| cot | undefined | $\sqrt{3}$ | $1$ | $\frac{1}{\sqrt{3}}$ | $0$ |

$\sin(n\pi \pm x) = [\,?\,]\sin x$, $\cos(n\pi \pm x) = [\,?\,]\cos x$, $\tan(n\pi \pm x) = [\,?\,]\tan x$ **,** the sign **?** is for plus or minus depending on the position of the terminal side. One may remember the four-quadrant rule: (**A**ll **S**tudents **T**ake **C**alculus: A = all, S = sine, T = tan, C = cosine)



Example: Find the value of $\sin 300^0$. We may write $\sin 300^0 = \sin(2 \cdot 180^0 - 60^0) = [-]\sin 60^0 = -\frac{\sqrt{3}}{2}$, in this case the terminal side is in quadrant four where sine is negative.

In the following diagram, each point on the unit circle is labeled first with its coordinates (exact values), then with the angle in degrees, then with the angle in radians. Points in the lower hemisphere have both positive and negative angles marked.

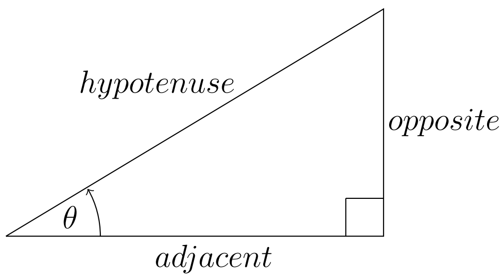


# Trigonometric Formula Sheet
## Definition of the Trig Functions

### Right Triangle Definition

Assume that:

$0 < \theta < \frac{\pi}{2}$ or $0^\circ < \theta < 90^\circ$



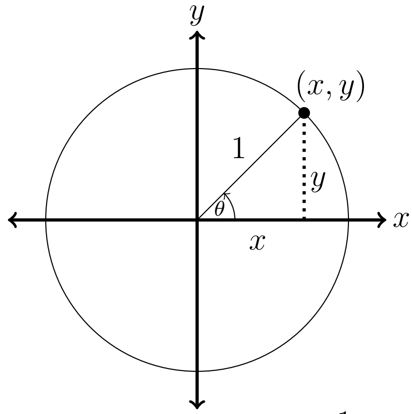

$$\sin\theta = \frac{opp}{hyp} \qquad \csc\theta = \frac{hyp}{opp}$$

$$\cos\theta = \frac{adj}{hyp} \qquad \sec\theta = \frac{hyp}{adj}$$

$$\tan\theta = \frac{opp}{adj} \qquad \cot\theta = \frac{adj}{opp}$$

### Unit Circle Definition

Assume $\theta$ can be any angle.




$$\sin\theta = \frac{y}{1} \qquad \csc\theta = \frac{1}{y}$$

$$\cos\theta = \frac{x}{1} \qquad \sec\theta = \frac{1}{x}$$

$$\tan\theta = \frac{y}{x} \qquad \cot\theta = \frac{x}{y}$$

## Domains of the Trig Functions

$\sin\theta, \quad \forall\ \theta \in (-\infty, \infty)$

$\cos\theta, \quad \forall\ \theta \in (-\infty, \infty)$

$\tan\theta, \quad \forall\ \theta \neq \left(n + \frac{1}{2}\right)\pi,\ where\ n \in \mathbb{Z}$

$\csc\theta, \quad \forall\ \theta \neq n\pi,\ where\ n \in \mathbb{Z}$

$\sec\theta, \quad \forall\ \theta \neq \left(n + \frac{1}{2}\right)\pi,\ where\ n \in \mathbb{Z}$

$\cot\theta, \quad \forall\ \theta \neq n\pi,\ where\ n \in \mathbb{Z}$

## Ranges of the Trig Functions

$$-1 \leq \sin\theta \leq 1$$
$$-1 \leq \cos\theta \leq 1$$
$$-\infty \leq \tan\theta \leq \infty$$

$$\csc\theta \geq 1\ and\ \csc\theta \leq -1$$
$$\sec\theta \geq 1\ and\ \sec\theta \leq -1$$
$$-\infty \leq \cot\theta \leq \infty$$

## Periods of the Trig Functions

The period of a function is the number, T, such that f ($\theta$ +T ) = f ($\theta$ ) .
So, if $\omega$ is a fixed number and $\theta$ is any angle we have the following periods.

$$\sin(\omega\theta) \ \Rightarrow\ T = \frac{2\pi}{\omega} \qquad \csc(\omega\theta) \ \Rightarrow\ T = \frac{2\pi}{\omega}$$

$$\cos(\omega\theta) \ \Rightarrow\ T = \frac{2\pi}{\omega} \qquad \sec(\omega\theta) \ \Rightarrow\ T = \frac{2\pi}{\omega}$$

$$\tan(\omega\theta) \ \Rightarrow\ T = \frac{\pi}{\omega} \qquad \cot(\omega\theta) \ \Rightarrow\ T = \frac{\pi}{\omega}$$

# Identities and Formulas

## Tangent and Cotangent Identities

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cot\theta = \frac{\cos\theta}{\sin\theta}$$

## Reciprocal Identities

$$\sin\theta = \frac{1}{\csc\theta} \qquad \csc\theta = \frac{1}{\sin\theta}$$

$$\cos\theta = \frac{1}{\sec\theta} \qquad \sec\theta = \frac{1}{\cos\theta}$$

$$\tan\theta = \frac{1}{\cot\theta} \qquad \cot\theta = \frac{1}{\tan\theta}$$

## Pythagorean Identities

$$\sin^2\theta + \cos^2\theta = 1$$

$$\tan^2\theta + 1 = \sec^2\theta$$

$$1 + \cot^2\theta = \csc^2\theta$$

## Even and Odd Formulas

$$\sin(-\theta) = -\sin\theta \qquad \csc(-\theta) = -\csc\theta$$

$$\cos(-\theta) = \cos\theta \qquad \sec(-\theta) = \sec\theta$$

$$\tan(-\theta) = -\tan\theta \qquad \cot(-\theta) = -\cot\theta$$

## Periodic Formulas

If n is an integer

$$\sin(\theta + 2\pi n) = \sin\theta \qquad \csc(\theta + 2\pi n) = \csc\theta$$

$$\cos(\theta + 2\pi n) = \cos\theta \qquad \sec(\theta + 2\pi n) = \sec\theta$$

$$\tan(\theta + \pi n) = \tan\theta \qquad \cot(\theta + \pi n) = \cot\theta$$

## Double Angle Formulas

$$\sin(2\theta) = 2\sin\theta\cos\theta$$

$$\begin{aligned}\cos(2\theta) &= \cos^2\theta - \sin^2\theta \\ &= 2\cos^2\theta - 1 \\ &= 1 - 2\sin^2\theta\end{aligned}$$

$$\tan(2\theta) = \frac{2\tan\theta}{1 - \tan^2\theta}$$

## Degrees to Radians Formulas

If $x$ is an angle in degrees and $t$ is an angle in radians then:

$$\frac{\pi}{180^\circ} = \frac{t}{x} \qquad \Rightarrow \qquad t = \frac{\pi x}{180^\circ} \quad \textbf{and} \quad x = \frac{180^\circ t}{\pi}$$

## Half Angle Formulas

$$\sin\theta = \pm\sqrt{\frac{1 - \cos(2\theta)}{2}}$$

$$\cos\theta = \pm\sqrt{\frac{1 + \cos(2\theta)}{2}}$$

$$\tan\theta = \pm\sqrt{\frac{1 - \cos(2\theta)}{1 + \cos(2\theta)}}$$

## Sum and Difference Formulas

$$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$$

$$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$$

$$\tan(\alpha \pm \beta) = \frac{\tan\alpha \pm \tan\beta}{1 \mp \tan\alpha\tan\beta}$$

## Product to Sum Formulas

$$\sin\alpha\sin\beta = \frac{1}{2}[\cos(\alpha - \beta) - \cos(\alpha + \beta)]$$

$$\cos\alpha\cos\beta = \frac{1}{2}[\cos(\alpha - \beta) + \cos(\alpha + \beta)]$$

$$\sin\alpha\cos\beta = \frac{1}{2}[\sin(\alpha + \beta) + \sin(\alpha - \beta)]$$

$$\cos\alpha\sin\beta = \frac{1}{2}[\sin(\alpha + \beta) - \sin(\alpha - \beta)]$$

## Sum to Product Formulas

$$\sin\alpha + \sin\beta = 2\sin\left(\frac{\alpha + \beta}{2}\right)\cos\left(\frac{\alpha - \beta}{2}\right)$$

$$\sin\alpha - \sin\beta = 2\cos\left(\frac{\alpha + \beta}{2}\right)\sin\left(\frac{\alpha - \beta}{2}\right)$$

$$\cos\alpha + \cos\beta = 2\cos\left(\frac{\alpha + \beta}{2}\right)\cos\left(\frac{\alpha - \beta}{2}\right)$$

$$\cos\alpha - \cos\beta = -2\sin\left(\frac{\alpha + \beta}{2}\right)\sin\left(\frac{\alpha - \beta}{2}\right)$$

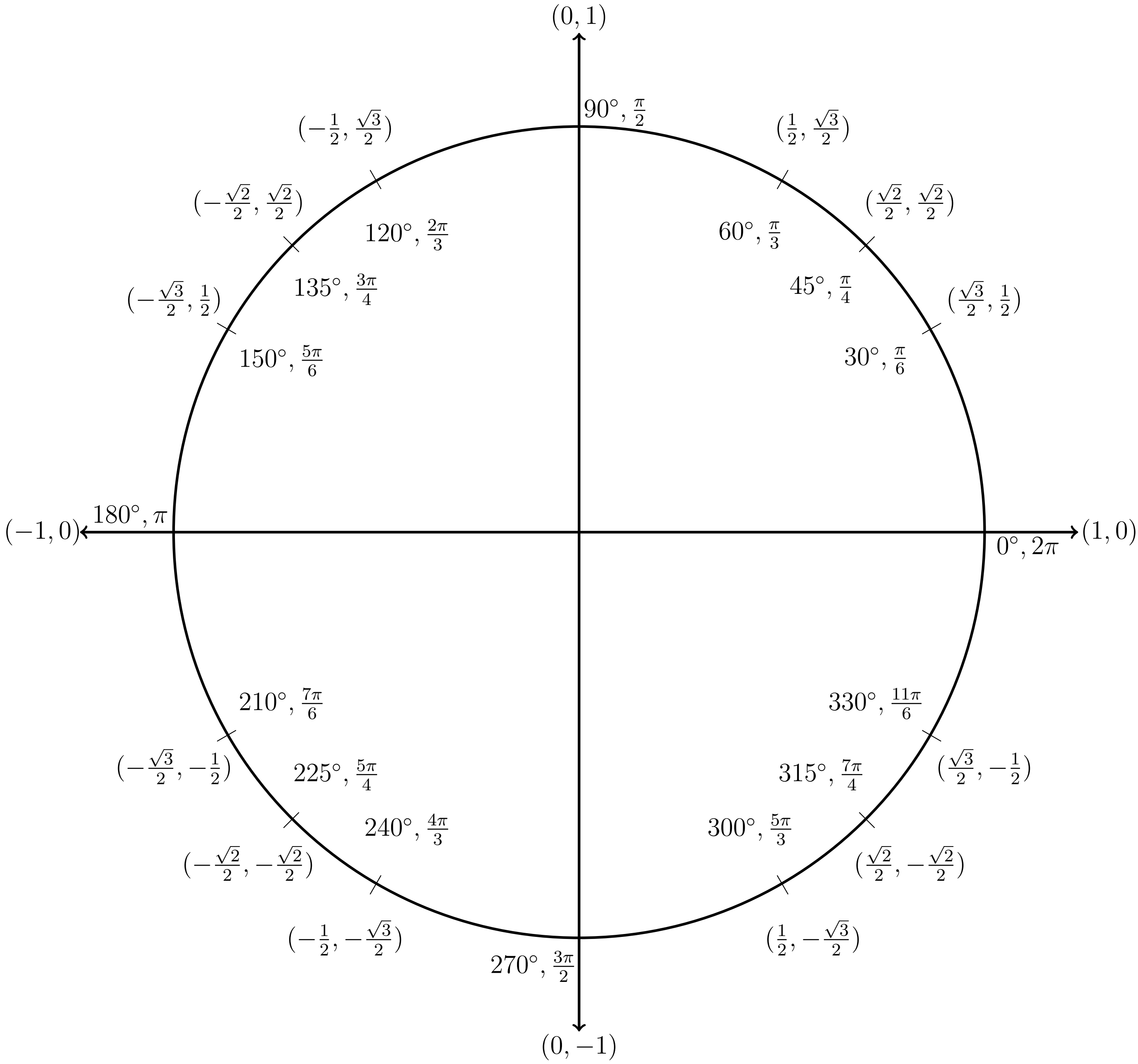
## Cofunction Formulas

$$\sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta \qquad \cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta$$

$$\csc\left(\frac{\pi}{2} - \theta\right) = \sec\theta \qquad \sec\left(\frac{\pi}{2} - \theta\right) = \csc\theta$$

$$\tan\left(\frac{\pi}{2} - \theta\right) = \cot\theta \qquad \cot\left(\frac{\pi}{2} - \theta\right) = \tan\theta$$

# Unit Circle




*For any ordered pair on the unit circle* $(x, y)$ : $\cos\theta = x$ *and* $\sin\theta = y$

# Example

$$\cos\left(\frac{7\pi}{6}\right) = -\frac{\sqrt{3}}{2} \qquad \sin\left(\frac{7\pi}{6}\right) = -\frac{1}{2}$$

# Inverse Trig Functions

### Definition

$\theta = \sin^{-1}(x)$ **is equivalent to** $x = \sin\theta$

$\theta = \cos^{-1}(x)$ **is equivalent to** $x = \cos\theta$

$\theta = \tan^{-1}(x)$ **is equivalent to** $x = \tan\theta$

### Domain and Range

| **Function** | **Domain** | **Range** |
|---|---|---|
| $\theta = \sin^{-1}(x)$ | $-1 \le x \le 1$ | $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$ |
| $\theta = \cos^{-1}(x)$ | $-1 \le x \le 1$ | $0 \le \theta \le \pi$ |
| $\theta = \tan^{-1}(x)$ | $-\infty \le x \le \infty$ | $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ |

### Inverse Properties

These properties hold for x in the domain and $\theta$ in the range

| | |
|---|---|
| $\sin(\sin^{-1}(x)) = x$ | $\sin^{-1}(\sin(\theta)) = \theta$ |
| $\cos(\cos^{-1}(x)) = x$ | $\cos^{-1}(\cos(\theta)) = \theta$ |
| $\tan(\tan^{-1}(x)) = x$ | $\tan^{-1}(\tan(\theta)) = \theta$ |

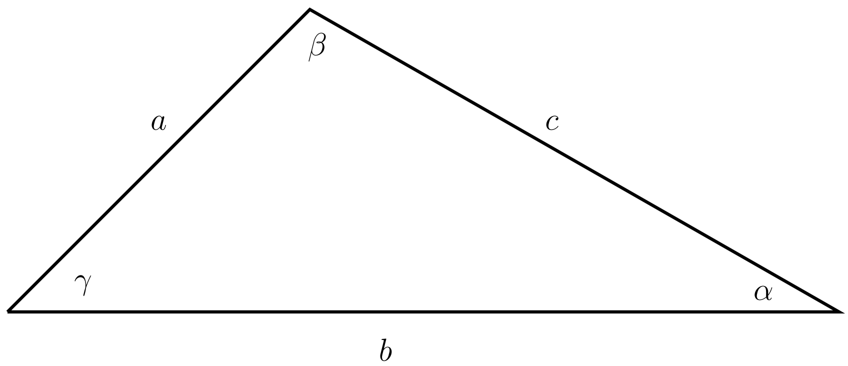
### Other Notations

$\sin^{-1}(x) = \arcsin(x)$

$\cos^{-1}(x) = \arccos(x)$

$\tan^{-1}(x) = \arctan(x)$

# Law of Sines, Cosines, and Tangents




### Law of Sines

$$\frac{\sin\alpha}{a} = \frac{\sin\beta}{b} = \frac{\sin\gamma}{c}$$

### Law of Cosines

$$a^2 = b^2 + c^2 - 2bc\cos\alpha$$

$$b^2 = a^2 + c^2 - 2ac\cos\beta$$

$$c^2 = a^2 + b^2 - 2ab\cos\gamma$$

### Law of Tangents

$$\frac{a-b}{a+b} = \frac{\tan\frac{1}{2}(\alpha-\beta)}{\tan\frac{1}{2}(\alpha+\beta)}$$

$$\frac{b-c}{b+c} = \frac{\tan\frac{1}{2}(\beta-\gamma)}{\tan\frac{1}{2}(\beta+\gamma)}$$

$$\frac{a-c}{a+c} = \frac{\tan\frac{1}{2}(\alpha-\gamma)}{\tan\frac{1}{2}(\alpha+\gamma)}$$

# Complex Numbers

$$i = \sqrt{-1} \qquad i^2 = -1 \qquad i^3 = -i \qquad i^4 = 1$$

$\sqrt{-a} = i\sqrt{a}, \quad a \geq 0$

$(a+bi)+(c+di) = a+c+(b+d)i$

$(a+bi)-(c+di) = a-c+(b-d)i$

$(a+bi)(c+di) = ac-bd+(ad+bc)i$

$(a+bi)(a-bi) = a^2+b^2$

$|a+bi| = \sqrt{a^2+b^2}$ **Complex Modulus**

$\overline{(a+bi)} = a-bi$ **Complex Conjugate**

$\overline{(a+bi)}(a+bi) = |a+bi|^2$

# DeMoivre's Theorem

Let $z = r(\cos\theta + i\sin\theta)$, and let $n$ be a positive integer.
Then:

$$z^n = r^n(\cos n\theta + i\sin n\theta).$$

**Example:** Let $z = 1-i$, find $z^6$.

Solution: First write $z$ in polar form.

$r = \sqrt{(1)^2+(-1)^2} = \sqrt{2}$

$\theta = arg(z) = \tan^{-1}\left(\frac{-1}{1}\right) = -\frac{\pi}{4}$

Polar Form: $z = \sqrt{2}\left(\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right)$

Applying DeMoivre's Theorem gives :

$$\begin{aligned} z^6 &= \left(\sqrt{2}\right)^6\left(\cos\left(6\cdot -\frac{\pi}{4}\right) + i\sin\left(6\cdot -\frac{\pi}{4}\right)\right) \\ &= 2^3\left(\cos\left(-\frac{3\pi}{2}\right) + i\sin\left(-\frac{3\pi}{2}\right)\right) \\ &= 8(0+i(1)) \\ &= 8i \end{aligned}$$

# Finding the $nth$ roots of a number using DeMoivre's Theorem

**Example:** Find all the complex fourth roots of 4. That is, find all the complex solutions of $x^4 = 4$.

We are asked to find all complex fourth roots of 4.
These are all the solutions (including the complex values) of the equation $x^4 = 4$.

For any positive integer $n$ , a nonzero complex number $z$ has exactly $n$ distinct $n$th roots. More specifically, if $z$ is written in the trigonometric form $r(\cos\theta + i\sin\theta)$, the $n$th roots of $z$ are given by the following formula.

$$(*)\ r^{\frac{1}{n}}\left(\cos\left(\frac{\theta}{n}+\frac{360^\circ k}{n}\right)+i\sin\left(\frac{\theta}{n}+\frac{360^\circ k}{n}\right)\right), \quad for \quad k = 0, 1, 2, ..., n-1.$$

Remember from the previous example we need to write 4 in trigonometric form by using:

$$r = \sqrt{(a)^2+(b)^2} \qquad \text{and} \qquad \theta = arg(z) = \tan^{-1}\left(\frac{b}{a}\right).$$

So we have the complex number $a + ib = 4 + i0$.

Therefore $a = 4$ and $b = 0$

So $r = \sqrt{(4)^2+(0)^2} = 4$ and
$\theta = arg(z) = \tan^{-1}\left(\frac{0}{4}\right) = 0$
Finally our trigonometric form is $4 = 4(\cos 0^\circ + i\sin 0^\circ)$

Using the formula $(*)$ above with $n = 4$, we can find the fourth roots of $4(\cos 0^\circ + i\sin 0^\circ)$

- For $k = 0$, $4^{\frac{1}{4}}\left(\cos\left(\frac{0^\circ}{4}+\frac{360^\circ*0}{4}\right)+i\sin\left(\frac{0^\circ}{4}+\frac{360^\circ*0}{4}\right)\right) = \sqrt{2}\left(\cos(0^\circ)+i\sin(0^\circ)\right) = \sqrt{2}$
- For $k = 1$, $4^{\frac{1}{4}}\left(\cos\left(\frac{0^\circ}{4}+\frac{360^\circ*1}{4}\right)+i\sin\left(\frac{0^\circ}{4}+\frac{360^\circ*1}{4}\right)\right) = \sqrt{2}\left(\cos(90^\circ)+i\sin(90^\circ)\right) = \sqrt{2}i$
- For $k = 2$, $4^{\frac{1}{4}}\left(\cos\left(\frac{0^\circ}{4}+\frac{360^\circ*2}{4}\right)+i\sin\left(\frac{0^\circ}{4}+\frac{360^\circ*2}{4}\right)\right) = \sqrt{2}\left(\cos(180^\circ)+i\sin(180^\circ)\right) = -\sqrt{2}$
- For $k = 3$, $4^{\frac{1}{4}}\left(\cos\left(\frac{0^\circ}{4}+\frac{360^\circ*3}{4}\right)+i\sin\left(\frac{0^\circ}{4}+\frac{360^\circ*3}{4}\right)\right) = \sqrt{2}\left(\cos(270^\circ)+i\sin(270^\circ)\right) = -\sqrt{2}i$

Thus all of the complex roots of $x^4 = 4$ are:

$\mathbf{\sqrt{2}, \sqrt{2}i, -\sqrt{2}, -\sqrt{2}i}$ .

# Formulas for the Conic Sections

## Circle

$$StandardForm : (\mathbf{x}-\mathbf{h})^{\mathbf{2}} + (\mathbf{y}-\mathbf{k})^{\mathbf{2}} = \mathbf{r}^{\mathbf{2}}$$

$$Where\ (\mathbf{h},\mathbf{k}) = \mathbf{center}\ and\ \mathbf{r} = \mathbf{radius}$$

## Ellipse

$$Standard\ Form\ for\ Horizontal\ Major\ Axis :$$

$$\frac{(\mathbf{x}-\mathbf{h})^{\mathbf{2}}}{\mathbf{a}^{\mathbf{2}}} + \frac{(\mathbf{y}-\mathbf{k})^{\mathbf{2}}}{\mathbf{b}^{\mathbf{2}}} = \mathbf{1}$$

$$Standard\ Form\ for\ Vertical\ Major\ Axis :$$

$$\frac{(\mathbf{x}-\mathbf{h})^{\mathbf{2}}}{\mathbf{b}^{\mathbf{2}}} + \frac{(\mathbf{y}-\mathbf{k})^{\mathbf{2}}}{\mathbf{a}^{\mathbf{2}}} = \mathbf{1}$$

Where $(\mathbf{h},\mathbf{k})$= center

$\mathbf{2a}$=length of major axis

$\mathbf{2b}$=length of minor axis

$(\mathbf{0} < \mathbf{b} < \mathbf{a})$

Foci can be found by using $\mathbf{c}^{\mathbf{2}} = \mathbf{a}^{\mathbf{2}} - \mathbf{b}^{\mathbf{2}}$

Where $\mathbf{c}$= foci length

# More Conic Sections

## Hyperbola

$$Standard\ Form\ for\ Horizontal\ Transverse\ Axis:$$

$$\frac{(\mathbf{x}-\mathbf{h})^{\mathbf{2}}}{\mathbf{a}^{\mathbf{2}}}-\frac{(\mathbf{y}-\mathbf{k})^{\mathbf{2}}}{\mathbf{b}^{\mathbf{2}}}=\mathbf{1}$$

$$Standard\ Form\ for\ Vertical\ Transverse\ Axis:$$

$$\frac{(\mathbf{y}-\mathbf{k})^{\mathbf{2}}}{\mathbf{a}^{\mathbf{2}}}-\frac{(\mathbf{x}-\mathbf{h})^{\mathbf{2}}}{\mathbf{b}^{\mathbf{2}}}=\mathbf{1}$$

Where $(\mathbf{h},\mathbf{k})$= center

**a**=distance between center and either vertex

Foci can be found by using $\mathbf{b^2} = \mathbf{c^2} - \mathbf{a^2}$

Where **c** is the distance between

center and either focus. $(\mathbf{b} > \mathbf{0})$

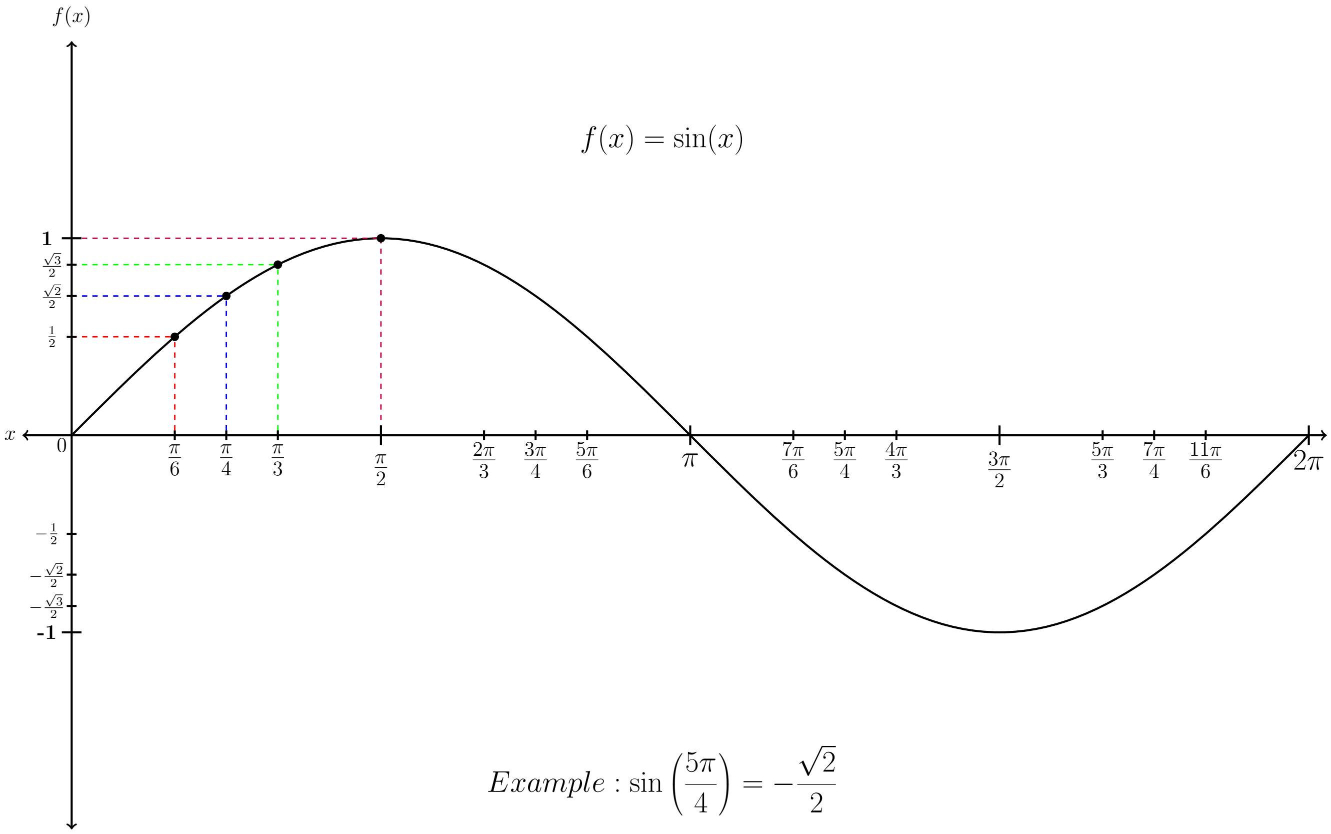
## Parabola

Vertical axis: $\mathbf{y} = \mathbf{a}(\mathbf{x} - \mathbf{h})^{\mathbf{2}} + \mathbf{k}$

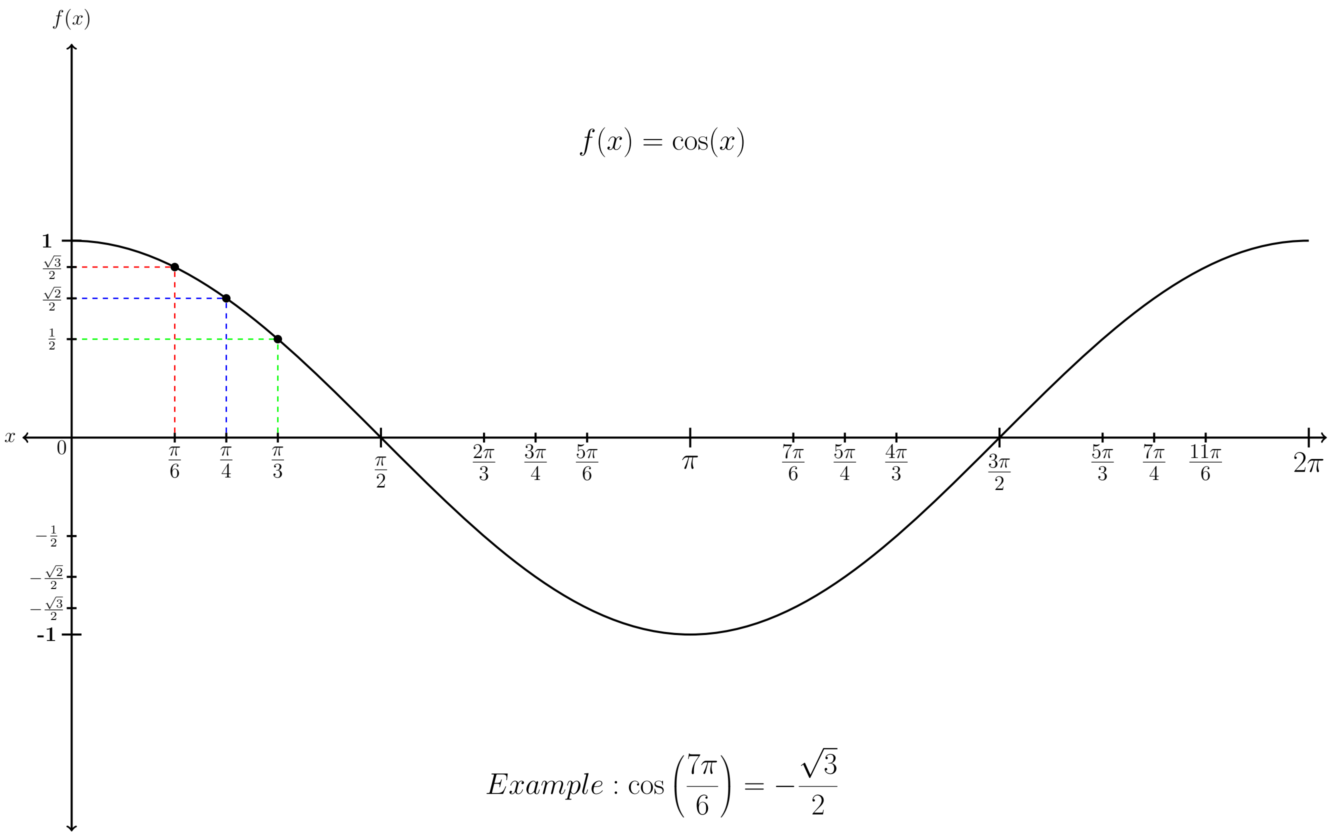
Horizontal axis: $\mathbf{x} = \mathbf{a}(\mathbf{y} - \mathbf{k})^{\mathbf{2}} + \mathbf{h}$

Where $(\mathbf{h},\mathbf{k})$= vertex

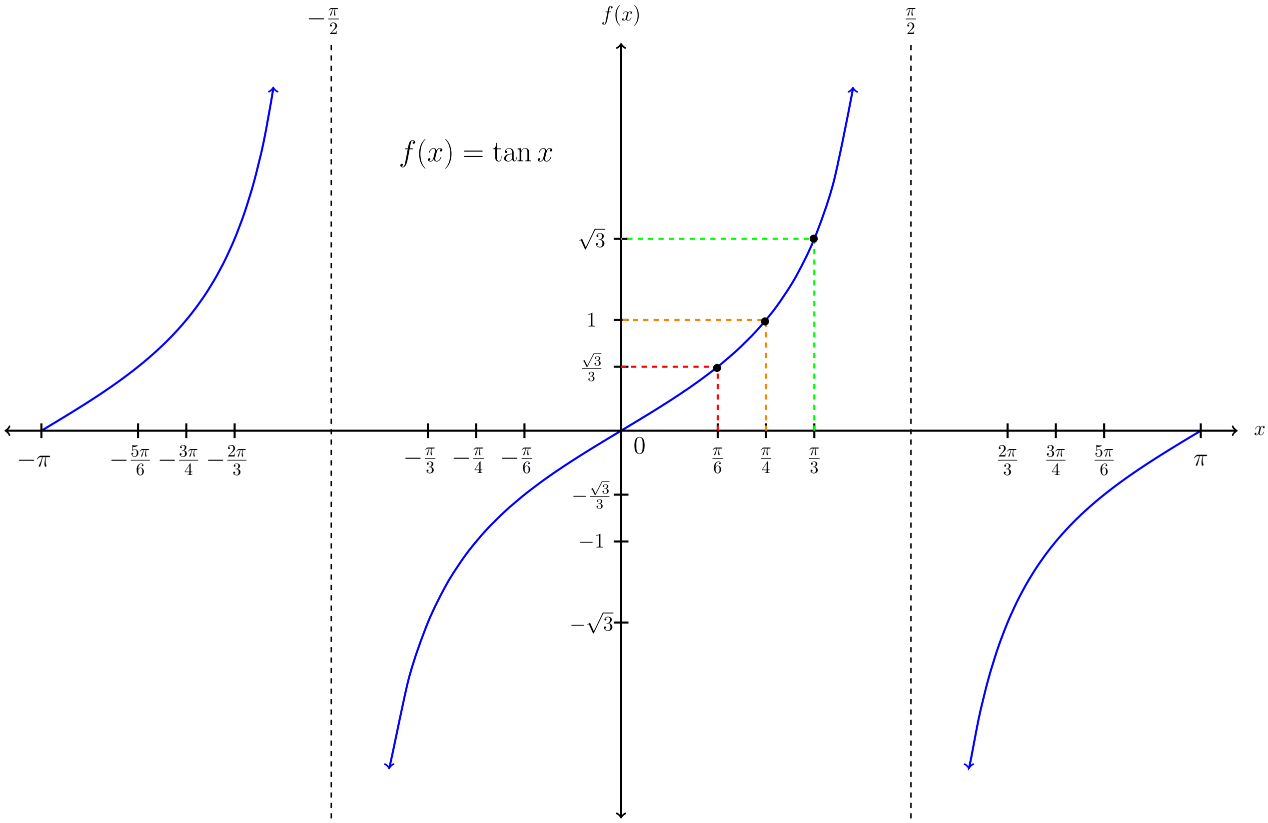
**a**=scaling factor




$$Example : \sin\left(\frac{5\pi}{4}\right) = -\frac{\sqrt{2}}{2}$$




$$Example : \cos\left(\frac{7\pi}{6}\right) = -\frac{\sqrt{3}}{2}$$



$f(x)$
$f(x) = \tan x$
$-\frac{\pi}{2}$
$\frac{\pi}{2}$
$x$
$\sqrt{3}$
$1$
$\frac{\sqrt{3}}{3}$
$0$
$-\frac{\sqrt{3}}{3}$
$-1$
$-\sqrt{3}$
$-\pi$
$-\frac{5\pi}{6}$
$-\frac{3\pi}{4}$
$-\frac{2\pi}{3}$
$-\frac{\pi}{3}$
$-\frac{\pi}{4}$
$-\frac{\pi}{6}$
$\frac{\pi}{6}$
$\frac{\pi}{4}$
$\frac{\pi}{3}$
$\frac{2\pi}{3}$
$\frac{3\pi}{4}$
$\frac{5\pi}{6}$
$\pi$

# Trigonometry

Dr. Caroline Danneels

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# 1 Angles

## 1.1 The trigonometric circle

Take an x-axis and an y-axis (orthonormal) and let O be the origin. A circle centered in O and with radius = 1, is called a trigonometric circle or unit circle. Turning counterclockwise is the positive orientation in trigonometry (fig. 1).

## 1.2 Oriented angles

An angle is the figure formed by two rays that have the same beginning point. That point is called the vertex and the two rays are called the sides of the angle (also legs). If we call [OA the initial side of the angle and [OB the terminal side, then we have an oriented angle. This angle is referred to as $\angle AOB$ and the orientation is indicated by an arrow from the initial side to the terminal side. We can draw the arrow also in the opposite direction, still starting from the initial side of the angle [OA. Both angles represent the same oriented angle. The angle $\angle BOA$ is a different oriented angle which we call the opposite angle of $\angle AOB$ (fig. 2).

*Remark*: an oriented angle is in fact the set of all angles which can be transformed to each other by a rotation and/or a translation.

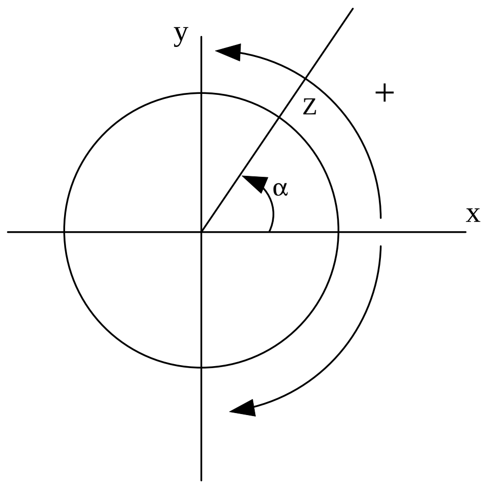



**fig. 1** : the trigonometric circle

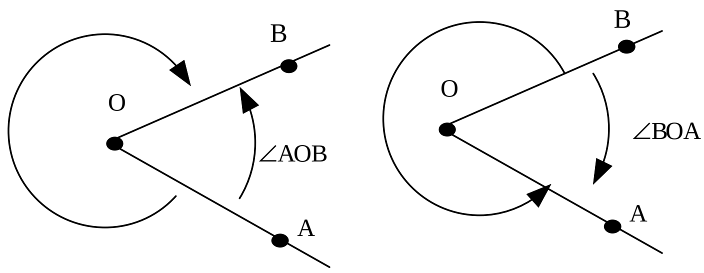



**fig. 2** : angles $\angle AOB$ and $\angle BOA$

The introduction of the trigonometric circle makes it possible to attach a value to each oriented angle $\angle AOB$, which we will call $\alpha$ from now on. Represent the oriented angle in the trigonometric circle and let the initial side of this angle coincide with the x-axis (see fig. 1). Then the terminal side intersects the trigonometric circle in point Z. Then Z is the representation of the oriented angle $\alpha$ on the trigonometric circle.

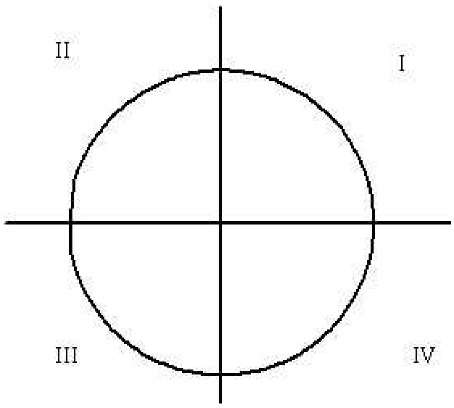



**fig. 3** : the four quadrants

If $Z \in$ I: angle $\alpha$ belongs to the first quadrant.

If $Z \in$ II: angle $\alpha$ belongs to the second quadrant.

If $Z \in$ III: angle $\alpha$ belongs to the third quadrant.

If $Z \in$ IV: angle $\alpha$ belongs to the fourth quadrant.

There are two commonly used units of measurement for angles. The more familiar unit of measurement is that of degrees. A circle is divided into 360 equal degrees, so that a right angle is 90°. Each degree is subdivided into 60 minutes and each minute into 60 seconds. The symbols °, ' and " are used for degrees, arcminutes and arcseconds.

In most mathematical work beyond practical geometry, angles are typically measured in radians rather than degrees.

An angle of 1 radian determines on the circle an arc with length the radius of the circle. Because the length of a full circle is $2\pi R$, a circle contains $2\pi$ radians. Contrariwise, if one draws in the centre of a circle with radius R an angle of $\theta$ radians, then this angle determines an arc on the circle with length $\theta \cdot R$. Subdivisions of radians are written in decimal form.

Next to Z you can put an infinite number of values which differ from each other by an integer multiple of 360° or $2\pi$, because you can make more turns in one or the other direction starting at the initial side of the angle and arriving at the terminal side of the angle (these angles are called coterminal). The set of all these values is called the measure of the oriented angle $\alpha$. The principal value of $\alpha$ is that value which belongs to ]- 180°, 180°], resp. $]-\pi,\pi]$.

## 1.3 Conversion between radians and degrees

Because $2\pi = 360°$ ,following conversion formulas can be applied:

$$\text{r rad} \rightarrow \left(\frac{360 \bullet \text{r}}{2\pi}\right)^{\circ}$$

$$\text{g}^{\circ} \rightarrow \left(\frac{2\pi \bullet g}{360}\right) \text{rad}$$

*Remark*: when an angle is represented in radians, one does only mention the value, not the term ‘rad’.

# 2 The trigonometric numbers

## 2.1 Definitons

Consider the construction of the oriented angle $\alpha$ as described in the previous paragraph. The terminal side of the angle $\alpha$ intersects the unit circle in the point Z. The $\sin\alpha$ can be defined as the y-coordinate of this point. The $\cos\alpha$ can be defined as the x-coordinate of this point. In this way, we can find the sine or cosine of any real value of $\alpha$ ($\alpha \in$ IR). Conversely, the choice of a cosine-value and a sine-value define in the interval $[0,2\pi[$ only one angle. Overall there are an infinite number of solutions, which one can find by adding on multiples of $2\pi$.

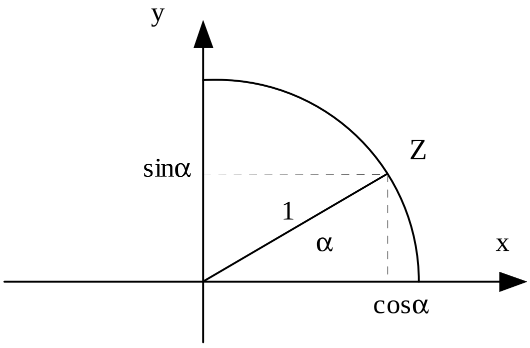



**fig. 4** : Sine and cosine in the  trigonometric circle

Beside sine and cosine other trigonometric numbers are defined as follows :

tangent : $\tan\alpha = \dfrac{\sin\alpha}{\cos\alpha}$

cotangent : $\cot\alpha = \dfrac{\cos\alpha}{\sin\alpha}$

secant : $\sec\alpha = \dfrac{1}{\cos\alpha}$

cosecant : $\csc\alpha = \dfrac{1}{\sin\alpha}$

Fig. 5 gives a graphical representation of the above trigonometric numbers in terms of distances associated with the unit circle.

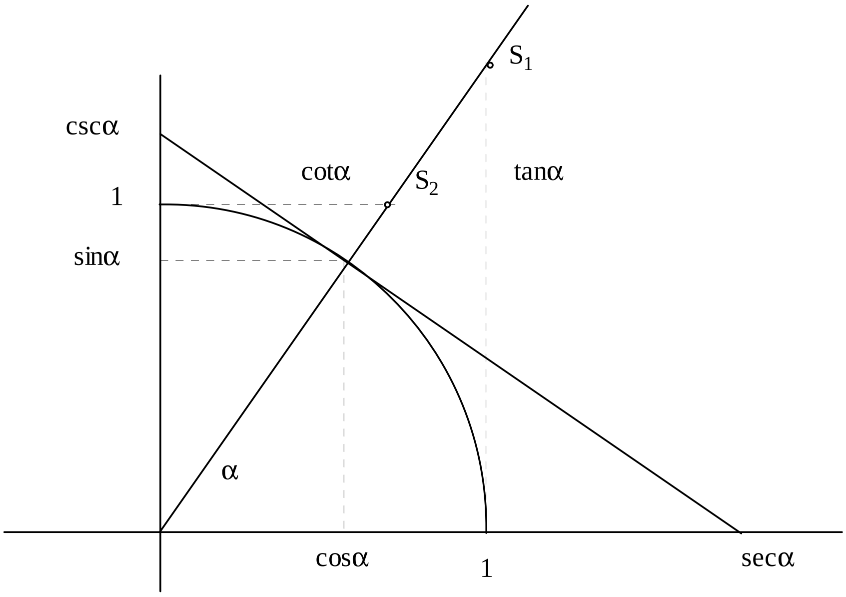



**fig. 5** : the graphical representation of the trigonometric numbers in terms of distances associated with the unit circle

Consequently, the trigonometric numbers have values which are in the following areas:

$\sin \alpha \in [\,-1,1\,]$ $\qquad$ $\cos \alpha \in [\,-1,1\,]$

$\tan \alpha \in \,]-\infty, +\infty[$ $\qquad$ $\cot \alpha \in \,]-\infty, +\infty[$

$\sec \alpha \in \,]-\infty, -1] \cup [1, +\infty[$ $\qquad$ $\csc \alpha \in \,]-\infty, -1] \cup [1, +\infty[$

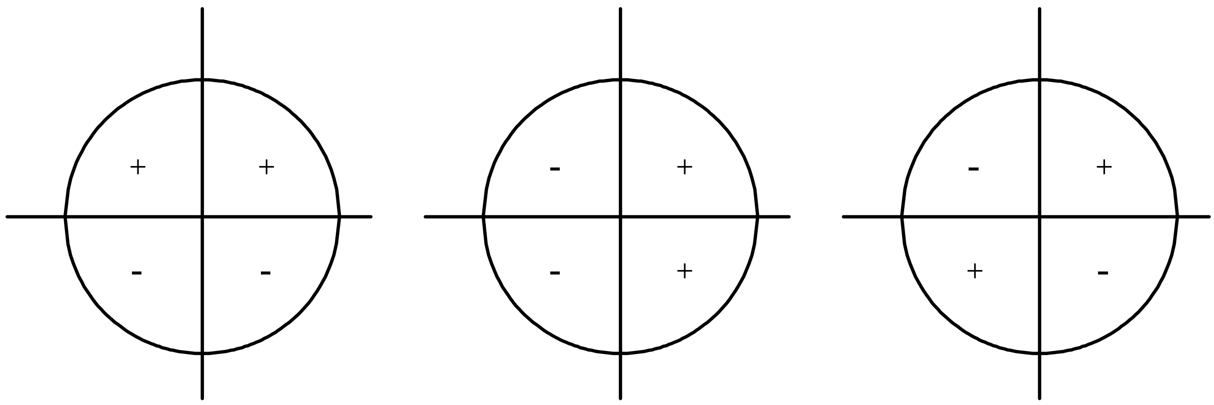
## 2.2 Some special angles and their trigonometric numbers

| $\alpha$ | 0 | $30° = \pi/6$ | $45° = \pi/4$ | $60° = \pi/3$ | $90° = \pi/2$ |
|---|---|---|---|---|---|
| $\sin \alpha$ | 0 | 1/2 | $\sqrt{2}/2$ | $\sqrt{3}/2$ | 1 |
| $\cos \alpha$ | 1 | $\sqrt{3}/2$ | $\sqrt{2}/2$ | 1/2 | 0 |
| $\tan \alpha$ | 0 | $1/\sqrt{3}$ | 1 | $\sqrt{3}$ | $\infty$ |
| $\cot \alpha$ | $\infty$ | $\sqrt{3}$ | 1 | $1/\sqrt{3}$ | 0 |
| $\sec \alpha$ | 1 | $2/\sqrt{3}$ | $\sqrt{2}$ | 2 | $\infty$ |
| $\csc \alpha$ | $\infty$ | 2 | $\sqrt{2}$ | $\sqrt{2}/3$ | 1 |

Trigonometric numbers of angles in the other quadrants we shell find through the use of the reference angle (see paragraph 2.6.2.)

## 2.3 Sign variation for the trigonometric numbers by quadrant

Inside a quadrant the trigonometric numbers keep the same sign (fig. 6).




sine
cosecant

cosine
secant

tangent
cotangent

**fig.6** : sign variation for the trigonometric numbers by quadrant

## 2.4 Pythagorean identities

The basic relationship between the sine and the cosine is the Pythagorean or fundamental trigonometric identity: $\cos^2\alpha + \sin^2\alpha = 1$

This can be viewed as a version of the Pythagorean theorem, and follows from the equation $x^2 + y^2 = 1$ for the unit circle (see fig. 7):

$\|OP\|^2 + \|PZ\|^2 = \|OZ\|^2$ with $\|OP\| = \cos\alpha$ ; $\|PZ\| = \sin\alpha$ ; $\|OZ\| = 1$

Dividing the Pythagorean identity by either $\cos^2\theta$ or $\sin^2\theta$ yields the following identities:

$1 + \tan^2\alpha = \sec^2\alpha$

$1 + \cot^2\alpha = \csc^2\alpha$

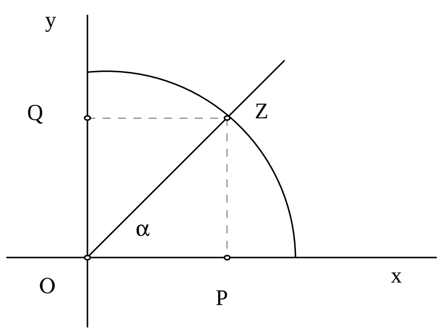



**fig. 7** : the triangle OPZ

## 2.5 Examples

### 2.5.1 Calculation of the trigonometric numbers

Given: $\sin\alpha = 5/13$

Asked: all other trigonometric numbers

Because the sine of this angle is positive, the angle is situated in the first or second quadrant. We determine the other trigonometric numbers as follows:

- from the Pythagorean trigonometric identity:

$$\cos^2\alpha = 1 - \sin^2\alpha = 1 - 25/169 = 144/169$$

we get: $\cos\alpha = \pm\sqrt{144/169} = \pm 12/13$

- $\tan\alpha = \sin\alpha/\cos\alpha = \pm 5/12$
- $\cot\alpha = 1/\tan\alpha = \pm 12/5$
- $\sec\alpha = 1/\cos\alpha = \pm 13/12$
- $\csc\alpha = 1/\sin\alpha = 13/5$

The two possible solutions for some of the trigonometric numbers correspond with the values of these numbers according to the quadrant in which the angle is situated.

Summary :

| quadrant | sin | cos | tan | cot | sec | csc |
|---|---|---|---|---|---|---|
| 1st | 5/13 | 12/13 | 5/12 | 12/5 | 13/12 | 13/5 |
| 2nd | 5/13 | -12/13 | -5/12 | -12/5 | -13/12 | 13/5 |

### 2.5.2 Proof the following identity

$\sec^2 \alpha + \csc^2 \alpha = \sec^2 \alpha \ \csc^2 \alpha$

Proof :

$$
\begin{aligned}
\sec^2 \alpha + \csc^2 \alpha &= \frac{1}{\cos^2 \alpha} + \frac{1}{\sin^2 \alpha} \\
&= \frac{\sin^2 \alpha + \cos^2 \alpha}{\cos^2 \alpha \ \sin^2 \alpha} \\
&= \frac{1}{\cos^2 \alpha \ \sin^2 \alpha} \\
&= \sec^2 \alpha \ \csc^2 \alpha
\end{aligned}
$$

## 2.6 Special pairs of angles

The sines, cosines and tangents, cotangents of some angles are equal to the sines, cosines and tangents, cotangents of other angles.

### 2.6.1 Formulas

a. Supplementary angles ( = sum is $\pi$ )

$\sin(\pi - \alpha) = \sin \alpha$ $\qquad$ $\cos(\pi - \alpha) = -\cos \alpha$

$\tan(\pi - \alpha) = -\tan \alpha$ $\qquad$ $\cot(\pi - \alpha) = -\cot \alpha$

b. Anti-supplementary angles ( = difference is $\pi$ )

$\sin(\pi + \alpha) = -\sin \alpha$ $\qquad$ $\cos(\pi + \alpha) = -\cos \alpha$

$\tan(\pi + \alpha) = \tan \alpha$ $\qquad$ $\cot(\pi + \alpha) = \cot \alpha$

c. Opposite angles ( = sum is $2\pi$ )

$\sin(2\pi - \alpha) = -\sin \alpha$ $\qquad$ $\cos(2\pi - \alpha) = \cos \alpha$

$\tan(2\pi - \alpha) = -\tan \alpha$ $\qquad$ $\cot(2\pi - \alpha) = -\cot \alpha$

d. Complementary angles ( = sum is $\pi / 2$ )

$\sin(\pi/2 - \alpha) = \cos \alpha$ $\qquad$ $\cos(\pi/2 - \alpha) = \sin \alpha$

$\tan(\pi/2 - \alpha) = \cot \alpha$ $\qquad$ $\cot(\pi/2 - \alpha) = \tan \alpha$

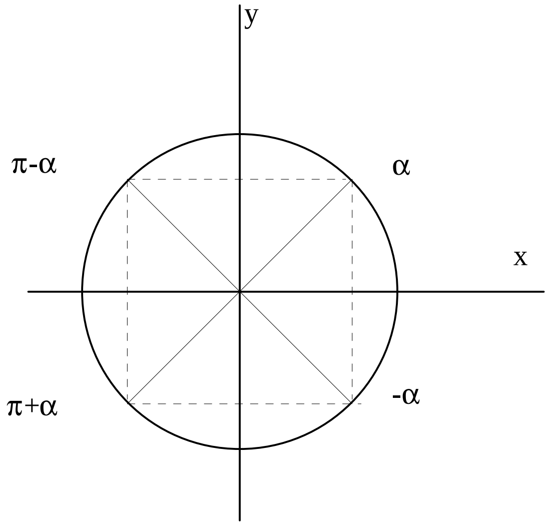



**fig. 8** : special pairs of angles

### 2.6.2 Reference angles

The use of reference angles is a way to simplify the calculation of the trigonometric numbers at various angles.

Associated with every angle drawn in standard position (which means that its vertex is located at the origin and the initial side is on the positive x-axis) (except angles of which the terminal side lies “on” the axes, called quadrantal angles) there is an angle called the reference angle. The reference angle is the acute angle formed by the terminal side of the given angle and the *x*-axis. Angles in quadrant I are their own reference angles. For angles in other quadrants, reference angles are calculated this way:

| Quadrant | β (reference angle) |
|---|---|
| I | $\beta = \alpha$ |
| II | $\beta = \pi - \alpha$ |
| III | $\beta = \alpha - \pi$ |
| IV | $\beta = - \alpha$ |

The reference angle and the given angle form a pair of angles to which you can apply the properties in the previous paragraph. Due to these properties, the value of a trigonometric number at a given angle is always the same as the value of that angle’s reference angle, except when there is a variation in sign. Because we know the signs of the numbers in different quadrants, we can simplify the calculation of a trigonometric number at any angle to the value of the number at the reference angle for that angle, to be found in the table in paragraph 2.2.

### 2.6.3 How to find all angles

To find the angle if given a certain trigonometric number, usually there are 2 solutions. Calculators give the most obvious solution, but in practical situations, there can be a second solution, or the second solution can be the only correct solution. In this case the user must adjust the solution given by the calculator.

The following table gives for positive and negative trigonometric numbers the quadrant in which the solution given by the calculator, is situated, and in the last column the quadrant of the second solution:

| Input | Calculator | Second solution |
|---|---|---|
| Positive sine or cosecant | 1 | 2 |
| Negative sine or cosecant | 4 | 3 |
| Positive cosine or secant | 1 | 4 |
| Negative cosine or secant | 2 | 3 |
| Positive tangent or cotangent | 1 | 3 |
| Negative tangent or cotangent | 4 | 2 |

## 2.7 Exercises

**2.7.1 Determine for the given trigonometric numbers the other trigonometric numbers; do not determine the angle before.**

1. $\sin\alpha = -\sqrt{6}/6$
2. $\csc\alpha = 4/3$
3. $\cot\alpha = -13/6$
4. $\sec\alpha = 25/24$

**2.7.2 Proof the following identities**

1. $\csc^2\alpha + \cot^2\beta = \csc^2\beta + \cot^2\alpha$

2. $\dfrac{(1-\sin\alpha)(1+\sin\alpha)}{(\sec\alpha+1)(\sec\alpha-1)} = \cos^2\alpha\cot^2\alpha$

3. $\dfrac{\sec\alpha+\tan\alpha}{\sec\alpha-\tan\alpha} = (\sec\alpha+\tan\alpha)^2 = \dfrac{1+\sin\alpha}{1-\sin\alpha}$

4. $(1+\cot\alpha)\left(\sec^2\alpha+2\tan\alpha\right) = \dfrac{(1+\tan\alpha)^3}{\tan\alpha}$

**2.7.3 Simplify the following expressions by applying the formulas of pairs of angles.**

1. $\dfrac{\cos\left(\frac{\pi}{2}+x\right)\cos(\pi-x)}{\sin\left(\frac{\pi}{2}-x\right)\sin(x-2\pi)} + \dfrac{\sin(\pi-x)\cos(\pi+x)}{\sin\left(\frac{\pi}{2}+x\right)\cos\left(\frac{3\pi}{2}+x\right)}$

2. $\dfrac{\csc(2\pi+x)\sec(\pi-x)}{\csc\left(\frac{\pi}{2}-x\right)\sec\left(x+\frac{\pi}{2}\right)} - \dfrac{\sec(2\pi-x)\csc(\pi-x)}{\sec\left(\frac{3\pi}{2}+x\right)\csc\left(\frac{3\pi}{2}-x\right)}$

**2.7.4 Determine the following trigonometric numbers. First find the reference angle, then apply the properties of special pairs of angles.**

1. $\sin 120°$
2. $\cos(-135°)$
3. $\tan 225°$
4. $\cot\left(-\frac{3\pi}{4}\right)$
5. $\tan\left(\frac{11\pi}{3}\right)$

**2.7.5 Solve in IR. Express the solution(s) in radians.**

1. $\cos 5x = -\frac{\sqrt{3}}{2}$
2. $\sin 5x = -\frac{\sqrt{3}}{2}$
3. $\sin 2x = \sqrt{3}\ \sin x$
4. $\sin x = \frac{1}{5}$ and $x \in \left]\frac{\pi}{2}, \pi\right[$;
   asked: $\sin 2x$
5. $2\sin^2 x = 3\cos x$
6. $\tan(3x + 2) = \sqrt{3}$

# 3 The trigonometric functions

## 3.1 Periodic functions

<u>Definiton :</u> a function $f : \mathbb{R} \to \mathbb{R}$ is periodic

$$\Leftrightarrow$$

$$\exists p \in \mathbb{R}_0 : \forall x \in \text{dom } f : x+p \in \text{dom } f \quad \wedge \quad f(x+p) = f(x)$$

If p satisfies this definition, then all positive and negative numbers which are an integer multiple of p also satisfy this definition. Therefore we call the smallest positive number which satisfies this definition the period P of the function. Graphically this periodicity means that the form of the graph of f(x) repeats itself over subsequent intervals with length P.

## 3.2 Even and odd functions

A function f is called EVEN if:

$$\forall x \in \text{dom } f : \quad -x \in \text{dom } f \quad \wedge \quad f(-x) = f(x)$$

Consequently two points with opposite x-values must have the same y-value. So the graph must be symmetric about the y-axis.

A function f is called ODD if:

$$\forall x \in \text{dom } f : \quad -x \in \text{dom } f \quad \wedge \quad f(-x) = -f(x)$$

Consequently two points with opposite x-values must have opposite y-values. So the graph is symmetric about the origin.

*Remark*: we consider the argument of trigonometric functions always in terms of radians.

## 3.3 Sine function

sin : $\mathbb{R} \rightarrow [-1,1]$ : x $\rightarrow$ sin x

The period of this function is $2\pi$. This function is odd, as opposite angles have opposite sines.

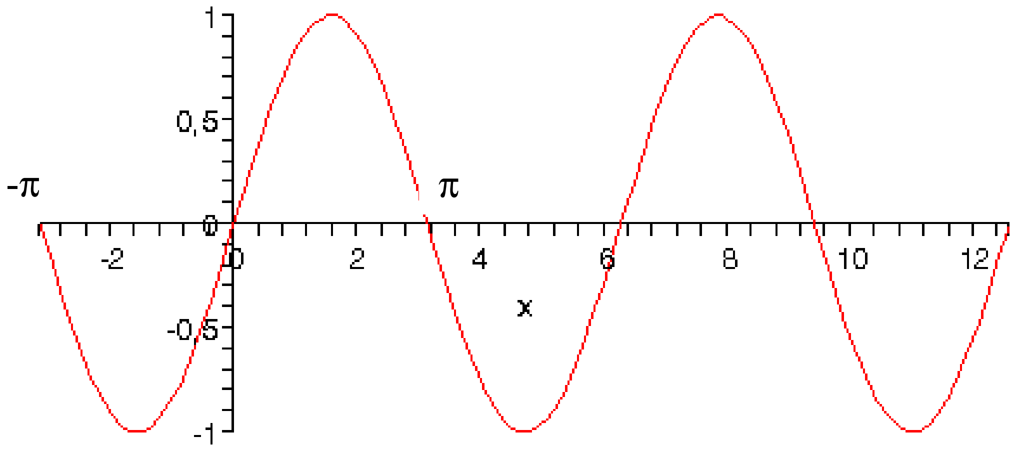



**fig. 9** : the sinusoïde

## 3.4 Cosine function

cos : $\mathbb{R} \rightarrow [-1,1]$ : x $\rightarrow$ cos x

The period of this function is $2\pi$. This function is even, as opposite angles have the same cosine.

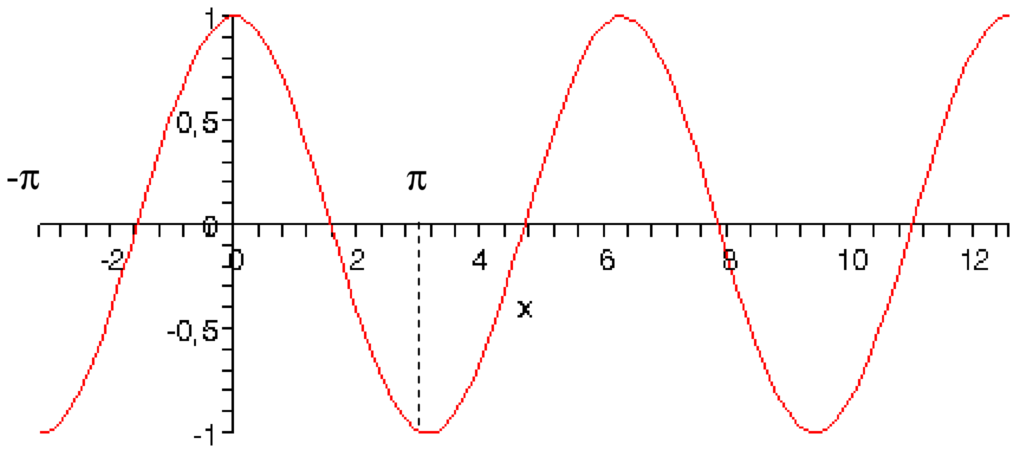



**fig. 10** : the cosinusoïde

## 3.5 Tangent function

$$\tan:\ \mathbb{R} \setminus \left\{ \frac{\pi}{2} + k\pi\ ,\ k \in \mathbb{Z} \right\} \to \mathbb{R}\ :\ x \to \tan x$$

The period of this function is $\pi$. This function is odd, as opposite angles have opposite tangents.

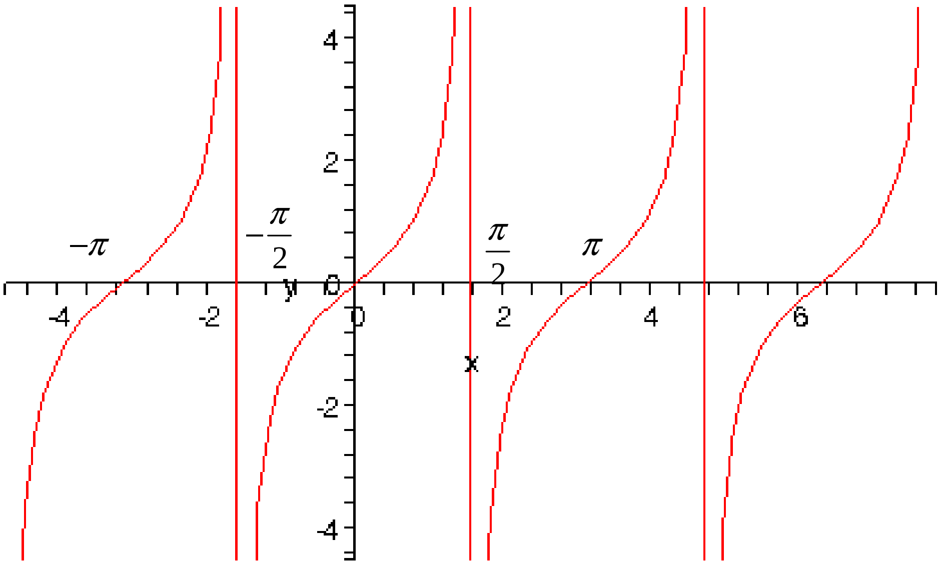



**fig. 11** : the tangent function

## 3.6 Cotangent function

$$\cot:\ \mathbb{R} \setminus \{ k\pi\ ,\ k \in \mathbb{Z} \} \to \mathbb{R}\ :\ x \to \cot x$$

The period of this function is $\pi$. This function is odd, as opposite angles have also opposite cotangents.

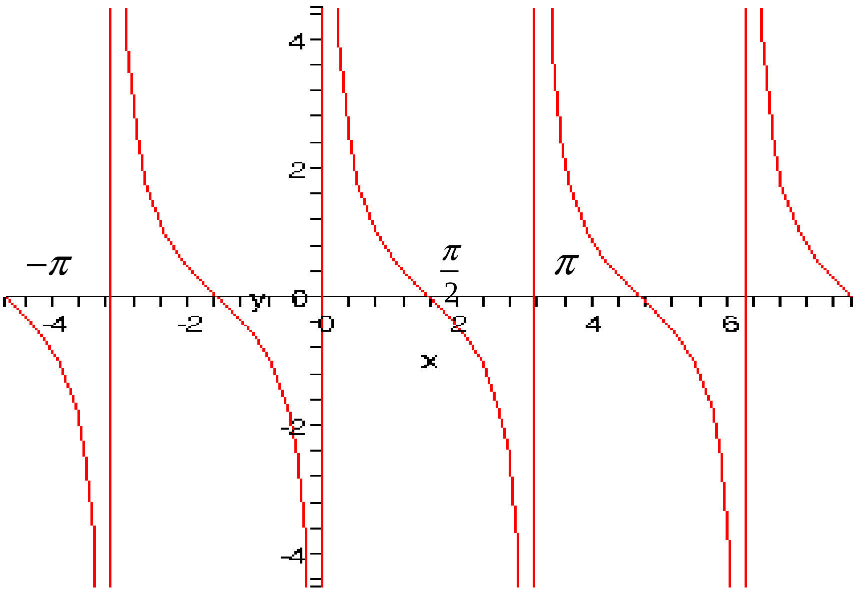



**fig. 12** : the cotangent function

## 3.7 The secant function

$$\sec : \mathbb{R} \setminus \left\{ \frac{\pi}{2} + k\pi \, , \, k \in \mathbb{Z} \right\} \to \, ]-\infty, -1] \cup [1, +\infty[ \, : x \to \sec x$$

The period of this function is $2\pi$. This function is even, as opposite angles have the same cosines and so the same secants.

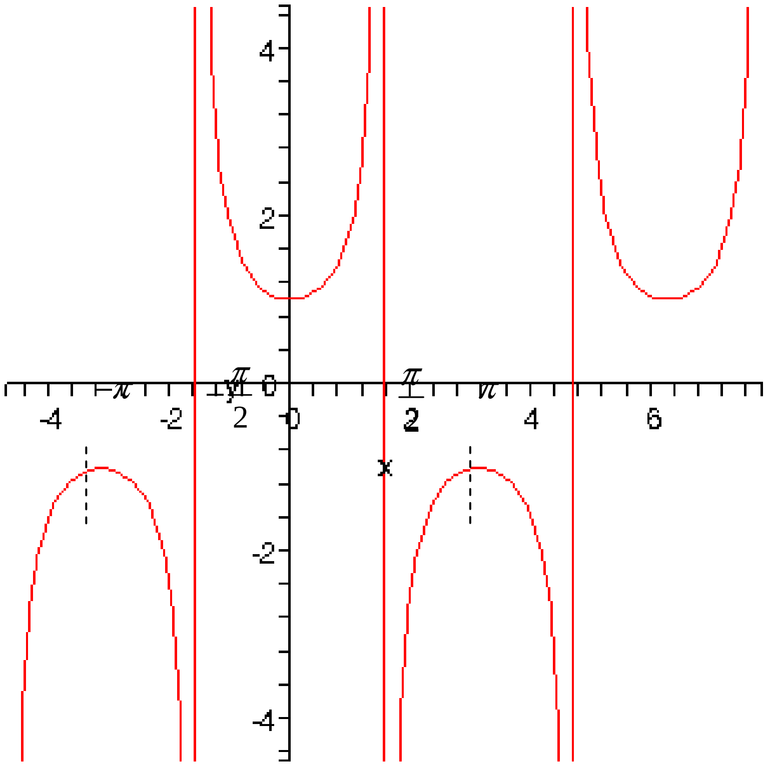



**fig. 13** : de secansfunctie

## 3.8 The cosecant function

$$\csc : \mathbb{R} \setminus \{ k\pi \, , \, k \in \mathbb{Z} \} \to \, ]-\infty, -1] \cup [1, +\infty[ \, : x \to \csc x$$

The period of this function is $2\pi$. This function is odd, as opposite angles have opposite sines and so opposite cosecants.

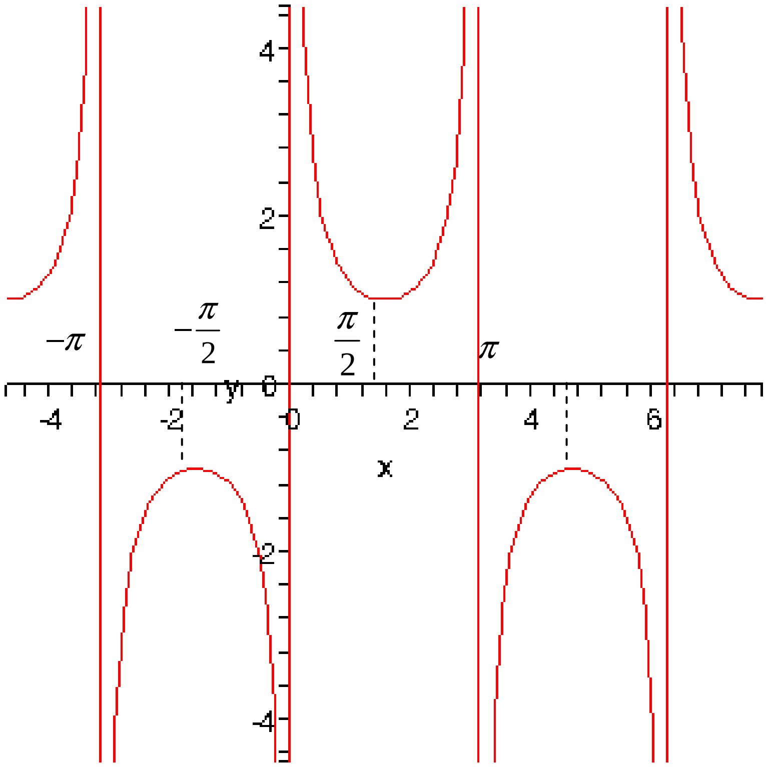



**fig. 14** : the cosecant function

## 3.9 Exercises

### 3.9.1 Determine the period of the following functions and draw their graph

1. $f(x) = \sin 2x$
2. $f(x) = \cos\left(\frac{x}{3}\right)$
3. $f(x) = \cos\left(\pi + \frac{x}{2}\right)$

# 4 Right triangles

## 4.1 Formulas

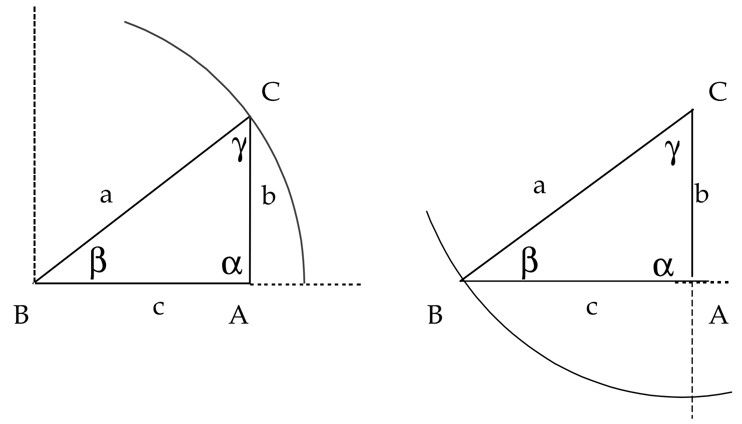



**fig. 15** : orthogonal triangles used to set up the formulas in this paragraph

In a right triangle with α as the right angle, the following formulas apply:

$\alpha = \frac{\pi}{2}$ $\beta + \gamma = \frac{\pi}{2}$ $a^2 = b^2 + c^2$

If we draw in the triangle above a circle segment with centre in B and radius a (see the first triangle in fig. 15), then we recognize a segment of a circle with radius a. The adjacent side of the right angle c and the opposite side b have resp. the following lengths:

$c = a \cos \beta$ and $b = a \sin \beta$

In a similar way, by considering a circle segment with centre in C and radius a (see second triangle in fig. 15), we find:

$b = a \cos \gamma$ and $c = a \sin \gamma$

In words :

*The cosine of an acute angle is the ratio of the length of the adjacent rectangle side and the length of the hypotenuse.*

*The sine of an acute angle is the ratio of the length of the opposite rectangle side and the length of the hypothenuse.*

By division of the first two formulas we get:

$b = c \tan \beta$ or $c = b \cot \beta$

If we do the same with the last two formulas, we get:

$c = b \tan \gamma$ or $b = c \cot \gamma$

In words :

*The tangent of an acute angle is the ratio of the length of the opposite rectangle side and the length of the adjacent rectangle side.*

*The cotangent of an acute angle is the ratio of the length of the adjacent rectangle side and the length of the opposite rectangle side.*

<u>*Example*</u>

Given : $\alpha = 90°$ ; $\beta = 13°$ ; $b = 10$

Asked : all missing angles and sides.

Solution:

$\gamma = 90° - \beta = 90° - 13° = 77°$

$$a = \frac{b}{\sin \beta} = \frac{10}{\sin 13°} = 44.5$$

$$c = \sqrt{a^2 - b^2} = 43.5$$

## 4.2 Exercises

1. Given : $\Delta ABC$ with $a = 45$, $\alpha = 90°$ , $\beta = 40°10'35''$

   Asked : the remaining sides and angles.

2. An extension ladder stands slantwise to a vertical wall on a horizontal floor. If the ladder is completely extended, it makes an angle of 53°18' with the floor; completely retracted the angle is 29°10', while the top at that moment leans at a heigth of 5 meter against the wall. If we assume that the foot of the ladder does not change, calculate

   - the maximal height one can reach
   - the maximal length of the ladder

3. An incident ray is a ray of light that strikes a surface. The angle between this ray and the perpendicular or normal to the surface is the angle of incidence ($\alpha$).
   The refracted ray or transmitted ray corresponding to a given incident ray represents the light that is transmitted through the surface. The angle between this ray and the normal is known as the angle of refraction ($\beta$).
   The relationship between the angles of incidence and refraction is given by Snell's Law:
   $$\frac{\sin\alpha}{\sin\beta} = n$$

   *Example* (see figure below):
   A ray of light strikes an air-water interface at an angle of 30 degrees from the normal ($\alpha = 30°$). The relative refractive index for the interface is 4/3. At which distance from P the ray of light hits the bottom if the water is 1 m deep.

   Remark: solve this exercise without calculating the angle $\beta$.

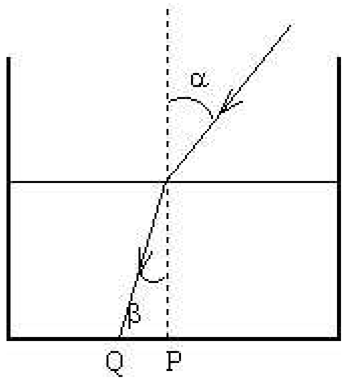



**fig. 16** : illustration for exercise 3

*In mechanics you will deal with exercises in which forces must be calculated. In the following exercises such situations will be sketched. We will confine to the calculation of angles between bars..*

4. Calculate:
   - the angle between FE and the horizontal plane
   - the angle between FC and the vertical plane

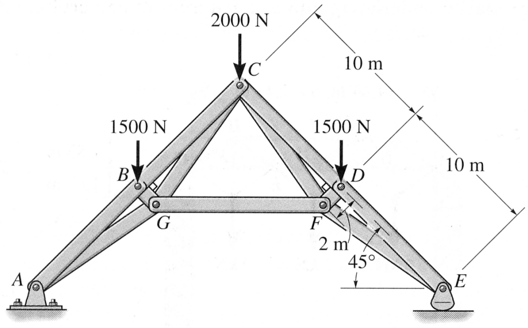



**fig. 17** : illustration for exercise 4

5. Calculate the angle between CD and DF

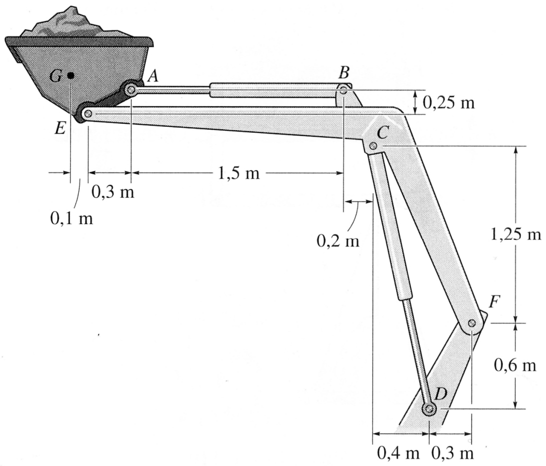



**fig. 18** : illustration for exercise 5

6. Calculate the angle between BC and CD

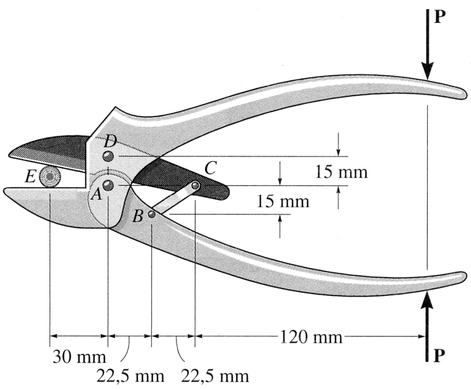



**fig. 19** : illustration for exercise 6

# 5 Oblique triangles

First, remember that also for oblique triangles the sum of angles is 180°.

An oblique triangle is any triangle that is not a right triangle. It could be an acute triangle (all three angles of the triangle are less than right angles) or it could be an obtuse triangle (one of the three angles is greater than a right angle).

Using the formulas for right triangles, we can set up formulas for oblique triangles. Let us consider an oblique triangle ΔABC with sides a, b and c and angles α, β and γ.

## 5.1 The sine rules

The altitude from A to the opposite side a intersects this side in point S. In this way the triangle is divided in two right angles with one common side AS, with length d. Use now the formulas of a right triangle in triangles ΔABS and ΔACS to calculate d.

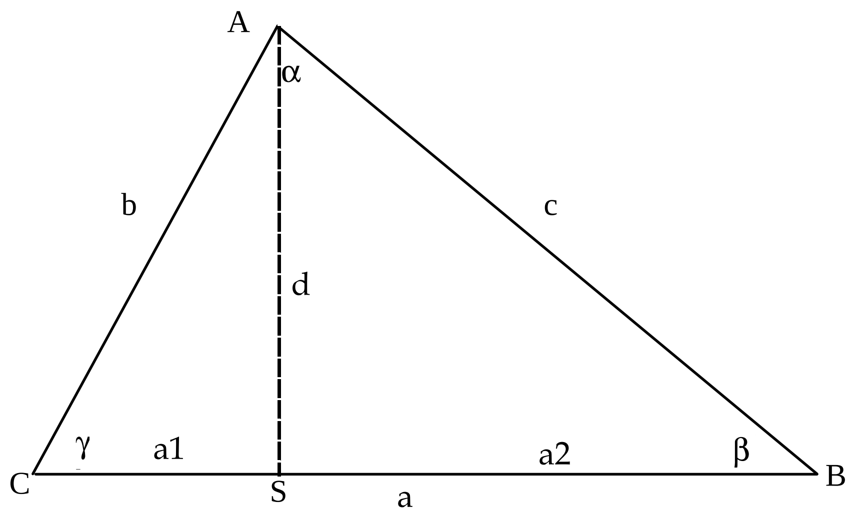



**fig. 20** : oblique triangle

$d = c \sin \beta$ and $d = b \sin \gamma$

So we get: $\frac{\sin \beta}{b} = \frac{\sin \gamma}{c}$

Apply the same reasoning with the altitude from B to the opposite side b to divide the triangle in two right triangles and derive similar formulas in which occur a and the opposite angle α. Then we get:

SINE RULES : $$\frac{\sin \alpha}{a} = \frac{\sin \beta}{b} = \frac{\sin \gamma}{c}$$

## 5.2 The cosine rules

This identity can be derived in different ways. In fig. 20 S divides the side a in two parts with length $a_1$ and $a_2$. Then $a_1$ and d can be written respectively as

$a1 = b \cos \gamma$

$d = b \sin \gamma$

In the triangle $\Delta ABS$ apply Pythagoras theorem:

$$c^2 = d^2 + a_2^2 = d^2 + (a - a_1)^2$$

$$= b^2 \sin^2 \gamma + a^2 + a_1^2 - 2 a a_1$$

$$= b^2 \sin^2 \gamma + a^2 + b^2 \cos^2 \gamma - 2 a b \cos \gamma$$

$$= b^2 + a^2 - 2 a b \cos \gamma$$

The same expression can be derived if S lies outside side a.

Then, similar expressions can be derived for the other angles.

Summarized, in this way we get:

COSINE RULES : $a^2 = b^2 + c^2 - 2 b c \cos \alpha$

$b^2 = a^2 + c^2 - 2 a c \cos \beta$

$c^2 = a^2 + b^2 - 2 a b \cos \gamma$

These statements relate the lengths of the sides of a triangle to the cosine of one of its angles. For example, the first statement states the relationship between the sides of lengths a , b and c, where $\alpha$ denotes the angle contained between sides of lengths *b* and *c* and opposite to the side with length a.

These rules look like the Pythagorean theorem except for the last term, and if you deal with a right triangle, that last term disappears, so these rules are actually a generalization of the Pythagorean theorem.

## 5.3 Solving oblique triangles

One of the most common applications of the trigonometry is solving triangles – finding missing sides and/or angles, given some information about a triangle. The process of solving triangles can be broken down into a number of cases.

In these situations we will use 3 sorts of formulas, applicable in all triangles:

- the sum of all angles is 180°
- the sine rule : relates two sides to their opposite angles
- the cosine rule : relates the three sides of the triangle to one of the angles.

Naturally, the given information must be such that the given elements allow a triangle:

- the sum of the given angles can not be larger than 180°,
- and the sides must meet the triangle inequality which states that for any triangle, the sum of the lengths of any two sides must be greater than the length of the remaining side.

a. If you know one angle and the two adjacent sides.

   Then, there is 1 solution:
   you can determine the opposite side by using the cosine rule, another angle by using the sine rule and the remaining angle as 180° minus the two already determined angles.

   Attention: the sine rule gives two solutions for the second angle (supplementary angles). Test the solutions by verifying the properties of a triangle (see exercises).

b. If you know one side and the two adjacent angles.

   Then there is 1 solution:
   the third angle is immediately known as 180° minus the two given angles; the two remaining sides can be determined by using the sine rule.

c. If you know all three sides of a triangle.

   Then there is 1 solution:
   determine one angle by using a cosine rule, the second angle can be determined by using another cosine rule or by using the sine rule. The last angle can be determined by the property of triangles that the sum of all angles must be 180°.

d. If you know sides a and b and $\beta$ (one of the adjacent angles). In this case, there can be 0, 1 or 2 solutions.

   Determine the angle $\alpha$ by using the sine rule. You will get 0 (if $\sin \alpha > 1$) or 2 solutions (supplementary angles have the same sine). For each solution determine the missing angle $\gamma$, and then the length of side c by using the sine rule. Finally you test if each solution which you find is acceptable: you can not have negative angles or sides (see exercises).

## 5.4 Exercises

1. A tower is seen from the ground under an angle of 21°. If one approaches the tower by 24 meter, then this angle becomes 35°. Determine the height of the tower.

2. Two planes depart from the same point, each in a different direction. The directions form an angle of 32°. The velocity of the first plane is 600 km/hour, the velocity of the second is 900 km/hour. Determine their mutual distance after one hour and a half.

3. The pole of a flag reaches up from a facade with an angle of 45° (see fig. 21). Five meter above the base point of the pole in the wall, there is a cable fixed to the wall with a length of 3,60 meter. At which distance of the base point, measured along the pole, the other end of the cable can be fixed.

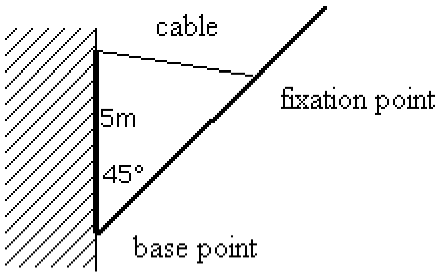



**fig. 21** : illustration for exercise 3

4. Solve the previous exercise for a cable with a length of 2 m, respectively with a cable with a length of 8 m.

5. Three observers are at mutual distances of 2, 3 and 4 meters. Determine for each observer the angle under which he sees the other 2 observers.

6. A boat sails north and sees a lighthouse 40° eastwards. After having sailed 20 km, this angle has increased to 80°. Determine at both positions the distance from the boat to the lighthouse.

7. The following figure demonstrates a situation in mechanics. Determine the angle between the ropes AC and AD (d = 1 m).

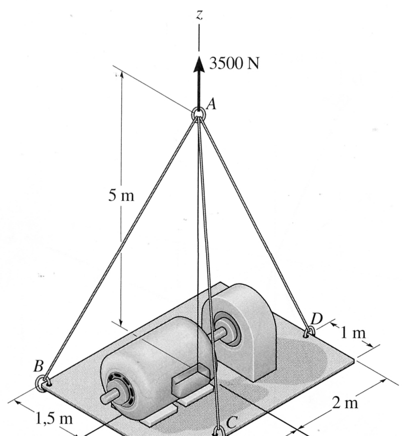



**fig. 22** : illustration of exercise 7

# 6 Extra’s

## 6.1 Special lines in a triangle

### 6.1.1 Altitude

An altitude of a triangle is a straight line through a vertex and perpendicular to (i.e. forming a right angle with) the opposite side.

This opposite side is called the base of the altitude, and the point where the altitude intersects the base (or its extension) is called the foot of the altitude. The three altitudes intersect in a single point, called the orthocenter of the triangle. The orthocenter lies inside the triangle if and only if the triangle is acute.

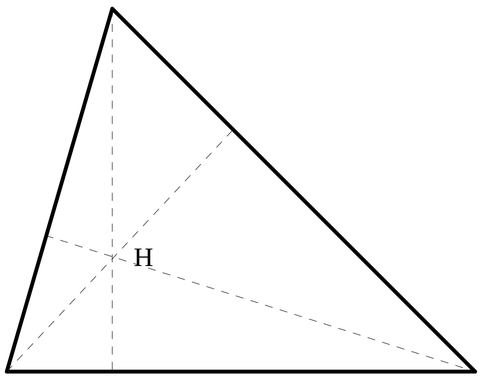



**fig. 23** : altitudes

### 6.1.2 Median

A median of a triangle is a straight line through a vertex and the midpoint of the opposite side, and divides the triangle into two equal areas.

The three medians intersect in a single point, the triangle's centroid. The centroid cuts every median in the ratio 2:1, i.e. the distance between a vertex and the centroid is twice the distance between the centroid and the midpoint of the opposite side.

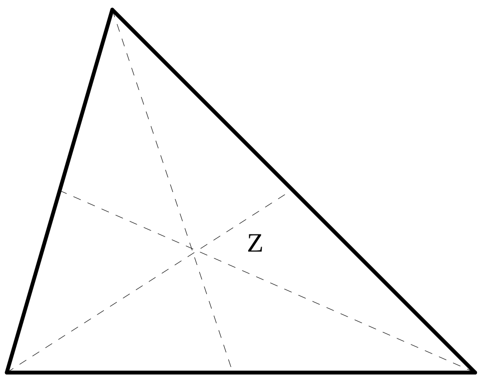



**fig. 24** : medians

### 6.1.3 Other lines

An angle bisector of a triangle is a straight line through a vertex which cuts the corresponding angle in half. The three angle bisectors intersect in a single point, which always lies inside the triangle.

A perpendicular bisector of a side of a triangle is a straight line passing through the midpoint of the side and being perpendicular to it, i.e. forming a right angle with it. The three perpendicular bisectors meet in a single point, the triangle's circumcenter; this point may also lie outside the triangle.

## 6.2 Isosceles triangles

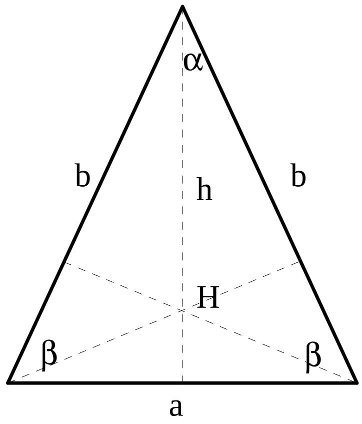



**fig. 25** : isosceles triangle

In an isosceles triangle, two sides are equal in length. The unequal side is called its base and the angle opposite the base is called the "vertex angle". The equal sides are called the legs of the triangle. The base angles of an isosceles triangle are always equal.

Property : the altitude and the median from the vertical angle coincide.

Let's call the altitude h, the legs b, the base a, the vertical angle $\alpha$ and the base-angle $\beta$:

then : $h = b \sin \beta$ and $\frac{a}{2} = b \cos \beta$

### 6.2.1 Exercises

1. Set up analog formulas which use the vertex angle.
2. Determine the value of the angles of an isosceles triangle with base 8 and legs 14.
3. Determine the length of the legs of an isosceles triangle with vertex angle 42° and base 12.
4. Determine the length of each side of an isosceles triangle with vertex angle 36° and vertex-altitude 28.
5. In an isosceles triangle with vertex angle 24° the orthocenter lies at a distance of 26 cm to the top. Determine all angles and sides.

## 6.3 Equilateral triangles

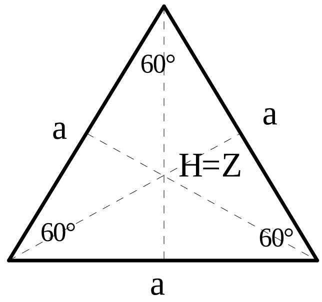



**fig. 26** : equilateral triangle

In an equilateral triangle all sides have the same length. Therefore all three angles are equal to each other, and thus 60°.

Property : the altitude from a certain angle coincides with the median from that angle. The orthocenter and the centroid coincide.

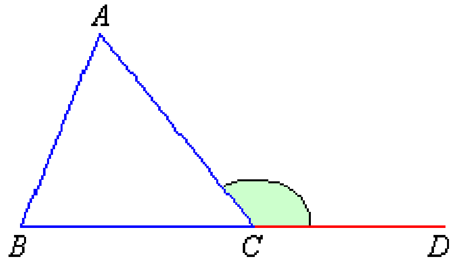
### 6.3.1 Exercises

1. Determine the distance from the orthocenter/centroid to one of the vertices in terms of the length of the side.
2. Determine the length of an altitude in an equilateral triangle with side 28 cm.
3. The altitude of an equilateral triangle has length 8 cm. Determine the length of the sides.

## 6.4 Exterior angles

The exterior angle of an angle in a triangle is formed by one side adjacent to that angle and a line extended from the other side adjacent to that angle. Clearly, the exterior angle *ACD* and the adjacent interior angle *ACB* are supplementary. That is:

$$\angle ACD + \angle ACB = 180°$$




The sum of the interior angle and the external angle on the same vertex is 180°. Therefore the sum of all exterior angles is 360° or 2π.

# 7 Trigonometric formulas

In this paragraph, we discuss formulas involving the trigonometric numbers of a sum or difference of two angles, of a double or half angle, conversions between sums and products of sines and cosines….

As we don't want you to learn these formulas by heart, it is important to understand their mutual connection, the way how one formula can be derived from another formula.

We also want to emphasize that the knowledge of these formulas facilitates solving integrals of trigonometric functions.

## 7.1 Sum and difference formulas

Let's start with the addition formula for the sine. Then the other formulas can be derived in an easy way.

$$\sin(\alpha + \beta) = \sin\alpha \cos\beta + \cos\alpha \sin\beta \qquad (1)$$

Replace $\beta$ by $-\beta$ , with $\sin(-\beta) = -\sin\beta$, $\cos(-\beta) = \cos\beta$, then we get:

$$\sin(\alpha - \beta) = \sin\alpha \cos\beta - \cos\alpha \sin\beta \qquad (2)$$

For the similar cosine formulas:

$$\begin{aligned}\cos(\alpha + \beta) &= \sin[\frac{\pi}{2} - (\alpha + \beta)] \\ &= \sin[(\frac{\pi}{2} - \alpha) - \beta] \\ &= \sin(\frac{\pi}{2} - \alpha)\cos\beta - \cos(\frac{\pi}{2} - \alpha)\sin\beta\end{aligned}$$

or: $\cos(\alpha + \beta) = \cos\alpha \cos\beta - \sin\alpha \sin\beta \qquad (3)$

Again, replace $\beta$ by $-\beta$, then we get:

$$\cos(\alpha - \beta) = \cos\alpha \cos\beta + \sin\alpha \sin\beta \qquad (4)$$

You see that the sine formulas keep the plus- or minus sign, but mix the trigonometric functions. The cosine formulas change the sign but hold the trigonometric functions together.

Let's divide (1) side by side by (3), and then divide the nominator and the denominator in the right hand side by ($\cos\alpha \cos\beta$) :

$$\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta} \qquad (5)$$

And replace $\beta$ by $-\beta$, then we get:

$$\tan(\alpha-\beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta} \qquad (6)$$

## 7.2 Double-angle formulas

Substitute $\alpha = \beta$ in the previous sum formulas, then we find the double-angle formulas: :

$$\cos 2\alpha = \cos^2\alpha - \sin^2\alpha \qquad (7)$$
$$\sin 2\alpha = 2\sin\alpha\cos\alpha \qquad (8)$$
$$\tan 2\alpha = \frac{2\tan\alpha}{1-\tan^2\alpha} \qquad (9)$$

Two useful forms of (7) are derived by replacing $\cos^2\alpha$ by $1 - \sin^2\alpha$, resp. $\sin^2\alpha$ by $1 - \cos^2\alpha$ :

$$\cos 2\alpha = 1 - 2\sin^2\alpha \qquad (10)$$
$$\cos 2\alpha = 2\cos^2\alpha - 1 \qquad (11)$$

And so:

$$\sin^2\alpha = \frac{1}{2}(1 - \cos 2\alpha) \qquad (12)$$

$$\cos^2\alpha = \frac{1}{2}(1 + \cos 2\alpha) \qquad (13)$$

## 7.3 Half-angle formulas

Replace $2\alpha$ by $\alpha$ in (10) and (11):

$$\cos\alpha = 1 - 2\sin^2\frac{\alpha}{2} \tag{14}$$

$$\cos\alpha = 2\cos^2\frac{\alpha}{2} - 1 \tag{15}$$

## 7.4 Trigonometric numbers in terms of tan $\alpha/2$

In (8) we divide and multiply the right hand side by $\sec^2\alpha$ . By replacing in the denominator $\sec^2\alpha = 1 + \tan^2\alpha$ (see $ 2.4.) and by simplifying the nominator (apply the definition of sec $\alpha$), we get:

$$\sin 2\alpha = \frac{2\tan\alpha}{1+\tan^2\alpha} \tag{16}$$

Replace $\alpha$ by $\frac{\alpha}{2}$:

$$\sin\alpha = \frac{2\tan\frac{\alpha}{2}}{1+\tan^2\frac{\alpha}{2}} \tag{17}$$

By performing the same operation on (7) we find :

$$\cos\alpha = \frac{1-\tan^2\frac{\alpha}{2}}{1+\tan^2\frac{\alpha}{2}} \tag{18}$$

and by replacing $\alpha$ by $\frac{\alpha}{2}$ in (9) , we get:

$$\tan\alpha = \frac{2\tan\frac{\alpha}{2}}{1-\tan^2\frac{\alpha}{2}} \tag{19}$$

## 7.5 Conversions sum/difference of angles into product of angles and vice versa

In the right hand side of the sum formulas (1) and (2) we notice the same term ($\sin\alpha\cos\beta$). By adding (1) and (2) side by side and bringing factor 2 to the other side, we get :

$$\sin\alpha\ \cos\beta = \frac{1}{2}\ [\ \sin(\alpha+\beta) + \sin(\alpha-\beta)\ ] \qquad (20)$$

In a similar way, by subtracting (2) from (1) side by side, we get :

$$\cos\alpha\sin\beta = \frac{1}{2}\ [\ \sin(\alpha+\beta) - \sin(\alpha-\beta)\ ] \qquad (21)$$

By doing the same with formulas (3) and (4) (adding, resp. subtracting side by side), we get:

$$\cos\alpha\cos\beta = \frac{1}{2}\ [\ \cos(\alpha+\beta) + \cos(\alpha-\beta)\ ] \qquad (22)$$

$$\sin\alpha\sin\beta = \frac{1}{2}\ [\ \cos(\alpha-\beta) - \cos(\alpha+\beta)\ ] \qquad (23)$$

These four formulas convert the product of two cosines and/or sines with a different argument into a sum. The reverse formulas we get by bringing factor ½ to the other side and by substitution:

$$\alpha = \frac{p+q}{2}$$

$$\beta = \frac{p-q}{2}$$

This leads us to the formulas of Simpson:

$$\sin p + \sin q = 2\sin\frac{p+q}{2}\ \cos\frac{p-q}{2} \qquad (24)$$

$$\sin p - \sin q = 2\cos\frac{p+q}{2}\ \sin\frac{p-q}{2} \qquad (25)$$

$$\cos p + \cos q = 2\cos\frac{p+q}{2}\ \cos\frac{p-q}{2} \qquad (26)$$

$$\cos p - \cos q = -2\sin\frac{p+q}{2}\ \sin\frac{p-q}{2} \qquad (27)$$

## 7.6 Exercises

1. Proof the following identity in a triangle:

$$\sin^2\alpha + \sin^2\beta - \sin^2\gamma = 2\sin\alpha\sin\beta\cos\gamma$$

2. Calculate and/or simplify:

a. $\tan\left(\alpha - \frac{\pi}{4}\right) + \cot\left(\alpha + \frac{\pi}{4}\right)$ $\qquad 0$

b. $\frac{\sin\alpha - \cos\alpha}{\sin\alpha + \cos\alpha}$ $\qquad \tan\left(\alpha - \frac{\pi}{4}\right)$

3. Write in terms of powers of sin α and/or cos α:

a. $\sin 3\alpha$ $\qquad 3\sin\alpha - 4\sin^3\alpha$

b. $\cos 4\alpha$ $\qquad 1 - 8\cos^2\alpha + 8\cos^4\alpha$

c. $\tan\frac{\alpha}{2}$ $\qquad \frac{\sin\alpha}{\cos\alpha + 1}$

d. $\frac{\sin\frac{\alpha}{2} + \cos\frac{\alpha}{2}}{\cos\frac{\alpha}{2} - \sin\frac{\alpha}{2}}$ $\qquad \frac{1 + \sin\alpha}{\cos\alpha}$

4. Factorize:

a. $\sin 3\alpha - \sin\alpha$ $\qquad 2\cos 2\alpha\sin\alpha$

b. $\cos 4\alpha + \cos 5\alpha + \cos 6\alpha$ $\qquad \cos 5\alpha(2\cos\alpha + 1)$

c. $\tan\alpha - \sin\alpha$ $\qquad 2\tan\alpha\sin^2\frac{\alpha}{2}$

d. $\cos^2\beta - \cos^2\alpha$ $\qquad \sin(\alpha + \beta)\sin(\alpha - \beta)$