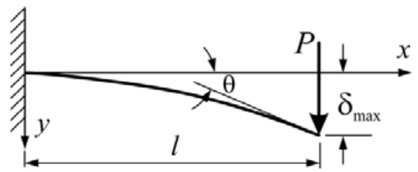
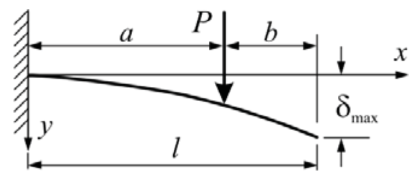
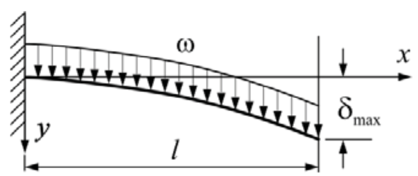
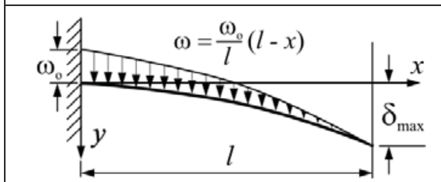
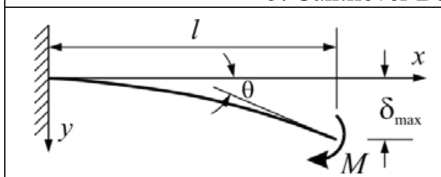
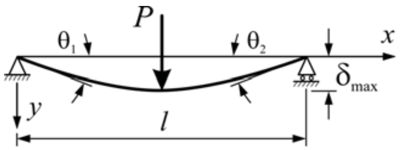
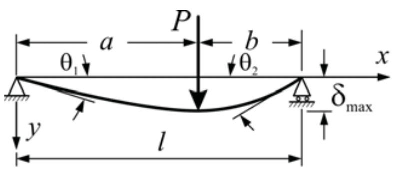
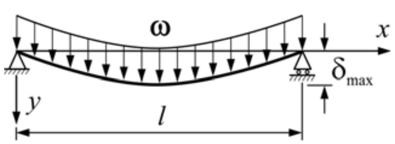
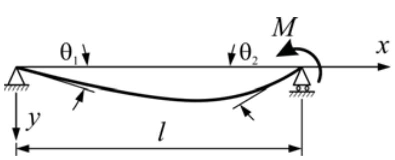
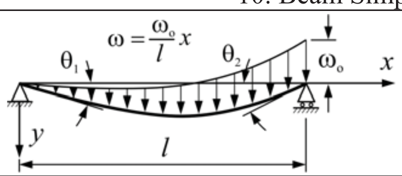
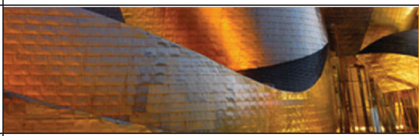
BEAM DEFLECTION FORMULAE

BEAM TYPE	SLOPE AT FREE END	DEFLECTION AT ANY SECTION IN TERMS OF *x*	MAXIMUM DEFLECTION
1. Cantilever Beam – Concentrated load *P* at the free end			
	$\theta = \frac{Pl^2}{2EI}$	$y = \frac{Px^2}{6EI}(3l - x)$	$\delta_{max} = \frac{Pl^3}{3EI}$
2. Cantilever Beam – Concentrated load *P* at any point			
	$\theta = \frac{Pa^2}{2EI}$	$y = \frac{Px^2}{6EI}(3a - x)$ for $0 < x < a$ $y = \frac{Pa^2}{6EI}(3x - a)$ for $a < x < l$	$\delta_{max} = \frac{Pa^2}{6EI}(3l - a)$
3. Cantilever Beam – Uniformly distributed load ω (N/m)			
	$\theta = \frac{\omega l^3}{6EI}$	$y = \frac{\omega x^2}{24EI}(x^2 + 6l^2 - 4lx)$	$\delta_{max} = \frac{\omega l^4}{8EI}$
4. Cantilever Beam – Uniformly varying load: Maximum intensity ω_o (N/m)			
	$\theta = \frac{\omega_o l^3}{24EI}$	$y = \frac{\omega_o x^2}{120lEI}(10l^3 - 10l^2 x + 5lx^2 - x^3)$	$\delta_{max} = \frac{\omega_o l^4}{30EI}$
5. Cantilever Beam – Couple moment *M* at the free end			
	$\theta = \frac{Ml}{EI}$	$y = \frac{Mx^2}{2EI}$	$\delta_{max} = \frac{Ml^2}{2EI}$

BEAM DEFLECTION FORMULAS

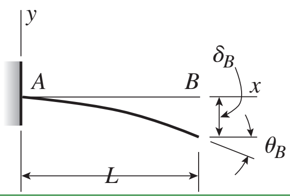
BEAM TYPE	SLOPE AT ENDS	DEFLECTION AT ANY SECTION IN TERMS OF x	MAXIMUM AND CENTER DEFLECTION
6. Beam Simply Supported at Ends – Concentrated load P at the center			
	$\theta_1 = \theta_2 = \dfrac{Pl^2}{16EI}$	$y = \dfrac{Px}{12EI}\left(\dfrac{3l^2}{4} - x^2\right)$ for $0 < x < \dfrac{l}{2}$	$\delta_{max} = \dfrac{Pl^3}{48EI}$
7. Beam Simply Supported at Ends – Concentrated load P at any point			
	$\theta_1 = \dfrac{Pb(l^2 - b^2)}{6lEI}$ $\theta_2 = \dfrac{Pab(2l - b)}{6lEI}$	$y = \dfrac{Pbx}{6lEI}\left(l^2 - x^2 - b^2\right)$ for $0 < x < a$ $y = \dfrac{Pb}{6lEI}\left[\dfrac{l}{b}(x-a)^3 + \left(l^2 - b^2\right)x - x^3\right]$ for $a < x < l$	$\delta_{max} = \dfrac{Pb\left(l^2 - b^2\right)^{3/2}}{9\sqrt{3}\, lEI}$ at $x = \sqrt{\left(l^2 - b^2\right)/3}$ $\delta = \dfrac{Pb}{48EI}\left(3l^2 - 4b^2\right)$ at the center, if $a > b$
8. Beam Simply Supported at Ends – Uniformly distributed load ω (N/m)			
	$\theta_1 = \theta_2 = \dfrac{\omega l^3}{24EI}$	$y = \dfrac{\omega x}{24EI}\left(l^3 - 2lx^2 + x^3\right)$	$\delta_{max} = \dfrac{5\omega l^4}{384EI}$
9. Beam Simply Supported at Ends – Couple moment M at the right end			
	$\theta_1 = \dfrac{Ml}{6EI}$ $\theta_2 = \dfrac{Ml}{3EI}$	$y = \dfrac{Mlx}{6EI}\left(1 - \dfrac{x^2}{l^2}\right)$	$\delta_{max} = \dfrac{Ml^2}{9\sqrt{3}\, EI}$ at $x = \dfrac{l}{\sqrt{3}}$ $\delta = \dfrac{Ml^2}{16EI}$ at the center
10. Beam Simply Supported at Ends – Uniformly varying load: Maximum intensity ω_o (N/m)			
	$\theta_1 = \dfrac{7\omega_o l^3}{360EI}$ $\theta_2 = \dfrac{\omega_o l^3}{45EI}$	$y = \dfrac{\omega_o x}{360lEI}\left(7l^4 - 10l^2x^2 + 3x^4\right)$	$\delta_{max} = 0.00652\dfrac{\omega_o l^4}{EI}$ at $x = 0.519\, l$ $\delta = 0.00651\dfrac{\omega_o l^4}{EI}$ at the center



H

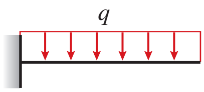
Deflections and Slopes of Beams

TABLE H-1 DEFLECTIONS AND SLOPES OF CANTILEVER BEAMS



v = deflection in the y direction (positive upward)
$v' = dv/dx$ = slope of the deflection curve
$\delta_B = -v(L)$ = deflection at end B of the beam (positive downward)
$\theta_B = -v'(L)$ = angle of rotation at end B of the beam (positive clockwise)
EI = constant

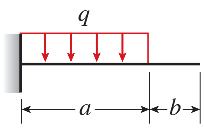
1



$$v = -\frac{qx^2}{24EI}(6L^2 - 4Lx + x^2) \qquad v' = -\frac{qx}{6EI}(3L^2 - 3Lx + x^2)$$

$$\delta_B = \frac{qL^4}{8EI} \qquad \theta_B = \frac{qL^3}{6EI}$$

2



$$v = -\frac{qx^2}{24EI}(6a^2 - 4ax + x^2) \qquad (0 \le x \le a)$$

$$v' = -\frac{qx}{6EI}(3a^2 - 3ax + x^2) \qquad (0 \le x \le a)$$

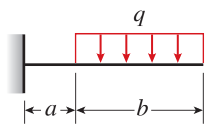
$$v = -\frac{qa^3}{24EI}(4x - a) \qquad v' = -\frac{qa^3}{6EI} \qquad (a \le x \le L)$$

$$\text{At } x = a\text{:} \quad v = -\frac{qa^4}{8EI} \qquad v' = -\frac{qa^3}{6EI}$$

$$\delta_B = \frac{qa^3}{24EI}(4L - a) \qquad \theta_B = \frac{qa^3}{6EI}$$

(Continued)

3



$$v = -\frac{qbx^2}{12EI}(3L + 3a - 2x) \qquad (0 \le x \le a)$$

$$v' = -\frac{qbx}{2EI}(L + a - x) \qquad (0 \le x \le a)$$

$$v = -\frac{q}{24EI}(x^4 - 4Lx^3 + 6L^2x^2 - 4a^3x + a^4) \qquad (a \le x \le L)$$

$$v' = -\frac{q}{6EI}(x^3 - 3Lx^2 + 3L^2x - a^3) \qquad (a \le x \le L)$$

$$\text{At } x = a\text{:} \quad v = -\frac{qa^2b}{12EI}(3L + a) \qquad v' = -\frac{qabL}{2EI}$$

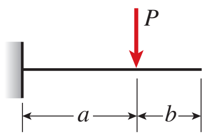
$$\delta_B = \frac{q}{24EI}(3L^4 - 4a^3L + a^4) \qquad \theta_B = \frac{q}{6EI}(L^3 - a^3)$$

4

$$v = -\frac{Px^2}{6EI}(3L - x) \qquad v' = -\frac{Px}{2EI}(2L - x)$$

$$\delta_B = \frac{PL^3}{3EI} \qquad \theta_B = \frac{PL^2}{2EI}$$

5



$$v = -\frac{Px^2}{6EI}(3a - x) \qquad v' = -\frac{Px}{2EI}(2a - x) \qquad (0 \le x \le a)$$

$$v = -\frac{Pa^2}{6EI}(3x - a) \qquad v' = -\frac{Pa^2}{2EI} \qquad (a \le x \le L)$$

$$\text{At } x = a\text{:} \quad v = -\frac{Pa^3}{3EI} \qquad v' = -\frac{Pa^2}{2EI}$$

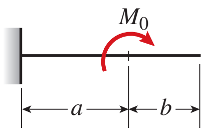
$$\delta_B = \frac{Pa^2}{6EI}(3L - a) \qquad \theta_B = \frac{Pa^2}{2EI}$$

6

$$v = -\frac{M_0x^2}{2EI} \qquad v' = -\frac{M_0x}{EI}$$

$$\delta_B = \frac{M_0L^2}{2EI} \qquad \theta_B = \frac{M_0L}{EI}$$

7



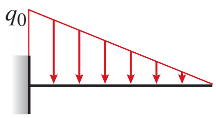
$$v = -\frac{M_0x^2}{2EI} \qquad v' = -\frac{M_0x}{EI} \qquad (0 \le x \le a)$$

$$v = -\frac{M_0a}{2EI}(2x - a) \qquad v' = -\frac{M_0a}{EI} \qquad (a \le x \le L)$$

$$\text{At } x = a\text{:} \quad v = -\frac{M_0a^2}{2EI} \qquad v' = -\frac{M_0a}{EI}$$

$$\delta_B = \frac{M_0a}{2EI}(2L - a) \qquad \theta_B = \frac{M_0a}{EI}$$

8

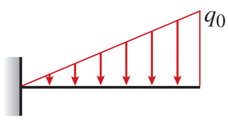


$$v = -\frac{q_0x^2}{120LEI}(10L^3 - 10L^2x + 5Lx^2 - x^3)$$

$$v' = -\frac{q_0x}{24LEI}(4L^3 - 6L^2x + 4Lx^2 - x^3)$$

$$\delta_B = \frac{q_0L^4}{30EI} \qquad \theta_B = \frac{q_0L^3}{24EI}$$

9

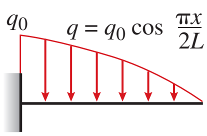


$$v = -\frac{q_0x^2}{120LEI}(20L^3 - 10L^2x + x^3)$$

$$v' = -\frac{q_0x}{24LEI}(8L^3 - 6L^2x + x^3)$$

$$\delta_B = \frac{11q_0L^4}{120EI} \qquad \theta_B = \frac{q_0L^3}{8EI}$$

10



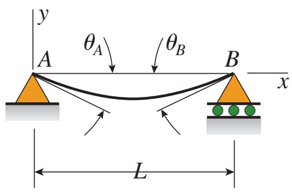
$$v = -\frac{q_0L}{3\pi^4EI}\left(48L^3 \cos \frac{\pi x}{2L} - 48L^3 + 3\pi^3Lx^2 - \pi^3x^3\right)$$

$$v' = -\frac{q_0L}{\pi^3EI}\left(2\pi^2Lx - \pi^2x^2 - 8L^2 \sin \frac{\pi x}{2L}\right)$$

$$\delta_B = \frac{2q_0L^4}{3\pi^4EI}(\pi^3 - 24) \qquad \theta_B = \frac{q_0L^3}{\pi^3EI}(\pi^2 - 8)$$

(*Continued*)

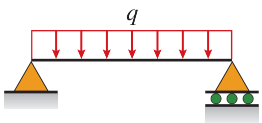
TABLE H-2 DEFLECTIONS AND SLOPES OF SIMPLE BEAMS



EI = constant

v = deflection in the y direction (positive upward)
$v' = dv/dx$ = slope of the deflection curve
$\delta_C = -v(L/2)$ = deflection at midpoint C of the beam (positive downward)
x_1 = distance from support A to point of maximum deflection
$\delta_{max} = -v_{max}$ = maximum deflection (positive downward)
$\theta_A = -v'(0)$ = angle of rotation at left-hand end of the beam (positive clockwise)
$\theta_B = v'(L)$ = angle of rotation at right-hand end of the beam (positive counterclockwise)

1

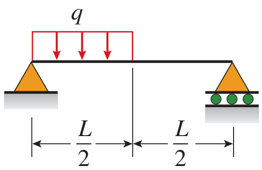


$$v = -\frac{qx}{24EI}(L^3 - 2Lx^2 + x^3)$$

$$v' = -\frac{q}{24EI}(L^3 - 6Lx^2 + 4x^3)$$

$$\delta_C = \delta_{max} = \frac{5qL^4}{384EI} \qquad \theta_A = \theta_B = \frac{qL^3}{24EI}$$

2



$$v = -\frac{qx}{384EI}(9L^3 - 24Lx^2 + 16x^3) \qquad \left(0 \le x \le \frac{L}{2}\right)$$

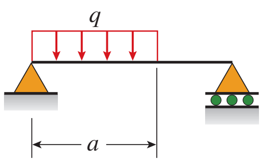
$$v' = -\frac{q}{384EI}(9L^3 - 72Lx^2 + 64x^3) \qquad \left(0 \le x \le \frac{L}{2}\right)$$

$$v = -\frac{qL}{384EI}(8x^3 - 24Lx^2 + 17L^2x - L^3) \qquad \left(\frac{L}{2} \le x \le L\right)$$

$$v' = -\frac{qL}{384EI}(24x^2 - 48Lx + 17L^2) \qquad \left(\frac{L}{2} \le x \le L\right)$$

$$\delta_C = \frac{5qL^4}{768EI} \qquad \theta_A = \frac{3qL^3}{128EI} \qquad \theta_B = \frac{7qL^3}{384EI}$$

3



$$v = -\frac{qx}{24LEI}(a^4 - 4a^3L + 4a^2L^2 + 2a^2x^2 - 4aLx^2 + Lx^3) \qquad (0 \le x \le a)$$

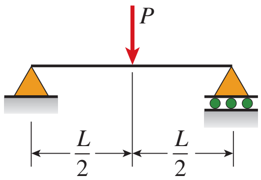
$$v' = -\frac{q}{24LEI}(a^4 - 4a^3L + 4a^2L^2 + 6a^2x^2 - 12aLx^2 + 4Lx^3) \qquad (0 \le x \le a)$$

$$v = -\frac{qa^2}{24LEI}(-a^2L + 4L^2x + a^2x - 6Lx^2 + 2x^3) \qquad (a \le x \le L)$$

$$v' = -\frac{qa^2}{24LEI}(4L^2 + a^2 - 12Lx + 6x^2) \qquad (a \le x \le L)$$

$$\theta_A = \frac{qa^2}{24LEI}(2L - a)^2 \qquad \theta_B = \frac{qa^2}{24LEI}(2L^2 - a^2)$$

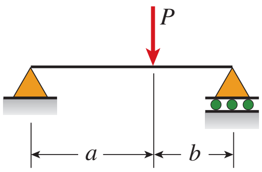
4



$$v = -\frac{Px}{48EI}(3L^2 - 4x^2) \qquad v' = -\frac{P}{16EI}(L^2 - 4x^2) \qquad \left(0 \le x \le \frac{L}{2}\right)$$

$$\delta_C = \delta_{\max} = \frac{PL^3}{48EI} \qquad \theta_A = \theta_B = \frac{PL^2}{16EI}$$

5



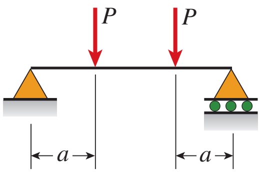
$$v = -\frac{Pbx}{6LEI}(L^2 - b^2 - x^2) \qquad v' = -\frac{Pb}{6LEI}(L^2 - b^2 - 3x^2) \qquad (0 \le x \le a)$$

$$\theta_A = \frac{Pab(L + b)}{6LEI} \qquad \theta_B = \frac{Pab(L + a)}{6LEI}$$

$$\text{If } a \ge b, \quad \delta_C = \frac{Pb(3L^2 - 4b^2)}{48EI} \qquad \text{If } a \le b, \quad \delta_C = \frac{Pa(3L^2 - 4a^2)}{48EI}$$

$$\text{If } a \ge b, \quad x_1 = \sqrt{\frac{L^2 - b^2}{3}} \quad \text{and} \quad \delta_{\max} = \frac{Pb(L^2 - b^2)^{3/2}}{9\sqrt{3}\,LEI}$$

6

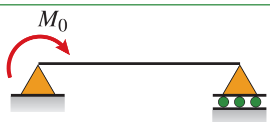


$$v = -\frac{Px}{6EI}(3aL - 3a^2 - x^2) \qquad v' = -\frac{P}{2EI}(aL - a^2 - x^2) \qquad (0 \le x \le a)$$

$$v = -\frac{Pa}{6EI}(3Lx - 3x^2 - a^2) \qquad v' = -\frac{Pa}{2EI}(L - 2x) \qquad (a \le x \le L - a)$$

$$\delta_C = \delta_{\max} = \frac{Pa}{24EI}(3L^2 - 4a^2) \qquad \theta_A = \theta_B = \frac{Pa(L - a)}{2EI}$$

7



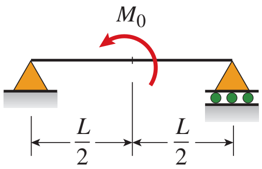
$$v = -\frac{M_0 x}{6LEI}(2L^2 - 3Lx + x^2) \qquad v' = -\frac{M_0}{6LEI}(2L^2 - 6Lx + 3x^2)$$

$$\delta_C = \frac{M_0 L^2}{16EI} \qquad \theta_A = \frac{M_0 L}{3EI} \qquad \theta_B = \frac{M_0 L}{6EI}$$

$$x_1 = L\left(1 - \frac{\sqrt{3}}{3}\right) \quad \text{and} \quad \delta_{\max} = \frac{M_0 L^2}{9\sqrt{3}\,EI}$$

(*Continued*)

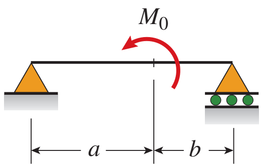
8



$$v = -\frac{M_0 x}{24LEI}(L^2 - 4x^2) \qquad v' = -\frac{M_0}{24LEI}(L^2 - 12x^2) \qquad \left(0 \le x \le \frac{L}{2}\right)$$

$$\delta_C = 0 \qquad \theta_A = \frac{M_0 L}{24EI} \qquad \theta_B = -\frac{M_0 L}{24EI}$$

9



$$v = -\frac{M_0 x}{6LEI}(6aL - 3a^2 - 2L^2 - x^2) \qquad (0 \le x \le a)$$

$$v' = -\frac{M_0}{6LEI}(6aL - 3a^2 - 2L^2 - 3x^2) \qquad (0 \le x \le a)$$

At $x = a$: $$v = -\frac{M_0 ab}{3LEI}(2a - L) \qquad v' = -\frac{M_0}{3LEI}(3aL - 3a^2 - L^2)$$

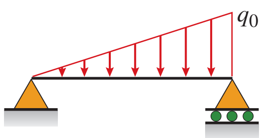
$$\theta_A = \frac{M_0}{6LEI}(6aL - 3a^2 - 2L^2) \qquad \theta_B = \frac{M_0}{6LEI}(3a^2 - L^2)$$

10

$$v = -\frac{M_0 x}{2EI}(L - x) \qquad v' = -\frac{M_0}{2EI}(L - 2x)$$

$$\delta_C = \delta_{max} = \frac{M_0 L^2}{8EI} \qquad \theta_A = \theta_B = \frac{M_0 L}{2EI}$$

11



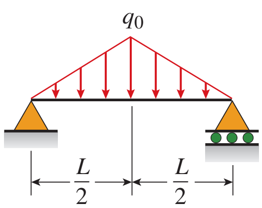
$$v = -\frac{q_0 x}{360LEI}(7L^4 - 10L^2x^2 + 3x^4)$$

$$v' = -\frac{q_0}{360LEI}(7L^4 - 30L^2x^2 + 15x^4)$$

$$\delta_C = \frac{5q_0L^4}{768EI} \qquad \theta_A = \frac{7q_0L^3}{360EI} \qquad \theta_B = \frac{q_0L^3}{45EI}$$

$$x_1 = 0.5193L \qquad \delta_{max} = 0.00652\frac{q_0L^4}{EI}$$

12

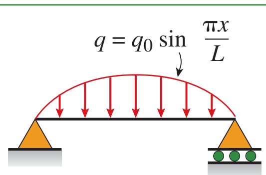


$$v = -\frac{q_0 x}{960LEI}(5L^2 - 4x^2)^2 \qquad \left(0 \le x \le \frac{L}{2}\right)$$

$$v' = -\frac{q_0}{192LEI}(5L^2 - 4x^2)(L^2 - 4x^2) \qquad \left(0 \le x \le \frac{L}{2}\right)$$

$$\delta_C = \delta_{max} = \frac{q_0L^4}{120EI} \qquad \theta_A = \theta_B = \frac{5q_0L^3}{192EI}$$

13



$$v = -\frac{q_0L^4}{\pi^4EI}\sin\frac{\pi x}{L} \qquad v' = -\frac{q_0L^3}{\pi^3EI}\cos\frac{\pi x}{L}$$

$$\delta_C = \delta_{max} = \frac{q_0L^4}{\pi^4EI} \qquad \theta_A = \theta_B = \frac{q_0L^3}{\pi^3EI}$$