THE EQUATIONS FOR INTRODUCTORY ASTRONOMY

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1. Confusion!!!!

Science is a game of measurement. Every number has, unlike pure math, something attached to it. The things that we attach to any number in science are called "units." You can drive a car on the highway at 80 miles per hour. A high school track star runs a mile in four minutes. When you make a cake, you start with 4 cups of flour. In each of these, the number is not just a pure number, but rather it measures a quantity of some real thing. You can have four cars, four miles and four hours. They are all fours, but they are quite different fours. While you can always add or subtract numbers in the most abstract sense, the real world tells you that there is no meaning at all to the question "What is four cars plus four hours minus four miles?" However, there can be a meaning to the measurement of "four cars per mile." Perhaps we are trying to measure the average usage of a road for a county road-repair planning committee. Someone could have measured over many days the total number of cars that pass along a road at various locations. This could be averaged over many days to give this useful number.

If we take a huge number of measurements of different aspects of the same types of things then, in addition to numbers with units attached to them, we might find relationships between different kinds of measurements. To riff on the above example, you can measure the speeds of cars on many different roads. You might find that cars go faster on one kind of road as opposed to another: (i.e. highways versus back county roads). This example shows a pretty basic relationship, but if you work for a city and want to help people get around more efficiently and get rid of traffic jams, then you'll do studies like these. Now, if we look at all of science, many such relationships have been discovered. Some are useful or profound. But Nature has a way of telling us those relationships. Nature always uses the language of mathematics to demonstrate the relationships. This in itself is extremely interesting, that Nature's Laws can be best expressed mathematically, and if we are lucky, in the form of a simple equation. Sometimes, the relationship is so complicated, that it's better to use a graph to show it. And sometimes the graphs are so complicated that we run big computer simulations and make movies out of them. All of these are valid ways of speaking mathematically. Let's keep it easy, and just start with equations. They are the first steps, but they cause the most confusion, because they are long thoughts compressed into a shorthand, compact form.

Equations have letters, numbers, symbols and implied operations. Each of these four objects have distinct meanings, and they can only play with each other in specific ways. For the uninitiated, it is often painfully confusing to know when a letter in an equation is a variable, and when it is a unit. Even worse, sometimes a unit is a combination of a letter or set of letters and an exponent. Fortunately, any equation can be read like a long and complicated sentence. Once you decode the equation into a sentence, then you cannot go wrong. Here is how you do it.

Start by reading equations as sentences. That is what they are, so you might as well begin by reading them like that. Once you get familiar with them, their meanings will become second nature. It takes time, but you can do it. One of Newton's Laws of Motion is simply: $F = m \cdot a$. At first glance, we might say "F is m times a." You don't need a Doctorate to know that that sentence has no meaning. We could say "Force is mass times acceleration." That's better, but still not very enlightening. Now let's try "A force is defined to be the product of a mass and its acceleration." This is now a complete thought. There are other ways to phrase it, to be sure, but this works well to understand a problem or to tell you that one measurement is related to another measurement to create a new thing, called a force, which we will presumably use later.

Once you read an equation to know what it is, you can then plug in the appropriate numbers and units. For our purposes, a force is measured in "Newtons", mass in "kilograms", and acceleration in "meters per second per second", then the equation which originally read as a definition can now be read as a question. Here is how.

If we wrote $m \cdot a = F$, it could then be read "If I push a one kilogram mass such that it accelerates at one meter per second per second, then I'm pushing on it with a force of one Newton." Now let's do it again, but now let's move the bits of the equation around. Let's make it look like this: $F/m = a$. This now reads: "If I push a mass of one kilogram with a force of 1 Newton, then the mass accelerates at one meter per second per second." We can even do the following: $F/a = m$. "If I push a mass with a force of 1 Newton and it accelerates at one meter per second per second, then it has a mass of one kilogram."

The next step is taking the equation and substituting the numbers with their correct units into the equation and evaluating. In all of these, I've used the most basic numbers and units. In the $F/m = a$ example, we would replace "F" with 1 Newton and "m" with 1 kilogram. Then the equation would look like this:

$$\frac{1 \text{ Newton}}{1 \text{ kilogram}} = 1 \text{ meters per second per second}$$

Now, at the risk of causing confusion, we need to again condense it down to shorthand. So, we use their commonly known abbreviations.

$$\frac{1 \text{ N}}{1 \text{ kg}} = 1 \text{ m per s per s}$$

Those "per"s are really irritating so let's do this instead.

$$1 \ \frac{\text{N}}{\text{kg}} = 1 \text{ m/s/s}$$

That's still a bit clumsy, so let's do this:

$$1 \ \frac{\text{N}}{\text{kg}} = 1 \text{ m/s}^2 = 1 \ \frac{\text{m}}{\text{s}^2}$$

Finally, we can end with a special way of writing it all:

$$1 \text{ N} \cdot \text{kg}^{-1} = 1 \text{ m} \cdot \text{s}^{-2}$$

Notice that anything in the denominator of a fraction gets a negative exponent. Also notice that you can square and cube and do whatever you need to units. OK, here's where you have to be careful. We started with a general equation, and we plugged in specific numbers. We just used ones, because we wanted to focus on the units themselves. The numbers just behave like any numbers. You add, multiply, divide and subtract as you need. We can move units around from one side of the equation to the other to get the original definition.

$$1 \text{ N} = 1 \text{ kg} \cdot \text{m} \cdot \text{s}^{-2}$$

We would read the above as the definition of the Newton. "A force of one Newton is defined to be 1 kilogram accelerated at one meter per second squared." We can really loosen up the language by saying "a Newton is a kilogram meter per second squared." You can see how this last loose way of phrasing it seems to wipe out the meaning. Notice, that from the beginning I chose the number one as our number. If we had other numbers then we would have to remember our second-grade multiplication tables. That would just add numbers, and hide the nature of the units and how they relate to each other. I could have easily have said "If I push a 5 kilogram mass such that it accelerates at 9.8 meters per second per second, then I am exerting 49 Newtons of force on the mass." See how the numbers fit in? They must have a meaning attached to them, with a unit of measurement.

Now, this is not just all circular thinking. There are other units for everything. Force can be measured in "dynes". Mass can be measured in grams or solar masses. Length can be measured in yards or miles or lightyears. Time can be measured in years or picoseconds or hummingbird heartbeats. The choice of units is completely arbitrary, and governed only by convenience. Would you measure the length of time it takes to drive across the US in microseconds or fortnights? Would you measure your height in miles? Would you measure your driving speed in millimeters per decade? You could, but everyone would be irritated with you. Choosing good units reduces irritation and makes you the life of the party.

Go back to the last equation. Did you see how when I put the "kilogram" on the other side, I removed the negative exponent? One big special rule about the game of units, is you can multiply and divide any units as many times as you like. You can create all sorts of new units if you like. But you cannot add numbers with different units on them; just like you can't add 4 miles to seven pounds. However, you can actually add 4 apples to 7 oranges, so long as you convert the units to fruit. Then it becomes "4 fruits plus 7 fruits is eleven fruits." Luckily, one apple is one fruit, as one orange is one fruit. That is, it would be odd to say "one apple is 3 fruits and one orange is 5 fruits." In a similar way, we convert units in different ways all the time. We convert feet to yards, hours to minutes and years to decades. We convert Fahrenheit to Celsius, miles to kilometers and the clothing sizes at one department store to the sizes in another. All of these are not one-to-one. Temperature even converts the zero point.

The magic of science is that we measure this thing or that thing, and if we are lucky or smart (and luck favors the diligent and persistent), we might notice that this thing's measurement is related to the thing's measurement of something else. The luminosities and masses of stars can be measured. It just so happens that there is a relationship between the mass and the luminosity of a star. As another example, there is a relationship between the brightness you measure of a light and how far away you are from the light. Finally, if you count the numbers of atoms in the tiny crystals of certain rocks, you find that sometimes the atoms transform into other atoms by fission. You can use the observed rate at which this process occurs naturally to tell you the age of the rock. Science is the process of discovering relationships like these between different measurements. The profound thoughts come after you've made many relationships. You might then discover that all these relationships are really different ways of saying the same thing. That is the deep quest of science: to find the hidden patterns in nature.

2. Why Measurement?

It is a common refrain by many students first encountering science; that the math is hard and they don't get it. This is because many students haven't considered that there are different kinds of knowledge. First and foremost, knowledge is something you gain while you live. You have many instinctual actions that you are born with, such as the will to breathe, fear of the dark, and knowing who mother is from all the other moms. These are things that you know from your birth, and don't reflect what you've gathered. These are instinctual and inherited knowledge, written into your DNA from tens or hundreds of millions of years of evolution. Because they are there from the start, they cannot be called bad or good, they are just there. Badness or goodness for these things are only relevant insofar as they are appropriate to the world around you. All other forms of knowledge, you get along the path of your life since birth. The sciences are concerned with things that you gather while living your life that affect your thinking or behavior. We'll call these acquired ideas "knowledge."

Generally speaking, there are two types of knowledge: empiricism and revelation. The first means knowledge acquired by experience with the world using the tools that you have around you. You don't instinctually run from fire, but you learn very fast as a baby or toddler what flame is if you're touched by it. You learn that fire can hurt, and is therefore "bad" or at least "scary." Such encounters during baby years are typical of gathering knowledge about the real world. Babies have no idea about much of these new sensations and inputs from their mouths, noses, eyes, ears and fingers, so they reach out to the world using all these "instruments" to learn about what's around them. They learn about what is harmful and what is pleasurable. They associate goodness and badness with things as they encounter them. But, occasionally, the encounter transcends goodness or badness and is fascinating for its own sake. The first campfire is transfixing, and is a mix of good and bad, requiring further knowledge about how to contain the badness (burning yourself or your things) and maximize the goodness (warmth and beauty). As the baby grows up, maybe he learns that some things burn with a nice smell and other things burn with a terrible smell. Maybe some things don't actually burn in a wood fire, and others grow green or blue. If you want to manage the location of the campfire, you use a tool like a steel tong, or a long sturdy log, to push around the embers. Studying the ways of the fire gives us control over not just the fire, but over other things. When we learn how to make it grow, how to keep it going after it's gone down to embers, we can learn how to cook with it, how to make pottery, how to start and stop it with ease, how to heat a home, how to melt iron. Each process takes different amounts of flammable material, and we learn how to do this in passing. Basically, using our hands, we fashion tools that help us control the fire, and we judge its behavior by our senses. These are the central ideas of empiricism: we harness the tools at hand to learn about and use things we find in the world around us.

Revelation is a completely different form of knowledge. It is knowledge acquired not through direct interaction with the natural world, but through some spiritual means. The spiritual knowledge we indicate here is frequently in the form of sacred books and long-held oral traditions. We learn these traditions and we hear or read the words, and we associate goodness or badness with various ideas, rather than physical objects. Revelation is nearly always in the form of a story. These stories that bind ideas together with metaphors. There are many revelatory stories from many cultures supporting many traditional ways and many new ways of life. However, when we discuss revelation, it is oftentimes obtained in spite of the natural world or opposition to it, or in an attempt to escape from it. Sometimes, revelation is discovered not by renouncing the natural world, but by isolation from other people or one's home society and immersing one-self into the natural world entirely in solitude. The number of stories abound that can be called revelation. However, in all cases, revelation seeks to acquire knowledge that supersedes the natural world, giving "reason" or "purpose" to the seemingly arbitrary or capricious nature of what happens to people on a day-to-day basis, or across one life or across many generations. Revelation is also strongly person-centric, meaning its goals are to provide knowledge for sentient being with "souls" to "show the way." The purpose of a science of the natural world is not to address any form of revelation, insofar as revelation refrains from speaking about the workings of the natural world. To wit, if a revelation stated that the world did not have mass or gravity, but rather it worked by the long arms of various spiritual beings holding things to the Earth, then it would be chatting about the natural world.

Here is where revelation and empiricism clash. When a revelation, which is defined to be true by its adherents, comes in conflict with an observation about the natural world, then only one of three paths can be taken. The first path is to state that the observation of the natural world is false, and that the revelation is true. This could be particularly tricky if the revelation states that gravity doesn't work if you're so high up on top of a cliff that a heaven-ward pull will spare you a fall. The second is that the adherent to the revelation realizes the falsity of the revelation and abandons it. The third is more common: to keep both. The revelatory statement is kept in the mind of the person at the same time as the data from the natural world comes in, even if they are completely contradictory. This is true of astrology. Astrology states that the heavens directly influence people, so it is a good example of what could be called "folk revelation" or "non-religious revelation", since no major religion incorporates it, and some actively deride it. The influence of the stars is demanded to be true by its adherents, and they try to make predictions based upon their "readings" but ultimately, they can't say anything other than general platitudes about the nature of human personalities or general behavior. When there is no predictive capacity, that means there is no physical mechanism that can be seen or felt or heard or touched or experienced or measured. Astrologists are not interested in *how* it works, just that it *feels* right. So the revelation about the nature of who you are clashes with the fact that no mechanism exists to demonstrate how the knowledge is obtained from the natural world. The goal of astrology is not to know how the planets actually move in the sky under gravity, or what it is like to be on that planet, or to know what resources might exist on that planet. The goal of astrology is not to learn how the stars formed, and whether or not they will ever stop shining or why the sky is dark or what the Milky Way is, or how they "stay up in the sky." The goal of astrology is to make sense of a person's experience in this natural world using spiritual means. In that, there's nothing wrong. Understanding why you do something or why things are the way they are is a strongly spiritual pursuit. However, astrology can be easily shown to lack any kind of predictive capacity and it cannot divulge the reasons for how the revelation is obtained from the stars and planets.

Since revelation has a huge aspect in our society, it regularly conflicts with empiricism. Empiricism has at its core, no real concern with "why" in the sense of "purpose" or "intention". It only cares about the kind of "why" whereby the physical processes that best explain the observed repeating occurrences. It removes the concept of "volition" or "will" from the natural world, replacing it with actions and reactions according to discovered rules and laws that are triggered if the components are present and their setup leads to an imbalance that is resolved by nature's application of those rules and laws. As an example, imagine you put a heavy rock at the edge of a cliff made of dirt and mud. Why you put it there is of no interest. Perhaps it's a convenient place to store the rock. Perhaps you didn't see the cliff. Perhaps there is a can of beans at the bottom of the cliff you cannot open and you want to use a falling rock to open it. None of these reasons matter to the rock or the dirt or the cliff or the force of gravity or the molecular binding of the dirt or the lack of cohesion of the dirt when it gets wet in the rain and then falls to crush the can of beans. The rock, the cliff, the can of beans, the force of gravity or the forces that make the dirt hard or soft all do not "care" why they were there or why they are doing it. They don't have will or volition, so they just behave according to natural laws. A conflicting revelation might be that the dirt "became tired of holding the rock" or that the rock "wants" to reach lower ground due to its dislike of the sky. (Of course rocks don't like the sky, they are on the ground!) These ideas ascribe thinking or will to objects that do not have it. This would be true even if the rock was in that location for a thousand years and its precarious position was remarkable and well-known. A revelation might be that if the rock fell, it would be due to the pleasure of some deity. Since, by definition, the thoughts and reasons of deities are incomprehensible to us, we should always choose to understand how the world works and manifests itself to us first in terms of physical laws. If no physical law or process can be discovered for why the rock fell, then it is more correct to explain that we don't understand why rocks fall to the ground when the ground holding them up is dampened rather than say that a Rain God pushed the rock off the cliff. The latter is satisfactory to many, especially people that have no responsibility for the rock's actions. The rock will have fallen and they will move on. However, if you happen to live in a place where there are a lot of rocks on the top of a lot of cliffs and people have chosen to live under those cliffs so that they don't have to walk very far to open their cans of beans using the frequently falling rocks, then someone in that community might actually need to know the real reason why the rocks fall off the cliffs and how to control them.

Once we need to know the true reasons why something acts the way it does in the real world, then revelations about the natural world are always completely ignored. Not that they want to be ignored, mind you. People will hold on very tight to their revelations they value. In the example above, an adherent might state that everything has a soul, even a rock, so it could be thinking, but on a "level we just don't understand." This is an example of the circular literalism of revelation; there can be as many revelations as one wishes. There is no end to them. The adherent to rock-thought might state that rocks do indeed think, but we don't understand them because we haven't learned to "talk like a rock." Any rebuttal to this thinking will create new reasons why rocks think and have souls. Therefore, it becomes necessary to learn a toolbox for getting away from such rhetorical traps that lead you away from your needed purpose: which is to assure that rocks only fall from the cliffs when the cans of beans are on their targets and all people are away from the falling zone. This requires seeking out predictable rules that show us how the world actually works, not how we would like it to work or how we believe it might be working on some spiritual plane of existence. In short, revelation-based explanations of the natural world are almost always "traps" that do not lead to understanding of why something works or behaves in the way it does. To wit, you could either just make sure that the cliff stays dry or you could spend years trying to talk to a rock. One of these paths provides the best chance of the needed outcome.

Therefore, empiricism acts to assist people in working with the world as it presents itself to them. It acts to solve the problems that are faced in the material world. Sometimes, these problems cannot be directly experienced by an individual because they happen on time scales that are too short or too long to easily act upon. Sometimes the problems occur in places that are extremely remote or too dangerous to venture. Or the problem is one that is chosen to be encountered rather than brought upon us by nature. This is the problem of exploration. Exploration is a fascinating human experience. When one explores, it may be for a particular purpose of gathering resources for your community. It can also be just the desire to find yourself in a new place. Exploration always contains the intent to return and show what you've found. This is different than a spiritual journey of discovery through isolation, since the actions undertaken on the journey are completely different. The spiritual journey seeks only to experience the voyage, but not document it. The experience itself is the sum and total of the journey, and the stories which are returned are the bounty. An exploration, however, is undertaken for a completely different reason; to document the Natural world as it presents itself and to explain it using things and processes observed to occur with your senses.

By why take these explorations into the Natural world? Why do such difficult and painstaking work that is almost never fun and easy? Perhaps there is an over-arching sense of duty or responsibility that is instilled upon us by the stories we hear in our youth and adulthood. Stories we can neither verify nor test, but that feel true anyway. There is a reason that much of the scientific studies in Medieval times was done by monks and priests in monasteries. There is a good reason why the Vatican has an astronomical observatory, and it is not to do astrology. Johannes Kepler, in studying the motions of the planets in the sky, become the first theoretical astrophysicist, but his intent was to understand the mind of God. It is not uncommon for deeply spiritual people to be intensely interested in the Natural world, wishing to understand it intimately. Perhaps some revelations compel us to understand the Great Work of Nature as a fundamental good. All great faiths wish to help guide us in the world as we live. Therefore, understanding, appreciating, mastering and valuing Nature as it truly is are always core elements to a faith worth living with.

3. Units

All of science has numbers. But the numbers are related to something you measure. So every number has units attached to it, such as 5 feet, or 17 minutes, or 120 Joules. Without the units, the numbers have no meaning at all. One may be the loneliest number, but the nature of "one" all by itself, with no units, has no meaning in science. Without exception, the thing that will trip you up 100% in this whole course will be units. Without exception, not all lengths are equal. If you see a length of 1.838, you should immediately ask 1.838 *whats*. Is it in meters, miles, Ångstroms, lightyears or Megaparsecs? How many of each is in each? Nearly all problems in this class will have some sort of unit conversion. Be completely sure that you know all of them. Here is a big list of all the ones worth chatting about. Each entry is a web link. It's expected that you'll be able to look up the actual numbers online or in your textbook.

There are odd rules to units. First, they tend to written down with an abbreviation that is one letter, but this is violated all the time. Next, the unit can have an exponent, just like a variable. This counts the number of times that unit is used for that number. If one unit is written next to another, then it's a "compound unit" made up of little sub-units. We combine sub-units together by "multiplying" them together and "dividing" them up and down. But we can't get rid of a unit just because it's long. If that compound unit is used a lot, then sometimes it's given its own name.

- Distances or lengths
  - meter or m
  - kilometer or km
  - Astronomical Unit or AU = 93,000,000 miles = 150,000,000 km = 1.5×10^8 km
  - Parsec or pc = 206,264.8 AU
  - Light year or ly = speed of light times one year = 9.4607×10^{12} km
  - Megaparsec or Gpc, which is 10^6 pc
  - Gigaparsec or nm, which is 10^9 pc
  - nanometer or nm which is 10^{-9} m
  - Ångstrom or Å which is 10^{-10} m
  - Schwarzschild Radius or R_s
  - Planck Length or l_{pl} = 1.6162×10^{-35} m
  - Hubble Length or c/H_o = 1.364×10^{26} m
- Area
  - Square meters or m^2
  - Square kilometers or km^2
- Volume
  - Cubic meters or m^3
  - Cubic kilometers or km^3
  - Cubic parcsecs or pc^3

  - Cubic Megaparcsecs or Mpc^3
- Density
  - Density is basically "how much stuff in a box".
  - Kilograms per cubic meter: $\mathrm{kg} \cdot \mathrm{m}^{-3}$
  - Number of stars per cubic parcsec: "stars" $\cdot\, \mathrm{pc}^{-3}$
  - Solar Masses per cubic Megaparcsec: $\mathrm{M}_\odot \cdot \mathrm{Mpc}^{-3}$
  - Number of galaxies per cubic Megaparcsec: "galaxies" $\cdot\, \mathrm{Mpc}^{-3}$
- Time
  - Second: s
  - Earth Year: yr, which is 31,622,400 s
  - Planck Time or $t_{pl} = 5.39106 \times 10^{-44}$ s
  - Hubble Time or $t_H = 4.55 \times 10^{17}$ s = 14.4 billion years
  - Frequency in units of inverse time, the Hertz, or $1/\mathrm{s} = \mathrm{s}^{-1}$
- Speed
  - kilometers per second: $\mathrm{km/s} = \mathrm{km} \cdot \mathrm{s}^{-1}$
  - meters per second: $\mathrm{m/s} = \mathrm{m} \cdot \mathrm{s}^{-1}$
  - The speed of light in a vacuum: $c = 299,792,458$ m/s
- Acceleration
  - kilometers per second per second: $\mathrm{km/s}^2 = \mathrm{km} \cdot \mathrm{s}^{-2}$
  - meters per second per second: $\mathrm{m/s}^2 = \mathrm{m} \cdot \mathrm{s}^{-2}$
  - Acceleration due to gravity at the surface of Earth or $g = 9.80665\ \mathrm{m} \cdot \mathrm{s}^{-2}$
- Mass
  - kilogram: kg
  - Solar Mass: $\mathrm{M}_\odot = 1.98855 \times 10^{30}$ kg.
  - Chandrasekhar Mass: 1.39 $\mathrm{M}_\odot$
- Force
  - Expressed in units called Newtons with the symbol "N"
  - one kilogram $\cdot$ meters per second per second: $\mathrm{N} = \mathrm{kg} \cdot \mathrm{m/s}^2 = \mathrm{kg} \cdot \mathrm{m} \cdot \mathrm{s}^{-2}$
- Energy
  - Joule or "J" which is a Newton-meter.
  - breaking it down to fundamentals: $\mathrm{J} = \mathrm{N} \cdot \mathrm{m} = \mathrm{m} \cdot \mathrm{kg} \cdot \mathrm{m/s}^2 = \mathrm{kg} \cdot \mathrm{m}^2/\mathrm{s}^2 = \mathrm{kg} \cdot \mathrm{m}^2 \cdot \mathrm{s}^{-2}$
  - Electron Volt or $\mathrm{eV} = 1.6 \times 10^{-19}$ J
  - Rest Mass Energy or $\mathrm{E} = \mathrm{mc}^2$. This should look familiar.

- Power
  - Watt or one Joule per second: $\mathrm{J/s} = \mathrm{J} \cdot \mathrm{s}^{-1}$
  - breaking it down to fundamentals: $\mathrm{W} = \mathrm{J} \cdot \mathrm{s}^{-1} = \mathrm{kg} \cdot \mathrm{m}^2 \cdot \mathrm{s}^{-2} \cdot \mathrm{s}^{-1} = \mathrm{kg} \cdot \mathrm{m}^2 \cdot \mathrm{s}^{-3}$
- Heat
  - Temperature or T in units called Kelvins, denoted “K”.
  - Boltzmann constant or $k = 1.38064853 \times 10^{-23}$ J/K.
  - Stephan-Boltzmann constant or $\sigma = 5.6704 \times 10^{8}\ \mathrm{W} \cdot \mathrm{m}^{-2} \cdot \mathrm{K}^{-4}$.

4. Converting Between Units

Here is how we do conversions between units. Let's convert 13.8 billion years to seconds.

$$\frac{13.8 \text{ billion years} \mid 10^9 \text{ years} \mid 365.25 \text{ days} \mid 24 \text{ hours} \mid 60 \text{ minutes} \mid 60 \text{ seconds}}{\mid 1 \text{ billion years} \mid 1 \text{ year} \mid 1 \text{ day} \mid 1 \text{ hour} \mid 1 \text{ minute}} = ?$$

We multiply numbers across and divide up and down. A unit above cancels a unit below, and we're left with only seconds across the top.

$$\frac{13.8 \mid 10^9 \mid 365.25 \mid 24 \mid 60 \mid 60 \text{ seconds}}{\mid 1 \mid 1 \mid 1 \mid 1 \mid 1}$$

Now just multiply and divide:

$$\frac{13.8 \times 10^9 \times 365.25 \times 24 \times 60 \times 60}{1 \times 1 \times 1 \times 1 \times 1} \text{ seconds} = 4.35494880 \times 10^{17} \text{ seconds}$$

We were lucky in that everything in the denominator was a 1. They could be other numbers. The treatment is the same.

Now let's convert the critical density of the universe to galaxies per cubic megaparsec.

$$\frac{9.47 \times 10^{-27} \text{ kg} \mid 1 \text{ M}_\odot \mid (3.1 \times 10^{16})^3 \text{ m}^3 \mid (10^6)^3 \text{ pc}^3}{\text{m}^3 \mid 2 \times 10^{30} \text{ kg} \mid 1 \text{ (pc)}^3 \mid 1 \text{ Mpc}^3} = 1.4 \times 10^{11} \text{ M}_\odot/\text{Mpc}^3$$

The Milky Way is about 2×10^{11} M$_\odot$, so the universe's critical density is a small Milky Way every megaparsec.

5. How to Read Equations and Units

Equations and units are both composed of letters, so how do you tell them apart? In the end, you'll just have to get out your vocabulary flash cards and do a bit of memorization.

But, there are always some big clues. Whenever we make an equation, let's choose to make it with "Equation Italics", like this:

$$F = \frac{mv^2}{r}$$

Each of these italicized things in the equation are called **variables**. Each variable stands for a number with a unit attached. All variables are always just one italicized letter. Now, we can read the equation as something named F is equal to m times v times v all divided by r. We said "times v" twice because of the exponent 2 above the v. If they have an exponent, meaning a number to the upper right of the letter, then you multiply it by that many times. As you might have gleaned, we multiply any two variables that are next to each other, and divide by any variable that's on the bottom of a ratio. If the variable is in the denominator, then you can replace it with a negative exponent like this:

$$F = mv^2r^{-1}$$

You can re-arrange the variables in many ways, so long as you keep the sense of exponent. (What's the exponent if we haven't written it out?) The last thing you want to do in any problem is plug in numbers and units. Perhaps we want the number associated with v instead of F. Then we rearrange the variables in the equation like this:

$$v^2 = Frm^{-1} \quad \longrightarrow \quad v = (Frm^{-1})^{1/2} = \sqrt{Frm^{-1}}$$

What we mean here is that to get v, you multiply all the numbers F, r and m^{-1} together, then take the square root of that number.

Now units are just things attached onto numbers. We might write the following numbers:

- First notice that if $m = 15$ kg, then $m^{-1} = \frac{1}{15\ \text{kg}} = \frac{1}{15} \cdot \frac{1}{\text{kg}} = 0.0666\ \text{kg}^{-1}$
- Therefore, if $F = 5$ N, $m^{-1} = 0.0666\ \text{kg}^{-1}$, and $r = 27$ m,
- then $v = 3$ m/s

Notice that the units have the normal font, and they can be more than one letter. You might be confused by the $r = 20$ m and $m = 15$ kg. You might be asking when m is "mass" and when is m "meters"? If it's in this exact equation and in italics, then it's "mass"; when it's attached onto a measurement as a unit and in normal font, then it's meters. You have to just become familiar with them. Also, we use m for magnitude. Remember that equations don't have units in them; they only have variables which represent numbers with units.

Finally, we sometimes encounter variables with subscripts, like these:

$$m_1 \cdot d_1 = m_2 \cdot d_2$$

This always means that the things with the subscripts are the same type of variable, but the subscript indicates that we're measuring the properties of two different objects. We might write a solution like this:

- If $m_1 = 5$ kg, $d_1 = 12$ m, and $m_2 = 20$ kg,

- then $d_2 = 3$ m

The $\cdot$ things you see in there stand for multiplication. If you think about it for a bit, this is the "teeter-totter equation". The big kid has to sit closer to the center of the teeter-totter so that the little kid can balance him by sitting farther away.

You've now seen that units on numbers can cause deep confusion as to which is which. Let's now think of solving physics-based math problems as different gears of a car being driven. Just like driving, if you skip a gear, the car has a tougher time going. We'll not skip gears. We'll read each thing on its own in its own way.

(1) Collect the equations you will need to get answers.
(2) Use algebra to manipulate the equations to make all input measurements on one side of the equation, and the result you are hunting for on the other.
(3) Convert all measurements to common units. The best common units are those of your universal constants. They are your guide to what units you need to use.
(4) Insert the numbers into the equations with the common units, to obtain the resulting derived measurement.
(5) Convert the derived measurement back to the required units, if necessary.

6. Basic Geometry

6.1. Length of the path around a circle.

$$C = 2\pi R$$

- C is the circumference of a circle.
- R is the radius of the circle.
- π is that famous number, 3.14159265359... Note the π is not a variable. It is a constant with no units attached.

6.2. Volume of a sphere.

$$V = \frac{4\pi}{3} R^3$$

- V is the volume of a sphere
- R is the radius of the sphere.
- π is that famous number, 3.14159265359...

6.3. Surface area of a sphere.

$$A = 4\pi R^2$$

- A is the surface area of a sphere
- R is the radius of the sphere.
- π is that famous number, 3.14159265359...

7. Measuring Angles

7.1. Angular Measurement.

$$\tan\theta = \frac{s}{D}$$

- D is the distance to something.
- s is the parallax baseline or the size of the distant object.
- θ is the parallax angle in degrees or radians.
- "tan" is the tangent function. For a right triangle, the tangent of an angle is always the length of the side across from the angle, divided by the length of the side touching the angle that's not the hypotenuse.

7.2. Parallax.

$$\theta = \frac{1}{d}$$

- This is only true for very small angles.
- d is now the distance to something in parsecs.
- s was replaced with a distance of 1 Astronomical Unit.
- θ is the parallax angle in arcseconds, because when the angle is very small, then $\tan\theta \approx \theta$
- There are 206,264.8 AU in one parsec.

7.3. Space Velocity.

$$v_{space} = \sqrt{v_{radial}^2 + (4.74\mu \cdot d)^2}$$

- v_{space} is the true speed that the distant star is going through space in $\mathrm{km \cdot s^{-1}}$.
- v_{radial} is the radial velocity that the distant star is approaching or receding in $\mathrm{km \cdot s^{-1}}$.
- μ is the proper motion in arcseconds per year.
- d is the distance in parsecs.

The "4.74" bit comes from the following, as we convert arcseconds per year to kilometers per second.

$$\begin{array}{c|c|c|c|c|c} \mu \text{ arcseconds} & d \text{ parsecs} & 1 \text{ year} & 1 \text{ degree} & \pi \text{ radians} & 3.086 \times 10^{13} \text{ km} \\ \hline \text{year} & & 31622400 \text{ seconds} & 3600 \text{ arcseconds} & 180 \text{ degrees} & 1 \text{ parsec} \end{array}$$
$$= 4.74\ \mu \cdot d$$

7.4. **Angular Resolution.**

$$\theta_{radians} = 1.220 \frac{\lambda_{meters}}{D} \quad \longrightarrow \quad \theta_{arcseconds} = 0.25 \frac{\lambda_{\mu m}}{D}$$

- $\theta_{radians}$ is the angular resolution in radians.
- λ_{meters} is the wavelength of light in meters.
- D is the diameter of the telescope's primary mirror or lens in meters.
- The factor of 1.220 comes from the fact that images in a telescope are always fuzzy due to diffraction. This comes only from measurement.
- $\theta_{arcseconds}$ is the angular resolution in arcseconds.
- $\lambda_{\mu m} = 10^{-6}\ \lambda_{meters}$ is the wavelength of light in micro-meters.

We can convert that to arcseconds shown below. Below, we must be sure to measure the wavelength in μm, and the diameter of the telescope's primary in meters. We could have left it in meters and meters, but then you'd be carrying around a lot of exponents. This way, it's built into the equation.

$$\begin{array}{c|c|c|c|c} 1.22 \text{ radians} & \lambda \text{ m} & 180 \text{ degrees} & 1\ \mu \text{ meter} & 3600 \text{ arcseconds} \\ \hline & D \text{ meters} & \pi \text{ radians} & 10^{-6} \text{ m} & 1 \text{ degree} \end{array} = 0.25 \frac{\lambda}{D} \text{ arcseconds}$$

8. Kepler's Laws

8.1. Ellipses.
All planets orbit the Sun on ellipses.

- a is the average distance between one focus of the ellipse and all points on the ellipse.
- e is the ellipticity of the ellipse.
- $a(1-e)$ is the distance of closest approach of the ellipse to one of the foci.
- $a(1+e)$ is the farthest distance of the ellipse to one of the foci.

8.2. For Just the Solar System.

$$P^2 = a^3$$

- P is the orbital period of a planet around the Sun in Earth-years.
- a is the average distance between a planet and the Sun in Astronomical Units.

8.3. For any orbiting bodies anywhere in Solar Units.

$$P^2 = \frac{a^3}{M_{total}}$$

- P is the orbital period of the two objects around their common center of mass in Earth-years.
- a is the average distance between the two orbiting objects in Astronomical Units.
- M_{total} is the total mass of the system in Solar masses: $\mathrm{M}_\odot$.

8.4. For any orbiting bodies anywhere in common units.

$$P^2 = \frac{4\pi^2}{G(M_1+M_2)}a^3$$

- P is the orbital period of the two objects around their common center of mass in seconds.
- a is the average distance between the two orbiting objects in meters.
- M_1 and M_2 are the masses of the two gravitating objects in kilograms.
- G is Newton's Gravitational Constant: $6.673 \times 10^{-11}\ \mathrm{N \cdot m^2/kg^2}$

8.5. Average orbital speed.

$$v = \frac{2\pi a}{P}$$

- P is the orbital period, usually in seconds or years.
- a is the average distance between the two objects in some convenient units, usually kilometers.
- v is the average speed of the orbiting body, in kilometers per second or per year.

9. Newton's Laws

9.1. Fundamental Definition of Force.

$$F = ma$$

- F is the force in Newtons.
- m is the affected mass in kilograms.
- a is the acceleration of the affected mass in meters per second per second ($\mathrm{m/s^2}$).

9.2. Momentum.

$$p = mv$$

- p is the momentum of the moving thing.
- m is the mass of the thing that's moving, typically in kg.
- v is the speed or velocity of the thing that is moving typically in m/s.

9.3. Angular momentum of something moving around in a circle.

$$L = rmv = rp$$

- L is the angular momentum.
- m is the mass of the object that's moving in a circle.
- v is the speed of the object as it goes around the circle.
- r is the distance the object is from the center of the circle. (i.e. the radius of the circle.)
- p is the momentum at one instant in its arc around the circle.

9.4. Force needed to make an object move in a circle.

$$F = \frac{mv^2}{r}$$

- F is the force in Newtons.
- m is the mass of the object moving in a circle.
- v is the speed of the object as it goes around in a circle.
- r is the distance the object is from the center of the circle. (i.e. the radius of the circle.)

10. Gravity

10.1. **Force between any two masses due to Gravity.**

$$F = \frac{Gm_1m_2}{d^2}$$

- F is the force in Newtons. Incidentally, this is your weight!
- G is Newton's Gravitational Constant: 6.673×10^{-11} N $\cdot$ m^2/kg^2
- m_1 is the mass of one object.
- m_2 is the mass of the other object.
- d is the distance between the two objects' centers of mass.

10.2. **Weight on Earth.**

$$F = mg$$

- F is the force in Newtons, or weight.
- m is the mass of the object near the surface of Earth.
- g is the acceleration due to gravity at the surface of Earth: 9.8 m/s^2

10.3. **Center of Mass.**

$$m_1 \cdot d_1 = m_2 \cdot d_2$$

- m_1 is the mass of one thing.
- m_2 is the mass of the other thing.
- d_1 is the distance from one thing to the center of mass.
- d_2 is the distance from the other thing to the center of mass.

11. Energy

11.1. Energy required to make a force move an object.

$$E = Fd = mad$$

- E is the Energy in Joules.
- F is force applied to the mass of the object that's being moved.
- d is the distance through which the force is applied.
- m is the mass of the object being accelerated.
- a is the acceleration that happens to the object when the force is applied.

11.2. Kinetic Energy of a moving thing.

$$E = \frac{1}{2}mv^2$$

- E is the Energy in Joules.
- m is the mass of the object that's moving.
- v is the speed of the object.

11.3. Energy required to lift an object against gravity near Earth's surface.

$$E = F_{gravity}H = mgH$$

- E is the Energy in J.
- $F_{gravity}$ is force of gravity on the object that's being moved in Newtons: N.
- H is the height through which the object is lifted in m.
- m is the mass of the object being accelerated in kg.
- g is the acceleration due to gravity at the surface of Earth, which is about $9.8\ \mathrm{m \cdot s^{-2}}$.

11.4. Energy required to lift an object against gravity.

$$E = \frac{GMm}{R} - \frac{GMm}{(R+H)}$$

- E is the Energy in Joules : J.
- G is Newton's Gravitational Constant: $6.673 \times 10^{-11}\ \mathrm{N \cdot m^2/kg^2}$
- m is the mass of the object that's being lifted in kg.
- M is the mass of the big thing that's being lifted against, like a planet or a star in kg.
- R is the original height from which the object is lifted in meters.
- H is the additional height that the object is lifted in meters.

11.5. **Escape speed.**

$$E_{esc} = \frac{GMm}{R} = \frac{1}{2}mv_{esc}^2 \implies v_{esc} = \sqrt{\frac{2GM}{R}}$$

You need a specific amount energy to lift or throw something to an infinite height. That means that H in the previous equation becomes huge. If an object is exactly at the escape speed from a planet or star, we mean that it never falls back down; always slowing down and finally stopping at an infinite distance.

- E_{esc} is the Energy required to escape the object in Joules.
- G is Newton's Gravitational Constant: 6.673×10^{-11} N $\cdot$ m^2/kg^2
- m is the mass of the object that's being lifted in kg.
- M is the mass of the big thing that's being lifted against, like a planet or a star in kg.
- R is the original height from which the object is lifted in meters. It can also be the size of the planet or star, if you're escaping from the surface of such an object.
- v_{esc} is the speed of the object in m/s.

11.6. **Energy of a mass at rest.**

$$E = mc^2$$

- E is the Energy in Joules.
- m is the mass of the object.
- c is the speed of light: 3×10^8 m/s
- Yes, this is Einstein's equivalence of matter and energy!

11.7. **Conservation of Energy in a Gravitational Field.**

$$E_{total} = \frac{1}{2}mv^2 - \frac{GMm}{D}$$

This is the Total Energy of a mass as it moves up or down in the gravitational field around a planet or star, such as a ball thrown upwards. It stays the same.

- E_{total} is the total energy of the object in Joules.
- m is the mass of little object.
- G is Newton's Gravitational Constant: 6.673×10^{-11} N $\cdot$ m^2/kg^2
- M is the mass of the big thing that's being lifted against, like a planet or a star in kg.
- D is the distance between the centers of mass of the two gravitating objects.
- v is the speed of the little object.

12. Light

12.1. Frequency.

$$F = \frac{1}{P}$$

- F is the frequency of the light in Hertz.
- P is the period of oscillation in seconds.

12.2. Speed of Light.

$$c = \lambda \cdot \nu = \omega \cdot f$$

- c is the speed of light: 3×10^8 m/s
- ν or f is the frequency of light in hertz.
- λ or ω is the wavelength of light in meters.
- Watch out, we use MANY different lengths for wavelengths in astronomy.

12.3. Doppler Effect and Redshift.

$$1 + z = \frac{\lambda_{observed}}{\lambda_{emitted}} = \frac{\nu_{emitted}}{\nu_{observed}} = \sqrt{\frac{1 + v/c}{1 - v/c}} \approx 1 + \frac{v}{c}$$

- z is the redshift or blueshift.
- λ is the wavelength of the observed or emitted light.
- ν is the frequency of the observed or emitted light.
- c is the speed of light: 3×10^8 m/s
- v is the speed relative to the observer (for speeds much less than that of light-speed.)
- The last approximation is true only for speeds much less than light-speed.

12.4. Energy of a photon.

$$E = hf = \frac{hc}{\lambda}$$

- E is the energy in Joules.
- h is Planck's constant: 6.626×10^{-34} J $\cdot$ s $= 4.135668 \times 10^{-15}$ eV $\cdot$ s
- c is the speed of light: 3×10^8 m/s
- λ is the wavelength of light in meters.
- f is the frequency of light in Hertz.
- Joules are not the usual units for energy of photons. Usually it's eV. Wavelengths are in many different units, so you'll have to be careful.

13. Blackbody Radiation

13.1. **Wien's Law.**

$$T = \frac{b}{\lambda_{max}}$$

- T is the temperature in Kelvins: K.
- λ_{max} is the peak wavelength of blackbody emission in meters.
- b is a number: $2.897773 \times 10^{-3}\ m \cdot K$

13.2. **Stefan-Boltzmann law.**

$$J = \sigma T^4$$

- J is the power emitted per square meter: $\mathrm{W} \cdot \mathrm{m}^{-2}$
- T is the temperature in Kelvins: K.
- σ is a number: $5.6704 \times 10^{8}\ \mathrm{W} \cdot \mathrm{m}^{-2} \cdot \mathrm{K}^{-4}$.

13.3. **Luminosity of a star.**

$$L = 4\pi R^2 \cdot J$$

- L is the luminosity in Watts
- R is the radius of the star in meters.
- J is the power emitted per square meter: $\mathrm{W} \cdot \mathrm{m}^{-2}$
- π is that old friend 3.14159...

13.4. **Brightness of a star.**

$$B = \frac{L}{4\pi d^2} = \sigma T^4 R^2 d^{-2}$$

- B is the brightness at the receiver in $\mathrm{Watts/m}^2$
- L is the luminosity in Watts: W.
- σ is a number: $5.6704 \times 10^{8}\ \mathrm{W} \cdot \mathrm{m}^{-2} \cdot \mathrm{K}^{-4}$.
- T is the temperature in Kelvins.
- R is the radius of the star in meters.
- d is the distance to the star in meters.

14. Magnitudes

14.1. Distance Modulus Equation.

$$m - M = 5\log_{10}(d) - 5$$

- m is the apparent magnitude, which is a measure of brightness.
- M is the absolute magnitude, which is a measure of luminosity.
- d is the distance in parsecs, which is operated on by the base-10 logarithm.
- To get this, we chose the standard star to be at $d_0 = 10$ parsecs, so that $M_0 = m_0$.

14.2. Brightness - Apparent Magnitude Relationship.

$$m_{star} - m_0 = -2.5\log_{10}\left(\frac{B_{star}}{B_0}\right)$$

- m_{star} is the apparent magnitude of the star, which is a unit-less measurement of brightness.
- m_0 is the apparent magnitude of a standard reference star, like the Sun or Vega..
- B_{star} is the brightness of the star measured in $\mathrm{W/m^2}$
- B_0 is the brightness of a standard reference star, like the Sun or Vega.

14.3. Luminosity - Apparent Magnitude Relationship.

$$m_{star} - m_0 = -2.5\log_{10}\left[\frac{L_{star}}{L_0}\left(\frac{d_0}{d_{star}}\right)^2\right]$$

- m_{star} is the apparent magnitude of the star, which is a measure of brightness.
- m_0 is the apparent magnitude of the reference star.
- L_{star} is the luminosity of the star.
- L_0 is the luminosity of the reference star.
- d_{star} is the distance of the star.
- d_0 is the distance to the reference star.

14.4. Luminosity - Absolute Magnitude Relationship.

$$M_{star} - M_0 = -2.5\log_{10}\left(\frac{L_{star}}{L_0}\right)$$

- M_{star} is the absolute magnitude of the star, which is a unit-less measurement of luminosity.
- M_0 is the absolute magnitude of the reference star.
- L_{star} is the luminosity of the star measured in Watts
- L_0 is the luminosity of the reference star.

15. Mass-Luminosity relationship of stars on Main Sequence

$$L = L_{\odot} \left(\frac{M}{M_{\odot}} \right)^{3.5}$$

- L is the luminosity of the star in Solar units.
- $L_{\odot}$ is the luminosity of the Sun.
- M is the mass of the main-sequence star in Solar units.
- $M_{\odot}$ is the mass of the Sun.

This is a strictly empirical relationship, seen from the data using a graph. It's not "derivable." It's also valid only for middle-mass stars. Same goes for the next section.

16. Main Sequence lifetime

$$t_{MS} = t_{\odot} \left(\frac{M}{M_{\odot}} \right)^{-2.5}$$

- t_{MS} is the main-sequence lifetime of the star in Solar units.
- $t_{\odot}$ is the main-sequence lifetime of the Sun.
- M is the mass of the main-sequence star in Solar units.
- $M_{\odot}$ is the mass of the Sun.

17. Average Kinetic Energy (Energy of Motion) of a Gas

$$\frac{1}{2}\, m_{ave} v_{gas}^2 = \frac{3}{2}\, kT$$

- v_{gas} is the average speed of the molecules in the gas in $\mathrm{m \cdot s^{-1}}$.
- k is Boltzmann's constant $= 1.38064853 \times 10^{-23}$ $\mathrm{J \cdot K}$
- m_{ave} is the average mass of the atoms and molecules of the gas.
- T is the average temperature of the entire gas in Kelvin.
- Note that if a gas is more than one thing, then they have different masses, and thus different speeds for the same temperature.

18. Hubble Law

$$v_{radial} = H_o \cdot d$$

- v_{radial} is the radial velocity that the distant galaxy is approaching or receding in $\mathrm{km} \cdot \mathrm{s}^{-1}$.
- d is the distance in Megaparsecs.
- H_o is the Hubble Constant, currently thought to be 68 $\mathrm{km} \cdot \mathrm{s}^{-1} \cdot \mathrm{Mpc}^{-1}$.

The universe's age can be estimated from the reciprocal of the Hubble constant.

$$\frac{1}{H_o} = \frac{\text{second} \cdot \text{Mpc}}{68\ \text{km}} \left| \frac{1\ \text{year}}{31622400\ \text{seconds}} \right| \frac{10^6\ \text{parsecs}}{1\ \text{Mpc}} \left| \frac{3.086 \times 10^{13}\ \text{km}}{1\ \text{parsec}} \right. = 14.3 \times 10^9\ \text{years}$$

Formula Sheet

Intensity Ratio: $\frac{I_A}{I_B} = 2.512^{M^B - M^A}$

Magnitude Difference: $M_A - M_B = 2.5 \log \frac{I_A}{I_B}$

Small Angle Formula: $\frac{angular\ diameter - arcseconds}{206265} = \frac{linear\ diameter}{distance}$

Circular Velocity: $V_C = \sqrt{\frac{GM}{r - meters}}$

M = mass of central body (kg)
G = 6.67×10^{-11} m^3/s^2kg
* Answer in m/s

Compare LGP: $\frac{LGP_A}{LGP_B} = \left(\frac{D_A}{D_B}\right)^2$

D = diameter
* Answer in times (×)

Resolving Power: $\alpha = \frac{11.6}{D(cm)}$

D = diameter (cm)
* Answer in arcseconds

Magnification: $M = \frac{F_O}{F_E}$

F_O = focal length of objective
F_e = focal length of eyepiece

Wien's Law: $\lambda_{max} = \frac{3{,}000{,}000}{T - degrees\ Kelvin}$ * Answer in nm

$\lambda_{max} = \frac{.2987}{T} \times 10^8 Å$

$T = \frac{2.9 \times 10^8 Å}{peak\ \lambda}$

Stefan-Boltzmann Law: $E = \sigma T^4 (J/s/m^2)$

T = K
$\sigma = 5.67 \times 10^{-8} J/m^2 s\ degree^4$
* Answer in J

Doppler Formula: $\frac{V_r}{c} = \frac{\Delta_\lambda}{\lambda_o}$

V_r = radial velocity
c = 300,000 km/s
Δ_λ = change in λ
λ_o = observed λ

Fusion Explained: $E = mc^2$

m = kg
c = 3×10^8 m/s
* Answer in Joules

Distance to Star: $d = \frac{206{,}265}{p - arcseconds}$

p = parallax * Answer in AU

F Ratio: $\frac{focal\ length(mm)}{objective\ diameter\ (mm)}$

Distance Modulus: $m_v - M_v = -5 + 5\log d \quad d = 10^{\frac{m_V - M_v + 5}{5}} = pc$

Luminosity of Star: $\frac{L}{L_\odot} = \left(\frac{R}{R_\odot}\right)\left(\frac{T}{T_\odot}\right)$ * Answer in times (×)

Mass of Binary System: $M_A + M_B = \frac{a^3}{p^2}$ M = solar masses
p = orbital period (yrs) a = AU

Kepler's 3rd Law: $p^2 = a^3$ p = orbital period (yrs) a = distance (AU)

Mass-Luminosity Relation: $L = M^{3.5}$ M = star mass in M_O * Answer in times (×)

Life Expectancy: $T = \frac{1}{M^{2.5}}$ M = star mass in M_O
* Answer in O lifetimes × 10 billion = years

Schwarzschild Radius: $R_S = \frac{2GM}{c^2}$ G = 6.67 × 10^{-11} m^3/s^2kg M = mass (kg)
C = 3× 10^8 m/s * Answer in m

Hubble Law: $V_r = \text{Hd}$ V_r = velocity of recession of galaxy (km/s)
H = 20km/s/Mpc d = distance (Mpc)

Redshift: $Z = \frac{\Delta\lambda}{\lambda_O}$ $\Delta\lambda$ = change in λ λ_O = unshifted λ

Age of Universe: $T_U = \frac{1}{H} \times 10^{12} years$ H = 70 km/s/Mpc
* Answer in years

Distance-Rate-Time: $d = rt$

$r = \frac{d}{t}$

$t = \frac{d}{r}$

Newton's Law of Gravity: $F = G\frac{m_1 m_2}{r^2}$ G = 6.67 × 10^{-11} m^3/s^2kg
$m_1 m_2$ = masses of objects in kg
r = distance between the two masses (m)
F = the strength of the gravitational force (N)

Kepler's 1st Law (Eccentricity): $e = \frac{c}{a}$

Ratio: $\frac{distance}{size/separation}$

Frequency: $v = \frac{c}{\lambda}$

Flux: $\frac{1}{d^2}$

$$L(M_V) = r^2$$

$$\frac{distance\ to\ star}{diameter\ of\ earth's\ orbit} = \frac{focal\ length\ of\ scope\ (mm)}{parallax\ shift}$$

Diameter of orbit: 300,000 km
* Answer in km

Dispersion Distance: $D = \frac{T_2 - T_1}{124.5\left(\left(\frac{1}{f_2}\right)^2 - \left(\frac{1}{f_1}\right)^2\right)}$

$$\left(\frac{1}{400}\right)^2 - \left(\frac{1}{600}\right)^2 = 3.472 \times 10^{-6}$$

$$\left(\frac{1}{400}\right)^2 - \left(\frac{1}{800}\right)^2 = 4.688 \times 10^{-6}$$

$$\left(\frac{1}{600}\right)^2 - \left(\frac{1}{800}\right)^2 = 1.215 \times 10^{-6}$$

Constants

$$1\ AU = 1.495979 \times 10^{11}\ m$$

$$1\ parsec = 206{,}265\ AU$$

$$= 3.085678 \times 10^{16}\ m$$

$$= 3.261633\ light\ years$$

$$1\ light\ year = 9.46053 \times 10^{15}\ m$$

$$c, or\ the\ speed\ of\ light = 2.997925 \times \frac{10^8 m}{s}$$

$$G, or\ the\ gravitational\ constant = \left(6.67 \times 10^{(-11)}\right) + (m^3/s^2\ )/kg$$

$$M_{\oplus} = 5.976 \times 10^{24} kg$$

$$R_{\oplus} = 6{,}378.164\ km$$

$$M_{\odot} = 1.989 \times 10^{30}\ kg$$

$$R_{\odot} = 6.9599 \times 10^{8}\ m$$

$$L_{\odot} = 3.826 \times 10^{26}\ kg$$

$$M\ of\ the\ Moon = 7.350 \times 10^{22}\ kg$$

$$R\ of\ the\ Moon = 1738\ km$$

$$M\ of\ H\ atom = 1.67352 \times 10^{-27} kg$$

$$1\ arc\ minute\ (1') = \frac{1}{60°}$$

$$1\ arc\ second\ (1") = \frac{1}{60'}$$

$$1\ Megaton = 1{,}000{,}000\ of\ TNT = 4.5 \times 10^{15} J$$

Astronomical terms and constants

Units of length

1 AU $\approx 1.5 \times 10^{13}$cm = one astronomical unit, i.e. the earth–sun distance.

1 pc $= 2.06 \times 10^5$AU $= 3.1 \times 10^{18}$cm = one parsec, i.e. a distance to a star with a parallax equal to one second of arc. A parallax is an angle at which the radius of earth's orbit around the sun is seen from a distance of the star. Notice: 2.06×10^5 is the number of seconds of arc in 1 radian.

1 kpc $= 10^3$ pc = one kilo-parsec,

1 Mpc $= 10^6$ pc = one mega-parsec,

1 Gpc $= 10^9$ pc = one giga-parsec,

$\mathrm{d_H = c/H_0} \approx 1.4 \times 10^{28}$cm ≈ 4 Gpc = Hubble distance, where $\mathrm{H_0} \approx 70$ km s^{-1} Mpc^{-1} is the Hubble constant; c $= 3 \times 10^{10}$ cm s^{-1} is the speed of light. The Hubble distance is approximately the radius of the observable universe with us at the "center".

1 $\mathrm{R_\odot} \approx 7 \times 10^{10}$cm = solar radius

Most stars have radii between $10^{-2}\mathrm{R_\odot}$ (white dwarfs) and $10^3\mathrm{R_\odot}$ (red supergiants); neutron stars have radii of about 10^6cm = 10 km .

Units of time

1 year $= 3 \times 10^7$s

$\mathrm{H_0^{-1} = d_H c^{-1}} \approx 1.4 \times 10^{10}$ years = Hubble time, approximate age of the universe.

Units of mass

$\mathrm{M_\odot} = 2 \times 10^{33}$ g = solar mass.

Known stars have masses in the range $0.08 - 100\ \mathrm{M_\odot}$. Below about $0.08\ \mathrm{M_\odot}$ the objects are *brown dwarfs*.

Units of luminosity, magnitudes

$\mathrm{L_\odot} = 4 \times 10^{33}$erg s^{-1} = solar luminosity.

Known stars have luminosity in the range $10^{-5} - 10^6\ \mathrm{L_\odot}$.

$\mathrm{M_{bol}} = 4.8 - 2.5 \log\ (\mathrm{L/L_\odot})$ = absolute bolometric magnitude of a star with a luminosity L . "Bolometric" means integrated over the entire stellar spectral energy distribution. $\mathrm{M_{bol,\odot}} = +4.74$.

$\mathrm{M_V = M_{bol} - BC}$ = absolute visual magnitude of a star; BC is a bolometric correction, and V indicates that we are referring to that part of the stellar radiation that is emitted in the "visual" part of the spectrum, i.e. at about 5×10^{-5} cm, 5000 Å . The BC depends on stellar temperature. $\mathrm{BC_\odot} = -0.08$.

$\mathrm{M_B}$ = absolute blue magnitude of a star; B indicates that we are referring to that part of stellar radiation that is emitted in the "blue" part of the spectrum, i.e. at about 4×10^{-5} cm, 4000 Å.

$\mathrm{m_{bol} = M_{bol}} + 5 \log\ (\mathrm{d/10pc})$ = apparent bolometric magnitude of a star at a distance d .

$\mathrm{V = M_V} + 5 \log\ (\mathrm{d/10pc})$ = apparent "visual" magnitude of a star as seen in the sky.

$\mathrm{B = M_B} + 5 \log\ (\mathrm{d/10pc})$ = apparent "blue" magnitude of a star as seen in the sky.

$\mathrm{B - V = M_B - M_V}$ = a difference between "visual" and "blue" magnitudes; it is called a "color index", and it is a measure of the color. i.e. of the shape of the stellar spectrum between 4×10^{-5} and 5×10^{-5} cm (4000 to 5000 Å). Very hot stars are blue, with $\mathrm{B - V} \approx -0.3$, whereas very cool stars are red and have $\mathrm{B - V} \approx +1.5$. In general, color index is a good indicator of the temperature of the stellar "surface", or photosphere.

UBVRI photometric system

	U	B	V	R	I
Effective wavelength [$\AA$]	3600	4400	5500	7100	9700
Flux for m= 0 [10^{-9}erg cm^{-2} s^{-1}Å^{-1}]	4.22	6.40	3.75	1.75	0.84

Temperature, spectra, and related concepts

Temperature is measured in Kelvins (K) . A unit area of a black body radiates a "flux" of energy given as:

$$F = \sigma T^4,$$

where $\sigma = 5.67 \times 10^{-5}$ erg s^{-1} cm^{-2} deg^{-4} is the Stefan-Boltzman constant. The flux of energy is measured in [erg s^{-1} cm^{-2}],

A star with a radius R and luminosity L has an "effective" temperature T_{eff} defined with the relation:

$$L = 4\pi R^2 \sigma T_{eff}^4.$$

The sun has $T_{eff,\odot} = 5.8 \times 10^3$K .

The coolest hydrogen-burning stars have $T_{eff} \approx 2 \times 10^3$K .

The hottest main sequence stars have $T_{eff} \approx 5 \times 10^4$K .

The hottest white dwarfs have $T_{eff} \approx 3 \times 10^5$K .

The hottest neutron stars have $T_{eff} \approx 3 \times 10^7$K .

The properties of stellar spectra gave rise to spectral classification, with spectra classified as O, B, A, F, G, K, M, with subclasses A0-A9 etc, e.g. A8, A9, F0, F1, F2. The exception is K stars, whose spectral subtypes are K0-K7. The following table gives approximate values of effective temperatures, bolometric corrections, and color indices of stars of various spectral types:

Spectral type	T_{eff}	BC	$B - V = M_B - M_V$
O5	40,000	-4.0	-0.35
B0	28,000	-2.8	-0.31
B5	15,500	-1.5	-0.16
A0	9,900	-0.4	0.00
A5	8,500	-0.12	+0.13
F0	7,400	-0.06	+0.27
F5	6,600	0.00	+0.42
G0	6,000	-0.03	+0.58
G5	5,500	-0.07	+0.70
K0	4,900	-0.2	+0.89
K5	4,100	-0.6	+1.18
M0	3,500	-1.2	+1.45
M5	2,800	-2.3	+1.63

Hertzsprung - Russell diagram

The original **Hertzsprung - Russell diagram** had spectral type of stars along the horizontal axis and absolute visual magnitude along the vertical axis, arranged so that bright stars were at the top, faint at the bottom, hot (blue) to the left, and cool (red) to the right.

It is more common now to use instead a **color - magnitude** diagram, with the $B - V$ color index along the horizontal axis, and either V or M_V along the vertical axis. This is the observer's diagram.

Theoreticians prefer to use a $\log T_{eff} - \log L$ diagram, with the logarithm of effective temperature plotted horizontally, and the logarithm of luminosity plotted vertically. In all these diagrams temperature increases to the left and the luminosity increases upwards.

Physical and Astronomical Constants

$c = 2.99792 \times 10^{10}$ cm s^{-1}
$G = 6.673 \times 10^{-8}$ dyne cm^2 gm^{-2}
$h = 6.626 \times 10^{-27}$ erg s
$e = 4.803 \times 10^{-10}$ esu
$m_e = 9.109 \times 10^{-28}$ gm
$m_p = 1.67 \times 10^{-24}$ gm
$k = 1.3806 \times 10^{-16}$erg K^{-1}
$\sigma = 5.67 \times 10^{-5}$ erg cm^{-2} K^{-4} s^{-1}
1 eV $= 1.602 \times 10^{-12}$ erg
1 A.U. $= 1.496 \times 10^{13}$ cm
1 pc $= 3.086 \times 10^{18}$ cm
1 $M_\odot = 1.989 \times 10^{33}$ gm
1 $R_\odot = 6.96 \times 10^{10}$ cm
1 $L_\odot = 3.826 \times 10^{33}$ erg s^{-1}
1 Jy(Jansky) $=$ 10^{-23} erg s^{-1} cm^{-2} Hz^{-1}
1 D(Debye) $=$ 10^{-18}esu.cm(statcoulomb)

DISTANCES, VOLUMES, TIME SCALES IN COSMOLOGY

Including Lambda CDM, but not yet quintessence…

A quick reference guide
Third revised Version, June 2005
(updated May 2010)

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Preface

Many years ago I made a compendium of cosmological formulas, since it was my feeling that I often and repeatedly wasted time in re-deriving distances, volume elements, look-back times, etc. in different model universes, while it was difficult to find a source in the then existing literature that presented such formulas in a satisfactory way. Some of the papers of Sandage in the early sixties came close, but were not complete (see for example Sandage 1961a and Sandage 1961b).

Although the collection of formulas was for personal use only and, to be honest, the very first version was actually more of an exercise for word processors (not yet on PCs but a kind of type-writers with a bit of memory) subsequent technological and software developments (for example LaTex, MSWord) made copying easy. I distributed an early version of the compendium among some colleagues of the Observatory and the Radio Astronomy Institute in Bologna.

A second version was made (in Microsoft Word and html) around 1996, which included now a discussion of flat models with non-zero cosmological constant. I thought that this would be fine for a long time, but new observations and theoretical work have provoked a rapid change in our view of the structure of the Universe. The main causes are certain observations of supernovae, which definitely suggest a re-acceleration of the expansion of the universe, and the incredibly fast evolving field of the CMB. Some other theoretical and observational developments are important as well. All evidence now points to the existence of components other than the "normal" baryonic material: the presence of dark matter is well established, but in particular the power spectrum of the CMB fluctuations can be reproduced by introducing yet another component, the Dark Energy (that is a non-zero cosmological constant, or alternatively a rather mysterious energy component exerting a negative pressure). This latter constituent also goes under the name of Quintessence; a modern, very extensive, discussion can be found in Peebles (2004). The equation of state of such a component is $p=w\rho$, with w negative. This kind of model is much liked by theoreticians, because a cosmological constant (i.e. truly constant) poses some serious problems of interpretation; and in particular there is an enormous discrepancy between value of the vacuum energy (thought to be the source of the cosmological constant) and the value measured for Lambda.
I rearranged this compendium a bit, but did not yet include the Quintessence models (lack of time). I limited the discussion to those models that have a flat space ($k=0$), although I decided to retain a selection of other models that are of interest for historical reasons only.

It is always a bit annoying -at least to me it is- if you have to use formulas without knowing where they come from. For that reason I first give a discussion of the derivation of the formulas, starting from the Robertson-Walker metric and the Einstein equations. It may indeed be useful to have a quick reference guide, where you can, for example, look up where this or that 1+z factor comes from.

The compendium itself is given in the second part (the appendices), and it is ordered according to the different cosmological models.

Obviously I do not at all pretend to replace a textbook or a lecture course on cosmology; what I do hope is that the compendium can provide a quick and easy way to find the right formula. As such it is intended to be a complement to the usual texts on cosmology, which normally deal with more important and fundamental matters like general relativity.

It would be very surprising if there were no errors, typographical or other, in the following list of equations. Please feel free to send me comments via E-mail (h.deruiter@ira.inaf.it); you can find this compendium on the WEB: see my Home Page, at:

http://www.ira.inaf.it/~deruiter.

You may find this compendium as a file in pdf format, which you can, if you so wish, freely download.

This version was written and completed in June 2005; note that my WEB and E-mail addresses have changed.

1 Introduction

An observational extra-galactic astronomer always needs to have at hand some cosmological tools, for example for the conversion of observed parameters like apparent magnitudes, fluxes, angular diameters to the corresponding intrinsic parameters. Obviously to do that you will have to adopt a particular cosmological model. Often you will see in an astronomical paper some remark like ``we used a model with H_0 = 50 km s^{-1} Mpc^{-1} and $q_0=1/2$''. In recent years non-standard models (that is with non-zero cosmological constant) have become very popular, in the first place because flat models with positive Λ are a natural consequence of the inflationary-universe theory, and second, because there are now also good observational reasons why such models are feasible. This makes it al the more important to know how to compute distances, look-back times.

This article is organized as follows. First, in this chapter I give some general background information on the various parameters that are relevant for our problem, and use as a starting point (1) the Robertson-Walker metric and (2) the Einstein equations. I decided these are good starting points: no discussion is given of their derivation (I would have to rewrite a textbook on gravitational cosmology, which is far beyond my capacity), nor do I discuss other than homogeneous and isotropic models.
Specific models are then described in Chapters 3 (the standard Friedmann model), 4 (flat models) and 5 (other models).
These chapters make this guide longer than it could have been, but they can be useful for retracing the origin of a formula.
A list of symbols used is given in Appendix A. The compendium of formulas (the ultimate reason for the existence of this guide) is given in Appendices B, C, and D.

1.1 The metric of Space-Time

I only use the metric of a homogeneous and isotropic space-time, often called the Robertson-Walker metric:

$$\boxed{ds^2 = c^2 dt^2 - R^2(t)\left\{\frac{dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2}{\left(1+\frac{kr^2}{4}\right)^2}\right\}}$$

The variables r, θ, ϕ are *co-moving* coordinates; this means that the expansion of the universe is represented by $R=R(t)$. The co-moving coordinates (r, θ, ϕ) do not depend on time and therefore an object will have fixed values of its co-moving coordinates.
The parameter k can be negative, zero, or positive, but without loss of generality we can assign it the possible values -1, 0, or +1. If $k=-1$ space is negatively curved, while $k=+1$ corresponds to a positive curvature; for $k=0$ space is flat.

There is another, more useful, form of the metric, obtained by changing the coordinate r to ω, according to:

$$\sin\omega = \frac{r}{1+\frac{r^2}{4}}$$

For k=+1, ω=r if k=0, while for k=–1, we set

$$\sinh\omega = \frac{r}{1-\frac{r^2}{4}}$$

This can be written more compactly as:

$$\frac{\sin\sqrt{k}\omega}{\sqrt{k}} = \frac{r}{1+\frac{kr^2}{4}}$$

Remembering that by definition sin(ix)/i = sinh x, and cos(ix) = cosh x. Moreover we can take the limit $k \to 0$ in order to find the correct equations for the case k=0. Using this compact form we can now write the equation for the metric as:

$$ds^2 = c^2dt^2 - R^2(t)\{d\omega^2 + \left(\frac{\sin\sqrt{k}\omega}{\sqrt{k}}\right)^2 (d\theta^2 + \sin^2\theta d\phi^2)\}$$

This is the form we will always use in the following.

1.2 The redshift

From the metric given in Chapter 2.1 we immediately can find an expression for the redshift as a function of R. Write R_0 for the value of the scale parameter at the present epoch and R_1 for the value it had at the time of emission of a photon. Then the photon will be redshifted by

$$1+z = \frac{R_0}{R_1}$$

Proof: for a photon ds=0, and since ω of the emitting source is fixed, while we can choose θ=φ=0, then:

$$\omega = c\int_{t_1}^{t_0} \frac{dt}{R}$$

where t_1 and t_0 are respectively the times of emission and reception of the beginning of the wave packet. For the end of the wave packet we get:

$$\boxed{\omega = c \int_{t_1+\Delta t_1}^{t_0+\Delta t_0} \frac{dt}{R}}$$

and $\Delta t_0/\Delta t_1 = R_0/R_1$. But Δt is related to the frequency of the photon ($\nu \propto 1/\Delta t$), and it follows that:

$$\boxed{1+z = \frac{\nu_{em}}{\nu_{obs}} = \frac{R_0}{R_1}}$$

1.3 Distances

It is up to us to define a coordinate distance, and good candidates would be r, or $\sin(\sqrt{k}\omega/\sqrt{k})$. Of course we would prefer a coordinate distance that has a close relation with the observations. A distance that makes sense is the luminosity distance, which is defined from $S=P/4\pi D^2$ (S is flux, P luminosity), and therefore we search for an expression for D in terms of a coordinate distance. A detailed discussion can be found in McVittie (1965), p. 163--165.
In the usual definition of flux (energy flow per unit area) we take the area of a sphere around the power source and divide the emitted power by this area. In Euclidean space the area of the sphere at distance D, where the observer is on the sphere[1], is found as:

$$\boxed{A = \int_0^{\pi}\int_0^{2\pi} D^2 \sin(\theta) d\theta d\phi = 4\pi D^2}$$

However, in the metric given in Chapter 2.1 space is not necessarily flat and we therefore must take the area of the pseudo-sphere around the source:

$$\boxed{A = \int_0^{\pi}\int_0^{2\pi} R_0^2 \left(\frac{\sin\sqrt{k}\omega}{\sqrt{k}}\right)^2 \sin\theta \cdot d\theta \cdot d\phi = 4\pi R_0^2 \left(\frac{\sin\sqrt{k}\omega}{\sqrt{k}}\right)^2}$$

where the ω factor takes account of the curvature of space. The above suggests that a relation between flux and power might be:

$$\boxed{S = \frac{P}{A} = \frac{P}{4\pi R_0^2 \sin(\sqrt{k}\omega)/\sqrt{k}}.}$$

Clearly a useful coordinate distance is $\sin\{\sqrt{(k)}\omega\}/\sqrt{(k)}$, and we call this the geometric distance r_g.

[1] We put the observer at ω and the source at zero, but in the end put back the observer at zero; this can be done because space-time is homogeneous and isotropic

We have not finished yet however, because there are two more effects we still have to take into account. Call $\varepsilon_{em} = h\nu_{em}$ the energy of an emitted photon. In a time interval Δt_{em} there are n photons emitted, so that the emitted power is $P_{em} = \varepsilon_{em} n/\Delta t_{em}$.
First effect (the energy effect): the wavelength of a photon is redshifted, or $\nu_{obs}=\nu_{em}/(1+z)$, so that $\varepsilon_{obs} = \varepsilon_{em}/(1+z)$.
Second effect (the number effect): photons will arrive at a slower rate. At the source n photons were counted in the interval Δt_{em}; but $\Delta t_{em} = \Delta t_{obs}/(1+z)$, so that the same photons are observed to arrive in an $(1+z)$ times longer interval.
Taking the two effects together and calling the ``observed'' power (from flux and geometric distance) P_{obs}, we have:

$$\boxed{P_{obs} = \varepsilon_{obs} \times \frac{n}{\Delta t_{obs}} = P_{em}/(1+z)^2.}$$

Finally:

$$\boxed{S = \frac{P_{em}}{4\pi(1+z)^2 R_o^2 r_g^2}.}$$

We can introduce here the luminosity distance D as:

$$\boxed{D = (1+z)R_0 r_g}$$

The derivation given above concerns bolometric fluxes and powers; if we observe in a limited frequency band we also have to take into account that the observed and emitted bandwidths are different by a factor $(1+z)$, and that the flux at frequency ν refers to the power emitted at $\nu(1+z)$.

1.4 Angular Size

The usual formula for angular size $\Delta\theta$ is: $\Delta\theta = L/d_\theta$, where L is the linear size and d_θ the distance, which we now have to specify.
Take two points A and B with co-moving coordinates (ω,θ,ϕ) and $(\omega,\theta+\Delta\theta,\phi)$. A and B could be for example the two components of a double radio source. We assume that a photon is emitted from A, and another one from B at time t_1; A and B are connected with the origin $(0,0,0)$ by null geodetics. The local separation L at time t_1 can be found by putting $dt=d\omega=d\phi=0$, so that $\Delta s^2 = -L^2 = -R_1^2 r_g^2 \Delta\theta^2$ and:

$$\Delta\theta = \frac{(1+z)L}{R_0 r_g}.$$

We can make the identification $d_\theta = R_0 r_g/(1+z)$ and, if we really want, use the standard formulas $S=P/4\pi D^2$ and $\Delta\theta = L/d_\theta$, remembering that:

$$R_0 r_g = (1+z)d_\theta = \frac{D}{(1+z)}$$

$$d_\theta = \frac{D}{(1+z)^2}$$

1.5 The Volume

The co-moving coordinates of an object are, by definition, fixed. Consequently the number of objects (if they are not created or destroyed) per co-moving volume remains constant and therefore this type of volume is the relevant one for computing e.g. a luminosity function. From the metric in Chapter 2.1 we have:

$$dV(\omega,\theta,\varphi) = R_0 d\omega R_0 \frac{\sin\sqrt{k}\omega}{\sqrt{k}} d\theta R_0 \frac{\sin\sqrt{k}\omega}{\sqrt{k}} \sin\theta d\phi$$

Integrating over ω, θ and ϕ we find the volume out to coordinate distance ω:

$$V(\omega) = 4\pi R_0^3 \int_0^\omega \left(\frac{\sin\sqrt{k}\omega}{\sqrt{k}}\right)^2 d\omega$$

and we can make the change $V(\omega)$ to $V(z)$ if we know $\omega = \omega(z)$; the differential volume in a redshift shell is then simply $(dV/dz)dz$.

1.6 How to compute ω

Let a photon be emitted at time t_1 and be received at t_0 and let it have fixed coordinates θ and ϕ. Since $ds=0$ we find

$$\omega = c\int_{t_1}^{t_0} \frac{dt}{R(t)}$$

where $\omega(t_0) = 0$.

If we can find, at the right-hand side, an expression in terms of H_0, q_0, z then we have solved $\omega = \omega(H_0, q_0, z)$ and distances, volumes, etc. are known in terms of H_0, q_0 and z.

1.7 The scale parameter R

We have already seen that in general $R_0/R_1=(1+z)$. The solution $R=R(t)$ is found from the Einstein equations which I give here without a derivation:

$$8\pi G\rho = \frac{3kc^2}{R^2} + \frac{3\dot{R}^2}{R^2} - \Lambda$$

$$8\pi G\frac{p}{c^2} = -\frac{2\ddot{R}}{R} - \frac{\dot{R}^2}{R^2} - \frac{kc^2}{R^2} + \Lambda$$

It follows directly that:

$$\frac{d}{dt}(\rho R^3) + \frac{p}{c^2}\frac{dR^3}{dt} = 0$$

where ρ is the density, p the pressure, and Λ the cosmological constant. The Einstein equations are obtained by equating the curvature tensor to the energy-momentum tensor.

In the most general case the equations are very difficult to solve because we need to know ρ and p (and thus an equation of state), as well as Λ. Although models with non-zero pressure have been calculated (for example assuming a polytropic relation $p \propto \rho^{\gamma}$), one normally takes $p=0$, so that the Einstein equations become:

$$8\pi G\rho = \frac{3kc^2}{R^2} + \frac{3\dot{R}^2}{R^2} - \Lambda$$

Also:

$$\frac{d}{dt}(\rho R^3) = \frac{d}{dR}(\rho R^3) = 0$$

Important parameters are the Hubble parameter and the deceleration parameter:

$$H = \frac{\dot{R}}{R}$$

$$q = -\frac{\ddot{R}R}{\dot{R}^2}$$

1.8 Time scales

Call the present cosmic time t_0; it can be found by measuring H_0, i.e. $t_0=t_0(H_0)$ the functional relation being different for different models. $(H_0)^{-1}$ often called the Hubble time, but it should be kept in mind that in Friedmann models $(\Lambda=0)$ always $t_0 \leq (H_0)^{-1}$.
Another important measure of time is the look-back time τ, defined as

$$\boxed{\tau = 1 - \frac{t_1}{t_0}}$$

We see that $t_1=t_0 \rightarrow \tau=0$, and $t_1=0 \rightarrow \tau=1$. The name look-back time is obvious.

2 The standard (Friedmann) model: $\Lambda=0$

2.1 General characteristics

The Friedmann models are the most important cosmological models and almost exclusively used in observational astronomy. Since $\Lambda=0$, we start from the simplified Einstein equations:

$$\boxed{\frac{8\pi G\rho}{3} = \frac{kc^2}{R^2} + \frac{\dot{R}^2}{R^2}}$$

$$\boxed{2\frac{\ddot{R}}{R} + \frac{\dot{R}^2}{R^2} + \frac{kc^2}{R^2} = 0}$$

As can be seen from above, the first equation states that $\Lambda=0$ implies $q=\sigma$, and therefore the Einstein equations reduce to one (the second). Note that the Friedmann models always have $q\geq 0$, because of course or σ or ρ are always ≥ 0[2].
The second Einstein equation provides a relation between R_0, H_0, q_0:

$$\boxed{R_0 = \sqrt{\frac{k}{2q_0 - 1}}\frac{c}{H_0}}$$

for $k=\pm 1$. It follows that

- $k = +1 \rightarrow q_0 > \frac{1}{2}$
- $k = 0 \rightarrow q_0 = \frac{1}{2}$
- $k = -1 \rightarrow 0 \leq q_0 < \frac{1}{2}$

2.2 The solution of R and t in parametric form

The solution of the Friedmann-Einstein equation is well known from equivalent differential equations used in mechanics. Introducing the ``development'' angle ψ we write $R=R(\psi)$ and $t=t(\psi)$.
Remembering that:

$$\boxed{\dot{R} = \frac{dR}{d\psi}\left(\frac{dt}{d\psi}\right)^{-1}}$$

[2] But negative pressure now has become a definite possibility, since the Quintessence models have been proposed. I should get around discussing these too.

$$\ddot{R} = \left(\frac{dt}{d\psi}\right)^{-2}\left(\frac{d^2R}{d\psi^2} - \frac{dR/d\psi}{dt/d\psi}\frac{d^2t}{d\psi^2}\right)$$

Taking $dt/d\psi = R/c$ we get:

$$-\left(\frac{dR}{d\psi}\right)^2 - 2\frac{d^2R}{d\psi^2}R + kR^2 = 0$$

The solution is:

$$R = \frac{a}{k}(1 - \cos\sqrt{k}\psi).$$

Since $t = \int(R/c)d\psi$:

$$t = \frac{a}{kc}\left(\psi - \frac{\sin\sqrt{k}\psi}{\sqrt{k}}\right).$$

The solutions are valid for all k:

- For $k = -1$ we use $\cos(ix) = \cosh(x)$ and $\sin(ix)/i = \sinh(x)$, then:
  - $R(k=-1) = a\{\cosh(\psi)-1\}$,
  - $t(k=-1) = (a/c)\{\sinh(\psi)-\psi\}$
- For $k = +1$ we have:
  - $R(k=+1) = a\{1-\cos(\psi)\}$
  - $t(k=+1) = (a/c)\{\psi-\sin(\psi)\}$
- For $k = 0$ we first write $\cos(\sqrt{k}\psi) = 1 - k\psi^2/2$ and $\sin(\sqrt{k}\psi)/\sqrt{k} = \psi + k\psi^3/6$, since higher terms in k will be zero when taking the limit $k \to 0$. Thus:
  - $R(k=0) = a\psi^2/2$
  - $t(k=0) = a\psi^3/(6c)$

  In this case, however, the scale parameter R cannot be tied directly to (c/H_0); in fact we might take anything we like. Eliminating ψ we find

$$R(k=0) = \frac{6^{2/3}}{2}a^{1/3}c^{2/3}t^{2/3} \propto t^{2/3}$$

As a next step we eliminate a, by using $H_0 = (2/3)t_0$, and imposing that $R_0 = c/H_0$. We find that $a = (3/4)ct_0$, so we can finally write:

$$R(k=0) = \left(\frac{c}{H_0}\right)\left(\frac{t}{t_0}\right)^{2/3}$$

2.3 The Hubble and deceleration parameters

Using the definitions of H and q given in Chapter 1 the above solutions lead to:

$$\boxed{H = \frac{c}{R^2}\frac{dR}{d\psi} = \frac{k^2 c}{a}\frac{\sin(\sqrt{k}\psi)/\sqrt{k}}{(1-\cos\sqrt{k}\psi)^2}}$$

After some more manipulation:

$$\boxed{q = \frac{1}{1+\cos\sqrt{k}\psi} \Leftrightarrow \cos\sqrt{k}\psi = \frac{1-q}{q}}$$

The explicit relations for $k=-1$ and $k=+1$ are obvious and will not be repeated here. For $k=0$ we can go back to the relation between R and t given in Chapter 2.2, or take the appropriate limits $k\to 0$ in the formulas given here. Both methods give: $H(k=0)=(2/3)t^{2/3}$ and $q(k=0)\equiv 1/2$.

2.4 Relation between ω and ψ

There is a very simple relation between the radial co-moving coordinate ω and the development angle ψ. Since $c/R=d\psi/dt$ (see Chapter 2.2), we can write:

$$\boxed{\omega = c\int_{t_1}^{t_0}\frac{dt}{R} = \int_{\psi_1}^{\psi_0} d\psi}$$

so that:

$$\boxed{\omega = \psi_0 - \psi_1}$$

Therefore ω and ψ are the same thing, up to a different zero-point: ω is measured with respect to the observer (at $\omega=0$), while ψ gives an absolute scale starting at $R=0$ with $\psi=0$.

2.5 Expressing ψ in the observables q0 and z

Since $\cos(\sqrt{k}\psi)=(1-q)/q$, ψ can be expressed in terms of R; using $R_1=R_0/(1+z)$, we can write:

$$\boxed{\cos\sqrt{k}\psi_1 = \frac{\cos\sqrt{k}\psi_0 + z}{1+z}}$$

or, for $q_0\neq 1/2$:

$$\cos\sqrt{k}\psi_1 = \frac{1-q_0+q_0 z}{q_0(1+z)}$$

With these formulas we have solved all our problems: ω is a difference between the development angles at t_1 and t_0, and therefore can be expressed in H_0, q_0, z. As a consequence this can also be done with the geometric distance, volume, etc. (see next sections), taking care that for $k = 0$ we take the appropriate limits only at the end of our calculations. The characteristic that all relevant cosmological quantities can be directly expressed in terms of observables is a very pleasant property of Friedmann models (and a few other non-standard models). It is not true in the general case $\Lambda \neq 0$.

2.6 The geometric distance

Going back to the original definition for r_g $(=\sin(\sqrt{k}\omega)/\sqrt{k})$ we can write:

$$r_g = \frac{\sin\sqrt{k}(\psi_0-\psi_1)}{\sqrt{k}} = \frac{\sin\sqrt{k}\psi_0\cos\sqrt{k}\psi_1 - \cos\sqrt{k}\psi_0\sin\sqrt{k}\psi_1}{\sqrt{k}}$$

With the knowledge of Chapter 2.5 this can be transformed into:

$$r_g = \left(\frac{2q_0-1}{k}\right)^{1/2} \frac{q_0 z+1-q_0+(q_0-1)(2q_0 z+1)^{1/2}}{q_0^2(1+z)}$$

but since $\sqrt{\{(2q_0-1)/k\}}=c/(R_0H_0)$:

$$r_g = \frac{c}{R_0 H_0}\frac{q_0 z+1-q_0+(q_0-1)(2q_0 z+1)^{1/2}}{q_0^2(1+z)}$$

This is the famous Mattig formula (Mattig 1958). Surprisingly enough, Mattig derived it many years after the original work of Friedmann in the early 1920's, and up till then one had to use cumbersome expansions in powers of z. It illustrates how the practical use of cosmology has often lagged behind the theory[3].

Although the limit for $q_0 \to 0$ exists, Mattig's formula is not very nice in case of very small q_0, because both numerator and denominator go to infinity, as q_0 tends to zero. Terrell (1977) obtained an alternative (and better) form of Mattig's formula (see also Peterson 1997). First we note that the luminosity distance D $(= R_0 r_g(1+z))$ is the difference two numbers:
$A = (1-q_0+q_0z)/q_0^2$ and $B = (1-q_0)(2q_0z+1)^{1/2}/q_0^2$.

[3] Mattig's formula was derived without any reference to particular values of k, or q_0, and indeed, it is valid for all k and all $q_0 \geq 0$. For $q_0 = 0$ we can take the limit by expanding the square-root term in powers of q_0z up to the second order. Zero- and first order terms in q_0 in the numerator cancel exactly, so the limit exists.

We want to write $A-B \propto z(1+C_z)$: since for $q_0=0$ we know that $D=(c/H_0)\,z(1+z/2)$ the latter form should be well behaved. Indeed, solving C_z in terms of A and B and writing $Q=(1+2q_0z)^{1/2}$ we get:

$$\boxed{D=\frac{cz}{H_0}\left\{1+\frac{1-q_0}{q_0^2 z}(1+q_0 z-Q)\right\}}$$

Multiplying the last term with $(1+q_0z+Q)/(1+q_0z+Q)$ we finally arrive at Terrell's alternative formula:

$$\boxed{D=\frac{cz}{H_0}\left\{1+\frac{z(1-q_0)}{1+q_0 z+(1+2q_0 z)^{1/2}}\right\}}$$

2.7 The co-moving volume

The general formula for the volume can be integrated immediately:

$$\boxed{V(\omega)=\frac{2\pi R_0^3}{k}\left(\omega-\frac{\sin 2\sqrt{k}\,\omega}{2\sqrt{k}}\right)}$$

We could leave it at that, since ω is known in terms of (H_0, q_0, z). Although even in the general case we could write the volume explicitly as a function of the observables, I think it is not worth the trouble of doing this (the formulas would be cumbersome): with present day computers (including pocket calculators), it is much easier to calculate ω first (using the formulas in Chapters 2.4 and 2.5) and then apply the above equation. However there are some special values of q_0 where the volume can be easily written in terms of redshift directly. We deal with these cases below.

- $q_0=0$ (the Milne model). Since $\sinh\omega = (z+z^2/2)/(1+z)$ and $\operatorname{arsinh}(x) = \ln\{\sqrt{(x^2+1)}+x\}$:

  $$\boxed{\omega=\ln(1+z)}$$

  Then, after some more manipulation [using $(1/2)\sinh\omega=\sinh\omega\cosh\omega$]:

  $$\boxed{V(z)=2\pi\left(\frac{c}{H_0}\right)^3\left\{\frac{1}{4}\frac{(1+z)^4-1}{(1+z)^2}-\ln(1+z)\right\}}$$

  $$\boxed{\frac{dV}{dz}=4\pi\left(\frac{c}{H_0}\right)^3\frac{(1+\frac{1}{2}z^2)^2}{(1+z)^3}}$$

- $q_0=1/2$ and $k=0$ (Einstein-de Sitter model). The volume in terms of ω is simply:

$$V(\omega)=\frac{4}{3}\pi R_0^3\omega^3$$

Using the fact that Mattig's formula gives

$$r_g=\omega=2\left\{1-(1+z)^{-1/2}\right\}$$

We have:

$$V(z)=\frac{32}{3}\pi\left(\frac{c}{H_0}\right)^3\left\{1-\frac{1}{(1+z)^{1/2}}\right\}^3$$

$$\frac{dV}{dz}=16\pi\left(\frac{c}{H_0}\right)^3\frac{\left\{(1+z)^{1/2}-1\right\}^2}{(1+z)^{5/2}}$$

- $q_0=1$. Using $\sin(2\omega)=2\sin\omega\,(1-\sin^2\omega)^{1/2}$:

$$V(z)=2\pi\left(\frac{c}{H_0}\right)^3\left\{\arcsin\left(\frac{z}{1+z}\right)-\frac{z(1+2z)^{1/2}}{(1+z)^2}\right\}$$

$$\frac{dV}{dz}=4\pi\left(\frac{c}{H_0}\right)^3\frac{z^2}{(1+z)^3(1+2z)^{1/2}}$$

In all other cases we use ω and $V(\omega)$.

2.8 Time Scales

2.8.1 The look-back time τ

By definition

$$\tau=\frac{t_0-t_1}{t_0}$$

It is normalized such that $t_1=t_0\rightarrow\tau=0$ and $t_1=0\rightarrow\tau=1$. From the solution of t in terms of ψ (Chapter 2.2) τ can be written as

Commento [HDR1]:

$$\tau=1-\frac{\psi_1-\sin\sqrt{k}\psi_1/\sqrt{k}}{\psi_0-\sin\sqrt{k}\psi_0/\sqrt{k}}$$

As for the volume the easiest way to proceed is to find ψ_0 and ψ_1 in terms of the observables H_0, q_0, z. Below I give the explicit expressions for τ as a function of the observables, for the cases $q_0=0, 1/2, 1$.

- $q_0 = 0$. Taking the limit $q_0 \to 0$ the above equation yields:

$$\tau = \frac{z}{1+z}$$

A much easier method, which obviously gives the same result: $q=0$ implies $d^2R/dt^2=0$, so that $R=ct$ and consequently:

$$\frac{R_0}{R_1} = \frac{t_0}{t_1} = 1+z$$

This leads immediately to the expression for τ.

- $q_0 = ½$. From the equation given in Chapter 2.2:

$$\tau = 1 - \left(\frac{\psi_1}{\psi_0}\right)^3$$

and therefore:

$$\tau = 1 - \frac{1}{(1+z)^{3/2}}$$

- $q_0 = +1$. Then $\cos\psi_0 = (1-q_0)/q_0 = 0$, or $\psi_0 = \pi/2$. Also $\cos\psi_1 = (\cos\psi_0+z)/(1+z)=z/(1+z)$, so that:

$$\tau = 1 - \frac{\arccos\left(\frac{z}{1+z}\right) - \left\{1-\left(\frac{z}{1+z}\right)^2\right\}^{1/2}}{\frac{\pi}{2}-1}$$

2.8.2 The age of the Universe (t_0) expressed in H_0 and q_0.

Taking together the expressions for t and H in terms of ψ (Chapters 2.2 and 2.3) and we get:

$$t_0 H_0 = \left(\frac{q_0}{k}\right)\left(\frac{k}{2q_0-1}\right)^{3/2}(\psi_0 - \sin\sqrt{k}\psi_0/\sqrt{k})$$

- $q_0=0$. Then clearly, since $R_0=ct_0$, and $R_0=c/H_0$

$$\boxed{t_0 H_0 = 1}$$

This can, as before, also be derived by taking the limit $q_0 \to 0$ in the general formula.

- $q_0=1/2$. Going back to Chapter 2.2 we directly have:

$$\boxed{t_0 H_0 = 2/3}$$

- $q_0=1$. Applying the general formula with $k=1$ and $q_0=1$:

$$\boxed{t_0 H_0 = \frac{\pi}{2} - 1}$$

With these formulas the description of the Friedmann models is complete.

3 Flat Models (k=0, $\Lambda \neq 0$)

3.1 Introductory Remarks

With the exception of the Einstein-de Sitter model, flat models ($k=0$) have never attracted much attention. The reasons are that a zero cosmological constant is appealing from a philosophical point of view, and anyway there has never been any compelling observational reason why we should introduce still another parameter.
Nevertheless every now and then non-standard models become popular with cosmologists. Usually this popularity quickly fades away, but in one occasion (the most recent one) a non-Friedmann model appears to have a good chance to survive for a long time. The introduction of the ``inflationary universe'' is the cause of renewed interest for a model with $\Lambda \neq 0$: according to the theory of inflation the cosmological constant might very well be positive: it is related to the properties of the vacuum, see Börner (1988). Also, the metric of space would be flat.

The matter is very complicated, because it is also claimed that very stringent observational limits can be set upon Λ, but a discussion of these points is obviously far beyond the scope of the present work. What makes the flat models with a positive cosmological constant so important is that there is now (from about the year 2000) serious observational evidence that this type of model may actually the correct description of the universe. Therefore I give the general equations for flat models, with particular emphasis on the ones with $\Lambda>0$; also I write the equations for a model with $\Omega_M = 1/3$ and $\Omega_\Lambda = 2/3$ (see for the definition below), since this is the model that is presently (2002) favoured by many cosmologists.

It is worth wile to look a bit closer at the Einstein equations, by writing them in terms of present day Hubble parameter, deceleration parameter and density parameter (defined as $\sigma_0 = 4\pi G\rho_0/(3H_0^2)$); then:

$$\Lambda = 3H_0^2(\sigma_0 - q_0)$$

$$k\left(\frac{c}{R_0 H_0}\right)^2 = 3\sigma_0 - q_0 - 1$$

It is appropriate to introduce at this point two often used parameters that replace in the modern literature σ_0 and Λ; they are $\Omega_M = 2\sigma_0$ and $\Omega_\Lambda = \Lambda/3H_0^2$. It is easy to see from the above equations that $\Omega_M + \Omega_\Lambda = 1$, for $k = 0$. The ratio of these two parameters will be used extensively in this chapter, and is denoted by $A = \Omega_M/\Omega_\Lambda$.

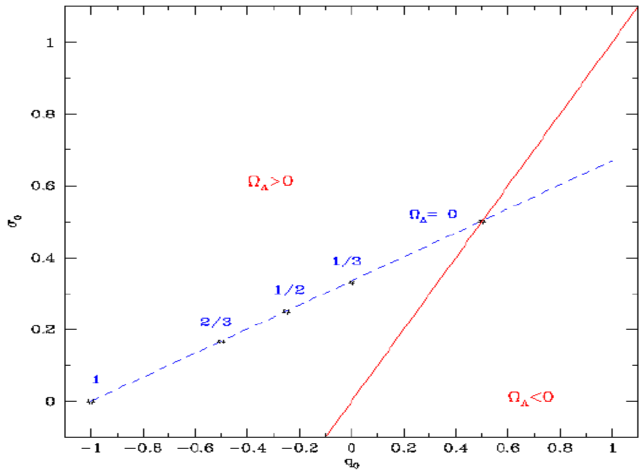

Figure 1 The q0-σ0 plane. The lines representing Friedmann models and flat models are drawn.

For a positive cosmological constant the value of *A* will be between zero (a pure deSitter model, see Chapter 4.2) and infinity (a pure Einstein-deSitter model, see Chapter 2).

In fig. 1 we show the q_0-σ_0 plane. The parts of the plane with $\Omega_M>0$ and $\Omega_\Lambda<0$ are indicated. We can recognize various distinct regions: the two lines $\sigma_0=q_0$ ($\Lambda=0$) and $3\sigma_0=q_0+1$ ($k=0$) give two special families of (relatively simple) solutions, discussed in detail in Chapter 2 and in this Chapter. More general solutions will only be described briefly (and limited to a few special cases), (see Chapter 4) not only because they can be complicated, but also because they are, with few exceptions, not very interesting. I looked a bit in more detail at models with $k=0$ and $\Lambda\neq0$. I will first show that a general solution exists for all flat models, regardless of the value of Λ, somewhat similar to the general solution for Friedmann models.

3.2 The general solution

As a starting point we use the Einstein equations with $k=0$:

$$\boxed{\frac{8\pi G\rho_0 R_0^3}{3R^3}=\frac{\dot{R}^2}{R^2}-\frac{\Lambda}{3}}$$

$$2\frac{\ddot{R}}{R}+\frac{\dot{R}^2}{R^2}=\Lambda$$

It can be easily verified that the solution of this system of two equations is:

$$R=\left(\varepsilon A^*\right)^{1/3}R_0\left(\frac{\sin\gamma^* t}{\sqrt{\varepsilon}}\right)^{2/3}$$

If $\Lambda>0 \rightarrow \varepsilon=-1$, if $\Lambda<0 \rightarrow \varepsilon=+1$; also:

$$A^*=\frac{8\pi G\rho_0}{-\Lambda}$$

$$\gamma^*=\frac{1}{2}\sqrt{-3\Lambda}$$

Going back to Chapter 3.1 we can see that:

- if $\Lambda < 0$: $\Lambda/3H^2=1-2\sigma$, so that $\sigma>1/2$, while $A^*=2\sigma/(2\sigma-1)>1$.
- if $\Lambda>0$: $\sigma<1/2$, and consequently $-A^*=2\sigma/(1-2\sigma)>0$.

We can in principle write general solutions for H, q, τ, etc., however, only the case $\Lambda>0$ is of interest and will be discussed in full below: for σ, q $>1/2$ the cosmological constant is an extra attractive force, and its effect is only to keep space flat, compared to the corresponding Friedmann model, for which $q_0>1/2$ implies a positive space curvature. Therefore from now on we will only consider $\Lambda>0$. We can now write:

$$R=A^{1/3}R_0\sinh^{2/3}(\gamma t)$$

Here we have used $A=-A^*$:

$$A=\frac{8\pi G\rho_0}{\Lambda}=\frac{\Omega_M}{\Omega_\Lambda}$$

and

$$\gamma=\frac{1}{2}\sqrt{3\Lambda}=\frac{3}{2}H_0\sqrt{\Omega_\Lambda}$$

So the general flat models with non-negative cosmological constant allow a direct solution of R in terms of t; but it can also be suspected that the rather complicated dependence (a hyperbolic sine to the two-third power) may create some problems in finding manageable expressions for geometric distance, volumes, etc. It will be

shown in the following that this is true only to some extent. Nevertheless the general properties of the models are quite simple.
A special case is the model that is nowadays favoured by many, with (roughly, the precise values may change as new observations become available): $\Omega_M=1/3$ and $\Omega_\Lambda=2/3$, so that $A = 1/2$. This model goes under the name of Concordance model. Then:

$$R = \left(\frac{1}{2}\right)^{1/3} R_0 \sinh^{2/3}\left(\sqrt{\frac{3}{2}} H_0 t\right)$$

3.3 H_0, q_0 and t_0 in terms of the parameter A

From the general solution we immediately find for $\Lambda>0$, by using $\sinh(\gamma t_0)=A^{-1/2}$ and $\cosh(\gamma t_0) = \{(1+A)/A\}^{1/2}$:

$$H_0 = \sqrt{\frac{\Lambda}{3}}(1+A)^{1/2} = \sqrt{\frac{\Lambda}{3\Omega_\Lambda}}$$

and

$$q_0 = \frac{1}{2}\left(\frac{A-2}{A+1}\right) = \frac{1}{2}(1-3\Omega_\Lambda) \Leftrightarrow A = 2\frac{1+q_0}{1-2q_0}$$

and

$$t_0 H_0 = \frac{2}{3}(1+A)^{1/2} \ln\{A^{-1/2} + (1+A^{-1})^{1/2}\} = \frac{2}{3}\Omega_\Lambda^{-1/2} \ln\left(\frac{1+\Omega_\Lambda^{1/2}}{\Omega_M^{1/2}}\right)$$

For $A=0.5$:

$$H_0 = \sqrt{\frac{\Lambda}{2}}$$

$$q_0 = -0.5$$

$$t_0 H_0 = 1.40 \times 2/3 = 0.933$$

It can be easily seen that for $A \to \infty$, that is the usual Einstein-deSitter universe, we find (as we should) $H^2 \to 8\pi G\rho_0/3$, $q0 \to 1/2$,and $t_0H_0 \to 2/3$. For $A \to 0$ we have $H_0 \to \sqrt{(\Lambda/3)}$, $q_0 \to -1$ and $t_0H_0 \to \infty$, which indeed are the equations for a deSitter universe (see Chapter 4.2). Similar formulas can be obtained for the case $\Lambda < 0$.

3.4 H, q and τ in terms of A and z

Again from the solution we can find some general expressions for H(z), q(z) and the look-back time τ(z); for $\Lambda > 0$:

$$\boxed{(1-2q)H^2 = \Lambda}$$

$$\boxed{H(z) = \sqrt{\frac{\Lambda}{3}}\{1 + A(1+z)^3\}^{1/2} = H_0\left(\frac{1 + A(1+z)^3}{1+A}\right)^{1/2}}$$

$$\boxed{1 - 2q(z) = \frac{3}{1 + A(1+z)^3}}$$

Note that there may have been a moment in the past in which q was zero; this represents a transition point in which the early stage of deceleration was changing into one of acceleration. In the model with A=0.5 this happened at a redshift z(q=0)=0.587.

For the look-back time we get:

$$\boxed{\tau = 1 - \frac{\ln[\{1 + A(1+z)^3\}^{1/2}] - \frac{3}{2}\ln(1+z) - \frac{1}{2}\ln A}{\ln\{1 + (1+A)^{1/2}\} - \frac{1}{2}\ln A}}$$

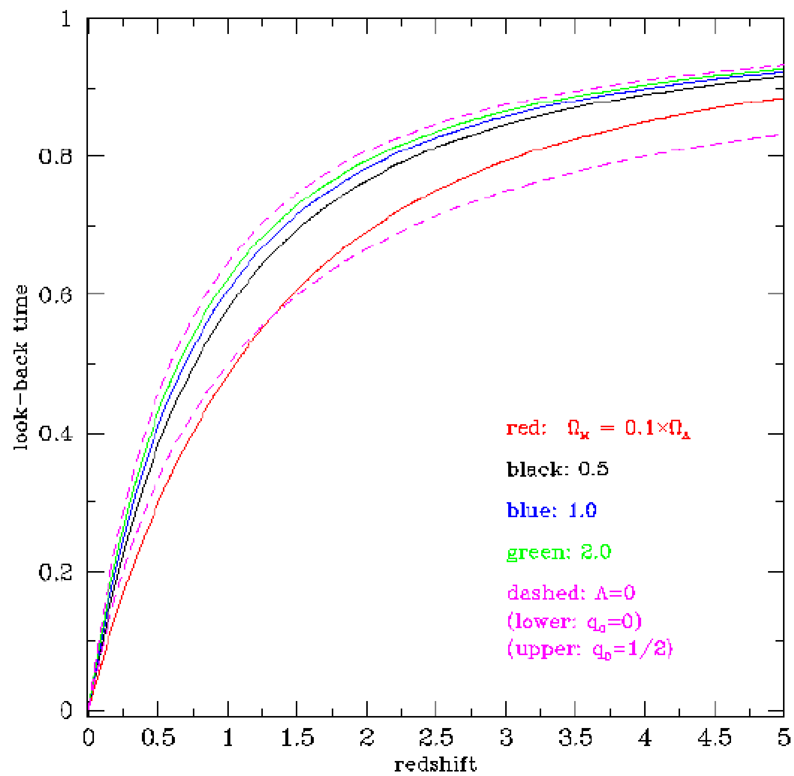

Figure 2 The look-back time for some flat models with positive cosmological constant. The full lines represent models with: A=0.1, 0.5, 1.0. The A=0.5 model is close to the Concordance model. Also shown are standard $q_0 = 0$ and $q_0 = 1/2$ models (lower and upper full lines respectively).

For $A=0.5$:

$$H(z)=H_0\left(\frac{1+\frac{1}{2}(1+z)^3}{\frac{3}{2}}\right)^{1/2}$$

$$q(z)=q_0\left(\frac{1-\frac{1}{4}(1+z)^3}{1+\frac{1}{2}(1+z)^3}\right)$$

Therefore $q(z=\infty) \to 0.5$, because $q_0 = -0.5$.
We give no separate equation for $\tau(z,A=0.5)$, as this does not clarify much. The dependence of τ on redshift for different values of A is given in fig. 2.
Similar formulas are found for the case $\Lambda < 0$. The appearance of the term $A(1+z)^3$ suggests that it is a fundamental term for flat universes. This will be clear from the following section, where we discuss the geometric distance.

3.5 The geometric distance and volume elements

By definition the geometric distance is the integral in time of the inverse of the scale parameter. In our case we find therefore:

$$r_g = c\int_{t_1}^{t_0}\frac{dt}{R} = c\int_{t_1}^{t_0}\frac{dt}{A^{1/3}R_0\sinh^{2/3}\gamma t}$$

Changing from t to z (using the fact that $(1+z)=R_0/R$), we find, after some manipulation:

$$r_g = \frac{c}{R_0 H_0}(1+A)^{1/2}\int_0^z\frac{d\zeta}{\{1+A(1+\zeta)^3\}^{1/2}}$$

It is quickly verified that for $A \to \infty$ (Einstein-De Sitter), we have

$$r_g \Rightarrow \frac{c}{R_0 H_0}2\{1-\frac{1}{(1+z)^{1/2}}\}$$

and for $A \to 0$:

$$\boxed{r_g \Rightarrow \frac{c}{R_0 H_0} z}$$

These are indeed the correct formulas. Although the general formula for the geometric distance has no analytical solution, it is simple enough that a numerical integration is straightforward. From that we arrive immediately at the solution for the volume ($V=4\pi\ R_0^3 r_g^3/3$) and the differential volume $dV/dz = 4\pi R_0^3 r_g^2 dr_g dz$. The dependence of r_g on z for different values of A is given in fig. 3.

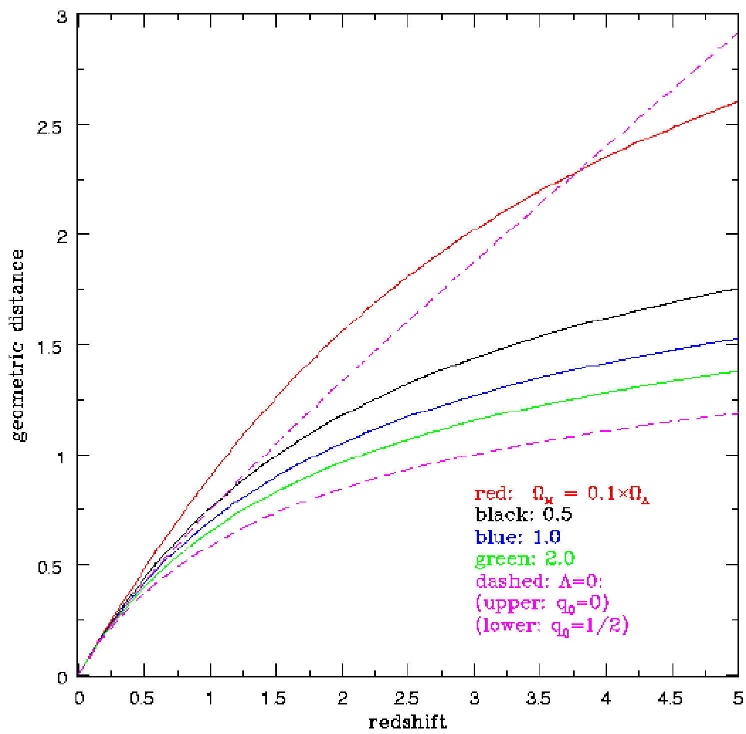

Figure 3: The geometric distance as a function of redshift, for A=0.1,0.5, 1.0 and 2.0. For comparison also the Friedmann models with q_0=0 and 0.5 are shown

4 A selection of Models with $k \neq 0$, $\Lambda \neq 0$

4.1 Zero-density model with q0 > 0

Although this model is unrealistic, because it presupposes $\sigma \equiv 0$, it is still of some interest: it allows q_0 to be of order one, while the matter density may be much smaller, such that $\sigma_0 = 0$ is a good first order approximation. As is immediately clear from the discussion in Chapter 3.1, a similar combination of σ and q, with $\sigma < q$, implies that $\Lambda < 0$. Therefore Λ provides a universal attractive force, which causes the deceleration parameter to be >0, even though no matter is present.

First we go back to the Einstein equations, putting $\rho = 0$:

$$\frac{3kc^2}{R^2} + \frac{3\dot{R}^2}{R^2} - \Lambda = 0$$

$$\frac{2\ddot{R}}{R} + \frac{\dot{R}^2}{R^2} + \frac{kc^2}{R^2} - \Lambda = 0$$

Since we require that $q_0 > 0$, it follows from the second Einstein equation as written in Chapter 3.1, that $3\sigma_0 - q_0 - 1 = -q_0 - 1 < 0$, so that $k = -1$.
Eliminating Λ from the two Einstein equations given above we get one equation:

$$\ddot{R}R - \dot{R}^2 + c^2 = 0$$

This can be written at present epoch in terms of the usual variables as:

$$R_0 = \frac{1}{(q_0+1)^{1/2}} \frac{c}{H_0}$$

The Einstein equation of this model allows an explicit solution $R = R(t)$, as can be easily verified by trying a form like $R = a \sin(bt)$. We find:

$$R = \frac{c}{H_0 q_0^{1/2}} \sin(H_0 q_0^{1/2} t)$$

with the boundary condition:

$$\sin(H_0 q_0^{1/2} t) = \left(\frac{q_0}{q_0+1}\right)^{1/2}$$

The Hubble and deceleration parameters as a function of t are:

$$\boxed{H = H_0 q_0^{1/2} \cot(H_0 q_0^{1/2} t)}$$

and:

$$\boxed{q = \tan^2(H_0 q_0^{1/2} t)}$$

Some comments:

- The model is closed, although it has a negatively curved metric ($k=-1$): R increases up to a maximum $c/(H_0 q_0^{1/2})$ at $t = \pi/(2H_0 q_0^{1/2})$, and goes to zero again at $t = \pi/(H_0 q_0^{1/2})$.
- The particular moment $q_0=1$ corresponds to $\sin(H_0 t_0)=(1/2)\sqrt{2}$, or $t_0 H_0 = \pi/4$.
- If one wishes to do so one can combine the solution and its boundary condition into one equation, to read:

$$\boxed{R = \frac{1}{(q_0+1)^{1/2}} \frac{c}{H_0} [\cos\{H_0 q_0^{1/2}(t-t_0)\} + q_0^{-1/2} \sin\{H_0 q_0^{1/2}(t-t_0)\}]}$$

Making use of the formulas $\int dx/\sin x = \ln\tan x/2$, $\tan x/2 = \sin x/(\cos x + 1)$ and $\operatorname{arcosh} x = \ln\{\sqrt{(x2-1)}+x\}$, and doing a bit of calculation we find:

$$\boxed{r_g = \sinh\omega = \frac{c}{R_0 H_0} \frac{\{(1+q_0)(1+z)^2 - q_0\}^{1/2} - (1+z)}{q_0}}$$

Therefore an equivalent of Mattig's formula does exist in this model. Note that for $q_0 \to 0$ we should recover the standard $q_0=0$ model of Milne, because $\sigma \equiv 0$, and thus $\Lambda \propto (\sigma_0 - q_0) \to 0$. It can be easily verified that taking the limit $q_0 \to 0$ in the above formula gives the correct expression for r_g.
The look-back time can be written as:

$$\boxed{\tau = 1 - \frac{\arcsin\left\{\dfrac{q_0^{1/2}}{(q_0+1)^{1/2}(1+z)}\right\}}{\arcsin\left(\dfrac{q_0}{q_0+1}\right)^{1/2}}}$$

and t_0 as:

$$\boxed{t_0 H_0 = \frac{\arctan q_0^{1/2}}{q_0^{1/2}}}$$

This completes our discussion of this model.

4.2 The Lemaître model: Λ>0 and k=+1

Before turning to the Lemaître model proper, two other very simple models should be briefly discussed, because they are closely related.

- The Einstein model. This is in fact the first cosmological model, given by Einstein. He tried to get a static solution (the recession of galaxies was only being discovered at that time by Hubble), and had to introduce the cosmological constant Λ to obtain this goal. Requiring R=constant we have $dR/dt=d^2R/dt^2=0$. The Einstein equations become:

$$\boxed{8\pi G\rho + \Lambda = \frac{3kc^2}{R^2}}$$

$$\boxed{\frac{kc^2}{R^2} = \Lambda}$$

It follows that $8\pi G\rho = 2kc^2/R^2$, or $k=1$ (since of course $\rho>0$) and $\Lambda>0$. We find

$$\boxed{R_E = \frac{c}{\sqrt{\Lambda}}}$$

and

$$\boxed{\frac{kc^2}{R^2} = \Lambda}$$

- The De Sitter model, which was already discussed in Chapter 3, as a limiting case of the flat models. Its relevance in this section comes from the fact that it is also a limiting case for the Lemaître models. We summarize some results:

$$\boxed{q \equiv -1}$$

$$\boxed{R = R_0 e^{H_0(t-t_0)}}$$

$$\boxed{H = \sqrt{\frac{\Lambda}{3}}}$$

The Lemaître model was one of the most famous models in the 1930's, and was revived for a short time after about 1965, as attempts were made to explain the

(apparently significant) redshift peak of quasars near $z = 2$. The reason for this is made clear below. For more information see for example Petrosian (1967).
In the Lemaître model we start with $k = +1$ and $\Lambda > 0$, just like in the Einstein model. However, we put slightly more matter in it: as a consequence the Lemaître model is unstable (as is, incidentally, also the Einstein model) against perturbations. The dynamics is determined by the competing forces of the density, which tries to slow down the expansion, and the cosmological constant which, being positive, acts as a universal repulsive force.
I follow the discussion of Weinberg (1972). It is convenient to write the scale parameter in terms of the Einstein radius R_E, and define $x \equiv R/R_E$. We put more matter in the universe than the Einstein value ρ_E, so that, since $\rho R^3 = \text{constant}$, we can write $R^3 = \rho x^3 R_E^3 = \alpha\rho_E R_E^3$, with $\alpha > 1$. Then:

$$\rho = \frac{\alpha\rho_E}{x^3} = \frac{\alpha\Lambda}{4\pi G x^3}$$

The Einstein equations expressed in x and α become:

$$\dot{x}^2 = \frac{\Lambda}{3x}(x^3 - 3x + 2\alpha)$$

$$\ddot{x} = \frac{\Lambda}{3x^2}(x^3 - \alpha)$$

A lot can be learned by studying the asymptotic behaviour of the equations:

- *For* $x \ll 1$ we find:

$$x = (3\alpha\Lambda)^{1/3} t^{2/3}$$

and thus:

$$H = \frac{2}{3t}$$

$$q = 1/2$$

This is of course nothing but the Einstein-De Sitter model: For $0 < x < 1$ the Lemaître model behaves as the Einstein-De Sitter model.}

- For $x \gg 1$ we find

$$\dot{x}^2 = \frac{1}{3}\Lambda x^2$$

or:

$$\boxed{H = \sqrt{\frac{\Lambda}{3}}}$$

and

$$\boxed{q = -1}$$

For $x >> 1$ the Lemaître model behaves as the De Sitter model. Note that it has these two features in common with the flat models with $\Lambda > 0$, discussed in Chapter 3. The characteristic properties of the Lemaître model are seen in the intermediate range of x:

- x of order unity. From the Einstein equations we see that dx/dt has a minimum:

$$\boxed{\dot{x}^2 = \Lambda(\alpha^{2/3} - 1)}$$

for

$$\boxed{x_{\min} = \alpha^{2/3}}$$

Then $q_{min} = 0$. So this is the phase where the deceleration (with $q = 1/2$), has stopped; after that q will eventually become -1 and the expansion will accelerate. This can be understood as follows: in the early phase the density is so high that it wins against the cosmological constant, until a radius $x_{min} = \alpha^{1/3}$ is reached. For some time the radius will stay close to this value. However, the matter density does not succeed in stopping the expansion definitively (dx/dt remains >0). After some time the density, which goes as R^{-3}, has dropped enough that the cosmological constant can take over. In the end $\rho \rightarrow 0$ and we have reached the De Sitter model stage.
We can force dx/dt (min) to become arbitrarily close to zero, by letting $\alpha \rightarrow 1$. For α close to unity we have:

$$\boxed{\dot{x}^2 - \dot{x}_{\min}^2 = \Lambda\left(\frac{x^2}{3} + \frac{2\alpha}{3x} - \alpha^{2/3}\right) \approx \frac{\Lambda}{3}(x - \alpha^{1/3})^2}$$

Then:

$$\dot{x}^2 \cong \Lambda(\alpha^{2/3}-1)+\frac{\Lambda}{3}(x-\alpha^{1/3})^2$$

and by trying a solution of the kind $x=\alpha^{1/3}+A\sinh B(t-t_m)$, we find that

$$A=\sqrt{3}(\alpha^{2/3}-1)^{1/2}$$

and

$$B=\left(\frac{\Lambda}{3}\right)^{1/2}$$

Since the redshift in this period (the ``coasting'' phase) can be written as:

$$\Delta z=\frac{R_0-R_1}{R_1}=\frac{x-\alpha^{1/3}}{\alpha^{1/3}}$$

we finally have

$$\dot{x}^2=\frac{\Lambda}{3}(\frac{x^3}{3}-x+\frac{2\alpha}{3})$$

$$\sinh\{\sqrt{\frac{\Lambda}{3}}(t-t_m)\}=\frac{\Delta z}{\sqrt{3}}(1-\alpha^{-2/3})^{-1/2}$$

Suppose we make $t-t_m = \Delta t$ very large. Then we still can make for small Δz the right hand side as large as we want, by letting $\alpha \to 1$. In other words during a very long period we can keep Δz very small.

It now becomes clear why it has been attempted to explain the redshift peak near $z=2$ with this model: one devises a model in which a coasting period occurs around $z=2$. Later observations have shown that such a redshift peak is in reality non-existent, so the need for introducing the Lemaître model has disappeared. One more comment: it will be obvious that in this model one has to take recourse to numerical integration in order to find the relevant parameters like geometric distance, time scales, etc.
This completes our discussion of the Lemaître model.

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Appendix A: Parameters and symbols used

General Symbols

- z: the redshift
- r_g: the geometric distance
- D: the luminosity distance
- $\Delta\theta$: the angular size (L is used for linear size)
- V: the volume in co-moving coordinates
- τ: the look-back time $(t_0 - t_1)/t_0$), t_0 present epoch, t_1 epoch at redshift z
- ψ: the development angle, used in Friedmann models (Λ=0)

A.1 The Metric

$$ds^2 = c^2 dt^2 - R^2(t)\left\{ d\omega^2 + \left(\frac{\sin\sqrt{k}\omega}{\sqrt{k}} \right)^2 (d\theta^2 + \sin^2\theta d\phi^2) \right\}$$

- ω: the radial co-moving coordinate
- θ: an angular co-moving coordinate
- ϕ: a second angular co-moving coordinate
- $R(t)$: the scale parameter

N.B. the co-moving coordinates are fixed once and for all for any galaxy.

A.2 The Einstein Equations

$$8\pi G\rho = 3\frac{kc^2}{R^2} + 3\frac{\dot{R}^2}{R^2} - \Lambda$$

$$8\pi G\frac{p}{c^2} = -2\frac{\ddot{R}}{R} - \frac{\dot{R}^2}{R^2} - \frac{kc^2}{R^2} + \Lambda$$

- ρ: the matter density; also often used $\sigma = 4\pi G\rho/(3H^2)$
- p: the pressure; we only consider $p = 0$
- Λ: the cosmological constant
- H: the Hubble parameter, $H = (dR/dt)(1/R)$
- q: the deceleration parameter, $q = -\{(d^2R/dt^2)R\}/(dR/dt)^2$

A.3 General Relations

- $1+z = R0/R1$: redshift
- $\omega = c\int(dt/R)$, from t_1 to t_0: radial coordinate
- $r_g = \sin\sqrt{(k)}\omega/\sqrt{(k)}$: geometric distance
- $D = (1 + z)\ R_0 r_g$: luminosity distance
- $\Delta\theta = (1+z)L/(R_0 r_g)$: angular size
- $V(\omega) = 4\pi R03\int[\sin\{\sqrt{(k)}\omega\}/\sqrt{(k)}]^2 d\omega$, from 0 to ω volume

Appendix B: The Friedmann model (Λ=0)

See Chapter 2 for information on the derivation of the formulas.

B.1 General Relations ($q_0 \geq 0$)

$$R = \frac{a}{k}(1 - \cos\sqrt{k}\psi)$$

$$t = \frac{a}{kc}\left(\psi - \frac{\sin\sqrt{k}\psi}{\sqrt{k}}\right)$$

$$\cos\sqrt{k}\psi_0 = \frac{1-q_0}{q_0}$$

$$\cos\sqrt{k}\psi_1 = \frac{\cos\sqrt{k}\psi_0 + z}{1+z}$$

$$\omega = \psi_0 - \psi_1$$

$$R_0 = \left(\frac{k}{2q_0 - 1}\right)^{1/2} \frac{c}{H_0}$$

$$H = \frac{k^2 c}{a} \frac{\sin\sqrt{k}\psi / \sqrt{k}}{(1-\cos\sqrt{k}\psi)^2} \Leftrightarrow H = H_0(1+z)(1+2q_0 z)^{1/2}$$

$$q = \frac{1}{1+\cos\sqrt{k}\psi} \Leftrightarrow q = q_0 \frac{1+z}{1+2q_0 z}$$

$$r_g = \frac{c}{R_0 H_0} \frac{q_0 z + 1 - q_0 + (q_0 - 1)\sqrt{(2q_0+1)}}{q_0^2(1+z)}$$

$$\Delta\theta = 57.3 \frac{(1+z)L}{R_0 r_g} (\text{deg})$$

$$V(\omega) = \frac{2\pi R^3}{k}\left(\omega - \frac{\sin 2\sqrt{k}\omega}{2\sqrt{k}}\right)$$

$$\frac{dV}{dz} = \frac{dV}{d\omega}\frac{d\omega}{dz}$$

$$\tau = 1 - \frac{\psi_1 - \sin\sqrt{k}\psi_1}{\psi_0 - \sin\sqrt{k}\psi_0}$$

$$t_0 H_0 = \frac{k^{3/2} q_0}{k(2q_0 - 1)^{3/2}}\left(\psi_0 - \frac{\sin\sqrt{k}\psi_0}{\sqrt{k}}\right)$$

B.2 k = –1; 0 ≤ q0 < 1/2

$$R = a(\cosh\psi - 1)$$

$$t = \frac{a}{c}(\sinh\psi - \psi)$$

$$\cosh\psi_0 = \frac{1 - q_0}{q_0}$$

$$\cosh\psi_1 = \frac{\cosh\psi_0 + z}{1 + z}$$

$$\omega = \psi_0 - \psi_1$$

$$R_0 = (1 - 2q_0)^{-1/2}\frac{c}{H_0}$$

$$H = \frac{c}{a}\frac{\sinh\psi}{(1 - \cosh\psi)^2}$$

$$q = \frac{1}{1 + \cosh\psi}$$

$$V(\omega) = 2\pi R_0^3\left(\frac{1}{2}\sinh 2\omega - \omega\right)$$

$$\frac{dV}{dz} = 2\pi R_0^3(\cosh 2\omega - 1)\frac{d\omega}{dz}$$

$$\tau = 1 - \frac{\sinh\psi_1 - \psi_1}{\sinh\psi_0 - \psi_0}$$

$$t_0 H_0 = \frac{q_0}{(1-2q_0)^{3/2}}(\sinh\psi_0 - \psi_0)$$

B.3 k = +1; q0 > 1/2

$$R = a(1-\cos\psi)$$

$$t = \frac{a}{c}(\psi - \sin\psi)$$

$$\cos\psi_0 = \frac{1-q_0}{q_0}$$

$$\cos\psi_1 = \frac{\cos\psi_0 + z}{1+z}$$

$$\omega = \psi_0 - \psi_1$$

$$R_0 = (2q_0 - 1)^{-1/2}\frac{c}{H_0}$$

$$H = \frac{c}{a}\frac{\sin\psi}{(1-\cos\psi)^2}$$

$$q = \frac{1}{1+\cos\psi}$$

$$v(\omega) = 2\pi R_0^3\left(\omega - \frac{1}{2}\sin 2\omega\right)$$

$$\frac{dV}{dz} = 2\pi R_0^3(1-\cos 2\omega)\frac{d\omega}{dz}$$

$$\tau = 1 - \frac{\psi_1 - \sin\psi_1}{\psi_0 - \sin\psi_0}$$

$$t_0 H_0 = \frac{q_0}{(2q_0-1)^{3/2}}(\psi_0 - \sin\psi_0)$$

B.4 k = –1; q_0 = 0

$$R = ct$$

$$\omega = \ln(1+z)$$

$$H = t^{-1}$$

$$r_g = \frac{z + \frac{1}{2} z^2}{1+z}$$

$$V(z) = 2\pi \left(\frac{c}{H_0} \right)^3 \left(\frac{1}{4} \frac{(1+z)^4 - 1}{(1+z)^2} - \ln(1+z) \right)$$

$$\frac{dV}{dz} = 4\pi \left(\frac{c}{H_0} \right)^3 \frac{(z + z^2/2)^2}{(1+z)^3}$$

$$\tau = \frac{z}{1+z}$$

$$t_0 H_0 = 1$$

B.5 k = 0 ; q_0 = 1/2

$$R = \frac{c}{H_0} \left(\frac{t}{t_0} \right)^{2/3}$$

$$\omega = 2 \left(1 - \frac{1}{(1+z)^{1/2}} \right)$$

$$H = \frac{2}{3} t^{-1}$$

$$r_g = \omega$$

$$V(z) = \frac{32}{3} \pi \left(\frac{c}{H_0} \right)^3 \left(1 - \frac{1}{(1+z)^{1/2}} \right)^3$$

$$\frac{dV}{dz} = 16\pi \left(\frac{c}{H_0} \right)^3 \frac{\{(1+z)^{1/2} - 1\}^2}{(1+z)^{5/2}}$$

$$\tau = 1 - \frac{1}{(1+z)^{3/2}}$$

$$t_0 H_0 = 2/3$$

B.6 k = +1; q_0 = 1

$$\omega = \frac{\pi}{2} - \arccos\left(\frac{z}{1+z}\right)$$

$$r_g = \frac{z}{1+z}$$

$$V(z) = 2\pi\left(\frac{c}{H_0}\right)^3 \left\{\arcsin\left(\frac{z}{1+z}\right) - \frac{z(1+2z)^{1/2}}{(1+z)^2}\right\}$$

$$\frac{dV}{dz} = 4\pi\left(\frac{c}{H_0}\right)^3 \frac{z^2}{(1+z)^3(1+2z)^{1/2}}$$

$$\tau = 1 - \frac{\arccos\left(\frac{z}{1+z}\right) - \left\{1 - \left(\frac{z}{1+z}\right)^2\right\}^{1/2}}{\frac{\pi}{2} - 1}$$

$$t_0 H_0 = \frac{\pi}{2} - 1$$

Appendix C: Flat Models (k = 0; Λ > 0)

See Chapter 3 for information on the derivation of the formulas.

$$R = A^{1/3} R_0 \sinh^{2/3} \gamma t$$

$$A = \frac{8\pi G \rho_0}{\Lambda}$$

$$\gamma = \frac{1}{2}\sqrt{3\Lambda}$$

$$H_0 = \sqrt{\frac{\Lambda}{3}(1+A)}$$

$$q_0 = \frac{1}{2}\left(\frac{A-2}{A+1}\right) \Leftrightarrow A = 2\left(\frac{1+q_0}{1-2q_0}\right)$$

$$t_0 H_0 = \frac{2}{3}(1+A)^{1/2} \ln\{A^{-1/2} + (1+A^{-1})^{1/2}\}$$

$$(1-2q)H^2 = \Lambda$$

$$H(z) = \sqrt{\frac{\Lambda}{3}\{1+A(1+z)^3\}}$$

$$1-2q = \frac{3}{1+A(1+z)^3}$$

$$r_g = \frac{c}{R_0 H_0}(1+A)^{1/2} \int_0^z \frac{d\zeta}{\{1+A(1+\zeta)^3\}^{1/2}}$$

$$\tau = 1 - \frac{\ln[\{1+A(1+z)^3\}^{1/2}+1] - \frac{3}{2}\ln(1+z) - \frac{1}{2}\ln A}{\ln\{1+(1+A)^{1/2}\} - \frac{1}{2}\ln A}$$

Appendix D: A Selection of Other Models

See Chapter 4 for information on the derivation of the formulas.

D.1 Zero-Density Model with $q_0 > 0$

$$\rho \equiv 0 \qquad \Lambda < 0$$

$$R = \frac{c}{H_0 q_0^{1/2}} \sin(H_0 q_0^{1/2} t)$$

$$\sin(H_0 q_0^{1/2} t_0) = \left(\frac{q_0}{q_0 + 1}\right)^{1/2}$$

These two equations can be written together as:

$$R = \frac{c}{H_0 (q_0 + 1)^{1/2}} [\cos\{H_0 q_0^{1/2}(t - t_0)\} + q_0^{-1/2} \sin\{H_0 q_0^{1/2}(t - t_0)\}]$$

$$\omega = \operatorname{arcosh}\left\{\left(\frac{q_0 + 1}{q_0}\right)^{1/2} (1 + z)\right\} - \operatorname{arcosh}\left(\frac{q_0 + 1}{q_0}\right)^{1/2}$$

$$H = H_0 q_0^{1/2} \cot(H_0 q_0^{1/2} t)$$

$$q = \tan^2(H_0 q_0^{1/2} t)$$

$$r_g = \left(\frac{c}{R_0 H_0}\right) \frac{\{(1 + q_0)(1 + z)^2 - q_0\}^{1/2} - (1 + z)}{q_0}$$

$$V(\omega) = 2\pi R_0^3 \left(\frac{1}{2} \sinh 2\omega - \omega\right)$$

$$\tau = 1 - \frac{\arcsin\left\{\frac{q_0^{1/2}}{(q_0 + 1)^{1/2}(1 + z)}\right\}}{\arcsin\left(\frac{q_0}{q_0 + 1}\right)^{1/2}}$$

$$t_0 H_0 = q_0^{-1/2} \arctan q_0^{1/2}$$

D.2 The Einstein model

$k = +1$

$\Lambda > 0$

$R_E = \frac{c}{\sqrt{\Lambda}}$

$\rho_E = \frac{\Lambda}{4\pi G}$

$\dot{R} = \ddot{R} = 0$

$H = q \equiv 0$

D.3 The De Sitter model

$k = 0$

$\Lambda > 0$

$\rho \equiv 0$

$R = R_0 \exp\{H_0(t - t_0)\}$

$\omega = \left(\frac{c}{R_0 H_0}\right) z$

$H = \sqrt{\frac{\Lambda}{3}}$

$q \equiv -1$

$r_g = \left(\frac{c}{R_0 H_0}\right) z$

$V(z) = \frac{4\pi}{3}\left(\frac{c}{H_0}\right)^3 z^3$

D.4 The Lemaître model

$k = +1$

$\Lambda > 0$

$x = \dfrac{R}{R_E}$ (definition)

$\alpha = x^3 \dfrac{\rho}{\rho_E}$ (definition)

The Einstein equations are then:

$$\dot{x}^2 = \frac{\Lambda}{3x}(x^3 - 3x + 2\alpha)$$

$$\ddot{x} = \frac{\Lambda}{3x^2}(x^3 - \alpha)$$

Three different phases can be distinguished:

- for $x \ll \alpha^{1/3}$, the behaviour is as in the Einstein-De Sitter model.

$$x = \left(\frac{3\alpha}{2}\right)^{1/3} \Lambda^{1/2} t^{2/3}$$

$$H = \frac{2}{3} t^{-1}$$

$$q = \frac{1}{2}$$

- for $x \gg \alpha^{1/3}$, the behaviour is as in the De Sitter model.

$$x = x_0 \exp\{H_0(t - t_0)\}$$

$$H_0 = \sqrt{\frac{\Lambda}{3}}$$

$$q \equiv -1$$

- $x \approx \alpha^{1/3}$, the *coasting phase*:

$$x \approx \alpha^{1/3} + (\alpha^{2/3} - 1)^{1/2} \sinh \sqrt{\frac{\Lambda}{3}}(t - t_0)$$

The duration of the coasting phase is:

$$\Delta t \approx \Lambda^{-1/2} \,|\{\ln(1 - \alpha^{-2/3})\,|$$

We can make this period arbitrarily long, because $\Delta t \to \infty$ for $\alpha \to 1$.

The End