Physics Notes for Class 12 Chapter 13 Nuclei

Nucleus

The entire positive charge and nearly the entire mass of atom is concentrated in a very small space called the nucleus of an atom.

The nucleus consists of protons and neutrons. They are called nucleons.

Terms Related to Nucleus

(i) **Atomic Number** The number of protons in the nucleus of an atom of the element is called atomic number (Z) of the element.

(ii) **Mass Number** The total number of protons and neutrons present inside the nucleus of an atom of the element is called mass number (A) of the element.

(iii) **Nuclear Size** The radius of the nucleus $R \propto A^{1/3}$

$\Rightarrow R = R_o A^{1/3}$

where, $R_o = 1.1 * 10^{-15}$ m is an empirical constant.

(iv) **Nuclear Density** Nuclear density is independent of mass number and therefore same for all nuclei.

ρ = mass of nucleus / volume of nucleus $\Rightarrow \rho = 3m / 4\pi R^3_o$

where, m = average mass of a nucleon.

(v) **Atomic Mass Unit** It is defined as 1 / 12th the mass of carbon nucleus.

It is abbreviated as arnu and often denoted by u. Thus

1 amu = $1.992678 * 10^{-26} / 12$ kg

= $1.6 * 10^{-27}$ kg = 931 Me V

Isotopes

The atoms of an element having same atomic number but different mass numbers. are called isotopes.

e.g., ${}_1H^1$, ${}_1H^2$, ${}_1H^3$ are isotopes of hydrogen.

Isobars

The atoms of different elements having same mass numbers but different atomic numbers, are called isobars.

e.g., ${}_1H^3$, ${}_2He^3$ and ${}_{10}Na^{22}$, ${}_{10}Ne^{22}$ are isobars.

Isotones

The atoms of different elements having different atomic numbers and different mass numbers but having same number of neutrons, are called isotones.

e.g., ${}_1H^3$, ${}_2He^4$ and ${}_6C^{14}$, ${}_8O^{16}$ are isobars.

Isomers

Atoms having the same mass number and the same atomic number but different radioactive properties are called isomers,

Nuclear Force

The force acting inside the nucleus or acting between nucleons is called nuclear force.

Nuclear forces are the strongest forces in nature.

- It is a very short range attractive force.
- It is non-central. non-conservative force.
- It is neither gravitational nor electrostatic force.
- It is independent of charge.
- It is 100 times that of electrostatic force and 10^{38} times that of gravitational force.

According to the Yukawa, the nuclear force acts between the nucleon due to continuous exchange of meson particles.

Mass Defect

The difference between the sum of masses of all nucleons (M) mass of the nucleus (m) is called mass defect.

Mass Defect $(\Delta m) = M - m = [Zm_p + (A - Z)m_n - m_n]$

Nuclear Binding Energy

The minimum energy required to separate the nucleons up to an infinite distance from the nucleus, is called nuclear binding energy.

Nuclear binding energy per nucleon = Nuclear binding energy / Total number of nucleons

Binding energy, $E_b = [Zm_p + (A – Z) m_n – m_N]c^2$

Packing Fraction (P)

p = (Exact nuclear mass) – (Mass number) / Mass number

= M – A / M

The larger the value of packing friction. greater is the stability of the nucleus.

[The nuclei containing even number of protons and even number of neutrons are **most stable.**

The nuclei containing odd number of protons and odd number of neutrons are **most instable.]**

Radioactivity

The phenomena of disintegration of heavy elements into comparatively lighter elements by the emission of radiations is called radioactivity. This phenomena was discovered by Henry Becquerel in 1896.

Radiations Emitted by a Radioactive Element

Three types of radiations emitted by radioactive elements

(i) α-rays

(ii) β-rays

(iii) γ – rays

α-rays consists of α-particles, which are doubly ionised helium ion.

β-rays are consist of fast moving electrons.

γ – rays are electromagnetic rays.

[When an α – particle is emitted by a nucleus its atomic number decreases by 2 and mass number decreases by 4.

$${}_{Z}X^{A} \xrightarrow{\alpha\text{-particle}} {}_{Z-2}Y^{A-4}$$

When a β -particle is emitted by a nucleus its atomic number is Increases by one and mass number remains unchanged.

$${}_Z X^A \xrightarrow{\beta\text{-particle}} {}_{Z+1}Y^A$$

When a γ – particle is emitted by a nucleus its atomic number and mass number remain unchanged

Radioactive Decay law

The rate of disintegration of radioactive atoms at any instant is directly proportional to the number of radioactive atoms present in the sample at that instant.

Rate of disintegration $(– dN / dt) \propto N$

$– dN / dt = \lambda N$

where λ is the decay constant.

The number of atoms present undecayed in the sample at any instant $N = N_o e^{-\lambda t}$

where, N_o is number of atoms at time $t = 0$ and N is number of atoms at time t.

Half-life of a Radioactive Element

The time is which the half number of atoms present initially in any sample decays, is called half-life (T) of that radioactive element.

Relation between half-life and disintegration constant is given by

$T = \log^2{}_e / \lambda = 0.6931 / \lambda$

Average Life or Mean Life(τ)

Average life or mean life (τ) of a radioactive element is the ratio of total life time of all the atoms and total number of atoms present initially in the sample.

Relation between average life and decay constant $\tau = 1 / \lambda$

Relation between half-life and average life $\tau = 1.44\ T$

The number of atoms left undecayed after n half-lifes is given by

$N = N_o (1 / 2)^n = N_o (1 / 2)^{t/T}$

where, $n = t / T$, here t = total time.

Activity of a Radioactive Element

The activity of a radioactive element is equal to its rate of disintegration.

Activity R = (– dN / dt)

Activity of the sample after time t,

$R = R_o e^{-\lambda t}$

Its SI unit is Becquerel (Bq).

Its other units are Curie and Rutherford.

1 Curie = $3.7 * 10^{10}$ decay/s

1 Rutherford = 10^6 decay/s

Nuclear Fission

The process of the splitting of a heavy nucleus into two or more lighter nuclei is called nuclear fission.

When a slow moving neutron strikes with a uranium nucleus ($_{92}U^{235}$), it splits into $_{56}Ba^{141}$ and $_{36}Kr^{92}$ along with three neutrons and a lot of energy.

$$_{92}U^{235} + {_0n^1} \longrightarrow {_{56}Ba^{141}} + {_{36}Kr^{92}} \doteq 3{_0n^1} + \text{energy}$$

Nuclear Chain Reaction

If the particle starting the nuclear fission reaction is produced as a product and further take part in the nuclear fission reaction, then a chain of fission reaction started, which is called nuclear chain reaction.

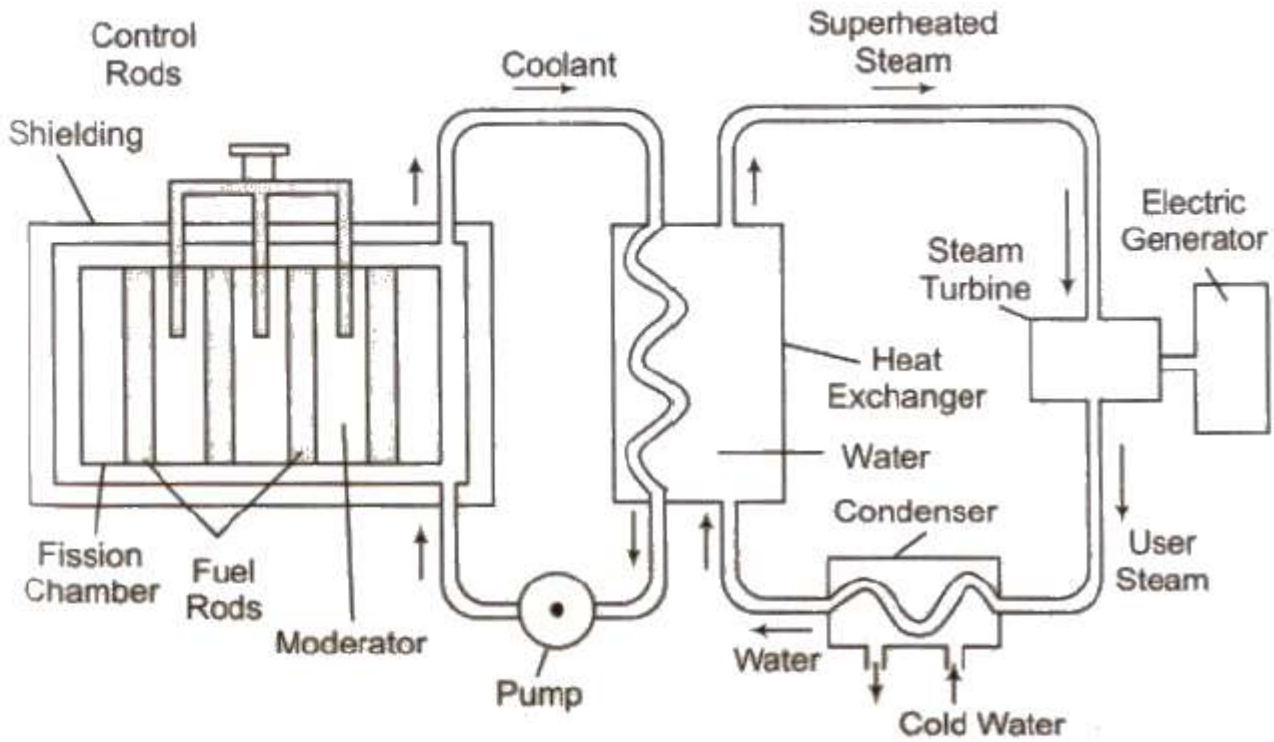
Nuclear chain reaction are of two types

(i) Controlled chain reaction

(ii) Uncontrolled chain reaction

Nuclear Reactor

The main parts of a nuclear reactor are following



(i) **Fuel** Fissionable materials like $_{92}U^{235}$, $_{92}U^{238}$, $_{94}U^{239}$ are used as fuel.

(ii) **Moderator** Heavy water, graphite and beryllium oxide are used to slower down fast moving neutrons.

(iii) **Coolant** The cold water, liquid oxygen, etc. are used to remove heat generated in the fission process.

(iv) **Control rods** Cadmium or boron rods are good absorber of neutrons and therefore used to control the fission reaction.

Atom bomb working is based on uncontrolled chain reaction.

Nuclear Fusion

The process of combining of two lighter nuclei to form one heavy nucleus, is called nuclear fusion.

Three deuteron nuclei ($_1H^2$) fuse, 21.6 MeV is energy released and nucleus of helium ($_2He^4$) is formed.

$$_1H^2 + {}_1H^2 + {}_1H^2 \longrightarrow {}_2He^4 + {}_1H^1 + {}_0n^1 + 21.6\ \text{MeV}$$

In this process, a large amount of energy is released.

Nuclear fusion takes place at very high temperature approximately about 10^7 K and at very high pressure 10^6 atmosphere.

Hydrogen bomb is based on nuclear fusion.

The source of Sun's energy is the nuclear fusion taking place at sun.

Thermonuclear Energy

The energy released during nuclear fusion is know as thermonuclear energy. Protons are needed for fusion while neutrons are needed for fission process.

Chapter 10

Nuclear Properties

Note to students and other readers: This Chapter is intended to supplement Chapter 3 of Krane's excellent book, "Introductory Nuclear Physics". Kindly read the relevant sections in Krane's book first. This reading is supplementary to that, and the subsection ordering will mirror that of Krane's, at least until further notice.

A nucleus, discovered by Ernest Rutherford in 1911, is made up of nucleons, a collective name encompassing both neutrons (n) and protons (p).

Name	**symbol**	**mass (MeV/c^2**	**charge**	**lifetime**	**magnetic moment**
neutron	n	939.565378(21)	0 e	881.5(15) s	-1.91304272(45) $\mu_{\rm N}$
proton	p	938.272046(21)	1 e	stable	2.792847356(23) $\mu_{\rm N}$

The neutron was theorized by Rutherford in 1920, and discovered by James Chadwick in 1932, while the proton was theorized by William Prout in 1815, and was discovered by Rutherford between 1917 and 1919m and named by him, in 1920.

Neutrons and protons are subject to all the four forces in nature, (strong, electromagnetic, weak, and gravity), but the strong force that binds nucleons is an intermediate-range force that extends for a range of about the nucleon diameter (about 1 fm) and then dies off very quickly, in the form of a decaying exponential. The force that keeps the nucleons in a nucleus from collapsing, is a short-range repulsive force that begins to get very large and repulsive for separations less than a nucleon radius, about $\frac{1}{2}$ fm. See Fig. 10.1 (yet to be created).

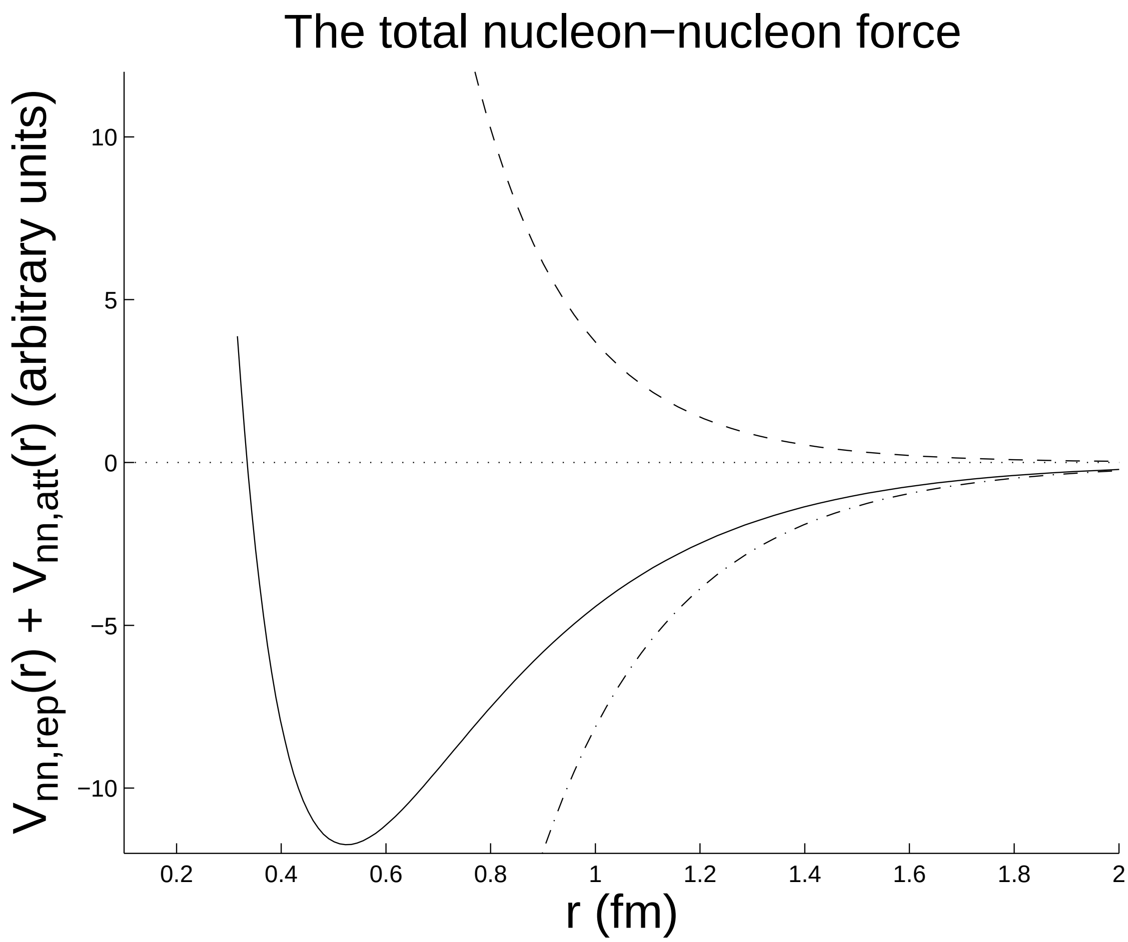


Figure 10.1: A sketch of the nucleon-nucleon potential.

The n-n, n-p, and p-p nuclear forces are all almost identical. (There are some important differences.) Of course, there is an additional p-p Coulombic repulsive potential, but that is separate from the nuclear force.

Owing to these nuclear forces between individual nucleons, a nucleus is tightly bound. The consequence is, from the attractive/repulsive form of the nuclear force, that the nucleons are in very close proximity. One can almost imagine a nucleus being made up of incompressible nucleonic spheres, sticking to one other, with a "contact" potential, like ping-pong balls smeared with petroleum jelly. A further consequence of the nuclear force is that nucleons in the nuclear core move, in what seems to be, a constant potential formed by the attraction of its nearby neighbors, only those that are in contact with it. A nucleon at the surface of a nucleus has fewer neighbors, and thus, is less tightly bound.

Nucleons are spin-$\frac{1}{2}$ particles (*i.e.* fermions). Hence the Pauli Exclusion Principle applies. That is, no two identical nucleons may possess the same set of quantum numbers. Consequently, we can "build" a nucleus, much as we built up an atom (in NERS311), by placing individual electrons into different quantum "orbitals", with orbitals being filled according to energy hierarchy, with a maximum of two electrons (spin up and spin down) to an orbital. Nucleons are formed in much the same way, except that all the force is provided by the other constituent nucleons, and there are two different "flavors" of nucleon, the neutron and the proton.

So, it seems that we could build a nucleus of almost any size, were it not for two physical facts that prevent this. The Pauli Exclusion Principle prevents the di-nucleon from being bound. Thus, uniform neutron matter does not exist in nature, except in neutron stars, where gravity, a long-range force, provides the additional binding energy to enable neutron matter to be formed. Thus, to build nuclei, we need to add in approximately an equal proportion of protons. However, this also breaks down because of Coulomb repulsion, for A (the total number of nucleons) greater than about 200 or so.

Moderate to large size nuclei also have more neutrons in the mix, thereby pushing the protons farther apart. It is all a matter of balance, between the Pauli Exclusion Principle and the Coulomb repulsion. And, that balance is remarkably delicate. The di-neutron is *not* bound, but *just* not bound. The deuteron *is* bound, but only *just* so. The alpha particle is tightly bound, but there are no stable $A = 5$ nuclei. ^{5}He ($2p + 3n$) has a half-life of only 7.9×10^{-22} seconds, while ^{5}Li ($3p + 2n$) has a half-life of only $\approx 3 \times 10^{-22}$ seconds. Those lifetimes are so short, that the unbalanced nucleon can only make a few orbits of the nucleus before it breaks away. Nature is delicately balanced, indeed.

Since we have argued that nuclei are held together by a "contact" potential, it follows that nuclei would tend to be spherical in "shape", and hence[1] it is reasonable to make mention of ...

[1] Admittedly, these are classical concepts. However, classical concepts tend to be very useful when discussing nuclei as these objects seem to straddle both the classical and quantum descriptions of its nature, with one foot set solidly in both.

10.1 The Nuclear Radius

Like the atom, the radius of a quantum object is not a precisely defined quantity; it depends on how that characteristic is measured. We can, with the proper tools, ask some very interesting things about the nucleus. Let us assume that the charge-independence of the nucleus means that the proton charge density and the neutron charge density are the same. Thus, a measure of the proton charge distribution yields direct knowledge of the neutron charge distribution. (In actual fact, the proton charge density distribution is forced to greater radius by Coulomb repulsion, but this effect is almost negligible.)

How may we measure the proton charge distribution?

In Nuclear and Particle Physics, the answer to this question usually takes some form of "Bang things together and see what happens!" In this case, we'll use electrons as the projectile and the nucleus as the target. The *scattering amplitude* is given by a proportionality (describing the constants necessary to convert the $\propto$ to an $=$ would be an unnecessary distraction):

$$F(\vec{k}_i, \vec{k}_f) \propto \langle e^{i\vec{k}_f\cdot\vec{x}}|V(\vec{x})|e^{i\vec{k}_i\cdot\vec{x}}\rangle \ , \tag{10.1}$$

where $e^{i\vec{k}_i\cdot\vec{x}}$ is the initial unscattered electron wavefunction, $e^{i\vec{k}_f\cdot\vec{x}}$ is the final scattered electron wavefunction, and $\vec{k}_i/\vec{k}_f$ are the initial/final wavenumbers.

Evaluating ...

$$\begin{aligned} F(\vec{k}_i, \vec{k}_f) &\propto \int \mathrm{d}\vec{x}\ e^{-i\vec{k}_f\cdot\vec{x}}V(\vec{x})e^{i\vec{k}_i\cdot\vec{x}} \\ &\propto \int \mathrm{d}\vec{x}\ V(\vec{x})e^{i(\vec{k}_i-\vec{k}_f)\cdot\vec{x}} \\ &\propto \int \mathrm{d}\vec{x}\ V(\vec{x})e^{i\vec{q}\cdot\vec{x}} \ , \end{aligned}$$

where $\vec{q} \equiv \vec{k}_i - \vec{k}_f$ is called the *momentum transfer*.

Thus, we see that scattering amplitude is proportional to the 3D Fourier Transform of the potential.

$$F(\vec{k}_i, \vec{k}_f) \equiv F(\vec{q}) \propto \int \mathrm{d}\vec{x}\ V(\vec{x})e^{i\vec{q}\cdot\vec{x}} \ , \tag{10.2}$$

For the present case, we apply the scattering amplitude to the case where the incident electron scatters from a much heavier nucleus that provides a scattering potential of the form:

$$V(\vec{x}) = -\frac{Ze^2}{4\pi\epsilon_0}\int \mathrm{d}\vec{x}'\ \frac{\rho_p(\vec{x}')}{|\vec{x}-\vec{x}'|}\ , \tag{10.3}$$

where $\rho_p(\vec{x}')$ is the number density of protons in the nucleus, normalized so that:

$$\int \mathrm{d}\vec{x}'\ \rho_p(\vec{x}') \equiv 1\ . \tag{10.4}$$

That is, the potential at $\vec{x}$ arises from the electrostatic attraction of the elemental charges in $\mathrm{d}\vec{x}$, integrated over all space. In order to probe the shape of the charge distribution, the reduced wavelength of the electron, $\lambda/2\pi$, must be less than the radius of the nucleus. Evaluating ...

$$\frac{\lambda}{2\pi} = \frac{\hbar}{p_e} = \frac{\hbar c}{p_e c} \approx \frac{\hbar c}{E_e} = \frac{197\ [\mathrm{MeV.fm}]}{E_e} < R_N\ ,$$

where R_N is the radius of the nucleus. The above is a relativistic approximation. (That is why the $\approx$ appears; $p_e c \approx E_e$.) The calculation is justified, however, since the inequality implies that the energy of the electron-projectile must be many 10s or 100s of MeV for the condition to hold. As we raise the electron energy even more, and it approaches 1 GeV or more, we can even begin to detect the individual charges of the constituent particles of the protons (and neutrons), the constituent quarks.

Proceeding with the calculation, taking the potential in (10.3) and putting it in (10.2), results in:

$$F(\vec{q}) \propto \left(-\frac{Ze^2}{4\pi\epsilon_0}\right)\int \mathrm{d}\vec{x}\ \int \mathrm{d}\vec{x}'\ \frac{\rho_p(\vec{x}')}{|\vec{x}-\vec{x}'|}e^{i\vec{q}\cdot\vec{x}}\ . \tag{10.5}$$

We choose the constant of proportionality in $F(\vec{q})$, to require that $F(0) \equiv 1$. The motivation for this choice is that, when $\vec{q} = 0$, the charge distribution is known to have no effect on the projectile. If a potential has no effect on the projectile, then we can rewrite (10.5) as

$$F(0) = 1 \propto \left(-\frac{Ze^2}{4\pi\epsilon_0}\right)\int \mathrm{d}\vec{x}\ \int \mathrm{d}\vec{x}'\ \frac{\rho_p(\vec{x}')}{|\vec{x}-\vec{x}'|}\ , \tag{10.6}$$

thereby determining the constant of proportionality. The details of this calculation will be left to enthusiastic students to discover for themselves. The final result is:

$$F(\vec{q}) = \int \mathrm{d}\vec{x}\ \rho_p(\vec{x})e^{i\vec{q}\cdot\vec{x}}\ . \tag{10.7}$$

Thus, we have determined, at least for charge distributions scattering other charges, that the scattering amplitude is the Fourier Transform of the charge distribution.

This realization is one of the most important discoveries of nuclear structure physics: namely, that a measurement of the scattering of electrons (or other charged particles) from charge distributions, yields a direct measure of the shape of that charge distribution. One merely has to invert the Fourier Transform.

We also note, from (10.4) that $F(0) = 1$.

10.1.1 Application to spherical charge distributions

Most nuclei are spherical in shape, so it behooves us to examine closely, the special case of spherical charge distributions. In this case, $\rho_p(\vec{x}) = \rho_p(r)$, and we write (10.7) more explicitly in spherical polar coordinates:

$$F(\vec{q}) = \int_0^{2\pi} \mathrm{d}\phi \int_0^\infty r^2 \mathrm{d}r\ \rho_p(r) \int_0^\pi \sin\theta \mathrm{d}\theta\ e^{iqr\cos\theta}\ . \tag{10.8}$$

The only "trick" we have used is to align our coordinate system so that $\vec{q} = q\hat{z}$. This is permissible since the charge distribution is spherically symmetric and there is no preferred direction. Hence, we choose a direction that makes the arithmetic easy. The remaining integrals are elementary, and one can easily show that:

$$F(q) = \frac{4\pi}{q} \int_0^\infty r \mathrm{d}r\ \rho_p(r) \sin qr\ . \tag{10.9}$$

Figure 10.2: From "Introductory Nuclear Physics", by Kenneth Krane

Figure 10.3: From “Introductory Nuclear Physics”, by Kenneth Krane

Figure 10.4: From “Introductory Nuclear Physics”, by Kenneth Krane

Conclusions from the data shown?

1. The central density, is (roughly) constant, almost independent of atomic number, and has a value about 0.13/fm^3. This is very close to the density nuclear in the infinite radius approximation,
$$\rho_0 = 3/(4\pi R_0^3) \ .$$

2. The "skin depth", s, is (roughly) constant as well, almost independent of atomic number, with a value of about 2.9 fm, typically. The skin depth is usually defined as the difference in radii of the nuclear densities at 90% and 10% of maximum value.

3. Measurements suggest a best fit to the radius of nuclei:
$$R_N = R_0 A^{1/3} \quad ; \quad R_0 \approx 1.22 \ [\mathrm{fm}], 1.20 \longrightarrow 1.25 \ \ \text{is common.} \tag{10.10}$$
however, values from 1.20⟶1.25 are commonly found

   A convenient parametric form of the nuclear density was psoposed by Woods and Saxon (*ca.* 1954).
$$\rho_N(r) = \frac{\rho_0}{1 + \exp\left(\frac{r-R_N}{t}\right)}$$
where t is a surface thickness parameter, related to s, by $s = 4t\log(3)$.

An example of this distribution is shown in Figure 10.5

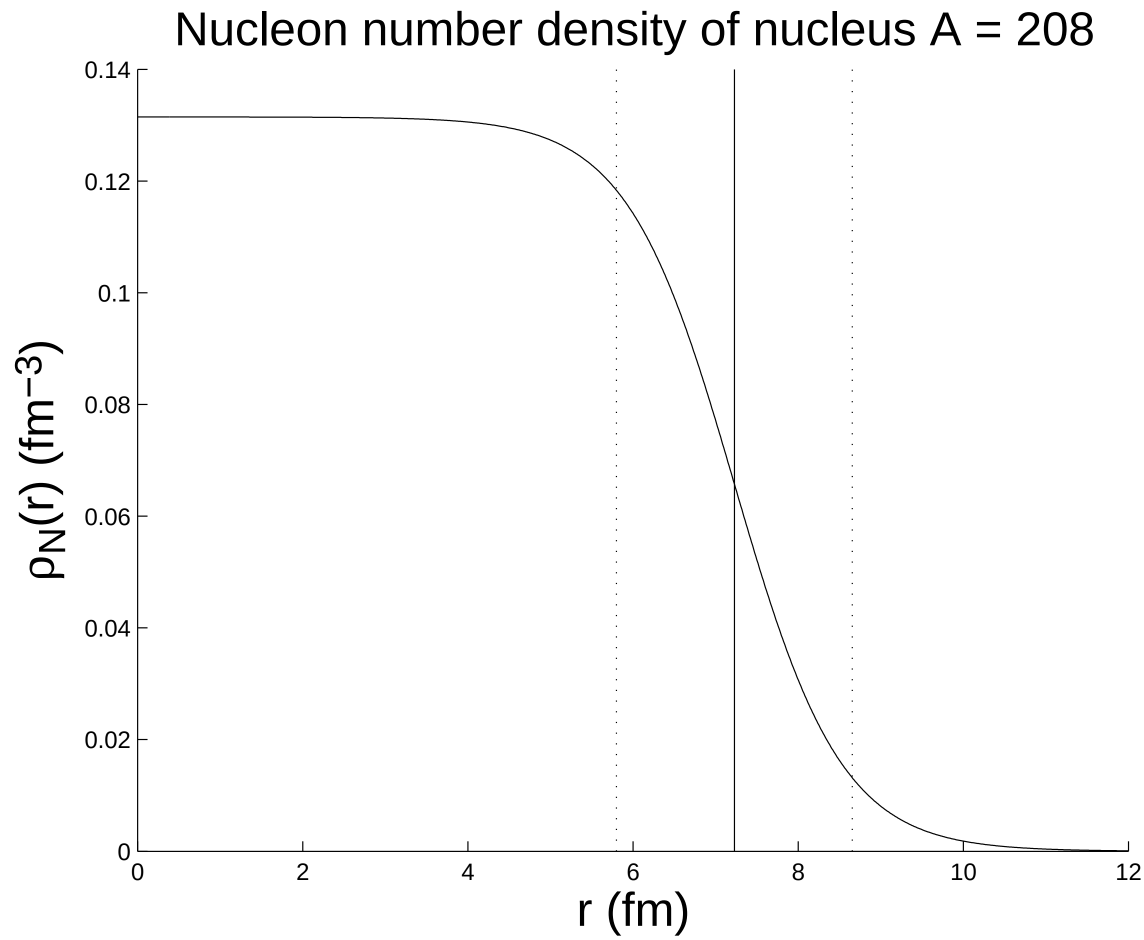


Figure 10.5: The Woods-Saxon model of the nucleon number density. In this figure, $A = 208$, $R_0 = 1.22$ (fm), and $t = 0.65$ (fm). The skin depth is shown, delimited by vertical dotted lines.

Let's work out a specific, but important realization of a charge distribution, namely, a uniform proton distribution, up to some radius R_N, the radius of the nucleus.

Example: Uniform nucleon charge density

In this case, the normalized proton density takes the form:

$$\rho_p(r) = \frac{3}{4\pi R_N^3}\Theta(R - r) \ . \tag{10.11}$$

Thus, combining (10.9) and (10.11), gives, after some reorganization:

$$F(q) = \frac{3}{(qR_N)^3}\int_0^{(qR_N)} \mathrm{d}z \ z \sin z \ , \tag{10.12}$$

which is easily evaluated to be,

$$F(q) = \frac{3[\sin(qR_N) - qR_N\cos(qR_N)]}{(qR_N)^3} \ , \tag{10.13}$$

for which $F(0) = 1$, as expected.

<u>Technical side note:</u>
The following `Mathematica` code was useful in deriving the above relations.

```
(* Here Z == q*R_N: *)

(3/Z^3)*Integrate[z Sin[z], {z,0,Z}]

Series[3*(Sin[Z] - Z*Cos[Z])/Z^3,{Z,0,2}]
```

Graphical output of (10.13) is given in Figure 10.6. We note, in particular, the zero minima when $\tan(qR_N) = qR_N$. The shape of the lobes is determined by the nuclear shape, while the minima are characteristic of the sharp edge. Measurements do not have such deep minima, since the nuclear edge is blurred, and the projectile energies are are not exact, but slightly distributed, and the detectors have imperfect resolution. However, the measurements do, unambiguously, reveal important details of the nuclear shape.

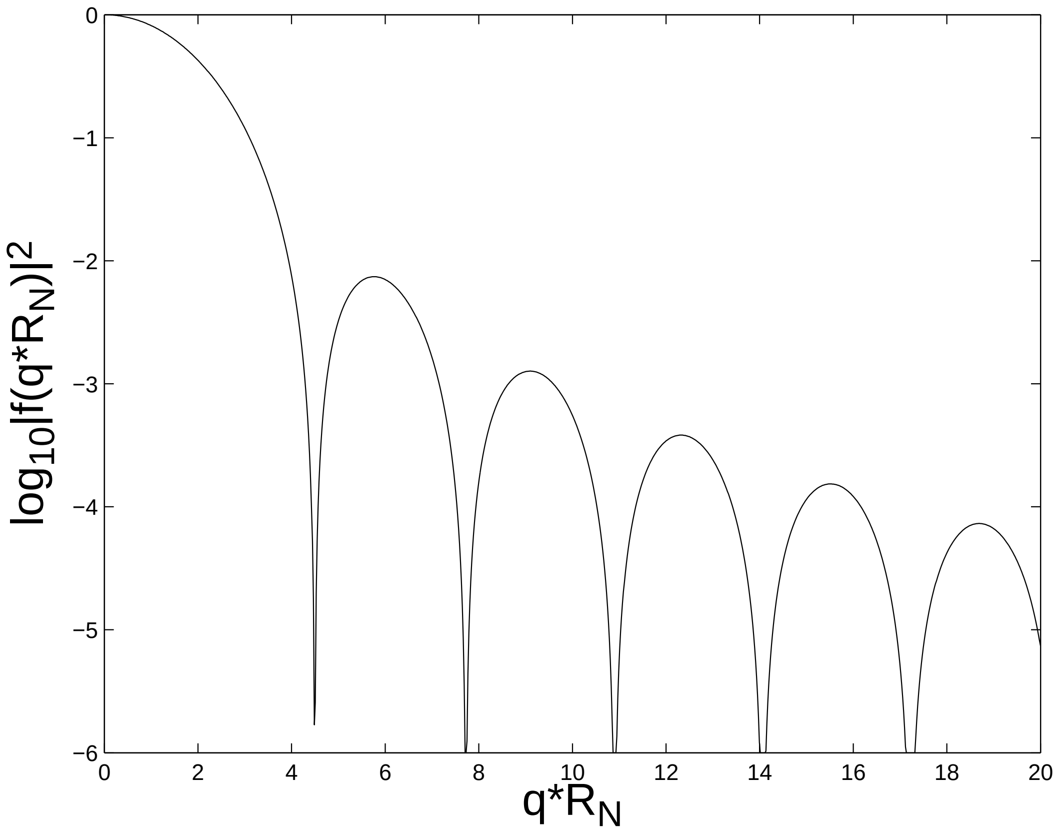


Figure 10.6: Graphical output corresponding to (10.13).

Technical side note:
The following `Matlab` code was useful in producing the above graph.

```
N = 1000; fMin = 1e-6; zMax = 20; % Graph data
z = linspace(0,zMax,N); f = 3*(sin(z) - z.*cos(z))./z.^3;
f(1) = 1; % Overcome the singularity at 0
f2 = f.*f;
for i = 1:N
   f2(i) = max(fMin,f2(i));
end
plot(z,log10(f2),'-k')
xlabel('\fontsize{20}q*R_N')
ylabel('\fontsize{20}log_{10}|f(q*R_N)|^2')
```

10.1.2 Nuclear shape data from electron scattering experiments

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Technical side note:
The mathematical details can be found in the supplemental notes.

Most of the mathematical detail is given in the supplementary notes to this lecture. Those notes obtain the following, very significant result.

What is measured in a scattering experiment is the relative intensity of deflected projectiles (e), scattered into different angles, by the nucleus (N). This is also known as the scattering cross section, differential in scattering angle. The result is that:

$$\frac{\mathrm{d}\sigma_{eN}}{\mathrm{d}\Omega} = \frac{\mathrm{d}\sigma_{eN}^{\mathrm{Ruth}}}{\mathrm{d}\Omega}|F(q)|^2 \ , \tag{10.14}$$

where $\mathrm{d}\sigma_{eN}^{\mathrm{Ruth}}/\mathrm{d}\Omega$ is the classical Rutherford cross section discussed in NERS311 (but re-derived in the supplemental notes to include relativistic kinematics, and $F(q)$ is the scattering amplitude we have been discussing so far. $|F(q)|^2$ is the scattering amplitude, modulus squared. (It can, in general, be complex.)

Hence, we have a direct experimental determination of the form factor, as a ratio of measurement data (the measured cross section), and a theoretical function, the Rutherford cross section.

$$|F(q)|^2 = \left(\frac{\mathrm{d}\sigma_{eN}^{\mathrm{meas}}}{\mathrm{d}\Omega}\right) \Bigg/ \left(\frac{\mathrm{d}\sigma_{eN}^{\mathrm{Ruth}}}{\mathrm{d}\Omega}\right) \ . \tag{10.15}$$

All that remains is to take the square root, and invert the Fourier Transform, to get $\rho(r)$. This is always done via a relatively simple numerical process.

Although the form factor $|F(q)|^2$ is given in terms of q, we may cast it into more recognizable kinematic quantities as follows. Recall,

$$q = \sqrt{q^2} = \sqrt{|\vec{k}_i - \vec{k}_f|^2} = \sqrt{2k^2(1-\cos\theta)} \ , \tag{10.16}$$

the final step above being obtained since this is an elastic scattering process, where $k = |\vec{k}_i| = |\vec{k}_f|$ and $\vec{k}_i \cdot \vec{k}_f = k^2\cos\theta$.

Thus, electron scattering experiments yield exquisitely detailed data on the shape of nuclei. Figure 3.11 in Krane depicts some very detailed data that shows the departure from the classical Rutherford scattering cross section, as the projectile's energy, α-particles in this case, is increased. The classical interpretation is that the projectile is penetrating the nucleus. The Quantum Mechanical picture is that the projectile's wave function has a wave number small enough to start resolving the finite size of the nucleus. We now examine another way that experiments can yield information about the nuclear shape.

10.1.3 Nuclear size from spectroscopy measurements

Nuclear and atomic spectroscopy, the technique of measuring the energies of nuclear and atomic transitions, is one of the most precise measurements in nuclear science. If that is the case, then spectroscopy ought to be able to measure differences in transition energies that arise from the finite nuclear size.

Assume, for the sake of argument, that the nucleus is a sphere of radius R_N. An ideal probe of the effect of a finite-sized nucleus *vs.* a point-nucleus (as in the Schrödinger atomic model), would be a $1s$ atomic state, since, of all the atomic electron wavefunctions, the $1s$ state has the most probability density in the vicinity of the nucleus.

The shift of energy of the $1s$ can be estimated as follows:

$$\Delta E_{1s} = \langle \psi_{1s} | V_{\circ}(r) - V_{\cdot}(r) | \psi_{1s} \rangle \ , \tag{10.17}$$

where the ψ_{1s} is the $1s$ wavefunction for the point-like nucleus, $V_{\circ}(r)$ is the Coulomb potential for the finite nucleus, and $V_{\cdot}(r)$ is the point-like Coulomb potential. This way of estimating energy shifts comes formally from "1st-order perturbation theory", where it is assumed that the difference in potential has only a small effect on the wavefunctions. For a uniform sphere of charge, we know from Classical Electrostatics, that $V_{\circ}(r) = V_{\cdot}(r)$ for $r \geq R_N$.

$$\begin{aligned}
V_{\circ}(r \leq R_N) &= -\frac{Ze^2}{4\pi\epsilon_0 R_N}\left[\frac{3}{2} - \frac{1}{2}\left(\frac{r}{R_N}\right)^2\right] \\
V_{\circ}(r \geq R_N) &\equiv V_{\cdot}(r) = -\frac{Ze^2}{4\pi\epsilon_0 r}
\end{aligned} \tag{10.18}$$

We evaluate this by combining (10.18) with (10.17) and using the hydrogenic wavefunctions given in NERS311 and also in Krane II (Tables 2.2 and 2.5), and obtain:

$$\Delta E_{1s} = \frac{Ze^2}{4\pi\epsilon_0 R_N}\frac{4Z^3}{a_0^3}\int_0^{R_N} \mathrm{d}r r^2 e^{-2Zr/a_0}\left[\frac{R_N}{r} - \frac{3}{2} + \frac{1}{2}\left(\frac{r}{R_N}\right)^2\right] \ . \tag{10.19}$$

In unitless quantities, we may rewrite the above as:

$$\Delta E_{1s} = Z^2\alpha^2(m_ec^2)\left(\frac{2ZR_N}{a_0}\right)^2\int_0^1 \mathrm{d}z e^{-(2ZR_N/a_0)z}\left[z-\frac{3}{2}z^2+\frac{z^4}{2}\right] . \quad (10.20)$$

Across all the elements, the dimensionless parameter $(2ZR_N/a_0)$ spans the range $2\times10^{-5} \longrightarrow\approx 10^{-2}$. Hence, the contribution to the exponential, in the integral, is inconsequential. The remaining integral is a pure number and evaluates to 1/10. Thus, we may write:

$$\Delta E_{1s} \approx \frac{1}{10}Z^2\alpha^2(m_ec^2)\left(\frac{2ZR_N}{a_0}\right)^2 . \quad (10.21)$$

This correction is about 1 eV for $Z = 100$ and much smaller for lighter nuclei.

Nuclear size determination from an isotope shift measurement

Let us imagine how we are to determine the nuclear size, by measuring the energy of the photon that is given off, from a $2p \to 1s$ transition.

The Schrödinger equation predicts that the energy of the photon will be given by:

$$(E_{2p\to1s})_\circ = (E_{2p\to1s})_\cdot + \langle\psi_{2p}|V_\circ(r) - V_\cdot(r)|\psi_{2p}\rangle - \langle\psi_{1s}|V_\circ(r) - V_\cdot(r)|\psi_{1s}\rangle , \quad (10.22)$$

or,

$$(\Delta E_{2p\to1s})_\circ = \langle\psi_{2p}|V_\circ(r) - V_\cdot(r)|\psi_{2p}\rangle - \langle\psi_{1s}|V_\circ(r) - V_\cdot(r)|\psi_{1s}\rangle , \quad (10.23)$$

expressing the change in the energy of the photon, due to the effect of finite nuclear size.

The latter term, $\langle\psi_{1s}|V_\circ(r) - V_\cdot(r)|\psi_{1s}\rangle$, has been calculated in (10.21). We now consider the former term, $\langle\psi_{2p}|V_\circ(r) - V_\cdot(r)|\psi_{2p}\rangle$. Figure 10.7 shows the $1s$ and $2p$ hydrogenic radial probabilities for the $1s$ and $2p$ states, each divided by their respective maxima. (This corresponds to having divided the $2p$ function by a factor of about 89.) The vertical line near the origin is the radius of an $A = 208$ nucleus, assuming $R_N = 1.22A^{1/3}$. That radius has been multiplied by a factor of 10 for display purposes. The actual value is $ZR_N/a_0 = 0.0112$, assuming further, that $Z = 82$.

As can be seen from this figure, the overlap of the $2p$ state is many orders of magnitude smaller than that of the $1s$ state. Hence, the term $\langle\psi_{2p}|V_\circ(r) - V_\cdot(r)|\psi_{2p}\rangle$ may be safely ignored in (10.23). Therefore, we can conclude, from (10.21), that the photon's energy is reduced by,

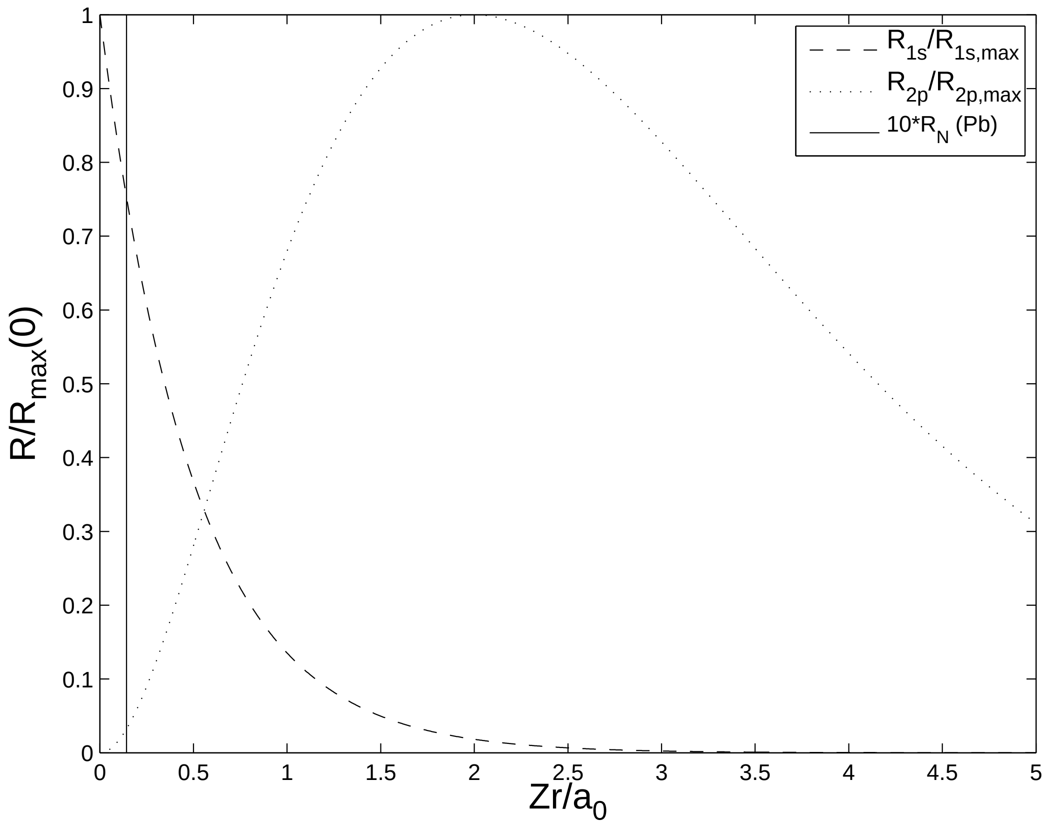


Figure 10.7: Overlap of $1s$ and $2p$ electronic orbitals with the nuclear radius. The nuclear radius depicted is for $A = 208$ and has been scaled upward by 10 for display purposes.

$$\Delta E_{2p\to 1s} \approx -\frac{1}{10} Z^2\alpha^2(m_ec^2)A^{2/3}\left(\frac{2ZR_0}{a_0}\right)^2 \ , \tag{10.24}$$

for a uniformly charged nucleus with radius $R_N = R_0A^{1/3}$.

However, we have yet to make the connection to a measurement, because the measurement of a photon's energy from a realistically shaped nucleus can not be compared with that of an identical atom with a point nucleus. That does not exist in nature. Instead, consider the following: the transition energy for two isotopes of the same element, A and A'. The difference in this transition energy may be determined experimentally, and we obtain:

$$\Delta E_{2p\to 1s}(A) - \Delta E_{2p\to 1s}(A') = \frac{1}{10} Z^2\alpha^2(m_ec^2)\left(\frac{2ZR_0}{a_0}\right)^2 (A'^{2/3} - A^{2/3}) \ . \tag{10.25}$$

The measured quantity is called the K X-ray *isotope shift*. The following few pages show measurements of isotope shifts, for K X-Rays and optical photon isotope shifts.

Figure 10.8: Fig 3.6 from Krane, K X-ray shifts for Hg.

Figure 10.9: Fig 3.7 from Krane, optical shifts for Hg.

A better probe of nuclear shape can be done by forming muonic atoms, formed from muons (usually from cosmic rays), that replace an inner K-shell electron, and has significant overlap of its wavefunction with the nucleus.

Figure 10.10: Fig 3.8 from Krane, K X-ray shifts for muonic Fe.

All these data are consistent with a nuclear size with a radius, $R_N = R_0 A^{/3}$, and a value for $R_0 \approx 1.2$ fm.

Charge radius from Coulomb energy in mirror nuclei

A mirror-pair of nuclei are two nuclei that have the same atomic mass, but the number of protons in one, is the number of neutrons in the other, and the number of protons and neutrons in one of the nuclei differs by only 1. So, if Z is the atomic number of the higher atomic number mirror nucleus, it has $Z-1$ neutrons. Its mirror pair has $Z-1$ protons and Z neutrons. The atomic mass of both is $2Z-1$. Examples of mirror pairs are: ^{3}H/^{3}He, and ^{39}Ca/^{39}K.

These mirror-pairs are excellent laboratories for investigating nuclear radius since the nuclear component of the binding energy of these nuclei ought to be the same, if the strong force does not distinguish between nucleons. The only remaining difference is the Coulomb self-energy. For a charge distribution with Z protons, the Coulomb self-energy is:

$$E_{\mathrm{C}} = \frac{1}{2}\frac{Z^2e^2}{4\pi\epsilon_0}\int \mathrm{d}\vec{x}_1 \rho_p(\vec{x}_1) \int \mathrm{d}\vec{x}_2 \rho_p(\vec{x}_2) \frac{1}{|\vec{x}_1 - \vec{x}_2|} \ . \tag{10.26}$$

The factor of 1/2 in front of (10.26) accounts for the double counting of repulsion that takes place when one integrates over the nucleus twice, as implied in (10.26).

For a uniform, spherical charge distribution of the form,

$$\rho_p(\vec{x}) = \frac{3}{4\pi R_N^3}\Theta(R_N - r) \ . \tag{10.27}$$

Figure 10.11: Fig 3.9 from Krane, composite K X-ray shift data.

As shown below:

$$E_{\rm C} = \frac{3}{5}\frac{Z^2e^2}{4\pi\epsilon_0 R_N} \ . \tag{10.28}$$

For a uniform, spherical charge distribution, given by (10.27):

$$\begin{aligned} E_{\rm C} &= \frac{1}{2}\frac{Z^2e^2}{4\pi\epsilon_0}\left(\frac{3}{4\pi R_N^3}\right)^2 \int_{|\vec{x}_1|\le R_N} {\rm d}\vec{x}_1 \int_{|\vec{x}_2|\le R_N} {\rm d}\vec{x}_2 \frac{1}{|\vec{x}_1-\vec{x}_2|} \\ &= \frac{1}{2}\frac{Z^2e^2}{4\pi\epsilon_0 R_N}\left(\frac{3}{4\pi}\right)^2 \int_{|\vec{u}_1|\le 1} {\rm d}\vec{u}_1 \int_{|\vec{u}_2|\le 1} {\rm d}\vec{u}_2 \frac{1}{|\vec{u}_1-\vec{u}_2|} \\ &= \frac{1}{2}\frac{Z^2e^2}{4\pi\epsilon_0 R_N} I \ , \end{aligned} \tag{10.29}$$

where

$$I = \left(\frac{3}{4\pi}\right)^2 \int_{|\vec{u}_1|\le 1} {\rm d}\vec{u}_1 \int_{|\vec{u}_2|\le 1} {\rm d}\vec{u}_2 \frac{1}{|\vec{u}_1-\vec{u}_2|} \ . \tag{10.30}$$

From (10.30), one sees that I has the interpretation as a pure number representing the average of $|\vec{u}_1-\vec{u}_2|^{-1}$, for two vectors, $\vec{u}_1$ and $\vec{u}_2$, integrated uniformly over the interior of a unit sphere. So, now it just remains, to calculate I. We'll work this out explicitly because the calculation is quite delicate. Features of this derivation are seen in several areas of Nuclear and Radiological Science.

Expanding the 3-dimensional integrals in (10.30) results in:

$$I = \left(\frac{3}{4\pi}\right)^2 \int_0^{2\pi} {\rm d}\phi_1 \int_0^{\pi} {\rm d}\theta_1 \sin\theta_1 \int_0^1 {\rm d}u_1 u_1^2 \int_0^{2\pi} {\rm d}\phi_2 \int_0^{\pi} {\rm d}\theta_2 \sin\theta_2 \int_0^1 {\rm d}u_2 u_2^2 \frac{1}{|\vec{u}_1-\vec{u}_2|} \ .$$

The following expression results from having done both azimuthal integrals, once having aligned the z-axis of the coordinate system with $\vec{u}_1$, when performing the 3 inner integrals. Then with the transformation $\cos\theta_1 \longrightarrow \mu_1$ and $\cos\theta_2 \longrightarrow \mu_2$, we obtain:

$$\begin{aligned} I &= \left(\frac{9}{2}\right) \int_0^1 {\rm d}u_1 u_1^2 \int_0^1 {\rm d}u_2 u_2^2 \int_{-1}^1 {\rm d}\mu_2 \frac{1}{\sqrt{u_1^2+u_2^2-2u_1u_2\mu_2}} \\ &= \left(\frac{9}{2}\right) \int_0^1 {\rm d}u_1 u_1 \int_0^1 {\rm d}u_2 u_2 [(u_1+u_2)-|u_1-u_2|] \end{aligned}$$

$$
\begin{aligned}
&= 9\int_0^1 \mathrm{d}u_1 u_1 \left[\int_0^{u_1} \mathrm{d}u_2 u_2^2 + u_1 \int_{u_1}^1 \mathrm{d}u_2 u_2\right] \\
&= 9\int_0^1 \mathrm{d}u_1 \left[\frac{u_1^2}{2} - \frac{u_1^4}{6}\right] \\
&= 9\left[\frac{1}{6} - \frac{1}{30}\right] \\
&= \frac{6}{5} \; . \qquad (10.31)
\end{aligned}
$$

A common error in performing the above integral results from ignoring the absolute value in the 2nd step. Recall that $\sqrt{a^2} = |a|$, not a.

Finally, combining (10.29) and (10.31) gives us the final result expressed in (10.28).

The Coulomb energy differences are measured through β-decay endpoint energies (more on this later in the course), which yield very good information on the nuclear radius. The difference in Coulomb energies is given by:

$$
\begin{aligned}
\Delta E_{\mathrm{C}} &= \frac{3}{5}\frac{e^2}{4\pi\epsilon_0 R_N}[Z^2 - (Z-1)^2] \\
&= \frac{3}{5}\frac{e^2}{4\pi\epsilon_0 R_N}(2Z-1) \\
&= \frac{3}{5}\frac{e^2}{4\pi\epsilon_0 R_0}A^{2/3} \; , \qquad (10.32)
\end{aligned}
$$

where, in the last step, we let $R_N = R_0 A^{1/3}$. (Recall, $A = 2Z - 1$ for mirror nuclei.)

Figure 10.12: Fig 3.10 from Krane, Coulomb energy differences.

10.2 Mass and Abundance of Nuclei

Note to students: Read 3.2 in Krane on your own. You are responsible for this material, but it will not be covered in class.

10.3 Nuclear Binding Energy

In this section , we discuss several ways that the binding energy of the nucleus is tabulated in nuclear data tables. Nuclear binding energy is always related to the atomic mass, an experimentally derived quantity, one that is obtained with great precision through spectroscopy measurements, at least for nuclei that are stable enough. We start by discussing the binding energy of an atom, and then draw the analogy with the binding energy of the nucleus.

The rest mass energy of a neutral atom, $m_{_A}c^2$, and the rest mass energy of its nucleus, $m_{_N}c^2$, are related by:

$$m_{_A}c^2 = m_{_N}c^2 + Zm_ec^2 - B_e(Z, A) \ , \tag{10.33}$$

where $B_e(Z, A)$ is the *electronic* binding, the sum of the binding energies of all the electrons in the atomic cloud. The total electronic binding energy can be as large as 1 MeV in the heavier atoms in the periodic table. However, this energy is swamped by factors of 10^5–10^6 by the rest mass energy of the nucleus, approximately $A \times 1000$ MeV. Hence, the contribution of the electronic binding is often ignored, particularly when mass *differences* are discussed, as the electronic binding component largely cancels out. We shall keep this in mind, however.

One may estimate the total electronic binding as done in the following example.

Technical aside: Estimating the electronic binding in Pb:
Lead has the following electronic configuration:
$1s^22s^22p^63s^23p^63d^{10}4s^24p^64d^{10}5s^25p^64f^{14}5d^{10}6s^26p^2$,
or, occupancies of 2, 8, 18, 32, 18, 4 in the $n = 1, 2, 3, 4, 5, 6$ atomic shells. Thus,

$$B_e(82, 208) \approx (82)^2(13.6 \text{ eV})\left(2 + \frac{8}{2^2} + \frac{18}{3^2} + \frac{32}{4^2} + \frac{18}{5^2} + \frac{4}{6^2}\right) = 0.8076 \text{ MeV} \ .$$

This is certainly an overestimate, since electron repulsion in the atomic shells has not been accounted for. However, the above calculation gives us some idea of the magnitude of the total electronic binding. (A more refined calculculation gives 0.2074 MeV, indicating that the overestimate is as much as a factor of 4.) So, for the time being, we shall ignore the total electronic binding but keep it in mind, should the need arise.

By analogy, and more apropos for our purposes, we state the formula for the *nuclear binding energy*, $B_N(Z, A)$, for atom X, with atomic mass $m(^AX)$:

$$B_N(Z, A) = \left\{Zm_p + Nm_n - \left[m(^AX) - Zm_e\right]\right\} c^2 \ . \tag{10.34}$$

Since

$$m_p + m_e \approx m(^1H) \ ,$$

we may rewrite (10.34) as

$$B_N(Z,A) = [Zm(^1H) + Nm_n - m(^AX)]c^2 \ . \tag{10.35}$$

We emphasize, however, that electron binding energy is being ignored, henceforth[2].

Thus, we have obtained the binding energy of the nucleus in terms of the atomic mass of its neutral atom, $m(^AX)$. Conventionally, atomic masses are quoted in terms of the *atomic mass unit*, u. The conversion factor is $uc^2 = 931.494028(23)$ MeV.

Occasionally, it is the nuclear binding energy that is tabulated (it may be listed as *mass defect* or *mass excess*), in which case that data may be used to infer the atomic mass. A word of warning, however. Don't assume that the uses of *mass defect* or *mass excess* are consistent in the literature. One must always consult with the detailed descriptions of the data tables, to see the exact definition employed in that document.

Separation energies

Other measured data of interest that shine some light on the binding energy, as well as the nuclear structure of a given nucleus, is the neutron separation energy, S_n. That is the energy required to liberate a neutron from the nucleus, overcoming the strong attractive force. From the binding energy expressed in (10.35), we see that S_n takes the form:

$$\begin{aligned} S_n &= B_N\left({}^A_Z X_N\right) - B_N\left({}^{A-1}_{\ \ Z} X_{N-1}\right) \\ &= \left[m\left({}^{A-1}_{\ \ Z} X_{N-1}\right) - m\left({}^A_Z X_N\right) + m_n\right] c^2 \ . \end{aligned} \tag{10.36}$$

The proton separation energy is a similar quantity, except that it also accounts for the repulsion by the other protons in the nucleus.

$$\begin{aligned} S_p &= B_N\left({}^A_Z X_N\right) - B_N\left({}^{A-1}_{Z-1} X_N\right) \\ &= \left[m\left({}^{A-1}_{Z-1} X_N\right) - m\left({}^A_Z X_N\right) + m\left({}^1H\right)\right] c^2 \ . \end{aligned} \tag{10.37}$$

Thus we see from (10.35), that measurement of atomic mass yields direct information on the binding energy. We also see from (10.36) and (10.37), that measurements of neutron and proton separation energies yield direct information on the difference in nuclear binding energy between two nuclei that differ in A by one neutron or proton.

[2] To adapt these equations to account for electronic binding, (10.34) would take the form:

$$B_N(Z,A) = [Z(m_p + m_e) + Nm_{_N} - m(^AX)]c^2 - B_N(Z,A) \ .$$

There are 82 stable[3] elements. $^{209}_{83}$Bi, the most stable isotope of Bi, has a measured half-life of $(19 \pm 2) \times 10^{18}$ years (α-decay). Those 82 stable elements have 256 stable isotopes. Tin[4] has 10 stable isotopes ranging from ^{112}Sn–^{126}Sn. These stable isotopes, plus the more than 1000 unstable but usable nuclei (from the standpoint of living long enough to provide a direct measurement of mass), can have their binding energy characterized by a universal fitting function, the semiempirical formula for $B(Z, A) \equiv B_N(Z, A)$, a five-parameter empirical fit to the 1000+ set of data points. (The subscript N is dropped to distinguish B as the formula derived from data fitting.

Semiempirical Mass Formula – Binding Energy per Nucleon

The formula for $B(Z, A)$ is given conventionally as:

$$B(Z,A) = a_{\rm V} A - a_{\rm S} A^{2/3} - a_{\rm C} Z(Z-1)A^{-1/3} - a_{\rm sym}\frac{(A-2Z)^2}{A} + a_{\rm p}\frac{(-1)^Z[1+(-1)^A]}{2}A^{-3/4} \ . \tag{10.38}$$

The numerical values of the filling constants and the meaning of each term are given in the following table:

a_i	[MeV]	Description	Source
$a_{\rm V}$	15.5	Volume attraction	Liquid Drop Model
$a_{\rm S}$	16.8	Surface repulsion	Liquid Drop Model
$a_{\rm C}$	0.72	Coulomb repulsion	Liquid Drop Model + Electrostatics
$a_{\rm sym}$	23	n/p symmetry	Shell model
$a_{\rm p}$	34	n/n, p/p pairing	Shell model

Table 10.1: Fitting parameters for the nuclear binding energy

The explanation of each term follows:

Volume attraction: This term represents the attraction of a core nucleon to its surrounding neighbors. The nuclear force is short-medium range, therefore, beyond the immediate neighbors, there is no further attraction. Thus we expect this term to be attractive, and proportional to the number of nucleons. Add one nucleon to the core, and the binding energy

[3] Let us use, as a working definition, that "stability" means "no measurable decay rate".

[4] Tin's remarkable properties arise from the fact that it has a "magic" number of protons (50). This "magic" number represents a major closed proton shell, in the "shell model" of the nucleus, that we shall study soon. Tin's remarkable properties don't stop there! Tin has 28 known additional unstable isotopes, ranging from ^{99}Sn–^{137}Sn! It even has a "doubly-magic" isotope, ^{100}Sn, with a half-life of about 1 s, discovered in 1994. Tin is <u>the</u> superstar of the "Chart of the Nuclides". And you thought tin was just for canning soup!

goes up by the same amount, regardless of what A is. Another way to see this is: the "bulk term" is proportional to the volume of material, thus it is proportional to R_N^3, or A, since $R_N \propto A^{1/3}$. This comes from considering the nucleus to be formed of an incompressible fluid of mutually-attracting nucleons, *i.e.* the Liquid Drop Model of the nucleus.

Surface attraction: The volume term overestimates the attraction, because the nucleons at the surface lack some of the neighbors that attract the core nucleons. Since the surface is proportional to R_N^2, this term is proportional to $A^{2/3}$, and is repulsive.

Coulomb repulsion: The Coulomb repulsion is estimated from (10.28). This term is proportional to $1/R_N$, or $A^{-1/3}$. The Z^2 is replaced by $Z(Z-1)$ since a proton does not repulse itself. As discussed previously, this term is derived from Electrostatics, but within the Liquid Drop Model, in which the electrostatic charge is considered to be spread continuously through the drop.

n/p symmetry: The Nuclear Shell Model predicts that nuclei like to form with equal numbers of protons and neutrons. This is reflected by the per nucleon factor of $[(A-2Z)/A]^2$. This "repulsion" minimizes (vanishes) when $Z = N$.

n/n, p/p pairing: The Nuclear Shell Model also predicts that nuclei prefer when protons or neutrons are paired up in $n-n$, $p-p$ pairs. This factor is attractive, for an even-even nucleus (both Z and N are even), repulsive for an odd-odd nucleus, and zero otherwise. The $A^{-3/4}$ term is not easy to explain, and different factors are seen in the literature.

For a graphical representation of B/A, see figure 3.17 in Krane.

Using the expression (10.38) and adapting (10.35), we obtain the *semiempirical mass formula:*

$$m(^AX) = Zm(^1H) + Nm_n - B(Z,A)/c^2 \ , \tag{10.39}$$

that one may use to estimate $m(^AX)$ from measured values of the binding energy, or vice-versa.

Application to β-decay

β-decay occurs when a proton or a neutron in a nucleus converts to the other form of nucleon, $n \to p$, or $p \to n$. (An unbound neutron will also β-decay.) This process preserves A. Therefore, one may characterize β-decay as an isobaric (*i.e.* same A) transition. For fixed A, (10.39) represents a parabola in Z, with the minimum occuring at (Note: There is a small error in Krane's formula below.):

$$Z_{\min} = \frac{[m_n - m(^1\mathrm{H})]c^2 + a_{\mathrm{C}}A^{-1/3} + 4a_{\mathrm{sym}}}{2a_{\mathrm{C}}A^{-1/3} + 8a_{\mathrm{sym}}A^{-1}} \ . \tag{10.40}$$

We have to use some caution when using this formula. When A is odd, there is no ambiguity. However, when the decaying nucleus is odd-odd, the transition picks up an additional loss in mass of $2a_{\rm p}A^{-3/4}$, because an odd-odd nucleus becomes an even-even one. Similarly, when an even-even nucleus decays to an odd-odd nucleus, it picks up a gain of $2a_{\rm p}A^{-3/4}$ in mass, that must be more than compensated for, by the energetics of the β-decay.

Figure 3.18 in Krane illustrates this for two different decays chains.

(10.40) can very nearly be approximated by:

$$Z_{\rm min} \approx \frac{A}{2}\frac{1}{1+(1/4)(a_{\rm C}/a_{\rm sym})A^{2/3}} \ . \tag{10.41}$$

This shows clearly the tendency for $Z \approx N$ for lighter nuclei. For heavier nuclei, $A \approx 0.41$.

Binding Energy per Nucleon

The binding energy per nucleon data is shown in Krane's Figure 3.16 and the parametric fit shown in Krane's Figure 3.17. There are interesting things to note. $B(Z,A)/A$ ...

- peaks at about $A = 56$ (Fe). Iron and nickel (the iron core of the earth) are natural endpoints of the fusion process.
- is about 8 MeV $\pm$ 10 % for $A > 10$.

10.4 Angular Momentum and Parity

The total angular momentum of a nucleus is formed from the sum of the individual constituents angular momentum, $\vec{l}$, and spin, $\vec{s}$, angular momentum. The symbol given to the nuclear angular momentum is I. Thus,

$$\vec{I} = \sum_{i=1}^{A}(\vec{l}_i + \vec{s}_i) \ . \tag{10.42}$$

These angular momenta add in the Quantum Mechanical sense. That is:

$$\begin{aligned}
\langle \vec{I}^2 \rangle &= \hbar^2 I(I+1) \\
I &= 0, \frac{1}{2}, 1, \frac{3}{2} \cdots
\end{aligned}$$

$$\begin{aligned} \langle I_z \rangle &= \hbar m_{_I} \\ m_{_I} &: -I \leq m_{_I} \leq I \\ \Delta m_{_I} &: \text{integral} \end{aligned} \tag{10.43}$$

Since neutron and proton spins are half-integral, and orbital angular momentum is integral, it follows that I is half-integral for odd-A nuclei, and integral for even-A nuclei.

Recall that parity is associated with a quantum number of ± 1, that is associated with the inversion of space. That is, if Π is the parity operator, acting on the composite nuclear wave function, $\Psi(\vec{x}; A, Z)$,

$$\Pi\Psi(\vec{x}; A, Z) = \pm\Psi(-\vec{x}; A, Z) \ . \tag{10.44}$$

The plus sign is associated with "even parity" and the minus sign with "odd parity".

Total spin and parity are measurable, and a nucleus is said to be in an I^π configuration. For example, ^{235}U has $I^\pi = \frac{7}{2}^-$, while ^{238}U has $I^\pi = 0^+$.

10.5 Nuclear Magnetic and Electric Moments

10.5.1 Magnetic Dipole Moments of Nucleons

We have learned from atomic physics, that the magnetic fields generated by moving charges, has a small but measurable effect on the energy levels of bound electrons in an atom. For example, the apparent motion of the nucleus about the electron (in the frame where the electron is at rest), leads to "fine structure" changes in atomic spectra. This arises because the nucleus can be thought of as a closed current loop, generating its own magnetic field, and that magnetic field exerts a torque on the spinning electron. Although the electron is a "point particle", that point charge is spinning, generating its own magnetic field. We know that two magnets exert torques on each other, attempting to anti-align the magnetic poles.

The nucleus itself, is made up of protons and neutrons that have intrinsic spin as well, generating their own "spin" magnetic fields, in addition to the orbital one. That provides an additional torque on the electron spins, resulting in the "hyperfine structure" of atomic energy levels.

"Superhyperfine structure" results from additional torques on the electron resulting from neighboring atoms in condensed materials, yet another set of forces on the electron.

These energy differences are small, but, nonetheless important, for interpreting atomic spectra. However, we are now concerned with nucleons, in a tightly-bound nucleus, all in close proximity to each other, all moving with velocities of about $0.001 \rightarrow 0.1$c. This is a radical departure from the leisurely orbit of an electron about a nucleus. This is a "mosh pit" of thrashing, slamming nucleons. The forces between them are considerable, and play a vital role in the determination of nuclear structure.

The orbital angular momentum can be characterized in classical electrodynamics in terms of a magnetic moment, $\vec{\mu}$:

$$\vec{\mu} = \frac{1}{2}\int \mathrm{d}\vec{x}\ \vec{x} \times J(\vec{x}) \ , \tag{10.45}$$

where $J(\vec{x})$ is the *current density*. For the purpose of determining the orbital angular momentum's contribution to the magnetic moment, the nucleons can be considered to be point-like particles. For point-like particles,

$$\mu = |\vec{\mu}| = g_l l \mu_N \ , \tag{10.46}$$

where l is the orbital angular momentum quantum number, g_l is the *g-factor or gyromagnetic ratio* ($g_l = 1$ for protons, $g_l = 0$ for neutrons, since the neutrons are neutral), and the nuclear magnetron, μ_N is:

$$\mu_N = \frac{e\hbar}{2m_p} \ , \tag{10.47}$$

defined in terms of the single charge of the proton, e, and its mass, m_p. Its current measured value is $\mu_N = 5.05078324(13) \times 10^{-27}$ J/T.

Intrinsic spins of the nucleons also result in magnetic moments. These are given by:

$$\mu = g_s s \mu_N \ , \tag{10.48}$$

where the *spin g factors* are known to be, for the electron, proton, neutron and muon:

Type	g_s (measured)	g_s (theory)
e	-2.002319043622(15)	agree!
p	5.585694713(90)	?
n	-3.82608545(46)	?
μ	-2.0023318414(12)	2.0023318361(10)

A simple(!) application of Dirac's *Relativistic Quantum Mechanics* and *Quantum Electrodynamics* (*aka* QED) leads to the prediction, $g_s = 2$ for the electron. The extra part comes from the *zitterbewegung* of the electron[5]. The fantastic agreement of g_s for the electron, between measurement and theory, 12 decimal places, is considered to be the most remarkable achievement of theoretical physics, and makes QED the most verified theory in existence.

I'm not aware of any theory for the determination of the nucleon g-factors. However, the measured values allow us to reach an important conclusion: The proton must be something very different from a point charge (else its g_s would be close to 2), and the neutron must be made up of internal charged constituents (else its g_s would be 0). These observations laid the groundwork for further investigation that ultimately led to the discovery (albeit indirectly), that neutrons and protons are made up of quarks. (Free quarks have never been observed.) This led to the development of *Quantum Chromodynamics* (*aka* QCD), that describes the the strong force in fundamental, theoretical terms. The unification of QCD, QED, and the weak force (responsible for β-decay) is called *The Standard Model* of particle physics.

Measurement and Theory differ, however, for the muon's g_s. It has been suggested that there is physics beyond The Standard Model that accounts for this.

Measurements of magnetic moments of nuclei abound in the literature. These magnetic moments are composites of intrinsic spin as well as the orbital component of the protons. Nuclear models provide estimates of these moments, and measured moments yield important information on nuclear structure. Table 3.2 in Krane provides some examples. Further exploration awaits our later discussions on nuclear models.

10.5.2 Quadrupole Moments of Nuclei

The electric quadrupole moment is derived from the following considerations.

The electrostatic potential of the nucleus is given by:

$$V(\vec{x}) = \frac{Ze}{4\pi\epsilon_0}\int \mathrm{d}\vec{x}' \; \frac{\rho_p(\vec{x}')}{|\vec{x}-\vec{x}'|} \; . \tag{10.49}$$

Now, imagine that we are probing the nucleus from a considerable distance, so far away from it, that we can only just discern the merest details of its shape. Given that $\rho_p(\vec{x}')$ is highly localized in the vicinity of the nucleus and our probe is far removed from it, we may expand (10.49) in a Taylor expansion in $|\vec{x}'|/|\vec{x}|$. Thus we obtain:

$$V(\vec{x}) = \frac{Ze}{4\pi\epsilon_0}\left[\frac{1}{|\vec{x}|}\int \mathrm{d}\vec{x}' \; \rho_p(\vec{x}') + \frac{\vec{x}}{|\vec{x}|^3}\cdot\int \mathrm{d}\vec{x}' \; \vec{x}'\rho_p(\vec{x}') + \frac{1}{2|\vec{x}|^5}\int \mathrm{d}\vec{x}' \; \left(3(\vec{x}\cdot\vec{x}')^2 - |\vec{x}|^2|\vec{x}'|^2\right)\rho_p(\vec{x}')\cdots\right] . \tag{10.50}$$

[5]According to Wikipedia, the term *zitterbewegung* is derived from German, meaning "trembling motion". According to Zack Ford (NERS312-W10 student), the word is derived from "cittern movements", a "cittern" (or "citter") being an old (Renaissance-era) instrument very similar to a guitar. I like Zack's definition better.

This simplifies to:

$$V(\vec{x}) = \frac{Ze}{4\pi\epsilon_0}\left[\frac{1}{|\vec{x}|} + \frac{Q}{2|\vec{x}|^3}\cdots\right] , \tag{10.51}$$

where

$$Q = \int \mathrm{d}\vec{x} \; \left(3z^2 - r^2\right)\rho_p(\vec{x}) \; . \tag{10.52}$$

We have used $\int \mathrm{d}\vec{x} \; \rho_p(\vec{x}) \equiv 1$ for the first integral in (10.50). This is simply a statement of our conventional normalization of $\rho_p(\vec{x})$. We also used $\int \mathrm{d}\vec{x} \; \vec{x}\rho_p(\vec{x}) \equiv 0$ in the second integral in (10.50). This is made possible by choosing the "center of charge" as the origin of the coordinate system for the integral. Finally, the third integral resulting in (10.52), arises from the conventional choice, when there is no preferred direction in a problem, and set the direction of $\vec{x}'$ to align with the z'-axis, for mathematical convenience.

Technical note: *The second integral can be made to vanish through the choice of a center of charge. This definition is made possible because the charge is of one sign. Generally, when charges of both signs are involved in an electrostatic configuration, and their respective centers of charge are different, the result is a non-vanishing term known as the electric dipole moment. In this case, the dipole moment is given by:*

$$\vec{d} = \int \mathrm{d}\vec{x} \; \vec{x}\rho(\vec{x}) \; .$$

Finally, when it is not possible to choose the z-axis to be defined by the direction of $\vec{x}$, but instead, by other considerations, the quadrupole becomes a tensor, with the form:

$$Q_{ij} = \int \mathrm{d}\vec{x} \; \left(3x_ix_j - |\vec{x}|^2\right)\rho_p(\vec{x}) \; .$$

The quantum mechanics analog to (10.52) is:

$$Q = \int \mathrm{d}\vec{x} \; \psi_N^*(\vec{x})(3z^2 - r^2)\psi_N(\vec{x}) \; , \tag{10.53}$$

where $\psi_N(\vec{x})$ is the composite nuclear wave function. The electric quadrupole moment of the nucleus is also a physical quantity that can be measured, and predicted by nuclear model theories. See Krane's Table 3.3.

Closed book "exam-type" problems

1. **Nuclear Form Factor**

   The nuclear form factor, $F(\vec{q})$, is defined as follows:

$$F(\vec{q}) = \int \mathrm{d}\vec{x}\, \rho_p(\vec{x}) e^{i\vec{q}\cdot\vec{x}} \ .$$

   If $\rho_p(\vec{x})$ is the proton density, normalized so that:

$$\int \mathrm{d}\vec{x}\, \rho_p(\vec{x}) \equiv 1 \ ,$$

   show:

   (a)

$$F(0) = 1$$

   (b)

$$F(\vec{q}) = F(q) = \frac{4\pi}{q} \int r\mathrm{d}r\, \rho_p(r) \sin(qr) \ ,$$

   for spherically symmetric nuclei ($\rho_p(\vec{x}) = \rho_p(r)$).

   (c) Given that:

$$\rho_p(\vec{x}) = N\Theta(R_N - r) \ ,$$

   where R_N is the radius of the nucleus, find an expression for the normalization constant, N above, in terms of R_N.

   (d) For the proton distribution implied by (c), show:

$$F(q) = \frac{3}{(qR_N)^3}[\sin(qR_N) - qR_N \cos(qR_N)] \ .$$

   Hint: $\int \mathrm{d}x\, x \sin x = \sin x - x \cos x$.

   (e) From the expression for $F(q)$ in part (d) above, show that,

$$\lim_{q\to 0} F(q) = 1 \ ,$$

   using either a Taylor expansion, or l'Hôpital's Rule.

2. **Nuclear Coulomb Repulsion Energy**

   (a) The potential felt by a proton due to a charge distribution made up of the $Z-1$ other protons in the nucleus, is given by

   $$V(\vec{x}) = \frac{(Z-1)e^2}{4\pi\epsilon_0} \int \mathrm{d}\vec{x}' \, \frac{\rho_p(\vec{x}')}{|\vec{x}-\vec{x}'|} \ ,$$

   where

   $$\int \mathrm{d}\vec{x}' \, \rho(\vec{x}') \equiv 1 \ .$$

   With the assumption that the protons are uniformly distributed through the nucleus up to radius R_N, that is, $\rho(\vec{x}') = \rho_0$, for $0 \le |\vec{x}'| \le R_N$, show,

   $$V(r) = \frac{(Z-1)e^2}{4\pi\epsilon_0 R_N}\left[\frac{3}{2} - \frac{1}{2}\left(\frac{r}{R_N}\right)^2\right] \ .$$

   From this expression, what can you tell about the force on a proton inside the nucleus?

   (b) The Coulomb self-energy of a charge distribution is given by

   $$E_c = \frac{e}{4\pi\epsilon_0}\left(\frac{1}{2}\right) \int \mathrm{d}\vec{x} \int \mathrm{d}\vec{x}' \, \frac{\rho(\vec{x})\rho(\vec{x}')}{|\vec{x}-\vec{x}'|} \ ,$$

   where

   $$Ze = \int \mathrm{d}\vec{x}' \, \rho(\vec{x}') \ .$$

   Justify the factor of (1/2) in the above expression?

   (c) Starting with the result of part a) or part b) of this problem, show, for a uniform charge distribution $\rho(\vec{x}') = \rho_0$, for $0 \le |\vec{x}'| \le R_N$, that,

   $$E_c = \frac{3}{5}\frac{Z(Z-1)e^2}{4\pi\epsilon_0}\frac{1}{R_N} \ .$$

3. **Nuclear Binding Energy**

   The semi-empirical formula for the total binding energy of the nucleus is:

   $$B(Z,A) = a_{\mathrm{v}}A - a_{\mathrm{s}}A^{2/3} - a_{\mathrm{c}}Z(Z-1)A^{-1/3} - a_{\mathrm{sym}}\frac{(A-2Z)^2}{A} + p(N,Z)a_{\mathrm{p}}A^{-3/4} \ ,$$

   where $p(N,Z)$ is 1 for even-N/even-Z, -1 for odd-N/odd-Z, and zero otherwise.

   (a) Identify the physical meaning of the 5 terms in $B(A,Z)$. Explain which terms would go up, or go down with the addition of one additional neutron or one additional proton.

(b) With A fixed, $B(Z,A)$ is a quadratic expression in Z. Find its minimum, and discuss.

(c) Show that $p_{\rm N,Z}$ can be written:

$$p(Z,N) = \frac{1}{2}[(-1)^Z + (-1)^N] \ ,$$

or equivalently,

$$p(Z,A) = \frac{1}{2}(-1)^Z[(-1)^A + 1] \ .$$

(d) The neutron separation energy is defined by:

$$S_{\rm n} = B(Z,A) - B(Z,A-1) \ ,$$

and the proton separation energy is defined by:

$$S_{\rm p} = B(Z,A) - B(Z-1,A-1) \ .$$

Using the large A approximation, namely:

$$(A-1)^n \approx A^n - nA^{n-1} \ ,$$

develop approximate expressions for $S_{\rm n}$ and $S_{\rm p}$.

4. **Quadrupole Moment**

The quadrupole moment, Q, in the liquid drop model of the nucleus is defined by:

$$Q = \int {\rm d}\vec{x}\rho_p(\vec{x})(3z^2 - r^2) \ ,$$

where $\rho_p(\vec{x})$ is the charge density per unit volume, normalized in the following way:

$$\int {\rm d}\vec{x}\rho_(\vec{x}) \equiv 1 \ .$$

We consider an ellipsoidal nucleus with a sharp nuclear edge, with its surface being given by:

$$\frac{x^2+y^2}{a^2} + \frac{z^2}{b^2} = 1 \ ,$$

where the larger of a or b is the semimajor axis, and the smaller of the two, the semiminor axis.

(a) Sketch the shape of this nucleus, for both the prolate and oblate cases.

(b) Show that the volume of this nucleus is $V = (4\pi/3)a^2b$.

(c) Find an expression for Q, that involves only Z, a, and b.

(d) Discuss the cases $a > b$, $a < b$, $a = b$. Even if you have not found an expression for Q, you should be able to discuss this effectively, given its integral form above, and say something about the sign of Q depending on the relative size of a and b.

5. **Nuclear Structure and Binding Energy**

(a) **Theoretical foundations**

i. What is the liquid drop model of the nucleus?

ii. What is the shell model of the nucleus?

iii. How is Classical Electrostatics employed in describing the structure of the nucleus? Cite two examples.

(b) **Binding energy of the nucleus**

The semi-empirical formula for the total binding energy of the nucleus is:

$$B(Z,A) = a_{\mathrm{V}}A - a_{\mathrm{S}}A^{2/3} - a_{\mathrm{C}}Z(Z-1)A^{-1/3} - a_{\mathrm{sym}}\frac{(A-2Z)^2}{A} + a_{\mathrm{p}}\frac{(-1)^Z[1+(-1)^A]}{2}A^{-3/4}\ .$$

i. In the table below...

A. In the 3rd column, identify the nuclear model that gives rise to this term. The answer is started for you, in the 2nd row. Complete the remaining rows.

B. In the 4th column, identify how this term arises (qualitative explanation) from the nuclear model identified in the 3rd column. The answer is started for you, in the 2nd row. Complete the remaining rows.

C. In the 5thth column, indicate with a “yes”, “no”, or “maybe”, if this term would cause $B(Z,A)$ to go up with the addition of one more proton.

D. In the 6thth column, indicate with a “yes”, “no”, or “maybe”, if this term would cause $B(Z,A)$ to go up with the addition of one more neutron.

ii. Justify the exact dependence on Z and A for the a_{V}, a_{S} and a_{C} terms. Qualitatively explain the dependence on Z and A for the a_{sym} term.

a_i	[MeV]	Theoretical origin	Description	$+p$?	$+n$?
$a_{\mathrm{V}}\cdots$	15.5	The theoretical model that suggests the term starting with a_{V} is the __________ model of the nucleus.	The term involving a_{V} comes from the idea that...		
$a_{\mathrm{S}}\cdots$	16.8				
$a_{\mathrm{C}}\cdots$	0.72				
$a_{\mathrm{sym}}\cdots$	23				
$a_{\mathrm{p}}\cdots$	34				

iii. With A fixed, $B(Z,A)$ is a quadratic expression in Z. Find its extremum, and discuss the relationship between the quadratic expression and β-decay.

6. **The Modeling of Protons in the Nucleus**

One approach to accounting for the effect of the electric charge of the protons, on the structure of the nucleus, is to combine the classical ideas of electrostatics with the liquid drop model of the nucleus. The approach starts with a calculation of the electrostatic potential of the nucleus, $V_{\mathrm{es}}(\vec{x})$, due to a distribution of protons, $\rho_p(\vec{x}')$.

$$V_{\mathrm{es}}(\vec{x}) = \frac{Ze}{4\pi\epsilon_0}\int \mathrm{d}\vec{x}'\ \frac{\rho_p(\vec{x}')}{|\vec{x}-\vec{x}'|} \quad ; \quad \int \mathrm{d}\,\vec{x}'\rho_p(\vec{x}') \equiv 1\ .$$

$\vec{x}$ is a vector that can be positioned anywhere. $\vec{x}'$, the vector over which we integrate, is positioned only within the confines of the nucleus. The origin of the coordinate system for $\vec{x}'$, is located at the *center of charge*. In the following, $\hat{n}$ is a unit vector pointing in the direction of $\vec{x}$.

(a) Show, for $|\vec{x}| >> |\vec{x}'|$, that;

$$\begin{aligned} V_{\rm es}(\vec{x}) &= \frac{Ze}{4\pi\epsilon_0|\vec{x}|}\left[1 + \frac{\hat{n}\cdot\vec{P}}{|\vec{x}|} + \frac{Q}{2|\vec{x}|^2} + \mathcal{O}\left(\frac{1}{|\vec{x}'|^3}\right)\right] \quad \text{where} \\ \vec{P} &\equiv \hat{n}\cdot\int \mathrm{d}\vec{x}'\ \vec{x}'\rho_p(\vec{x}') \\ Q &\equiv \int \mathrm{d}\vec{x}'\ \left[3(\hat{n}\cdot\vec{x}')^2 - |\vec{x}'|^2\right]\rho_p(\vec{x}') \ . \end{aligned}$$

(b) Whether or not you derived the above expression, interpret the meaning of the 3 terms, $(1, \vec{P}, Q)$ in

$$\left[1 + \frac{\hat{n}\cdot\vec{P}}{|\vec{x}|} + \frac{Q}{2|\vec{x}|^2}\right] \ .$$

(c) Why does $\vec{P} = 0$ for the nucleus.

(d) If $\hat{n}$ is aligned with $\hat{z}$, show that:

$$Q = \int \mathrm{d}\vec{x}\ \rho_p(\vec{x})(3z^2 - r^2) \ .$$

(e) Show that $Q = 0$ if $\rho_p(\vec{x})$ is spherically symmetric, but otherwise arbitrary. That is, show:

$$Q = \int \mathrm{d}\vec{x}\ \rho_p(r)(3z^2 - r^2) = 0 \ .$$

(f) The connection between the nuclear wavefunction, $\psi_N(\vec{x})$ and the proton density, in the liquid drop model of the nucleus is given by:

$$|\psi_N(\vec{x})|^2 = \rho_p(\vec{x}) \ .$$

Justify this assumption.

(g) If the nucleus is a uniformly charged ellipsoid of the form:

$$\frac{x^2+y^2}{a^2} + \frac{z^2}{b^2} = 1 \ ,$$

Show that

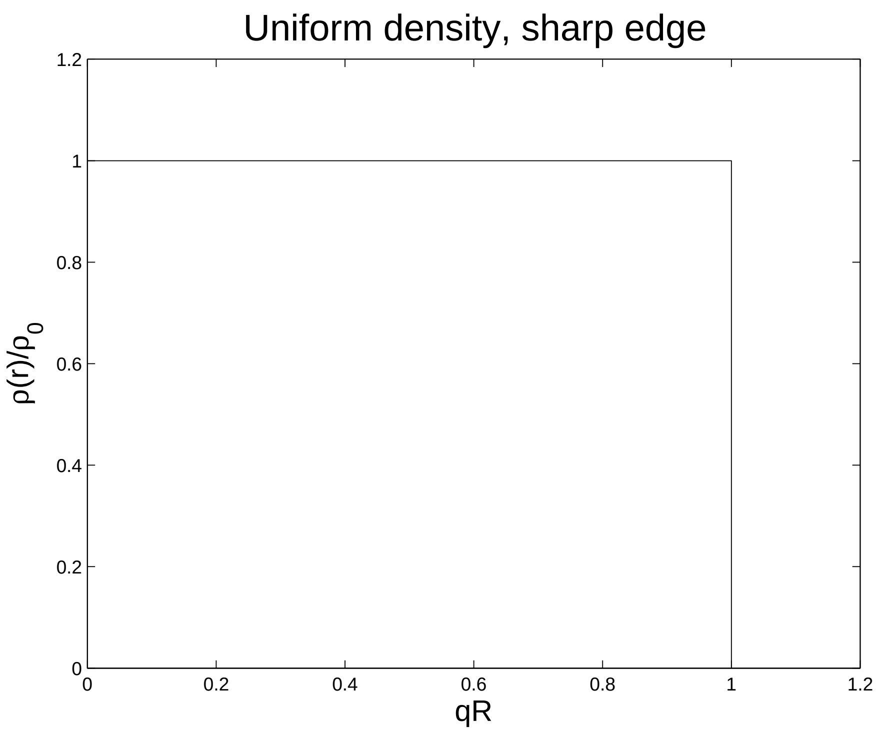
$$Q = \frac{2}{5}(b^2 - a^2) \ .$$

Assignment-type problems

1. If all the matter on earth collapsed to a sphere with the same density as the interior of a nucleus, what would the radius of the earth be? Cite all sources of data you used.

2. **The effect of the nuclear edge on $F(\vec{q})$**

   (a) Find $F(\vec{q})$ for a uniformly charged sphere. For a uniform sphere, $\rho_{\rm p}(r) = \rho_0\Theta(R-r)$, where R is the radius of the nucleus.[6]



   Plot[7] $\log(|F(\vec{q})|^2)$ *vs.* qR.

[6]Mathematical note:

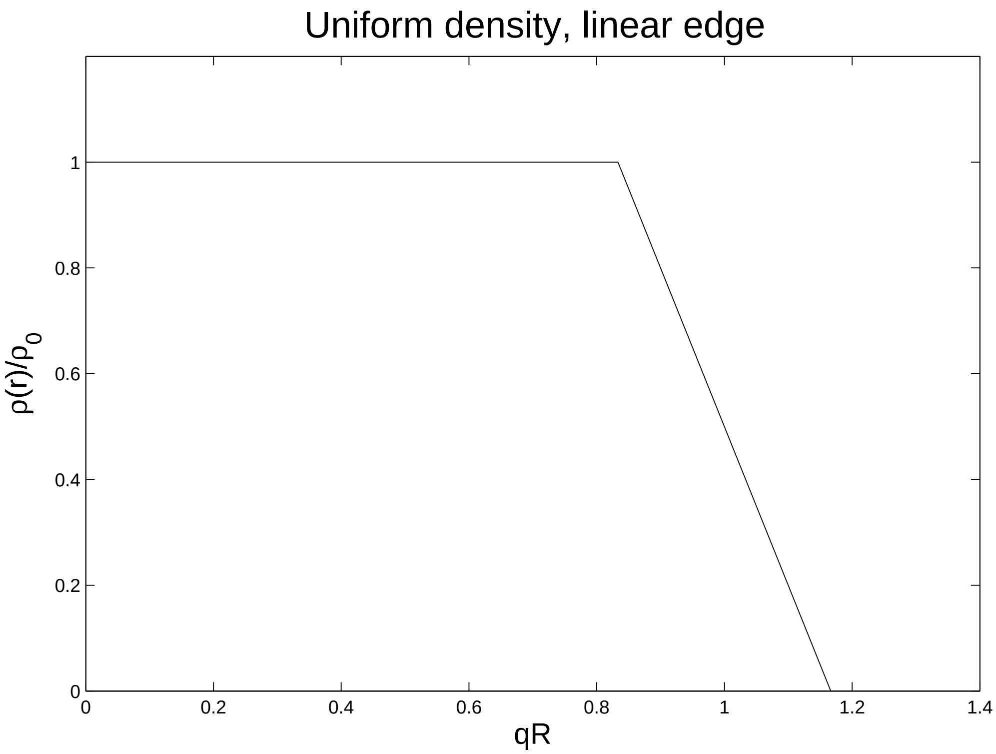
$$\begin{aligned}\Theta(z) &= 1\ ;\ z>0\\ &= 0\ ;\ z<0\end{aligned}$$

[7]Technical note: $|F(\vec{q})|)^2$ can be zero, so its logarithm would be $-\infty$, causing the plots to look strange. We need the logarithm to see the full structure of $|F(\vec{q})|^2$. So, in order to make the plots look reasonable, you will probably have to adjust the scale on the y-axis in an appropriate fashion.

(b) Do the same as in (a) but include a linear edge in the model of the nucleus. That is,

$$\begin{aligned} \rho_{\rm p}(r) &= \rho_0 \Theta([R-t/2]-r) \\ &= \rho_0[1/2-(r-R)/t]\ ;\ R-t/2 \le r \le R+t/2\ , \end{aligned}$$

where t is the nuclear "skin depth", and, mathematically, can take any value between 0 and $2R$. Note that $\rho_{\rm p}(R-t/2) = \rho_0$ and $\rho_{\rm p}(R+t/2) = 0$. Note also, that when $t = 0$, the nuclear shape is the same as in part (a).



Plot $\log(|F(\vec{q})|^2)$ *vs.* qR as in part (a) showing several values of t over the entire domain of t, $0 \le t \le 2R$. Compare with the result of part (a).

(c) Compare and discuss your results. Does the elimination of the sharp edge eliminate the sharp minima in $\log(|F(\vec{q})|^2)$? What else could be contributing to the reduction of the sharpness of these minima?

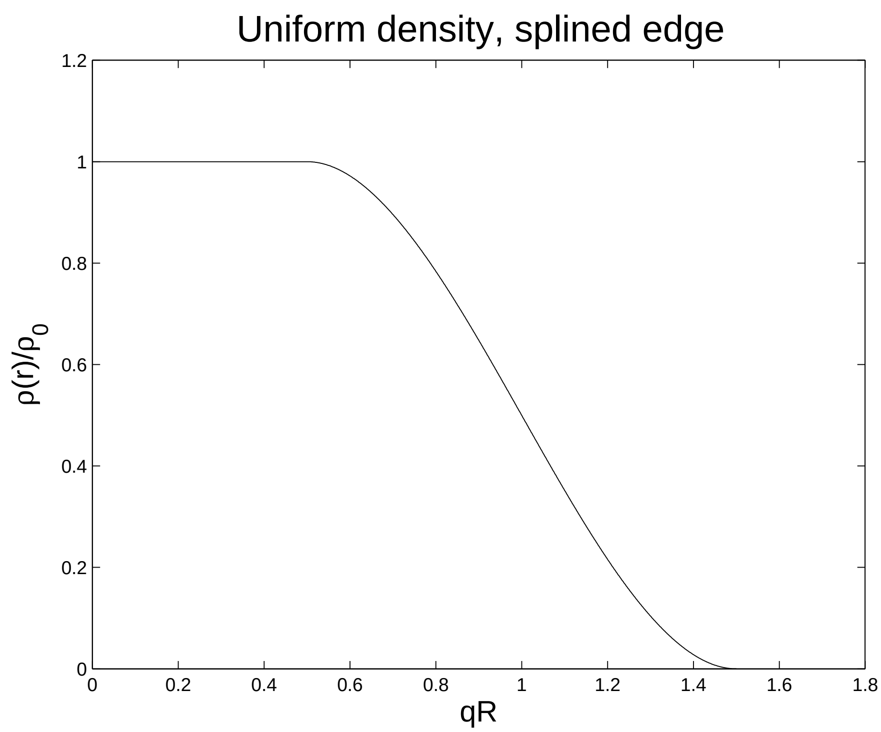
(d) Do the same as in (b) and (c) but include a cubic edge in the model of the nucleus. That is,

$$\begin{aligned}\rho_{\rm p}(r) &= \rho_0\Theta([R-t/2]-r) \\ &= \rho_0(A+Br+Cr^2+Dr^3)\ ;\ R-t/2 \le r \le R+t/2\ .\end{aligned}$$

Arrange the four constants A, B, C, D such that:

$$\begin{aligned}\rho_{\rm p}(R-t/2) &= \rho_0 \\ \rho_{\rm p}'(R-t/2) &= 0 \\ \rho_{\rm p}(R+t/2) &= 0 \\ \rho_{\rm p}'(R+t/2) &= 0\end{aligned}$$

.



Plot $\log(|F(\vec{q})|^2)$ *vs.* qR as in part (b) showing several values of t over the entire domain of t, $0 \le t \le 2R$. Compare with the result of parts (a) and (b).

3. **Perturbation of atomic energy levels**

   (a) Read carefully and understand the text on pages 49–55 in Krane on this topic. Krane makes the assertion that ΔE_{2p} can be ignored for atomic electron transitions. Verify this assertion by repeating the calculation for ΔE_{2p} and obtain a relationship for $\Delta E_{2p}/\Delta E_{1s}$ for atomic electrons and muons. Evaluate numerically for ^{12}C and ^{208}Pb.

   (b) The results in (a), for muons, is suspect, because the muon's wavefunctions have significant overlap with the physical location of the nucleus, and the shape of the edge of the nucleus may play a significant role. Repeat the analysis of part (a), but introduce a skin depth using one of the models in Question 2) or some other model of your choosing. What do you conclude?

4. Show:

$$\frac{6}{5} = \left(\frac{3}{4\pi}\right)^2 \int_{|\vec{u}|\le 1} \mathrm{d}\vec{u} \int_{|\vec{u}'|\le 1} \mathrm{d}\vec{u}' \left(\frac{1}{|\vec{u}-\vec{u}'|}\right) ,$$

   which is used to find the energy of assembly of the protons in a nucleus.

   Then, consider:

$$I(n) = \left(\frac{3}{4\pi}\right)^2 \int_{|\vec{u}|\le 1} \mathrm{d}\vec{u} \int_{|\vec{u}'|\le 1} \mathrm{d}\vec{u}' \left(\frac{1}{|\vec{u}-\vec{u}'|^n}\right) ,$$

   where $n \geq 0$, and is an integer. From this, make conclusions as to the fundamental forces in nature, and their forms as applied to classical and quantum physics.

5. **"Pop Quiz" Review questions on Quantum Mechanics**

   (a) In words, describe "The Compton interaction".

   (b) Derive the relationship between the scattered γ energy and its scattering angle in the Compton interaction.

   (c) What is the time dependent Schrödinger equation in 1D? 3D? What is the time independent Schrödinger equation in 1D? 3D? What is the main application of solutions to the time independent Schrödinger equation?

   (d) Describe and state the expressions (1D and 3D) for the i) probability density, and ii) the probability current density.

   (e) What is the Heisenberg Uncertainty Principle and what does it mean?

   (f) What is the Pauli Exclusion Principle and what does it mean?

   (g) For a particle of mass m with velocity $\vec{v}$, what is its i) momentum, ii) total energy, iii) kinetic energy in both non-relativistic and relativistic formalisms.

   (h) For a massless particle with momentum $\vec{p}$, what is its i) total energy, ii) kinetic energy.

(i) State the Conservation of Energy and the Conservation of Momentum equations for a 2-body interaction involving two masses, m_1 and m_2 with initial velocities $\vec{v}_1$ and $\vec{v}_2$. Perform this in both non-relativistic and relativistic formalisms.

6. We have seen that the calculation of the scattering rate due to electron scattering from a bare nucleus approaches the Rutherford scattering law in the limit of small q. Discuss the onset of the departure of this change, via the nuclear form factor, as q gets larger. Adopt a more realistic model for the distribution of charge in the nucleus and see if you can match the data for the experiments shown in Figures 3.1 and 3.2 in Krane.

7. For low q, the electrons impinging on a nucleus have the positive charge of the nucleus screened by the orbital electrons. Develop an approximate model for the screening of the nuclei by the orbital electrons and demonstrate the effect on the Rutherford scattering law. Show that the forward scattering amplitude is finite and calculate its numerical value using sensible numbers for the parameters of your model.

8. The numerical data of the average shift given in Figure 3.8 of Krane can be matched very closely by introducing a realistic positive charge distribution in the nucleus. Introduce such a model and demonstrate that you obtain the correct numerical answer.

9. Calculate the muonic K X-ray shift for Fe using one of the following nuclear shapes:

$$
\begin{aligned}
\frac{\rho(r)}{\rho_0} &= 1 \text{ for } r \leq R - t_{\rm min}/2 \\
&= \frac{R + t_{\rm min}/2 - r)}{t_{\rm min}} \text{ for } R - t_{\rm min}/2 \leq r \leq R + t_{\rm min}/2 \\
&= 0 \text{ for } r \geq R + t_{\rm min}/2
\end{aligned}
$$

or, the Fermi distribution,

$$
\frac{\rho(r)}{\rho_0} = \frac{1}{1 + \exp[(r - R)/a]} \ .
$$

Compare with data. You may use $R = R_0 A^{1/3}$ with any value between 1.20 and 1.25 fm to got the K X-rays in the right place. Use $t_{\rm min} = 2.3$ fm, and make sure to interpret a in the right way with respect to $t_{\rm min}$, if you use the Fermi distribution.

5. Strange nuclear material

The contents of a nucleus

So far, an atomic nucleus has been described as a collection of Z protons and N neutrons. The protons each carry one unit of positive charge, whereas the neutrons are uncharged and are very slightly heavier than protons. A nucleus has been assumed to contain only protons and neutrons. However, enough quantum mechanical surprises have been introduced to suggest that, when it comes to nuclei, nothing is as simple as it seems.

When nuclei fall apart

One way of finding the constituents of nuclei is to observe what is emitted from them. In radioactivity, some nuclei spontaneously eject particles and change into different kinds of nuclei without any prompting. Stable nuclei, however, need inducement before they will eject particles. In fact, any nucleus can be broken apart by hitting it hard enough with another nucleus. This can be done in the laboratory by using high energy particles from accelerators, but it also happens naturally in stars.

Nuclear reactions were important even before stars existed; most of the nuclei in the Universe were created a few minutes after the Big Bang. At that time, the Universe was an extremely hot and crowded place, with nuclei being formed and, almost as fast, broken apart. Almost immediately, the Universe expanded and cooled sufficiently for nuclei not to be broken apart as soon as they were created. These surviving nuclei formed the primordial material which eventually condensed into stars.

Radioactivity and the constitution of nuclei

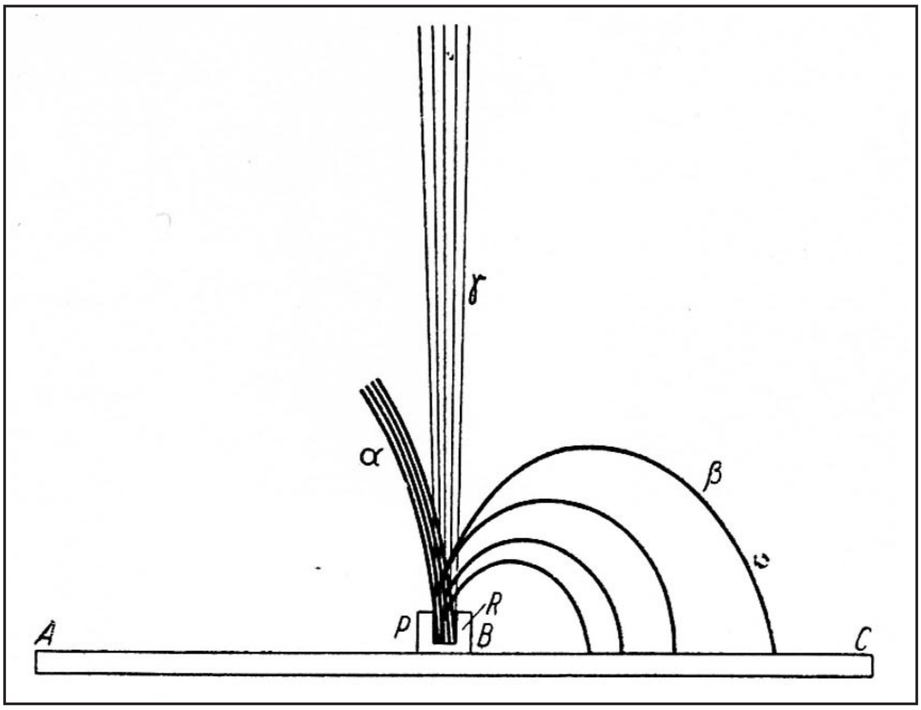
The idea that most of the mass of atoms resides in compact nuclei was still years in the future when Rutherford first noticed that radioactivity involved two kinds of particles. It soon became clear that beta rays were J.J. Thomson's recently discovered electrons, since their paths were bent by magnetic fields in exactly the same way. They also penetrated matter much more readily than alpha particles. It took a little longer to identify alpha particles as helium nuclei.

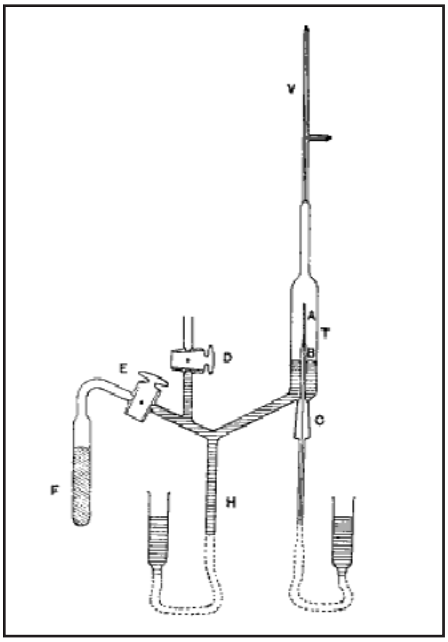
One possible conclusion to be drawn from radioactivity is that atomic nuclei are made of alpha particles and electrons, but this would not work for hydrogen since the hydrogen nucleus has much less mass than an alpha particle. To solve this problem, a very old idea was revived: that all atoms are composed of multiples of hydrogen. This idea originated with William Prout in 1815, and is known as Prout's hypothesis. It was applied a hundred years later to nuclei rather than atoms; all nuclei were predicted to be composed of various numbers of hydrogen nuclei (protons).

This theory cannot be quite right as, for example, the nucleus of a nitrogen atom has the mass of fourteen protons, but the charge of just seven protons. The text-books of the 1920s had a simple answer to this: the nitrogen nucleus has fourteen

Do nuclei have alpha particles within them? By far the most common isotope of all the even Z elements from carbon to sulphur have the same number of protons and neutrons as three, four, five etc. alpha particles. It is not true for argon for which ^{36}Ar has an abundance of less than 1%. These facts are best explained in terms of the the energy valley (discussed in Chapter 6) and the tendency of protons and neutrons to pair up. Before the discovery of the neutron, it was natural to suppose that nuclei are composed of protons, electrons and alpha particles.

A figure from Marie Curie's thesis illustrating the way that alpha rays are bent very little by a magnetic field whereas beta rays are bent much more. Gamma rays are not bent at all. (ACJC-Archives Curie and Joliot-Curie.)





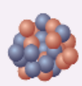
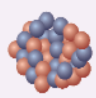
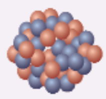
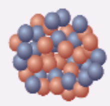
The apparatus used by Rutherford and Royds in 1908 to prove that alpha particles were helium ions (in modern terms: helium nuclei). Alpha particles penetrated through a thin window into a vacuum chamber. When an electric discharge was passed through this chamber the light given out was analysed by a spectroscope, which confirmed the presence of helium.

protons and also seven electrons to balance out the charge of half the protons. After all, there must be electrons inside atomic nuclei since they come out of them in beta decay.

In this proton–electron model, alpha particles would consist of four protons and two electrons. Many nuclei decay by ejecting this particular structure from the nucleus as a complete particle, so there must be something special about this configuration. Another feature also suggests the helium nucleus is somehow special: the most common isotopes of many light elements are apparently multiples of the alpha particle. By far the most common isotope of carbon, for example, is ^{12}C which has the same charge and mass as three alpha particles. ^{13}C is very uncommon and ^{14}C is unstable. From this, the alpha particle appears to be special in some way, and indeed some nuclei do contain alpha particles but not in a straightforward way, as we see at the end of this chapter.

Do nuclei contain electrons?

Like alpha particles, electrons appear to be ejected from nuclei. Unlike alpha particles, there is a very straightforward answer to the question of whether they exist in nuclei. They do not, and they cannot.

^{12}C	^{16}O	^{20}Ne	^{24}Mg	^{28}Si	^{32}S	^{36}Ar	^{40}Ca
							
3α	4α	5α	6α	7α	8α	9α	10α
98.9%	99.8%	90.5%	79.0%	92.2%	95.0%	<1%	96.9%

Firstly, the absence of electrons within nuclei can be understood from Heisenberg's Uncertainty Principle. This principle is commonly, and wrongly, taken to mean that everything on the microscopic level is uncertain. In fact, it implies that the more we try to localize a quantum particle such as an electron – that is to confine it to a smaller and smaller region of space – the larger the fluctuations in its motion due to the uncertainty in its momentum. The result is that for an electron to be confined to the tiny nucleus, its motion would be too violent for it to stay there for long. Protons on the other hand, which are nearly two thousand times heavier than electrons, can easily be confined to the volume of a nucleus since they move about more slowly and the uncertainty in their motion is much less.

James Chadwick, 1891–1974. His discovery of the neutron in 1932 was a turning point in nuclear physics. Chadwick was well aware that the neutron had been predicted by Rutherford in 1920. (Copyright the Nobel Foundation.)

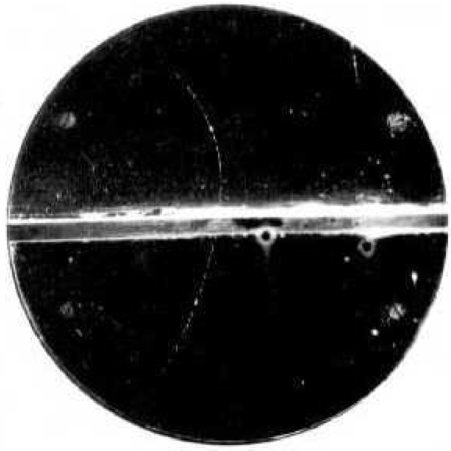
Another argument against atomic nuclei having any electrons in them comes from the light emitted when an electric discharge is passed through nitrogen gas. The fine details in the pattern of spectral lines indicate clearly, according to certain quantum rules, that the nitrogen nucleus has an even number of particles, but if this nucleus is made of protons and electrons it would have to be an odd number: fourteen protons plus seven electrons.

In 1932, Chadwick's discovery of the neutron, which is a neutral particle with about the same mass as a proton, solved this problem. Werner Heisenberg then suggested, correctly, that atomic nuclei consist purely of protons and neutrons. Chadwick's discovery was to mark the beginning of modern nuclear physics. Now the nitrogen nucleus would have seven protons and seven neutrons: an even number of particles in total.

The question now arose that if nuclei consist of just protons and neutrons, where do the beta decay electrons come from? These electrons also posed another problem, one of the greatest problems of physics in the twentieth century, and a problem which drove physicists to desperate remedies.

Missing particles

Beta decay electrons appeared to break the law of conservation of energy. The nucleus emitting the electron has a definite energy, and the nucleus produced in the decay has a definite energy, but the electrons emerge with a range of energies always less than the difference in energy between the initial and final nuclei. The missing amount of energy suggested energy is not always conserved in beta decay, but as the law of conservation of energy is so sacred in physics, few physicists dared to speculate that there was an exception to the rule. Despite this, Niels Bohr did just that – and turned out to be wrong.



The first evidence for the positron, the anti-particle of the electron. The cloud chamber track bends the wrong way in a magnetic field for it to be caused by an electron, and it is not a proton track. Anderson's unprecedented discovery was highly controversial until it was confirmed a year later. (Courtesy C.D. Anderson.)

Another proposal, made by Wolfgang Pauli in 1930, was that a second, virtually undetectable particle was emitted at the same time as the electron. This new particle, later called a neutrino (meaning 'little neutral one'), shared energy with the electron in a random way such that the total energy of the two particles was equal to the difference in energy between the parent and daughter nuclei. In this manner, energy is conserved after all.

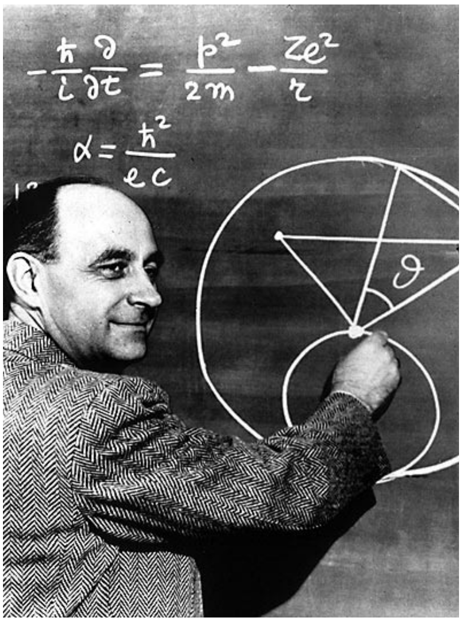
Pauli's idea was radical, but it became a key part of a theory of beta decay proposed by the brilliant Italian physicist, Enrico Fermi, in 1934. Over the years, Fermi's theory has been greatly extended and incorporated into more modern models, but still successfully accounts for a wide range of phenomena connected with beta decay. According to this theory, in a nucleus that has more than the ideal number of neutrons, a neutron turns into a proton, creating both an electron and a neutrino.

Wolfgang Pauli, 1900–1958, and Niels Bohr, 1885–1962. One of Pauli's many contributions to quantum physics concerned the spin of elementary particles such as electrons. The spin of much larger objects like this top was clearly also of great interest. (AIP Emilio Segrè Visual Archives.)

Neutrinos are exceptionally hard to detect. More than 150 million neutrinos pass through every square centimetre of your body every second, and even a body the size of the Earth will very rarely stop a neutrino. It is therefore not surprising that it was 1956 before Reines and Cowan could announce neutrinos had been positively detected. This was long after Pauli's speculation (1930) and Fermi's theory (1934), but proved that they had been correct. Today, neutrinos play a vital role in astronomy and cosmology as well as in nuclear physics.

Antimatter

Fermi's theory embodies one of the most revolutionary ideas of twentieth century physics: the number of fundamental particles in the world is not fixed. Electrons are not emitted from nuclei in beta decay, but are created, along with neutrinos. The idea that particles can be created was one consequence of combining quantum mechanics with Einstein's special theory of relativity, a feat achieved by Paul Dirac.



Enrico Fermi, 1901–1954, was unique in the 20th century for his eminence in both theoretical and experimental physics. (AIP Emilio Segrè Visual Archives.)

Combining relativity and quantum mechanics had another consequence: the existence of anti-particles. Dirac's theory predicted that for every kind of particle there is a corresponding anti-particle having the same mass. If the particle has an electric charge, the anti-particle has the opposite charge, but even neutral particles like neutrons have anti-particles. The anti-particle for the electron, called the positron, was discovered in 1932 by Carl Anderson.

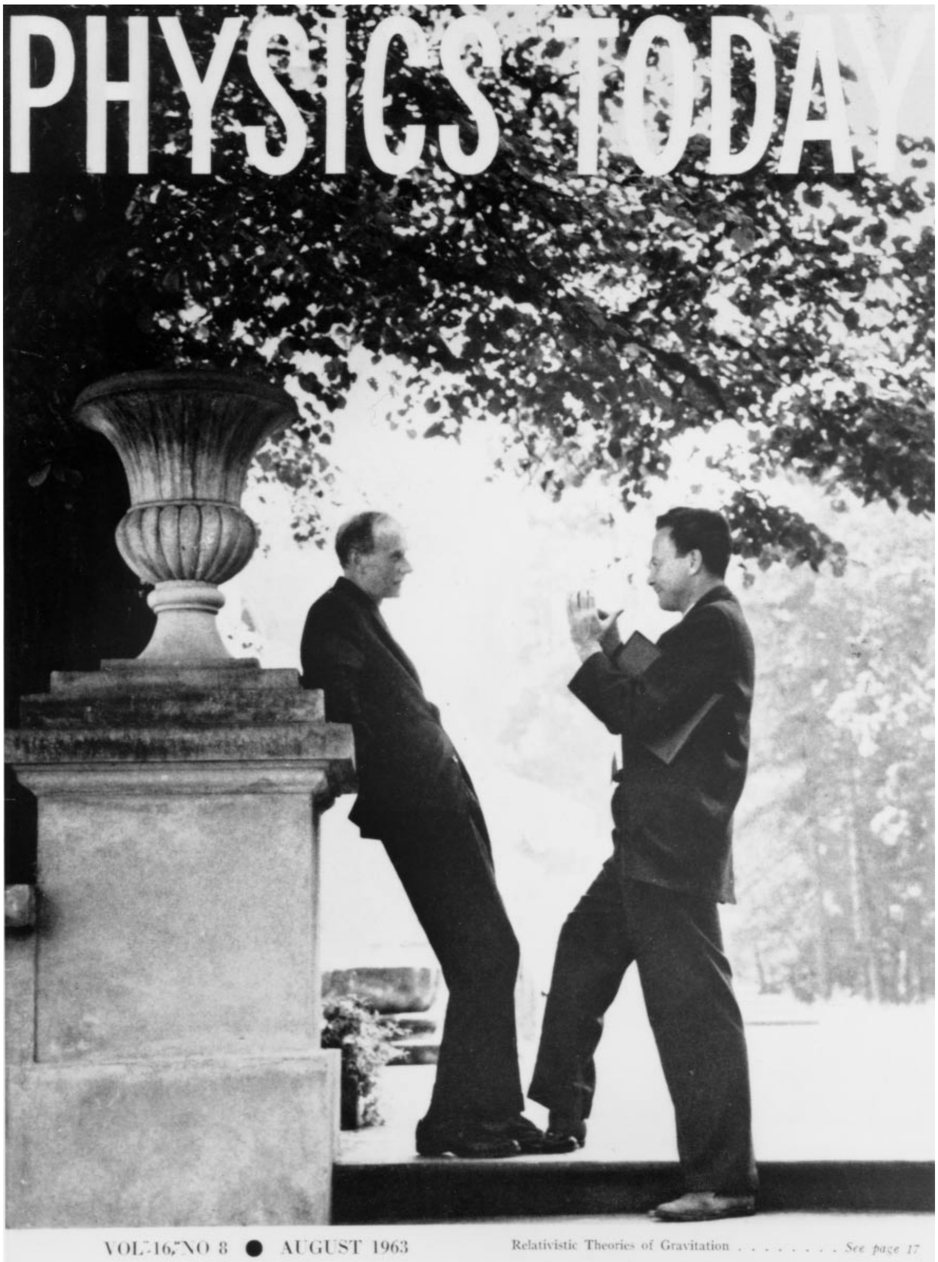
When a particle meets its anti-particle, they annihilate each other releasing pure energy that is carried off as radiation. The energy released is that predicted by Einstein's famous equation, $E=mc^2$. When an electron and a positron annihilate each other, for example, the energy released, E, is equal to their combined mass, m, multiplied by the square of the speed of light. The process of annihilation is reversible and when a high energy photon, maybe from cosmic rays, interacts with a nucleus it quite often disappears, with its energy reappearing as two particles, an electron–positron pair.

Beta decay as it was originally discovered takes place when nuclei with too many neutrons correct the imbalance by transmuting a neutron into a proton, an electron and an anti-neutrino. Nuclei with the opposite imbalance, too many protons, decay in the opposite fashion: a proton is converted into a neutron, a positron and a neutrino. This kind of beta radioactivity was first discovered by Irène Joliot-Curie (Marie Curie's daughter) and Frederic Joliot in 1934 and is now of immense importance in modern medicine, particularly in positron emission tomography (known as PET for short). PET is a much used diagnostic tool, especially important for diagnosing brain disease.

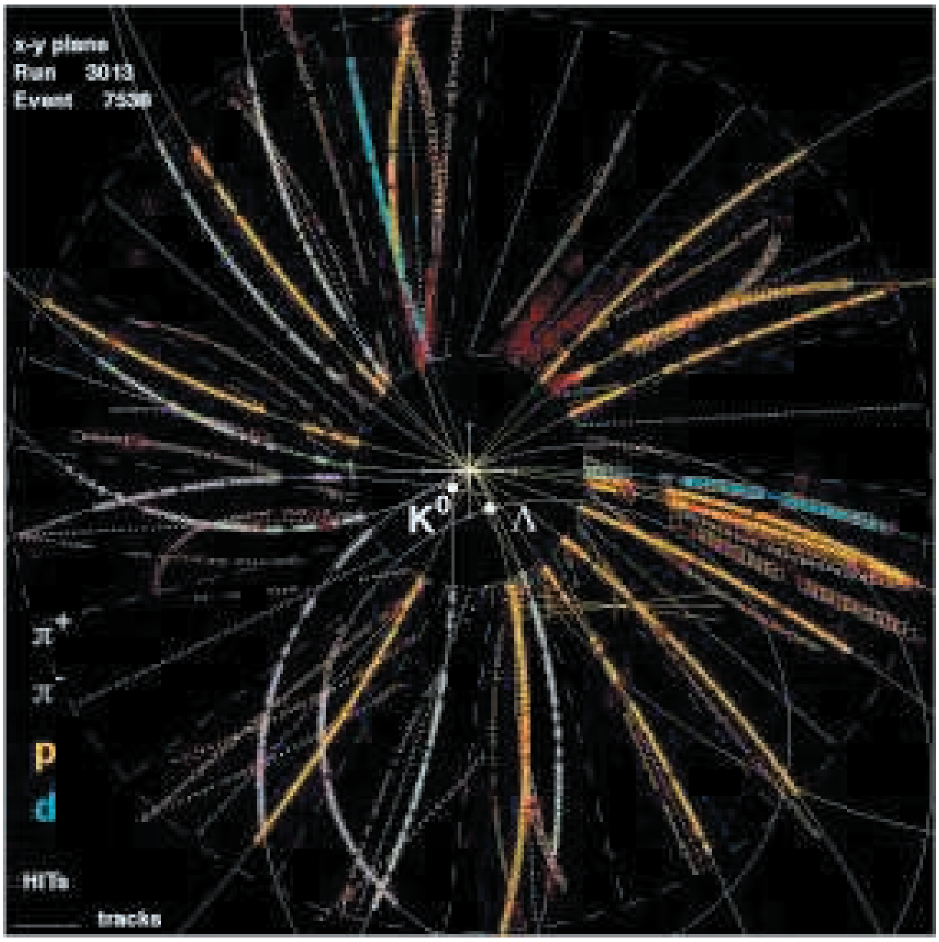
Chadwick's discovery of the neutron provided the foundations for a very successful picture of nuclear structure which has been the cornerstone of nuclear physics for many years. In this model, nuclei consist of protons and neutrons, and the emission of electrons and positrons does not mean that these particles are contained within them.

Carl D. Anderson, 1905–1991, discoverer of the positron. (AIP Emilio Segrè Visual Archives.)

Owen Chamberlain, 1920–, together with Emilio Segrè, discovered the antiproton, the negatively charged anti-particle of the proton, in 1955. (Copyright the Nobel Foundation.)



Paul Adrien Maurice Dirac, 1902–1984, left, and Richard P. Feynman, 1918–1988. Dirac was the first to successfully combine quantum theory and relativity; in doing so, he predicted the existence of positrons. Feynman developed a formalism for doing the very complex calculations involving electrons, positrons and light. (AIP Emilio Segrè Visual Archives.)



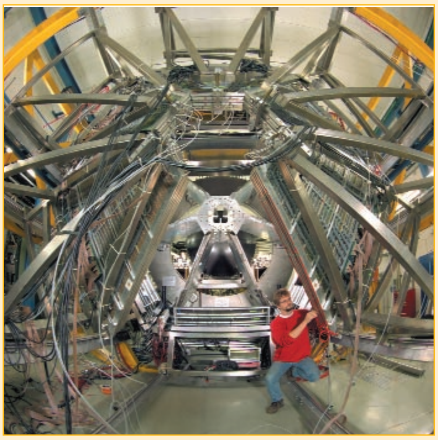
When two nickel nuclei collide at 89% of the speed of light, many particles are released. This picture, taken using a 'drift chamber' at GSI in Darmstadt, Germany, allows them to be identified. Two of these particles are exotic: a K-zero (K^0) and a lambda (Λ). They did not exist in the incoming nuclei, and very quickly decay into more familiar particles. Such experiments yield vital clues concerning neutron stars and supernova explosions. (Courtesy GSI.)

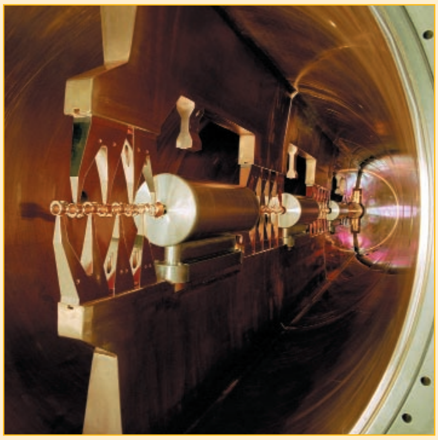
When nuclei collide

In 1919, Rutherford discovered that when alpha particles from a radioactive source travel through air, a proton is sometimes produced. The alpha particles had knocked protons out of the nuclei of nitrogen atoms in the air – the first time a nuclear reaction had been observed.

Nuclear reaction experiments are no longer performed with alpha particles from radioactive sources. Large accelerators produce beams of a wide variety of particles, ranging from electrons and protons to uranium nuclei. The energy of these projectiles is very much higher than alpha particles emitted in radioactive decay and can be precisely controlled. In addition, the number of particles per second is vastly greater than a radioactive source could possibly produce. Beams of high energy particles can be directed onto targets containing any chosen nuclei, where they collide and nuclear reactions take place.

When two nuclei collide at high energies, many different processes occur, leading to the production of a variety of final products. Sometimes the resulting nuclei have a great deal of internal energy as a result of the collision. By choosing suitable beams and targets, it is possible to select what new types of nuclei will be produced in order for these to be studied. Some of these products have found a vital role in modern medicine, and many have quite different properties to the nuclei of atoms found naturally on Earth. However, as long as the energy of the incident particle is not too high, all the transmutations are consistent with the proton–neutron model: the total numbers of protons and neutrons in the products of the collision are the same as those in the original. If the energy of the collision is higher, other particles can be produced.





Taking nuclei apart

One particular kind of reaction is known as photo-disintegration which, as the name suggests, involves the breaking up of nuclei by light. Just as the ultraviolet radiation in sunlight breaks up dye molecules and, in so doing, fades clothes and fine tapestries, so high energy photons can break up nuclei. These are not photons of visible light, but high energy gamma rays, the same radiation that nuclei emit when they lose energy by jumping down from one quantum energy level to another.

Just as there are machines for making high energy particles of matter, so there are machines that produce high energy gamma ray photons. However, when Chadwick and Goldhaber carried out photo-disintegration for the first time, breaking up a deuteron into a proton and a neutron, they used gamma rays from

Top: Aerial view of the Relativistic Heavy Ion Collider (RHIC) at Brookhaven National Laboratory in the United States. Heavy nuclei collide head on at close to the speed of light to produce a quark–gluon plasma. (Brookhaven National Laboratory)

Below: The outer pictures show part of the HADES spectrometer being built at GSI near Darmstadt in Germany to study the electron–positron pairs produced when heavy nuclei collide at high energies. This will help us to understand nuclear matter at high density. The central picture is a view inside the linear accelerator which gives the nuclei at GSI their first boost towards high energy. (Courtesy A. Zschau, GSI.)

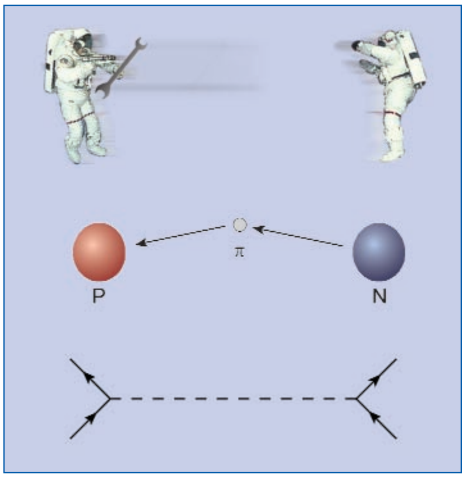
radioactivity. Today, with a machine like the one at Mainz, in Germany, it is possible to knock a proton or a neutron out of any nucleus with photons or electrons. It is easy to imagine doing this repeatedly to a nucleus, one nucleon at a time, until ending up with a deuteron, and then finally just a proton or neutron. If it is possible to take a nucleus apart one nucleon at a time in this way it suggests that nuclei must contain only protons and neutrons.

Quantum mechanics, being quantum mechanics, implies matters are not so simple.

Top: The Mainz Microtron (MAMI), an accelerator at Mainz in Germany, produces high energy electrons for probing fine details of nuclear structure. (Johannes Gutenberg-Universität Mainz)

Below: Three images from GANIL, a complex of accelerators at Caen in France devoted to experiments with beams of heavy nuclei. The central picture shows the main cyclotron (CSS), on the left is a resonator, and on the right is the CIME cyclotron which accelerates exotic nuclei produced using the CSS. (Copyright M. Desaunay, J.M. Enguerrand/GANIL.)

Hideki Yukawa, 1907–1981, put forward a theory for the force that holds protons and neutrons together based on the exchange of a new kind of particle, the meson. The first meson to be discovered is now known as a pion. (AIP Emilio Segrè Visual Archives, W.F. Meggers Collection.)

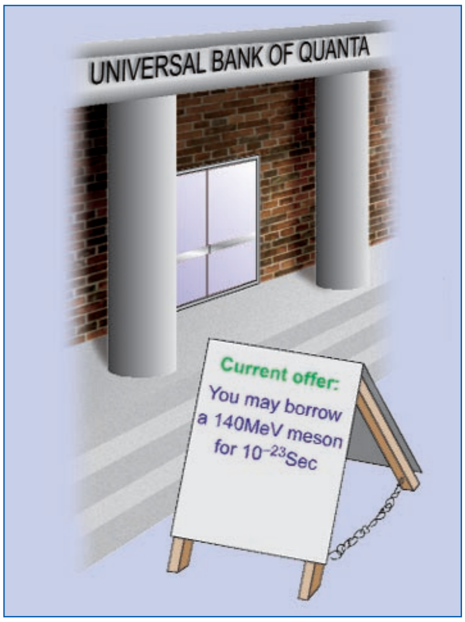


If two astronauts in space were to throw spanners to each other, it would tend to force them apart. Nucleons are not like this. The reason why particle exchange causes attraction lies in the depths of the mathematics. The bottom section, with the dashed line indicating the exchanged meson, shows the diagram invented by Feynman that symbolizes the exchange process.

Something else inside nuclei

Particles can be created and destroyed, like the electrons of beta decay which are created in the decay process, or the electron–positron pairs created from high energy photons. In 1935 the Japanese physicist Hideki Yukawa proposed a radical, and Nobel prize winning, theory for the attractive force between nucleons. This is the force that binds a proton and a neutron into a deuteron and holds together all nuclei. Yukawa proposed that the origin of this force is an entirely new kind of particle. It is created at one of a pair of neighbouring nucleons and jumps across at near the speed of light to be absorbed by the other nucleon. It may seem surprising that this gives rise to an attractive force, but such an attraction was a clear prediction of Yukawa's calculations. There are also other examples. The electric force between charged particles (repulsive for like charges, attractive for unlike charges) is also due to exchange of particles, but in this case the particles are photons.

Yukawa's particle was eventually discovered more than a decade later, and is now known as the pion; it is the lightest of a class of particles known collectively as mesons. Originally, mesons were so-called because they were intermediate in mass between electrons and protons, the pion being about 273 times as heavy as an electron, whereas a proton has about 1835 times the mass of an electron. Over the years many other mesons have been found, including some so heavy they have the mass of a large nucleus.



Nature allows energy to be borrowed briefly to allow an exchange meson to be created out of nothing. The law of energy conservation can be challenged only for a VERY brief period; the bigger the energy loan, the more quickly it must be paid back.

According to the modern versions of Yukawa's theory, the nucleons in the nucleus are held together by mesons coming into a very brief existence during which they jump from one particle to another before vanishing again. The energy to create the meson is a loan from a quantum bank and has to be very quickly repaid. So, at any given time nuclei must contain some mesons as well as protons and neutrons.

Mesons cannot explain everything about the force between a pair of nucleons, especially when they are very close together. A fuller understanding involves the fact that protons and neutrons, like all the mesons, are themselves composite objects. They consist of even more fundamental building blocks, called 'quarks', which are held together within the nucleons and mesons by the exchange of particles known as gluons. Modern nuclear research has opened up the whole question of how nuclei can provide a new window into Nature at the quark–gluon level.

No simple answer

When very high energy projectiles collide with other nuclei, they sometimes apparently knock pions out. However, this is not strictly evidence that there are pions inside nuclei since pions can also be created out of the energy of the incident projectile. As has been seen previously, the fact that particle 'X' comes out of nuclei is not proof that nuclei have particle 'X' inside them. There are, however, some rather difficult experiments involving beams of electrons and photons which can only be understood if it is assumed that there are pions within nuclei.

It is quite typical of the quantum world that the answer you get to any question depends on how you ask it. If you ask what is in the nucleus, and ask the question with low energy particles, then the answer is: just protons and neutrons; in such a low energy experiment if a nucleus is taken apart nucleon by nucleon, it can never end up with just a pion. Asking the question with high energy particles, the answer will be different and will include a whole variety of other particles.

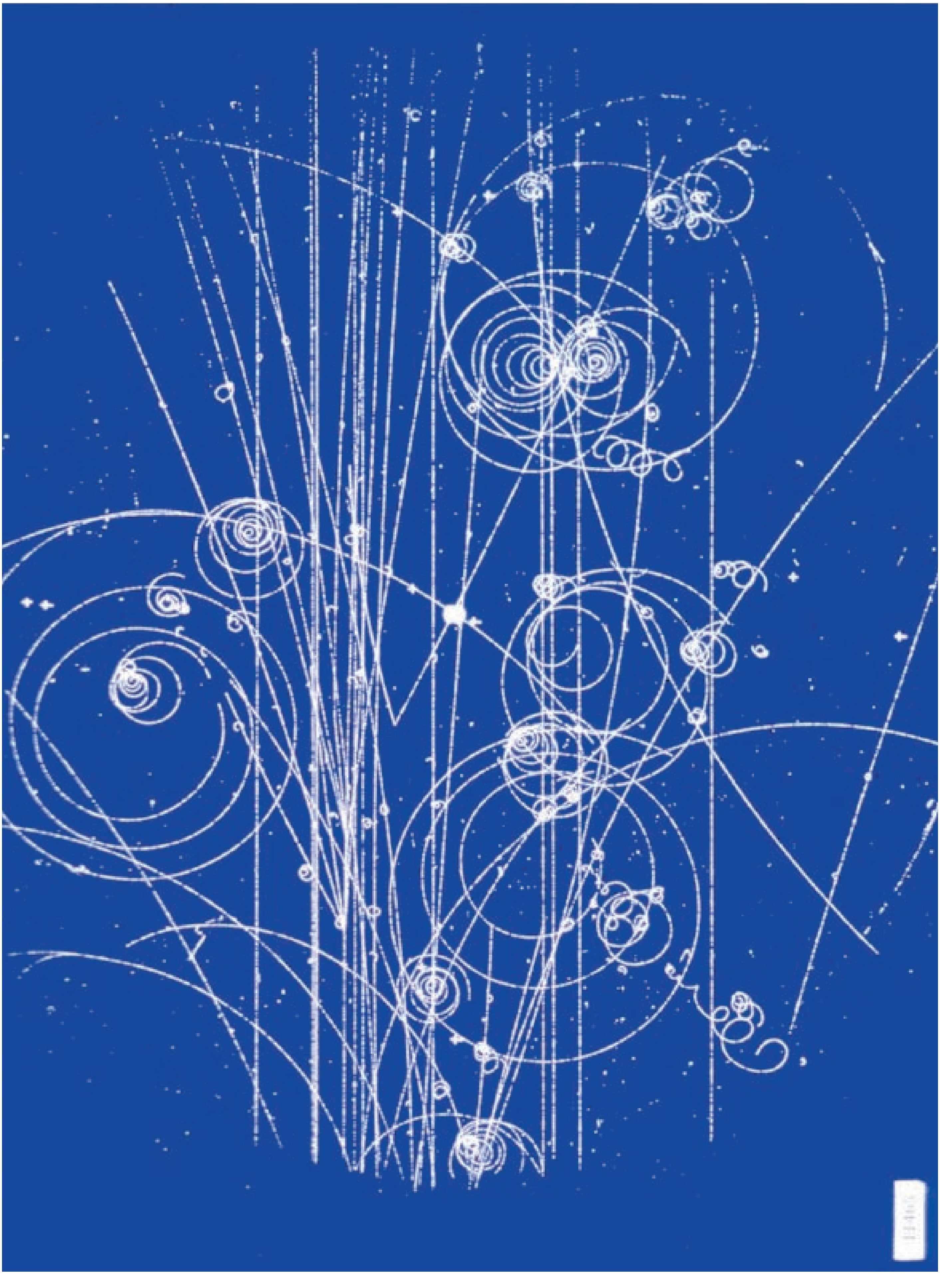
Mesons do not end the list of what might be in nuclei. Atomic nuclei, like atoms, have excited states and if a nucleus is in one of its excited states, it will quickly lose its excess energy by emitting gamma rays and jumping down to the state with the least energy, the ground state. In the 1950s it was found that protons and neutrons can also be excited to higher energy states whereby their properties are changed and they become new particles.

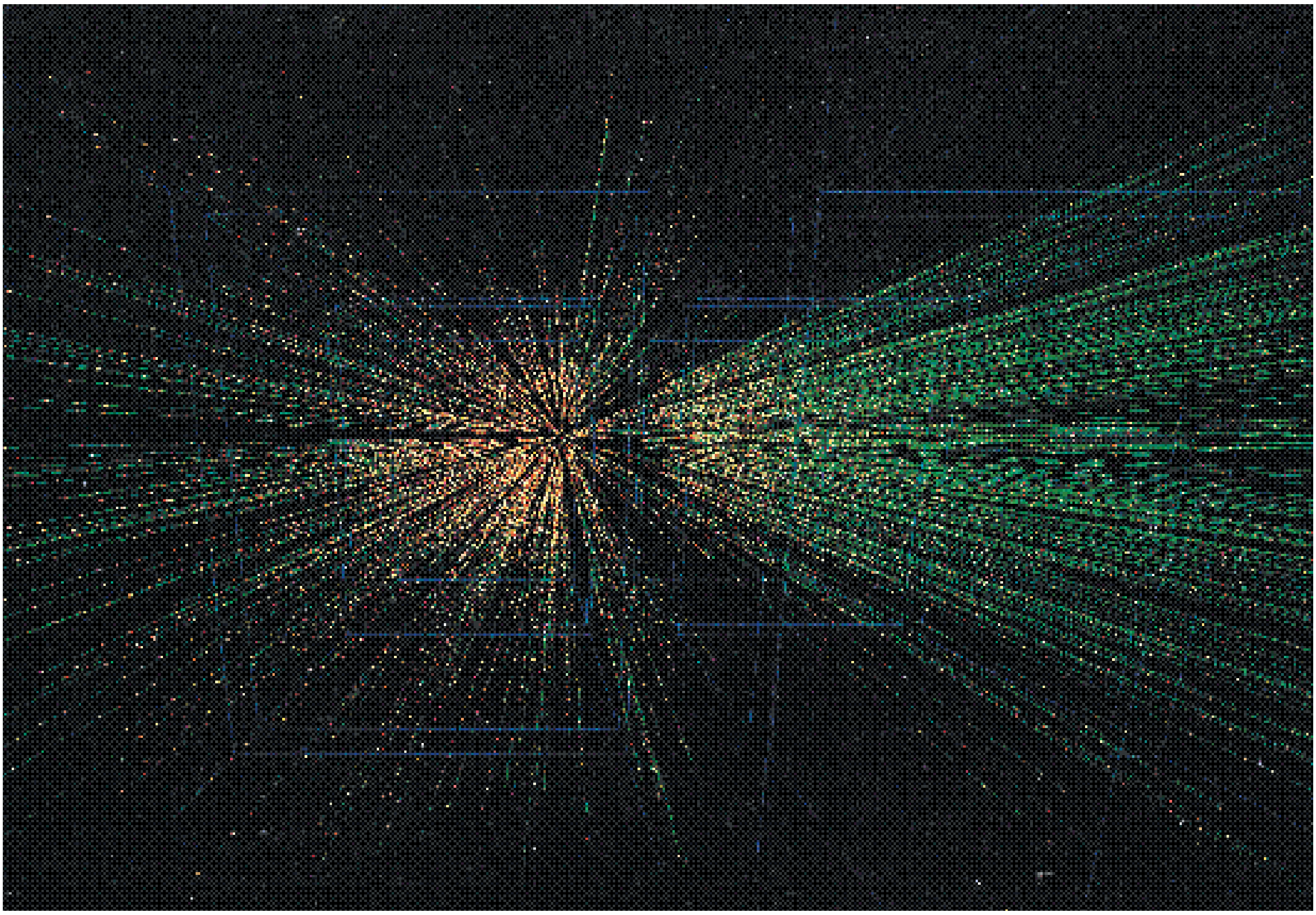
The most common state that an excited nucleon can be in is known as a 'delta resonance', or a delta. Outside nuclei, deltas live for a very short time (about the time it takes for light to cross from one side of a proton to the other), but inside a nucleus, a proton can take a very short-term loan of energy from the same mysterious quantum bank that provides energy for mesons. Protons and neutrons use this energy to fund a brief transformation into delta particles. The result is that the proton within a deuteron is actually a delta for around one percent of the time.

It is possible to knock deltas out of a deuteron, but only by hitting the deuteron with enough energy to turn the nucleon into a delta. But only pions come out to be seen in detectors. This is not evidence that there are deltas ready-formed inside deuterons. The evidence for deltas is either indirect or due to the fact that certain nuclear theories depend for their success on deltas, and other types of particles, being formed from excited states of the nucleon, and existing fleetingly in nuclei.

Nuclear matter at high energies

We have seen that ordinary nuclei contain not only protons and neutrons but also mesons and other particles which exist very briefly before disappearing altogether when the nuclei are dismantled with low energy reactions. Examining reactions at high energies is also important because there are locations in the Universe where nuclei, or the matter of which nuclei are composed, exist in extraordinary states. Neutron stars, for example, are composed of nuclear matter, but not quite the





When two lead nuclei collide at the highest possible energy, a quark–gluon plasma is fleetingly produced. It then disappears in this spectacular shower of particles as registered in a track chamber. The beam direction is towards the observer. The Universe consisted of a quark–gluon plasma a few microseconds after the Big Bang. (CERN)

same type of nuclear matter as the ingredients of ordinary nuclei in their ground states. The stuff of neutron stars cannot be made on Earth, but clues about it can be gained from the study of what happens when very high energy nuclei collide. Not surprisingly, many new particles are created.

When two complex nuclei collide at high energy, many different kinds of mesons besides pions emerge, and these give clues as to the behaviour of nuclear matter at high energy and pressure. This information is needed by physicists who are trying to understand neutron stars and the processes which take place in supernovae, the explosions of stars which are the source of many of the elements found on Earth.

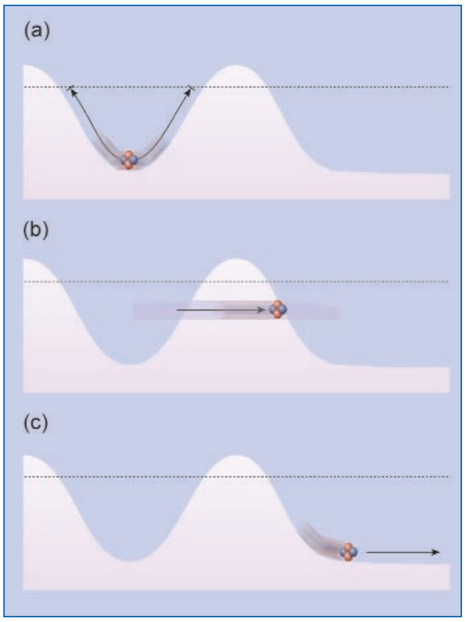
There does not appear to be a limit to the variety of particles that can be made to come out of nuclei if enough energy is pumped in. The vast array of particles which emerge in the highest energy collisions has recently provided evidence that in very hot nuclei the individual protons and neutrons melt away leaving a soup of quarks and gluons, the so-called 'quark–gluon plasma'. It is important to understand this state of matter since conditions in the very early Universe were once fleetingly extreme enough for it to exist.

Tracks left by high energy particles in a bubble chamber. Some particles are bent much more easily by a magnetic field than others, just as in the diagram from Marie Curie's thesis on page 57. A little to the left of the centre, a 'V' shows a particle–antiparticle pair created from a neutral particle that has left no track. (CERN)

Quantum tunnelling

The suggestion that whatever comes out of nuclei must be inside them has already been shown not to be true for electrons. The question of whether this is true for alpha particles reveals, yet again, that the quantum world makes the answer (both yes and no) extremely intriguing. Alpha particles are tightly bound nuclei with their two protons and two neutrons fitting together very snugly. It is for this reason that they often have an independent existence, of a sort, within many nuclei.

The nucleus of ^{20}Ne, the isotope of neon composed of ten protons and ten neutrons, has a structure similar to five alpha particles. This is predicted even by models which assume the twenty nucleons move freely throughout the nucleus. Although the nucleons within certain nuclei do tend to group themselves into alpha-like clusters, the clusters do not consist of four particular nucleons, or always the same four nucleons. In quantum mechanics, every proton in the nucleus is partly in every alpha-like cluster and the same is true for every neutron. The fact that all protons are identical in the special quantum mechanical sense means that every proton is equally part of every cluster. Thus the simple intuitive picture of alpha particles bouncing around in nuclei, each with its own unique set of nucleons, is not quite accurate.



(a) shows a particle (imagine a marble) rolling back and forth simply does not have enough energy to get out of the well. However, quantum mechanics allows something amazing to happen: a particle with too little energy to get over the top can occasionally tunnel through the well (b) in a way that has no parallel in the world of human-sized objects. After tunnelling through, the particle accelerates 'down the hill' just as everyday experience would suggest (c).

A heavy nucleus like the uranium isotope ^{238}U, which decays by emitting alpha particles, has many more neutrons (146) than protons (92). It therefore cannot be thought of as a collection of alpha particles. Nevertheless, alpha particle-like structures do occur in the nucleus and sooner or later these appear outside the nucleus as alpha particles, and can be picked up in detectors.

The greater the probability that the alpha particles are emitted, the shorter the half-life of the nucleus. The half-life of a nucleus such as ^{238}U is determined by two factors. The first is the likelihood that the protons and neutrons arrange themselves into structures that look like a nucleus of the thorium isotope ^{234}Th, plus a single alpha particle. The second factor is how rapidly the alpha particle can tunnel out of the nucleus.

In 1928 George Gamow calculated the tunnelling probability of an alpha particle. It is the different rates of alpha tunnelling that determines the huge difference in half-lives: 4.5 billion years for ^{238}U and a third of a microsecond for the polonium isotope ^{212}Po. The essential point of quantum mechanical tunnelling is that quantum theory makes it possible for particles to tunnel through a barrier which it does not have enough energy to jump over. Such a barrier would be absolutely impenetrable according to pre-quantum physics.

The ease with which a particle gets through this barrier depends extremely sensitively on the amount of energy possessed by the particle. Just how sensitively can be seen from the huge difference in the half-lives of the uranium and polonium isotopes. The vast ratio of half-lives corresponds to a factor of just two in energy: the alpha particle emitted by ^{212}Po has about twice the energy of the alpha particle emitted by ^{238}U.

The importance of tunnelling in nuclear physics extends far beyond understanding alpha decay. The fusion reactions by which stars get their energy and by which elements are built up from hydrogen, depend on tunnelling inwards through otherwise impenetrable barriers. The sensitivity of the rate of tunnelling to the energy of the particles is a crucial ingredient in governing the rate at which stars burn and evolve. Tunnelling makes this possible and governs not only the energy from the Sun, but is a key to producing the very elements from which we are made.

The Thomas Jefferson National Accelerator Facility in Virginia, United States, represents a new step in electron accelerators. It is already producing new information about nuclear structure at the quark level. (Courtesy the Thomas Jefferson National Accelerator Facility.)