# 6.8 Laplace Transform: General Formulas

| Formula | Name, Comments | Sec. |
|---|---|---|
| $F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$<br>$f(t) = \mathcal{L}^{-1}\{F(s)\}$ | Definition of Transform<br>Inverse Transform | 6.1 |
| $\mathcal{L}\{af(t) + bg(t)\} = a\mathcal{L}\{f(t)\} + b\mathcal{L}\{g(t)\}$ | Linearity | 6.1 |
| $\mathcal{L}\{e^{at} f(t)\} = F(s - a)$<br>$\mathcal{L}^{-1}\{F(s - a)\} = e^{at} f(t)$ | *s*-Shifting<br>(First Shifting Theorem) | 6.1 |
| $\mathcal{L}(f') = s\mathcal{L}(f) - f(0)$<br>$\mathcal{L}(f'') = s^2\mathcal{L}(f) - sf(0) - f'(0)$<br>$\mathcal{L}(f^{(n)}) = s^n\mathcal{L}(f) - s^{(n-1)}f(0) - \cdots$<br>$\cdots - f^{(n-1)}(0)$<br>$\mathcal{L}\left\{\int_0^t f(\tau)\, d\tau\right\} = \frac{1}{s}\mathcal{L}(f)$ | Differentiation of Function<br>Integration of Function | 6.2 |
| $(f * g)(t) = \int_0^t f(\tau) g(t - \tau)\, d\tau$<br>$= \int_0^t f(t - \tau) g(\tau)\, d\tau$<br>$\mathcal{L}(f * g) = \mathcal{L}(f)\mathcal{L}(g)$ | Convolution | 6.5 |
| $\mathcal{L}\{f(t - a)\, u(t - a)\} = e^{-as} F(s)$<br>$\mathcal{L}^{-1}\{e^{-as} F(s)\} = f(t - a)\, u(t - a)$ | *t*-Shifting<br>(Second Shifting Theorem) | 6.3 |
| $\mathcal{L}\{tf(t)\} = -F'(s)$<br>$\mathcal{L}\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(\tilde{s})\, d\tilde{s}$ | Differentiation of Transform<br>Integration of Transform | 6.6 |
| $\mathcal{L}(f) = \frac{1}{1 - e^{-ps}} \int_0^p e^{-st} f(t)\, dt$ | *f* Periodic with Period *p* | 6.4<br>Project<br>16 |

# 6.9 Table of Laplace Transforms

***For more extensive tables, see Ref.* [A9] *in Appendix 1.***

| | $F(s) = \mathcal{L}\{f(t)\}$ | $f(t)$ | Sec. |
|---|---|---|---|
| 1 | $1/s$ | $1$ | 6.1 |
| 2 | $1/s^2$ | $t$ | |
| 3 | $1/s^n \quad (n = 1, 2, \cdots)$ | $t^{n-1}/(n-1)!$ | |
| 4 | $1/\sqrt{s}$ | $1/\sqrt{\pi t}$ | |
| 5 | $1/s^{3/2}$ | $2\sqrt{t/\pi}$ | |
| 6 | $1/s^a \quad (a > 0)$ | $t^{a-1}/\Gamma(a)$ | |
| 7 | $\dfrac{1}{s-a}$ | $e^{at}$ | 6.1 |
| 8 | $\dfrac{1}{(s-a)^2}$ | $te^{at}$ | |
| 9 | $\dfrac{1}{(s-a)^n} \quad (n = 1, 2, \cdots)$ | $\dfrac{1}{(n-1)!} t^{n-1}e^{at}$ | |
| 10 | $\dfrac{1}{(s-a)^k} \quad (k > 0)$ | $\dfrac{1}{\Gamma(k)} t^{k-1}e^{at}$ | |
| 11 | $\dfrac{1}{(s-a)(s-b)} \quad (a \neq b)$ | $\dfrac{1}{a-b}(e^{at} - e^{bt})$ | |
| 12 | $\dfrac{s}{(s-a)(s-b)} \quad (a \neq b)$ | $\dfrac{1}{a-b}(ae^{at} - be^{bt})$ | |
| 13 | $\dfrac{1}{s^2+\omega^2}$ | $\dfrac{1}{\omega}\sin \omega t$ | 6.1 |
| 14 | $\dfrac{s}{s^2+\omega^2}$ | $\cos \omega t$ | |
| 15 | $\dfrac{1}{s^2-a^2}$ | $\dfrac{1}{a}\sinh at$ | |
| 16 | $\dfrac{s}{s^2-a^2}$ | $\cosh at$ | |
| 17 | $\dfrac{1}{(s-a)^2+\omega^2}$ | $\dfrac{1}{\omega}e^{at}\sinh \omega t$ | |
| 18 | $\dfrac{s-a}{(s-a)^2+\omega^2}$ | $e^{at}\cos \omega t$ | |
| 19 | $\dfrac{1}{s(s^2+\omega^2)}$ | $\dfrac{1}{\omega^2}(1-\cos \omega t)$ | 6.2 |
| 20 | $\dfrac{1}{s^2(s^2+\omega^2)}$ | $\dfrac{1}{\omega^3}(\omega t - \sin \omega t)$ | |

***(continued)***

**Table of Laplace Transforms (*continued*)**

| | $F(s) = \mathcal{L}\{f(t)\}$ | $f(t)$ | Sec. |
|---|---|---|---|
| 21 | $\dfrac{1}{(s^2+\omega^2)^2}$ | $\dfrac{1}{2\omega^3}(\sin\omega t - \omega t\cos\omega t)$ | 6.6 |
| 22 | $\dfrac{s}{(s^2+\omega^2)^2}$ | $\dfrac{t}{2\omega}\sin\omega t$ | 6.6 |
| 23 | $\dfrac{s^2}{(s^2+\omega^2)^2}$ | $\dfrac{1}{2\omega}(\sin\omega t + \omega t\cos\omega t)$ | 6.6 |
| 24 | $\dfrac{s}{(s^2+a^2)(s^2+b^2)}$ $(a^2 \neq b^2)$ | $\dfrac{1}{b^2-a^2}(\cos at - \cos bt)$ | |
| 25 | $\dfrac{1}{s^4+4k^4}$ | $\dfrac{1}{4k^3}(\sin kt\cos kt - \cos kt\sinh kt)$ | |
| 26 | $\dfrac{s}{s^4+4k^4}$ | $\dfrac{1}{2k^2}\sin kt\sinh kt$ | |
| 27 | $\dfrac{1}{s^4-k^4}$ | $\dfrac{1}{2k^3}(\sinh kt - \sin kt)$ | |
| 28 | $\dfrac{s}{s^4-k^4}$ | $\dfrac{1}{2k^2}(\cosh kt - \cos kt)$ | |
| 29 | $\sqrt{s-a} - \sqrt{s-b}$ | $\dfrac{1}{2\sqrt{\pi t^3}}(e^{bt} - e^{at})$ | |
| 30 | $\dfrac{1}{\sqrt{s+a}\sqrt{s+b}}$ | $e^{-(a+b)t/2}I_0\left(\dfrac{a-b}{2}t\right)$ | I 5.5 |
| 31 | $\dfrac{1}{\sqrt{s^2+a^2}}$ | $J_0(at)$ | J 5.4 |
| 32 | $\dfrac{s}{(s-a)^{3/2}}$ | $\dfrac{1}{\sqrt{\pi t}}e^{at}(1+2at)$ | |
| 33 | $\dfrac{1}{(s^2-a^2)^k}$ $(k>0)$ | $\dfrac{\sqrt{\pi}}{\Gamma(k)}\left(\dfrac{t}{2a}\right)^{k-1/2} I_{k-1/2}(at)$ | I 5.5 |
| 34 | $e^{-as}/s$ | $u(t-a)$ | 6.3 |
| 35 | $e^{-as}$ | $\delta(t-a)$ | 6.4 |
| 36 | $\dfrac{1}{s}e^{-k/s}$ | $J_0(2\sqrt{kt})$ | J 5.4 |
| 37 | $\dfrac{1}{\sqrt{s}}e^{-k/s}$ | $\dfrac{1}{\sqrt{\pi t}}\cos 2\sqrt{kt}$ | |
| 38 | $\dfrac{1}{s^{3/2}}e^{k/s}$ | $\dfrac{1}{\sqrt{\pi k}}\sinh 2\sqrt{kt}$ | |
| 39 | $e^{-k\sqrt{s}}$ $(k>0)$ | $\dfrac{k}{2\sqrt{\pi t^3}}e^{-k^2/4t}$ | |

***(continued)***

**Table of Laplace Transforms (*continued*)**

| | $F(s) = \mathcal{L}\{f(t)\}$ | $f(t)$ | Sec. |
|---|---|---|---|
| 40 | $\dfrac{1}{s}\ln s$ | $-\ln t - \gamma \quad (\gamma \approx 0.5772)$ | $\gamma$ 5.5 |
| 41 | $\ln \dfrac{s-a}{s-b}$ | $\dfrac{1}{t}(e^{bt} - e^{at})$ | |
| 42 | $\ln \dfrac{s^2+\omega^2}{s^2}$ | $\dfrac{2}{t}(1 - \cos \omega t)$ | 6.6 |
| 43 | $\ln \dfrac{s^2-a^2}{s^2}$ | $\dfrac{2}{t}(1 - \cosh at)$ | |
| 44 | $\arctan \dfrac{\omega}{s}$ | $\dfrac{1}{t}\sin \omega t$ | |
| 45 | $\dfrac{1}{s}\operatorname{arccot} s$ | $\mathrm{Si}(t)$ | App. A3.1 |

## CHAPTER 6 REVIEW QUESTIONS AND PROBLEMS

1. State the Laplace transforms of a few simple functions from memory.
2. What are the steps of solving an ODE by the Laplace transform?
3. In what cases of solving ODEs is the present method preferable to that in Chap. 2?
4. What property of the Laplace transform is crucial in solving ODEs?
5. Is $\mathcal{L}\{f(t) + g(t)\} = \mathcal{L}\{f(t)\} + \mathcal{L}\{g(t)\}$? $\mathcal{L}\{f(t)g(t)\} = \mathcal{L}\{f(t)\}\mathcal{L}\{g(t)\}$? Explain.
6. When and how do you use the unit step function and Dirac's delta?
7. If you know $f(t) = \mathcal{L}^{-1}\{F(s)\}$, how would you find $\mathcal{L}^{-1}\{F(s)/s^2\}$?
8. Explain the use of the two shifting theorems from memory.
9. Can a discontinuous function have a Laplace transform? Give reason.
10. If two different continuous functions have transforms, the latter are different. Why is this practically important?

### 11–19 LAPLACE TRANSFORMS

Find the transform, indicating the method used and showing the details.

**11.** $5\cosh 2t - 3\sinh t$

**12.** $e^{-t}(\cos 4t - 2\sin 4t)$

**13.** $\sin^2(\tfrac{1}{2}\pi t)$

**14.** $16t^2 u(t - \tfrac{1}{4})$

**15.** $e^{t/2}u(t-3)$

**16.** $u(t-2\pi)\sin t$

**17.** $t\cos t + \sin t$

**18.** $(\sin \omega t) * (\cos \omega t)$

**19.** $12t * e^{-3t}$

### 20–28 INVERSE LAPLACE TRANSFORM

Find the inverse transform, indicating the method used and showing the details:

**20.** $\dfrac{7.5}{s^2 - 2s - 8}$

**21.** $\dfrac{s+1}{s^2}e^{-s}$

**22.** $\dfrac{\frac{1}{16}}{s^2 + s + \frac{1}{2}}$

**23.** $\dfrac{\omega\cos\theta + s\sin\theta}{s^2+\omega^2}$

**24.** $\dfrac{s^2 - 6.25}{(s^2+6.25)^2}$

**25.** $\dfrac{6(s+1)}{s^4}$

**26.** $\dfrac{2s - 10}{s^3}e^{-5s}$

**27.** $\dfrac{3s+4}{s^2+4s+5}$

**28.** $\dfrac{3s}{s^2 - 2s + 2}$

### 29–37 ODEs AND SYSTEMS

Solve by the Laplace transform, showing the details and graphing the solution:

**29.** $y'' + 4y' + 5y = 50t, \quad y(0) = 5, \quad y'(0) = -5$

**30.** $y'' + 16y = 4\delta(t - \pi), \quad y(0) = -1, \quad y'(0) = 0$

**31.** $y'' - y' - 2y = 12u(t - \pi)\sin t, \quad y(0) = 1,$
$y'(0) = -1$

**32.** $y'' + 4y = \delta(t - \pi) - \delta(t - 2\pi), \quad y(0) = 1,$
$y'(0) = 0$

**33.** $y'' + 3y' + 2y = 2u(t - 2), \quad y(0) = 0, \quad y'(0) = 0$

**34.** $y_1' = y_2, \quad y_2' = -4y_1 + \delta(t - \pi), \quad y_1(0) = 0,$
$y_2(0) = 0$

**35.** $y_1' = 2y_1 - 4y_2, \quad y_2' = y_1 - 3y_2, \quad y_1(0) = 3,$
$y_2(0) = 0$

**36.** $y_1' = 2y_1 + 4y_2, \quad y_2' = y_1 + 2y_2, \quad y_1(0) = -4,$
$y_2(0) = -4$

**37.** $y_1' = y_2 + u(t - \pi), \quad y_2' = -y_1 + u(t - 2\pi),$
$y_1(0) = 1, \quad y_2(0) = 0$

## 38–45 MASS–SPRING SYSTEMS, CIRCUITS, NETWORKS

Model and solve by the Laplace transform:

**38.** Show that the model of the mechanical system in Fig. 149 (no friction, no damping) is

$$m_1 y_1'' = -k_1 y_1 + k_2(y_2 - y_1)$$
$$m_2 y_2'' = -k_2(y_2 - y_1) - k_3 y_2.$$

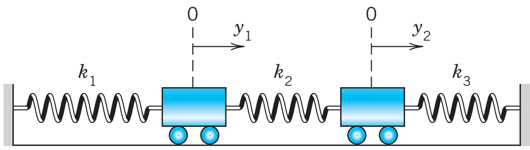


**Fig. 149.** System in Problems 38 and 39

**39.** In Prob. 38, let $m_1 = m_2 = 10$ kg, $k_1 = k_3 = 20$ kg/sec$^2$, $k_2 = 40$ kg/sec$^2$. Find the solution satisfying the initial conditions $y_1(0) = y_2(0) = 0$, $y_1'(0) = 1$ meter/sec, $y_2'(0) = -1$ meter/sec.

**40.** Find the model (the system of ODEs) in Prob. 38 extended by adding another mass $m_3$ and another spring of modulus $k_4$ in series.

**41.** Find the current $i(t)$ in the $RC$-circuit in Fig. 150, where $R = 10\ \Omega$, $C = 0.1$ F, $v(t) = 10t$ V if $0 < t < 4$, $v(t) = 40$ V if $t > 4$, and the initial charge on the capacitor is 0.

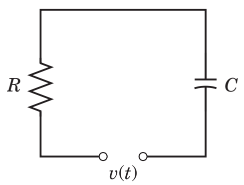


**Fig. 150.** *RC*-circuit

**42.** Find and graph the charge $q(t)$ and the current $i(t)$ in the $LC$-circuit in Fig. 151, assuming $L = 1$ H, $C = 1$ F, $v(t) = 1 - e^{-t}$ if $0 < t < \pi$, $v(t) = 0$ if $t > \pi$, and zero initial current and charge.

**43.** Find the current $i(t)$ in the $RLC$-circuit in Fig. 152, where $R = 160\ \Omega$, $L = 20$ H, $C = 0.002$ F, $v(t) = 37 \sin 10t$ V, and current and charge at $t = 0$ are zero.

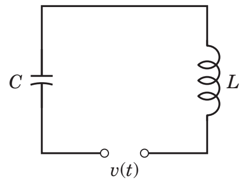


**Fig. 151.** *LC*-circuit

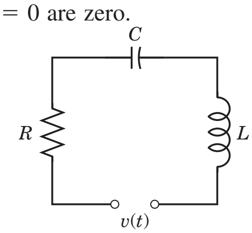


**Fig. 152.** *RLC*-circuit

**44.** Show that, by Kirchhoff's Voltage Law (Sec. 2.9), the currents in the network in Fig. 153 are obtained from the system

$$Li_1' + R(i_1 - i_2) = v(t)$$
$$R(i_2' - i_1') + \frac{1}{C} i_2 = 0.$$

Solve this system, assuming that $R = 10\ \Omega$, $L = 20$ H, $C = 0.05$ F, $v = 20$ V, $i_1(0) = 0$, $i_2(0) = 2$ A.

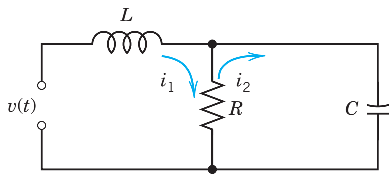


**Fig. 153.** Network in Problem 44

**45.** Set up the model of the network in Fig. 154 and find the solution, assuming that all charges and currents are 0 when the switch is closed at $t = 0$. Find the limits of $i_1(t)$ and $i_2(t)$ as $t \to \infty$, (i) from the solution, (ii) directly from the given network.

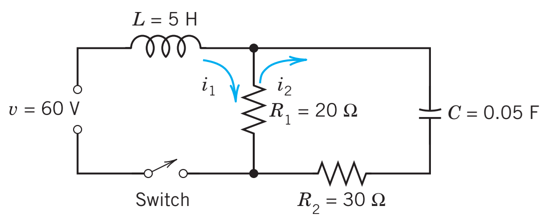


**Fig. 154.** Network in Problem 45

## SUMMARY OF CHAPTER 6

# Laplace Transforms

The main purpose of Laplace transforms is the solution of differential equations and systems of such equations, as well as corresponding initial value problems. The **Laplace transform** $F(s) = \mathcal{L}(f)$ of a function $f(t)$ is defined by

$$F(s) = \mathcal{L}(f) = \int_0^\infty e^{-st} f(t)\, dt \qquad \text{(Sec. 6.1).} \tag{1}$$

This definition is motivated by the property that the differentiation of $f$ with respect to $t$ corresponds to the multiplication of the transform $F$ by $s$; more precisely,

$$\begin{aligned} \mathcal{L}(f') &= s\mathcal{L}(f) - f(0) \\ \mathcal{L}(f'') &= s^2\mathcal{L}(f) - sf(0) - f'(0) \end{aligned} \qquad \text{(Sec. 6.2)} \tag{2}$$

etc. Hence by taking the transform of a given differential equation

$$y'' + ay' + by = r(t) \qquad (a, b \text{ constant}) \tag{3}$$

and writing $\mathcal{L}(y) = Y(s)$, we obtain the **subsidiary equation**

$$(s^2 + as + b)Y = \mathcal{L}(r) + sf(0) + f'(0) + af(0). \tag{4}$$

Here, in obtaining the transform $\mathcal{L}(r)$ we can get help from the small table in Sec. 6.1 or the larger table in Sec. 6.9. This is the first step. In the second step we solve the subsidiary equation *algebraically* for $Y(s)$. In the third step we determine the **inverse transform** $y(t) = \mathcal{L}^{-1}(Y)$, that is, the solution of the problem. This is generally the hardest step, and in it we may again use one of those two tables. $Y(s)$ will often be a rational function, so that we can obtain the inverse $\mathcal{L}^{-1}(Y)$ by partial fraction reduction (Sec. 6.4) if we see no simpler way.

The Laplace method avoids the determination of a general solution of the homogeneous ODE, and we also need not determine values of arbitrary constants in a general solution from initial conditions; instead, we can insert the latter directly into (4). Two further facts account for the practical importance of the Laplace transform. First, it has some basic properties and resulting techniques that simplify the determination of transforms and inverses. The most important of these properties are listed in Sec. 6.8, together with references to the corresponding sections. More on the use of unit step functions and Dirac's delta can be found in Secs. 6.3 and 6.4, and more on convolution in Sec. 6.5. Second, due to these properties, the present method is particularly suitable for handling right sides $r(t)$ given by different expressions over different intervals of time, for instance, when $r(t)$ is a square wave or an impulse or of a form such as $r(t) = \cos t$ if $0 \leqq t \leqq 4\pi$ and 0 elsewhere.

The application of the Laplace transform to systems of ODEs is shown in Sec. 6.7. (The application to PDEs follows in Sec. 12.12.)

# TABLE OF LAPLACE TRANSFORM FORMULAS

$$\mathcal{L}\left[t^n\right] = \frac{n!}{s^{n+1}} \qquad \mathcal{L}^{-1}\left[\frac{1}{s^n}\right] = \frac{1}{(n-1)!}\, t^{n-1}$$

$$\mathcal{L}\left[e^{at}\right] = \frac{1}{s-a} \qquad \mathcal{L}^{-1}\left[\frac{1}{s-a}\right] = e^{at}$$

$$\mathcal{L}\left[\sin at\right] = \frac{a}{s^2+a^2} \qquad \mathcal{L}^{-1}\left[\frac{1}{s^2+a^2}\right] = \frac{1}{a}\sin at$$

$$\mathcal{L}\left[\cos at\right] = \frac{s}{s^2+a^2} \qquad \mathcal{L}^{-1}\left[\frac{s}{s^2+a^2}\right] = \cos at$$

**Differentiation and integration**

$$\mathcal{L}\left[\frac{d}{dt}f(t)\right] = s\mathcal{L}[f(t)] - f(0)$$

$$\mathcal{L}\left[\frac{d^2}{dt^2}f(t)\right] = s^2\mathcal{L}[f(t)] - sf(0) - f'(0)$$

$$\mathcal{L}\left[\frac{d^n}{dt^n}f(t)\right] = s^n\mathcal{L}[f(t)] - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - f^{(n-1)}(0)$$

In the following formulas $F(s) = \mathcal{L}[f(t)]$, so $f(t) = \mathcal{L}^{-1}[F(s)]$.

$$\mathcal{L}\left[\int_0^t f(u)\ du\right] = \frac{1}{s}\mathcal{L}[f(t)] \qquad \mathcal{L}^{-1}\left[\frac{1}{s}F(s)\right] = \int_0^t f(u)\ du$$

$$\mathcal{L}\left[t^n f(t)\right] = (-1)^n \frac{d^n}{ds^n}\mathcal{L}[f(t)] \qquad \mathcal{L}^{-1}\left[\frac{d^n F(s)}{ds^n}\right] = (-1)^n t^n f(t)$$

**Shift formulas**

$$\mathcal{L}\left[e^{at}f(t)\right] = F(s-a) \qquad \mathcal{L}^{-1}[F(s)] = e^{at}\mathcal{L}^{-1}[F(s+a)]$$

$$\mathcal{L}\left[u_a(t)f(t)\right] = e^{-as}\mathcal{L}[f(t+a)] \qquad \mathcal{L}^{-1}\left[e^{-as}F(s)\right] = u_a(t)f(t-a)$$

Here $u_a(t) = \begin{cases} 0, & t < a, \\ 1, & t \geq a. \end{cases}$