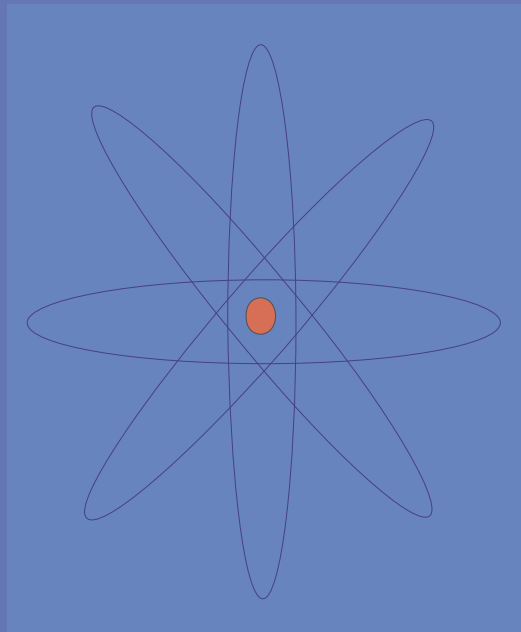


Physics pocket diary of concepts and formulas



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Kinematics

Note: - Bold letter are used to denote vector quantity
 $\mathbf{i}, \mathbf{j}, \mathbf{z}$ are the unit vector along x, y and z axis

Quick review of Kinematics formulas

S.No.	Type of Motion	Formula
1	Motion in one dimension	$\mathbf{r} = x\mathbf{i}$ $\mathbf{v} = (dx/dt)\mathbf{i}$ $\mathbf{a} = dv/dt = (d^2x/dt^2)\mathbf{i}$ and $\mathbf{a} = vdv/dr$ $\mathbf{v} = \mathbf{u} + \mathbf{at}$ $\mathbf{s} = \mathbf{ut} + 1/2\mathbf{at}^2$ $\mathbf{v}^2 = \mathbf{u}^2 + 2\mathbf{as}$ In integral form $\mathbf{r} = \int \mathbf{v} dt$ $\mathbf{v} = \int \mathbf{a} dt$
2	Motion in two dimension	$\mathbf{r} = x\mathbf{i} + y\mathbf{j}$ $\mathbf{v} = d\mathbf{r}/dt = (dx/dt)\mathbf{i} + (dy/dt)\mathbf{j}$ $\mathbf{a} = dv/dt = (d^2x/dt^2)\mathbf{i} + (d^2y/dt^2)\mathbf{j}$ and $\mathbf{a} = vdv/dr$ Constant accelerated equation same as above
3	Motion in three dimension	$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ $\mathbf{v} = d\mathbf{r}/dt = (dx/dt)\mathbf{i} + (dy/dt)\mathbf{j} + (dz/dt)\mathbf{k}$ $\mathbf{a} = dv/dt = (d^2x/dt^2)\mathbf{i} + (d^2y/dt^2)\mathbf{j} + (d^2z/dt^2)\mathbf{k}$ $\mathbf{a} = vdv/dr$ Constant accelerated equation same as above
4	Projectile Motion	$x = (v_0 \cos \theta_0)t$ $y = (v_0 \sin \theta_0)t - gt^2/2$ $v_x = v_0 \cos \theta_0$ and $v_y = v_0 \sin \theta_0 t - gt$, where θ_0 is the angle initial velocity makes with the positive x axis.
5	Uniform circular motion	$a = v^2/R$, where a is centripetal acceleration whose direction of is always along radius of the circle towards the centre and $a = 4\pi^2 R/T^2$ acceleration in uniform circular motion in terms of time period T

Concept of relative velocity

For two objects A and B moving with the uniform velocities \mathbf{V}_A and \mathbf{V}_B .

Relative velocity is defined as

$$\mathbf{V}_{BA} = \mathbf{V}_B - \mathbf{V}_A$$

where \mathbf{V}_{BA} is relative velocity of B relative to A

Similarly relative velocity of A relative to B

$$\mathbf{V}_{AB} = \mathbf{V}_A - \mathbf{V}_B$$

Special cases: -

S.No.	Case	Description
1	For straight line motion	If the objects are moving in the same direction, relative velocity can be get by subtracting other. If they are moving in opposite direction ,relative velocity will be get by adding the velocities example like train problems
2	For two dimensions motion	if $\mathbf{v}_a = v_{xa}\mathbf{i} + v_{ya}\mathbf{j}$ $\mathbf{v}_b = v_{xb}\mathbf{i} + v_{yb}\mathbf{j}$ Relative velocity of B relative to A $= v_{xb}\mathbf{i} + v_{yb}\mathbf{j} - (v_{xa}\mathbf{i} + v_{ya}\mathbf{j})$ $= \mathbf{i}(v_{xb} - v_{xa}) + \mathbf{j}(v_{yb} - v_{ya})$
3	For three dimensions motion	$\mathbf{v}_a = v_{xa}\mathbf{i} + v_{ya}\mathbf{j} + v_{za}\mathbf{z}$ $\mathbf{v}_b = v_{xb}\mathbf{i} + v_{yb}\mathbf{j} + v_{zb}\mathbf{z}$ Relative velocity of B relative to A $= v_{xb}\mathbf{i} + v_{yb}\mathbf{j} + v_{zb}\mathbf{z} - (v_{xa}\mathbf{i} + v_{ya}\mathbf{j} + v_{za}\mathbf{z})$ $= \mathbf{i}(v_{xb} - v_{xa}) + \mathbf{j}(v_{yb} - v_{ya}) + \mathbf{z}(v_{zb} - v_{za})$

Free fall acceleration

S.No.	Point
1	Freely falling motion of any body under the effect of gravity is an example of uniformly accelerated motion.
2	Kinematics equation of motion under gravity can be obtained by replacing acceleration ' \mathbf{a} ' in equations of motion by acceleration due to gravity ' \mathbf{g} '.
3	Thus kinematics equations of motion under gravity are $v = v_0 + gt$, $x = v_0t + \frac{1}{2}(gt^2)$ and $v^2 = (v_0)^2 + 2gx$
4	Value of g is equal to 9.8 m.s^{-2} . The value of g is taken positive when the body falls vertically downwards and negative when the body is projected up against gravity.

Laws of motion

S.No.	Term	Description
1	Newton's first law of motion	'A body continues to be in state of rest or uniform motion unless it is acted upon by some external force to act otherwise'
2	Newton's second law of motion	'Rate of change of momentum of a body is proportional to the applied force and takes place in the direction of action of force applied Mathematically, $\mathbf{F} = \frac{d\mathbf{p}}{dt} = m\mathbf{a}$ where, $\mathbf{p} = m\mathbf{v}$, momentum of the body \mathbf{a} =acceleration
3	Impulse	Impulse is the product of force and time which is equal to the change in momentum Impulse = $\mathbf{F}\Delta t = \Delta\mathbf{p}$
4	Newton's third law of motion	'To every action there is always an equal and opposite reaction' $\mathbf{F}_{AB} = -\mathbf{F}_{BA}$
5	Law of conservation of linear momentum	Initial momentum = final momentum $m_1\mathbf{v}_1 + m_2\mathbf{v}_2 = m_1\mathbf{v}_1' + m_2\mathbf{v}_2'$ For equilibrium of a body $\mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 = 0$

Some points to note

S.No.	Point
1	An accelerated frame is called non inertial frame while an non accelerated frame is called inertial frame
2	Newton first law are valid in inertial frame only
3	Apparent weight of a body in the lift Going Upward with acceleration a $W = m(g + a)$ Going Down with acceleration a $W = m(g - a)$
4	Always draw free body diagram to solve the force related problems

Friction and Frame of reference

S.No.	Term	Description
1	Friction	Frictional force acts between the bodies whenever there is a relative motion between them. When bodies slip, frictional force is called static frictional force and when the bodies do not slip, it is called kinetic frictional force.
2	Kinetic Frictional force	When bodies slip over each other $f = \mu_k N$ Where N is the normal contact force between the surface and μ_k is the coefficient of kinetic Friction. Direction of frictional force is such that relative slipping is opposed by the friction
3	Static Frictional force	Frictional force can also act even if there is no relative motion. Such force is called static Frictional force. Maximum Static friction that a body can exert on other body in contact with it is called limiting Friction. $f_{\max} = \mu_s N$ Where N is the normal contact force between the surface And μ_s is the coefficient of static Friction f_{\max} is the maximum possible force of static Friction. Note that $\mu_s > \mu_k$ and Angle of friction $\tan\lambda = \mu_s$
4	Inertial Frame Of reference	Inertial frame of references is those attached to objects which are at rest or moving at constant Velocity. Newton's law are valid in inertial frame of reference. Example person standing in a train moving at constant velocity.
5	Non Inertial Frame Of reference	Inertial frame of references is attached to accelerated objects for example: A person standing in a train moving with increasing speed. Newton's law are not valid. To apply Newton's law ,pseudo force has to be introduced in the equation whose value will be $F = -ma$

Work, Energy and Power

S.No.	Term	Description
1	Work	<p>1. Work done by the force is defined as dot product of force and displacement vector. For constant Force $W = \mathbf{F} \cdot \mathbf{s}$ where \mathbf{F} is the force vector and \mathbf{s} is displacement Vector</p> <p>2. For variable Force $dW = \mathbf{F} \cdot d\mathbf{s}$ or $W = \int \mathbf{F} \cdot d\mathbf{s}$ It is a scalar quantity</p>
2	Conservative And Non Conservative Forces	<p>1. If the work done by the force in a closed path is zero, then it is called conservative Force</p> <p>2. If the work done by the force in a closed path is not zero, then it is called non conservative Force</p> <p>3. Gravitational ,electrical force are Conservative Forces and Non Conservative Forces are frictional forces</p>
3	Kinetic Energy	<p>1. It is the energy possessed by the body in motion. It is defined as $K.E = (1/2)mv^2$</p> <p>2. Net work done by the external force is equal to the change in the kinetic energy of the system $W = K_f - K_i$</p>
4	Potential Energy	<p>1. It is the kind of energy possessed due to configuration of the system. It is due to conservative force. It is defined as $dU = -\mathbf{F} \cdot d\mathbf{r}$ $U_f - U_i = -\int \mathbf{F} \cdot d\mathbf{r}$ Where \mathbf{F} is the conservative force $\mathbf{F} = -(\partial U / \partial x)\mathbf{i} - (\partial U / \partial y)\mathbf{j} - (\partial U / \partial z)\mathbf{k}$ For gravitational Force</p> <p>2. Change in Potential Energy = mgh where h is the height between the two points</p> <p>3. Mechanical Energy is defined as $E = K.E + P.E$</p>
5	Law Of conservation of Energy	<p>In absence of external forces, internal forces being conservative, total energy of the system remains constant. $K.E_1 + P.E_1 = K.E_2 + P.E_2$</p>
6	Power	<p>Power is rate of doing work i.e., $P = \text{work} / \text{Time}$. Unit of power is Watt. $1W = 1Js^{-1}$. In terms of force $P = \mathbf{F} \cdot \mathbf{v}$ and it is a scalar quantity.</p>

Momentum and Collision

S.No.	Term	Description
1	Linear Momentum	<p>The linear momentum \mathbf{p} of an object of mass m moving with velocity \mathbf{v} is defined as $\mathbf{p} = m\mathbf{v}$</p> <p>Impulse of a constant force delivered to an object is equal to the change in momentum of the object $\mathbf{F}\Delta t = \Delta \mathbf{p} = m\mathbf{v}_f - m\mathbf{v}_i$</p> <p>Momentum of system of particles is the vector sum of individual momentum of the particle $\mathbf{p}_{\text{total}} = \sum \mathbf{p}_i$</p>
2	Conservation of momentum	<p>When no net external force acts on an isolated system, the total momentum of the system is constant. This principle is called conservation of momentum. if $\sum \mathbf{F}_{\text{ext}} = 0$ then $\sum \mathbf{p}_i = \text{constant}$</p>
3	Collision	<p>Inelastic collision - the momentum of the system is conserved, but kinetic energy is not.</p> <p>Perfectly inelastic collision - the colliding objects stick together.</p> <p>Elastic collision - both the momentum and the kinetic energy of the system are conserved.</p>
4	Inelastic collision	<p>While colliding if two bodies stick together then speed of the</p>

		composite body is $v = \frac{m_1 u_1 + m_2 u_2}{m_1 + m_2}$ Kinetic energy of the system after collision is less than that before collision
5	Elastic collision in one dimension	Final velocities of bodies after collision are $v_1 = \left(\frac{m_1 - m_2}{m_1 + m_2} \right) u_1 + \left(\frac{2m_2}{m_1 + m_2} \right) u_2$ $v_2 = \left(\frac{2m_1}{m_1 + m_2} \right) u_1 + \left(\frac{m_2 - m_1}{m_1 + m_2} \right) u_2$ also $u_1 - u_2 = v_2 - v_1$

Special cases of Elastic Collision

S.No.	Case	Description
1	$m_1 = m_2$	$v_1 = u_2$ and $v_2 = u_1$
2	When one of the bodies is at rest say $u_2 = 0$	$v_1 = \left(\frac{m_1 - m_2}{m_1 + m_2} \right) u_1$ and $v_2 = \left(\frac{2m_1}{m_1 + m_2} \right) u_1$
3	When $m_1 = m_2$ and $u_2 = 0$ i.e., m_2 is at rest	$v_1 = 0$ and $v_2 = u_1$
4	When body in motion has negligible mass i.e. $m_1 < m_2$	$v_1 = -u_1$ and $v_2 = 0$
5	When body at rest has negligible mass i.e. $m_1 \gg m_2$	$v_1 = u_1$ and $v_2 = 2u_1$

Mechanics of system of particles

S.No.	Term	Description
1	Centre of mass	It is that point where entire mass of the system is imagined to be concentrated, for consideration of its translational motion.
2	position vector of centre of mass	$\mathbf{R}_{cm} = \sum \mathbf{r}_i M_i / \sum M_i$ where \mathbf{r}_i is the coordinate of element i and M_i is mass of element i
3	In coordinate system	$x_{cm} = \sum x_i M_i / \sum M_i$ $y_{cm} = \sum y_i M_i / \sum M_i$ $z_{cm} = \sum z_i M_i / \sum M_i$
4	Velocity of CM	$\mathbf{v}_{cm} = \sum \mathbf{v}_i M_i / \sum M_i$ The total momentum of a system of particles is equal to the total mass times the velocity of the centre of mass
5	Force	When Newton's second law of motion is applied to the system of particles we find $\mathbf{F}_{tot} = M \mathbf{a}_{cm}$ with $\mathbf{a}_{cm} = d^2 \mathbf{R}_{cm} / dt^2$ Thus centre of mass of the system moves as if all the mass of the system were concentrated at the centre of mass and external force were applied to that point.
6	Momentum conservation in COM motion	$\mathbf{P} = M \mathbf{v}_{cm}$ which means that total linear momentum of system of particles is equal to the product of the total mass of the system and the velocity of its centre of mass.

Rigid body dynamics

S.No.	Term	Description
1	Angular Displacement	-When a rigid body rotates about a fixed axis, the angular displacement is the angle $\Delta\theta$ swept out by a line passing through any point on the body and intersecting the axis of rotation perpendicularly. -It can be positive (counter clockwise) or negative (clockwise). -Analogous to a component of the displacement vector. -SI unit: radian (rad). Other units: degree, revolution.
2	Angular velocity	-Average angular velocity, is equal to $\Delta\theta / \Delta t$. Instantaneous Angular Velocity $\omega = d\theta / dt$ -Angular velocity can be positive or negative.

		<ul style="list-style-type: none"> -It is a vector quantity and direction is perpendicular to the plane of rotation -Angular velocity of a particle is different about different points -Angular velocity of all the particles of a rigid body is same about a point.
3	Angular Acceleration	Average angular acceleration = $\Delta\omega/\Delta t$ Instantaneous Angular Acceleration $\alpha = d\omega/dt$
4	Vector Nature of Angular Variables	<ul style="list-style-type: none"> -The direction of an angular variable vector is along the axis. - positive direction defined by the right hand rule. - Usually we will stay with a fixed axis and thus can work in the scalar form. -angular displacement cannot be added like vectors. Angular velocity and acceleration are vectors
5	Kinematics of rotational Motion	$\omega = \omega_0 + \alpha t$ $\theta = \omega_0 t + 1/2 \alpha t^2$ $\omega \cdot \omega = \omega_0 \cdot \omega_0 + 2 \alpha \cdot \theta$; Also $\alpha = d\omega/dt = \omega(d\omega/d\theta)$
6	Relation Between Linear and angular variables	$\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}$ Where r is vector joining the location of the particle and point about which angular velocity is being computed $\mathbf{a} = \boldsymbol{\alpha} \times \mathbf{r}$
7	Moment of Inertia	Rotational Inertia (Moment of Inertia) about a Fixed Axis For a group of particles, $I = \sum mr^2$ For a continuous body, $I = \int r^2 dm$ For a body of uniform density $I = \rho \int r^2 dV$
8	Parallel Axis Theorem	$I_{xx} = I_{cc} + Md^2$ Where I_{cc} is the moment of inertia about the centre of mass
9	Perpendicular Axis Theorem	$I_{xx} + I_{yy} = I_{zz}$ It is valid for plane laminas only.
10	Torque	$\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}$ also $\tau = I\alpha$ where α is angular acceleration of the body.
11	Rotational Kinetic Energy	$KE = (1/2)I\omega^2$ where ω is angular acceleration of the body
12	Rotational Work Done	<ul style="list-style-type: none"> -If a force is acting on a rotating object for a tangential displacement of $s = r\theta$ (with θ being the angular displacement and r being the radius) and during which the force keeps a tangential direction and a constant magnitude of F, and with a constant perpendicular distance r (the lever arm) to the axis of rotation, then the work done by the force is: $W = \tau\theta$ -W is positive if the torque τ and θ are of the same direction, otherwise, it can be negative.
13	Power	$P = dW/dt = \tau\omega$
14	Angular Momentum	$\mathbf{L} = \mathbf{r} \times \mathbf{p}$ $= \mathbf{r} \times (m\mathbf{v})$ $= m(\mathbf{r} \times \mathbf{v})$ For a rigid body rotating about a fixed axis $L = I\omega$ and $dL/dt = \tau$ if $\tau = 0$ and L is constant For rigid body having both translational motion and rotational motion $\mathbf{L} = \mathbf{L}_1 + \mathbf{L}_2$ \mathbf{L}_1 is the angular momentum of Centre mass about an stationary axis \mathbf{L}_2 is the angular momentum of the rigid body about Centre of mass.
15	Law of Conservation On Angular Momentum	If the external torque is zero on the system then Angular momentum remains constant $dL/dt = \tau_{ext}$ if $\tau_{ext} = 0$ then $dL/dt = 0$

16	Equilibrium of a rigid body	$F_{\text{net}}=0$ and $\tau_{\text{ext}}=0$
17	Angular Impulse	$\int \tau dt$ term is called angular impulse. It is basically the change in angular momentum
18	Pure rolling motion of sphere/cylinder/disc	-Relative velocity of the point of contact between the body and platform is zero -Friction is responsible for pure rolling motion -If friction is non dissipative in nature $E = (1/2)mv_{\text{cm}}^2 + (1/2)I\omega^2 + mgh$

Gravitation

S.No.	Term	Description	
1	Newton's Law of gravitation	$F = \frac{Gm_1m_2}{r^2}$ where G is the universal gravitational constant $G=6.67 \times 10^{-11}\text{Nm}^2\text{Kg}^{-2}$	
2	Acceleration due to gravity	$g=GM/R^2$ where M is the mass of the earth and R is the radius of the earth	
3	Gravitational potential energy	PE of mass m at point h above surface of earth is $PE = -\frac{GmM}{(R+h)}$	
4	Gravitational potential	$V = -\frac{GM}{(R+h)}$	
5	Kepler's Law of planetary motion	Law of orbits	Each planet revolves round the sun in an elliptical orbit with sun at one of the foci of elliptical orbit.
		Law of areas	The straight line joining the sun and the planet sweeps equal area in equal interval of time.
		Law of periods	The squares of the periods of the planet are proportional to the cubes of their mean distance from sun i.e., $T^2 \propto R^3$
6	Escape velocity	Escape velocity is the minimum velocity with which a body must be projected in order that it may escape earth's gravitational pull. Its magnitude is $v_e=\sqrt{(2MG/R)}$ and in terms of g $v_e=\sqrt{(2gR)}$	
7	Satellites	Orbital Velocity	The velocity which is imparted to an artificial satellite few hundred Km above the earth's surface so that it may start orbiting the earth $v_0=\sqrt{(gR)}$
		Periodic Time	$T=2\pi\sqrt{[(R+h)^3/gR^2]}$
8	Variation of g	With altitude	$g_h = g\left(1-\frac{2h}{R}\right)$
		With depth	$g_d = g\left(1-\frac{d}{R}\right)$
		With latitude	$g_\phi = g - 0.037 \cos^2 \phi$

Elasticity

S.No.	Term	Description
1	Elasticity	The ability of a body to regain its original shape and size when deforming force is withdrawn
2	Stress	$\text{Stress} = F/A$ where F is applied force and A is area over which it acts.
3	Strain	It is the ratio of the change in size or shape to the original size or shape. Longitudinal strain = $\Delta l/l$ volume strain = $\Delta V/V$ and shear strain is due to change in shape of the body.
4	Hook's Law	Hook's law is the fundamental law of elasticity and is stated as "for small deformations stress is proportional to strain".

		Thus, stress proportional to strain or, stress/strain = constant This constant is known as modulus of elasticity of a given material	
5	Elastic Modulus	Young's Modulus of Elasticity	$Y = F\ell / A\Delta l$
		Bulk Modulus of Elasticity	$K = -V\Delta P / \Delta V$
		Modulus of Rigidity	$\eta = F / A\theta$
6	Poisson's Ratio	The ratio of lateral strain to the longitudinal strain is called Poisson's ratio which is constant for material of that body. $\sigma = \Delta D / D\Delta l$	
7	Strain energy	Energy stored per unit volume in a strained wire is $E = \frac{1}{2}(\text{stress}) \times (\text{strain})$	

Hydrostatics

S.No.	Term	Description
1	Fluid pressure	It is force exerted normally on a unit area of surface of fluid $P = F/A$. Its unit is Pascal $1\text{Pa} = 1\text{Nm}^{-2}$.
2	Pascal's Law	Pressure in a fluid in equilibrium is same everywhere.
3	Density	Density of a substance is defined as the mass per unit volume.
4	Atmospheric pressure	Weight of all the air above the earth causes atmospheric pressure which exerts pressure on the surface of earth. Atmospheric pressure at sea level is $P_0 = 1.01 \times 10^5 \text{Pa}$
5	Hydrostatic pressure	At depth h below the surface of the fluid is $P = \rho gh$ where ρ is the density of the fluid and g is acceleration due to gravity.
6	Gauge pressure	$P = P_0 + \rho gh$, pressure at any point in fluid is sum of atmospheric pressure and pressure due to all the fluid above that point.
7	Archimedes principle	When a solid body is fully or partly immersed in a fluid it experience a buoyant force equal to the weight of fluid displaced by it.
8	Upthrust	It is the weight of the displaced liquid.
9	Boyle's law	$PV = \text{constant}$
10	Charle's law	$V/T = \text{constant}$

Hydrodynamics

S.No.	Term	Description
1	Streamline flow	In such a flow of liquid in a tube each particle follows the path of its preceding particle.
2	Turbulent flow	It is irregular flow which does not obey above condition.
3	Bernoulli's principle	$p + \frac{1}{2}\rho v^2 + \rho gh = \text{constant}$
4	Continuity of flow	$A_1 v_1 = A_2 v_2$ where A_1 and A_2 are the area of cross section of tube of variable cross section and v_1 and v_2 are the velocity of flow of liquids crossing these areas.
5	Viscosity	Viscous force between two layers of fluid of area A and velocity gradient dv/dx is $F = -\eta A \frac{dv}{dx}$ where η is the coefficient of viscosity.
6	Stokes' law	Viscous force on a spherical body of radius r falling through a liquid of viscosity η is $F = 6\pi\eta r v$ where v is the velocity of the sphere.
7	Poiseuille's equation	Volume of a liquid flowing per second through a capillary tube of radius r when its end are maintained at a pressure difference P is given by $Q = \frac{\pi P r^4}{8\eta l}$ where l is the length of the tube.

Simple Harmonic Motion

S.No.	Term	Description
1	SHM	In SHM the restoring force is proportional to the displacement from the mean position and opposes its increase. Restoring force is $F = -Kx$ Where K =Force constant, x =displacement of the system from its mean or equilibrium position

		Differential Equation of SHM is $d^2x/dt^2 + \omega^2x=0$ Solutions of this equation can both be sine or cosine functions .We conveniently choose $x=A\cos(\omega t+\phi)$ where A , ω and ϕ all are constants
2	Amplitude	Quantity A is known as amplitude of SHM which is the magnitude of maximum value of displacement on either sides from the equilibrium position
3	Time period	Time period (T) of SHM the time during which oscillation repeats itself i.e, repeats its one cycle of motion and it is given by $T=2\pi/\omega$ where ω is the angular frequency
4	Frequency	Frequency of the SHM is the number of the complete oscillation per unit time i.e., frequency is reciprocal of the time period $f=1/T$. Thus angular frequency $\omega=2\pi f$
5	Velocity	Velocity of a system executing SHM as a function of time is $v=-\omega A\sin(\omega t+\phi)$
6	Acceleration	Acceleration of particle executing SHM is $a=-\omega^2 A\cos(\omega t+\phi)$. So $a=-\omega^2 x$ This shows that acceleration is proportional to the displacement but in opposite direction
7	Kinetic energy	At any time t KE of system in SHM is $KE=(1/2)mv^2=(1/2)m\omega^2 A^2 \sin^2(\omega t+\phi)$ which is a function varying periodically in time
8	Potential energy	PE of system in SHM at any time t is $PE=(1/2)Kx^2=(1/2)m\omega^2 A^2 \cos^2(\omega t+\phi)$
9	Total energy	Total Energy in SHM $E=KE+PE=(1/2)m\omega^2 A^2$ and it remain constant in absence of dissipative forces like frictional forces
10	Oscillations of a Spring mass system	In this case particle of mass m oscillates under the influence of hook's law restoring force given by $F=-Kx$ where K is the spring constant Angular Frequency $\omega=\sqrt{K/m}$ Time period $T=2\pi\sqrt{m/K}$ frequency is $=(1/2\pi)\sqrt{K/m}$ Time period of both horizontal and vertical oscillation are same but spring constant have different value for horizontal and vertical motion
11	Simple pendulum	-Motion of simple pendulum oscillating through small angles is a case of SHM with angular frequency given by $\omega=\sqrt{g/L}$ and Time period $T=2\pi\sqrt{L/g}$ Where L is the length of the string. -Here we notice that period of oscillation is independent of the mass m of the pendulum
12	Compound Pendulum	- Compound pendulum is a rigid body of any shape, capable of oscillating about the horizontal axis passing through it. -Such a pendulum swinging with small angle executes SHM with the timeperiod $T=2\pi\sqrt{I/mgL}$ Where I =Moment of inertia of pendulum about the axis of suspension L is the length of the pendulum
13	Damped Oscillation	-SHM which continues indefinitely without the loss of the amplitude are called free oscillation or undamped and it is not a real case - In real physical systems energy of the oscillator gradually decreases with time and oscillator will eventually come to rest. This happens because in actual physical systems, friction(or damping) is always present -The reduction in amplitude or energy of the oscillator is called damping and oscillation are call damped
14	Forced Oscillations and Resonance	- Oscillations of a system under the influence of an external periodic force are called forced oscillations - If frequency of externally applied driving force is equal to the natural frequency of the oscillator resonance is said to occur

Waves

S.No.	Term	Description
1	Wave	-It is a disturbance which travels through the medium due to repeated periodic motion of particles of the medium about their equilibrium position.

		-Examples are sound waves travelling through an intervening medium, water waves, light waves etc.
2	Mechanical waves	Waves requiring material medium for their propagation are MECHANICAL WAVES. These are governed by Newton's law of motion. -Sound waves are mechanical waves in atmosphere between source and the listener and require medium for their propagation.
3	Non mechanical waves	-Those waves which does not require material medium for their propagation are called NON MECHANICAL WAVES. -Examples are waves associated with light or light waves , radio waves, X-rays, micro waves, UV light, visible light and many more.
4	Transverse waves	These are such waves where the displacements or oscillations are perpendicular to the direction of propagation of wave.
5	Longitudinal waves	Longitudinal waves are those waves in which displacement or oscillations in medium are parallel to the direction of propagation of wave for example sound waves
6	Equation of harmonic wave	-At any time t , displacement y of the particle from it's equilibrium position as a function of the coordinate x of the particle is $y(x,t)=A \sin(\omega t - kx)$ where, A is the amplitude of the wave k is the wave number ω is angular frequency of the wave and $(\omega t - kx)$ is the phase.
7	Wave number	Wavelength λ and wave number k are related by the relation $k = 2\pi/\lambda$
8	Frequency	Time period T and frequency f of the wave are related to ω by $\omega/2\pi = f = 1/T$
9	Speed of wave	speed of the wave is given by $v = \omega/k = \lambda/T = \lambda f$
10	Speed of a transverse wave	Speed of a transverse wave on a stretched string depends on tension and the linear mass density of the string not on frequency of the wave i.e, $v = \sqrt{T/\mu}$ T=Tension in the string μ =Linear mass density of the string.
11	Speed of longitudinal waves	Speed of longitudinal waves in a medium is given by $v = \sqrt{B/\rho}$ B=bulk modulus ρ =Density of the medium Speed of longitudinal waves in ideal gas is $v = \sqrt{\gamma P/\rho}$ P=Pressure of the gas , ρ =Density of the gas and $\gamma = C_p/C_v$
12	Principle of superposition	When two or more waves traverse through the same medium,the displacement of any particle of the medium is the sum of the displacement that the individual waves would give it. $y = \sum y_i(x,t)$
13	Interference of waves	If two sinusoidal waves of the same amplitude and wavelength travel in the same direction they interfere to produce a resultant sinusoidal wave travelling in that direction with resultant wave given by the relation $y'(x,t) = [2A_m \cos(u/2)] \sin(\omega t - kx + u/2)$ where u is the phase difference between two waves. -If $u=0$ then interference would be fully constructive. -If $u=n$ then waves would be out of phase and there interference would be destructive.
14	Reflection of waves	When a pulse or travelling wave encounters any boundary it gets reflected. If an incident wave is represented by $y_i(x,t) = A \sin(\omega t - kx)$ then reflected wave at rigid boundary is $y_r(x,t) = A \sin(\omega t + kx + \pi) = -A \sin(\omega t + kx)$ and for reflections at open boundary reflected wave is given by $y_r(x,t) = A \sin(\omega t + kx)$
15	Standing waves	The interference of two identical waves moving in opposite directions produces standing waves. The particle displacement in standing wave is given by $y(x,t) = [2A \cos(kx)] \sin(\omega t)$ In standing waves amplitude of waves is different at different points i.e., at nodes amplitude is zero and at antinodes amplitude is maximum which is equal to sum of amplitudes of constituting waves.

16	Normal modes of stretched string	Frequency of transverse motion of stretched string of length L fixed at both the ends is given by $f = nv/2L$ where $n=1,2,3,4,\dots$ -The set of frequencies given by above relation are called normal modes of oscillation of the system. Mode $n=1$ is called the fundamental mode with frequency $f_1 = v/2L$. Second harmonic is the oscillation mode with $n=2$ and so on. -Thus the string has infinite number of possible frequency of vibration which are harmonics of fundamental frequency f_1 such that $f_n = nf_1$
17	Beats	Thus beats arises when two waves having slightly differing frequencies v_1 and v_2 and comparable amplitude are superposed. -Here interfering waves have slightly differing frequencies v_1 and v_2 such that $ v_1 - v_2 < (v_1 + v_2)/2$ The beat frequency is $v_{\text{beat}} = v_1 \square v_2$
18	Doppler effect	-Doppler effect is a change in the observed frequency of the wave when the source S and the observer O move relative to the medium. -There are three different ways where we can analyse this change in frequency. (1) When observer is stationary and source is approaching observer is $v = v_0(1 + v_s/v)$ where, v_s = velocity of source relative to the medium v = velocity of wave relative to the medium v = observed frequency of sound waves in term of source frequency v_0 = source frequency -Change in frequency when source recedes from stationary observer is $v = v_0(1 - v_s/v)$ -Observer at rest measures higher frequency when source approaches it and it measures lower frequency when source recedes from the observer. (2) Observer is moving with a velocity V_o towards source and the source is at rest is $v = v_0(1 + V_o/v)$ (3) Both source and observer are moving then frequency observed by observer is $v = v_0(V + V_o)/(V + v_s)$ and all the symbols have respective meanings as told earlier

Thermal expansion

S.No.	Term	Description
1	Coefficient of linear expansion	$\alpha = \frac{l_t - l_0}{l_0 t}$ where α = coefficient of linear expansion, l_t = length at $t^\circ\text{C}$ and l_0 is length at 0°C .
2	Length at temperature $t^\circ\text{C}$	$l_t = l_0(1 + \alpha t)$
3	Coefficient of superficial expansion	$\beta = \frac{A_t - A_0}{A_0 t}$
4	Area at temperature $t^\circ\text{C}$	$A_t = A_0(1 + \beta t)$
5	Coefficient of volume expansion	$\gamma = \frac{V_t - V_0}{V_0 t}$
6	Volume at temperature $t^\circ\text{C}$	$V_t = V_0(1 + \gamma t)$
7	Coefficient of apparent expansion of a liquid	$\gamma_a = \frac{V_a - V_0}{V_0 t}$ where γ_a = coefficient of apparent expansion, V_0 = volume at 0°C and V_a = apparent volume at $t^\circ\text{C}$
8	Coefficient of real expansion of a liquid	$\gamma_r = \frac{V_r - V_0}{V_0 t}$ where γ_r = coefficient of real expansion, V_0 = volume at 0°C and V_r = real volume at $t^\circ\text{C}$
9	Density variation with temperature	$d_t = \frac{d_0}{1 + \gamma t}$ where d_t = density at temperature $t^\circ\text{C}$, d_0 = density at 0°C .
10	Pressure coefficient of gas	$\gamma_p = \frac{P_t - P_0}{P_0 t}$

11	Volume coefficient of gas	$\gamma_v = \frac{V_t - V_0}{V_0 t}$
12	Ideal gas equation	PV=nRT where n is number of moles of gas and R is universal constant.

Measurement of heat and temperature

S.No.	Term	Description
1	Relation between Celsius, Fahrenheit and Kelvin scale of temperature	$\frac{C}{100} = \frac{F - 32}{180} = \frac{K - 273}{100}$
2	Principle of electrical resistance thermometer	$R_t = R_0(1 + \alpha t)$ where R_t is resistance at $t^\circ\text{C}$, R_0 is resistance at 0°C and α is temperature coefficient of resistance.
3	Joule's mechanical equivalent to heat(J)	$W = JH$ where value of J is $J = 4.2\text{J/cal}$.
4	Heat capacity	$C = \Delta Q / \Delta t$ where ΔQ is the amount of heat absorbed and Δt is rise in temperature.
5	Specific heat	It is the amount of heat required to raise the temperature of 1Kg of substance by 1°C
6	Molar specific heat	Molar specific heat of the substance is the amount of heat required to raise the temperature of 1 mole of the substance through 1°C .
7	Relation between C_p and C_v	$C_p - C_v = R$ where R is universal gas constant.
8	Latent heat of fusion	Heat energy required to convert a unit mass of substance from the solid to the liquid state without change in temperature.
9	Latent heat of sublimation	Heat energy required to convert a unit mass of substance from the liquid to the gaseous state without change in temperature.
10	Principle of calorimetry	Heat lost = Heat gain

Thermodynamics

S.No.	Term	Description
1	Thermodynamic state	It refers to the state of the system that is completely defined by pressure, volume and temperature of the system.
2	Zeroth law of thermodynamics	If two systems A and B are separately in equilibrium with the third system C then system A and B are in thermal equilibrium with each other
3	First law of thermodynamics	Heat energy given to the system is equal to the increase in internal energy of the system and the work done. $Q = \Delta U + W$
4	Second law of thermodynamics	Heat can not flow from a colder body to a hotter body without some work being done by the external agency.
5	Work in volume changes	If pressure remains constant while the volume changes, then work is $W = P(V_2 - V_1)$
6	Quasi static Processes	In Quasi static process deviation of system from its thermodynamic equilibrium is infinitesimally small.
7	Isothermal Process	temperature of the system remains constant throughout the process and thus $\Delta Q = \Delta W$
8	Adiabatic Process	no heat enters or leaves a system thus $\Delta U = U_2 - U_1 = - \Delta W$
9	Isochoric process	volume of the system remain unchanged throughout and $U_2 - U_1 = \Delta U = \Delta Q$
10	Isobaric Process	This process takes place at constant pressure.
11	Work done in Isothermal process	$W = nRT \ln(V_2/V_1)$ Where n is number of moles in sample of gas taken
12	Ideal gas equation for adiabatic process	$PV^\gamma = K$ (Constant) Where γ is the ratio of specific heat (ordinary or molar) at constant pressure and at constant volume $\gamma = C_p/C_v$
13	Work done in an Adiabatic process	$W = (P_1 V_1 - P_2 V_2) / (\gamma - 1)$ In an adiabatic process if $W > 0$ i.e., work is done by the gas then $T_2 < T_1$. If work is done on the gas ($W < 0$) then $T_2 > T_1$ i.e., temperature of gas rises
14	Thermal efficiency of heat engine	$\eta = 1 - (Q_2/Q_1)$

15	Coefficient of performance of refrigerator	$\beta = Q_2/W = Q_2/(Q_1 - Q_2)$
16	Efficiency of Carnot engine	$\beta = Q_2/W = Q_2/(Q_1 - Q_2)$
17	Carnot Theorem	Carnot's theorem consists of two parts (i) No engine working between two given temperatures can be more efficient than a reversible Carnot engine working between same source and sink. (ii) All reversible engines working between same source and sink (same limits or temperature) have the same efficiency irrespective of the working substance.

Heat transfer

S.No.	Term	description	
1	Thermal Conductivity	$H = \frac{kA(T_2 - T_1)}{L}$ Where H is the quantity of heat flowing through the slab and k is the constant called thermal conductivity of material of slab.	
2	Convection	Convection is transfer of heat by actual motion of matter	
3	Radiation	Radiation process does not need any material medium for heat transfer	
4	Stefan Boltzmann law	The rate u_{rad} at which an object emits energy via EM radiation depends on object's surface area A and temperature T in kelvin of that area and is given by $u_{\text{rad}} = \sigma \epsilon AT^4$ Where $\sigma = 5.6703 \times 10^{-8} \text{ W/m}^2\text{K}^4$ is Stefan Boltzmann constant and ϵ is emissivity of object's surface with value between 0 and 1	
5	Wein's displacement law	$\lambda_m T = b$ Where $b = 0.2896 \times 10^{-2} \text{ mK}$ for black body and is known as Wien's constant	
6	Kirchoff's law	Emissive Power	It is the energy radiated per unit area per unit solid angle normal to the area. $E = \Delta u / [(\Delta A) (\Delta \omega) (\Delta t)]$ where, Δu is the energy radiated by area ΔA of surface in solid angle $\Delta \omega$ in time Δt .
		Absorptive Power	is defined as the fraction of the incident radiation that is absorbed by the body $a(\text{absorptive power}) = \text{energy absorbed} / \text{energy incident}$
		Kirchoff's Law	"It states that at any given temperature the ratio of emissive power to the absorptive power is constant for all bodies and this constant is equal to the emissive power of perfect B.B. at the same temperature. $E/a_{\text{body}} = E_{\text{B.B}}$
7	Newton's Law of Cooling	For small temperature difference between the body and surrounding rate of cooling is directly proportional to the temperature difference and surface area exposed i.e., $dT/dt = -bA(T_1 - T_2)$. This is known as Newton's law of cooling	

Kinetic theory of gases

S.No.	Term	Description	
1	Gas laws	Boyle's law	At constant temperature, the volume of a given mass of gas is inversely proportional to pressure. Thus $PV = \text{constant}$
		Charles's Law	When pressure of a gas is constant the volume of a given mass of gas is directly proportional to its absolute temperature. $V/T = \text{Constant}$
		Dalton's law of partial pressures	The total pressure of mixture of ideal gases is sum of partial pressures of individual gases of which mixture is made of
2	Ideal gas equation	$PV = nRT$ where n is number of moles of gas	
3	Pressure of gas	$P = (1/3)\rho v_{\text{mg}}^2$ or $PV = (1/3)Nm v_{\text{mg}}^2$ where v_{mg} known as mean	

		square speed
4	rms speed	$v_{rms} = \sqrt{(3P/\rho)} = \sqrt{(3PV/M)} = \sqrt{(3RT/M)}$
5	Law of Equipartition of energy	each velocity component has, on the average, an associated kinetic energy $(1/2)KT$
6	Specific Heat Capacity	Monatomic gases $C_V = (3/2)R$, $C_P = 5/2 R$ and $\gamma_{mono} = C_P/C_V = 5/3$
		Diatomic gases $C_V = (5/2)R$, $C_P = (9/2)R$ and $\gamma = C_P/C_V = 9/7$
7	Specific heat Capacity of Solids	$C = 3R$ this is Dulong and Petit law
8	Mean free path	If v is the distance traversed by molecule in one second then mean free path is given by λ = total distance traversed in one second /no. of collision suffered by the molecules $= v/n\sigma^2vn$ $= 1/n\sigma^2n$

Electric Charge, Force and Field

S.No.	Term	Description
1	Charge	Charges are of two types (a) positive charge (b) negative charge like charges repel each other and unlike charges attract each other.
2	Properties of charge	1. Quantisation : - $q = ne$ where $n = 0, 1, 2, \dots$ and e is charge of an electron. 2. Additivity : - $q_{net} = \sum q$ 3. conservation :- total charge of an isolated system is constant
3	Coulomb's law	The mutual electrostatic force between the charges q_1 and q_2 separated by a distance r is given by Force on the charge q_1 $F_1 = Kq_1q_2r_{12}/r^2$ where r_{12} is the unit vector in the direction from q_2 to q_1 For more than two charges in the system the force acting on any charges is vector sum of the coulomb force from each of the other charges. This is called principle of superposition for $q_1, q_2, q_3, \dots, q_n$ charges are present in the system. Force on charge q_1 $F = Kq_1q_2r_{12}/r_{12}^2 + Kq_1q_3r_{13}/r_{13}^2 + Kq_1q_4r_{14}/r_{14}^2 \dots Kq_1q_nr_{1n}/r_{1n}^2$ Similarly for the other charges...
4	Electric field	-The region around a particular charge in which its electrical effects can be observed is called the electric field of the charge -Electric field has its own existence and is present even if there is no charge to experience the electric force.
5	Electric field Intensity	$E = F/q_0$ Where F is the electric force experience by the test charge q_0 at this point. It is the vector quantity Some point to note on this 1. Electric field lines extend away from the positive charge and towards the negative charges 2. Electric field produces the force so if a charge q is placed in the electric field E the force experience by the charge is $F = qE$ 3 Principle of superposition also applies to electric field so $E = E_1 + E_2 + E_3 + E_4 + \dots$ Electric field intensity due to point charge $E = KQ/r^2$ Where r is the distance from the point charge and r is the unit vector along the direction from source to point.
6	Some useful Formula	Electric field for the Uniformly charged ring $E = KQx/(r^2 + x^2)^{3/2}$ Where x is the distance from the centre of the ring. At $x=0$ $E=0$
		Electric Field due to uniformly charged disc $E = (\sigma/2\epsilon_0)(1 - x/(\sqrt{R^2 + x^2}))$ σ =Surface charge density of the disc. At $x=0$ $E = \sigma/2\epsilon_0$
		Electric Field Intensity due to Infinite sheet of the charge $E = \sigma/2\epsilon_0$
7	Charge density	Linear charge density $\lambda = Q/L = dQ/dL$ Surface charge density $\sigma = Q/A = dQ/dA$

		Volume charge density $\rho = Q/V = dQ/dV$
8	Electrostatics in Conductor	1. $E=0$ inside the conductor 2. All charge resides on the outer surface of the conductor 3. Electric at the surface is Perpendicular to the surface
9	Electric Flux	$d\phi = \mathbf{E} \cdot d\mathbf{a}$ $d\mathbf{a}$ is the area vector to the surface and it is taken positive along the outward normal to the surface $d\phi = E d\mathbf{a} \cos\theta$ $\phi = \int \mathbf{E} \cdot d\mathbf{a}$
10	Gauss's Theorem	Flux in closed surface is equal net charge inside divided by ϵ $\int \mathbf{E} \cdot d\mathbf{a} = q_{in}/\epsilon$
11	Some points	a. \mathbf{E} is the electric field present due to all charges in the system not just the charge inside b. Flux crossing a closed surface does not depend on the shapes and size of Gaussian surface

Electric potential

S.No.	Term	Description
1	Electric Potential energy	$\Delta U = -W$ Where ΔU = Change in Potential energy and W = Work done by the electric lines of forces For a system of two particles $U(r) = q_1 q_2 / 4\pi\epsilon r$ where r is the separation between the charges. We assume U to be zero at infinity Similarly for a system of n charges U = Sum of potential energy of all the distinct pairs in the system For example for three charges $U = (1/4\pi\epsilon)(q_1 q_2 / r_{12} + q_2 q_3 / r_{23} + q_1 q_3 / r_{13})$
2	Electric PE of a charge	$= qV$ where V is the potential there
3	Electric Potential	Liken Electric field intensity is used to define the electric field; we can also use Electric Potential to define the field. Potential at any point P is equal to the work done per unit test charge by the external agent in moving the test charge from the reference point (without Change in KE) $V_p = W_{ext}/q$ So for a point charge $V_p = Q/4\pi\epsilon r$ where r is the distance of the point from charge
4	Some points about Electric potential	1. It is scalar quantity 2. Potential at point due to system of charges will be obtained by the summation of potential of each charge at that point $V = V_1 + V_2 + V_3 + V_4$ 3. Electric forces are conservative force so work done by the electric force between two point is independent of the path taken 4. $V_2 - V_1 = -\int \mathbf{E} \cdot d\mathbf{r}$ 5. In Cartesian coordinates system $dV = -\mathbf{E} \cdot d\mathbf{r}$ $dV = -(E_x dx + E_y dy + E_z dz)$ So $E_x = -\partial V / \partial x$, $E_y = -\partial V / \partial y$ and $E_z = -\partial V / \partial z$ Also $\mathbf{E} = -[(\partial V / \partial x)\mathbf{i} + (\partial V / \partial y)\mathbf{j} + (\partial V / \partial z)\mathbf{k}]$ 6. Surface where electric potential is same everywhere is call equipotential surface Electric field components parallel to equipotential surface is always zero
5	Electric dipole	A combination of two charge $+q$ and $-q$ separated by the distance d $\mathbf{p} = q\mathbf{d}$ Where d is the vector joining negative to positive charge
6	Electric potential due to dipole	$V = (1/4\pi\epsilon)(p \cos\theta / r^2)$ where r is the distance from the center and θ is angle made by the line from the axis of dipole
7	Electric field due to dipole	$E_\theta = (1/4\pi\epsilon)(p \sin\theta / r^3)$ $E_r = (1/4\pi\epsilon)(2p \cos\theta / r^3)$ Total $E = \sqrt{E_\theta^2 + E_r^2}$

		$=(p/4\pi\epsilon r^3)(\sqrt{(3\cos^2\theta+1)})$ Torque on dipole= $\mathbf{p}\times\mathbf{E}$ Potential Energy $U=-\mathbf{p}\cdot\mathbf{E}$
8	Few more points	1. $\oint \mathbf{E}\cdot d\mathbf{l}$ over closed path is zero 2. Electric potential in the spherical charge conductor is $Q/4\pi\epsilon R$ where R is the radius of the shell and the potential is same everywhere in the conductor 3. Conductor surface is a equipotential surface

Capacitance

S.No.	Term	Description
1	Capacitance C of the capacitor	$C=q/V$ or $q=CV$ -Unit of capacitance is Farads or CV^{-1} capacitance of a capacitor is constant and depends on shape, size and separation of the two conductors and also on insulating medium being used for making capacitor.
2	Capacitance of parallel plate cap	$C=(\epsilon_0 A)/d$ where, C = capacitance of capacitor A = area of conducting plate d = distance between plates of the capacitor $\epsilon_0=8.854\times 10^{-12}$ and is known as electric permittivity in vacuum.
3	parallel plate air capacitor in presence of dielectric medium	$C=\epsilon A/d$
4	Capacitance of spherical capacitor having radii a, b ($b>a$)	(a) air as dielectric between them $C=(4\pi\epsilon_0 ab)/(b-a)$ (b) dielectric with relative permittivity ϵ $C=(4\pi\epsilon ab)/(b-a)$
5	Parallel combination of capacitors	$C=Q/V= C_1+C_2+C_3$, resultant capacitance C is greater then the capacitance of greatest individual one.
6	Series combination of capacitors	$1/C=1/C_1+1/C_2+1/C_3$, resultant capacitance C is less then the capacitance of smallest individual capacitor.
7	Energy stored in capacitor	Energy stored in capacitor is $E=QV/2$ or $E=CV^2/2$ or $E=Q^2/2C$ factor 1/2 is due to average potential difference across the capacitor while it is charged.
8	Force between plates of capacitor	$F = \frac{Q^2}{2K\epsilon_0 A}$
9	Force per unit area of plates	$f = \frac{\sigma^2}{2K\epsilon_0}$ Where σ is charge per unit area.

Electric current and D.C. circuits

S.No.	Term	Description
1	Electromotive force	EMF of a cell is the total energy per unit charge when the cell is on an open circuit i.e., when the current through the cell is zero.
2	Electric current	$I=q/t$ it is the rate of flow of electric charge. Unit of current is ampere.
3	Drift speed of electron in a conductor	$v_d = \frac{I}{neA}$ Where I is the current, n is the number of electrons per unit volume and A is the area of cross section of conductor.
4	Resistivity of conductor	$\rho = \frac{m}{ne^2\tau}$ where m is the mass of electron and τ is the relaxation time
5	Ohm's law	$V=IR$ where R is the resistance of the given conductor and unit of resistance is ohm(Ω)

6	Electrical resistivity	$P=RA/l$ where l is the length of the wire and A is its area of cross-section
7	Resistors in series	$R=R_1+R_2+R_3+....$
8	Resistors in parallel	$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} +$
9	Terminal voltage	It is equal to emf of battery minus potential drop across internal resistance r across the battery. Terminal voltage = $E-Ir$
10	Kirchoff's first law	The algebraic sum of current at any junction in a circuit is zero.
11	Kirchoff's second law	The algebraic sum of the products of the current and resistances and the emf in a closed loop is zero.
12	Heating effect of current	Heat energy delivered by current when it flows through resistance of R ohm for t sec. maintained at potential difference V is $H=V^2t/R$
13	Electrical power	$P = VI = I^2R = V^2/R$
14	Variation of resistance with temperature	$R=R_0(1+\alpha(T-T_0))$
15	Variation of resistivity with temperature	$\rho=\rho_0(1+\alpha(T-T_0))$

Magnetic effect of current

S.No.	Term	Description
1	Biot-Savart law	Magnetic field dB at any point whose position vector is \mathbf{r} wrt current element $d\mathbf{l}$ is given by $d\mathbf{B} = \frac{\mu_0 I}{4\pi} \frac{(d\mathbf{l} \times \mathbf{r})}{r^3}$
2	Magnetic field due to long current carrying conductor	$B = \frac{\mu_0 2I}{4\pi r}$
3	Magnetic field at centre of a circular loop	$B = \frac{\mu_0 I}{2r}$
4	Magnetic field at centre of coil of n turns	$B = \frac{\mu_0 In}{2r}$
5	Magnetic field on the axis of a circular loop	$B = \frac{\mu_0 I r^2}{2(r^2 + x^2)^{3/2}}$ if there are n turns in the coil then $B = \frac{\mu_0 In r^2}{2(r^2 + x^2)^{3/2}}$
6	Ampere's circuital law	$\oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 I$
7	Field due to toroidal solenoid	$B = \mu_0 nI$
8	Field inside straight solenoid	$B = \mu_0 nI$ and direction of field is parallel to the axis of solenoid
9	Force on moving charge in magnetic field	$\mathbf{F} = q(\mathbf{v} \times \mathbf{B})$ Direction of force is perpendicular to both \mathbf{v} and \mathbf{B}
10	Force on current carrying conductor in the magnetic field	$\mathbf{F} = I(\mathbf{l} \times \mathbf{B})$ where l is the length of the conductor in the direction of current in it
11	Force between two parallel wires carrying current	$F = \frac{\mu_0 I_1 I_2}{2\pi R}$
12	Torque on a current carrying loop	$\boldsymbol{\tau} = (\mathbf{m} \times \mathbf{B})$ Where \mathbf{m} is the magnetic moment of the dipole and magnitude of magnetic moment is $m=NIB$ where A is the area of the loop and N is the number of turns in the loop.
13	Lorentz force	Force on electron moving with velocity \mathbf{v} in presence of both uniform electric and magnetic field is $\mathbf{F} = -e(\mathbf{E} + \mathbf{v} \times \mathbf{B})$
14	Magnetic dipole moment of bar magnet	$\mathbf{m}=q(2\mathbf{a})$ where q is the pole strength and $(2\mathbf{a})$ is the length of the bar magnet. It is the vector pointing from south to north pole of the magnet.
15	torque on the bar magnet	$\boldsymbol{\tau} = \mathbf{m} \times \mathbf{B}$
16	Potential energy of a magnetic dipole	$U=-mB\cos\theta$

Electromagnetic induction

S.No.	Term	Description
1	Magnetic flux	$\phi = \mathbf{B} \cdot \mathbf{A} = BA \cos \theta$ where θ is the angle between normal to the plane of the coil and magnetic field s.I. unit of flux is Weber
2	Faraday's law	$e = -\frac{d\phi}{dt}$ and for a closely packed coil of N turns $e = -N \frac{d\phi}{dt}$ where e is the induced EMF
3	Lenz's law	The induced current has such a direction such that magnetic field of the current opposes the change in the magnetic flux that produces current.
4	Inductance L of inductor	$L = \frac{N\phi}{i}$ Where N is windings of inductor, I is the current associated with each winding of inductor.
5	Self induction	Phenomenon by which an opposing EMF is introduced in the coil because of varying current in coil itself. Self induction EMF is $e = -L \frac{di}{dt}$
6	Series RL circuit	Rise of current $i = \frac{e}{R} (1 - e^{-t/\tau_L})$ where $\tau_L = L/R$ is inductive time constant of the circuit. On removing emf current decays from a value i_0 according to equation $i = i_0 e^{-t/\tau_L}$
7	Magnetic energy	Energy stored by inductor's magnetic field is $U = \frac{1}{2} Li^2$.
8	Density of stored magnetic energy	$u = B^2/2\mu_0$
9	Mutual induction	It refers to the phenomenon by which a current I is induced in a coil when current in a neighbouring coil circuit is changed. It is described by $e_2 = -M \frac{di_1}{dt}$ and $e_1 = -M \frac{di_2}{dt}$ where M is the mutual induction for the coil arrangement.

Magnetism of matter

S.No.	Term	Description
1	Gauss's law for magnetic fields	$\phi = \oint \mathbf{B} \cdot d\mathbf{A} = 0$ i.e., net magnetic flux through any closed Gaussian surface is zero.
2	Spin magnetic dipole moment	$\mu_s = -\frac{e}{m} \mathbf{S}$ where \mathbf{S} is spin angular momentum.
3	Orbital magnetic dipole moment	$\mu_L = -\frac{e}{2m} \mathbf{L}$ where \mathbf{L} is orbital angular momentum
4	Diamagnetism	Diamagnetic materials are those materials which on being placed in magnetic field get feebly magnetised in the direction opposite to the magnetic field.
5	Paramagnetism	In Paramagnetic materials each atom has permanent magnetic moment but dipole moments are randomly oriented and material as a whole lacks property of magnetism but dipoles can be aligned in the presence of external magnetic field to give net dipole moment and material gets feebly magnetised in the direction of the field.
6	Ferromagnetism	Ferromagnetic materials when placed in external magnetic field gets strongly magnetised in the direction of the magnetic field.
7	Maxwell's extension of ampere's law	$\oint \mathbf{B} \cdot d\mathbf{s} = \mu_0 \epsilon_0 \frac{d\phi}{dt}$

Transient Currents

S.No.	Term	Description
1	Growth of charge in CR circuit	$q = q_0 (1 - e^{-\frac{t}{CR}})$ and current in CR circuit is $i = i_0 e^{-\frac{t}{CR}}$

2	Decay of charge in CR circuit	$q = q_0 e^{-\frac{t}{CR}}$ and current in CR circuit is $i = -i_0 e^{-\frac{t}{CR}}$
3	Capacitive time constant	CR has dimensions of time and is called capacitive time constant for circuit
4	Energy stored in inductor	$U = \frac{1}{2}(Li^2)$
5	Energy stored in capacitor	$U = \frac{1}{2}(CE^2) = \frac{1}{2}(q_0 E)$ Where E is the maximum value of potential difference set up across the plates.
6	LC oscillations	Frequency of oscillations is $f = \frac{1}{2\pi\sqrt{LC}}$

Alternating Current

S.No.	Term	Description
1	Alternating current	It is current whose magnitude changes with time and direction reverses periodically. $I = I_0 \sin \omega t$ where I_0 is the peak value of a.c. and $\omega = 2\pi/T$ is the frequency
2	Mean value of a.c.	$I_m = 2I_0/\pi = 0.636I_0$
3	RMS value	$I_{rms} = I_0/\sqrt{2}$
4	a.c. through resistor	Alternating emf is in phase with current
5	a.c. through inductor	Emf leads the current by an phase angle $\pi/2$
6	a.c. through capacitor	Emf lags behind the current by an phase angle $\pi/2$
7	Inductive reactance	Opposition offered by inductor to the flow of current mathematically, $X_L = \omega L = 2\pi fL$
8	Capacitive reactance	Opposition offered by capacitor to the flow of current mathematically, $X_C = \frac{1}{\omega C} = \frac{1}{2\pi fC}$
9	a.c. through series LR circuit	Emf leads the current by an phase angle ϕ given by $\tan \phi = \frac{\omega L}{R}$ and impedance of circuit is $Z = \sqrt{R^2 + (\omega^2 L^2)}$
10	a.c. through series CR circuit	Emf lags behind the current by an phase angle ϕ given by $\tan \phi = \frac{1/\omega C}{R}$ and impedance of circuit is $Z = \sqrt{R^2 + \left(\frac{1}{\omega^2 C^2}\right)}$
11	a.c. through series LCR circuit	Emf leads/lags behind the current by an phase angle ϕ given by $\tan \phi = \frac{\omega L - 1/\omega C}{R}$ emf leads the current when $\omega L > \frac{1}{\omega C}$ and lags behind when $\omega L < \frac{1}{\omega C}$ and impedance of circuit is $Z = \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}$
12	Average power of an a.c. circuit	$P_{avg} = I_{rms}^2 R = E_{rms} I_{rms} \cos \phi$ Where ϕ is called power factor of the circuit.
13	Transformer	It is a device used to change low alternating voltage at high current into high voltage at low current and vice-versa. Primary and secondary voltage for a transformer are related as $V_s = V_p \frac{N_s}{N_p}$ and current through the coils is related as $I_s = I_p \frac{N_p}{N_s}$

Electromagnetic waves

S.No.	Term	Description
1	Conduction current	It is the current due to the flow of electrons through the connecting wires in an electric circuit

2	Displacement current	It arises due to time rate of change of electric flux in some part of circuit $I_D = \epsilon_0 \frac{d\phi}{dt}$	
3	Modified ampere's circuital law	$\oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 (I_C + I_D) = \mu_0 (I_C + \epsilon_0 \frac{d\phi_E}{dt})$ where I_C is the conduction current.	
4	Maxwell's Equations	Gauss's law in electrostatics	$\oint \mathbf{E} \cdot d\mathbf{s} = \frac{q}{\epsilon_0}$
		Gauss's law in magnetism	$\oint \mathbf{B} \cdot d\mathbf{s} = 0$
		Faraday's law of EM induction	$\oint \mathbf{E} \cdot d\mathbf{l} = -\frac{d\phi_B}{dt}$
		Ampere-Maxwell's circuital law	$\oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 (I_C + \epsilon_0 \frac{d\phi_E}{dt})$
5	Velocity of EM waves in free space	$v = \frac{1}{\epsilon_0 \mu_0} = 3 \times 10^8 \text{ m/s}$	

Huygens' Principle and Interference of Light

S.No.	Term	Description
1	Wave front	It is the locus of points in the medium which at any instant are vibrating in the same phase.
2	Huygens' Principle	1 Each point on the given primary Wavefront acts as a source of secondary wavelets spreading out disturbance in all direction.
		2 The tangential plane to these secondary wavelets constitutes the new wave front
3	Interference	It is the phenomenon of non uniform distribution of energy in the medium due to superposition of two light waves.
4	Condition of maximum intensity	$\phi = 2n\pi$ or $x = n\lambda$ where $n=0,1,2,3,\dots$
6	Condition of minimum intensity	$\phi = (2n+1)\pi$ or $x = (2n+1)\lambda/2$ where $n=0,1,2,3,\dots$
7	Ratio of maximum and minimum intensity	$\frac{I_{\max}}{I_{\min}} = \frac{(a_1 + a_2)^2}{(a_1 - a_2)^2}$
8	Distance of nth bright fringe from centre of the screen	$y_n = \frac{nD\lambda}{d}$ where d is the separation distance between two coherent source of light, D is the distance between screen and slit, λ is the wavelength of light used.
9	Angular position of nth bright fringe	$\theta_n = \frac{y_n}{D} = \frac{n\lambda}{d}$
10	Distance of nth dark fringe from centre of the screen	$y_n' = \frac{(2n+1)D\lambda}{2d}$
11	Angular position of nth dark fringe	$\theta_n' = \frac{y_n'}{D} = \frac{(2n+1)\lambda}{2d}$
12	Fringe width	$\beta = \frac{D\lambda}{d}$

Diffraction and polarisation of light

S.No.	Term	Description
1	Diffraction	It is the phenomenon of bending of light waves round the sharp corners and spreading into the regions of geometrical shadow of the object.
2	Single slit diffraction	Condition for dark fringes is $\sin \theta = \frac{n\lambda}{a}$ where $n = \pm 1, \pm 2, \pm 3, \pm 4, \dots$, a is the width of slit and θ is angle of diffraction. Condition for bright fringes is $\sin \theta = \frac{(2n+1)\lambda}{2a}$
3	Width of central maximum is	$\theta_0 = \frac{2\lambda D}{a}$ where D is the distance between slit and screen.
4	Diffraction	The arrangement of large number of narrow rectangular slits of equal

	grating	width placed side by side parallel to each other. the condition for maxima in the interference pattern at the angle θ is $d \sin \theta = n\lambda$ where $n=0,1,2,3,4,\dots$
6	resolving power of the grating	For two nearly equal wavelengths λ_1 and λ_2 between which a diffraction grating can just barely distinguish resolving power is $R = \frac{\lambda}{\lambda_2 - \lambda_1} = \frac{\lambda}{\Delta\lambda}$ where $\lambda = (\lambda_1 + \lambda_2)/2$
7	Diffraction of X-Rays by crystals	the condition for constructive interference is $2d \sin \theta = n\lambda$ where $n=1,2,3,4,\dots$
8	Polarisation	It is the phenomenon due to which vibrations of light are restricted in a particular plane.
9	Brewster's law	$\mu = \tan p$ where μ is refractive index of medium and p is angle of polarisation.
10	Law of Malus	$I = I_0 \cos^2 \theta$ where I is the intensity of emergent light from analyser, I_0 is the intensity of incident plane polarised light and θ is the angle between planes of transmission of analyser and the polarizer.

Reflection and Refraction of light

S.No.	Term	Description
1	Laws of reflection	1. The incident ray, the normal at a point of incidence and reflected ray all lie on the same plane.
		2. Angle of incidence is always equal to angle of reflection.
2	Relation between f and R	$f = R/2$ both f and R are positive for concave mirror and negative for convex mirror.
3	Mirror formula	$\frac{1}{u} + \frac{1}{v} = \frac{1}{f}$
4	Magnification	$m = \frac{I}{O} = \frac{f-v}{f}$
5	Refraction	The phenomenon of change in path of light as it goes from one medium to another
6	Laws of refraction	1. The incident ray, the normal at a point of incidence and refracted ray all lie on the same plane.
		2. The ratio of sine of angle of incidence to the sine of angle of refraction is constant for any two given media. This is known as Snell's law. Mathematically $\frac{\sin i}{\sin r} = \mu_b^a$ where μ_b^a is relative refractive index of medium b w.r.t. medium a.
7	Lateral shift	$d = \frac{t \sin(i-r)}{\cos r}$
8	Spherical refracting surface	$-\frac{\mu_1}{u} + \frac{\mu_2}{v} = \frac{\mu_2 - \mu_1}{R}$ for object situated in rarer medium $-\frac{\mu_2}{u} + \frac{\mu_1}{v} = \frac{\mu_1 - \mu_2}{R}$ for object situated in denser medium
9	Power of spherical refracting surface	$P = \frac{\mu_2 - \mu_1}{R}$
10	Lens maker's formula	$\frac{1}{f} = (\mu - 1) \left(\frac{1}{R_1} - \frac{1}{R_2} \right)$ focal length of convex lens is positive and that of concave lens is negative
11	Lens equation	$-\frac{1}{u} + \frac{1}{v} = \frac{1}{f}$
12	Linear magnification	$m = \frac{I}{O} = \frac{f-v}{f}$
13	Power of lens	$P = \frac{1}{f} = (\mu - 1) \left(\frac{1}{R_1} - \frac{1}{R_2} \right)$
14	For thin lenses placed in contact	Focal length $\frac{1}{f} = \frac{1}{f_1} + \frac{1}{f_2}$

		Power of equivalent lens	$P = P_1 + P_2$
		Magnification of equivalent lens	$m = m_1 + m_2$
15	Spherical aberration	The inability of lenses of aperture to bring all the rays in wide beam of light falling on it to focus a single point is called spherical aberration.	

Dual nature of waves and matter

S.No.	Term	Description
1	Energy of a photon	$\epsilon = h\nu$ where h is the plank's constant.
2	Photoelectric effect	$h\nu = KE_{\max} + \phi$ where $\phi = h\nu_0$ is the work function of a metal and ν_0 is the critical frequency for that metal.
3	Compton effect	$\lambda' - \lambda = \lambda_c(1 - \cos\phi)$ where $\lambda_c = h/m_0c$ is Compton wavelength
4	De Brogli wavelength	$\lambda = \frac{h}{mv}$
5	De Brogli phase velocity	$v_p = v\lambda = \frac{c^2}{v}$
6	Wave formula	$y = A \cos 2\pi\left(\nu t - \frac{x}{\lambda}\right)$ or $y = A \cos(\omega t - kx)$ where $\omega = 2\pi\nu$ is angular frequency and $k = \frac{2\pi}{\lambda} = \frac{\omega}{v_p}$ is the wave number.
7	Phase velocity	$v_p = \frac{\omega}{k}$
8	Group velocity	$v_g = \frac{d\omega}{dk}$
9	Uncertainty principle	It is impossible to know both the exact position and exact momentum of an object at same time. Mathematically , $\Delta x \Delta p \geq \frac{h}{4\pi}$
10	Uncertainty in energy and time	$\Delta E \Delta t \geq \frac{h}{4\pi}$

Atomic structure

S.No.	Term	Description	
1	Velocity of electron in an orbit	$v = \frac{e}{\sqrt{4\pi\epsilon_0 mr}}$ where r is the orbit radius	
2	Total energy of hydrogen atom	$E = -\frac{e^2}{8\pi\epsilon_0 r}$	
3	Atomic spectra	Lyman	$\frac{1}{\lambda} = R\left(\frac{1}{1^2} - \frac{1}{n^2}\right)$ where R is Rydberg constant
		Balmer	$\frac{1}{\lambda} = R\left(\frac{1}{2^2} - \frac{1}{n^2}\right)$ R=1.097x10 ⁻⁷ m ⁻¹
		Paschen	$\frac{1}{\lambda} = R\left(\frac{1}{3^2} - \frac{1}{n^2}\right)$
		Brackett	$\frac{1}{\lambda} = R\left(\frac{1}{4^2} - \frac{1}{n^2}\right)$
		Pfund	$\frac{1}{\lambda} = R\left(\frac{1}{5^2} - \frac{1}{n^2}\right)$

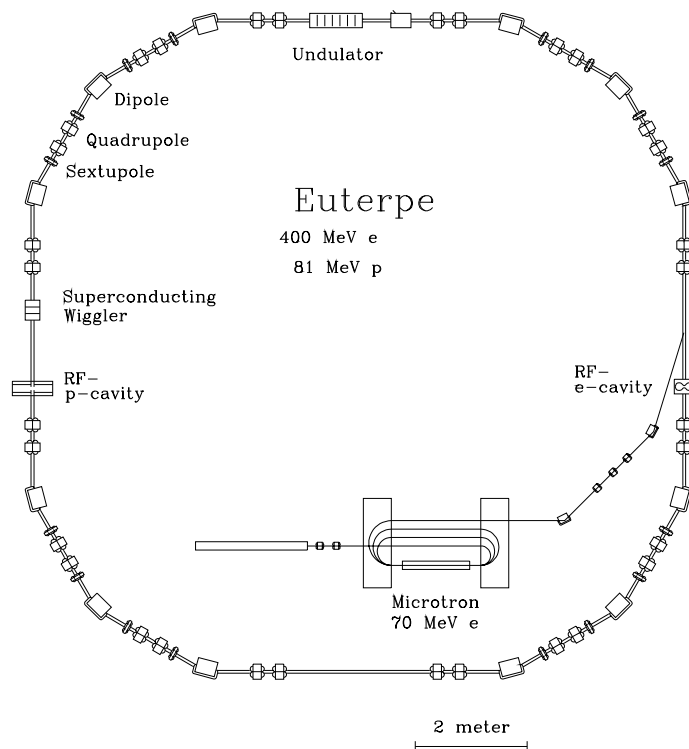
4	Orbital electron wavelength according to Bohr atom model	$\lambda = \frac{h}{e} \sqrt{\frac{4\pi\epsilon_0 r}{m}}$
5	Condition for orbital stability	$n\lambda = 2\pi r_n$ where $n=1,2,3,\dots$ And r_n is radius of orbit that contains n wavelengths.
6	Orbital radii in Bohr atom	$r_n = \frac{n^2 h^2 \epsilon_0}{\pi m e^2}$

Atomic nucleus and nuclear energy

S.No.	Term	Description
1	Isotopes	Nuclei having same atomic number but different mass number.
2	Isobars	Nuclei having different atomic number but same mass number.
3	Atomic mass unit	One atomic mass unit is defined as one twelfth part of the mass of $^{12}\text{C}_6$ atom. Value of a mass unit is $1\text{u} = 1.66054 \times 10^{-27} \text{Kg} = 931 \text{ MeV}$
4	Nuclear radius	$R = R_0 A^{1/3}$ Where value of $R_0 \approx 1.2 \times 10^{-15} \approx 1.2 \text{ fm}$ and is known as nuclear radius parameter
5	Nuclear density	The density of nuclear matter is approximately of the order of 10^{17} Kg/m^3 and is very large compared to the density of ordinary matter.
6	Nuclear forces	It is the force which holds the nucleons together inside the nucleus.
7	Mass defect	$\Delta m = [Zm_p + (A-Z)m_n] - m$
8	Binding energy	$BE = \Delta mc^2 = [Zm_p + (A-Z)m_n] - m]c^2$
9	Binding energy per nucleon	$= BE/A$
10	Radioactive decay law	$N = N_0 e^{-\lambda t}$
11	Half life	$T = \ln 2 / \lambda$
12	Average life	$T_{av} = 1/\lambda$

Physics Formulary

By ir. J.C.A. Wevers



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This, and a Dutch version of this file, can be obtained from the author, Johan Wevers (johanw@vulcan.xs4all.nl).

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If you find any errors or have any comments, please let me know. I am always open for suggestions and possible corrections to the physics formulary.

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Physical Constants

Name	Symbol	Value	Unit
Number π	π	3.14159265358979323846	
Number e	e	2.718281828459	
Euler's constant	$\gamma = \lim_{n \rightarrow \infty} \left(\sum_{k=1}^n 1/k - \ln(n) \right)$	= 0.5772156649	
Elementary charge	e	$1.60217733 \cdot 10^{-19}$	C
Gravitational constant	G, κ	$6.67259 \cdot 10^{-11}$	$\text{m}^3\text{kg}^{-1}\text{s}^{-2}$
Fine-structure constant	$\alpha = e^2/2hc\varepsilon_0$	$\approx 1/137$	
Speed of light in vacuum	c	$2.99792458 \cdot 10^8$	m/s (def)
Permittivity of the vacuum	ε_0	$8.854187 \cdot 10^{-12}$	F/m
Permeability of the vacuum	μ_0	$4\pi \cdot 10^{-7}$	H/m
$(4\pi\varepsilon_0)^{-1}$		$8.9876 \cdot 10^9$	Nm^2C^{-2}
Planck's constant	h	$6.6260755 \cdot 10^{-34}$	Js
Dirac's constant	$\hbar = h/2\pi$	$1.0545727 \cdot 10^{-34}$	Js
Bohr magneton	$\mu_B = e\hbar/2m_e$	$9.2741 \cdot 10^{-24}$	Am^2
Bohr radius	a_0	0.52918	Å
Rydberg's constant	Ry	13.595	eV
Electron Compton wavelength	$\lambda_{Ce} = h/m_e c$	$2.2463 \cdot 10^{-12}$	m
Proton Compton wavelength	$\lambda_{Cp} = h/m_p c$	$1.3214 \cdot 10^{-15}$	m
Reduced mass of the H-atom	μ_H	$9.1045755 \cdot 10^{-31}$	kg
Stefan-Boltzmann's constant	σ	$5.67032 \cdot 10^{-8}$	Wm^2K^{-4}
Wien's constant	k_W	$2.8978 \cdot 10^{-3}$	mK
Molar gasconstant	R	8.31441	J/mol
Avogadro's constant	N_A	$6.0221367 \cdot 10^{23}$	mol^{-1}
Boltzmann's constant	$k = R/N_A$	$1.380658 \cdot 10^{-23}$	J/K
Electron mass	m_e	$9.1093897 \cdot 10^{-31}$	kg
Proton mass	m_p	$1.6726231 \cdot 10^{-27}$	kg
Neutron mass	m_n	$1.674954 \cdot 10^{-27}$	kg
Elementary mass unit	$m_u = \frac{1}{12}m(^{12}_6\text{C})$	$1.6605656 \cdot 10^{-27}$	kg
Nuclear magneton	μ_N	$5.0508 \cdot 10^{-27}$	J/T
Diameter of the Sun	D_\odot	$1392 \cdot 10^6$	m
Mass of the Sun	M_\odot	$1.989 \cdot 10^{30}$	kg
Rotational period of the Sun	T_\odot	25.38	days
Radius of Earth	R_A	$6.378 \cdot 10^6$	m
Mass of Earth	M_A	$5.976 \cdot 10^{24}$	kg
Rotational period of Earth	T_A	23.96	hours
Earth orbital period	Tropical year	365.24219879	days
Astronomical unit	AU	$1.4959787066 \cdot 10^{11}$	m
Light year	lj	$9.4605 \cdot 10^{15}$	m
Parsec	pc	$3.0857 \cdot 10^{16}$	m
Hubble constant	H	$\approx (75 \pm 25)$	$\text{km}\cdot\text{s}^{-1}\cdot\text{Mpc}^{-1}$

Chapter 1

Mechanics

1.1 Point-kinetics in a fixed coordinate system

1.1.1 Definitions

The position \vec{r} , the velocity \vec{v} and the acceleration \vec{a} are defined by: $\vec{r} = (x, y, z)$, $\vec{v} = (\dot{x}, \dot{y}, \dot{z})$, $\vec{a} = (\ddot{x}, \ddot{y}, \ddot{z})$.
The following holds:

$$s(t) = s_0 + \int |\vec{v}(t)| dt; \quad \vec{r}(t) = \vec{r}_0 + \int \vec{v}(t) dt; \quad \vec{v}(t) = \vec{v}_0 + \int \vec{a}(t) dt$$

When the acceleration is constant this gives: $v(t) = v_0 + at$ and $s(t) = s_0 + v_0 t + \frac{1}{2} at^2$.
For the unit vectors in a direction \perp to the orbit \vec{e}_t and parallel to it \vec{e}_n holds:

$$\vec{e}_t = \frac{\vec{v}}{|\vec{v}|} = \frac{d\vec{r}}{ds} \quad \dot{\vec{e}}_t = \frac{v}{\rho} \vec{e}_n; \quad \vec{e}_n = \frac{\dot{\vec{e}}_t}{|\dot{\vec{e}}_t|}$$

For the *curvature* k and the *radius of curvature* ρ holds:

$$\vec{k} = \frac{d\vec{e}_t}{ds} = \frac{d^2\vec{r}}{ds^2} = \left| \frac{d\varphi}{ds} \right|; \quad \rho = \frac{1}{|k|}$$

1.1.2 Polar coordinates

Polar coordinates are defined by: $x = r \cos(\theta)$, $y = r \sin(\theta)$. So, for the unit coordinate vectors holds:
 $\dot{\vec{e}}_r = \dot{\theta} \vec{e}_\theta$, $\dot{\vec{e}}_\theta = -\dot{\theta} \vec{e}_r$.

The velocity and the acceleration are derived from: $\vec{r} = r \vec{e}_r$, $\vec{v} = \dot{r} \vec{e}_r + r \dot{\theta} \vec{e}_\theta$, $\vec{a} = (\ddot{r} - r \dot{\theta}^2) \vec{e}_r + (2\dot{r} \dot{\theta} + r \ddot{\theta}) \vec{e}_\theta$.

1.2 Relative motion

For the motion of a point D w.r.t. a point Q holds: $\vec{r}_D = \vec{r}_Q + \frac{\vec{\omega} \times \vec{v}_Q}{\omega^2}$ with $\vec{QD} = \vec{r}_D - \vec{r}_Q$ and $\omega = \dot{\theta}$.

Further holds: $\alpha = \ddot{\theta}$. ' means that the quantity is defined in a moving system of coordinates. In a moving system holds:

$$\vec{v} = \vec{v}_Q + \vec{v}' + \vec{\omega} \times \vec{r}' \quad \text{and} \quad \vec{a} = \vec{a}_Q + \vec{a}' + \vec{\alpha} \times \vec{r}' + 2\vec{\omega} \times \vec{v}' - \vec{\omega} \times (\vec{\omega} \times \vec{r}')$$

with $|\vec{\omega} \times (\vec{\omega} \times \vec{r}')| = \omega^2 \vec{r}'_n$

1.3 Point-dynamics in a fixed coordinate system

1.3.1 Force, (angular)momentum and energy

Newton's 2nd law connects the force on an object and the resulting acceleration of the object where the *momentum* is given by $\vec{p} = m\vec{v}$:

$$\vec{F}(\vec{r}, \vec{v}, t) = \frac{d\vec{p}}{dt} = \frac{d(m\vec{v})}{dt} = m \frac{d\vec{v}}{dt} + \vec{v} \frac{dm}{dt} \stackrel{m=\text{const}}{=} m\vec{a}$$

Newton's 3rd law is given by: $\vec{F}_{\text{action}} = -\vec{F}_{\text{reaction}}$.

For the power P holds: $P = \dot{W} = \vec{F} \cdot \vec{v}$. For the total energy W , the kinetic energy T and the potential energy U holds: $W = T + U$; $\dot{T} = -\dot{U}$ with $T = \frac{1}{2}mv^2$.

The *kick* \vec{S} is given by: $\vec{S} = \Delta\vec{p} = \int \vec{F} dt$

The work A , delivered by a force, is $A = \int_1^2 \vec{F} \cdot d\vec{s} = \int_1^2 F \cos(\alpha) ds$

The torque $\vec{\tau}$ is related to the angular momentum \vec{L} : $\vec{\tau} = \dot{\vec{L}} = \vec{r} \times \vec{F}$; and $\vec{L} = \vec{r} \times \vec{p} = m\vec{v} \times \vec{r}$, $|\vec{L}| = mr^2\omega$. The following equation is valid:

$$\tau = -\frac{\partial U}{\partial \theta}$$

Hence, the conditions for a mechanical equilibrium are: $\sum \vec{F}_i = 0$ and $\sum \vec{\tau}_i = 0$.

The *force of friction* is usually proportional to the force perpendicular to the surface, except when the motion starts, when a threshold has to be overcome: $F_{\text{fric}} = f \cdot F_{\text{norm}} \cdot \vec{e}_t$.

1.3.2 Conservative force fields

A conservative force can be written as the gradient of a potential: $\vec{F}_{\text{cons}} = -\vec{\nabla}U$. From this follows that $\nabla \times \vec{F} = \vec{0}$. For such a force field also holds:

$$\oint \vec{F} \cdot d\vec{s} = 0 \Rightarrow U = U_0 - \int_{r_0}^{r_1} \vec{F} \cdot d\vec{s}$$

So the work delivered by a conservative force field depends not on the trajectory covered but only on the starting and ending points of the motion.

1.3.3 Gravitation

The Newtonian law of gravitation is (in GRT one also uses κ instead of G):

$$\vec{F}_g = -G \frac{m_1 m_2}{r^2} \vec{e}_r$$

The gravitational potential is then given by $V = -Gm/r$. From Gauss law it then follows: $\nabla^2 V = 4\pi G\rho$.

1.3.4 Orbital equations

If $V = V(r)$ one can derive from the equations of Lagrange for ϕ the conservation of angular momentum:

$$\frac{\partial \mathcal{L}}{\partial \phi} = \frac{\partial V}{\partial \phi} = 0 \Rightarrow \frac{d}{dt}(mr^2\dot{\phi}) = 0 \Rightarrow L_z = mr^2\dot{\phi} = \text{constant}$$

For the radial position as a function of time can be found that:

$$\left(\frac{dr}{dt}\right)^2 = \frac{2(W - V)}{m} - \frac{L^2}{m^2 r^2}$$

The angular equation is then:

$$\phi - \phi_0 = \int_0^r \left[\frac{mr^2}{L} \sqrt{\frac{2(W - V)}{m} - \frac{L^2}{m^2 r^2}} \right]^{-1} dr \stackrel{\text{field}}{=} r^{-2} \arccos \left(1 + \frac{\frac{1}{r} - \frac{1}{r_0}}{\frac{1}{r_0} + km/L_z^2} \right)$$

If $F = F(r)$: $L = \text{constant}$, if F is conservative: $W = \text{constant}$, if $\vec{F} \perp \vec{v}$ then $\Delta T = 0$ and $U = 0$.

Kepler's orbital equations

In a force field $F = kr^{-2}$, the orbits are conic sections with the origin of the force in one of the foci (Kepler's 1st law). The equation of the orbit is:

$$r(\theta) = \frac{\ell}{1 + \varepsilon \cos(\theta - \theta_0)}, \text{ or: } x^2 + y^2 = (\ell - \varepsilon x)^2$$

with

$$\ell = \frac{L^2}{G\mu^2 M_{\text{tot}}}; \quad \varepsilon^2 = 1 + \frac{2WL^2}{G^2\mu^3 M_{\text{tot}}^2} = 1 - \frac{\ell}{a}; \quad a = \frac{\ell}{1 - \varepsilon^2} = \frac{k}{2W}$$

a is half the length of the long axis of the elliptical orbit in case the orbit is closed. Half the length of the short axis is $b = \sqrt{a\ell}$. ε is the *excentricity* of the orbit. Orbits with an equal ε are of equal shape. Now, 5 types of orbits are possible:

1. $k < 0$ and $\varepsilon = 0$: a circle.
2. $k < 0$ and $0 < \varepsilon < 1$: an ellipse.
3. $k < 0$ and $\varepsilon = 1$: a parabole.
4. $k < 0$ and $\varepsilon > 1$: a hyperbole, curved towards the centre of force.
5. $k > 0$ and $\varepsilon > 1$: a hyperbole, curved away from the centre of force.

Other combinations are not possible: the total energy in a repulsive force field is always positive so $\varepsilon > 1$.

If the surface between the orbit covered between t_1 and t_2 and the focus C around which the planet moves is $A(t_1, t_2)$, Kepler's 2nd law is

$$A(t_1, t_2) = \frac{L_C}{2m}(t_2 - t_1)$$

Kepler's 3rd law is, with T the period and M_{tot} the total mass of the system:

$$\frac{T^2}{a^3} = \frac{4\pi^2}{GM_{\text{tot}}}$$

1.3.5 The virial theorem

The virial theorem for one particle is:

$$\langle m\vec{v} \cdot \vec{r} \rangle = 0 \Rightarrow \langle T \rangle = -\frac{1}{2} \left\langle \vec{F} \cdot \vec{r} \right\rangle = \frac{1}{2} \left\langle r \frac{dU}{dr} \right\rangle = \frac{1}{2} n \langle U \rangle \text{ if } U = -\frac{k}{r^n}$$

The virial theorem for a collection of particles is:

$$\langle T \rangle = -\frac{1}{2} \left\langle \sum_{\text{particles}} \vec{F}_i \cdot \vec{r}_i + \sum_{\text{pairs}} \vec{F}_{ij} \cdot \vec{r}_{ij} \right\rangle$$

These propositions can also be written as: $2E_{\text{kin}} + E_{\text{pot}} = 0$.

1.4 Point dynamics in a moving coordinate system

1.4.1 Apparent forces

The total force in a moving coordinate system can be found by subtracting the apparent forces from the forces working in the reference frame: $\vec{F}' = \vec{F} - \vec{F}_{\text{app}}$. The different apparent forces are given by:

1. Transformation of the origin: $F_{\text{or}} = -m\vec{a}_a$
2. Rotation: $\vec{F}_{\alpha} = -m\vec{\alpha} \times \vec{r}'$
3. Coriolis force: $F_{\text{cor}} = -2m\vec{\omega} \times \vec{v}$
4. Centrifugal force: $\vec{F}_{\text{cf}} = m\omega^2 \vec{r}_n' = -\vec{F}_{\text{cp}}; \quad \vec{F}_{\text{cp}} = -\frac{mv^2}{r} \vec{e}_r$

1.4.2 Tensor notation

Transformation of the Newtonian equations of motion to $x^\alpha = x^\alpha(x)$ gives:

$$\frac{dx^\alpha}{dt} = \frac{\partial x^\alpha}{\partial \bar{x}^\beta} \frac{d\bar{x}^\beta}{dt};$$

The chain rule gives:

$$\frac{d}{dt} \frac{dx^\alpha}{dt} = \frac{d^2 x^\alpha}{dt^2} = \frac{d}{dt} \left(\frac{\partial x^\alpha}{\partial \bar{x}^\beta} \frac{d\bar{x}^\beta}{dt} \right) = \frac{\partial x^\alpha}{\partial \bar{x}^\beta} \frac{d^2 \bar{x}^\beta}{dt^2} + \frac{d\bar{x}^\beta}{dt} \frac{d}{dt} \left(\frac{\partial x^\alpha}{\partial \bar{x}^\beta} \right)$$

so:

$$\frac{d}{dt} \frac{\partial x^\alpha}{\partial \bar{x}^\beta} = \frac{\partial}{\partial \bar{x}^\gamma} \frac{\partial x^\alpha}{\partial \bar{x}^\beta} \frac{d\bar{x}^\gamma}{dt} = \frac{\partial^2 x^\alpha}{\partial \bar{x}^\beta \partial \bar{x}^\gamma} \frac{d\bar{x}^\gamma}{dt}$$

This leads to:

$$\frac{d^2 x^\alpha}{dt^2} = \frac{\partial x^\alpha}{\partial \bar{x}^\beta} \frac{d^2 \bar{x}^\beta}{dt^2} + \frac{\partial^2 x^\alpha}{\partial \bar{x}^\beta \partial \bar{x}^\gamma} \frac{d\bar{x}^\gamma}{dt} \left(\frac{d\bar{x}^\beta}{dt} \right)$$

Hence the Newtonian equation of motion

$$m \frac{d^2 x^\alpha}{dt^2} = F^\alpha$$

will be transformed into:

$$m \left\{ \frac{d^2 x^\alpha}{dt^2} + \Gamma_{\beta\gamma}^\alpha \frac{dx^\beta}{dt} \frac{dx^\gamma}{dt} \right\} = F^\alpha$$

The apparent forces are taken from the origin to the effect side in the way $\Gamma_{\beta\gamma}^\alpha \frac{dx^\beta}{dt} \frac{dx^\gamma}{dt}$.

1.5 Dynamics of masspoint collections

1.5.1 The centre of mass

The velocity w.r.t. the centre of mass \vec{R} is given by $\vec{v} - \dot{\vec{R}}$. The coordinates of the centre of mass are given by:

$$\vec{r}_m = \frac{\sum m_i \vec{r}_i}{\sum m_i}$$

In a 2-particle system, the coordinates of the centre of mass are given by:

$$\vec{R} = \frac{m_1 \vec{r}_1 + m_2 \vec{r}_2}{m_1 + m_2}$$

With $\vec{r} = \vec{r}_1 - \vec{r}_2$, the kinetic energy becomes: $T = \frac{1}{2} M_{\text{tot}} \dot{\vec{R}}^2 + \frac{1}{2} \mu \dot{\vec{r}}^2$, with the *reduced mass* μ given by:

$$\frac{1}{\mu} = \frac{1}{m_1} + \frac{1}{m_2}$$

The motion within and outside the centre of mass can be separated:

$$\begin{aligned} \dot{\vec{L}}_{\text{outside}} &= \vec{\tau}_{\text{outside}} ; \quad \dot{\vec{L}}_{\text{inside}} = \vec{\tau}_{\text{inside}} \\ \vec{p} &= m \vec{v}_m ; \quad \vec{F}_{\text{ext}} = m \vec{a}_m ; \quad \vec{F}_{12} = \mu \vec{u} \end{aligned}$$

1.5.2 Collisions

With collisions, where B are the coordinates of the collision and C an arbitrary other position, holds: $\vec{p} = m \vec{v}_m$ is constant, and $T = \frac{1}{2} m \vec{v}_m^2$ is constant. The changes in the *relative velocities* can be derived from: $\vec{S} = \Delta \vec{p} = \mu (\vec{v}_{\text{aft}} - \vec{v}_{\text{before}})$. Further holds $\Delta \vec{L}_C = \vec{C}\vec{B} \times \vec{S}$, $\vec{p} \parallel \vec{S}$ = constant and \vec{L} w.r.t. B is constant.

1.6 Dynamics of rigid bodies

1.6.1 Moment of Inertia

The angular momentum in a moving coordinate system is given by:

$$\vec{L}' = I\vec{\omega} + \vec{L}'_n$$

where I is the *moment of inertia* with respect to a central axis, which is given by:

$$I = \sum_i m_i \vec{r}_i^2 ; \quad T' = W_{\text{rot}} = \frac{1}{2} \omega I_{ij} \vec{e}_i \vec{e}_j = \frac{1}{2} I \omega^2$$

or, in the continuous case:

$$I = \frac{m}{V} \int r'_n dV = \int r'_n dm$$

Further holds:

$$L_i = I^{ij} \omega_j ; \quad I_{ii} = I_i ; \quad I_{ij} = I_{ji} = - \sum_k m_k x'_i x'_j$$

Steiner's theorem is: $I_{\text{w.r.t.D}} = I_{\text{w.r.t.C}} + m(DM)^2$ if axis C \parallel axis D.

Object	I	Object	I
Cavern cylinder	$I = mR^2$	Massive cylinder	$I = \frac{1}{2} mR^2$
Disc, axis in plane disc through m	$I = \frac{1}{4} mR^2$	Halter	$I = \frac{1}{2} \mu R^2$
Cavern sphere	$I = \frac{2}{3} mR^2$	Massive sphere	$I = \frac{2}{5} mR^2$
Bar, axis \perp through c.o.m.	$I = \frac{1}{12} ml^2$	Bar, axis \perp through end	$I = \frac{1}{3} ml^2$
Rectangle, axis \perp plane thr. c.o.m.	$I = \frac{1}{12} (a^2 + b^2)$	Rectangle, axis $\parallel b$ thr. m	$I = ma^2$

1.6.2 Principal axes

Each rigid body has (at least) 3 principal axes which stand \perp to each other. For a principal axis holds:

$$\frac{\partial I}{\partial \omega_x} = \frac{\partial I}{\partial \omega_y} = \frac{\partial I}{\partial \omega_z} = 0 \quad \text{so} \quad L'_n = 0$$

The following holds: $\dot{\omega}_k = -a_{ijk} \omega_i \omega_j$ with $a_{ijk} = \frac{I_i - I_j}{I_k}$ if $I_1 \leq I_2 \leq I_3$.

1.6.3 Time dependence

For torque of force $\vec{\tau}$ holds:

$$\vec{\tau}' = I\ddot{\theta} ; \quad \frac{d'' \vec{L}'}{dt} = \vec{\tau}' - \vec{\omega} \times \vec{L}'$$

The *torque* \vec{T} is defined by: $\vec{T} = \vec{F} \times \vec{d}$.

1.7 Variational Calculus, Hamilton and Lagrange mechanics

1.7.1 Variational Calculus

Starting with:

$$\delta \int_a^b \mathcal{L}(q, \dot{q}, t) dt = 0 \quad \text{with} \quad \delta(a) = \delta(b) = 0 \quad \text{and} \quad \delta \left(\frac{du}{dx} \right) = \frac{d}{dx} (\delta u)$$

the equations of Lagrange can be derived:

$$\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}_i} = \frac{\partial \mathcal{L}}{\partial q_i}$$

When there are additional conditions applying to the variational problem $\delta J(u) = 0$ of the type $K(u) = \text{constant}$, the new problem becomes: $\delta J(u) - \lambda \delta K(u) = 0$.

1.7.2 Hamilton mechanics

The *Lagrangian* is given by: $\mathcal{L} = \sum T(\dot{q}_i) - V(q_i)$. The *Hamiltonian* is given by: $H = \sum \dot{q}_i p_i - \mathcal{L}$. In 2 dimensions holds: $\mathcal{L} = T - U = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2) - U(r, \phi)$.

If the used coordinates are *canonical* the Hamilton equations are the equations of motion for the system:

$$\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}; \quad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i}$$

Coordinates are canonical if the following holds: $\{q_i, q_j\} = 0$, $\{p_i, p_j\} = 0$, $\{q_i, p_j\} = \delta_{ij}$ where $\{, \}$ is the *Poisson bracket*:

$$\{A, B\} = \sum_i \left[\frac{\partial A}{\partial q_i} \frac{\partial B}{\partial p_i} - \frac{\partial A}{\partial p_i} \frac{\partial B}{\partial q_i} \right]$$

The Hamiltonian of a Harmonic oscillator is given by $H(x, p) = p^2/2m + \frac{1}{2}m\omega^2 x^2$. With new coordinates (θ, I) , obtained by the canonical transformation $x = \sqrt{2I/m\omega} \cos(\theta)$ and $p = -\sqrt{2Im\omega} \sin(\theta)$, with inverse $\theta = \arctan(-p/m\omega x)$ and $I = p^2/2m\omega + \frac{1}{2}m\omega x^2$ it follows: $H(\theta, I) = \omega I$.

The Hamiltonian of a charged particle with charge q in an external electromagnetic field is given by:

$$H = \frac{1}{2m} (\vec{p} - q\vec{A})^2 + qV$$

This Hamiltonian can be derived from the Hamiltonian of a free particle $H = p^2/2m$ with the transformations $\vec{p} \rightarrow \vec{p} - q\vec{A}$ and $H \rightarrow H - qV$. This is elegant from a relativistic point of view: this is equivalent to the transformation of the momentum 4-vector $p^\alpha \rightarrow p^\alpha - qA^\alpha$. A gauge transformation on the potentials A^α corresponds with a canonical transformation, which make the Hamilton equations the equations of motion for the system.

1.7.3 Motion around an equilibrium, linearization

For natural systems around equilibrium the following equations are valid:

$$\left(\frac{\partial V}{\partial q_i} \right)_0 = 0; \quad V(q) = V(0) + V_{ik} q_i q_k \quad \text{with} \quad V_{ik} = \left(\frac{\partial^2 V}{\partial q_i \partial q_k} \right)_0$$

With $T = \frac{1}{2}(M_{ik} \dot{q}_i \dot{q}_k)$ one receives the set of equations $M\ddot{q} + Vq = 0$. If $q_i(t) = a_i \exp(i\omega t)$ is substituted, this set of equations has solutions if $\det(V - \omega^2 M) = 0$. This leads to the eigenfrequencies of the problem:

$\omega_k^2 = \frac{a_k^T V a_k}{a_k^T M a_k}$. If the equilibrium is stable holds: $\forall k$ that $\omega_k^2 > 0$. The general solution is a superposition of eigenfrequencies.

1.7.4 Phase space, Liouville's equation

In phase space holds:

$$\nabla = \left(\sum_i \frac{\partial}{\partial q_i}, \sum_i \frac{\partial}{\partial p_i} \right) \quad \text{so} \quad \nabla \cdot \vec{v} = \sum_i \left(\frac{\partial}{\partial q_i} \frac{\partial H}{\partial p_i} - \frac{\partial}{\partial p_i} \frac{\partial H}{\partial q_i} \right)$$

If the equation of continuity, $\partial_t \varrho + \nabla \cdot (\varrho \vec{v}) = 0$ holds, this can be written as:

$$\{\varrho, H\} + \frac{\partial \varrho}{\partial t} = 0$$

For an arbitrary quantity A holds:

$$\frac{dA}{dt} = \{A, H\} + \frac{\partial A}{\partial t}$$

Liouville's theorem can than be written as:

$$\frac{d\rho}{dt} = 0; \quad \text{or:} \quad \int p dq = \text{constant}$$

1.7.5 Generating functions

Starting with the coordinate transformation:

$$\begin{cases} Q_i = Q_i(q_i, p_i, t) \\ P_i = P_i(q_i, p_i, t) \end{cases}$$

one can derive the following Hamilton equations with the new Hamiltonian K :

$$\frac{dQ_i}{dt} = \frac{\partial K}{\partial P_i}; \quad \frac{dP_i}{dt} = -\frac{\partial K}{\partial Q_i}$$

Now, a distinction between 4 cases can be made:

1. If $p_i \dot{q}_i - H = P_i \dot{Q}_i - K(P_i, Q_i, t) - \frac{dF_1(q_i, Q_i, t)}{dt}$, the coordinates follow from:

$$p_i = \frac{\partial F_1}{\partial q_i}; \quad P_i = \frac{\partial F_1}{\partial Q_i}; \quad K = H + \frac{dF_1}{dt}$$

2. If $p_i \dot{q}_i - H = -\dot{P}_i Q_i - K(P_i, Q_i, t) + \frac{dF_2(q_i, P_i, t)}{dt}$, the coordinates follow from:

$$p_i = \frac{\partial F_2}{\partial q_i}; \quad Q_i = \frac{\partial F_2}{\partial P_i}; \quad K = H + \frac{\partial F_2}{\partial t}$$

3. If $-\dot{p}_i q_i - H = P_i \dot{Q}_i - K(P_i, Q_i, t) + \frac{dF_3(p_i, Q_i, t)}{dt}$, the coordinates follow from:

$$q_i = -\frac{\partial F_3}{\partial p_i}; \quad P_i = -\frac{\partial F_3}{\partial Q_i}; \quad K = H + \frac{\partial F_3}{\partial t}$$

4. If $-\dot{p}_i q_i - H = -\dot{P}_i Q_i - K(P_i, Q_i, t) + \frac{dF_4(p_i, P_i, t)}{dt}$, the coordinates follow from:

$$q_i = -\frac{\partial F_4}{\partial p_i}; \quad Q_i = \frac{\partial F_4}{\partial P_i}; \quad K = H + \frac{\partial F_4}{\partial t}$$

The functions F_1 , F_2 , F_3 and F_4 are called *generating functions*.

Chapter 2

Electricity & Magnetism

2.1 The Maxwell equations

The classical electromagnetic field can be described by the *Maxwell equations*. Those can be written both as differential and integral equations:

$$\begin{aligned}\oint (\vec{D} \cdot \vec{n}) d^2 A &= Q_{\text{free, included}} & \nabla \cdot \vec{D} &= \rho_{\text{free}} \\ \oint (\vec{B} \cdot \vec{n}) d^2 A &= 0 & \nabla \cdot \vec{B} &= 0 \\ \oint \vec{E} \cdot d\vec{s} &= -\frac{d\Phi}{dt} & \nabla \times \vec{E} &= -\frac{\partial \vec{B}}{\partial t} \\ \oint \vec{H} \cdot d\vec{s} &= I_{\text{free, included}} + \frac{d\Psi}{dt} & \nabla \times \vec{H} &= \vec{J}_{\text{free}} + \frac{\partial \vec{D}}{\partial t}\end{aligned}$$

For the fluxes holds: $\Psi = \iint (\vec{D} \cdot \vec{n}) d^2 A$, $\Phi = \iint (\vec{B} \cdot \vec{n}) d^2 A$.

The electric displacement \vec{D} , polarization \vec{P} and electric field strength \vec{E} depend on each other according to:

$$\vec{D} = \varepsilon_0 \vec{E} + \vec{P} = \varepsilon_0 \varepsilon_r \vec{E}, \quad \vec{P} = \sum \vec{p}_0 / \text{Vol}, \quad \varepsilon_r = 1 + \chi_e, \quad \text{with } \chi_e = \frac{np_0^2}{3\varepsilon_0 kT}$$

The magnetic field strength \vec{H} , the magnetization \vec{M} and the magnetic flux density \vec{B} depend on each other according to:

$$\vec{B} = \mu_0 (\vec{H} + \vec{M}) = \mu_0 \mu_r \vec{H}, \quad \vec{M} = \sum \vec{m} / \text{Vol}, \quad \mu_r = 1 + \chi_m, \quad \text{with } \chi_m = \frac{\mu_0 n m_0^2}{3kT}$$

2.2 Force and potential

The force and the electric field between 2 point charges are given by:

$$\vec{F}_{12} = \frac{Q_1 Q_2}{4\pi \varepsilon_0 \varepsilon_r r^2} \vec{e}_r; \quad \vec{E} = \frac{\vec{F}}{Q}$$

The Lorentz force is the force which is felt by a charged particle that moves through a magnetic field. The origin of this force is a relativistic transformation of the Coulomb force: $\vec{F}_L = Q(\vec{v} \times \vec{B}) = I(\vec{l} \times \vec{B})$.

The magnetic field in point P which results from an electric current is given by the *law of Biot-Savart*, also known as the law of Laplace. In here, $d\vec{l} \parallel \vec{I}$ and \vec{r} points from $d\vec{l}$ to P :

$$d\vec{B}_P = \frac{\mu_0 I}{4\pi r^2} d\vec{l} \times \vec{e}_r$$

If the current is time-dependent one has to take *retardation* into account: the substitution $I(t) \rightarrow I(t - r/c)$ has to be applied.

The potentials are given by: $V_{12} = - \int_1^2 \vec{E} \cdot d\vec{s}$ and $\vec{A} = \frac{1}{2} \vec{B} \times \vec{r}$.

Here, the freedom remains to apply a *gauge transformation*. The fields can be derived from the potentials as follows:

$$\vec{E} = -\nabla V - \frac{\partial \vec{A}}{\partial t}, \quad \vec{B} = \nabla \times \vec{A}$$

Further holds the relation: $c^2 \vec{B} = \vec{v} \times \vec{E}$.

2.3 Gauge transformations

The potentials of the electromagnetic fields transform as follows when a gauge transformation is applied:

$$\begin{cases} \vec{A}' = \vec{A} - \nabla f \\ V' = V + \frac{\partial f}{\partial t} \end{cases}$$

so the fields \vec{E} and \vec{B} do not change. This results in a canonical transformation of the Hamiltonian. Further, the freedom remains to apply a limiting condition. Two common choices are:

1. Lorentz-gauge: $\nabla \cdot \vec{A} + \frac{1}{c^2} \frac{\partial V}{\partial t} = 0$. This separates the differential equations for \vec{A} and V : $\square V = -\frac{\rho}{\varepsilon_0}$, $\square \vec{A} = -\mu_0 \vec{J}$.
2. Coulomb gauge: $\nabla \cdot \vec{A} = 0$. If $\rho = 0$ and $\vec{J} = 0$ holds $V = 0$ and follows \vec{A} from $\square \vec{A} = 0$.

2.4 Energy of the electromagnetic field

The energy density of the electromagnetic field is:

$$\frac{dW}{dVol} = w = \int H dB + \int E dD$$

The energy density can be expressed in the potentials and currents as follows:

$$w_{\text{mag}} = \frac{1}{2} \int \vec{J} \cdot \vec{A} d^3x, \quad w_{\text{el}} = \frac{1}{2} \int \rho V d^3x$$

2.5 Electromagnetic waves

2.5.1 Electromagnetic waves in vacuum

The wave equation $\square \Psi(\vec{r}, t) = -f(\vec{r}, t)$ has the general solution, with $c = (\varepsilon_0 \mu_0)^{-1/2}$:

$$\Psi(\vec{r}, t) = \int \frac{f(\vec{r}', t - |\vec{r} - \vec{r}'|/c)}{4\pi |\vec{r} - \vec{r}'|} d^3r'$$

If this is written as: $\vec{J}(\vec{r}, t) = \vec{J}(\vec{r}) \exp(-i\omega t)$ and $\vec{A}(\vec{r}, t) = \vec{A}(\vec{r}) \exp(-i\omega t)$ with:

$$\vec{A}(\vec{r}) = \frac{\mu}{4\pi} \int \vec{J}(\vec{r}') \frac{\exp(ik|\vec{r} - \vec{r}'|)}{|\vec{r} - \vec{r}'|} d^3r', \quad V(\vec{r}) = \frac{1}{4\pi\varepsilon} \int \rho(\vec{r}') \frac{\exp(ik|\vec{r} - \vec{r}'|)}{|\vec{r} - \vec{r}'|} d^3r'$$

A derivation via multipole expansion will show that for the radiated energy holds, if $d, \lambda \gg r$:

$$\frac{dP}{d\Omega} = \frac{k^2}{32\pi^2\varepsilon_0 c} \left| \int J_{\perp}(\vec{r}') e^{i\vec{k} \cdot \vec{r}} d^3r' \right|^2$$

The energy density of the electromagnetic wave of a vibrating dipole at a large distance is:

$$w = \varepsilon_0 E^2 = \frac{p_0^2 \sin^2(\theta) \omega^4}{16\pi^2 \varepsilon_0 r^2 c^4} \sin^2(kr - \omega t), \quad \langle w \rangle_t = \frac{p_0^2 \sin^2(\theta) \omega^4}{32\pi^2 \varepsilon_0 r^2 c^4}, \quad P = \frac{ck^4 |\vec{p}|^2}{12\pi \varepsilon_0}$$

The radiated energy can be derived from the *Poynting vector* \vec{S} : $\vec{S} = \vec{E} \times \vec{H} = cW\vec{e}_v$. The *irradiance* is the time-averaged of the Poynting vector: $I = \langle |\vec{S}| \rangle_t$. The radiation pressure p_s is given by $p_s = (1 + R)|\vec{S}|/c$, where R is the coefficient of reflection.

2.5.2 Electromagnetic waves in matter

The wave equations in matter, with $c_{\text{mat}} = (\varepsilon\mu)^{-1/2}$ the lightspeed in matter, are:

$$\left(\nabla^2 - \varepsilon\mu \frac{\partial^2}{\partial t^2} - \frac{\mu}{\rho} \frac{\partial}{\partial t}\right) \vec{E} = 0, \quad \left(\nabla^2 - \varepsilon\mu \frac{\partial^2}{\partial t^2} - \frac{\mu}{\rho} \frac{\partial}{\partial t}\right) \vec{B} = 0$$

give, after substitution of monochromatic plane waves: $\vec{E} = E \exp(i(\vec{k} \cdot \vec{r} - \omega t))$ and $\vec{B} = B \exp(i(\vec{k} \cdot \vec{r} - \omega t))$ the dispersion relation:

$$k^2 = \varepsilon\mu\omega^2 + \frac{i\mu\omega}{\rho}$$

The first term arises from the displacement current, the second from the conductance current. If k is written in the form $k := k' + ik''$ it follows that:

$$k' = \omega \sqrt{\frac{1}{2}\varepsilon\mu} \sqrt{1 + \sqrt{1 + \frac{1}{(\rho\varepsilon\omega)^2}}} \quad \text{and} \quad k'' = \omega \sqrt{\frac{1}{2}\varepsilon\mu} \sqrt{-1 + \sqrt{1 + \frac{1}{(\rho\varepsilon\omega)^2}}}$$

This results in a damped wave: $\vec{E} = E \exp(-k''\vec{n} \cdot \vec{r}) \exp(i(k'\vec{n} \cdot \vec{r} - \omega t))$. If the material is a good conductor, the wave vanishes after approximately one wavelength, $k = (1 + i)\sqrt{\frac{\mu\omega}{2\rho}}$.

2.6 Multipoles

Because $\frac{1}{|\vec{r} - \vec{r}'|} = \frac{1}{r} \sum_0^\infty \left(\frac{r'}{r}\right)^l P_l(\cos\theta)$ the potential can be written as: $V = \frac{Q}{4\pi\varepsilon} \sum_n \frac{k_n}{r^n}$

For the lowest-order terms this results in:

- Monopole: $l = 0, k_0 = \int \rho dV$
- Dipole: $l = 1, k_1 = \int r \cos(\theta) \rho dV$
- Quadrupole: $l = 2, k_2 = \frac{1}{2} \sum_i (3z_i^2 - r_i^2)$

1. The electric dipole: dipole moment: $\vec{p} = Ql\vec{e}$, where \vec{e} goes from \oplus to \ominus , and $\vec{F} = (\vec{p} \cdot \nabla) \vec{E}_{\text{ext}}$, and $W = -\vec{p} \cdot \vec{E}_{\text{out}}$.

Electric field: $\vec{E} \approx \frac{Q}{4\pi\varepsilon r^3} \left(\frac{3\vec{p} \cdot \vec{r}}{r^2} - \vec{p} \right)$. The torque is: $\vec{\tau} = \vec{p} \times \vec{E}_{\text{out}}$

2. The magnetic dipole: dipole moment: if $r \gg \sqrt{A}$: $\vec{\mu} = \vec{I} \times (A\vec{e}_\perp)$, $\vec{F} = (\vec{\mu} \cdot \nabla) \vec{B}_{\text{out}}$

$|\mu| = \frac{mv_\perp^2}{2B}$, $W = -\vec{\mu} \cdot \vec{B}_{\text{out}}$

Magnetic field: $\vec{B} = \frac{-\mu}{4\pi r^3} \left(\frac{3\vec{\mu} \cdot \vec{r}}{r^2} - \vec{\mu} \right)$. The moment is: $\vec{\tau} = \vec{\mu} \times \vec{B}_{\text{out}}$

2.7 Electric currents

The continuity equation for charge is: $\frac{\partial \rho}{\partial t} + \nabla \cdot \vec{J} = 0$. The *electric current* is given by:

$$I = \frac{dQ}{dt} = \iint (\vec{J} \cdot \vec{n}) d^2A$$

For most conductors holds: $\vec{J} = \vec{E}/\rho$, where ρ is the *resistivity*.

If the flux enclosed by a conductor changes this results in an *induced voltage* $V_{\text{ind}} = -N \frac{d\Phi}{dt}$. If the current flowing through a conductor changes, this results in a self-inductance which opposes the original change: $V_{\text{selfind}} = -L \frac{dI}{dt}$. If a conductor encloses a flux Φ holds: $\Phi = LI$.

The magnetic induction within a coil is approximated by: $B = \frac{\mu NI}{\sqrt{l^2 + 4R^2}}$ where l is the length, R the radius and N the number of coils. The energy contained within a coil is given by $W = \frac{1}{2} LI^2$ and $L = \mu N^2 A/l$.

The *capacity* is defined by: $C = Q/V$. For a capacitor holds: $C = \varepsilon_0 \varepsilon_r A/d$ where d is the distance between the plates and A the surface of one plate. The electric field strength between the plates is $E = \sigma/\varepsilon_0 = Q/\varepsilon_0 A$ where σ is the surface charge. The accumulated energy is given by $W = \frac{1}{2} CV^2$. The current through a capacity is given by $I = -C \frac{dV}{dt}$.

For most PTC resistors holds approximately: $R = R_0(1 + \alpha T)$, where $R_0 = \rho l/A$. For a NTC holds: $R(T) = C \exp(-B/T)$ where B and C depend only on the material.

If a current flows through two different, connecting conductors x and y , the contact area will heat up or cool down, depending on the direction of the current: the *Peltier effect*. The generated or removed heat is given by: $W = \Pi_{xy} It$. This effect can be amplified with semiconductors.

The *thermic voltage* between 2 metals is given by: $V = \gamma(T - T_0)$. For a Cu-Konstantane connection holds: $\gamma \approx 0.2 - 0.7$ mV/K.

In an electrical net with only stationary currents, *Kirchhoff's* equations apply: for a knot holds: $\sum I_n = 0$, along a closed path holds: $\sum V_n = \sum I_n R_n = 0$.

2.8 Depolarizing field

If a dielectric material is placed in an electric or magnetic field, the field strength within and outside the material will change because the material will be polarized or magnetized. If the medium has an ellipsoidal shape and one of the principal axes is parallel with the external field \vec{E}_0 or \vec{B}_0 then the depolarizing is field homogeneous.

$$\begin{aligned}\vec{E}_{\text{dep}} &= \vec{E}_{\text{mat}} - \vec{E}_0 = -\frac{\mathcal{N}\vec{P}}{\varepsilon_0} \\ \vec{H}_{\text{dep}} &= \vec{H}_{\text{mat}} - \vec{H}_0 = -\mathcal{N}\vec{M}\end{aligned}$$

\mathcal{N} is a constant depending only on the shape of the object placed in the field, with $0 \leq \mathcal{N} \leq 1$. For a few limiting cases of an ellipsoid holds: a thin plane: $\mathcal{N} = 1$, a long, thin bar: $\mathcal{N} = 0$, a sphere: $\mathcal{N} = \frac{1}{3}$.

2.9 Mixtures of materials

The average electric displacement in a material which is inhomogeneous on a mesoscopic scale is given by: $\langle D \rangle = \langle \varepsilon E \rangle = \varepsilon^* \langle E \rangle$ where $\varepsilon^* = \varepsilon_1 \left(1 - \frac{\phi_2(1-x)}{\Phi(\varepsilon^*/\varepsilon_2)} \right)^{-1}$ where $x = \varepsilon_1/\varepsilon_2$. For a sphere holds: $\Phi = \frac{1}{3} + \frac{2}{3}x$. Further holds:

$$\left(\sum_i \frac{\phi_i}{\varepsilon_i} \right)^{-1} \leq \varepsilon^* \leq \sum_i \phi_i \varepsilon_i$$

Chapter 3

Relativity

3.1 Special relativity

3.1.1 The Lorentz transformation

The Lorentz transformation $(\vec{x}', t') = (\vec{x}'(\vec{x}, t), t'(\vec{x}, t))$ leaves the wave equation invariant if c is invariant:

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} = \frac{\partial^2}{\partial x'^2} + \frac{\partial^2}{\partial y'^2} + \frac{\partial^2}{\partial z'^2} - \frac{1}{c^2} \frac{\partial^2}{\partial t'^2}$$

This transformation can also be found when $ds^2 = ds'^2$ is demanded. The general form of the Lorentz transformation is given by:

$$\vec{x}' = \vec{x} + \frac{(\gamma - 1)(\vec{x} \cdot \vec{v})\vec{v}}{|\vec{v}|^2} - \gamma \vec{v} t, \quad t' = \frac{\gamma(t - \vec{x} \cdot \vec{v}/c^2)}{c^2}$$

where

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

The velocity difference \vec{v}' between two observers transforms according to:

$$\vec{v}' = \left(\gamma \left(1 - \frac{\vec{v}_1 \cdot \vec{v}_2}{c^2} \right) \right)^{-1} \left(\vec{v}_2 + (\gamma - 1) \frac{\vec{v}_1 \cdot \vec{v}_2}{v_1^2} \vec{v}_1 - \gamma \vec{v}_1 \right)$$

If the velocity is parallel to the x -axis, this becomes $y' = y, z' = z$ and:

$$\begin{aligned} x' &= \gamma(x - vt), \quad x = \gamma(x' + vt') \\ t' &= \gamma \left(t - \frac{xv}{c^2} \right), \quad t = \gamma \left(t' + \frac{x'v}{c^2} \right), \quad v' = \frac{v_2 - v_1}{1 - \frac{v_1 v_2}{c^2}} \end{aligned}$$

If $\vec{v} = v\vec{e}_x$ holds:

$$p'_x = \gamma \left(p_x - \frac{\beta W}{c} \right), \quad W' = \gamma(W - vp_x)$$

With $\beta = v/c$ the electric field of a moving charge is given by:

$$\vec{E} = \frac{Q}{4\pi\epsilon_0 r^2} \frac{(1 - \beta^2)\vec{e}_r}{(1 - \beta^2 \sin^2(\theta))^{3/2}}$$

The electromagnetic field transforms according to:

$$\vec{E}' = \gamma(\vec{E} + \vec{v} \times \vec{B}), \quad \vec{B}' = \gamma \left(\vec{B} - \frac{\vec{v} \times \vec{E}}{c^2} \right)$$

Length, mass and time transform according to: $\Delta t_r = \gamma \Delta t_0, m_r = \gamma m_0, l_r = l_0/\gamma$, with $_0$ the quantities in a co-moving reference frame and $_r$ the quantities in a frame moving with velocity v w.r.t. it. The proper time τ is defined as: $d\tau^2 = ds^2/c^2$, so $\Delta\tau = \Delta t/\gamma$. For energy and momentum holds: $W = m_r c^2 = \gamma W_0$,

$W^2 = m_0^2 c^4 + p^2 c^2$. $p = m_\tau v = \gamma m_0 v = Wv/c^2$, and $pc = W\beta$ where $\beta = v/c$. The force is defined by $\vec{F} = d\vec{p}/dt$.

4-vectors have the property that their modulus is independent of the observer: their components *can* change after a coordinate transformation but not their modulus. The difference of two 4-vectors transforms also as a 4-vector. The 4-vector for the velocity is given by $U^\alpha = \frac{dx^\alpha}{d\tau}$. The relation with the “common” velocity $u^i := dx^i/dt$ is: $U^\alpha = (\gamma u^i, i c \gamma)$. For particles with nonzero restmass holds: $U^\alpha U_\alpha = -c^2$, for particles with zero restmass (so with $v = c$) holds: $U^\alpha U_\alpha = 0$. The 4-vector for energy and momentum is given by: $p^\alpha = m_0 U^\alpha = (\gamma p^i, i W/c)$. So: $p_\alpha p^\alpha = -m_0^2 c^2 = p^2 - W^2/c^2$.

3.1.2 Red and blue shift

There are three causes of red and blue shifts:

1. Motion: with $\vec{e}_v \cdot \vec{e}_r = \cos(\varphi)$ follows: $\frac{f'}{f} = \gamma \left(1 - \frac{v \cos(\varphi)}{c} \right)$.

This can give both red- and blueshift, also \perp to the direction of motion.

2. Gravitational redshift: $\frac{\Delta f}{f} = \frac{\kappa M}{rc^2}$.

3. Redshift because the universe expands, resulting in e.g. the cosmic background radiation:

$$\frac{\lambda_0}{\lambda_1} = \frac{R_0}{R_1}.$$

3.1.3 The stress-energy tensor and the field tensor

The stress-energy tensor is given by:

$$T_{\mu\nu} = (\rho c^2 + p)u_\mu u_\nu + pg_{\mu\nu} + \frac{1}{c^2} (F_{\mu\alpha} F_\nu^\alpha + \frac{1}{4} g_{\mu\nu} F^{\alpha\beta} F_{\alpha\beta})$$

The conservation laws can than be written as: $\nabla_\nu T^{\mu\nu} = 0$. The electromagnetic field tensor is given by:

$$F_{\alpha\beta} = \frac{\partial A_\beta}{\partial x^\alpha} - \frac{\partial A_\alpha}{\partial x^\beta}$$

with $A_\mu := (\vec{A}, iV/c)$ and $J_\mu := (\vec{J}, ic\rho)$. The Maxwell equations can than be written as:

$$\partial_\nu F^{\mu\nu} = \mu_0 J^\mu, \quad \partial_\lambda F_{\mu\nu} + \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} = 0$$

The equations of motion for a charged particle in an EM field become with the field tensor:

$$\frac{dp_\alpha}{d\tau} = q F_{\alpha\beta} u^\beta$$

3.2 General relativity

3.2.1 Riemannian geometry, the Einstein tensor

The basic principles of general relativity are:

1. The geodesic postulate: free falling particles move along geodesics of space-time with the proper time τ or arc length s as parameter. For particles with zero rest mass (photons), the use of a free parameter is required because for them holds $ds = 0$. From $\delta \int ds = 0$ the equations of motion can be derived:

$$\frac{d^2 x^\alpha}{ds^2} + \Gamma_{\beta\gamma}^\alpha \frac{dx^\beta}{ds} \frac{dx^\gamma}{ds} = 0$$

2. The *principle of equivalence*: inertial mass \equiv gravitational mass \Rightarrow gravitation is equivalent with a curved space-time were particles move along geodesics.
3. By a proper choice of the coordinate system it is possible to make the metric locally flat in each point x_i : $g_{\alpha\beta}(x_i) = \eta_{\alpha\beta} := \text{diag}(-1, 1, 1, 1)$.

The *Riemann tensor* is defined as: $R^\mu_{\nu\alpha\beta} T^\nu := \nabla_\alpha \nabla_\beta T^\mu - \nabla_\beta \nabla_\alpha T^\mu$, where the covariant derivative is given by $\nabla_j a^i = \partial_j a^i + \Gamma^i_{jk} a^k$ and $\nabla_j a_i = \partial_j a_i - \Gamma^k_{ij} a_k$. Here,

$$\Gamma^i_{jk} = \frac{g^{il}}{2} \left(\frac{\partial g_{lj}}{\partial x^k} + \frac{\partial g_{lk}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^l} \right), \text{ for Euclidean spaces this reduces to: } \Gamma^i_{jk} = \frac{\partial^2 \bar{x}^l}{\partial x^j \partial x^k} \frac{\partial x^i}{\partial \bar{x}^l},$$

are the *Christoffel symbols*. For a second-order tensor holds: $[\nabla_\alpha, \nabla_\beta] T^\mu_\nu = R^\mu_{\sigma\alpha\beta} T^\sigma_\nu + R^\sigma_{\nu\alpha\beta} T^\mu_\sigma$, $\nabla_k a^i_j = \partial_k a^i_j - \Gamma^l_{kj} a^i_l + \Gamma^i_{kl} a^l_j$, $\nabla_k a_{ij} = \partial_k a_{ij} - \Gamma^l_{ki} a_{lj} - \Gamma^l_{kj} a_{il}$ and $\nabla_k a^{ij} = \partial_k a^{ij} + \Gamma^i_{kl} a^{lj} + \Gamma^j_{kl} a^{il}$. The following holds: $R^\alpha_{\beta\mu\nu} = \partial_\mu \Gamma^\alpha_{\beta\nu} - \partial_\nu \Gamma^\alpha_{\beta\mu} + \Gamma^\alpha_{\sigma\mu} \Gamma^\sigma_{\beta\nu} - \Gamma^\alpha_{\sigma\nu} \Gamma^\sigma_{\beta\mu}$.

The *Ricci tensor* is a contraction of the Riemann tensor: $R_{\alpha\beta} := R^\mu_{\alpha\mu\beta}$, which is symmetric: $R_{\alpha\beta} = R_{\beta\alpha}$. The *Bianchi identities* are: $\nabla_\lambda R_{\alpha\beta\mu\nu} + \nabla_\nu R_{\alpha\beta\lambda\mu} + \nabla_\mu R_{\alpha\beta\nu\lambda} = 0$.

The *Einstein tensor* is given by: $G^{\alpha\beta} := R^{\alpha\beta} - \frac{1}{2} g^{\alpha\beta} R$, where $R := R^\alpha_\alpha$ is the *Ricci scalar*, for which holds: $\nabla_\beta G_{\alpha\beta} = 0$. With the variational principle $\delta \int (\mathcal{L}(g_{\mu\nu}) - Rc^2/16\pi\kappa) \sqrt{|g|} d^4x = 0$ for variations $g_{\mu\nu} \rightarrow g_{\mu\nu} + \delta g_{\mu\nu}$ the *Einstein field equations* can be derived:

$$\boxed{G_{\alpha\beta} = \frac{8\pi\kappa}{c^2} T_{\alpha\beta}}, \text{ which can also be written as } R_{\alpha\beta} = \frac{8\pi\kappa}{c^2} (T_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} T^\mu_\mu)$$

For empty space this is equivalent to $R_{\alpha\beta} = 0$. The equation $R_{\alpha\beta\mu\nu} = 0$ has as only solution a flat space.

The Einstein equations are 10 independent equations, which are of second order in $g_{\mu\nu}$. From this, the Laplace equation from Newtonian gravitation can be derived by stating: $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, where $|h| \ll 1$. In the stationary case, this results in $\nabla^2 h_{00} = 8\pi\kappa \varrho / c^2$.

The most general form of the field equations is: $R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R + \Lambda g_{\alpha\beta} = \frac{8\pi\kappa}{c^2} T_{\alpha\beta}$

where Λ is the *cosmological constant*. This constant plays a role in inflatory models of the universe.

3.2.2 The line element

The *metric tensor* in an Euclidean space is given by: $g_{ij} = \sum_k \frac{\partial \bar{x}^k}{\partial x^i} \frac{\partial \bar{x}^k}{\partial x^j}$.

In general holds: $ds^2 = g_{\mu\nu} dx^\mu dx^\nu$. In special relativity this becomes $ds^2 = -c^2 dt^2 + dx^2 + dy^2 + dz^2$. This metric, $\eta_{\mu\nu} := \text{diag}(-1, 1, 1, 1)$, is called the *Minkowski metric*.

The *external Schwarzschild metric* applies in vacuum outside a spherical mass distribution, and is given by:

$$ds^2 = \left(-1 + \frac{2m}{r} \right) c^2 dt^2 + \left(1 - \frac{2m}{r} \right)^{-1} dr^2 + r^2 d\Omega^2$$

Here, $m := M\kappa/c^2$ is the *geometrical mass* of an object with mass M , and $d\Omega^2 = d\theta^2 + \sin^2 \theta d\varphi^2$. This metric is singular for $r = 2m = 2\kappa M/c^2$. If an object is smaller than its event horizon $2m$, that implies that its escape velocity is $> c$, it is called a *black hole*. The Newtonian limit of this metric is given by:

$$ds^2 = -(1 + 2V) c^2 dt^2 + (1 - 2V) (dx^2 + dy^2 + dz^2)$$

where $V = -\kappa M/r$ is the Newtonian gravitation potential. In general relativity, the components of $g_{\mu\nu}$ are associated with the potentials and the derivatives of $g_{\mu\nu}$ with the field strength.

The Kruskal-Szekeres coordinates are used to solve certain problems with the Schwarzschild metric near $r = 2m$. They are defined by:

- $r > 2m$:

$$\begin{cases} u &= \sqrt{\frac{r}{2m} - 1} \exp\left(\frac{r}{4m}\right) \cosh\left(\frac{t}{4m}\right) \\ v &= \sqrt{\frac{r}{2m} - 1} \exp\left(\frac{r}{4m}\right) \sinh\left(\frac{t}{4m}\right) \end{cases}$$

- $r < 2m$:

$$\begin{cases} u &= \sqrt{1 - \frac{r}{2m}} \exp\left(\frac{r}{4m}\right) \sinh\left(\frac{t}{4m}\right) \\ v &= \sqrt{1 - \frac{r}{2m}} \exp\left(\frac{r}{4m}\right) \cosh\left(\frac{t}{4m}\right) \end{cases}$$

- $r = 2m$: here, the Kruskal coordinates are singular, which is necessary to eliminate the coordinate singularity there.

The line element in these coordinates is given by:

$$ds^2 = -\frac{32m^3}{r} e^{-r/2m} (dv^2 - du^2) + r^2 d\Omega^2$$

The line $r = 2m$ corresponds to $u = v = 0$, the limit $x^0 \rightarrow \infty$ with $u = v$ and $x^0 \rightarrow -\infty$ with $u = -v$. The Kruskal coordinates are only singular on the hyperbole $v^2 - u^2 = 1$, this corresponds with $r = 0$. On the line $dv = \pm du$ holds $d\theta = d\varphi = ds = 0$.

For the metric outside a rotating, charged spherical mass the Newman metric applies:

$$\begin{aligned} ds^2 &= \left(1 - \frac{2mr - e^2}{r^2 + a^2 \cos^2 \theta}\right) c^2 dt^2 - \left(\frac{r^2 + a^2 \cos^2 \theta}{r^2 - 2mr + a^2 - e^2}\right) dr^2 - (r^2 + a^2 \cos^2 \theta) d\theta^2 - \\ &\quad \left(r^2 + a^2 + \frac{(2mr - e^2)a^2 \sin^2 \theta}{r^2 + a^2 \cos^2 \theta}\right) \sin^2 \theta d\varphi^2 + \left(\frac{2a(2mr - e^2)}{r^2 + a^2 \cos^2 \theta}\right) \sin^2 \theta (d\varphi)(cdt) \end{aligned}$$

where $m = \kappa M/c^2$, $a = L/Mc$ and $e = \kappa Q/\varepsilon_0 c^2$.

A rotating charged black hole has an event horizon with $R_S = m + \sqrt{m^2 - a^2 - e^2}$.

Near rotating black holes frame dragging occurs because $g_{t\varphi} \neq 0$. For the Kerr metric ($e = 0$, $a \neq 0$) then follows that within the surface $R_E = m + \sqrt{m^2 - a^2 \cos^2 \theta}$ (de ergosphere) no particle can be at rest.

3.2.3 Planetary orbits and the perihelium shift

To find a planetary orbit, the variational problem $\delta \int ds = 0$ has to be solved. This is equivalent to the problem $\delta \int ds^2 = \delta \int g_{ij} dx^i dx^j = 0$. Substituting the external Schwarzschild metric yields for a planetary orbit:

$$\frac{du}{d\varphi} \left(\frac{d^2 u}{d\varphi^2} + u \right) = \frac{du}{d\varphi} \left(3mu + \frac{m}{h^2} \right)$$

where $u := 1/r$ and $h = r^2 \dot{\varphi} = \text{constant}$. The term $3mu$ is not present in the classical solution. This term can in the classical case also be found from a potential $V(r) = -\frac{\kappa M}{r} \left(1 + \frac{h^2}{r^2} \right)$.

The orbital equation gives $r = \text{constant}$ as solution, or can, after dividing by $du/d\varphi$, be solved with perturbation theory. In zeroth order, this results in an elliptical orbit: $u_0(\varphi) = A + B \cos(\varphi)$ with $A = m/h^2$ and B an arbitrary constant. In first order, this becomes:

$$u_1(\varphi) = A + B \cos(\varphi - \varepsilon \varphi) + \varepsilon \left(A + \frac{B^2}{2A} - \frac{B^2}{6A} \cos(2\varphi) \right)$$

where $\varepsilon = 3m^2/h^2$ is small. The perihelion of a planet is the point for which r is minimal, or u maximal. This is the case if $\cos(\varphi - \varepsilon \varphi) = 0 \Rightarrow \varphi \approx 2\pi n(1 + \varepsilon)$. For the perihelion shift then follows: $\Delta\varphi = 2\pi\varepsilon = 6\pi m^2/h^2$ per orbit.

3.2.4 The trajectory of a photon

For the trajectory of a photon (and for each particle with zero restmass) holds $ds^2 = 0$. Substituting the external Schwarzschild metric results in the following orbital equation:

$$\frac{du}{d\varphi} \left(\frac{d^2u}{d\varphi^2} + u - 3mu \right) = 0$$

3.2.5 Gravitational waves

Starting with the approximation $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ for weak gravitational fields and the definition $h'_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h^\alpha_\alpha$ it follows that $\square h'_{\mu\nu} = 0$ if the gauge condition $\partial h'_{\mu\nu}/\partial x^\nu = 0$ is satisfied. From this, it follows that the loss of energy of a mechanical system, if the occurring velocities are $\ll c$ and for wavelengths \gg the size of the system, is given by:

$$\frac{dE}{dt} = -\frac{G}{5c^5} \sum_{i,j} \left(\frac{d^3Q_{ij}}{dt^3} \right)^2$$

with $Q_{ij} = \int \varrho(x_i x_j - \frac{1}{3}\delta_{ij}r^2)d^3x$ the mass quadrupole moment.

3.2.6 Cosmology

If for the universe as a whole is assumed:

1. There exists a global time coordinate which acts as x^0 of a Gaussian coordinate system,
2. The 3-dimensional spaces are isotrope for a certain value of x^0 ,
3. Each point is equivalent to each other point for a fixed x^0 .

then the *Robertson-Walker metric* can be derived for the line element:

$$ds^2 = -c^2 dt^2 + \frac{R^2(t)}{r_0^2 \left(1 - \frac{kr^2}{4r_0^2} \right)} (dr^2 + r^2 d\Omega^2)$$

For the *scalefactor* $R(t)$ the following equations can be derived:

$$\frac{2\ddot{R}}{R} + \frac{\dot{R}^2 + kc^2}{R^2} = -\frac{8\pi\kappa p}{c^2} + \Lambda \quad \text{and} \quad \frac{\dot{R}^2 + kc^2}{R^2} = \frac{8\pi\kappa\varrho}{3} + \frac{\Lambda}{3}$$

where p is the pressure and ϱ the density of the universe. If $\Lambda = 0$ can be derived for the *deceleration parameter* q :

$$q = -\frac{\ddot{R}R}{\dot{R}^2} = \frac{4\pi\kappa\varrho}{3H^2}$$

where $H = \dot{R}/R$ is *Hubble's constant*. This is a measure of the velocity with which galaxies far away are moving away from each other, and has the value $\approx (75 \pm 25) \text{ km} \cdot \text{s}^{-1} \cdot \text{Mpc}^{-1}$. This gives 3 possible conditions for the universe (here, W is the total amount of energy in the universe):

1. **Parabolical universe:** $k = 0$, $W = 0$, $q = \frac{1}{2}$. The expansion velocity of the universe $\rightarrow 0$ if $t \rightarrow \infty$. The hereto related *critical density* is $\varrho_c = 3H^2/8\pi\kappa$.
2. **Hyperbolical universe:** $k = -1$, $W < 0$, $q < \frac{1}{2}$. The expansion velocity of the universe remains positive forever.
3. **Elliptical universe:** $k = 1$, $W > 0$, $q > \frac{1}{2}$. The expansion velocity of the universe becomes negative after some time: the universe starts collapsing.

Chapter 4

Oscillations

4.1 Harmonic oscillations

The general form of a harmonic oscillation is: $\Psi(t) = \hat{\Psi}e^{i(\omega t \pm \varphi)} \equiv \hat{\Psi} \cos(\omega t \pm \varphi)$,

where $\hat{\Psi}$ is the *amplitude*. A superposition of several harmonic oscillations *with the same frequency* results in another harmonic oscillation:

$$\sum_i \hat{\Psi}_i \cos(\alpha_i \pm \omega t) = \hat{\Phi} \cos(\beta \pm \omega t)$$

with:

$$\tan(\beta) = \frac{\sum_i \hat{\Psi}_i \sin(\alpha_i)}{\sum_i \hat{\Psi}_i \cos(\alpha_i)} \quad \text{and} \quad \hat{\Phi}^2 = \sum_i \hat{\Psi}_i^2 + 2 \sum_{j>i} \sum_i \hat{\Psi}_i \hat{\Psi}_j \cos(\alpha_i - \alpha_j)$$

For harmonic oscillations holds: $\int x(t)dt = \frac{x(t)}{i\omega}$ and $\frac{d^n x(t)}{dt^n} = (i\omega)^n x(t)$.

4.2 Mechanic oscillations

For a construction with a spring with constant C parallel to a damping k which is connected to a mass M , to which a periodic force $F(t) = \hat{F} \cos(\omega t)$ is applied holds the equation of motion $m\ddot{x} = F(t) - k\dot{x} - Cx$. With complex amplitudes, this becomes $-m\omega^2 x = F - Cx - ik\omega x$. With $\omega_0^2 = C/m$ follows:

$$x = \frac{F}{m(\omega_0^2 - \omega^2) + ik\omega}, \quad \text{and for the velocity holds: } \dot{x} = \frac{F}{i\sqrt{Cm}\delta + k}$$

where $\delta = \frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}$. The quantity $Z = F/\dot{x}$ is called the *impedance* of the system. The *quality* of the system is given by $Q = \frac{\sqrt{Cm}}{k}$.

The frequency with minimal $|Z|$ is called *velocity resonance frequency*. This is equal to ω_0 . In the *resonance curve* $|Z|/\sqrt{Cm}$ is plotted against ω/ω_0 . The width of this curve is characterized by the points where $|Z(\omega)| = |Z(\omega_0)|\sqrt{2}$. In these points holds: $R = X$ and $\delta = \pm Q^{-1}$, and the width is $2\Delta\omega_B = \omega_0/Q$.

The *stiffness* of an oscillating system is given by F/x . The *amplitude resonance frequency* ω_A is the frequency where $i\omega Z$ is minimal. This is the case for $\omega_A = \omega_0 \sqrt{1 - \frac{1}{2}Q^2}$.

The *damping frequency* ω_D is a measure for the time in which an oscillating system comes to rest. It is given by $\omega_D = \omega_0 \sqrt{1 - \frac{1}{4Q^2}}$. A weak damped oscillation ($k^2 < 4mC$) dies out after $T_D = 2\pi/\omega_D$. For a *critical damped* oscillation ($k^2 = 4mC$) holds $\omega_D = 0$. A strong damped oscillation ($k^2 > 4mC$) drops like (if $k^2 \gg 4mC$) $x(t) \approx x_0 \exp(-t/\tau)$.

4.3 Electric oscillations

The *impedance* is given by: $Z = R + iX$. The phase angle is $\varphi := \arctan(X/R)$. The impedance of a resistor is R , of a capacitor $1/i\omega C$ and of a self inductor $i\omega L$. The quality of a coil is $Q = \omega L/R$. The total impedance in case several elements are positioned is given by:

1. Series connection: $V = IZ$,

$$Z_{\text{tot}} = \sum_i Z_i, \quad L_{\text{tot}} = \sum_i L_i, \quad \frac{1}{C_{\text{tot}}} = \sum_i \frac{1}{C_i}, \quad Q = \frac{Z_0}{R}, \quad Z = R(1 + iQ\delta)$$

2. parallel connection: $V = IZ$,

$$\frac{1}{Z_{\text{tot}}} = \sum_i \frac{1}{Z_i}, \quad \frac{1}{L_{\text{tot}}} = \sum_i \frac{1}{L_i}, \quad C_{\text{tot}} = \sum_i C_i, \quad Q = \frac{R}{Z_0}, \quad Z = \frac{R}{1 + iQ\delta}$$

$$\text{Here, } Z_0 = \sqrt{\frac{L}{C}} \text{ and } \omega_0 = \frac{1}{\sqrt{LC}}.$$

The power given by a source is given by $P(t) = V(t) \cdot I(t)$, so $\langle P \rangle_t = \hat{V}_{\text{eff}} \hat{I}_{\text{eff}} \cos(\Delta\phi)$
 $= \frac{1}{2} \hat{V} \hat{I} \cos(\phi_v - \phi_i) = \frac{1}{2} \hat{I}^2 \text{Re}(Z) = \frac{1}{2} \hat{V}^2 \text{Re}(1/Z)$, where $\cos(\Delta\phi)$ is the work factor.

4.4 Waves in long conductors

These cables are in use for signal transfer, e.g. coax cable. For them holds: $Z_0 = \sqrt{\frac{dL}{dx} \frac{dx}{dC}}$.

The transmission velocity is given by $v = \sqrt{\frac{dx}{dL} \frac{dx}{dC}}$.

4.5 Coupled conductors and transformers

For two coils enclosing each others flux holds: if Φ_{12} is the part of the flux originating from I_2 through coil 2 which is enclosed by coil 1, than holds $\Phi_{12} = M_{12}I_2$, $\Phi_{21} = M_{21}I_1$. For the coefficients of mutual induction M_{ij} holds:

$$M_{12} = M_{21} := M = k\sqrt{L_1 L_2} = \frac{N_1 \Phi_1}{I_2} = \frac{N_2 \Phi_2}{I_1} \sim N_1 N_2$$

where $0 \leq k \leq 1$ is the *coupling factor*. For a transformer is $k \approx 1$. At full load holds:

$$\frac{V_1}{V_2} = \frac{I_2}{I_1} = -\frac{i\omega M}{i\omega L_2 + R_{\text{load}}} \approx -\sqrt{\frac{L_1}{L_2}} = -\frac{N_1}{N_2}$$

4.6 Pendulums

The oscillation time $T = 1/f$, and for different types of pendulums is given by:

- Oscillating spring: $T = 2\pi\sqrt{m/C}$ if the spring force is given by $F = C \cdot \Delta l$.
- Physical pendulum: $T = 2\pi\sqrt{I/\tau}$ with τ the moment of force and I the moment of inertia.
- Torsion pendulum: $T = 2\pi\sqrt{I/\kappa}$ with $\kappa = \frac{2lm}{\pi r^4 \Delta\varphi}$ the constant of torsion and I the moment of inertia.
- Mathematical pendulum: $T = 2\pi\sqrt{l/g}$ with g the acceleration of gravity and l the length of the pendulum.

Chapter 5

Waves

5.1 The wave equation

The general form of the wave equation is: $\square u = 0$, or:

$$\nabla^2 u - \frac{1}{v^2} \frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} - \frac{1}{v^2} \frac{\partial^2 u}{\partial t^2} = 0$$

where u is the disturbance and v the *propagation velocity*. In general holds: $v = f\lambda$. By definition holds: $k\lambda = 2\pi$ and $\omega = 2\pi f$.

In principle, there are two types of waves:

1. Longitudinal waves: for these holds $\vec{k} \parallel \vec{v} \parallel \vec{u}$.
2. Transversal waves: for these holds $\vec{k} \parallel \vec{v} \perp \vec{u}$.

The *phase velocity* is given by $v_{\text{ph}} = \omega/k$. The *group velocity* is given by:

$$v_g = \frac{d\omega}{dk} = v_{\text{ph}} + k \frac{dv_{\text{ph}}}{dk} = v_{\text{ph}} \left(1 - \frac{k}{n} \frac{dn}{dk} \right)$$

where n is the refractive index of the medium. If v_{ph} does not depend on ω holds: $v_{\text{ph}} = v_g$. In a dispersive medium it is possible that $v_g > v_{\text{ph}}$ or $v_g < v_{\text{ph}}$, and $v_g \cdot v_{\text{ph}} = c^2$. If one wants to transfer information with a wave, e.g. by modulation of an EM wave, the information travels with the velocity at which a change in the electromagnetic field propagates. This velocity is often almost equal to the group velocity.

For some media, the propagation velocity follows from:

- Pressure waves in a liquid or gas: $v = \sqrt{\kappa/\rho}$, where κ is the modulus of compression.
- For pressure waves in a gas also holds: $v = \sqrt{\gamma p/\rho} = \sqrt{\gamma RT/M}$.
- Pressure waves in a solid bar: $v = \sqrt{E/\rho}$
- waves in a string: $v = \sqrt{F_{\text{span}} l/m}$
- Surface waves on a liquid: $v = \sqrt{\left(\frac{g\lambda}{2\pi} + \frac{2\pi\gamma}{\rho\lambda} \right) \tanh\left(\frac{2\pi h}{\lambda} \right)}$

where h is the depth of the liquid and γ the surface tension. If $h \ll \lambda$ holds: $v \approx \sqrt{gh}$.

5.2 Solutions of the wave equation

5.2.1 Plane waves

In n dimensions a harmonic plane wave is defined by:

$$u(\vec{x}, t) = 2^n \hat{u} \cos(\omega t) \sum_{i=1}^n \sin(k_i x_i)$$

The equation for a harmonic traveling plane wave is: $u(\vec{x}, t) = \hat{u} \cos(\vec{k} \cdot \vec{x} \pm \omega t + \varphi)$

If waves reflect at the end of a spring this will result in a change in phase. A fixed end gives a phase change of $\pi/2$ to the reflected wave, with boundary condition $u(l) = 0$. A loose end gives no change in the phase of the reflected wave, with boundary condition $(\partial u / \partial x)_l = 0$.

If an observer is moving w.r.t. the wave with a velocity v_{obs} , he will observe a change in frequency: the *Doppler effect*. This is given by: $\frac{f}{f_0} = \frac{v_f - v_{\text{obs}}}{v_f}$.

5.2.2 Spherical waves

When the situation is spherical symmetric, the homogeneous wave equation is given by:

$$\frac{1}{v^2} \frac{\partial^2(ru)}{\partial t^2} - \frac{\partial^2(ru)}{\partial r^2} = 0$$

with general solution:

$$u(r, t) = C_1 \frac{f(r - vt)}{r} + C_2 \frac{g(r + vt)}{r}$$

5.2.3 Cylindrical waves

When the situation has a cylindrical symmetry, the homogeneous wave equation becomes:

$$\frac{1}{v^2} \frac{\partial^2 u}{\partial t^2} - \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u}{\partial r} \right) = 0$$

This is a Bessel equation, with solutions which can be written as Hankel functions. For sufficient large values of r these are approximated by:

$$u(r, t) = \frac{\hat{u}}{\sqrt{r}} \cos(k(r \pm vt))$$

5.2.4 The general solution in one dimension

Starting point is the equation:

$$\frac{\partial^2 u(x, t)}{\partial t^2} = \sum_{m=0}^N \left(b_m \frac{\partial^m}{\partial x^m} \right) u(x, t)$$

where $b_m \in \mathbb{R}$. Substituting $u(x, t) = A e^{i(kx - \omega t)}$ gives two solutions $\omega_j = \omega_j(k)$ as dispersion relations. The general solution is given by:

$$u(x, t) = \int_{-\infty}^{\infty} \left(a(k) e^{i(kx - \omega_1(k)t)} + b(k) e^{i(kx - \omega_2(k)t)} \right) dk$$

Because in general the frequencies ω_j are non-linear in k there is dispersion and the solution cannot be written any more as a sum of functions depending only on $x \pm vt$: the wave front transforms.

5.3 The stationary phase method

Usually the Fourier integrals of the previous section cannot be calculated exactly. If $\omega_j(k) \in \mathbb{R}$ the stationary phase method can be applied. Assuming that $a(k)$ is only a slowly varying function of k , one can state that the parts of the k -axis where the phase of $kx - \omega(k)t$ changes rapidly will give no net contribution to the integral because the exponent oscillates rapidly there. The only areas contributing significantly to the integral are areas with a stationary phase, determined by $\frac{d}{dk}(kx - \omega(k)t) = 0$. Now the following approximation is possible:

$$\int_{-\infty}^{\infty} a(k) e^{i(kx - \omega(k)t)} dk \approx \sum_{i=1}^N \sqrt{\frac{2\pi}{\frac{d^2 \omega(k_i)}{dk_i^2}}} \exp \left[-i \frac{1}{4} \pi + i(k_i x - \omega(k_i)t) \right]$$

5.4 Green functions for the initial-value problem

This method is preferable if the solutions deviate much from the stationary solutions, like point-like excitations. Starting with the wave equation in one dimension, with $\nabla^2 = \partial^2/\partial x^2$ holds: if $Q(x, x', t)$ is the solution with initial values $Q(x, x', 0) = \delta(x - x')$ and $\frac{\partial Q(x, x', 0)}{\partial t} = 0$, and $P(x, x', t)$ the solution with initial values $P(x, x', 0) = 0$ and $\frac{\partial P(x, x', 0)}{\partial t} = \delta(x - x')$, then the solution of the wave equation with arbitrary initial conditions $f(x) = u(x, 0)$ and $g(x) = \frac{\partial u(x, 0)}{\partial t}$ is given by:

$$u(x, t) = \int_{-\infty}^{\infty} f(x')Q(x, x', t)dx' + \int_{-\infty}^{\infty} g(x')P(x, x', t)dx'$$

P and Q are called the *propagators*. They are defined by:

$$\begin{aligned} Q(x, x', t) &= \frac{1}{2}[\delta(x - x' - vt) + \delta(x - x' + vt)] \\ P(x, x', t) &= \begin{cases} \frac{1}{2v} & \text{if } |x - x'| < vt \\ 0 & \text{if } |x - x'| > vt \end{cases} \end{aligned}$$

Further holds the relation: $Q(x, x', t) = \frac{\partial P(x, x', t)}{\partial t}$

5.5 Waveguides and resonating cavities

The boundary conditions for a perfect conductor can be derived from the Maxwell equations. If \vec{n} is a unit vector \perp the surface, pointed from 1 to 2, and \vec{K} is a surface current density, than holds:

$$\begin{aligned} \vec{n} \cdot (\vec{D}_2 - \vec{D}_1) &= \sigma & \vec{n} \times (\vec{E}_2 - \vec{E}_1) &= 0 \\ \vec{n} \cdot (\vec{B}_2 - \vec{B}_1) &= 0 & \vec{n} \times (\vec{H}_2 - \vec{H}_1) &= \vec{K} \end{aligned}$$

In a waveguide holds because of the cylindrical symmetry: $\vec{E}(\vec{x}, t) = \vec{E}(x, y)e^{i(kz - \omega t)}$ and $\vec{B}(\vec{x}, t) = \vec{B}(x, y)e^{i(kz - \omega t)}$. From this one can now deduce that, if B_z and E_z are not $\equiv 0$:

$$\begin{aligned} \mathcal{B}_x &= \frac{i}{\varepsilon\mu\omega^2 - k^2} \left(k \frac{\partial \mathcal{B}_z}{\partial x} - \varepsilon\mu\omega \frac{\partial \mathcal{E}_z}{\partial y} \right) & \mathcal{B}_y &= \frac{i}{\varepsilon\mu\omega^2 - k^2} \left(k \frac{\partial \mathcal{B}_z}{\partial y} + \varepsilon\mu\omega \frac{\partial \mathcal{E}_z}{\partial x} \right) \\ \mathcal{E}_x &= \frac{i}{\varepsilon\mu\omega^2 - k^2} \left(k \frac{\partial \mathcal{E}_z}{\partial x} + \varepsilon\mu\omega \frac{\partial \mathcal{B}_z}{\partial y} \right) & \mathcal{E}_y &= \frac{i}{\varepsilon\mu\omega^2 - k^2} \left(k \frac{\partial \mathcal{E}_z}{\partial y} - \varepsilon\mu\omega \frac{\partial \mathcal{B}_z}{\partial x} \right) \end{aligned}$$

Now one can distinguish between three cases:

1. $B_z \equiv 0$: the Transversal Magnetic modes (TM). Boundary condition: $\mathcal{E}_z|_{\text{surf}} = 0$.
2. $E_z \equiv 0$: the Transversal Electric modes (TE). Boundary condition: $\frac{\partial \mathcal{B}_z}{\partial n} \Big|_{\text{surf}} = 0$.

For the TE and TM modes this gives an eigenvalue problem for \mathcal{E}_z resp. \mathcal{B}_z with boundary conditions:

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \psi = -\gamma^2 \psi \text{ with eigenvalues } \gamma^2 := \varepsilon\mu\omega^2 - k^2$$

This gives a discrete solution ψ_ℓ with eigenvalue γ_ℓ^2 : $k = \sqrt{\varepsilon\mu\omega^2 - \gamma_\ell^2}$. For $\omega < \omega_\ell$, k is imaginary and the wave is damped. Therefore, ω_ℓ is called the *cut-off frequency*. In rectangular conductors the following expression can be found for the cut-off frequency for modes $\text{TE}_{m,n}$ of $\text{TM}_{m,n}$:

$$\lambda_\ell = \frac{2}{\sqrt{(m/a)^2 + (n/b)^2}}$$

3. E_z and B_z are zero everywhere: the Transversal electromagnetic mode (TEM). Then holds: $k = \pm\omega\sqrt{\varepsilon\mu}$ and $v_f = v_g$, just as if here were no waveguide. Further $k \in \mathbb{R}$, so there exists no cut-off frequency.

In a rectangular, 3 dimensional resonating cavity with edges a , b and c the possible wave numbers are given by: $k_x = \frac{n_1\pi}{a}$, $k_y = \frac{n_2\pi}{b}$, $k_z = \frac{n_3\pi}{c}$. This results in the possible frequencies $f = vk/2\pi$ in the cavity:

$$f = \frac{v}{2} \sqrt{\frac{n_x^2}{a^2} + \frac{n_y^2}{b^2} + \frac{n_z^2}{c^2}}$$

For a cubic cavity, with $a = b = c$, the possible number of oscillating modes N_L for longitudinal waves is given by:

$$N_L = \frac{4\pi a^3 f^3}{3v^3}$$

Because transversal waves have two possible polarizations holds for them: $N_T = 2N_L$.

5.6 Non-linear wave equations

The *Van der Pol* equation is given by:

$$\frac{d^2x}{dt^2} - \varepsilon\omega_0(1 - \beta x^2)\frac{dx}{dt} + \omega_0^2x = 0$$

βx^2 can be ignored for very small values of the amplitude. Substitution of $x \sim e^{i\omega t}$ gives: $\omega = \frac{1}{2}\omega_0(i\varepsilon \pm 2\sqrt{1 - \frac{1}{2}\varepsilon^2})$. The lowest-order instabilities grow as $\frac{1}{2}\varepsilon\omega_0$. While x is growing, the 2nd term becomes larger and diminishes the growth. Oscillations on a time scale $\sim \omega_0^{-1}$ can exist. If x is expanded as $x = x^{(0)} + \varepsilon x^{(1)} + \varepsilon^2 x^{(2)} + \dots$ and this is substituted one obtains, besides periodic, *secular terms* $\sim \varepsilon t$. If it is assumed that there exist timescales τ_n , $0 \leq \tau \leq N$ with $\partial\tau_n/\partial t = \varepsilon^n$ and if the secular terms are put 0 one obtains:

$$\frac{d}{dt} \left\{ \frac{1}{2} \left(\frac{dx}{dt} \right)^2 + \frac{1}{2}\omega_0^2 x^2 \right\} = \varepsilon\omega_0(1 - \beta x^2) \left(\frac{dx}{dt} \right)^2$$

This is an energy equation. Energy is conserved if the left-hand side is 0. If $x^2 > 1/\beta$, the right-hand side changes sign and an increase in energy changes into a decrease of energy. This mechanism limits the growth of oscillations.

The *Korteweg-De Vries* equation is given by:

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} - \underbrace{au \frac{\partial u}{\partial x}}_{\text{non-lin}} + \underbrace{b^2 \frac{\partial^3 u}{\partial x^3}}_{\text{dispersive}} = 0$$

This equation is for example a model for ion-acoustic waves in a plasma. For this equation, soliton solutions of the following form exist:

$$u(x - ct) = \frac{-d}{\cosh^2(e(x - ct))}$$

with $c = 1 + \frac{1}{3}ad$ and $e^2 = ad/(12b^2)$.

Chapter 6

Optics

6.1 The bending of light

For the refraction at a surface holds: $n_i \sin(\theta_i) = n_t \sin(\theta_t)$ where n is the *refractive index* of the material. Snell's law is:

$$\frac{n_2}{n_1} = \frac{\lambda_1}{\lambda_2} = \frac{v_1}{v_2}$$

If $\Delta n \leq 1$, the change in phase of the light is $\Delta\varphi = 0$, if $\Delta n > 1$ holds: $\Delta\varphi = \pi$. The refraction of light in a material is caused by scattering from atoms. This is described by:

$$n^2 = 1 + \frac{n_e e^2}{\varepsilon_0 m} \sum_j \frac{f_j}{\omega_{0,j}^2 - \omega^2 - i\delta\omega}$$

where n_e is the electron density and f_j the *oscillator strength*, for which holds: $\sum_j f_j = 1$. From this follows that $v_g = c/(1 + (n_e e^2 / 2\varepsilon_0 m \omega^2))$. From this the equation of Cauchy can be derived: $n = a_0 + a_1/\lambda^2$. More general, it is possible to expand n as: $n = \sum_{k=0}^n \frac{a_k}{\lambda^{2k}}$.

For an electromagnetic wave in general holds: $n = \sqrt{\varepsilon_r \mu_r}$.

The path, followed by a light ray in material can be found from *Fermat's principle*:

$$\delta \int_1^2 dt = \delta \int_1^2 \frac{n(s)}{c} ds = 0 \Rightarrow \delta \int_1^2 n(s) ds = 0$$

6.2 Paraxial geometrical optics

6.2.1 Lenses

The Gaussian lens formula can be deduced from Fermat's principle with the approximations $\cos \varphi = 1$ and $\sin \varphi = \varphi$. For the refraction at a spherical surface with radius R holds:

$$\frac{n_1}{v} - \frac{n_2}{b} = \frac{n_1 - n_2}{R}$$

where $|v|$ is the distance of the object and $|b|$ the distance of the image. Applying this twice results in:

$$\frac{1}{f} = (n_1 - 1) \left(\frac{1}{R_2} - \frac{1}{R_1} \right)$$

where n_1 is the refractive index of the lens, f is the focal length and R_1 and R_2 are the curvature radii of both surfaces. For a double concave lens holds $R_1 < 0$, $R_2 > 0$, for a double convex lens holds $R_1 > 0$ and $R_2 < 0$. Further holds:

$$\frac{1}{f} = \frac{1}{v} - \frac{1}{b}$$

$D := 1/f$ is called the dioptric power of a lens. For a lens with thickness d and diameter D holds to a good approximation: $1/f = 8(n - 1)d/D^2$. For two lenses placed on a line with distance d holds:

$$\frac{1}{f} = \frac{1}{f_1} + \frac{1}{f_2} - \frac{d}{f_1 f_2}$$

In these equations the following signs are being used for refraction at a spherical surface, as is seen by an incoming light ray:

Quantity	+	−
R	Concave surface	Convex surface
f	Converging lens	Diverging lens
v	Real object	Virtual object
b	Virtual image	Real image

6.2.2 Mirrors

For images of mirrors holds:

$$\frac{1}{f} = \frac{1}{v} + \frac{1}{b} = \frac{2}{R} + \frac{h^2}{2} \left(\frac{1}{R} - \frac{1}{v} \right)^2$$

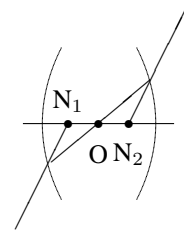
where h is the perpendicular distance from the point the light ray hits the mirror to the optical axis. Spherical aberration can be reduced by not using spherical mirrors. A parabolical mirror has no spherical aberration for light rays parallel with the optical axis and is therefore often used for telescopes. The used signs are:

Quantity	+	−
R	Concave mirror	Convex mirror
f	Concave mirror	Convex mirror
v	Real object	Virtual object
b	Real image	Virtual image

6.2.3 Principal planes

The *nodal points* N of a lens are defined by the figure on the right. If the lens is surrounded by the same medium on both sides, the nodal points are the same as the principal points H . The plane \perp the optical axis through the principal points is called the *principal plane*. If the lens is described by a matrix m_{ij} than for the distances h_1 and h_2 to the boundary of the lens holds:

$$h_1 = n \frac{m_{11} - 1}{m_{12}}, \quad h_2 = n \frac{m_{22} - 1}{m_{12}}$$



6.2.4 Magnification

The *linear magnification* is defined by: $N = -\frac{b}{v}$

The *angular magnification* is defined by: $N_\alpha = -\frac{\alpha_{\text{syst}}}{\alpha_{\text{none}}}$

where α_{syst} is the size of the retinal image with the optical system and α_{none} the size of the retinal image without the system. Further holds: $N \cdot N_\alpha = 1$. For a telescope holds: $N = f_{\text{objective}}/f_{\text{ocular}}$. The *f-number* is defined by $f/D_{\text{objective}}$.

6.3 Matrix methods

A light ray can be described by a vector $(n\alpha, y)$ with α the angle with the optical axis and y the distance to the optical axis. The change of a light ray interacting with an optical system can be obtained using a matrix multiplication:

$$\begin{pmatrix} n_2\alpha_2 \\ y_2 \end{pmatrix} = M \begin{pmatrix} n_1\alpha_1 \\ y_1 \end{pmatrix}$$

where $\text{Tr}(M) = 1$. M is a product of elementary matrices. These are:

1. Transfer along length l : $M_R = \begin{pmatrix} 1 & 0 \\ l/n & 1 \end{pmatrix}$
2. Refraction at a surface with dioptric power D : $M_T = \begin{pmatrix} 1 & -D \\ 0 & 1 \end{pmatrix}$

6.4 Aberrations

Lenses usually do not give a perfect image. Some causes are:

1. **Chromatic aberration** is caused by the fact that $n = n(\lambda)$. This can be partially corrected with a lens which is composed of more lenses with different functions $n_i(\lambda)$. Using N lenses makes it possible to obtain the same f for N wavelengths.
2. **Spherical aberration** is caused by second-order effects which are usually ignored; a spherical surface does not make a perfect lens. Incoming rays far from the optical axis will more bent.
3. **Coma** is caused by the fact that the principal planes of a lens are only flat near the principal axis. Further away of the optical axis they are curved. This curvature can be both positive or negative.
4. **Astigmatism**: from each point of an object not on the optical axis the image is an ellipse because the thickness of the lens is not the same everywhere.
5. **Field curvature** can be corrected by the human eye.
6. **Distorsion** gives aberrations near the edges of the image. This can be corrected with a combination of positive and negative lenses.

6.5 Reflection and transmission

If an electromagnetic wave hits a transparent medium part of the wave will reflect at the same angle as the incident angle, and a part will be refracted at an angle according to Snell's law. It makes a difference whether the \vec{E} field of the wave is \perp or \parallel w.r.t. the surface. When the coefficients of reflection r and transmission t are defined as:

$$r_{\parallel} \equiv \left(\frac{E_{0r}}{E_{0i}} \right)_{\parallel}, \quad r_{\perp} \equiv \left(\frac{E_{0r}}{E_{0i}} \right)_{\perp}, \quad t_{\parallel} \equiv \left(\frac{E_{0t}}{E_{0i}} \right)_{\parallel}, \quad t_{\perp} \equiv \left(\frac{E_{0t}}{E_{0i}} \right)_{\perp}$$

where E_{0r} is the reflected amplitude and E_{0t} the transmitted amplitude. Then the Fresnel equations are:

$$r_{\parallel} = \frac{\tan(\theta_i - \theta_t)}{\tan(\theta_i + \theta_t)}, \quad r_{\perp} = \frac{\sin(\theta_t - \theta_i)}{\sin(\theta_t + \theta_i)}$$

$$t_{\parallel} = \frac{2 \sin(\theta_t) \cos(\theta_i)}{\sin(\theta_t + \theta_i) \cos(\theta_t - \theta_i)}, \quad t_{\perp} = \frac{2 \sin(\theta_t) \cos(\theta_i)}{\sin(\theta_t + \theta_i)}$$

The following holds: $t_{\perp} - r_{\perp} = 1$ and $t_{\parallel} + r_{\parallel} = 1$. If the coefficient of reflection R and transmission T are defined as (with $\theta_i = \theta_r$):

$$R \equiv \frac{I_r}{I_i} \quad \text{and} \quad T \equiv \frac{I_t \cos(\theta_t)}{I_i \cos(\theta_i)}$$

with $I = \langle |\vec{S}| \rangle$ it follows: $R + T = 1$. A special case is $r_{\perp} = 0$. This happens if the angle between the reflected and transmitted rays is 90° . From Snell's law it then follows: $\tan(\theta_i) = n$. This angle is called *Brewster's angle*. The situation with $r_{\parallel} = 0$ is not possible.

6.6 Polarization

The polarization is defined as: $P = \frac{I_p}{I_p + I_u} = \frac{I_{\max} - I_{\min}}{I_{\max} + I_{\min}}$

where the intensity of the polarized light is given by I_p and the intensity of the unpolarized light is given by I_u . I_{\max} and I_{\min} are the maximum and minimum intensities when the light passes a polarizer. If polarized light passes through a polarizer *Malus law* applies: $I(\theta) = I(0) \cos^2(\theta)$ where θ is the angle of the polarizer.

The state of a light ray can be described by the *Stokes-parameters*: start with 4 filters which each transmits half the intensity. The first is independent of the polarization, the second and third are linear polarizers with the transmission axes horizontal and at $+45^\circ$, while the fourth is a circular polarizer which is opaque for L -states. Then holds $S_1 = 2I_1$, $S_2 = 2I_2 - 2I_1$, $S_3 = 2I_3 - 2I_1$ and $S_4 = 2I_4 - 2I_1$.

The state of a *polarized* light ray can also be described by the *Jones vector*:

$$\vec{E} = \begin{pmatrix} E_{0x} e^{i\varphi_x} \\ E_{0y} e^{i\varphi_y} \end{pmatrix}$$

For the horizontal P -state holds: $\vec{E} = (1, 0)$, for the vertical P -state $\vec{E} = (0, 1)$, the R -state is given by $\vec{E} = \frac{1}{2}\sqrt{2}(1, -i)$ and the L -state by $\vec{E} = \frac{1}{2}\sqrt{2}(1, i)$. The change in state of a light beam after passage of optical equipment can be described as $\vec{E}_2 = M \cdot \vec{E}_1$. For some types of optical equipment the Jones matrix M is given by:

$$\text{Horizontal linear polarizer:} \quad \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\text{Vertical linear polarizer:} \quad \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\text{Linear polarizer at } +45^\circ \quad \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

$$\text{Linear polarizer at } -45^\circ \quad \frac{1}{2} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

$$\frac{1}{4}\lambda \text{ plate, fast axis vertical} \quad e^{i\pi/4} \begin{pmatrix} 1 & 0 \\ 0 & -i \end{pmatrix}$$

$$\frac{1}{4}\lambda \text{ plate, fast axis horizontal} \quad e^{i\pi/4} \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$$

$$\text{Homogene circular polarizer right} \quad \frac{1}{2} \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}$$

$$\text{Homogene circular polarizer left} \quad \frac{1}{2} \begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix}$$

6.7 Prisms and dispersion

A light ray passing through a prism is refracted twice and acquires a deviation from its original direction $\delta = \theta_i + \theta_{i'} + \alpha$ w.r.t. the incident direction, where α is the apex angle, θ_i is the angle between the incident angle and a line perpendicular to the surface and $\theta_{i'}$ is the angle between the ray leaving the prism and a line perpendicular to the surface. When θ_i varies there is an angle for which δ becomes minimal. For the refractive index of the prism now holds:

$$n = \frac{\sin(\frac{1}{2}(\delta_{\min} + \alpha))}{\sin(\frac{1}{2}\alpha)}$$

The dispersion of a prism is defined by:

$$D = \frac{d\delta}{d\lambda} = \frac{d\delta}{dn} \frac{dn}{d\lambda}$$

where the first factor depends on the shape and the second on the composition of the prism. For the first factor follows:

$$\frac{d\delta}{dn} = \frac{2 \sin(\frac{1}{2}\alpha)}{\cos(\frac{1}{2}(\delta_{\min} + \alpha))}$$

For visible light usually holds $dn/d\lambda < 0$: shorter wavelengths are stronger bent than longer. The refractive index in this area can usually be approximated by Cauchy's formula.

6.8 Diffraction

Fraunhofer diffraction occurs far away from the source(s). The Fraunhofer diffraction of light passing through multiple slits is described by:

$$\frac{I(\theta)}{I_0} = \left(\frac{\sin(u)}{u} \right)^2 \cdot \left(\frac{\sin(Nv)}{\sin(v)} \right)^2$$

where $u = \pi b \sin(\theta)/\lambda$, $v = \pi d \sin(\theta)/\lambda$. N is the number of slits, b the width of a slit and d the distance between the slits. The maxima in intensity are given by $d \sin(\theta) = k\lambda$.

The diffraction through a spherical aperture with radius a is described by:

$$\frac{I(\theta)}{I_0} = \left(\frac{J_1(ka \sin(\theta))}{ka \sin(\theta)} \right)^2$$

The diffraction pattern of a rectangular aperture at distance R with length a in the x -direction and b in the y -direction is described by:

$$\frac{I(x, y)}{I_0} = \left(\frac{\sin(\alpha')}{\alpha'} \right)^2 \left(\frac{\sin(\beta')}{\beta'} \right)^2$$

where $\alpha' = kax/2R$ and $\beta' = kby/2R$.

When X rays are diffracted at a crystal holds for the position of the maxima in intensity *Bragg's relation*: $2d \sin(\theta) = n\lambda$ where d is the distance between the crystal layers.

Close at the source the Fraunhofermodel is invalid because it ignores the angle-dependence of the reflected waves. This is described by the *obliquity* or *inclination factor*, which describes the directionality of the secondary emissions: $E(\theta) = \frac{1}{2}E_0(1 + \cos(\theta))$ where θ is the angle w.r.t. the optical axis.

Diffraction limits the *resolution* of a system. This is the minimum angle $\Delta\theta_{\min}$ between two incident rays coming from points far away for which their refraction patterns can be detected separately. For a circular slit holds: $\Delta\theta_{\min} = 1.22\lambda/D$ where D is the diameter of the slit.

For a grating holds: $\Delta\theta_{\min} = 2\lambda/(Na \cos(\theta_m))$ where a is the distance between two peaks and N the number of peaks. The minimum difference between two wavelengths that gives a separated diffraction pattern in a multiple slit geometry is given by $\Delta\lambda/\lambda = n/N$ where N is the number of lines and n the order of the pattern.

6.9 Special optical effects

- **Birefringe and dichroism.** \vec{D} is not parallel with \vec{E} if the polarizability \vec{P} of a material is not equal in all directions. There are at least 3 directions, the *principal axes*, in which they are parallel. This results in 3 refractive indices n_i which can be used to construct Fresnel's ellipsoid. In case $n_2 = n_3 \neq n_1$, which happens e.g. at trigonal, hexagonal and tetragonal crystals there is one optical axis in the direction of n_1 . Incident light rays can now be split up in two parts: the *ordinary wave* is linear polarized \perp the plane through the transmission direction and the optical axis. The *extraordinary wave* is linear polarized

in the plane through the transmission direction and the optical axis. *Dichroism* is caused by a different absorption of the ordinary and extraordinary wave in some materials. *Double images* occur when the incident ray makes an angle with the optical axis: the extraordinary wave will refract, the ordinary will not.

- **Retarders: waveplates and compensators.** Incident light will have a phase shift of $\Delta\varphi = 2\pi d(|n_o - n_e|)/\lambda_0$ if an uniaxial crystal is cut in such a way that the optical axis is parallel with the front and back plane. Here, λ_0 is the wavelength in vacuum and n_o and n_e the refractive indices for the ordinary and extraordinary wave. For a quarter-wave plate holds: $\Delta\varphi = \pi/2$.
- **The Kerr-effect:** isotropic, transparent materials can become birefringent when placed in an electric field. In that case, the optical axis is parallel to \vec{E} . The difference in refractive index in the two directions is given by: $\Delta n = \lambda_0 K E^2$, where K is the *Kerr constant* of the material. If the electrodes have an effective length ℓ and are separated by a distance d , the retardation is given by: $\Delta\varphi = 2\pi K \ell V^2/d^2$, where V is the applied voltage.
- **The Pockels** or linear electro-optical effect can occur in 20 (from a total of 32) crystal symmetry classes, namely those without a centre of symmetry. These crystals are also *piezoelectric*: their polarization changes when a pressure is applied and vice versa: $\vec{P} = pd + \epsilon_0 \chi \vec{E}$. The retardation in a Pockels cell is $\Delta\varphi = 2\pi n_0^3 r_{63} V/\lambda_0$ where r_{63} is the 6-3 element of the electro-optic tensor.
- **The Faraday effect:** the polarization of light passing through material with length d and to which a magnetic field is applied in the propagation direction is rotated by an angle $\beta = \mathcal{V} B d$ where \mathcal{V} is the *Verdet constant*.
- **Čerenkov radiation** arises when a charged particle with $v_q > v_f$ arrives. The radiation is emitted within a cone with an apex angle α with $\sin(\alpha) = c/c_{\text{medium}} = c/nv_q$.

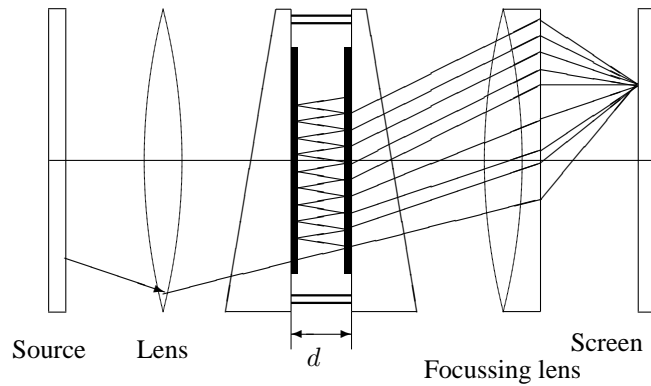
6.10 The Fabry-Perot interferometer

For a Fabry-Perot interferometer holds in general: $T + R + A = 1$ where T is the transmission factor, R the reflection factor and A the absorption factor. If F is given by $F = 4R/(1 - R)^2$ it follows for the intensity distribution:

$$\frac{I_t}{I_i} = \left[1 - \frac{A}{1 - R} \right]^2 \frac{1}{1 + F \sin^2(\theta)}$$

The term $[1 + F \sin^2(\theta)]^{-1} := \mathcal{A}(\theta)$ is called the *Airy function*.

The width of the peaks at half height is given by $\gamma = 4/\sqrt{F}$. The *finesse* \mathcal{F} is defined as $\mathcal{F} = \frac{1}{2}\pi\sqrt{F}$. The maximum resolution is then given by $\Delta f_{\min} = c/2nd\mathcal{F}$.



Chapter 7

Statistical physics

7.1 Degrees of freedom

A molecule consisting of n atoms has $s = 3n$ degrees of freedom. There are 3 translational degrees of freedom, a linear molecule has $s = 3n - 5$ vibrational degrees of freedom and a non-linear molecule $s = 3n - 6$. A linear molecule has 2 rotational degrees of freedom and a non-linear molecule 3.

Because vibrational degrees of freedom account for both kinetic and potential energy they count double. So, for linear molecules this results in a total of $s = 6n - 5$. For non-linear molecules this gives $s = 6n - 6$. The average energy of a molecule in thermodynamic equilibrium is $\langle E_{\text{tot}} \rangle = \frac{1}{2} s k T$. Each degree of freedom of a molecule has in principle the same energy: the *principle of equipartition*.

The rotational and vibrational energy of a molecule are:

$$W_{\text{rot}} = \frac{\hbar^2}{2I} l(l+1) = B l(l+1), \quad W_{\text{vib}} = (v + \frac{1}{2}) \hbar \omega_0$$

The vibrational levels are excited if $kT \approx \hbar \omega$, the rotational levels of a heteronuclear molecule are excited if $kT \approx 2B$. For homonuclear molecules additional selection rules apply so the rotational levels are well coupled if $kT \approx 6B$.

7.2 The energy distribution function

The general form of the equilibrium velocity distribution function is

$P(v_x, v_y, v_z) dv_x dv_y dv_z = P(v_x) dv_x \cdot P(v_y) dv_y \cdot P(v_z) dv_z$ with

$$P(v_i) dv_i = \frac{1}{\alpha \sqrt{\pi}} \exp\left(-\frac{v_i^2}{\alpha^2}\right) dv_i$$

where $\alpha = \sqrt{2kT/m}$ is the *most probable velocity* of a particle. The average velocity is given by $\langle v \rangle = 2\alpha/\sqrt{\pi}$, and $\langle v^2 \rangle = \frac{3}{2} \alpha^2$. The distribution as a function of the absolute value of the velocity is given by:

$$\frac{dN}{dv} = \frac{4N}{\alpha^3 \sqrt{\pi}} v^2 \exp\left(-\frac{mv^2}{2kT}\right)$$

The general form of the energy distribution function then becomes:

$$P(E) dE = \frac{c(s)}{kT} \left(\frac{E}{kT}\right)^{\frac{1}{2}s-1} \exp\left(-\frac{E}{kT}\right) dE$$

where $c(s)$ is a normalization constant, given by:

1. Even s : $s = 2l$: $c(s) = \frac{1}{(l-1)!}$
2. Odd s : $s = 2l + 1$: $c(s) = \frac{2^l}{\sqrt{\pi}(2l-1)!}$

7.3 Pressure on a wall

The number of molecules that collides with a wall with surface A within a time τ is given by:

$$\iiint d^3N = \int_0^\infty \int_0^\pi \int_0^{2\pi} nAv\tau \cos(\theta)P(v, \theta, \varphi)dv d\theta d\varphi$$

From this follows for the particle flux on the wall: $\Phi = \frac{1}{4}n \langle v \rangle$. For the pressure on the wall then follows:

$$d^3p = \frac{2mv \cos(\theta)d^3N}{A\tau}, \text{ so } p = \frac{2}{3}n \langle E \rangle$$

7.4 The equation of state

If intermolecular forces and the volume of the molecules can be neglected then for gases from $p = \frac{2}{3}n \langle E \rangle$ and $\langle E \rangle = \frac{3}{2}kT$ can be derived:

$$pV = n_sRT = \frac{1}{3}Nm \langle v^2 \rangle$$

Here, n_s is the number of *moles* particles and N is the total number of particles within volume V . If the own volume and the intermolecular forces cannot be neglected the *Van der Waals* equation can be derived:

$$\left(p + \frac{an_s^2}{V^2}\right)(V - bn_s) = n_sRT$$

There is an isotherme with a horizontal point of inflection. In the Van der Waals equation this corresponds with the *critical temperature, pressure and volume* of the gas. This is the upper limit of the area of coexistence between liquid and vapor. From $dp/dV = 0$ and $d^2p/dV^2 = 0$ follows:

$$T_{\text{cr}} = \frac{8a}{27bR}, \quad p_{\text{cr}} = \frac{a}{27b^2}, \quad V_{\text{cr}} = 3bn_s$$

For the critical point holds: $p_{\text{cr}}V_{m,\text{cr}}/RT_{\text{cr}} = \frac{3}{8}$, which differs from the value of 1 which follows from the general gas law.

Scaled on the critical quantities, with $p^* := p/p_{\text{cr}}$, $T^* = T/T_{\text{cr}}$ and $V_m^* = V_m/V_{m,\text{cr}}$ with $V_m := V/n_s$ holds:

$$\left(p^* + \frac{3}{(V_m^*)^2}\right)(V_m^* - \frac{1}{3}) = \frac{8}{3}T^*$$

Gases behave the same for equal values of the reduced quantities: the *law of the corresponding states*. A *virial expansion* is used for even more accurate views:

$$p(T, V_m) = RT \left(\frac{1}{V_m} + \frac{B(T)}{V_m^2} + \frac{C(T)}{V_m^3} + \dots \right)$$

The *Boyle temperature* T_B is the temperature for which the 2nd virial coefficient is 0. In a Van der Waals gas, this happens at $T_B = a/Rb$. The *inversion temperature* $T_i = 2T_B$.

The equation of state for solids and liquids is given by:

$$\frac{V}{V_0} = 1 + \gamma_p \Delta T - \kappa_T \Delta p = 1 + \frac{1}{V} \left(\frac{\partial V}{\partial T} \right)_p \Delta T + \frac{1}{V} \left(\frac{\partial V}{\partial p} \right)_T \Delta p$$

7.5 Collisions between molecules

The collision probability of a particle in a gas that is translated over a distance dx is given by $n\sigma dx$, where σ is the *cross section*. The mean free path is given by $\ell = \frac{v_1}{nu\sigma}$ with $u = \sqrt{v_1^2 + v_2^2}$ the relative velocity between the particles. If $m_1 \ll m_2$ holds: $\frac{u}{v_1} = \sqrt{1 + \frac{m_1}{m_2}}$, so $\ell = \frac{1}{n\sigma}$. If $m_1 = m_2$ holds: $\ell = \frac{1}{n\sigma\sqrt{2}}$. This means that the average time between two collisions is given by $\tau = \frac{1}{n\sigma v}$. If the molecules are approximated by hard spheres the cross section is: $\sigma = \frac{1}{4}\pi(D_1^2 + D_2^2)$. The average distance between two molecules is $0.55n^{-1/3}$. Collisions between molecules and small particles in a solution result in the *Brownian motion*. For the average motion of a particle with radius R can be derived: $\langle x_i^2 \rangle = \frac{1}{3} \langle r^2 \rangle = kTt/3\pi\eta R$.

A gas is called a *Knudsen gas* if $\ell \gg$ the dimensions of the gas, something that can easily occur at low pressures. The equilibrium condition for a vessel which has a hole with surface A in it for which holds that $\ell \gg \sqrt{A/\pi}$ is: $n_1\sqrt{T_1} = n_2\sqrt{T_2}$. Together with the general gas law follows: $p_1/\sqrt{T_1} = p_2/\sqrt{T_2}$.

If two plates move along each other at a distance d with velocity w_x the *viscosity* η is given by: $F_x = \eta \frac{Aw_x}{d}$. The velocity profile between the plates is in that case given by $w(z) = zw_x/d$. It can be derived that $\eta = \frac{1}{3}\rho\ell \langle v \rangle$ where v is the *thermal velocity*.

The heat conductance in a non-moving gas is described by: $\frac{dQ}{dt} = \kappa A \left(\frac{T_2 - T_1}{d} \right)$, which results in a temperature profile $T(z) = T_1 + z(T_2 - T_1)/d$. It can be derived that $\kappa = \frac{1}{3}C_{mV}n\ell \langle v \rangle / N_A$. Also holds: $\kappa = C_V\eta$. A better expression for κ can be obtained with the *Eucken correction*: $\kappa = (1 + 9R/4c_{mV})C_V \cdot \eta$ with an error $< 5\%$.

7.6 Interaction between molecules

For dipole interaction between molecules can be derived that $U \sim -1/r^6$. If the distance between two molecules approaches the molecular diameter D a repulsing force between the electron clouds appears. This force can be described by $U_{\text{rep}} \sim \exp(-\gamma r)$ or $V_{\text{rep}} = +C_s/r^s$ with $12 \leq s \leq 20$. This results in the *Lennard-Jones* potential for intermolecular forces:

$$U_{\text{LJ}} = 4\epsilon \left[\left(\frac{D}{r} \right)^{12} - \left(\frac{D}{r} \right)^6 \right]$$

with a minimum ϵ at $r = r_m$. The following holds: $D \approx 0.89r_m$. For the Van der Waals coefficients a and b and the critical quantities holds: $a = 5.275N_A^2 D^3 \epsilon$, $b = 1.3N_A D^3$, $kT_{\text{kr}} = 1.2\epsilon$ and $V_{m,\text{kr}} = 3.9N_A D^3$.

A more simple model for intermolecular forces assumes a potential $U(r) = \infty$ for $r < D$, $U(r) = U_{\text{LJ}}$ for $D \leq r \leq 3D$ and $U(r) = 0$ for $r \geq 3D$. This gives for the potential energy of one molecule: $E_{\text{pot}} = \int_D^{3D} U(r)F(r)dr$.

with $F(r)$ the spatial distribution function in spherical coordinates, which for a homogeneous distribution is given by: $F(r)dr = 4n\pi r^2 dr$.

Some useful mathematical relations are:

$$\int_0^\infty x^n e^{-x} dx = n! \quad , \quad \int_0^\infty x^{2n} e^{-x^2} dx = \frac{(2n)! \sqrt{\pi}}{n! 2^{2n+1}} \quad , \quad \int_0^\infty x^{2n+1} e^{-x^2} dx = \frac{1}{2} n!$$

Chapter 8

Thermodynamics

8.1 Mathematical introduction

If there exists a relation $f(x, y, z) = 0$ between 3 variables, one can write: $x = x(y, z)$, $y = y(x, z)$ and $z = z(x, y)$. The *total differential* dz of z is then given by:

$$dz = \left(\frac{\partial z}{\partial x} \right)_y dx + \left(\frac{\partial z}{\partial y} \right)_x dy$$

By writing this also for dx and dy it can be obtained that

$$\left(\frac{\partial x}{\partial y} \right)_z \cdot \left(\frac{\partial y}{\partial z} \right)_x \cdot \left(\frac{\partial z}{\partial x} \right)_y = -1$$

Because dz is a total differential holds $\oint dz = 0$.

A homogeneous function of degree m obeys: $\varepsilon^m F(x, y, z) = F(\varepsilon x, \varepsilon y, \varepsilon z)$. For such a function Euler's theorem applies:

$$mF(x, y, z) = x \frac{\partial F}{\partial x} + y \frac{\partial F}{\partial y} + z \frac{\partial F}{\partial z}$$

8.2 Definitions

- The isochoric pressure coefficient: $\beta_V = \frac{1}{p} \left(\frac{\partial p}{\partial T} \right)_V$
- The isothermal compressibility: $\kappa_T = -\frac{1}{V} \left(\frac{\partial V}{\partial p} \right)_T$
- The isobaric volume coefficient: $\gamma_p = \frac{1}{V} \left(\frac{\partial V}{\partial T} \right)_p$
- The adiabatic compressibility: $\kappa_S = -\frac{1}{V} \left(\frac{\partial V}{\partial p} \right)_S$

For an ideal gas follows: $\gamma_p = 1/T$, $\kappa_T = 1/p$ and $\beta_V = -1/V$.

8.3 Thermal heat capacity

- The specific heat at constant X is: $C_X = T \left(\frac{\partial S}{\partial T} \right)_X$
- The specific heat at constant pressure: $C_p = \left(\frac{\partial H}{\partial T} \right)_p$
- The specific heat at constant volume: $C_V = \left(\frac{\partial U}{\partial T} \right)_V$

For an ideal gas holds: $C_{mp} - C_{mV} = R$. Further, if the temperature is high enough to thermalize all internal rotational and vibrational degrees of freedom, holds: $C_V = \frac{1}{2}sR$. Hence $C_p = \frac{1}{2}(s+2)R$. For their ratio now follows $\gamma = (2+s)/s$. For a lower T one needs only to consider the thermalized degrees of freedom. For a Van der Waals gas holds: $C_{mV} = \frac{1}{2}sR + ap/RT^2$.

In general holds:

$$C_p - C_V = T \left(\frac{\partial p}{\partial T} \right)_V \cdot \left(\frac{\partial V}{\partial T} \right)_p = -T \left(\frac{\partial V}{\partial T} \right)_p^2 \left(\frac{\partial p}{\partial V} \right)_T \geq 0$$

Because $(\partial p/\partial V)_T$ is always < 0 , the following is always valid: $C_p \geq C_V$. If the coefficient of expansion is 0, $C_p = C_V$, and also at $T = 0K$.

8.4 The laws of thermodynamics

The zeroth law states that heat flows from higher to lower temperatures. The first law is the conservation of energy. For a closed system holds: $Q = \Delta U + W$, where Q is the total added heat, W the work done and ΔU the difference in the internal energy. In differential form this becomes: $\vec{d}Q = dU + \vec{d}W$, where \vec{d} means that it is not a differential of a quantity of state. For a quasi-static process holds: $\vec{d}W = pdV$. So for a reversible process holds: $\vec{d}Q = dU + pdV$.

For an open (flowing) system the first law is: $Q = \Delta H + W_i + \Delta E_{\text{kin}} + \Delta E_{\text{pot}}$. One can extract an amount of work W_t from the system or add $W_t = -W_i$ to the system.

The second law states: for a closed system there exists an additive quantity S , called the entropy, the differential of which has the following property:

$$dS \geq \frac{\vec{d}Q}{T}$$

If the only processes occurring are reversible holds: $dS = \vec{d}Q_{\text{rev}}/T$. So, the entropy difference after a reversible process is:

$$S_2 - S_1 = \int_1^2 \frac{\vec{d}Q_{\text{rev}}}{T}$$

So, for a reversible cycle holds: $\oint \frac{\vec{d}Q_{\text{rev}}}{T} = 0$.

For an irreversible cycle holds: $\oint \frac{\vec{d}Q_{\text{irr}}}{T} < 0$.

The third law of thermodynamics is (Nernst):

$$\lim_{T \rightarrow 0} \left(\frac{\partial S}{\partial X} \right)_T = 0$$

From this it can be concluded that the thermal heat capacity $\rightarrow 0$ if $T \rightarrow 0$, so absolute zero temperature cannot be reached by cooling through a finite number of steps.

8.5 State functions and Maxwell relations

The quantities of state and their differentials are:

Internal energy:	U	$dU = TdS - pdV$
Enthalpy:	$H = U + pV$	$dH = TdS + Vdp$
Free energy:	$F = U - TS$	$dF = -SdT - pdV$
Gibbs free enthalpy:	$G = H - TS$	$dG = -SdT + Vdp$

From this one can derive Maxwell's relations:

$$\left(\frac{\partial T}{\partial V}\right)_S = -\left(\frac{\partial p}{\partial S}\right)_V, \quad \left(\frac{\partial T}{\partial p}\right)_S = \left(\frac{\partial V}{\partial S}\right)_p, \quad \left(\frac{\partial p}{\partial T}\right)_V = \left(\frac{\partial S}{\partial V}\right)_T, \quad \left(\frac{\partial V}{\partial T}\right)_p = -\left(\frac{\partial S}{\partial p}\right)_T$$

From the total differential and the definitions of C_V and C_p it can be derived that:

$$TdS = C_V dT + T \left(\frac{\partial p}{\partial T}\right)_V dV \quad \text{and} \quad TdS = C_p dT - T \left(\frac{\partial V}{\partial T}\right)_p dp$$

For an ideal gas also holds:

$$S_m = C_V \ln \left(\frac{T}{T_0}\right) + R \ln \left(\frac{V}{V_0}\right) + S_{m0} \quad \text{and} \quad S_m = C_p \ln \left(\frac{T}{T_0}\right) - R \ln \left(\frac{p}{p_0}\right) + S'_{m0}$$

Helmholtz' equations are:

$$\left(\frac{\partial U}{\partial V}\right)_T = T \left(\frac{\partial p}{\partial T}\right)_V - p, \quad \left(\frac{\partial H}{\partial p}\right)_T = V - T \left(\frac{\partial V}{\partial T}\right)_p$$

for an enlarged surface holds: $\delta W_{\text{rev}} = -\gamma dA$, with γ the surface tension. From this follows:

$$\gamma = \left(\frac{\partial U}{\partial A}\right)_S = \left(\frac{\partial F}{\partial A}\right)_T$$

8.6 Processes

The *efficiency* η of a process is given by: $\eta = \frac{\text{Work done}}{\text{Heat added}}$

The *Cold factor* ξ of a cooling down process is given by: $\xi = \frac{\text{Cold delivered}}{\text{Work added}}$

Reversible adiabatic processes

For adiabatic processes holds: $W = U_1 - U_2$. For reversible adiabatic processes holds Poisson's equation: with $\gamma = C_p/C_V$ one gets that $pV^\gamma = \text{constant}$. Also holds: $TV^{\gamma-1} = \text{constant}$ and $T^\gamma p^{1-\gamma} = \text{constant}$. Adiabatics exhibit a greater steepness p - V diagram than isothermics because $\gamma > 1$.

Isobaric processes

Here holds: $H_2 - H_1 = \int_1^2 C_p dT$. For a reversible isobaric process holds: $H_2 - H_1 = Q_{\text{rev}}$.

The throttle process

This is also called the *Joule-Kelvin* effect and is an adiabatic expansion of a gas through a porous material or a small opening. Here H is a conserved quantity, and $dS > 0$. In general this is accompanied with a change in temperature. The quantity which is important here is the *throttle coefficient*:

$$\alpha_H = \left(\frac{\partial T}{\partial p}\right)_H = \frac{1}{C_p} \left[T \left(\frac{\partial V}{\partial T}\right)_p - V \right]$$

The *inversion temperature* is the temperature where an adiabatically expanding gas keeps the same temperature. If $T > T_i$ the gas heats up, if $T < T_i$ the gas cools down. $T_i = 2T_B$, with for T_B : $[\partial(pV)/\partial p]_T = 0$. The throttle process is e.g. applied in refrigerators.

The Carnot process

The system undergoes a reversible cycle with 2 isothermics and 2 adiabatics:

1. Isothermic expansion at T_1 . The system absorbs a heat Q_1 from the reservoir.
2. Adiabatic expansion with a temperature drop to T_2 .

3. Isothermic compression at T_2 , removing Q_2 from the system.
4. Adiabatic compression to T_1 .

The efficiency for Carnot's process is:

$$\eta = 1 - \frac{|Q_2|}{|Q_1|} = 1 - \frac{T_2}{T_1} := \eta_C$$

The *Carnot efficiency* η_C is the maximal efficiency at which a heat machine can operate. If the process is applied in reverse order and the system performs a work $-W$ the cold factor is given by:

$$\xi = \frac{|Q_2|}{W} = \frac{|Q_2|}{|Q_1| - |Q_2|} = \frac{T_2}{T_1 - T_2}$$

The Stirling process

Stirling's cycle exists of 2 isothermics and 2 isochorics. The efficiency in the ideal case is the same as for Carnot's cycle.

8.7 Maximal work

Consider a system that changes from state 1 into state 2, with the temperature and pressure of the surroundings given by T_0 and p_0 . The maximum work which can be obtained from this change is, when all processes are reversible:

1. Closed system: $W_{\max} = (U_1 - U_2) - T_0(S_1 - S_2) + p_0(V_1 - V_2)$.
2. Open system: $W_{\max} = (H_1 - H_2) - T_0(S_1 - S_2) - \Delta E_{\text{kin}} - \Delta E_{\text{pot}}$.

The minimal work needed to attain a certain state is: $W_{\min} = -W_{\max}$.

8.8 Phase transitions

Phase transitions are isothermic and isobaric, so $dG = 0$. When the phases are indicated by α , β and γ holds: $G_m^\alpha = G_m^\beta$ and

$$\Delta S_m = S_m^\alpha - S_m^\beta = \frac{r_{\beta\alpha}}{T_0}$$

where $r_{\beta\alpha}$ is the transition heat of phase β to phase α and T_0 is the transition temperature. The following holds: $r_{\beta\alpha} = r_{\alpha\beta}$ and $r_{\beta\alpha} = r_{\gamma\alpha} - r_{\gamma\beta}$. Further

$$S_m = \left(\frac{\partial G_m}{\partial T} \right)_p$$

so G has a twist in the transition point. In a two phase system Clapeyron's equation is valid:

$$\frac{dp}{dT} = \frac{S_m^\alpha - S_m^\beta}{V_m^\alpha - V_m^\beta} = \frac{r_{\beta\alpha}}{(V_m^\alpha - V_m^\beta)T}$$

For an ideal gas one finds for the vapor line at some distance from the critical point:

$$p = p_0 e^{-r_{\beta\alpha}/RT}$$

There exist also phase transitions with $r_{\beta\alpha} = 0$. For those there will occur only a discontinuity in the second derivatives of G_m . These second-order transitions appear at *organization phenomena*.

A phase-change of the 3rd order, so with e.g. $[\partial^3 G_m / \partial T^3]_p$ non continuous arises e.g. when ferromagnetic iron changes to the paramagnetic state.

8.9 Thermodynamic potential

When the number of particles within a system changes this number becomes a third quantity of state. Because addition of matter usually takes place at constant p and T , G is the relevant quantity. If a system exists of more components this becomes:

$$dG = -SdT + Vdp + \sum_i \mu_i dn_i$$

where $\mu = \left(\frac{\partial G}{\partial n_i} \right)_{p,T,n_j}$ is called the thermodynamic potential. This is a *partial quantity*. For V holds:

$$V = \sum_{i=1}^c n_i \left(\frac{\partial V}{\partial n_i} \right)_{n_j,p,T} := \sum_{i=1}^c n_i V_i$$

where V_i is the partial volume of component i . The following holds:

$$\begin{aligned} V_m &= \sum_i x_i V_i \\ 0 &= \sum_i x_i dV_i \end{aligned}$$

where $x_i = n_i/n$ is the molar fraction of component i . The molar volume of a mixture of two components can be a concave line in a V - x_2 diagram: the mixing contracts the volume.

The thermodynamic potentials are not independent in a multiple-phase system. It can be derived that $\sum_i n_i d\mu_i = -SdT + Vdp$, this gives at constant p and T : $\sum_i x_i d\mu_i = 0$ (Gibbs-Duhmen).

Each component has as much μ 's as there are phases. The number of free parameters in a system with c components and p different phases is given by $f = c + 2 - p$.

8.10 Ideal mixtures

For a mixture of n components holds (the index 0 is the value for the pure component):

$$U_{\text{mixture}} = \sum_i n_i U_i^0, \quad H_{\text{mixture}} = \sum_i n_i H_i^0, \quad S_{\text{mixture}} = n \sum_i x_i S_i^0 + \Delta S_{\text{mix}}$$

where for ideal gases holds: $\Delta S_{\text{mix}} = -nR \sum_i x_i \ln(x_i)$.

For the thermodynamic potentials holds: $\mu_i = \mu_i^0 + RT \ln(x_i) < \mu_i^0$. A mixture of two liquids is rarely ideal: this is usually only the case for chemically related components or isotopes. In spite of this holds Raoult's law for the vapour pressure holds for many binary mixtures: $p_i = x_i p_i^0 = y_i p$. Here is x_i the fraction of the i th component in liquid phase and y_i the fraction of the i th component in gas phase.

A solution of one component in another gives rise to an increase in the boiling point ΔT_k and a decrease of the freezing point ΔT_s . For $x_2 \ll 1$ holds:

$$\Delta T_k = \frac{RT_k^2}{r_{\beta\alpha}} x_2, \quad \Delta T_s = -\frac{RT_s^2}{r_{\gamma\beta}} x_2$$

with $r_{\beta\alpha}$ the evaporation heat and $r_{\gamma\beta} < 0$ the melting heat. For the *osmotic pressure* Π of a solution holds: $\Pi V_{m1}^0 = x_2 RT$.

8.11 Conditions for equilibrium

When a system evolves towards equilibrium the only changes that are possible are those for which holds: $(dS)_{U,V} \geq 0$ or $(dU)_{S,V} \leq 0$ or $(dH)_{S,p} \leq 0$ or $(dF)_{T,V} \leq 0$ or $(dG)_{T,p} \leq 0$. In equilibrium for each component holds: $\mu_i^\alpha = \mu_i^\beta = \mu_i^\gamma$.

8.12 Statistical basis for thermodynamics

The number of possibilities P to distribute N particles on n possible energy levels, each with a g -fold degeneracy is called the thermodynamic probability and is given by:

$$P = N! \prod_i \frac{g_i^{n_i}}{n_i!}$$

The most probable distribution, that with the maximum value for P , is the *equilibrium state*. When Stirling's equation, $\ln(n!) \approx n \ln(n) - n$ is used, one finds for a discrete system the Maxwell-Boltzmann distribution. The occupation numbers in equilibrium are then given by:

$$n_i = \frac{N}{Z} g_i \exp\left(-\frac{W_i}{kT}\right)$$

The *state sum* Z is a normalization constant, given by: $Z = \sum_i g_i \exp(-W_i/kT)$. For an ideal gas holds:

$$Z = \frac{V(2\pi mkT)^{3/2}}{h^3}$$

The entropy can then be defined as: $S = k \ln(P)$. For a system in thermodynamic equilibrium this becomes:

$$S = \frac{U}{T} + kN \ln\left(\frac{Z}{N}\right) \approx \frac{U}{T} + k \ln\left(\frac{Z^N}{N!}\right)$$

For an ideal gas, with $U = \frac{3}{2}kT$ then holds: $S = \frac{5}{2}kN + kN \ln\left(\frac{V(2\pi mkT)^{3/2}}{Nh^3}\right)$

8.13 Application to other systems

Thermodynamics can be applied to other systems than gases and liquids. To do this the term $\bar{d}W = pdV$ has to be replaced with the correct work term, like $\bar{d}W_{\text{rev}} = -Fdl$ for the stretching of a wire, $\bar{d}W_{\text{rev}} = -\gamma dA$ for the expansion of a soap bubble or $\bar{d}W_{\text{rev}} = -BdM$ for a magnetic system.

A rotating, non-charged black hole has a temperature of $T = \hbar c/8\pi km$. It has an entropy $S = Akc^3/4\hbar\kappa$ with A the area of its event horizon. For a Schwarzschild black hole A is given by $A = 16\pi m^2$. Hawking's area theorem states that $dA/dt \geq 0$.

Hence, the lifetime of a black hole $\sim m^3$.

Chapter 9

Transport phenomena

9.1 Mathematical introduction

An important relation is: if X is a quantity of a volume element which travels from position \vec{r} to $\vec{r} + d\vec{r}$ in a time dt , the total differential dX is then given by:

$$dX = \frac{\partial X}{\partial x} dx + \frac{\partial X}{\partial y} dy + \frac{\partial X}{\partial z} dz + \frac{\partial X}{\partial t} dt \Rightarrow \frac{dX}{dt} = \frac{\partial X}{\partial x} v_x + \frac{\partial X}{\partial y} v_y + \frac{\partial X}{\partial z} v_z + \frac{\partial X}{\partial t}$$

This results in general to: $\frac{dX}{dt} = \frac{\partial X}{\partial t} + (\vec{v} \cdot \nabla)X$.

From this follows that also holds: $\frac{d}{dt} \iiint V X d^3V = \frac{\partial}{\partial t} \iiint V X d^3V + \oint X (\vec{v} \cdot \vec{n}) d^2A$

where the volume V is surrounded by surface A . Some properties of the ∇ operator are:

$$\begin{aligned} \text{div}(\phi \vec{v}) &= \phi \text{div} \vec{v} + \text{grad} \phi \cdot \vec{v} & \text{rot}(\phi \vec{v}) &= \phi \text{rot} \vec{v} + (\text{grad} \phi) \times \vec{v} & \text{rot grad} \phi &= \vec{0} \\ \text{div}(\vec{u} \times \vec{v}) &= \vec{v} \cdot (\text{rot} \vec{u}) - \vec{u} \cdot (\text{rot} \vec{v}) & \text{rot rot} \vec{v} &= \text{grad div} \vec{v} - \nabla^2 \vec{v} & \text{div rot} \vec{v} &= 0 \\ \text{div grad} \phi &= \nabla^2 \phi & \nabla^2 \vec{v} &\equiv (\nabla^2 v_1, \nabla^2 v_2, \nabla^2 v_3) \end{aligned}$$

Here, \vec{v} is an arbitrary vector field and ϕ an arbitrary scalar field. Some important integral theorems are:

Gauss: $\oint (\vec{v} \cdot \vec{n}) d^2A = \iiint (\text{div} \vec{v}) d^3V$

Stokes for a scalar field: $\oint (\phi \cdot \vec{e}_t) ds = \iint (\vec{n} \times \text{grad} \phi) d^2A$

Stokes for a vector field: $\oint (\vec{v} \cdot \vec{e}_t) ds = \iint (\text{rot} \vec{v} \cdot \vec{n}) d^2A$

This results in: $\oint (\text{rot} \vec{v} \cdot \vec{n}) d^2A = 0$

Ostrogradsky: $\oint (\vec{n} \times \vec{v}) d^2A = \iiint (\text{rot} \vec{v}) d^3V$

$$\oint (\phi \vec{n}) d^2A = \iiint (\text{grad} \phi) d^3V$$

Here, the orientable surface $\iint d^2A$ is limited by the Jordan curve $\oint ds$.

9.2 Conservation laws

On a volume work two types of forces:

1. The force \vec{f}_0 on each volume element. For gravity holds: $\vec{f}_0 = \rho \vec{g}$.
2. Surface forces working only on the margins: \vec{t} . For these holds: $\vec{t} = \vec{n} \mathbf{T}$, where \mathbf{T} is the *stress tensor*.

\mathbf{T} can be split in a part $p\mathbf{l}$ representing the normal tensions and a part \mathbf{T}' representing the shear stresses: $\mathbf{T} = \mathbf{T}' + p\mathbf{l}$, where \mathbf{l} is the unit tensor. When viscous aspects can be ignored holds: $\text{div}\mathbf{T} = -\text{grad}p$.

When the flow velocity is \vec{v} at position \vec{r} holds on position $\vec{r} + d\vec{r}$:

$$\vec{v}(d\vec{r}) = \underbrace{\vec{v}(\vec{r})}_{\text{translation}} + \underbrace{d\vec{r} \cdot (\text{grad}\vec{v})}_{\text{rotation, deformation, dilatation}}$$

The quantity $\mathbf{L} := \text{grad}\vec{v}$ can be split in a symmetric part \mathbf{D} and an antisymmetric part \mathbf{W} . $\mathbf{L} = \mathbf{D} + \mathbf{W}$ with

$$D_{ij} := \frac{1}{2} \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right), \quad W_{ij} := \frac{1}{2} \left(\frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i} \right)$$

When the rotation or *vorticity* $\vec{\omega} = \text{rot}\vec{v}$ is introduced holds: $W_{ij} = \frac{1}{2}\epsilon_{ijk}\omega_k$. $\vec{\omega}$ represents the local rotation velocity: $d\vec{r} \cdot \mathbf{W} = \frac{1}{2}\vec{\omega} \times d\vec{r}$.

For a *Newtonian liquid* holds: $\mathbf{T}' = 2\eta\mathbf{D}$. Here, η is the dynamical viscosity. This is related to the shear stress τ by:

$$\tau_{ij} = \eta \frac{\partial v_i}{\partial x_j}$$

For compressible media can be stated: $\mathbf{T}' = (\eta' \text{div}\vec{v})\mathbf{l} + 2\eta\mathbf{D}$. From equating the thermodynamical and mechanical pressure it follows: $3\eta' + 2\eta = 0$. If the viscosity is constant holds: $\text{div}(2\mathbf{D}) = \nabla^2\vec{v} + \text{grad}\text{div}\vec{v}$.

The conservation laws for mass, momentum and energy for continuous media can be written in both integral and differential form. They are:

Integral notation:

1. Conservation of mass: $\frac{\partial}{\partial t} \iiint \rho d^3V + \oint \rho(\vec{v} \cdot \vec{n}) d^2A = 0$
2. Conservation of momentum: $\frac{\partial}{\partial t} \iiint \rho \vec{v} d^3V + \oint \rho \vec{v}(\vec{v} \cdot \vec{n}) d^2A = \iiint f_0 d^3V + \oint \vec{n} \cdot \mathbf{T} d^2A$
3. Conservation of energy: $\frac{\partial}{\partial t} \iiint (\frac{1}{2}v^2 + e)\rho d^3V + \oint (\frac{1}{2}v^2 + e)\rho(\vec{v} \cdot \vec{n}) d^2A =$
 $-\oint (\vec{q} \cdot \vec{n}) d^2A + \iiint (\vec{v} \cdot \vec{f}_0) d^3V + \oint (\vec{v} \cdot \vec{n}) \mathbf{T} d^2A$

Differential notation:

1. Conservation of mass: $\frac{\partial \rho}{\partial t} + \text{div} \cdot (\rho \vec{v}) = 0$
2. Conservation of momentum: $\rho \frac{\partial \vec{v}}{\partial t} + (\rho \vec{v} \cdot \nabla) \vec{v} = \vec{f}_0 + \text{div}\mathbf{T} = \vec{f}_0 - \text{grad}p + \text{div}\mathbf{T}'$
3. Conservation of energy: $\rho T \frac{ds}{dt} = \rho \frac{de}{dt} - \frac{p}{\rho} \frac{d\rho}{dt} = -\text{div}\vec{q} + \mathbf{T}' : \mathbf{D}$

Here, e is the internal energy per unit of mass E/m and s is the entropy per unit of mass S/m . $\vec{q} = -\kappa \vec{\nabla}T$ is the heat flow. Further holds:

$$p = -\frac{\partial E}{\partial V} = -\frac{\partial e}{\partial (1/\rho)}, \quad T = \frac{\partial E}{\partial S} = \frac{\partial e}{\partial s}$$

so

$$C_V = \left(\frac{\partial e}{\partial T} \right)_V \quad \text{and} \quad C_p = \left(\frac{\partial h}{\partial T} \right)_p$$

with $h = H/m$ the enthalpy per unit of mass.

From this one can derive the *Navier-Stokes* equations for an incompressible, viscous and heat-conducting medium:

$$\begin{aligned}\operatorname{div} \vec{v} &= 0 \\ \varrho \frac{\partial \vec{v}}{\partial t} + \varrho(\vec{v} \cdot \nabla) \vec{v} &= \varrho \vec{g} - \operatorname{grad} p + \eta \nabla^2 \vec{v} \\ \varrho C \frac{\partial T}{\partial t} + \varrho C(\vec{v} \cdot \nabla) T &= \kappa \nabla^2 T + 2\eta \mathbf{D} : \mathbf{D}\end{aligned}$$

with C the thermal heat capacity. The force \vec{F} on an object within a flow, when viscous effects are limited to the boundary layer, can be obtained using the momentum law. If a surface A surrounds the object outside the boundary layer holds:

$$\vec{F} = - \oint [p\vec{n} + \varrho \vec{v}(\vec{v} \cdot \vec{n})] d^2 A$$

9.3 Bernoulli's equations

Starting with the momentum equation one can find for a non-viscous medium for stationary flows, with

$$(\vec{v} \cdot \operatorname{grad}) \vec{v} = \frac{1}{2} \operatorname{grad}(v^2) + (\operatorname{rot} \vec{v}) \times \vec{v}$$

and the potential equation $\vec{g} = -\operatorname{grad}(gh)$ that:

$$\frac{1}{2} v^2 + gh + \int \frac{dp}{\varrho} = \text{constant along a streamline}$$

For compressible flows holds: $\frac{1}{2} v^2 + gh + p/\varrho = \text{constant}$ along a line of flow. If also holds $\operatorname{rot} \vec{v} = 0$ and the entropy is equal on each streamline holds $\frac{1}{2} v^2 + gh + \int dp/\varrho = \text{constant}$ everywhere. For incompressible flows this becomes: $\frac{1}{2} v^2 + gh + p/\varrho = \text{constant}$ everywhere. For ideal gases with constant C_p and C_V holds, with $\gamma = C_p/C_V$:

$$\frac{1}{2} v^2 + \frac{\gamma}{\gamma-1} \frac{p}{\varrho} = \frac{1}{2} v^2 + \frac{c^2}{\gamma-1} = \text{constant}$$

With a velocity potential defined by $\vec{v} = \operatorname{grad} \phi$ holds for instationary flows:

$$\frac{\partial \phi}{\partial t} + \frac{1}{2} v^2 + gh + \int \frac{dp}{\varrho} = \text{constant everywhere}$$

9.4 Characterising of flows by dimensionless numbers

The advantage of dimensionless numbers is that they make model experiments possible: one has to make the dimensionless numbers which are important for the specific experiment equal for both model and the real situation. One can also deduce functional equalities without solving the differential equations. Some dimensionless numbers are given by:

$$\begin{array}{lll} \text{Strouhal: } \operatorname{Sr} = \frac{\omega L}{v} & \text{Froude: } \operatorname{Fr} = \frac{v^2}{gL} & \text{Mach: } \operatorname{Ma} = \frac{v}{c} \\ \text{Fourier: } \operatorname{Fo} = \frac{a}{\omega L^2} & \text{Péclet: } \operatorname{Pe} = \frac{vL}{a} & \text{Reynolds: } \operatorname{Re} = \frac{vL}{\nu} \\ \text{Prandtl: } \operatorname{Pr} = \frac{\nu}{a} & \text{Nusselt: } \operatorname{Nu} = \frac{L\alpha}{\kappa} & \text{Eckert: } \operatorname{Ec} = \frac{v^2}{c\Delta T} \end{array}$$

Here, $\nu = \eta/\varrho$ is the *kinematic viscosity*, c is the speed of sound and L is a characteristic length of the system. α follows from the equation for heat transport $\kappa \partial_y T = \alpha \Delta T$ and $a = \kappa/\varrho c$ is the thermal diffusion coefficient.

These numbers can be interpreted as follows:

- Re : (stationary inertial forces)/(viscous forces)

- Sr: (non-stationary inertial forces)/(stationary inertial forces)
- Fr: (stationary inertial forces)/(gravity)
- Fo: (heat conductance)/(non-stationary change in enthalpy)
- Pe: (convective heat transport)/(heat conductance)
- Ec: (viscous dissipation)/(convective heat transport)
- Ma: (speed of sound)/(velocity): objects moving faster than approximately $Ma = 0,8$ produce shock-waves which propagate with an angle θ with the velocity of the object. For this angle holds $Ma = 1/\arctan(\theta)$.
- Pr and Nu are related to specific materials.

Now, the dimensionless Navier-Stokes equation becomes, with $x' = x/L$, $\vec{v}' = \vec{v}/V$, $\text{grad}' = L\text{grad}$, $\nabla'^2 = L^2\nabla^2$ and $t' = t\omega$:

$$Sr \frac{\partial \vec{v}'}{\partial t'} + (\vec{v}' \cdot \nabla') \vec{v}' = -\text{grad}' p + \frac{\vec{g}}{Fr} + \frac{\nabla'^2 \vec{v}'}{Re}$$

9.5 Tube flows

For tube flows holds: they are laminar if $Re < 2300$ with dimension of length the diameter of the tube, and turbulent if Re is larger. For an incompressible laminar flow through a straight, circular tube holds for the velocity profile:

$$v(r) = -\frac{1}{4\eta} \frac{dp}{dx} (R^2 - r^2)$$

For the volume flow holds: $\Phi_V = \int_0^R v(r) 2\pi r dr = -\frac{\pi}{8\eta} \frac{dp}{dx} R^4$

The *entrance length* L_e is given by:

1. $500 < Re_D < 2300$: $L_e/2R = 0.056 Re_D$
2. $Re > 2300$: $L_e/2R \approx 50$

For gas transport at low pressures (Knudsen-gas) holds: $\Phi_V = \frac{4R^3 \alpha \sqrt{\pi}}{3} \frac{dp}{dx}$

For flows at a small Re holds: $\nabla p = \eta \nabla^2 \vec{v}$ and $\text{div} \vec{v} = 0$. For the total force on a sphere with radius R in a flow then holds: $F = 6\pi\eta R v$. For large Re holds for the force on a surface A : $F = \frac{1}{2} C_W A \rho v^2$.

9.6 Potential theory

The *circulation* Γ is defined as: $\Gamma = \oint (\vec{v} \cdot \vec{e}_t) ds = \iint (\text{rot} \vec{v}) \cdot \vec{n} d^2 A = \iint (\vec{\omega} \cdot \vec{n}) d^2 A$

For non viscous media, if $p = p(\varrho)$ and all forces are conservative, Kelvin's theorem can be derived:

$$\frac{d\Gamma}{dt} = 0$$

For rotationless flows a velocity potential $\vec{v} = \text{grad} \phi$ can be introduced. In the incompressible case follows from conservation of mass $\nabla^2 \phi = 0$. For a 2-dimensional flow a flow function $\psi(x, y)$ can be defined: with Φ_{AB} the amount of liquid flowing through a curve s between the points A and B:

$$\Phi_{AB} = \int_A^B (\vec{v} \cdot \vec{n}) ds = \int_A^B (v_x dy - v_y dx)$$

and the definitions $v_x = \partial\psi/\partial y$, $v_y = -\partial\psi/\partial x$ holds: $\Phi_{AB} = \psi(B) - \psi(A)$. In general holds:

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} = -\omega_z$$

In polar coordinates holds:

$$v_r = \frac{1}{r} \frac{\partial\psi}{\partial\theta} = \frac{\partial\phi}{\partial r}, \quad v_\theta = -\frac{\partial\psi}{\partial r} = \frac{1}{r} \frac{\partial\phi}{\partial\theta}$$

For source flows with power Q in $(x, y) = (0, 0)$ holds: $\phi = \frac{Q}{2\pi} \ln(r)$ so that $v_r = Q/2\pi r$, $v_\theta = 0$.

For a dipole of strength Q in $x = a$ and strength $-Q$ in $x = -a$ follows from superposition: $\phi = -Qax/2\pi r^2$ where Qa is the dipole strength. For a vortex holds: $\phi = \Gamma\theta/2\pi$.

If an object is surrounded by an uniform main flow with $\vec{v} = v\vec{e}_x$ and such a large Re that viscous effects are limited to the boundary layer holds: $F_x = 0$ and $F_y = -\rho\Gamma v$. The statement that $F_x = 0$ is d'Alembert's paradox and originates from the neglect of viscous effects. The lift F_y is also created by η because $\Gamma \neq 0$ due to viscous effects. Hence rotating bodies also create a force perpendicular to their direction of motion: the *Magnus effect*.

9.7 Boundary layers

9.7.1 Flow boundary layers

If for the thickness of the boundary layer holds: $\delta \ll L$ holds: $\delta \approx L/\sqrt{Re}$. With v_∞ the velocity of the main flow it follows for the velocity $v_y \perp$ the surface: $v_y L \approx \delta v_\infty$. Blasius' equation for the boundary layer is, with $v_y/v_\infty = f(y/\delta)$: $2f''' + ff'' = 0$ with boundary conditions $f(0) = f'(0) = 0$, $f'(\infty) = 1$. From this follows: $C_W = 0.664 Re_x^{-1/2}$.

The momentum theorem of Von Karman for the boundary layer is: $\frac{d}{dx}(\vartheta v^2) + \delta^* v \frac{dv}{dx} = \frac{\tau_0}{\rho}$

where the displacement thickness $\delta^* v$ and the momentum thickness ϑv^2 are given by:

$$\vartheta v^2 = \int_0^\infty (v - v_x) v_x dy, \quad \delta^* v = \int_0^\infty (v - v_x) dy \quad \text{and} \quad \tau_0 = -\eta \left. \frac{\partial v_x}{\partial y} \right|_{y=0}$$

The boundary layer is released from the surface if $\left(\frac{\partial v_x}{\partial y} \right)_{y=0} = 0$. This is equivalent with $\frac{dp}{dx} = \frac{12\eta v_\infty}{\delta^2}$.

9.7.2 Temperature boundary layers

If the thickness of the temperature boundary layer $\delta_T \ll L$ holds:

1. If $Pr \leq 1$: $\delta/\delta_T \approx \sqrt{Pr}$.
2. If $Pr \gg 1$: $\delta/\delta_T \approx \sqrt[3]{Pr}$.

9.8 Heat conductance

For non-stationary heat conductance in one dimension without flow holds:

$$\frac{\partial T}{\partial t} = \frac{\kappa}{\rho c} \frac{\partial^2 T}{\partial x^2} + \Phi$$

where Φ is a source term. If $\Phi = 0$ the solutions for harmonic oscillations at $x = 0$ are:

$$\frac{T - T_\infty}{T_{\max} - T_\infty} = \exp\left(-\frac{x}{D}\right) \cos\left(\omega t - \frac{x}{D}\right)$$

with $D = \sqrt{2\kappa/\omega\rho c}$. At $x = \pi D$ the temperature variation is in anti-phase with the surface. The one-dimensional solution at $\Phi = 0$ is

$$T(x, t) = \frac{1}{2\sqrt{\pi at}} \exp\left(-\frac{x^2}{4at}\right)$$

This is mathematical equivalent to the diffusion problem:

$$\frac{\partial n}{\partial t} = D\nabla^2 n + P - A$$

where P is the production of and A the discharge of particles. The flow density $J = -D\nabla n$.

9.9 Turbulence

The time scale of turbulent velocity variations τ_t is of the order of: $\tau_t = \tau\sqrt{\text{Re}}/\text{Ma}^2$ with τ the molecular time scale. For the velocity of the particles holds: $v(t) = \langle v \rangle + v'(t)$ with $\langle v'(t) \rangle = 0$. The Navier-Stokes equation now becomes:

$$\frac{\partial \langle \vec{v} \rangle}{\partial t} + (\langle \vec{v} \rangle \cdot \nabla) \langle \vec{v} \rangle = -\frac{\nabla \langle p \rangle}{\rho} + \nu \nabla^2 \langle \vec{v} \rangle + \frac{\text{div} \mathbf{S}_R}{\rho}$$

where $\mathbf{S}_{Rij} = -\rho \langle v_i v_j \rangle$ is the turbulent stress tensor. Boussinesq's assumption is: $\tau_{ij} = -\rho \langle v'_i v'_j \rangle$. It is stated that, analogous to Newtonian media: $\mathbf{S}_R = 2\rho\nu_t \langle \mathbf{D} \rangle$. Near a boundary holds: $\nu_t = 0$, far away of a boundary holds: $\nu_t \approx \nu \text{Re}$.

9.10 Self organization

For a (semi) two-dimensional flow holds: $\frac{d\omega}{dt} = \frac{\partial \omega}{\partial t} + J(\omega, \psi) = \nu \nabla^2 \omega$

With $J(\omega, \psi)$ the Jacobian. So if $\nu = 0$, ω is conserved. Further, the kinetic energy/ mA and the enstrofy V are conserved: with $\vec{v} = \nabla \times (\vec{k}\psi)$

$$E \sim (\nabla \psi)^2 \sim \int_0^\infty \mathcal{E}(k, t) dk = \text{constant} \quad , \quad V \sim (\nabla^2 \psi)^2 \sim \int_0^\infty k^2 \mathcal{E}(k, t) dk = \text{constant}$$

From this follows that in a two-dimensional flow the energy flux goes towards large values of k : larger structures become larger at the expense of smaller ones. In three-dimensional flows the situation is just the opposite.

Chapter 10

Quantum physics

10.1 Introduction to quantum physics

10.1.1 Black body radiation

Planck's law for the energy distribution for the radiation of a black body is:

$$w(f) = \frac{8\pi h f^3}{c^3} \frac{1}{e^{hf/kT} - 1}, \quad w(\lambda) = \frac{8\pi hc}{\lambda^5} \frac{1}{e^{hc/\lambda kT} - 1}$$

Stefan-Boltzmann's law for the total power density can be derived from this: $P = A\sigma T^4$. Wien's law for the maximum can also be derived from this: $T\lambda_{\max} = k_W$.

10.1.2 The Compton effect

For the wavelength of scattered light, if light is considered to exist of particles, can be derived:

$$\lambda' = \lambda + \frac{h}{mc}(1 - \cos \theta) = \lambda + \lambda_C(1 - \cos \theta)$$

10.1.3 Electron diffraction

Diffraction of electrons at a crystal can be explained by assuming that particles have a wave character with wavelength $\lambda = h/p$. This wavelength is called the Broglie-wavelength.

10.2 Wave functions

The wave character of particles is described by a wavefunction ψ . This wavefunction can be described in normal or momentum space. Both definitions are each others Fourier transform:

$$\Phi(k, t) = \frac{1}{\sqrt{h}} \int \Psi(x, t) e^{-ikx} dx \quad \text{and} \quad \Psi(x, t) = \frac{1}{\sqrt{h}} \int \Phi(k, t) e^{ikx} dk$$

These waves define a particle with group velocity $v_g = p/m$ and energy $E = \hbar\omega$.

The wavefunction can be interpreted as a measure for the probability P to find a particle somewhere (Born): $dP = |\psi|^2 d^3V$. The expectation value $\langle f \rangle$ of a quantity f of a system is given by:

$$\langle f(t) \rangle = \iiint \Psi^* f \Psi d^3V, \quad \langle f_p(t) \rangle = \iiint \Phi^* f \Phi d^3V_p$$

This is also written as $\langle f(t) \rangle = \langle \Phi | f | \Phi \rangle$. The normalizing condition for wavefunctions follows from this: $\langle \Phi | \Phi \rangle = \langle \Psi | \Psi \rangle = 1$.

10.3 Operators in quantum physics

In quantum mechanics, classical quantities are translated into operators. These operators are hermitian because their eigenvalues must be real:

$$\int \psi_1^* A \psi_2 d^3V = \int \psi_2 (A \psi_1)^* d^3V$$

When u_n is the eigenfunction of the eigenvalue equation $A\Psi = a\Psi$ for eigenvalue a_n , Ψ can be expanded into a basis of eigenfunctions: $\Psi = \sum_n c_n u_n$. If this basis is taken orthonormal, then follows for the coefficients:

$c_n = \langle u_n | \Psi \rangle$. If the system is in a state described by Ψ , the chance to find eigenvalue a_n when measuring A is given by $|c_n|^2$ in the discrete part of the spectrum and $|c_n|^2 da$ in the continuous part of the spectrum between a and $a + da$. The *matrix element* A_{ij} is given by: $A_{ij} = \langle u_i | A | u_j \rangle$. Because $(AB)_{ij} = \langle u_i | AB | u_j \rangle = \langle u_i | A \sum_n | u_n \rangle \langle u_n | B | u_j \rangle$ holds: $\sum_n | u_n \rangle \langle u_n | = 1$.

The time-dependence of an operator is given by (Heisenberg):

$$\frac{dA}{dt} = \frac{\partial A}{\partial t} + \frac{[A, H]}{i\hbar}$$

with $[A, B] \equiv AB - BA$ the *commutator* of A and B . For hermitian operators the commutator is always complex. If $[A, B] = 0$, the operators A and B have a common set of eigenfunctions. By applying this to p_x and x follows (Ehrenfest): $m d^2 \langle x \rangle_t / dt^2 = - \langle dU(x)/dx \rangle$.

The first order approximation $\langle F(x) \rangle_t \approx F(\langle x \rangle)$, with $F = -dU/dx$ represents the classical equation.

Before the addition of quantummechanical operators which are a product of other operators, they should be made symmetrical: a classical product AB becomes $\frac{1}{2}(AB + BA)$.

10.4 The uncertainty principle

If the uncertainty ΔA in A is defined as: $(\Delta A)^2 = \langle \psi | A_{\text{op}} - \langle A \rangle |^2 \psi \rangle = \langle A^2 \rangle - \langle A \rangle^2$ it follows:

$$\Delta A \cdot \Delta B \geq \frac{1}{2} | \langle \psi | [A, B] | \psi \rangle |$$

From this follows: $\Delta E \cdot \Delta t \geq \frac{1}{2} \hbar$, and because $[x, p_x] = i\hbar$ holds: $\Delta p_x \cdot \Delta x \geq \frac{1}{2} \hbar$, and $\Delta L_x \cdot \Delta L_y \geq \frac{1}{2} \hbar L_z$.

10.5 The Schrödinger equation

The momentum operator is given by: $p_{\text{op}} = -i\hbar \nabla$. The position operator is: $x_{\text{op}} = i\hbar \nabla_p$. The energy operator is given by: $E_{\text{op}} = i\hbar \partial / \partial t$. The Hamiltonian of a particle with mass m , potential energy U and total energy E is given by: $H = p^2 / 2m + U$. From $H\psi = E\psi$ then follows the *Schrödinger equation*:

$$\boxed{-\frac{\hbar^2}{2m} \nabla^2 \psi + U\psi = E\psi = i\hbar \frac{\partial \psi}{\partial t}}$$

The linear combination of the solutions of this equation give the general solution. In one dimension it is:

$$\psi(x, t) = \left(\sum + \int dE \right) c(E) u_E(x) \exp \left(-\frac{iEt}{\hbar} \right)$$

The current density J is given by: $J = \frac{\hbar}{2im} (\psi^* \nabla \psi - \psi \nabla \psi^*)$

The following conservation law holds: $\frac{\partial P(x, t)}{\partial t} = -\nabla J(x, t)$

10.6 Parity

The parity operator in one dimension is given by $\mathcal{P}\psi(x) = \psi(-x)$. If the wavefunction is split in even and odd functions, it can be expanded into eigenfunctions of \mathcal{P} :

$$\psi(x) = \underbrace{\frac{1}{2}(\psi(x) + \psi(-x))}_{\text{even: } \psi^+} + \underbrace{\frac{1}{2}(\psi(x) - \psi(-x))}_{\text{odd: } \psi^-}$$

$[\mathcal{P}, H] = 0$. The functions $\psi^+ = \frac{1}{2}(1 + \mathcal{P})\psi(x, t)$ and $\psi^- = \frac{1}{2}(1 - \mathcal{P})\psi(x, t)$ both satisfy the Schrödinger equation. Hence, parity is a conserved quantity.

10.7 The tunnel effect

The wavefunction of a particle in an ∞ high potential step from $x = 0$ to $x = a$ is given by $\psi(x) = a^{-1/2} \sin(kx)$. The energy levels are given by $E_n = n^2 \hbar^2 / 8a^2 m$.

If the wavefunction with energy W meets a potential well of $W_0 > W$ the wavefunction will, unlike the classical case, be non-zero within the potential well. If 1, 2 and 3 are the areas in front, within and behind the potential well, holds:

$$\psi_1 = Ae^{ikx} + Be^{-ikx}, \quad \psi_2 = Ce^{ik'x} + De^{-ik'x}, \quad \psi_3 = A'e^{ikx}$$

with $k'^2 = 2m(W - W_0)/\hbar^2$ and $k^2 = 2mW$. Using the boundary conditions requiring continuity: ψ is continuous and $\partial\psi/\partial x$ is continuous at $x = 0$ and $x = a$ gives B , C and D and A' expressed in A . The amplitude T of the transmitted wave is defined by $T = |A'|^2/|A|^2$. If $W > W_0$ and $2a = n\lambda' = 2\pi n/k'$ holds: $T = 1$.

10.8 The harmonic oscillator

For a harmonic oscillator holds: $U = \frac{1}{2}bx^2$ and $\omega_0^2 = b/m$. The Hamiltonian H is then given by:

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2 = \frac{1}{2}\hbar\omega + \omega A^\dagger A$$

with

$$A = \sqrt{\frac{1}{2}m\omega}x + \frac{ip}{\sqrt{2m\omega}} \quad \text{and} \quad A^\dagger = \sqrt{\frac{1}{2}m\omega}x - \frac{ip}{\sqrt{2m\omega}}$$

$A \neq A^\dagger$ is non hermitian. $[A, A^\dagger] = \hbar$ and $[A, H] = \hbar\omega A$. A is a so called *raising ladder operator*, A^\dagger a *lowering ladder operator*. $HAu_E = (E - \hbar\omega)Au_E$. There is an eigenfunction u_0 for which holds: $Au_0 = 0$. The energy in this ground state is $\frac{1}{2}\hbar\omega$: the zero point energy. For the normalized eigenfunctions follows:

$$u_n = \frac{1}{\sqrt{n!}} \left(\frac{A^\dagger}{\sqrt{\hbar}} \right)^n u_0 \quad \text{with} \quad u_0 = \sqrt{\frac{m\omega}{\pi\hbar}} \exp\left(-\frac{m\omega x^2}{2\hbar}\right)$$

with $E_n = (\frac{1}{2} + n)\hbar\omega$.

10.9 Angular momentum

For the angular momentum operators L holds: $[L_z, L^2] = [L_z, H] = [L^2, H] = 0$. However, cyclically holds: $[L_x, L_y] = i\hbar L_z$. Not all components of L can be known at the same time with arbitrary accuracy. For L_z holds:

$$L_z = -i\hbar \frac{\partial}{\partial \varphi} = -i\hbar \left(x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} \right)$$

The ladder operators L_\pm are defined by: $L_\pm = L_x \pm iL_y$. Now holds: $L^2 = L_+L_- + L_z^2 - \hbar L_z$. Further,

$$L_\pm = \hbar e^{\pm i\varphi} \left(\pm \frac{\partial}{\partial \theta} + i \cot(\theta) \frac{\partial}{\partial \varphi} \right)$$

From $[L_+, L_z] = -\hbar L_+$ follows: $L_z(L_+ Y_{lm}) = (m+1)\hbar(L_+ Y_{lm})$.

From $[L_-, L_z] = \hbar L_-$ follows: $L_z(L_- Y_{lm}) = (m-1)\hbar(L_- Y_{lm})$.

From $[L^2, L_\pm] = 0$ follows: $L^2(L_\pm Y_{lm}) = l(l+1)\hbar^2(L_\pm Y_{lm})$.

Because L_x and L_y are hermitian (this implies $L_\pm^\dagger = L_\mp$) and $|L_\pm Y_{lm}|^2 > 0$ follows: $l(l+1) - m^2 - m \geq 0 \Rightarrow -l \leq m \leq l$. Further follows that l has to be integral or half-integral. Half-odd integral values give no unique solution ψ and are therefore dismissed.

10.10 Spin

For the spin operators are defined by their commutation relations: $[S_x, S_y] = i\hbar S_z$. Because the spin operators do not act in the physical space (x, y, z) the uniqueness of the wavefunction is not a criterium here: also half odd-integer values are allowed for the spin. Because $[L, S] = 0$ spin and angular momentum operators do not have a common set of eigenfunctions. The spin operators are given by $\vec{S} = \frac{1}{2}\hbar\vec{\sigma}$, with

$$\vec{\sigma}_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \vec{\sigma}_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \vec{\sigma}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

The eigenstates of S_z are called *spinors*: $\chi = \alpha_+\chi_+ + \alpha_-\chi_-$, where $\chi_+ = (1, 0)$ represents the state with spin up ($S_z = \frac{1}{2}\hbar$) and $\chi_- = (0, 1)$ represents the state with spin down ($S_z = -\frac{1}{2}\hbar$). Then the probability to find spin up after a measurement is given by $|\alpha_+|^2$ and the chance to find spin down is given by $|\alpha_-|^2$. Of course holds $|\alpha_+|^2 + |\alpha_-|^2 = 1$.

The electron will have an intrinsic magnetic dipole moment \vec{M} due to its spin, given by $\vec{M} = -eg_S\vec{S}/2m$, with $g_S = 2(1 + \alpha/2\pi + \dots)$ the gyromagnetic ratio. In the presence of an external magnetic field this gives a potential energy $U = -\vec{M} \cdot \vec{B}$. The Schrödinger equation then becomes (because $\partial\chi/\partial x_i \equiv 0$):

$$i\hbar \frac{\partial\chi(t)}{\partial t} = \frac{eg_S\hbar}{4m} \vec{\sigma} \cdot \vec{B}\chi(t)$$

with $\vec{\sigma} = (\vec{\sigma}_x, \vec{\sigma}_y, \vec{\sigma}_z)$. If $\vec{B} = B\vec{e}_z$ there are two eigenvalues for this problem: χ_{\pm} for $E = \pm eg_S\hbar B/4m = \pm\hbar\omega$. So the general solution is given by $\chi = (ae^{-i\omega t}, be^{i\omega t})$. From this can be derived: $\langle S_x \rangle = \frac{1}{2}\hbar \cos(2\omega t)$ and $\langle S_y \rangle = \frac{1}{2}\hbar \sin(2\omega t)$. Thus the spin precesses about the z -axis with frequency 2ω . This causes the normal Zeeman splitting of spectral lines.

The potential operator for two particles with spin $\pm\frac{1}{2}\hbar$ is given by:

$$V(r) = V_1(r) + \frac{1}{\hbar^2}(\vec{S}_1 \cdot \vec{S}_2)V_2(r) = V_1(r) + \frac{1}{2}V_2(r)[S(S+1) - \frac{3}{2}]$$

This makes it possible for two states to exist: $S = 1$ (triplet) or $S = 0$ (Singlet).

10.11 The Dirac formalism

If the operators for p and E are substituted in the relativistic equation $E^2 = m_0^2c^4 + p^2c^2$, the *Klein-Gordon* equation is found:

$$\left(\nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \frac{m_0^2c^2}{\hbar^2} \right) \psi(\vec{x}, t) = 0$$

The operator $\square - m_0^2c^2/\hbar^2$ can be separated:

$$\nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \frac{m_0^2c^2}{\hbar^2} = \left\{ \gamma_\lambda \frac{\partial}{\partial x_\lambda} - \frac{m_0^2c^2}{\hbar^2} \right\} \left\{ \gamma_\mu \frac{\partial}{\partial x_\mu} + \frac{m_0^2c^2}{\hbar^2} \right\}$$

where the Dirac matrices γ are given by: $\gamma_\lambda\gamma_\mu + \gamma_\mu\gamma_\lambda = 2\delta_{\lambda\mu}$. From this it can be derived that the γ are hermitian 4×4 matrices given by:

$$\gamma_k = \begin{pmatrix} 0 & -i\sigma_k \\ i\sigma_k & 0 \end{pmatrix}, \quad \gamma_4 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$$

With this, the Dirac equation becomes:

$$\left(\gamma_\lambda \frac{\partial}{\partial x_\lambda} + \frac{m_0^2c^2}{\hbar^2} \right) \psi(\vec{x}, t) = 0$$

where $\psi(x) = (\psi_1(x), \psi_2(x), \psi_3(x), \psi_4(x))$ is a spinor.

10.12 Atomic physics

10.12.1 Solutions

The solutions of the Schrödinger equation in spherical coordinates if the potential energy is a function of r alone can be written as: $\psi(r, \theta, \varphi) = R_{nl}(r)Y_{l,m_l}(\theta, \varphi)\chi_{m_s}$, with

$$Y_{lm} = \frac{C_{lm}}{\sqrt{2\pi}} P_l^m(\cos \theta) e^{im\varphi}$$

For an atom or ion with one electron holds: $R_{lm}(\rho) = C_{lm} e^{-\rho/2} \rho^l L_{n-l-1}^{2l+1}(\rho)$

with $\rho = 2rZ/na_0$ with $a_0 = \varepsilon_0 \hbar^2 / \pi m_e e^2$. The L_i^j are the associated Laguerre functions and the P_l^m are the associated Legendre polynomials:

$$P_l^{|m|}(x) = (1-x^2)^{m/2} \frac{d^{|m|}}{dx^{|m|}} [(x^2-1)^l] \quad , \quad L_n^m(x) = \frac{(-1)^m n!}{(n-m)!} e^{-x} x^{-m} \frac{d^{n-m}}{dx^{n-m}} (e^{-x} x^n)$$

The parity of these solutions is $(-1)^l$. The functions are $2 \sum_{l=0}^{n-1} (2l+1) = 2n^2$ -folded degenerated.

10.12.2 Eigenvalue equations

The eigenvalue equations for an atom or ion with with one electron are:

Equation	Eigenvalue	Range
$H_{\text{op}}\psi = E\psi$	$E_n = \mu e^4 Z^2 / 8\varepsilon_0^2 \hbar^2 n^2$	$n \geq 1$
$L_{z\text{op}}Y_{lm} = L_z Y_{lm}$	$L_z = m_l \hbar$	$-l \leq m_l \leq l$
$L_{\text{op}}^2 Y_{lm} = L^2 Y_{lm}$	$L^2 = l(l+1) \hbar^2$	$l < n$
$S_{z\text{op}}\chi = S_z \chi$	$S_z = m_s \hbar$	$m_s = \pm \frac{1}{2}$
$S_{\text{op}}^2 \chi = S^2 \chi$	$S^2 = s(s+1) \hbar^2$	$s = \frac{1}{2}$

10.12.3 Spin-orbit interaction

The total momentum is given by $\vec{J} = \vec{L} + \vec{S}$. The total magnetic dipole moment of an electron is then $\vec{M} = \vec{M}_L + \vec{M}_S = -(e/2m_e)(\vec{L} + g_S \vec{S})$ where $g_S = 2.0023$ is the gyromagnetic ratio of the electron. Further holds: $J^2 = L^2 + S^2 + 2\vec{L} \cdot \vec{S} = L^2 + S^2 + 2L_z S_z + L_+ S_- + L_- S_+$. J has quantum numbers j with possible values $j = l \pm \frac{1}{2}$, with $2j+1$ possible z -components ($m_J \in \{-j, \dots, 0, \dots, j\}$). If the interaction energy between S and L is small it can be stated that: $E = E_n + E_{SL} = E_n + a \vec{S} \cdot \vec{L}$. It can then be derived that:

$$a = \frac{|E_n| Z^2 \alpha^2}{\hbar^2 n l (l+1) (l + \frac{1}{2})}$$

After a relativistic correction this becomes:

$$E = E_n + \frac{|E_n| Z^2 \alpha^2}{n} \left(\frac{3}{4n} - \frac{1}{j + \frac{1}{2}} \right)$$

The *fine structure* in atomic spectra arises from this. With $g_S = 2$ follows for the average magnetic moment: $\vec{M}_{\text{av}} = -(e/2m_e) g \hbar \vec{J}$, where g is the Landé-factor:

$$g = 1 + \frac{\vec{S} \cdot \vec{J}}{J^2} = 1 + \frac{j(j+1) + s(s+1) - l(l+1)}{2j(j+1)}$$

For atoms with more than one electron the following limiting situations occur:

1. $L - S$ coupling: for small atoms the electrostatic interaction is dominant and the state can be characterized by L, S, J, m_J . $J \in \{|L - S|, \dots, L + S - 1, L + S\}$ and $m_J \in \{-J, \dots, J - 1, J\}$. The spectroscopic notation for this interaction is: $^{2S+1}L_J$. $2S + 1$ is the multiplicity of a multiplet.
2. $j - j$ coupling: for larger atoms the electrostatic interaction is smaller than the $L_i \cdot s_i$ interaction of an electron. The state is characterized by $j_1 \dots j_n, J, m_J$ where only the j_i of the not completely filled subshells are to be taken into account.

The energy difference for larger atoms when placed in a magnetic field is: $\Delta E = g\mu_B m_J B$ where g is the Landé factor. For a transition between two singlet states the line splits in 3 parts, for $\Delta m_J = -1, 0 + 1$. This results in the normal Zeeman effect. At higher S the line splits up in more parts: the anomalous Zeeman effect.

Interaction with the spin of the nucleus gives the hyperfine structure.

10.12.4 Selection rules

For the dipole transition matrix elements follows: $p_0 \sim |\langle l_2 m_2 | \vec{E} \cdot \vec{r} | l_1 m_1 \rangle|$. Conservation of angular momentum demands that for the transition of an electron holds that $\Delta l = \pm 1$.

For an atom where $L - S$ coupling is dominant further holds: $\Delta S = 0$ (but not strict), $\Delta L = 0, \pm 1$, $\Delta J = 0, \pm 1$ except for $J = 0 \rightarrow J = 0$ transitions, $\Delta m_J = 0, \pm 1$, but $\Delta m_J = 0$ is forbidden if $\Delta J = 0$.

For an atom where $j - j$ coupling is dominant further holds: for the jumping electron holds, except $\Delta l = \pm 1$, also: $\Delta j = 0, \pm 1$, and for all other electrons: $\Delta j = 0$. For the total atom holds: $\Delta J = 0, \pm 1$ but no $J = 0 \rightarrow J = 0$ transitions and $\Delta m_J = 0, \pm 1$, but $\Delta m_J = 0$ is forbidden if $\Delta J = 0$.

10.13 Interaction with electromagnetic fields

The Hamiltonian of an electron in an electromagnetic field is given by:

$$H = \frac{1}{2\mu} (\vec{p} + e\vec{A})^2 - eV = -\frac{\hbar^2}{2\mu} \nabla^2 + \frac{e}{2\mu} \vec{B} \cdot \vec{L} + \frac{e^2}{2\mu} A^2 - eV$$

where μ is the reduced mass of the system. The term $\sim A^2$ can usually be neglected, except for very strong fields or macroscopic motions. For $\vec{B} = B\vec{e}_z$ it is given by $e^2 B^2 (x^2 + y^2) / 8\mu$.

When a gauge transformation $\vec{A}' = \vec{A} - \nabla f$, $V' = V + \partial f / \partial t$ is applied to the potentials the wavefunction is also transformed according to $\psi' = \psi e^{iqef/\hbar}$ with qe the charge of the particle. Because $f = f(x, t)$, this is called a *local* gauge transformation, in contrast with a *global* gauge transformation which can always be applied.

10.14 Perturbation theory

10.14.1 Time-independent perturbation theory

To solve the equation $(H_0 + \lambda H_1)\psi_n = E_n \psi_n$ one has to find the eigenfunctions of $H = H_0 + \lambda H_1$. Suppose that ϕ_n is a complete set of eigenfunctions of the non-perturbed Hamiltonian H_0 : $H_0 \phi_n = E_n^0 \phi_n$. Because ϕ_n is a complete set holds:

$$\psi_n = N(\lambda) \left\{ \phi_n + \sum_{k \neq n} c_{nk}(\lambda) \phi_k \right\}$$

When c_{nk} and E_n are being expanded into λ : $c_{nk} = \lambda c_{nk}^{(1)} + \lambda^2 c_{nk}^{(2)} + \dots$
 $E_n = E_n^0 + \lambda E_n^{(1)} + \lambda^2 E_n^{(2)} + \dots$

and this is put into the Schrödinger equation the result is: $E_n^{(1)} = \langle \phi_n | H_1 | \phi_n \rangle$ and

$c_{nm}^{(1)} = \frac{\langle \phi_m | H_1 | \phi_n \rangle}{E_n^0 - E_m^0}$ if $m \neq n$. The second-order correction of the energy is then given by:

$$E_n^{(2)} = \sum_{k \neq n} \frac{|\langle \phi_k | H_1 | \phi_n \rangle|^2}{E_n^0 - E_k^0}. \text{ So to first order holds: } \psi_n = \phi_n + \sum_{k \neq n} \frac{\langle \phi_k | \lambda H_1 | \phi_n \rangle}{E_n^0 - E_k^0} \phi_k.$$

In case the levels are degenerated the above does not hold. In that case an orthonormal set eigenfunctions ϕ_{ni} is chosen for each level n , so that $\langle \phi_{mi} | \phi_{nj} \rangle = \delta_{mn} \delta_{ij}$. Now ψ is expanded as:

$$\psi_n = N(\lambda) \left\{ \sum_i \alpha_i \phi_{ni} + \lambda \sum_{k \neq n} c_{nk}^{(1)} \sum_i \beta_i \phi_{ki} + \dots \right\}$$

$E_{ni} = E_n^0 + \lambda E_{ni}^{(1)}$ is approximated by $E_{ni}^0 := E_n^0$. Substitution in the Schrödinger equation and taking dot product with ϕ_{ni} gives: $\sum_i \alpha_i \langle \phi_{nj} | H_1 | \phi_{ni} \rangle = E_n^{(1)} \alpha_j$. Normalization requires that $\sum_i |\alpha_i|^2 = 1$.

10.14.2 Time-dependent perturbation theory

From the Schrödinger equation $i\hbar \frac{\partial \psi(t)}{\partial t} = (H_0 + \lambda V(t))\psi(t)$

and the expansion $\psi(t) = \sum_n c_n(t) \exp\left(\frac{-iE_n^0 t}{\hbar}\right) \phi_n$ with $c_n(t) = \delta_{nk} + \lambda c_n^{(1)}(t) + \dots$

follows: $c_n^{(1)}(t) = \frac{\lambda}{i\hbar} \int_0^t \langle \phi_n | V(t') | \phi_k \rangle \exp\left(\frac{i(E_n^0 - E_k^0)t'}{\hbar}\right) dt'$

10.15 N-particle systems

10.15.1 General

Identical particles are indistinguishable. For the total wavefunction of a system of identical indistinguishable particles holds:

1. Particles with a half-odd integer spin (Fermions): ψ_{total} must be antisymmetric w.r.t. interchange of the coordinates (spatial and spin) of each pair of particles. The Pauli principle results from this: two Fermions cannot exist in an identical state because then $\psi_{\text{total}} = 0$.
2. Particles with an integer spin (Bosons): ψ_{total} must be symmetric w.r.t. interchange of the coordinates (spatial and spin) of each pair of particles.

For a system of two electrons there are 2 possibilities for the spatial wavefunction. When a and b are the quantum numbers of electron 1 and 2 holds:

$$\psi_S(1, 2) = \psi_a(1)\psi_b(2) + \psi_a(2)\psi_b(1) \quad , \quad \psi_A(1, 2) = \psi_a(1)\psi_b(2) - \psi_a(2)\psi_b(1)$$

Because the particles do not approach each other closely the repulsion energy at ψ_A in this state is smaller. The following spin wavefunctions are possible:

$$\begin{aligned} \chi_A &= \frac{1}{2}\sqrt{2}[\chi_+(1)\chi_-(2) - \chi_+(2)\chi_-(1)] \quad m_s = 0 \\ \chi_S &= \begin{cases} \chi_+(1)\chi_+(2) & m_s = +1 \\ \frac{1}{2}\sqrt{2}[\chi_+(1)\chi_-(2) + \chi_+(2)\chi_-(1)] & m_s = 0 \\ \chi_-(1)\chi_-(2) & m_s = -1 \end{cases} \end{aligned}$$

Because the total wavefunction must be antisymmetric it follows: $\psi_{\text{total}} = \psi_S \chi_A$ or $\psi_{\text{total}} = \psi_A \chi_S$.

For N particles the symmetric spatial function is given by:

$$\psi_S(1, \dots, N) = \sum \psi(\text{all permutations of } 1..N)$$

The antisymmetric wavefunction is given by the determinant $\psi_A(1, \dots, N) = \frac{1}{\sqrt{N!}} |u_{E_i}(j)|$

10.15.2 Molecules

The wavefunctions of atom a and b are ϕ_a and ϕ_b . If the 2 atoms approach each other there are two possibilities: the total wavefunction approaches the bonding function with lower total energy $\psi_B = \frac{1}{2}\sqrt{2}(\phi_a + \phi_b)$ or approaches the anti-bonding function with higher energy $\psi_{AB} = \frac{1}{2}\sqrt{2}(\phi_a - \phi_b)$. If a molecular-orbital is symmetric w.r.t. the connecting axis, like a combination of two s-orbitals it is called a σ -orbital, otherwise a π -orbital, like the combination of two p-orbitals along two axes.

The energy of a system is: $E = \frac{\langle \psi | H | \psi \rangle}{\langle \psi | \psi \rangle}$.

The energy calculated with this method is always *higher* than the real energy if ψ is only an approximation for the solutions of $H\psi = E\psi$. Also, if there are more functions to be chosen, the function which gives the lowest energy is the best approximation. Applying this to the function $\psi = \sum c_i \phi_i$ one finds: $(H_{ij} - ES_{ij})c_i = 0$. This equation has only solutions if the *secular determinant* $|H_{ij} - ES_{ij}| = 0$. Here, $H_{ij} = \langle \phi_i | H | \phi_j \rangle$ and $S_{ij} = \langle \phi_i | \phi_j \rangle$. $\alpha_i := H_{ii}$ is the Coulomb integral and $\beta_{ij} := H_{ij}$ the exchange integral. $S_{ii} = 1$ and S_{ij} is the overlap integral.

The first approximation in the molecular-orbital theory is to place both electrons of a chemical bond in the bonding orbital: $\psi(1, 2) = \psi_B(1)\psi_B(2)$. This results in a large electron density between the nuclei and therefore a repulsion. A better approximation is: $\psi(1, 2) = C_1\psi_B(1)\psi_B(2) + C_2\psi_{AB}(1)\psi_{AB}(2)$, with $C_1 = 1$ and $C_2 \approx 0.6$.

In some atoms, such as C, it is energetical more suitable to form orbitals which are a linear combination of the s, p and d states. There are three ways of hybridization in C:

1. SP-hybridization: $\psi_{sp} = \frac{1}{2}\sqrt{2}(\psi_{2s} \pm \psi_{2p_z})$. There are 2 hybrid orbitals which are placed on one line under 180° . Further the $2p_x$ and $2p_y$ orbitals remain.
2. SP² hybridization: $\psi_{sp^2} = \psi_{2s}/\sqrt{3} + c_1\psi_{2p_z} + c_2\psi_{2p_y}$, where $(c_1, c_2) \in \{(\sqrt{2/3}, 0), (-1/\sqrt{6}, 1/\sqrt{2}), (-1/\sqrt{6}, -1/\sqrt{2})\}$. The 3 SP² orbitals lay in one plane, with symmetry axes which are at an angle of 120° .
3. SP³ hybridization: $\psi_{sp^3} = \frac{1}{2}(\psi_{2s} \pm \psi_{2p_z} \pm \psi_{2p_y} \pm \psi_{2p_x})$. The 4 SP³ orbitals form a tetraheder with the symmetry axes at an angle of $109^\circ 28'$.

10.16 Quantum statistics

If a system exists in a state in which one has not the disposal of the maximal amount of information about the system, it can be described by a *density matrix* ρ . If the probability that the system is in state ψ_i is given by a_i , one can write for the expectation value a of A : $\langle a \rangle = \sum_i r_i \langle \psi_i | A | \psi_i \rangle$.

If ψ is expanded into an orthonormal basis $\{\phi_k\}$ as: $\psi^{(i)} = \sum_k c_k^{(i)} \phi_k$, holds:

$$\langle A \rangle = \sum_k (A\rho)_{kk} = \text{Tr}(A\rho)$$

where $\rho_{lk} = c_k^* c_l$. ρ is hermitian, with $\text{Tr}(\rho) = 1$. Further holds $\rho = \sum_i r_i |\psi_i\rangle \langle \psi_i|$. The probability to find eigenvalue a_n when measuring A is given by ρ_{nn} if one uses a basis of eigenvectors of A for $\{\phi_k\}$. For the time-dependence holds (in the Schrödinger image operators are not explicitly time-dependent):

$$i\hbar \frac{d\rho}{dt} = [H, \rho]$$

For a macroscopic system in equilibrium holds $[H, \rho] = 0$. If all quantumstates with the same energy are equally probable: $P_i = P(E_i)$, one can obtain the distribution:

$$P_n(E) = \rho_{nn} = \frac{e^{-E_n/kT}}{Z} \quad \text{with the state sum } Z = \sum_n e^{-E_n/kT}$$

The thermodynamic quantities are related to these definitions as follows: $F = -kT \ln(Z)$, $U = \langle H \rangle = \sum_n p_n E_n = -\frac{\partial}{\partial kT} \ln(Z)$, $S = -k \sum_n P_n \ln(P_n)$. For a mixed state of M orthonormal quantum states with probability $1/M$ follows: $S = k \ln(M)$.

The distribution function for the internal states for a system in thermal equilibrium is the most probable function. This function can be found by taking the maximum of the function which gives the number of states with Stirling's equation: $\ln(n!) \approx n \ln(n) - n$, and the conditions $\sum_k n_k = N$ and $\sum_k n_k W_k = W$. For identical, indistinguishable particles which obey the Pauli exclusion principle the possible number of states is given by:

$$P = \prod_k \frac{g_k!}{n_k!(g_k - n_k)!}$$

This results in the *Fermi-Dirac statistics*. For indistinguishable particles which *do not* obey the exclusion principle the possible number of states is given by:

$$P = N! \prod_k \frac{g_k^{n_k}}{n_k!}$$

This results in the *Bose-Einstein statistics*. So the distribution functions which explain how particles are distributed over the different one-particle states k which are each g_k -fold degenerate depend on the spin of the particles. They are given by:

1. Fermi-Dirac statistics: integer spin. $n_k \in \{0, 1\}$, $n_k = \frac{N}{Z_g} \frac{g_k}{\exp((E_k - \mu)/kT) + 1}$
with $\ln(Z_g) = \sum g_k \ln[1 + \exp((E_i - \mu)/kT)]$.
2. Bose-Einstein statistics: half odd-integer spin. $n_k \in \mathbb{N}$, $n_k = \frac{N}{Z_g} \frac{g_k}{\exp((E_k - \mu)/kT) - 1}$
with $\ln(Z_g) = -\sum g_k \ln[1 - \exp((E_i - \mu)/kT)]$.

Here, Z_g is the large-canonical state sum and μ the chemical potential. It is found by demanding $\sum n_k = N$, and for it holds: $\lim_{T \rightarrow 0} \mu = E_F$, the Fermi-energy. N is the total number of particles. The Maxwell-Boltzmann distribution can be derived from this in the limit $E_k - \mu \gg kT$:

$$n_k = \frac{N}{Z} \exp\left(-\frac{E_k}{kT}\right) \quad \text{with } Z = \sum_k g_k \exp\left(-\frac{E_k}{kT}\right)$$

With the Fermi-energy, the Fermi-Dirac and Bose-Einstein statistics can be written as:

1. Fermi-Dirac statistics: $n_k = \frac{g_k}{\exp((E_k - E_F)/kT) + 1}$.
2. Bose-Einstein statistics: $n_k = \frac{g_k}{\exp((E_k - E_F)/kT) - 1}$.

Chapter 11

Plasma physics

11.1 Introduction

The *degree of ionization* α of a plasma is defined by: $\alpha = \frac{n_e}{n_e + n_0}$

where n_e is the electron density and n_0 the density of the neutrals. If a plasma contains also negative charged ions α is not well defined.

The probability that a test particle collides with another is given by $dP = n\sigma dx$ where σ is the *cross section*. The collision frequency $\nu_c = 1/\tau_c = n\sigma v$. The *mean free path* is given by $\lambda_v = 1/n\sigma$. The *rate coefficient* K is defined by $K = \langle \sigma v \rangle$. The number of collisions per unit of time and volume between particles of kind 1 and 2 is given by $n_1 n_2 \langle \sigma v \rangle = K n_1 n_2$.

The potential of an electron is given by:

$$V(r) = \frac{-e}{4\pi\epsilon_0 r} \exp\left(-\frac{r}{\lambda_D}\right) \quad \text{with} \quad \lambda_D = \sqrt{\frac{\epsilon_0 k T_e T_i}{e^2 (n_e T_i + n_i T_e)}} \approx \sqrt{\frac{\epsilon_0 k T_e}{n_e e^2}}$$

because charge is shielded in a plasma. Here, λ_D is the *Debye length*. For distances $< \lambda_D$ the plasma cannot be assumed to be quasi-neutral. Deviations of charge neutrality by thermic motion are compensated by oscillations with frequency

$$\omega_{pe} = \sqrt{\frac{n_e e^2}{m_e \epsilon_0}}$$

The distance of closest approximation when two equal charged particles collide for a deviation of $\pi/2$ is $2b_0 = e^2/(4\pi\epsilon_0 \frac{1}{2} m v^2)$. A “neat” plasma is defined as a plasma for which holds: $b_0 < n_e^{-1/3} \ll \lambda_D \ll L_p$. Here $L_p := |n_e/\nabla n_e|$ is the gradient length of the plasma.

11.2 Transport

Relaxation times are defined as $\tau = 1/\nu_c$. Starting with $\sigma_m = 4\pi b_0^2 \ln(\Lambda_C)$ and with $\frac{1}{2} m v^2 = kT$ it can be found that:

$$\tau_m = \frac{4\pi\epsilon_0^2 m^2 v^3}{n e^4 \ln(\Lambda_C)} = \frac{8\sqrt{2}\pi\epsilon_0^2 \sqrt{m} (kT)^{3/2}}{n e^4 \ln(\Lambda_C)}$$

For momentum transfer between electrons and ions holds for a Maxwellian velocity distribution:

$$\tau_{ee} = \frac{6\pi\sqrt{3}\epsilon_0^2 \sqrt{m_e} (kT_e)^{3/2}}{n_e e^4 \ln(\Lambda_C)} \approx \tau_{ei} \quad , \quad \tau_{ii} = \frac{6\pi\sqrt{3}\epsilon_0^2 \sqrt{m_i} (kT_i)^{3/2}}{n_i e^4 \ln(\Lambda_C)}$$

The energy relaxation times for identical particles are equal to the momentum relaxation times. Because for e-i collisions the energy transfer is only $\sim 2m_e/m_i$ this is a slow process. Approximately holds: $\tau_{ee} : \tau_{ei} : \tau_{ie} : \tau_{ii}^E = 1 : 1 : \sqrt{m_i/m_e} : m_i/m_e$.

The relaxation for e-o interaction is much more complicated. For $T > 10$ eV holds approximately: $\sigma_{eo} = 10^{-17} v_e^{-2/5}$, for lower energies this can be a factor 10 lower.

The resistivity $\eta = E/J$ of a plasma is given by:

$$\eta = \frac{n_e e^2}{m_e \nu_{ei}} = \frac{e^2 \sqrt{m_e} \ln(\Lambda_C)}{6\pi\sqrt{3}\epsilon_0^2 (kT_e)^{3/2}}$$

The diffusion coefficient D is defined by means of the flux Γ by $\vec{\Gamma} = n\vec{v}_{\text{diff}} = -D\nabla n$. The equation of continuity is $\partial_t n + \nabla(nv_{\text{diff}}) = 0 \Rightarrow \partial_t n = D\nabla^2 n$. One finds that $D = \frac{1}{3}\lambda_v v$. A rough estimate gives $\tau_D = L_p/D = L_p^2\tau_c/\lambda_v^2$. For magnetized plasma's λ_v must be replaced with the cyclotron radius. In electrical fields also holds $\vec{J} = ne\mu\vec{E} = e(n_e\mu_e + n_i\mu_i)\vec{E}$ with $\mu = e/mv_c$ the mobility of the particles. The Einstein ratio is:

$$\frac{D}{\mu} = \frac{kT}{e}$$

Because a plasma is electrically neutral electrons and ions are strongly coupled and they don't diffuse independent. The *coefficient of ambipolar diffusion* D_{amb} is defined by $\vec{\Gamma} = \vec{\Gamma}_i = \vec{\Gamma}_e = -D_{\text{amb}}\nabla n_{e,i}$. From this follows that

$$D_{\text{amb}} = \frac{kT_e/e - kT_i/e}{1/\mu_e - 1/\mu_i} \approx \frac{kT_e\mu_i}{e}$$

In an external magnetic field B_0 particles will move in spiral orbits with *cyclotron radius* $\rho = mv/eB_0$ and with cyclotron frequency $\Omega = B_0e/m$. The helical orbit is perturbed by collisions. A plasma is called *magnetized* if $\lambda_v > \rho_{e,i}$. So the electrons are magnetized if

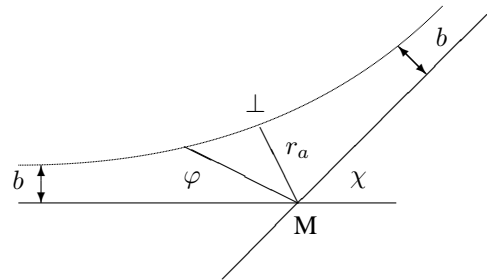
$$\frac{\rho_e}{\lambda_{ee}} = \frac{\sqrt{m_e}e^3n_e \ln(\Lambda_C)}{6\pi\sqrt{3}\varepsilon_0^2(kT_e)^{3/2}B_0} < 1$$

Magnetization of only the electrons is sufficient to confine the plasma reasonable because they are coupled to the ions by charge neutrality. In case of magnetic confinement holds: $\nabla p = \vec{J} \times \vec{B}$. Combined with the two stationary Maxwell equations for the B -field these form the ideal magneto-hydrodynamic equations. For a uniform B -field holds: $p = nkT = B^2/2\mu_0$.

If both magnetic and electric fields are present electrons and ions will move in the same direction. If $\vec{E} = E_r\vec{e}_r + E_z\vec{e}_z$ and $\vec{B} = B_z\vec{e}_z$ the $\vec{E} \times \vec{B}$ drift results in a velocity $\vec{u} = (\vec{E} \times \vec{B})/B^2$ and the velocity in the r, φ plane is $\dot{r}(r, \varphi, t) = \vec{u} + \dot{\vec{\rho}}(t)$.

11.3 Elastic collisions

11.3.1 General



The scattering angle of a particle in interaction with another particle, as shown in the figure at the right is:

$$\chi = \pi - 2b \int_{r_a}^{\infty} \frac{dr}{r^2 \sqrt{1 - \frac{b^2}{r^2} - \frac{W(r)}{E_0}}}$$

Particles with an impact parameter between b and $b + db$, moving through a ring with $d\sigma = 2\pi b db$ leave the scattering area at a solid angle $d\Omega = 2\pi \sin(\chi) d\chi$. The *differential cross section* is then defined as:

$$I(\Omega) = \left| \frac{d\sigma}{d\Omega} \right| = \frac{b}{\sin(\chi)} \frac{\partial b}{\partial \chi}$$

For a potential energy $W(r) = kr^{-n}$ follows: $I(\Omega, v) \sim v^{-4/n}$.

For low energies, $\mathcal{O}(1 \text{ eV})$, σ has a *Ramsauer minimum*. It arises from the interference of matter waves behind the object. $I(\Omega)$ for angles $0 < \chi < \lambda/4$ is larger than the classical value.

11.3.2 The Coulomb interaction

For the Coulomb interaction holds: $2b_0 = q_1 q_2 / 2\pi\epsilon_0 m v_0^2$, so $W(r) = 2b_0/r$. This gives $b = b_0 \cot(\frac{1}{2}\chi)$ and

$$I(\Omega) = \frac{b}{\sin(\chi)} \frac{\partial b}{\partial \chi} = \frac{b_0^2}{4 \sin^2(\frac{1}{2}\chi)}$$

Because the influence of a particle vanishes at $r = \lambda_D$ holds: $\sigma = \pi(\lambda_D^2 - b_0^2)$. Because $dp = d(mv) = mv_0(1 - \cos \chi)$ a cross section related to momentum transfer σ_m is given by:

$$\sigma_m = \int (1 - \cos \chi) I(\Omega) d\Omega = 4\pi b_0^2 \ln \left(\frac{1}{\sin(\frac{1}{2}\chi_{\min})} \right) = 4\pi b_0^2 \ln \left(\frac{\lambda_D}{b_0} \right) := 4\pi b_0^2 \ln(\Lambda_C) \sim \frac{\ln(v^4)}{v^4}$$

where $\ln(\Lambda_C)$ is the *Coulomb-logarithm*. For this quantity holds: $\Lambda_C = \lambda_D/b_0 = 9n(\lambda_D)$.

11.3.3 The induced dipole interaction

The induced dipole interaction, with $\vec{p} = \alpha \vec{E}$, gives a potential V and an energy W in a dipole field given by:

$$V(r) = \frac{\vec{p} \cdot \vec{e}_r}{4\pi\epsilon_0 r^2}, \quad W(r) = -\frac{|e|p}{8\pi\epsilon_0 r^2} = -\frac{\alpha e^2}{2(4\pi\epsilon_0)^2 r^4}$$

with $b_a = \sqrt[4]{\frac{2e^2\alpha}{(4\pi\epsilon_0)^2 \frac{1}{2} m v_0^2}}$ holds: $\chi = \pi - 2b \int_{r_a}^{\infty} \frac{dr}{r^2 \sqrt{1 - \frac{b^2}{r^2} + \frac{b_a^4}{4r^4}}}$

If $b \geq b_a$ the charge would hit the atom. Repulsing nuclear forces prevent this to happen. If the scattering angle is a lot times 2π it is called capture. The cross section for capture $\sigma_{\text{orb}} = \pi b_a^2$ is called the Langevin limit, and is a lowest estimate for the total cross section.

11.3.4 The centre of mass system

If collisions of two particles with masses m_1 and m_2 which scatter in the centre of mass system by an angle χ are compared with the scattering under an angle θ in the laboratory system holds:

$$\tan(\theta) = \frac{m_2 \sin(\chi)}{m_1 + m_2 \cos(\chi)}$$

The energy loss ΔE of the incoming particle is given by:

$$\frac{\Delta E}{E} = \frac{\frac{1}{2} m_2 v_2^2}{\frac{1}{2} m_1 v_1^2} = \frac{2m_1 m_2}{(m_1 + m_2)^2} (1 - \cos(\chi))$$

11.3.5 Scattering of light

Scattering of light by free electrons is called Thomson scattering. The scattering is free from collective effects if $k\lambda_D \ll 1$. The cross section $\sigma = 6.65 \cdot 10^{-29} \text{ m}^2$ and

$$\frac{\Delta f}{f} = \frac{2v}{c} \sin(\frac{1}{2}\chi)$$

This gives for the scattered energy $E_{\text{scat}} \sim n\lambda_0^4/(\lambda^2 - \lambda_0^2)^2$ with n the density. If $\lambda \gg \lambda_0$ it is called Rayleigh scattering. Thomson scattering is a limit of Compton scattering, which is given by $\lambda' - \lambda = \lambda_C(1 - \cos \chi)$ with $\lambda_C = h/mc$ and cannot be used any more if relativistic effects become important.

11.4 Thermodynamic equilibrium and reversibility

Planck's radiation law and the Maxwellian velocity distribution hold for a plasma in equilibrium:

$$\rho(\nu, T)d\nu = \frac{8\pi h\nu^3}{c^3} \frac{1}{\exp(h\nu/kT) - 1} d\nu, \quad N(E, T)dE = \frac{2\pi n}{(\pi kT)^{3/2}} \sqrt{E} \exp\left(-\frac{E}{kT}\right) dE$$

“Detailed balancing” means that the number of reactions in one direction equals the number of reactions in the opposite direction because both processes have equal probability if one corrects for the used phase space. For the reaction

$$\sum_{\text{forward}} X_{\text{forward}} \rightleftharpoons \sum_{\text{back}} X_{\text{back}}$$

holds in a plasma in equilibrium *microscopic* reversibility:

$$\prod_{\text{forward}} \hat{\eta}_{\text{forward}} = \prod_{\text{back}} \hat{\eta}_{\text{back}}$$

If the velocity distribution is Maxwellian, this gives:

$$\hat{\eta}_x = \frac{n_x}{g_x} \frac{h^3}{(2\pi m_x kT)^{3/2}} e^{-E_{\text{kin}}/kT}$$

where g is the statistical weight of the state and $n/g := \eta$. For electrons holds $g = 2$, for excited states usually holds $g = 2j + 1 = 2n^2$.

With this one finds for the Boltzmann balance, $X_p + e^- \rightleftharpoons X_1 + e^- + (E_{1p})$:

$$\frac{n_p^B}{n_1} = \frac{g_p}{g_1} \exp\left(\frac{E_p - E_1}{kT_e}\right)$$

And for the Saha balance, $X_p + e^- + (E_{pi}) \rightleftharpoons X_1^+ + 2e^-$:

$$\frac{n_p^S}{g_p} = \frac{n_1^+}{g_1^+} \frac{n_e}{g_e} \frac{h^3}{(2\pi m_e kT_e)^{3/2}} \exp\left(\frac{E_{pi}}{kT_e}\right)$$

Because the number of particles on the left-hand side and right-hand side of the equation is different, a factor g/V_e remains. This factor causes the *Saha-jump*.

From microscopic reversibility one can derive that for the rate coefficients $K(p, q, T) := \langle \sigma v \rangle_{pq}$ holds:

$$K(q, p, T) = \frac{g_p}{g_q} K(p, q, T) \exp\left(\frac{\Delta E_{pq}}{kT}\right)$$

11.5 Inelastic collisions

11.5.1 Types of collisions

The kinetic energy can be split in a part *of* and a part *in* the centre of mass system. The energy *in* the centre of mass system is available for reactions. This energy is given by

$$E = \frac{m_1 m_2 (v_1 - v_2)^2}{2(m_1 + m_2)}$$

Some types of inelastic collisions important for plasma physics are:

1. Excitation: $A_p + e^- \rightleftharpoons A_q + e^-$
2. Decay: $A_q \rightleftharpoons A_p + hf$

3. Ionisation and 3-particles recombination: $A_p + e^- \rightleftharpoons A^+ + 2e^-$
4. radiative recombination: $A^+ + e^- \rightleftharpoons A_p + hf$
5. Stimulated emission: $A_q + hf \rightarrow A_p + 2hf$
6. Associative ionisation: $A^{**} + B \rightleftharpoons AB^+ + e^-$
7. Penning ionisation: b.v. $Ne^* + Ar \rightleftharpoons Ar^+ + Ne + e^-$
8. Charge transfer: $A^+ + B \rightleftharpoons A + B^+$
9. Resonant charge transfer: $A^+ + A \rightleftharpoons A + A^+$

11.5.2 Cross sections

Collisions between an electron and an atom can be approximated by a collision between an electron and one of the electrons of that atom. This results in

$$\frac{d\sigma}{d(\Delta E)} = \frac{\pi Z^2 e^4}{(4\pi\epsilon_0)^2 E (\Delta E)^2}$$

Then follows for the transition $p \rightarrow q$: $\sigma_{pq}(E) = \frac{\pi Z^2 e^4 \Delta E_{q,q+1}}{(4\pi\epsilon_0)^2 E (\Delta E)_{pq}^2}$

For ionization from state p holds to a good approximation: $\sigma_p = 4\pi a_0^2 Ry \left(\frac{1}{E_p} - \frac{1}{E} \right) \ln \left(\frac{1.25\beta E}{E_p} \right)$

For resonant charge transfer holds: $\sigma_{ex} = \frac{A[1 - B \ln(E)]^2}{1 + CE^{3.3}}$

11.6 Radiation

In equilibrium holds for radiation processes:

$$\underbrace{n_p A_{pq}}_{\text{emission}} + \underbrace{n_p B_{pq} \rho(\nu, T)}_{\text{stimulated emission}} = \underbrace{n_q B_{qp} \rho(\nu, T)}_{\text{absorption}}$$

Here, A_{pq} is the matrix element of the transition $p \rightarrow q$, and is given by:

$$A_{pq} = \frac{8\pi^2 e^2 \nu^3 |r_{pq}|^2}{3\hbar\epsilon_0 c^3} \quad \text{with} \quad r_{pq} = \langle \psi_p | \vec{r} | \psi_q \rangle$$

For hydrogenic atoms holds: $A_p = 1.58 \cdot 10^8 Z^4 p^{-4.5}$, with $A_p = 1/\tau_p = \sum_q A_{pq}$. The intensity I of a line is given by $I_{pq} = hf A_{pq} n_p / 4\pi$. The Einstein coefficients B are given by:

$$B_{pq} = \frac{c^3 A_{pq}}{8\pi h \nu^3} \quad \text{and} \quad \frac{B_{pq}}{B_{qp}} = \frac{g_q}{g_p}$$

A spectral line is broadened by several mechanisms:

1. Because the states have a finite life time. The natural life time of a state p is given by $\tau_p = 1/\sum_q A_{pq}$.

From the uncertainty relation then follows: $\Delta(h\nu) \cdot \tau_p = \frac{1}{2}\hbar$, this gives

$$\Delta\nu = \frac{1}{4\pi\tau_p} = \frac{\sum_q A_{pq}}{4\pi}$$

The natural line width is usually \ll than the broadening due to the following two mechanisms:

2. The Doppler broadening is caused by the thermal motion of the particles:

$$\frac{\Delta\lambda}{\lambda} = \frac{2}{c} \sqrt{\frac{2 \ln(2) k T_i}{m_i}}$$

This broadening results in a Gaussian line profile:

$$k_\nu = k_0 \exp(-[2\sqrt{\ln 2}(\nu - \nu_0)/\Delta\nu_D]^2), \text{ with } k \text{ the coefficient of absorption or emission.}$$

3. The Stark broadening is caused by the electric field of the electrons:

$$\Delta\lambda_{1/2} = \left[\frac{n_e}{C(n_e, T_e)} \right]^{2/3}$$

with for the H- β line: $C(n_e, T_e) \approx 3 \cdot 10^{14} \text{ \AA}^{-3/2} \text{ cm}^{-3}$.

The natural broadening and the Stark broadening result in a Lorentz profile of a spectral line:

$k_\nu = \frac{1}{2} k_0 \Delta\nu_L / [(\frac{1}{2} \Delta\nu_L)^2 + (\nu - \nu_0)^2]$. The total line shape is a convolution of the Gauss- and Lorentz profile and is called a *Voigt profile*.

The number of transitions $p \rightarrow q$ is given by $n_p B_{pq} \rho$ and by $n_p n_{hf} \langle \sigma_a c \rangle = n_p (\rho d\nu / h\nu) \sigma_a c$ where $d\nu$ is the line width. Then follows for the cross section of absorption processes: $\sigma_a = B_{pq} h\nu / c d\nu$.

The background radiation in a plasma originates from two processes:

1. Free-Bound radiation, originating from radiative recombination. The emission is given by:

$$\varepsilon_{fb} = \frac{C_1}{\lambda^2} \frac{z_i n_i n_e}{\sqrt{k T_e}} \left[1 - \exp\left(-\frac{hc}{\lambda k T_e}\right) \right] \xi_{fb}(\lambda, T_e)$$

with $C_1 = 1.63 \cdot 10^{-43} \text{ W m}^4 \text{ K}^{1/2} \text{ sr}^{-1}$ and ξ the *Biberman factor*.

2. Free-free radiation, originating from the acceleration of particles in the EM-field of other particles:

$$\varepsilon_{ff} = \frac{C_1}{\lambda^2} \frac{z_i n_i n_e}{\sqrt{k T_e}} \exp\left(-\frac{hc}{\lambda k T_e}\right) \xi_{ff}(\lambda, T_e)$$

11.7 The Boltzmann transport equation

It is assumed that there exists a distribution function F for the plasma so that

$$F(\vec{r}, \vec{v}, t) = F_r(\vec{r}, t) \cdot F_v(\vec{v}, t) = F_1(x, t) F_2(y, t) F_3(z, t) F_4(v_x, t) F_5(v_y, t) F_6(v_z, t)$$

Then the BTE is: $\frac{dF}{dt} = \frac{\partial F}{\partial t} + \nabla_r \cdot (F \vec{v}) + \nabla_v \cdot (F \vec{a}) = \left(\frac{\partial F}{\partial t} \right)_{\text{coll-rad}}$

Assuming that v does not depend on r and a_i does not depend on v_i , holds $\nabla_r \cdot (F \vec{v}) = \vec{v} \cdot \nabla F$ and $\nabla_v \cdot (F \vec{a}) = \vec{a} \cdot \nabla_v F$. This is also true in magnetic fields because $\partial a_i / \partial x_i = 0$. The velocity is separated in a thermal velocity \vec{v}_t and a drift velocity \vec{w} . The total density is given by $n = \int F d\vec{v}$ and $\int \vec{v} F d\vec{v} = n \vec{w}$.

The balance equations can be derived by means of the moment method:

1. Mass balance: $\int (\text{BTE}) d\vec{v} \Rightarrow \frac{\partial n}{\partial t} + \nabla \cdot (n \vec{w}) = \left(\frac{\partial n}{\partial t} \right)_{\text{cr}}$
2. Momentum balance: $\int (\text{BTE}) m \vec{v} d\vec{v} \Rightarrow m n \frac{d\vec{w}}{dt} + \nabla \cdot \mathbf{T}' + \nabla p = m n \langle \vec{a} \rangle + \vec{R}$
3. Energy balance: $\int (\text{BTE}) m v^2 d\vec{v} \Rightarrow \frac{3}{2} \frac{dp}{dt} + \frac{5}{2} p \nabla \cdot \vec{w} + \nabla \cdot \vec{q} = Q$

Here, $\langle \vec{a} \rangle = e/m(\vec{E} + \vec{w} \times \vec{B})$ is the average acceleration, $\vec{q} = \frac{1}{2}nm \langle \vec{v}_t^2 \vec{v}_t \rangle$ the heat flow,

$Q = \int \frac{mv_t^2}{r} \left(\frac{\partial F}{\partial t} \right)_{\text{cr}} d\vec{v}$ the source term for energy production, \vec{R} is a friction term and $p = nkT$ the pressure.

A thermodynamic derivation gives for the total pressure: $p = nkT = \sum_i p_i - \frac{e^2(n_e + z_i n_i)}{24\pi\epsilon_0\lambda_D}$

For the electrical conductance in a plasma follows from the momentum balance, if $w_e \gg w_i$:

$$\eta \vec{J} = \vec{E} - \frac{\vec{J} \times \vec{B} + \nabla p_e}{en_e}$$

In a plasma where only elastic e-a collisions are important the equilibrium energy distribution function is the *Druyvesteyn distribution*:

$$N(E)dE = Cn_e \left(\frac{E}{E_0} \right)^{3/2} \exp \left[-\frac{3m_e}{m_0} \left(\frac{E}{E_0} \right)^2 \right] dE$$

with $E_0 = eE\lambda_v = eE/n\sigma$.

11.8 Collision-radiative models

These models are first-moment equations for excited states. One assumes the Quasi-steady-state solution is valid, where $\forall_{p>1} [(\partial n_p / \partial t = 0) \wedge (\nabla \cdot (n_p \vec{w}_p) = 0)]$. This results in:

$$\left(\frac{\partial n_{p>1}}{\partial t} \right)_{\text{cr}} = 0, \quad \frac{\partial n_1}{\partial t} + \nabla \cdot (n_1 \vec{w}_1) = \left(\frac{\partial n_1}{\partial t} \right)_{\text{cr}}, \quad \frac{\partial n_i}{\partial t} + \nabla \cdot (n_i \vec{w}_i) = \left(\frac{\partial n_i}{\partial t} \right)_{\text{cr}}$$

with solutions $n_p = r_p^0 n_p^S + r_p^1 n_p^B = b_p n_p^S$. Further holds for all collision-dominated levels that $\delta b_p := b_p - 1 = b_0 p_{\text{eff}}^{-x}$ with $p_{\text{eff}} = \sqrt{Ry/E_{pi}}$ and $5 \leq x \leq 6$. For systems in ESP, where only collisional (de)excitation between levels p and $p \pm 1$ is taken into account holds $x = 6$. Even in plasma's far from equilibrium the excited levels will eventually reach ESP, so from a certain level up the level densities can be calculated.

To find the population densities of the lower levels in the stationary case one has to start with a macroscopic equilibrium:

Number of populating processes of level p = Number of depopulating processes of level p ,

When this is expanded it becomes:

$$\underbrace{n_e \sum_{q<p} n_q K_{qp}}_{\text{coll. excit.}} + \underbrace{n_e \sum_{q>p} n_q K_{qp}}_{\text{coll. deexcit.}} + \underbrace{\sum_{q>p} n_q A_{qp}}_{\text{rad. deex. to}} + \underbrace{n_e^2 n_i K_{+p}}_{\text{coll. recomb.}} + \underbrace{n_e n_i \alpha_{\text{rad}}}_{\text{rad. recomb}} =$$

$$\underbrace{n_e n_p \sum_{q<p} K_{pq}}_{\text{coll. deexcit.}} + \underbrace{n_e n_p \sum_{q>p} K_{pq}}_{\text{coll. excit.}} + \underbrace{n_p \sum_{q<p} A_{pq}}_{\text{rad. deex. from}} + \underbrace{n_e n_p K_{p+}}_{\text{coll. ion.}}$$

11.9 Waves in plasma's

Interaction of electromagnetic waves in plasma's results in scattering and absorption of energy. For electromagnetic waves with complex wave number $k = \omega(n + i\kappa)/c$ in one dimension one finds:

$E_x = E_0 e^{-\kappa\omega x/c} \cos[\omega(t - nx/c)]$. The refractive index n is given by:

$$n = c \frac{k}{\omega} = \frac{c}{v_f} = \sqrt{1 - \frac{\omega_p^2}{\omega^2}}$$

For disturbances in the z -direction in a cold, homogeneous, magnetized plasma: $\vec{B} = B_0 \vec{e}_z + \vec{B} e^{i(kz - \omega t)}$ and $n = n_0 + \hat{n} e^{i(kz - \omega t)}$ (external E fields are screened) follows, with the definitions $\alpha = \omega_p/\omega$ and $\beta = \Omega/\omega$ and $\omega_p^2 = \omega_{pi}^2 + \omega_{pe}^2$:

$$\vec{J} = \vec{\sigma} \vec{E}, \text{ with } \vec{\sigma} = i\varepsilon_0 \omega \sum_s \alpha_s^2 \begin{pmatrix} \frac{1}{1 - \beta_s^2} & \frac{-i\beta_s}{1 - \beta_s^2} & 0 \\ \frac{i\beta_s}{1 - \beta_s^2} & \frac{1}{1 - \beta_s^2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

where the sum is taken over particle species s . The dielectric tensor \mathcal{E} , with property:

$$\vec{k} \cdot (\vec{\mathcal{E}} \cdot \vec{E}) = 0$$

is given by $\vec{\mathcal{E}} = \vec{I} - \vec{\sigma}/i\varepsilon_0 \omega$.

With the definitions $S = 1 - \sum_s \frac{\alpha_s^2}{1 - \beta_s^2}$, $D = \sum_s \frac{\alpha_s^2 \beta_s}{1 - \beta_s^2}$, $P = 1 - \sum_s \alpha_s^2$

follows:

$$\vec{\mathcal{E}} = \begin{pmatrix} S & -iD & 0 \\ iD & S & 0 \\ 0 & 0 & P \end{pmatrix}$$

The eigenvalues of this hermitian matrix are $R = S + D$, $L = S - D$, $\lambda_3 = P$, with eigenvectors $\vec{e}_r = \frac{1}{\sqrt{2}}(1, i, 0)$, $\vec{e}_l = \frac{1}{\sqrt{2}}(1, -i, 0)$ and $\vec{e}_3 = (0, 0, 1)$. \vec{e}_r is connected with a right rotating field for which $iE_x/E_y = 1$ and \vec{e}_l is connected with a left rotating field for which $iE_x/E_y = -1$. When k makes an angle θ with \vec{B} one finds:

$$\tan^2(\theta) = \frac{P(n^2 - R)(n^2 - L)}{S(n^2 - RL/S)(n^2 - P)}$$

where n is the refractive index. From this the following solutions can be obtained:

A. $\theta = 0$: transmission in the z -direction.

1. $P = 0$: $E_x = E_y = 0$. This describes a longitudinal linear polarized wave.
2. $n^2 = L$: a left, circular polarized wave.
3. $n^2 = R$: a right, circular polarized wave.

B. $\theta = \pi/2$: transmission \perp the B -field.

1. $n^2 = P$: the ordinary mode: $E_x = E_y = 0$. This is a transversal linear polarized wave.
2. $n^2 = RL/S$: the extraordinary mode: $iE_x/E_y = -D/S$, an elliptical polarized wave.

Resonance frequencies are frequencies for which $n^2 \rightarrow \infty$, so $v_f = 0$. For these holds: $\tan(\theta) = -P/S$. For $R \rightarrow \infty$ this gives the electron cyclotron resonance frequency $\omega = \Omega_e$, for $L \rightarrow \infty$ the ion cyclotron resonance frequency $\omega = \Omega_i$ and for $S = 0$ holds for the extraordinary mode:

$$\alpha^2 \left(1 - \frac{m_i}{m_e} \frac{\Omega_i^2}{\omega^2} \right) = \left(1 - \frac{m_i}{m_e} \frac{\Omega_i^2}{\omega^2} \right) \left(1 - \frac{\Omega_i^2}{\omega^2} \right)$$

Cut-off frequencies are frequencies for which $n^2 = 0$, so $v_f \rightarrow \infty$. For these holds: $P = 0$ or $R = 0$ or $L = 0$.

In the case that $\beta^2 \gg 1$ one finds Alfvén waves propagating parallel to the field lines. With the Alfvén velocity

$$v_A = \frac{\Omega_e \Omega_i}{\omega_{pe}^2 + \omega_{pi}^2} c^2$$

follows: $n = \sqrt{1 + c/v_A}$, and in case $v_A \ll c$: $\omega = kv_A$.

Chapter 12

Solid state physics

12.1 Crystal structure

A lattice is defined by the 3 translation vectors \vec{a}_i , so that the atomic composition looks the same from each point \vec{r} and $\vec{r}' = \vec{r} + \vec{T}$, where \vec{T} is a translation vector given by: $\vec{T} = u_1\vec{a}_1 + u_2\vec{a}_2 + u_3\vec{a}_3$ with $u_i \in \mathbb{N}$. A lattice can be constructed from primitive cells. As a primitive cell one can take a parallelepiped, with volume

$$V_{\text{cell}} = |\vec{a}_1 \cdot (\vec{a}_2 \times \vec{a}_3)|$$

Because a lattice has a periodical structure the physical properties which are connected with the lattice have the same periodicity (neglecting boundary effects):

$$n_e(\vec{r} + \vec{T}) = n_e(\vec{r})$$

This periodicity is suitable to use Fourier analysis: $n(\vec{r})$ is expanded as:

$$n(\vec{r}) = \sum_{\vec{G}} n_{\vec{G}} \exp(i\vec{G} \cdot \vec{r})$$

with

$$n_{\vec{G}} = \frac{1}{V_{\text{cell}}} \iiint_{\text{cell}} n(\vec{r}) \exp(-i\vec{G} \cdot \vec{r}) dV$$

\vec{G} is the *reciprocal lattice vector*. If \vec{G} is written as $\vec{G} = v_1\vec{b}_1 + v_2\vec{b}_2 + v_3\vec{b}_3$ with $v_i \in \mathbb{N}$, it follows for the vectors \vec{b}_i , cyclically:

$$\vec{b}_i = 2\pi \frac{\vec{a}_{i+1} \times \vec{a}_{i+2}}{\vec{a}_i \cdot (\vec{a}_{i+1} \times \vec{a}_{i+2})}$$

The set of \vec{G} -vectors determines the Röntgen diffractions: a maximum in the reflected radiation occurs if: $\Delta\vec{k} = \vec{G}$ with $\Delta\vec{k} = \vec{k} - \vec{k}'$. So: $2\vec{k} \cdot \vec{G} = G^2$. From this follows for parallel lattice planes (Bragg reflection) that for the maxima holds: $2d \sin(\theta) = n\lambda$.

The Brillouin zone is defined as a Wigner-Seitz cell in the reciprocal lattice.

12.2 Crystal binding

A distinction can be made between 4 binding types:

1. Van der Waals bond
2. Ion bond
3. Covalent or homopolar bond
4. Metallic bond.

For the ion binding of NaCl the energy per molecule is calculated by:

$$E = \text{cohesive energy}(\text{NaCl}) - \text{ionization energy}(\text{Na}) + \text{electron affinity}(\text{Cl})$$

The interaction in a covalent bond depends on the relative spin orientations of the electrons constituting the bond. The potential energy for two parallel spins is higher than the potential energy for two antiparallel spins. Furthermore the potential energy for two parallel spins has sometimes no minimum. In that case binding is not possible.

12.3 Crystal vibrations

12.3.1 A lattice with one type of atoms

In this model for crystal vibrations only nearest-neighbour interactions are taken into account. The force on atom s with mass M can then be written as:

$$F_s = M \frac{d^2 u_s}{dt^2} = C(u_{s+1} - u_s) + C(u_{s-1} - u_s)$$

Assuming that all solutions have the same time-dependence $\exp(-i\omega t)$ this results in:

$$-M\omega^2 u_s = C(u_{s+1} + u_{s-1} - 2u_s)$$

Further it is postulated that: $u_{s\pm 1} = u \exp(isKa) \exp(\pm iKa)$.

This gives: $u_s = \exp(iKsa)$. Substituting the later two equations in the first results in a system of linear equations, which has only a solution if their determinant is 0. This gives:

$$\omega^2 = \frac{4C}{M} \sin^2(\frac{1}{2}Ka)$$

Only vibrations with a wavelength within the first Brillouin Zone have a physical significance. This requires that $-\pi < Ka \leq \pi$.

The group velocity of these vibrations is given by:

$$v_g = \frac{d\omega}{dK} = \sqrt{\frac{Ca^2}{M}} \cos(\frac{1}{2}Ka) .$$

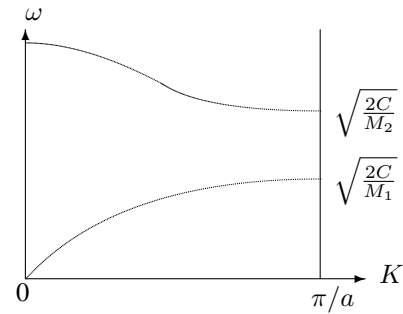
and is 0 on the edge of a Brillouin Zone. Here, there is a standing wave.

12.3.2 A lattice with two types of atoms

Now the solutions are:

$$\omega^2 = C \left(\frac{1}{M_1} + \frac{1}{M_2} \right) \pm C \sqrt{\left(\frac{1}{M_1} + \frac{1}{M_2} \right)^2 - \frac{4 \sin^2(Ka)}{M_1 M_2}}$$

Connected with each value of K are two values of ω , as can be seen in the graph. The upper line describes the optical branch, the lower line the acoustical branch. In the optical branch, both types of ions oscillate in opposite phases, in the acoustical branch they oscillate in the same phase. This results in a much larger induced dipole moment for optical oscillations, and also a stronger emission and absorption of radiation. Furthermore each branch has 3 polarization directions, one longitudinal and two transversal.



12.3.3 Phonons

The quantum mechanical excitation of a crystal vibration with an energy $\hbar\omega$ is called a *phonon*. Phonons can be viewed as quasi-particles: with collisions, they behave as particles with momentum $\hbar K$. Their total momentum is 0. When they collide, their momentum need not be conserved: for a normal process holds: $K_1 + K_2 = K_3$, for an umklapp process holds: $K_1 + K_2 = K_3 + G$. Because phonons have no spin they behave like bosons.

12.3.4 Thermal heat capacity

The total energy of the crystal vibrations can be calculated by multiplying each mode with its energy and sum over all branches K and polarizations P :

$$U = \sum_K \sum_P \hbar\omega \langle n_{k,p} \rangle = \sum_\lambda \int D_\lambda(\omega) \frac{\hbar\omega}{\exp(\hbar\omega/kT) - 1} d\omega$$

for a given polarization λ . The thermal heat capacity is then:

$$C_{\text{lattice}} = \frac{\partial U}{\partial T} = k \sum_\lambda \int D(\omega) \frac{(\hbar\omega/kT)^2 \exp(\hbar\omega/kT)}{(\exp(\hbar\omega/kT) - 1)^2} d\omega$$

The dispersion relation in one dimension is given by:

$$D(\omega)d\omega = \frac{L}{\pi} \frac{dK}{d\omega} d\omega = \frac{L}{\pi} \frac{d\omega}{v_g}$$

In three dimensions one applies periodic boundary conditions to a cube with N^3 primitive cells and a volume L^3 : $\exp(i(K_x x + K_y y + K_z z)) \equiv \exp(i(K_x(x+L) + K_y(y+L) + K_z(z+L)))$.

Because $\exp(2\pi i) = 1$ this is only possible if:

$$K_x, K_y, K_z = 0; \pm \frac{2\pi}{L}; \pm \frac{4\pi}{L}; \pm \frac{6\pi}{L}; \dots \pm \frac{2N\pi}{L}$$

So there is only one allowed value of \vec{K} per volume $(2\pi/L)^3$ in K -space, or:

$$\left(\frac{L}{2\pi}\right)^3 = \frac{V}{8\pi^3}$$

allowed \vec{K} -values per unit volume in \vec{K} -space, for each polarization and each branch. The total number of states with a wave vector $< K$ is:

$$N = \left(\frac{L}{2\pi}\right)^3 \frac{4\pi K^3}{3}$$

for each polarization. The density of states for each polarization is, according to the Einstein model:

$$D(\omega) = \frac{dN}{d\omega} = \left(\frac{V K^2}{2\pi^2}\right) \frac{dK}{d\omega} = \frac{V}{8\pi^3} \iint \frac{dA_\omega}{v_g}$$

The *Debye model* for thermal heat capacities is a low-temperature approximation which is valid up to $\approx 50\text{K}$. Here, only the acoustic phonons are taken into account (3 polarizations), and one assumes that $v = \omega K$, independent of the polarization. From this follows: $D(\omega) = V\omega^2/2\pi^2 v^3$, where v is the speed of sound. This gives:

$$U = 3 \int D(\omega) \langle n \rangle \hbar\omega d\omega = \int_0^{\omega_D} \frac{V\omega^2}{2\pi^2 v^3} \frac{\hbar\omega}{\exp(\hbar\omega/kT) - 1} d\omega = \frac{3V k^2 T^4}{2\pi^2 v^3 \hbar^3} \int_0^{x_D} \frac{x^3 dx}{e^x - 1}.$$

Here, $x_D = \hbar\omega_D/kT = \theta_D/T$. θ_D is the *Debye temperature* and is defined by:

$$\theta_D = \frac{\hbar v}{k} \left(\frac{6\pi^2 N}{V}\right)^{1/3}$$

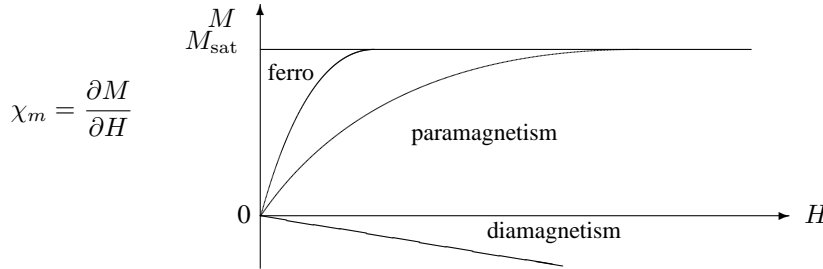
where N is the number of primitive cells. Because $x_D \rightarrow \infty$ for $T \rightarrow 0$ it follows from this:

$$U = 9NkT \left(\frac{T}{\theta_D}\right)^3 \int_0^\infty \frac{x^3 dx}{e^x - 1} = \frac{3\pi^4 NkT^4}{5\theta_D} \sim T^4 \quad \text{and} \quad C_V = \frac{12\pi^4 NkT^3}{5\theta_D^3} \sim T^3$$

In the Einstein model for the thermal heat capacity one considers only phonons at one frequency, an approximation for optical phonons.

12.4 Magnetic field in the solid state

The following graph shows the magnetization versus fieldstrength for different types of magnetism:



12.4.1 Dielectrics

The quantum mechanical origin of diamagnetism is the Larmorprecession of the spin of the electron. Starting with a circular electron orbit in an atom with two electrons, there is a Coulomb force F_c and a magnetic force on each electron. If the magnetic part of the force is not strong enough to significantly deform the orbit holds:

$$\omega^2 = \frac{F_c(r)}{mr} \pm \frac{eB}{m}\omega = \omega_0^2 \pm \frac{eB}{m}(\omega_0 + \delta) \Rightarrow \omega = \sqrt{\left(\omega_0 \pm \frac{eB}{2m}\right)^2 + \dots} \approx \omega_0 \pm \frac{eB}{2m} = \omega_0 \pm \omega_L$$

Here, ω_L is the *Larmor frequency*. One electron is accelerated, the other decelerated. Hence there is a net circular current which results in a magnetic moment $\vec{\mu}$. The circular current is given by $I = -Ze\omega_L/2\pi$, and $\langle \mu \rangle = IA = I\pi \langle \rho^2 \rangle = \frac{2}{3}I\pi \langle r^2 \rangle$. If N is the number of atoms in the crystal it follows for the susceptibility, with $\vec{M} = \vec{\mu}N$:

$$\chi = \frac{\mu_0 M}{B} = -\frac{\mu_0 N Z e^2}{6m} \langle r^2 \rangle$$

12.4.2 Paramagnetism

Starting with the splitting of energy levels in a weak magnetic field: $\Delta U_m - \vec{\mu} \cdot \vec{B} = m_J g \mu_B B$, and with a distribution $f_m \sim \exp(-\Delta U_m/kT)$, one finds for the average magnetic moment $\langle \mu \rangle = \sum f_m \mu / \sum f_m$. After linearization and because $\sum m_J = 0$, $\sum J = 2J + 1$ and $\sum m_J^2 = \frac{2}{3}J(J+1)(J + \frac{1}{2})$ it follows that:

$$\chi_p = \frac{\mu_0 M}{B} = \frac{\mu_0 N \langle \mu \rangle}{B} = \frac{\mu_0 J(J+1)g^2 \mu_B^2 N}{3kT}$$

This is the *Curie law*, $\chi_p \sim 1/T$.

12.4.3 Ferromagnetism

A ferromagnet behaves like a paramagnet above a critical temperature T_c . To describe ferromagnetism a field B_E parallel with M is postulated: $\vec{B}_E = \lambda \mu_0 \vec{M}$. From there the treatment is analogous to the paramagnetic case:

$$\mu_0 M = \chi_p (B_a + B_E) = \chi_p (B_a + \lambda \mu_0 M) = \mu_0 \left(1 - \lambda \frac{C}{T}\right) M$$

From this follows for a ferromagnet: $\chi_F = \frac{\mu_0 M}{B_a} = \frac{C}{T - T_c}$ which is *Weiss-Curie's law*.

If B_E is estimated this way it results in values of about 1000 T. This is clearly unrealistic and suggests another mechanism. A quantum mechanical approach from Heisenberg postulates an interaction between two neighbouring atoms: $U = -2J \vec{S}_i \cdot \vec{S}_j \equiv -\vec{\mu} \cdot \vec{B}_E$. J is an overlap integral given by: $J = 3kT_c/2zS(S+1)$, with z the number of neighbours. A distinction between 2 cases can now be made:

1. $J > 0$: S_i and S_j become parallel: the material is a ferromagnet.

2. $J < 0$: S_i and S_j become antiparallel: the material is an antiferromagnet.

Heisenberg's theory predicts quantized spin waves: magnons. Starting from a model with only nearest neighbouring atoms interacting one can write:

$$U = -2J\vec{S}_p \cdot (\vec{S}_{p-1} + \vec{S}_{p+1}) \approx \vec{\mu}_p \cdot \vec{B}_p \quad \text{with} \quad \vec{B}_p = \frac{-2J}{g\mu_B}(\vec{S}_{p-1} + \vec{S}_{p+1})$$

The equation of motion for the magnons becomes: $\frac{d\vec{S}}{dt} = \frac{2J}{\hbar}\vec{S}_p \times (\vec{S}_{p-1} + \vec{S}_{p+1})$

From here the treatment is analogous to phonons: postulate traveling waves of the type $\vec{S}_p = \vec{u} \exp(i(pka - \omega t))$. This results in a system of linear equations with solution:

$$\hbar\omega = 4JS(1 - \cos(ka))$$

12.5 Free electron Fermi gas

12.5.1 Thermal heat capacity

The solution with period L of the one-dimensional Schrödinger equation is: $\psi_n(x) = A \sin(2\pi x/\lambda_n)$ with $n\lambda_n = 2L$. From this follows

$$E = \frac{\hbar^2}{2m} \left(\frac{n\pi}{L} \right)^2$$

In a linear lattice the only important quantum numbers are n and m_s . The *Fermi level* is the uppermost filled level in the ground state, which has the *Fermi-energy* E_F . If n_F is the quantum number of the Fermi level, it can be expressed as: $2n_F = N$ so $E_F = \hbar^2\pi^2 N^2/8mL$. In 3 dimensions holds:

$$k_F = \left(\frac{3\pi^2 N}{V} \right)^{1/3} \quad \text{and} \quad E_F = \frac{\hbar^2}{2m} \left(\frac{3\pi^2 N}{V} \right)^{2/3}$$

The number of states with energy $\leq E$ is then: $N = \frac{V}{3\pi^2} \left(\frac{2mE}{\hbar^2} \right)^{3/2}$.

and the density of states becomes: $D(E) = \frac{dN}{dE} = \frac{V}{2\pi^2} \left(\frac{2m}{\hbar^2} \right)^{3/2} \sqrt{E} = \frac{3N}{2E}$.

The heat capacity of the electrons is approximately 0.01 times the classical expected value $\frac{3}{2}Nk$. This is caused by the Pauli exclusion principle and the Fermi-Dirac distribution: only electrons within an energy range $\sim kT$ of the Fermi level are excited thermally. There is a fraction $\approx T/T_F$ excited thermally. The internal energy then becomes:

$$U \approx NkT \frac{T}{T_F} \quad \text{and} \quad C = \frac{\partial U}{\partial T} \approx Nk \frac{T}{T_F}$$

A more accurate analysis gives: $C_{\text{electrons}} = \frac{1}{2}\pi^2 NkT/T_F \sim T$. Together with the T^3 dependence of the thermal heat capacity of the phonons the total thermal heat capacity of metals is described by: $C = \gamma T + AT^3$.

12.5.2 Electric conductance

The equation of motion for the charge carriers is: $\vec{F} = m d\vec{v}/dt = \hbar d\vec{k}/dt$. The variation of \vec{k} is given by $\delta\vec{k} = \vec{k}(t) - \vec{k}(0) = -e\vec{E}t/\hbar$. If τ is the characteristic collision time of the electrons, $\delta\vec{k}$ remains stable if $t = \tau$. Then holds: $\langle \vec{v} \rangle = \mu \vec{E}$, with $\mu = e\tau/m$ the *mobility* of the electrons.

The current in a conductor is given by: $\vec{J} = nq\vec{v} = \sigma \vec{E} = \vec{E}/\rho = ne\mu \vec{E}$. Because for the collision time holds: $1/\tau = 1/\tau_L + 1/\tau_i$, where τ_L is the collision time with the lattice phonons and τ_i the collision time with the impurities follows for the resistivity $\rho = \rho_L + \rho_i$, with $\lim_{T \rightarrow 0} \rho_L = 0$.

12.5.3 The Hall-effect

If a magnetic field is applied \perp to the direction of the current the charge carriers will be pushed aside by the Lorentz force. This results in a magnetic field \perp to the flow direction of the current. If $\vec{J} = J\vec{e}_x$ and $\vec{B} = B\vec{e}_z$ than $E_y/E_x = \mu B$. The Hall coefficient is defined by: $R_H = E_y/J_x B$, and $R_H = -1/ne$ if $J_x = ne\mu E_x$. The Hall voltage is given by: $V_H = Bvb = IB/neh$ where b is the width of the material and h de height.

12.5.4 Thermal heat conductivity

With $\ell = v_F\tau$ the mean free path of the electrons follows from $\kappa = \frac{1}{3}C\langle v\rangle\ell$: $\kappa_{\text{electrons}} = \pi^2 nk^2 T\tau/3m$. From this follows for the *Wiedemann-Franz ratio*: $\kappa/\sigma = \frac{1}{3}(\pi k/e)^2 T$.

12.6 Energy bands

In the *tight-bond* approximation it is assumed that $\psi = e^{ikna}\phi(x - na)$. From this follows for the energy: $\langle E \rangle = \langle \psi | H | \psi \rangle = E_{\text{at}} - \alpha - 2\beta \cos(ka)$. So this gives a cosine superimposed on the atomic energy, which can often be approximated by a harmonic oscillator. If it is assumed that the electron is nearly free one can postulate: $\psi = \exp(ik \cdot \vec{r})$. This is a traveling wave. This wave can be decomposed into two standing waves:

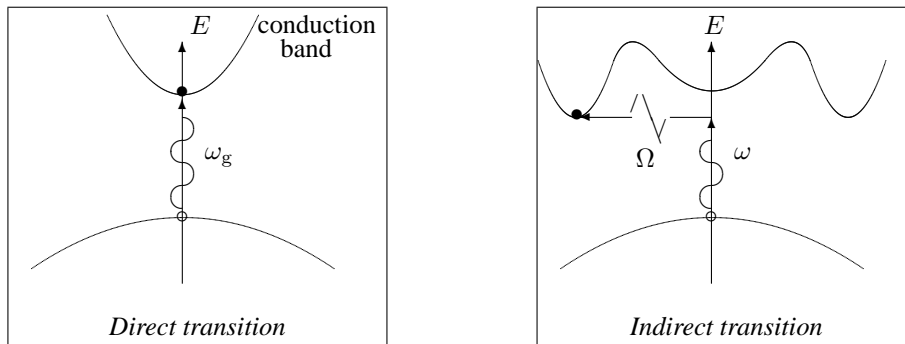
$$\begin{aligned}\psi(+) &= \exp(i\pi x/a) + \exp(-i\pi x/a) = 2\cos(\pi x/a) \\ \psi(-) &= \exp(i\pi x/a) - \exp(-i\pi x/a) = 2i\sin(\pi x/a)\end{aligned}$$

The probability density $|\psi(+)|^2$ is high near the atoms of the lattice and low in between. The probability density $|\psi(-)|^2$ is low near the atoms of the lattice and high in between. Hence the energy of $\psi(+)$ is also lower than the energy of $\psi(-)$. Suppose that $U(x) = U \cos(2\pi x/a)$, than the bandgap is given by:

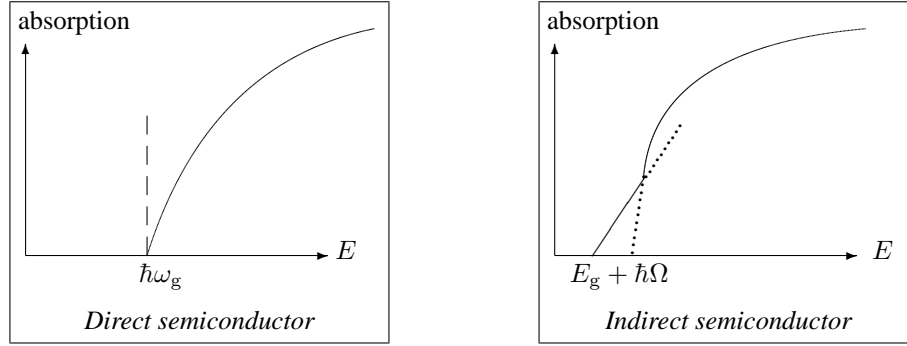
$$E_{\text{gap}} = \int_0^1 U(x) [|\psi(+)|^2 - |\psi(-)|^2] dx = U$$

12.7 Semiconductors

The band structures and the transitions between them of direct and indirect semiconductors are shown in the figures below. Here it is assumed that the momentum of the absorbed photon can be neglected. For an indirect semiconductor a transition from the valence- to the conduction band is also possible if the energy of the absorbed photon is smaller than the band gap: then, also a phonon is absorbed.



This difference can also be observed in the absorption spectra:



So indirect semiconductors, like Si and Ge, cannot emit any light and are therefore not usable to fabricate lasers. When light is absorbed holds: $\vec{k}_h = -\vec{k}_e$, $E_h(\vec{k}_h) = -E_e(\vec{k}_e)$, $\vec{v}_h = \vec{v}_e$ and $m_h = -m_e^*$ if the conduction band and the valence band have the same structure.

Instead of the normal electron mass one has to use the *effective mass* within a lattice. It is defined by:

$$m^* = \frac{F}{a} = \frac{dp/dt}{dv_g/dt} = \hbar \frac{dK}{dv_g} = \hbar^2 \left(\frac{d^2 E}{dk^2} \right)^{-1}$$

with $E = \hbar\omega$ and $v_g = d\omega/dk$ and $p = \hbar k$.

With the distribution function $f_e(E) \approx \exp((\mu - E)/kT)$ for the electrons and $f_h(E) = 1 - f_e(E)$ for the holes the density of states is given by:

$$D(E) = \frac{1}{2\pi^2} \left(\frac{2m^*}{\hbar^2} \right)^{3/2} \sqrt{E - E_c}$$

with E_c the energy at the edge of the conductance band. From this follows for the concentrations of the holes p and the electrons n :

$$n = \int_{E_c}^{\infty} D_e(E) f_e(E) dE = 2 \left(\frac{m^* kT}{2\pi\hbar^2} \right)^{3/2} \exp \left(\frac{\mu - E_c}{kT} \right)$$

For the product np follows: $np = 4 \left(\frac{kT}{2\pi\hbar^2} \right)^3 \sqrt{m_e^* m_h} \exp \left(-\frac{E_g}{kT} \right)$

For an intrinsic (no impurities) semiconductor holds: $n_i = p_i$, for a n -type holds: $n > p$ and in a p -type holds: $n < p$.

An exciton is a bound electron-hole pair, rotating on each other as in positronium. The excitation energy of an exciton is smaller than the bandgap because the energy of an exciton is lower than the energy of a free electron and a free hole. This causes a peak in the absorption just under E_g .

12.8 Superconductivity

12.8.1 Description

A superconductor is characterized by a zero resistivity if certain quantities are smaller than some critical values: $T < T_c$, $I < I_c$ and $H < H_c$. The BCS-model predicts for the transition temperature T_c :

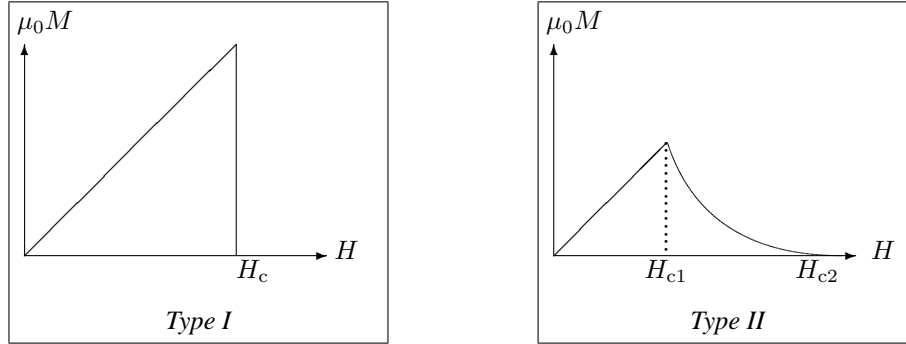
$$T_c = 1.14\Theta_D \exp \left(\frac{-1}{UD(E_F)} \right)$$

while experiments find for H_c approximately:

$$H_c(T) \approx H_c(T_c) \left(1 - \frac{T^2}{T_c^2} \right).$$

Within a superconductor the magnetic field is 0: the *Meissner effect*.

There are type I and type II superconductors. Because the Meissner effect implies that a superconductor is a perfect diamagnet holds in the superconducting state: $\vec{H} = \mu_0 \vec{M}$. This holds for a type I superconductor, for a type II superconductor this only holds to a certain value H_{c1} , for higher values of H the superconductor is in a *vortex state* to a value H_{c2} , which can be 100 times H_{c1} . If H becomes larger than H_{c2} the superconductor becomes a normal conductor. This is shown in the figures below.



The transition to a superconducting state is a second order thermodynamic state transition. This means that there is a twist in the $T - S$ diagram and a discontinuity in the $C_X - T$ diagram.

12.8.2 The Josephson effect

For the Josephson effect one considers two superconductors, separated by an insulator. The electron wavefunction in one superconductor is ψ_1 , in the other ψ_2 . The Schrödinger equations in both superconductors is set equal:

$$i\hbar \frac{\partial \psi_1}{\partial t} = \hbar T \psi_2, \quad i\hbar \frac{\partial \psi_2}{\partial t} = \hbar T \psi_1$$

$\hbar T$ is the effect of the coupling of the electrons, or the transfer interaction through the insulator. The electron wavefunctions are written as $\psi_1 = \sqrt{n_1} \exp(i\theta_1)$ and $\psi_2 = \sqrt{n_2} \exp(i\theta_2)$. Because a Cooper pair exist of *two* electrons holds: $\psi \sim \sqrt{n}$. From this follows, if $n_1 \approx n_2$:

$$\frac{\partial \theta_1}{\partial t} = \frac{\partial \theta_2}{\partial t} \quad \text{and} \quad \frac{\partial n_2}{\partial t} = -\frac{\partial n_1}{\partial t}$$

The Josephson effect results in a current density through the insulator depending on the phase difference as: $J = J_0 \sin(\theta_2 - \theta_1) = J_0 \sin(\delta)$, where $J_0 \sim T$. With an AC-voltage across the junction the Schrödinger equations become:

$$i\hbar \frac{\partial \psi_1}{\partial t} = \hbar T \psi_2 - eV \psi_1 \quad \text{and} \quad i\hbar \frac{\partial \psi_2}{\partial t} = \hbar T \psi_1 + eV \psi_2$$

This gives: $J = J_0 \sin \left(\theta_2 - \theta_1 - \frac{2eVt}{\hbar} \right)$.

Hence there is an oscillation with $\omega = 2eV/\hbar$.

12.8.3 Flux quantisation in a superconducting ring

For the current density in general holds: $\vec{J} = q\psi^* \vec{v} \psi = \frac{nq}{m} [\hbar \vec{\nabla} \theta - q \vec{A}]$

From the Meissner effect, $\vec{B} = 0$ and $\vec{J} = 0$, follows: $\hbar \vec{\nabla} \theta = q \vec{A} \Rightarrow \oint \vec{\nabla} \theta dl = \theta_2 - \theta_1 = 2\pi s$ with $s \in \mathbb{N}$. Because: $\oint \vec{A} dl = \iint (\text{rot} \vec{A}, \vec{n}) d\sigma = \iint (\vec{B}, \vec{n}) d\sigma = \Psi$ follows: $\Psi = 2\pi \hbar s / q$. The size of a flux quantum follows by setting $s = 1$: $\Psi = 2\pi \hbar / e = 2.0678 \cdot 10^{-15} \text{ Tm}^2$.

12.8.4 Macroscopic quantum interference

From $\theta_2 - \theta_1 = 2e\Psi/\hbar$ follows for two parallel junctions: $\delta_b - \delta_a = \frac{2e\Psi}{\hbar}$, so

$$J = J_a + J_b = 2J_0 \sin\left(\delta_0 \cos\left(\frac{e\Psi}{\hbar}\right)\right) \quad \text{This gives maxima if } e\Psi/\hbar = s\pi.$$

12.8.5 The London equation

A current density in a superconductor proportional to the vector potential \vec{A} is postulated:

$$\vec{J} = \frac{-\vec{A}}{\mu_0 \lambda_L^2} \quad \text{or} \quad \text{rot} \vec{J} = \frac{-\vec{B}}{\mu_0 \lambda_L^2}$$

where $\lambda_L = \sqrt{\varepsilon_0 m c^2 / n q^2}$. From this follows: $\nabla^2 \vec{B} = \vec{B} / \lambda_L^2$.

The Meissner effect is the solution of this equation: $\vec{B}(x) = B_0 \exp(-x/\lambda_L)$. Magnetic fields within a superconductor drop exponentially.

12.8.6 The BCS model

The BCS model can explain superconductivity in metals. (So far there is no explanation for high- T_c superconductance).

A new ground state where the electrons behave like independent fermions is postulated. Because of the interaction with the lattice these pseudo-particles exhibit a mutual attraction. This causes two electrons with opposite spin to combine to a *Cooper pair*. It can be proved that this ground state is perfect diamagnetic.

The infinite conductivity is more difficult to explain because a ring with a persisting current is not a real equilibrium: a state with zero current has a lower energy. Flux quantization prevents transitions between these states. Flux quantization is related to the existence of a coherent many-particle wavefunction. A flux quantum is the equivalent of about 10^4 electrons. So if the flux has to change with one flux quantum there has to occur a transition of many electrons, which is very improbable, or the system must go through intermediary states where the flux is not quantized so they have a higher energy. This is also very improbable.

Some useful mathematical relations are:

$$\int_0^\infty \frac{x dx}{e^{ax} + 1} = \frac{\pi^2}{12a^2}, \quad \int_{-\infty}^\infty \frac{x^2 dx}{(e^x + 1)^2} = \frac{\pi^2}{3}, \quad \int_0^\infty \frac{x^3 dx}{e^x + 1} = \frac{\pi^4}{15}$$

And, when $\sum_{n=0}^\infty (-1)^n = \frac{1}{2}$ follows: $\int_0^\infty \sin(px) dx = \int_0^\infty \cos(px) dx = \frac{1}{p}.$

Chapter 13

Theory of groups

13.1 Introduction

13.1.1 Definition of a group

\mathcal{G} is a group for the operation \bullet if:

1. $\forall A, B \in \mathcal{G} \Rightarrow A \bullet B \in \mathcal{G}$: \mathcal{G} is *closed*.
2. $\forall A, B, C \in \mathcal{G} \Rightarrow (A \bullet B) \bullet C = A \bullet (B \bullet C)$: \mathcal{G} obeys the *associative law*.
3. $\exists E \in \mathcal{G}$ so that $\forall A \in \mathcal{G} A \bullet E = E \bullet A = A$: \mathcal{G} has a *unit element*.
4. $\forall A \in \mathcal{G} \exists A^{-1} \in \mathcal{G}$ so that $A \bullet A^{-1} = E$: Each element in \mathcal{G} has an *inverse*.

If also holds:

5. $\forall A, B \in \mathcal{G} \Rightarrow A \bullet B = B \bullet A$ the group is called *Abelian* or *commutative*.

13.1.2 The Cayley table

Each element arises only once in each row and column of the Cayley- or multiplication table: because $E A_i = A_k^{-1} (A_k A_i) = A_i$ each A_i appears once. There are h positions in each row and column when there are h elements in the group so each element appears only once.

13.1.3 Conjugated elements, subgroups and classes

B is *conjugate* to A if $\exists X \in \mathcal{G}$ such that $B = X A X^{-1}$. Then A is also conjugate to B because $B = (X^{-1}) A (X^{-1})^{-1}$.

If B and C are conjugate to A , B is also conjugate with C .

A *subgroup* is a subset of \mathcal{G} which is also a group w.r.t. the same operation.

A *conjugacy class* is the maximum collection of conjugated elements. Each group can be split up in conjugacy classes. Some theorems:

- All classes are completely disjoint.
- E is a class itself: for each other element in this class would hold: $A = X E X^{-1} = E$.
- E is the only class which is also a subgroup because all other classes have no unit element.
- In an Abelian group each element is a separate class.

The physical interpretation of classes: elements of a group are usually symmetry operations which map a symmetrical object into itself. Elements of one class are then the same kind of operations. The opposite need not to be true.

13.1.4 Isomorfism and homomorfism; representations

Two groups are *isomorphic* if they have the same multiplication table. The mapping from group \mathcal{G}_1 to \mathcal{G}_2 , so that the multiplication table remains the same is a homomorphic mapping. It need not be isomorphic.

A *representation* is a homomorphic mapping of a group to a group of square matrices with the usual matrix multiplication as the combining operation. This is symbolized by Γ . The following holds:

$$\Gamma(E) = \mathbb{I} \quad , \quad \Gamma(AB) = \Gamma(A)\Gamma(B) \quad , \quad \Gamma(A^{-1}) = [\Gamma(A)]^{-1}$$

For each group there are 3 possibilities for a representation:

1. A *faithful* representation: all matrices are different.
2. The representation $A \rightarrow \det(\Gamma(A))$.
3. The identical representation: $A \rightarrow 1$.

An *equivalent representation* is obtained by performing an unitary base transformation: $\Gamma'(A) = S^{-1}\Gamma(A)S$.

13.1.5 Reducible and irreducible representations

If *the same* unitary transformation can bring all matrices of a representation Γ in the same block structure the representation is called *reducible*:

$$\Gamma(A) = \begin{pmatrix} \Gamma^{(1)}(A) & 0 \\ 0 & \Gamma^{(2)}(A) \end{pmatrix}$$

This is written as: $\Gamma = \Gamma^{(1)} \oplus \Gamma^{(2)}$. If this is not possible the representation is called *irreducible*.

The number of irreducible representations equals the number of conjugacy classes.

13.2 The fundamental orthogonality theorem

13.2.1 Schur's lemma

Lemma: Each matrix which commutes with all matrices of an irreducible representation is a constant $\times \mathbb{I}$, where \mathbb{I} is the unit matrix. The opposite is (of course) also true.

Lemma: If there exists a matrix M so that for two irreducible representations of group \mathcal{G} , $\gamma^{(1)}(A_i)$ and $\gamma^{(2)}(A_i)$, holds: $M\gamma^{(1)}(A_i) = \gamma^{(2)}(A_i)M$, then the representations are equivalent, or $M = \underline{\underline{0}}$.

13.2.2 The fundamental orthogonality theorem

For a set of unequivalent, irreducible, unitary representations holds that, if h is the number of elements in the group and ℓ_i is the dimension of the i -th representation:

$$\sum_{R \in \mathcal{G}} \Gamma_{\mu\nu}^{(i)*}(R) \Gamma_{\alpha\beta}^{(j)}(R) = \frac{h}{\ell_i} \delta_{ij} \delta_{\mu\alpha} \delta_{\nu\beta}$$

13.2.3 Character

The *character* of a representation is given by the trace of the matrix and is therefore invariant for base transformations: $\chi^{(j)}(R) = \text{Tr}(\Gamma^{(j)}(R))$

Also holds, with N_k the number of elements in a conjugacy class: $\sum_k \chi^{(i)*}(C_k) \chi^{(j)}(C_k) N_k = h \delta_{ij}$

Theorem: $\sum_{i=1}^n \ell_i^2 = h$

13.3 The relation with quantum mechanics

13.3.1 Representations, energy levels and degeneracy

Consider a set of symmetry transformations $\vec{x}' = R\vec{x}$ which leave the Hamiltonian \mathcal{H} invariant. These transformations are a group. An isomorphic operation on the wavefunction is given by: $P_R\psi(\vec{x}) = \psi(R^{-1}\vec{x})$. This is considered an *active rotation*. These operators commute with \mathcal{H} : $P_R\mathcal{H} = \mathcal{H}P_R$, and leave the volume element unchanged: $d(R\vec{x}) = d\vec{x}$.

P_R is the symmetry group of the physical system. It causes degeneracy: if ψ_n is a solution of $\mathcal{H}\psi_n = E_n\psi_n$ then also holds: $\mathcal{H}(P_R\psi_n) = E_n(P_R\psi_n)$. A degeneracy which is not the result of a symmetry is called an *accidental degeneracy*.

Assume an ℓ_n -fold degeneracy at E_n : then choose an orthonormal set $\psi_\nu^{(n)}$, $\nu = 1, 2, \dots, \ell_n$. The function

$$P_R\psi_\nu^{(n)} \text{ is in the same subspace: } P_R\psi_\nu^{(n)} = \sum_{\kappa=1}^{\ell_n} \psi_\kappa^{(n)} \Gamma_{\kappa\nu}^{(n)}(R)$$

where $\Gamma^{(n)}$ is an *irreducible, unitary* representation of the symmetry group \mathcal{G} of the system. Each n corresponds with another energy level. One can purely mathematical derive irreducible representations of a symmetry group and label the energy levels with a quantum number this way. A fixed choice of $\Gamma^{(n)}(R)$ defines the base functions $\psi_\nu^{(n)}$. This way one can also label each separate base function with a quantum number.

Particle in a periodical potential: the symmetry operation is a cyclic group: note the operator describing one translation over one unit as A . Then: $\mathcal{G} = \{A, A^2, A^3, \dots, A^h = E\}$.

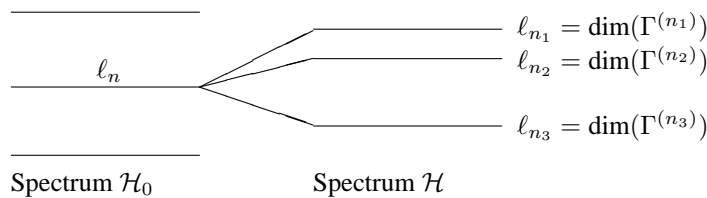
The group is Abelian so all irreducible representations are one-dimensional. For $0 \leq p \leq h-1$ follows:

$$\Gamma^{(p)}(A^n) = e^{2\pi i p n / h}$$

If one defines: $k = -\frac{2\pi p}{ah} \left(\text{mod } \frac{2\pi}{a} \right)$, so: $P_A\psi_p(x) = \psi_p(x-a) = e^{2\pi i p/h} \psi_p(x)$, this gives *Bloch's theorem*: $\psi_k(x) = u_k(x)e^{ikx}$, with $u_k(x \pm a) = u_k(x)$.

13.3.2 Breaking of degeneracy by a perturbation

Suppose the unperturbed system has Hamiltonian \mathcal{H}_0 and symmetry group \mathcal{G}_0 . The perturbed system has $\mathcal{H} = \mathcal{H}_0 + \mathcal{V}$, and symmetry group $\mathcal{G} \subset \mathcal{G}_0$. If $\Gamma^{(n)}(R)$ is an irreducible representation of \mathcal{G}_0 , it is also a representation of \mathcal{G} but not all elements of $\Gamma^{(n)}$ in \mathcal{G}_0 are also in \mathcal{G} . The representation then usually becomes *reducible*: $\Gamma^{(n)} = \Gamma^{(n_1)} \oplus \Gamma^{(n_2)} \oplus \dots$. The degeneracy is then (possibly partially) removed: see the figure below.



Theorem: The set of ℓ_n degenerated eigenfunctions $\psi_\nu^{(n)}$ with energy E_n is a basis for an ℓ_n -dimensional irreducible representation $\Gamma^{(n)}$ of the symmetry group.

13.3.3 The construction of a base function

Each function F in configuration space can be decomposed into *symmetry types*: $F = \sum_{j=1}^n \sum_{\kappa=1}^{\ell_j} f_\kappa^{(j)}$

The following operator extracts the symmetry types:

$$\left(\frac{\ell_j}{h} \sum_{R \in \mathcal{G}} \Gamma_{\kappa\kappa}^{(j)*}(R) P_R \right) F = f_\kappa^{(j)}$$

This is expressed as: $f_\kappa^{(j)}$ is the part of F that transforms according to the κ -th row of $\Gamma^{(j)}$.

F can also be expressed in base functions φ : $F = \sum_{a,j,\kappa} c_{a,j,\kappa} \varphi_\kappa^{(aj)}$. The functions $f_\kappa^{(j)}$ are in general not transformed into each other by elements of the group. However, this does happen if $c_{ja\kappa} = c_{ja}$.

Theorem: Two wavefunctions transforming according to non-equivalent unitary representations or according to different rows of an unitary irreducible representation are orthogonal:

$\langle \varphi_\kappa^{(i)} | \psi_\lambda^{(j)} \rangle \sim \delta_{ij} \delta_{\kappa\lambda}$, and $\langle \varphi_\kappa^{(i)} | \psi_\kappa^{(i)} \rangle$ is independent of κ .

13.3.4 The direct product of representations

Consider a physical system existing of two subsystems. The subspace $D^{(i)}$ of the system transforms according to $\Gamma^{(i)}$. Basefunctions are $\varphi_\kappa^{(i)}(\vec{x}_i)$, $1 \leq \kappa \leq \ell_i$. Now form all $\ell_1 \times \ell_2$ products $\varphi_\kappa^{(1)}(\vec{x}_1) \varphi_\lambda^{(2)}(\vec{x}_2)$. These define a space $D^{(1)} \otimes D^{(2)}$.

These product functions transform as:

$$P_R(\varphi_\kappa^{(1)}(\vec{x}_1) \varphi_\lambda^{(2)}(\vec{x}_2)) = (P_R \varphi_\kappa^{(1)}(\vec{x}_1))(P_R \varphi_\lambda^{(2)}(\vec{x}_2))$$

In general the space $D^{(1)} \otimes D^{(2)}$ can be split up in a number of invariant subspaces:

$$\Gamma^{(1)} \otimes \Gamma^{(2)} = \sum_i n_i \Gamma^{(i)}$$

A useful tool for this reduction is that for the characters hold:

$$\chi^{(1)}(R) \chi^{(2)}(R) = \sum_i n_i \chi^{(i)}(R)$$

13.3.5 Clebsch-Gordan coefficients

With the reduction of the direct-product matrix w.r.t. the basis $\varphi_\kappa^{(i)} \varphi_\lambda^{(j)}$ one uses a new basis $\varphi_\mu^{(a\kappa)}$. These base functions lie in subspaces $D^{(a\kappa)}$. The unitary base transformation is given by:

$$\varphi_\mu^{(a\kappa)} = \sum_{\kappa\lambda} \varphi_\kappa^{(i)} \varphi_\lambda^{(j)} (i\kappa j \lambda | a\kappa \mu)$$

and the inverse transformation by: $\varphi_\kappa^{(i)} \varphi_\lambda^{(j)} = \sum_{a\kappa\mu} \varphi_\mu^{(a\kappa)} (a\kappa \mu | i\kappa j \lambda)$

In essence the Clebsch-Gordan coefficients are dot products: $(i\kappa j \lambda | a\kappa \mu) := \langle \varphi_\kappa^{(i)} \varphi_\lambda^{(j)} | \varphi_\mu^{(a\kappa)} \rangle$

13.3.6 Symmetric transformations of operators, irreducible tensor operators

Observables (operators) transform as follows under symmetry transformations: $A' = P_R A P_R^{-1}$. If a set of operators $A_\kappa^{(j)}$ with $0 \leq \kappa \leq \ell_j$ transform into each other under the transformations of \mathcal{G} holds:

$$P_R A_\kappa^{(j)} P_R^{-1} = \sum_\nu A_\nu^{(j)} \Gamma_{\nu\kappa}^{(j)}(R)$$

If $\Gamma^{(j)}$ is irreducible they are called *irreducible tensor operators* $A^{(j)}$ with components $A_\kappa^{(j)}$.

An operator can also be decomposed into symmetry types: $A = \sum_{j\kappa} a_\kappa^{(j)}$, with:

$$a_\kappa^{(j)} = \left(\frac{\ell_j}{h} \sum_{R \in \mathcal{G}} \Gamma_{\kappa\kappa}^{(j)*}(R) \right) (P_R A P_R^{-1})$$

Theorem: Matrix elements H_{ij} of the operator \mathcal{H} which is invariant under $\forall_{A \in \mathcal{G}}$, are 0 between states which transform according to non-equivalent irreducible unitary representations or according to different rows of such a representation. Further $\langle \varphi_\kappa^{(i)} | \mathcal{H} | \psi_\kappa^{(i)} \rangle$ is independent of κ . For $\mathcal{H} = 1$ this becomes the previous theorem.

This is applied in quantum mechanics in *perturbation theory* and *variational calculus*. Here one tries to diagonalize \mathcal{H} . Solutions can be found within each category of functions $\varphi_\kappa^{(i)}$ with common i and κ : \mathcal{H} is already diagonal in categories as a whole.

Perturbation calculus can be applied independent within each category. With variational calculus the try function can be chosen within a separate category because the exact eigenfunctions transform according to a row of an irreducible representation.

13.3.7 The Wigner-Eckart theorem

Theorem: The matrix element $\langle \varphi_\lambda^{(i)} | A_\kappa^{(j)} | \psi_\mu^{(k)} \rangle$ can only be $\neq 0$ if $\Gamma^{(j)} \otimes \Gamma^{(k)} = \dots \oplus \Gamma^{(i)} \oplus \dots$. If this is the case holds (if $\Gamma^{(i)}$ appears only once, otherwise one has to sum over a):

$$\langle \varphi_\lambda^{(i)} | A_\kappa^{(j)} | \psi_\mu^{(k)} \rangle = (i\lambda | j\kappa k\mu) \langle \varphi^{(i)} || A^{(j)} || \psi^{(k)} \rangle$$

This theorem can be used to determine selection rules: the probability of a dipole transition is given by ($\vec{\epsilon}$ is the direction of polarization of the radiation):

$$P_D = \frac{8\pi^2 e^2 f^3 |r_{12}|^2}{3\hbar \epsilon_0 c^3} \quad \text{with } r_{12} = \langle l_2 m_2 | \vec{\epsilon} \cdot \vec{r} | l_1 m_1 \rangle$$

Further it can be used to determine intensity ratios: if there is only one value of a the ratio of the matrix elements are the Clebsch-Gordan coefficients. For more a -values relations between the intensity ratios can be stated. However, the intensity ratios are also dependent on the occupation of the atomic energy levels.

13.4 Continuous groups

Continuous groups have $h = \infty$. However, not all groups with $h = \infty$ are continuous, e.g. the translation group of an spatially infinite periodic potential is not continuous but does have $h = \infty$.

13.4.1 The 3-dimensional translation group

For the translation of wavefunctions over a distance a holds: $P_a \psi(x) = \psi(x - a)$. Taylor expansion near x gives:

$$\psi(x - a) = \psi(x) - a \frac{d\psi(x)}{dx} + \frac{1}{2} a^2 \frac{d^2\psi(x)}{dx^2} - + \dots$$

Because the momentum operator in quantum mechanics is given by: $p_x = \frac{\hbar}{i} \frac{\partial}{\partial x}$, this can be written as:

$$\psi(x - a) = e^{-iap_x/\hbar} \psi(x)$$

13.4.2 The 3-dimensional rotation group

This group is called SO(3) because a faithful representation can be constructed from orthogonal 3×3 matrices with a determinant of +1.

For an infinitesimal rotation around the x -axis holds:

$$\begin{aligned} P_{\delta\theta_x} \psi(x, y, z) &\approx \psi(x, y + z\delta\theta_x, z - y\delta\theta_x) \\ &= \psi(x, y, z) + \left(z\delta\theta_x \frac{\partial}{\partial y} - y\delta\theta_x \frac{\partial}{\partial z} \right) \psi(x, y, z) \\ &= \left(1 - \frac{i\delta\theta_x L_x}{\hbar} \right) \psi(x, y, z) \end{aligned}$$

Because the angular momentum operator is given by: $L_x = \frac{\hbar}{i} \left(z \frac{\partial}{\partial y} - y \frac{\partial}{\partial z} \right)$.

So in an arbitrary direction holds: Rotations: $P_{\alpha, \vec{n}} = \exp(-i\alpha(\vec{n} \cdot \vec{J})/\hbar)$
Translations: $P_{a, \vec{n}} = \exp(-ia(\vec{n} \cdot \vec{p})/\hbar)$

J_x, J_y and J_z are called the *generators* of the 3-dim. rotation group, p_x, p_y and p_z are called the generators of the 3-dim. translation group.

The commutation rules for the generators can be derived from the properties of the group for multiplications: translations are interchangeable $\leftrightarrow p_x p_y - p_y p_x = 0$.

Rotations are not generally interchangeable: consider a rotation around axis \vec{n} in the xz -plane over an angle α . Then holds: $P_{\alpha, \vec{n}} = P_{-\theta, y} P_{\alpha, x} P_{\theta, y}$, so:

$$e^{-i\alpha(\vec{n} \cdot \vec{J})/\hbar} = e^{i\theta J_y/\hbar} e^{-i\alpha J_x/\hbar} e^{-i\theta J_y/\hbar}$$

If α and θ are very small and are expanded to second order, and the corresponding terms are put equal with $\vec{n} \cdot \vec{J} = J_x \cos \theta + J_z \sin \theta$, it follows from the $\alpha\theta$ term: $J_x J_y - J_y J_x = i\hbar J_z$.

13.4.3 Properties of continuous groups

The elements $R(p_1, \dots, p_n)$ depend continuously on parameters p_1, \dots, p_n . For the translation group this are e.g. an_x, an_y and an_z . It is demanded that the multiplication and inverse of an element R depend continuously on the parameters of R .

The statement that each element arises only once in each row and column of the Cayley table holds also for continuous groups. The notion conjugacy class for continuous groups is defined equally as for discrete groups. The notion representation is fitted by demanding continuity: each matrix element depends continuously on $p_i(R)$.

Summation over all group elements is for continuous groups replaced by an integration. If $f(R)$ is a function defined on \mathcal{G} , e.g. $\Gamma_{\alpha\beta}(R)$, holds:

$$\int_{\mathcal{G}} f(R) dR := \int_{p_1} \cdots \int_{p_n} f(R(p_1, \dots, p_n)) g(R(p_1, \dots, p_n)) dp_1 \cdots dp_n$$

Here, $g(R)$ is the *density function*.

Because of the properties of the Cayley table is demanded: $\int f(R) dR = \int f(SR) dR$. This fixes $g(R)$ except for a constant factor. Define new variables p' by: $SR(p_i) = R(p'_i)$. If one writes: $dV := dp_1 \cdots dp_n$ holds:

$$g(S) = g(E) \frac{dV}{dV'}$$

Here, $\frac{dV}{dV'}$ is the *Jacobian*: $\frac{dV}{dV'} = \det \left(\frac{\partial p_i}{\partial p'_j} \right)$, and $g(E)$ is constant.

For the translation group holds: $g(\vec{a}) = \text{constant} = g(\vec{0})$ because $g(a\vec{n}) d\vec{a}' = g(\vec{0}) d\vec{a}$ and $d\vec{a}' = d\vec{a}$.

This leads to the fundamental orthogonality theorem:

$$\int_{\mathcal{G}} \Gamma_{\mu\nu}^{(i)*}(R) \Gamma_{\alpha\beta}^{(j)}(R) dR = \frac{1}{\ell_i} \delta_{ij} \delta_{\mu\alpha} \delta_{\nu\beta} \int_{\mathcal{G}} dR$$

and for the characters hold:

$$\int_{\mathcal{G}} \chi^{(i)*}(R) \chi^{(j)}(R) dR = \delta_{ij} \int_{\mathcal{G}} dR$$

Compact groups are groups with a finite group volume: $\int_{\mathcal{G}} dR < \infty$.

13.5 The group SO(3)

One can take 2 parameters for the direction of the rotational axis and one for the angle of rotation φ . The parameter space is a collection points $\varphi\vec{n}$ within a sphere with radius π . The diametrical points on this sphere are equivalent because $R_{\vec{n},\pi} = R_{-\vec{n},-\pi}$.

Another way to define parameters is by means of *Eulers angles*. If α , β and γ are the 3 Euler angles, defined as:

1. The spherical angles of axis 3 w.r.t. xyz are θ , $\varphi := \beta$, α . Now a rotation around axis 3 remains possible.
2. The spherical angles of the z -axis w.r.t. 123 are θ , $\varphi := \beta$, $\pi - \gamma$.

then the rotation of a quantum mechanical system is described by:

$$\psi \rightarrow e^{-i\alpha J_z/\hbar} e^{-i\beta J_y/\hbar} e^{-i\gamma J_z/\hbar} \psi. \text{ So } P_R = e^{-i\varepsilon(\vec{n}\cdot\vec{J})/\hbar}.$$

All irreducible representations of SO(3) can be constructed from the behaviour of the spherical harmonics $Y_{lm}(\theta, \varphi)$ with $-l \leq m \leq l$ and for a fixed l :

$$P_R Y_{lm}(\theta, \varphi) = \sum_{m'} Y_{lm'}(\theta, \varphi) D_{mm'}^{(l)}(R)$$

$D^{(l)}$ is an irreducible representation of dimension $2l + 1$. The character of $D^{(l)}$ is given by:

$$\chi^{(l)}(\alpha) = \sum_{m=-l}^l e^{im\alpha} = 1 + 2 \sum_{k=1}^l \cos(k\alpha) = \frac{\sin([l + \frac{1}{2}]\alpha)}{\sin(\frac{1}{2}\alpha)}$$

In the performed derivation α is the rotational angle around the z -axis. This expression is valid for all rotations over an angle α because the classes of SO(3) are rotations around the same angle around an axis with an arbitrary orientation.

Via the fundamental orthogonality theorem for characters one obtains the following expression for the density function (which is normalized so that $g(0) = 1$):

$$g(\alpha) = \frac{\sin^2(\frac{1}{2}\alpha)}{(\frac{1}{2}\alpha)^2}$$

With this result one can see that the given representations of SO(3) are the only ones: the character of another representation χ' would have to be \perp to the already found ones, so $\chi'(\alpha) \sin^2(\frac{1}{2}\alpha) = 0 \forall \alpha \Rightarrow \chi'(\alpha) = 0 \forall \alpha$. This is contradictory because the dimension of the representation is given by $\chi'(0)$.

Because fermions have an half-odd integer spin the states ψ_{sm_s} with $s = \frac{1}{2}$ and $m_s = \pm \frac{1}{2}$ constitute a 2-dim. space which is invariant under rotations. A problem arises for rotations over 2π :

$$\psi_{\frac{1}{2}m_s} \rightarrow e^{-2\pi i S_z/\hbar} \psi_{\frac{1}{2}m_s} = e^{-2\pi i m_s} \psi_{\frac{1}{2}m_s} = -\psi_{\frac{1}{2}m_s}$$

However, in SO(3) holds: $R_{z,2\pi} = E$. So here holds $E \rightarrow \pm \mathbb{I}$. Because observable quantities can always be written as $\langle \phi | \psi \rangle$ or $\langle \phi | A | \psi \rangle$, and are bilinear in the states, they do not change sign if the states do. If only one state changes sign the observable quantities do change.

The existence of these half-odd integer representations is connected with the topological properties of SO(3): the group is two-fold coherent through the identification $R_0 = R_{2\pi} = E$.

13.6 Applications to quantum mechanics

13.6.1 Vectormodel for the addition of angular momentum

If two subsystems have angular momentum quantum numbers j_1 and j_2 the only possible values for the total angular momentum are $J = j_1 + j_2, j_1 + j_2 - 1, \dots, |j_1 - j_2|$. This can be derived from group theory as follows: from $\chi^{(j_1)}(\alpha) \chi^{(j_2)}(\alpha) = \sum_J n_J \chi^{(J)}(\alpha)$ follows:

$$D^{(j_1)} \otimes D^{(j_2)} = D^{(j_1+j_2)} \oplus D^{(j_1+j_2-1)} \oplus \dots \oplus D^{(|j_1-j_2|)}$$

The states can be characterized by quantum numbers in two ways: with j_1, m_1, j_2, m_2 and with j_1, j_2, J, M . The Clebsch-Gordan coefficients, for SO(3) called the *Wigner coefficients*, can be chosen real, so:

$$\begin{aligned}\psi_{j_1 j_2 J M} &= \sum_{m_1 m_2} \psi_{j_1 m_1 j_2 m_2} (j_1 m_1 j_2 m_2 | J M) \\ \psi_{j_1 m_1 j_2 m_2} &= \sum_{J M} \psi_{j_1 j_2 J M} (j_1 m_1 j_2 m_2 | J M)\end{aligned}$$

13.6.2 Irreducible tensor operators, matrixelements and selection rules

Some examples of the behaviour of operators under SO(3)

1. Suppose $j = 0$: this gives the identical representation with $\ell_j = 1$. This state is described by a scalar operator. Because $P_R A_0^{(0)} P_R^{-1} = A_0^{(0)}$ this operator is invariant, e.g. the Hamiltonian of a free atom. Then holds: $\langle J' M' | \mathcal{H} | J M \rangle \sim \delta_{M M'} \delta_{J J'}$.
2. A vector operator: $\vec{A} = (A_x, A_y, A_z)$. The cartesian components of a vector operator transform equally as the cartesian components of \vec{r} by definition. So for rotations around the z -axis holds:

$$D(R_{\alpha, z}) = \begin{pmatrix} \cos \alpha & -\sin \alpha & 0 \\ \sin \alpha & \cos \alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The transformed operator has the same matrix elements w.r.t. $P_R \psi$ and $P_R \phi$:

$\langle P_R \psi | P_R A_x P_R^{-1} | P_R \phi \rangle = \langle \psi | A_x | \phi \rangle$, and $\chi(R_{\alpha, z}) = 1 + 2 \cos(\alpha)$. According to the equation for characters this means one can choose base operators which transform like $Y_{1m}(\theta, \varphi)$. These turn out to be the spherical components:

$$A_{+1}^{(1)} = -\frac{1}{\sqrt{2}}(A_x + iA_y), \quad A_0^{(1)} = A_z, \quad A_{-1}^{(1)} = \frac{1}{\sqrt{2}}(A_x - iA_y)$$

3. A cartesian tensor of rank 2: T_{ij} is a quantity which transforms under rotations like $U_i V_j$, where \vec{U} and \vec{V} are vectors. So T_{ij} transforms like $P_R T_{ij} P_R^{-1} = \sum_{kl} T_{kl} D_{ki}(R) D_{lj}(R)$, so like $D^{(1)} \otimes D^{(1)} = D^{(2)} \oplus D^{(1)} \oplus D^{(0)}$. The 9 components can be split in 3 invariant subspaces with dimension 1 ($D^{(0)}$), 3 ($D^{(1)}$) and 5 ($D^{(2)}$). The new base operators are:

- I. $\text{Tr}(\underline{T}) = T_{xx} + T_{yy} + T_{zz}$. This transforms as the scalar $\vec{U} \cdot \vec{V}$, so as $D^{(0)}$.
- II. The 3 antisymmetric components $A_z = \frac{1}{2}(T_{xy} - T_{yx})$, etc. These transform as the vector $\vec{U} \times \vec{V}$, so as $D^{(1)}$.
- III. The 5 independent components of the traceless, symmetric tensor \underline{S} :
 $S_{ij} = \frac{1}{2}(T_{ij} + T_{ji}) - \frac{1}{3}\delta_{ij}\text{Tr}(\underline{T})$. These transform as $D^{(2)}$.

Selection rules for dipole transitions

Dipole operators transform as $D^{(1)}$: for an electric dipole transfer is the operator $e\vec{r}$, for a magnetic $e(\vec{L} + 2\vec{S})/2m$.

From the Wigner-Eckart theorem follows: $\langle J' M' | A_{\kappa}^{(1)} | J M \rangle = 0$ except $D^{(J')}$ is a part of $D^{(1)} \otimes D^{(J)} = D^{(J+1)} \oplus D^{(J)} \oplus D^{(|J-1|)}$. This means that $J' \in \{J+1, J, |J-1|\}$: $J' = J$ or $J' = J \pm 1$, except $J' = J = 0$.

Landé-equation for the anomalous Zeeman splitting

According to Landé's model the interaction between a magnetic moment with an external magnetic field is determined by the projection of \vec{M} on \vec{J} because \vec{L} and \vec{S} precess fast around \vec{J} . This can also be understood

from the Wigner-Eckart theorem: from this follows that the matrix elements from all vector operators show a certain proportionality. For an arbitrary operator \vec{A} follows:

$$\langle \alpha j m' | \vec{A} | \alpha j m \rangle = \frac{\langle \alpha j m | \vec{A} \cdot \vec{J} | \alpha j m \rangle}{j(j+1)\hbar^2} \langle \alpha j m' | \vec{J} | \alpha j m \rangle$$

13.7 Applications to particle physics

The physics of a system does not change after performing a transformation $\psi' = e^{i\delta}\psi$ where δ is a constant. This is a *global gauge transformation*: the phase of the wavefunction changes everywhere by the same amount.

There exists some freedom in the choice of the potentials \vec{A} and ϕ at the same \vec{E} and \vec{B} : *gauge transformations* of the potentials do not change \vec{E} and \vec{B} (See chapter 2 and 10). The solution ψ' of the Schrödinger equation with the transformed potentials is: $\psi' = e^{-iqf(\vec{r},t)}\psi$.

This is a *local gauge transformation*: the phase of the wavefunction changes different at each position. The physics of the system does not change if \vec{A} and ϕ are also transformed. This is now stated as a guide principle: *the “right of existence” of the electromagnetic field is to allow local gauge invariance*.

The gauge transformations of the EM-field form a group: U(1), unitary 1×1 -matrices. The split-off of charge in the exponent is essential: it allows one gauge field for all charged particles, independent of their charge.

This concept is generalized: particles have a “special charge” Q . The group elements now are $P_R = \exp(-iQ\Theta)$.

Other force fields than the electromagnetic field can also be understood this way. The weak interaction together with the electromagnetic interaction can be described by a force field that transforms according to $U(1) \otimes SU(2)$, and consists of the photon and three intermediary vector bosons. The colour force is described by SU(3), and has a gauge field that exists of 8 types of gluons.

In general the group elements are given by $P_R = \exp(-i\vec{T} \cdot \vec{\Theta})$, where Θ_n are real constants and T_n operators (generators), like Q . The commutation rules are given by $[T_i, T_j] = i \sum_k c_{ijk} T_k$. The c_{ijk} are the *structure constants* of the group. For SO(3) these constants are $c_{ijk} = \varepsilon_{ijk}$, here ε_{ijk} is the complete antisymmetric tensor with $\varepsilon_{123} = +1$.

These constants can be found with the help of group product elements: because \mathcal{G} is closed holds: $e^{i\vec{\Theta} \cdot \vec{T}} e^{i\vec{\Theta}' \cdot \vec{T}} = e^{-i\vec{\Theta} \cdot \vec{T}} e^{-i\vec{\Theta}' \cdot \vec{T}} = e^{-i\vec{\Theta}'' \cdot \vec{T}}$. Taylor expansion and setting equal $\Theta^n \Theta'^m$ -terms results in the commutation rules.

The group SU(2) has 3 free parameters: because it is unitary there are 4 real conditions over 4 complex parameters, and the determinant has to be +1, remaining 3 free parameters.

Each unitary matrix U can be written as: $U = e^{-iH}$. Here, H is a Hermitian matrix. Further it always holds that: $\det(U) = e^{-i\text{Tr}(H)}$.

For each matrix of SU(2) holds that $\text{Tr}(H)=0$. Each Hermitian, traceless 2×2 matrix can be written as a linear combination of the 3 *Pauli-matrices* σ_i . So these matrices are a choice for the operators of SU(2). One can write: $SU(2) = \{\exp(-\frac{i}{2}\vec{\sigma} \cdot \vec{\Theta})\}$.

In abstraction, one can consider an isomorphic group where only the commutation rules are considered to be known regarding the operators T_i : $[T_1, T_2] = iT_3$, etc.

In elementary particle physics the T_i can be interpreted e.g. as the *isospin* operators. Elementary particles can be classified in isospin-multiplets, these are the irreducible representations of SU(2). The classification is:

1. The isospin-singlet \equiv the identical representation: $e^{-i\vec{T} \cdot \vec{\Theta}} = 1 \Rightarrow T_i = 0$
2. The isospin-doublet \equiv the faithful representation of SU(2) on 2×2 matrices.

The group $SU(3)$ has 8 free parameters. (The group $SU(N)$ has $N^2 - 1$ free parameters). The Hermitian, traceless operators are 3 $SU(2)$ -subgroups in the $\vec{e}_1\vec{e}_2$, $\vec{e}_1\vec{e}_3$ and the $\vec{e}_2\vec{e}_3$ plane. This gives 9 matrices, which are not all 9 linear independent. By taking a linear combination one gets 8 matrices.

In the Lagrange density for the colour force one has to substitute $\frac{\partial}{\partial x} \rightarrow \frac{D}{Dx} := \frac{\partial}{\partial x} - \sum_{i=1}^8 T_i A_x^i$

The terms of 3rd and 4th power in A show that the colour field interacts with itself.

Chapter 14

Nuclear physics

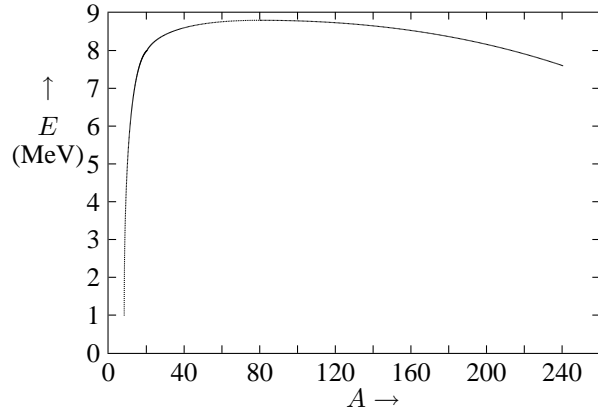
14.1 Nuclear forces

The mass of a nucleus is given by:

$$M_{\text{nuc}} = Zm_p + Nm_n - E_{\text{bind}}/c^2$$

The binding energy per nucleon is given in the figure at the right. The top is at ${}^{56}_{26}\text{Fe}$, the most stable nucleus. With the constants

$$\begin{aligned} a_1 &= 15.760 \text{ MeV} \\ a_2 &= 17.810 \text{ MeV} \\ a_3 &= 0.711 \text{ MeV} \\ a_4 &= 23.702 \text{ MeV} \\ a_5 &= 34.000 \text{ MeV} \end{aligned}$$



and $A = Z + N$, in the *droplet* or *collective model* of the nucleus the binding energy E_{bind} is given by:

$$\frac{E_{\text{bind}}}{c^2} = a_1 A - a_2 A^{2/3} - a_3 \frac{Z(Z-1)}{A^{1/3}} - a_4 \frac{(N-Z)^2}{A} + \epsilon a_5 A^{-3/4}$$

These terms arise from:

1. a_1 : Binding energy of the strong nuclear force, approximately $\sim A$.
2. a_2 : Surface correction: the nucleons near the surface are less bound.
3. a_3 : Coulomb repulsion between the protons.
4. a_4 : Asymmetry term: a surplus of protons or neutrons has a lower binding energy.
5. a_5 : Pair off effect: nuclei with an even number of protons or neutrons are more stable because groups of two protons or neutrons have a lower energy. The following holds:

$$\begin{aligned} \underline{Z \text{ even}, N \text{ even}}: \epsilon &= +1, \underline{Z \text{ odd}, N \text{ odd}}: \epsilon = -1. \\ \underline{Z \text{ even}, N \text{ odd}}: \epsilon &= 0, \underline{Z \text{ odd}, N \text{ even}}: \epsilon = 0. \end{aligned}$$

The Yukawa potential can be derived if the nuclear force can to first approximation, be considered as an exchange of virtual pions:

$$U(r) = -\frac{W_0 r_0}{r} \exp\left(-\frac{r}{r_0}\right)$$

With $\Delta E \cdot \Delta t \approx \hbar$, $E_\gamma = m_0 c^2$ and $r_0 = c \Delta t$ follows: $r_0 = \hbar / m_0 c$.

In the shell model of the nucleus one assumes that a nucleon moves in an average field of other nucleons. Further, there is a contribution of the spin-orbit coupling $\sim \vec{L} \cdot \vec{S}$: $\Delta V_{ls} = \frac{1}{2}(2l+1)\hbar\omega$. So each level (n, l) is split in two, with $j = l \pm \frac{1}{2}$, where the state with $j = l + \frac{1}{2}$ has the lowest energy. This is just the opposite for electrons, which is an indication that the $L-S$ interaction is not electromagnetic. The energy of a 3-dimensional harmonic oscillator is $E = (N + \frac{3}{2})\hbar\omega$. $N = n_x + n_y + n_z = 2(n-1) + l$ where $n \geq 1$ is the main oscillator number. Because $-l \leq m \leq l$ and $m_s = \pm \frac{1}{2}\hbar$ there are $2(2l+1)$

substates which exist independently for protons and neutrons. This gives rise to the so called *magical numbers*: nuclei where each state in the outermost level are filled are particularly stable. This is the case if N or $Z \in \{2, 8, 20, 28, 50, 82, 126\}$.

14.2 The shape of the nucleus

A nucleus is to first approximation spherical with a radius of $R = R_0 A^{1/3}$. Here, $R_0 \approx 1.4 \cdot 10^{-15}$ m, constant for all nuclei. If the nuclear radius is measured including the charge distribution one obtains $R_0 \approx 1.2 \cdot 10^{-15}$ m. The shape of oscillating nuclei can be described by spherical harmonics:

$$R = R_0 \left[1 + \sum_{lm} a_{lm} Y_l^m(\theta, \varphi) \right]$$

$l = 0$ gives rise to monopole vibrations, density vibrations, which can be applied to the theory of neutron stars. $l = 1$ gives dipole vibrations, $l = 2$ quadrupole, with $a_{2,0} = \beta \cos \gamma$ and $a_{2,\pm 2} = \frac{1}{2} \sqrt{2} \beta \sin \gamma$ where β is the deformation factor and γ the shape parameter. The multipole moment is given by $\mu_l = Z e r^l Y_l^m(\theta, \varphi)$. The parity of the electric moment is $\Pi_E = (-1)^l$, of the magnetic moment $\Pi_M = (-1)^{l+1}$.

There are 2 contributions to the magnetic moment: $\vec{M}_L = \frac{e}{2m_p} \vec{L}$ and $\vec{M}_S = g_S \frac{e}{2m_p} \vec{S}$.

where g_S is the *spin-gyromagnetic ratio*. For protons holds $g_S = 5.5855$ and for neutrons $g_S = -3.8263$. The z -components of the magnetic moment are given by $M_{L,z} = \mu_N m_l$ and $M_{S,z} = g_S \mu_N m_S$. The resulting magnetic moment is related to the nuclear spin I according to $\vec{M} = g_I (e/2m_p) \vec{I}$. The z -component is then $M_z = \mu_N g_I m_I$.

14.3 Radioactive decay

The number of nuclei decaying is proportional to the number of nuclei: $\dot{N} = -\lambda N$. This gives for the number of nuclei N : $N(t) = N_0 \exp(-\lambda t)$. The *half life time* follows from $\tau_{\frac{1}{2}} \lambda = \ln(2)$. The average life time of a nucleus is $\tau = 1/\lambda$. The probability that N nuclei decay within a time interval is given by a Poisson distribution:

$$P(N)dt = N_0 \frac{\lambda^N e^{-\lambda}}{N!} dt$$

If a nucleus can decay into more final states then holds: $\lambda = \sum \lambda_i$. So the fraction decaying into state i is $\lambda_i / \sum \lambda_i$. There are 5 types of natural radioactive decay:

1. α -decay: the nucleus emits a He^{2+} nucleus. Because nucleons tend to order themselves in groups of $2p+2n$ this can be considered as a tunneling of a He^{2+} nucleus through a potential barrier. The tunnel probability P is

$$P = \frac{\text{incoming amplitude}}{\text{outgoing amplitude}} = e^{-2G} \quad \text{with} \quad G = \frac{1}{\hbar} \sqrt{2m} \int [V(r) - E] dr$$

G is called the *Gamow factor*.

2. β -decay. Here a proton changes into a neutron or vice versa:
 $p^+ \rightarrow n^0 + W^+ \rightarrow n^0 + e^+ + \nu_e$, and $n^0 \rightarrow p^+ + W^- \rightarrow p^+ + e^- + \bar{\nu}_e$.
3. Electron capture: here, a proton in the nucleus captures an electron (usually from the K-shell).
4. Spontaneous fission: a nucleus breaks apart.
5. γ -decay: here the nucleus emits a high-energetic photon. The decay constant is given by

$$\lambda = \frac{P(l)}{\hbar \omega} \sim \frac{E_\gamma}{(\hbar c)^2} \left(\frac{E_\gamma R}{\hbar c} \right)^{2l} \sim 10^{-4l}$$

where l is the quantum number for the angular momentum and P the radiated power. Usually the decay constant of electric multipole moments is larger than the one of magnetic multipole moments. The energy of the photon is $E_\gamma = E_i - E_f - T_R$, with $T_R = E_\gamma^2/2mc^2$ the recoil energy, which can usually be neglected. The parity of the emitted radiation is $\Pi^l = \Pi^i \cdot \Pi^f$. With I the quantum number of angular momentum of the nucleus, $L = \hbar\sqrt{I(I+1)}$, holds the following selection rule: $|\vec{I}_i - \vec{I}_f| \leq \Delta l \leq |\vec{I}_i + \vec{I}_f|$.

14.4 Scattering and nuclear reactions

14.4.1 Kinetic model

If a beam with intensity I hits a target with density n and length x (Rutherford scattering) the number of scatterings R per unit of time is equal to $R = Inx\sigma$. From this follows that the intensity of the beam decreases as $-dI = In\sigma dx$. This results in $I = I_0 e^{-n\sigma x} = I_0 e^{-\mu x}$.

Because $dR = R(\theta, \varphi)d\Omega/4\pi = Inxd\sigma$ it follows: $\frac{d\sigma}{d\Omega} = \frac{R(\theta, \varphi)}{4\pi n x I}$

If N particles are scattered in a material with density n then holds: $\frac{\Delta N}{N} = n \frac{d\sigma}{d\Omega} \Delta\Omega \Delta x$

For Coulomb collisions holds: $\left. \frac{d\sigma}{d\Omega} \right|_C = \frac{Z_1 Z_2 e^2}{8\pi\epsilon_0 \mu v_0^2} \frac{1}{\sin^4(\frac{1}{2}\theta)}$

14.4.2 Quantum mechanical model for n-p scattering

The initial state is a beam of neutrons moving along the z -axis with wavefunction $\psi_{\text{init}} = e^{ikz}$ and current density $J_{\text{init}} = v|\psi_{\text{init}}|^2 = v$. At large distances from the scattering point they have approximately a spherical wavefunction $\psi_{\text{scat}} = f(\theta)e^{ikr}/r$ where $f(\theta)$ is the *scattering amplitude*. The total wavefunction is then given by

$$\psi = \psi_{\text{in}} + \psi_{\text{scat}} = e^{ikz} + f(\theta)\frac{e^{ikr}}{r}$$

The particle flux of the scattered particles is $v|\psi_{\text{scat}}|^2 = v|f(\theta)|^2 d\Omega$. From this it follows that $\sigma(\theta) = |f(\theta)|^2$. The wavefunction of the incoming particles can be expressed as a sum of angular momentum wavefunctions:

$$\psi_{\text{init}} = e^{ikz} = \sum_l \psi_l$$

The impact parameter is related to the angular momentum with $L = bp = b\hbar k$, so $b\hbar k \approx l$. At very low energy only particles with $l = 0$ are scattered, so

$$\psi = \psi'_0 + \sum_{l>0} \psi_l \quad \text{and} \quad \psi_0 = \frac{\sin(kr)}{kr}$$

If the potential is approximately rectangular holds: $\psi'_0 = C \frac{\sin(kr + \delta_0)}{kr}$

The cross section is then $\sigma(\theta) = \frac{\sin^2(\delta_0)}{k^2}$ so $\sigma = \int \sigma(\theta) d\Omega = \frac{4\pi \sin^2(\delta_0)}{k^2}$

At very low energies holds: $\sin^2(\delta_0) = \frac{\hbar^2 k^2 / 2m}{W_0 + W}$

with W_0 the depth of the potential well. At higher energies holds: $\sigma = \frac{4\pi}{k^2} \sum_l \sin^2(\delta_l)$

14.4.3 Conservation of energy and momentum in nuclear reactions

If a particle P_1 collides with a particle P_2 which is in rest w.r.t. the laboratory system and other particles are created, so

$$P_1 + P_2 \rightarrow \sum_{k>2} P_k$$

the total energy Q gained or required is given by $Q = (m_1 + m_2 - \sum_{k>2} m_k)c^2$.

The minimal required kinetic energy T of P_1 in the laboratory system to initialize the reaction is

$$T = -Q \frac{m_1 + m_2 + \sum m_k}{2m_2}$$

If $Q < 0$ there is a threshold energy.

14.5 Radiation dosimetry

Radiometric quantities determine the strength of the radiation source(s). *Dosimetric quantities* are related to the energy transfer from radiation to matter. Parameters describing a relation between those are called *interaction parameters*. The intensity of a beam of particles in matter decreases according to $I(s) = I_0 \exp(-\mu s)$. The deceleration of a *heavy* particle is described by the *Bethe-Bloch equation*:

$$\frac{dE}{ds} \sim \frac{q^2}{v^2}$$

The *fluence* is given by $\Phi = dN/dA$. The *flux* is given by $\phi = d\Phi/dt$. The energy loss is defined by $\Psi = dW/dA$, and the energy flux density $\psi = d\Psi/dt$. The *absorption coefficient* is given by $\mu = (dN/N)/dx$. The *mass absorption coefficient* is given by μ/ρ .

The *radiation dose* X is the amount of charge produced by the radiation per unit of mass, with unit C/kg. An old unit is the Röntgen: $1\text{Ro} = 2.58 \cdot 10^{-4} \text{ C/kg}$. With the energy-absorption coefficient μ_E follows:

$$X = \frac{dQ}{dm} = \frac{e\mu_E}{W\rho} \Psi$$

where W is the energy required to disjoin an elementary charge.

The *absorbed dose* D is given by $D = dE_{\text{abs}}/dm$, with unit Gy=J/kg. An old unit is the rad: $1 \text{ rad} = 0.01 \text{ Gy}$. The *dose tempo* is defined as \dot{D} . It can be derived that

$$D = \frac{\mu_E}{\rho} \Psi$$

The *Kerma* K is the amount of kinetic energy of secondary produced particles which is produced per mass unit of the radiated object.

The *equivalent dose* H is a weight average of the absorbed dose per type of radiation, where for each type radiation the effects on biological material is used for the weight factor. These weight factors are called the quality factors. Their unit is Sv. $H = QD$. If the absorption is not equally distributed also weight factors w per organ need to be used: $H = \sum w_k H_k$. For some types of radiation holds:

Radiation type	Q
Röntgen, gamma radiation	1
β , electrons, mesons	1
Thermic neutrons	3 to 5
Fast neutrons	10 to 20
protons	10
α , fission products	20

Chapter 15

Quantum field theory & Particle physics

15.1 Creation and annihilation operators

A state with more particles can be described by a collection occupation numbers $|n_1 n_2 n_3 \dots\rangle$. Hence the vacuum state is given by $|000 \dots\rangle$. This is a complete description because the particles are indistinguishable. The states are orthonormal:

$$\langle n_1 n_2 n_3 \dots | n'_1 n'_2 n'_3 \dots \rangle = \prod_{i=1}^{\infty} \delta_{n_i n'_i}$$

The time-dependent state vector is given by

$$\Psi(t) = \sum_{n_1 n_2 \dots} c_{n_1 n_2 \dots}(t) |n_1 n_2 \dots\rangle$$

The coefficients c can be interpreted as follows: $|c_{n_1 n_2 \dots}|^2$ is the probability to find n_1 particles with momentum \vec{k}_1 , n_2 particles with momentum \vec{k}_2 , etc., and $\langle \Psi(t) | \Psi(t) \rangle = \sum |c_{n_i}(t)|^2 = 1$. The expansion of the states in time is described by the Schrödinger equation

$$i \frac{d}{dt} |\Psi(t)\rangle = H |\Psi(t)\rangle$$

where $H = H_0 + H_{\text{int}}$. H_0 is the Hamiltonian for free particles and keeps $|c_{n_i}(t)|^2$ constant, H_{int} is the interaction Hamiltonian and can increase or decrease a c^2 at the cost of others.

All operators which can change occupation numbers can be expanded in the a and a^\dagger operators. a is the *annihilation operator* and a^\dagger the *creation operator*, and:

$$\begin{aligned} a(\vec{k}_i) |n_1 n_2 \dots n_i \dots\rangle &= \sqrt{n_i} |n_1 n_2 \dots n_i - 1 \dots\rangle \\ a^\dagger(\vec{k}_i) |n_1 n_2 \dots n_i \dots\rangle &= \sqrt{n_i + 1} |n_1 n_2 \dots n_i + 1 \dots\rangle \end{aligned}$$

Because the states are normalized holds $a|0\rangle = 0$ and $a(\vec{k}_i) a^\dagger(\vec{k}_i) |n_i\rangle = n_i |n_i\rangle$. So aa^\dagger is an occupation number operator. The following commutation rules can be derived:

$$[a(\vec{k}_i), a(\vec{k}_j)] = 0, \quad [a^\dagger(\vec{k}_i), a^\dagger(\vec{k}_j)] = 0, \quad [a(\vec{k}_i), a^\dagger(\vec{k}_j)] = \delta_{ij}$$

Hence for free spin-0 particles holds: $H_0 = \sum_i a^\dagger(\vec{k}_i) a(\vec{k}_i) \hbar \omega_{k_i}$

15.2 Classical and quantum fields

Starting with a real field $\Phi^\alpha(x)$ (complex fields can be split in a real and an imaginary part), the *Lagrange density* \mathcal{L} is a function of the position $x = (\vec{x}, ict)$ through the fields: $\mathcal{L} = \mathcal{L}(\Phi^\alpha(x), \partial_\nu \Phi^\alpha(x))$. The Lagrangian is given by $L = \int \mathcal{L}(x) d^3x$. Using the variational principle $\delta I(\Omega) = 0$ and with the action-integral $I(\Omega) = \int \mathcal{L}(\Phi^\alpha, \partial_\nu \Phi^\alpha) d^4x$ the field equation can be derived:

$$\frac{\partial \mathcal{L}}{\partial \Phi^\alpha} - \frac{\partial}{\partial x_\nu} \frac{\partial \mathcal{L}}{\partial (\partial_\nu \Phi^\alpha)} = 0$$

The *conjugated field* is, analogous to momentum in classical mechanics, defined as:

$$\Pi^\alpha(x) = \frac{\partial \mathcal{L}}{\partial \dot{\Phi}^\alpha}$$

With this, the Hamilton density becomes $\mathcal{H}(x) = \Pi^\alpha \dot{\Phi}^\alpha - \mathcal{L}(x)$.

Quantization of a classical field is analogous to quantization in point mass mechanics: the field functions are considered as operators obeying certain commutation rules:

$$[\Phi^\alpha(\vec{x}), \Phi^\beta(\vec{x}')] = 0, \quad [\Pi^\alpha(\vec{x}), \Pi^\beta(\vec{x}')] = 0, \quad [\Phi^\alpha(\vec{x}), \Pi^\beta(\vec{x}')] = i\delta_{\alpha\beta}(\vec{x} - \vec{x}')$$

15.3 The interaction picture

Some equivalent formulations of quantum mechanics are possible:

1. Schrödinger picture: time-dependent states, time-independent operators.
2. Heisenberg picture: time-independent states, time-dependent operators.
3. Interaction picture: time-dependent states, time-dependent operators.

The interaction picture can be obtained from the Schrödinger picture by an unitary transformation:

$$|\Phi(t)\rangle = e^{iH_0^S} |\Phi^S(t)\rangle \quad \text{and} \quad O(t) = e^{iH_0^S} O^S e^{-iH_0^S}$$

The index ^S denotes the Schrödinger picture. From this follows:

$$i \frac{d}{dt} |\Phi(t)\rangle = H_{\text{int}}(t) |\Phi(t)\rangle \quad \text{and} \quad i \frac{d}{dt} O(t) = [O(t), H_0]$$

15.4 Real scalar field in the interaction picture

It is easy to find that, with $M := m_0^2 c^2 / \hbar^2$, holds:

$$\frac{\partial}{\partial t} \Phi(x) = \Pi(x) \quad \text{and} \quad \frac{\partial}{\partial t} \Pi(x) = (\nabla^2 - M^2) \Phi(x)$$

From this follows that Φ obeys the Klein-Gordon equation $(\square - M^2)\Phi = 0$. With the definition $k_0^2 = \vec{k}^2 + M^2 := \omega_k^2$ and the notation $\vec{k} \cdot \vec{x} - ik_0 t := kx$ the general solution of this equation is:

$$\Phi(x) = \frac{1}{\sqrt{V}} \sum_{\vec{k}} \frac{1}{\sqrt{2\omega_k}} \left(a(\vec{k}) e^{ikx} + a^\dagger(\vec{k}) e^{-ikx} \right), \quad \Pi(x) = \frac{i}{\sqrt{V}} \sum_{\vec{k}} \sqrt{\frac{1}{2}\omega_k} \left(-a(\vec{k}) e^{ikx} + a^\dagger(\vec{k}) e^{-ikx} \right)$$

The field operators contain a volume V , which is used as normalization factor. Usually one can take the limit $V \rightarrow \infty$.

In general it holds that the term with e^{-ikx} , the positive frequency part, is the creation part, and the negative frequency part is the annihilation part.

the coefficients have to be each others hermitian conjugate because Φ is hermitian. Because Φ has only one component this can be interpreted as a field describing a particle with spin zero. From this follows that the commutation rules are given by $[\Phi(x), \Phi(x')] = i\Delta(x - x')$ with

$$\Delta(y) = \frac{1}{(2\pi)^3} \int \frac{\sin(ky)}{\omega_k} d^3k$$

$\Delta(y)$ is an odd function which is invariant for proper Lorentz transformations (no mirroring). This is consistent with the previously found result $[\Phi(\vec{x}, t), \Phi(\vec{x}', t)] = 0$. In general holds that $\Delta(y) = 0$ outside the light cone. So the equations obey the locality postulate.

The Lagrange density is given by: $\mathcal{L}(\Phi, \partial_\nu \Phi) = -\frac{1}{2}(\partial_\nu \Phi \partial_\nu \Phi + m^2 \Phi^2)$. The energy operator is given by:

$$H = \int \mathcal{H}(x) d^3x = \sum_{\vec{k}} \hbar \omega_k a^\dagger(\vec{k}) a(\vec{k})$$

15.5 Charged spin-0 particles, conservation of charge

The Lagrange density of charged spin-0 particles is given by: $\mathcal{L} = -(\partial_\nu \Phi \partial_\nu \Phi^* + M^2 \Phi \Phi^*)$.

Noether's theorem connects a continuous symmetry of \mathcal{L} and an additive conservation law. Suppose that $\mathcal{L}((\Phi^\alpha)', \partial_\nu(\Phi^\alpha)') = \mathcal{L}(\Phi^\alpha, \partial_\nu \Phi^\alpha)$ and there exists a continuous transformation between Φ^α and $\Phi^{\alpha'}$ such as $\Phi^{\alpha'} = \Phi^\alpha + \epsilon f^\alpha(\Phi)$. Then holds

$$\frac{\partial}{\partial x_\nu} \left(\frac{\partial \mathcal{L}}{\partial(\partial_\nu \Phi^\alpha)} f^\alpha \right) = 0$$

This is a continuity equation \Rightarrow conservation law. Which quantity is conserved depends on the symmetry. The above Lagrange density is invariant for a change in phase $\Phi \rightarrow \Phi e^{i\theta}$: a global gauge transformation. The conserved quantity is the current density $J_\mu(x) = -ie(\Phi \partial_\mu \Phi^* - \Phi^* \partial_\mu \Phi)$. Because this quantity is 0 for real fields a complex field is needed to describe charged particles. When this field is quantized the field operators are given by

$$\Phi(x) = \frac{1}{\sqrt{V}} \sum_{\vec{k}} \frac{1}{\sqrt{2\omega_k}} \left(a(\vec{k}) e^{ikx} + b^\dagger(\vec{k}) e^{-ikx} \right), \quad \Phi^\dagger(x) = \frac{1}{\sqrt{V}} \sum_{\vec{k}} \frac{1}{\sqrt{2\omega_k}} \left(a^\dagger(\vec{k}) e^{ikx} + b(\vec{k}) e^{-ikx} \right)$$

Hence the energy operator is given by:

$$H = \sum_{\vec{k}} \hbar \omega_k \left(a^\dagger(\vec{k}) a(\vec{k}) + b^\dagger(\vec{k}) b(\vec{k}) \right)$$

and the charge operator is given by:

$$Q(t) = -i \int J_4(x) d^3x \Rightarrow Q = \sum_{\vec{k}} e \left(a^\dagger(\vec{k}) a(\vec{k}) - b^\dagger(\vec{k}) b(\vec{k}) \right)$$

From this follows that $a^\dagger a := N_+(\vec{k})$ is an occupation number operator for particles with a positive charge and $b^\dagger b := N_-(\vec{k})$ is an occupation number operator for particles with a negative charge.

15.6 Field functions for spin- $\frac{1}{2}$ particles

Spin is defined by the behaviour of the solutions ψ of the Dirac equation. A *scalar* field Φ has the property that, if it obeys the Klein-Gordon equation, the rotated field $\tilde{\Phi}(x) := \Phi(\Lambda^{-1}x)$ also obeys it. Λ denotes 4-dimensional rotations: the proper Lorentz transformations. These can be written as:

$$\tilde{\Phi}(x) = \Phi(x) e^{-i\vec{n} \cdot \vec{L}} \quad \text{with} \quad L_{\mu\nu} = -i\hbar \left(x_\mu \frac{\partial}{\partial x_\nu} - x_\nu \frac{\partial}{\partial x_\mu} \right)$$

For $\mu \leq 3, \nu \leq 3$ these are rotations, for $\nu = 4, \mu \neq 4$ these are Lorentz transformations.

A rotated field $\tilde{\psi}$ obeys the Dirac equation if the following condition holds: $\tilde{\psi}(x) = D(\Lambda)\psi(\Lambda^{-1}x)$. This results in the condition $D^{-1}\gamma_\lambda D = \Lambda_{\lambda\mu}\gamma_\mu$. One finds: $D = e^{i\vec{n} \cdot \vec{S}}$ with $S_{\mu\nu} = -i\frac{1}{2}\hbar\gamma_\mu\gamma_\nu$. Hence:

$$\tilde{\psi}(x) = e^{-i(S+L)}\psi(x) = e^{-iJ}\psi(x)$$

Then the solutions of the Dirac equation are given by:

$$\psi(x) = u_{\pm}^r(\vec{p}) e^{-i(\vec{p} \cdot \vec{x} \pm Et)}$$

Here, r is an indication for the direction of the spin, and \pm is the sign of the energy. With the notation $v^r(\vec{p}) = u_{\pm}^r(-\vec{p})$ and $u^r(\vec{p}) = u_{\pm}^r(\vec{p})$ one can write for the dot products of these spinors:

$$u_{+}^r(\vec{p}) u_{+}^{r'}(\vec{p}) = \frac{E}{M} \delta_{rr'}, \quad u_{-}^r(\vec{p}) u_{-}^{r'}(\vec{p}) = \frac{E}{M} \delta_{rr'}, \quad u_{+}^r(\vec{p}) u_{-}^{r'}(\vec{p}) = 0$$

Because of the factor E/M this is not relativistic invariant. A Lorentz-invariant dot product is defined by $\bar{a}b := a^\dagger \gamma_4 b$, where $\bar{a} := a^\dagger \gamma_4$ is a row spinor. From this follows:

$$\overline{u^r(\vec{p})} u^{r'}(\vec{p}) = \delta_{rr'} \quad , \quad \overline{v^r(\vec{p})} v^{r'}(\vec{p}) = -\delta_{rr'} \quad , \quad \overline{u^r(\vec{p})} v^{r'}(\vec{p}) = 0$$

Combinations of the type $a\bar{a}$ give a 4×4 matrix:

$$\sum_{r=1}^2 u^r(\vec{p}) \overline{u^r(\vec{p})} = \frac{-i\gamma_\lambda p_\lambda + M}{2M} \quad , \quad \sum_{r=1}^2 v^r(\vec{p}) \overline{v^r(\vec{p})} = \frac{-i\gamma_\lambda p_\lambda - M}{2M}$$

The Lagrange density which results in the Dirac equation and having the correct energy normalization is:

$$\mathcal{L}(x) = -\overline{\psi(x)} \left(\gamma_\mu \frac{\partial}{\partial x_\mu} + M \right) \psi(x)$$

and the current density is $J_\mu(x) = -ie\overline{\psi}\gamma_\mu\psi$.

15.7 Quantization of spin- $\frac{1}{2}$ fields

The general solution for the field operators is in this case:

$$\psi(x) = \sqrt{\frac{M}{V}} \sum_{\vec{p}} \frac{1}{\sqrt{E}} \sum_r (c_r(\vec{p}) u^r(\vec{p}) e^{ipx} + d_r^\dagger(\vec{p}) v^r(\vec{p}) e^{-ipx})$$

and

$$\overline{\psi(x)} = \sqrt{\frac{M}{V}} \sum_{\vec{p}} \frac{1}{\sqrt{E}} \sum_r (c_r^\dagger(\vec{p}) \overline{u^r(\vec{p})} e^{-ipx} + d_r(\vec{p}) \overline{v^r(\vec{p})} e^{ipx})$$

Here, c^\dagger and c are the creation respectively annihilation operators for an electron and d^\dagger and d the creation respectively annihilation operators for a positron. The energy operator is given by

$$H = \sum_{\vec{p}} E_{\vec{p}} \sum_{r=1}^2 (c_r^\dagger(\vec{p}) c_r(\vec{p}) - d_r(\vec{p}) d_r^\dagger(\vec{p}))$$

To prevent that the energy of positrons is negative the operators must obey anti commutation rules in stead of commutation rules:

$$[c_r(\vec{p}), c_{r'}^\dagger(\vec{p})]_+ = [d_r(\vec{p}), d_{r'}^\dagger(\vec{p})]_+ = \delta_{rr'} \delta_{pp'} \quad , \quad \text{all other anti commutators are 0.}$$

The field operators obey

$$[\psi_\alpha(x), \psi_\beta(x')] = 0 \quad , \quad [\overline{\psi_\alpha(x)}, \overline{\psi_\beta(x')}] = 0 \quad , \quad [\psi_\alpha(x), \overline{\psi_\beta(x')}]_+ = -iS_{\alpha\beta}(x - x')$$

$$\text{with } S(x) = \left(\gamma_\lambda \frac{\partial}{\partial x_\lambda} - M \right) \Delta(x)$$

The anti commutation rules give besides the positive-definite energy also the Pauli exclusion principle and the Fermi-Dirac statistics: because $c_r^\dagger(\vec{p}) c_r^\dagger(\vec{p}) = -c_r^\dagger(\vec{p}) c_r^\dagger(\vec{p})$ holds: $\{c_r^\dagger(\vec{p})\}^2 = 0$. It appears to be impossible to create two electrons with the same momentum and spin. This is the exclusion principle. Another way to see this is the fact that $\{N_r^+(\vec{p})\}^2 = N_r^+(\vec{p})$: the occupation operators have only eigenvalues 0 and 1.

To avoid infinite vacuum contributions to the energy and charge the *normal product* is introduced. The expression for the current density now becomes $J_\mu = -ieN(\overline{\psi}\gamma_\mu\psi)$. This product is obtained by:

- Expand all fields into creation and annihilation operators,
- Keep all terms which have no annihilation operators, or in which they are on the right of the creation operators,
- In all other terms interchange the factors so that the annihilation operators go to the right. By an interchange of two fermion operators add a minus sign, by interchange of two boson operators not. Assume hereby that all commutators are zero.

15.8 Quantization of the electromagnetic field

Starting with the Lagrange density $\mathcal{L} = -\frac{1}{2} \frac{\partial A_\nu}{\partial x_\mu} \frac{\partial A_\nu}{\partial x_\mu}$

it follows for the field operators $A(x)$:

$$A(x) = \frac{1}{\sqrt{V}} \sum_{\vec{k}} \frac{1}{\sqrt{2\omega_k}} \sum_{m=1}^4 \left(a_m(\vec{k}) \epsilon^m(\vec{k}) e^{ikx} + a^\dagger(\vec{k}) \epsilon^m(\vec{k})^* e^{-ikx} \right)$$

The operators obey $[a_m(\vec{k}), a_{m'}^\dagger(\vec{k}')] = \delta_{mm'} \delta_{\vec{k}\vec{k}'}$. All other commutators are 0. m gives the polarization direction of the photon: $m = 1, 2$ gives transversal polarized, $m = 3$ longitudinal polarized and $m = 4$ timelike polarized photons. Further holds:

$$[A_\mu(x), A_\nu(x')] = i\delta_{\mu\nu} D(x - x') \text{ with } D(y) = \Delta(y)|_{m=0}$$

In spite of the fact that $A_4 = iV$ is imaginary in the classical case, A_4 is still defined to be hermitian because otherwise the sign of the energy becomes incorrect. By changing the definition of the inner product in configuration space the expectation values for $A_{1,2,3}(x) \in \mathbb{R}$ and for $A_4(x)$ become imaginary.

If the potentials satisfy the Lorentz gauge condition $\partial_\mu A_\mu = 0$ the E and B operators derived from these potentials will satisfy the Maxwell equations. However, this gives problems with the commutation rules. It is now demanded that only those states are permitted for which holds

$$\frac{\partial A_\mu}{\partial x_\mu} |\Phi\rangle = 0$$

This results in:

$$\left\langle \frac{\partial A_\mu}{\partial x_\mu} \right\rangle = 0.$$

From this follows that $(a_3(\vec{k}) - a_4(\vec{k}))|\Phi\rangle = 0$. With a local gauge transformation one obtains $N_3(\vec{k}) = 0$ and $N_4(\vec{k}) = 0$. However, this only applies to free EM-fields: in intermediary states in interactions there can exist longitudinal and timelike photons. These photons are also responsible for the stationary Coulomb potential.

15.9 Interacting fields and the S-matrix

The S (scattering)-matrix gives a relation between the initial and final states of an interaction: $|\Phi(\infty)\rangle = S|\Phi(-\infty)\rangle$. If the Schrödinger equation is integrated:

$$|\Phi(t)\rangle = |\Phi(-\infty)\rangle - i \int_{-\infty}^t H_{\text{int}}(t_1) |\Phi(t_1)\rangle dt_1$$

and perturbation theory is applied one finds that:

$$S = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int \cdots \int T \{ \mathcal{H}_{\text{int}}(x_1) \cdots \mathcal{H}_{\text{int}}(x_n) \} d^4x_1 \cdots d^4x_n \equiv \sum_{n=0}^{\infty} S^{(n)}$$

Here, the T -operator means a *time-ordered product*: the terms in such a product must be ordered in increasing time order from the right to the left so that the earliest terms act first. The S -matrix is then given by: $S_{ij} = \langle \Phi_i | S | \Phi_j \rangle = \langle \Phi_i | \Phi(\infty) \rangle$.

The interaction Hamilton density for the interaction between the electromagnetic and the electron-positron field is: $\mathcal{H}_{\text{int}}(x) = -J_\mu(x) A_\mu(x) = ieN(\bar{\psi} \gamma_\mu \psi A_\mu)$

When this is expanded as: $\mathcal{H}_{\text{int}} = ieN \left((\bar{\psi}^+ + \bar{\psi}^-) \gamma_\mu (\psi^+ + \psi^-) (A_\mu^+ + A_\mu^-) \right)$

eight terms appear. Each term corresponds with a possible process. The term $ie\bar{\psi}^+\gamma_\mu\psi^+A_\mu^-$ acting on $|\Phi\rangle$ gives transitions where A_μ^- creates a photon, ψ^+ annihilates an electron and $\bar{\psi}^+$ annihilates a positron. Only terms with the correct number of particles in the initial and final state contribute to a matrix element $\langle\Phi_i|S|\Phi_j\rangle$. Further the factors in \mathcal{H}_{int} can create and thereafter annihilate particles: the *virtual particles*.

The expressions for $S^{(n)}$ contain time-ordered products of normal products. This can be written as a sum of normal products. The appearing operators describe the minimal changes necessary to change the initial state into the final state. The effects of the virtual particles are described by the (anti)commutator functions. Some time-ordered products are:

$$\begin{aligned} T\{\Phi(x)\Phi(y)\} &= N\{\Phi(x)\Phi(y)\} + \frac{1}{2}\Delta^F(x-y) \\ T\{\psi_\alpha(x)\bar{\psi}_\beta(y)\} &= N\{\psi_\alpha(x)\bar{\psi}_\beta(y)\} - \frac{1}{2}S_{\alpha\beta}^F(x-y) \\ T\{A_\mu(x)A_\nu(y)\} &= N\{A_\mu(x)A_\nu(y)\} + \frac{1}{2}\delta_{\mu\nu}D_{\mu\nu}^F(x-y) \end{aligned}$$

Here, $S^F(x) = (\gamma_\mu\partial_\mu - M)\Delta^F(x)$, $D^F(x) = \Delta^F(x)|_{m=0}$ and

$$\Delta^F(x) = \begin{cases} \frac{1}{(2\pi)^3} \int \frac{e^{ikx}}{\omega_{\vec{k}}} d^3k & \text{if } x_0 > 0 \\ \frac{1}{(2\pi)^3} \int \frac{e^{-ikx}}{\omega_{\vec{k}}} d^3k & \text{if } x_0 < 0 \end{cases}$$

The term $\frac{1}{2}\Delta^F(x-y)$ is called the contraction of $\Phi(x)$ and $\Phi(y)$, and is the expectation value of the time-ordered product in the vacuum state. Wick's theorem gives an expression for the time-ordered product of an arbitrary number of field operators. The graphical representation of these processes are called *Feynman diagrams*. In the x -representation each diagram describes a number of processes. The contraction functions can also be written as:

$$\Delta^F(x) = \lim_{\epsilon \rightarrow 0} \frac{-2i}{(2\pi)^4} \int \frac{e^{ikx}}{k^2 + m^2 - i\epsilon} d^4k \quad \text{and} \quad S^F(x) = \lim_{\epsilon \rightarrow 0} \frac{-2i}{(2\pi)^4} \int e^{ipx} \frac{i\gamma_\mu p_\mu - M}{p^2 + M^2 - i\epsilon} d^4p$$

In the expressions for $S^{(2)}$ this gives rise to terms $\delta(p+k-p'-k')$. This means that energy and momentum is conserved. However, virtual particles do not obey the relation between energy and momentum.

15.10 Divergences and renormalization

It turns out that higher orders contribute infinite terms because only the sum $p+k$ of the four-momentum of the virtual particles is fixed. An integration over one of them becomes ∞ . In the x -representation this can be understood because the product of two functions containing δ -like singularities is not well defined. This is solved by discounting all divergent diagrams in a renormalization of e and M . It is assumed that an electron, if there would not be an electromagnetical field, would have a mass M_0 and a charge e_0 unequal to the observed mass M and charge e . In the Hamilton and Lagrange density of the free electron-positron field appears M_0 . So this gives, with $M = M_0 + \Delta M$:

$$\mathcal{L}_{e-p}(x) = -\bar{\psi}(x)(\gamma_\mu\partial_\mu + M_0)\psi(x) = -\bar{\psi}(x)(\gamma_\mu\partial_\mu + M)\psi(x) + \Delta M\bar{\psi}(x)\psi(x)$$

and $\mathcal{H}_{\text{int}} = ieN(\bar{\psi}\gamma_\mu\psi A_\mu) - i\Delta eN(\bar{\psi}\gamma_\mu\psi A_\mu)$.

15.11 Classification of elementary particles

Elementary particles can be categorized as follows:

1. **Hadrons:** these exist of quarks and can be categorized in:

I. **Baryons:** these exist of 3 quarks or 3 antiquarks.

II. **Mesons:** these exist of one quark and one antiquark.

2. **Leptons:** $e^\pm, \mu^\pm, \tau^\pm, \nu_e, \nu_\mu, \nu_\tau, \bar{\nu}_e, \bar{\nu}_\mu, \bar{\nu}_\tau$.

3. **Field quanta:** γ, W^\pm, Z^0 , gluons, gravitons (?).

An overview of particles and antiparticles is given in the following table:

Particle	spin (\hbar)	B	L	T	T_3	S	C	B^*	charge (e)	m_0 (MeV)	antipart.
u	1/2	1/3	0	1/2	1/2	0	0	0	+2/3	5	\bar{u}
d	1/2	1/3	0	1/2	-1/2	0	0	0	-1/3	9	\bar{d}
s	1/2	1/3	0	0	0	-1	0	0	-1/3	175	\bar{s}
c	1/2	1/3	0	0	0	0	1	0	+2/3	1350	\bar{c}
b	1/2	1/3	0	0	0	0	0	-1	-1/3	4500	\bar{b}
t	1/2	1/3	0	0	0	0	0	0	+2/3	173000	\bar{t}
e^-	1/2	0	1	0	0	0	0	0	-1	0.511	e^+
μ^-	1/2	0	1	0	0	0	0	0	-1	105.658	μ^+
τ^-	1/2	0	1	0	0	0	0	0	-1	1777.1	τ^+
ν_e	1/2	0	1	0	0	0	0	0	0	0(?)	$\bar{\nu}_e$
ν_μ	1/2	0	1	0	0	0	0	0	0	0(?)	$\bar{\nu}_\mu$
ν_τ	1/2	0	1	0	0	0	0	0	0	0(?)	$\bar{\nu}_\tau$
γ	1	0	0	0	0	0	0	0	0	0	γ
gluon	1	0	0	0	0	0	0	0	0	0	gluon
W^+	1	0	0	0	0	0	0	0	+1	80220	W^-
Z	1	0	0	0	0	0	0	0	0	91187	Z
graviton	2	0	0	0	0	0	0	0	0	0	graviton

Here B is the baryon number and L the lepton number. It is found that there are three different lepton numbers, one for e, μ and τ , which are separately conserved. T is the isospin, with T_3 the projection of the isospin on the third axis, C the charmness, S the strangeness and B^* the bottomness. The anti particles have quantum numbers with the opposite sign except for the total isospin T. The composition of (anti)quarks of the hadrons is given in the following table, together with their mass in MeV in their ground state:

π^0	$\frac{1}{2}\sqrt{2}(u\bar{u}+d\bar{d})$	134.9764	J/ Ψ	$c\bar{c}$	3096.8	$\bar{\Sigma}^+$	$\bar{d}\bar{d}\bar{s}$	1197.436
π^+	$u\bar{d}$	139.56995	Υ	$b\bar{b}$	9460.37	Ξ^0	$u s s$	1314.9
π^-	$d\bar{u}$	139.56995	p^+	$u u d$	938.27231	Ξ^0	$\bar{u} \bar{s} \bar{s}$	1314.9
K^0	$s\bar{d}$	497.672	p^-	$\bar{u} \bar{u} \bar{d}$	938.27231	Ξ^-	$d s s$	1321.32
K^0	$d\bar{s}$	497.672	n^0	$u d d$	939.56563	Ξ^+	$\bar{d} \bar{s} \bar{s}$	1321.32
K^+	$u\bar{s}$	493.677	\bar{n}^0	$\bar{u} \bar{d} \bar{d}$	939.56563	Ω^-	$s s s$	1672.45
K^-	$s\bar{u}$	493.677	Λ	$u d s$	1115.684	Ω^+	$\bar{s} \bar{s} \bar{s}$	1672.45
D^+	$c\bar{d}$	1869.4	$\bar{\Lambda}$	$\bar{u} \bar{d} \bar{s}$	1115.684	Λ_c^+	$u d c$	2285.1
D^-	$d\bar{c}$	1869.4	Σ^+	$u u s$	1189.37	Δ^{2-}	$\bar{u} \bar{u} \bar{u}$	1232.0
D^0	$c\bar{u}$	1864.6	$\bar{\Sigma}^-$	$\bar{u} \bar{u} \bar{s}$	1189.37	Δ^{2+}	$u u u$	1232.0
D^0	$u\bar{c}$	1864.6	Σ^0	$u d s$	1192.55	Δ^+	$u u d$	1232.0
F^+	$c\bar{s}$	1969.0	Σ^0	$\bar{u} \bar{d} \bar{s}$	1192.55	Δ^0	$u d d$	1232.0
F^-	$s\bar{c}$	1969.0	Σ^-	$d d s$	1197.436	Δ^-	$d d d$	1232.0

Each quark can exist in two spin states. So mesons are bosons with spin 0 or 1 in their ground state, while baryons are fermions with spin $\frac{1}{2}$ or $\frac{3}{2}$. There exist excited states with higher internal L . Neutrino's have a helicity of $-\frac{1}{2}$ while antineutrino's have only $+\frac{1}{2}$ as possible value.

The quantum numbers are subject to conservation laws. These can be derived from symmetries in the Lagrange density: continuous symmetries give rise to additive conservation laws, discrete symmetries result in multiplicative conservation laws.

Geometrical conservation laws are invariant under Lorentz transformations and the CPT-operation. These are:

1. Mass/energy because the laws of nature are invariant for translations in time.

2. Momentum because the laws of nature are invariant for translations in space.
3. Angular momentum because the laws of nature are invariant for rotations.

Dynamical conservation laws are invariant under the CPT-operation. These are:

1. Electrical charge because the Maxwell equations are invariant under gauge transformations.
2. Colour charge is conserved.
3. Isospin because QCD is invariant for rotations in T-space.
4. Baryon number and lepton number are conserved but not under a possible SU(5) symmetry of the laws of nature.
5. Quarks type is only conserved under the colour interaction.
6. Parity is conserved except for weak interactions.

The elementary particles can be classified into three families:

	leptons	quarks	antileptons	antiquarks
1st generation	e^- ν_e	d u	e^+ $\bar{\nu}_e$	\bar{d} \bar{u}
2nd generation	μ^- ν_μ	s c	μ^+ $\bar{\nu}_\mu$	\bar{s} \bar{c}
3rd generation	τ^- ν_τ	b t	τ^+ $\bar{\nu}_\tau$	\bar{b} \bar{t}

Quarks exist in three colours but because they are *confined* these colours cannot be seen directly. The color force does *not* decrease with distance. The potential energy will become high enough to create a quark-antiquark pair when it is tried to disjoin an (anti)quark from a hadron. This will result in two hadrons and not in free quarks.

15.12 P and CP-violation

It is found that the weak interaction violates P-symmetry, and even CP-symmetry is not conserved. Some processes which violate P symmetry but conserve the combination CP are:

1. μ -decay: $\mu^- \rightarrow e^- + \nu_\mu + \bar{\nu}_e$. Left-handed electrons appear more than $1000\times$ as much as right-handed ones.
2. β -decay of spin-polarized ^{60}Co : $^{60}\text{Co} \rightarrow ^{60}\text{Ni} + e^- + \bar{\nu}_e$. More electrons with a spin parallel to the Co than with a spin antiparallel are created: (parallel–antiparallel)/(total)=20%.
3. There is no connection with the neutrino: the decay of the Λ particle through: $\Lambda \rightarrow p^+ + \pi^-$ and $\Lambda \rightarrow n^0 + \pi^0$ has also these properties.

The CP-symmetry was found to be violated by the decay of neutral Kaons. These are the lowest possible states with a s-quark so they can decay only weakly. The following holds: $C|K^0\rangle = \eta|\bar{K}^0\rangle$ where η is a phase factor. Further holds $P|K^0\rangle = -|K^0\rangle$ because K^0 and \bar{K}^0 have an intrinsic parity of -1 . From this follows that K^0 and \bar{K}^0 are not eigenvalues of CP: $CP|K^0\rangle = |\bar{K}^0\rangle$. The linear combinations

$$|K_1^0\rangle := \frac{1}{\sqrt{2}}\sqrt{2}(|K^0\rangle + |\bar{K}^0\rangle) \text{ and } |K_2^0\rangle := \frac{1}{\sqrt{2}}\sqrt{2}(|K^0\rangle - |\bar{K}^0\rangle)$$

are eigenstates of CP: $CP|K_1^0\rangle = +|K_1^0\rangle$ and $CP|K_2^0\rangle = -|K_2^0\rangle$. A base of K_1^0 and K_2^0 is practical while describing weak interactions. For colour interactions a base of K^0 and \bar{K}^0 is practical because then the number u–number \bar{u} is constant. The expansion postulate must be used for weak decays:

$$|K^0\rangle = \frac{1}{\sqrt{2}}(\langle K_1^0|K^0\rangle + \langle K_2^0|K^0\rangle)$$

The probability to find a final state with $CP = -1$ is $\frac{1}{2} |\langle K_2^0 | K^0 \rangle|^2$, the probability of $CP = +1$ decay is $\frac{1}{2} |\langle K_1^0 | K^0 \rangle|^2$.

The relation between the mass eigenvalues of the quarks (unaccented) and the fields arising in the weak currents (accented) is $(u', c', t') = (u, c, t)$, and:

$$\begin{pmatrix} d' \\ s' \\ b' \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \theta_2 & \sin \theta_2 \\ 0 & -\sin \theta_2 & \cos \theta_2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{i\delta} \end{pmatrix} \begin{pmatrix} \cos \theta_1 & \sin \theta_1 & 0 \\ -\sin \theta_1 & \cos \theta_1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \theta_3 & \sin \theta_3 \\ 0 & -\sin \theta_3 & \cos \theta_3 \end{pmatrix} \begin{pmatrix} d \\ s \\ b \end{pmatrix}$$

$\theta_1 \equiv \theta_C$ is the *Cabibbo angle*: $\sin(\theta_C) \approx 0.23 \pm 0.01$.

15.13 The standard model

When one wants to make the Lagrange density which describes a field invariant for local gauge transformations from a certain group, one has to perform the transformation

$$\frac{\partial}{\partial x_\mu} \rightarrow \frac{D}{Dx_\mu} = \frac{\partial}{\partial x_\mu} - i \frac{g}{\hbar} L_k A_\mu^k$$

Here the L_k are the generators of the gauge group (the “charges”) and the A_μ^k are the gauge fields. g is the matching coupling constant. The Lagrange density for a scalar field becomes:

$$\mathcal{L} = -\frac{1}{2} (D_\mu \Phi^* D^\mu \Phi + M^2 \Phi^* \Phi) - \frac{1}{4} F_{\mu\nu}^a F_a^{\mu\nu}$$

and the field tensors are given by: $F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g c_{lm}^a A_\mu^l A_\nu^m$.

15.13.1 The electroweak theory

The electroweak interaction arises from the necessity to keep the Lagrange density invariant for local gauge transformations of the group $SU(2) \otimes U(1)$. Right- and left-handed spin states are treated different because the weak interaction does not conserve parity. If a fifth Dirac matrix is defined by:

$$\gamma_5 := \gamma_1 \gamma_2 \gamma_3 \gamma_4 = - \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

the left- and right- handed solutions of the Dirac equation for neutrino's are given by:

$$\psi_L = \frac{1}{2} (1 + \gamma_5) \psi \quad \text{and} \quad \psi_R = \frac{1}{2} (1 - \gamma_5) \psi$$

It appears that neutrino's are always left-handed while antineutrino's are always right-handed. The *hypercharge* Y , for quarks given by $Y = B + S + C + B^* + T'$, is defined by:

$$Q = \frac{1}{2} Y + T_3$$

so $[Y, T_k] = 0$. The group $U(1)_Y \otimes SU(2)_T$ is taken as symmetry group for the electroweak interaction because the generators of this group commute. The multiplets are classified as follows:

	e_R^-	ν_{eL} e_L^-	u_L d_L'	u_R	d_R
T	0	$\frac{1}{2}$	$\frac{1}{2}$	0	0
T_3	0	$\frac{1}{2}$ $-\frac{1}{2}$	$\frac{1}{2}$ $-\frac{1}{2}$	0	0
Y	-2	-1	$\frac{1}{3}$	$\frac{4}{3}$	$-\frac{2}{3}$

Now, 1 field $B_\mu(x)$ is connected with gauge group U(1) and 3 gauge fields $\vec{A}_\mu(x)$ are connected with SU(2). The total Lagrange density (minus the fieldterms) for the electron-fermion field now becomes:

$$\begin{aligned} \mathcal{L}_{0,\text{EW}} = & -(\overline{\psi_{\nu e,L}}, \overline{\psi_{eL}}) \gamma^\mu \left(\partial_\mu - i \frac{g}{\hbar} \vec{A}_\mu \cdot \left(\frac{1}{2} \vec{\sigma} \right) - \frac{1}{2} i \frac{g'}{\hbar} B_\mu \cdot (-1) \right) \begin{pmatrix} \psi_{\nu e,L} \\ \psi_{eL} \end{pmatrix} - \\ & \overline{\psi_{eR}} \gamma^\mu \left(\partial_\mu - \frac{1}{2} i \frac{g'}{\hbar} (-2) B_\mu \right) \psi_{eR} \end{aligned}$$

Here, $\frac{1}{2} \vec{\sigma}$ are the generators of T and -1 and -2 the generators of Y .

15.13.2 Spontaneous symmetry breaking: the Higgs mechanism

All leptons are massless in the equations above. Their mass is probably generated by *spontaneous symmetry breaking*. This means that the dynamic equations which describe the system have a symmetry which the ground state does not have. It is assumed that there exists an isospin-doublet of scalar fields Φ with electrical charges $+1$ and 0 and potential $V(\Phi) = -\mu^2 \Phi^* \Phi + \lambda (\Phi^* \Phi)^2$. Their antiparticles have charges -1 and 0 . The extra terms in \mathcal{L} arising from these fields, $\mathcal{L}_H = (D_{L\mu} \Phi)^* (D_{L\mu} \Phi) - V(\Phi)$, are globally U(1) \otimes SU(2) symmetric. Hence the state with the lowest energy corresponds with the state $\Phi^*(x) \Phi(x) = v = \mu^2 / 2\lambda = \text{constant}$. The field can be written (were ω^\pm and z are Nambu-Goldstone bosons which can be transformed away, and $m_\phi = \mu\sqrt{2}$) as:

$$\Phi = \begin{pmatrix} \Phi^+ \\ \Phi^0 \end{pmatrix} = \begin{pmatrix} i\omega^+ \\ (v + \phi - iz)/\sqrt{2} \end{pmatrix} \text{ and } \langle 0 | \Phi | 0 \rangle = \begin{pmatrix} 0 \\ v/\sqrt{2} \end{pmatrix}$$

Because this expectation value $\neq 0$ the SU(2) symmetry is broken but the U(1) symmetry is not. When the gauge fields in the resulting Lagrange density are separated one obtains:

$$\begin{aligned} W_\mu^- &= \frac{1}{2} \sqrt{2} (A_\mu^1 + i A_\mu^2), \quad W_\mu^+ = \frac{1}{2} \sqrt{2} (A_\mu^1 - i A_\mu^2) \\ Z_\mu &= \frac{g A_\mu^3 - g' B_\mu}{\sqrt{g^2 + g'^2}} \equiv A_\mu^3 \cos(\theta_W) - B_\mu \sin(\theta_W) \\ A_\mu &= \frac{g' A_\mu^3 + g B_\mu}{\sqrt{g^2 + g'^2}} \equiv A_\mu^3 \sin(\theta_W) + B_\mu \cos(\theta_W) \end{aligned}$$

where θ_W is called the *Weinberg angle*. For this angle holds: $\sin^2(\theta_W) = 0.255 \pm 0.010$. Relations for the masses of the field quanta can be obtained from the remaining terms: $M_W = \frac{1}{2} v g$ and $M_Z = \frac{1}{2} v \sqrt{g^2 + g'^2}$, and for the elementary charge holds: $e = \frac{g g'}{\sqrt{g^2 + g'^2}} = g' \cos(\theta_W) = g \sin(\theta_W)$

Experimentally it is found that $M_W = 80.022 \pm 0.26 \text{ GeV}/c^2$ and $M_Z = 91.187 \pm 0.007 \text{ GeV}/c^2$. According to the weak theory this should be: $M_W = 83.0 \pm 0.24 \text{ GeV}/c^2$ and $M_Z = 93.8 \pm 2.0 \text{ GeV}/c^2$.

15.13.3 Quantumchromodynamics

Coloured particles interact because the Lagrange density is invariant for the transformations of the group SU(3) of the colour interaction. A distinction can be made between two types of particles:

1. “White” particles: they have no colour charge, the generator $\vec{T} = 0$.
2. “Coloured” particles: the generators \vec{T} are $8 \times 3 \times 3$ matrices. There exist three colours and three anti-colours.

The Lagrange density for coloured particles is given by

$$\mathcal{L}_{\text{QCD}} = i \sum_k \overline{\Psi}_k \gamma^\mu D_\mu \Psi_k + \sum_{k,l} \overline{\Psi}_k M_{kl} \Psi_l - \frac{1}{4} F_{\mu\nu}^a F_a^{\mu\nu}$$

The gluons remain massless because this Lagrange density does not contain spinless particles. Because left- and right- handed quarks now belong to the same multiplet a mass term can be introduced. This term can be brought in the form $M_{kl} = m_k \delta_{kl}$.

15.14 Path integrals

The development in time of a quantum mechanical system can, besides with Schrödinger's equation, also be described by a *path integral* (Feynman):

$$\psi(x', t') = \int F(x', t', x, t) \psi(x, t) dx$$

in which $F(x', t', x, t)$ is the amplitude of probability to find a system on time t' in x' if it was in x on time t . Then,

$$F(x', t', x, t) = \int \exp\left(\frac{iS[x]}{\hbar}\right) d[x]$$

where $S[x]$ is an action-integral: $S[x] = \int L(x, \dot{x}, t) dt$. The notation $d[x]$ means that the integral has to be taken over all possible paths $[x]$:

$$\int d[x] := \lim_{n \rightarrow \infty} \frac{1}{N} \prod_n \left\{ \int_{-\infty}^{\infty} dx(t_n) \right\}$$

in which N is a normalization constant. To each path is assigned a probability amplitude $\exp(iS/\hbar)$. The classical limit can be found by taking $\delta S = 0$: the average of the exponent vanishes, except where it is stationary. In quantum field theory, the probability of the transition of a field operator $\Phi(\vec{x}, -\infty)$ to $\Phi'(\vec{x}, \infty)$ is given by

$$F(\Phi'(\vec{x}, \infty), \Phi(\vec{x}, -\infty)) = \int \exp\left(\frac{iS[\Phi]}{\hbar}\right) d[\Phi]$$

with the action-integral

$$S[\Phi] = \int_{\Omega} \mathcal{L}(\Phi, \partial_{\nu} \Phi) d^4x$$

15.15 Unification and quantum gravity

The strength of the forces varies with energy and the reciprocal coupling constants approach each other with increasing energy. The SU(5) model predicts complete unification of the electromagnetic, weak and colour forces at 10^{15} GeV. It also predicts 12 extra X bosons which couple leptons and quarks and are i.g. responsible for proton decay, with dominant channel $p^+ \rightarrow \pi^0 + e^+$, with an average lifetime of the proton of 10^{31} year. This model has been experimentally falsified.

Supersymmetric models assume a symmetry between bosons and fermions and predict partners for the currently known particles with a spin which differs $\frac{1}{2}$. The supersymmetric SU(5) model predicts unification at 10^{16} GeV and an average lifetime of the proton of 10^{33} year. The dominant decay channels in this theory are $p^+ \rightarrow K^+ + \bar{\nu}_{\mu}$ and $p^+ \rightarrow K^0 + \mu^+$.

Quantum gravity plays only a role in particle interactions at the Planck dimensions, where $\lambda_C \approx R_S$: $m_{Pl} = \sqrt{\hbar c/G} = 3 \cdot 10^{19}$ GeV, $t_{Pl} = \hbar/m_{Pl}c^2 = \sqrt{\hbar G/c^5} = 10^{-43}$ sec and $r_{Pl} = ct_{Pl} \approx 10^{-35}$ m.

Chapter 16

Astrophysics

16.1 Determination of distances

The *parallax* is mostly used to determine distances in nearby space. The parallax is the angular difference between two measurements of the position of the object from different view-points. If the annual parallax is given by p , the distance R of the object is given by $R = a/\sin(p)$, in which a is the radius of the Earth's orbit. The *clusterparallax* is used to determine the distance of a group of stars by using their motion w.r.t. a fixed background. The tangential velocity v_t and the radial velocity v_r of the stars along the sky are given by

$$v_r = V \cos(\theta) \quad , \quad v_t = V \sin(\theta) = \omega R$$

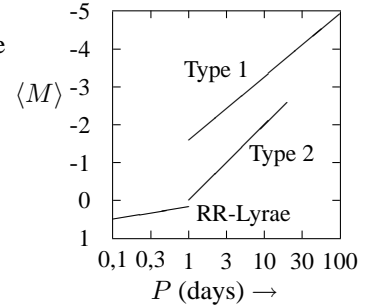
where θ is the angle between the star and the *point of convergence* and \hat{R} the distance in pc. This results, with $v_t = v_r \tan(\theta)$, in:

$$R = \frac{v_r \tan(\theta)}{\omega} \Rightarrow \hat{R} = \frac{1''}{p}$$

where p is the parallax in arc seconds. The parallax is then given by

$$p = \frac{4.74\mu}{v_r \tan(\theta)}$$

with μ de proper motion of the star in ''/yr. A method to determine the distance of objects which are somewhat further away, like galaxies and star clusters, uses the period-Brightness relation for Cepheids. This relation is shown in the above figure for different types of stars.



16.2 Brightness and magnitudes

The *brightness* is the total radiated energy per unit of time. Earth receives $s_0 = 1.374 \text{ kW/m}^2$ from the Sun. Hence, the brightness of the Sun is given by $L_\odot = 4\pi r^2 s_0 = 3.82 \cdot 10^{26} \text{ W}$. It is also given by:

$$L_\odot = 4\pi R_\odot^2 \int_0^\infty \pi F_\nu d\nu$$

where πF_ν is the monochromatic radiation flux. At the position of an observer this is πf_ν , with $f_\nu = (R/r)^2 F_\nu$ if absorption is ignored. If A_ν is the fraction of the flux which reaches Earth's surface, the transmission factor is given by R_ν and the surface of the detector is given by πa^2 , then the apparent brightness b is given by:

$$b = \pi a^2 \int_0^\infty f_\nu A_\nu R_\nu d\nu$$

The *magnitude* m is defined by:

$$\frac{b_1}{b_2} = (100)^{\frac{1}{5}(m_2 - m_1)} = (2.512)^{m_2 - m_1}$$

because the human eye perceives lightintensities logarithmical. From this follows that $m_2 - m_1 = 2.5 \cdot^{10} \log(b_1/b_2)$, or: $m = -2.5 \cdot^{10} \log(b) + C$. The apparent brightness of a star if this star would be at a distance of 10 pc is called the *absolute brightness* B : $B/b = (\hat{r}/10)^2$. The absolute magnitude is then given by $M = -2.5 \cdot^{10} \log(B) + C$, or: $M = 5 + m - 5 \cdot^{10} \log(\hat{r})$. When an interstellar absorption of $10^{-4}/\text{pc}$ is taken into account one finds:

$$M = (m - 4 \cdot 10^{-4} \hat{r}) + 5 - 5 \cdot^{10} \log(\hat{r})$$

If a detector detects all radiation emitted by a source one would measure the *absolute bolometric magnitude*. If the *bolometric correction* BC is given by

$$BC = 2.5 \cdot^{10} \log \left(\frac{\text{Energy flux received}}{\text{Energy flux detected}} \right) = 2.5 \cdot^{10} \log \left(\frac{\int f_\nu d\nu}{\int f_\nu A_\nu R_\nu d\nu} \right)$$

holds: $M_b = M_V - BC$ where M_V is the visual magnitude. Further holds

$$M_b = -2.5 \cdot^{10} \log \left(\frac{L}{L_\odot} \right) + 4.72$$

16.3 Radiation and stellar atmospheres

The radiation energy passing through a surface dA is $dE = I_\nu(\theta, \varphi) \cos(\theta) d\nu d\Omega dA dt$, where I_μ is the *monochromatical intensity* [$\text{Wm}^{-2}\text{sr}^{-1}\text{Hz}^{-1}$]. When there is no absorption the quantity I_ν is independent of the distance to the source. Planck's law holds for a black body:

$$I_\nu(T) \equiv B_\nu(T) = \frac{c}{4\pi} w_\nu(T) = \frac{2h\nu^3}{c^2} \frac{1}{\exp(h\nu/kT) - 1}$$

The radiation transport through a layer can then be written as:

$$\frac{dI_\nu}{ds} = -I_\nu \kappa_\nu + j_\nu$$

Here, j_ν is the *coefficient of emission* and κ_ν the *coefficient of absorption*. $\int ds$ is the thickness of the layer. The *optical thickness* τ_ν of the layer is given by $\tau_\nu = \int \kappa_\nu ds$. The layer is optically thin if $\tau_\nu \ll 1$, the layer is optically thick if $\tau_\nu \gg 1$. For a stellar atmosphere in LTE holds: $j_\nu = \kappa_\nu B_\nu(T)$. Then also holds:

$$I_\nu(s) = I_\nu(0)e^{-\tau_\nu} + B_\nu(T)(1 - e^{-\tau_\nu})$$

16.4 Composition and evolution of stars

The structure of a star is described by the following equations:

$$\begin{aligned} \frac{dM(r)}{dr} &= 4\pi \varrho(r) r^2 \\ \frac{dp(r)}{dr} &= -\frac{GM(r)\varrho(r)}{r^2} \\ \frac{L(r)}{dr} &= 4\pi \varrho(r) \varepsilon(r) r^2 \\ \left(\frac{dT(r)}{dr} \right)_{\text{rad}} &= -\frac{3}{4} \frac{L(r)}{4\pi r^2} \frac{\kappa(r)}{4\sigma T^3(r)}, \text{ (Eddington), or} \\ \left(\frac{dT(r)}{dr} \right)_{\text{conv}} &= \frac{T(r)}{p(r)} \frac{\gamma - 1}{\gamma} \frac{dp(r)}{dr}, \text{ (convective energy transport)} \end{aligned}$$

Further, for stars of the solar type, the composing plasma can be described as an ideal gas:

$$p(r) = \frac{\varrho(r)kT(r)}{\mu m_{\text{H}}}$$

The ∇ -operator

In cartesian coordinates (x, y, z) holds:

$$\begin{aligned}\vec{\nabla} &= \frac{\partial}{\partial x} \vec{e}_x + \frac{\partial}{\partial y} \vec{e}_y + \frac{\partial}{\partial z} \vec{e}_z, \quad \text{grad} f = \vec{\nabla} f = \frac{\partial f}{\partial x} \vec{e}_x + \frac{\partial f}{\partial y} \vec{e}_y + \frac{\partial f}{\partial z} \vec{e}_z \\ \text{div } \vec{a} &= \vec{\nabla} \cdot \vec{a} = \frac{\partial a_x}{\partial x} + \frac{\partial a_y}{\partial y} + \frac{\partial a_z}{\partial z}, \quad \nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \\ \text{rot } \vec{a} &= \vec{\nabla} \times \vec{a} = \left(\frac{\partial a_z}{\partial y} - \frac{\partial a_y}{\partial z} \right) \vec{e}_x + \left(\frac{\partial a_x}{\partial z} - \frac{\partial a_z}{\partial x} \right) \vec{e}_y + \left(\frac{\partial a_y}{\partial x} - \frac{\partial a_x}{\partial y} \right) \vec{e}_z\end{aligned}$$

In cylinder coordinates (r, φ, z) holds:

$$\begin{aligned}\vec{\nabla} &= \frac{\partial}{\partial r} \vec{e}_r + \frac{1}{r} \frac{\partial}{\partial \varphi} \vec{e}_\varphi + \frac{\partial}{\partial z} \vec{e}_z, \quad \text{grad} f = \frac{\partial f}{\partial r} \vec{e}_r + \frac{1}{r} \frac{\partial f}{\partial \varphi} \vec{e}_\varphi + \frac{\partial f}{\partial z} \vec{e}_z \\ \text{div } \vec{a} &= \frac{\partial a_r}{\partial r} + \frac{a_r}{r} + \frac{1}{r} \frac{\partial a_\varphi}{\partial \varphi} + \frac{\partial a_z}{\partial z}, \quad \nabla^2 f = \frac{\partial^2 f}{\partial r^2} + \frac{1}{r} \frac{\partial f}{\partial r} + \frac{1}{r^2} \frac{\partial^2 f}{\partial \varphi^2} + \frac{\partial^2 f}{\partial z^2} \\ \text{rot } \vec{a} &= \left(\frac{1}{r} \frac{\partial a_z}{\partial \varphi} - \frac{\partial a_\varphi}{\partial z} \right) \vec{e}_r + \left(\frac{\partial a_r}{\partial z} - \frac{\partial a_z}{\partial r} \right) \vec{e}_\varphi + \left(\frac{\partial a_\varphi}{\partial r} + \frac{a_\varphi}{r} - \frac{1}{r} \frac{\partial a_r}{\partial \varphi} \right) \vec{e}_z\end{aligned}$$

In spherical coordinates (r, θ, φ) holds:

$$\begin{aligned}\vec{\nabla} &= \frac{\partial}{\partial r} \vec{e}_r + \frac{1}{r} \frac{\partial}{\partial \theta} \vec{e}_\theta + \frac{1}{r \sin \theta} \frac{\partial}{\partial \varphi} \vec{e}_\varphi \\ \text{grad} f &= \frac{\partial f}{\partial r} \vec{e}_r + \frac{1}{r} \frac{\partial f}{\partial \theta} \vec{e}_\theta + \frac{1}{r \sin \theta} \frac{\partial f}{\partial \varphi} \vec{e}_\varphi \\ \text{div } \vec{a} &= \frac{\partial a_r}{\partial r} + \frac{2a_r}{r} + \frac{1}{r} \frac{\partial a_\theta}{\partial \theta} + \frac{a_\theta}{r \tan \theta} + \frac{1}{r \sin \theta} \frac{\partial a_\varphi}{\partial \varphi} \\ \text{rot } \vec{a} &= \left(\frac{1}{r} \frac{\partial a_\varphi}{\partial \theta} + \frac{a_\theta}{r \tan \theta} - \frac{1}{r \sin \theta} \frac{\partial a_\theta}{\partial \varphi} \right) \vec{e}_r + \left(\frac{1}{r \sin \theta} \frac{\partial a_r}{\partial \varphi} - \frac{\partial a_\varphi}{\partial r} - \frac{a_\varphi}{r} \right) \vec{e}_\theta + \\ &\quad \left(\frac{\partial a_\theta}{\partial r} + \frac{a_\theta}{r} - \frac{1}{r} \frac{\partial a_r}{\partial \theta} \right) \vec{e}_\varphi \\ \nabla^2 f &= \frac{\partial^2 f}{\partial r^2} + \frac{2}{r} \frac{\partial f}{\partial r} + \frac{1}{r^2} \frac{\partial^2 f}{\partial \theta^2} + \frac{1}{r^2 \tan \theta} \frac{\partial f}{\partial \theta} + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 f}{\partial \varphi^2}\end{aligned}$$

General orthonormal curvilinear coordinates (u, v, w) can be obtained from cartesian coordinates by the transformation $\vec{x} = \vec{x}(u, v, w)$. The unit vectors are then given by:

$$\vec{e}_u = \frac{1}{h_1} \frac{\partial \vec{x}}{\partial u}, \quad \vec{e}_v = \frac{1}{h_2} \frac{\partial \vec{x}}{\partial v}, \quad \vec{e}_w = \frac{1}{h_3} \frac{\partial \vec{x}}{\partial w}$$

where the factors h_i set the norm to 1. Then holds:

$$\begin{aligned}\text{grad} f &= \frac{1}{h_1} \frac{\partial f}{\partial u} \vec{e}_u + \frac{1}{h_2} \frac{\partial f}{\partial v} \vec{e}_v + \frac{1}{h_3} \frac{\partial f}{\partial w} \vec{e}_w \\ \text{div } \vec{a} &= \frac{1}{h_1 h_2 h_3} \left(\frac{\partial}{\partial u} (h_2 h_3 a_u) + \frac{\partial}{\partial v} (h_3 h_1 a_v) + \frac{\partial}{\partial w} (h_1 h_2 a_w) \right) \\ \text{rot } \vec{a} &= \frac{1}{h_2 h_3} \left(\frac{\partial (h_3 a_w)}{\partial v} - \frac{\partial (h_2 a_v)}{\partial w} \right) \vec{e}_u + \frac{1}{h_3 h_1} \left(\frac{\partial (h_1 a_u)}{\partial w} - \frac{\partial (h_3 a_w)}{\partial u} \right) \vec{e}_v + \\ &\quad \frac{1}{h_1 h_2} \left(\frac{\partial (h_2 a_v)}{\partial u} - \frac{\partial (h_1 a_u)}{\partial v} \right) \vec{e}_w \\ \nabla^2 f &= \frac{1}{h_1 h_2 h_3} \left[\frac{\partial}{\partial u} \left(\frac{h_2 h_3}{h_1} \frac{\partial f}{\partial u} \right) + \frac{\partial}{\partial v} \left(\frac{h_3 h_1}{h_2} \frac{\partial f}{\partial v} \right) + \frac{\partial}{\partial w} \left(\frac{h_1 h_2}{h_3} \frac{\partial f}{\partial w} \right) \right]\end{aligned}$$

The SI units

Basic units

Quantity	Unit	Sym.
Length	metre	m
Mass	kilogram	kg
Time	second	s
Therm. temp.	kelvin	K
Electr. current	ampere	A
Luminous intens.	candela	cd
Amount of subst.	mol	mol

Extra units

Plane angle	radian	rad
solid angle	sterradian	sr

Derived units with special names

Quantity	Unit	Sym.	Derivation
Frequency	hertz	Hz	s^{-1}
Force	newton	N	$kg \cdot m \cdot s^{-2}$
Pressure	pascal	Pa	$N \cdot m^{-2}$
Energy	joule	J	$N \cdot m$
Power	watt	W	$J \cdot s^{-1}$
Charge	coulomb	C	$A \cdot s$
El. Potential	volt	V	$W \cdot A^{-1}$
El. Capacitance	farad	F	$C \cdot V^{-1}$
El. Resistance	ohm	Ω	$V \cdot A^{-1}$
El. Conductance	siemens	S	$A \cdot V^{-1}$
Mag. flux	weber	Wb	$V \cdot s$
Mag. flux density	tesla	T	$Wb \cdot m^{-2}$
Inductance	henry	H	$Wb \cdot A^{-1}$
Luminous flux	lumen	lm	$cd \cdot sr$
Illuminance	lux	lx	$lm \cdot m^{-2}$
Activity	becquerel	Bq	s^{-1}
Absorbed dose	gray	Gy	$J \cdot kg^{-1}$
Dose equivalent	sievert	Sv	$J \cdot kg^{-1}$

Prefixes

exa	E	10^{18}	mega	M	10^6	deci	d	10^{-1}	nano	n	10^{-9}
peta	P	10^{15}	kilo	k	10^3	centi	c	10^{-2}	pico	p	10^{-12}
tera	T	10^{12}	hecto	h	10^2	milli	m	10^{-3}	femto	f	10^{-15}
giga	G	10^9	deca	da	10	micro	μ	10^{-6}	atto	a	10^{-18}

FUNDAMENTAL PHYSICAL CONSTANTS

Peter J. Mohr and Barry N. Taylor

These tables give the 1998 self-consistent set of values of the basic constants and conversion factors of physics and chemistry recommended by the Committee on Data for Science and Technology (CODATA) for international use. The 1998 set replaces the previous set of constants recommended by CODATA in 1986; assigned uncertainties have been reduced by a factor of 1/5 to 1/12 (and sometimes even greater) relative to the 1986 uncertainties. The recommended set is based on a least-squares adjustment involving all of the relevant experimental and theoretical data available through December 31, 1998. Full details of the input data and the adjustment procedure are given in Reference 1.

The 1998 adjustment was carried out by P. J. Mohr and B. N. Taylor of the National Institute of Standards and Technology (NIST) under the auspices of the CODATA Task Group on Fundamental Constants. The Task Group was established in 1969 with the aim of periodically providing the scientific and technological communities with a self-consistent set of internationally recommended values of the fundamental physical constants based on all applicable information available at a given point in time. The first set was published in 1973 and was followed by a revised set first published in 1986; the current 1998 set first appeared in 1999. In the future, the CODATA Task Group plans to take advantage of the high level of automation developed for the current set in order to issue a new set of recommended values at least every four years.

At the time of completion of the 1998 adjustment, the membership of the Task Group was as follows:

F. Cabiati, Istituto Elettrotecnico Nazionale "Galileo Ferraris," Italy
E. R. Cohen, Science Center, Rockwell International (retired), United States of America
T. Endo, Electrotechnical Laboratory, Japan
R. Liu, National Institute of Metrology, China (People's Republic of)
B. A. Mamyrin, A. F. Ioffe Physical-Technical Institute, Russian Federation
P. J. Mohr, National Institute of Standards and Technology, United States of America
F. Nez, Laboratoire Kastler-Brossel, France
B. W. Petley, National Physical Laboratory, United Kingdom
T. J. Quinn, Bureau International des Poids et Mesures
B. N. Taylor, National Institute of Standards and Technology, United States of America
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B. M. Wood, National Research Council, Canada

REFERENCES

1. Mohr, Peter J., and Taylor, Barry N., *J. Phys. Chem. Ref. Data* **28**, 1713, 1999; *Rev. Mod. Phys.* **72**, 351, 2000. The 1998 set of recommended values is also available at the Web site of the Fundamental Constants Data Center of the NIST Physics Laboratory: <http://physics.nist.gov/constants>.

Fundamental Physical Constants

Quantity	Symbol	Value	Unit	Relative std. uncert. u_r
UNIVERSAL				
speed of light in vacuum	c, c_0	299 792 458	m s^{-1}	(exact)
magnetic constant	μ_0	$4\pi \times 10^{-7}$ $= 12.566\,370\,614\dots \times 10^{-7}$	N A^{-2} N A^{-2}	(exact)
electric constant $1/\mu_0 c^2$	ϵ_0	$8.854\,187\,817\dots \times 10^{-12}$	F m^{-1}	(exact)
characteristic impedance of vacuum $\sqrt{\mu_0/\epsilon_0} = \mu_0 c$	Z_0	376.730 313 461...	Ω	(exact)
Newtonian constant of gravitation	G $G/\hbar c$	$6.673(10) \times 10^{-11}$ $6.707(10) \times 10^{-39}$	$\text{m}^3 \text{kg}^{-1} \text{s}^{-2}$ $(\text{GeV}/c^2)^{-2}$	1.5×10^{-3} 1.5×10^{-3}
Planck constant in eV s	h	$6.626\,068\,76(52) \times 10^{-34}$ $4.135\,667\,27(16) \times 10^{-15}$	J s eV s	7.8×10^{-8} 3.9×10^{-8}
$\hbar/2\pi$ in eV s	\hbar	$1.054\,571\,596(82) \times 10^{-34}$ $6.582\,118\,89(26) \times 10^{-16}$	J s eV s	7.8×10^{-8} 3.9×10^{-8}
Planck mass $(\hbar c/G)^{1/2}$	m_{P}	$2.1767(16) \times 10^{-8}$	kg	7.5×10^{-4}
Planck length $\hbar/m_{\text{P}} c = (\hbar G/c^3)^{1/2}$	l_{P}	$1.6160(12) \times 10^{-35}$	m	7.5×10^{-4}
Planck time $l_{\text{P}}/c = (\hbar G/c^5)^{1/2}$	t_{P}	$5.3906(40) \times 10^{-44}$	s	7.5×10^{-4}
ELECTROMAGNETIC				
elementary charge	e e/h	$1.602\,176\,462(63) \times 10^{-19}$ $2.417\,989\,491(95) \times 10^{14}$	C A J ⁻¹	3.9×10^{-8} 3.9×10^{-8}
magnetic flux quantum $h/2e$	Φ_0	$2.067\,833\,636(81) \times 10^{-15}$	Wb	3.9×10^{-8}
conductance quantum $2e^2/h$	G_0	$7.748\,091\,696(28) \times 10^{-5}$	S	3.7×10^{-9}
inverse of conductance quantum	G_0^{-1}	12 906.403 786(47)	Ω	3.7×10^{-9}
Josephson constant ^a $2e/h$	K_{J}	$483\,597.898(19) \times 10^9$	Hz V ⁻¹	3.9×10^{-8}
von Klitzing constant ^b $h/e^2 = \mu_0 c/2\alpha$	R_{K}	25 812.807 572(95)	Ω	3.7×10^{-9}
Bohr magneton $e\hbar/2m_{\text{e}}$ in eV T ⁻¹	μ_{B} μ_{B}/h μ_{B}/hc μ_{B}/k	$927.400\,899(37) \times 10^{-26}$ $5.788\,381\,749(43) \times 10^{-5}$ $13.996\,246\,24(56) \times 10^9$ 46.686 4521(19)	J T ⁻¹ eV T ⁻¹ Hz T ⁻¹ $\text{m}^{-1} \text{T}^{-1}$	4.0×10^{-8} 7.3×10^{-9} 4.0×10^{-8} 4.0×10^{-8}
nuclear magneton $e\hbar/2m_{\text{p}}$ in eV T ⁻¹	μ_{N} μ_{N}/h μ_{N}/hc μ_{N}/k	$5.050\,783\,17(20) \times 10^{-27}$ $3.152\,451\,238(24) \times 10^{-8}$ $7.622\,593\,96(31)$ $2.542\,623\,66(10) \times 10^{-2}$ $3.658\,2638(64) \times 10^{-4}$	J T ⁻¹ eV T ⁻¹ MHz T ⁻¹ $\text{m}^{-1} \text{T}^{-1}$ K T ⁻¹	4.0×10^{-8} 7.6×10^{-9} 4.0×10^{-8} 4.0×10^{-8} 1.7×10^{-6}
ATOMIC AND NUCLEAR				
General				
fine-structure constant $e^2/4\pi\epsilon_0\hbar c$	α	$7.297\,352\,533(27) \times 10^{-3}$		3.7×10^{-9}
inverse fine-structure constant	α^{-1}	137.035 999 76(50)		3.7×10^{-9}

Fundamental Physical Constants

Quantity	Symbol	Value	Unit	Relative std. uncert. u_r
Rydberg constant $\alpha^2 m_e c / 2h$	R_∞	10 973 731.568 549(83)	m^{-1}	7.6×10^{-12}
	$R_\infty c$	$3.289\,841\,960\,368(25) \times 10^{15}$	Hz	7.6×10^{-12}
	$R_\infty h c$	$2.179\,871\,90(17) \times 10^{-18}$	J	7.8×10^{-8}
$R_\infty h c$ in eV		13.605 691 72(53)	eV	3.9×10^{-8}
Bohr radius $\alpha / 4\pi R_\infty = 4\pi\epsilon_0 \hbar^2 / m_e e^2$	a_0	$0.529\,177\,2083(19) \times 10^{-10}$	m	3.7×10^{-9}
Hartree energy $e^2 / 4\pi\epsilon_0 a_0 = 2R_\infty h c$				
$= \alpha^2 m_e c^2$	E_h	$4.359\,743\,81(34) \times 10^{-18}$	J	7.8×10^{-8}
in eV		27.211 3834(11)	eV	3.9×10^{-8}
quantum of circulation	$h/2m_e$	$3.636\,947\,516(27) \times 10^{-4}$	$\text{m}^2 \text{s}^{-1}$	7.3×10^{-9}
	h/m_e	$7.273\,895\,032(53) \times 10^{-4}$	$\text{m}^2 \text{s}^{-1}$	7.3×10^{-9}
Electroweak				
Fermi coupling constant ^c	$G_F/(\hbar c)^3$	$1.166\,39(1) \times 10^{-5}$	GeV^{-2}	8.6×10^{-6}
weak mixing angle ^d θ_W (on-shell scheme)				
$\sin^2 \theta_W = s_W^2 \equiv 1 - (m_W/m_Z)^2$	$\sin^2 \theta_W$	0.2224(19)		8.7×10^{-3}
Electron, e^-				
electron mass	m_e	$9.109\,381\,88(72) \times 10^{-31}$	kg	7.9×10^{-8}
in u, $m_e = A_r(e) \text{ u}$ (electron relative atomic mass times u)		$5.485\,799\,110(12) \times 10^{-4}$	u	2.1×10^{-9}
energy equivalent	$m_e c^2$	$8.187\,104\,14(64) \times 10^{-14}$	J	7.9×10^{-8}
in MeV		0.510 998 902(21)	MeV	4.0×10^{-8}
electron-muon mass ratio	m_e/m_μ	$4.836\,332\,10(15) \times 10^{-3}$		3.0×10^{-8}
electron-tau mass ratio	m_e/m_τ	$2.875\,55(47) \times 10^{-4}$		1.6×10^{-4}
electron-proton mass ratio	m_e/m_p	$5.446\,170\,232(12) \times 10^{-4}$		2.1×10^{-9}
electron-neutron mass ratio	m_e/m_n	$5.438\,673\,462(12) \times 10^{-4}$		2.2×10^{-9}
electron-deuteron mass ratio	m_e/m_d	$2.724\,437\,1170(58) \times 10^{-4}$		2.1×10^{-9}
electron to alpha particle mass ratio	m_e/m_α	$1.370\,933\,5611(29) \times 10^{-4}$		2.1×10^{-9}
electron charge to mass quotient	$-e/m_e$	$-1.758\,820\,174(71) \times 10^{11}$	C kg^{-1}	4.0×10^{-8}
electron molar mass $N_A m_e$	$M(e), M_e$	$5.485\,799\,110(12) \times 10^{-7}$	kg mol^{-1}	2.1×10^{-9}
Compton wavelength $h/m_e c$	λ_C	$2.426\,310\,215(18) \times 10^{-12}$	m	7.3×10^{-9}
$\lambda_C/2\pi = \alpha a_0 = \alpha^2/4\pi R_\infty$	$\tilde{\lambda}_C$	$386.159\,2642(28) \times 10^{-15}$	m	7.3×10^{-9}
classical electron radius $\alpha^2 a_0$	r_e	$2.817\,940\,285(31) \times 10^{-15}$	m	1.1×10^{-8}
Thomson cross section $(8\pi/3)r_e^2$	σ_e	$0.665\,245\,854(15) \times 10^{-28}$	m^2	2.2×10^{-8}
electron magnetic moment	μ_e	$-928.476\,362(37) \times 10^{-26}$	J T^{-1}	4.0×10^{-8}
to Bohr magneton ratio	μ_e/μ_B	$-1.001\,159\,652\,1869(41)$		4.1×10^{-12}
to nuclear magneton ratio	μ_e/μ_N	$-1\,838.281\,9660(39)$		2.1×10^{-9}
electron magnetic moment anomaly $ \mu_e /\mu_B - 1$	a_e	$1.159\,652\,1869(41) \times 10^{-3}$		3.5×10^{-9}
electron g-factor $-2(1 + a_e)$	g_e	$-2.002\,319\,304\,3737(82)$		4.1×10^{-12}
electron-muon magnetic moment ratio	μ_e/μ_μ	206.766 9720(63)		3.0×10^{-8}

Fundamental Physical Constants

Quantity	Symbol	Value	Unit	Relative std. uncert. u_r
electron-proton magnetic moment ratio	μ_e/μ_p	$-658.210\,6875(66)$		1.0×10^{-8}
electron to shielded proton magnetic moment ratio (H ₂ O, sphere, 25 °C)	μ_e/μ'_p	$-658.227\,5954(71)$		1.1×10^{-8}
electron-neutron magnetic moment ratio	μ_e/μ_n	$960.920\,50(23)$		2.4×10^{-7}
electron-deuteron magnetic moment ratio	μ_e/μ_d	$-2\,143.923\,498(23)$		1.1×10^{-8}
electron to shielded helion ^e magnetic moment ratio (gas, sphere, 25 °C)	μ_e/μ'_h	$864.058\,255(10)$		1.2×10^{-8}
electron gyromagnetic ratio $2 \mu_e /\hbar$	γ_e	$1.760\,859\,794(71) \times 10^{11}$	$s^{-1} T^{-1}$	4.0×10^{-8}
	$\gamma_e/2\pi$	$28\,024.9540(11)$	$MHz T^{-1}$	4.0×10^{-8}
Muon, μ^-				
muon mass	m_μ	$1.883\,531\,09(16) \times 10^{-28}$	kg	8.4×10^{-8}
in u, $m_\mu = A_r(\mu) u$ (muon relative atomic mass times u)		$0.113\,428\,9168(34)$	u	3.0×10^{-8}
energy equivalent	$m_\mu c^2$	$1.692\,833\,32(14) \times 10^{-11}$	J	8.4×10^{-8}
in MeV		$105.658\,3568(52)$	MeV	4.9×10^{-8}
muon-electron mass ratio	m_μ/m_e	$206.768\,2657(63)$		3.0×10^{-8}
muon-tau mass ratio	m_μ/m_τ	$5.945\,72(97) \times 10^{-2}$		1.6×10^{-4}
muon-proton mass ratio	m_μ/m_p	$0.112\,609\,5173(34)$		3.0×10^{-8}
muon-neutron mass ratio	m_μ/m_n	$0.112\,454\,5079(34)$		3.0×10^{-8}
muon molar mass $N_A m_\mu$	$M(\mu), M_\mu$	$0.113\,428\,9168(34) \times 10^{-3}$	kg mol ⁻¹	3.0×10^{-8}
muon Compton wavelength $h/m_\mu c$	$\lambda_{C,\mu}$	$11.734\,441\,97(35) \times 10^{-15}$	m	2.9×10^{-8}
$\lambda_{C,\mu}/2\pi$	$\lambda_{C,\mu}$	$1.867\,594\,444(55) \times 10^{-15}$	m	2.9×10^{-8}
muon magnetic moment	μ_μ	$-4.490\,448\,13(22) \times 10^{-26}$	J T ⁻¹	4.9×10^{-8}
to Bohr magneton ratio	μ_μ/μ_B	$-4.841\,970\,85(15) \times 10^{-3}$		3.0×10^{-8}
to nuclear magneton ratio	μ_μ/μ_N	$-8.890\,597\,70(27)$		3.0×10^{-8}
muon magnetic moment anomaly $ \mu_\mu /(e\hbar/2m_\mu) - 1$	a_μ	$1.165\,916\,02(64) \times 10^{-3}$		5.5×10^{-7}
muon g-factor $-2(1 + a_\mu)$	g_μ	$-2.002\,331\,8320(13)$		6.4×10^{-10}
muon-proton magnetic moment ratio	μ_μ/μ_p	$-3.183\,345\,39(10)$		3.2×10^{-8}
Tau, τ^-				
tau mass ^f	m_τ	$3.167\,88(52) \times 10^{-27}$	kg	1.6×10^{-4}
in u, $m_\tau = A_r(\tau) u$ (tau relative atomic mass times u)		$1.907\,74(31)$	u	1.6×10^{-4}
energy equivalent	$m_\tau c^2$	$2.847\,15(46) \times 10^{-10}$	J	1.6×10^{-4}
in MeV		$1\,777.05(29)$	MeV	1.6×10^{-4}

Fundamental Physical Constants

Quantity	Symbol	Value	Unit	Relative std. uncert. u_r
tau-electron mass ratio	m_τ/m_e	3 477.60(57)		1.6×10^{-4}
tau-muon mass ratio	m_τ/m_μ	16.8188(27)		1.6×10^{-4}
tau-proton mass ratio	m_τ/m_p	1.893 96(31)		1.6×10^{-4}
tau-neutron mass ratio	m_τ/m_n	1.891 35(31)		1.6×10^{-4}
tau molar mass $N_A m_\tau$	$M(\tau), M_\tau$	$1.907\,74(31) \times 10^{-3}$	kg mol ⁻¹	1.6×10^{-4}
tau Compton wavelength $h/m_\tau c$	$\lambda_{C,\tau}$	$0.697\,70(11) \times 10^{-15}$	m	1.6×10^{-4}
$\lambda_{C,\tau}/2\pi$	$\tilde{\lambda}_{C,\tau}$	$0.111\,042(18) \times 10^{-15}$	m	1.6×10^{-4}
Proton, p				
proton mass	m_p	$1.672\,621\,58(13) \times 10^{-27}$	kg	7.9×10^{-8}
in u, $m_p = A_r(p)$ u (proton relative atomic mass times u)		1.007 276 466 88(13)	u	1.3×10^{-10}
energy equivalent	$m_p c^2$	$1.503\,277\,31(12) \times 10^{-10}$	J	7.9×10^{-8}
in MeV		938.271 998(38)	MeV	4.0×10^{-8}
proton-electron mass ratio	m_p/m_e	1 836.152 6675(39)		2.1×10^{-9}
proton-muon mass ratio	m_p/m_μ	8.880 244 08(27)		3.0×10^{-8}
proton-tau mass ratio	m_p/m_τ	0.527 994(86)		1.6×10^{-4}
proton-neutron mass ratio	m_p/m_n	0.998 623 478 55(58)		5.8×10^{-10}
proton charge to mass quotient	e/m_p	$9.578\,834\,08(38) \times 10^7$	C kg ⁻¹	4.0×10^{-8}
proton molar mass $N_A m_p$	$M(p), M_p$	$1.007\,276\,466\,88(13) \times 10^{-3}$	kg mol ⁻¹	1.3×10^{-10}
proton Compton wavelength $h/m_p c$	$\lambda_{C,p}$	$1.321\,409\,847(10) \times 10^{-15}$	m	7.6×10^{-9}
$\lambda_{C,p}/2\pi$	$\tilde{\lambda}_{C,p}$	$0.210\,308\,9089(16) \times 10^{-15}$	m	7.6×10^{-9}
proton magnetic moment	μ_p	$1.410\,606\,633(58) \times 10^{-26}$	J T ⁻¹	4.1×10^{-8}
to Bohr magneton ratio	μ_p/μ_B	$1.521\,032\,203(15) \times 10^{-3}$		1.0×10^{-8}
to nuclear magneton ratio	μ_p/μ_N	2.792 847 337(29)		1.0×10^{-8}
proton g -factor $2\mu_p/\mu_N$	g_p	5.585 694 675(57)		1.0×10^{-8}
proton-neutron				
magnetic moment ratio	μ_p/μ_n	-1.459 898 05(34)		2.4×10^{-7}
shielded proton magnetic moment (H ₂ O, sphere, 25 °C)	μ'_p	$1.410\,570\,399(59) \times 10^{-26}$	J T ⁻¹	4.2×10^{-8}
to Bohr magneton ratio	μ'_p/μ_B	$1.520\,993\,132(16) \times 10^{-3}$		1.1×10^{-8}
to nuclear magneton ratio	μ'_p/μ_N	2.792 775 597(31)		1.1×10^{-8}
proton magnetic shielding correction $1 - \mu'_p/\mu_p$ (H ₂ O, sphere, 25 °C)	σ'_p	$25.687(15) \times 10^{-6}$		5.7×10^{-4}
proton gyromagnetic ratio $2\mu_p/\hbar$	γ_p	$2.675\,222\,12(11) \times 10^8$	s ⁻¹ T ⁻¹	4.1×10^{-8}
	$\gamma_p/2\pi$	42.577 4825(18)	MHz T ⁻¹	4.1×10^{-8}
shielded proton gyromagnetic ratio $2\mu'_p/\hbar$ (H ₂ O, sphere, 25 °C)	γ'_p	$2.675\,153\,41(11) \times 10^8$	s ⁻¹ T ⁻¹	4.2×10^{-8}
	$\gamma'_p/2\pi$	42.576 3888(18)	MHz T ⁻¹	4.2×10^{-8}
Neutron, n				

Fundamental Physical Constants

Quantity	Symbol	Value	Unit	Relative std. uncert. u_r
neutron mass	m_n	$1.674\,927\,16(13) \times 10^{-27}$	kg	7.9×10^{-8}
in u, $m_n = A_r(n)$ u (neutron relative atomic mass times u)		1.008 664 915 78(55)	u	5.4×10^{-10}
energy equivalent	$m_n c^2$	$1.505\,349\,46(12) \times 10^{-10}$	J	7.9×10^{-8}
in MeV		939.565 330(38)	MeV	4.0×10^{-8}
neutron-electron mass ratio	m_n/m_e	1 838.683 6550(40)		2.2×10^{-9}
neutron-muon mass ratio	m_n/m_μ	8.892 484 78(27)		3.0×10^{-8}
neutron-tau mass ratio	m_n/m_τ	0.528 722(86)		1.6×10^{-4}
neutron-proton mass ratio	m_n/m_p	1.001 378 418 87(58)		5.8×10^{-10}
neutron molar mass $N_A m_n$	$M(n), M_n$	$1.008\,664\,915\,78(55) \times 10^{-3}$	kg mol ⁻¹	5.4×10^{-10}
neutron Compton wavelength $h/m_n c$	$\lambda_{C,n}$	$1.319\,590\,898(10) \times 10^{-15}$	m	7.6×10^{-9}
$\lambda_{C,n}/2\pi$	$\tilde{\lambda}_{C,n}$	$0.210\,019\,4142(16) \times 10^{-15}$	m	7.6×10^{-9}
neutron magnetic moment	μ_n	$-0.966\,236\,40(23) \times 10^{-26}$	J T ⁻¹	2.4×10^{-7}
to Bohr magneton ratio	μ_n/μ_B	$-1.041\,875\,63(25) \times 10^{-3}$		2.4×10^{-7}
to nuclear magneton ratio	μ_n/μ_N	-1.913 042 72(45)		2.4×10^{-7}
neutron g -factor $2\mu_n/\mu_N$	g_n	-3.826 085 45(90)		2.4×10^{-7}
neutron-electron magnetic moment ratio	μ_n/μ_e	$1.040\,668\,82(25) \times 10^{-3}$		2.4×10^{-7}
neutron-proton magnetic moment ratio	μ_n/μ_p	-0.684 979 34(16)		2.4×10^{-7}
neutron to shielded proton magnetic moment ratio (H ₂ O, sphere, 25 °C)	μ_n/μ'_p	-0.684 996 94(16)		2.4×10^{-7}
neutron gyromagnetic ratio $2 \mu_n /\hbar$	γ_n	$1.832\,471\,88(44) \times 10^8$	s ⁻¹ T ⁻¹	2.4×10^{-7}
	$\gamma_n/2\pi$	29.164 6958(70)	MHz T ⁻¹	2.4×10^{-7}
Deuteron, d				
deuteron mass	m_d	$3.343\,583\,09(26) \times 10^{-27}$	kg	7.9×10^{-8}
in u, $m_d = A_r(d)$ u (deuteron relative atomic mass times u)		2.013 553 212 71(35)	u	1.7×10^{-10}
energy equivalent	$m_d c^2$	$3.005\,062\,62(24) \times 10^{-10}$	J	7.9×10^{-8}
in MeV		1 875.612 762(75)	MeV	4.0×10^{-8}
deuteron-electron mass ratio	m_d/m_e	3 670.482 9550(78)		2.1×10^{-9}
deuteron-proton mass ratio	m_d/m_p	1.999 007 500 83(41)		2.0×10^{-10}
deuteron molar mass $N_A m_d$	$M(d), M_d$	$2.013\,553\,212\,71(35) \times 10^{-3}$	kg mol ⁻¹	1.7×10^{-10}
deuteron magnetic moment	μ_d	$0.433\,073\,457(18) \times 10^{-26}$	J T ⁻¹	4.2×10^{-8}
to Bohr magneton ratio	μ_d/μ_B	$0.466\,975\,4556(50) \times 10^{-3}$		1.1×10^{-8}
to nuclear magneton ratio	μ_d/μ_N	0.857 438 2284(94)		1.1×10^{-8}
deuteron-electron magnetic moment ratio	μ_d/μ_e	$-4.664\,345\,537(50) \times 10^{-4}$		1.1×10^{-8}
deuteron-proton magnetic moment ratio	μ_d/μ_p	0.307 012 2083(45)		1.5×10^{-8}

Fundamental Physical Constants

Quantity	Symbol	Value	Unit	Relative std. uncert. u_r
deuteron-neutron magnetic moment ratio	μ_d/μ_n	$-0.448\,206\,52(11)$		2.4×10^{-7}
Helion, h				
helion mass ^e in u, $m_h = A_r(h)$ u (helion relative atomic mass times u)	m_h	$5.006\,411\,74(39) \times 10^{-27}$	kg	7.9×10^{-8}
energy equivalent in MeV	$m_h c^2$	$3.014\,932\,234\,69(86)$ $4.499\,538\,48(35) \times 10^{-10}$ $2\,808.391\,32(11)$	u J MeV	2.8×10^{-10} 7.9×10^{-8} 4.0×10^{-8}
helion-electron mass ratio	m_h/m_e	$5\,495.885\,238(12)$		2.1×10^{-9}
helion-proton mass ratio	m_h/m_p	$2.993\,152\,658\,50(93)$		3.1×10^{-10}
helion molar mass $N_A m_h$	$M(h), M_h$	$3.014\,932\,234\,69(86) \times 10^{-3}$	kg mol ⁻¹	2.8×10^{-10}
shielded helion magnetic moment (gas, sphere, 25 °C)	μ'_h	$-1.074\,552\,967(45) \times 10^{-26}$	J T ⁻¹	4.2×10^{-8}
to Bohr magneton ratio	μ'_h/μ_B	$-1.158\,671\,474(14) \times 10^{-3}$		1.2×10^{-8}
to nuclear magneton ratio	μ'_h/μ_N	$-2.127\,497\,718(25)$		1.2×10^{-8}
shielded helion to proton magnetic moment ratio (gas, sphere, 25 °C)	μ'_h/μ_p	$-0.761\,766\,563(12)$		1.5×10^{-8}
shielded helion to shielded proton magnetic moment ratio (gas/H ₂ O, spheres, 25 °C)	μ'_h/μ'_p	$-0.761\,786\,1313(33)$		4.3×10^{-9}
shielded helion gyromagnetic ratio $2 \mu'_h /\hbar$ (gas, sphere, 25 °C)	γ'_h	$2.037\,894\,764(85) \times 10^8$	s ⁻¹ T ⁻¹	4.2×10^{-8}
	$\gamma'_h/2\pi$	$32.434\,1025(14)$	MHz T ⁻¹	4.2×10^{-8}
Alpha particle, α				
alpha particle mass in u, $m_\alpha = A_r(\alpha)$ u (alpha particle relative atomic mass times u)	m_α	$6.644\,655\,98(52) \times 10^{-27}$	kg	7.9×10^{-8}
energy equivalent in MeV	$m_\alpha c^2$	$4.001\,506\,1747(10)$ $5.971\,918\,97(47) \times 10^{-10}$ $3\,727.379\,04(15)$	u J MeV	2.5×10^{-10} 7.9×10^{-8} 4.0×10^{-8}
alpha particle to electron mass ratio	m_α/m_e	$7\,294.299\,508(16)$		2.1×10^{-9}
alpha particle to proton mass ratio	m_α/m_p	$3.972\,599\,6846(11)$		2.8×10^{-10}
alpha particle molar mass $N_A m_\alpha$	$M(\alpha), M_\alpha$	$4.001\,506\,1747(10) \times 10^{-3}$	kg mol ⁻¹	2.5×10^{-10}
PHYSICO-CHEMICAL				
Avogadro constant	N_A, L	$6.022\,141\,99(47) \times 10^{23}$	mol ⁻¹	7.9×10^{-8}
atomic mass constant $m_u = \frac{1}{12}m(^{12}\text{C}) = 1$ u $= 10^{-3}$ kg mol ⁻¹ / N_A	m_u	$1.660\,538\,73(13) \times 10^{-27}$	kg	7.9×10^{-8}
energy equivalent in MeV	$m_u c^2$	$1.492\,417\,78(12) \times 10^{-10}$ $931.494\,013(37)$	J MeV	7.9×10^{-8} 4.0×10^{-8}
Faraday constant ^g $N_A e$	F	$96\,485.3415(39)$	C mol ⁻¹	4.0×10^{-8}

Fundamental Physical Constants

Quantity	Symbol	Value	Unit	Relative std. uncert. u_r
molar Planck constant	$N_A h$	$3.990\,312\,689(30) \times 10^{-10}$	J s mol^{-1}	7.6×10^{-9}
	$N_A h c$	$0.119\,626\,564\,92(91)$	J m mol^{-1}	7.6×10^{-9}
molar gas constant	R	$8.314\,472(15)$	$\text{J mol}^{-1} \text{K}^{-1}$	1.7×10^{-6}
Boltzmann constant R/N_A	k	$1.380\,6503(24) \times 10^{-23}$	J K^{-1}	1.7×10^{-6}
in eV K^{-1}		$8.617\,342(15) \times 10^{-5}$	eV K^{-1}	1.7×10^{-6}
	k/h	$2.083\,6644(36) \times 10^{10}$	Hz K^{-1}	1.7×10^{-6}
	k/hc	$69.503\,56(12)$	$\text{m}^{-1} \text{K}^{-1}$	1.7×10^{-6}
molar volume of ideal gas RT/p				
$T = 273.15 \text{ K}, p = 101.325 \text{ kPa}$	V_m	$22.413\,996(39) \times 10^{-3}$	$\text{m}^3 \text{mol}^{-1}$	1.7×10^{-6}
Loschmidt constant N_A/V_m	n_0	$2.686\,7775(47) \times 10^{25}$	m^{-3}	1.7×10^{-6}
$T = 273.15 \text{ K}, p = 100 \text{ kPa}$	V_m	$22.710\,981(40) \times 10^{-3}$	$\text{m}^3 \text{mol}^{-1}$	1.7×10^{-6}
Sackur-Tetrode constant (absolute entropy constant) ^h $\frac{5}{2} + \ln[(2\pi m_u k T_1 / h^2)^{3/2} k T_1 / p_0]$				
$T_1 = 1 \text{ K}, p_0 = 100 \text{ kPa}$	S_0/R	$-1.151\,7048(44)$		3.8×10^{-6}
$T_1 = 1 \text{ K}, p_0 = 101.325 \text{ kPa}$		$-1.164\,8678(44)$		3.7×10^{-6}
Stefan-Boltzmann constant $(\pi^2/60)k^4/\hbar^3 c^2$	σ	$5.670\,400(40) \times 10^{-8}$	$\text{W m}^{-2} \text{K}^{-4}$	7.0×10^{-6}
first radiation constant $2\pi\hbar c^2$	c_1	$3.741\,771\,07(29) \times 10^{-16}$	W m^2	7.8×10^{-8}
first radiation constant for spectral radiance $2hc^2$	c_{1L}	$1.191\,042\,722(93) \times 10^{-16}$	$\text{W m}^2 \text{sr}^{-1}$	7.8×10^{-8}
second radiation constant $\hbar c/k$	c_2	$1.438\,7752(25) \times 10^{-2}$	m K	1.7×10^{-6}
Wien displacement law constant $b = \lambda_{\max} T = c_2/4.965\,114\,231\dots$	b	$2.897\,7686(51) \times 10^{-3}$	m K	1.7×10^{-6}

^a See the “Adopted values” table for the conventional value adopted internationally for realizing representations of the volt using the Josephson effect.

^b See the “Adopted values” table for the conventional value adopted internationally for realizing representations of the ohm using the quantum Hall effect.

^c Value recommended by the Particle Data Group, Caso et al., Eur. Phys. J. C **3**(1-4), 1-794 (1998).

^d Based on the ratio of the masses of the W and Z bosons m_W/m_Z recommended by the Particle Data Group (Caso et al., 1998). The value for $\sin^2\theta_W$ they recommend, which is based on a particular variant of the modified minimal subtraction ($\overline{\text{MS}}$) scheme, is $\sin^2\hat{\theta}_W(M_Z) = 0.231\,24(24)$.

^e The helion, symbol h, is the nucleus of the ^3He atom.

^f This and all other values involving m_τ are based on the value of $m_\tau c^2$ in MeV recommended by the Particle Data Group, Caso et al., Eur. Phys. J. C **3**(1-4), 1-794 (1998), but with a standard uncertainty of 0.29 MeV rather than the quoted uncertainty of $-0.26 \text{ MeV}, +0.29 \text{ MeV}$.

^g The numerical value of F to be used in coulometric chemical measurements is $96\,485.3432(76)$ [7.9×10^{-8}] when the relevant current is measured in terms of representations of the volt and ohm based on the Josephson and quantum Hall effects and the internationally adopted conventional values of the Josephson and von Klitzing constants K_{J-90} and R_{K-90} given in the “Adopted values” table.

^h The entropy of an ideal monoatomic gas of relative atomic mass A_r is given by $S = S_0 + \frac{3}{2}R \ln A_r - R \ln(p/p_0) + \frac{5}{2}R \ln(T/K)$.

Fundamental Physical Constants — Adopted values

Quantity	Symbol	Value	Unit	Relative std. uncert. u_r
molar mass of ^{12}C	$M(^{12}\text{C})$	12×10^{-3}	kg mol^{-1}	(exact)
molar mass constant ^a $M(^{12}\text{C})/12$	M_{u}	1×10^{-3}	kg mol^{-1}	(exact)
conventional value of Josephson constant ^b	$K_{\text{J-90}}$	483 597.9	GHz V^{-1}	(exact)
conventional value of von Klitzing constant ^c	$R_{\text{K-90}}$	25 812.807	Ω	(exact)
standard atmosphere		101 325	Pa	(exact)
standard acceleration of gravity	g_{n}	9.806 65	m s^{-2}	(exact)

^a The relative atomic mass $A_{\text{r}}(\text{X})$ of particle X with mass $m(\text{X})$ is defined by $A_{\text{r}}(\text{X}) = m(\text{X})/m_{\text{u}}$, where $m_{\text{u}} = m(^{12}\text{C})/12 = M_{\text{u}}/N_{\text{A}} = 1 \text{ u}$ is the atomic mass constant, N_{A} is the Avogadro constant, and u is the atomic mass unit. Thus the mass of particle X in u is $m(\text{X}) = A_{\text{r}}(\text{X}) \text{ u}$ and the molar mass of X is $M(\text{X}) = A_{\text{r}}(\text{X})M_{\text{u}}$.

^b This is the value adopted internationally for realizing representations of the volt using the Josephson effect.

^c This is the value adopted internationally for realizing representations of the ohm using the quantum Hall effect.

Energy Equivalents

	J	kg	m ⁻¹	Hz
1 J	(1 J) = 1 J	(1 J)/c ² = 1.112 650 056 × 10 ⁻¹⁷ kg	(1 J)/hc = 5.034 117 62(39) × 10 ²⁴ m ⁻¹	(1 J)/h = 1.509 190 50(12) × 10 ³³ Hz
1 kg	(1 kg)c ² = 8.987 551 787 × 10 ¹⁶ J	(1 kg) = 1 kg	(1 kg)c/h = 4.524 439 29(35) × 10 ⁴¹ m ⁻¹	(1 kg)c ² /h = 1.356 392 77(11) × 10 ⁵⁰ Hz
1 m ⁻¹	(1 m ⁻¹)hc = 1.986 445 44(16) × 10 ⁻²⁵ J	(1 m ⁻¹)h/c = 2.210 218 63(17) × 10 ⁻⁴² kg	(1 m ⁻¹) = 1 m ⁻¹	(1 m ⁻¹)c = 299 792 458 Hz
1 Hz	(1 Hz)h = 6.626 068 76(52) × 10 ⁻³⁴ J	(1 Hz)h/c ² = 7.372 495 78(58) × 10 ⁻⁵¹ kg	(1 Hz)/c = 3.335 640 952 × 10 ⁻⁹ m ⁻¹	(1 Hz) = 1 Hz
1 K	(1 K)k = 1.380 6503(24) × 10 ⁻²³ J	(1 K)k/c ² = 1.536 1807(27) × 10 ⁻⁴⁰ kg	(1 K)k/hc = 69.503 56(12) m ⁻¹	(1 K)k/h = 2.083 6644(36) × 10 ¹⁰ Hz
1 eV	(1 eV) = 1.602 176 462(63) × 10 ⁻¹⁹ J	(1 eV)/c ² = 1.782 661 731(70) × 10 ⁻³⁶ kg	(1 eV)/hc = 8.065 544 77(32) × 10 ⁵ m ⁻¹	(1 eV)/h = 2.417 989 491(95) × 10 ¹⁴ Hz
1 u	(1 u)c ² = 1.492 417 78(12) × 10 ⁻¹⁰ J	(1 u) = 1.660 538 73(13) × 10 ⁻²⁷ kg	(1 u)c/h = 7.513 006 658(57) × 10 ¹⁴ m ⁻¹	(1 u)c ² /h = 2.252 342 733(17) × 10 ²³ Hz
1 E _h	(1 E _h) = 4.359 743 81(34) × 10 ⁻¹⁸ J	(1 E _h)/c ² = 4.850 869 19(38) × 10 ⁻³⁵ kg	(1 E _h)/hc = 2.194 746 313 710(17) × 10 ⁷ m ⁻¹	(1 E _h)/h = 6.579 683 920 735(50) × 10 ¹⁵ Hz

Derived from the relations $E = mc^2 = hc/\lambda = h\nu = kT$, and based on the 1998 CODATA adjustment of the values of the constants;

1 eV = (e/C) J, 1 u = $m_u = \frac{1}{12}m(^{12}\text{C}) = 10^{-3} \text{ kg mol}^{-1}/N_A$, and $E_h = 2R_\infty hc = \alpha^2 m_e c^2$ is the Hartree energy (hartree).

Energy Equivalents

	K	eV	u	E_h
1 J	(1 J)/ k = $7.242\,964(13) \times 10^{22}$ K	(1 J) = $6.241\,509\,74(24) \times 10^{18}$ eV	(1 J)/ c^2 = $6.700\,536\,62(53) \times 10^9$ u	(1 J) = $2.293\,712\,76(18) \times 10^{17}$ E_h
1 kg	(1 kg) c^2/k = $6.509\,651(11) \times 10^{39}$ K	(1 kg) c^2 = $5.609\,589\,21(22) \times 10^{35}$ eV	(1 kg) = $6.022\,141\,99(47) \times 10^{26}$ u	(1 kg) c^2 = $2.061\,486\,22(16) \times 10^{34}$ E_h
1 m ⁻¹	(1 m ⁻¹) hc/k = $1.438\,7752(25) \times 10^{-2}$ K	(1 m ⁻¹) hc = $1.239\,841\,857(49) \times 10^{-6}$ eV	(1 m ⁻¹) h/c = $1.331\,025\,042(10) \times 10^{-15}$ u	(1 m ⁻¹) hc = $4.556\,335\,252\,750(35) \times 10^{-8}$ E_h
1 Hz	(1 Hz) h/k = $4.799\,2374(84) \times 10^{-11}$ K	(1 Hz) h = $4.135\,667\,27(16) \times 10^{-15}$ eV	(1 Hz) h/c^2 = $4.439\,821\,637(34) \times 10^{-24}$ u	(1 Hz) h = $1.519\,829\,846\,003(12) \times 10^{-16}$ E_h
1 K	(1 K) = 1 K	(1 K) k = $8.617\,342(15) \times 10^{-5}$ eV	(1 K) k/c^2 = $9.251\,098(16) \times 10^{-14}$ u	(1 K) k = $3.166\,8153(55) \times 10^{-6}$ E_h
1 eV	(1 eV)/ k = $1.160\,4506(20) \times 10^4$ K	(1 eV) = 1 eV	(1 eV)/ c^2 = $1.073\,544\,206(43) \times 10^{-9}$ u	(1 eV) = $3.674\,932\,60(14) \times 10^{-2}$ E_h
1 u	(1 u) c^2/k = $1.080\,9528(19) \times 10^{13}$ K	(1 u) c^2 = $931.494\,013(37) \times 10^6$ eV	(1 u) = 1 u	(1 u) c^2 = $3.423\,177\,709(26) \times 10^7$ E_h
1 E_h	(1 E_h)/ k = $3.157\,7465(55) \times 10^5$ K	(1 E_h) = $27.211\,3834(11)$ eV	(1 E_h)/ c^2 = $2.921\,262\,304(22) \times 10^{-8}$ u	(1 E_h) = 1 E_h

Derived from the relations $E = mc^2 = hc/\lambda = h\nu = kT$, and based on the 1998 CODATA adjustment of the values of the constants;
1 eV = (e/C) J, 1 u = $m_u = \frac{1}{12}m(^{12}\text{C}) = 10^{-3}$ kg mol⁻¹/ N_A , and $E_h = 2R_\infty hc = \alpha^2 m_e c^2$ is the Hartree energy (hartree).