THE STANDARD MODEL

Contents

I. INTRODUCTION TO THE STANDARD MODEL

A. Interactions and particles

The Standard Model of particle physics is a mathematical description of four types of interactions: The strong interactions, the electromagnetic interactions, the weak interactions, and the Yukawa interactions. The first three types of interactions are mediated by vector-boson (spin-1) force carriers: eight massless gluons mediate the strong interactions, one massless photon mediates the electromagnetic interactions, and the three massive W^+, W^- and Z^0 bosons mediate the weak interactions. The existence of these three types of interactions, their mediation by spin-1 force carriers, and the dependence of each of these on a single parameters (the couplings constants α_s, α_W and α) are predictions that follow from imposing a certain *gauge symmetry* on the model,

$$G_{\rm SM} = SU(3)_C \times SU(2)_L \times U(1)_Y . \tag{1}$$

The Yukawa interactions are mediated by a single scalar (spin-0) particle, the Higgs boson. The bosons described by the SM are presented in Table I.

The matter particles of the Standard Model are all fermions (spin-1/2). They come in three generations, namely three sets of particles that carry the same gauge quantum

TABLE I: The SM bosons

boson	force	spin	$SU(3)_C$	$U(1)_{\rm EM}$	mass [GeV]
g	strong	1	8	0	0
γ	electromagnetic	1	1	0	0
$W^\pm$	weak	1	1	±1	80
Z^0	weak	1	1	0	91
h^0	Yukawa	0	1	0	125

TABLE II: The SM fermions

quark	$SU(3)$	$U(1)_{\rm EM}$	mass [GeV]	lepton	$SU(3)$	$U(1)_{\rm EM}$	mass [GeV]
u	3	$+2/3$	0.002	ν_1	1	0	$\lesssim 10^{-11}$
d	3	$-1/3$	0.005	e	1	-1	0.0005
c	3	$+2/3$	1.3	ν_2	1	0	$\sim 10^{-11}$
s	3	$-1/3$	0.1	μ	1	-1	0.1
t	3	$+2/3$	173	ν_3	1	0	$\sim 10^{-10}$
b	3	$-1/3$	4.2	τ	1	-1	1.8

numbers, and differ only in mass. In each generation there are four types of particles: an up-type quark, a down-type quark, a charged lepton, and a neutrino. The list of the SM fermions is given in Table II.

The SM is defined by its symmetries and fermionic and scalar particle content. The renormalizable part of the most general Lagrangian that is consistent with this definition depends on eighteen independent parameters. All phenomena related to the strong, weak, electromagnetic and Yukawa interactions depend, in principle, on just these eighteen parameters. The model has successfully predicted numerous experimental results.

B. Problems of the Standard Model

In spite of the enormous experimental success of the Standard Model (SM), it is commonly believed that it is not the full picture of Nature and that there exists New Physics (NP) beyond the SM at an energy scale higher than the electroweak (EW) breaking scale ($\Lambda_{\rm EW} \sim 10^2$ GeV).

It is indeed clear that the SM cannot describe physics above a scale $m_{\rm Pl} \sim 10^{19}\ GeV$. At this scale, gravitational effects become as important as the known gauge interactions and cannot be neglected. But there are good reasons to believe that there is additional NP between $\Lambda_{\rm EW}$ and $m_{\rm Pl}$. Let us first list the relevant problems related to experiments and observations.

(i) There are two, related, pieces of experimental evidence for such NP. Both suggest that the neutrinos are massive, in contrast to the Standard Model prediction that they are massless. First, measurements of the flux of atmospheric neutrions find that the ratio of ν_μ-to-ν_e fluxes is different from expectations and that the ν_μ flux has an up-down asymmetry. Both facts can be beautifully explained by neutrino masses and mixing which lead to $\nu_\mu - \nu_\tau$ oscillations. Second, measurements of the solar neutrino flux find that the flux of electron-neutrinos is significantly smaller than the total flux of active ($\nu_a = \nu_e, \nu_\mu, \nu_\tau$) neutrinos. This puzzle (the Sun produces only ν_e's) can be beautifully explained by $\nu_e - \nu_{\mu,\tau}$ mixing.

(ii) There exists also an 'observational' evidence for NP. The features of the Cosmic Microwave Background Radiation (CMBR) imply a certain baryon asymmetry of the Universe. Similarly, the standard Big Bang Nucleosynthesis (BBN) scenario is consistent with the observed abundance of light elements only for a certain range of baryon asymmetry, consistent with the CMBR constraint. To generate a baryon asymmetry, CP violation is required. The SM CP violation generates baryon asymmetry that is smaller by at least twelve orders of magnitude than the 'observed' asymmetry. This implies that there are new sources of CP violation, beyond the SM.

(iii) Another observation that cannot be explained within the Standard Model is the requirement for dark matter.

(iv) The three gauge couplings of the strong, weak and electromagnetic interactions seem to converge to a unified value at a high energy scale. The Standard Model cannot explain this fact, which is just accidental within this model.

There are other deficiencies in the SM. The most serious ones are related to problems of 'Naturalness': there are small parameters in the SM and it requires miraculous fine-tuning to explain them.

(i) The mass-squared of the Higgs gets quadratically divergent radiative corrections. This means that, if there is no New Physics below $m_{\rm Pl}$, the bare mass-squared term and the loop contributions have to cancel each other to an accuracy of about thirty four orders of magnitude. Supersymmetry (SUSY) can solve this fine-tuning problem in that is stabilizes the ratio $\Lambda_{\rm EW}/m_{\rm Pl}$. Dynamical SUSY breaking (DSB) can even explain this ratio.

(ii) The CP violating $\theta_{\rm QCD}$ parameter contributes to the electric dipole moment of the neutron. For $\theta_{\rm QCD} = \mathcal{O}(1)$ this contribution exceeds the experimental upper bound by about nine orders of magnitude. This fine-tuning problem can be solved by a Peccei-Quinn symmetry, by making CP a spontaneously broken symmetry or if $m_u = 0$.

(iii) The Yukawa couplings are small (except for the top Yukawa) and hierarchical. For example, the electron Yukawa is of $\mathcal{O}(10^{-5})$. Horizontal symmetries can explain the smallness and hierarchy in the flavor parameters.

Finally, there are considerable theoretical efforts into finding a theory that is more 'aesthetic' and capable of answering more questions than the SM. For example, string theory has, in principle, one free parameter (compared to the eighteen of the SM). It can explain, in principle, why our Universe is four-dimensional, why there are three fermion generations, *etc.*

Several aspects of the SM have not been tested well yet. In particular, we are only beginning to have direct experimental information on the mechanism of spontaneous symmetry breaking. The ATLAS and CMS experiments have recently discovered a Higgs-like boson. The program for the coming years is to measure various properties of this particle, testing whether it is indeed the Higgs boson, and whether its couplings are consistent with the SM predictions.

C. The Scale of New Physics

As mentioned above, the SM cannot be valid at a scale higher than the Planck scale, $m_{\rm Pl} \sim 10^{19}$ GeV. The existence of neutrino masses requires that there is yet a lower scale of new physics, that is the "seesaw scale", $\Lambda_\nu \lesssim 10^{15}$ GeV. This scale is also intriguingly close to the scale where the three gauge couplings unify within the supersymmetric extension of the SM, $\Lambda_{\rm GUT} \sim 10^{16}$ GeV. There are, however, reasons to believe that there is a much lower scale of new physics. One motivation comes from the dark matter puzzle. If the DM particles are weakly interacting massive particles (WIMPs), the cross section of their annihilation that is required to explain the DM abundance suggests a scale of $\Lambda_{\rm DM} \sim 1$ TeV. A similar scale is suggested by the fine tuning problem.

The fine-tuning problem arises from the fact that there are quadratically divergent loop contributions to the Higgs mass which drive the Higgs mass to unacceptably large values unless the tree level mass parameter is finely tuned to cancel the large quantum corrections.

The most significant of these divergences come from three sources. They are one loop diagrams involving - in order of decreasing magnitude - the top quark, the electroweak gauge bosons, and the Higgs itself.

For the sake of concreteness (and, also, because this is the scale that will be probed by the LHC), let us assume that the SM is valid up to a cut-off scale of 10 TeV. Then, the contributions from the three diagrams are

$$-\frac{3}{8\pi^2}Y_t^2\Lambda^2 \sim -(2\ TeV)^2 \tag{2}$$

from the top loop,

$$\frac{1}{16\pi^2}g^2\Lambda^2 \sim (700\ GeV)^2 \tag{3}$$

from the gauge loop, and

$$\frac{1}{16\pi^2}\lambda^2\Lambda^2 \sim (500\ GeV)^2 \tag{4}$$

from the Higgs loop. Thus the total Higgs mass is approximately

$$m_h^2 \simeq m_{\rm tree}^2 - [250 - 30 - 16](125\ GeV)^2. \tag{5}$$

In order for this to add up to a Higgs mass of order a hundred GeV as required in the SM, fine tuning of order one part in 200 is required. This is the hierarchy problem.

Is the SM already fine tuned given that we have experimentally probed to near 1 TeV and found no NP? Setting $\Lambda = 1$ TeV in the above formulas we find that the most dangerous contribution from the top loop is only about $(200\ GeV)^2$. Thus no fine tuning is required, the SM with no NP up to 1 TeV is perfectly natural, and we should not be surprised that we have not yet seen deviations from it at colliders.

We can now turn the argument around and use the hierarchy problem to predict what forms of new physics exist at what scale in order to solve the fine tuning problem. Consider for example the contribution to the Higgs mass from the top loop. Limiting the contribution to be no larger than about 10 times the Higgs mass-squared (limiting fine tuning to less than 1 part in 10) we find a maximum cut-off for $\Lambda = 2$ TeV. In other words, we predict the existence of new particles with masses less than 2 TeV which cancel the quadratically divergent contribution to the Higgs mass from the top loop. In order for this cancelation to occur naturally, the new particles must be related to the top quark by a symmetry. In practice, this means that the new particles have to carry similar quantum numbers to top quarks. In supersymmetry, these new particles are the top squarks.

The contributions from gauge loops also need to be canceled by new particles which are related to the SM $SU(2) \times U(1)$ gauge bosons by a symmetry. The masses of these states are predicted to be at or below 5 TeV for the cancelation to be natural. Similarly, the Higgs loop requires new states related to the Higgs at 8 TeV. We see that the hierarchy problem can be used to obtain specific predictions.

II. LAGRANGIANS

All fundamental laws of particle physics interactions can be encoded in a mathematical construct called the *action* S. The action is an integral over spacetime of another mathematical construct called the "Lagrange density" or *Lagrangian* $\mathcal{L}$, for short. For most of our purposes, we need to consider just the Lagrangian.

During these lectures we will explain (i) how we "construct" Lagrangians, (ii) how we determine their parameters, and (iii) how we test whether they describe nature correctly. We do so by the example of the Standard Model Lagrangian

The action is given by

$$S = \int d^4x \; \mathcal{L}[\phi(x), \partial_\mu \phi(x)] \,, \tag{6}$$

where $d^4x = dx^0 dx^1 dx^2 dx^3$ is the integration measure in four-dimensional Minkowski space, $\mathcal{L}$ is the Lagrangian and ϕ is a field. A field $\phi(x)$ is a mathematical construct which carries certain quantum numbers and is able to annihilate or create particles with these quantum numbers at the space-time point x. There could be several different fields, in which case ϕ carries an index that runs from 1 to the number of fields. We denote a generic field by $\phi(x)$.

Later, we use $\phi(x)$ for a scalar field, $\psi(x)$ for a fermion field, and $V(x)$ for a vector field.

The action S has units of ML^2T^{-1} or, equivalently, of $\hbar$. In a natural unit system, where $\hbar = 1$, S is taken to be "dimensionless." Then in four dimensions $\mathcal{L}$ has natural dimensions of $L^{-4} = M^4$. The requirement of the variation of the action with respect to variation of the fields vanishes, $\delta S = 0$, leads to the equations of motion (EOM):

$$\frac{\delta \mathcal{L}}{\delta \phi} = \partial_\mu \left(\frac{\delta \mathcal{L}}{\delta(\partial_\mu \phi)} \right) , \tag{7}$$

where the x dependence of ϕ is omitted. When there are several fields, the above equation should be satisfied for each of them.

In general, we require the following properties for the Lagrangian:

(i) It is a function of the fields and their derivatives only, so as to ensure translational invariance.

(ii) It depends on the fields taken at one space-time point x^μ only, leading to a local field theory.

(iii) It is real in order to have the total probability conserved.

(iv) It is invariant under the Poincaré group.

(v) It is invariant under certain internal symmetry groups. The symmetries of S (or of $\mathcal{L}$) are in correspondence with conserved quantities and therefore reflect the basic symmetries of the physical system.

Often, we add another requirement:

(vi) Naturalness: Every term in the Lagrangian that is not forbidden by a symmetry should appear.

We did not include renormalizability in our list of properties. Indeed, the Lagrangian that corresponds to the full theory of nature should be renormalizable. This means that it contains only terms that are of dimension less or equal to four (in the fields and their derivatives). In particular, this requirement ensures that it contains at most two ∂_μ operations, so it leads to classical equations of motion that are no higher than second order derivatives. However, the theories that we consider and, in particular, the Standard Model, are only low energy effective theories, valid up to some energy scale Λ. Thereofore, we must include also non-renormalizable terms. These terms have coefficients with inverse mass dimensions, $1/\Lambda^n$, $n = 1, 2, \ldots$. For most purposes, however, the renormalizable terms constitute the leading terms in an expansion in E/Λ, where E is the energy scale of the physical processes under study. Therefore, the renormalizable part of the Lagrangian is a good starting point for our study.

Properties (i)-(iv) are not the subject of this book. You must be familiar with them from your QFT course(s). Here we consider only Lagrangians which fulfill these requirements and let textbooks explain to you why they are needed. We do, however, deal intensively with the last two requirements. Actually, the most important message that we would like to convey in this course is the following: *(Almost) all experimental data for elementary particles and their interactions can be explained by the standard model of a spontaneously broken $SU(3) \times SU(2) \times U(1)$ gauge symmetry.*[1]

A. Scalars

The renormalizable Lagrangian for a free real scalar field is

$$\mathcal{L} = \frac{1}{2}\left[\partial^\mu\phi\partial_\mu\phi - m^2\phi^2\right] . \tag{8}$$

We work in the "canonically normalized" basis where the coefficient of the kinetic term is one. This part of the Lagrangian is necessary if we want to describe free propagation in spacetime. Additional terms describe interactions. The most general $\mathcal{L}(\phi)$ we can write for a single scalar field includes trilinear and quartic interaction terms:

$$\mathcal{L}_S = \frac{1}{2}\left[\partial^\mu\phi\partial_\mu\phi - m^2\phi^2 + \mu\phi^3 + \lambda\phi^4\right] . \tag{9}$$

We do not write a constant term since it does not enter the equation of motion. In principle we could write a linear term but it is not physical, that is, we can always redefine the field such that the linear term vanishes.

[1] Actually, the great hope of all high-energy physics community is to prove this statement wrong!

B. Fermions

The renormalizable Lagrangian for a single Dirac fermion field is

$$\mathcal{L}_F = \bar{\psi}(i\partial\!\!\!/ - m)\psi\,. \tag{10}$$

Again, we work in the canonically normalized basis. Note that this is the most general renormalizable $\mathcal{L}(\psi)$ we can write, so is satisfies the naturalness principle.[2] Terms with three fermions violate Lorentz symmetry, while terms with four fermions are non-renormalizable. We treat ψ and $\bar{\psi}$ as independent fields. The reason is that a fermion field is complex, namely, with two independent degrees of freedom which we choose to be ψ and $\bar{\psi}$.

C. Fermions and scalars

The renormalizable Lagrangian for a single Dirac fermion and a single real scalar field includes, in addition to the terms written in Eqs. (9) and (10), the following term:

$$\mathcal{L}_Y = Y\bar{\psi}_R\psi_L\phi + \text{h.c.}. \tag{11}$$

Such a term is called a Yukawa interaction and Y is the Yukawa coupling.

III. SYMMETRIES AND CONSERVED CHARGES

A. Introduction

Particle physicists seek deeper reasons for the rules they have discovered. A major role in these answers in modern theories of particle physics is played by symmetries. In the physicists's language, the term *symmetry* refers to an invariance of the equations that describe a physical system. The fact that a symmetry and an invariance are related concepts is obvious enough–a smooth ball has a spherical symmetry and its appearance in invariant under rotation.

Symmetries are built into QFT as invariance properties of the Lagrangian. If we construct our theories to encode various empirical facts and, in particular, the observed conservation

[2] There is a subtlety involved in this statement. By saying that the fermion in question is of the Dirac type, we are implicitly imposing a symmetry that forbids Majorana mass terms. We discuss this issue later.

laws, then the equations turn out to exhibit certain invariance properties. For example, if we want the theory to give the same physics at all places, then the form of the Lagrangian cannot depend on the coordinates that we use to describe the position. It does depend on the values of the fields at each position, but the products of the fields that define this dependence are the same for every location. Furthermore, the form does not change when we decide to measure all our distances with respect to a different zero point.

Conversely, if we take the symmetries to be the fundamental rules that determine the theory we can write, then various observed features of particles and their interactions are a necessary consequence of the symmetry principle. In this sense, symmetries provide an explanation of these features.

There are several types of symmetries. First, we distinguish between *spacetime* and *internal* symmetries. Spacetime symmetries include the Poincaré group. They give us the energy–momentum and angular momentum conservation laws. In additional they also include the C, P and T operators.

Internal symmetries act on the fields, not directly on spacetime. That is, they work in mathematical spaces that are generated by the fields. These symmetries are divided into two: *global* and *local* (the latter are also called *gauge* symmetries). Global symmetries can be *discrete* or *continuous*. The word global means that the transformation operators are constant in space. These symmetries give us conservation laws. There are reasons to think that there can be no exact global symmetries in nature (they are likely to be violated by gravitational effects). Thus, we usually do not *impose* global symmetries on our Lagrangians, they are *accidental*. An accidental symmetry arises at the renormalizable level as a result of other, imposed, symmetries, and specific particle content. They can be broken explicitly by higher dimension operators, and can also be broken by a small parameter at the renormalizable level, in which case the symmetry is approximate. Actually, it is likely that all conservation laws that are results of global symmetries are only *approximate*. For example, isospin-$SU(2)$ and its extension to flavor-$SU(3)$ are broken by the quark masses. Baryon number $U(1)_B$ and lepton number $U(1)_L$ are expected to be broken by higher dimension operators. When a continuous global symmetry is *broken spontaneously*, we get a massless boson called a *Goldstone boson*. When it is broken both explicitly and spontaneously, and the spontaneous breaking occurs even if all the explicit breaking parameters are put to zero, we get a light scalar, a pseudo-Goldstone boson. For example, the pions are pseudo-Goldstone

TABLE III: Symmetries

Type	Consequences
Spacetime	Conservation of energy, momentum, angular momentum
Discrete	Selection rules
Global (exact)	Conserved charges
Global (spon. broken)	Massless scalars
Local (exact)	Interactions, massless spin-1 mediators
Local (spon. broken)	Interactions, massive spin-1 mediators

bosons that correspond to the spontaneous breaking of the chiral symmetry.

Local, or gauge, symmetries are symmetries where the transformation operators are not constant in space. Symmetries of this type are the ones we impose on $\mathcal{L}$. For example, in the SM we impose a local $SU(3) \times SU(2) \times U(1)$ symmetry. In addition to conservation laws, local symmetries require the existence of gauge fields. A gauge symmetry cannot be broken explicitly. When it is broken spontaneously the gauge bosons acquire masses. For example, the W and Z bosons are massive due to the spontaneous breaking of $SU(2) \times U(1)$ into a $U(1)$ subgroup.

The main consequences of the various types of symmetries are summarized in Table III.

B. Noether's theorem

The Noether's theorem relates internal global continuous symmetries to conserved charges. We will first prove it, and then demonstrate it with the cases of free massless scalars and free massless fermions.

Let $\phi_i(x)$ be a set of fields, $i = 1, 2, \ldots, N$, on which the Lagrangian $\mathcal{L}(\phi)$ depends. Consider an infinitesimal change $\delta\phi_i$ in the fields. This is a symmetry if

$$\mathcal{L}(\phi + \delta\phi) = \mathcal{L}(\phi). \tag{12}$$

Since $\mathcal{L}$ depends only on ϕ and $\partial_\mu\phi$, we have

$$\delta\mathcal{L}(\phi) = \mathcal{L}(\phi + \delta\phi) - \mathcal{L}(\phi) = \frac{\delta\mathcal{L}}{\delta\phi_j}\delta\phi_j + \frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi_j)}\delta(\partial_\mu\phi_j). \tag{13}$$

The relation between symmetries and conserved quantities is expressed by Noether's theorem: *To every symmetry in the Lagrangian there corresponds a conserved current.* To prove the theorem, one uses the equation of motion:

$$\partial_\mu \frac{\delta \mathcal{L}}{\delta(\partial_\mu \phi_j)} = \frac{\delta \mathcal{L}}{\delta \phi_j}. \tag{14}$$

The condition for a symmetry is then

$$\partial_\mu \left[\frac{\delta \mathcal{L}}{\delta(\partial_\mu \phi_j)}\right] \delta\phi_j + \frac{\delta \mathcal{L}}{\delta(\partial_\mu \phi_j)} \delta(\partial_\mu \phi_j) = \partial_\mu \left[\frac{\delta \mathcal{L}}{\delta(\partial_\mu \phi_j)} \delta\phi_j\right] = 0. \tag{15}$$

Thus, the conserved current – $\partial_\mu J^\mu = 0$ – is

$$J^\mu = \frac{\delta \mathcal{L}}{\delta(\partial_\mu \phi_j)} \delta\phi_j. \tag{16}$$

The conserved charge – $\dot{Q} = 0$ – is given by

$$Q = \int d^3x \; J_0(x). \tag{17}$$

We will be interested in unitary transformations,

$$\phi \to \phi' = U\phi, \qquad UU^\dagger = \mathbf{1}. \tag{18}$$

(ϕ is here a vector with N components, so U is an $N \times N$ matrix, and $\mathbf{1}$ stands for the $N \times N$ unit matrix.) The reason that we are interested in unitary transformation is that they keep the canonical form of the kinetic terms. A unitary matrix can always be written as

$$U = e^{i\epsilon_a T^a}, \tag{19}$$

where ϵ_a are numbers and T^a are *hermitian* matrices. For infinitesimal transformation ($\epsilon_a \ll 1$),

$$\phi' \approx (1 + i\epsilon_a T^a)\phi \implies \delta\phi = i\epsilon_a T^a \phi. \tag{20}$$

A *global* symmetry is defined by $\epsilon_a = \text{const}(x)$. For *internal* symmetry, $\delta(\partial_\mu \phi) = \partial_\mu(\delta\phi)$. For an internal global symmetry,

$$\delta(\partial_\mu \phi) = i\epsilon_a T^a \partial_\mu \phi. \tag{21}$$

In the physics jargon, we say that $\partial_\mu \phi$ transforms like ϕ. The conserved current is

$$J^a_\mu = i \frac{\delta \mathcal{L}}{\delta(\partial_\mu \phi)} T^a \phi. \tag{22}$$

The matrices T^a form am algebra of the symmetry group,

$$[T^a, T^b] = if^{abc}T^c. \tag{23}$$

The charges that are associated with these symmetry also satisfy the algebra:

$$[Q^a, Q^b] = if^{abc}Q^c. \tag{24}$$

Note that T^a are $N \times N$ matrices, while Q^a are operators in the Hilbert space where the theory lives.

All this is very abstract. Let us see some (admittedly, abstract) examples.

1. *Example I: Free massless scalars*

Consider N real, free, massless scalar fields ϕ_i:

$$\mathcal{L}(\phi) = \frac{1}{2}(\partial_\mu \phi_j)(\partial^\mu \phi_j) = \frac{1}{2}(\partial_\mu \phi^T)(\partial^\mu \phi). \tag{25}$$

The theory is invariant under the group of orthogonal $N \times N$ matrices, which is the group of rotations in an N-dimensional real vector space. This group is called $SO(N)$. The generators T^a are the $N(N-1)/2$ independent antisymmetric imaginary matrices, that is

$$\delta\phi = i\epsilon_a T^a \phi, \tag{26}$$

with T^a antisymmetric and imaginary. (It must be imaginary so that $\delta\phi$ is real.) Then,

$$\delta\mathcal{L} = \frac{\delta\mathcal{L}}{\delta\phi}\delta\phi + \frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi)}\delta(\partial_\mu\phi) = (\partial_\mu\phi)_m i\epsilon_a T^a_{mn}(\partial_\mu\phi)_n = 0, \tag{27}$$

where we used the antisymmetry of T^a. The associated current is

$$J^a_\mu = i(\partial_\mu\phi)T^a\phi. \tag{28}$$

The $SO(N)$ groups have no important role in the SM. We will mention $SO(4)$ when we discuss the Higgs mechanism. In a more advanced course, you may encounter $SO(10)$ as a grand unifying group.

2. Example II: Free massless fermions

Consider N free, massless, spin-$\frac{1}{2}$, four-component fermion fields ψ_i:

$$\mathcal{L}(\psi) = i\bar{\psi}_j \partial\!\!\!/ \psi_j \tag{29}$$

The ψ's are necessarily complex because of the Dirac structure. The theory is invariant under the group of unitary $N \times N$ matrices. This group is called $U(N) = SU(N) \times U(1)$. The generators are the independent N^2 Hermitian matrices, where the $N^2 - 1$ traceless ones generate the $SU(N)$ group:

$$\delta\psi_j = i\epsilon_a T^a_{jk}\psi_k \qquad \text{or} \qquad \delta\psi = i\epsilon_a T^a \psi \tag{30}$$

where T^a is a general Hermitian matrix. The transformation law of $\bar{\psi}$ is as follows:

$$\delta\bar{\psi} = \delta(\psi^\dagger\gamma^0) = (i\epsilon_a T^a\psi)^\dagger\gamma^0 = \psi^\dagger(-i)\epsilon_a^* T^{a*}\gamma^0 = -i\psi^\dagger\gamma^0\epsilon_a T^a = -i\bar{\psi}\epsilon_a T^a \tag{31}$$

because T^a are Hermitian. Note that T^a and the γ^μ matrices commute because they act on different spaces. We also need to derive the transformation property of the derivative. For an internal symmetry,

$$\delta\partial\!\!\!/\psi = \partial\!\!\!/\delta\psi. \tag{32}$$

For an internal global symmetry (ϵ_a independent of x)

$$\delta\partial\!\!\!/\psi = i\epsilon_a T^a \partial\!\!\!/\psi. \tag{33}$$

Using the fact that $\mathcal{L}$ does not depend on ψ and on $\partial_\mu\bar{\psi}$, we have

$$\delta\mathcal{L} = \delta\bar{\psi}\frac{\delta\mathcal{L}}{\delta\bar{\psi}} + \frac{\delta\mathcal{L}}{\delta\partial\!\!\!/\psi}\delta\partial\!\!\!/\psi\,. \tag{34}$$

We use

$$\begin{aligned} \delta\bar{\psi}\frac{\delta\mathcal{L}}{\delta\bar{\psi}} &= (-i\bar{\psi}\epsilon_a T^a)(i\partial\!\!\!/\psi) = \bar{\psi}\epsilon_a T^a\partial\!\!\!/\psi, \\ \frac{\delta\mathcal{L}}{\delta\partial\!\!\!/\psi}\delta\partial\!\!\!/\psi &= (i\bar{\psi})(i\epsilon_a T^a\partial\!\!\!/\psi) = -\bar{\psi}\epsilon_a T^a\partial\!\!\!/\psi, \end{aligned} \tag{35}$$

and find that $\delta\mathcal{L} = 0$. The corresponding conserved current is

$$J^a_\mu = \bar{\psi}\gamma_\mu T^a\psi. \tag{36}$$

The charge associated with $U(1)$, $\int d^3x\psi^\dagger\psi$, is the fermion number operator.

IV. GLOBAL DISCRETE SYMMETRIES

We start with a simple example of an internal discrete global Z_2 symmetry.

Consider a real scalar field ϕ. The most general Lagrangian we can write is given in (9). We now impose a symmetry: we demand that $\mathcal{L}$ is invariant under a Z_2 symmetry, $\phi \to -\phi$, namely

$$\mathcal{L}(\phi) = \mathcal{L}(-\phi)\,. \tag{37}$$

$\mathcal{L}$ is invariant under this symmetry if $\mu = 0$. Thus, by imposing the symmetry we force $\mu = 0$: The most general $\mathcal{L}(\phi)$ that we can write that also respects the Z_2 symmetry is

$$\mathcal{L} = \frac{1}{2}\left[\partial^\mu\phi\partial_\mu\phi - m^2\phi^2 + \lambda\phi^4\right]\,, \tag{38}$$

What conservation law corresponds to this symmetry? We can call it ϕ parity. The number of particles in a system can change, but always by an even number. Therefore, if we define parity as $(-1)^n$, where n is the number of particles in the system, we see that this parity is conserved. When we do not impose the symmetry and $\mu \neq 0$, the number of particle can change by any integer and ϕ parity is not conserved. When μ is very small (in the appropriate units), ϕ parity is an approximate symmetry.

While this is a simple example, it is a useful exercise to describe it in terms of group theory. Recall that Z_2 has two elements that we call even $(+)$ and odd $(-)$. The multiplication table is very simple:

$$(+)\cdot(+) = (-)\cdot(-) = (+), \qquad (+)\cdot(-) = (-)\cdot(+) = (-). \tag{39}$$

When we say that we impose a Z_2 symmetry on $\mathcal{L}$, we mean that $\mathcal{L}$ belongs to the even representation of Z_2. By saying that $\phi \to -\phi$ we mean that ϕ belong to the odd representation of Z_2. Since $\mathcal{L}$ is even, all terms in $\mathcal{L}$ must be even. The field ϕ, however, is odd. Thus, we need to ask how we can get even terms from products of odd fields? This can be done, of course, by keeping only even powers of ϕ. Then we can construct the most general $\mathcal{L}$ and it is given by Eq. (38).

V. GLOBAL CONTINUOUS SYMMETRIES

We now extend our "model building" ideas to continuous symmetries. The idea is that we demand that $\mathcal{L}$ is invariant under rotation in some internal space. That is, while (some of) the fields are not invariant under rotation in that space, the combinations that appear in the Lagrangian are invariant.

A. Scalars

Consider a Lagrangian that depends on two real scalar fields, $\mathcal{L}(\phi_1, \phi_2)$:

$$\mathcal{L} = \frac{1}{2}\left[\delta_{ij}\partial^\mu\phi_i\partial_\mu\phi_j - m_{ij}^2\phi_i\phi_j + \mu_{ijk}\phi_i\phi_j\phi_k + \lambda_{ijk\ell}\phi_i\phi_j\phi_k\phi_\ell\right] , \tag{40}$$

with m^2, μ and λ real.[3] Note that we can always choose a basis where m^2 is diagonal, $m_{ij}^2 = \delta_{ij}m_i^2$. We impose an SO(2) symmetry under which the scalars transform as follows:

$$\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix} \to O\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}, \tag{41}$$

where O is a general orthogonal matrix (note that ϕ are real fields). Imposing this symmetry leads to a much simpler Lagrangian:

$$\mathcal{L} = \frac{1}{2}\left[\delta_{ij}\partial^\mu\phi_i\partial_\mu\phi_j - m^2\delta_{ij}\phi_i\phi_j + \frac{\lambda}{4}\left(\phi_1^4 + \phi_2^4 + 2\phi_1^2\phi_2^2\right)\right] . \tag{42}$$

It can be written in an even simpler way by taking advantage of the fact that SO(2) and U(1) are equivalent. Then instead of considering two real scalar fields, we can consider a single complex scalar field

$$\phi \equiv \frac{1}{\sqrt{2}}\left(\phi_1 + i\phi_2\right) , \tag{43}$$

with the following $U(1)$ transformation:

$$\phi \to \exp(2\pi i\theta)\phi, \qquad \phi^* \to \exp(-2\pi i\theta)\phi^*. \tag{44}$$

Then we rewrite (42) as

$$\mathcal{L} = \partial^\mu\phi\partial_\mu\phi^* - m^2\phi\phi^* + \lambda(\phi\phi^*)^2. \tag{45}$$

We would like to emphasize the following points regarding Eq. (45):

- All three terms that appear in this equation and, in particular, the mass term, do not violate any internal symmetry. Thus, there is no way to forbid them by imposing a symmetry.

[3] In order for the potential to be bounded from below, we require that some combinations of the λ are positive. For simplicity, we will take *all* the parameters as positive.

- The conserved charge is very similar in nature to an electric charge. We can think of ϕ as a charged field that carries a positive charge and then ϕ^* carries negative charge. This is the source of the statement that only complex fields can be charged.
- The normalization of a $U(1)$ charge is arbitrary.

Let us next consider a model with four real scalar field. We group them into two complex fields and assign them charges +1 and +2 under a $U(1)$ symmetry. Then most general $U(1)$-symmetric Lagrangian is

$$\mathcal{L} = \partial^\mu \phi_i \partial_\mu \phi_i^* - m_1^2 \phi_1 \phi_1^* - m_2^2 \phi_2 \phi_2^* + \lambda_{ij} (\phi_i \phi_i^*)(\phi_j \phi_j^*) + (\mu \phi_1^2 \phi_2^* + \text{h.c.}). \tag{46}$$

We now examine the symmetry properties of the various terms of $\mathcal{L}$. The symmetry is largest for the kinetic term, become smaller when the mass terms are included, and even smaller with interaction terms added. Explicitly, the kinetic term has an SO(4) symmetry. The mass (m^2) and the quartic interaction (λ) terms have a $U(1)^2$ symmetry. The trilinear interaction (μ) term reduces the symmetry to a single $U(1)$.

There are cases where we can think about the μ terms as small. In this case the $\mathrm{U}(1)^2$ symmetry is an approximate symmetry.

Consider a similar model, but now we assign ϕ_2 charge of 3. In this case there is no trilinear scalar interaction, but the new four-scalar interaction terms, $\lambda_{1112}\phi_1^3\phi_2^* + \text{h.c.}$, break the $\mathrm{U}(1)^2$ down to U(1). Note, however, $\mathcal{L}$ has a Z_2 "scalar parity", $\phi_i \to -\phi_i$. This Z_2 is an accidental symmetry: We did not impose, we get it as a consequence of the $U(1)$ symmetry and particle content (the charge assignments of the scalar fields). Accidental symmetries can be broken if we add other fields, for example a field with charge 2.

Consider a similar model, but now we assign ϕ_2 charge of 4. The renormalizable terms in the Lagrangian have a $\mathrm{U}(1)^2$ symmetry. Yet, the dimension-5 term of the form $\phi_1^4\phi_2^*$ breaks the symmetry down to the one we imposed. In the full UV model this NR operators arise by adding other fields, so in a way this case is not different from the one we discussed earlier.

B. Fermions

Let us define the following projection operators:

$$P_\pm = \frac{1}{2}(1 \pm \gamma_5). \tag{47}$$

The four-component Dirac fermion can be decomposed to a left-handed and a right-handed (L and R) Weyl spinor fields,

$$\psi_L \;=\; P\psi, \quad \psi_R = P_+\psi, \quad \overline{\psi_L} = \overline{\psi}P_+, \quad \overline{\psi_R} = \overline{\psi}P_-. \tag{48}$$

The L and R states are (for massless fields) *helicity* eigenstates. To see that, consider a plane wave traveling in the z direction, $p^0 = p^3$, and $p^1 = p^2 = 0$. The Dirac equation in momentum space is $\not{p}\psi = 0$, so $p(\gamma^0 - \gamma^3)\psi = 0$, or

$$\gamma^0\psi = \gamma^3\psi. \tag{49}$$

The spin angular momentum in the z direction is

$$J^3 = \sigma^{12}/2 = i\gamma^1\gamma^2/2. \tag{50}$$

Then

$$\begin{aligned} J^3\psi_L \;&=\; \frac{i}{2}\gamma^1\gamma^2\psi_L = \frac{i}{2}\gamma^0\gamma^0\gamma^1\gamma^2\psi_L = \frac{i}{2}\gamma^0\gamma^1\gamma^2\gamma^0\psi_L = \frac{i}{2}\gamma^0\gamma^1\gamma^2\gamma^3\psi_L = \\ &=\; \frac{1}{2}\gamma_5\psi_L = -\frac{1}{2}\psi_L. \end{aligned} \tag{51}$$

We learn that ψ_L describes a massless particle with helicity $-1/2$. Similarly, ψ_R describes a massless particle with helicity $+1/2$.

The introduction of ψ_L and ψ_R allows yet another classification of symmetries. A *chiral symmetry* is defined as a symmetry where the LH fermion transforms differently from the RH fermion. A *vectorial symmetry* is one under which ψ_L and ψ_R transform in the same way. Denoting the charge under a $U(1)$ symmetry as Q, we thus define

$$\begin{aligned} \text{chiral symmetry :} \quad & Q(\psi_L) = Q(\psi_R), \\ \text{vectorial symmetry :} \quad & Q(\psi_L) \neq Q(\psi_R). \end{aligned} \tag{52}$$

There are two possible mass terms for fermions: Dirac and Majorana. Dirac masses couple left- and right-handed fields,

$$m_D\overline{\psi_L}\psi_R + \text{h.c.}. \tag{53}$$

Here m_D is the Dirac mass. Majorana masses couple a left-handed or a right-handed field to itself. Consider ψ_R, a SM singlet right-handed field. Its Majorana mass term is

$$m_M\overline{\psi_R^c}\,\psi_R, \qquad \psi^c = C\,\overline{\psi}^T, \tag{54}$$

where m_M is the Majorana mass and C is the charge conjugation matrix. Note that ψ_R and $\overline{\psi_R^c}$ transform in the same way under all symmetries. A similar expression holds for left handed fields.

We emphasize the following points regarding Eqs. (53) and (54):

- Since ψ_L and ψ_R are different fields, there are four degrees of freedom with the same Dirac mass, m_D. In contrast, since only one Weyl fermion field is needed in order to generate a Majorana mass term, there are only two degrees of freedom that have the same Majorana mass, m_M.

- Consider a theory with one or more exact $U(1)$ symmetries. To allow a Dirac mass, the charges of $\overline{\psi_L}$ and ψ_R under these symmetries must be opposite. In particular, the two fields can carry electric charge as long as $Q(\psi_L) = Q(\psi_R)$. Thus, to have a Dirac mass term, the fermion has to be in a *vector representation* of the symmetry group.

- The additive quantum numbers of $\overline{\psi_R^c}$ and ψ_R are the same. Thus, a fermion field can have a Majorana mass only if it is neutral under all unbroken local and global $U(1)$ symmetries. In particular, fields that carry electric charges cannot acquire Majorana masses. If we include any non-Abelian group the condition is that the fermion cannot be in a complex representation.

- When there are m left-handed fields and n right-handed fields with the same quantum numbers, the Dirac mass terms for these fields form an $m \times n$ general complex matrix m_D:
$$(m_D)_{ij}(\overline{\psi_L})_i(\psi_R)_j + \text{h.c.}. \tag{55}$$
In the SM, fermion fields are present in three copies with the same quantum numbers, and the Dirac mass matrices are 3×3. In general, however, m_D does not have to be a square matrix.

- When there are n neutral fermion fields, the Majorana mass terms form an $n \times n$ symmetric, complex matrix m_M:
$$(m_M)_{ij}(\overline{\psi_R^c})_i(\psi_R)_j. \tag{56}$$
In the SM, neutrinos are the only neutral fermions. If they have Majorana masses, then their mass matrix is 3×3.

TABLE IV: Dirac and Majorana masses

	Dirac	Majorana
# of degrees of freedom	4	2
Representation	vector	neutral
Mass matrix	$m \times n$, general	$n \times n$, symmetric
SM fermions	quarks, charged leptons	neutrinos (?)

We summarize these differences between Dirac and Majorana masses in Table IV.

The main lesson that we can draw from these observations is the following: *Charged fermions in a chiral representation are massless.* In other words, if we encounter massless fermions in Nature, there is a way to explain their masslessness from symmetry principles.

We now discuss the case of many Dirac fields and their accidental symmetries. Consider N Dirac fermions charged under a $U(1)$ fermion number. Such a theory has $2N$ chiral fields and thus the kinetic terms have a $[U(N)]^2$ symmetry. If we give the left- and right-handed fields different charges under the $U(1)$ symmetry, the mass term is forbidden and all we have is a theory of free massless fermions.

To allow masses, we assign left- and right-handed fields the same charge under $U(1)$. The case of universal mass is of particular interest:

$$\mathcal{L} = i\bar{\psi}\partial\!\!\!/\psi - m\bar{\psi}\psi = i\bar{\psi}_L\partial\!\!\!/\psi_L + i\bar{\psi}_R\partial\!\!\!/\psi_R - m\bar{\psi}_L\psi_R - m\bar{\psi}_R\psi_L, \tag{57}$$

where the "flavor" index j is omitted. This Lagrangian is invariant under the symmetry in which the L and R fields rotate together, $U(N) = SU(N) \times U(1)$. We learn that a universal mas term breaks $[U(N)]^2 \to U(N)$.

For a general, non-universal, mass term the symmetry is smaller. We can always choose a basis where the mass matrix is diagonal:

$$m^i\bar{\psi}^i_L\psi^i_R + h.c.\,. \tag{58}$$

In this case the symmetry is $[U(1)]^N$. Within the SM, this is the case of lepton flavor symmetry, which ensures that the flavor of the leptons (namely, e, μ and τ) is conserved. This is also the approximate flavor symmetry of the quark sector that is conserved by the strong and the EM forces.

Next consider a model with N left-handed and N right-handed fermions, and a single scalar field:

$$\mathcal{L} = \bar{\psi}_i[i\partial\!\!\!/\delta_{ij} - m_{ij} + Y_{ij}\phi]\psi_j + \mathcal{L}_S\,, \tag{59}$$

where $\mathcal{L}_S$ includes the kinetic term for the scalar field and Y_{ij} are the Yukawa couplings. In general we can diagonalize only m or only λ but not both. We see that the symmetry is even smaller. The only exact symmetry is $U(1)$, which is the fermion number symmetry. This is the case in the SM for the quarks, where the only exact[4] global symmetry is baryon number.

VI. LOCAL SYMMETRIES

So far we discussed global symmetries, that is, symmetries that transform the field in the same way over all space-time. Now we discuss local symmetries, that is, symmetries where the transformation can be different in different space-time points. The space-time dependence of the phase of charged fields should not be observable. Therefore, we would now let the infinitesimal parameter ϵ_a depend on x.

Before proceeding, we introduce the following notation:

$$\tilde{O} \equiv T_a O_a. \tag{60}$$

$\tilde{O}$ is an $N \times N$ matrix. Knowing $\tilde{O}$ allows us to easily recover the O_a's. Take the T_a's to be orthogonal:

$$\mathrm{tr}(T_a T_b) = \delta_{ab}. \tag{61}$$

Then

$$O_a = \mathrm{tr}(T_a \tilde{O}). \tag{62}$$

Consider the effect of a local transformation,

$$\phi \to e^{i\tilde{\epsilon}(x)}\phi(x) \quad \Longrightarrow \quad \delta\phi(x) = i\tilde{\epsilon}(x)\phi(x) \tag{63}$$

on a Lagrangian

$$\mathcal{L}(\phi, \partial_\mu\phi). \tag{64}$$

[4] at the renormalizable level!

Note that a global transformation is a special case of the local transformation. However, when we apply the local transformation on a globally invariant $\mathcal{L}$, we encounter a problem with the derivative term:

$$\delta\partial^\mu\phi = \partial^\mu\delta\phi = i\tilde{\epsilon}\partial^\mu\phi + i(\partial^\mu\tilde{\epsilon}(x))\phi\,. \tag{65}$$

The second term breaks the local symmetry. Take, for example, free massless fermions:

$$\delta\mathcal{L} = \delta\bar{\psi}\frac{\delta\mathcal{L}}{\delta\bar{\psi}} + \frac{\delta\mathcal{L}}{\delta\partial\!\!\!/\psi}\delta\partial\!\!\!/\psi\,. \tag{66}$$

We have, as before,

$$\delta\bar{\psi}\frac{\delta\mathcal{L}}{\delta\bar{\psi}} = (-i\bar{\psi}\epsilon_a T^a)(i\partial\!\!\!/\psi) = \bar{\psi}\tilde{\epsilon}\partial\!\!\!/\psi, \tag{67}$$

but now

$$\frac{\delta\mathcal{L}}{\delta\partial\!\!\!/\psi}\delta\partial\!\!\!/\psi = (i\bar{\psi})(i\epsilon_a T^a\partial\!\!\!/\psi) = -\bar{\psi}\tilde{\epsilon}\partial\!\!\!/\psi + i\bar{\psi}(\partial\!\!\!/\tilde{\epsilon})\psi\,. \tag{68}$$

Thus, the symmetry is violated:

$$\delta\mathcal{L} = i\bar{\psi}(\partial\!\!\!/\tilde{\epsilon})\psi \neq 0 \tag{69}$$

How can we "correct" for the extra term? For the global symmetry case, $\delta\mathcal{L}$ vanishes since ϕ and $\partial_\mu\phi$ transform in the same way, and we constructed all the terms in $\mathcal{L}$ as products of ϕ and $\phi^\dagger$ or their derivatives. (Recall, ϕ and $\phi^\dagger$ transform in the opposite way). The way to solve the situation for the local case is to generalize the derivative, such that its generalized form transforms as the field: We need to replace $\partial^\mu\phi$ with a "covariant" derivative $D^\mu\phi$ such that

$$\delta D^\mu\phi = i\tilde{\epsilon}D^\mu\phi. \tag{70}$$

The D^μ should have a term which cancels the $\partial^\mu\tilde{\epsilon}$ piece in (69). This is the case if D^μ transforms as

$$D^\mu \to e^{i\tilde{\epsilon}(x)}D^\mu e^{-i\tilde{\epsilon}(x)}\,. \tag{71}$$

Let us try

$$D^\mu = \partial^\mu + ig\tilde{A}^\mu\,, \tag{72}$$

where g is a fixed constant called "the coupling constant" and the transformation of A_a^μ is designed to cancel the extra piece in (69).

The construction that leads to a non-trivial local symmetry is to take A_a^μ to be a set of adjoint vector fields. We do not give here the full proof but only a brief explanation. Note that T_a are the generators of the symmetry group. Thus, the index a runs from 1 to the dimension of the group. For example, for $SU(N)$ the index a runs from 1 to N^2-1. Namely, there are N^2-1 copies of A^μ. This suggest that A^μ belongs to the adjoint representation.

The transformation law for A_a is directly obtained from Eq. (71):

$$\delta(\partial^\mu + ig\tilde{A}^\mu) = (1+i\tilde{\epsilon})(\partial^\mu + ig\tilde{A}^\mu)(1-i\tilde{\epsilon}) - (\partial^\mu + ig\tilde{A}^\mu) = ig\left(i[\tilde{\epsilon},\tilde{A}] - \frac{1}{g}\partial^\mu\tilde{\epsilon}\right). \tag{73}$$

Thus, $\tilde{A}^\mu$ transforms as follows:

$$\delta\tilde{A}^\mu = i[\tilde{\epsilon},\tilde{A}^\mu] - \frac{1}{g}\partial^\mu\tilde{\epsilon}. \tag{74}$$

Using the algebra of the group,

$$[T_a, T_b] = if_{abc}T_c \tag{75}$$

we can rewrite Eq. (74) as

$$\delta A_a^\mu = -f_{abc}\epsilon_b A_c^\mu - \frac{1}{g}\partial^\mu\epsilon_a. \tag{76}$$

Now we can check that our "guess" (72) indeed works. Remember:

$$\delta\phi = i\tilde{\epsilon}\phi\,. \tag{77}$$

Then

$$\begin{aligned}\delta D^\mu\phi &= \partial^\mu(\delta\phi) + ig\delta(\tilde{A}^\mu\phi) \\ &= i\tilde{\epsilon}\partial^\mu\phi + i(\partial^\mu\tilde{\epsilon})\phi + ig\tilde{A}^\mu i\tilde{\epsilon}\phi + ig\{i[\tilde{\epsilon},\tilde{A}^\mu]\phi - \frac{1}{g}(\partial^\mu\tilde{\epsilon})\phi\} = i\tilde{\epsilon}D^\mu\phi.\end{aligned} \tag{78}$$

The covariant derivative of a field transforms in the same way as the field. We conclude that replacing ∂^μ with D^μ gives $\mathcal{L}$ that is invariant under the local symmetry.

The field A^μ is called a gauge field. The constant g is the gauge coupling constant. We next find the kinetic term of A^μ. We define

$$[D^\mu, D^\nu] = ig\tilde{F}^{\mu\nu}. \tag{79}$$

Then

$$\tilde{F}^{\mu\nu} = \partial^\mu\tilde{A}^\nu - \partial^\nu\tilde{A}^\mu + ig[\tilde{A}^\mu,\tilde{A}^\nu]. \tag{80}$$

Using the algebra, we can rewrite Eq. (80) as follows:

$$F_a^{\mu\nu} = \partial^\mu A_a^\nu - \partial^\nu A_a^\mu - gf_{abc}A_b^\mu A_c^\nu\,. \tag{81}$$

Using the transformation law of A^μ (74) we find the transformation law for $F^{\mu\nu}$:

$$\delta(\tilde{F}^{\mu\nu}) = i[\tilde{\epsilon}, \tilde{F}^{\mu\nu}]\,. \tag{82}$$

This transformation law implies that $F^{\mu\nu}$ belongs to the adjoint representation. We can thus obtain a singlet by multiplying it with $F_{\mu\nu}$. Since this is also a Lorentz singlet, we get the locally invariant kinetic term,

$$-\frac{1}{4}F_a^{\mu\nu}F_{a\mu\nu}, \tag{83}$$

where the $-1/4$ factor is a normalization factor. While a kinetic term is gauge invariant, a mass term $\frac{1}{2}m^2 A_a^\mu A_{a\mu}$ is not. You will prove it in your homework. Here we just emphasize the result: *Local invariance implies massless gauge fields.* These gauge bosons have only two degree of freedom.

If the symmetry decomposes into several commuting factors, each factor has its own independent coupling constant. For example, if the symmetry is $SU(2)\times U(1)$, we have two independent coupling constants that we can denote as g for the $SU(2)$ and g' for the $U(1)$.

A. QED

As our first example consider QED. This theory has an Abelian local symmetry, that is $U(1)$. This is the simplest case as $\tilde{\epsilon}$ is a commuting number and $\tilde{A}$ is a commuting field. Actually, A^μ is the photon field, and

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu \tag{84}$$

is the familiar field strength tensor of EM. The Lagrangian for free photon fields is then

$$\mathcal{L}_{\rm kin} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu}. \tag{85}$$

Using the Euler–Lagrange equation, $\mathcal{L}$ gives the Maxwell equations.

Adding charged fermions to the theory, we have

$$\mathcal{L}_{\rm QED} = \bar{\psi}(i\not{D} - m)\psi - \frac{1}{4}F^{\mu\nu}F_{\mu\nu}, \tag{86}$$

where

$$D^\mu = \partial^\mu + ieqA^\mu\,. \tag{87}$$

Note that we identify the coupling constant $g = eq$, where q the electric charge of the fermions in units of the positron charge. For the electron $q = -1$. That is, in the units of the positron charge the "representation" of the electron under $U(1)_{\text{EM}}$ is -1.

Expanding D^μ, we obtain the photon–fermion interaction term:

$$\mathcal{L}_{\text{int}} = -eq\bar{\psi}\not{A}\psi \tag{88}$$

We learn that the coupling is proportional to the fermion charge and that the interaction is vector-like.

B. QCD

For non-Abelian symmetries the situation is more complicated. The gauge bosons have self-interactions, namely, they are charged under the symmetry group. In QCD the gauge group is $SU(3)$. The gluon field G_a^μ is in the adjoint (octet) representation of the group, and

$$F_a^{\mu\nu} = \partial^\mu G_a^\nu - \partial^\nu G_a^\mu - g_s f_{abc} G_b^\mu G_c^\nu \,. \tag{89}$$

where g_s is the strong interaction constant. Note the extra term compared to the photon case. This term gives rise to self interactions of the gluons. To see this, we inspect the kinetic term:

$$\mathcal{L}_{\text{kin}} = -\frac{1}{4} F_a^{\mu\nu} F_{a\mu\nu} = \mathcal{L}_0 + g_s f_{abc} (\partial^\mu G_a^\nu) G_b^\mu G_c^\nu \; + g_s^2 (f_{abc} G_b^\mu G_c^\nu)(f_{ade} G_d^\mu G_e^\nu) \,, \tag{90}$$

where $\mathcal{L}_0$ is the free field Lagrangian. The last two terms are the 3-point and 4-point gluon self interactions.

Adding fermions to the theory we have

$$\mathcal{L}_{\text{QCD}} = \bar{\psi}(i\not{D} - m)\psi - \frac{1}{4} F_a^{\mu\nu} F_{a\mu\nu}, \tag{91}$$

where

$$D^\mu = \partial^\mu + i g_s T^a G_a^\mu \,. \tag{92}$$

Expanding D^μ we obtain the gluon–fermion interaction terms:

$$\mathcal{L}_{\text{int}} = -g_s \bar{\psi} T_a \not{G}^a \psi. \tag{93}$$

We learn that the coupling is proportional to the fermion representation, T_a, and that the strong interaction is a vector-like interaction. Note that fermions that are singlets under $SU(3)_C$ have $T_a = 0$ and thus they do not interact with the gluons.

We return to QCD later in the course.

VII. SPONTANEOUS SYMMETRY BREAKING

Symmetries can be broken explicitly or spontaneously. By explicit breaking we refer to breaking by terms in the Lagrangian that is characterized by a small parameter (either a small dimensionless coupling, or small ratio between mass scales), so the symmetry is approximate. Spontaneous breaking, however, refers to the case where the Lagrangian is symmetric, but the vacuum state is not. Before we get to the formal discussion, let us first explain this concept in more detail.

A symmetry of a interactions is determined by the symmetry of the Lagrangian. The states, however, do not have to obey the symmetries. Consider, for example, the hydrogen atom. While the Lagrangian is invariant under rotations, an eigenstate does not have to be. Any state with a finite m quantum number is not invariant under rotation around the z axis. This is a general case when we have degenerate states. We can always find a basis of states that preserve the symmetry but there is the possibility to have another set that does not.

In QFT we always expand around the lowest state. This lowest state is called the "vacuum" state. When the vacuum state is degenerate, we can end up expanding around a state that does not conserve the initial symmetry of the theory. Then, it may seems that the symmetry is not there. Yet, there are features that testify to the fact that the symmetry is only spontaneously broken.

The name "spontaneously broken" indicates that there is no preference as to which of the states is chosen. The classical example is that of the hungry donkey. A donkey is in the middle between two stacks of hay. Symmetry tells us that it costs the same to go to any of the stacks. Thus, the donkey cannot choose and would not go anywhere! Yet, a real donkey would arbitrarily choose one side and go there. We say that the donkey spontaneously breaks the symmetry between the two sides.

A. Global Discrete symmetries

Consider the following Lagrangian for a single real scalar field:

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{1}{2}\mu^2\phi^2 - \frac{1}{4}\lambda\phi^4. \tag{94}$$

It is invariant under the transformation

$$\phi \to -\phi. \tag{95}$$

This symmetry would have been broken if we had a ϕ^3 term. The potential should be physically relevant, so we take $\lambda > 0$. But we can still have either $\mu^2 > 0$ or $\mu^2 < 0$. (μ^2 should be real for hermiticity of $\mathcal{L}$.) For $\mu^2 > 0$ we have an ordinary ϕ^4 theory with μ^2=(mass)^2 of ϕ. The case of interest for our purposes is

$$\mu^2 < 0. \tag{96}$$

The potential has two minima. They satisfy

$$0 = \frac{\partial V}{\partial \phi} = \phi(\mu^2 + \lambda\phi^2). \tag{97}$$

The solutions are

$$\phi_\pm = \pm\sqrt{\frac{-\mu^2}{\lambda}} \equiv \pm v. \tag{98}$$

The classical solution would be either ϕ_+ or ϕ_-. We say that ϕ acquires a *vacuum expectation value* (VEV):

$$\langle\phi\rangle \equiv \langle 0|\phi|0\rangle \neq 0. \tag{99}$$

Perturbative calculations should involve expansions around the classical minimum. Let us choose ϕ_+ (the two solutions are physically equivalent). Define a field ϕ' with a vanishing VEV:

$$\phi' = \phi - v. \tag{100}$$

In terms of ϕ', the Lagrangian is

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi')(\partial^\mu \phi') - \frac{1}{2}(2\lambda v^2)\phi'^2 - \lambda v\phi'^3 - \frac{1}{4}\lambda\phi'^4. \tag{101}$$

We used $\mu^2 = -\lambda v^2$ and discarded a constant term. Let us make several points:

a. The symmetry is *hidden.* It is *spontaneously broken* by our choice of the ground state $\langle\phi\rangle = +v$.

b. The theory is still described by two parameters only. The two parameters can be μ^2 and λ or v and λ.

c. The field ϕ' corresponds to a massive scalar field of mass $\sqrt{2}|\mu|$.

d. The existence of the other possible vacuum does not show up in perturbation theory.

The fact that the three terms - the mass term, the trilinear terms and the quartic term - depend on only two parameters, means that there is a relation between the three couplings. This relation is the clue that the symmetry is spontaneously, rather than explicitly, broken.

B. Global Continuous Symmetries

Consider a Lagrangian for a complex scalar field ϕ that is invariant under $U(1)$ transformations

$$\phi \to e^{i\theta}\phi. \tag{102}$$

It is given by

$$\mathcal{L} = (\partial_\mu\phi^*)(\partial^\mu\phi) - \mu^2\phi^*\phi - \lambda(\phi^*\phi)^2. \tag{103}$$

We can rewrite it in terms of two real scalar fields, π and σ, such that

$$\phi = (\sigma + i\pi)/\sqrt{2}. \tag{104}$$

Then

$$\mathcal{L} = \frac{1}{2}[(\partial_\mu\sigma)(\partial^\mu\sigma) + (\partial_\mu\pi)(\partial^\mu\pi)] - \frac{1}{2}\mu^2(\sigma^2+\pi^2) - \frac{1}{4}\lambda(\sigma^2+\pi^2)^2 \tag{105}$$

In term of the two real fields, the invariance is under $SO(2)$ transformations:

$$\begin{pmatrix}\sigma\\ \pi\end{pmatrix} \to \begin{pmatrix}\sigma\\ \pi\end{pmatrix}' = \begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}\sigma\\ \pi\end{pmatrix}. \tag{106}$$

The symmetry would have been broken if we had *e.g.* a $\sigma(\sigma^2+\pi^2)$ term. Again, we take $\mu^2 < 0$. In the (σ,π) plane, there is a circle of radius v of minima of the potential:

$$\langle\sigma^2+\pi^2\rangle = v^2 = -\frac{\mu^2}{\lambda}. \tag{107}$$

Without loss of generality, we choose

$$\langle\sigma\rangle = v, \qquad \langle\pi\rangle = 0. \tag{108}$$

In terms of

$$\sigma' = \sigma - v, \qquad \pi' = \pi, \tag{109}$$

the scalar is written as

$$\phi = (\sigma' - v + i\pi')/\sqrt{2}. \tag{110}$$

The Lagrangian is then

$$\mathcal{L} = \frac{1}{2}[(\partial_\mu\sigma')(\partial^\mu\sigma') + (\partial_\mu\pi')(\partial^\mu\pi')] - \lambda v^2\sigma'^2 - \lambda v\sigma'(\sigma'^2 + \pi'^2) - \frac{1}{4}\lambda(\sigma'^2 + \pi'^2)^2. \tag{111}$$

We used $\mu^2 = -\lambda v^2$ and discarded a constant term.

Note the following points:

a. The $SO(2)$ symmetry is spontaneously broken.

b. The Lagrangian describes one massive scalar σ' and one massless scalar π'.

c. In the symmetry limit we could not tell the two components of the complex scalar field. After the breaking they are different. For example, they have different masses.

d. The spontaneous breaking of a continuous global symmetry is always accompanied by the appearance of a massless scalar called *Goldstone Boson*.

e. Note that we chose a basis by assigning the vev to the real component of the field. This is an arbitrary choice. We made it since it is convenient.

Again, the Lagrangian (111) is not the most general Lagrangian without an $SO(2)$ symmetry. The three couplings obey a relation that signals spontaneous symmetry breaking.

C. The Goldstone Theorem

The spontaneous breaking of a global continuous symmetry is accompanied by massless scalars. Their number and QN's equal those of the broken generators.

Consider the Lagrangian

$$\mathcal{L}(\phi) = \frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - V(\phi) \tag{112}$$

where ϕ is some multiplet of scalar fields, and $\mathcal{L}(\phi)$ is invariant under some symmetry group:

$$\delta\phi = i\epsilon_a T^a \phi, \tag{113}$$

where the T^a are imaginary antisymmetric matrices.

We want to perturb around a minimum of the potential $V(\phi)$. We expect the ϕ field to have a VEV, $\langle\phi\rangle = v$, which minimizes V. We define

$$V_{j_1\cdots j_n}(\phi) = \frac{\partial^n}{\partial\phi_{j_1}\cdots\partial\phi_{j_n}} V(\phi). \tag{114}$$

The condition that v is an *extremum* of $V(\phi)$ reads

$$V_j(v) = 0. \tag{115}$$

The condition for a *minimum* at v is, in addition to (115),

$$V_{jk}(v) \geq 0. \tag{116}$$

The second derivative matrix $V_{jk}(v)$ is the scalar mass-squared matrix. We can see that by expanding $V(\phi)$ in a Taylor series in the shifted fields $\phi' = \phi - v$ and noting that the mass term is $\frac{1}{2}V_{jk}(v)\phi'_j\phi'_k$.

Now we check for the behavior of the VEV v under the transformation (113). There are two cases. If

$$T_a v = 0 \tag{117}$$

for all a, the symmetry is not broken. This is certainly what happens if $v = 0$. But (117) is the more general statement that the vacuum does not carry the charge T_a, so the charge cannot disappear into the vacuum. However, it is also possible that

$$T_a v \neq 0 \quad \text{for some } a. \tag{118}$$

Then the charge T_a can disappear into the vacuum even though the associated current is conserved. This is spontaneous symmetry breaking.

Often there are some generators of the original symmetry that are spontaneously broken while others are not. The set of generators satisfying (117) is closed under commutation (because $T_a v = 0$ and $T_b v = 0 \Longrightarrow [T_a, T_b]v = 0$) and generates the unbroken subgroup of the original symmetry group.

Because V is invariant under (113), we can write

$$V(\phi + \delta\phi) - V(\phi) = iV_k(\phi)\epsilon_a(T^a)_{kl}\phi_l = 0. \tag{119}$$

If we differentiate with respect to ϕ_j, we get

$$V_{jk}(\phi)(T^a)_{kl}\phi_l + V_k(\phi)(T^a)_{kj} = 0. \tag{120}$$

Setting $\phi = v$ in (120), we find that the second term drops out because of (115), and we obtain

$$V_{jk}(v)(T^a)_{kl}v_l = 0. \tag{121}$$

But $V_{jk}(v)$ is the mass-squared matrix M^2_{jk} for the scalar fields, so we can rewrite (121) in a matrix form as

$$M^2 T^a v = 0. \tag{122}$$

For T^a in the unbroken subgroup, (122) is trivially satisfied. But if $T^a v \neq 0$, (122) requires that $T^a v$ is an eigenvector of M^2 with eigenvalue zero. It corresponds to a massless boson field given by

$$\phi^T T^a v \tag{123}$$

which is called a Goldstone boson.

D. Fermion Masses

Spontaneous symmetry breaking can give masses to chiral fermions, provided that these fermions are in a vector-like representation of the unbroken subgroup. Consider a model with a $U(1)$ symmetry. The particle content consists of two chiral fermions and a complex scalar with the following $U(1)$ charges:

$$q\,(\psi_L) = 1, \qquad q\,(\psi_R) = 2, \qquad q\,(\phi) = 1. \tag{124}$$

The most general Lagrangian we can write is

$$\mathcal{L} = \mathcal{L}_{\text{kin}} + V(\phi) + Y\phi\bar{\psi}_R\psi_L + \text{h.c.}, \tag{125}$$

where $V(\phi)$ is the "scalar potential" that describes the mass and self interaction terms of the scalar. We assume that the scalar potential is such that $\langle\phi\rangle = v \neq 0$, and define

$$\phi = (h - v + i\xi)/\sqrt{2}, \tag{126}$$

so that h and ξ do not acquire vevs. Expanding around the vacuum we find

$$\mathcal{L} = \mathcal{L}_{\rm kin} + V(h) - \frac{Yv}{\sqrt{2}}\bar{\psi}_R\psi_L + \frac{Y(h+i\xi)}{\sqrt{2}}\bar{\psi}_R\psi_L + \text{h.c.}. \tag{127}$$

Note the following points:

a. The fermion has a mass $m_\psi = Yv/\sqrt{2}$. This mass is proportional to the Yukawa coupling and to the vev of the scalar.

b. The two real scalar fields, h and ξ couple to the fermion in the same way. Moreover, their coupling is proportional to the fermion mass.

E. Local symmetries: the Higgs mechanism

In this subsection we discuss spontaneous breaking of local symmetries. We demonstrate it by a breaking of a $U(1)$ gauge symmetry. We will find out that a breaking of a local symmetry results in mass terms for the gauge bosons that correspond to the broken generators. It is a somewhat surprising result, since the spontaneous breaking of a global symmetry gives massless Goldstone boson. In the case of a local symmetry, these would-be Goldstone bosons are "eaten" by the gauge bosons such that the gauge bosons have longitudinal components.

Consider the following Lagrangian for a single complex scalar field ϕ:

$$\mathcal{L} = [(\partial_\mu - igV_\mu)\phi^*][(\partial^\mu + igV^\mu)\phi] - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \mu^2\phi^*\phi - \lambda(\phi^*\phi)^2. \tag{128}$$

This Lagrangian is invariant under a local $U(1)$ symmetry,

$$\phi \to e^{i\epsilon(x)}\phi, \qquad V_\mu \to V_\mu - \frac{1}{g}\partial_\mu\epsilon(x). \tag{129}$$

Both λ and μ^2 are real, with $\lambda > 0$ and $\mu^2 < 0$. Consequently, ϕ acquires a VEV,

$$\langle\phi\rangle = \frac{v}{\sqrt{2}}, \qquad v^2 = -\frac{\mu^2}{\lambda}. \tag{130}$$

Up to a constant term, the scalar potential can be written as follows:

$$V = \lambda\left(\phi^*\phi - v^2\right)^2 . \tag{131}$$

We choose the real component of ϕ to carry the VEV, $\langle \text{Im } \phi\rangle = 0$, and define

$$\phi = \frac{1}{\sqrt{2}}(v + \eta + i\zeta) \tag{132}$$

with

$$\langle\eta\rangle = \langle\zeta\rangle = 0. \tag{133}$$

Furthermore, it is convenient to choose a gauge $\epsilon(x) = -\zeta(x)/v$. Since the symmetry is broken, a gauge choice does change the way we write the Lagrangian. It is this gauge choice that is best suited for our purposes. In this gauge

$$\phi \to \phi' = \frac{1}{\sqrt{2}}(\eta + v), \qquad V_\mu \to V'_\mu = V_\mu + \frac{1}{gv}\partial_\mu\zeta. \tag{134}$$

Then

$$\begin{aligned}\mathcal{L} \; = \; & -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}(\partial_\mu\eta)(\partial^\mu\eta) + \frac{1}{2}(g^2v^2)V'_\mu V'^\mu - \frac{1}{2}(2\lambda v^2)\eta^2 \\ & + \frac{1}{2}g^2V'_\mu V'^\mu\eta(2v+\eta) - \lambda v\eta^3 - \frac{1}{4}\lambda\eta^4.\end{aligned} \tag{135}$$

Note the following points:

1. The $U(1)$ symmetry is spontaneously broken.
2. The Lagrangian describes a massive vector boson with $m_V = gv$. In the limit $g \to 0$ we have $m_V \to 0$. That is, the longitudinal component is the Goldstone boson as expected.
3. The ζ field was "eaten" in order to give mass to the gauge boson. (Note that there is no kinetic term for ζ.) The number of degrees of freedom did not change: instead of the scalar ζ, we have the longitudinal component of a massive vector boson.
4. η is a massive scalar with $m_\eta = \sqrt{2\lambda}\, v$.

Spontaneous symmetry breaking gives masses to the gauge bosons related to the broken generators. Gauge bosons related to an unbroken subgroup will remain massless, because their masslessness is protected by the symmetry. Similarly, the Higgs boson, that is the field that acquires a VEV, must be a scalar. Otherwise it would break Lorentz invariance. Spontaneous breaking of local symmetry can give masses also to fermions, as is the case for global symmetry. In the physical gauge, the coupling of the longitudinal part of the gauge boson to the fermion is proportional to the mass, while that of the transverse component is proportional to the gauge coupling.

APPENDIX A: LIE GROUPS

A crucial role in model building is played by symmetries. You are already familiar with symmetries and with some of their consequences. For example, nature seems to have the symmetry of the Lorentz group which implies conservation of energy, momentum and angular momentum. In order to understand the interplay between symmetries and interactions, we need a mathematical tool called *Lie groups*. These are the groups that describe all continuous symmetries. There are many texts about Lie group. Three that are very useful for particle physics purposes are the book by Howard Georgi ("Lie Algebras in particle physics"), the book by Robert Cahn ("Semi-simple Lie algebras and their representations") and the physics report by Richard Slansky ("Group Theory for Unified Model Building", Phys. Rept. 79 (1981) 1).

1. Groups and representations

We start by presenting a series of definitions.

Definition: A *group* G is a set x_i (finite or infinite), with a multiplication law $\cdot$, subject to the following four requirements:

- Closure:
$$x_i \cdot x_j \in G \quad \forall\, x_i. \tag{A1}$$
- Associativity:
$$x_i \cdot (x_j \cdot x_k) = (x_i \cdot x_j) \cdot x_k. \tag{A2}$$
- Identity element I (or e):
$$I \cdot x_i = x_i \cdot I = x_i \quad \forall\, x_i. \tag{A3}$$
- Inverse element x_i^{-1}:
$$x_i \cdot x_i^{-1} = x_i^{-1} \cdot x_i = I. \tag{A4}$$

Definition: A group is *Abelian* if all its elements commute:
$$x_i \cdot x_j = x_j \cdot x_i \quad \forall\, x_i. \tag{A5}$$

A *non-Abelian* group is a group that is not Abelian, that is, at least one pair of elements does not commute.

Let us give a few examples:

- Z_2, also known as parity, is a group with two elements, I and P, such that I is the identity and $P^{-1} = P$. This completely specifies the multiplication table. This group is finite and Abelian.

- Z_N, with N=integer, is a generalization of Z_2. It contains N elements labeled from zero until $N-1$. The multiplication law is the same as addition modulo N: $x_i x_j = x_{(i+j) \text{mod } N}$. The identity element is x_0, and the inverse element is given by $x_i^{-1} = x_{N-i}$. This group is also finite and Abelian.

- Multiplication of positive numbers. It is an infinite Abelian group. The identity is the number one and the multiplication law is just a standard multiplication.

- S_3, the group that describes permutation of 3 elements. It contains 6 elements. This group is non-Abelian. Work for yourself the 6 elements and the multiplication table.

Definition: A *representation* is a realization of the multiplication law among matrices.

Definition: Two representations are *equivalent* if they are related by a similarity transformation.

Definition: A representation is *reducible* if it is equivalent to a representation that is block diagonal.

Definition: An *irreducible* representation (irrep) is a representation that is not reducible.

Definition: An irrep that contains matrices of size $n \times n$ is said to be of *dimension* n.

Statement: Any reducible representation can be written as a direct sum of irreps, *e.g.* $D = D_1 + D_2$.

Statement: The dimension of all irreps of an Abelian group is one.

Statement: Any finite group has a finite number of irreps R_i. If N is the number of elements in the group, the irreps satisfy

$$\sum_{R_i} [\dim(R_i)]^2 = N. \tag{A6}$$

Statement: For any group there exists a *trivial* representation such that all the matrices are just the number 1. This representation is also called the *singlet* representation. It is of particular importance for us.

Let us give some examples for the statements that we made here.

- Z_2: Its trivial irrep is $I = 1$, $P = 1$. The other irrep is $I = 1$, $P = -1$. Clearly these two irreps satisfy Eq. (A6).
- Z_N: An example of a non-trivial irrep is $x_k = \exp(i2\pi k/N)$.
- S_3: In your homework you will work out its properties.

The groups that we are interested in are *transformation groups of physical systems.* Such transformations are associated with *unitary operators* in the Hilbert space. We often describe the elements of the group by the way that they transform physical states. When we refer to representations of the group, we mean either the appropriate set of unitary operators, or, equivalently, by the matrices that operate on the vector states of the Hilbert space.

2. Lie groups

While finite groups are very important, the ones that are most relevant to particle physics and, in particular, to the Standard Model, are infinite groups, in particular continuous groups, that is of cardinality $\aleph_1$. These groups are called Lie groups.

Definition: A *Lie group* is an infinite group whose elements are labeled by a finite set of N continuous real parameters α_ℓ, and whose multiplication law depends smoothly on the α_ℓ's. The number N is called the dimension of the group.

Statement: An Abelian Lie group has $N = 1$. A non-Abelian Lie group has $N > 1$.

The first example is a group we denote by $U(1)$. It represents addition of real numbers modulo 2π, that is, rotation on a circle. Such a group has an infinite number of elements that are labeled by a single continuous parameter α. We can write the group elements as $M = \exp(i\alpha)$. We can also represent it by $M = \exp(2i\alpha)$ or, more generally, as $M = \exp(iX\alpha)$ with X real. Each X generates an irrep of the group.

We are mainly interested in *compact* Lie groups. We do not define this term formally here, but we can use the $U(1)$ example to give an intuitive explanation of what it means. A group of adding with a modulo is compact, while just adding (without the modulo) would be non-compact. In the first, if you repeat the same addition a number of times, you may return to your starting point, while in the latter this would never happen. In other words, in a compact Lie group, the parameters have a finite range, while in a non-compact group, their range is infinite. (Do not confuse that with the number of elements, which is infinite in either case.) Another example is rotations and boosts: Rotations represent a compact group while boosts do not.

Statement: The elements of any compact Lie group can be written as

$$M_i = \exp(i\alpha_\ell X_\ell) \tag{A7}$$

such that X_ℓ are Hermitian matrices that are called *generators*. (We use the standard summation convention, that is $\alpha_\ell X_\ell \equiv \sum_\ell \alpha_\ell X_\ell$.)

Let us perform some algebra before we turn to our next definition. Consider two elements of a group, A and B, such that in A only $\alpha_a \neq 0$, and in B only $\alpha_b \neq 0$ and, furthermore, $\alpha_a = \alpha_b = \lambda$:

$$A \equiv \exp(i\lambda X_a), \qquad B \equiv \exp(i\lambda X_b). \tag{A8}$$

Since A and B are in the group, each of them has an inverse. Thus also

$$C = BAB^{-1}A^{-1} \equiv \exp(i\beta_c X_c) \tag{A9}$$

is in the group. Let us take λ to be a small parameter and expand around the identity. Clearly, if λ is small, also all the β_c are small. Keeping the leading order terms, we get

$$C = \exp(i\beta_c X_c) \approx I + i\beta_c X_c, \qquad C = BAB^{-1}A^{-1} \approx I + \lambda^2[X_a, X_b]. \tag{A10}$$

In the $\lambda \to 0$ limit, we have

$$[X_a, X_b] = i\frac{\beta_c}{\lambda^2}X_c. \tag{A11}$$

Clearly, the combinations

$$f_{abc} \equiv \lambda^{-2}\beta_c \tag{A12}$$

should be independent of λ. Furthermore, while λ and β_c are infinitesimal, the f_{abc}-constants do not diverge. This brings us to a new set of definitions.

Definition: f_{abc} are called the *structure constants* of the group.

Definition: The commutation relations [see Eq. (A11)]

$$[X_a, X_b] = if_{abc}X_c, \tag{A13}$$

constitute the *algebra* of the Lie group.

Note the following points regarding the Lie Algebra:

- The algebra defines the local properties of the group but not its global properties. Usually, this is all we care about.

- The Algebra is closed under the commutation operator.

- Similar to our discussion of groups, one can define representations of the algebra, that is, matrix representations of X_ℓ. In particular, each representation has its own dimension. (Do not confuse the dimension of the representation with the dimension of the group!)

- The generators satisfy the Jacoby identity

$$[X_a,[X_b,X_c]]+[X_b,[X_c,X_a]]+[X_c,[X_a,X_b]]=0. \tag{A14}$$

- For each algebra there is the trivial (singlet) representation which is $X_\ell=0$ for all ℓ. The trivial representation of the algebra generates the trivial representation of the group.

- Since an Abelian Lie group has only one generator, its algebra is always trivial. Thus, the algebra of $U(1)$ is the only Abelian Lie algebra.

- Non-Abelian Lie groups have non-trivial algebras.

The example of $SU(2)$ algebra is well-known from QM courses:

$$[X_a,X_b]=i\varepsilon_{abc}X_c. \tag{A15}$$

Usually, in QM, X is called L or S or J. The $SU(2)$ group represents non-trivial rotations in a two-dimensional complex space. Its algebra is the same as the algebra of the $SO(3)$ group, which represents rotations in the three-dimensional real space.

We should explain what we mean when we say that "the group represents rotations in a space." The QM example makes it clear. Consider a finite Hilbert space of, say, a particle with spin S. The matrices that rotate the direction of the spin are written in terms of exponent of the S_i operators. For a spin-half particle, the S_i operators are written in terms of the Pauli matrices. For particles with spin different from $1/2$, the S_i operators will be written in terms of different matrices. We learn that the group represents rotations in some space, while the various representations correspond to different objects that can "live" in that space.

There are three important irreps that have special names. The first one is the trivial – or *singlet* – representation that we already mentioned. Its importance stems from the fact that it corresponds to something that is symmetric under rotations. While that might sound confusing it is really trivial. Rotation of a singlet does not change its representation. Rotation of a spin half does change its representation.

The second important irrep is the *fundamental* representation. This is the smallest irrep. For $SU(2)$, this is the spinor representation. An important property of the fundamental representation is that it can be used to get all other representations. We return to this point later. Here we just remind you that this statement is well familiar from QM. One can get spin-1 by combining two spin-1/2, and you can get spin-3/2 by combining three spin-1/2. Any Lie group has a fundamental irrep.

The third important irrep is the *Adjoint* representation. It is made out of the structure constants themselves. Think of a matrix representation of the generators. Each entry, T^c_{ij} is labelled by three indices. One is the c index of the generator itself, that runs from 1 to N, such that N depends on the group. The other two indices, i and j, are the matrix indices that run from 1 to the dimension of the representation. One can show that each Lie group has one representation where the dimension of the representation is the same as the dimension of the group. This representation is obtained by defining

$$(X_c)_{ab} \equiv -if_{abc}. \tag{A16}$$

In other words, the structure constants themselves satisfy the algebra of their own group. In $SU(2)$, the Adjoint representation is that of spin-1. It is easy to see that the ε_{ijk} are just the set of the three 3×3 representations of spin 1.

3. More formal developments

Definition: A *subalgebra* M is a set of generators that are closed under commutation.

Definition: Consider an algebra L with a subalgebra M. M is an *ideal* if for any $x \in M$ and $y \in L$, $[x, y] \in M$. (For a subalgebra that is not ideal we still have $[x, y] \in L$.)

Definition: A *simple* Lie algebra is an algebra without a non-trivial ideal. (Any algebra has a trivial ideal, the algebra itself.)

Definition: A *semi-simple* Lie algebra is an algebra without a $U(1)$ ideal.

Any algebra can be written as a direct product of simple lie algebras. Thus, we can think about each of the simple algebras separately. You are familiar with this. For example, consider the hydrogen atom. We can think about the Hilbert space as a direct product of the spin of the electron and the spin of the proton.

A useful example is that of the $U(2)$ group, which is not semi-simple:

$$U(2) = SU(2) \times U(1). \tag{A17}$$

A $U(2)$ transformation corresponds to a rotation in two-dimensional complex space. Think, for example, about the rotation of a spinor. It can be separated into two: The trivial rotation is just a $U(1)$ transformation, that is, a phase multiplication of the spinor. The non-trivial rotation is the $SU(2)$ transformation, that is, an internal rotation between the two spin components.

Definition: The *Cartan subalgebra* is the largest subset of generators whose matrix representations can all be diagonalized at once.

Obviously, these generators all commute with each other and thus they constitute a subalgebra.

Definition: The number of generators in the Cartan subalgebra is called the *rank* of the algebra.

Let us consider a few examples. Since the $U(1)$ algebra has only a single generator, it is of rank one. $SU(2)$ is also rank one. You can make one of its three generators, say S_z, diagonal, but not two of them simultaneously. $SU(3)$ is rank two. We later elaborate on $SU(3)$ in much more detail. (We have to, because the Standard Model has an $SU(3)$ symmetry.)

Our next step is to introduce the terms roots and weights. We do that via an example. Consider the $SU(2)$ algebra. It has three generators. We usually choose S_3 to be in the Cartan subalgebra, and we can combine the two other generators, S_1 and S_2, to a raising and a lowering operator, $S^\pm = S_1 \pm iS_2$. Any representation can be defined by the eigenvalues under the operation of the generators in the Cartan subalgebra, in this case S_3. For example, for the spin-1/2 representation, the eigenvalues are $-1/2$ and $+1/2$; For the spin-1 representation, the eiganvalues are -1, 0, and $+1$. Under the operation of the raising (S^+) and lowering (S^-) generators, we "move" from one eigenstate of S_3 to another. For example, for a spin-1 representation, we have $S^+|-1\rangle \propto |0\rangle$.

Let us now consider a general Lie group of rank n. Any representation is characterized by the possible eigenvalues of its eigenstates under the operation of the Cartan subalgebra: $|e_1, e_2 ..., e_n\rangle$. We can assemble all the operators that are not in the Cartan subalgebra into "lowering" and "raising" operators. That is, when they act on an eigenstate they either move it to another eigenstate or annihilate it.

Definition: The *weight vectors* (weights) of a representation are the possible eigenvalues of the generators in the Cartan subalgebra.

Definition: The *roots* of the algebra are the various ways in which the generators move a state between the possible weights.

Statement: The weights completely describe the representation.

Statement: The roots completely describe the Lie algebra.

Note that both roots and weights live in an n-dimensional vector space, where n is the rank of the group. The number of roots is the dimension of the group. The number of weights is the dimension of the irrep.

Let us return to our $SU(2)$ example. The vector space of roots and weights is one-dimensional. The three roots are $0, \pm 1$. The trivial representation has only one weight, zero; The fundamental has two, $\pm 1/2$; The adjoint has three, $0, \pm 1$ (the weights of the adjoint representations are just the roots); and so on.

4. $SU(3)$

In this section we discuss the $SU(3)$ group. It is more complicated than $SU(2)$. It allows us to demonstrate few aspects of Lie groups that cannot be demonstrated with $SU(2)$. Of course, it is also important since it is relevant to particle physics.

$SU(3)$ is a generalization of $SU(2)$. It may be useful to think about it as rotations in three-dimensional complex space. Similar to $SU(2)$, the full symmetry of the rotations is called $U(3)$, and it can be written as a direct product of simple groups, $U(3) = SU(3) \times U(1)$. The $SU(3)$ algebra has eight generators. (There are nine independent Hermitian 3×3 matrices. They can be separated to a unit matrix, which corresponds to the $U(1)$ part, and eight traceless matrices, which correspond to the $SU(3)$ part.)

Similar to the use of the Pauli matrices for the fundamental representation of $SU(2)$, the fundamental representation of $SU(3)$ is usually written in terms of the Gell-Mann matrices,

$$X_a = \lambda_a/2, \tag{A18}$$

with

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$
$$\lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$
$$\lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \qquad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$
$$\lambda_7 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \qquad \lambda_8 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}. \tag{A19}$$

We would like to emphasize the following points:

1. The Gell-Mann matrices are traceless, as they should.

2. There are three $SU(2)$ subalgebras. One of them is manifest and it is given by λ_1, λ_2 and λ_3. Can you find the other two?

3. It is manifest that $SU(3)$ is of rank two: λ_3 and λ_8 are in the Cartan subalgebra.

Having explicit expressions of fundamental representation in our disposal, we can draw the weight diagram. In order to do so, let us recall how we do it for the fundamental (spinor) representation of $SU(2)$. We have two basis vectors (spin-up and spin-down); we apply S_z on them and obtain the two weights, $+1/2$ and $-1/2$. Here we follow the same steps. We take the three vectors,

$$(1,0,0,)^T, \quad (0,1,0)^T, \quad (0,0,1)^T, \tag{A20}$$

and apply to them the two generators in the Cartan subalgebra, X_3 and X_8. We find the three weights

$$\left(+\frac{1}{2},+\frac{1}{2\sqrt{3}}\right), \qquad \left(-\frac{1}{2},+\frac{1}{2\sqrt{3}}\right), \qquad \left(0,-\frac{1}{\sqrt{3}}\right). \tag{A21}$$

We can plot this in a weight diagram in the $X_3 - X_8$ plane. Please do it.

Once we have the weights we can get the roots. They are just the combination of generators that move us between the weights. Clearly, the two roots that are in the Cartan are at the origin. The other six are those that move us between the three weights. It is easy to find that they are

$$\left(\pm\frac{1}{2}, \pm\frac{\sqrt{3}}{2}\right), \qquad (\pm 1, 0)\,. \tag{A22}$$

Again, it is a good idea to plot it. This root diagram is also the weight diagram of the Adjoint representation.

5. Dynkin diagrams

The $SU(3)$ example allows us to obtain more formal results. In the case of $SU(2)$, it is clear what are the raising and lowering operators. The generalization to groups with higher rank is as follows.

Definition: A *positive (negative) root* is a root whose first non-zero component is positive (negative). A raising (lowering) operator correspond to a positive (negative) root.

Definition: A *simple root* is a positive root that is not the sum of other positive roots.

Statement: Every rank-k algebra has k simple roots. Which ones they are is a matter of convention, but their relative lengths and angles are fixed.

In fact, it can be shown that the simple roots fully describe the algebra. It can be further shown that there are only four possible angles and corresponding relative length between simple roots:

$$\begin{array}{r||c|c|c|c} \text{angle} & 90^\circ & 120^\circ & 135^\circ & 150^\circ \\ \text{relative length} & 1:1 & 1:1 & 1:\sqrt{2} & 1:\sqrt{3}. \end{array} \tag{A23}$$

The above rules can be visualized using Dynkin diagrams. Each simple root is described by a circle. The angle between two roots is described by the number of lines connecting the circles:

(A24)

where the solid circle in a link represent the largest root.

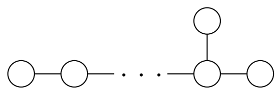
There are seven classes of Lie groups. Four classes are infinite and three classes, called the exceptional groups, have each only a finite number of Lie groups. below you can find all the sets. The number of circles is the rank of the group. Note that different names for the infinite groups are used in the physics and mathematics communities. Below we give both names, but we use only the physics name from now on.

SU$(k+1)$ $[A_k]$

Sp$(2k)$ $[B_k]$

SO$(2k+1)$ $[C_k]$

SO$(2k)$ $[D_k]$



(A25)

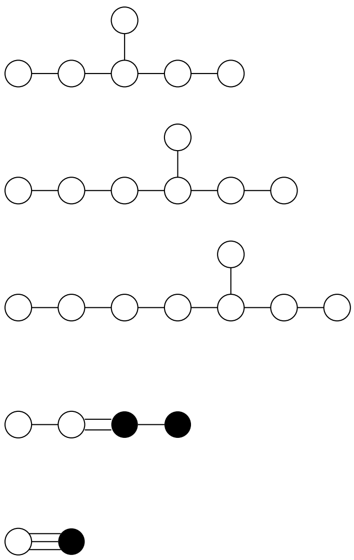
E_6

E_7

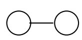
E_8

F_4

G_2



(A26)

Consider, for example, $SU(3)$. The two simple roots are equal in length and have an angle of 120° between them. Thus, the Dynkin diagram is just  .

Dynkin diagrams provide a very good tool to tell us also about what are the subalgebras of a given algebra. We do not describe the procedure in detail here, and you are encouraged to read it for yourself in one of the books. One simple point to make is that removing a simple root always corresponds to a subalgebra. For example, removing simple roots you can see the following breaking pattern:

$$E_6 \to SO(10) \to SU(5) \to SU(3) \times SU(2). \tag{A27}$$

You may find such a breaking pattern in the context of Grand Unified Theories (GUTs).

Finally, we would like to mention that the algebras of some small groups are the same. For example, the algebras of $SU(2)$ and $SO(3)$ are the same, as are those of $SU(4)$ and $SO(6)$.

6. Naming representations

How do we name a representation? In the context of $SU(2)$, which is rank one, there are three different ways to do so.

(i) We denote a representation by its highest weight. For example, spin-0 denotes the singlet representation, spin-1/2 refers to the fundamental representation, where the highest weight is 1/2, and spin-1 refers to the adjoint representation, where the highest weight is 1.

(ii) We can define the representation according to the dimension of the representation-matrices. Then the singlet representation is denoted by 1, the fundamental by 2, and the adjoint by 3.

(iii) We can name the representation by the number of times we can apply S_- to the highest weight without annihilating it. In this notation, the singlet is denoted as (0), the fundamental as (1), and the adjoint as (2).

Before we proceed, let us explain in more detail what we mean by "annihilating the state". Let us examine the weight diagram. In $SU(2)$, which is rank-one, this is a one dimensional diagram. For example, for the fundamental representation, it has two entries, at $+1/2$ and $-1/2$. We now take the highest weight (in our example, $+1/2$), and move away from it by applying the root that corresponds to the lowering operator, -1. When we apply it once, we move to the lowest weight, $-1/2$. When we apply it once more, we move out of the weight diagram, and thus "annihilate the state". Thus, for the spin-1/2 representation, we can apply the root corresponding to S_- once to the highest weight before moving out of the weight diagram, and – in the naming scheme (iii) – we call the representation (1).

We are now ready to generalize this to general Lie algebras. Either of the methods (ii) and (iii) are used. Method (ii) is straightforward, but somewhat problematic. For example, for $SU(3)$, the singlet, fundamental and adjoint representations are denoted by, respectively, 1, 3, and 8. The problem lies in the fact that there could be several different representations with the same dimension, in which case they are distinguihsed by other ways (*e.g.* m and m', or m_1 and m_2).

To use the scheme (iii), we must order the simple roots in a well-defined (even if arbitrary) order. Then we have a unique highest weight. We denote a representation of a rank-k algebra as a k-tuple, such that the first entry is the maximal number of times that we can apply the first simple root on the highest weight before the state is annihilated, the second entry refers to the maximal number of times that we can apply the second simple root on the highest weight before annihilation, and so on. Take again $SU(3)$ as an example. We order the Cartan subalgebra as X_3, X_8 and the two simple roots as

$$S_1 = \left(+\frac{1}{2}, +\frac{\sqrt{3}}{2}\right), \qquad S_2 = \left(+\frac{1}{2}, -\frac{\sqrt{3}}{2}\right). \tag{A28}$$

Consider the fundamental representation where the highest weight can be chosen to be $\left(+1/2, +1/(2\sqrt{3})\right)$. Subtracting S_1 twice or subtracting S_2 once from the highest weight would annihilate it. Thus the fundamental representation is denoted by $(1, 0)$. You can work out the case of the adjoint representation and find that it should be denoted as $(1, 1)$. In fact, it can be shown that any pair of non-negative integers forms a different irrep. (For $SU(2)$ with the naming scheme (iii), any non-negative integer defines a different irrep.)

From now on we limit our discussion to $SU(N)$.

Statement: For any $SU(N)$ algebra, the fundamental representation is $(1, 0, 0, ..., 0)$.

Statement: For any $SU(N \geq 3)$ algebra, the adjoint representation is $(1, 0, 0, ..., 1)$.

Definition: The *conjugate representation* is the one where the order of the k-tuple is reversed.

For example, $(0, 1)$ is the conjugate of the fundamental representation, which is usually called the anti-fundamental representation. Note that some representations are self–conjugate, *e.g.*, the adjoint representation. An irrep and its conjugate have the same dimension. In the naming scheme (ii), they are called m and $\bar{m}$.

7. Particle representations

We now return to the notion that the groups that we are dealing with are transformation groups of physical states. These physical states are often just particles. For example, when we talk about the $SU(2)$ group that is related to the spin transformations, the physical system that is being transformed is often that of a single particle with well-defined spin. In this context, particle physicists often abuse the language by saying that the particle is, for example, in the spin-1/2 representation of $SU(2)$. What they mean is that, as a state in the Hilbert space, it transforms by the spin operator in the 1/2 representation of $SU(2)$. Similarly, when we say that the proton and the neutron form a doublet of isospin-$SU(2)$ (we later define the isospin group), we mean that we represent p by the vector-state $(1,0)^T$ and n by the vector-state $(0,1)^T$, so that the appropriate representation of the isospin generators is by the 2×2 Pauli matrices. In other words, we loosely speak on "particles in a representation" when we mean "the representation of the group generators acting on the vector states that describe these particles." Now, that we explained how physicists abuse the language, we feel free to do so ourselves; We will often talk about "particles in a representation."

How many particles there are in a given irrep? Let us consider a few examples.

- Consider an (α) representation of $SU(2)$. It has

$$N = \alpha + 1, \tag{A29}$$

particles. The singlet (0), fundamental (1) and adjoint (2) representations have, respectively, 1, 2, and 3 particles.

- Consider an (α, β) representation of $SU(3)$. It has

$$N = (\alpha+1)(\beta+1)\,\frac{\alpha+\beta+2}{2} \tag{A30}$$

particles. The singlet $(0,0)$, fundamental $(1,0)$ and adjoint $(1,1)$ representations have, respectively, 1, 3, and 8 particles.

- Consider an (α,β,γ) representation of $SU(4)$. It has

$$N = (\alpha+1)(\beta+1)(\gamma+1)\,\frac{\alpha+\beta+2}{2}\,\frac{\beta+\gamma+2}{2}\,\frac{\alpha+\beta+\gamma+3}{3} \tag{A31}$$

particles. The singlet $(0,0,0)$, fundamental $(1,0,0)$ and adjoint $(1,0,1)$ representations have, respectively, 1, 4, and 15 particles. Note that there is no $\alpha+\gamma+2$ factor. Only a consecutive sequence of the label integers appears in any factor.

- The generalization to any $SU(N)$ is straightforward. It is easy to see that the fundamental of $SU(N)$ is an N and the adjoint is N^2-1.

In $SU(2)$, the number of particles in a representation is unique. In a general Lie group, however, the case may be different. Yet, it is often used to identify irreps. For example, in $SU(3)$ we usually call the fundamental 3, and the adjoint 8. For the anti-fundamental we use $\bar{3}$. In cases where there are several irreps with the same number of particles we often use a prime to distinguish them. For example, in $SU(3)$, both $(4,0)$ and $(2,1)$ contain 15 particles. We denote them by 15 and $15'$.

Two more definitions: For an $SU(N)$ group, a *real* representation is a one that is equal to its conjugate one. $SU(2)$ has only real irreps. The adjoint of any $SU(N)$ is real, while the fundamental for $N \geq 3$ is complex.

8. Combining representations

When we study spin, we learn how to combine $SU(2)$ representations. The canonical example is to combine two spin-1/2 to generate a singlet (spin-0) and a triplet (spin-1). We need to learn how to combine representations in $SU(N > 2)$ as well. The basic idea is, just like in $SU(2)$, that we need to find all the possible ways to combine the indices and then assign it to the various irreps. That way we know what irreps are in the product representation and the corresponding CG-coefficients. This is explained in many textbooks and we do not explain it any further here.

Often, however, all we want to know is what irreps appear in the product representation, without the need to get all the CG-coefficients. There is a simple way to do just this for a general $SU(N)$. This method is called *Young Tableaux*, or Young Diagrams. The details of the method are well explained in the PDG, pdg.lbl.gov/2007/reviews/youngrpp.pdf.

With this comment we conclude our very brief introduction to Lie groups. We are now ready to start the physics part of the course.