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ON THE ROLE OF DIVISION, JORDAN AND RELATED ALGEBRAS IN PARTICLE PHYSICS

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Bir katredir ancak aldigim hep,
Derya yine durmada lebalep.

The Things I've chosen are a drop, no more;
The undiminished sea still crowds the shore

Ziya Pasha

PREFACE

In the past two decades we have witnessed a breathtaking movement toward ever greater algebrization and geometrization of particle physics. Thus powerful algebras such as division algebras, specifically quaternions and octonions, Jordan and related algebras, have arisen in a concerted manner in unified theories of basic interactions. This book attempts a personal survey of these exceptional structures and some of their physical applications. Our presentation begins with the specific structures associated with quaternions, octonions, with Jordan algebras over them. It closes with more general aspects of division, Jordan and related algebras. These topics are necessarily interwoven with supersymmetry, realized linearly and nonlinearly, with superstrings and supermembranes. As outlined in our table of contents, we first detail various algebraic, group theoretical, functional, geometrical and topological aspects of quaternions and octonions. We then proceed to the Jordan and related algebras. The subject matter being a vast one, our focus will be on selected mathematical formulations and applications familiar to us in gauge theories and theories of extended objects. While possible ramifications of certain topics are pointed out, we will not dwell upon them. To add more flavor and color to the text and to follow the lead of Bourbaki's Elements of Mathematics, a historical sketch is attached to each main topic.

While our themes center around physical applications of the above mentioned algebras, we cannot help touching on connections to other branches of mathematics such as function theory, differential geometry, topology and even number theory. Their interconnectedness mirrors the unity of Nature, which cares not for the artificial, shifting boundaries between scientific disciplines and subdisciplines. The map is not the territory.

One major motivation of this volume is to assist interested advanced students and researchers who have found it hard to locate a unified treatment of topics which are scattered in the physics and mathematics literature. Here we hope to have gathered into an organic whole selected materials from our notes, papers, lectures and review articles, published and unpublished. At places, we have drawn freely from related works by others. The balance of the book is broadly divided into three parts: 1) quaternions, 2) octonions, 3) division, Jordan and connected algebras. The varying scope and depth of coverage are somewhat arbitrary and reflect partly our familiarity (or lack of it) with the topics presented. As such, this volume offers our own limited perspective of an exciting domain of research in contemporary particle and mathematical physics. Hopefully, our extensive, though not exhaustive, reference list will help fill the many gaps and omissions. In locating these references, we were struck by their number, diversity and, in some instances, by their high degree of mathematical sophistication. This realization convinced us early on of the futility of achieving any degree of completeness, if this book is to be kept within sensible bounds.

We thank our many friends, former students and colleagues, particularly Itzhak Bars, Sultan Catto, Resit Dündarer, Mark Evans, Marek Grabowski, Murat Günaydin, Wenxin Jiang, Mehmet Koca, Soonkeon Nam, Victor Ogievetski, Pierre Ramond, Meral Serdaroglu and Pierre Sikivie. They have collaborated with one or both of us over the years on the topics presented here and have contributed much to our own understanding of these matters. To those authors whose works are briefly reviewed here, we apologize for any omission of details, difficulties and subtleties. We are specially grateful to Suha Gürsey and Laura Tze for their love, for their unwavering support and encouragements through the years it took to carry this project to completion. At last but not least, we acknowledge the partial support from the U.S. Department of Energy.

Originally this volume was conceived and written as a Physics Report C, whence its present format. May its perusal ease the way to all persons curious about the nature and workings of quaternions, octonions, Jordan algebras and the likes. May the reader share with us the sense of beauty, of mystery on encountering these exceptional structures, a strong feeling for their ultimate physical relevance.

April 1992

Feza Gürsey and Chia-Hsiung Tze

Note Added: Shortly after the completion of the first draft of the manuscript, Feza Gürsey passed away on April 13, 1992, after a short illness. Those of us who had the good fortune of knowing him, of doing physics with him, will miss his wonderful human qualities, his deep insights, the unique sense of beauty and poetry with which he lived life and pursued physics. To paraphrase what was said about Hermann Weyl, Feza was truly a physicist with the soul of a mathematician with the soul of poet.

In this book, Feza had wanted to see some fruits of his research in the past two decades gathered in a unified exposition. Hopefully, his wish has been fulfilled within the limited scope of this volume. Had Feza been able to see this project to its conclusion, he would surely have made a number of improvements to the present volume, in both style and content. I apologize for any errors and shortcomings of this work, they are entirely my own. I wish to warmly thank Ms Janet Sanders, Dr. Hoseong La and especially World Scientific editor Ms E H Chionh for their careful reading of the manuscript, their helpful suggestions and for spotting many typos.

July 1996

Chia-Hsiung Tze

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ON THE ROLE OF DIVISION, JORDAN AND RELATED ALGEBRAS IN PARTICLE PHYSICS

1. Introduction

Symmetries: their roles, their mathematics

In the past two decades we have witnessed a renaissance in the interaction between mathematics and physics [1]. Notably, a remarkably rich cross fertilization has been forged between two frontier areas of physics and mathematics, between relativistic, classical and quantized, field theory and the theory of complex manifolds [2, 3].

On the one hand, in their search for a unified theory of all basic forces including gravity, particle theorists have come upon structures of great mathematical beauty and depth. On the other hand, complex analysis, rooted in the theory of Riemann surfaces and algebraic geometry, have developed vigorously through the injection of ideas from algebraic topology, global differential geometry and the theory of partial differential equations. The connection between physics and complex manifolds rests primarily on the following facts: first, supersymmetry and complex manifolds go hand in hand [4]. Second, the solutions of the conformally invariant field equations, be they the Maxwell or Yang-Mills equations, can be recast as Cauchy-Riemann equations over a suitably chosen background space [5, 6, 7]. Specifically, in Penrose Twistor's Program, these massless field equations can be expressed entirely in terms of cohomology classes on complex manifolds with coefficients in certain holomorphic vector bundles [2, 8]. Yet seen in the all encompassing scenario of string theories [9], these field equations merely describe the dynamics of the massless modes of the infinite spectrum of Planckian excitations of a 10-dimensional superstring.

For Hermann Weyl [10, 11] , the origin of symmetry in nature lies in the very mathematical character of physical laws. In the past thirty or so years, the developments, at times feverous, of particle physics have been dominated by one theme, the exploitation of exact or broken symmetry principles [12]. With its unqualified successes the use of symmetries has become synonymous with that of Lie algebras and groups. We are here reminded of an insightful remark of Poincaré. He wrote that " the group concept pre-exists in our minds, at least potentially; it comes to us not as an aspect of our feelings but as an aspect of our understanding."

Since the 1960s physicists have gone a long way from finite parameter Lie algebras and groups, applied to flavor and local gauge symmetries of point particles, to infinite Virasoro-Kac-Moody algebras and groups arising in superstring theories. In this renewed cross-fertilization between pure mathematics and theoretical physics, the mathematical standards in the latter discipline have be raised to ever greater heights. Along with the realization via supersymmetry and the pursuit of anyons that symmetries in quantum theory are more general than groups [13], the range of applicable mathematics has widened to include the whole gamut of all the major arteries of modern mathematics, from infinite groups to non-commutative geometry, to the theory of knots and links, to quantum groups to p-adic numbers.

Since the underlying themes of this volume are symmetries, how fundamental, may we ask, are the conjectured and discovered algebraic symmetry structures ? Can we ever tell whether they constitute the ultimate essence of the basic theories or they arise as effective, and hence approximate symmetries, from more fundamental theories in which they are themselves absent ?

What are the symmetries of a certain given field theory ? The latter's action functional, aside from any of its explicit symmetry (ies), can have additional, hidden symmetries. While some of these symmetries of both the action and the vacuum may survive in the Hilbert space of states, others will be spontaneously broken by the asymmetric vacuum and disappear from the physical spectrum. Moreover, the latter could exhibit effective symmetries which, while absent in the original action, are at work in a new effective action, describing some subset of physical states.

The exact or approximate nature of a symmetry can also be a function of the resolution of experimental measurements. One classic example is the accidental O(4) symmetry of the hydrogen atom [14], an exact symmetry only if one neglects the fine structure. It is thus a symmetry not of the underlying QED but of an effective theory given by the non-relativistic Schrödinger equation with a Coulomb potential. A second example is the exact color SU(3) symmetry of QCD [15]; it is totally absent from the observed hadronic spectrum composed solely of color singlets. Its existence was indirectly inferred 1) by noting that, in the context of a fermionic quark model with the Pauli principle enforced, the spin $s = 3/2$ and $s = 1/2$ baryons lie in a symmetric representation of the spin-flavor group; 2) then by relating the new color degrees of freedom to the total $e^+ - e^-$ cross-sections. Spontaneously broken symmetries are also realized in QCD in the limit of vanishing u and d quark masses. There is then a $SU(2)_L \times SU(2)_R$ chiral symmetry. The latter is spontaneously broken by a non-symmetric vacuum and shows up in the physical spectrum through the small mass of the pions, the Nambu-Goldstone bosons. We note the phenomenological successes of both the soft pion theorems of current algebras and of the revived Skyrmion approach to low energy hadronic physics [16]. The other symmetry of QCD is the U(1) symmetry connected to the η meson. As 't Hooft [17, 18] showed, its absence from the hadron spectrum arises from nonperturbative instanton induced tunnelling in the degenerate QCD vacuum.

Where could supersymmetry [19] find its phenomenological niche within the standard model and beyond ? It is clearly absent at the level of the QCD lagrangian; the latter is not invariant under any transformations mixing quarks and gluons. Yet, at the level of the asymptotic hadronic states, a kind of supersymmetry [20, 21] is suggested by the near equality of the slopes of baryonic and mesonic Regge trajectories [22, 23] (for more details, see Section 3.f.3.). Is such a hadronic supersymmetry rooted at some deeper level ? or is it, like the O(4) symmetry of the hydrogen atom, a mere dynamical accident ? Of course, one can readily supersymmetrize QCD by doubling its basic fields. However, if they exist at all, the additional gluinos and squarks must have a high mass and bear no relation to the observed low energy hadronic supersymmetry.

On the other hand, if supersymmetry is a feature of the basic action, it cannot hold true in nature for the vacuum state and must then be, like chiral symmetry, spontaneously broken at a higher mass scale. At present the compelling reasons for having any kind of fundamental supersymmetry still appear to be purely theoretical. Long ago Pauli [24] observed that, since fermions and bosons contribute to loop diagrams with opposite signs, mutual cancellations could be contemplated. Supersymmetric field theories realize such divergence cancellations. Thus the addition of the spin 3/2 gravitino to Einstein gravity, while not resolving the problems of quantum gravity, does improve on its quantum behavior. Gauging supersymmetry automatically incorporates gravity within the folds of unification of all the other basic forces and puts it on the same footing as the latter. The cosmological constant is also naturally zero in a supersymmetric theory.

Now a prevalent strategy in particle theory consists in packing in one's theories as many fundamental symmetries as conceivable, even spontaneously broken ones. In so doing, one improves the chance of ending up with a renormalizable, finite relativistic quantum theory of all interactions. Therein lies the true significance of theories which are symmetric at a fundamental level, even though these symmetries do not show up in the physical spectrum. Considerations of compatibility of seemingly disparate requirements such as general covariance, gauge invariance, quantum mechanics, finiteness and unification have led us to models with ever higher symmetries. There may well be no solution at the end of this journey. However, if a solution does exist, it is unlikely to be of the generic type. In fact, if the recent experience with superstrings is any guide, we expect the emergence of a very special, maximally symmetric theory (or hierarchy of theories) with an underlying mathematical structure which is equally exceptional and symmetrical. What could that structure be?

In modern mathematics, notable examples are the exceptional Lie groups and their affine or hyperbolic extensions, their arithmetics and associated Tits geometries, the sporadic finite groups such as the Monster, self-dual lattices in higher dimensions, modular invariant automorphic functions and their generalizations. The discoveries of these linked patterns have sent mathematicians on the search for a unified theory of all exceptional structures.

In physics, the quest for a finite, unified relativistic quantum theory has also brought physicists to precisely the same exceptional structures. The present volume attempts a survey of these exceptional structures. Such uncanny convergence of vistas in fundamental mathematics and physics would be too extraordinary a coincidence in the absence of a deeper *raison d'être*. The latter still eludes us and, for lack of further experimental inputs, spurs us on to even greater leaps of the imagination. In the following chapters, we shall explore some of the algebraic stepping stones for such pushes forward.

2. Quaternions

After a short review of some relevant algebraic notions, we lay down the elements of the quaternion algebra. A detailed treatment is given since a comprehensive presentation seems lacking in the literature. We next proceed to a discussion of quaternionic Hilbert spaces and manifolds, to Fueter's theory of holomorphic functions and the number theory of quaternions. Selected physical applications and a historical note follow.

2.a. Algebraic Structures

2.a.1. Basic properties and identities

We first recall the definitions of some recurring mathematical entities [25, 26] such as groups, fields, rings and algebras. The simplest among them are *groups.*

A *group* G is a set of objects with one operation defined on it obeying:

1) associativity.

2) the existence of an identity element $e \in G$ such that for any $a \in G$, $a\,e = e\,a = a$.

3) the existence of an inverse a^{-1} for any $a \in G$ such that $a\,a^{-1} = a^{-1}\,a = e$.

The group is called *Abelian* if $a\,b = b\,a$ for any $a, b \in G$.

A *field* F is a more complex entity, a set on which *two* operations, addition ($a + b$) and multiplication ($a\,b$) are defined, with the following conditions:

1) F is an abelian group under addition with the identity element 0.

2) multiplication is distributive over addition: $a\,(b + c) = a\,b + a\,c$, $a, b, c \in F$

3) $F-\{0\}$ is an abelian group under multiplication with the identity element 1.

A *ring* $\boldsymbol{R}$ is an additive abelian group where a second law of composition, multiplication, is defined along with the distributive laws,

$$(x + y) z = x z + y z \quad , \quad z (x + y) = z x + z y \qquad \text{for all } x, y, z \in \mathcal{R}. \tag{2a.1}$$

An *algebra* $\mathcal{A}$ over a field F is a ring, a vector space over F with a bilinear multiplication, i.e. one for which Eq. (2a.1) and

$$\alpha (x y) = (\alpha x) y = x (\alpha y) \quad \text{for all } \alpha \in F, x, y \in \mathcal{A} \tag{2a.2}$$

hold. Furthermore, if the associative law

$$(x y) z = x (y z) \quad \text{for all } x, y, z \in \mathcal{R} \text{ (or } \mathcal{A} \text{)} \tag{2a.3}$$

holds for a ring $\mathcal{R}$ or an algebra $\mathcal{A}$, we have an *associative ring* or an *associative algebra*, respectively.

$\mathcal{A}$ is a *division algebra* if the equations $a x = b$ and $x a = b$ always have solutions for $a \neq 0$. Let $\{ e_i, i = 1, ...n \}$ be a basis of $\mathcal{A}$ over F so that for any element $a \in \mathcal{A}$, $a = \sum_{i=1}^{n} a_i e_i$, $a_i \in F$. The *norm* of $\mathcal{A}$ is defined by $|a|^2 = \sum_{i=1}^{n} a_i^2$. The algebra $\mathcal{A}$ is a *normed algebra* if for all $a, b \in \mathcal{A}$, $|a b|^2 = |a|^2 |b|^2$ in some basis $\{ e_i \}$. Next, we specifically recall the essential properties of the quaternion algebra $\mathcal{A} = \mathbf{H}$ [25, 27, 28].

The algebra of the real quaternions **H** [29] is a non-commutative division ring over **R**, the field of real numbers. Its canonical basic units e_μ, $\mu = 0, 1, 2, 3$, obey Hamilton's multiplication table:

$$\begin{gathered} e_i e_j = - \delta_{ij} e_0 + \varepsilon_{ijk} e_k \ , \ e_0 e_j = e_j e_0 \quad 1 \leq i < j \, , \ k \leq 3 \\ e_i^2 = - e_0 \, . \end{gathered} \tag{2a.4}$$

e_0, the unit element or *modulus* of **H**, behaves exactly like the real number unity. The Levi-Civita symbol ε_{ijk} is totally antisymmetric with $\varepsilon_{123} = \varepsilon_{231} = \varepsilon_{312} = 1$. Unless otherwise specified, Einstein's summation convention for repeated indices will be adopted throughout.

We can define on **H**, of rank 4 over **R**, a topology homeomorphic to that of the euclidean 4-space R^4. In fact, we *identify* **H** with R^4 and its canonical basis $\{ e_\mu \}$ with the canonical vectorial basis of R^4. The quaternion product defined in (2a.4) on R^4 is then the **R**-bilinear product $R^4 \otimes R^4 \to R^4$. The topology of **H** is even compatible with its division ring structure; namely, for a non-zero $q \in$ **H** the coordinates of q^{-1} are rational functions of those of q, with non-zero denominators. We observe further that the notion of continuity is the very essence of topology and that the operations of addition and multiplication in an algebra are *continuous* processes. Thus let two quaternions q and p be points in R^4, and let q' and p' be nearby points, then p' q' is close to p q and p' + q' is close to p + q. In this manner topological structures are already built into the very basic rules of algebra. As we will see (Sect.4b), especially in the works of Hopf [30, 31] , this connection has fundamental consequences for both differential topology and division algebras.

In view of subsequent applications to relativistic field theories, a covariant notation is highly desirable, if not mandatory. To that end, we write any quaternion q uniquely as the linear form $q = q_\mu e^\mu$ with $q_\mu \in$ **R**. Addition and multiplication of quaternions are defined through the rules (2a.4) and the usual distributivity law. Since the 27 equations $(e_i\, e_j)\, e_k = e_i\, (e_j\, e_k)$ $(0 \leq i, j, k \leq 3)$ can be verified to hold, **H** is an associative but a noncommutative algebra (i.e. $e_1 e_2 = - e_2 e_1$ etc.).

The quaternionic *conjugate* of q is defined by

$$\bar{q} = q_\mu \bar{e}^\mu = q_0 e_0 - \mathbf{q} \cdot \mathbf{e} \tag{2a.5}$$

with $q \to \bar{q}$ being an involutive anti-automorphism or anti-involution of **H**.

The quadratic reduced square **norm** of q is defined as

$$N(q) \equiv \langle q, q \rangle \equiv | q |^2 = q \bar{q} = \bar{q} q$$
$$= q_0^2 + q_1^2 + q_2^2 + q_3^2 \ , \qquad (2a.6)$$

$|q|$ is called the *modulus* of q (or its *Euclidean norm*) when q is viewed as an element of R^4 (under the identification $R \times R^3 \approx R^4$). Clearly, N(q) is a positive scalar, it is zero only if q = 0. If q and q' $\in$ **H**, then the norm theorem : N(q q') = N(q) N(q') holds. This key multiplicative factorization property implies that the set of all unit quaternions forms a compact subgroup $SU(2) \approx S^3$, the unit 3-sphere of the multiplicative group $\mathbf{H}^*$ of non-zero quaternions. And if u $\in$ **H** with N(u) = 1, u is called a *unit* quaternion. So to every non-zero quaternion corresponds a unit quaternion $\frac{q}{\sqrt{N(q)}}$. From the existence of the norm, it follows that **H** is a *division algebra* ; every q has an inverse $q^{-1} = N(q)^{-1}\bar{q}$ such that $q^{-1}q = qq^{-1} = e_0$.

The inner product $\langle q, q' \rangle$ of two quaternions q and q' is easily obtained from the above definition of the square norm by a procedure [32] called *polarization or linearization*, which is most handy in the derivation of useful identities for normed algebras. It consists in substituting q by the sum (q + q') in Eq. (2a.6), which then gives

$$\langle q, q' \rangle = \frac{1}{2}\left(| q+q' |^2 - | q |^2 - | q' |^2 \right) . \qquad (2a.7)$$

Similarly, the polarization of $| pq |^2 = | p |^2 | q |^2$ gives rise to the following useful, equivalent identities

$$\langle pw, qw \rangle = \langle p, q \rangle | w |^2 \ , \qquad (2a.8a)$$

$$\langle wp, wq \rangle = | w |^2 \langle p, q \rangle \ , \qquad (2a.8b)$$

$$\langle pv, qw \rangle + \langle qv, pw \rangle = 2 \langle p, q \rangle \langle v, w \rangle \langle pv , qw \rangle + \langle qv, pw \rangle = 2 \langle p, q \rangle \langle v, w \rangle \ . \qquad (2a.8c)$$

In fact, the above identities, being solely dependent on the factorization property of the norm, are valid for all normed algebras.

Any quaternion q solves the characteristic equation

$$q^2 - 2q_0\, q + N(q) = 0 \ , \tag{2a.9}$$

which follows from the relation $(q - q)(q - \bar{q}) = 0$. Subsequently, such a quadratic equation will be our natural starting point for the number theory of quaternions.

Before proceeding further, we shall first breathe more life into the definitions above. We do so by recalling some equivalent, useful representations of real quaternions [33, 34].

In two dimensions, we are all familiar with the geometric representation of complex numbers as *dilatative rotations*, namely as products of a rotation with a dilatation. Addition and multiplication of complex numbers translate into addition of vectors in the Gauss plane and multiplication of mappings, respectively. Yet the construction of complex numbers z is nevertheless a purely *algebraic* procedure. Thus, we can associate to the dilatative rotations matrices of degree 2:

$$z = a + i\,b \quad \leftrightarrow \quad A_z = \begin{pmatrix} a & b \\ -b & a \end{pmatrix} . \tag{2a.10}$$

In particular, to every real number a and to the imaginary unit i correspond the matrices $\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, respectively. To the norm of z corresponds the determinant $\det(A_z) = c \equiv a^2 + b^2$. Being closed under addition, subtraction and multiplication, the matrices (2a.10) form a ring. They even form a field, except for the zero matrix which has no inverse. The field property derives from the fact that, for real numbers, if $a \neq 0$ or $b \neq 0$, then $c \neq 0$. Our algebraic definition of complex numbers in terms of real numbers only requires the ring property for the field **R**. Complex numbers can in fact be introduced for any field K obeying Eq. (2a.10). The result is an extension field K(i) of K, a 2-dimensional vector space over K with canonical basis (1, i).

While the matrix representation (2a.10) allows greater calculational concision with complex numbers, K(i) can also be seen as the space of pairs or couples (a_1, a_2), a_1, b_2 $\in$ K, as first emphasized by Hamilton. Complex multiplication then reads

$$(a_1 , a_2) (b_1 , b_2) = (a_1b_1 - a_2b_2 , a_1b_2 + a_2b_1) \ . \qquad (2a.11)$$

It is this pairing procedure which readily generalizes to give the hypercomplex systems of quaternions and octonions. This Cayley-Dickson process will be detailed in full subsequently.

We shall also encounter the notion of algebraic closedness of a field K. To define it , we need the *Intermediate value theorem.* The latter states that if a real function f(x) is defined and is continuous at every x such that $a \le x \le b$ ($a < b$) and if $f(a) < C < f(b)$, then there exists a value c such that $a < c < b$ for which $f(c) = C$. For our purposes, it is enough to recall that K is *real - closed* if it can be so ordered that such a theorem holds for polynomials.

Moving on to four dimensions, consider as an analog of the Gauss plane, an affine complex plane with a Hermitian metric. Let ($z_1\bar{z}_1 + z_2\bar{z}_2$) be the length squared of a vector labelled by two complex numbers z_1 and z_2. The length and origin preserving affine mapping (with unit determinant) is

$$\begin{pmatrix} z'_1 \\ z'_2 \end{pmatrix} = \begin{pmatrix} \alpha & -\bar{\beta} \\ \beta & \bar{\alpha} \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} , \qquad \alpha\,\bar{\alpha} + \beta\,\bar{\beta} = 1 \ . \qquad (2a.12)$$

In general, every mapping (2a.12), a Hermitian dilative rotation, corresponds to a real quaternion q. It can be generated by a hermitian rotation and a dilatation (with a real dilatation factor $\alpha\bar{\alpha}+\beta\bar{\beta} \neq 1$). We then have the endomorphism of **H** in the map

$$q = q_\mu e_\mu \quad \leftrightarrow \quad \begin{pmatrix} z_1 & -\bar{z}_2 \\ z_2 & \bar{z}_1 \end{pmatrix} . \qquad (2a.13)$$

Algebraically, just as **C** is an extension of **R**, Eq. (2a.13) means that the quaternion algebra $\mathbf{H} = \mathbf{C} \times \mathbf{C}$, i.e. it is a 2-dimensional algebra over **C**, the field of complex numbers. Consequently, any q is a hypercomplex number $q = v + e_2 w$ with $v, w \in \mathbf{C}$ and $v = q_0 + e_1 q_1$, $w = q_2 - e_1 q_3$. The basic multiplication (2a.4) translates into $v e_2 = - e_2 \bar{v}$.

The mapping (2a.13) is just the familiar representation of **H** by (2 x 2) complex matrices. A real quaternion may be written as a (2 x 2) matrix $q = q_0 - i\boldsymbol{\sigma}\cdot\mathbf{q}$, with $e_0 = \sigma_0 = I$, the unit (2 x 2) matrix and $e_i = - i\sigma_i$ ($i = 1, 2, 3$), σ_i being Pauli's matrices. Also in its matrix form, we have $q = \sigma_2 q^T \sigma_2$, q^T being the transpose of q and σ_2, σ_3 chosen pure imaginary and diagonal, respectively. Through this isomorphism, **H** is manifestly an associative division algebra over **R**. The characteristic equation (2a.7) is thus a special case of the Cayley-Hamilton theorem of matrix algebra. As we will see, the representation (2a.13) brings out best the parallel between the analytic function theories of a complex and a quaternionic variable.

Next, we gather some basic properties of quaternions, along with a few useful lemmas, most of them derived by Hamilton.

In practice, it is often necessary to make the basis independent splitting of a quaternion q into its "scalar" and "vectorial" parts:

$$q = \mathrm{Sc}(q) + \mathrm{Vec}(q) \tag{2a.14}$$

where

$$\mathrm{Sc}(q) = \frac{1}{2}(q + \bar{q}) = \frac{1}{2}\mathrm{Tr}(q) = q_0 \ , \tag{2a.15a}$$

$$\mathrm{Vec}(q) = \frac{1}{2}(q - \bar{q}) = \mathbf{e}\cdot\ \mathbf{q} \equiv \mathbf{Q} \ . \tag{2a.15b}$$

Topologically, we have the corresponding natural decomposition $R^4 \approx R \oplus R^3$.

Besides the inner or scalar product for two quaternions p and q

$$\langle p, q \rangle \equiv \frac{1}{2} (p\bar{q} + q\bar{p}) = \mathrm{Sc} (p \bar{q}) \;, \tag{2a.16}$$

a natural associated entity is the cross product

$$p \times q \equiv \frac{1}{2} (\bar{q} p - \bar{p} q) \equiv \mathrm{Vec} (\bar{q} p) \;. \tag{2a.17}$$

It generalizes to $R^4 \approx \mathbf{H}$ the well-known Gibbs vector cross product of R^3 [35, 36]. Its conjugate is $\overline{p \times q} \equiv \frac{1}{2}(q\bar{p} \; - p\bar{q}) \equiv \mathrm{Vec}(q\bar{p})$.

A quaternion p is called *pure* or *purely vectorial* if Sc(p) = 0. Since $p^2 = - N(p)$, for a pure unit quaternion, the polynomial ($p^2 + 1$) has two roots, the analogs of ± i, the roots of the complex polynomial ($z^2 + 1$). It is the second root – i of the latter which provides the only automorphism $z \to z^*$ of **C**. Due to this automorphism, if the function of a complex variable f(z) is a polynomial with *real* coefficients, then $f(z^*) = [f(z)]^*$ i.e. Schwarz's Reflection Principle is satisfied. The latter principle will be seen to hold also for an important class of holomorphic functions of a quaternionic variable. In the same vein, we note that, for any unit vector η, a pure unit quaternion ($\eta^2 = -1$), the set of quaternions ($x + \eta y$) for $x, y \in \mathbf{R}$ is a subfield of **H** and is isomorphic to **C** under the function $x + \eta y \to x + i y$.

To make contact with vector analysis, 3-space R^3 is identified with the imaginary subspace Im **H** of $\mathbf{H} \approx R^4$. Then the usual scalar and vector products are contained in the quaternion product of two purely vectorial quaternions u and v:

$$uv = (\mathbf{e} \cdot \; \mathbf{u}) (\mathbf{e} \cdot \; \mathbf{v}) = - \mathbf{u} \cdot \; \mathbf{v} + \mathbf{e} \cdot \; (\mathbf{u} \times \mathbf{v}) \;. \tag{2a.18}$$

The latter is a special case of the product of any two quaternions q and q'

$$qq' = \{ (q_0 q'_0 - \mathbf{Q} \cdot \mathbf{Q}') + (q_0 \mathbf{Q}' + q'_0 \mathbf{Q}) \} + \mathbf{Q} \mathrm{x} \mathbf{Q}' \;. \tag{2a.19}$$

We observe that, on the right hand side of Eq. (2a.19), the combination in the first curly bracket is just the anticommutator $\frac{1}{2}\{ q, q' \}$. It only vanishes if $q_0 q_0' = \mathbf{Q} \cdot \mathbf{Q}'$ and $q_0'\mathbf{Q} = -q_0\mathbf{Q}'$. $\mathbf{Q}\mathrm{x}\mathbf{Q}'$ is the commutator $\frac{1}{2}[q, q']$. So two quaternions commute if and only if the vector product of their vectorial parts vanishes.

Hamilton [Hamilton, 1853 #44] also introduced the three-dimensional "del" or "Nabla" operator $\nabla = \mathbf{e} \cdot \vec{\nabla}$. When acting on a vector function $\mathbf{u}(\mathbf{x})$ of $\mathbf{x}$, it gives

$$\nabla u = -\vec{\nabla} \cdot \mathbf{u} + \mathbf{e} \cdot (\vec{\nabla} \times \mathbf{u}) = -\operatorname{div} \mathbf{u} + \mathbf{e} \cdot \operatorname{curl} \mathbf{u} \tag{2a.20a}$$

and

$$N(\nabla) = \nabla \overline{\nabla} = -\vec{\nabla}^2 = \Delta \tag{2a.20b}$$

is Laplace's operator. These formulae connect quaternion analysis to Gibbs' vector calculus [35]. No further elaboration is needed on this familiar subject; numerous treatments are found in the literature [37, 38, 39].

In recent years, just as the ubiquitous complex phase factor $e^{i\theta} \in S^1$ of monopole and vortex theories [40], the unit quaternion $u = q\,|q|^{-1}$ has similarly played a key role in topological phenomena in quantum field theories [41, 42]. Its polar representation is

$$u = e^{\mathbf{n}\theta} = \cos\theta + \mathbf{n}\sin\theta \tag{2a.21}$$

with $\cos\theta = q_0\,|q|^{-1}$, $\sin\theta = |\mathbf{q}|\,|q|^{-1}$ and $\mathbf{n} = \mathbf{e}\cdot\mathbf{q}\,|\mathbf{q}|^{-1}$ ($\mathbf{n}^2 = -1$). Indeed, any quaternion has the polar decomposition $q = |q|\,u$ with a quaternionic ($\mathbf{H}$-) counterpart to DeMoivre's theorem:

$$q^m = |q|^m (\cos(m\theta) + \mathbf{n}\sin(m\theta)), \quad m = \text{an integer} . \tag{2a.22}$$

Having in mind later applications to winding numbers of topological gauge field configurations, we merely quote the fundamental theorem of algebra for quaternions. It states that, given P(q) a polynomial over **H** of degree n such that P = f + g , f being a monomial of degree n and g a polynomial of degree < n, then the quaternionic mapping P: **H** $\rightarrow$ **H** is surjective, and in particular it has zeros in **H**. The first proof of such a theorem was given in 1944 by Eilenberg and Niven [43], the topological proof came later in a text of Eilenberg and Steenrod [44]. In essence it says that the equation P(x) = 0 has n solutions. Furthermore, by compactifying R^4 to S^4 with the addition of the point at infinity and setting $P(\infty) = \infty$, one gets a mapping P: $S^4 \rightarrow S^4$ of Brower degree n [45], homotopic to the mapping $g(x) = x^n : S^4 \rightarrow S^4$.

Geometrically, the multiplication of a complex number z by the phase $e^{i\theta}$ rotates z counterclockwise through an angle θ. Its **H**-counterpart is the "conical rotation" [34]

$$\mathbf{V}_1 = \exp(\tfrac{\mathbf{n}}{2}\theta)\,\mathbf{V}_0\,\exp(\tfrac{\mathbf{n}}{2}\theta)\,, \tag{2a.23}$$

$$\mathbf{V}_1 = \mathbf{V}_0 \cos\theta + \mathbf{n}\mathrm{x}\mathbf{V}_0 \sin\theta + \mathbf{n}\,(\,\mathbf{n}\cdot\mathbf{V}_0\,)\left[\,1-\cos\theta\right] \tag{2a.24}$$

where $\mathbf{V}_1$ and $\mathbf{V}_0$ are pure vectorial quaternions. To visualize such a rotation, let $\mathbf{V}_0$ extend from the tip T of a right circular cone with **n**, a unit vector pointing from T along the axis of the cone at an acute angle with $\mathbf{V}_0$. $\mathbf{V}_1$ is then obtained as the final locus of $\mathbf{V}_0$ after the plane spanned by **n** and $\mathbf{V}_0$ is rotated through an angle θ.

Alternatively, with a unit quaternion u represented by a unitary unimodular (2 x 2) matrix, we have the correspondence between unit quaternions and SU(2), the universal covering group of O(3). If U denotes the matrix representation of u, then

$$q' = U\,q\,, \qquad N(q') = N(q)\ . \tag{2a.25}$$

And, if we now define the complex valued 2-spinors

$$\psi = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}, \qquad \psi' = \begin{bmatrix} \alpha' \\ \beta' \end{bmatrix}, \tag{2a.26}$$

where α and β (with $|\alpha|^2 + |\beta|^2 = 1$) are the Cayley-Klein rotation parameters, we have the transformation law

$$\psi' = U\,\psi \quad , \quad \psi'^{\dagger}\psi' = \psi^{\dagger}\psi \tag{2a.27}$$

of an O(3) 2-spinor under the rotation U.

A 3-dimensional vector $\vec{v}$ can be represented either by the purely vectorial quaternion $q\, e_3\, \bar{q}$ or by the traceless (2 x 2) matrix

$$\boldsymbol{\Omega} = -\, i\,\vec{\boldsymbol{\tau}} \cdot \vec{v} \;=\; -\, iQ\,\tau_3\, Q^{\dagger} \ . \tag{2a.28}$$

Under the rotation (2a.25), it transforms as $\boldsymbol{\Omega}' = U\boldsymbol{\Omega}\, U^{\dagger}$, $\mathbf{e} \cdot \boldsymbol{\omega}' = U\,(\, q\, e_3\, \bar{q}\,)\, \overline{U}$ is the transformation law of a vector under rotation.

The scalar product (q, q') ≡ Sc ($\overline{qq'}$) of two quaternions q and q' is left invariant by the SU(2) x SU(2) transformation

$$q' = a\, q\, b \tag{2a.29}$$

where a and b are unit quaternions, $|a|^2 = |b|^2 = 1$. So N(q') = N(q). Via Eq. (2a.19), Eq. (2a.29) also reads

$$\begin{aligned} q' = {} & (\, a_0 b_0 - \mathbf{a} \cdot \mathbf{b})\, q_0 - (\, a_0\mathbf{b} + b_0\mathbf{a} - \mathbf{a} \times \mathbf{b}\,) \cdot \mathbf{q} + (\, a_0\mathbf{b} + b_0\mathbf{a} + \mathbf{a} \times \mathbf{b}\,)\, q_0 \\ & + (\, a_0\, b_0 + \mathbf{a} \cdot \mathbf{b}\,)\,\mathbf{q} - (\,\mathbf{a} \cdot \mathbf{q}\,)\,\mathbf{b} - (\,\mathbf{b}\,\mathbf{q}\,)\,\mathbf{a} + (\,\mathbf{a}\, b_0 - a_0\,\mathbf{b}\,) \times \mathbf{q} \ . \end{aligned} \tag{2a.30}$$

Since **H** is associative, the above left and right multiplications commute, which means simply that SU(2) x SU(2) ≈ O(4), the group of orthogonal rotations in R^4. Clearly, the semi-simple group O(4) is the norm (preserving) group of the quaternions, its O(3) subgroup of transformations

$$q' = a\, q\, \bar{a}\,, \quad |a| = 1\ , \tag{2a.31}$$

not only leaves invariant the above scalar product but also the multiplication table (2a.4). So O(3) only affects the vectorial parts of the q's; it is their automorphism group.

We now list some more lemmas involving unit and/or pure quaternions, as defined above

1) For any quaternion q we can find a unit quaternion u such that $q\,u = u\,\bar{q}$. More generally, if q and r are two quaternions with identical norms and scalar parts, then there is a unit quaternion u such that $q\,u = u\,r$.

2) Any unit quaternion u is expressible in the form $a\,b\,a^{-1}\,b^{-1}$ where a and b are non-zero quaternions.

3) Any quaternion is expressible as a product of a pair of pure quaternions, indeed as a power of a pure quaternion.

For more details and for the proofs of these lemmas, the interested reader can consult a fine article of Coxeter [46]. Also to be found there are elegant quaternionic representations of reflexion and rotations in R^3 and R^4.

2.a.2. Covariant O(4) and (anti-) self-dual tensors

We have seen how the existence of the algebra **H** in $R^4 \approx \mathbf{H}$ sheds light on some unique features of four dimensional space [7, 47]. While the group SO(n) for all $n \neq 4$ is simple, SO(4) is semi-simple due to its unique local isomorphism SO(4) ≈ SO(3) x SO(3) or Spin(4) ≈ SU(2) x SU(2). It corresponds to the isomorphism $R^4 \otimes \mathbf{C} \cong S^+ \otimes S^-$ with S^+ and S^- denoting the two spin $\frac{1}{2}$ representations of Spin(4). Since pure vectorial quaternions $x = i\,\mathbf{e} \cdot \mathbf{x}$ satisfy $x^2 = 1$, the points of the 2-sphere S^2, they parametrize the length and orientation preserving *complex structure* on $R^4 \cong C^2$. This point will be of primary importance in connection with Fueter's function theory over a quaternion and its connection to the twistor [48] and harmonic superspace [49, 50] approaches.

When the space of antisymmetric 2-tensors or 2-forms $\Lambda^2(R^4)$ is seen as the Lie algebra of SO(4), it is natural to introduce in R^4 the duality, Hodge star operator * : $\Lambda^2(R^4) \to \Lambda^2(R^4)$, which sends $e_\mu \Lambda e_\nu \to e_\rho \Lambda e_\sigma$, with e_μ, e_ν, e_ρ and e_σ forming an orthogonal basis of R^4. While definable for any R^d, it is only in a four dimensional space that the * operation maps 2-forms into 2-forms. Since $(*)^2 = 1$, duality decomposes $\Lambda^2(R^4)$ into $\Lambda^2(R^4) \cong \Lambda^+ \oplus \Lambda^-$, its self-dual ($\Lambda^+$) and anti-self-dual ($\Lambda^-$) components correspond to the (+) and (–) eigenspaces of *. Consequently, in differential geometry, the curvature 2-form F of any principal bundle with connection over a Riemannian 4-manifold M admits an intrinsic splitting into self-dual and anti-self-dual parts $F^\pm = \frac{1}{2}(F \pm {*F})$. For $M \approx S^4 \approx HP(1)$, the quaternionic projective line, we therefore expect quaternions to play an essential role in the algebraic construction of (anti-) self-dual (or (anti-) instantons) vector bundles over S^4. Such an application of quaternions will be discussed in Section 2.f.

The duality structure of R^4 is encoded in the following self-dual and anti-self-dual antisymmetric tensors [51].

$$e_{\mu\nu} = \frac{1}{2}\left(\bar{e}_\mu e_\nu - \bar{e}_\nu e_\mu\right) = \mathrm{Vec}\,(\bar{e}_\mu e_\nu) = \frac{1}{2}\varepsilon_{\mu\nu\alpha\beta}e_{\alpha\beta} \equiv e^*_{\mu\nu}\,, \tag{2a.32}$$

$$e'_{\mu\nu} = \frac{1}{2}\left(e_\mu \bar{e}_\nu - e_\nu \bar{e}_\mu\right) = \mathrm{Vec}\,(e_\mu \bar{e}_\nu) \equiv -e'^*_{\mu\nu}\,. \tag{2a.33}$$

The corresponding scalar part is

$$e_\mu \bar{e}_\nu + e_\nu \bar{e}_\mu = 2\,\mathrm{Sc}\,(e_\mu \bar{e}_\nu) = 2\delta_{\mu\nu}\,. \tag{2a.34}$$

These tensors are natural structures in R^4; they form the kernels of the fundamental cross-product (2a.17) with Eq. (2a.34) being the scalar product (2a.16) of basis vectors.

We readily check the following O(4) commutation relations:

$$\frac{1}{2}\left[\, e_{\mu\alpha}\,,\, e_{\nu\beta}\,\right] = \delta_{\alpha\nu}\, e_{\mu\beta} - \delta_{\mu\nu}\, e_{\alpha\beta} + \delta_{\alpha\beta}\, e_{\nu\mu} - \delta_{\mu\beta}\, e_{\nu\alpha} \;, \tag{2a.35a}$$

$$\frac{1}{2}\left[\, e'_{\mu\alpha}\,,\, e'_{\nu\beta}\,\right] = \delta_{\alpha\nu}\, e'_{\mu\beta} - \delta_{\mu\nu}\, e'_{\alpha\beta} + \delta_{\alpha\beta}\, e'_{\nu\mu} - \delta_{\mu\beta}\, e'_{\nu\alpha} \tag{2a.35b}$$

while the anticommutation relations are also of interest

$$\frac{1}{2}\left\{\, e_{\mu\alpha},\, e_{\nu\beta}\,\right\} = \frac{1}{2}\left\{ e'_{\mu\alpha},\, e'_{\nu\beta}\right\} = -\,\varepsilon_{\mu\nu\alpha\beta} - \delta_{\mu\alpha}\,\delta_{\nu\beta} + \delta_{\nu\alpha}\,\delta_{\mu\beta} \tag{2a.36}$$

and we also have $e_{\mu\nu}\, e'_{\mu\nu} = 0$; i.e. the $e_{\mu\nu}$'s and $e'_{\mu\nu}$'s are not independent.

At this juncture, we recall the connection between the $e_{\mu\nu}$'s and 't Hooft symbol $\eta_{a\mu\nu}$ (a = 1, 2, 3) [17, 18]. it is a simple one: $e'_{\mu\nu} = \eta_{a\mu\nu} e_a$. For completeness, we gather below the definition of the 't Hooft tensor, of wide use in the instanton calculus, and list some associated identities:

$$\eta_{\,a\mu\sigma} = \varepsilon_{0a\mu\sigma} + \delta_{\mu a}\delta_{\sigma 0} \;-\delta_{\mu 0}\delta_{\,a\sigma} \;, \tag{2a.37a}$$

$$\eta_{\,a\mu\sigma}\,\eta_{\,a\nu k} = \;\delta_{\mu\nu}\delta_{\sigma k} - \delta_{\mu\sigma}\delta_{\nu k} - \varepsilon_{\nu\sigma\mu k} \;\;, \tag{2a.37b}$$

$$\eta_{\,a\mu\sigma}\,\eta_{\,b\mu k} = \;\varepsilon_{abc}\eta_{c\sigma k} + \delta_{ab}\delta_{k\sigma} \;, \tag{2a.37c}$$

$$\varepsilon_{abc}\eta_{\,b\mu k}\,\eta_{\,c\nu\sigma} = \;\eta_{\,a\mu\nu}\delta_{k\sigma} - \eta_{\,a\mu\sigma}\delta_{k\nu} + \eta_{\,ak\sigma}\delta_{\mu\nu} - \eta_{\,ak\nu}\delta_{\mu\sigma} \;\;, \tag{2a.37d}$$

$$\varepsilon_{\rho\nu\kappa\sigma}\eta_{\,a\mu\kappa} = \;\eta_{\,a\nu\kappa}\delta_{\mu\sigma} + \eta_{\,a\sigma\nu}\delta_{\mu\kappa} + \eta_{\,a\kappa\sigma}\delta_{\mu\nu} \;\;. \tag{2a.37e}$$

In working with quaternions, algebraic manipulations often simplify thanks to the following identities:

$$e'^{\mu\nu}\, e^{\alpha} = \delta^{\nu\alpha}\, e^{\mu} - \delta^{\mu\alpha}\, e^{\nu} + \varepsilon^{\mu\nu\alpha\beta}\, e_{\beta} \ , \tag{2a.38a}$$

$$e^{\mu\nu}\, \bar{e}^{\alpha} = \delta^{\nu\alpha}\, \bar{e}^{\mu} - \delta^{\mu\alpha}\, \bar{e}^{\nu} - \varepsilon^{\mu\nu\alpha\beta}\, \bar{e}_{\beta} s \ , \tag{2a.38b}$$

$$e_{\mu}\, q\, e_{\mu} = \bar{e}_{\mu}\, q\, \bar{e}_{\mu} = -2\ \bar{q} \ , \tag{2a.38c}$$

$$e_{\mu} q\, \bar{e}_{\mu} = \bar{e}_{\mu} q\, e_{\mu} = 4\ \mathrm{Sc}\,(q) = 2\,(\, q + \bar{q}\,) \ , \tag{2a.38d}$$

$$e_{\mu\nu}\, q\, e_{\mu\nu} = e'_{\mu\nu}\, q\, e'_{\mu\nu} = -4\,(\, q + 2\bar{q}\,) \ , \tag{2a.38e}$$

$$e_{\mu\nu}\, q\, e'_{\mu\nu} = e'_{\mu\nu}\, q\, e_{\mu\nu} = 0 \ , \tag{2a.38f}$$

$$e_{\mu\nu}\, q\, e_{\nu} = -(\, q\, e_{\mu} + 2\, \bar{e}_{\mu}\, q\,) \ , \tag{2a.38g}$$

$$e_{\mu\nu}\, q\, \bar{e}_{\nu} = (\, q + 2\, \bar{q}\,)\, \bar{e}_{\mu} \ , \tag{2a.38h}$$

$$\frac{1}{2}\,(\, \bar{e}_{\mu}\, q\, \bar{e}_{\nu} + \bar{e}_{\nu}\, q\, \bar{e}_{\mu}\,) = \bar{e}_{\mu}\, q_{\nu} + \bar{e}_{\nu}\, q_{\mu} - \bar{q}\, \delta_{\mu\nu} \ , \tag{2a.38i}$$

$$\frac{1}{2}\,(\, \bar{e}_{\mu}\, q\, \bar{e}_{\nu} - \bar{e}_{\nu}\, q\, \bar{e}_{\mu}\,) = -\frac{1}{2}\, \varepsilon_{\mu\nu\alpha\beta}\,(\, \bar{e}_{\alpha}\, q_{\beta} - \bar{e}_{\beta}\, q_{\alpha}\,) \ , \tag{2a.38j}$$

$$\bar{e}_{\mu}\, \bar{q}\, e_{\mu\sigma}\, q\, e_{\sigma} = \mathrm{Sc}\,(\, \bar{q}\, e_{\mu\sigma}\ q\, e'_{\mu\sigma}\,) = 0 \ , \tag{2a.38k}$$

$$e_{\mu}\, \bar{q}\, e'_{\mu\sigma}\ q\, e_{\sigma} = 0 \ . \tag{2a.38m}$$

We should also mention that the algebra **H** is isomorphic to a subset of (4 x 4) real matrices under the mapping:

$$q \rightarrow \begin{pmatrix} q_0 & q_1 & q_2 & q_3 \\ -q_1 & q_0 & -q_3 & q_2 \\ -q_2 & q_3 & q_0 & -q_1 \\ -q_3 & -q_2 & q_1 & q_0 \end{pmatrix} . \tag{2a.39}$$

To illustrate the usefulness of this representation, we consider the following transformation X' = L X of a quaternion X. It gives a linear function of x with coefficients given by the elements of the quaternion L represented by the matrix (2a.39), namely

$$X' = \begin{pmatrix} x'_0 \\ x'_1 \\ x'_2 \\ x'_3 \end{pmatrix} = \begin{pmatrix} L_0 & -L_1 & L_2 & -L_3 \\ L_1 & L_0 & -L_3 & L_2 \\ L_2 & L_3 & L_0 & -L_1 \\ L_3 & -L_2 & L_1 & L_0 \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix} = L\,X \ . \qquad (2a.40)$$

Similarly, a matrix representation for the right action of a quaternion R, X" = X R, reads

$$\begin{pmatrix} x''_0 \\ x''_1 \\ x''_2 \\ x''_3 \end{pmatrix} = \begin{pmatrix} R_0 & -R_1 & -R_2 & -R_3 \\ R_1 & R_0 & R_3 & R_2 \\ R_2 & R_3 & R_0 & R_1 \\ R_3 & R_2 & -R_1 & R_0 \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix} \ . \qquad (2a.41)$$

Here L and R represent the left and the right action of a quaternion, respectively. Since the last two operations commute, we have

$$L\,R = R\,L \ . \qquad (2a.42)$$

In the case of pure quaternions ($L_0 = R_0 = 0$), the matrices L and R are antisymmetric as they should be.

Next, we represent the pure quaternions $\vec{e_L} \cdot \vec{f}$ and $\vec{e_L} \cdot \vec{g}$ by the matrices F and G where

$$F = \begin{pmatrix} 0 & -f_1 & -f_2 & -f_3 \\ f_1 & 0 & -f_3 & f_2 \\ f_2 & f_3 & 0 & -f_1 \\ f_3 & -f_2 & f_1 & 0 \end{pmatrix} \quad , \quad G = \begin{pmatrix} 0 & g_1 & g_2 & g_3 \\ -g_1 & 0 & g_3 & -g_2 \\ -g_2 & -g_3 & 0 & g_1 \\ -g_3 & g_2 & -g_1 & 0 \end{pmatrix} . \qquad (2a.43)$$

The elements of such a (4x4) antisymmetric matrix are $F_{\alpha\beta}$ with $F_{01} = F_{23} = -f_1$ Consequently, F is self-dual and G is anti-selfdual. Again [F, G] = 0; the product F G =

G F is symmetric and traceless. In fact, it is recognized as a quaternionic representation of Maxwell's energy-momentum tensor in R^4 with $\vec{f} = \frac{1}{2}(\vec{E} + \vec{B})$ and $\vec{g} = \frac{1}{2}(\vec{E} - \vec{B})$.

T = F(x)G(x) = G(x)F(x)

$$= \begin{pmatrix} \vec{f}\cdot\vec{g} & (\vec{f}\times\vec{g})_1 & (\vec{f}\times\vec{g})_2 & (\vec{f}\times\vec{g})_3 \\ (\vec{f}\times\vec{g})_1 & 2f_1g_1 - \vec{f}\cdot\vec{g} & (f_1g_2 + f_2g_1) & (f_3g_1 + f_1g_3) \\ (\vec{f}\times\vec{g})_2 & (f_1g_2 + f_2g_1) & 2f_2g_2 - \vec{f}\cdot\vec{g} & (f_2g_3 + f_3g_2) \\ (\vec{f}\times\vec{g})_3 & (f_3g_1 + f_1g_3) & (f_2g_3 + f_3g_2) & 2f_3g_3 - \vec{f}\cdot\vec{g} \end{pmatrix}. \qquad (2a.44)$$

This tensor is associated with the linear transformation induced by the quaternionic equation

$$X' = (\vec{\mathbf{e}}\cdot\vec{\mathbf{f}})\, X\, (\vec{\mathbf{e}}\cdot\vec{\mathbf{g}}) \qquad (2a.45)$$

giving $X'_\mu = T_{\mu\nu}X^\mu$.

We obtain

$$[F, G] = 0 \ , \ F^2 = -(\vec{\mathbf{f}}\cdot\vec{\mathbf{f}})\, I \ , \ G^2 = -(\vec{\mathbf{g}}\cdot\vec{\mathbf{g}})I \ . \qquad (2a.46)$$

It follows that, for a Maxwellian tensor T,

$$T^2 = (\vec{\mathbf{f}}\cdot\vec{\mathbf{f}})(\vec{\mathbf{g}}\cdot\vec{\mathbf{g}})\,\mathbf{I}\,. \qquad (2a.47)$$

In Section 2.d.4, this factorized form of the Maxwellian tensor will be seen as a D = 4 counterpart of the Sugawara-Sommerfield representation of the energy-momentum tensor of D = 2 conformal field theories.

From the common properties of **C** and **H** as extensions of K, we are led to the notion of a *(associative) division algebra* A_D *of finite rank n over K* [25]. A_D is a skew extension field of K with $a\,\alpha = \alpha\, a$ for all $a \in K$, $\alpha \in A_D$, an n-dimensional vector space over K. In fact, the unique algebraic importance of **C** and **H** rests with the following Theorem of Frobenius:

A (associative) division algebra A_D of finite rank over a real-closed field K is either K itself or is, up to isomorphism, the field K(i) ($i^2 = -1$) the quaternion skew field over K.

C and **H** are therefore the *only associative* division algebras of finite rank > 1 over **R** with **H** being the only non-commutative division ring of finite range over $\mathbf{R}^4$. One can define a topology on **H** homeomorphic to $\mathbf{R}^4$ by the identification of e_μ (μ = 0, 1, 2, 3) with the canonical $\mathbf{j}_\mu$ basis of $\mathbf{R}^4$. This topology is compatible not only with the ring structure of **H** but also with its division algebra structure as well.

2.a.3. Clifford and Grassmann algebras

A survey of the basic properties of quaternions would be incomplete without a brief discussion of their connection to related algebraic structures. The latter are a) the Clifford, Heisenberg, Grassmann, Jordan and symplectic (Lie) algebras, b) complex (specifically Hermitian) quaternions, which are of physical relevance to the description of Lorentzian spacetime [52].

A (odd) Clifford algebra C_r [53, 54, 55] of rank r, is generated by n = (2^r x 2^r) anticommuting hermitian matrices $\Gamma_a = \Gamma_a^\dagger$, a = 1, 2, ..., n, n = (2r+1) , with $\{\Gamma_a, \Gamma_b\} = 2\delta_{ab}$. Γ_a are either real and symmetric or imaginary and anti-symmetric matrices depending on whether a is odd or even. Clearly, C_1 is given by the three Pauli matrices while C_2 is just the algebra of Dirac γ-matrices extended to include γ_5 . Next, we recall how, in the Dirac theory of the electron, the four (4 x 4) hermitian matrices γ_α ($\alpha = 1,\ldots,4$) can be constructed from two commuting sets of Pauli matrices σ_i and ρ_i or, equivalently, from two commuting sets of quaternionic imaginary units $e_j = -i\,\sigma_j$, $\tilde{e}_j = -i\,\rho_j$. One chooses $\gamma_4 = \rho_1$, $\gamma_n = \rho_2\,\sigma_n$ then $\{\gamma_\alpha, \gamma_\beta\} = 2\,\delta_{\alpha\beta}$. As γ_5 anticommutes with γ_α and $\gamma_5{}^2 = 1$, it realizes the Clifford algebra C_2 with the other four γ's. Generally, the Clifford algebra C_N can be so constructed from N sets of commuting quaternion units { $e_i^{(n)}$ }, n = 1, 2, ..., N. From C_N we can then construct all the vector and spinor representations of its automorphism group $SO(2N+1) \approx B_N$.

To see further substructures of C_N and $SO(2N + 1)$ in a unified way, we go over to a new basis:

$$\left\{ a_j = \frac{\Gamma_{2j-1} + i\Gamma_{2j}}{2} \,,\; a_j^\dagger = \frac{\Gamma_{2j-1} - i\Gamma_{2j}}{2} \,,\; \Gamma_{2r+1} \right\} \;,\quad j = 1, 2, \ldots, r \tag{2a.48}$$

so that a_j, $a_j^\dagger$ and Γ_{2r+1} are all real with

$$\{a_i \,,\, a_j\} = 0, \quad \left\{a_i^\dagger, a_j^\dagger\right\} = 0, \quad \left\{a_i, a_j^\dagger\right\} = \delta_{ij} \;, \tag{2a.49}$$

$$\{\Gamma_{2r+1}, a_i\} = \left\{\Gamma_{2r+1}, a_i^\dagger\right\} = 0 \;,\; \frac{1}{2}\left\{\Gamma_{2r+1}, \Gamma_{2r+1}\right\} = 1 \;. \tag{2a.50}$$

By considering the subalgebra consisting of the set $\{a_j, a_j^\dagger\}$, i, j = 1, 2, ..., N, we obtain the even Clifford algebra $\{\ \Gamma_a$, a = 1, ..., 2N $\}$ of rank N, the Heisenberg algebra $\boldsymbol{H}_N$

$$\left\{a_i, a_j^\dagger\right\} = \delta_{ij} \quad , \quad \{a_i, a_j\} = 0 \quad \text{and} \quad \left\{a_i^\dagger, a_j^\dagger\right\} = 0 \;. \tag{2a.51}$$

Its automorphism group is $SO(2N) \approx D_N$.

If we consider the subalgebras $\{a_i, a_j\} = 0$ and $\left\{a_i^\dagger, a_j^\dagger\right\} = 0$ made up independently by the a_i and $a_j^\dagger$ respectively, we have Gr_N, the Grassmann algebra of order N, and its conjugate algebra Gr_N'. Their automorphism group is $SU(N) \approx A_{N-1}$. So to the nesting of the algebras $Gr_N \subset \boldsymbol{H}_N \subset C_N$ corresponds to that of their respective automorphism groups $SU(N) \subset SO(2N) \subset SO(2N+1)$.

2.a.4. Complex and hermitian quaternions

Ever since the early days of the special theory of relativity, Klein [56], Conway [57, 58], Weiss [59] and Kilmister [60, 61] among others have studied the properties of complex and hermitian quaternions. These investigations were done in connections with the Lorentz and other transformations, with electrodynamics and relativistic wave equations in Minkowski spacetime. We begin with a few definitions and some additional notation [62].

By a complex quaternion, we understand a quaternion $q = q_\mu e_\mu$ with *complex* coefficients $q_\mu = a_\mu + ib_\mu$ where a_μ, $b_\mu \in \mathbf{R}$. Thus we must distinguish two kinds of conjugation: $\overline{q} = q_\mu \overline{e_\mu}$, the quaternion conjugate and $q^* = q_\mu^* e_\mu$, the complex conjugate of q. Since the operations $q \to \overline{q}$ and $q \to q^*$ commute, we may also define an entity $q^\dagger \equiv \overline{q^*}$, the hermitian conjugate of q.

If qp is the product of two quaternions q and p according to the rules (2a.4), then besides $\overline{pq} = \overline{q}\,\overline{p}$, there are the products $(pq)^* = p^* q^*$ and $(pq)^\dagger = q^\dagger p^\dagger$. In addition to the splitting into scalar and vector parts Sc(q) and Vec(q), there are also the real and imaginary parts of q: $\mathrm{Re}(q) = \frac{1}{2}(q + q^*)$ and $\mathrm{Im}(q) = \frac{1}{2}(q - q^*)$. For notational simplicity, we will denote in this subsection the scalar product $\mathrm{Sc}(pq) \equiv \frac{1}{2}(p\,q + \overline{q}\,\overline{p})$ by p#q.

For any three quaternions p, q and r, we have the identity

$$\mathrm{Sc}\,(pqr) = q\#(pr) = q\#(r\,p) = r\#(p\,q)\ . \tag{2a.52}$$

Note that, unlike the case of real quaternions, here the norm of q, $N(q) \equiv q\overline{q}$, can vanish for $q \neq 0$; so complex quaternions do *not* form a division algebra.

It is also natural to define hermitian ($q = q^\dagger$) and anti-hermitian ($q = -q^\dagger$) quaternion q. Hermiticity of q implies Im(Sc (q)) = Re(Vec (q)) = 0, while anti-hermicity means Re(Sc(q)) = Im(Vec(q)) = 0. Then any q may be decomposed into its hermitian and anti-hermitian parts, denoted below by $q_\pm \equiv \frac{1}{2}(q \pm q^\dagger)$. By using hermitian quaternionic units $I_\mu \equiv (e_0\,, i\,\mathbf{e})$, any *hermitian* quaternion A can be expressed as $A = a_\mu I_\mu$ with all the coefficients a_μ being <u>real</u>.

The following identities among hermitian quaternions A, B, C and D are easily verified:

$$(A\,\overline{B}\,C)_{+} = (A\#\overline{B})\,C + (B\#\overline{C})\,A - (C\#\overline{A})\,B\ , \qquad (2a.53)$$

$$(A\,\overline{B}\,C)_{-} = -\,i\,\varepsilon_{\alpha\beta\rho\sigma}\bar{I}_{\alpha}a_{\beta}b_{\rho}c_{\sigma}\ , \qquad (2a.54)$$

$$\overline{D}\#\left[i(A\overline{B}C)_{-}\right] = i\,\mathrm{Im}\,(\overline{D}\#\,A\overline{B}C) = \varepsilon_{\alpha\beta\mu\nu}d_{\alpha}a_{\beta}b_{\mu}c_{\nu} \qquad (2a.55)$$

where the right hand side of Eq. (2a.55) is simply the determinant formed by the components of the four hermitian quaternions.

The last identity (2a.55) implies that the hermitian quaternion $H \equiv i(A\overline{B}C)_{-}$ satisfies the relations $\overline{H}\#A = \overline{H}\#B = \overline{H}\#C = 0$. So we have a generalization of the alternate product of two ordinary 3-vectors. Due to the determinantal form (2a.55) of this new alternate product, $(A\,\overline{B}\,C)_{-} \neq 0$ is the condition for three hermitian quaternions A, B and C to be linearly independent.

With the Lorentz orthogonality of two hermitian quaternions A and B being given by $A\#\overline{B} = 0$, it follows from Eq. (2a.53) that $(A\,\overline{B}\,C)_{+} = 0$ is the condition for three hermitian quaternions A, B and C to be mutually orthogonal.

As with real quaternions, we have various useful matrix representations of complex quaternions and the correspondence between quaternionic and matrix operations. The two familiar regular representations of $q = q_{\mu}e_{\mu}$ are given by the complex matrices

$$\Psi_R = \begin{pmatrix} q_0 & -q_1 & -q_2 & -q_3 \\ q_1 & q_0 & -q_3 & q_2 \\ q_2 & q_3 & q_0 & -q_1 \\ q_3 & -q_2 & q_1 & q_0 \end{pmatrix} \quad \text{and} \quad \Psi_{R'} = \begin{pmatrix} q_0 & q_1 & q_2 & q_3 \\ -q_1 & q_0 & -q_3 & q_2 \\ -q_2 & q_3 & q_0 & -q_1 \\ -q_3 & -q_2 & q_1 & q_0 \end{pmatrix}. \qquad (2a.56)$$

They belong to the matrix equivalents of the quaternionic equations QS = P and SQ =

P', i.e. $\Psi_R s = p$ and $\Psi_{R'} s = p'$. s , p and p' are column matrices formed by the complex coefficients of the corresponding quaternions S, P and P'.

In the above representations, the quaternion units e_i (i = 1, 2, 3) themselves are given by *real* matrices, so the complex conjugate matrice $\Psi_R{}^*$ or $\Psi_{R'}{}^*$ corresponds to Q^*, the complex conjugate of Q. However, Eq. (2a.56) are no longer irreducible if the e_i 's are represented by $e_i \rightarrow -i\,\sigma_i$, so that the hermitian units I_μ are the Pauli matrices. Then, denoting a real and purely imaginary quaternion by $R = r_\mu e_\mu$ and $iS = i\, s_\mu e_\mu$, r_μ , $s_\mu \in \mathbf{R}$, respectively, we have

$$R = \begin{pmatrix} \psi_1 & -\psi_2^* \\ \psi_2 & \psi_1^* \end{pmatrix} \qquad \text{and} \qquad iS = \begin{pmatrix} \psi_3 & \psi_4^* \\ \psi_4 & -\psi_3^* \end{pmatrix} \tag{2a.57}$$

with $\psi_1 = r_0 - i\,r_3$, $\psi_2 = r_2 - i\,r_1$, $\psi_3 = is_0 + s_3$ and $\psi_4 = is_2 + s_1$.

The general complex quaternion then reads $Q = R + iS$, $R, S \in \mathbf{H}$ and Q has the complex matrix representation

$$\Psi = \begin{pmatrix} \psi_1 + \psi_2 & -\psi_2^* + \psi_4^* \\ \psi_2 + \psi_4 & \psi_1^* - \psi_3^* \end{pmatrix} ; \tag{2a.58}$$

the pairs (ψ_1, ψ_2), (ψ_3, ψ_4) are determined by the real and imaginary parts of Q, respectively. Equivalently, we can associate to Q uniquely the 4-spinor

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix} . \tag{2a.59}$$

With no further elaboration, it suffices to say that this connection between quaternions and Dirac (and Weyl) spinors has been extensively analysed in several references [58, 63]. The important relation between real quaternions, spinors and twistors will be taken up again when we come to the Euclidean conformal group.

With the norm of Q given by $N(Q) = \overline{Q}\, Q = \mathrm{Det}\,(\Psi)$, the matrix form of $\overline{Q}$ is

$$\overline{\Psi} = \Psi^{-1}\,\mathrm{Det}(\Psi) = \begin{pmatrix} \psi_1^* - \psi_3^* & \psi_2^* - \psi_4^* \\ -\psi_2 - \psi_4 & \psi_1 + \psi_3 \end{pmatrix} . \tag{2a.60}$$

It can be verified that $Q^\dagger$ is given by $\Psi^\dagger$, the hermitian conjugate of Ψ. In this fashion, all the identities (2a.53–2a.55) readily translate into their (2x2) matrix counterparts. We also observe that a distinct advantage of a matrix representation of spinors is the possibility of taking their inverses.

In physics applications such as in performing Lorentz transformations, it is often convenient to use the exponential form of a unit normed, hermitian quaternion

$$H = \exp\,(\, i\, \mathbf{a}\theta) = \cosh\theta + i\,\mathbf{a}\,\sinh\theta\,, \tag{2a.61}$$

the unit vectorial quaternion $\mathbf{a}$ and the angle θ being real.

It can further be proved that any normed quaternion $V = |Q|^{-1}Q$ can be written as a product of a real quaternion and a hermitian quaternion, either as $V = HR$ or as $V = RH'$ with $|\,V\,| = |\,H\,| = |\,H'\,| = 1$. Generally, any complex quaternion Q with nonzero norm admits the polar representation

$$Q = s\,\exp(i\,\mathbf{a}\theta)\,\exp(\mathbf{b}\varphi) \qquad \text{or} \qquad Q = s\,\exp(\mathbf{b}\varphi)\,\exp(i\,\mathbf{a}'\theta') \tag{2a.62}$$

with complex s but real $\mathbf{a}$, $\mathbf{b}$, $\mathbf{a}'$, θ, φ and θ'. In relativity, the above factorization simply corresponds to the decomposition of a general Lorentz transformation into a spatial rotation followed by a boost or vice versa. It mirrors in the matrix language the well-known decomposition of any non-singular matrix Ψ, into a product of a hermitian matrix P or P' and a unitary matrix U, i.e. $\Psi = PU$ or $\Psi = UP'$.

We close this subsection with a quaternionic representation of proper and improper

Lorentz transformations. The practical importance of hermitian quaternions rests with the fact that $x = x^{\mu}I_{\mu} = x^{\dagger}$ represents the position vector in D = 4 Minkowski spacetime. If the metric tensor $g^{\mu\nu}$ is chosen such that its non-zero components $g_{00} = -g_{11} = -g_{22} = -g_{33} = 1$, the covariant and contravariant hermitian units are simply related: $I_{\mu} = g^{\mu\nu}I_{\nu} = \overline{I_{\mu}}$, so that

$$g^{\mu\nu} = g_{\mu\nu} = I_{\mu}\#I_{\nu} = \frac{1}{2}\left(I_{\mu}\overline{I_{\nu}} + I_{\nu}\overline{I_{\mu}} \right) . \tag{2a.63}$$

Consequently, the norm $N(x) = x\overline{x} = x_{\mu}x^{\mu}$ is just the spacetime interval, which is either positive (time-like), zero (light-like) or negative (space-like). In a quaternionic form a Lorentz transformation $x \to x'$ preserving the norm $N(x) = N(x')$ is $x' = QxQ^{+}$, or more explicitly

$$x' = \exp(i\,\mathbf{a}\theta)\exp(-\,\mathbf{b}\varphi)\;x\;\exp(\mathbf{b}\varphi)\exp(-\,i\mathbf{a}\theta) \quad , \tag{2a.64}$$

namely, the transformation is given by a 2φ rotation around the **b** axis followed by a boost in the **a** direction with a translation velocity of magnitude $\tanh(2\theta)$.

A more general transformation reads

$$x' = L\,x\,(L^{*})^{-1} \tag{2a.65}$$

with $L\overline{L} = (L\overline{L})^{*}$ due to hermiticity of x and x'. Equation (2a.65) is invariant under the scaling $L \to kL$, $k \in \mathbf{R}$. If $N(L) > 0$, L can be normalized to one; it is then equivalent to Eq. (2a.64), a proper Lorentz transformation. However, if $N(L) < 0$ and is normalized to -1 by choosing $L = i\,Q$, $|\,Q\,| = 1$, then $x' = -\,Q\,x\,Q^{\dagger}$, which is therefore a product of a proper Lorentz transformation and a space-time reversal. So to implement the full Lorentz group, we need to adjoint to Eq. (2a.65) the space reflection $x' = R_s\{x\} = -\,x^{*}$.

2.a.5. Symplectic Lie algebras and Quaternionic Jordan algebras

Consider an n-component quaternionic vector v in a quaternionic Hilbert space *H*. The components of v are the quaternions v_i (i = 1, 2, ..., n) . The hermitian conjugate vector $v^\dagger$ has components $\bar{v}_i$. Let v undergo a homogeneous linear transformation induced by the left action of a (n x n) quaternionic matrix S. The latter is then a quaternionic unitary matrix or a symplectic matrix if the norm $v^\dagger v$ of v is preserved under the S-transformation.

If v is a (n x 1) quaternionic column matrix, then

$$v' = S\, v \quad , \quad v'^\dagger = v^\dagger S^\dagger \quad , \quad (S^\dagger = \overline{S^T}) \quad . \tag{2a.66}$$

For the invariance $v'^\dagger v' = v^\dagger v$ to hold, S must be unitary

$$S^\dagger S = I \ , \tag{2a.67}$$

which is solved by setting

$$S = \exp(\Omega) \ . \tag{2a.68}$$

Then S must satisfy the constraint

$$\Omega^\dagger = -\,\Omega \ . \tag{2a.69}$$

Hence Ω is a quaternionic antihermitian (n x n) matrix with n purely the vectorial diagonal elements and $\frac{n(n-1)}{2}$ off-diagonal elements. The number of real parameters of S is therefore

$$N = 3n + 4\ \frac{n(n-1)}{2} = n\ (2n+1)\ , \tag{2a.70}$$

which is the dimension of the symplectic group Sp(n, **H**). Since quaternions can be viewed as (2 x 2) complex matrices, the matrix S is a (2n x 2n) complex matrix. So the transformation group can also be denoted as Sp(2n, **C**).

The group property of the quaternionic matrices are easily checked. If S and T obey the unitarity condition (2a.67), by setting W = S T, we may write

$$W^\dagger = T^\dagger S^\dagger \quad , \quad W^\dagger W = (T^\dagger S^\dagger)(ST) = T^\dagger(S^\dagger S)T = I\ , \tag{2a.71}$$

where the associativity of quaternionic matrices was used. Multiplying Eq. (2a.67) to the left by $(S^\dagger)^{-1}$ and to the right by S^{-1}, we obtain

$$S^{\dagger -1} = S^{-1\dagger} S^{-1} = I\ , \tag{2a.72}$$

hence the inverse matrix is also symplectic. Note that the inverse always exists, since by writing S as a (2n x 2n) complex matrix, we have from Eq. (2a.67)

$$|\det S|^2 = 1\ . \tag{2a.73}$$

We also note the isomorphisms

$$Sp(1, \mathbf{H}) \approx SU(2) \quad , \quad Sp(2, \mathbf{H}) \approx Sp(5) \approx O(5)\ . \tag{2a.74}$$

The elements of the Lie algebra Sp(n, **H**) are represented by (n x n) antihermitian quaternionic matrices Ω. Let

$$\Omega^\dagger = -\Omega\ \ , \ \ \Omega'^\dagger = -\Omega'\ \ , \ \ \Omega'' = [\Omega\ , \Omega']\qquad , \tag{2a.75}$$

we find

$$\Omega''^{\dagger} = \left(\Omega\Omega' - \Omega'\Omega\right)^{\dagger} = -\,\Omega'' \ ; \tag{2a.76}$$

so Ω'' is also antihermitian.

In summary, we have found that the Lie algebra of Sp(n, **H**) is an algebra of antihermitian quaternionic matrices Ω, closed under the Lie bracket multiplication. This procedure generalizes the properties of the unitary groups SU(n) whose su(n) algebras have antihermitian complex (n x n) matrices as elements and are also closed under the anticommutative Lie bracket multiplication. The Jacobi identity for the latter multiplication follows from the associativity of the group elements, the complex matrices for SU(n) and quaternionic matrices for Sp(n, **H**) obeying Eq. (2a.67).

Similarly, antihermitian matrices are closed under the anticommutative Lie bracket multiplication, the algebra of antihermitian matrices requires a commutative multiplication for closure given by an anticommutator. Thus the Jordan product of two (nxn) hermitian matrices J and K is defined by

$$M \equiv J \cdot K = K \cdot J = \frac{1}{2}\{J, K\} = \frac{1}{2}(JK+KJ) \ . \tag{2a.77}$$

From

$$J = J^{\dagger} \quad , \quad K = K^{\dagger} \tag{2a.78}$$

it follows that

$$M^{\dagger} = \frac{1}{2}\,(\, JK + KJ \,)^{\dagger} = M \ . \tag{2a.79}$$

Since the Jordan product is not associative, one introduces the *associator* of 3 hermitian matrices J, K and L

$$(\, JKL \,) \equiv \left[\, J, K, L \right] \equiv (\, J \cdot K \,) \cdot L - J \cdot (\, K \cdot L \,) \, . \tag{2a.80}$$

From the abelian property of Jordan products, we find

$$[\, L \, , K \, , J \,] = (\, L \cdot K) \cdot J - L \cdot (\, K \cdot J \,) = -[\, J, K, L \,] \ . \tag{2a.81}$$

The useful identity

$$[[L, J], K] = (LJ - JL)K - K(LJ - JL) = 4[J, K, L] \tag{2a.82}$$

relating the associator to the double commutator leads to

$$[J, K, J^2] = \frac{1}{4}[[J^2, J], K] = 0 \ . \tag{2a.83}$$

Two other identities

$$J \cdot K = K \cdot J \tag{2a.84a}$$

and

$$[J, K, J^2] = (J \cdot K) \cdot J^2 - J \cdot (K \cdot J^2) = 0 \tag{2a.84b}$$

also hold for (n x n) complex or quaternionic hermitian matrices. In fact, when taken as axioms, Eqs. (2a.84a) and (2a.84b) by themselves characterize a general Jordan algebra. Moreover, a most remarkable fact is the following: the only solution of these two identities, other than a (nxn) real, complex or quaternionic matrices, is the exceptional Jordan algebra element, given by a (3 x 3) octonionic hermitian matrix. The latter algebra will be discussed after we come to octonions, in Section 3.b.

We note that, as in the case of Lie algebras which can be exponentiated to Lie groups with elements $\exp(\Omega)$, we can similarly exponentiate the Jordan algebra elements to

$$R = \exp(J) \quad , \quad R^{-1} = \exp(-J) \ ; \tag{2a.85}$$

R^{-1} always exists since

$$\mathrm{Det}\, R = \exp(\mathrm{Tr}\, J) \neq 0 \ . \tag{2a.86}$$

However, since R, like J, is also hermitian, the exponentiated elements also close only under the Jordan product. They form a non-associative Jordan division algebra.

The quaternionic Jordan algebra with its hermitian elements can represent the algebra of quaternionic observables in a quaternionic Hilbert space of **Q**-quantum mechanics. In such a space, a vector v will be a ket $|v\rangle$ representing a state. The conjugate state is then the bra $\langle v| = v^\dagger$ defined as a row vector. The state is normalized, $\langle v|v\rangle = v^\dagger v = 1$. The phase transformed vector $v' = v\,\varphi$ with $\varphi = \exp(\vec{e}\cdot\vec{\omega})$, $\bar{\varphi}\varphi = 1$, is an arbitrary unit quaternion. So v' will represent the same state. We have just defined a projective geometry HP_n, associated with n-dimensional quantum mechanics.

If we now represent the quaternionic state by the projection operator

$$P(v) = |v\rangle\langle v| = P^\dagger \ , \tag{2a.87}$$

P(v) is automatically invariant under the quaternionic phase transformation $v' = v\,\varphi$. We have

$$P(v)^2 = P(v) \ , \tag{2a.88}$$

showing that the state can also be represented by an element P of the Jordan algebra of observables, provided that it is idempotent.

We defer till later applications of **Q**-quantum mechanics represented by a **H**-Jordan algebra. Here we only note that **Q**-quantum mechanics associated with a **H**-projective geometry is <u>not</u> equivalent to complex quantum mechanics. A state v in an n-dimensional state can be represented by a point P in an (n – 1) dimensional complex or quaternionic space. The n-components of v are its homogeneous coordinates. When states $|v_1\rangle$, $|v_2\rangle$, $|v_3\rangle$ are such that $|v_3\rangle$ is a linear superposition of $|v_1\rangle$ and $|v_2\rangle$ the three points P_1, P_2 and P_3 lie on a straight line. The superposition properties in the n-dimensional Hilbert spaces can be translated into geometric properties in the (n – 1) dimensional complex or quaternionic projective spaces. Indeed it can be proved that, in the complex case, the Pappus theorem holds unlike in the quaternionic case, the superposition properties are therefore altered by the non-commutativity of quaternions. They are then different in complex and quantum mechanics.

2.b. Jordan Formulation, H-Hilbert Spaces and Groups

During the exuberant first decade after the advent of Quantum Mechanics, its creators explored all conceivable extensions of quantum mechanical concepts and methods. Thus, in 1933, Jordan [64] offered a novel formulation of Quantum Mechanics on what became known subsequently in mathematics as Jordan algebras. In the standard formulation of quantum theory, states and observables are represented by vectors (kets) in a Hilbert space and Hermitian matrices acting on the states respectively. In the Jordan formulation, an algebra of observables is so defined that observables can be combined to give other observables. If p and q are observables in classical mechanics, so are p^2, q^2 and pq etc. However, if p and q are replaced by hermitian matrices P and Q, then only their symmetric combination, namely their Jordan product (2a.77) is hermitian and hence an observable representing the c-number product pq. A Jordan algebra is then a symmetric, non-associative algebra of (n x n) Hermitian matrices over $\mathbf{R}$, $\mathbf{C}$ and $\mathbf{H}$. In the exceptional case, it is the algebra of (3 x 3) hermitian matrices over octonions ($\mathbf{\Omega}$). In the following year, 1934, in an attempt to generalize quantum mechanics to incorporate Heisenberg's isospin symmetry, Jordan, von Neumann and Wigner [65] developed the representation theory of this algebra, defined axiomatically by the product $A \cdot B = B \cdot A$ along with a measure of the lack of associativity, the associator

$$(ABC) = - (CBA) \equiv (A \cdot B) \cdot C - A \cdot (B \cdot C) \qquad (2b.1)$$

with the Jordan identity $(A\, B\, A^2) = 0$. This identity makes the algebra of observables power associative i.e. P^n is an unambiguous expression representing the classical observable p^n. Albert [66] immediately proved the uniqueness of this construction.

In the Jordan formulation of ordinary quantum mechanics, the commutative but non-associative Jordan algebra of hermitian matrix observables over $\mathbf{C}$ replaces the non-commutative but associative matrix algebra of Heisenberg and Dirac. In the following, we proceed from the latter to the former case.

2.b.1. The Jordan form of quantum mechanics

Consider a finite quantum mechanical state space, such as an internal symmetry charge space at a given space-time point. In Dirac's formulation of quantum mechanics [67], a state is represented by a ket $|\alpha\rangle$, a column vector with complex components in conventional Hilbert spaces. A conjugate vector $\langle\beta|$ is a bra and the scalar product $\langle\alpha|\beta\rangle$ is the amplitude T_{AB}. The associated probability reads

$$\Pi_{\alpha\beta} = |\langle\beta|\alpha\rangle|^2 = \langle\beta|\alpha\rangle\langle\alpha|\beta\rangle \ . \tag{2b.2}$$

States are normalized such that $\langle\alpha|\alpha\rangle = 1$. With the vectors $|\alpha\rangle$ and $\lambda|\alpha\rangle$ ($|\lambda| = 1$) representing the same physical state in the ray representation, we can write

$$|\alpha\rangle = \frac{a}{\sqrt{a^\dagger a}} \tag{2b.3}$$

a being a complex column vector.

By introducing in this space a complete orthonormal set of states $|i\rangle$, $\langle i|j\rangle = \delta_{ij}$, we have

$$\langle\beta|\alpha\rangle = \langle\beta|i\rangle\langle i|\alpha\rangle \equiv \sum_i \langle\beta|i\rangle\langle i|\alpha\rangle \ . \tag{2b.4}$$

So

$$\Pi_{\alpha\beta} = \langle\beta|i\rangle\langle i|\alpha\rangle\langle\alpha|\beta\rangle = \langle\beta|\alpha\rangle\langle\alpha|i\rangle\langle i|\beta\rangle \tag{2b.5}$$

or

$$\Pi_{\alpha\beta} = \langle\beta|i\rangle\langle i|\alpha\rangle\langle\alpha|j\rangle\langle j|\beta\rangle \ . \tag{2b.6}$$

Since the Hilbert space is a linear vector space, the superposition principle implies that

$$|\gamma\rangle = a|\alpha\rangle + b|\beta\rangle \ , \tag{2b.7}$$

also represents a possible physical state. a and b are complex numbers constrained by a normalization condition on $|\gamma\rangle$. In particular $|\gamma\rangle = c_i\,|i\rangle$ with the c_i being the components of $|\gamma\rangle$ in the $|i\rangle$ basis.

Observables are represented by hermitian matrices having real eigenvalues. To each nondegenerate eigenvalue ν corresponds the physical state $|\nu\rangle$.

In the Jordan formulation, the state is also represented by the hermitian matrix

$$P_\alpha = P_\alpha^\dagger = |\alpha\rangle\langle\alpha| = \frac{a\,a^\dagger}{\left(\sqrt{a\,a^\dagger}\right)^2} \tag{2b.8}$$

obeying $P_\alpha{}^2 = P_\alpha$. So P_α is a projection operator, an idempotent operator with eigenvalue 0 or 1. For normalized states, we also have

$$\mathrm{Tr}\,P_\alpha = \mathrm{Tr}\,|\alpha\rangle\langle\alpha| = \langle\alpha|\alpha\rangle = 1 \ . \tag{2b.9}$$

So, to each set $|i\rangle$ corresponds the set P_i such that

$$\mathrm{Tr}\,P_i\,P_j = \mathrm{Tr}\,P_i \cdot P_j = \delta_{ij} \ . \tag{2b.10}$$

The dot in (2b.10) denotes the Jordan product (2a.77). The probability $\Pi_{\alpha\beta}$ then reads

$$\Pi_{\alpha\beta} = \mathrm{Tr}\,P_i \cdot P_j = \mathrm{Tr}\big(|\alpha\rangle\langle\alpha||\beta\rangle\langle\beta|\big) = |\langle\alpha|\beta\rangle|^2 \ . \tag{2b.11}$$

All the measurable quantities in Quantum Mechanics are transition probabilities of the form

$$\begin{aligned}\Pi_{\alpha\beta}(\Omega) &= \left|\langle\beta|\Omega|\alpha\rangle\right|^2 \\ &= \langle\alpha|\Omega|\beta\rangle\langle\beta|\Omega|\alpha\rangle = \mathrm{Tr}\big(\Omega\,P_\beta\,\Omega\,P_\alpha\big)\end{aligned} \tag{2b.12}$$

where Ω is hermitian. To show that the last expression between brackets can be written solely in terms of the Jordan product, we let $U(\Omega)P_\beta = \Omega P_\beta \Omega$, then

$$\Pi_{\alpha\beta}(\Omega) = \mathrm{Tr}\left\{P_\alpha \cdot U(\Omega)P_\beta\right\} , \qquad (2b.13)$$

with the dot again denoting the Jordan product. By way of the identity

$$\{ABC\} = \frac{1}{2}(ABC + CBA) = (A \cdot B) \cdot C + (A \cdot B) \cdot C - (A \cdot C) \cdot B \quad (2b.14)$$

satisfied by the *ternary product* $\{ABC\}$. As a special case, Eq. (2b.14) gives

$$\Omega P_\beta \Omega = \left\{ \Omega P_\beta \Omega \right\} = 2\,(\Omega \cdot P_\beta) \cdot \Omega - \Omega^2 \cdot P_\beta \qquad (2b.15)$$

from which $U(\Omega)P_\beta = \left\{\Omega P_\beta \Omega\right\}$ the right hand side of which is therefore expressed purely in terms of the Jordan product. So the absolute square of the matrix element $\Omega_{\alpha\beta}$ reads

$$\Pi_{\alpha\beta}(\Omega) = \mathrm{Tr}\left\{ P_\alpha \cdot U(\Omega)P_\beta \right\} = \mathrm{Tr}\left(P_\alpha \cdot \left\{\Omega P_\beta \Omega\right\} \right) = \mathrm{Tr}\left(U(\Omega)P_\alpha \cdot P_\beta\right) . \quad (2b.16)$$

On the other hand, the non-measurable probability amplitude $\langle \beta | \alpha \rangle$ cannot be expressed by way of the Jordan product and the arbitrary phase in $| \alpha \rangle$ disappears in P_α.

The insertion of a complete set of states (2b.5) translates into

$$\Pi_{\alpha\beta} = \sum_i \mathrm{Tr}\left(P_\alpha P_i P_\beta \right) = \sum_i \mathrm{Tr}\left(P_\beta P_i P_\alpha\right) = \frac{1}{2} \sum_i \mathrm{Tr}\left(P_\alpha P_i P_\beta + P_\beta P_i P_\alpha \right) \quad (2b.17a)$$

or equivalently, in terms of the ternary product (2b.14),

$$\Pi_{\alpha\beta} = \sum_i \mathrm{Tr}\left\{ P_\beta P_i P_\alpha \right\} . \qquad (2b.17b)$$

Similarly, in lieu of Eq. (2b.6), we have

$$\Pi_{\alpha\beta} = \sum_i \sum_j \mathrm{Tr}\left(\{P_i\, P_\alpha\, P_j\} \cdot P_\beta \right) . \tag{2b.18}$$

In this fashion a complete set of states can be inserted in a twofold way using only the Jordan product.

Further properties of the complete set of projection operators P_i are

$$\sum_i P_i = I , \tag{2b.19}$$

$$P_i \cdot P_j = \Delta_{ij}^{k} P_k \tag{2b.20}$$

where Δ_{ij}^{k} is unity for $i = j = k$ and zero otherwise.

To transcribe the superposition principle in the Jordan formulation, we define yet another product, the *Freudenthal product*

$$A \times B \equiv A \cdot B - \frac{1}{2} A\, \mathrm{Tr}B - \frac{1}{2} B\, \mathrm{Tr}\, A - \frac{1}{2} I\, (\mathrm{Tr}\, A \cdot B - \mathrm{Tr}\, A\, \mathrm{Tr}\, B) . \tag{2b.21}$$

We also record here the useful equality

$$\mathrm{Tr}\left[(A \times B) \cdot C\right] = \mathrm{Tr}\left[(B \times C) \cdot A\right] = \mathrm{Tr}\left[(C \times A) \cdot B\right] . \tag{2b.22}$$

If $|\gamma\rangle = a\,|\alpha\rangle + b\,|\beta\rangle$, we obtain

$$P_\gamma = |a|^2 P_\alpha + |b|^2 P_\beta + a\, b^* |\alpha\rangle\langle\beta| + a^* b\, |\beta\rangle\langle\alpha| \tag{2b.23}$$

after some algebra, the linear dependence of the states $|\alpha\rangle$, $|\beta\rangle$ and $|\gamma\rangle$ translates into the relation

$$\mathrm{Tr}\left[\left(P_\alpha \times P_\beta\right)\cdot P_\gamma\right] = 0\,. \tag{2b.24}$$

In the usual quantum mechanics, transition amplitudes are invariant under unitary transformations as $\langle \alpha' \mid \beta' \rangle = \langle \alpha \mid \beta \rangle$ with the transformed states $\mid \alpha' \rangle = U \mid \alpha \rangle$, $\mid \beta' \rangle = U \mid \beta \rangle$, $UU^\dagger = 1$. Then the projection operators transform as

$$P'_\alpha = U\, P_\alpha U^{-1}\,. \tag{2b.25}$$

The observables Ω, which are linear combinations of projection operators, also transform in the same way : $\Omega' = U\ \Omega\ U^\dagger$. So the Jordan product Ω of two observables Ω_1 and Ω_2 also transform like P_α since

$$\Omega' = \Omega'_1 \cdot \Omega'_2 = \frac{1}{2}\left(\Omega'_1\Omega'_2 + \Omega'_2\Omega'_1\right)$$

$$= \frac{1}{2} U\left(\Omega_1\Omega_2 + \Omega_2\Omega_1\right)U^\dagger = U\,\Omega\ U^\dagger\,. \tag{2b.26}$$

It follows that, in an n-dimensional Hilbert space, with (n x n) hermitian matrices associated with observables and projection operators for states, the automorphism group of the Jordan algebra of observables is U(n) or SU(n).

Since the corresponding infinitesimal transformation is $U = I + f$, ($f = -f^\dagger$), f is an element of the U(n) Lie algebra, we have

$$P'_\alpha = P_\alpha + \delta P_\alpha \tag{2b.27}$$

with

$$\delta P_\alpha = [\, f\,, P_\alpha]\,. \tag{2b.28}$$

To cast this transformation law solely in terms of the Jordan product, we set

$$f = \frac{1}{4}[\, h_1\,, h_2] \tag{2b.29}$$

where h_1 and h_2 are hermitian, so that they are elements of the algebra of observables. Equation (2b.28) becomes

$$[f, P_\alpha] = \frac{1}{4}[[h_1, h_2], P_\alpha] . \tag{2b.30}$$

The latter can be recast in forms involving either the associator (2b.1) or the triple product (2b.14); by way of Eq. (2a.82) we obtain

$$\delta P_\alpha = \frac{1}{4}[[h_1, h_2], P_\alpha] = (h_2 P_\alpha h_1) . \tag{2b.31a}$$

Alternatively,

$$\delta P_\alpha = \frac{1}{2}\{ h_1 h_2 P_\alpha \} - \frac{1}{2}\{ h_2 h_1 P_\alpha \} . \tag{2b.31b}$$

The finite transformation becomes a series of multiply-nested associators

$$P_\alpha' = P_\alpha + (h_2 P_\alpha h_1) + \frac{1}{2}(h_2 (h_2 P_\alpha h_1) h_1) + \ldots \tag{2b.32}$$

which, upon integration, becomes

$$P_\alpha' = \exp\left(\frac{1}{4}[h_1, h_2]\right) P_\alpha \exp\left(-\frac{1}{4}[h_1, h_2]\right) = U P_\alpha U^{-1} . \tag{2b.33}$$

We remark that, in this case, the transformation U, an automorphism of the Jordan algebra, depends only on the antihermitian combination $[h_1, h_2]$ of h_1 and h_2. While the formulae (2b.17b), (2b.18), Eqs. (2b.24) and (2b.32) do *not* in fact appear in the usual Jordan formulation of quantum mechanics, they are nevertheless essential to the latter's octonionic extension. There, due to the non-associativity of octonions, one actually must forego the ordinary matrix product of the ket formulation.

Finally, we should mention that Jordan also found in an infinite Jordan algebra a corresponding formulation of the Heisenberg algebra of infinite hermitian matrices. To the familiar equal time commutation relations

$$\left[q_\alpha , q_\beta\right] = 0 \quad , \quad \left[p_\alpha , p_\beta\right] = 0 \quad , \quad \left[q_\alpha , p_\beta\right] = \delta_{\alpha\beta} \tag{2b.34}$$

correspond the following cubic associator conditions

$$\left(q_\alpha\, q_\gamma\, p_\beta\right) = \left(p_\beta\, p_\gamma\, q_\alpha\right) = 0 \ ,$$

$$\left(p_\alpha^2\, q_\gamma\, q_\alpha\right) = \left(q_\beta^2\, p_\gamma\, p_\alpha\right) = \frac{1}{2}\,\delta_{\beta\gamma}\delta_{\alpha\beta} \ . \tag{2b.35}$$

Indeed, more generally, in the Jordan formulation, compatible (i.e. simultaneously measurable) observables O_1 and O_2 are such that, if Y is any element of the Jordan algebra, we have the cubic associator condition $(O_1\ Y\ O_2) = 0$. The latter implies via the identity (2a.82) the expected result : $[O_1 , O_2] = 0$ (but not vice versa!) for non-exceptional (i.e. special) Jordan algebras of **R**, **C** and **H**.

Using Eqs. (2b.11) and (2b.13), we can write

$$d_{\alpha\beta}^2 = \frac{1}{2}\,\mathrm{Tr}\,(\,P_\alpha - P_\beta\,)^2 = 1 - \Pi_{\alpha\beta} \ . \tag{2b.36}$$

In projective geometry the normalized idempotent P_α represents a point with its n homogeneous or (n –1) inhomogeneous coordinates. $d_{\alpha\beta}$ would be the invariant distance between two such points α and β. Through Eq. (2b.13) we see its close relation to the transition probability. When $\alpha = \beta$ their relative distance vanishes while the transition probability equals unity, as it should. A subgroup H of U(n) (e.g. H ≈ U(n –1) x U(1)) will leave P_α invariant; it is the stability group in the corresponding projective geometry leaving the point unchanged. The transformation of the coset changes the point and therefore the state. The set of all transformed states is a homogeneous space of dimension (dimG – dimH) and forms our quantum mechanical space. Positive definite transition probabilities can be defined from the invariant distance between two distinct points.

Geometrically, a ket in an n-dimensional complex Hilbert space may then be viewed as the set of homogeneous coordinates of a point in an (n – 1)-dimensional projective geometry. The vectors **a** and λ**a** represent the same point P_α when $\lambda \in \mathbf{C}$; there exists a one to one correspondence between the physical state associated with the projection operator P_α and a point of the projective sphere. The point is an element of the Jordan algebra obeying the Freudenthal condition (2b.21). Let us take the set of physical states P_γ that are linear combinations of two different states P_α and P_β; the corresponding kets and the projection operators obey the superposition principle, (2b.7) and (2b.24), respectively. As P_γ varies, this is the equation of a line passing through the points P_α and P_β.

As an illustration of the above geometrical picture, we work out the simple example of a 2-dimensional quantum mechanical projective space. Further explicit illustrations will be given in connection with quaternionic and octonionic Hilbert spaces.

For a 2-dimensional projective space the homogeneous coordinates are three-dimensional and are represented by 3 component vectors. Let us represent the homogeneous coordinate of the point P_γ. From the definition (2b.21) the Freudenthal product reduces to

$$\begin{aligned} A \times B &= \frac{1}{2}(A+B)\times(A+B) - \frac{1}{2}A\times A - \frac{1}{2}B\times B \\ &= \frac{1}{2}(A+B)^{-1}\,\mathrm{Det}(A+B) - \frac{1}{2}A^{-1}\,\mathrm{Det}\,A - \frac{1}{2}B^{-1}\,\mathrm{Det}\,B \;, \end{aligned} \tag{2b.37}$$

where the fact that $A \times A = A^{-1}\mathrm{Det}A$ for 3 x 3 matrices has been used. We find

$$P_{\vec{a}} \times P_{\vec{b}} = \lambda(\vec{a},\vec{b})\,P_{\vec{a}^* \times \vec{b}^*} \;. \tag{2b.38}$$

$\vec{a}^* \times \vec{b}^*$ denotes the vector product of $\vec{a^*}$ and $\vec{b^*}$ and $\lambda(\vec{a},\vec{b})$ is the normalization factor

$$\lambda(\vec{a},\vec{b}) = \frac{1}{2}\left[1 - \frac{|\vec{a}^* \cdot \vec{b}|^2}{(\vec{a}^* \cdot \vec{a})(\vec{b}^* \cdot \vec{b})}\right] = \frac{1}{2}\frac{|\vec{a}\times\vec{b}|^2}{|\vec{a}|^2\,|\vec{b}|^2} \;, \tag{2b.39}$$

$\vec{a}^{*}\cdot\vec{b}$ being the scalar product of $\vec{a}^{*}$ and $\vec{b}$. Also

$$\mathrm{Tr}\,(P_{\vec{a}}\cdot P_{\vec{b}})=\frac{|a^{+}b|^{2}}{(a^{+}a)\,(b^{+}b)}=\frac{|\vec{a}^{*}\cdot\vec{b}|^{2}}{\left(\vec{a}^{*}\cdot\vec{a}\right)\left(\vec{b}^{*}\cdot\vec{b}\right)}\quad . \tag{2b.40}$$

It follows that

$$\mathrm{Tr}\left[\,(P_{\vec{a}}\cdot P_{\vec{b}})\cdot P_{\vec{c}}\,\right]=\frac{1}{2}\,\frac{|(\vec{a}\times\vec{b})\cdot\vec{c}|^{2}}{|\vec{a}|^{2}\,|\vec{b}|^{2}\,|\vec{c}|^{2}}\quad . \tag{2b.41}$$

Consequently, the linear dependence condition (2b.24) is equivalent to the vanishing of the volume of the parallelpiped ($\vec{a}$, $\vec{b}$, $\vec{c}$), or to the condition that the ket vector $\vec{c}$ is a linear combination of $\vec{a}$ and $\vec{b}$. If $\vec{c}$ is a point on the line determined by the points $\vec{a}$ and $\vec{b}$, then the equation of the line is

$$\left(\vec{a}\times\vec{b}\right)\cdot\vec{c}=0\ , \tag{2b.42}$$

which is equivalent to Eq. (2b.24).

Consider two lines determined respectively by the pairs of points ($\vec{a}$, $\vec{b}$) and ($\vec{a'}$, $\vec{b'}$). Then, the intersection $\vec{f}$ of the two lines satisfies

$$\left(\vec{a}\times\vec{b}\right)\cdot\vec{f}=0 \qquad , \qquad \left(\vec{a}'\times\vec{b}'\right)\cdot\vec{f}=0\ . \tag{2b.43}$$

These equations are solved by choosing

$$\vec{f}=\left(\vec{a}\times\vec{b}\right)\times\left(\vec{a}'\times\vec{b}'\right)\ . \tag{2b.44}$$

By way of Eq. (2b.38) we represent this point by the projection operator

$$P_{\vec{f}}=\lambda^{-1}(\,\vec{a}\times\vec{b}\,,\,\vec{a}'\times\vec{b}'\,)\,P_{\vec{a}^{*}\times\vec{b}^{*}}\times P_{\vec{a}'^{*}\times\vec{b}'^{*}}=k\left(P_{\vec{a}}\times P_{\vec{b}}\right)\times\left(P_{\vec{a}'}\times P_{\vec{b}'}\right) \tag{2b.45}$$

where

$$k\equiv\lambda^{-1}(\vec{a}\times\vec{b}\,,\,\vec{a}'\times\vec{b}')\,\lambda^{-1}\,(\vec{a}\,,\vec{b})\,\lambda^{-1}\,(\vec{a}'\,,\,\vec{b}')\ . \tag{2b.46}$$

Thus two lines determine a point by the same algorithm that two points determine a line, namely through the Freudenthal product of the characteristic projection operators.

Next we introduce the distance $d(\vec{a}\ ,\vec{b})$ between two points $\vec{a}$ and $\vec{b}$ on the projective 2-sphere by way of

$$\cos^2\left[d(\vec{a}\ ,\vec{b}\ )\right] = \mathrm{Tr}\left(P_{\vec{a}} \cdot P_{\vec{b}}\right) \tag{2b.47}$$

or using Eq. (2b.40)

$$\cos\left[d(\vec{a}\ ,\vec{b}\ )\right] = \frac{\left|\vec{a}^{*} \cdot \vec{b}\right|}{|\vec{a}|\ |\vec{b}|}\ . \tag{2b.48}$$

This distance is invariant under the action of SU(3).

We can write

$$\frac{1}{2}\sin^2\left[d(\vec{a}\ ,\vec{b}\ )\right] = \lambda(\vec{a}\ ,\vec{b}\ ) \tag{2b.49}$$

with λ given by Eq. (2b.39) or

$$d(\vec{a}\ ,\vec{b}\ ) = \sin^{-1}\left(\frac{\left|\vec{a}\times\vec{b}\right|}{|\vec{a}|\ \left|\vec{b}\right|}\right)\ . \tag{2b.50}$$

So if $\vec{b} = \vec{a} + d\vec{a}$ then the angular distance is

$$d(\vec{a}, \vec{a} + d\vec{a}) = ds \tag{2b.51}$$

and Eq. (2b.47) gives the line element

$$ds^2 = (\vec{a}^{*} \cdot \vec{a})^{-1}\left[d\vec{a}^{*} \cdot d\vec{a} - \frac{\vec{a} \cdot d\vec{a}^{*}\ \vec{a}^{*} \cdot d\vec{a}}{\left(\vec{a}^{*} \cdot \vec{a}\right)}\right]\ . \tag{2b.52}$$

Letting $a_1 = \alpha_1$, $a_2 = \alpha_2$, $a_3 = 1$, we get

$$d^2s = \left(1 + \alpha^{\dagger}\alpha\right)^{-1}\left[d\alpha^{\dagger}\, d\alpha - \left(1 + \alpha^{\dagger}\alpha\right)^{-1}(d\alpha)^{\dagger}\alpha\alpha^{\dagger}\, d\alpha\right]\ . \tag{2b.53}$$

For real and complex α, ds^2 reduces to the well-known Riemannian S^2 line element. This check justifies the choice of Eq. (2b.47) for the definition of distance. The probability $\Pi_{\alpha\beta}$ and the distance $d_{\alpha\beta}$ are then connected by

$$\Pi_{\alpha\beta} = \cos^2 d_{\alpha\beta} = \mathrm{Tr}\,(\, P_\alpha \cdot P_\beta \,) \tag{2b.54}$$

and are invariant under the group SU(3).

We close our discussion of the geometrical interpretation of the quantum mechanical formalism with Desargues' theorem (see Pedoe in Ref. [29]) in the 3-dimensional quantum mechanical space.

Let S and A, B, C represent points (physical states) such that A, B, C are not on a line. In yet a more vivid depiction, S may be conceived as a light source and the triangle A' B' C' the shadow of ABC. The planes of (ABC) and (A'B'C') intersect on a line containing A", B", C" where A" is the intersection of BC and B'C', with cyclic permutations giving B" and C". So A", B", C" are linearly dependent and we have Desargues' theorem

$$\left(\vec{a}'' \times \vec{b}''\right) \cdot \vec{c}'' = 0 \tag{2b.55}$$

where

$$\vec{a}'' = \left(\vec{b}\times\vec{c}\right)\times\left(\vec{b}'\times\vec{c}'\right) \;,\quad \left(\vec{b}\times\vec{b}'\right)\cdot\vec{s} = \left(\vec{c}\times\vec{c}'\right)\cdot\vec{s} = 0 \;, \tag{2b.56}$$

with similar relations obtained by cyclic permutations of $\vec{a}$, $\vec{b}$ and $\vec{c}$.

In terms of the Jordan algebra, Eq. (2b.55) translates into

$$\mathrm{Tr}\left(P_{\alpha''} \times P_{\beta''}\right) \cdot P_{\gamma''} = 0 \tag{2b.57}$$

if

$$P_{\alpha''} = \lambda_{\alpha''}\left(P_\beta \times P_\gamma\right)\times\left(P_{\beta'}\times P_{\gamma'}\right) \quad , \quad \mathrm{Tr}\left[\left(P_{\alpha'}\times P_{\alpha'}\right)\cdot P_\sigma\right] = 0 \tag{2b.58}$$

with $P_{\beta''}$ and $P_{\gamma''}$ obtained through cyclic permutations. Consequently, Desargues' theorem in quantum mechanics derives from the relation (2b.38). Next, we specialize on certain aspects of quaternionic Hilbert spaces and their associated groups of transformations.

2.b.2. H-Hilbert spaces and symplectic groups

Let us consider an n-dimensional space over **H**. Its elements are quaternionic vectors $\mathbf{q}$ and their H-conjugates $\overline{\mathbf{q}}$, namely

$$\mathbf{q} = \begin{pmatrix} q_1 \\ \cdot \\ \cdot \\ \cdot \\ q_n \end{pmatrix} \quad \text{and} \quad \overline{\mathbf{q}} = \begin{pmatrix} \overline{q_1} \\ \cdot \\ \cdot \\ \cdot \\ \overline{q_n} \end{pmatrix} \in \mathbf{H}^n \tag{2b.59}$$

with as *coordinates* $q_i \in \mathbf{H}$. We also define

$$\tilde{\mathbf{q}} = \overline{\left(\mathbf{q}^{\mathrm{T}}\right)} = (\overline{\mathbf{q}})^{\mathrm{T}} = (\,\overline{q}_1, \overline{q}_2, \ldots, \overline{q}_n\,) \ . \tag{2b.60}$$

Addition of vectors is defined by that of their corresponding coordinates. Left-multiplication of $\mathbf{q}$ by a quaternion $a \in \mathbf{H}$ is given by $\mathbf{q}^{\mathrm{T}}a = (\,q_1 a, q_2 a, \ldots, q_n a\,)$. And if $\mathbf{q}_1$ and $\mathbf{q}_2$ are vectors, $a, a_1, a_2 \in \mathbf{H}$, then

$$(\,\mathbf{q}_1 + \mathbf{q}_2)\,a = \mathbf{q}_1 a + \mathbf{q}_2 a \quad ; \quad \mathbf{q}\,(\,a_1 + a_2) = \mathbf{q}a_1 + \mathbf{q}a_2 \ , \tag{2b.61}$$

$$\mathbf{q}\,(\,a_1\,a_2) = (\mathbf{q}\,a_1)a_2 \ . \tag{2b.62}$$

The quadratic form or *symplectic product* $(\mathbf{q}_1, \mathbf{q}_2)$ of two vectors is the quaternion

$$(\mathbf{q}_1, \mathbf{q}_2) \equiv \tilde{\mathbf{q}}_1 \cdot \mathbf{q}_2 = \sum_{i=1}^{n} \overline{q_{1i}}\, q_{2i} \tag{2b.63}$$

with the following properties

$$(\mathbf{q}_1 + \mathbf{q}_2\, ,\, \mathbf{q}) = (\mathbf{q}_1, \mathbf{q}) + (\mathbf{q}_2, \mathbf{q})\ , \tag{2b.64}$$

$$(\mathbf{q}\ , \mathbf{q}_1 + \mathbf{q}_2) = (\mathbf{q}, \mathbf{q}_1) + (\mathbf{q}, \mathbf{q}_2)\ , \tag{2b.65}$$

$$(\mathbf{q}\ , \mathbf{q}_1 a) = (\mathbf{q}\ , \mathbf{q}_1)\, a\ ;\ (\mathbf{q}\, a\, , \mathbf{q}_1) = \bar{a}(\mathbf{q}\, , \mathbf{q}_1)\ . \tag{2b.66}$$

In particular, the norm of the vector $(\mathbf{q}\ ,\ \mathbf{q}) = \sum_{i=1}^{n} \overline{q_i}\, q_i$ is real and is nonzero if $\mathbf{q} \neq 0$. So a left vector space over **H**, together with an inner product making the resulting normed linear space complete, is a quaternionic Hilbert space.

Under a linear transformation S

$$\mathbf{r}' = S\, \mathbf{r} \qquad , \qquad \mathbf{q}' = S\mathbf{q}\ . \tag{2b.67}$$

Hence, $(\mathbf{r}', \mathbf{q}') = \mathbf{r}' \cdot \mathbf{q}' = \mathbf{r} S\, S\, \mathbf{q} = \mathbf{r} \cdot \mathbf{q} = (\mathbf{r}, \mathbf{q}))$. From the invariance of the norm, it follows that $\tilde{S}\, S = 1$, Det $S = \pm 1$. Taking Det $S = +1$ and writing S as

$$S = \exp(M) \qquad , \qquad \tilde{S} = \exp(\tilde{M})\ ; \tag{2b.68}$$

$$M + \tilde{M} = M + \overline{M^T} = 0\ . \tag{2b.69}$$

So

$$M_{kk} + \tilde{M}_{kk} = 0\ \rightarrow \mathrm{Sc}(\ M_{kk}) = 0\ , \tag{2b.70}$$

namely the diagonal elements of M have zero real parts; they are pure quaternions. Therefore only 3m real parameters are needed to specify the diagonal elements of S.

Since $\overline{M}_{ij} = -M_{ji}$, $i \neq j$, we need to specify only one set of subdiagonal elements for which $4\,\frac{n(n-1)}{2} = 2n(n-1)$ real parameters are required. Hence, $3n + 2n(n-1) = n(2n+1)$ real parameters are necessary to define a linear transformation in the space. The group of transformations $\{ S \mid \mathrm{Det}\, S = +1 \}$ constitutes the n-dimensional *symplectic group* Sp(n) .

Now a set of matrices U forms an n-dimensional group Sp(n), if each U is a 2n x 2n unitary matrix obeying the relation

$$Z = U\,Z\,U^T \; ; \tag{2b.71}$$

the matrix Z is a banded diagonal matrix with + 1 in the subdiagonal and – 1 in the superdiagonal, i.e. $Z_{ij} = \delta_{i,j+1} - \delta_{i,j-1}$. In quaternionic language, if I is the (nxn) unit matrix, then simply $Z = e_2 I$.

Indeed any (2n x 2n) matrix U can be written as an (n x n) matrix Q with quaternionic elements q_{ij} (i, j=1, 2,..., n). Since any 2 x 2 matrix A over **C** can be cast as a quaternion Q

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = A_\mu e^\mu \tag{2b.72}$$

with

$$A_0 = \frac{1}{2}(a+d)\,,\; A_1 = \frac{i}{2}(a-d)\,,\; A_2 = -\frac{1}{2}(b-c)\,,\; A_3 = \frac{i}{2}(b+c)\;, \tag{2b.73}$$

all (2n x 2n) matrices can be partitioned into n^2 (2 x 2) blocks of (2 x 2) with each block expressed in terms of quaternions. The usual matrix operations such as transposition and hermitian conjugation on U translate into the corresponding ones on Q, namely

$$\left(Q^T\right)_{ij} = -\,e_2 q_{ji}^{*} e_2 \tag{2b.74}$$

and

$$\left(Q^\dagger\right)_{ij} = q_{ji}^{\dagger}\;. \tag{2b.75}$$

More details on these **H**-valued matrices will be presented in connection with Yang-Mills instantons (Sect.2f.3) and with the Berry phases arising in half-integral spin systems with time-reversal invariant Hamiltonians (Sect.4.a).

2.c. Vector Products, Parallelisms and Quaternionic Manifolds

2.c.1. Vector products on manifolds

Before getting to (almost) complex then to quaternionic structures (and subsequently, to octonionic structures) on manifolds, it is natural to first define more general objects called *vector cross products* [68, 69].

Indeed vector products on manifolds are important structures for at least three reasons: Firstly, they naturally generalize the fundamental notion of almost complex structure [70] . Secondly, the existence of a vector product on a manifold M induces on its submanifolds unusual almost complex structures. Thirdly, they provide a way to study the less well understood Riemannian manifolds with holonomy group G_2 or Spin(7).

Following Eckmann [68] we define a r-fold vector cross product in R^n as a continuous map $P_r : R^{nr} \approx R^n \times R^n \times R^n \times \ldots \times R^n \to R^n$ such that, if $\langle\ ,\ \rangle$ denotes a positive definite bilinear form in R^n, for every set of vectors $(a_1, a_2, \ldots, a_r) \in R^{nr}$, we have

$$\text{a)}\ \langle P(a_1, \ldots, a_r) , a_i \rangle = 0 \quad (1 \le i \le r) \ , \tag{2c.1}$$

$$\text{b)}\ \| P(a_1, \ldots, a_r)\|^2 = \det (\langle a_i, a_j \rangle) \ . \tag{2c.2}$$

It can be deduced that R^n admits a r-fold vector product only in the following cases; 1) r = 1 and n even, 2) r = (n – 1), 3) r = 2 and n = 3, 7 and 4) r = 3 and n = 8.

Explicit expressions for these vector products are known in various forms. For example, the usual Gibbsian " vector product" in R^n corresponds to r = 2, n = 3 in cases (2) and (3). For any two vectors in R^3 represented by purely vectorial quaternions **a** and **b**, this 2-fold product is $P_2(\mathbf{a}, \mathbf{b}) = \mathrm{Vec}(\mathbf{ab})$, e.g. $P_2(e_i, e_j) = \varepsilon_{ijk}\, e_k$ if we consider two imaginary basic units e_i and e_j. It merely reflects the existence of the multiplication table of the imaginary quaternions $e_i\, e_j = -\delta_{ij} + \varepsilon_{ijk}\, e_k$ (i, j, k = 1, 2, 3) on R^3, the tangent space of S^3. The kernel of the cross product lies in the ε_{ijk}, the structure constants of SU(2) or in the parallelizable torsion of the 3-sphere S^3. In fact the existence of the vector cross product is connected in a one-to-one way by certain celebrated theorems such as Adams' Theorem in K-theory. The latter theorem will be a center piece in our discussion on the link between Hopf fibrations and division algebras (Section 4.b). As a didactic prelude to the parallelism on S^7, the Clifford parallelism on S^3 [71] will be considered next.

2.c.2. Absolute parallelisms on Lie groups and S^3

According to E.Cartan [72] , a Riemannian manifold M is *parallelizable* manifold or a space of *absolute parallelism* (AP) if the parallelism of two directions in two different points of M can be defined in a coordinate, path independent (and therefore absolute) way. So in such a space, parallel transport of tangent vectors or segments is such that the latter stay parallel to themselves.

As trivial Riemannian spaces, the euclidean spaces R^n are clearly the only simply connected, complete manifolds where the parallel transport of tangent vectors is path independent. Yet seen from Cartan's theory of affine connections, this absolute parallelism is simply a matter of vanishing curvature i.e. one of the type of connection defining parallel transport of geometrical objects in a given space.

Besides R^n, Cartan and Schouten [72] discovered that, besides compact semisimple Lie groups, the <u>only</u> other parallelizable compact Riemannian spaces is the coset space $S^7 \approx \dfrac{SO(8)}{SO(7)} \approx \dfrac{SO(7)}{G_2}$. Specifically, in addition to the symmetric, curvature-full and torsion free Riemannian connection, they can define uniquely on all these manifolds two curvature-free but torsion-full connections. So these manifolds can be alternatively seen as spaces with two possible absolute parallelisms, connected to the left and right group translations respectively. Historically, this result generalizes the absolute parallelism of Clifford on $S^3 \approx SU(2)$, the space of unit quaternions.

Before focusing on a quaternionic formulation of the parallelism on SU(2) or the unit 3-sphere S^3, it may be worthwhile to dwell on the better known absolute parallelism of general Lie groups. An old fashioned, explicit treatment is absent in modern texts on group theory and differential geometry; so it may be instructive to recall it below [73].

Consider a compact semi-simple Lie group G of rank r. In some local coordinate patch ($x_1, x_2, \ldots, x_r$) of G, the Cartan-Schouten parallelism [72] can be defined via two complete sets of left (+) and right (–) orthonormal frames $h^i_\alpha(\pm)$, $h^i_\alpha(\pm)\, h^k_\alpha(\pm) = \delta^{ik}$. ($i, \alpha = 1, 2, \ldots, r$). The Riemannian metric reads

$$ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta = h^i_\alpha(\pm) h^i_\beta(\pm) dx^\alpha dx^\beta \ . \tag{2c.3}$$

The legs $h^i_\alpha(\pm)$ obey Cartan's 1st structural equations

$$\nabla_\gamma\, h^i_\beta(\pm) = \frac{1}{2}\left(\frac{\partial h^i_\beta(\pm)}{\partial x_\gamma} - \frac{\partial h^i_\gamma(\pm)}{\partial x_\beta}\right) = \pm\, S^\alpha_{\beta\gamma} h^i_\alpha(\pm) \tag{2c.4}$$

where

$$\nabla_\gamma\, h^i_\beta(\pm) = \left(\partial_\alpha \delta_{\gamma\beta} - \overset{0}{\Gamma}{}^\gamma_{\alpha\beta}\right) h^i_\gamma(\pm) \tag{2c.5}$$

is the covariant derivative w.r.t. the symmetric Riemannian connection,

$$\overset{0}{\Gamma}{}^{\gamma}_{\alpha\beta} \equiv \frac{1}{2}\left(\Gamma^{\gamma}_{\alpha\beta} + \Gamma^{\gamma}_{\beta\alpha}\right) , \tag{2c.6}$$

and

$$S^{\gamma}_{\alpha\beta} = \frac{1}{2}\left(\Gamma^{\gamma}_{\alpha\beta} - \Gamma^{\gamma}_{\beta\alpha}\right) = - S^{\gamma}_{\beta\alpha} \tag{2c.7}$$

is the torsion tensor, $\Gamma^{\gamma}_{\alpha\beta}$ is the asymmetric affine connection. Alternatively, Eq. (2c.4) reads

$$\nabla_{\gamma}(\pm) h^{i}_{\beta}(\pm) = 0 , \tag{2c.8}$$

the covariant derivative $\nabla_{\gamma}(\pm)$ is taken w.r.t. the (+) or (–) parallelizing Cartan-Schouten connection:

$$\Gamma^{\gamma}_{\alpha\beta}(\pm) = h^{\gamma}_{i}(\pm) \frac{\partial h^{i}_{\alpha}(\pm)}{\partial x_{\beta}} \equiv \frac{1}{2}\left(\overset{0}{\Gamma}{}^{\gamma}_{\alpha\beta} \pm S^{\gamma}_{\alpha\beta}\right) . \tag{2c.9}$$

A statement of path independent parallel transport, Eq. (2c.8) is standardly taken as the defining equation of absolute parallelism. It implies:

a) The Killing property of the r-legs

$$\nabla_{\alpha} h^{i}_{\beta}(\pm) + \nabla_{\beta} h^{i}_{\alpha}(\pm) = 0 \tag{2c.10}$$

and that the $h^{i}_{\alpha}(\pm)$ form sets of *global* Killing fields.

b) The covariant constancy of the metric tensor w.r.t. all three types of connections

$$\nabla_{\gamma}(\pm)\, g_{\alpha\beta} = \nabla_{\gamma} g_{\alpha\beta} = 0 . \tag{2c.11}$$

So $\overset{0}{\Gamma}{}^{\gamma}_{\alpha\beta}$ is the Levi-Civita connection $\left\{ {\gamma \atop \alpha\beta} \right\}$ and the torsion tensor $S^{\rho}_{\alpha\beta}$ is skew-symmetric in *all* its three indices. Thus $g_{\gamma\rho} S^{\rho}_{\alpha\beta} = S_{\alpha\beta\gamma} = - S_{\gamma\beta\alpha}$, an important property in applications to the Kaluza-Klein compactifications of gravity and supergravity [74].

The Maurer-Cartan equations on G are

$$[L_i(\pm), L_j(\pm)] = \pm f_{ijk} L_k(\pm) \tag{2c.12}$$

where $L_i(\pm) \equiv h^i_\alpha(\pm)\frac{\partial}{\partial x^\alpha}$ is the left (right)-invariant displacement operator on G. They form the 2r *nowhere* vanishing vector fields defining the (+) and (−) absolute parallelisms on G. f_{ijk} are the $\frac{1}{2} r^2 (r-1)$ completely antisymmetric structure constants of G, $f^k_{ij} + f^k_{ji} = 0$. The associativity of the algebra is reflected in the Jacobi identity

$$f^k_{ij} f^m_{lk} + f^k_{jr} f^m_{ik} + f^k_{ri} f^m_{jk} = 0\,, \tag{2c.13}$$

which are the necessary and sufficient conditions for the left and right connections $\Gamma^\gamma_{\beta\alpha}(\pm)$ to be *integrable*.

In terms of the legs

$$S^\rho_{\alpha\beta} = \frac{1}{2} f^k_{ij} h^i_\alpha h^j_\beta h^\rho_k\ , \tag{2c.14}$$

namely the torsion coefficients are *constants* and $S^\rho_{\alpha\beta}(+) = -S^\rho_{\alpha\beta}(-)$. Due to the Jacobi identity or the associativity of Lie algebras,

$$\nabla_\rho S_{\alpha\beta\gamma} = S_{\lambda[\alpha\beta} S^\lambda_{\gamma]\rho} = 0\ ; \tag{2c.15}$$

the torsion is covariantly constant. This fact will be contrasted later with the non-zero covariant derivative of the torsion tensor on the round 7-sphere S^7, a signature of the lack of associativity of octonions.

Besides Eq. (2c.4), the remaining defining 2nd structural equations are

$$[\nabla_\alpha, \nabla_\beta] h^i_\gamma(\pm) = R_{\alpha\beta\gamma\sigma} h^{i\sigma}(\pm)\ , \tag{2c.16}$$

where $R_{\alpha\beta\gamma\sigma} = - S_{\alpha\beta\rho}S^{\rho}_{\gamma\sigma}$ is the non-vanishing Riemannian curvature since the symmetric connection is non-integrable. From Eq. (2c.15), it follows that $\nabla_\rho R_{\alpha\beta\gamma\sigma} = 0$, namely a Lie group is a locally symmetric space. We get for the Ricci curvature

$$R_{\rho\sigma} = - S_{\alpha\beta\rho}S^{\alpha\beta}_{\sigma} = -\left[\frac{(r-1)}{R_0^2}\right] g_{\rho\sigma} \; ; \tag{2c.17}$$

so G is locally an Einstein space.

Finally, two relevant geometric bilinear and cubic invariants of the torsion are the Cartan-Schouten equations:

$$S_{\alpha\beta\lambda}S^{\alpha\beta\lambda} = \frac{r(r-1)}{R_0^2} \quad , \tag{2c.18}$$

$$S^{\beta}_{\alpha\lambda}\, S^{\gamma}_{\beta\mu}\, S^{\alpha}_{\gamma\sigma} = -\frac{(r-1)}{2R_0^2} S_{\lambda\mu\sigma} \; . \tag{2c.19}$$

Since the spheres $S^1 \approx U(1)$ and $S^3 \approx SU(2)$ are compact groups spaces, they are therefore parallelizable. Yet their parallelizability is also linked to the existence of the complex algebra in E^2 and the quaternion algebra in E^4, respectively. As a prelude to discussing the nontrivial AP property of S^7 in Section 3.c.2, we now detail the connection between the Clifford's AP property of S^3 and the $\mathbf{H}$ algebra. Our presentation closely follows the lucid and elegant work of Rooman [75].

Let P be a point on S^3, parametrized by the unit quaternion $X = e_\mu x^\mu$, $|X| = 1$. The 3 unit quaternions $e_j X$ (Xe_j) obtained by left (+) (right (−)) multiplication of X by the basis quaternion e_j induces a mapping from S^3 to S^3. Since by the definition of the scalar product $\langle A \cdot B\rangle = \frac{1}{2}(\bar{A}B + \bar{B}A)$ and the $\mathbf{H}$-algebra (2a.4) we have

$$\langle e_iX \cdot e_jX \rangle = \delta_{ij} \quad , \quad \langle e_iX \cdot X \rangle = 0 \quad , \quad i, j = 1, 2, 3 \tag{2c.20}$$

and

$$\left\langle Xe_i \cdot Xe_j \right\rangle = \delta_{ij} \quad , \quad \left\langle Xe_i \cdot X \right\rangle = 0 \quad , \ i, j = 1, 2, 3 \ , \qquad (2c.21)$$

the { e_jX } ({ Xe_j }) therefore define tangent **H**-valued vectors $E_i^{(\pm)}(P) = E_i^{(\pm)\,\mu}(P)\, e_\mu$ at P. In components

$$E_i^{(\pm)\,m}(P) = \pm\, \varepsilon_{inm} x_n + \delta_{i\,m} x_0 \quad , \quad E_i^{(\pm)\,0}(P) = -\, x_0 \quad , \ i, n, m = 1, 2, 3. \qquad (2c.22)$$

which explicitly show that the (+) and (–) dreibeins or triads are different except at the north and south poles of S^3 ($X = \pm x_0$).

Since the metric on S^3 has the form $g^{-1} = E_i^{(\pm)} \otimes E_i^{(\pm)}$, the set { $e_j\, X$ } ({Xe_j}) forms a nowhere vanishing, global, orthonormal basis, a global Killing frame on S^3. The above construction then defines the following natural law of global parallelism : two vectors V and V', tangent to S^7 at X and X' respectively, are parallel if and only if V has the same coordinates w.r.t. the basis e_iX (Xe_i) as V' w.r.t. its basis e_iX' ($X'e_i$) i.e.

$$\left\langle V \cdot e_i X \right\rangle = \left\langle V' \cdot e_i X' \right\rangle \qquad (2c.23)$$

and

$$\left\langle V \cdot X e_i \right\rangle = \left\langle V' \cdot X' e_i \right\rangle \quad , \ i = 1, 2, 3. \qquad (2c.24)$$

More elegantly, they can be equivalently stated as

$$V\,\overline{X} = V'\,\overline{X'} \quad \text{and} \quad \overline{X}\,V = \overline{X'}\,V' \ . \qquad (2c.25)$$

An infinitesimal V at X may be parametrized as $V = [\, \exp(\eta^i e_i) - 1 \,]\, X$, η^i being the infinitesimal coordinates of V w.r.t. e_iX. Writing $Y = \exp(\eta^i e_i)\, X$, $Y' = \exp(\eta'^i e_i)\, X'$, the condition (2c.25) becomes

$$Y\,\overline{X} = Y'\,\overline{X'} \quad \text{and} \quad \overline{X}\,Y = \overline{X'}\,Y' \ . \qquad (2c.26)$$

For finite η^i and η'^i, Y and Y' are points on S^3, Eq. (2c.26) then defines the (+) and (–)

Cartan-Schouten parallelisms between two segments $\overrightarrow{XY}$ and $\overrightarrow{X'Y'}$ of two geodesics on S^3.

2.c.3. Quaternionic, H-Kählerian structures

In the last thirty years significant progress has been made on the subject of quaternionic manifolds [76]. There are several motivating forces behind their study. The Riemannian and particularly complex structures of manifolds [2, 8] have come to occupy a central place in modern differential geometry, in general relativity and more recently in supersymmetric and massless field theories [77]. The hierarchy of structures from almost hermitian to hermitian, to almost complex, to complex, to Kählerian and hyperkählerian structures testify to the richness of complex manifold theory. Just as it was natural to seek quaternionic analogs of Cauchy-Riemann analyticity in D = 4n real manifolds, it was natural to extend the above notions of (almost) complex structures to the quaternionic case of D=4n real manifolds M. In fact these two endeavors are closely intertwined when seen from the standpoint of self-duality structures in gauge theories and Einstein gravity.

Every Riemannian space M (dim M = n) is locally Euclidean with the standard inner product left invariant by O(n), the structure group of M. The *holonomy group* H, a subgroup of O(n), is the group generated by all closed loops (based at any point of M) obtained by parallel transport of the canonical connection on M. Whenever there is a reduction of the group O(n) of the orthonormal frame bundle to a proper subgroup of O(n), the corresponding spaces M and their metrics are called " special". The latter turn out to be very rich in structure and frequently they satisfy Einstein vacuum equations, whereby their relevance to physics.

A space with a defining holonomy group U(n) ≈ SU(n) x U(1) is by definition a complex Kähler manifold. One readily suspects that its **H**-counterpart, the quaternionic projective space HP^n with holonomy Sp(n) x Sp(1), should be a quaternionic Kähler manifold. This turns out to be the case. Secondly, the holonomy group of HP^n being

$Sp(n) \times Sp(1) \subset SO(4n)$ occurs in Berger's [78] short list of possible holonomy groups of irreducible Riemannian manifolds which are *not* locally symmetric. These groups are

$$SO(n),\ U(n),\ SU(n),\ Sp(n),\ Sp(n)xSp(1),\ Spin(7) \text{ and } G_2\ . \qquad (2c.27)$$

Yet, there arose a problem in many past attempts to suitably define on M a quaternionic structure [79, 80, 81, 82]. In contrast to complex manifolds, the implementation of the most general notion of transition functions in quaternionic coordinates results only in the local equivalence to HP^n. So the key question was : can a less restrictive and hence possibly richer theory of quaternionic manifolds be formulated with weaker restrictions ?

Recently, such a richer theory came about by exploiting an inherent almost complex structure. It takes roots in the work of Atiyah et al [47] on self-duality on Riemannian manifolds and on the twistorial ideas of Penrose [6], [83]. Indeed the constructions of Yang-Mills instantons on compactified spacetime $S^4 \approx HP(1)$ and that of self-dual 4-manifolds have brought to the fore, perhaps for the very first time, the essential importance of spacetime quaternions and topological mappings between quaternionic manifolds. These developments have injected greater vitality and new insights into the fundamental issues of complex and algebraic geometries. In detailing the properties of the prototype quaternionic Kähler HP^n space, we shall place it in the context of a general theory of quaternionic manifolds due to Salomon [84]. We shall assume some knowledge of fibre bundles and complex manifolds. The results will be reviewed here without proofs. Greater details are to be found in several recent papers [85, 86], in the extensive works of Salomon [86, 87, 88] and particularly in his recent beautiful book [76].

It is clear at the outset that any sensible definition of a quaternionic manifold should include as the canonical example of HP^n, the quaternionic projective space. To define the latter space, consider a quaternionic D = (n+1) linear space H^{n+1} over **H**. We label any of its points by a quaternionic vector, an ordered (n+1)-tuple, $q = (q_0, q_1, \ldots, q_n)$;

the $q_i \in \mathbf{H}$ ($i = 0, 1, ..., n$) are the homogeneous coordinates. By excluding the origin $\{0\}$ in H^{n+1}, we then have the space $*H^{n+1}$, the set $\{ q = (q_0, q_1, ..., q_n), q \neq 0 \}$. If for a nonzero $\lambda \in \mathbf{H}$, any elements q and $r = \lambda q$ of $*H^{n+1}$ are considered *equivalent*, HP^n is the resulting quotient space $*H^{n+1} / H^*$, H^* being the group of non-zero quaternions acting by *right* multiplication.

As a compact space HP^n has to be covered by at least (n+1) local open coordinate patches P_μ, $\mu = 0, ..., n$, given by the set of points $q \in HP^n$ with their $q_\mu \neq 0$. The mapping

$$(q_0, q_1, ..., q_n) \rightarrow (t_v^0, t_v^1, ..., t_v^n) \tag{2c.28}$$

with $t_v^i = q_i q_v^{-1}$ is a holomorphic isomorphism of P_μ onto $*H^{n+1}$ and define the local inhomogeneous or affine coordinate in P_μ. Indeed the transition functions of local coordinates in the overlap $P_\mu \cap P_v$ given by $t_v^h = t_\mu^h (t_v^\mu)^{-1}$ are holomorphic. These coordinates t_v^h being rational in the q_i 's are *genuine* functions on HP^n.

Next by assigning to a point of $*H^{n+1}$ the point it defines in the quotient space HP^n, we have a projection map $\pi : *H^{n+1} \rightarrow HP^n$ which defines a holomorphic line bundle. To a point of HP^n the coordinates of a point of π^{-1} are its homogeneous coordinates. With only the ratios of coordinates determined, HP^n is thus homeomorphic to the factor space, the sphere S^{4n+3} defined by $\sum_{i=0}^{n} q_i \overline{q_i} = 1$ by identification of the end points of each diameter. The map π corresponds to the maps $S^{4n+3} \rightarrow HP^n$, the 3-sphere principle bundles over HP^n or the quaternionic Hopf fibrations with fiber $S^3 \approx SU(2)$, the multiplicative group of unit quaternions.

As to various group actions, we have the symplectic group Sp(n+1) as the group of linear **H**-valued transformations acting on the q_i's from the left and preserving the constraint $\sum_{i=0}^{n} q_i\overline{q_i} = 1$. The group Sp(1) act freely on S^{4n+3} by right translation. So with the sphere $S^{4n+3} \approx \frac{Sp(n+1)}{Sp(n)}$, we have $HP^n \approx \frac{Sp(n+1)}{Sp(n) \times Sp(1)}$. With HP^n as our guide, we proceed next to a characterization of quaternionic manifolds.

Since the imaginary part of **H** is isomorphic to the Sp(1) algebra, so **H** $\approx$ **R**$\oplus$Sp(1). A real 4n-dimensional manifold M_{4n} ($n \geq 2$) is termed *almost quaternionic or almost hermitian quaternionic* if it admits a G-structure, namely if the structural group of its tangent bundle is reduced from GL(4n, **R**) to G$\approx$GL(n, **H**) x GL(1, **H**). Equivalently, we have G $\approx$ Sp(n) x Sp(1) where Sp(1) corresponds to right multiplication by imaginary quaternions. Let V be the real 3-dimensional vector bundle connected to the adjoint representation of Sp(1). So if $v \in$ V , then $v^2 = - \| v \|^2 1$, the inner product is w.r.t. the Killing form of Sp(1). Any oriented orthonormal basis **J** = { $J^i{}_\alpha{}^\beta$, i = 1, 2, 3 ; α, β =1, 2, ..., 4n } of V, which acts on each tangent space of M_{4n}, defines an almost quaternionic structure as it satisfies the quaternion algebra

$$J^i{}_\alpha{}^\beta \, J^j{}_\beta{}^\gamma = - \delta_{ij}\delta_\alpha{}^\gamma + \varepsilon_{ijk} \, J^k{}_\alpha{}^\gamma \ . \tag{2c.29}$$

Another basis **J'** will be connected to **J** by a SO(3) transformation at each point.

Following Salamon [84, 86], we define a quaternionic Kähler manifold as a Riemannian manifold M_{4n} whose holonomy group is *contained* in Sp(n) x Sp(1), a manifold endowed with an almost quaternionic structure <u>and</u> a quaternionic-Hermitian metric. The latter condition implies that $J^i{}_{\alpha\beta} = - J^i{}_{\beta\alpha}$, thus it defines a covariantly closed 2-form J^i

$$\nabla_\alpha \, J^i{}_{\beta\sigma} + \varepsilon_{ijk} A^j{}_\alpha \, J^k{}_{\beta\sigma} = 0 \tag{2c.30}$$

and a closed 4-form $\Omega_4 = J^i \wedge J^i$, $d\Omega_4 = 0$, which can be taken as a defining property of a quaternionic Kähler structure.

The local torsion free SU(2)-valued connection 1-forms A^i coincides with the Sp(1) part of the Sp(n) x Sp(1) Riemannian connection. Its curvature 2-form is a Yang-Mills field strength $F^i = dA^i + \frac{1}{2}\varepsilon_{ijk} A^j \wedge A^k$, which is simply related to the Riemann curvature of M_{4n},

$$F^i{}_{\alpha\beta} = \frac{1}{2n} J^i{}_\rho{}^\sigma R^\rho{}_{\sigma\alpha\beta} \quad . \tag{2c.31}$$

While as in the case of HP(1) $\approx S^4$ a quaternionic Kähler manifold is not necessarily a complex Kähler manifold, the above structures are quite analogous to those of complex Kähler manifolds with holonomy U(n) $\approx$ SU(n) x U(1). There the corresponding almost complex structure $J^\alpha{}_\beta$ and Hermitian metric define a *closed, covariantly constant* 2-form J, the standard defining property of complex Kähler structure. We note here the parallel with the more familiar case of complex Kählerian manifolds. Similarly the U(1) part of the SU(n) x U(1) connection has as its analog in (2c.31), a curvature given by the Ricci form $\frac{1}{2} J_\rho{}^\sigma R^\rho{}_{\sigma\alpha\beta}$.

We also remark that *hyperkähler manifolds* with defining holonomy Sp(n) are particular cases of quaternionic Kähler manifolds, for which $F^i{}_{\alpha\beta} = 0$, or equivalently, which are Ricci flat.

Just as all two dimensional manifolds are complex Kähler, all four dimensional manifolds are quaternionic Kähler. Moreover for $n \geq 2$, the holonomy condition restricts severely the properties of the Riemannian curvature: any irreducible, quaternionic Kähler manifold is necessarily Einstein , $R_{\alpha\beta} = \lambda\, g_{\alpha\beta}$. Thus there are three classes of quaternionic Kähler manifolds according to whether the associated constant scalar curvature $\kappa = 0$, $\kappa > 0$ or $\kappa < 0$.

1) For $\kappa = 0$, besides the trivial case of $R^{4n} \approx H^n$, non-trivial examples are Calabi's hyper-Kähler manifolds.

2) For $\kappa > 0$, the known ones are all symmetric Wolf spaces, they come in three infinite families of 4n-dimensional manifolds

$$HP^n \approx \frac{Sp(n+1)}{Sp(n) \times Sp(1)} \quad , X^n \approx \frac{SU(n+2)}{S(U(n) \times U(2))} \quad , Y^n \approx \frac{SO(n+4)}{S(O(n) \times O(4))} \qquad (2c.32)$$

with $X^2 \approx Y^2$, $HP^1 \approx Y^1 \approx S^4$ and $X^1 \approx CP^2$, which is also complex Kählerian. In contrast to HP^n which has an *integrable* quaternionic structure, the real and complex Grassmannian manifolds X^n and Y^n are examples of non-integrable quaternionic manifolds.

There are also 5 exceptional coset spaces

$$\frac{G_2}{SO(4)} \; , \; \frac{F_4}{Sp(3) \times Sp(1)} \; , \; \frac{E_6}{SU(6) \times Sp(1)} \; ,$$

$$\frac{E_7}{Spin(12) \times Sp(1)} \; , \; \frac{E_8}{E_7 \times Sp(1)} \; ; \qquad (2c.33)$$

their linear holonomy groups obtain via the inclusions, $Sp(3) \subset Sp(7)$, $\frac{SU(6)}{Z_3} \subset Sp(10)$, $\frac{Spin(12)}{Z_2} \subset Sp(16)$ and $E_7 \subset Sp(28)$.

3) For $\kappa < 0$, ready examples are simply the non-compact duals of the above listed $\kappa > 0$ spaces.

2.d Quaternionic Function Theory

2.d.1. Fueter's quaternion analysis

In the past century, with the realization that, besides **R** and **C**, the algebra of quaternions **H** is the only other nontrivial, real, associative division algebra, many mathematicians [89] naturally sought a distinguished class of functions on **H** with properties akin to complex analytic functions. Basically, we can define the latter in three distinct but equivalent ways [90]:

1) through the notion of a complex derivative

$$f'(z) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \quad ,$$

2) in the way of Weierstrass, through a power series expansion

$$f(z) = \sum_{m=0}^{\infty} a_m (z - z_0)^m \quad ,$$

3) finally, as solutions to the Cauchy-Riemann equations

$$\partial_{\bar{z}} f = 0 \quad .$$

However, life is richer in higher dimensions with more options opened. The above threefold equivalence *no* longer holds for quaternions and, more generally, for Clifford algebras.

While several previous attempts [91] were sterile, notable success was achieved in 1940s by R.Fueter [92, 93, 94, 95] and his Zurich school [96, 97]. Their approach extends to real quaternions the familiar Cauchy-Riemann equations through the left-analytic and right-analytic conditions, $DR = e_\mu \frac{\partial R(x)}{\partial x_\mu} = 0$, $\underset{\leftarrow}{LD} = \frac{\partial L(x)}{\partial x_\mu} e_\mu = 0$ (R, L, x $\in$ **H**), respectively. There, one can prove various theorems for the quaternionic functions R(x), L(x) such as the Taylor-Laurent expansion theorem, the Cauchy, the Morera and the Liouville theorems. Even fourfold periodic Abelian functions of a quaternionic variable were explicitly constructed.

Yet, due perhaps to the general disregard for quaternions coupled with the spectacular advances of algebraic geometry, of the theory of complex manifolds in particular, this brief chapter of hypercomplex analysis quickly eclipsed into the realm of mathematical curiosities. In fact, it seems little known even today among mathematicians and physicists.

As far as comprehensive presentations of Fueter's work, there have been very few through the years. We know of Bareiss' notes [98] of Fueter's comprehensive 1948-49 lecture course, delivered and recorded in German. In the post-war years, besides a handful publications by Fueter's students, there had been only sporadic interest on analytic functions of a quaternion [99], mostly among a few Soviet mathematicians [100, 101, 102, 103, 104, 105]. As for physical applications, the relevance of Fueter theory to Maxwellian electrodynamics was primarily discussed by K. Imaeda [106, 107, 108] in a series of papers. Henceforth, we will often refer to Fueter theory simply as quaternionic analysis.

It was not until 1979 that quaternionic analysis was recast in a more rigorous modern setting by A.Sudbery [109], who also contributed some new results. Seen today in a broader context, that theory has been subsumed under the umbrella of Clifford analysis or the general function theory over Clifford algebras [90, 110, 111].

In the following, we will follow primarily our own approach to quaternionic analysis [51]. The essential elements are covered in this and the following subsections. More details are deferred to specific physical applications later.

There have been many attempts [89, 112] at extending function theory to finite dimensional, linear associative algebras over **R** or **C**. Typically, one takes an algebra A with $x = x_i\, e_i$ as one of its elements and $\{e_i\}$ as a basis. Let $F(x) = f_i(x_1, x_2, ..., x_n)\, e_i$ be a function from A to A with the f_i's being ordinary functions of $(x_1, x_2, ..., x_n)$. Further algebraic structure of A can be incorporated into $F(x)$. For example, one may demand Hausdorff's differentiability, i.e. $dF(x) = (\partial f_i/\partial x_k)\, dx_k\, e_i = \rho_i\, dx\, \theta_i$ where ρ_i, θ_i

$\in \boldsymbol{A}$ are independent of dx. While such a condition leads to the Cauchy-Riemann equations for $\boldsymbol{A} = \mathbf{C}$, it does *not* impose enough restriction on F(x) in the more general cases of semi-simple algebras. In contrast to such fruitless attempts at a function theory over algebras, Fueter's theory [92, 93, 94, 98] is endowed with quite rich a structure. His success is due primarily to the injection into his theory of a sufficient dose of complex and algebraic structures, the **ring** structure, in particular. The basic motivation is as follows:

Let $\boldsymbol{A}$ and $\boldsymbol{A}'$ be (anti-) isomorphic algebras over the same number field with $\{e_i\}$ and $\{e'_i\}$ as their respective, isomorphic bases. Let the function $F(\mathbf{x}) = F_i(x_\alpha)\, e_i$ be defined on $\boldsymbol{A}$. It is natural to define the same function F on $\boldsymbol{A}'$ with $\mathbf{x}' = x_i e'_i$ as $F(\mathbf{x}') = F_i(x_\alpha) e'_i$. So if $\boldsymbol{A} = \boldsymbol{A}'$, an immediate restriction on F is $F(\Omega\, \mathbf{x}) = \Omega\, F(\mathbf{x})$, Ω being any (anti-) automorphism of $\boldsymbol{A}$. Such F's are called <u>intrinsic</u> functions on $\boldsymbol{A}$.

First, let $\boldsymbol{A} = \mathbf{C}$ with its canonical basis (1, i) and $z = x + i\, y \in \mathbf{C}$. Consider the function $F(z) = u(x,y) + i\, v(x,y)$. Since conjugation is the only involutive (anti-) automorphism of $\mathbf{C}$, the automorphism group of $\mathbf{C} \approx \mathbf{Z}_2$, $\Omega(x + i\, y) = z^* = x - i\, y$. Then the conditions for F(z), being intrinsic, are: a) that $F(z^*)$ be defined whenever F(z) is defined, b) the Schwarz's reflection principle holds, namely $[\, F(z)\,]^* = F(z^*)$. For $\boldsymbol{A} = \mathbf{H}$, the group G of automorphisms and anti-automorphisms consists of all linear transformations leaving $e_0 = 1$ fixed and effecting an orthogonal transformation on the vector space V_3, spanned by the basis (e_1, e_2, e_3), a subspace of $\mathbf{H}$. The 4-space $\mathbf{H}$ has the natural topology of a direct sum of a 1-dimensional space V_1 and a 3-dimensional space V_3, isomorphic to a R^3. An automorphism induces a rigid rotation of V_3 while an antiautomorphism produces a rotation plus a reflection. So an element of G is either a SO(3) rotation or a rotation plus a reflection, of the space $\mathbf{H}$ about the real axis, leaving the latter axis fixed.

2.d.2. H-holomorphic functions from C-analytic functions

Any quaternion $x = x_i e_i \in \mathbf{H}$ has the decomposition $x = x_0 + \mu\, r$, $\mu\, r \in V_3$ with the unit quaternion $\mu \equiv \frac{\mathbf{e}\cdot\mathbf{r}}{r}$, $r = N(\mathbf{e}.\mathbf{x}) = (\mathbf{x}^2)^{1/2} > 0$. Since $\mu^2 = -1$, μ is the 3-space analog of the complex imaginary unit $i = \sqrt{-1}$, i.e. it defines a *complex structure* in R^4.

Let $F(x) = f(x_0, \mathbf{x}) + \mu\, g(x_0, \mathbf{x})$ be an intrinsic function on $\mathbf{H}$ of domain H [89]. Next, let $x^c = x_0{}^c + \mu^c r^c$ be a constant of $\mathbf{H}$, $r^c \neq 0$. Any element $g \in G$ then generates a rotation about the axis μ^c leaving x^c invariant, i.e. $g\, F(x^c) = F(g\, x^c)$ = a constant. We have $\mu^c = \pm\, \mu$ since the only vectors in $\mathbf{H}$ left fixed by all such transformations g are either parallel or antiparallel to μ^c. Under a mapping of an intrinsic function, the unit vector of the argument is thus invariant.

For fixed x_0 and for any $\Omega \in G$, we necessarily have

$$\begin{aligned}\Omega\, F(x) &= f_0(x_0, \mathbf{x}) + g(x_0, \mathbf{x})\, \Omega\mu \\ &= F(\Omega x) = f_0(x_0, \mathbf{x}') + g(x_0, \mathbf{x}')\, \Omega\mu \ .\end{aligned} \tag{2d.1}$$

Therefore, for fixed x, f_0 and g_0 must be invariant functions of $\mu\, r = \mathbf{e}.\mathbf{x}$ and so a function of r only, r being a complete set of SO(3) invariants for $\mathbf{H}$. It follows that F(x) must be of the form

$$F(x) = F(x_0 + \mu\, r) = u(x_0, r) + \mu\, v(x_0, r) \ . \tag{2d.2}$$

Moreover, under the anti-automorphism $\Gamma: \mu \rightarrow -\mu$, the condition $F(\Gamma\, x) = \Gamma\, F(x)$ implies that $u(x_0, -r) = u(x_0, r)$ and $v(x_0, -r) = -v(x_0, -r)$. Therefore the function $u(x_0, r) + \mu\, v(x_0, r)$ is an intrinsic function of the complex variable $z = x_0 + i\, r$. F(x) is also called a *primary* function since it arises from the extension of a given scalar function, a *stem function* F(z) of a complex variable z.

It can be readily proved that a) the classes of intrinsic functions and primary functions on **H** are in fact identical, b) if $F(z) = u(x_0, r) + i\, v(x_0, r)$ is the complex valued stem function; then $F(x)$, the corresponding primary extension to **H**, is given by $F(x_0 + \mu r) = u(x_0, r) + \mu v(x_0, r)$. Finally, we may add that, in its general formulation, the above theory of intrinsic functions works not just for associative algebras, but it is extendable to the nonassociative case of octonions, as well as to arbitrary rings. Such an octonionic extension will be the subject of a later discussion.

Having summarized the key elements of the theory of intrinsic functions on algebras, we proceed specifically to Fueter's theory of function over a quaternion [94, 98]. Due to the latter's relevance to self-dual structure in field theories, we begin our presentation by way of some special quaternionic power series.

It is well-known that relativistic field theories have their euclidean continuation in R^4 through a Wick rotation $ct \rightarrow x_0 = ict$. Furthermore, under appropriate field boundary conditions, R^4 is effectively compactified into S^4, the 4-dimensional sphere, by the addition of the points at infinity. S^4 can be viewed as the coset manifolds $\frac{SO(5)}{SO(4)}$ or $\frac{Spin(5)}{Spin(4)} \approx \frac{Sp(2)}{Sp(1)xSp(1)}$ while Minkowski space may be represented by the coset space $\frac{SO(4,\,1)}{SO(3,\,1)}$, of the Poincaré group over the Lorentz group.

Since the projective quaternionic space HP_n is realized by the coset space $\frac{Sp(n+1)}{Sp(n)xSp(1)}$, we have the identification $S^4 \approx HP_1$, the quaternionic projective line. According to Sect.2c.3, this makes explicit the quaternionic structure of S^4, the **H**-counterpart of the Riemann sphere $S^2 \approx SU(2) / U(1) \approx CP_1$, the complex projective line. Now S^4 does not have a complex structure, since a celebrated theorem states that, of all spheres S^n, only S^2 and S^6 have complex and almost complex structure, respectively. As we will see, this corollary to Adams' theorem, uniquely tied to the existence of the four division algebras over **R**, lies at the basis of the complex structure and analyticity of some special classes of quaternionic and octonionic Fueter holomorphic functions in four and eight dimensions, respectively.

A point on S^2 is compactly represented by a complex number $z = x + iy$, likewise a point on S^4 is represented by a real quaternion $x = e_\mu x^\mu$. This position quaternion x transforms as a 4- vector, namely as a $\left(\frac{1}{2}, \frac{1}{2}\right)$ representation of the euclidean Lorentz group SO(4) ≈ Sp(1) x Sp(1). Under SO(4), the transformed of x, $x' = m\, x\, \bar{n}$, ($|m| = |n| = 1$). Let y be another 4-vector, then

$$\overline{x'}\, y' = n\, \bar{x}\, y\, \bar{n} \quad , \; x'\, \overline{y'} = m\, x\, \bar{y}\, \bar{m} \; . \tag{2d.3}$$

It readily follows that scalar parts of these bilinears

$$Sc(\overline{x'}\, y') = Sc\, (x'\overline{y'}) = Sc(x\bar{y}) = x_\mu y_\mu \; , \tag{2d.4}$$

are O(4) invariant, belonging to the (0, 0) representation of O(4), while their vectorial parts

$$Vec(\bar{x}y) = \frac{1}{2}\, e_{\nu\mu} x_{\nu\mu} = \frac{1}{4}\, e_{\nu\mu}\left(x_{\nu\mu} + \tilde{x}_{\nu\mu}\right) \; , \tag{2d.5}$$

$$Vec(x\bar{y}) = \frac{1}{2}\, e'_{\nu\mu} x_{\nu\mu} = \frac{1}{4}\, e'_{\nu\mu}\left(x_{\nu\mu} - \tilde{x}_{\nu\mu}\right) \tag{2d.6}$$

give a self-dual antisymmetric (0, 1) tensor and an antiself-dual (1, 0) tensor, respectively, with $x_{\mu\nu} \equiv x_\mu y_\nu - x_\nu y_\mu$ and $x_{\mu\nu} \equiv \frac{1}{2}\epsilon_{\mu\nu\alpha\beta} x^{\alpha\beta}$. They transform under their respective SU(2) as

$$\left(Vec(\overline{x}'y') = n\; [Vec(\bar{x}y)]\; \bar{n}\right) \qquad \text{and} \qquad Vec(x'\overline{y'}) = m\; [Vec(x\bar{y})]\; \bar{m} \; . \tag{2d.7}$$

Contraction with $e_{\mu\nu}$ and $e'_{\mu\nu}$ selects the self-dual and the antiself-dual parts of $x_{\mu\nu}$, respectively. As we will see, the above transformation properties hold the key to the O(4) covariantization of Fueter's theory and to the O(4) covariant construction of chiral coordinates in four dimensions. These coordinates are the natural D = 4 counterparts of D = 2 analytic (left-moving) and anti-analytic (right-moving) coordinates, familiar in string and conformal field theories. As such, they provide the proper variables on which to formulate a D = 4 analog of the Virasoro algebra.

The conformal SO(5, 1) ≈ SL(2, **H**) transformations of R^4 are given by the special linear group in 2-dimensional quaternionic space (q_1, q_2)

$$q' = \begin{pmatrix} q'_1 \\ q'_2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} q'_1 \\ q'_2 \end{pmatrix} \\ = A\, q\,. \tag{2d.8}$$

With the quaternionic 2-spinor q cast in a (2 x 2) matrix form, A becomes, in the Pauli representation, a (4 x 4) complex matrix with unit determinant. Then the quaternion x admits the projective representation $x = q_1\, q_2^{\,-1}$ with q_1 (1/2, 0) and q_2 (0, 1/2) being quaternionic spinors. They transform like $q'_1 = m\, q_1$ and $q'_2 = n\, q_2$. The 15-parameter group SL(2, **H**) then acts nonlinearly on x via the Mobius map:

$$x' = q'_1\, (q'_2)^{-1} = \frac{(ax + b)}{(cx + d)} = a\,(x + a^{-1} b\,)\,(x + c^{-1} d)^{-1}\, c^{-1}\,, \tag{2d.9}$$

$$\det A = \,|\, c\,|^2 \,|\, a c^{-1} d - b\,|^2 = 1\ , \tag{2d.10}$$

a, b, c, d being real constant quaternions. Alternatively, this linear fractional transformation, a rational non-linear realization of SL(2, **H**), reads

$$x' = \alpha + \beta\,(x - s)^{-1} \gamma\ , \tag{2d.11}$$

$\alpha = ac^{-1}$, $\gamma = c^{-1}$, $s = -\,c^{-1}d$ and $\beta = b - a\, c^{-1}d$. In turn the factor $(x - s)^{-1}$ may be written as

$$(x - s)^{-1} = -\, s^{-1} \sum_{n=0}^{\infty} \xi^n = \sum_{n=0}^{\infty} \eta^n s^{-1}\ , \tag{2d.12}$$

a power series in $\xi = x\, s^{-1}$ or $\eta = s^{-1}\, x$. It generalizes to the series

$$B(x) = \sum_{n=0}^{\infty} \overline{c_n}\,(x\,\overline{c_n})^n \quad , \; c_n \in \mathbf{H}\,. \tag{2d.13}$$

We note in passing that if c_n transform like x under SO(4), the euclidean Lorentz and norm group of **H**, and if $x'' = v_1 x v_2$ and $c_n'' = v_1 c_n v_2$ ($|v_1| = |v_2| = 1$), then $B'' = v_1 B v_2$ and B is also a 4-vector. In fact, for physical applications, the O(4) covariance of the hypercomplex function theory is of paramount importance. It will be the focal point of the next subsection.

Our first observation is that the product of any two series of type B(x) (2.d.13) does **not** result in a third series of the same kind, namely they do not form a ring. This is to be contrasted with the cases of real and complex power series. However, if all the c_n's are restricted to be *scalar* quaternions, i.e. $\mathrm{Vec}(c_n) = 0$, then the Weierstrassian series

$$W(x) = \sum_{n=0}^{\infty} \alpha_n x^n \;\; , \quad \mathrm{Vec}(\alpha_n) = 0 \tag{2d.14}$$

do form a ring. The resulting functions are therefore "intrinsic" in the sense discussed above. They can be generated from their "stem functions", i.e. from complex analytic functions in the upper half plane $z = x_0 + i\,r$, $r = (\mathbf{x}^2)^{1/2} \geq 0$. Since they leave the real axis invariant and obey Schwarz's reflection principle $\overline{f(x)} = f(\bar{x})$, these **H**-intrinsic functions generalize ordinary C-analytic functions.

To be more specific about intrinsic functions, let us consider the following useful projection operators

$$E_{\pm} = \frac{1}{2}(\,1 \pm i\mu\,)\;, \tag{2d.15}$$

$$E_{\pm}^2 = E_{\pm},\, E_+ + E_- = I\,,\;\; E_+ E_- = E_- E_+ = 0\;\;, \tag{2d.16}$$

where μ, defined previously, is a unit normed, vectorial quaternion.

With $z = x_0 + i\,r$ ($r \geq 0$), a point in the Poincaré upper-half plane, the quaternion x has the decomposition $x = zE_- + zE_+$. Since $x^m = z^m E_- + z^{*m} E_+$, W(x) has the corresponding decomposition

$$\begin{aligned} W(x) &= W(z)\, E_- + W(\bar{z})\, E_+ \\ &= \mathrm{Re}\, W(z) + \mu\, \mathrm{Im}\, W(z) \quad . \end{aligned} \tag{2d.17}$$

Therefore, to any W(x) corresponds uniquely the **C**-analytic stem function W(z). We shall see that just such a function W(x) arises in the 't Hooft, the Witten-Peng instantons and the BPS monopole solutions and thus reveals their underlying quaternionic structure.

Admittedly, the above class of intrinsic functions W(x) is quite special. Due to the non-commutativity of **H** and thereby to the necessary distinction between left and right multiplication for quaternions, there is in fact a threefold generalizations of Eq. (2d.14). They are the series:

$$L(x) = \sum_{n=0}^{\infty} \xi^n a_n \quad , \tag{2d.18}$$

$$R(x) = \sum_{n=0}^{\infty} b_n \xi^n \quad , \tag{2d.19}$$

and

$$C(x) = \sum_{n=0}^{\infty} b_n \xi^n a_n \quad . \tag{2d.20}$$

a_n and b_n are quaternionic valued coefficients with $\xi = \bar{c}x$ and $\eta = x\bar{c}$, with $\xi = x$ as a special case. While B(x) is both left and right holomorphic, L(x), R(x) and C(x) are left, right and cross- holomorphic, respectively. Through the involution $x \to \bar{x}$, these series become anti-holomorphic while functions with both arguments x and $\bar{x}$ are called mixed holomorphic. Instead of the above Taylor series, we could just as easily consider Laurent series (i.e. with negative and positive n).

As to the most general polynomial function over quaternions,

$$p(x) = a_0 + a_1x + xa_2 + a_3xa_4 + b_1x^2 + x^2b_2 + b_3x^2b_4 + b_5xb_6xb_7 + \ldots, \qquad (2d.21)$$

it consists of a finite sum of generic **H**-monomials $m(x) = \alpha_0 x \alpha_1 x \ldots \alpha_{s-1} x \alpha_s$. The index s denotes a non-negative integral degree. α_i, $i = 0, 1, \ldots, s$, are constant quaternions.

With an eye for hypercomplex analytic structure, the general polynomial p(x) is actually not so interesting because it has no quaternionic holomorphic property. To see this, the following observation suffices. By repeated use of the identity $e_\mu\, q\, e_\mu = -2\bar{q}$ (2a.38e), antiholomorphic functions can be generated from holomorphic ones by inserting suitable quaternionic coefficients within the monomials. For a simple proof, let us take any quaternionic polynomial $p(x) = p_\mu e_\mu$ with their components p_μ ($\mu = 0, 1, 2, 3$) being real valued polynomials in x_0, x, y and z. By the reconstruction theorem and upon substituting the arguments $x_\mu = Sc(e_\mu x)$ back into p_μ, we obtain a sum of quaternionic monomials m(x) i.e. a general quaternionic polynomial. So p(x) is nothing more than the quaternionic representation of a general real analytic general coordinate transformations in R^4. Yet, as we will show, through further refinements of the Fueter theory, there does exist a quaternionic (anti-) holomorphic structure associated with a suitable infinite *subset*, of general coordinate transformations. The details of such a *Fueter group* of transformations will occupy us in the next subsection.

To seek defining equations for the above holomorphic functions à la Cauchy-Riemann, we consider Hamilton's left and right acting differential operators $\vec{D} = e_\mu \vec{\partial}_\mu$ and $\overleftarrow{D} = e_\mu \overleftarrow{\partial}_\mu$ and their conjugates (or adjoints) $\bar{\vec{D}} = \bar{e}_\mu \vec{\partial}^\mu$, $\bar{\overleftarrow{D}} = \bar{e}_\mu \overleftarrow{\partial}^\mu$. Then $\bar{D}\, D = D\, \bar{D} = \Box$ is the 4-dimensional Laplacian. Thus Fueter defines left, right and left-right analytic functions as those which obey the quaternionic Cauchy-Riemann equations:

$$\overrightarrow{D} L = 0 \ , \ R \overleftarrow{D} = 0 \ , \qquad\qquad (2d.22)$$
$$\overrightarrow{D} B = B\overleftarrow{D} = 0 \ ,$$

respectively.

To see the close parallel with complex analysis, we look at another form of (2d.22). Let $\mathbf{H} = \mathbf{C} \times \mathbf{C}$, then $x = z_1 + z_2 e_2$ with $z_1 = x_0 + e_1 x_1$ and $z_2 = x_2 + e_1 x_3$. This splitting of a quaternion x into two complex numbers reflects the isomorphic representation of quaternions as (2 x 2) matrices over $\mathbf{C}$:

$$x = \begin{pmatrix} z_1 & z_2 \\ -z_2^* & z_1^* \end{pmatrix} . \qquad (2d.23)$$

The superscript * denotes complex conjugation. For real z_1 and z_2, Eq. (2d.23) reduces to the matrix form of a complex number. The $\mathbf{H}$-multiplication rule translates into

$$(z_1 + z_2 e_2)(w_1 + w_2 e_2) = z_1 w_1 - z_2 \overline{w_2} + (z_1 w_2 + z_2 \overline{w_1}) e_2 \ . \qquad (2d.24)$$

So Eq. (2d.23) makes a quaternion a hypercomplex number; it illustrates the so called Cayley-Dickson construction.

Defining the quaternion conjugate to q as $\overline{x} = z_1^* - z_2 e_2$, we may rewrite [104, 105, 109]

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial z_1} - \frac{\partial}{\partial z_2} e_2 \qquad , \qquad \frac{\partial}{\partial \overline{x}} = \frac{\partial}{\partial \overline{z_1}} + \frac{\partial}{\partial z_2} e_2 \qquad (2d.25)$$

in terms of the usual operators of complex differentiation. The D = 4 Laplacian operator is $\frac{\partial^2}{\partial x \partial \overline{x}} = \frac{\partial^2}{\partial z_1 \partial \overline{z_1}} + \frac{\partial^2}{\partial z_2 \partial \overline{z_2}} = \frac{1}{4} \Box$, the box operator.

A function $f(x) = f_1 + f_2 e_2$ is quaternion holomorphic if

$$\frac{\partial f}{\partial \bar{x}} = 0, \tag{2d.26}$$

or

$$\frac{\partial f_1}{\partial \overline{z_1}} = \frac{\partial \bar{f}_2}{\partial \overline{z_2}} \quad , \quad \frac{\partial f_2}{\partial \overline{z_1}} = -\frac{\partial \bar{f}_1}{\partial \overline{z_2}} \ . \tag{2d.27}$$

When f_1 and f_2, z_1 and z_2 are all real, these equations reduce, as they should, to the complex Cauchy-Riemann equations.

Returning to the **H**-analytic function $F(x) = F_\mu e^\mu$ defined by $\mathcal{D}F = 0$ or $F\overleftarrow{\mathcal{D}} = 0$. In the special case of pure vectorial F and x, $\mathcal{D}F = 0$ reduces to the vectorial equation $\nabla \cdot (\mathbf{e} \cdot F) = \nabla \times (\mathbf{e} \cdot F) = 0$. The latter only differs from the gradient of harmonic functions in three real variables in the symmetry property $(F_1, F_2, F_3) \rightarrow (F_1, F_2, -F_3)$.

As to the Fueter series, the left, right, left-right and cross-holomorphic functions L(x), R(x), B(x) and C(x) can be easily shown to be all biharmonic, i.e. they all solve the generic equation:

$$\Box\,\Box\,\Phi = 0 \ . \tag{2d.28}$$

Moreover, they satisfy third order linear equations:

$$\overrightarrow{\mathcal{D}}\,\Box\,L = 0 \quad , \quad \Box\,R\,\overleftarrow{\mathcal{D}} = 0 \tag{2d.29}$$

$$\Box\,\overrightarrow{\mathcal{D}}B(x) = \Box\,B(x)\,\overleftarrow{\mathcal{D}} = 0 \quad . \tag{2d.30}$$

For example, the reader is encouraged to check [93] that $\Box\,\Box\,x^n = 0$ and $\Box\,\Box\left\{\frac{ax+b}{cx+d}\right\} = 0$.

Mere inspection of Eqs. (2d.29)-(2d.30) suggests an easy construction of left, right and left-right Fueter analytic functions from these holomorphic functions. They are given by

$$l(x) = \Box L(x) \ , \ r(x) = \Box R(x) \ , \quad b(x) = \Box B(x) \ . \tag{2d.31}$$

Similarly, cross-holomorphic functions are used indirectly. In fact, the following combinations:

$$l(x) = \Box \overrightarrow{D} c(x) \qquad , \quad r(x) = \Box c(x) \overleftarrow{D} \ , \tag{2d.32}$$

are left and right analytic. Two explicit examples are:

$$l_1(x) = \Box \overrightarrow{D} \sum_{n=0}^{\infty} (a_n x + b_n)(c_n x + d_n)^{-1} \tag{2d.33}$$

and

$$l_2(x) = \Box (x - \beta_n)^{-1} \gamma_n \ . \tag{2d.34}$$

The proof that $b(x) = \Box\, \omega(x)$ with $\omega(x)$ being an intrinsic function (2d.14) can be done simply. We need the relations

$$\overrightarrow{D}(x_0 \pm ir) = 2 E_{\pm} \ , \tag{2d.35a}$$

$$\overrightarrow{D} E_{\pm} = \pm \frac{(-i)}{r} \ . \tag{2d.35b}$$

They allow $\overrightarrow{D}\, \omega(x)$ to be rewritten as

$$\overrightarrow{D}\, \omega(x) = \left(\overrightarrow{D}\, \omega_+(x)\right) E_- + \left(\overrightarrow{D}\, \omega_-(x)\right) E_+ + \frac{2i}{r}\left(\omega_+(x) - \omega_-(x)\right) \ . \tag{2d.36}$$

Since $\overrightarrow{D}\, \omega_{\pm}(x) = \omega'_{\pm} \overrightarrow{D}(x_0 \pm i\, r) = 2\omega'_{\pm} E_{\pm}$ and by Eq. (2d.16) we get

$$\vec{D}\,\omega(x) = \frac{i}{r}\,[\omega(x_0 + ir) - \omega(x_0 - ir)\,] \quad . \tag{2d.37}$$

However, as is familiar from the theory of wave propagation

$$\Box\left[\frac{\omega(x_0 + ir) - \omega(x_0 - ir)}{r}\right] = 0 \quad . \tag{2d.38}$$

Therefore

$$\Box\vec{D}\,\omega = \vec{D}\,\Box\,\omega = 0 \ . \tag{2d.39}$$

Similarly,

$$(\Box\,\omega)\overleftarrow{D} = 0 \tag{2d.40}$$

and $\omega(x)$ is biharmonic, $\Box\,\Box\,\omega = 0$.

The above definition of **H**-analyticity, Eq. (2d.22), provides a D = 4 generalization of the Cauchy-Morera integral theorem. To see this, we fist consider the system (2d.22) in a bounded domain Δ of R^4 with a smooth 3-dimensional boundary $\Sigma \approx S^3$. The associated quaternionic area element is $d\Sigma = e^\mu d\sigma_\mu$. By way of the Gauss-Ostrogradskii-Green formula

$$\iiint_\Sigma d\Sigma\, l\,(x) = \iiiint_\Delta D\, l\,(x) d^4x \ , \tag{2d.41}$$

Fueter's left-analyticity in Eq. (2d.22) translates into

$$\oiiint_{S^3} d\Sigma\, l\,(x) = 0 \ , \tag{2d.42}$$

and $\oiiint_{S^3} r\,(x)\, d\Sigma = 0$ for right analytic functions. There exists a unified integral formula for both left and right analytic functions:

$$\oiiint_{\Sigma} r\,(x)\, d\Sigma\, l\,(x) = \oiiiint_{\Omega} \left(r\; D\, l + \overline{(\overline{D\, r}\,)}\, l\right) d^4x \;= 0\ . \qquad (2d.43)$$

As an application, we take as a kernel the left and right analytic function

$$r\,(x) = -\frac{1}{4}\,\Box\,(\,x - \rho\,)^{-1} = |x - \rho\,|^{-2}\,(\,x - \rho\,)^{-1}\ . \qquad (2d.44)$$

We now enclose a point ρ in Δ by a small 3-sphere of radius R of boundary Ξ while x is a moving point on Σ and σ. So if $l(x)$ is analytic in the region (Σ + Ω), Eq. (2d.43) becomes

$$\iiint_{\Sigma} \frac{1}{x-\rho}\,\frac{d\Sigma}{|x-\rho|^2}\, l\,(x) + \iiint_{\Xi} \frac{1}{x-\rho}\,\frac{d\Sigma}{|x-\rho|^2}\, l\,(x) = 0\ . \qquad (2d.45)$$

Since on Ξ, $d\Sigma = -\,(x-\rho)|\,x-\rho|^{-1}|d\Sigma|$ with $\iiint_{\sigma} |d\Sigma| = 2\pi^2 R^3$, as $R \to 0$, and making use of Eq. (2d.44), we obtain the **H**-counterpart of Cauchy's integral theorem or the Fueter-Moisil fundamental formula:

$$l\,(\rho) = \frac{-1}{8\pi^2} \oiiint_{S^3} \Box\,\frac{1}{x-\rho}\, d\Sigma\, l\,(x)\ . \qquad (2d.46a)$$

Similarly,

$$r\,(\rho) = \frac{-1}{8\pi^2} \oiiint_{S^3} r\,(x) d\Sigma\, \Box\,\frac{1}{x-\rho}\ . \qquad (2d.46b)$$

We pause to note the pleasing one-to-one correspondence

$$2\pi \leftrightarrow 8\pi^2 \quad , \; i\,dz \leftrightarrow d\Sigma, \; (z-\xi) \leftrightarrow \square\,(x-\rho)^{-1} \tag{2d.47}$$

with the Cauchy formula of complex analysis

$$f(\xi) = \frac{1}{2\pi i}\oint_{S^1} \frac{f(z)\,dz}{z-\xi} \; . \tag{2d.48}$$

Next, in setting r(x) = 1, Eq. (2d.46b) gives

$$\frac{1}{8\pi^2}\oiiint_{S^3} d\Sigma\,\square\,(\xi - x)^{-1} = 1 \tag{2d.49}$$

or the residue of the function $(\xi\text{-}x)^{-1}$ at its pole. By way of the identity

$$(a - x)^{-1} = \frac{1}{2}\bar{D}\ln|a-x|^2 \;, \tag{2d.50}$$

Eq. (2d.49) gives

$$\frac{1}{16\pi^2}\oiiint_{S^3} d\Sigma\,\square\,\bar{D}\ln|a-x|^2 = 1 \;, \tag{2d.51}$$

a useful expression in a later application to 't Hooft instantons.

In light of the Cayley-Dickson construction, Eqs. (2d.23)-(2d.27), we may naively expect that a function theory of a quaternion should share more features with the theory of two complex variables. However, one finds that a basic difference exists between them: unlike the holomorphism condition for the latter theory, the Fueter equations (2d.22) are <u>not</u> overdetermined. Rather, as we have seen, the function theory of one quaternionic variable actually resembles in many respects the function theory of a single complex variable.

2.d.3. Fourfold periodic Weierstrassian functions

One of the most important properties of circular functions f(z), e.g. sinz, tanz, etc. is their periodicity : f(z + 2nπ) = f(z) for all integer n. If ω_1 and ω_2 are two real or complex numbers whose ratio is not purely real, then a function f(z) obeying f(z + 2ω_1) = f(z) and f (z + 2ω_2) = f(z) is a doubly periodic function of z, with periods 2ω_1 and 2ω_2. Such a f(z) which is holomorphic in the finite plane is an *elliptic function.* A canonical example is the Weierstrass function [113]

$$P\ (z) = \frac{1}{z^2} + \sum_{m,n}{}' \left\{ \left(z - \Omega_{m,n} \right)^{-2} - \Omega_{m,n}^{-2} \right\} \tag{2d.52}$$

where $\Omega_{m,n} = 2\, m\omega_1 + 2\, n\, \omega_2$, m and n being integers. While P(z) is an even function of z, another Weierstrassian function, the Zeta function ζ(z) defined by

$$P\ (z) \equiv -\frac{d\zeta(z)}{dz} \tag{2d.53}$$

is odd in z, $\zeta(-z) = -\zeta(z)$ and is *quasi-periodic,* namely

$$\zeta(\, z + 2\omega_2) = \zeta(z) + 2\,\eta_2 \qquad \text{and} \qquad \zeta(\, z + 2\omega_1) = \zeta(z) + 2\,\eta_1\ , \tag{2d.54}$$

η_1 and η_2 being complex constants. In particular, we have the Legendre relation:

$$\eta_1\omega_2 - \eta_2\omega_1 = \frac{i\,\pi}{2}\ . \tag{2d.55}$$

As part of his function theory over **H**, Fueter also extended the above ζ function to the quaternion algebra [98, 114]. We give below our compact reformulation of his construction.

Consider the specific **H**-valued power series

$$F_q = \frac{1}{x-q} + \frac{1}{q} + \left(\frac{1}{q} x \frac{1}{q}\right) + \left(\frac{1}{q} x \frac{1}{q} x \frac{1}{q}\right) + \left(\frac{1}{q} x \frac{1}{q} x \frac{1}{q} x \frac{1}{q}\right) + \ldots \qquad (2d.56)$$

where $q = n_0 q_{(0)} + n_1 q_{(1)} + n_2 q_{(2)} + n_3 q_{(3)}$, n_0, n_1, n_2, n_3 are integers and q_0, q_1, q_2, q_3 are linearly independent, fixed quaternions i.e. if $q^{(\alpha)} = q^{(\alpha)}{}_\beta\, n_\beta$, then $\det\left(q_\beta^{(\alpha)}\right) \neq 0$.

By way of the series expansion

$$\frac{1}{x-q} = -\frac{1}{q} \sum_{m=0}^{\infty} (x\, q^{-1})^m \qquad (2d.57)$$

$F_q(x)$ can be recast as

$$F_q(x) = -\frac{1}{q}\left[(xq^{-1})^4 + (xq^{-1})^5 + ..\right] \qquad (2d.58)$$

$$= \left(\frac{1}{q} x\right)^4 \frac{1}{(xq^{-1} - 1)\, q} = (q^{-1}x)^4 \frac{1}{x-q} \quad . \qquad (2d.59)$$

With $F_q(x)$ being Fueter holomorphic, the function $f_q(x) = \Box\, F_q(x)$ is then left-right Fueter analytic, $Df_q(x) = f_q(x) \overleftarrow{D} = 0$. This property is readily verified from the following form of $f_q(x)$:

$$f_q(x) = 4\left\{ \frac{1}{|q-x|^2} \frac{1}{q-x} - \frac{1}{|q|^2} \frac{1}{q} - \frac{2}{|q|^2} \frac{1}{q} x \frac{1}{q} - \frac{1}{|q|^4} \bar{x} \right\} \qquad (2d.60)$$

which is obtained after some algebra.

Next we define after Fueter the quaternionic $\zeta(x)$ function as

$$\zeta(x) = q_{000}(x) + \sum_q{}' \zeta_q(x) \qquad (2d.61)$$

where

$$\zeta_q(x) = q_{000}(z+q) - q_{000}(q) + q_{100}(q)\, p_{100}(x) + q_{010}(q)\, p_{010}(x) + q_{001}(q) p_{001}(x) \tag{2d.62}$$

with

$$q_{000}(q) = -\frac{4}{|q|^2}\frac{1}{q} \quad , q_{100}(q) = -\frac{\partial q_{000}(q)}{\partial q_1} \ ,$$

$$q_{010}(q) = -\frac{\partial q_{000}(q)}{\partial q_2} \quad , q_{001}(q) = -\frac{\partial q_{000}(q)}{\partial q_3}$$

$$p_{100}(x) = x_1 - e_1\, x_0 \ , \ p_{010}\,(x) = x_2 - e_2\, x_0 \ , p_{001}(x) = x_3 - e_3\, x_0 \ . \tag{2d.63}$$

Explicitly,

$$q_{100}(x) = -\frac{4}{|x|^2}\left(2\,\frac{1}{x}\,e_1\frac{1}{x} - \frac{e_1}{|x|^2}\right) , \tag{2d.64}$$

$$q_{010}(x) = -\frac{4}{|x|^2}\left(2\,\frac{1}{x}\,e_2\,\frac{1}{x} - \frac{e_2}{|x|^2}\right) , \tag{2d.65}$$

$$q_{001}(x) = -\frac{4}{|x|^2}\left(2\,\frac{1}{x}\,e_3\,\frac{1}{x} - \frac{e_3}{|x|^2}\right) , \tag{2d.66}$$

$$q_{100}(q)\; p_{100}(x) = -\frac{4}{|q|^2}\left(2\,\frac{1}{q}\,e_1\,\frac{1}{q} - \frac{e_1 x_1}{|q|^2}\right) + \frac{4}{|q|^2}\left(2\,\frac{1}{q}\,e_1\,\frac{1}{q}e_1\; x_0 + \frac{1}{|q|^2}\right)x_0 \ . \tag{2d.67}$$

Hence

$$R(\,q,\,x) \equiv \; q_{100}(q)\, p_{100}(x) + q_{010}(q)\, p_{010}(x) + q_{001}(q)\, p_{001}(x)$$

$$= \; -\frac{4}{|q|^2}\left(2\,\frac{1}{q}\mathbf{e}\cdot\mathbf{x}\,\frac{1}{q} - \frac{\mathbf{e}\cdot\mathbf{x}}{|q|^2}\right) + \frac{4}{|q|^2}\left(2\,\frac{1}{q}e_n\,\frac{1}{q}\,e_n x_0 + \frac{3}{|q|^2}\,x_0\right) . \tag{2d.68a}$$

Using the identity $e_\mu \frac{1}{q}\, e_\mu = -\frac{2}{\bar{q}}$ and $e_n\frac{1}{q}\, e_n = \; -\frac{2}{\bar{q}} - \frac{1}{q}$, we further simplify R(q,x) to

$$R(q,\,x) = \frac{-4}{|q|^2}\left(2\,\frac{1}{q}\,x\,\frac{1}{q} + \frac{1}{|q|^2}\,\bar{x}\right) . \tag{2d.68b}$$

Then

$$\zeta_q(x) = -\frac{4}{|x+q|^2}\frac{1}{x+q} + \frac{4}{|q|^2}\frac{1}{q} - \frac{4}{|q|^2}\left(2\frac{1}{q}x\frac{1}{q} + \frac{1}{|q|^2}\bar{x}\right) . \tag{2d.69a}$$

On the other hand,

$$\zeta_q(x) = f_{-q}(x) = \Box F_{-q}(x) \tag{2d.69b}$$

and since $\sum_q \zeta_q = \sum_q \zeta_{-q}$, alternatively

$$\zeta = \Box Z(x) = \Box\left(\frac{1}{x} + \sum_q{}' F_q(x)\right) \tag{2d.70}$$

with

$$F_q(x) = \frac{1}{x-q} + \sum_{\alpha=0}^{3} q^{-1}(xq^{-1})^\alpha = (q^{-1}x)^4 \frac{1}{x-q} \tag{2d.71}$$

and $D\zeta(x) = \Box DZ(x) = 0$. Thus $\zeta(x)$ is right analytic and is odd $\zeta(-x) = -\zeta(x)$ since as is clear from their explicit expressions above, $q_{000}(x)$ is odd while $q_{100}(x)$, $q_{010}(x)$ and $q_{001}(x)$ are even. The series $\zeta(x)$ can be shown [98] to be absolutely and uniformly convergent in every finite domain containing no lattice point.

As Fueter showed, resulting from the integral theorem there exists the generalized Legendre relation

$$\frac{1}{8\pi^2}\oiiint_{\partial C = S^3} d\Sigma\, \zeta(x) = \frac{1}{8\pi^2}\oiiint_{\partial C = S^3} d\Sigma\, \Box Z(x) = \frac{1}{8\pi^2}\oiiint_{\partial C = S^3} d\Sigma \left(\Box\frac{1}{x}\right) = 1 , \tag{2d.72}$$

the integration here is over the 3-sphere boundary of a period 4-cell.

Paralleling the complex case, <u>derivatives</u> of $\zeta(x)$ are then fourfold periodic in q_0, q_1, q_2, q_3; they in fact define the **H**-analog of the Weierstrassian P function. $\zeta(x)$ is pseudo-periodic in that

$$\zeta(z + q_h) = \zeta(z) + \eta_h \tag{2d.73}$$

where the η_h are certain constant quaternions of dimension (length)^3. Putting the qseudo-periodicity. (2d.75) in Eq. (2d.74) will give the second Legendre relation. The latter is derived by performing the same integral as in Eq. (2d.74) differently. First, we shall denote the D=3 sides of the periodic 4-cell or parallelpiped such that a side adjoining a corner α always carries the index of the periodic vector not used...So the side S_1 is spanned by q_2, q_3 and q_4, etc. The whole hypersurface S is then given by $S = 2 \sum_{h=1}^{4} S_h$. Now we return to the first integral in Eq. (2d.74) and integrate $\zeta(x)$ over every side individually. Noting that opposite sides of the cell have oppositely oriented normal and differ only by the period q_h, the surface integral over all the 8 sides gives

$$\sum_{h=1}^{4} \left\{ \frac{1}{8\pi^2} \iiint_{(s_h)} d\Sigma\, \zeta(x) - \frac{1}{8\pi^2} \iiint_{(s_h+\omega_h)} d\Sigma\, \zeta(x) \right\} . \tag{2d.74}$$

Substituting $x \to x + q_h$ in the second integral, thus integrating over the original sides S_h, then using Eq. (2d.75), we get

$$\sum_{h=1}^{4} \frac{1}{8\pi^2} \iiint_{(s_h)} \{\zeta(x) - \zeta(x+\omega_h)\}\, d\Sigma = \left\{ -\sum_{n=1}^{\infty} \frac{\eta_h}{8\pi^2} \iiint_{(s_h)} d\Sigma \right\} . \tag{2d.75}$$

Writing the orientable hypersurface in the well-known determinantal form of $Q_h \equiv \iiint_{S_h} d\Sigma = e_\alpha \varepsilon_{\alpha\beta\gamma\delta} q_\beta^{(m)} q_\gamma^{(n)} q_\delta^{(r)}$ (h, m, n, r cyclic), we readily deduce [51, 115] from Eqs. (2d.75) and (2d.72) that

$$\sum_h \eta_h Q_h = 8\pi^2 . \tag{2d.76}$$

These second Legendre relations thus generalize their complex counterparts (2d.50) and play an important role in the general theory of quaternionic $\zeta(x)$ functions.

2.d.4. Recent developments of Fueter's theory : O(4) covariance, conformal and quasi-conformal structures

In certain respects the function theory of a quaternionic variable developed by Fueter and his school is quite incomplete. In particular, as it stands, it is rather unsuitable for applications in relativistic field theories. In this subsection, we report mainly on the progress [116, 117] made through the incorporation of an O(4) covariant, (anti-) self-dual structure into the Fueter theory of functions. The connections to the twistorial and harmonic space approaches are also briefly discussed.

By current standards of rigor and generality, the old proofs of many of Fueter's theorems [98] leave much to be desired. In recent years, these deficiencies have been remedied. Notably, A. Sudbery [109] not only reformulated Fueter's theory in the more compact language of differential forms but also contributed some new results. A brief summary of the highlights of the latter results, is a sufficient complement to our presentation. The reader should consult ref.[109] for proofs and further details. The more encompassing subject of Clifford Analysis, the function theory over Clifford algebras, should also be of interest [111, 118] .

First, we define some elements of the calculus of quaternionic valued differential forms. A function f: **H** $\rightarrow$ **H** is *real differentiable* if it is differentiable in the usual sense. Its differential at a point $q \in \mathbf{H}$ is an **R**-linear map df_q: **H** $\rightarrow$ **H**, i.e. an **H**-valued 1-form

$$df = \frac{\partial f}{\partial x_\mu} dx^\mu = e_\alpha df^\alpha \tag{2d.77}$$

with df^α being real forms. Similarly, an **H**-valued r-form θ is an alternating **R**-multilinear map θ: **H**x**H**...x**H** (r times) $\rightarrow$ **H**. The exterior product of an r-form θ and an s-form ϕ is an (r+s)-form

$$\theta \wedge \phi(h_1, \ldots, h_{r+s}) = \frac{1}{r!\, s!} \sum_{\rho} \varepsilon(\rho)\, \theta(h_{\rho(1)}, \ldots, h_{\rho(r)})\, \phi(h_{\rho(r+1)}, \ldots, h_{\rho(r+s)}) \ , \qquad (2d.78)$$

the summation being over all permutations ρ of (r+s) objects, $\varepsilon(\rho)$ denoting the sign of ρ. The set of all r-forms is then a two-sided quaternionic vector space where for any $a \in \mathbf{H}$, r-form θ and s-form ϕ

$$a\,(\theta \wedge \phi) = (a\,\theta \wedge \phi) \ , \qquad (2d.79)$$

$$(\theta \wedge \phi)a = (\theta \wedge \phi a) \ , \qquad (2d.80)$$

$$(a\,\theta) \wedge \phi = \theta \wedge (a\,\phi) \ . \qquad (2d.81)$$

The conjugate forms are $\bar{\theta} = \bar{e}_\alpha \theta^\alpha$ and $\bar{\phi} = \bar{e}_\alpha \phi^\alpha$. We further record the useful formulae

$$d\,(\theta \wedge \phi) = d\theta \wedge \phi + (-1)^r\, \theta \wedge d\phi \qquad (2d.82)$$

and

$$\overline{(\theta \wedge \phi)} = (-1)^{rs}\, \bar{\phi} \wedge \bar{\theta} \ . \qquad (2d.83)$$

Clearly, for Fueter's quaternion analysis, the relevant **H**-valued forms are respectively the spacetime position 1-form $dq = e_\mu\, dx^\mu$, the area 2-form $dq \wedge dq = \frac{1}{2}\varepsilon_{ijk}\, e_i dx_j \wedge dx_k$ and the volume 3-form Dq ($\equiv d\,\Sigma$ defined in connection with Eq. (2d.41))

$$\begin{aligned} Dq &= dx_1 \wedge dx_2 \wedge dx_3 - \tfrac{1}{2}\varepsilon_{ijk} e_i dx_0 \wedge dx_j \wedge dx_k \\ &= e_0 dx_1 \wedge dx_2 \wedge dx_3 - e_1 dx_0 \wedge dx_2 \wedge dx_3 \\ &\quad - e_2 dx_0 \wedge dx_3 \wedge dx_1 - e_3 dx_0 \wedge dx_1 \wedge dx_2 \ . \end{aligned} \qquad (2d.84)$$

Geometrically, the quaternion $D(a, b, c) = \frac{1}{2}(c\bar{a}b - b\bar{a}c)$ is perpendicular to a, b and $c \in \mathbf{H}$. Its magnitude is the volume of a tridimensional parallelepiped of sides a, b and c. Finally, there is the constant real volume 4-form $v = dx_0 \wedge dx_1 \wedge dx_2 \wedge dx_3$.

In the theory of function of a complex variable, for a function f (z): **C** $\rightarrow$ **C**, the Cauchy-Riemann equation $\frac{\partial f}{\partial \bar{z}} = 0$ is equivalent to the existence of a function f '(z) such that df = f '(z) dz. Similarly, it can be shown that the Cauchy-Riemann-Fueter definition of a left-regular function f: **H** $\rightarrow$ **H**, $\vec{D}$ f = 0, is equivalent to the existence of a quaternion $f_L'(q)$ such that

$$d\left(dq \wedge dq\right) f = Dq\, f_L'(q) \; . \qquad (2d.85)$$

A function f is right regular, f $\overleftarrow{D}$ = 0, if there exists a quaternion $f_R'(q)$ such that

$$d\left(f\, dq \wedge dq\right) = f_R'(q)\, Dq \; . \qquad (2d.86)$$

For the Fueter-Cauchy theorem (2d.41-2d.42) and related integral formulae to hold, Sudbery showed, by way of Coursat's method, that the contour Ξ needs only be a rectifiable boundary and that the derivative of $l(x)$ or $r(x)$ needs *not* even be continuous.

Progress has also been made in the construction of regular quaternionic functions. The latter can now be obtained not just from analytic functions of a complex variable, but also from harmonic functions of four real variables. Extending Fueter's work, McCarthy and Sudbery [119] initiated the study of regular homogeneous polynomials by way of harmonic analysis on SU(2) $\approx S^3$. Such procedure is to quaternionic analysis what Fourier series on U(1) $\approx S^1$ are to complex analysis. As we shall see, the connection between harmonic analysis on S^3 and Fueter analysis arises naturally in the harmonic analyticity approach.

A few important remarks could be drawn from the works of Fueter [98] and of his followers [120]. In spite of many similarities, quaternionic analysis turns out to be algebraically and geometrically different from complex analysis. Being non-commutative, general regular functions of a quaternion variable can neither be composed nor multiplied to generate further regular functions; they do not form a ring. Moreover, both mathematical analysis and physical applications thus far have involved primarily functions which are rational in the quaternion. One has yet to confront the general singularity structure of quaternionic regular functions.

Thus we may wish to inquire on the possible kinds of singular points. For example, they can be isolated (essential or non-essential) singular points, forming generally a point set lying on a two-dimensional surface or distributed in a highly complex pattern on several surfaces. They can also be concentrated on a curve or at a point, even accumulate at that point. For example, it is known that the zeroes of a quaternionic regular function are not necessarily isolated, nor is its range necessarily open; in fact neither of these sets needs even be a submanifold of **H**. The whole subject is intricate, it is neither well studied nor understood at the present time. Next, we discuss in some details the O(4) covariant Fueter theory, its connection to twistors and harmonic analyticity, to conformal and quasi-conformal structures

The growing interest in higher dimensional analogs [121, 122] of D = 2 conformal field theories has led many [48, 49, 50, 123, 124, 125, 126] to seek the 4-dimensional counterpart(s) to the powerful description of complex analytic functions as conformal mappings. From the standpoint of quaternionic function theory, a parallel may exist in a context necessarily larger than that of the D = 4, 15 parameter conformal group. We have argued [127] that one needs a Fueter group, namely an infinite parameter quaternionic subgroup (with the conformal group as its subgroup) of the general covariance group. All this leads to our next topic on recent extensions of Fueter's theory. These refinements [116, 117] bring to it the desired O(4) covariance, hence they open the door for applications to D = 4 relativistic field theories, particularly to their conformal and quasi-conformal structures in space-time, to the analysis of D = 4 S-duality [128, 129]. We begin our discussion with a quaternionic formulation of the D = 4 conformal group and of its associated twistors. We first briefly recall the situation in two dimensions.

In two dimensions, we may start from the 2-dimensional representations of SL(2, **C**), i.e. the complex spinor ψ_L with components ψ^α ($\alpha = 1, 2$). It transforms under that group as

$$\psi'_L = L\,\psi_L \qquad (\det L = 1) \tag{2d.87}$$

where L is a (2 x 2) complex, unimodular matrix, ψ_L is a $(\frac{1}{2}, 0)$ representation. The right-handed spinor ψ_R associated with the $(0, \frac{1}{2})$ representation transforms as $\psi'_R = L^{\dagger-1}\,\psi_R$. Due to the identity

$$L^{\dagger-1} = (i\sigma_2)^{-1} L^* i\sigma_2 = \sigma_2 L^* \sigma_2 \ , \tag{2d.88}$$

$\hat{\psi}_L \equiv -\,i\sigma_2\psi^* = \begin{pmatrix} -\psi_2^* \\ \psi_1^* \end{pmatrix}$, the CP conjugate of ψ_L, transforms like a right handed spinor ψ_R, and $\hat{\psi}'_L = L^{\dagger-1}\hat{\psi}_L$.

With SL(2, **C**) seen as the D = 2 conformal group, ψ_L is then called a *twistor*, transforming linearly under SO(3, 1). The latter is generated by translations, rotations, dilatation and inversion. The components ψ^α can be viewed as the homogeneous coordinates of a point in the projective space $CP^1 \approx S^2$, the inhomogeneous position coordinate is the ratio

$$z \equiv \frac{\psi_1}{\psi_2} \tag{2d.89}$$

which transforms nonlinearly under SL(2, **C**), namely $z' = L\,z = \frac{az+b}{cz+d}$ with as special cases giving

a) dilatation

$$z \to kz$$

b) rotation

$$z \to e^{i\alpha} z$$

c) translation

$$z \to z + b$$

d) inversion

$$z \to \frac{1}{z}$$

e) special conformal transformation (translation of the inverse)

$$\frac{1}{z} \to \frac{1}{z} - \zeta^* \quad (\zeta \text{ complex}) .$$

In two dimensions, a field would be a function of z with definite SL(2, **C**) transformation properties. As a generalization of the field concept, we can consider functions $F(\psi_1, \psi_2)$ of the spinor or $D = 2$ twistor ψ_L. The field concept is recovered if F is homogeneous of degree δ in ψ_1 and ψ_2, i.e.

$$F(\lambda\psi_1, \lambda\psi_2) = \lambda^\delta F(\psi_1, \psi_2) . \tag{2d.90}$$

Setting $\lambda = \psi_2^{-1}$, we have

$$F(\psi_1, \psi_2) = \psi_2^\delta f(z) , \tag{2d.91}$$

$$f(z) = F(\frac{\psi_1}{\psi_2}, 1) = F(z,1) . \tag{2d.92}$$

The homogeneous function of ψ can be replaced by a field, a function of the position with a factor that is a definite power of ψ_2, namely the field f(z) is given through the homogeneous function F by the relation

$$f(z) = \psi_2^\delta F(\psi_1, \psi_2) . \tag{2d.93}$$

Under SL(2, **C**)

$$F \to F (a\psi_1 + b\psi_2, c\psi_1 + d\psi_2) = \psi_2^\delta (cz + d)^\delta f\left(\frac{az + b}{cz + d}\right) . \tag{2d.94}$$

So that

$$T_G f(z) = (cz + d)^\delta f\left(\frac{az + b}{cz + d}\right) \tag{2d.95}$$

corresponding to the (δ, 0) representation of SL(2, **C**) in terms of a homogeneous analytic function $f^{\delta}(z)$. δ denotes the conformal weight of the analytic field. The (0, $\bar{\delta}$) representations are obtained through the transformation

$$\psi \to \hat{\psi} \quad \text{or} \quad z \to z^{-1} \text{ and } L = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \to L^{\dagger\,-1} = \begin{pmatrix} \bar{d} & -\bar{c} \\ -\bar{b} & \bar{a} \end{pmatrix} \text{.(2d.96)}$$

So that

$$G^{\bar{\delta}}(-\overline{\psi_2 \lambda}, \overline{\psi_1 \lambda}) = \bar{\lambda}^{\bar{\delta}}\, G^{\bar{\delta}}(-\overline{\psi_2}, \overline{\psi_1}) \tag{2d.97}$$

or taking $\lambda = \overline{\psi_2}^{\,-1}$,

$$G^{\bar{\delta}}(-\overline{\psi_2}, \overline{\psi_1}) = \overline{\psi_2}^{\,\bar{\delta}}\, G^{\bar{\delta}}(-1, \bar{z}) \tag{2d.98}$$

giving

$$g(\bar{z}) = G^{\bar{\delta}}(-1, \bar{z}) = \overline{\psi_2}^{\,-\bar{\delta}}\, G^{\bar{\delta}}(-\overline{\psi_2}, \overline{\psi_1}) \;. \tag{2d.99}$$

Hence

$$g'(\bar{z}') = (\bar{c}\bar{z} + \bar{d})^{\bar{\delta}}\, g\left(\frac{\bar{a}\bar{z} + \bar{b}}{\bar{c}\bar{z} + \bar{d}}\right) \;. \tag{2d.100}$$

For the (δ, $\bar{\delta}$) representation, we have

$$h'^{(\delta,\bar{\delta})}(z', \bar{z}') = (cz + d)^{\delta}\, (\bar{c}\bar{z} + \bar{d})^{\bar{\delta}}\, h^{(\delta,\bar{\delta})}\left(\frac{az + b}{cz + d}, \frac{\bar{a}\bar{z} + \bar{b}}{\bar{c}\bar{z} + \bar{d}}\right) \;. \tag{2d.101}$$

Such primary fields transform like $(dz)^{-\delta/2}\,(d\bar{z})^{-\bar{\delta}/2}$ as

$$(dz')^{-\delta/2}\,(d\bar{z}')^{-\bar{\delta}/2} = (cz+d)^{\delta}\,(\bar{c}\bar{z}+\bar{d})^{\bar{\delta}}\,(dz)^{-\delta/2}(d\bar{z})^{-\bar{\delta}/2} \;. \tag{2d.102}$$

Under a more general conformal mapping $z \to w(z)$, $dz \to w'(z)dz$, we get

$$h^{(\lambda,\bar{\lambda})}(z,\bar{z}) \to [w'(z)]^{\lambda}\, \overline{w'(z)}^{\,\bar{\lambda}}\, h^{(\lambda,\bar{\lambda})}(w(z), \bar{w}(\bar{z})) \;. \tag{2d.103}$$

The transformation (2d.103) corresponds to the infinite dimensional D = 2 Virasoro group generalizing the Mobiüs group SL(2, **C**).

Generalizing SL(2, **C**) to the D = 4 conformal group SO(5, 1) or better the latter's covering group SL(2, **H**). These groups generalize the euclidean Lorentz group in R^4, SO(3, 1) ≈ SL(2, **C**). Recall that the latter's (2 x 2) quaternionic representation is

$$\Lambda = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad a, b, c, d \in \mathbf{H} \quad , \quad \mathrm{Det}\,\Lambda = |ac^{-1}dc - bc|^2 = 1 \qquad (2d.104)$$

with acting on a 2-dimensional **H**-valued ket vector w in quaternionic Hilbert space

$$w = \begin{pmatrix} V \\ \overline{U} \end{pmatrix} \quad , w' = \Lambda\, w \qquad (2d.105)$$

with the (2 x 2) complex representation

$$V = \begin{pmatrix} v_1 & -v_2^* \\ v_2 & v_1^* \end{pmatrix} \quad , \quad \overline{U} = \begin{pmatrix} u_1 & -u_2^* \\ u_2 & u_1^* \end{pmatrix}. \qquad (2d.106)$$

Thus W can be represented by the (4 x 2) complex matrix with, as its first column, ψ and as its second column, $\widehat{\psi}$ where

$$\psi = \begin{pmatrix} v_1 \\ v_2 \\ u_1 \\ u_1 \end{pmatrix} \quad , \quad \widehat{\psi} = -i\sigma_2\psi^* = \begin{pmatrix} -v_2^* \\ v_1^* \\ -u_2^* \\ u_1^* \end{pmatrix}. \qquad (2d.107)$$

Λ^{-1}, the inverse SL(2, **H**) matrix, can also act on a (1 x 2) quaternionic row $s^\dagger = (\overline{T}, R)$ from the right, $s'^\dagger = s^\dagger\Lambda^{-1}$ with

$$\Lambda^{-1} = \begin{pmatrix} (a-bd^{-1}c)^{-1} & (c-db^{-1}a) \\ (b-ac^{-1}d)^{-1} & (d-ca^{-1}b)^{-1} \end{pmatrix}. \qquad (2d.108)$$

The inhomogeneous coordinate in the quaternionic projective line $HP^1 \approx S^4$ is the ratio of V and U; we write

$$w = \begin{pmatrix} x\overline{U} \\ \overline{U} \end{pmatrix}, \quad x = V\overline{U}^{-1} . \tag{2d.109}$$

The above 4-spinor ψ is the twistor representation of the conformal group SO(5, 1) ≈ SL(2, **H**), we find

$$x = \begin{pmatrix} u_1^* v_1 + v_2^* u_2 & v_1 u_2^* - v_2^* u_1 \\ v_2 u_1^* - v_1^* u_2 & v_2 u_2^* + v_1^* u_1 \end{pmatrix} \left(|u_1|^2 + |u_2|^2 \right)^{-1} . \tag{2d.110}$$

The twistor ψ then reads

$$\psi = \begin{pmatrix} |u|\, x_+ \\ u \end{pmatrix} = \begin{pmatrix} x\, u \\ u \end{pmatrix} , \quad u \equiv \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = U \frac{1+\sigma_3}{2} \tag{2d.111}$$

with

$$x_+ = |u|^{-1}\, x\, u = (x_0 - i\,\vec{\sigma}\cdot\vec{x})\; u\, |u|^{-1} . \tag{2d.112}$$

We also have the O(4) invariant quaternion

$$w = \overline{U}^{-1} V = \left(|u_1|^2 + |u_2|^2 \right)^{-1} \begin{pmatrix} u_1^* & -u_2^* \\ -u_2 & u_1 \end{pmatrix} \begin{pmatrix} v_1 & -v_2^* \\ v_2 & v_1^* \end{pmatrix} \tag{2d.113}$$

$$w = \left(|u_1|^2 + |u_2|^2 \right)^{-1} \begin{pmatrix} u^\dagger v & u^\dagger \hat{v} \\ \hat{u}^\dagger v & v^\dagger u \end{pmatrix} . \tag{2d.114}$$

Under a conformal transformation $\Lambda \in$ SL(2, **H**),

$$V' = aV + b\,\overline{U} = (ax + b)\,\overline{U} , \tag{2d.115}$$

$$\overline{U}' = cV + d\,\overline{U} = (cx + d)\,\overline{U} \tag{2d.116}$$

where a, b, c, d $\in$ $\mathbf{H}$, are the elements of Λ. Then the position quaternion x transforms under the Mobiüs transformation as

$$x' = (ax + b)(cx + d)^{-1} \tag{2d.117}$$

while

$$y' = (cy + d)^{-1}(ay + b) \tag{2d.118}$$

where y is defined through $s \equiv (\overline{T}, R) \equiv (R\overline{y}, R)$.

Special cases, along with their infinitesimal forms, are:

a) Translations

$$x \to x + b \quad (\delta_\varepsilon x = \varepsilon) \tag{2d.119}$$

b) Dilatations

$$x \to \lambda\, x\,, \;\; \mathrm{Vec}\lambda = 0\,, \;\; \lambda > 0 \quad (\delta_\kappa x = \kappa\, x) \tag{2d.120}$$

c) Left-rotations

$$x \to mx\,, \;\; |m| = 1 \;\; (\delta_\mu x = \mu x\,, \; \mathrm{Sc}(\mu) = 0) \tag{2d.121}$$

d) Right-rotations

$$x \to x\overline{n}\;\;,\;\; |n|{=}1 \;\; (\delta_\nu x = \nu x\,, \mathrm{Sc}(\nu) = 0) \tag{2d.122}$$

e) Inversion

$$x \to x^{-1} \quad \text{(no infinitesimal form)} \tag{2d.123}$$

f) Special conformal transformations

$$x^{-1} \to x^{-1} - \bar{c} \;\; \text{or} \;\; x \to x(1 - \bar{c}x)^{-1}\,,\; (\delta_c x = x\bar{c}x)\;. \tag{2d.124}$$

Just as with SL(2, $\mathbf{C}$), we now consider homogeneous functions of the SL(2, $\mathbf{H}$) twistor. They can be functions of x^μ and the spinor u, or of the spinors x_+ and u, or of the quaternions $V = x\,\overline{U}$ and $\overline{U}$.

Under the subgroup O(4) $\approx$ SU(2) x SU(2) of SL(2, $\mathbf{H}$), we have

$$x' = m x \bar{n} \quad , V \to mV \ , \bar{U} \to n\bar{U} \ , \tag{2d.125}$$

$$x_+' = m\, x_+ \quad , u' = n\, u \ . \tag{2d.126}$$

Similarly,

$$y' = n\, y\, m \quad \text{or} \quad y' = m\, y\, n \ . \tag{2d.127}$$

It follows that

$$\omega_L = x\bar{y} \qquad \text{with} \quad \omega_L' = m\omega_L\bar{m} \ , \tag{2d.128}$$

$$\omega_R = \bar{y}x \qquad \text{with} \quad \omega_R' = n\omega_R\bar{n} \ . \tag{2d.129}$$

Therefore, from a right twistor and a left twistor, we can form two 4-vectors ω_L and ω_R transforming like a (0, 0) + (1, 0) and (0, 0) + (0, 1) representations of O(4), respectively. So $\lambda_L = \mathrm{Vec}(x\bar{y}) = \frac{1}{2}(x\bar{y} - y\bar{x})$ and $\lambda_R = \mathrm{Vec}(\bar{y}x) = \frac{1}{2}(\bar{y}x - \bar{x}y)$ are self-dual and antiself-dual antisymmetrical tensors, respectively, while $\mathrm{Sc}(\omega_L) = \frac{1}{2}(x\bar{y} + y\bar{x}) = \mathrm{Sc}(\omega_R)$ is an O(4) invariant scalar.

In analogy to the D = 2 primary fields, which are homogeneous functions of the spinor representations of SL(2, **C**), we shall construct in D = 4 certain functions of the lowest dimensional representations of SL(2, **H**). We saw that, in D = 2, homogeneous functions of (ψ_1, ψ_2) can be viewed as analytic functions f(z) of $z = \psi_1\psi_2^{-1}$, once a power of ψ_2 is factored out. In turn, f(z) represents a conformal mapping of the z-plane. So we consider a function F of two quaternionic components V and $\bar{U}$ of the fundamental representation of SL(2, **H**), or equivalently a function G of the first columns v and u of V and $\bar{U}$ making up the spinorial components of a twistor. Introducing the position quaternion $x = V\bar{U}^{-1}$, we have

$$F \equiv F(\, V, \bar{U}\,) = F(\, x\,\bar{U}, \bar{U}\,) \tag{2d.130}$$

or

$$G \equiv G(\, v, u) = G(\, x\, u, u\,) = G(\, x_+, u\,) \ . \tag{2d.131}$$

These functions are homogeneous w.r.t. dilatation if we write

$$F(\lambda V, \lambda \overline{U}) = \lambda^k F(V, \overline{U}) = \lambda^k F(x\overline{U}, \overline{U}) \tag{2d.132}$$

or

$$G(\lambda v, \lambda u) = \lambda^k G(v, u) = \lambda^k G(x_+ |u|, u) \ . \tag{2d.133}$$

By choosing $\lambda = |U|^{-1} = (U\overline{U})^{-\frac{1}{2}} = |u|^{-1}$ so that

$$F(V, \overline{U}) = \lambda^{-k} F(V|U|^{-1}, \overline{U}|U|^{-1}) \ , \tag{2d.134}$$

$$G(v, u) = \lambda^{-k} G\left(x_+, \frac{u}{|u|}\right) \ . \tag{2d.135}$$

Setting

$$w \equiv \frac{u}{|u|} \quad , \quad |w| = 1 \quad , \quad W \equiv |U|^{-1} U \quad , \quad W\overline{W} = 1 \tag{2d.136}$$

then W is a unit quaternion, a point on the sphere S^3 parametrized by the normalized spinor w. So we also have

$$F(V, \overline{U}) = |U|^k F(x\overline{W}, \overline{W}) \ , \tag{2d.137}$$

$$G(v, u) = |u|^k G(x_+, w) \quad , \quad x_+ \equiv xw \ . \tag{2d.138}$$

To proceed further, let us consider the representation of W by a unitary matrix

$$W = Z \exp(-i\sigma_3 \frac{\alpha}{2}) \quad , \quad Z \, \varepsilon \, \frac{SU(2)}{U(1)} \approx S^2 \tag{2d.139}$$

where U(1) is associated with rotations around the 3rd axis. Since the (2 x 2) matrix representation of W is

$$W = \begin{pmatrix} w_1 & -w_2^* \\ w_2 & w_1^* \end{pmatrix} , \tag{2d.140}$$

its general parametrization is

$$W = \frac{1}{\sqrt{1+|\zeta|^2}} \begin{pmatrix} 1 & -\zeta^* \\ \zeta & 1 \end{pmatrix} \begin{pmatrix} e^{i\alpha/2} & 0 \\ 0 & e^{-i\alpha/2} \end{pmatrix} . \tag{2d.141}$$

The complex number $\zeta = \frac{w_1}{w_2} = \frac{u_1}{u_2}$ thus denotes a point on the unit 2-sphere S^2.

Taking $\lambda = u_1^{-1}$ in Eq. (2d.135) complex, we may write

$$G(v, u) = u_1^k \, G(x_+ , \zeta) \tag{2d.142}$$

with

$$F(V, \overline{U}) = | U |^k \, F(x\overline{W}, \overline{W}) \; x_+ = (x_0 - i\vec{\sigma}\cdot\vec{x}) \begin{pmatrix} 1 \\ \zeta \end{pmatrix} \frac{1}{\sqrt{1 + |\zeta|^2}} . \tag{2d.143}$$

From the above, we readily observe that, with the functions of v_1, v_2, u_1, u_2 being reduced to functions of $\frac{v_1}{u_1}$, $\frac{v_2}{u_1}$ and $\frac{u_2}{u_1}$, we thus obtain the projective (twistor) space CP^3. First introduced in physics by Penrose, the space CP^3 was subsequently used by Atiyah and Ward in their work on Yang-Mills instantons [130]. On the other hand, these functions can also be interpreted as functions of x_+ and of a point on S^2 parametrized by ζ. This approach is recognized as the method of harmonic analyticity of Ogievetsky et al. [49, 50].

Next, we take up a third category of functions associated with a quaternionic homogeneity corresponding to HP^1, the quaternionic projective line. Consider functions $F(V, \overline{U})$, transforming like some representation of the O(4) subgroup of SL(2, **H**), such that

$$F(V\lambda, \overline{U}\lambda) = \mu(\lambda) \, F(V, \overline{U}) \, \nu(\lambda) \tag{2d.144}$$

or

$$F(\,V, \overline{U}\,) = \mu^{-1}(\lambda)\, F(\,V\lambda, \overline{U}\lambda\,)\, \nu^{-1}(\lambda) \qquad (2d.145)$$

where $\mu(\lambda)$ and $\nu(\lambda)$ quaternionic multipliers. Taking $\lambda = \overline{U}^{-1}$ and using the variable $x = V\overline{U}^{-1}$, we get

$$F(\,V,\overline{U}\,) = \mu^{-1}(\overline{U}^{-1})\, F(\,x,\,1)\, \nu^{-1}(\overline{U}^{-1})\ . \qquad (2d.146)$$

Next, we relate the latter method to Fueter's analytic functions. More generally, V and $\overline{U}$ may undergo an O(4) transformation

$$V \rightarrow mV \quad , \quad \overline{U} \rightarrow n\overline{U}\ . \qquad (2d.147)$$

So $F(\,mV\lambda, n\overline{U}\lambda\,) = F(\,mV\lambda, mp\overline{U}\lambda\,)$, if we set $n = mp$, we have

$$\begin{aligned} F(\,v, U\,) &= M^{-1}\, F(\,mV\lambda, mpU\lambda\,)\, N^{-1}\ , \\ &= M^{-1}\, F(\,mxp^{-1}\overline{m}\,)\, N^{-1} \end{aligned} \qquad (2d.148)$$

after taking $\lambda = \overline{U}^{-1}p^{-1}\overline{m}$, ($\overline{m} = m^{-1}$, $\overline{n} = n^{-1}$). The more general multipliers M and N are now functions of m, p and $\overline{U}$. So if F is scalar, then M and N cannot depend on m and n, or alternatively on m and p. They will depend only on the scalar $|\,U\,|$. In this case, for a function F homogeneous of degree k, we have

$$F(\,V, \overline{U}\,) = |\,U\,|^k\, F(\,mxp^{-1}\overline{m}\,)\ . \qquad (2d.149)$$

If F is quaternionic and transforms like (0, 0) + (1, 0) under O(4), then

$$F_L^{(k)}(\,V, \overline{U}\,) = |\,U\,|^k\, \overline{m}\, F(\,mxp^{-1}\overline{m}\,)\, m\ . \qquad (2d.150)$$

Such a function is a power series in $Z = xp^{-1}$, namely

$$F_L^{(k)}(Z) = \sum_n c_n Z^n, \qquad (\mathrm{Vec}(c_n) = 0) \ . \tag{2d.151}$$

We recognize the $F_L^{(k)}(Z)$'s as just left-right holomorphic Fueter functions of Z (2d.14).

On the other hand, if F_R transforms like a (0, 0) + (0, 1) representation of O(4), then it is a function of $S = p^{-1}x$, with p transforming like O(4)-vector, $p \to mp\bar{n}$, so that $S \to nS\bar{n}$. Then

$$F_R^{(k)}(V, \overline{U}) = |U|^k \bar{n} F(np^{-1}x\bar{n})\, n \ . \tag{2d.152}$$

If F transforms like a 4-vector, then it is clearly of the form

$$F = p^{-1}\sum_n c_n (xp^{-1})^n = p^{-1}F_L(Z) = F_R(S)p^{-1} \ . \tag{2d.153}$$

So $F \to nF\bar{m}$, which is the transformation law for x^{-1}. Then

$$x'^{-1} = n\left[p^{-1}\sum_n c_n (xp^{-1})^n + h^{-1}\right]\bar{m} \tag{2d.154}$$

where use is made of an O(4) transformation. We note that, unlike the D = 2 case, this mapping (for $n \to \infty$) is <u>not</u> conformal. We shall call it *quasi-conformal* for the following reasons.

Take the special case where $c_n = -c$ and $n \to \infty$; the mapping (2d.154) then reads

$$x' = m\left(\frac{c}{x-p} + h^{-1}\right)^{-1}\bar{n} \ , \tag{2d.155}$$

which is equivalent to a SL(2, **H**) transformation (see Eq. (2d.11)). Therefore, the infinite parameter mapping (2d.154) admits the conformal group as a subgroup. It is the quaternionic generalization to D = 4 of the infinite parameter holomorphic mapping $z' = f(z) = \sum_n b_n z^n$ with $z, c_n \in \mathbf{C}$ in D=2. The latter admits as a finite parameter subgroup of Mobiüs transformations $z' = \frac{az+b}{cz+d}$, $ad - bc = 1$, which is a nonlinear realization of $SO(3, 1) \approx SL(2, \mathbf{C})$ on the coset $\frac{SL(2, \mathbf{C})}{\Delta \times E_2}$, Δ being the dilatation and E_2, the D = 2 euclidean group.

In two dimensions, it is well-known that the holomorphic group and its Möbius subgroups are both conformal; we have the line element

$$ds^2 = dz'd\bar{z}' = |\, f'(z)\, |^2 dz d\bar{z} \; . \tag{2d.156}$$

This is no longer the case in four dimensions. The infinite group (2d.154) does not lead to a conformally flat metric though its Möbius subgroup SL(2, **H**) does. Indeed we then have

$$dx' = m\left(\frac{c}{x-p} + h^{-1}\right)^{-1} dx \left(\frac{c}{x-p} + h^{-1}\right)^{-1} \bar{n} \tag{2d.157}$$

and so

$$dx' d\overline{x'} = \left|\left(\frac{c}{x-p} + h^{-1}\right)\right|^{-2} dx d\bar{x} \; . \tag{2d.158}$$

Performing the O(4) transformation $X' = mx'\,\bar{n}$, $x' = mx\,\bar{n}$, $P = mp\bar{n}$ and $H = mh\bar{n}$, then Eq. (2d.154) becomes

$$X' = \sum_n c_n \left(P^{-1}x\right)^n P^{-1} + H^{-1} \tag{2d.159}$$

or putting $Z' = P^{-1}X'$, $Z = P^{-1}X = Z_0 - i\vec{\sigma}\cdot\vec{Z}$ then

$$Z^{-1} = \sum_n c_n Z^n + H^{-1}P \; , \tag{2d.160}$$

which is recognized as a left-right holomorphic Fueter transformation.

If we write $ds^2 = dZ' d\overline{Z} = g_{\mu\nu} dZ^\mu dZ^\nu$, it will be shown subsequently that $g_{0n} = 0$, n = 1, 2, 3 and that, using the upper half-plane variable $z = Z_0 + i\left|\vec{Z}\right|$, the line element takes the form

$$ds^2 = \Phi^2(z, \bar{z})\, dz d\bar{z} + \rho^2(z, \bar{z}) d\Omega^2(\theta, \varphi) \tag{2d.161}$$

with $d\Omega^2$ being the line element of S^2. This Kruskal form, a generalization of the conformal metric, deserves therefore the name of quasi-conformal metric. Before going into much greater details on the quasi-conformal structure of spacetime, we first establish the connection between Fueter analytic functions and harmonic analyticity.

We remarked before that, as they are, the left-right holomorphic quaternionic Fueter series $F(x) = \sum_n \alpha_n x^n$, $(\mathrm{Vec}(\alpha_n) = 0)$, Eq. (2d.14), cannot be used in physics. Indeed if x is a 4-vector transforming under O(4), its powers x^n then transform like tensor elements, so that F(x) will not have a definite tensorial property. However, we found that the appropriate Weierstrassian series are $F_L(Z) = \sum_n a_n Z^n$ and $F_R(U) = \sum_n b_n U^n$ with a_n and b_n being O(4) scalars and $Z = xp^{-1}$ or $U = p^{-1}x$. They are such that $\mathrm{Sc}F_L$ and $\mathrm{Sc}F_R$ are O(4) scalars while $\mathrm{Vec}F_L$ and $\mathrm{Vec}F_R$ are respectively self-dual and antiself-dual. Then the functions $G_L = \Box F_L$ and $G_R = \Box F_R$ are Fueter left-right analytic. Under the action of O(4)

$$G_L \to m\, G_L\, \overline{m} \quad , \quad G_R \to n\, G_L\, \overline{n} \; , \tag{2d.162}$$

$$D \to m\, D \overline{n} \quad , \quad \overline{D} \to n\, \overline{D} \overline{m} \; . \tag{2d.163}$$

Direct differentiation gives

$$D G_R(x) = 0 \quad , \quad G_L \overleftarrow{D} = 0 \; , \tag{2d.164}$$

since each term $g_n^R(x)$ of the series G_R (and similarly for G_L) satisfies the analyticity equation $Dg_n^R(x) = 0$, $g_n^R(x) \equiv \frac{1}{4}\Box\,(\bar{c}x)^n$. Now a solution to $Df = 0$ also solves

$$\bar{x}\,Df = 0 \quad . \tag{2d.165}$$

We may write

$$\bar{x}\,D = x^{\mu}\partial_{\mu} - \vec{\sigma}\cdot\vec{L} = \Delta - \vec{\sigma}\cdot\vec{L} \ , \tag{2d.166}$$

$\Delta = x^{\mu}\partial_{\mu}$ being the dilatation operator and $\vec{L} = i\left(\vec{x}\times\nabla + \vec{x}\partial_0 - x_0\vec{\nabla}\right)$ being the self-dual part of the angular momentum tensor associated with the $SU(2)_L$ subgroup of O(4). Hence f satisfies $\Delta f(x) = \vec{\sigma}\cdot\vec{L}f(x)$. So, if as a solution to Eq. (2d.165), $f^{(j)}(x)$ is homogeneous of degree j, $\Delta f^{(j)}(x) = j\,f^{(j)}(x)$. Iterating and using the commutation relations $\vec{L}\times\vec{L} = i\,\vec{L}$, we find $\left(\vec{\sigma}\cdot\vec{L}\right)^2 = \vec{L}\cdot\vec{L} - \vec{\sigma}\cdot\vec{L}$ and

$$\vec{L}^2 f^{(j)}(x) = j\,(j+1)\,f^{(j)}(x) \ . \tag{2d.167}$$

Consequently, $f^{(j)}(x)$ is proportional to a Wigner $D^j_{m\,m}$ function on $SU(2) \approx S^3$. Thus the above formulae give the relation between S^3 functions and the basis functions of Fueter analytic mappings.

To show the connection to the harmonic analyticity of Ogievetsky et al. [49, 50], we rewrite Eq. (2d.165) in (2 x 2) matrix form with

$$x = \begin{pmatrix} \overline{\chi_2} & \chi_1 \\ -\overline{\chi_1} & \chi_2 \end{pmatrix} , \quad \bar{x} = \begin{pmatrix} \chi_2 & -\chi_1 \\ \overline{\chi_1} & \overline{\chi_2} \end{pmatrix} \tag{2d.168}$$

where $\chi_1 \equiv -\left(x^2+ix^1\right)$, $\chi_2 \equiv \left(x^0+ix^3\right)$. In these new coordinates,

$$\partial_0 - i\vec{\sigma}\cdot\vec{\nabla} = D = \begin{pmatrix} \dfrac{\partial}{\partial\chi_2} & \dfrac{\partial}{\partial\overline{\chi_1}} \\ -\dfrac{\partial}{\partial\chi_1} & \dfrac{\partial}{\partial\overline{\chi_2}} \end{pmatrix} \tag{2d.169}$$

and

$$i\,\bar{x}D = \Delta + \vec{\sigma}\cdot\vec{L} = i\begin{pmatrix} \chi_1\dfrac{\partial}{\partial\chi_1}+\chi_2\dfrac{\partial}{\partial\chi_2} & -\chi_2\dfrac{\partial}{\partial\overline{\chi_1}}-\chi_1\dfrac{\partial}{\partial\overline{\chi_2}} \\ \overline{\chi_1}\dfrac{\partial}{\partial\chi_2}-\overline{\chi_2}\dfrac{\partial}{\partial\chi_1} & \overline{\chi_1}\dfrac{\partial}{\partial\overline{\chi_1}}+\overline{\chi_2}\dfrac{\partial}{\partial\overline{\chi_2}} \end{pmatrix}$$

$$\equiv i\begin{pmatrix} D^{+-} & -D^{++} \\ D^{--} & D^{-+} \end{pmatrix}. \qquad (2d.170)$$

So with $f = \begin{pmatrix} f_1 & -\overline{f_2} \\ f_2 & \overline{f_1} \end{pmatrix}$, Eq. (2d.165) reads

$$\begin{cases} D^{+-}f_1 - D^{++}f_2 = 0\ , \\ D^{--}f_1 - D^{-+}f_2 = 0\ . \end{cases} \qquad (2d.171)$$

With $L_{\mu\nu} = i\left(x_\mu\partial_\nu - x_\nu\partial_\mu\right)$, $D^{++} = L_{20} + L_{31} + i\,(L_{23} + L_{10})$ is an operator associated with the self-dual angular momentum generating left rotations. The U(1) subgroup is generated by

$$L_3 = \frac{i}{2}\left(D^{+-} - D^{-+}\right) = \frac{1}{2}D_0 \qquad (2d.172)$$

whereas the dilatations, giving the degree of homogeneity, are generated by

$$\Delta \equiv i x_\mu\partial^\mu = \frac{i}{2}\left(D^{+-} + D^{-+}\right)\ . \qquad (2d.173)$$

On S^2, for a given representation, L_3 and Δ are given eigenvalues so that the self-dual analyticity equation (2d.165) reduces to an equation in D^{++}. This operator and the variable x_+ defined earlier in Eq. (2d.112) are the cornerstones of harmonic analyticity. The latter involves functions of x_+ and u; u being a spinor or a unit quaternion. If u is defined up to a U(1) phase, it represents a point on S^2. We saw that functions of x_+ and u are equivalent to functions of a twistor or functions of two quaternions. So twistor analyticity, harmonic analyticity and Fueter 's quaternionic analyticity are all related. For greater details on these connections and on several possible variants of quaternionic analyticity, we refer the reader to a 1992 paper of Evans et al. [131].

Just as the operator $\bar{x}\,D$ represents the dilatation and the left rotation subgroups of SL(2, **H**), $Dx = x_0 D - i\vec{x}\cdot D\vec{\sigma}$ represents the combination of dilatations and right rotations. D represents the translations. In a special conformal transformation, we have from Eq. (2d.124), $\delta_c x = c_\mu(x\overline{e^\mu x})$. Hence for a function $\phi(x)$

$$\delta_c\phi(x) = \delta_c x^\mu \partial_\mu \phi = c_\mu \mathrm{Sc}\left(x\overline{e^\mu x}\overline{D}\right)\phi(x) \ . \tag{2d.174}$$

So the generators of special conformal transformation are $g^\mu = \mathrm{Sc}\left(x\overline{e^\mu x}\overline{D}\right)$, or in quaternionic form $g \equiv e_\mu g^\mu = e_\mu \mathrm{Sc}\left(x\overline{e^\mu x}\overline{D}\right)$. Thus, the (2 x 2) quaternionic matrix operator for SL(2, **H**) takes the form:

$$\mathcal{L} = \begin{pmatrix} D\bar{x} & D \\ \bar{x}D\bar{x} & -\bar{x}D \end{pmatrix} . \tag{2d.175}$$

In two dimensions, the group SL(2, **C**) is characterized by a differential equation involving the Schwarzian derivative [132]

$$\{f, z\} \equiv f'''\,(f')^{-1} - \frac{3}{2}\left[f''\,(f')^{-1}\right]^2 \tag{2d.176}$$

or equivalently, the quadratic Schwarz differential

$$S\left[f(z)\right] \equiv dz\,\{f, z\} \equiv d^3f\,(df)^{-1} - \frac{3}{2}\left[d^2f\,(df)^{-1}\right]^2 \ . \tag{2d.177}$$

Already known to Lagrange [133], such a differential S[f(z)] provides a bridge between geometry and calculus. Its presence is ubiquitous; it shows up as the invariant curvature in the geometry of curves [132] as well as in conformal field and string theories. Its universality is rooted in two properties:

a) It is projectively invariant

$$S\left[\frac{af(z)+b}{cf(z)+d}\right] = S\,(f(z))\ . \tag{2d.178}$$

b) It is the unique continuous 1-cocycle on the group of diffeomorphisms of the line with values in the space of quadratic differentials. Namely, if f and g are diffeomorphisms of **R**, then

$$S\,[f(g)] = S\,[f]\,(g)\,(g')^2 + S\,[g]\ . \tag{2d.179}$$

So f is a Mobiüs transformation $f(z) = \phi(z) = \frac{az+b}{cz+d}$, then

$$S\,[\phi(z)] = 0\ . \tag{2d.180}$$

Geometrically, the Schwarz differential can also be obtained as the limit of the cross-ratio of 4 points when they all tend to a common point.

The property (2d.180) easily generalizes to the D = 4 Mobiüs transformation in Euclidean space-time. Setting $dy = dx^\mu\, \partial_\mu\, y$ and accounting for lack of commutativity, we can similarly define two quaternionic quadratic differentials

$$S_L\,[y] \equiv d^3y\,(dy)^{-1} - \frac{3}{2}\left[d^2y\,(dy)^{-1}\right]^2 \tag{2d.181}$$

and

$$S_R\,[y] \equiv (dy)^{-1}d^3y - \frac{3}{2}\left[(dy)^{-1}d^2y\right]^2\ . \tag{2d.182}$$

Under O(4) they transform like left and right quators (to be defined latter), respectively. If y is a SL(2, **H**) transformation (2d.138), then $S_L\,[y] = S_R\,[y] = 0$. To do so, we rewrite the SL(2, **H**) transformation $y = (ax+b)(cx+d)^{-1}$ as Eq. (2d.11), i.e.

$$\begin{aligned} y &= a\left(x + a^{-1}b\right)\left(x + c^{-1}d\right)^{-1}c^{-1} \\ &= ac^{-1} + \Delta\left(x + c^{-1}d\right)^{-1}c^{-1}\ ,\quad \Delta \equiv \left(b - ac^{-1}d\right)\ . \end{aligned} \tag{2d.183}$$

Letting higher infinitesimals $d^2x = d^3x = \ldots = 0$ and using repeatedly the relation $\partial_\mu (x+a)^{-1} = -(x+a)^{-1} e_\mu (x+a)^{-1}$, we find that

$$dy = -\Delta \left(x + c^{-1}d\right)^{-1} dx \left(x + c^{-1}d\right)^{-1} c^{-1} \ , \tag{2d.184}$$

$$d^2y = 2\,\Delta \left(x + c^{-1}d\right)^{-1} dx \left(x + c^{-1}d\right)^{-1} dx \left(x + c^{-1}d\right)^{-1} c^{-1} \ , \tag{2d.185}$$

$$d^3y = -6\,\Delta \left(x + c^{-1}d\right)^{-1} dx \left(x + c^{-1}d\right)^{-1} dx \left(x + c^{-1}d\right)^{-1} dx \left(x + c^{-1}d\right)^{-1} c^{-1} \ , \tag{2d.186}$$

$$dy^{-1} = -c\left(x + c^{-1}d\right) dx^{-1} \left(x + c^{-1}d\right) \Delta^{-1} \ . \tag{2d.187}$$

From the above, $S_L[y] = S_R[y] = 0$ is readily checked.

However, for a general Fueter mapping, these **H**-Schwarzian differentials are non-zero. To handle the more general pseudo-conformal mappings (2d.154), we seek a generalization of Eq. (2d.179). To that end, we introduce an infinite matrix T

$$T = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & \rightarrow \\ 0 & 0 & 1 & 0 & 0 & \\ 0 & 0 & 0 & 1 & 0 & \\ 0 & 0 & 0 & 0 & 1 & \\ 0 & 0 & 0 & 0 & 0 & \\ \downarrow & & & & & \end{pmatrix} , \tag{2d.188}$$

with elements equal to unity in the upper line parallel to the diagonal, with all other elements being zero. Next we consider the quaternionic matrix

$$\frac{1}{1-TZ} = 1 + TZ + T^2Z^2 + \ldots = \begin{pmatrix} 1 & Z & Z^2 & Z^3 & Z^4 & \cdots \\ 0 & 1 & Z & Z^2 & Z^3 & \cdots \\ 0 & 0 & 1 & Z & Z^2 & \cdots \\ 0 & 0 & 0 & 1 & Z & \cdots \\ 0 & 0 & 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix} \tag{2d.189}$$

namely a triangular Toeplitz matrix. The latter's finite rank (nxn) counterparts were first introduced in connection with "fanned derivatives" in our quaternion analysis of the Yang-Mills instantons [51]. This is perhaps not really surprising since Toeplitz matrices have arisen in many areas of mathematics particularly in complex analysis [134] and K theory. Define the ket vectors

$$|\alpha\rangle = \begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \vdots \end{pmatrix} \quad , \quad |0\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \end{pmatrix} \tag{2d.190}$$

whose basis is provided by the harmonic oscillator annihilation and creation operators a, $a^\dagger$ with $[a\, ,\, a^\dagger] = 1,\ [a\, ,\, a] = [a^\dagger,\, a^\dagger\,] = 0$. Then

$$\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = a^\dagger |0\rangle = |1\rangle \quad , \quad \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \frac{(a^\dagger)^2}{\sqrt{2!}} |0\rangle = |2\rangle \quad , \text{ etc.} \tag{2d.191}$$

It is readily checked that

$$T = \sum_{n=0}^{\infty} |n\rangle\langle n+1| = \sum_{n=0}^{\infty} \frac{a^{\dagger n}}{\sqrt{n!}} |0\rangle\langle 0| \frac{a^{n+1}}{\sqrt{(n+1)!}} \quad , \tag{2d.192}$$

$$U = \langle 0 | (1 - TZ)^{-1} | \alpha \rangle = \sum_{n}^{\infty} \alpha_n Z^n = F(Z) \quad , \tag{2d.193}$$

displaying the Fueter mapping as a matrix element of the operator $(1 - TZ)^{-1}$. The other matrix elements are also Fueter functions. In this fashion, we can now evaluate d^nU, the nth-order differential of u; for example

$$dU = \langle 0 \, | (1 - TZ)^{-1}\, T\, dZ \,(1 - TZ)^{-1} | \, \alpha \rangle \tag{2d.194}$$

and

$$d^2U = 2\left\langle 0 \left| (1-TZ)^{-1}\, T\, dZ\, (1-TZ)^{-1} T\, dZ\, (1-TZ)^{-1} \right| \alpha \right\rangle, \text{ etc.} \tag{2d.195}$$

Thus if we have W = W(U) and U = U(Z), we can write

$$dW = \left\langle 0 \left| (1-TU)^{-1}\, T\, dU\, (1-TU)^{-1} \right| \beta \right\rangle, \tag{2d.196}$$

$$\begin{aligned} d^2W = \Big\langle 0 \,\Big|\, & 2\,(1-TU)^{-1}\, T\, dU\, (1-TU)^{-1}\, T\, dU\, (1-TU)^{-1} \\ & + (1-TU)^{-1}\, T\, d^2U (1-TU)^{-1} \Big|\, \beta \Big\rangle \end{aligned} \tag{2d.197}$$

etc.

These formulae thus allow the evaluation of quaternionic quadratic Schwarz differentials for Fueter mappings. We further note that, while the Schwarzian differential of the operator $H(Z) = (1-TZ)^{-1}$ vanishes, i.e. $S_R(H(Z)) = 0$, $S_L(H(Z)) = 0$, this is not the case for its matrix elements; e.g. $S_L(\langle 0 \mid H(Z) \mid \alpha \rangle) \neq 0$.

Next, we display the algebra of the generators of Fueter mappings in the classical case. The result will be a quaternionic analog of the centerless Virasoro algebra generating the complex holomorphic mappings. Recall that, in the latter case, the generators of the classical, complex Virasoro algebra have the representation $V^{(k)} = -\, z^{k+1}\frac{d}{dz}$, $k \in Z_\infty$ (the additive group of integers). They close on the loop algebra: $[V^{(k)}, V^{(m)}] = (k-m)\, V^{(k+m)}$. The SL(2, **C**) Lie subalgebra is obtained by taking k = 0, ±1. Equivalently, we may write the commutators of the Lie algebra elements by contracting the generators with powers of a complex parameter $\bar{c}$ (**C**-conjugate of c) and define the operators $t^{(k)} \equiv -(\bar{c}z)^k z\frac{d}{dz}$. They close on the same Witt algebra

$$\left[t^{(k)}, t^{(m)}\right] = (k-m)\, t^{(k+m)} . \tag{2d.198}$$

It is these $t^{(k)}$ which admit a quaternionic generalization in

$$T^{(k)} = -\mathrm{Sc}\left[(x\bar{c})^k x\bar{D}\right] = -\,\mathrm{Sc}\left[(x\bar{c})^k x\bar{e}_\mu\right] \partial_\mu \tag{2d.199}$$

where $\bar{c}$ is the **H**-conjugate of a constant quaternion c. To show that the $T^{(k)}$ similarly close on the Witt algebra:

$$\left[T^{(k)}, T^{(m)}\right] = (k-m)\, T^{(k+m)}, \quad (k, m \in Z_\infty) \ , \tag{2d.200}$$

we appeal to the identity

$$\mathrm{Sc}\left(A\bar{e}^\mu\right)\mathrm{Sc}\left(e^\mu B\right) = \mathrm{Sc}\left(\overline{A}\,\overline{B}\right) \ , \tag{2d.201}$$

good for any two quaternionic differential operators A and B. This lemma is proved by explicit computation of its left hand side, namely

$$\frac{1}{4}\left(A\bar{e}^\mu + e^\mu\overline{A}\right)\left(e^\mu B + \overline{B}\bar{e}^\mu\right) = \frac{1}{4}\left(4AB + e^\mu\overline{A}\overline{B}\bar{e}^\mu + A\bar{e}^\mu\overline{B}\bar{e}^\mu + e^\mu\overline{A}e^\mu B\right) . \tag{2d.202}$$

Next, making use of the identities (2a.38e) and (2a.38f), we can then write

$$\left[\, T^{(j)}, T^{(k)} \,\right] = \left\{\mathrm{Sc}\left((x\bar{c})^j\bar{e}^\mu\right)\mathrm{Sc}\,\partial_\mu\!\left((x\bar{c})^k x\bar{e}^\nu\right) - \mathrm{Sc}\left((x\bar{c})^k\bar{e}^\mu\right)\mathrm{Sc}\,\partial_\mu\!\left((x\bar{c})^j x\bar{e}^\nu\right)\right\}\partial_\nu \ . \tag{2d.203}$$

On the other hand, we have

$$\partial_\mu\!\left((x\bar{c})^k x\right) = \sum_{n=0}^{k} (x\bar{c})^n e_\mu (\bar{c}x)^{k-n} \ , \tag{2d.204}$$

whence

$$\mathrm{Sc}\,\partial_\mu\!\left((x\bar{c})^k x\bar{e}^\nu\right) = \mathrm{Sc}\left\{ e_\mu \sum_{n=0}^{k} (\bar{c}x)^{k-n}\bar{e}^\nu (x\bar{c})^n \right\} . \tag{2d.205}$$

Through (2d.201) with $A = (x\bar{c})^j$ and $B = \sum_{n=0}^{k} (\bar{c}x)^{k-n}\bar{e}^\nu(x\bar{c})^n$, we find

$$\mathrm{Sc}\left\{(x\bar{c})^j\bar{e}^\mu\right\}\mathrm{Sc}\left\{\partial_\mu\!\left((x\bar{c})^k x\bar{e}^\nu\right)\right\} = (k+1)\,\mathrm{Sc}\left\{(x\bar{c})^{j+k} x\bar{e}^\nu\right\} \tag{2d.206}$$

which, upon insertion into Eq. (2d.202), proves Eq. (2d.200).

Note that we have

$$T^{(0)} = -\,Sc\,(x\bar{D}) = x^{\mu}\partial_{\mu} = -\,i\,\Delta \ , \tag{2d.207}$$

$$T^{(0)} = -\,Sc\,T^{(-1)} = -\frac{1}{|\,c\,|^{2}}\,Sc\,(c\bar{D}) = -\,i\frac{1}{|\,c\,|^{2}}\,c^{\mu}P_{\mu}(x\bar{D}) = x^{\mu}\partial_{\mu} = -\,i\,\Delta \ , \tag{2d.208}$$

$$T^{(1)} = -\,Sc\,(x\bar{c}x\bar{D}) = -\,ic^{\mu}C_{\mu} \tag{2d.209}$$

where $P_{\mu} \equiv -\,i\partial_{\mu}$ and $C_{\mu} \equiv -\,ix^{\nu}x_{\nu}\partial_{\mu} + 2\,ix_{\mu}x^{\nu}\partial_{\nu}$. These are just the generator of special conformal transformations. So for $k = 0, \pm 1$, the subalgebra of Eq. (2d.200) is related to the space $\frac{SO(5,1)}{SO(4)}$. The latter coset involves dilatations, translations and special transformations. The parameter free form of the infinite algebra (2d.200) is more complicated and is given in Ref.[117].

It should be noted that the differential operators (2d.199) can also be gotten directly from the change induced in a Fueter function $f(u) = \sum_{n} a_k u^k$ through the increment of the quaternionic variable $u = \bar{c}x$

$$df = f(u+du) - f(u), \qquad du = u + \varepsilon_n u^{n+1} \ . \tag{2d.210}$$

So, expanding to first order in ε_n

$$df = \varepsilon_n u^{n+1} \sum_{n} k u^{k+1} a_k = \varepsilon_n u^{n+1} f'(u) \tag{2d.211}$$

where $f'(u) \equiv f'\left(\tau + \vec{e}\cdot\vec{\xi}\right) = \frac{\partial}{\partial\tau} f\left(\tau + \vec{e}\cdot\vec{\xi}\right)$. Therefore, we can write

$$\begin{aligned} df &= -\,\varepsilon_n L_n\, f = -\,\varepsilon_n \left(u^{n+1}\right)^{\alpha}\partial_{\alpha}\, f \\ &= -\,\varepsilon_n\, Sc\left(u^{n+1}\bar{D}_u\right) f \end{aligned} \tag{2d.212}$$

with $\overline{D}_u = \frac{\partial}{\partial \tau} - \vec{e} \cdot \frac{\partial}{\partial \vec{\xi}}$. It follows that the L_n satisfy the Virasoro algebra

$$[\, L_n \,,\ L_m \,]\, f(u) = (m-n)\, L_{m+n}\, f(u) \ . \tag{2d.213}$$

From the above tools, the possibility is now opened for defining a left or right pseudo-conformal field as a quaternionic Fueter function transforming like a power of S_L or S_R. Due to the inhomogeneous term in the transformation that vanishes for a Mobiüs mapping, such fields will represent a centrally extended Fueter algebra, the extension being quaternionic in general. The determination of the exact nature of such an extension is very much an open problem in the representation theory of infinite dimensional algebras [135]. The general problem, first tackled long ago by Gelfand and Fucks [136, 137], has in recent years been studied by Fradkin and collaborators [121] among others from various angles.

Next, we elaborate on some material in Section 2.a.2, on the (4 x 4) matrix representation of quaternions It has a natural place in this section as it relates to the factorization of the energy-momentum tensor as the D = 4 analog of the Sugawara-Sommerfield factorization.

In a large class of D = 2 chiral field theories, one is led to the Sugawara-Sommerfield form of the traceless energy momentum tensor:

$$T_{\alpha\beta} = J^i_\alpha J_{i\beta} - \frac{1}{2}\, \delta_{\alpha\beta} J^{i\gamma} J_{i\gamma} \tag{2d.214}$$

where J^i_α and their duals $\tilde{J}^i_\alpha \equiv \varepsilon_{\alpha\beta} J^{i\beta}$ are conserved currents, $\partial_\alpha J^{\alpha i} = \partial_\alpha \tilde{J}^{\alpha i} = 0$. Then,

$$\partial_\alpha T^\alpha_\beta = \left(\partial_\alpha J^{i\alpha}\right) J_{i\beta} + J^{i\alpha}_{\ \alpha}\, \partial_\alpha J^i_\beta - J^{i\alpha}\, \partial_\beta J_{i\alpha} = 0 \ . \tag{2d.215}$$

It is known [138] that the moments of J^i_α generate a Kac-Moody algebra, whereas those of $T_{\alpha\beta}$ satisfy a Virasoro algebra.

In conformally invariant D = 4 theories, the energy momentum tensor $T_{\alpha\beta}$ is traceless and symmetric. It can be written as a traceless symmetric (4x4) matrix associated with the representation (1, 1) of O(4). So it should admit a factorization into (1, 0) self-dual fields $F^i_{\alpha\beta}$ and its anti-self dual fields $G^i_{\alpha\beta}$ (i being an internal symmetry index), namely $T_{\alpha\beta} = F^i_{\alpha\gamma} G^\gamma_{\beta i}$.

From Section 2.a.2, specifically Eqs. (2a.39 - 2a.47), we know that the quaternions F and G have two (4 x 4) matrix representations, associated with left and right quaternionic multiplications, respectively. Since the latter operations commute, so do the matrix representations of F and G. In Eq. (2a.44), we already presented the quaternionic representation T of $T_{\alpha\beta}$. When $\vec{f}$ and $\vec{g}$ are labelled by an internal symmetry index, i.e. we have $\vec{f^i}$ (or F^i) and $\vec{g^i}$ (or G^i), they are realized by the self-dual and anti-self-dual parts of a Yang-Mills field strength tensor in R^4. Then self-duality reads $\vec{g^i} = 0$, ($G^i = 0$) .

The currents J^i_μ are represented by (4 x 4) quaternionic matrices with trace $4J^i_0$ and they transform like vectors under O(4). In the Maxwellian case, we have

$$J = F\underset{\leftarrow}{\overline{D}} = DG = \frac{1}{2}\left(DG + F\,\underset{\leftarrow}{\overline{D}}\right) \quad . \tag{2d.216}$$

So the homogeneous set of Maxwell equations reads

$$DG - F\,\underset{\leftarrow}{\overline{D}} = 0 \quad . \tag{2d.217}$$

The current $J = J_0 + \vec{\sigma_R}\cdot\vec{J}$ transforms like the $\left(\frac{1}{2}, \frac{1}{2}\right)$ representation of O(4). Its conservation is given by $Sc\left(\overline{D}G\right) = Sc\left(\Box\, G\right) = 0$, since G is traceless. In the self-dual case G = 0, hence J = 0 as expected. The same result follows for anti-self-duality.

The decomposition of T in terms of the currents is

$$T = (D^{-1}J)(J\overleftarrow{\bar{D}}^{-1}) \, , \tag{2d.218}$$

D^{-1} is the massless Dirac Green's function. For J = 0, we can obtain F and G as quaternion analytic functions. Thus DG = 0, then $G = \Box \sum_n c_n (p^{-1}x)^n$, which is also a solution of $\bar{x}\,DG = 0$, associated with the harmonic analyticity discussed previously.

The energy momentum tensor T is not the only tensor that can be constructed from F and G. In fact we can obtain a scalar, (0, 0), representation of O(4), a self-dual (2, 0) and an anti-self-dual (0, 2) Weyl tensors W_L and W_R, represented by traceless (3 x 3) matrices. W_L can be combined with a (0, 0) scalar into a (4 x 4) matrix L while W_R is combined with a (0, 0) scalar r into a (4 x 4) matrix R:

$$L = L^T = \begin{pmatrix} s & 0 \\ 0 & W_L \end{pmatrix} = F\Delta F$$

$$= \begin{pmatrix} -\frac{1}{3}\vec{f}\cdot\vec{f} & 0 & 0 & 0 \\ 0 & f_1^2 - \frac{1}{3}\vec{f}\cdot\vec{f} & f_1 f_2 & f_3 f_1 \\ 0 & 0 & f_2^2 - \frac{1}{3}\vec{f}\cdot\vec{f} & f_2 f_3 \\ 0 & 0 & 0 & f_3^2 - \frac{1}{3}\vec{f}\cdot\vec{f} \end{pmatrix} \tag{2d.219}$$

and similarly for R

$$R = \begin{pmatrix} r & 0 \\ 0 & W_R \end{pmatrix} = G \, \Delta \, G \tag{2d.220}$$

where we have introduced the diagonal matrix Δ

$$\Delta \equiv \begin{pmatrix} -\frac{2}{3} & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & \frac{1}{3} \end{pmatrix} \tag{2d.221}$$

and where we have $-3s = \vec{f}\cdot\vec{f} = \frac{1}{4}\left(\vec{E}+\vec{B}\right)^2$ and $-3r = \vec{g}\cdot\vec{g} = \frac{1}{4}\left(\vec{E}-\vec{B}\right)^2$, giving the two invariants, $\vec{E}^2 + \vec{B}^2$ and $2\,\vec{E}\cdot\vec{B}$ of the Euclidean Maxwell's theory.

Here we also note that the curvature tensor in a Riemannian euclidean tensor may be decomposed into five pieces with the following O(4) transformation properties [139] : 1) $R = (0,0)$, the scalar curvature; 2) $R_{\mu\nu} - \frac{1}{4}g_{\mu\nu} = (1,1)$, the traceless Ricci tensor; 3) $C^L_{\alpha\beta\mu\nu} = (2, 0)$, the self-dual Weyl tensor; 4) $C^R_{\alpha\beta\mu\nu} = (0, 2)$, the anti-self-dual Weyl tensor. Here these 20 functions can be expressed quadratically in terms of the six components and its pieces can be factorized into FG, FΔF and GΔG. If G = 0 (self-duality), then the Ricci tensor as well as W_R vanish; we then have a half-flat space. All the curvature may be expressed in terms of F, which has the quaternionic analytic form of $F(x) = \Box \sum_n \alpha_n (x\bar{c})^n$. When F is meromorphic, we have a gravitational instanton solution. These results generalize to the Yang-Mills case with $F\Delta F \rightarrow F^i \Delta F_i$, etc.

All this shows the factorization of the self-dual Weyl tensor to be a better analog to the factorization of Sugawara and Sommerfield [140, 141] of T in D = 2. There both factors F can be analytic, whereas T in D = 4 requires one analytic factor (1, 0) and one anti-analytic factor (0, 1).

For an operator product expansion of L for example, we would have

$$L(x)L(x') = \begin{pmatrix} s(x)s(x') & 0 \\ 0 & W_L(x)W_L(x') \end{pmatrix} \tag{2d.222}$$

$W_L(x)W_L(x')$ is a (3 x 3) matrix that decomposes into a multiple of unity (spin 0), an antisymmetric matrix (spin 1) and a symmetric traceless matrix. Thus we can write

$$W_L(x)W_L(x') = K(x-x')\, I + u(x-x')\, F(x) + v(x-x')\, W_L(x) \qquad (2d.223)$$

with singular c-number coefficients which can be determined by dimensional arguments. The product $T(x)T(x')$ is more complicated and bears less resemblance to the D = 2 case. The analog of $T(z)$ in D = 2 then appears to be $W_L(x)$, which has the simplest operator product expansion.

To test its true usefulness and power, the above quaternionic formalism must still await further progress in the representation theory of D = 4 diffeomorphism and current algebras and the discovery of D = 4 counterpart of the Wess-Zumino-Novikov-Witten model.

To approach the problem of D = 4 gravity, we elaborate further on the quasi-conformal structure of spacetime. For this, we follow closely the treatment of Ref.[117].

On a D = 4 Riemannian space, consider a general coordinate transformation $y^\mu = f(x^\mu)$, regular around some coordinate origin x = 0. So

$$y^\mu = f(x^\mu) = a^\mu + b^\mu_\alpha x^\alpha + c^\mu_{\alpha\beta}\, x^\alpha x^\beta + \ldots \qquad (2d.224)$$

where the coefficients of the nth degree monomials are tensors of order (n+1). In quaternionic form, Eq. (2d.224) reads

$$y = a + b_1\, x\, b_2 + b_3 \bar{x} b_4 + c_1 x c_2 x c_3 + c_4 \bar{x} c_5 \bar{x} c_6 \qquad (2d.225)$$

where not all the constants b_n and c_n are independent since $e_\mu\, x\, e_\mu = -2x$. As noted previously, Eq. (2d.224) is equivalent to the most general quaternionic polynomial (2d.21).

Furthermore, by way of the **R**-algebra isomorphism $e_j = -\, i\, \sigma_j$, it can be rewritten in a (2 x 2) complex or (4 x 4) real matrix form, i.e.

$$\vec{e}\cdot\vec{\omega} = \vec{J_L}\cdot\vec{\omega} = \begin{pmatrix} 0 & -\omega_3 & \omega_2 & \omega_1 \\ \omega_3 & 0 & -\omega_1 & \omega_2 \\ -\omega_2 & \omega_1 & 0 & \omega_3 \\ -\omega_1 & -\omega_2 & -\omega_3 & 0 \end{pmatrix}, \tag{2d.226}$$

$$\vec{e'}\cdot\vec{\xi} = \vec{J_R}\cdot\vec{\xi} = \begin{pmatrix} 0 & -\xi_3 & \xi_2 & -\xi_1 \\ \xi_3 & 0 & -\xi_1 & -\xi_2 \\ -\xi_2 & \xi_1 & 0 & -\xi_3 \\ \xi_1 & \xi_2 & \xi_3 & 0 \end{pmatrix}. \tag{2d.227}$$

Observe that, in the (4 x 4) representations, the J_L^i and J_R^i are respectively self-dual and anti-self-dual real matrices.

Since under an O(4) ≈ SU(2) x SU(2) rotation, the position quaternion x transforms as

$$x' = m\, x\, \bar{n} \qquad \left(|m|^2 = m\,\bar{m} = 1\,,\, |n|^2 = n\,\bar{n} = 1\right). \tag{2d.228}$$

Any quaternion y behaving as a 4-vector under O(4) transforms as $y' = m\, y\, \bar{n}$. So the transformation properties of the quaternions in Eq. (2d.225) are fixed:

$$a' = m\, a\, \bar{n}\ ,\quad b_1' = m\, b_1\, \bar{m}\ ,\quad b_2' = n\, b_2\, \bar{n}\,,$$

$$b_3' = m\, b_3\, \bar{n}\ ,\quad b_4' = m\, b_4\, \bar{n}\quad \text{etc.} \tag{2d.229}$$

The quaternions in the series (2d.225) fall therefore into three categories labelled by the set of two indices (λ_1, λ_2) of the representations of SU(2) x SU(2):

a) V-type $\qquad v' = m\, V\, \bar{n}\,, \qquad v \in \left(\frac{1}{2}, \frac{1}{2}\right)$ (2d.230)

b) L-type $\qquad L = n\, L\, \bar{n}\,, \qquad L \in |\,(0,0) + (0,1)\,|$ (2d.231)

c) R-type $\qquad r' = n\,r\,\bar{n}\,, \qquad r \in |\,(0,0) + (0,1)\,|\,. \qquad$ (2d.232)

The V-type quaternions are 4-vectors. The L and R types have O(4) invariant scalar parts while their vectorial parts (i.e. the traceless parts in the matrix representations) are associated with the self-dual (1, 0) and antiself-dual (0, 1) tensors, respectively. Such quaternions with L- and R-type rotation or handedness properties constitute the fundamental elements of the improved Fueter theory. Because they play a basic role in field theory applications, we have given them the special name of *quators*, specifically *left-quators* and *right-quators*.

We now remark that, in a general coordinate mapping, the new coordinates y^α need not behave like a 4-vector. A familiar example is the transformation of a 3-position vector in going over to a polar coordinate basis. While its magnitude is an O(3) scalar, the corresponding polar angles θ and ϕ transform nonlinearly under rotation. If we now perform a diffeomorphism map $x \to y(x)$, such that the O(4) subgroup is represented linearly (i.e. where x and y are both real 4-dimensional representations of O(4)), then x and y can be either four-vectors <u>or</u> quators.

To associate a position vector x with a quator, it is clear from Eqs. (2d.3)-(2d.7) that we need one more vector p, invariant under translations. In fact such an auxiliary vector can be interpreted as an arbitrary time direction in the canonical Hamiltonian formalism. Its direction runs parallel to the space-like hyperplane in the Minkowski case, a familiar parametrization in the covariant quantization of relativistic fields. We can form the following left and right quators

$$u = \bar{p}\,x = \tau + \vec{e}\cdot\vec{\xi} \quad , \quad v = x\,\bar{p} = \tau + \vec{e}\cdot\vec{\omega} \tag{2d.233}$$

since p is a V-type quaternion (2d.230). They are related by

$$v = p\,u\,p^{-1} \tag{2d.234}$$

so that

$$\tau = Sc(u) = Sc(v) = \frac{1}{2}(\,u + \bar{u}\,) = \frac{1}{2}(\,v + \bar{v}\,) = p^0x^0 + \vec{p}\cdot\vec{x} = p_\mu x^\mu\,. \tag{2d.235}$$

This new time τ is an O(4) scalar, so is the τ-translation generator, the Hamiltonian operator. In the literature, with v^μ being a unit vector along p^μ and representing the normal to the hyperplane of equal τ events, an invariant Hamiltonian

$$H = v^\mu P_\mu = i\, v^\mu\, \partial_\mu \tag{2d.236}$$

is introduced for the purpose covariant field quantization. However, the three momenta P_i (i =1, 2, 3) are not given in a covariant form. In Euclidean spacetime, in the place of the P_i's we can use two kinds of covariant entities:

$$P^R_{\alpha\beta} = v_\alpha\, P_\beta - v_\beta\, P_\alpha + \varepsilon_{\alpha\beta\gamma\delta}\; v^\gamma\, P^\delta \tag{2d.237}$$

and

$$P^L_{\alpha\beta} = v_\alpha\, P_\beta - v_\beta\, P_\alpha - \varepsilon_{\alpha\beta\gamma\delta}\; v^\gamma\, P^\delta\ . \tag{2d.238}$$

When v^μ is along the time direction ($v^\mu = (1, 0, 0, 0)$), we recover

$$H = P_o\,,\quad P^L_{oi} = \frac{1}{2}\,\varepsilon_{ijk}\, P^L_{jk} = P_i\ ,\quad P^R_{oi} = -\frac{1}{2}\,\varepsilon_{ijk}\, P^L_{jk} = -\,P_i\ . \tag{2d.239}$$

Consequently, H coincides with the usual Hamiltonian and both P^L_{oi} and P^R_{oi} collapse into the three components of the usual momentum.

Using Eq. (2a.4), we obtain

$$\vec{\xi} = p^o\,\vec{x} - \vec{p}\,x^o - \vec{p}\times\vec{x}\,,\quad \vec{\omega} = p^o\,\vec{x} - \vec{p}\,x^o + \vec{p}\times\vec{x}\ . \tag{2d.240}$$

Alternatively,

$$\omega_1 = \omega_{01} = \omega_{23} = L_{01} + \widetilde{L_{01}} = L_{23} + \widetilde{L_{23}}\ , \tag{2d.241}$$

$$\xi_1 = \xi_{01} = -\xi_{23} = L_{01} - \widetilde{L_{01}} = -\left(L_{23} - \widetilde{L_{23}} \right) , \tag{2d.242}$$

the other components being given by cyclic permutations of (1, 2, 3). The tensor L and its dual are defined by $L^{\mu\nu} = p^\mu x^\nu - p^\nu x^\mu$, $\widetilde{L^{\mu\nu}} = \frac{1}{2}\epsilon^{\mu\nu\alpha\beta} L_{\alpha\beta}$.

Clearly, the ω_i and ξ_i (i =1, 2, 3) are just the independent components of the self-dual and anti-self-dual tensor $\xi_{\mu\nu}$:

$$\omega_{\mu\nu} = \widetilde{\omega}_{\mu\nu} = L_{\mu\nu} + \widetilde{L}_{\mu\nu} , \quad \xi_{\mu\nu} = \widetilde{\xi}_{\mu\nu} = L_{\mu\nu} - \widetilde{L}_{\mu\nu} . \tag{2d.243}$$

They are the (1, 0) and (0, 1) representations of O(4) , respectively. The left quator v and right quator u then take the desirable form of

$$u = \tau + \frac{1}{2} e'_{\mu\nu} \xi_{\mu\nu} \qquad , \qquad v = \tau + \frac{1}{2} e_{\mu\nu} \omega_{\mu\nu} \tag{2d.244}$$

where $e_{\mu\nu}$ and $e'_{\mu\nu}$ are the self-dual and anti-self-dual quaternion units. As such they are the appropriate D = 4 counterparts of left and right movers of two-dimensional conformal field theory.

If p is a unit quaternion, $p\bar{p} = |p|^2 = 1$, a suitable O(4) transformation can reduce p to unity. In such a "rest" frame

$$v(\vec{p}=0) = u(\vec{p}=0) = x^0 + \vec{e}\cdot\vec{x} = x \ ; \tag{2d.245}$$

hence τ coincides with the time coordinate and both $\vec{\omega}$ and $\vec{\xi}$ collapse into the 3-position vector. As with the vector x, the transformed coordinate y in Eq. (2d.225) is expressible as a quaternionic vector $y = e_\alpha y_\alpha$ or by the quators $u = \bar{p}y$, $v = y\bar{p} = p\,u\,p^{-1}$, according to their behavior under the local O(4) subgroup of diffeomorphisms.

Fueter, we recall, did in fact consider various subgroups of the general coordinate mappings (2d.225). Specifically, he did so by constructing the left, right and left-right holomorphic series L(x) (2d.18), R(x) (2d.19) and W(x) (2d.14). In light of the above discussion, these mappings as they stand clearly cannot be subgroups of (2d.225) if both x and y transform linearly under O(4). Thus it is easy to check that L(x) will transform like the superposition of tensors of different ranks when x transforms like a vector (2d.228). Consequently, a serious problem arises since the physically required O(4) covariance is not maintained.

However, O(4) covariance does hold if we are willing to replace the argument in the Fueter series by a right or left-quator u and v. Indeed, if we take the function

$$U = f(u) = \sum_n c_n x^n \quad , \ \mathrm{Vec}(c_n) = 0 \tag{2d.246}$$

with the O(4) transformation laws $u' = n\,u\,\bar{n}$, $U' = n\,U\,\bar{n}$ or

$$V = p\,U\,p^{-1} = f\,(pup^{-1}) = f(v) \tag{2d.247}$$

with $v' = m\,v\,\bar{m}$, $V' = m\,V\,\bar{m}$, the function (2d.246) can be put in vector form by way of the vector

$$Z = p\bar{U} = \bar{V}\,p = p\,\bar{Y}\,p\ . \tag{2d.248}$$

From Eqs. (2d.246) and (2d.247) we get

$$\bar{Z} = f(u)\bar{p} = f(\bar{p}\,x)\,\bar{p} = \bar{p}\,f(x\bar{p})\ . \tag{2d.249}$$

By way of the quator operators

$$D_u \equiv \frac{\partial}{\partial\tau} + \vec{e}\cdot\frac{\partial}{\partial\vec{\xi}}\ ,\ D_v \equiv \frac{\partial}{\partial\tau} + \vec{e}\cdot\frac{\partial}{\partial\vec{\omega}}\ , \tag{2d.250}$$

and the O(4) invariant operators $D_u\overline{D}_u = D_v\overline{D}_v = |p|^{-2}\,\square$, we get

$$\square\, D_v\, f(u) = 0 \tag{2d.251}$$

and hence the covariant equation

$$\square\, D\,\overline{Z} = 0\ . \tag{2d.252}$$

Since the coefficients c_n are real in Eq. (2d.246), f(u) arises from a complex stem function f(z) obeying Schwarz's reflection principle. So $f(\overline{u}) = \overline{f(u)}$.

Furthermore, we can construct quaternionic function from more general power series of a complex variable with complex coefficients. To do so, we first recast Eq. (2d.246) as

$$U = A(\tau,\ \xi) + j\,B(\tau,\xi) = f(\tau + j\xi) \tag{2d.253}$$

with $\xi = |\vec{\xi}|$, $j \equiv \frac{\vec{e}\cdot\vec{\xi}}{\xi}$ such that $\tau^2 + \xi^2 = |x|^2|p|^2$, $j^2 = -1$. Moreover, we have

$$\partial_\tau\, j = 0 \quad , \quad \partial_\xi\, j = \xi^i \frac{\partial}{\partial \xi^i} j = 0 \quad , \quad \frac{\partial}{\partial \xi^i}\xi^i = \frac{\xi^i}{\xi} \quad , \tag{2d.254}$$

$$\frac{\partial}{\partial \xi^i} j = \frac{1}{\xi}\left(\delta_{ik} - \frac{\xi_i \xi_k}{\xi^2}\right) e_k\ . \tag{2d.255}$$

Since under O(4), τ, ξ are invariant with ξ^i transforming like an anti-self-dual tensor, with $j' = n\,j\,\overline{n}$, ($j'^2 = -1$) being an anti-self-dual complex structure, all the above relations are O(4) covariant.

The action of the quator Dirac operator on U gives

$$D_u U = A_\tau + j\,B_\tau + e_i \frac{\xi^i}{\xi}\left(A_\xi + j\,B_\xi\right) + B e_i \frac{\partial}{\partial \xi^i} j\ , \qquad (2d.256)$$

the suffixes τ and ξ denote partial derivation w.r.t. τ and ξ. Using the relation $e_i \frac{\partial}{\partial \xi^i} j = -\frac{2}{\xi}$, derived from Eq. (2d.255), we get

$$D_u U = \left(A_\tau - B_\xi\right) + j\left(B_\tau + A_\xi\ \right) - \frac{2B}{\xi}\ . \qquad (2d.257)$$

From Eq. (2d.253) it follows that

$$A(\tau,\xi) + i\,B(\tau,\xi) = f\,(\tau + i\,\xi\,) \qquad (2d.258)$$

giving the Cauchy-Riemann relations

$$A_\tau - B_\xi = B_\tau + A_\xi = 0 \qquad (2d.259)$$

or

$$\left(\partial_\tau + j\,\partial_\xi\right)(\,A + j\,B) = 0\ . \qquad (2d.260)$$

Consequently,

$$D_u U = -\frac{2B}{\xi}, \qquad \mathrm{Vec}\,(D_u\,U) = 0\ . \qquad (2d.261)$$

Next, consider another quator function

$$g(u) = L(\tau,\xi) + j\,M(\tau,\xi) = g\,(\,\tau + j\,\xi) = \sum_n k_n u^n\ . \qquad (2d.262)$$

Since $\left[\,j\,,\ \left(\partial_\tau + j\,\partial_\xi\right)\right] = 0$,

$$A_\tau - B_\xi = B_\tau + A_\xi = (\partial_\tau + j\partial_\xi)\,j\,g = (\partial_\tau + j\partial_\xi)(-M + j\,L\,) = 0\ , \qquad (2d.263)$$

so that

$$D_u(f(u) + j\,g(u)) = D_u \sum_n (c_n + j\,k_n)\,u^n) = -\frac{2}{\xi}(B + L) \qquad (2d.264)$$

and

$$\mathrm{Vec}\,D_u\,F(u) = 0 \quad . \qquad (2d.265)$$

The series

$$F(u) = \sum_n (c_n + j\,k_n)\,u^n = \sum_n u^n(c_n + j\,k_n) \qquad (2d.266)$$

has quator coefficients, and is associated with the general complex holomorphic stem function $f(z) = \sum_n a_n z^n$ with $a_n \equiv (c_n + i\,k_n) \in \mathbf{C}$.

By way of the harmonicity of $B(\tau, \xi)$ and $L(\tau, \xi)$ i.e. $\left(\partial_\tau^2 + \partial_\xi^2\right) B = \left(\partial_\tau^2 + \partial_\xi^2\right) L = 0$, it also follows that

$$\Box\, D_u F(u) = -2\;\Box\left(\frac{B+L}{\xi}\right) = -2\,\frac{B_{\tau\tau} + L_{\tau\tau} + B_{\xi\xi} + L_{\xi\xi}}{\xi} = 0 \quad . \qquad (2d.267)$$

So the above O(4) covariant quator mappings F(u) with their associated complex stem functions f(z) are more general than Fueter mappings.

We now return to functions with argument x, namely to

$$W(x) = F(\bar{p}\,x)\,\bar{p} = \sum_n (\bar{p}x)^n \left(c_n + \frac{\mathrm{Vec}(\bar{p}x)}{|\mathrm{Vec}(\bar{p}x)|}\,k_n\right) p \quad . \qquad (2d.268)$$

Like Z (2d.252), it satisfies $\Box\, D\, W(x) = 0$, which is translation and O(4) invariant. From the first invariance, it follows that W(x-a) is also a solution; from the second, another solution $W' = m\,W\,\bar{n}$ $\left(\overline{W'} = n\,\overline{W}\,\bar{m}\right)$ is generated by an O(4) rotation $x' = m\,x\,\bar{n}$, $D' = m\,D\,\bar{n}$. We are then naturally led to the infinite parameter mappings

$$\overline{W}(x) = n\, F\left(\bar{p}\,(x\text{-}a)\right)\bar{p}\,\overline{m} \quad . \tag{2d.269}$$

They form a subgroup of the general diffeomorphism group and arise from complex analytic stem functions. For reasons to be made clear, such functions will be called *quasi-conformal* quaternionic functions.

Finally, Fueter's left holomorphic series can also be given a covariant form by allowing the coefficients c_n and k_n in Eq. (2d.266) to be constant quators. For that purpose, we consider the mapping

$$Y_L = \sum_n u^n \,(C_n + j\, K_n) \tag{2d.270}$$

where

$$u = \bar{p}\, x = \tau + \vec{e}\cdot\vec{\xi} \quad , \quad Y_L = \bar{p}\, y \, ,$$
$$C_n = c_n + \vec{e}\cdot\vec{c_n} \qquad , \quad K_n = k_n + \vec{e}\cdot\vec{k_n} \, , \tag{2d.271}$$

here $\vec{c_n}$ and $\vec{k_n}$ are independent components of anti-self-dual tensors, so that C_n and K_n are left quators. Then

$$\Box\, D_u\, Y_L = 0 \tag{2d.272}$$

still holds although Vec $D_u Y_l$ is anti-self-dual and nonzero.

Next, we show that, notably, the conformal group forms a subgroup of the above infinite quasi-conformal group. To that end, we set

$$\overline{W} = \overline{b'}\; y\; \overline{b'} \quad , \quad p = b \, , \quad b' = m\, b\, \bar{n} \quad , \tag{2d.273}$$

in the quasi-conformal transformation (2d.269). Such a substitution yields

$$y = m\left[\bar{b}^{-1}\, F\left(\bar{b}\left(x - a\right)\right)\right]\bar{n} \quad . \tag{2d.274}$$

Next, by choosing

$$F(u) = \sum_{k=1}^{\infty} \lambda^{-k} u^{k} = \sum_{k=1}^{\infty} \left[\frac{\bar{b}(x-a)}{\lambda}\right]^{k} = \frac{\bar{b}(x-a)}{\lambda}\left(1 - \frac{\bar{b}(x-a)}{\lambda}\right) \quad (2d.275)$$

which satisfies $\Box D_u F(u) = 0$, we get

$$y = m\left(\frac{\lambda}{x-a} - \bar{b}\right)^{-1} \bar{n} \ , \quad (2d.276)$$

which is biharmonic $\Box\Box\, y(x) = 0$. Equation (2d.276) is readily recognized as the standard nonlinear, Mobiüs representation of the euclidean conformal group SO(5, 1) $\approx$ SL(2, **H**). Hence, up to an O(4) rotation, the covariant Fueter mappings (2d.269) form an infinite group having the conformal group as a subgroup.

The form (2d.276) arises from a coset decomposition of the SL(2, **H**) element in its unimodular (2 x 2) quaternionic matrix form

$$M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \quad , \quad \mathrm{Det}\, M = \left| \alpha\delta - \beta\delta^{-1}\gamma\delta \right| = 1 \ . \quad (2d.277)$$

Moreover, M admits the following decomposition into two triangular and one diagonal unimodular matrices:

$$M = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \bar{c} & 1 \end{pmatrix} \begin{pmatrix} k & 0 \\ 0 & r \end{pmatrix} \quad , \qquad (|k|\,|r| = 1) \quad (2d.278)$$

with

$$x = \beta\,\delta^{-1} \ , \ \bar{c} = \left(\alpha\gamma^{-1} - \beta\,\delta^{-1}\right) \ ,$$
$$k = \alpha - \beta\,\delta^{-1}\gamma \ , \ r = \delta \ . \quad (2d.279)$$

The quaternionic diagonal part (k, r) stands for the Weyl subgroup of rotation and dilatation O(4)xO(1,1), commuting with the Dirac matrix $\rho_3 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$, $I = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}$. So the coset element $K \approx \dfrac{SO(5,10)}{O(4) \times O(1,1)}$ is represented by the unimodular matrix

$$K = M\rho_3 M^{-1}\rho = \begin{pmatrix} 1+2\,x\bar{c} & 2(x+x\bar{c}x) \\ 2\bar{c} & 1+2\bar{c}x \end{pmatrix} . \tag{2d.280}$$

Under a general SL(2, **H**) transformation,

$$M' = G\,M = \begin{pmatrix} \alpha' & \beta' \\ \gamma & \delta' \end{pmatrix} \tag{2d.281}$$

with

$$G = \begin{pmatrix} f & a \\ \bar{b} & g \end{pmatrix} , \ \text{Det}\, G = 1 , \tag{2d.282}$$

a (2 x 2) quaternionic matrix. The coset elements x and c transform as

$$x' = \beta'\,\delta'^{-1} \ , \ \bar{c}' = \left(\alpha'\gamma - \beta'\,\delta'\right)^{-1} \ ; \tag{2d.283}$$

namely, linearly when G belongs to the Weyl subgroup and nonlinearly when $G \in \dfrac{SO(5,1)}{O(4) \times O(1,1)}$. As for K, $K' = G\,K\,\overline{G}$ with $G = \rho_3\, G^{-1}\, \rho_3$. For the Weyl group

$$W = \begin{pmatrix} \frac{1}{\sqrt{\lambda}}\, m & 0 \\ 0 & \sqrt{\lambda}\, n \end{pmatrix} , \ (\,|m| = |n| = 1\,) , \tag{2d.284}$$

$$K' = W\,K\,W^{-1} \tag{2d.285}$$

giving

$$x' = \lambda^{-1}\, m\, x\, \bar{n} \ , \quad \bar{c'} = \lambda\, n\, \bar{c}\, \bar{m} \ , \tag{2d.286}$$

$$x'\,\bar{c'} = m\,(x\,\bar{c})\,\bar{m} \ , \quad \bar{c'}\,x' = n\,\bar{c}\,x\,\bar{n} \ . \tag{2d.287}$$

If G belongs to the translation subgroup $G = T = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$, $T = \overline{T}$, we have

$$x' = x + a \ , \ c' = c \ . \tag{2d.288}$$

So the vector c is translation invariant and can be taken as the auxiliary direction p discussed previously. The diagonal elements of K are seen to be left and right quators while its off diagonal elements are 4-vectors. Since any power of K behaves like K under the Weyl group, for any formal power series Q = f(K), we have

$$Q' = W\, Q\, W^{-1} = f(K') \ . \tag{2d.289}$$

This result shows the diagonal elements of Q to be functions of the quators $v = x\,\bar{c}$ and $u = \bar{c}\,x$, while the off-diagonal elements are vectors obtained by multiplying x or c by quator functions. So the covariant Fueter mappings can be interpreted as functions on the coset of the conformal group mod its Weyl subgroup.

As to how the coset elements transform under special conformal mappings $G = S = \begin{pmatrix} 1 & 0 \\ \bar{b} & 1 \end{pmatrix}$, $\overline{S} = S$, $K' = S\,K\,S$, one finds

$$x' = x\left(1 + \bar{b}\,x\right)^{-1} = \left(\frac{1}{x} + \bar{b}\right)^{-1} \ , \tag{2d.290}$$

$$c' = \left[\left(\frac{1}{x + c^{-1}} + \bar{b}\right)^{-1} - \left(\frac{1}{x} + \bar{b}\right)^{-1}\right]^{-1} \ . \tag{2d.291}$$

In terms of the quators u and v,

$$u' = u + \bar{b}\,\bar{c}^{-1}\left(u + u^2\right) \quad , \quad v' = v + \left(v + v^2\right)\bar{c}^{-1}\,b \ . \tag{2d.292}$$

Under a transformation belonging to the coset $N = T\,S$, $\overline{N} = \overline{S}\,\overline{T} = S\,T$, the SL(2, **H**) map reads

$$K' = W\, N\, K\, \overline{N}\, W^{-1} \ ; \tag{2d.293}$$

under SO(5, 1) then transforms like Eq. (2d.276) while the coset function Q transforms as

$$Q' = W\, Q\left(N\, K\, \overline{N}\right) W^{-1} \ . \tag{2d.294}$$

And since the matrix elements of K^n are quators and vector functions of x, Q is biharmonic $\Box\Box Q(X) = 0$, and so is Q′, $\Box\Box Q' = 0$. Consequently, all quaternionic elements of the coset function Q are biharmonic.

Next, we consider the tetrad and the metric induced by quasi-conformal Fueter mappings.

Let us take the functions of the right quator u given by Eqs. (2d.233) and (2d.244):

$$u' = \tau' + \vec{e}\cdot\vec{\xi'} = f(u) = f\left(\tau + \vec{e}\cdot\vec{\xi}\right) \ . \tag{2d.295}$$

Alternatively, using $j \equiv \frac{\vec{e}\cdot\vec{\xi}}{\xi}$, we can recast u′ as

$$u' = A\,(\tau,\xi) + j\, B\,(\tau, \xi) \ . \tag{2d.296}$$

From the Cauchy-Riemann relations (2d.260) and using

$$dj = \frac{d\xi^i}{\xi}\left(\delta_{ik} - \frac{\xi_i\xi_k}{\xi^2}\right) e^k \tag{2d.297}$$

obtained from Eq. (2d.255), as well as $d\xi = \frac{\vec{\xi}}{\xi}\cdot d\vec{\xi} \equiv \vec{\kappa}\cdot d\vec{\xi}$, we get

$$d\tau' = A_\tau\, d\tau - B_\tau\, \vec{\kappa}\cdot d\vec{\xi} \tag{2d.298}$$

$$d\vec{\xi} = \left(B_\tau\, d\tau + A_\tau\, \vec{\kappa}\cdot d\,\vec{\xi}\right)\vec{\kappa} + \frac{B}{\xi}\left(d\vec{\xi} - \vec{\kappa}\left(\vec{\kappa}\cdot d\,\vec{\xi}\right)\right). \tag{2d.299}$$

In tensorial form du'^{μ} reads

$$du'^{\mu} = E^{\mu}{}_{\nu}\, d\,u^{\nu} \tag{2d.300}$$

with

$$E = \begin{pmatrix} A_\tau & -B_\tau \kappa^T \\ B_\tau \kappa & \frac{B}{\xi} I + \left(A_\tau - \frac{B}{\xi}\right)\kappa\,\kappa^T \end{pmatrix}. \tag{2d.301}$$

The metric tensor Γ is given by

$$\Gamma = E\,E^T = \begin{pmatrix} A_\tau^2 + B_\tau^2 & 0 \\ 0 & \frac{B^2}{\xi^2} I + \left(A_\tau^2 + B_\tau^2 - \frac{B^2}{\xi^2}\right)\kappa\,\kappa^T \end{pmatrix}. \tag{2d.302}$$

The line element is given by

$$ds^2 = du^{\mu}\Gamma_{\mu\nu}du^{\nu} = H^2 d\tau^2 + \left|\frac{B^2}{\xi^2}\,\delta_{ij} + \left(H^2 - \frac{B^2}{\xi^2}\right)\frac{\xi^i\xi^j}{\xi^2}\right| d\xi^i d\xi^j \tag{2d.303}$$

with $H^2 \equiv A_\tau^2 + B_\tau^2$. It follows that

$$\Gamma_{oi} = 0\ , \tag{2d.304}$$

an O(4) invariant condition as the indices o and i refer respectively to the invariant τ direction and the three independent components of the tensor $\xi_{\alpha\beta}$.

Being O(4) invariant, ds^2 maintains its form under a Weyl transformation. We note that, unlike the D = 2 conformal transformations, $\Gamma_{\mu\nu}$ is not proportional to the flat metric $\delta_{\mu\nu}$. However, due to the condition (2d.304), it is quasi-conformal and becomes conformally flat for a SL(2, **H**) transformation given by Eq. (2d.277). This quasi-conformal structure is preserved under the more general mapping

$$\tau' = f(\tau) \qquad , \qquad \xi'^{\,i} = g^i(\xi^i) \tag{2d.305}$$

consisting of a regraduation of the O(4) invariant time τ and a three-dimensional diffeomorphism of the spatial variables ξ_i. In fact the transformations (2d.305) and Fueter mappings (2d.295) form quasi-conformal subgroups of the D = 4 diffeomorphisms of the quator variables τ and ξ^i, collectively denoted by u^μ.

Next, we proceed to the general situation of a Riemannian space parametrized by the coordinates τ and ξ^i. By choosing three of the four arbitrary functions in a general coordinate transformation, we can go to a frame where $g_{oi} = 0$. The latter form is preserved by the transformations (2d.305). In this coordinate frame, under an O(4) transformation, g_{oo} is a scalar and g_{ij} behaves like a rank 4 tensor $h_{\alpha\beta\mu\nu}$ with

$$h_{\alpha\beta\mu\nu} = h_{\mu\nu\alpha\beta} = -\,h_{\beta\alpha\mu\nu} = -\,h_{\alpha\beta\nu\mu} \quad , \tag{2d.306}$$

$$h_{\alpha\beta\mu\nu} + \frac{1}{2}\,\varepsilon_{\alpha\beta\rho\sigma} h^{\rho\sigma}{}_{\mu\nu} = h_{\alpha\beta\mu\nu} + \frac{1}{2}\,\varepsilon_{\mu\nu\rho\sigma} h_{\alpha\beta}{}^{\rho\sigma} = 0 \quad . \tag{2d.307}$$

These properties are those of a self-dual or anti-self-dual curvature tensor.

Specifically, the space-space metric components are

$$g_{ij} = k(\tau,\xi)\delta_{ij} + q(\tau,\xi)\,(s_i t_j + s_j t_i) \tag{2d.308}$$

where k and q are arbitrary functions of the isotropic coordinates τ and ζ, the s_i and t_i are taken to be self-dual or antiself-dual skew tensors. If g_{io} is to vanish upon a Fueter transformation, the new metric takes the form of $G' = E\,G\,E^T$, $g'_{oi} = 0$ where G has elements $g_{\mu\nu}$ and E is given by Eq. (2d.308).

One finds that the quasi-conformal structure is preserved provided that both s_i and t_i are proportional to the quator coordinates ξ^i, and one obtain

$$g_{oo} = k + q \quad , \quad g_{oi} = 0 \ , \tag{2d.309}$$

$$g_{ij} = k\, \delta_{ij} + q\, \frac{\xi_i \xi_j}{\xi^2} \ . \tag{2d.310}$$

They are solved for

$$g_{oo} = \xi^{-2}\, \xi^i\, \xi^j\, g_{ij} \ , \tag{2d.311}$$

$$\left(g_{ij} - \ \xi^{-2}\, \xi^r\, \xi^s\, g_{rs} \delta_{ij}\right) \xi^j = 0 \tag{2d.312}$$

which follow from the relations $G' = E\, G\, E^T$, $g'_{oi} = 0$. So the corresponding line element is

$$ds^2 = (\ k + q\ )\ d\tau^2 + k\ d\xi^i d\xi_i + q \xi^{-2} \left(\xi^i\ d\xi_i\right)^2 \ . \tag{2d.313}$$

It is exactly of the form of a Schwarzschild line element in Kruskal coordinates. The preservation of the Kruskal form by Fueter transformations in Euclidean gravity was first noted in [127]. A summary of that work will be the subject of Section 2.f.4.

The line element can also be diagonalized by way of two complex coordinates $z = \tau + i\xi$ and $\varsigma \equiv \varsigma_1 + i\ \varsigma_2$. The latter projectively parametrizes a 2-sphere S^2 through the relations

$$\frac{\xi_1 + i\xi_2}{\xi} = \frac{\varsigma}{1 + \frac{|\varsigma|^2}{4}} \quad , \tag{2d.314}$$

$$\frac{\xi_3}{\xi} = \frac{1 - \frac{|\varsigma|^2}{4}}{1 + \frac{|\varsigma|^2}{4}} \quad . \tag{2d.315}$$

We then have

$$ds^2 = (\,k + q)\; dz\, dz^* + k \left(\frac{z - z^*}{2i}\right)^2 \left(1 + \frac{|\varsigma|^2}{4}\right)^{-2} d\varsigma\; d\varsigma^* \tag{2d.316}$$

$$= (\,k + q)\;(d\tau^2 + d\xi^2) + k\,\xi^2 \left(1 + \frac{\varsigma_1{}^2 + \varsigma_2{}^2}{4}\right)^{-2} d\varsigma_1{}^2\; d\varsigma_2{}^2\,, \tag{2d.317}$$

which is form invariant under analytic transformations of z and ς.

When $q = 0$, the spacetime is conformally flat and k is the Weyl factor. When $q \neq 0$, it is quasi-conformally flat with two Weyl factors k and $k' = k + q$. The quasi-conformal structure appears not only for the black hole metric but also for the Gowdy cosmological line element with two Killing vectors. In $D = 4$, the conformal Weyl form is preserved by the 15 parameter group while the quasi-conformal form is preserved by the infinite parameter Fueter group. This was also shown previously with the Schwarzschild metric with Minkowski signature.

Finally, we should mention that the more general quasi-conformal structure of the metric can be cast in a canonical quaternionic form. To do so, we start with the 4-vector diffeomorphism transformation

$$V'_\mu = g_{\mu\nu}\, V^\nu \quad . \tag{2d.318}$$

Its quaternionic equivalent is

$$V' = \lambda^2\, V + \vec{e} \cdot \vec{a}\; V \vec{e} \cdot \vec{b} + c\,\overline{V}\,c \quad , \tag{2d.319}$$

where λ is a scalar, a and b are respectively self- and anti-self-dual tensors, and $c = c^o + \vec{e} \cdot \vec{c}$ is a 4-vector.

In fact, this expression is just the quaternionic version of the Euclidean counterpart of Lichnerowicz's form [142] for a symmetric tensor in space-time; such a tensor always has a threefold decomposition into a multiple of unity, a Maxwell tensor made out of the curvature tensor $F_{\mu\nu}$ and $t_{\mu\nu} = v_\mu v_\nu - \frac{1}{4}\eta_{\mu\nu} v^\lambda v_\lambda$, a traceless symmetric tensor bilinear in a 4-vector v_μ.

Due to the invariance of V' under the scaling transformations

$$\vec{a} \rightarrow k\,\vec{a} \quad , \quad \vec{b} \rightarrow k^{-1}\,\vec{b} \; , \tag{2d.320}$$

the pair actually stand for five degrees of freedom. Combined with one degree of freedom for λ and four for c, we recover the expected 10 degrees of freedom for the metric $g_{\mu\nu}$. And if V and V' transform like left quators, so must a, b, c transform like left quators.

Specializing onto the quasi-conformal case ($g_{oi} = 0$) of Eq. (2d.312), we obtain

$$V' = \lambda^2 V + \frac{1}{2}\left(\vec{e} \cdot \vec{a}\; V \vec{e} \cdot \vec{b}\right) + \frac{1}{2}\left(\vec{e} \cdot \vec{b}\; V \vec{e} \cdot \vec{a}\right) \; . \tag{2d.321}$$

For covariance of the condition (2d.304) for g_{oi}, V and V' must transform like left (right) quators. Then $g_{\alpha\beta}$ depends on seven functions, but invariance under the scaling (2d.320) leaves six independent quantities transforming like the (0, 0) and (2, 0) (or (0, 2) if V is a right quator) representations of O(4). The quasi-conformal matrix G takes the canonical form

$$G = \lambda^2 I + \frac{1}{2}\left(AB' + BA' \right) \tag{2d.322}$$

where, by way of Eq. (2d.227), we obtain the following expressions for A, A', B and B'

$$A = \vec{J}_L \cdot \vec{a} \, , \; A' = \vec{J}_R \cdot \vec{a} \, , \; B = \vec{J}_L \cdot \vec{b} \, , \; B' = \vec{J}_L \cdot \vec{b} \quad , \tag{2d.323}$$

$$A' = \eta \, A \, \eta^{-1} \, , \; \eta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} . \tag{2d.324}$$

We observe that the matrix AB' obtained from Eq. (2d.326) when λ and c vanish represents the euclidean form of Maxwell's stress tensor. It can assume the form of the traceless symmetric matrix M

$$M = AB' = \begin{pmatrix} M_{ij} & M_{oi} \\ M_{io} & M_{oo} \end{pmatrix} \tag{2d.325}$$

where

$$M_{oi} = M_{io} = M_i = - \varepsilon_{ijk} a^j \, b^k \, , \;\; M_{oo} = a_i \, b^i \tag{2d.326}$$

$$M_{ij} = a_i \, b_i + a_j \, b_i - a^k b_k \delta_{ij} \quad , \tag{2d.327}$$

with M_i corresponding to the components of the Poynting vector.

In the quasi-conformal case, the matrix elements of G in Eq. (2d.322) read

$$G_{ij} = a_i \, b_i + a_j \, b_i + \left(\lambda^2 - a^k b_k\right) \delta_{ij} \tag{2d.328}$$

$$G_{oo} = \lambda^2 + a^k b_k \quad , \quad G_{oi} = 0 \quad . \tag{2d.329}$$

As such they generalize Eq. (2d.308).

From the above account, the potential physical relevance of quaternionic analysis is apparent. This is particularly the case in the study of infinite dimensional symmetries of D = 4 euclidean gravity and topological field theories. Clearly, we have only scratched the surface, and the above developments are a far cry from being a complete formalism. Greater elaboration is needed on the connections with quaternionic, twistorial and harmonic space analyticities. Hopefully, the search for new tools with which to tackle the implications of $D \geq 4$ duality, conformal and topological field theories will bring forth further advances on this topic.

2.e. Arithmetics of Quaternions

Historically, the theory of quadratic forms has been the point of departure of the number theory of quadratic extensions. Being an element of a division algebra, $h = h_\mu e_\mu$ obeys a characteristic quadratic (or "rank equation") with real coefficients

$$h^2 - 2\,Sc(h)\,h + N(h) = 0 \ , \qquad (2e.1)$$

($2Sc(h) \equiv h + \bar{h}$, $N(h) \equiv h\bar{h}$), it is natural to consider the number theory over all these algebras.

Integer real numbers are the ordinary integers ($\mathbf{Z}$). Integer complex numbers and quaternions were extensively studied by Gauss, Lipschitz and particularly by Hurwitz. Integral complex numbers are called *Gaussian integers* ($\mathbf{Z}\oplus\mathbf{Z}$), ($z = m + i\,n$, $m, n \in \mathbf{Z}$). By definition, integer elements of the Hurwitz algebras must obey eq.(2e.1) with integer coefficients so that 1) $2\,Sc(h)$ and 2) $N(h)$ are both integers. The first condition implies that the scalar part of the integer elements is an integer or half integer. However, the second, norm condition cannot be satisfied for complex numbers, so that both real and imaginary parts of a Gaussian integer must be integers. However, for $\mathbf{H}$ the scalar and vectorial parts can be integers *or* half odd integers.

We begin with real integer numbers of unit norm, they are $r = \pm 1$. They correspond to the roots of $SU(2)$ and real integers form the root lattice of this group.

Geometrically, complex integers are represented by lattice points of the regular tessalation of squares. The integral complex numbers of unit norm, also called *units*, are ± 1 and $\pm i$. The corresponding lattice points ($\pm 1, 0$) and ($0, \pm 1$) form a square or *vertex figure* of the tessalation. They also represent the four roots of $O(4) \approx SU(2) \times SU(2)$.

Similarly, integral quaternions are $q = q_\mu e_\mu$ where the q_μ's are either integers or all halves of odd integers. They form a maximum ring. The integer quaternions of unit norm lie on S^3 and lead to 24 units of which 8, $\pm e_\mu$ ($\mu = 0, 1, 2, 3$) correspond to the solution with integer q_0 and $2^4 = 16$ combinations

$$\pm \frac{1}{2}(e_0+e_1+e_2+e_3) \tag{2e.2}$$

to half integer q_0. Together they form the 24 roots of O(8), closed under Weyl's reflections $x'=-a\,\bar{x}\,a$, with $\pm e_\mu$ and (2e.2) being the roots of the $O(4) \times O(4)$ subgroup and those in the coset $\frac{0(8)}{0(4)x0(4)}$, respectively.

Alternatively, these units also constitute the elements of the " binary tetrahedral group". They are the vertices

$$(\pm 1, 0, 0, 0)\ ,\quad (0, \pm 1, 0, 0)\ ,\quad (0, 0, \pm 1, 0)\ ,$$

$$(0, 0, 0, \pm 1)\ ,\quad (\pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}) \tag{2e.3}$$

of a 4-dimensional object, a 24-cell honeycomb in R^4 with 96 faces and 96 edges. It is one of Schläfli's six possible regular polytopes on S^3, its Schläfli symbol being {3, 4, 3}. Several applications of integer quaternions can be found for instance in the works of Linnik [143, 144, 145].

2.f. Selected Physical Applications

2.f.1. Quaternionic quantum mechanics and all that

Past works on quaternionic quantum mechanics have been amply reviewed in the literature [146, 147, 148, 149]. And the recent attempts by Adler at a new quaternionic quantum mechanics and field theory are well covered in his papers, lectures [150] and his comprehensive book [151] (see also [152, 153]). We will therefore not go over these developments here. Our limited coverage of hypercomplex extensions of quantum mechanics, included here for completeness, is solely conceptual and kept at a minimum.

In quantum mechanics, in contrast to classical mechanics, one superposes the c-number probability amplitudes rather than probabilities. As pointed out by Birkhoff and von Neumann [154], the interpretation of quantum mechanics requires the c-numbers to form a field, more precisely a topological, algebraic number field, since quantum theory is to be built continuously from infinitesimals. Now a fundamental theorem states that the only such number fields are the real numbers (**R**), complex numbers (**C**) and quaternions (**H**). In fact, it was shown that a generalized Hilbert space over scalar coefficients be they real, complex or quaternionic fulfills the general lattice theory axioms or "quantum logic" for quantum mechanics. That one may forego in quantum mechanics the commutativity of multiplication of scalars lies in the following fact: in the Dirac bra-ket notation, the superposition of amplitudes reads as

$$\Phi_{ac} = \langle c|a\rangle = \sum_b \langle c|b\rangle\langle b|a\rangle = \sum_b \Phi_{bc}\Phi_{ab} \tag{2f.1}$$

and carries with it a *natural factor ordering.*

Ever since the early attempt of Jordan, von Neumann and Wigner [65] to incorporate isospin kinematically within the framework of quaternionic quantum mechanics, there have been periodically renewed interest on this subject. Thus Stueckelberg [155, 156, 157] analysed in details real quantum mechanics. He showed that, to have a reasonable physical interpretation, **R**-quantum mechanics must be endowed with a complex structure, i.e. an anti-symmetric operator J, $J^T = -J$ and $J^2 = -1$, e.g. $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, a real matrix representation of i. Consequently, real quantum mechanics is a mere disguise of complex quantum mechanics.

As to the quaternionic quantum mechanics as formulated by Finkelstein, Jauch, Schiminovitch and Speiser [146, 158, 159, 160], except for the discovery of the natural emergence of a Yang-Mills gauge structure, primarily kinematical results were obtained. The two stubborn stumbling blocks are the lack of 1) intrinsically quaternionic dynamics and 2) a commutative tensor product for handling composite systems, many-body systems and hence for implementing second quantization in some **H**-quantum field theory. We refer the curious reader to the quoted references and particularly to the recent treatise of Adler [151] and references therein for a possible resolution of these issues.

On the mathematical side, we also note that the 1959 renewed interest [158, 161] in quaternionic quantum mechanics has resulted in various spectral theorems for unitary and skew-hermitian operators on quaternionic Hilbert spaces. It has spurred studies [162] of the unitary representations of groups and operator theory in these Hilbert spaces. In later sections, we will illustrate one use of quaternionic quantum mechanical spaces in the construction of Yang-Mills instantons and in the Berry phase of half integral spin systems.

2.f.2. Maxwell and Dirac-Kähler equations

As recounted in our historical notes, during the second half of the last century, the usefulness of quaternions in mathematical physics was the center of much controversy.

With Gibbs' vector calculus ultimately gaining the upper hand, the resurgence of quaternions must await the advent of special relativity. Dirac gave the reason simply when he wrote: "the fact that quaternions have four components and the physical world has four dimensions has led people to think that there ought to be a close correspondence between them."

We recall that the pseudo-euclidean Minkowski metric $\eta_{\mu\nu}$ with nonzero diagonal elements

$$- \eta_{00} = \eta_{11} = \eta_{22} = \eta_{33} = 1 \tag{2f.2}$$

cannot define the norm of a quaternion q, $q\bar{q} = q^\alpha \delta_{\alpha\beta} q^\beta$ with euclidean metric $\delta_{\alpha\beta}$. Hence to represent a 4-vector in Minkowski space we introduce the Hermitian quaternion

$$x = x^0 + i\,\vec{e}\cdot\vec{x} \quad , \tag{2f.3}$$

also given by the hermitian 2 x 2 matrix

$$X = x^0 + \vec{\sigma}\cdot\vec{x} = X^\dagger = \sigma_\mu x^\mu \quad . \tag{2f.4}$$

Its quaternion conjugate is

$$\bar{X} = x^0 - \vec{\sigma}\cdot\vec{x} = \sigma_2 X^T \sigma_2 = \bar{\sigma}_\mu x^\mu \tag{2f.5}$$

where

$$\bar{\sigma}_0 = \sigma_0 \quad , \quad \bar{\sigma}_i = - \sigma_i \quad (i = 1, 2, 3) \; . \tag{2f.6}$$

We have

$$\bar{\sigma}_\mu = - \eta^{\mu\nu} \sigma_\nu = - \sigma^\mu \tag{2f.7}$$

and

$$\bar{X}\,X = (x^0)^2 - \vec{x}\cdot\vec{x} = - x^\mu \eta_{\mu\nu} x^\nu \tag{2f.8}$$

so that

$$N(X) = x\bar{x} = X\,\bar{X} = \mathrm{Det}\,X = - x^\mu x_\mu \quad . \tag{2f.9}$$

For hermitian quaternion units, we have

$$\sigma_\mu \bar{\sigma}_\nu + \sigma_\nu \bar{\sigma}_\mu = 2\,\eta_{\mu\nu}$$
$$\sigma_\mu \sigma_\nu + \sigma_\nu \sigma_\mu = \bar{\sigma}_\mu \bar{\sigma}_\nu + \bar{\sigma}_\nu \bar{\sigma}_\mu = 2\delta_{\mu\nu} \;. \tag{2f.10}$$

However hermitian quaternions do not close under quaternion multiplication but only under the symmetric Jordan product. To obtain a closed associative algebra, we must embed them in the algebra of complex quaternions (also called biquaternions) given by

$$z = r + is = X + iY \,, \tag{2f.11}$$

with r and s being real quaternions, X and Y being hermitian quaternions. Then we can write

$$\sigma_\mu \bar{\sigma}_\nu = \delta_{\mu\nu} + \sigma_{\mu\nu} \quad , \quad \bar{\sigma}_\mu \sigma_\nu = \delta_{\mu\nu} + \sigma'_{\mu\nu} \tag{2f.12}$$

where

$$\sigma_{k0} = -\,\sigma_{0k} = \sigma_k = ie_k$$
$$\sigma_{ij} = -\,\sigma_i\,\sigma_j = e_i\,e_j = \varepsilon_{ijk} e_k = -\,i\,\varepsilon_{ijk}\sigma_k \;\;, \;\; (\,i \neq k) \;. \tag{2f.13}$$

So σ_{k0} are hermitian (imaginary quaternions) while σ_{ij} are antihermitian (real quaternions). It follows that

$$x\bar{y} = \sigma_\mu \bar{\sigma}_\nu\, x^\mu y^\nu = x^\mu y_\mu + \frac{1}{2}\sigma_{\mu\nu}\,(x^\mu y^\nu - x^\nu y^\mu) \;\;, \tag{2f.14}$$

$$\bar{y}x = \bar{\sigma}_\mu \sigma_\nu\, x^\mu y^\nu = x^\mu y_\mu - \frac{1}{2}\sigma'_{\mu\nu}\,(x^\mu y^\nu - x^\nu y^\mu) \;\;. \tag{2f.15}$$

We shall now show the analogs of self-duality and anti-self-duality for $\sigma_{\mu\nu}$ and $\sigma'_{\mu\nu}$, respectively. Now

$$\sigma_{23} = -\,i\,\sigma_{10} = i\sigma_{01} \;\;, \;\; \sigma_{31} = i\,\sigma_{02} \,, \;\; \sigma_{12} = i\,\sigma_{03} \tag{2f.16}$$

or, in covariant form, using the dual tensor

$$\tilde{\sigma}_{\alpha\beta} = \frac{1}{2}\varepsilon_{\alpha\beta\mu\nu}\,\sigma^{\mu\nu} \;, \; (\varepsilon_{0123} = 1) \tag{2f.17}$$

we can write

$$\tilde{\sigma}_{\alpha\beta} = i\,\sigma_{\alpha\beta} \quad , \quad \tilde{\sigma}'_{\alpha\beta} = i\,\sigma'_{\alpha\beta} \;. \tag{2f.18}$$

Let

$$a^{\mu\nu} = x^{\mu}y^{\nu} - x^{\nu}y^{\mu} \;. \tag{2f.19}$$

Then

$$\frac{1}{2}\,\sigma_{\mu\nu}a^{\mu\nu} = \frac{1}{4}\,(\,\sigma_{\mu\nu} - i\tilde{\sigma}_{\mu\nu}\,)\,a^{\mu\nu} \tag{2f.20}$$

or using

$$\tilde{\sigma}_{\mu\nu}\,a^{\mu\nu} = \frac{1}{2}\,\varepsilon_{\mu\nu\alpha\beta}\sigma_{\alpha\beta}a^{\mu\nu} = \sigma_{\mu\nu}\tilde{a}^{\mu\nu} \;, \tag{2f.21}$$

$$\frac{1}{2}\,\sigma_{\mu\nu}a^{\mu\nu} = \frac{1}{2}\,\sigma_{\mu\nu}\,\frac{1}{2}\,(\,a^{\mu\nu} - i\,\tilde{a}^{\mu\nu}) \;, \tag{2f.22}$$

we define the complex vector $\vec{\alpha}$ by

$$\vec{\alpha} = a^{23} - i\,a^{01} \;. \tag{2f.23}$$

Then

$$x\bar{y} = x_{\mu}y^{\mu} - i\,\vec{\sigma}\cdot\vec{a} = x_{\mu}y^{\mu} - \frac{1}{2}\sigma_{\mu\nu}\alpha^{\mu\nu} \tag{2f.24}$$

with

$$\alpha^{\mu\nu} = \frac{1}{2}\,(\,a^{\mu\nu} - i\,\tilde{a}^{\mu\nu}) \;, \tag{2f.25}$$

$$\tilde{\alpha}^{\mu\nu} = \frac{1}{2}\,(\,a^{\mu\nu} + i\,\tilde{a}^{\mu\nu}) = i\,\alpha^{\mu\nu} \;. \tag{2f.26}$$

Similarly,

$$\bar{y}x = x_{\mu}y^{\mu} + \frac{1}{2}\sigma_{\mu\nu}a^{\mu\nu} = x_{\mu}y^{\mu} - \frac{1}{2}\sigma'_{\mu\nu}\alpha^{\mu\nu} \;. \tag{2f.27}$$

We are now ready to discuss transformation properties of vectors and antisymmetric tensors under the Lorentz group SL(2, **C**) ≈ SO(3, 1). An element of the Lorentz group is the (2 x 2) complex matrix L representing a complex quaternion. It obeys the constraint

$$\mathrm{Det}\, L = L\,\overline{L} = 1 \ . \tag{2f.28}$$

The vector x is represented by the hermitian quaternion or (2 x 2) matrix X. Under the Lorentz group X transforms as

$$X' = L\,X\,L^{\dagger} \quad , \quad (\overline{L} = L^{-1}) \ , \tag{2f.29}$$

preserving both its hermiticity and norm (determinant). Also

$$Y' = L\,Y\,L^{\dagger} \quad , \quad \overline{Y'} = \overline{L^{\dagger}}\,\overline{Y}\,\overline{L} = (L^{\dagger})^{-1}\overline{Y}\,\overline{L} \ . \tag{2f.30}$$

Therefore

$$X'\,\overline{Y'} = L\,X\,\overline{Y}\,\overline{L} \ , \tag{2f.31}$$

$$\overline{Y'}\,\overline{X'} = \overline{L^{\dagger}}\,\overline{Y}\,X\,L^{\dagger} \ . \tag{2f.32}$$

We have

$$\mathrm{Sc}\left(X'\,\overline{Y'}\right) = \mathrm{Sc}(\overline{Y'}\,X') = x'_{\mu}y'^{\mu} - x^{\mu}y_{\mu} \ , \tag{2f.33}$$

$$\vec{\sigma} \cdot \vec{\alpha}' = L\,\vec{\sigma} \cdot \vec{\alpha}\,\overline{L} \quad , \quad \vec{\sigma} \cdot \vec{\alpha^{*}} = \overline{L^{\dagger}}\,\vec{\sigma} \cdot \vec{\alpha}\,\overline{L^{\dagger}} \ . \tag{2f.34}$$

The Lorentz invariant norm of the tensor $\alpha^{\mu\nu}$ is given by the complex quantity

$$\left(\vec{\sigma} \cdot \vec{\alpha}\right)^2 = \vec{\sigma} \cdot \vec{\alpha} = c_1 + i\,c_2 \ , \tag{2f.35}$$

c_1 and c_2 are q scalar and pseudoscalar, respectively. Any odd power of $\vec{\sigma} \cdot \vec{\alpha}$ will have the same transformation properties as $\vec{\alpha}$ and the same duality property as $\alpha^{\mu\nu}$.

According to Eq. (2f.11), a complex quaternion z can be decomposed in two ways, either into two real quaternions or into its hermitian and anti-hermitian parts. Then we use hermitian and quaternion conjugations on z represented by the complex matrix Z

$$Z = r_0 - \vec{\sigma} \cdot \vec{r} + i\, s_0 + \vec{\sigma} \cdot \vec{s} = (r_0 + is_0) + \vec{\sigma} \cdot (\vec{s} - i\,\vec{r}) \ . \qquad (2f.36)$$

Then

$$\overline{Z} = (r_0 + is_0) - \vec{\sigma} \cdot (\vec{s} - i\,\vec{r}) \ , \qquad (2f.37)$$

$$Z^\dagger = (r_0 - is_0) - \vec{\sigma} \cdot (\vec{s} + i\,\vec{r}) \ , \qquad (2f.38)$$

$$\overline{Z}^\dagger = Z^* = (r_0 - is_0) - \vec{\sigma} \cdot (\vec{s} + i\,\vec{r}) = r - i\,s \ . \qquad (2f.39)$$

The last operation corresponds to complex conjugation of a complex quaternion. If Z and W are two complex quaternions, we have

$$\overline{Z\,W} = \overline{W}\,\overline{Z} \ , \quad (ZW)^\dagger = W^\dagger Z^\dagger , \quad (ZW)^* = Z^* W^* . \qquad (2f.40)$$

Since $Z\overline{Z}$ is not positive definite, complex quaternions do not form a division algebra; we can have null quaternions with vanishing norms. In the case of hermitian quaternions Λ with zero norms, we have $\Lambda\,\overline{\Lambda} = \mathrm{Det}\,\Lambda = 0$, so that Λ is a lightlike vector. It factorizes

$$\Lambda = \lambda\,\lambda^\dagger \ , \qquad \lambda = \begin{pmatrix}\lambda_1\\ \lambda_2\end{pmatrix} . \qquad (2f.41)$$

Under a Lorentz transformation,

$$\Lambda' = L\Lambda\, L^\dagger = (L\,\lambda)\,(\lambda^\dagger\, L^\dagger) = \lambda'\,\lambda'^\dagger \qquad (2f.42)$$

is generated by $\lambda' = L\,\lambda$, which is the law for a left handed spinor.

Under the CP operation combining parity with charge conjugation, the vector Λ is transformed into Λ^* so that

$$\Lambda \to \Lambda^* = \overline{\Lambda}^\dagger = (\sigma_2\,\Lambda\,\sigma_2)^\dagger = -i\sigma_2\Lambda^*(i\sigma_2) = (-i\sigma_2\lambda^*)(\lambda^T i\sigma_2) \qquad (2f.43)$$

or

$$\lambda = \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} \rightarrow \widehat{\lambda} = -i\,\sigma_2 \lambda^* = \begin{pmatrix} -\lambda_2^* \\ \lambda_1^* \end{pmatrix} \tag{2f.44}$$

behaving like a right handed spinor ρ,

$$\rho' = \overline{L}^\dagger \rho = L^* \rho \quad , \quad \widehat{\lambda}' = L^* \widehat{\lambda} \ . \tag{2f.45}$$

Finally, an antihermitian matrix B must represent a vector since, to preserve antihermiticity, it must transform as

$$B' = L\, B\, L^\dagger \ . \tag{2f.46}$$

Clearly, we can set

$$B = i\,A \quad , \quad A = A^\dagger \ . \tag{2f.47}$$

Under CP we get

$$B \rightarrow B^* = \overline{B}^\dagger = -\,i\,A^* \ ; \tag{2f.48}$$

hence an antihermitian matrix with the transformation law (2f.46) represents a pseudo 4-vector. The combination

$$W = V + i\,A \qquad (V = V^\dagger , A = A^\dagger) \tag{2f.49}$$

of a vector and a pseudovector A is a complex matrix such that

$$W' = L\, W\, L^\dagger \ . \tag{2f.50}$$

On the other hand, a complex matrix Z

$$Z = \varphi \ + \vec{\sigma} \cdot \vec{f} = \ = s + i\,p + \vec{\sigma} \cdot (\vec{E} + i\vec{B}) \tag{2f.51}$$

such that $W' = L\,W\,L^{\dagger}$ represents a scalar s, a pseudoscalar p and a self-dual complex antisymmetric tensor with three complex components $\vec{f}$ associated with a skew symmetric tensor $f_{\mu\nu}$.

Finally, a complex matrix Ψ with

$$\Psi' = L\,\Psi \tag{2f.52}$$

can be viewed as consisting of two columns ψ_L and $\widehat{\psi}_R$, so that

$$\Psi = \begin{pmatrix} \psi_L & \widehat{\psi}_R \end{pmatrix} = \begin{pmatrix} \psi_L^1 & -\left(\psi_R^2\right)^* \\ \psi_L^2 & \left(\psi_R^1\right)^* \end{pmatrix} . \tag{2f.53}$$

The transformation (2f.52) is equivalent to

$$\psi_L = L\,\psi_L \quad , \quad \psi'_R = \overline{L}^{\dagger}\,\psi_R = L^*\,\psi_R \ , \tag{2f.54}$$

showing that a Dirac spinor Ψ with chiral components ψ_L and ψ_R can also be represented by a complex quaternion associated with a complex matrix and with relativistic properties given by Eq. (2f.52). We have

$$\overline{\Psi'} = \overline{\Psi}\,\overline{L} \tag{2f.55}$$

where

$$\overline{\Psi} = \ \sigma_2 \Psi^T \sigma_2 = -\,i\,\sigma_2 \begin{pmatrix} \psi_L^T i\sigma_2 \\ \widehat{\psi}_R^T i\sigma_2 \end{pmatrix} , \tag{2f.56}$$

$$= \begin{pmatrix} -\,\widehat{\psi}_R^T i\sigma_2 \\ \widehat{\psi}_L^T i\sigma_2 \end{pmatrix} = \begin{pmatrix} \psi_R^{\dagger} \\ \psi_L^T i\,\sigma_2 \end{pmatrix} . \tag{2f.57}$$

Hence the complex number $\overline{\Psi}\,\Psi$ is Lorentz invariant. We find

$$\overline{\Psi}\,\Psi = \begin{pmatrix} \psi_R^{\dagger} \\ \psi_L^{T} \end{pmatrix} \begin{pmatrix} \psi_L & -i\,\sigma_2\psi_R^{*} \end{pmatrix} = \begin{pmatrix} \psi_R^{\dagger}\psi_L & -\psi_R^{\dagger} i\sigma_2\psi_R^{*} \\ \psi_L^{T} i\sigma_2\psi_L & \psi_L^{T}\psi_R^{*} \end{pmatrix} . \tag{2f.58}$$

If the components of Ψ commute ,

$$\overline{\Psi}\,\Psi = \psi_R^{\dagger}\,\psi_L\,I \ . \tag{2f.59}$$

In terms of the Dirac spinor ψ, we get

$$\overline{\Psi}\Psi = \frac{1}{2}\,\overline{\psi}\,(1+\gamma_5)\,\psi \quad , \quad (\overline{\psi} = \psi^{\dagger}\,\gamma_4). \tag{2f.60}$$

If Ψ has anticommuting components, $\overline{\Psi}\,\Psi$ is given by the traceless matrix C with Lorentz invariant components

$$C_{11} = -\,C_{22} = \psi_R^{\dagger}\psi_L = \frac{1}{2}\overline{\psi}\,(\,1+\gamma_5)\,\psi \quad , \tag{2f.61}$$

$$C_{21} = \frac{1}{2}\overline{\psi^{C}}\,(\,1+\gamma_5)\,\psi \quad , \quad C_{12} = \frac{1}{2}\overline{\psi}\,(\,1+\gamma_5)\,\psi^{C} \ , \ (\psi^{C} = \gamma_2\psi^{*}) \ . \tag{2f.62}$$

We can also represent the Dirac spinor ψ by the matrix

$$\Psi^{*} = \overline{\Psi}^{\dagger} = \sigma_2\Psi^{*}\,\sigma_2 = \begin{pmatrix} \psi_R & \hat{\psi}_L \end{pmatrix} \ . \tag{2f.63}$$

It transforms like a right handed spinor

$$\Psi^{*\prime} = \overline{L}^{\dagger}\,\Psi^{*} \ . \tag{2f.64}$$

Then

$$\overline{\Psi}^{*}\Psi^{*} = \Psi^{\dagger}\,\overline{\Psi}^{\dagger} = \psi_L^{\dagger}\psi_R\,I \tag{2f.65}$$

in the bosonic (commuting) case and

$$\Psi^{\dagger}\overline{\Psi}^{\dagger} = \begin{pmatrix} \psi_L^{\dagger}\psi_R & -\psi_L^{\dagger} i\sigma_2 \psi_L^{*} \\ \psi_R^{T} i\sigma_2 \psi_R & -\psi_L^{\dagger}\psi_R \end{pmatrix} \tag{2f.66}$$

in the fermionic (anticommuting components) case giving the invariants

$$\frac{1}{2}\overline{\psi}\,(1+\gamma_5)\,\psi \quad , \quad \frac{1}{2}\overline{\psi^C}\,(1-\gamma_5)\,\psi \quad \text{and} \quad \frac{1}{2}\overline{\psi}\,(1-\gamma_5)\,\psi^C \;. \tag{2f.67}$$

We note that, out of the Ψ, we can form the vector

$$V = \Psi\,\Psi^{\dagger} \quad , \quad V \to L\,V\,L^{\dagger} \;, \tag{2f.68}$$

the pseudovector

$$B = \Psi\, i\sigma_3\, \Psi^{\dagger} \quad , \quad B \to L\,B\,L^{\dagger} \tag{2f.69}$$

and the antisymmetric tensor

$$F = \Psi\,\sigma_3\,\overline{\Psi} \qquad , \quad F \to L\,F\,\overline{L} \tag{2f.70}$$

in the bosonic case or

$$\Phi = \Psi\,\overline{\Psi} \qquad , \quad \Phi \to L\,\Phi\,\overline{L} \tag{2f.71}$$

in the fermionic case. Both F and Φ are traceless.

With these preliminaries, we now proceed to the quaternionic forms of the Maxwell and Dirac-Kähler equations. In terms of the following chiral Dirac operators

$$D = \partial_0 + \vec{\sigma}\cdot\vec{\nabla} \quad , \quad \overline{D} = \partial_0 - \vec{\sigma}\cdot\vec{\nabla} \quad , \; D = D^{\dagger} \;, \tag{2f.72}$$

the massless Dirac equation reads

$$\overline{D}\,\Psi = 0 \ . \tag{2f.73}$$

It decomposes into

$$\overline{D}\,\psi_L = 0 \qquad \text{and} \qquad D\,\psi_R = 0 \ . \tag{2f.74}$$

Since

$$D' = L\,D\,L^{\dagger} \quad , \ \overline{D'} = \overline{L}^{\dagger}\,\overline{D}\,\overline{L} \ , \tag{2f.75}$$

$$\psi'_L = L\,\psi_L \qquad , \qquad \psi'_r = \overline{L}^{\dagger}\,\psi_R \ , \tag{2f.76}$$

the Lorentz invariance of the equation is manifest .

In the massive case, we have the Dirac equation

$$\overline{D}\,\psi_L = m\,\psi_R \ , \tag{2f.77}$$
$$D\,\psi_R = -\,m\,\psi_L \tag{2f.78}$$

leading to the second order Klein-Gordon equation

$$D\,\overline{D}\,\psi_L = \Box\,\psi_L = (\,\partial_0^2 - \nabla^2)\,\psi_L = -\,m^2\psi_L \ . \tag{2f.79}$$

Equation (2f.78) also reads as

$$\overline{D}\,\widehat{\psi}_R = -\,m\,\widehat{\psi}_L \tag{2f.80}$$

so that the Dirac equation has the following (2 x 2) matrix form

$$\overline{D}\,\Psi = m\,(\,\psi_R - \widehat{\psi}_L\,)\,, \tag{2f.81}$$

and since $\Psi^* \sigma_3 = \overline{\Psi}^\dagger \sigma_3 = (\psi_R - \hat{\psi}_L)$, we obtain

$$\overline{D}\,\Psi = m\,\overline{\Psi}^\dagger \sigma_3 = i\,m\,\sigma_2 \Psi^* \sigma_1 \tag{2f.82}$$

or

$$D\,\overline{\Psi}^\dagger = -\,m\,\Psi\,\sigma_3\ . \tag{2f.83}$$

We next turn to the Maxwell equations. With the vector potential A_μ represented by the hermitian quaternion A, we find

$$\overline{D}\,A = \omega + F \tag{2f.84}$$

where

$$\omega = \partial_\mu A^\mu = \mathrm{Sc}\left(\overline{D}\,A\right)\ ,\ \ F = \vec{\sigma}\cdot\vec{f} = \vec{\sigma}\cdot\left(\vec{E} + i\vec{B}\right) = \mathrm{Vec}\,(\,\overline{D}\,A\,)\ , \tag{2f.85}$$

the complex vector $\vec{f}$ representing the electromagnetic tensor $F_{\mu\nu}$. Thus

$$D\,F = \Box\,A - D\,\omega = -\frac{4\pi}{c}\,J\ , \tag{2f.86}$$

$J \equiv \sigma_\mu J^\mu$ is the hermitian quaternionic current density.

In the absence of charges and with the Lorentz condition satisfied:

$$\omega = \partial_\mu\,A^\mu = 0\ , \tag{2f.87}$$

we have

$$DF = \Box\,A = 0\ . \tag{2f.88}$$

If $\omega \neq 0$,

$$D\,(\,\omega + F\,) = 0 \ \text{ or }\ \overline{D}\,(\,\omega - \vec{\sigma}\cdot\vec{f}^{\,*}\,) = 0 \tag{2f.89}$$

has the same form as the Dirac equation for a massless particle. Note that, in the massless case, the Dirac equation

$$\overline{D}\,\Psi = 0 \tag{2f.90}$$

is invariant under a SL(2, **C**) x SL(2, **C**) transformation with

$$\overline{D} \to \overline{L}^{\dagger}\,\overline{D}\,\overline{L} \quad , \quad \Psi \to L\,\Psi\,M \ . \tag{2f.91}$$

Here M is another unimodular complex matrix whose right multiplication action on Ψ commutes with the left action of the Lorentz group.

Let us consider a complex vector potential $C = A + iB$. Then, we can write

$$\Phi = \overline{D}\,C = \omega - i\,\kappa - \vec{\sigma}\vec{\varphi}^{*} \tag{2f.92}$$

where the complex $\vec{\varphi}$ is associated with the antisymmetrical tensor $\varphi_{\mu\nu}$ and κ is the pseudoscalar.

$$\kappa = -\,\partial_{\mu}\,B^{\mu} \ . \tag{2f.93}$$

In the massless case we have

$$\begin{aligned} &D\,\Phi = 0\,, \\ &\overline{D}\,\Phi^{\dagger} = \overline{D}\,(\,\omega + i\,\kappa + \vec{\sigma}\cdot\vec{\varphi}) = 0 \end{aligned} \tag{2f.94}$$

which should be compared with Eq. (2f.90). The difference is that under Lorentz transformations

$$\Psi \to L\,\Psi\,M \ , \tag{2f.95}$$

$$\overline{\Phi}^{\dagger} \to L\,\overline{\Phi}^{\dagger}\,\overline{L} \ . \tag{2f.96}$$

So we have gone from the Dirac equation for a (Dirac) spinor to the field Φ consisting of a scalar ω, a pseudoscalar κ and an antisymmetrical tensor $\vec{\sigma}\cdot\vec{\varphi}$ by taking the diagonal SL(2, **C**) subgroup of SL(2, **C**) x SL(2, **C**). This process is achieved by requiring that $M = \bar{L}$. Then we find the following correspondence between the components ψ_1, ψ_2, ψ_3, ψ_4 of the Dirac field and the bosonic fields ω, κ and $\varphi_{\mu\nu}$

$$\Psi = \begin{pmatrix} \psi_1 & -\psi_4^* \\ \psi_2 & \psi_3^* \end{pmatrix} \longleftrightarrow \Phi = \begin{pmatrix} \omega + i\kappa + \varphi_3 & \varphi_1 - i\varphi_2 \\ \varphi_1 + i\varphi_2 & \omega + i\kappa - \varphi_3 \end{pmatrix} . \tag{2f.97}$$

Such a correspondence carries over to the massive case. Then the Maxwell equations give way to the Proca equations for a scalar, a pseudoscalar and a vector ; the neutrino equations get replaced by the massive Dirac equation. The resulting bosonic equations, the equivalent of the Dirac equation, are called the Dirac-Kähler equations.

The scalar equations are

$$D\,\omega = m\,G \quad , \qquad \bar{D}\,G = -\,m\,\omega\ . \tag{2f.98}$$

The pseudoscalar equations are

$$D\,\kappa = m\,B \quad , \qquad \bar{D}\,B = -\,m\,\kappa\ , \tag{2f.99}$$

while the vector equations are

$$D\,\vec{\sigma}\cdot\vec{\varphi} = m\,A\ ,\ \ \bar{D}\,A = i\,m\,\vec{\sigma}\cdot\vec{\varphi}\ . \tag{2f.100}$$

So

$$\begin{cases} D\,(\,\omega + i\,\kappa + \vec{\sigma}\cdot\vec{\varphi}\,) = m\,(\,A + G + i\,B\,) \\ \bar{D}(\,A + G + i\,B\,) = -\,m\,(\,\omega + i\,\kappa + \vec{\sigma}\cdot\vec{\varphi}\,)\ , \end{cases} \tag{2f.101}$$

correspond to

$$D\,\Xi = m\,\Psi \quad , \qquad \bar{D}\,\Psi = -\,m\,\Xi \tag{2f.102}$$

where $\Xi \equiv \overline{\Psi}^{\dagger} u$, $u\overline{u}^{\dagger}=1$, u being Lorentz invariant.

2.f.3. Self-duality and Yang-Mills and gravitational instantons

In D = 4 Minkowski spacetime, a 4-vector cannot be directly identified with a real quaternion; the latter's norm squared being positive definite. As noted earlier, a Lorentzian spacetime calls for the use of complexified or hermitian quaternions [62, 163]. However, these bi-quaternions no longer satisfy the division algebra axiom, nor do they carry any fundamental property distinguishing them from the plethora of known hypercomplex systems. On the other hand, Euclidean quantum field theory [164] opens the way for applying real quaternions to spacetime. An important illustration arises in the instanton solutions to Yang-Mills theories [165]. In recent years, the relevance and ubiquity of real quaternions in particle physics have also emerged as part of a unique correspondence uncovered between basic structural features of supersymmetric theories and division algebras [166, 167]. This connection will be detailed in Section 3.

As a prelude, we consider the conformal group SO(5, 1) in Euclidean spacetime R^4, more specifically, its covering group SL(2, **H**) [168, 169]. First we recall [51] how this group acts on a quaternionic 2-vector Ψ:

$$\Psi' = \begin{pmatrix} \Psi'_1 \\ \Psi'_2 \end{pmatrix} = L\,\Psi = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix}, \quad \det L = 1 \ , \tag{2f.103}$$

a, b, c and d are quaternions and det L denotes the determinant of a 15-parameter (4 x 4) matrix L, obtained through the matrix representation $e_n = -i\,\sigma_n$, n = 1, 2, 3. Let M be an element of SL(2, **H**)

$$M = \begin{pmatrix} k & l \\ m & n \end{pmatrix}, \ \mathrm{Det}\, M = |k|^2\,|n|^2 + |m|^2\,|\,l\,|^2 - 2\,Sc\,(\,k\,\overline{m}n\,\bar{l}\,) = 1 \ . \tag{2f.104}$$

It may be decomposed into

$$\begin{pmatrix} k & l \\ m & n \end{pmatrix} = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \lambda^{-1} & 0 \\ 0 & \lambda \end{pmatrix} \begin{pmatrix} 1 & 0 \\ Y & 1 \end{pmatrix} \begin{pmatrix} v_1 & 0 \\ 0 & v_2 \end{pmatrix} \tag{2f.105}$$

with $\mathrm{Vec}\lambda = 0$, $|v_1| = |v_2| = 1$. Provided that $n \neq 0$,

$$\begin{aligned} &x = l\, n^{-1},\ \lambda = |n|,\ v_2 = n|n|^{-1}\ , \\ &y = m\,(k - l\, n^{-1}\, m)^{-1}\, |n|^{-2}\ ,\ v_1 = |n|\,(k - l\, n^{-1}\, m)\ . \end{aligned} \tag{2f.106}$$

Under a conformal transformation

$$M\,' = L\,M = \begin{pmatrix} k\,' & l\,' \\ m\,' & n\,' \end{pmatrix} \tag{2f.107}$$

so that

$$x\,' = l\,'\, n\,'^{-1} = (a\,x + b)\,(c\,x + d)^{-1}\ , \tag{2f.108}$$

x labels a spacetime point and is a coset representative of the conformal group w.r.t. the triangular group of matrices

$$T = \begin{pmatrix} v_1\, \lambda^{-1} & 0 \\ Y\, \lambda\, v_1 & \lambda\, v_2 \end{pmatrix}\ . \tag{2f.109}$$

Here T corresponds to the inhomogeneous Weyl group of D = 4 rotations, dilatation and translations. Also x transforms like the ratio $\psi_1 \psi_2^{-1}$ of the two components of the quaternionic 2-spinor (2f.103), perhaps better known as a *twistor* [3, 5, 170].

The SO(5) subgroup of SO(5, 1) takes the matrix form of

$$\Theta = \frac{1}{\sqrt{1 + |h|^2}} \begin{pmatrix} 1 & -h \\ h & 1 \end{pmatrix} \begin{pmatrix} v_1 & 0 \\ 0 & v_2 \end{pmatrix} \tag{2f.110}$$

corresponding to the representation of Sp(2, **H**) obtained previously with n = 1. h labels the coset $SO(5)/SO(4) \approx S^4$, the 4-sphere. Setting $h = x$ provides a mapping of S^4 onto the coset space $SL(2, \mathbf{H}) / T$.

Next, we seek the metric tensor associated with a conformal transformation. From Eq. (2f.108), we get

$$d\,x' = a\left[dx\,(x + c^{-1}d) - (x + a^{-1}b)\,(x + c^{-1}d)^{-1}\,dx\,(x + c^{-1}d)^{-1}\right]c^{-1} \tag{2f.111}$$

and, using the identity $\det A = |\,c\,|^2\,|\,ac^{-1}d - b\,|^2 = 1$,

$$|dx'|^2 = |c|^{-4}\,|x + c^{-1}d|^{-4}\,|dx|^2 = |cx + d|^{-4}\,|dx|^2. \tag{2f.112}$$

So after a conformal reparametrization $x \rightarrow x'$, the line element of a conformally flat spacetime

$$ds^2 = \phi^2\,dx\,d\bar{x} \tag{2f.113}$$

becomes

$$ds^2 = \phi'^{\,2}\,dx'\,\overline{dx'}, \quad \phi' = \phi\,|\,cx + d\,|^{-2}, \tag{2f.114}$$

thus preserving its conformal flatness. It differs from the D = 2 line-element $ds^2 = \phi^2 dz\,d\bar{z}$, which remains invariant under an infinite parameter transformation $z' = f(z)$, a **C**-analytic function. Next, we proceed to matrix and vector derivatives of quaternionic functions and their connections to the ADHM n-instanton construction [165].

Covariant quaternionic functions may be put in a matrix form. Consider the following rational function $Y(x)$ with N poles

$$Y = \sum_{i=1}^{N} \Gamma_i(a_i x + b_i)\,(c_i\,x + d_i)^{-1} \tag{2f.115}$$

where $\Gamma_i = (\,a_i\,c_i^{-1}\,d_i - b_i\,)^{-1}$. We now introduce the diagonal quaternionic (N x N) matrices A, B, C and D with elements a_n, b_n, c_n, d_n respectively. Set $\Gamma = (AC^{-1}D - B)$

and let

$$M_Y = \Gamma(A\,x + B)(Cx + D)^{-1} \ , \tag{2f.116}$$

the function Y can then be written as the quaternionic trace

$$Y = \mathrm{Tr}\left(M_Y\right) = \mathrm{Tr}\left(\Gamma(A\,x + B)\,(Cx + D)^{-1}\right) \ . \tag{2f.117}$$

The rational matrix function M_Y satisfies the Fueter equations $\Box D\, M_Y = 0$, which remains valid for $N \to \infty$. Similarly, Y obeys in this limit the same equation, provided that the meromorphic series Tr M_Y converges to Y.

Since we also have

$$M_y = \Gamma A C^{-1} + (\,xI + C^{-1}D\,)^{-1}C^{-1} \ , \tag{2f.118}$$

I being the (Nx N) unit matrix, it follows that

$$dM_Y = -\left(xI + C^{-1}D\right)^{-1}dx\left(xI + C^{-1}D\right)^{-1}C^{-1} \equiv L\,dx\,R \tag{2f.119}$$

and therefore $dY = \mathrm{Tr}(\,LdxR\,)$.

Although a derivative cannot be defined for an *arbitrary* quaternionic function, we see that a meromorphic function of the type (2f.115) does admit left and right matrix derivatives L(x) and R(x) with $\Box\, D\, R = 0$, $\Box\, L \underset{\leftarrow}{D} = 0$.

For an alternative matrix formulation, we introduce the following entities:

1) a row vector $n^T = (1,\ 1,\ldots,\ 1)$ with unit components and the superscript T denoting transposition.
2) a quaternionic column vectors α and μ with respective components $\Gamma_i\, a_i$ and c_i.

3) a diagonal matrix Q with elements $c_i^{-1}d_i$.

Then we may write Y as

$$Y = n^T\left[\, \alpha + (x + Q)^{-1}\mu\right] = n^T m_Y \ . \tag{2f.120}$$

Differentiation leads to the definition of a left row derivative l^T and a right column derivative r:

$$dY = l^T(x)\ \ dx\, r\ (x) \ = \ n^T\, dm_Y \ , \tag{2f.121}$$

$$l^T(x) = - \, n^T\, (x + Q)^{-1}, r\,(x) = (x + Q)^{-1}\mu \tag{2f.122}$$

$\square\, r$ is left-regular and $\square\, l^T$ is right-regular since $\square\, \mathrm{D}r = 0$, $\square\, l^T \overleftarrow{\mathrm{D}} = 0$; evidently they are the same equations as those previously given for the matrices R and L.

Next, we consider the column matrix

$$m_Y = \alpha + (x + Q)^{-1}\, \mu \tag{2f.123}$$

which is left-holomorphic, $\square\, \mathrm{D}m_Y = 0$. It follows that

$$dm_Y = - \,(x + Q)^{-1} dx\, (x + Q)^{-1}\mu \ , \tag{2f.124}$$

so that

$$dY = n^T dm_Y = d(n^T m_Y) \ . \tag{2f.125}$$

Let δm_Y stand for the change in the column induced by $x \rightarrow x+\delta x$, then

$$\delta\, \overline{m_Y} = \ - \mu^\dagger (\,\overline{x} + \overline{Q}\,)^{-1} \delta\overline{x}\, (\,\overline{x} + \overline{Q}\,)^{-1} \ . \tag{2f.126}$$

Here the dagger means transposition together with **H**-conjugation. Let G be a hermitian metric in the N-dimensional quaternionic space such that $G = G^\dagger = \overline{G}^T$, we can define the following Riemannian line element:

$$d^2s = d\overline{m}_Y G\, dm_Y \equiv (d\overline{m}_Y, dm_Y) \quad . \tag{2f.127}$$

By way of the scalar product

$$\left(\delta\,\overline{m_Y}\,,\, d\, m_Y\right) = \mathrm{Sc}\left\{\delta\,\overline{m_Y} G\, d\, m_Y\right\} \tag{2f.128}$$

and the 2-form

$$\delta\,\overline{m_Y}\,\Lambda\, d\, m_Y = \mathrm{Vec}\left\{\,\overline{\delta m_Y\, G\, d\, m_Y}\right\} \quad , \tag{2f.129}$$

we find

$$\left(\delta\,\overline{m_Y},\, d\, m_Y\right) = \mu^{\dagger}\,(\,\overline{x} + \overline{Q}\,)^{-1}\,\delta\overline{x}\;\; dx\left[(x+Q)(\,\overline{x}+\overline{Q}\,)\right]^{-1}(x+Q)^{-1}\mu \quad , \tag{2f.130}$$

namely

$$ds^2 = \phi^2\, dx_\mu\, dx_\mu \quad , \tag{2f.131a}$$

and

$$\delta\,\overline{m_Y}\,\Lambda\, dm_Y = s^{\dagger} e_{\mu\nu} t\, dx_\mu\, \delta x_\nu \tag{2f.131b}$$

i.e. the line element is conformally flat and the 2-form is self-dual.

The following generalization readily comes to mind. Let Q be a symmetric quaternionic matrix instead of a diagonal matrix and G a given hermitian metric. Assume that, for every x,

$$\left[(\,\overline{x} + \overline{Q}\,)^{-1} G(\,x + Q\,)^{-1},\, dx\right] = 0 \tag{2f.132}$$

or

$$\mathrm{Vec}\left[(x+Q)\, G^{-1}(\overline{x} + \overline{Q})\right] = 0 \quad , \tag{2f.133}$$

then d^2s will still be conformally flat and the 2-form $\delta\,\overline{m_Y}\,\Lambda\, dm_Y$ self-dual. Mappings which preserve these properties will be called ADHM maps after the celebrated n-instanton construction [165]. They generalize the conformal and the Fueter mappings. We will shortly show their usefulness in HP_n σ-models. For the moment, we observe that Eq. (2f.133) implies

$$\mathrm{Vec}\, G^{-1} = 0 \ , \quad QG^{-1} = (QG^{-1})^T \ , \quad \mathrm{Vec}(QG^{-1}\overline{Q}) = 0 \ . \qquad (2f.134)$$

Such constraints lead automatically to the expressions (2f.131a) and (2f.131b) for the metric and its associated 2-form.

We encountered previously the projective spaces HP_n, in the context of quaternionic quantum mechanics. To obtain instanton solutions, we need to explicitly construct HP_n spaces seen as the coset space Sp(n+1) / Sp(n) x Sp(1). To parametrize the latter, we use n standard quaternionic parameters t_i, the components of an n-dimensional column vector t, transforming linearly under Sp(n) x Sp(1). Then an element U of Sp(n+1) may be written as

$$U=V(t)W, \qquad W \in Sp(1)xSp(n) \ . \qquad (2f.135)$$

In terms of the quaternion r and the (n x n) H-matrix R,

$$W = \begin{pmatrix} r & 0 \\ 0 & R \end{pmatrix} \quad , \ |r| = 1 \ , \ RR^{-1} = I \ , \qquad (2f.136)$$

I being the (nxn) unit matrix. The diagonal matrix

$$\eta = \begin{pmatrix} 1 & 0 \\ 0 & I \end{pmatrix} \quad , \quad \eta^2 = E = (\ n+1\ )(n+1) \text{ unit matrix} \qquad (2f.137)$$

is left invariant by W since

$$W\eta W^{\dagger} = \eta \ . \qquad (2f.138)$$

Next, we introduce an hermitian matrix N defined by

$$N(t) = U\eta U^\dagger = V(t)\eta V^\dagger(t) \ , \ (N^2 = E) \ . \tag{2f.139}$$

Under Sp(n) x Sp(1) it transforms as

$$N' = W\,N\,W^\dagger = V'\,\eta\,V'^\dagger \, , \ V' = W\,V\,W^\dagger \, . \tag{2f.140}$$

The parameters of W can be arbitrary functions of the quaternion position x of the Euclidean spacetime R^4. Explicitly, the symplectic matrix V(t) reads

$$V(t) = \left(1 + t^\dagger t \right)^{1/2} \begin{pmatrix} 1 & -t^\dagger \\ t & \Lambda(t) \end{pmatrix} \tag{2f.141}$$

with

$$\Lambda(t) = \Lambda(t)^\dagger = \left(1 + t^\dagger t \right)^{1/2} \left(I + t\,t^\dagger \right)^{-1/2} \tag{2f.142}$$

and

$$\eta V(t) \eta = V(t)^\dagger = V(-t), \ V = (N\eta)^{-1/2} \tag{2f.143}$$

Under the action of the subgroup W,

$$V'=V(t') = W\,V(t)\,W^\dagger \ , \tag{2f.144}$$

so that $t' = R\,t\,\bar{r}$, showing that t can be taken as the standard parametrization of HP_n .

Alternatively, we could also introduce the (n+1) component unit **H**-ket v with components v_0 and k, obtained by transforming V to the right by W ($r = \frac{n_0}{|n_0|}$, $R = 1$). Then

$$v_0 = \left(\bar{n}_0 n_0 + n^\dagger n \right)^{-1/2} n_0 \quad , \quad k = \left(\bar{n}_0 n_0 + n^\dagger n \right)^{-1/2} n \tag{2f.145}$$

with $t \rightarrow t$, $n_0 \rightarrow n_0 r$ under the action of Sp(1).

In the special case of n = 1, the vector t reduces to a single quaternion q, then

$$N = V^2\eta = (1 + q\bar{q})^{-1}\begin{pmatrix} 1 - q\bar{q} & 2\bar{q} \\ 2q & -1 + q\bar{q} \end{pmatrix} \; ; \tag{2f.146}$$

q is the stereographic projection parameter of $HP_1 \approx S^4$; namely it is the D = 4 counterpart of the complex number z, the stereographic projection parameter of $CP_1 \approx S^2$, the Riemann sphere. In HP_1, the subgroup W reduces to Sp(1) x Sp(1) ≈ SO(4), i.e. the Euclidean Lorentz group

The $S^4 \rightarrow S^4$ map is represented by q = q(x) while the mapping $S^4 \rightarrow HP_n$ corresponds to t = t(x), an embedding of S^4 in HP_n. As x varies, t(x) traces out a quaternionic curve. Its tangent space can be characterized by the pair (K, L) if dt reads

$$dt = K \, dx \, L \tag{2f.147}$$

where L is a quaternionic column and K a quaternionic matrix. This geometrical picture is the analog of a holomorphic curve parametrized by a complex parameter z [171].

Next, we introduce the anti-hermitian 1-form

$$\omega = -\,\omega^\dagger = \Omega_\mu dx_\mu = V^{-1}\, d\, V = \begin{pmatrix} \alpha & -\varphi^\dagger \\ \varphi & \beta \end{pmatrix} . \tag{2f.148}$$

Here $\alpha = a_\mu \, dx_\mu$ is a vectorial quaternion, φ is an (nx1) column and β an (nxn) matrix. We have

$$\varphi = \varphi_\mu dx_\mu \quad , \quad \varphi_\mu = G^{-1/2}\, \partial_\mu \, t \tag{2f.149}$$

with $G \equiv (1+t^\dagger t)^{-1}(I + tt^\dagger)^{-1}$, the Fubini-Study metric of HP_n [82, 172, 173],

$$a_\mu = \frac{1}{2}\,\mathbf{e}\cdot\mathbf{A}_\mu = \frac{1}{2}\,\frac{t^\dagger \overleftrightarrow{\partial}_\mu t}{1 + t^\dagger t} \quad . \tag{2f.150}$$

So if R is constant and $r = r(\mathrm{x})$, under a Sp(1) transformation

$$a_\mu \to \bar{r}\, a_\mu\, r = \bar{r}\, \partial_\mu r \, . \tag{2f.151}$$

Choosing $r = \frac{n_0}{|n_0|}$, we find

$$a_\mu = \frac{1}{2}\left(\overline{n_0}\, n_0 + n^\dagger n\right)^{-1}\left[\overline{n_0}\, \overleftrightarrow{\partial}_\mu n_0 + n^\dagger\, \overleftrightarrow{\partial}_\mu n\right] = \mathrm{Vec}\,(\, v^\dagger \partial_\mu v \,)\, . \tag{2f.152}$$

Thus $\mathbf{A}_\mu$ transforms like a Sp(1) ≈ SU(2) Yang-Mills connection. And the Maurer-Cartan integrability condition

$$\partial_\mu \Omega_\nu - \partial_\nu \Omega_\mu + \left[\Omega_\mu \, , \Omega_\nu\,\right] = 0 \quad , \tag{2f.153}$$

for the connection form ω, gives a Sp(1) Yang-Mills field

$$f_{\mu\nu} = \frac{1}{2}\,\mathbf{e}\cdot\; \mathbf{F}_{\mu\nu} = \partial_\mu a_\nu - \;\partial_\nu a_\mu + \left[a_\mu \, , \, a_\nu\,\right] = 2\,\mathrm{Vec}(\varphi_\mu^\dagger \varphi_\nu) \quad . \tag{2f.154}$$

Under a local Sp(1) transformation,

$$f_{\mu\nu} \to r\, f_{\mu\nu}\, r^{-1} \quad . \tag{2f.155}$$

As to the self-dual Yang-Mills field, we consider the quaternion product

$$\left(\varphi_\mu^\dagger dx_\mu\right)\left(\varphi_\nu dx_\nu\right) = \mathrm{Sc}(\varphi_\mu^\dagger \varphi_\nu)dx_\mu\, dx_\nu + \mathrm{Vec}(\varphi_\mu^\dagger \varphi_\nu)\, dx_\mu\, dx_\nu \quad . \tag{2f.156}$$

Through this splitting, it naturally associates a scalar product form with a 2-form. So going hand in hand with the Riemannian metric

$$ds^2 = g_{\mu\nu}\, dx_\mu\, dx_\nu\,, \quad g_{\mu\nu} = Sc(\varphi_\mu^\dagger \varphi_\nu) \tag{2f.157}$$

is the Yang-Mills 2-form $f = f_{\mu\nu}\, dx_\mu \Lambda\, dx_\nu$. In fact, they are parts of the following hermitian quaternionic Kähler metric

$$K_{\mu\nu} = \varphi_\mu^\dagger \varphi_\nu = g_{\mu\nu} + \frac{1}{2} f_{\mu\nu} = \partial_\mu t^\dagger G\, \partial_\nu t \quad . \tag{2f.158}$$

The associated manifold is Kählerian since $f_{\mu\nu}$ derives from a gauge potential and hence is covariantly closed: the curvature form f satisfies the Bianchi identity

$$df + \alpha \Lambda f = 0 \quad . \tag{2f.159}$$

The quaternionic scalar product of dx and δx reads

$$\langle dx, \delta x \rangle = dx_\mu K_{\mu\nu} \delta\, x_\nu = dt^\dagger G\, dt \quad . \tag{2f.160}$$

Consequently, if the mapping t(x) is tangent admissible in the sense of Eq. (2f.147), we find

$$\frac{1}{2} f_{\mu\nu}\, dx_\mu\, \delta x_\nu = Vec\, \langle dx_\mu\, , \delta x_\nu \rangle \tag{2f.161}$$

$$= Vec\left(l^\dagger d\bar{x}\; K^\dagger G\; K \delta x\; l \right) \quad , \tag{2f.162}$$

and

$$ds^2 = Sc\left(l^\dagger d\bar{x}\; K^\dagger G K \delta x\; l \right) \quad . \tag{2f.163}$$

The Yang-Mills equations for $f_{\mu\nu}$ are automatically satisfied if $f_{\mu\nu}$ is self-dual or antiself-dual. This result follows from the analyticity condition of Atiyah et al. [165] of the mapping t(x) according to Eq. (2f.124). The requirement of finiteness of the action implies a rational function t(x) of the form (2f.123)

$$t(x) = \alpha + (x+Q)^{-1} \mu \quad , \quad (Q = Q^\dagger) \tag{2f.164}$$

and the conformal flatness of d^2s, together with the (anti-) self-duality of $f_{\mu\nu}$, derives from the analyticity condition of Atiyah et al. [165, 174], Eq. (2f.132) or Eq. (2f.133). The latter reads

$$\mathrm{Vec}\left\{(x+Q)(I+t\,t^{\dagger})(\bar{x}+\bar{Q})\right\} = 0 \quad . \tag{2f.165}$$

In the special case when $\lim_{x\to\infty} G = I$, we must have $\alpha = 0$, so that Eq. (2f.165) reduces to

$$\lim_{x\to\infty} \mathrm{Vec}\left(Q\bar{Q} + \mu^{\dagger}\mu\right) = 0 \,, \;\; (\,Q = Q^{\dagger}\,) \;= I \;, \tag{2f.166}$$

which is the ADHM condition. The 't Hooft solutions [17, 175], yet to be discussed, correspond to the special case of a *diagonal* Q and a *scalar* μ.

However, the condition $\lim_{x\to\infty} G = I$ or Eq. (2f.166) are not invariant under conformal transformations (2f.108); the choice of

$$\mu = (aI - Q)\alpha \,, \; a \in \mathbf{H} \;, \tag{2f.167}$$

gives rise to a more general, conformally invariant form of $t(x)$:

$$t = \alpha + (x + Q)^{-1}(aI - Q)\alpha = (x + Q)^{-1}(x+a)\alpha \;. \tag{2f.168}$$

The constraint (2f.165) is again solved by taking Q diagonal and α purely scalar. We thus find the Jackiw-Rebbi-Nohl solution [176]. The resulting a_μ takes the form of (2f.52) with $n_0 = (x + a)^{-1}$, $n = (x + Q)^{-1}\alpha$. We get

$$a_\mu = -\frac{\mathrm{Vec}\left(e_\mu \dfrac{1}{|x+a|^2}\dfrac{1}{|x+a|} + \alpha^{\dagger}\dfrac{1}{|x+Q|^2}\dfrac{1}{|x+Q|}\alpha\right)}{\dfrac{1}{|x+a|^2} + \alpha^{\dagger}\dfrac{1}{|x+Q|^2}\alpha} \;. \tag{2f.169}$$

Next, by introducing the Schwarzian Fueter function

$$F(x) = (x+a)^{-1} + \alpha^{\dagger}(x+Q)^{-1}\alpha \ , \tag{2f.170}$$

we get

$$\left[\overline{n_0}\,\overleftrightarrow{\partial_\mu} n_0 + n^{\dagger}\,\overleftrightarrow{\partial_\mu} n\right] = \frac{1}{4}\Box F \ , \tag{2f.171a}$$

$$\left(\overline{n_0}\, n_0 + n^{\dagger} n\right) \equiv \sigma = \frac{1}{2} D\, F \ , \tag{2f.171b}$$

so that

$$a_\mu = \frac{1}{2}\,\mathrm{Vec}\left[\, e_\mu(\Box F)\,(DF)^{-1}\right] = \frac{1}{2}\, e'_{\mu\nu}\,\partial_\nu \ln\sigma \ . \tag{2f.172}$$

Denoting the self-dual and anti-self-dual parts of $f_{\mu\nu}$ by

$$\phi_{\mu\nu} = f_{\mu\nu} + \tilde{f}_{\mu\nu}\ , \quad \phi'_{\mu\nu} = f_{\mu\nu} - \tilde{f}_{\mu\nu} \ , \tag{2f.173}$$

we may write

$$\phi'_{\mu\nu} = -\frac{1}{2}\, e'_{\mu\nu}\,(DF)^{-1}\,\Box\, DF = -\frac{1}{2}\, e'_{\mu\nu}\,\sigma^{-1}\Box\,\sigma \ , \tag{2f.174}$$

$$\phi_{\mu\nu} = -\frac{1}{2}(DF)\left(De_{\mu\nu}\overline{D}\right)(DF)^{-1} = \sigma\left(\partial_\mu\partial_\lambda\,\sigma^{-1}\right)e'_{\lambda\nu} - \sigma\left(\partial_\nu\partial_\lambda\,\sigma^{-1}\right)e'_{\lambda\mu} \ . \tag{2f.175}$$

Furthermore, the self-duality condition simplifies to

$$\sigma^{-1}\,\Box\,\sigma = 0 \quad \text{or} \quad \Box\, D\, F = 0 \ , \tag{2f.176}$$

namely to a *linear* equation, which is just Fueter 's holomorphy condition for the function F. Thus Eq. (2f.176) generalizes to quaternions and to D = 4 Yang-Mills gauge theory the Cauchy-Riemann equations. The latter are the (anti-)self-duality equations of the 2-dimensional CP_n σ-models [177], albeit only in the restricted case of the 't Hooft et al SU(2) instantons.

In the same vein, for the above special solutions, we expect and find a simple, natural interpretation of the Pontryagin index as a winding number of quaternionic analysis. The Pontryagin density is easily found to be proportional to

$$\Pi = -\,\mathrm{Sc}\; f_{\mu\nu}\, \tilde{f}_{\mu\nu} = \partial_\mu k_\mu = \frac{1}{2}\,\Box\Box \ln\sigma - \frac{1}{2}\;\partial_\mu \sigma^2\, \partial_\mu\!\left(\frac{\Box\,\sigma}{\sigma^3}\right) \tag{2f.177}$$

with

$$k_\mu = \frac{1}{2}\,\partial_\mu \Box \ln\sigma - \frac{1}{2}\,\sigma^2\, \partial_\mu\!\left(\frac{1}{\sigma^3}\,\Box\,\sigma\right) \tag{2f.178}$$

as the associated conserved topological current.

For self-dual fields where Eq. (2f.176) holds; only the first terms in Eqs. (2f.177) and (2f.178) survive; Π is then the Yang-Mills Lagrangian density. The Chern-Pontryagin index simplifies to

$$C_2 = \frac{1}{8\pi^2} \oint\!\!\!\iiint d^4x\, \Pi = \frac{1}{16\pi^2} \oint\!\!\!\iint_{S^3} dS\; \Box\, \bar{D} \ln\sigma\ . \tag{2f.179}$$

Since C_2 is diffeomorphism invariant, a topological deformation of the S^3-contour reduces it to the sum of small hyperspheres (or cycles) S^3, one surrounding each pole. By way of Eqs. (2d.4)-(2d.51) we obtain

$$C_2 = \frac{1}{8\pi^2} \oint\!\!\!\iint_{S^3} d\Sigma\; \Box \sum_{i=0}^{n} \frac{1}{\beta_i - x}\ , \tag{2f.180}$$

with β_i being the elements of the diagonal matrix Q and $\beta_0 = -\,a$. By way of a suitable conformal transformation, it is always possible to remove one pole of $\bar{D}\ln\sigma$. Consequently, we get the expected result of $C_2 = n$. The quantity $(1{+}C_2)$ then counts the number of poles of F(x) as well as the dimension of the associated embedding HP_n space. Again, this is the exact parallel with the first Chern index of the D = 2 complex

CP_1 instantons. There, the instanton number, or number of poles in the rational, complex analytic solution, is given by the argument principle of complex analysis. Here it is given by the argument principle of quaternion analysis. Next, we apply quaternion analysis to probe further the mentioned correspondence between Einstein and Yang-Mills theories.

Gravitational instantons are classical solutions to Euclidean Einstein's equations [178]. As in Yang-Mills theories, these pseudoparticles are associated with tunnelling amplitudes between vacua with two definite topological quantum numbers, the Euler characteristics and the Hirzebruch signature, respectively. The well-known vacuum Einstein equations with cosmological constant Λ are

$$R^{\alpha}_{\beta\alpha\nu} = R_{\beta\nu} = \Lambda\, g_{\beta\nu} \ . \tag{2f.181}$$

$R^{\alpha\beta}{}_{\mu\nu}$, $R_{\mu\nu}$ and $g_{\mu\nu}$ denote the curvature, Ricci and metric tensors, respectively. In terms of the vierbeins h^a_μ, the SO(4) invariant metric reads

$$g_{\mu\nu} = h^a_\mu h^a_\nu \qquad , \qquad (h^a = h^a_\mu dx^\mu) \ . \tag{2f.182}$$

The connection 1-form ω^{ab} is defined through Cartan's structural equations

$$d\, h^a + \omega^{ab} \wedge h^b = 0 \, , \quad (\, \omega^{ab} = -\omega^{ba} \,) \ . \tag{2f.183}$$

We recall that two gauge potentials (or spin connections) [47] are associated to the two SU(2) ≈ Sp(1) gauge groups of the SO(4) holonomy of a 4-Riemannian manifold. They are

$$a_\mu = \frac{1}{2}\, e'_{ab}\, \omega^{ab}_\mu \quad \text{and} \quad b_\mu = \frac{1}{2}\, e_{ab}\, \omega^{ab}_\mu \ , \tag{2f.184}$$

their curvatures are simply related to $R^{ab}{}_{\mu\nu}$ through

$$f_{\mu\nu} = \frac{1}{2}\, e'_{ab}\, R^{ab}_{\mu\nu} \quad \text{and} \quad b_{\mu\nu} = \frac{1}{2}\, e_{ab}\, R^{ab}_{\mu\nu} \ . \tag{2f.185}$$

We note that, while a_μ and b_μ are fundamental local fields in the usual Yang-Mills theory, they are composite in the fields h_μ^a in Einstein gravity. Moreover the Lagrangian is linear in the curvature for the latter theory while it is quadratic in the Yang-Mills case. A gravity Lagrangian quadratic in the curvature would lead to a Weyl-Yang theory [179], the field equations of which are third order equations in the metric.

To a doubly self-dual gravitational curvature corresponds an anti-self-dual SU(2) gauge field and a self-dual one

$$\phi_{\mu\nu} = f_{\mu\nu} + \tilde{f}_{\mu\nu} = \frac{1}{2} e'_{ab} \left(R^{ab}_{\mu\nu} + \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} R^{ab}_{\rho\sigma} \right) , \tag{2f.186}$$

$$\phi'_{\mu\nu} = f_{\mu\nu} - \tilde{f}_{\mu\nu} = \frac{1}{2} e'_{ab} \left(R^{ab}_{\mu\nu} - \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} R^{ab}_{\rho\sigma} \right) . \tag{2f.187}$$

The part of $R^{ab}_{\mu\nu}$ which is self-dual in the indices a and b gives similar equations with e_{ab} replaced by e_{ab}'.

As shown by Belavin and Burlankov [180] and by Eguchi and Hanson [181], the fourth order equations of motion coming from the Weyl-Yang action are automatically satisfied by a doubly self-dual curvature. This double self-duality corresponds to the nine equations:

$$S_{\mu\nu} = R_{\mu\nu} - \frac{1}{4} g_{\mu\nu} R = 0 . \tag{2f.188}$$

These are solved by

$$R_{\mu\nu} = \Lambda \, g_{\mu\nu} , \tag{2f.189}$$

Λ being an arbitrary cosmological constant. While of second order in $g_{\mu\nu}$, (2f.189) is first order in $a_\mu = \frac{1}{2} e'_{ab} \omega^{ab}_\mu$ as it can be cast into the form of

$$\vec{F}_{\mu\nu} + \frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}\vec{F}_{\rho\sigma} = 0 \ . \tag{2f.190}$$

Consequently, Einstein's equations can be viewed either as

a) the field equations derived from the variation of the linear Lagrangian

$$L^{E} = \sqrt{g}\ (R^{\alpha}_{\alpha} - \Lambda) \tag{2f.191}$$

or as

b) the defining equations of the doubly self-dual sector of a theory derived from the quadratic Weyl-Yang Lagrangian

$$L^{W} = \sqrt{g}\ R_{\alpha\beta\rho\sigma}\, R^{\alpha\beta\rho\sigma} \ . \tag{2f.192}$$

Using the mapping (2f.186)-(2f.187) for the SO(4) gauge field, we may also select the doubly self-dual or anti-self-dual sectors by requiring finiteness of the Weyl action. Furthermore, the anti-self-dual Yang-Mills equations $\phi_{\mu\nu} = 0$ lead to the Einstein equation (2f.199) or (2f.188).

In fact, using the well-known decomposition of $R_{ab\,\mu\nu}$

$$R_{ab\,\mu\nu} = C_{ab\mu\nu} + \frac{1}{2}\left(g_{a\mu}\, R_{b\nu} + g_{b\nu}\, R_{a\mu} - g_{b\mu}\, R_{a\nu} - g_{a\nu}\, R_{b\mu}\right)$$

$$- \frac{1}{6}\left(g_{a\mu} g_{b\nu} - g_{b\mu} g_{a\nu} \right) R \ , \tag{2f.193}$$

we have split $R_{ab\mu\nu}$ into $C_{ab\mu\nu}$, the Weyl conformal tensor (10 components), the traceless part of the Ricci tensor (9 components) and the scalar curvature R (1 component), we get

$$\phi_{\mu\nu} = \frac{1}{2}\left(e'_{\mu\lambda} S_{\lambda\nu} - e'_{\nu\lambda} S_{\lambda\mu}\right) , \tag{2f.194}$$

$$\phi'_{\mu\nu} = \frac{1}{2}\, e'_{ab}\left(C^{ab}_{\mu\nu} - \frac{1}{2}\,\varepsilon_{\mu\nu\kappa\lambda}\, C^{ab}_{\kappa\lambda}\right) - \frac{1}{12}\, e'_{\mu\nu} R \tag{2f.195}$$

and

$$\beta_{\mu\nu} = b_{\mu\nu} + \tilde{b}_{\mu\nu} = \frac{1}{2} e_{ab} \left(C^{ab}_{\mu\nu} + \frac{1}{2} \varepsilon_{\mu\nu\kappa\lambda} C^{ab}_{\kappa\lambda} \right) + \frac{1}{12} e_{\mu\nu} R \ , \tag{2f.196}$$

$$\beta'_{\mu\nu} = b_{\mu\nu} - \tilde{b}_{\mu\nu} = \frac{1}{2} \left(e_{\mu\lambda} S_{\lambda\nu} - e_{\nu\lambda} S_{\lambda\mu} \right) \ . \tag{2f.197}$$

We note the one-to-one correspondence between $S_{\lambda\mu}$ and either $\phi_{\mu\nu}$ or $\beta_{\mu\nu}$ since each tensor has 9 components. On the other hand, a general field $\phi'_{\mu\nu}$ (or $\beta_{\mu\nu}$) has 9 components, but through Eq. (2f.155) we can always select a special gauge which puts these field strengths in the form Eqs. (2f.195) and (2f.196). In such a SO(4) gauge, $f_{\mu\nu}$ has 15 components, made up of $R_{\mu\nu}$ and the anti-self-dual part C^- of $C^{ab}_{\mu\nu}$, while $b_{\mu\nu}$ consists of $R_{\mu\nu}$ and the self-dual part C^+ of the Weyl tensor.

In terms of the Weyl tensor, the topological indices are respectively the Euler-Poincaré characteristics

$$\chi = \frac{1}{32\pi^2} \iiiint \sqrt{g}\, d^4x \left(C_{ab\mu\nu} C^{ab\mu\nu} + \frac{8}{3} \Lambda^2 \right) \tag{2f.198}$$

and the Pontryagin index

$$\tau = \frac{1}{48\pi^2} \iiiint \sqrt{g}\, d^4x \left(C_{ab\mu\nu} {}^*C^{ab\mu\nu} \right) \ , \tag{2f.199}$$

with ${}^*C^{ab\mu\nu} \equiv \frac{1}{2} \varepsilon^{\mu\nu\rho\sigma} C^{ab\rho\sigma}$. Λ is given by

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R\, g_{\mu\nu} + \Lambda\, g_{\mu\nu} = 0 \ . \tag{2f.200a}$$

At this juncture, we also record two useful identities

$$\iiiint \sqrt{g}\, d^4x \left(R_{ab\mu\nu} R^{ab\mu\nu} - 4 R_{\mu\nu} R^{\mu\nu} + R^2 \right) = 0 \tag{2f.200b}$$

and

$$C_{\alpha\beta\gamma\delta}C^{\alpha\beta\gamma\delta} = R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu} - 2R_{\alpha\beta}R^{\alpha\beta} + \frac{1}{3}R^2 \tag{2f.200c}$$

as well as the following inequalities. Denoting the volume V of a compact instanton by $V = \int \sqrt{g}\,d^4x$, the action is just $A = -\frac{\Lambda V}{8\pi}$. The Weyl tensor satisfies the inequality

$$C_{abcd}C^{abcd} \geq \frac{1}{2}\left|C_{abcd}C^{cdef}\varepsilon^{ab}{}_{ef}\right| . \tag{2f.201}$$

Upon integration, it gives

$$2\chi - 3|\tau| \geq \frac{\Lambda^2 V}{6\pi^2} , \tag{2f.202}$$

with the equality sign holding provided that

$$C = \pm * C \tag{2f.203}$$

i.e. when the metric is conformally self-dual. Examples are the complex manifold CP_2 and the Kummer surface K_3 with both spaces being quaternionic manifolds.

In the case of a Kähler metric, the invariants χ and τ are further linked by the first Chern number c_1 through the relation $2\chi + 3\tau = c_1^2$. Therefore, if the metric is also Einstein, then

$$c_1^2 = \frac{V\Lambda^2}{4\pi^4} \tag{2f.204}$$

and

$$\chi \geq \frac{V\Lambda^2}{6\pi^2} , \tag{2f.205}$$

the equality holds provided that the Weyl tensor is self-dual. Bounds were obtained for algebraic subvarieties of CP^n, the upper bound is saturated by products of pairs of D=2 spaces of constant curvatures. Subsequently, we will see an example of a quaternionic four manifold given by such a product space.

From Eq. (2f.194) to Eq. (2f.197), we see that, when the tensors C and R all vanish, $f_{\mu\nu}$ is self-dual while $b_{\mu\nu}$ is anti-self-dual. The resulting metric is then conformally flat:

$$ds^2 = \sigma^2\, dx\, d\bar{x}\,, \qquad \Box\,\sigma \equiv \Box\,(DF) = 0 \tag{2f.206}$$

with

$$R = 6\,\frac{1}{\sigma^3}\,\Box\,\sigma\ . \tag{2f.207}$$

We have here an Einstein theory with a scalar field. $S_{\mu\nu}$ is nonvanishing;

$$S_{\mu\nu} = -\,2\,\sigma(\,\partial_\mu\partial_\nu\sigma^{-1} - \frac{1}{4}\,\delta_{\mu\nu}\,\Box\,\sigma^{-1})\ . \tag{2f.208}$$

It is the improved energy-momentum tensor of the scalar field σ, with the $\phi_{\mu\nu}$ in Eq. (2f.194) being precisely the one occurring in the 't Hooft instanton ansatz.

Double self-duality of the curvature yields $\phi'_{\mu\nu} = 0$ or $R = 0$. It corresponds to the half flat case with $C^- \neq 0$, $C^+ = 0$, as is exemplified by the Kummer surface K_3. Double anti-self-duality gives $\phi_{\mu\nu} = 0$ and $S_{\mu\nu} = 0$. In the former case, σ is harmonic; F is Fueter analytic. In the latter case, the traceless part of the energy-momentum tensor vanishes. In general, not only the 't Hooft instantons but also the general ADHM SU(2) instantons will solve for $\phi_{\mu\nu}' = 0$. They therefore provide new solutions of the Einstein equations with non-conformally flat metrics.

2.f.4. H-analyticity and Milne's regraduation of clocks

The Euclidean conformal group SL(2, **H**) is the largest finite parameter subgroup of the general covariance group (GCG) leaving the conformally flat metric $ds^2 = \chi^2|dx|^2$ form invariant. This situation is to be contrasted with the two-dimensional case where *all* 2-metrics are conformally flat. There, $ds^2 = \chi^2 dz\, d\bar{z}$ is left invariant, not just by SL(2, **R**), but by the infinite parameter group of analytic transformations $z' = f(z)$, the Virasoro group.

In this connection, we note the existence of a remarkable theorem of V.I.Ogievetsky [182]. It states that the infinite dimensional D = 4 GCG algebra can be viewed as the closure of the finite-dimensional algebras of the affine SL(4, **R**) and the conformal groups. Namely, any generator $L^{n_1n_2n_3n_4}{}_{\mu}$ (n_i = positive integers) of a GCG transformation $\delta x_\mu = f_\mu(x_1, x_2, x_3, x_4)$, $(\mu = 1, 2, 3, 4)$

$$L^{n_1n_2n_3n_4}{}_{\mu} \equiv -\, i x_1^{n_1} x_2^{n_2} x_3^{n_3} x_4^{n_4} \frac{\partial}{\partial x_\mu} \tag{2f.209}$$

can be represented as a linear combination of repeated commutators of the generators of SL(4, **R**) and SO(5, 1) ≈ SL(2, **H**). On this basis one might think that, asides from these finite parameter subgroups and the GCG, no distinctive intervening invariance group exists. That it is not so was pointed out in Section 2d.4. There the quaternionic approach does single out a (albeit infinite) subgroup of the Riemannian GCG, which we called the *Fueter group*. What follows illustrates the workings of such a group in D = 4 Euclidean gravity [127].

First, we shall switch to a complex representation of the position quaternion x of a Riemannian spacetime M_4. We introduce two complex numbers z and ζ defined by

$$\begin{cases} z = x_4 + i\,(\, x_1^2 + x_2^2 + x_3^2\,)^{1/2} = \tau + i\, r\,, \\ \zeta = 2\,(\,x_1 + i x_2\,)(\,x_3 + r\,)^{-1} = 2\tan\dfrac{\theta}{2}\exp i\varphi\,. \end{cases} \tag{2f.210}$$

$\tau = x_4$ and $r = |\mathbf{e}\cdot \mathbf{x}|$, $\mathbf{e}\cdot\mathbf{x} = r\, U\, e_3\, U^{-1}$, θ and φ are the usual polar angles.

$$U = \exp\left(e_3 \frac{\varphi}{2}\right) \exp\left(e_2 \frac{\theta}{2}\right) \exp\left(e_3 \frac{\varphi}{2}\right) \tag{2f.211}$$

$$= \left(1 + \frac{1}{4}\zeta\bar{\zeta}\right)^{-1/2} \begin{pmatrix} 1 & \frac{-\bar{\zeta}}{2} \\ \frac{\zeta}{2} & 1 \end{pmatrix} , \tag{2f.212}$$

ζ is the projective coordinate of the 2-sphere S^2. A point $x \in M_4$ may then be parametrized as

$$x = U\,(\,\tau + e_3 r\,)\,U^{-1} = U \begin{pmatrix} \bar{z} & 0 \\ 0 & z \end{pmatrix} U^{-1} \; . \tag{2f.213}$$

Here $z = \tau + ir$ ($r > 0$) labels a point in the Poincaré upper half-plane H^2. It also follows that

$$x^n = U\,(\tau + e_3 r)^n\, U^{-1} \; . \tag{2f.214}$$

We have

$$|dx|^2 = dx\, d\bar{x} = dz\, d\bar{z} + (\mathrm{Im}\, z)^2 d\Omega^2 = (\mathrm{Im} z)^2\, dl^2 \; , \tag{2f.215}$$

with

$$dl^2 \equiv dh^2 + d\Omega^2 \; , \tag{2f.216}$$

$$d\Omega^2 = \left(1 + \frac{\zeta\bar{\zeta}}{4}\right)^{-2} d\zeta\, d\bar{\zeta} = d\theta^2 + \sin^2\theta\, d\varphi^2 \, , \tag{2f.217}$$

$$dh^2 = \frac{dz\, d\bar{z}}{r^2} \; . \tag{2f.218}$$

Clearly, the space with the specific metric (2f.215) is conformally related to the product space $H^2 \times S^2$ with, as its coordinates, the complex variables $z \in H^2$ and $\zeta \in S^2$. Its conformally Kählerian property is manifest from the following alternative form of the line element dl^2:

$$dl^2 = \frac{\partial^2 \Xi}{\partial z \partial \bar{z}}\, dz\, d\bar{z} + \frac{\partial^2 \Xi}{\partial \zeta \partial \bar{\zeta}}\, d\zeta\, d\bar{\zeta} \tag{2f.219}$$

where

$$\Xi(z,\zeta) = \ln\left[\frac{z-\bar{z}}{2i}\right]^{-4} + \ln\left[1+\frac{\zeta\bar{\zeta}}{4}\right]^{-4} \qquad (2f.220)$$

is a potential additive in the Kähler potential of H^2 and S^2.

Next we study the action of a specific class of general coordinate transformations on the metric (2f.215).

We recall from a previous discussion that a theory of general quaternionic power series (2d.88) is no more than a theory of real analytic functions on R^4. So the equation $x' = F(x)$ is merely a quaternionic form of a general coordinate transformation in R^4. This fact induces us to consider instead a more restricted class of coordinate transformations, ones defined by a power series with *scalar* coefficients

$$y = F(x) = \sum_{n=0}^{\infty} a_n x^n \qquad (\mathrm{Vec}\, a_n = 0)\ . \qquad (2f.221)$$

Using the polar coordinates (2f.214), it reads

$$F(x) = UF(z)U^{-1}\ , \qquad (2f.222)$$

which establishes the correspondence between the function $F(x)$ obeying the quaternionic Schwarz reflection principle $F(\bar{x})=\overline{F(x)}$ with $F(z) = \{F(z^*)\}^*$, the complex analytic stem function in the upper half-plane with $\mathrm{Im}\, F(z) \geq 0$.

Since $F(x)$ maps τ into $\tau' = Sc(y)$, it also maps the real time axis into itself. Such a transformation is seen as the Euclidean counterpart of a regraduation of clocks introduced long ago by Milne [183] in a Lorentzian spacetime. From Sect. 2d, due the ring property of such Fueter series, we know that the sum and the product of the series $F(x)$ and $G(x)$ are again of the same form $H(x)$. So, if the inverse transformation $x = F^{-1}(y)$ also exists, then the set of functions $F(x)$ forms an infinite parameter group. We shall refer to it as the clock regraduation group (CRG) or the Fueter group.

Writing

$$y' = U(\theta', \varphi')\,(\tau' + e_3 r')\,U^{-1}(\theta', \varphi') = F(x) \tag{2f.223}$$

with

$$z' = \tau' + i\,r' \;,\quad \zeta' = 2\exp(i\varphi')\tan\left(\frac{\theta'}{2}\right) = 2\,\frac{x'_1 + i\,x'_2}{x'_3 + r'} \tag{2f.224}$$

the mapping $x \rightarrow F(x)$ gives

$$z' = F(z) \quad , \quad \zeta' = \zeta \;. \tag{2f.225}$$

Hence

$$\begin{aligned}|dx|^2 &= (\mathrm{Im} z')^2 \left[\frac{dz' d\bar{z}'}{(\mathrm{Im} z')^2} + d\Omega^2\right] \\ &= (\mathrm{Im}F(z))^2 \frac{|F'(z)|^2}{(\mathrm{Im}\,F(z))^2}\, dz d\bar{z} + \frac{d\zeta\, d\bar{\zeta}}{\left(1 + \frac{\zeta\bar{\zeta}}{4}\right)^2} \;.\end{aligned} \tag{2f.226}$$

So starting from a Riemannian metric having as line element

$$ds^2 = \phi^2(z', \bar{z}')\left[\,\chi^2(z', \bar{z}')\, dz'\, d\bar{z}' + \psi^2(\zeta', \bar{\zeta}')\, d\zeta'\, d\bar{\zeta}'\,\right] \;, \tag{2f.227}$$

a CRG transformation then gives

$$ds^2 = \phi^2(F(z), \overline{F(z)})\left[\,|F'(z)|^2 \chi^2(F, \bar{F})\, dz\, d\bar{z} + \psi^2(\zeta, \bar{\zeta})\, d\zeta d\bar{\zeta}\,\right] \;. \tag{2f.228}$$

The latter reproduces the same doubly conformal Kruskal form of the metric. Consequently, the line element (2f.227) is form invariant under the infinite parameter CRG transformations.

We can generalize the above transformations by defining an extended clock regraduation group through

$$z' = F(z) \quad , \mathrm{Im}\, z' \geq 0 , \qquad \zeta' = G(\zeta) \quad . \tag{2f.229}$$

The angle mapping group $G(\zeta)$ can be expressed as a quaternionic function by noting that

$$\zeta'_1 + e_1\zeta'_2 = G\left(\frac{x - \bar{x} + e_3| x - \bar{x}|}{x - \bar{x} - e_3| x - \bar{x}|} e_1 \right) \quad . \tag{2f.230}$$

Hence

$$\tau' \to \tau \quad , \quad r' \to r \quad , \quad 2\,\frac{x'_1 + i\, xv_2}{x'_3 + r} = G\left(2\,\frac{x_1 + i\, x_2}{x_3 + r} \right) , \tag{2f.231}$$

$$U(\zeta) \to U\left[G(\zeta)\right] \quad . \tag{2f.232}$$

In particular, if $G = \zeta^n$, such a mapping leaves H^2 invariant and wraps S^2 n times around itself.

The analytic functions $F(z)$ and $G(\zeta)$ solve the Cauchy-Riemann equations

$$\left(\frac{\partial}{\partial\tau} + i\,\frac{\partial}{\partial r}\right) F(\,\tau, r) = 0 \quad , \quad \left(\frac{\partial}{\partial\zeta_1} + i\,\frac{\partial}{\partial\zeta_2}\right) G(\zeta_1, \zeta_2) = 0 \quad . \tag{2f.233}$$

We then observe that a Riemannian metric in its Kruskal form

$$ds^2 = \xi^2(\, z, \bar{z}, \zeta, \bar{\zeta}\,)\, dz\, d\bar{z} + \Theta^2(\, z, \bar{z}, \zeta, \bar{\zeta}\,)\, d\zeta d\bar{\zeta} \tag{2f.234}$$

is in fact form invariant under the combination of the conformal and Fueter transformations $y = F\left(\frac{ax + b}{cx + d} \right)$, $F(h)$ being a power series in h with scalar coefficients.

By way of Eq. (2f.233), it can be checked that

$$D\, F(x) = \rho \quad , \ \mathrm{Vec}\, \rho = 0 \ , \tag{2f.235}$$

$$\Box DF = \Box \rho = 0 \quad , \tag{2f.236}$$

$$\Box \Box W_{(i)} = 0 \quad , \quad W_{(i)} \equiv (a_i x + b_i) (c_i x + d_i)^{-1} \quad , \tag{2f.237}$$

$$\Box \Box \sum_i W_{(i)} = 0 \quad , \quad \Box \Box F (W_{(i)} c_i) = 0 \quad . \tag{2f.238}$$

So combining Fueter and conformal mappings gives rise to biharmonic functions.

Let $f(x) = \Box F$, then $Df = 0$; if we write the quaternion $f(x)$ in its Pauli's (2 x 2) matrix representation

$$f(x) = e_\mu f_\mu = \begin{pmatrix} \psi_1 & -\overline{\psi}_2 \\ \psi_2 & \overline{\psi}_1 \end{pmatrix} , \tag{2f.239}$$

the spinor $\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$ then solves for the Dirac equation of a free, massless neutrino.

We also note that the meromorphic functions

$$Y = \sum_i \gamma_i W_{(i)} \quad , \quad Z = \sum_i W_{(i)} c_i \ , \tag{2f.240}$$

with $\gamma_i = \left(a_i c_i^{-1} d_i - b_i \right)^{-1}$, obey respectively

$$\Box D Y = 0 \quad , \quad \Box Z \overleftarrow{D} = 0 \ . \tag{2f.241}$$

Hence $\Box Y$ and $\Box Z$ are left and right regular Fueter functions respectively. Under a SO(4) transformation, Y transforms like (0,1) while Z transforms like (1, 0). So the entries $U = l\,Y$ and $V = Z\,r$ transform like 4-vectors if l and r are arbitrary 4-vectors. So if we choose $l = \sum_n c_n^{-1}$ and $r = \sum_n (a_n c_n^{-1} d_n - b_n)$, U(x) and V(x) are Lorentz covariant 4-vectors.

As a more specific application of the above formalism, we consider the metric of the form

$$ds^2 = \phi^2(z,\bar{z})\, dz d\bar{z} + \rho^2(z,\bar{z})\, d\Omega^2(\theta,\varphi) \tag{2f.242}$$

where $d\Omega^2$ is the S^2 line element (2f.217). The Einstein energy momentum tensor is $G_\mu{}^\nu = R_\mu{}^\nu - \frac{1}{2} R\, \delta_\mu{}^\nu$ and its traceless part is the same as the traceless part of $R_\mu{}^\nu$. The only non-zero elements of $G_\mu{}^\nu$ are its diagonal elements and G_r^τ. Hence the double anti-self-duality equations for the curvature reduces to four equations which are equivalent to Eq. (2f.188):

$$A = G^\theta{}_\theta - G^\varphi{}_\varphi = 0\ , \tag{2f.243}$$

$$B = G^\theta{}_\theta + G^\varphi{}_\varphi - G^\tau{}_\tau - G^r{}_r \tag{2f.244}$$

$$= 8\,\phi^{-2}\left[\frac{1}{2}\frac{\partial^2}{\partial z\,\partial\bar{z}}\ln\phi^2 + \rho^{-2}\left(\frac{1}{4}\phi^2 - \frac{\partial\rho}{\partial z}\frac{\partial\rho}{\partial\bar{z}}\right)\right] = 0\ , \tag{2f.245}$$

$$C = \frac{1}{2}\left(G^\tau{}_\tau - G^r{}_r\right) + i\, G^\tau{}_r = 4\,\rho^{-1}\frac{\partial}{\partial\bar{z}}\left(\phi^{-2}\frac{\partial\rho}{\partial\bar{z}}\right) = 0\ . \tag{2f.246}$$

For nonzero cosmological constant Λ, the Einstein equation $G^\alpha_\alpha = -R$ gives the additional equation

$$K = G^\theta{}_\theta + G^\varphi{}_\varphi = 8\,\phi^{-2}\left[\frac{1}{2}\frac{\partial^2}{\partial z\,\partial\bar{z}}\ln\phi^2 + \rho^{-1}\frac{\partial^2\rho}{\partial z\,\partial\bar{z}}\right] = \text{const.} \tag{2f.247}$$

and K=0 if $R_{\mu\nu}=0$. In the latter case,

$$B - K = 8\,\phi^{-2}\left[\rho^{-2}\left(\frac{1}{4}\phi^2 - \frac{\partial\rho}{\partial z}\frac{\partial\rho}{\partial\bar{z}}\right) - \rho^{-1}\frac{\partial^2\rho}{\partial z\,\partial\bar{z}}\right] = 0 \tag{2f.248}$$

is solved by

$$\phi^2 = 2\frac{\partial^2\rho}{\partial z\,\partial\bar{z}} = \frac{1}{2}\,\Delta\rho^2 \quad , \quad (\Delta \equiv \partial_\tau^{\,2} + \partial_r^{\,2}) \ . \tag{2f.249}$$

Equation (2f.246) has the general solution

$$\phi^2\left[\frac{\partial\rho}{\partial\bar{z}}\right]^{-1} = \frac{1}{2}\left(\Delta\,\rho^2\right)\left[\frac{\partial\rho}{\partial\bar{z}}\right]^{-1} = f(z) \tag{2f.250}$$

where f(z) is an arbitrary analytic function of $\tau + ir$ in the upper half-plane. The Fueter holomorphic function

$$f(z) = U(\,\theta,\,\varphi)\,f\,(\,\tau + e_3\,r\,)\,U(\,\theta,\,\varphi)^{-1} \tag{2f.251}$$

where U given by (2f.211-212) is then a Schwarzian Fueter function. Einstein's equations then become the analyticity condition

$$\Box\,D\,f(x) = 0 \ . \tag{2f.252}$$

Either Eq. (2f.244) or (2f.247) is now a nonlinear equation for $\rho(z\,,\bar{z})$. Since $\ln f(z)$ is harmonic, we find

$$\Delta\ln\phi^2 - \Delta\,(\,\ln\frac{\partial\rho}{\partial\bar{z}}) = 0 \tag{2f.253}$$

so that K = 0 gives

$$\Delta\,\rho + \frac{\rho}{2}\,\Delta\,\ln\frac{\partial\rho}{\partial\bar{z}} = 0 \ . \tag{2f.254}$$

One solution is provided by a harmonic $\rho = \mathrm{Im}\,h(z)$ with h(z) analytic. It results in the flat metric

$$ds^2 = \frac{\partial h}{\partial z}\overline{\left(\frac{\partial h}{\partial z}\right)}\,dz\,d\bar{z} + (\,\mathrm{Im}\,h(z)\,)\,d\Omega^2 \tag{2f.255}$$

$$= dh\, d\bar{h} + (\,\mathrm{Im}\, h\,)^2\, d\Omega^2 \; . \tag{2f.256}$$

Another (non-flat) solution is the Schwarzschild solution of Eq. (2f.254), in the implicit form:

$$L(\rho) = \frac{\rho}{2\,m} + \ln\left(\frac{\rho}{2\,m} - 1\right) = 2\,\mathrm{Re}\,\ln z = \ln(z\bar{z})\,. \tag{2f.257}$$

Differentiation with respect to $\bar{z}$ and z gives

$$\frac{\partial\rho}{\partial\bar{z}} = \frac{2m}{\bar{z}}\left(1 - \frac{2m}{\rho}\right) \quad , \quad \frac{\partial\rho}{\partial z} = \frac{2m}{z}\left(1 - \frac{2m}{\rho}\right) \; , \tag{2f.258}$$

$$\Delta\,\rho = 4\,\frac{\partial\rho}{\partial\bar{z}\partial z} = \frac{32\,m^3}{z\,\bar{z}}\,\frac{1}{\rho^2}\left(1 - \frac{2m}{\rho}\right) \; , \tag{2f.259}$$

$$\Delta\;\ln\frac{\partial\rho}{\partial\bar{z}} = -\,\frac{64\,m^3}{z\,\bar{z}}\,\frac{1}{\rho^3}\left(1 - \frac{2m}{\rho}\right) . \tag{2f.260}$$

For this solution, we find

$$\phi^2 = \frac{1}{2}\,\Delta\,\rho^2 = \frac{64\,m^3}{z\,\bar{z}}\left(1 - \frac{2m}{\rho}\right) = \frac{32\,m^3}{\rho}\exp\left(-\frac{\rho}{2m}\right) \; . \tag{2f.261}$$

Hence $f(z) = \phi^2\left(\frac{\partial\rho}{\partial\bar{z}}\right)^{-1} = \frac{8m}{z}$ with one pole at z = 0. The transformation $z \rightarrow g(z)$ leads to another Kruskal coordinate system for which

$$f(z) = \; 8m\,\frac{g'(z)}{g(z)} \quad , \quad \phi^2 = 16\,m^2\left|\frac{g'}{g}\right|^2\left(1 - \frac{2m}{\rho}\right) \; . \tag{2f.262}$$

It can be checked that ρ is implicitly given by

$$\frac{2m}{\rho} + \ln\left(\frac{\rho}{2m} - 1\right) = \ln|g(z)|^2 \quad . \tag{2f.263}$$

Each choice of g(z) gives a new Kruskal coordinate system preserving the form (2f.227) of the metric. We find

$$\phi^2 = 16\, m^2 \left|\frac{g'}{g}\right|^2 \left(1 - \frac{2m}{\rho} \right) \tag{2f.264}$$

and

$$f(z) = \phi^2 \left(\frac{\partial\rho}{\partial\bar{z}}\right)^{-1} = 8m\, \frac{g'(z)}{g(z)} \quad . \tag{2f.265}$$

The metric reads

$$ds^2 = \rho^2 \left[\frac{2}{m} \frac{\partial^2 \rho}{\partial z\, \partial\bar{z}}\, dz\, d\bar{z} + d\Omega^2 \right] \quad . \tag{2f.266}$$

The choice of g(z) = z gives the standard Kruskal solution

$$ds^2 = 32\, m^3\, \rho^{-1} \exp\left(-\frac{\rho}{2m}\right) dz\, d\bar{z} + \rho^2 d\Omega^2 \tag{2f.267}$$

while that of $g(z) = \exp\left(\frac{z}{4m}\right)$ leads to

$$ds^2 = \left(1 - \frac{2m}{\rho}\right) dz\, d\bar{z} + \rho^2 d\Omega^2 = \frac{2m}{\rho} \exp\left(\frac{\tau - \rho}{2m}\right) dz\, d\bar{z} + \rho^2\, d\Omega^2 \tag{2f.268}$$

with ρ given by $\left(\frac{\rho}{2m} - 1 \right) \exp\left(\frac{\rho}{2m} \right) = \exp\left(\frac{\tau}{2m} \right)$ and $f(z) = 2z$.

If the cosmological constant Λ is non-zero, K in Eq. (2f.247) is a constant. Instead of Eq. (2f.249), we find

$$\phi^2 = \frac{1}{2} \left(1 - 3\lambda\rho^2 \right)^{-1} \Delta\rho^2 \quad , \quad G_{\mu\nu} = 3\lambda g_{\mu\nu} \quad . \tag{2f.269}$$

The S^4 metric in one Kruskal system is obtained by taking

$$f(z) = \frac{\phi^2}{\frac{\partial \rho}{\partial \bar{z}}} = -2\,i\,(\,1 + \frac{\lambda}{4} z^2)^{-1} \tag{2f.270}$$

corresponding to

$$\rho = (\,1 + \frac{\lambda}{4} z\,\bar{z})^{-1}\,\mathrm{Im}\,z \quad ,$$
$$\phi^2 = (\,1 + \frac{\lambda}{4}\,z\,\bar{z})^{-2}\,. \tag{2f.271}$$

In consequence, we recover the well-known conformally flat S^4 metric associated with the BPST 1-instanton:

$$d^2s = (\,1 + \frac{\lambda}{4}\,z\,\bar{z})^{-2}\left[dz\,d\bar{z} + (\,\mathrm{Im}z)^2 d\Omega^2\right]\;. \tag{2f.272}$$

Another S^4-Kruskal system is obtained by way of the Milne mapping $z \to g(z)$, which gives

$$f(z) = -2i\;g'(z)(\,1 + \frac{\lambda}{4}\,g(z)^2)^{-2} \tag{2f.273}$$

and

$$d^2s = \frac{1}{2}\,(\,1 - 3\lambda\rho^2\,)^{-1}\Delta\rho^2 dz\,d\bar{z} \;+ \rho^2 d\Omega^2 \tag{2f.274}$$

or

$$ds^2 = \left(1 + \frac{\lambda}{4}\,g\bar{g}\right)^{-2}\left[g'\;\bar{g'}\right]\,dg\;dg' + (\mathrm{Img})^2\,d\Omega^2 \tag{2f.275}$$

with $\rho^2 = (\,1 + \frac{\lambda}{4} g\;\bar{g})^{-2}\,(\mathrm{Im}\,g)^2$ and $\phi^2 = (1 + \frac{\lambda}{4}\,g\;\bar{g})^{-2}\;g'\,\bar{g}$.

By direct computation the choice of the stem function $g(z) = z^n$, namely $g(x) = x^n$ leads to multi-S^4 gravitational instantons centered at the origin with Pontryagin number n.

Just as in the instantons of the $D = 2$ CP_1 σ-model, the analyticity of the complex function f(z) given by Eq. (2f.270) in the Kruskal system leads to the integral conservation law

$$\oint dz\, f(z) = 0 \quad . \tag{2f.276}$$

Thus

$$\oint dz\, \phi^{2n}\left[\frac{\partial\rho}{\partial\bar{z}}\right]^{-n} = 0 \tag{2f.277}$$

implying that, due to the invariance of the Kruskal form under Fueter transformations, the quantities

$$\Pi^{(n)} = \int dr\, \mathrm{Re}\left[\phi^{2n}\left(\frac{\partial\rho}{\partial\bar{z}}\right)^{-n}\right] \quad , \quad \frac{d\,\Pi^{(n)}}{d\tau} = 0 \tag{2f.278}$$

are conserved in Euclidean time τ. By way of the Fueter function F(x) , Eq. (2f.251), these conservation laws can be set in covariant forms:

$$\frac{d}{d\tau} H^{(n)} = 0 \tag{2f.279}$$

with

$$H^{(n)} = \frac{1}{8\pi^2} \oiiint_{S^3} d^3x\, \mathrm{Sc}\{\square\, [F(x)]^n\} \quad . \tag{2f.280}$$

Equation (2f.279) follows from

$$\oiiint_{S^3} d\Sigma\, \square\, F(x) = 0 \tag{2f.281}$$

in the domain of analyticity of F(x).

A systematic quaternionic analysis of multi-gravitational instantons with topologies

S^4, CP^2, $S^2 \times S^2$ and the Schwarzschild solutions by analytic mappings has been done in Refs.[184, 185, 186].

In the above example based on Ref.[127], one main defect is the lack of manifest O(4) covariance, not built in the original Fueter theory of quaternionic functions. The welcome cure found recently for this problem was in fact discussed previously in subsection 2e.

Another application of quaternion analysis concerns periodic instantons and their superpositions [187]. Such solutions are counterparts of the finite gaps or Bloch wave solitons of integrable systems such as the KdV equations [188]. These solutions are given in terms of elliptic, hyperelliptic and, in the general case, by theta functions characterizing Abelian varieties. Our point of departure is the particular Yang-Mills connection

$$a_\mu = \frac{1}{2}\,\mathrm{Vec}\left\{e_\mu(\square F)\,(DF)^{-1}\right\} \tag{2f.282}$$

associated with the trigonometric Fueter function

$$F(x) = \frac{1}{2}\cot\left(\frac{\beta}{2}x\right) = \frac{1}{2\pi}\sum_{n=-\infty}^{+\infty}\left(\frac{\beta}{2\pi}x - n\right)^{-1} . \tag{2f.283}$$

Since F(x) has an infinite number of poles, it results in an infinite Pontryagin number. (2f.283) can be seen as an infinite superposition of 't Hooft instantons along the τ axis at periodic interval separated by the period $4\pi\beta^{-1}$. The corresponding Jackiw-Rebbi-Nohl function σ is

$$\sigma = \frac{1}{2}DF = \beta^{-1}\sum_{n=-\infty}^{\infty}\left[\,r^2 + (\tau - 2\pi\beta^{-1}n)^2\,\right]^{-2} . \tag{2f.284}$$

The τ dependence of a_μ can be removed by the gauge transformation

$$a'_\mu = S^{-1}\left[a_\mu + (\partial_\mu S)\, S^{-1}\right] S \tag{2f.285}$$

with

$$S = \exp\left(- e \cdot \frac{\mathbf{r}}{r}\gamma\right) , \tag{2f.286}$$

the angle γ is given by

$$\tan\gamma = \frac{\mathrm{Im}\cos\beta z}{1 - \mathrm{Re}\cos\beta z} \; . \tag{2f.287}$$

It is readily checked that Eq. (2f.285) is the connection of the standard Prasad-Sommerfield solution for a monopole. The action has value one when the spacetime volume integration is over all space and between the time interval τ and $\tau = a{+}4\pi\,\beta^{-1}$. This temporal period has the interpretation of the inverse temperature of an associated statistical mechanical system.

By noting that

$$\partial_\tau F = F'(x) = -\frac{1}{4}\,\beta\left[\sin\frac{\beta x}{2}\right]^{-2} \tag{2f.288}$$

is also a meromorphic function with period $4\pi\beta^{-1}$, new solutions to the Yang-Mills equations suggest themselves. For example, we could consider other functions F(x)'s such that F' (x) are meromorphic functions with more than one period, e.g. functions with 4 periods along all four space-time directions. One example is the quaternionic Weierstrass *P* function. Its associated ζ-function $\zeta(x) = \Box\, Z(x)$ (2d.72) was explicitly constructed in Sect. 2d.3. We recall its fundamental property

$$\frac{1}{8\pi^2}\int_{\partial C} d\Sigma\, \zeta(x) = \frac{1}{8\pi^2}\int_{\partial C} d\Sigma\, \Box\, Z(x) = 1 \tag{2f.289}$$

with ∂C being the boundary of the period cell. Clearly the choice of the ansatz

$$a_\mu = \frac{1}{2}\,\mathrm{Vec}\left\{ e_\mu \Box\, Z(x)\,(DZ\,)^{-1}\right\} \tag{2f.290}$$

corresponds to a Yang-Mills solution with unit instanton number per period 4-cell. The quantity $|q_i|^{-1}$, $i = 1, 2, 3$ can be associated with a momentum cut-off along the direction $q^{(i)}$. The function $D\,Z = \sigma(x, q)$ may be seen as a distorted instanton solution when S^4 is mapped onto a periodic cell. Indeed the BPST 1-instanton [189, 190] is recovered in the $\lim_{q \to \infty} \sigma(x, q) = \sigma(x)$, in the same way that the soliton solutions (say, of the KdV) solitons are recovered from the corresponding finite zone Bloch wave solutions when the latter's periods go to infinity [191]. Similarly, n-instanton solutions can be superposed periodically along four directions to generate distorted n-instantons solutions $\sigma^{(n)}(x\,,\,q)$.

As there is a linear relation between the Ricci tensor and the SU(2) Yang-Mills field given by Eqs. (2f.194-2f.195), we can consider a metric associated with the Einstein equations with a source term

$$C_{\alpha\beta\mu\nu} = 0 \quad , \quad R = 0 \quad , R_{\mu\nu} = \sigma \partial_\mu \partial_\nu \sigma - \frac{1}{2}\, \delta_{\mu\nu} \sigma^{-3} \partial_\lambda \sigma \partial_\lambda \sigma \ . \tag{2f.291}$$

The solution to this coupled gravitational-scalar field theory is given by the metric

$$d^2 s = \sigma^2 dx\, d\bar{x} \quad , \ \Box \sigma = 0 \ . \tag{2f.292}$$

As there is a one-to-one correspondence between this system and the SU(2) Yang-Mills theory with the 't Hooft ansatz, the gravitational Pontryagin number associated with the metric (2f.292) is the same as the instanton number of the SU(2) Yang-Mills theory in the 't Hooft sector. Thus we can take $\sigma = \frac{1}{2} D\, F$ where F(x) is a Fueter function. Hence the choice Eq. (2f.170) leads to a Pontryagin number n for the scalar field coupled to conformal gravity. Similarly, a configuration with unit instanton number per time period is obtained by selecting Eq. (2f.283). A ready generalization is

$$F(x) = \frac{1}{2} \sum_{i=1}^{\infty} \cot \frac{\beta(\, x - x_i)}{2} \ . \tag{2f.293}$$

The corresponding generalized monopole solution solves both the Yang-Mills and equations (2f.191). Finally, the choice of F(x) = Z(x) (2d.72) provides a solution with unit topological number per period spacetime cell. In the limit of $q \rightarrow 0$, the Weyl-Yang action becomes proportional to the enclosed spacetime volume. As to the possible physical relevance of the above fourfold periodic solutions, the most enticing application is in the semi-classical approach to quark confinement [51]. The renewed interest in such an approach was triggered by the remarkable works of Seiberg and Witten [192]. Exploiting the powerful analyticity of a generalized electric-magnetic duality, they proved that, as conjectured long ago by 't Hooft and Mandelstam, confinement and chiral symmetry breaking result in certain D = 4 SUSY QCD models from a vacuum condensate of monopoles. More generally, over the years, there have been compelling arguments [193, 194, 195] and, more recently, realistic non-perturbative computational schemes in QCD [196] linking confinement to the random vacuum gauge fields of, say, a Copenhagen spaghetti vacuum. Thus a confining variational wave functional for the QCD vacuum has been modelled by a gas of dyons and anti-dyons [197]. In that same spirit, the solution (2f.290) has been proposed as a concrete realization of a monopole condensate in pure Yang-Mills theory [115].

We next close our discussion of quaternions with some historical remarks; they are confessedly a mere sketch with no pretense to completeness.

2.g. Historical Notes

About quaternions J.C. Maxwell, the unifier of the electric and magnetic interactions, expectantly wrote in 1869 [198]:

> The invention of the calculus of quaternions is a step towards the knowledge of quantities related to space which can only be compared for its importance, with the invention of triple coordinates by Descartes. The ideas of this calculus, as distinguished from its operations and symbols, are fitted to be the greatest use in all parts of science.

Yet such early high hope did not soon materialized. Instead the past 100 years have witnessed the practical triumphs of vector analysis championed by Gibbs [35] and of matrix theory invented by Cayley. In fact most physicists, even today, are still reluctant to use quaternions; possibly because the latter still "exude an air of 19th century decay, as a rather unsuccessful species in the struggle-for-life of mathematical ideas." [34] To better appreciate the growing relevance of quaternions to physical theories, we should discern an emerging pattern in the many ups and downs in their tumultuous history. In the following brief and incomplete account, this history is divided roughly into six distinct main episodes.

2.g.1. Birth and high expectations (1843-1873)

The life and works of Sir William Rowan Hamilton (1805-1865) have been minutely portrayed in the three volumes of John T. Graves [199]. A centenary celebration of the birth of quaternions spans the pages of the Proceedings of the Royal Irish Academy, vol. A**50** in 1943 and of *Scripta Mathematica*, vol.**10**, 1944. There exist also excellent new biographies [200, 201] along with many articles [28, 202].

In 1833, independently of Gauss, Hamilton first showed complex numbers to realize an algebra of couples. Given the real and imaginary units such that $e_0^2 = 1$ and $i^2 = -1$, the elements of the algebra are then the complex number $z = (x , y) = xe_0 + iy$, x and y being real. During the next ten years, Hamilton tried unsuccessfully to extend this notion as he sought to define "multiplication of triplets", with one real unit, 1 and two imaginary units, $\mathbf{i}$ and $\mathbf{j}$. Then, on the fateful Monday of October the 16th, 1843, during a walk with his wife in Dublin, he realized, in a flash of inspiration, that, not two, but three imaginary units are needed with the fundamental defining properties $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1$. He recorded this discovery at once in his pocket-book. Nor could he resist the unphilosophical impulse to carve these formulae on a stone of the nearby Brougham bridge, so did a century later the Irish patriot and future President of Ireland (1959-1973), Eamon de Valera, on his prison's wall. Hamilton called *quaternion* the hypercomplex number $q = a + \mathbf{i}\, b + \mathbf{j}\, c + \mathbf{k}\, d$. As he wrote to Graves the very next day "... and there dawned on me the notion that we must admit, in some sense, a *fourth dimension* of space for the purpose of calculating with triplets." [203] In this work he may have introduced the word "associative" for the very first time in mathematics: "... the commutative character is lost... However, it will be found that another important property of the old multiplication is preserved, or extended to the new, namely, that which may be called the *associative* character of the operation..."

As to the origin of the word quaternion [34], Tait wrote that "Sir W.R. Hamilton was probably influenced by the recollection of its Greek equivalent the Pythagorean Tetractys (i.e. τετραχτυ'ς, a *Set of Four)...*, the mystic source of all things...." In the New Testament, one finds that the word quaternion refers to the four groups of four soldiers used by King Herod to guard Peter: " ...he put him in prison, and delivered him to four quaternions of soldiers to keep him."

Actually, unbeknown to anyone, as early as 1748, Euler already had communicated in a letter to Goldbach [204] the product rule for quaternions, in the form of the "four squares theorem." Indeed, in showing that every positive integer is the sum of, at most, four integer squares, Euler used the following formula

$$\left(a^2+b^2+c^2+d^2\right)\left(q_0^2+q_1^2+q_2^2+q_3^2\right)=\left(Q_0^2+Q_1^2+Q_2^2+Q_3^2\right) \qquad (2g.1)$$

where

$$\begin{aligned} Q_0 &= aq_0 - bq_1 - cq_2 - dq_3 \ , \\ Q_1 &= bq_0 + aq_1 + dq_2 - cq_3 \ , \\ Q_2 &= cq_0 - dq_1 + aq_2 + bq_3 \ , \\ Q_3 &= dq_0 + cq_1 - bq_2 + aq_3 \ . \end{aligned} \qquad (2g.2)$$

Written as a matrix formula, (2g.1) reads as $\mathbf{Q} = \mathbf{q}\mathrm{A}$ with the vectors $\mathbf{Q} = (Q_0, Q_1, Q_2, Q_3)$ and $\mathbf{q} = (q_0, q_1, q_2, q_3)$ and the matrix $\mathbf{A}$

$$A = \begin{pmatrix} a & -b & -c & -d \\ b & a & d & -c \\ c & -d & a & b \\ d & c & -b & a \end{pmatrix} \quad . \tag{2g.3}$$

Of course, with $\mathbf{Q}$, $\mathbf{q}$ and A seen Hamilton's quaternions, (2g.1) is simply the factorization of the norms. In Eq. (2g.2), we also find proof for the (4 x 4) matrix representation (2a.39) of a quaternion. In a short note on the "Mutations of space" written in 1819 and published posthumously, Gauss had also written down the quaternionic multiplication formulae [205].

During the period 1843-1873 Hamilton wrote his *Lectures on Quaternions* (1853) [27]. His other treatise, *Elements of Quaternions* [203] were published posthumously in 1866. The champion of the new cult of quaternions was Hamilton's only real pupil, P.G. Tait, whose book *Treatise on Quaternions* [37] appeared in 1867. In 1873, in his *Treatise on Electricity and Magnetism* [206], Maxwell crowned his fundamental equations of electromagnetism in quaternionic form.

A Kantian and a follower of the then popular *Naturephilosophie* of Central Europe, Hamilton sensed in mathematical structures a deep affinity to the physical world. In his quaternions he saw the key to the physical universe. He believed that "the design of physical science is ... to learn the language and interpret the oracles of the universe." So after 1843, to the great sorrow of scientists and mathematicians, he devoted himself almost exclusively to advancing quaternion algebra and analysis. Since the divergence and curl operators derive naturally from the square root of the Laplace operator, applications to hydrodynamics and later to Maxwell's equations were immediate. The recognition of the natural relation between quaternions and rotations led to applications to the kinematics of the rigid body and to spherical trigonometry. Yet in these applications centered on rotational covariance, the scalar and vectorial parts of the quaternion have to be split. Such a partitioning therefore robs the algebra of its larger O(4) symmetry. The proper physical setting for quaternions was yet to be found.

In fact, Hamilton had higher ambitions for his favorite brainchild, a creation he put on par with that of Newton's infinitesimal calculus. As early as 1835, he had prophetically discoursed on the role of "algebra as a science of pure time," contrasting it with geometry seen as a science of pure space. So, upon realizing the relation of quaternions with three dimensional space, and having interpreted the quaternion as the ratio of two vectors, he pondered on the physical interpretation of the scalar part. In a letter to Humphrey Lloyd, he asserted: " **i**, **j**, **k** terms of the quaternion probably represent the three dimensions of space while the real term represents time". On a previous occasion he also wrote to Graves: "There seems to me to be something analogous to polarized intensity in the pure imaginary part and to unpolarized energy... in the real part of a quaternion." These insightful musings foreshadow the space-time and momentum-energy quaternions of special relativity, provided, as we now know, time and energy are taken as imaginary entities.

By the end of the 19th century Lord Kelvin (1892) concluded [207] : "quaternions came from Hamilton after his really good work had been done; and though beautifully ingenious, have been an unmixed evil to those who have touched them in any way, including Clerk Maxwell... have been of the slightest use to any creature." In fact Maxwell himself had no further use for quaternions after only casting his equations aesthetically in quaternionic form. In the preface to the 3rd edition of his *Elements of Quaternions*, even Tait, the acknowledged champion of quaternions, conceded that little progress had been achieved in their developments. Befitting applications must await the advent of special relativity, quantum mechanics and gauge field theories.

2.g.2. Their demise from physics and a haven in mathematics (1873-1900)

While teaching electromagnetism at Yale in 1870, Gibbs had abstracted vector calculus from the quaternionic form of Maxwell's equations. In his famous lecture notes [35], vector and scalar products were introduced as independent entities with no reference to quaternions. Indeed guided by Grassmann [208], who in 1844 discovered vector multiplication as a special case of his "exterior product," Gibbs realized that, unlike quaternions, vectors can be generalized to arbitrary integer dimensions. In 1882, vector calculus was independently introduced by Heaviside whose book *Electromagnetic Theory* [209] as well as Foppl's *Geometry of Vortex Field* [210] established vector calculus as the langua franca of classical field theories.

Much has been documented (such as in the angry exchange of letters to *Nature*) on the controversy over the relative merits of quaternions vs vectors. Thus Tait accused Gibbs as "... one of the retarders of quaternion progress, by virtue of his pamphlet on *Vector Analysis*, a sort of hermaphrodite monster, compounded of the notations of Hamilton and of Grassmann." His protagonist Heaviside pronounced that quaternions are " a positive evil of no inconsiderable magnitude." With the passing of Tait, the early life of quaternions came to an end, in spite of the valiant efforts of his student A. M. Macfarlane in Texas and of Kimura in Japan, through the International Association for Promoting the Study of Quaternions and Allied Systems of Mathematics (1900-1923). The latter was founded in New Haven no less in 1895, at Yale, by a colleague and opponent of Gibbs. Taking the place of quaternions are linear algebras such as vector, tensor and matrix calculus, much of which is rooted in the earlier works of Grassmann and Cayley.

However, during their period of eclipse in physics, quaternions, which spurted unprecedented developments in algebra, found a haven in mathematics. They were infused with new life mainly through the works of Frobenius and Clifford. Subsequently, some highlights came with the theorem on division algebras, the theory of integer quaternions both due to Hurwitz, and with Klein's [211] quaternionic theory of the top, a 19th century forerunner of Donaldson's [212] quaternionic formulation of Nahm's equations for the BPS monopoles. Indeed in 1878, in a celebrated theorem, Ferdinand Georg Frobenius (and independently, the American mathematician Charles Sanders Pierce [28] showed that the only associative division algebras having the real numbers as their center are the real numbers, the complex numbers and the quaternions. Twenty years later, by dropping the associativity requirement, Hurwitz [213] extended this

Squares Theorem and showed that the only additional solution is the algebra of the octonions. These theorems give the quaternions and octonions, along with the real and complex numbers, a new and unique status with many deep echoes in the great halls of modern mathematics. Since vector spaces such as Hilbert spaces and Lie groups are constructed over associative division algebras, we expect and do find them to come in three varieties: real, complex and quaternionic spaces. In some special cases there are also solutions over the non-associative division algebras i.e. over octonions. Section 3b.2 will deal with these exceptional vector spaces and Lie groups. A historical sketch of octonions will be given in Sect. 3g.

During this same period mathematics was spotlighted by Felix Klein's *Erlanger Programm* (1872) linking geometries with transformation groups and their invariants [214]. Accordingly, the classification of real, complex, quaternionic and exceptional algebras leads to the corresponding classification of groups and geometries. This happy period saw the advent of continuous groups in the hands of Lie, Poincaré, Klein, Frobenius, Schur, Killing and Cartan. The tools were being forged for the understanding of symmetries which constitute a dominant theme of 20th century physics.

A related key development came with the introduction of Clifford algebras [71]. The latter mathematical structure, a synthesis of the works of Hamilton and Grassmann, makes quaternions relevant in dimensions higher than four and gives an explicit construction of Grassmann's anticommuting numbers with vanishing squares. Both discoveries will be instrumental in the physics of the 20th century. Clifford (1845-1879) even prophetically wrote on the role of mathematics in the physics of the future. Along with Riemann, he envisioned matter as a manifestation of the curvature of space-time and foresaw novel applications of hypercomplex numbers to the natural world.

2.g.3. Complex quaternions in relativity (1911-1926)

During the first quarter of this century the quaternion algebra underwent further developments within the then emerging abstract algebra, in connection with discrete groups, rings, ideals and modules. Quaternions appeared to make a further retreat from reality when a number theory of quaternions was created by Lipschitz, Hurwitz, Wedderburn, Study and Dickson.

As reviewed in Sect. 2a.4, the advent of the special theory of relativity led to the natural application of biquaternions introduced previously by Hamilton and Clifford. This formalism was later reviewed and expanded by F. Klein [211] and Lanczos [215]. Later, quaternions were used in the same context by Oppenheimer, Laporte and Wigner. It is fair to say that they had re-entered physics, albeit through the backdoor. The quaternionic treatment of special relativity [52] was at most incidental since the majority of physicists opted instead to master tensor calculus after the arrival of general relativity.

2.g.4. A deeper role in quantum mechanics, function theory (1927-1950)

After the discovery of quantum mechanics by Heisenberg, Dirac, Schrödinger, Born and Jordan, some of the founding fathers of the new mechanics were led to rediscover quaternions, Clifford and Grassmann algebras and quaternionic Hilbert spaces. Thus, while Hamilton's dream for a key role of his quaternions in physics was partly realized, most practicing physicists were blissfully unaware of these developments. Here are some of the quaternionic structures making their way into the physics of that golden era.

First came the connection with spin. To describe quantum mechanically the double valuedness of the electron spin, Pauli introduced a two-complex component wave function (a spinor). He defined three spin operators σ_m along the three spatial directions. As operators on spinors and physical observables, the σ_m 's must be Hermitian 2 x 2 matrices. The commutations and anticommutation relations $[\sigma_m , \sigma_n] = 2i\, \varepsilon_{mnk} \sigma_k$ and $\{\sigma_m , \sigma_n\} = 2\, \delta_{mn}$ are characteristic of the O(3) rotation group. In a footnote to his paper Pauli remarked that $\boldsymbol{\sigma} = i\,(\mathbf{i}, \mathbf{j}, \mathbf{k})$ i.e. they are just the quaternionic units, and he thanked Jordan for pointing this out to him. Except for this passing remark, quaternions showed up rarely in the subsequent physics literature.

In 1932, upon the discovery of the neutron, Heisenberg put forth the concept of the nucleon as a spinor in an abstract charge space, the two isospin states being the proton and the neutron. The isospinor is acted upon by Hermitian quaternion units. Later Kemmer developed the isospace formalism. Rotational invariance in charge space became subsequently known as charge independence.

Spurred by the phenomenon of spin, the calculus of spinors was formalized by Van der Waerden in the 1930s. In essence it is a quaternion calculus in disguise. A quaternion Ψ:

$$\Psi = \begin{pmatrix} \psi_1 & -\psi_2{}^* \\ \psi_2 & \psi_1{}^* \end{pmatrix} \quad ,\overline{\Psi} = \begin{pmatrix} \psi_1{}^* & \psi_2{}^* \\ -\psi_2 & \psi_1 \end{pmatrix} \tag{2g.4}$$

(in (2x2) matrix form) corresponds to the spinor $\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$. The difference with vector calculus is as follows: While the latter splits a quaternion into its scalar and vector parts, spinor calculus split it into its first and second columns. The general theory of spinors was extensively developed geometrically by E. Cartan in his *Leçons sur la Theorie des Spineurs.* [216]

In 1928 Dirac [217] wrote down his relativistic electron equation :

$$\gamma^{\mu}(\partial_{\mu} + i\, e\, A_{\mu})\, \psi = m\, \psi \qquad (\,\mu = 1,\ldots,4\,)\ . \tag{2g.5}$$

The γ's are 4 x 4 Hermitian matrices, obeying $\{\gamma_m, \gamma_n\} = 2\,\delta_{nm}$, they are just Clifford numbers with r = 2; requiring two sets of quaternion units. These Hermitian quaternion units were called $\boldsymbol{\sigma}$ and $\boldsymbol{\rho}$ by Dirac, who unknowingly rediscovered a special case of Clifford's construction.

Subject to the exclusion principle, fermions obey Fermi-Dirac statistics. So if $a_s^{\dagger}$ and a_s denote fermionic creation and annihilation operators, they must satisfy the anticommutation relations, already known to Clifford. The latter had shown how to construct them out of r sets of quaternions if there are r Grassmann numbers. The same construction was later given by Jordan and Wigner, and also by Cartan while working on spinors in n dimensions.

Since overall phases of wave functions are not observables, the quantum mechanical space of ray representations corresponds to the complex projective space CP(n). Thus a ket vector $| \boldsymbol{\alpha} >$ labels a point (physical state) by its homogeneous coordinates and $\lambda | \boldsymbol{\alpha} >$ with $|\lambda| = 1$ represents the same point (physical state). This correspondence came to light in the works of Weyl, Birkhoff and von Neumann. Similarly quaternionic projective geometry provides a generalization of quantum mechanics, quaternionic quantum mechanics. Here the Pappus theorem [218], valid in the real and complex projective cases, fails. Consequently, the superposition principle no longer holds for some configurations of the physical states. However, there was no application of quaternionic quantum mechanics during that period. As noted in Sect.2.f.1, quaternionic quantum mechanics was revived in the fifties, again without significant success, and surfaced more sporadically ever since.

As far as the development of a function theory over a quaternion, Hamilton himself had studied two kinds of functions. They are the fractional linear (Möbius) or SL(2, **Q**) conformal transformation

$$y = (ax + b)(cx + d)^{-1} \tag{2g.6}$$

and the Taylor series

$$y = \sum_{n=0}^{\infty} \alpha_n x^n \qquad , \mathrm{Vec}(a_n) = 0 \quad . \tag{2g.7}$$

In a series of papers spanning a whole decade in the 1930s, Fueter [94] (and also independently Moisil [95]) studied left and right quaternionic series L(x) and R(x). He showed that the functions $\lambda = \Box L$ and $\rho = \Box R$, called left and right regular functions of the quaternion x, solve the equation

$$\partial \lambda = 0 \quad , \quad \partial \rho = 0 \ . \tag{2g.8}$$

They generalize the Cauchy-Riemann equations in the Gauss plane. He further proved that many theorems of complex function theory, such as the Cauchy integral theorem, Morera and Liouville theorems, have their counterparts for left and right regular functions. While the latter do not form a ring, the special, simultaneously left and right analytic functions do form a ring as they are complex analytic in $z = x_4 + i\,|\mathbf{x}|$. Equation (2g.8) is just the massless Dirac (or Weyl) equation, which historically first motivated Fueter's and others' quest [219] for a quaternionic counterpart to Cauchy-Riemann analyticity. Yet, until the end of this episode, very few physicists and mathematicians took notice of these developments.

We recalled earlier how the linear representations of the groups SU(2) and O(4) can be realized by way of quaternions. We also point out that the conformal group of Minkowski space-time is O(4, 2) ≈ SU(2, 2). In a beautiful paper [168], Dirac showed that the conformal group of Euclidean space O(5,1) is locally isomorphic to SL(2, **H**). He obtained a kinematic representation of the fractional linear transformation (2g.6). When restricted to its O(3, 1) subgroup, a nonlinear representation of the Lorentz group obtains in terms of real quaternions. Such was Dirac's original motivation in this paper. It will only find application much later, in quantum gravity viewed as a spontaneously broken gauge theory.

2.g.5. New hopes and disappointments (1950-1975)

This period marks the coming of age of high energy physics, a time when symmetry principles and Lie groups assumed a dominant guiding role in classifying the observed particle states, predicting new ones and understanding their dynamics. The isospin group SU(2) has a twofold nesting in chiral SU(2) x SU(2) and in flavor SU(3), the group of the Eighfoldway. In particular, it was observed that if isospin symmetry is fundamental, it can be nicely accounted for as the phase group of quaternionic quantum mechanics. In his *Collected papers* [220] C.N. Yang recalls:

> In 1954-55 Lee and I spent considerable time trying to develop a field theory based on quaternions rather than complex numbers... The phase in complex algebra is related to electromagnetism. The phase in quaternion algebra would be related to isotopic gauge fields. If quaternions are the fundamental basis of field theory, the very existence of isotopic spin symmetry would be understood.

Yet, unlike the U(1) phase group which is phenomenologically exact, the isospin group, be it the strong or weak interaction group, is now understood as a broken symmetry at the level of the quarks. The phase group of quaternionic quantum mechanics on the other hand should be exact. Furthermore, as Finkelstein, Jauch, Schiminovich and Speiser [158], and Yang [220] soon realized, there is a fundamental stumbling block in quaternionic quantum mechanics: its SU(2) symmetry cannot be preserved for the many-particle states. Up to a factor, the representation of the Poincaré group (including translations) reduces this SU(2) symmetry down to U(1). As Yang remembered:

> We did not succeed in developing a quaternionic theory that is more than a rewriting of the usual theory in quaternion form. Nor did I succeed in many trials later. But I continue to believe the basic direction is right.... What is the difficulty ? I do not know. Some key ideas are missing obviously. Is it because we do not have an understanding of the theory of a function of a quaternion variable ? ... Is it because spacetime itself should be described by a quaternion variable ? Maybe. Or is it something much simpler ?

In recent years however, a systematic quaternionic extension of standard quantum mechanics and quantum fields has been vigorously undertaken by S. L. Adler. The latter has particularly sought the relevance of quaternionic Hilbert spaces in the modern unification program. Many new and important results along with outstanding issues are discussed in his comprehensive treatise [160]. There have also been a few other quaternionic formulations or variants of the standard electroweak model [221, 222, 223, 224].

During this period, attempts to apply quaternions primarily beget a new but nevertheless useful formalism. For example, the quaternionic approach suggested two fertile lines of developments: the chiral SU(2)xSU(2) symmetry of the strong interactions [225] and the representation of the conformal group in Minkowski space-time in terms of Hermitian quaternions. Such a representation shows manifestly the conformal invariance of the Maxwell (later, of the Yang-Mills) equations. It makes contact with complex analytic geometry [77] and the emerging edifice of Penrose's twistor theory [6]. Moreover, it carries the seed of deeper connections yet to come.

2.g.6. Comeback in Euclidean QFT (1978-Present)

The analytic properties of quantum field theories [164] allow, through a Wick rotation, the continuation of the time coordinate into purely imaginary values. Therein lies the physical basis of relativistic Euclidean field theories where spacetime points are parametrized by real quaternions. Here the field theory is invariant under the inhomogeneous O(4) group, the Euclidean Poincaré group. In the massless limit, it is invariant under $O(5, 1) \approx SL(2, \mathbf{H})$, the Euclidean conformal group. So, from this perspective, it takes the notion of quantization (and massless fields) in a 4-dimensional space to finally bring out the true relevance of real quaternions and their associated groups to field theories. Hamilton's intuition about the connection between the scalar part of a quaternion and time is realized in a way he could hardly have imagined in his day.

One therefore expects and finds the natural occurrence of real quaternions in the solutions of unified gauge theories and in quantum gravity. Familiar examples are the Yang-Mills instantons of Belavin et al., the gravitational instantons of Hawking et al. [226] and Donaldson's formulation of Nahm equations for BPS monopoles [227]. Specifically, the ADHM construction (1978) of the general Yang-Mills multi-instantons maps the physical problem into the solution of a set of linear algebraic quaternion equations. In 1980, self-duality was shown [51] in the context of the SU(2) 't Hooft-Rebbi-Jackiw-Nohl [17, 176] solutions to be simply given by the Fueter-Cauchy-Riemann equations of quaternionic analysis. There, the instantons correspond to mappings of the type (2g.4) for **H**-holomorphic functions, their Chern-Pontryagin index reduces to the winding number of quaternionic analysis.

As for the general SU(2) n-instanton, its gauge connection reads

$$A_\mu = \frac{1}{2}\, \mathbf{e} \cdot \mathbf{A}_\mu = v^\dagger \partial_\mu v = \frac{\left[u^\dagger \partial_\mu u - \partial u_\mu^\dagger u\right]}{1 + u^\dagger u} \tag{2g.9}$$

where v(x) is an (n+1)-dimensional quaternionic vector normalized to one. The phase of v is a function of x and an arbitrary SU(2) group element. Changing the phase of u induces a gauge transformation on A_μ, u(x) is a point in quaternionic projective space HP(n) with (n+1) homogeneous coordinates. Finiteness of the action and self-duality constrain the map u(x): $S^4 \rightarrow$ HP(n) to be meromorphic in x and to satisfy certain algebraic quaternionic conditions. The number n, the dimension of HP(n), is the instanton number, a topological invariant. Here one has a geometrical interpretation of instantons as quaternionic rational curves in von Neumann's quaternionic Hilbert space. While the final verdict is still out as to the full extent of their practical utility, instanton solutions have certainly played an important role in our semi-classical understanding of the gauge, gravitational, string and membrane theory vacua [228, 229]. Furthermore, in this age of fruitful cross-fertilization between modern mathematics and physics, the instanton problem has spurred, in addition to fundamental discoveries by Donaldson [230] on the topology of four manifolds, renewed interests and advances in hypercomplex structures, specifically in quaternionic Kähler manifolds, hyperkähler and twistor spaces [84, 86, 87, 88]. A purported goal of this research into quaternionic structures is the study of generalized conformal structures, the integration of a large class of non-linear differential equations and the integrability of higher dimensional field theories [231]. As illustrated in Sect.2f, remarkably it appears that every quaternionic theory developed so far has found an application and physical interpretation in quantum gauge field theories.

Quaternions, indeed division algebras in general, as the recent history of the octonions (Sect. 3g) testifies, seem to turn up only at the deepest level of physical laws ranging from self-dual gauge fields, unified (supersymmetric) gauge theories, to superstrings and supermembranes. This renaissance takes place at a time of breathtaking developments in mathematical physics. There has also been a great impetus to physically understand the very special nature of four dimensional spacetime [232]. The importance of generalized electric-magnetic duality and holomorphy in exactly solving for D = 4 supersymmetric gauge field theories [128, 129, 233] , the discovery of nontrivial D = 4 conformal [192] and integrable field theories [234] calls for further developments of quaternionic analysis [235]. One may thus witness the end to the periods of neglect, of estrangement and misunderstanding between this branch of mathematics and physics. In this process quaternions may have been demystified. Only the future will show how deeply relevant they are to the portrayal of fundamental physics in four dimensions, whether they have finally come of age [236].

3. Octonions

This chapter deals with octonions and related structures. It parallels our treatment of quaternions. After a summary of the essential algebraic properties of octonions, we discuss their associated Hilbert spaces and quantum mechanics. Then, in the physical context of unified gauge theories, we detail some octonionic aspects of exceptional algebras, groups and coset spaces, of the Jordan and related algebras. After a brief presentation of exceptional manifolds, we proceed to the elements of a Fueter function theory of an octonionic variable. A brief discussion of the number theory of octonions will follow. We next review a few selected physical applications then close with some historical notes.

3.a. Algebraic Structures

3.a.1. Basic properties, Moufang and other identities

The real octonion algebra $\mathbf{\Omega}$ is an 8-dimensional division algebra over $\mathbf{R}$ [237]. A canonical basis for $\mathbf{\Omega}$ is given by the octonionic units e_μ, $\mu = 0, 1, ..., 7$, obeying the multiplication rules:

$$e_0 = 1, \quad e_n e_0 = e_0 e_n = e_n \,, \qquad n = 0, 1, ..., 7,$$

$$e_\alpha e_\beta = -\delta_{\alpha\beta} + \psi_{\alpha\beta\gamma} e_\gamma \,, \qquad \alpha, \beta, \gamma = 1, 2, ..., 7 \,. \tag{3a.1}$$

e_0 is the multiplicative unit element and the e_α 's are the imaginary octonion units. The completely antisymmetric structure constants $\psi_{\alpha\beta\gamma}$ can be chosen nonzero and equal to one for the following seven combinations or triads

$$\alpha\beta\gamma = (123), (246), (435), (367), (651), (572) \text{ and } (714) \,. \tag{3a.2}$$

The values of $\psi_{\alpha\beta\gamma}$ are readily read off by cyclic permutations from a mnemonic triangle (123) (Fig. 1). The latter is inscribed in a circle with seven points labelled in the order (1243657). A triangle $\alpha\beta\gamma$ is obtained from (123) by six successive rotations of angle $\frac{2\pi}{7}$. We also note that, along with the identity $e_0 = 1$, the elements corresponding to the corners of a triangle form a basis of a SU(2) quaternion subalgebra.

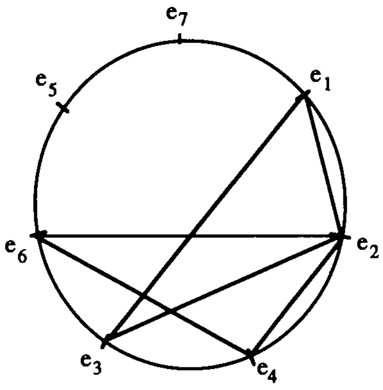

Fig. 1

An alternative graphical representation of the multiplication table (3a.1) is the following triangular diagram [238] :

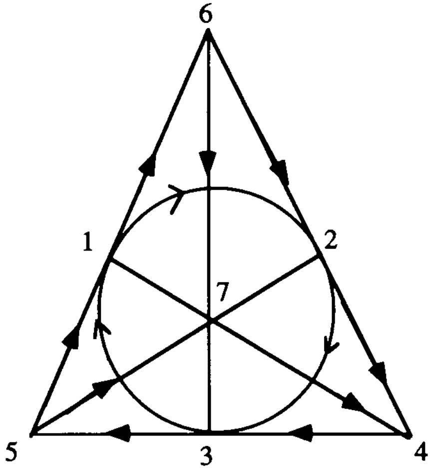

Fig. 2

The arrows indicate the directions along which the multiplication has a positive sign. For example, $e_5\, e_1 = e_6$, $e_7\, e_5 = -\, e_2$, $e_6 e_5 = e_1$ etc. In fact, there are nontrivial variations among the 480 ways of exhibiting the multiplication table! Subsequently, yet a third variant will also be used.

The algebra $\mathbf{\Omega}$ is non-associative. Indeed, from Eq. (3a.1), it follows that $(e_i e_j) e_k = e_i(e_j e_k) = -\psi_{ijk}$ if (ijk) belongs to the above triads (3a.2), e.g. $e_4 (e_3 e_5) = (e_4 e_3) e_5$, or if any two indices are the same, but for all other triads, $(e_i e_j) e_k = - e_i (e_j e_k)$, for example $(e_5 e_1) e_2 = - e_5 (e_1 e_2)$.

Any real octonion $\omega \in \mathbf{\Omega}$ can be written in the covariant form of $\omega = \omega_\mu e^\mu$. The anti-automorphism is the conjugation $\omega \to \bar{\omega} = \bar{e}_\mu \omega^\mu$, the conjugate of ω, with the conjugate basis units $\bar{e}_\mu = (e_0, -\mathbf{e})$. It is an anti-involution, i.e. $\bar{\bar{\omega}} = \omega$ and $\overline{\omega \eta} = \bar{\eta}\,\bar{\omega}$.

Topologically, as in the $4 \to 1 + 3$ splitting for quaternions, we have the $8 \to 1 + 7$ splitting $\mathbf{\Omega} = \mathbf{R} + \mathbf{\Omega}'$. $\mathbf{R}$, the identity element, is a 1-dimensional subspace and $\mathbf{\Omega}'$ is a 7-dimensional subspace whose elements are pure imaginary octonions. As in the $\mathbf{H}$-algebra, there is a basis independent splitting of any octonion into its scalar and vector part of ω: $\mathrm{Sc}(\omega) = 1/2\,(\omega + \bar{\omega}) = \omega_0$ and $\mathrm{Vec}(\omega) = 1/2\,(\omega - \bar{\omega}) = \mathbf{e}\cdot\boldsymbol{\omega}$.

The scalar or inner product of two octonions ω and ω' is given as

$$\langle \omega , \omega' \rangle \equiv \omega_n \omega_n = \frac{1}{2}\,(\omega\overline{\omega'} + \omega'\bar{\omega}) = \frac{1}{2}\,(\bar{\omega}\omega' + \overline{\omega'}\omega) \ . \tag{3a.3}$$

Applied to the octonionic units, it reads

$$\langle e_n , e_m \rangle = \delta_{nm} \ . \tag{3a.4}$$

A particular case of Eq. (3a.3) is the positive definite norm of ω

$$N(\omega) = \omega\,\bar{\omega} = \bar{\omega}\,\omega = \omega_\mu \omega_\mu = \omega_0^2 + \boldsymbol{\omega}^2 \ . \tag{3a.5}$$

The multiplicative inverse ω^{-1} of a nonzero $\omega \in \mathbf{\Omega}$ is

$$\omega^{-1} = \frac{\bar{\omega}}{N(\omega)} \tag{3a.6}$$

then $(\omega\,\omega')^{-1} = \omega'^{-1}\omega^{-1}$.

Next, in analogy to the quaternionic case in R^4, we introduce two antisymmetric tensors [239]:

$$e_{nm} = \frac{1}{2}\,(\bar{e}_n e_m - \bar{e}_m e_n) \ , \qquad e'_{nm} = \frac{1}{2}\,(e_n \bar{e}_m - e_m \bar{e}_n) \ . \tag{3a.7}$$

In components, we get $e_{0\alpha} = -e'_{0\alpha} = e_\alpha$, $e_{\alpha\beta} = e'_{\alpha\beta} = -\psi_{\alpha\beta\gamma} e_\gamma$. These octonionic tensors will enter in our covariant formulation of various cross-products in R^8.

Frobenius' Theorem [28] came about by dropping the commutativity of multiplication. The result is a new, quaternion skew number field **H**. So may one discover new structures by dropping or merely weakening other basic algebraic properties? A complete answer was given by weakening the associativity of multiplication by an *alternativity* condition. The new algebra is then that of the octonions [240, 241].

If x, y, a $\in$ **Ω**, they obey the left and right alternativity conditions:

$$a\,(a\,x) = a^2\,x \qquad \text{and} \qquad (x\,a)a = x\,a^2 \tag{3a.8}$$

respectively. The resulting algebraic structures are called *alternative fields.* **Ω** is also a *flexible* algebra as a (x a) = (a x) a, the latter relation is called the flexibility property.

A measure of the lack of associativity, the *associator* [a, b, c] of three elements a, b and c is defined as

$$[a, b, c] = (a\,b)\,c - a\,(b\,c)\ . \tag{3a.9}$$

From the multiplication table, **Ω** is readily checked to be an alternative algebra, i.e. the associator is an alternating function of its arguments:

$$[a, b, c] = [c, a, b] = [b, c, a] = -[b, a, c] = -[a, c, b]\ . \tag{3a.10}$$

This property derives from alternativity, trivially satisfied by the associative algebras **R**, **C**, and **H**. The associator is purely vectorial (or imaginary): $\overline{[a, b, c]} = -[a, b, c]$, since the real numbers in **Ω** commute and associate with all other elements, only the purely vectorial parts of the elements contribute to commutators and associators.

We begin by writing down the three fundamental, quadrilinear Moufang identities:

$$(x\,a)\,(b\,x) = x\,(a\,b)x\ , \tag{3a.11}$$

$$(y\,a\,y)\,z = y\,(a\,(y\,z)) \tag{3a.12}$$

$$z\,(y\,a\,y) = ((z\,y)\,a)\,y \tag{3a.13}$$

for a, x, y, z $\in \mathbf{\Omega}$.

A special case of the Moufang identities reads

$$(x\,a)\,x^2 = x\,(a\,x)\,x = x\,(a\,x^2)\ . \tag{3a.14}$$

It implies that expressions such as $a\,x\,\bar{a}$ and $a\,x\,a^2$ are unambiguous. For the proofs of the above relations, of essential importance in octonionic analysis, and much more, the reader should consult the book by Harvey [32].

Next, we compile a list of handy formulae some of which are given by Yokota [242], others may be found in Refs.[29, 32, 55, 243].

For any x, y, a, b $\in \mathbf{\Omega}$

$$\mathrm{Sc}\,(x\,y) = \mathrm{Sc}\,(y\,x)\quad ,\quad \mathrm{Sc}\,(\,x(y\,z)\,) = \mathrm{Sc}\,((x\,y)\,z)\quad , \tag{3a.15}$$

$$2\,(x,y) = 2\,(\bar{x},\bar{y}\,) = x\bar{y} + y\bar{x} = \bar{x}y + \bar{y}x\quad , \tag{3a.16}$$

$$2\,(\,a,b\,)\,x\ = \bar{a}\,(\,b\,x\,)\ +\ \bar{b}\,(\,a\,x\,) = a\,(\,\bar{b}\,x\,)\ +\ b\,(\,\bar{a}\,x\,)\ \ , \tag{3a.17a}$$

$$2\,(\,a,b\,)\,x\ = (\,x\,\bar{b}\,)\,a + (\,x\,\bar{a}\,)\,b\,, \tag{3a.17b}$$

$$a\,(a\,x\,) = (\,a\,a\,)\,x\,,\ \ a\,(\,x\,a) = (\,a\,x\,)\,a\,,\ \ x\,(\,a\,a\,) = (\,x\,a\,)\,a\ \ , \tag{3a.18}$$

$$a\,(\,\bar{a}\,x\,) = (\,a\,\bar{a}\,)\,x\,,\ \ a\,(\,x\,\bar{a}\,) = (\,a\,x\,)\,\bar{a}\,,\ x\,(\,a\,\bar{a}\,) = (\,x\,a\,)\,\bar{a}\ , \tag{3a.19}$$

$$a\,(\,a\,x\,) = (\,a\,a\,)\,x\,,\ a\,(\,x\,a\,) = (\,a\,x\,)\,a\,,\ x\,(\,a\,a\,) = (\,x\,a\,)\,a\ . \tag{3a.20}$$

Due to non-associativity, brackets must therefore be handled with special care to avoid ambiguities in the order of multiplication. However, since $\mathbf{\Omega}$ is <u>power associative</u>, there is no ambiguity in raising the same octonion to any power. Generally, for any m, n, r, s $\in Z_+$, the positive integers, we have

$$a^m\,(\,a^n\,b\,) = a\,(\,a^{m+n-1}\,b\,)\ \ , \tag{3a.21}$$

$$a^m\, b\, a^n = (\, a^m\, b\,)\, a^n = a^m\, (\, b\, a^n\,) = a^{m-r}(\, a^r\, b\, a^s\,)\, a^{n-s}\ . \tag{3a.22}$$

Since any associator [a, b, c] is completely antisymmetric in any three octonions a, b and c, then [a, b, b] = 0. From [e_i, e_j, e_j] = 0, we deduce the bilinear relations

$$\psi_{irs}\, \psi_{jrs} = 6\, \delta_{ij}\ . \tag{3a.23}$$

By way of the first Moufang identities (3a.11), we similarly find

$$\psi_{ris}\, \psi_{sjt}\, \psi_{tkr} = 3\, \psi_{ijk}\ . \tag{3a.24}$$

The basic associator of any three imaginary units:

$$\begin{aligned} [e_i,\, e_j,\, e_k] &= 2\, \varphi_{ijkr} e_r \\ &= 2\, \psi^m_{[\,i\,j}\, \psi^m_{k\,]\,r}\ e_r \end{aligned} \tag{3a.25}$$

defines a fully antisymmetric 4 index object ϕ_{ijkr}. Reflecting the natural 7 = (4+3) splitting of 7-dimensional space, the ϕ_{ijkr} is dual to the ψ_{ijk} since

$$\psi_{ijk} = -\,(1/4!)\ \varepsilon_{ijkmnpq}\ \phi_{mnpqm}\ , \tag{3a.26}$$

$$= -\,(1/4!)\ \varepsilon_{ijkmnpq}\ \psi_{mnr} \psi_{pqr}\ . \tag{3a.27}$$

The values of these numerical mutually dual tensors can be read off from the following variant of Cayley's multiplication table:

$$\psi_{ijk} = 1 \quad \rightarrow \begin{bmatrix} 1 & 2 & 4 & 3 & 6 & 5 & 7 \\ 2 & 4 & 3 & 6 & 5 & 7 & 1 \\ 3 & 6 & 5 & 7 & 1 & 2 & 4 \end{bmatrix} \tag{3a.28}$$

and

$$\phi_{ijkm} = -1 \ \rightarrow \begin{bmatrix} 4 & 3 & 6 & 5 & 7 & 1 & 2 \\ 5 & 7 & 1 & 2 & 4 & 3 & 6 \\ 6 & 5 & 7 & 1 & 2 & 4 & 3 \\ 7 & 1 & 2 & 4 & 3 & 6 & 5 \end{bmatrix} \tag{3a.29}$$

The rule for the above tables is: each line is a cyclic permutation of the first line, the columns of the first three lines give the structure constants or half the commutators of the octonionic imaginary units e_i, e.g. $e_1 e_2 = e_3$, $e_2 e_4 = e_6$, etc., while the columns of the remaining four lines give half the associator of the e_i. For example, we read off the following: $1/2\,[\,e_3, e_7, e_5\,] = -\,e_1$, $1/2\,[\,e_4, e_5, e_6\,] = -\,e_7$, etc... So they determine the values of the ϕ_{ijkr}, e.g. if $\psi_{123} = 1$, $\phi_{4567} = -1$ etc.

It is no surprise that the *associator* plays a central role in the derivation of various properties of octonions. As an illustration, we give a short proof of the composition property of the norm for real octonions. Since $[\,e_i, e_j, e_k\,] = 2\,\phi_{ijkr}\,e_r$ is purely vectorial, we deduce that $Sc\,(e_i(e_j\,e_k)) = Sc\,((e_i\,e_j)\,e_k)$ or $Sc\,[\,a\,(\,b\,c\,)\,] = Sc\,[\,(\,a\,b\,)\,c\,]$, therefore

$$N(\,a\,b\,) = Sc\,[\,(\,b\,a\,)\,(\,a\,b\,)\,] = Sc\,[\,(\,b\,(\,a\,(\,a\,b\,)\,)\,)\,] = Sc\,[\,(\,b\,a\,a\,)\,b\,]$$

$$= N(a)\,Sc\,(b\,b\,) = N(\,a\,)\,N(\,b\,)\ . \tag{3a.30}$$

As mentioned previously, there exists a unified construction of the division algebras **R**, **C**, **H** and **Ω**. It is the Cayley-Dickson process. We saw this construction twice before in our algebraic constructions of **C** and **H**. It consists of three successive applications of the following doubling procedure applied to the real numbers **R**:

Let F be a number field, consider the system Γ made up of a direct sum of two copies of F. For any $Q = (\,q_1\,, q_2\,) \in \Gamma$, addition and multiplication of the Q's are defined as:

$$Q + Q' = (\,q_1, q_2\,) + (\,q_1', q_2'\,) = (\,q_1 + q_1'\,, q_2 + q_2'\,)\ , \tag{3a.31}$$

$$Q\,Q' = (\,q_1 q_1' - q_2'\overline{q_2}\,, q_1' q_2 + \overline{q_1} q_2'\,)\ . \tag{3a.32}$$

So this doubling of real numbers gives the complex numbers which, upon doubling, gives quaternions. In turn, upon doubling, the latter give octonions. Making the above construction explicit, we decompose an element $\omega \in \mathbf{\Omega}$ as

$$\omega = \left(\sum_{i=0}^{3} \omega_i e_i\right) + e_7(\,\omega_7 e_0 + \omega_4 e_1 + \omega_5 e_2 + \omega_6 e_3)\ , \tag{3a.33}$$

with $\omega = p + e_7 q$ where p and q are quaternions. In this form, the multiplication of two octonions ω and χ reads

$$\omega\,\chi = (\,p + e_7 q)\,(\,r + e_7 s)$$

$$= (pr - s\bar{q}) + e_7(rq + \bar{p}s) \tag{3a.34}$$

with the overbar now denoting <u>quaternion</u> conjugation.

An explicit check of Eq. (3a.34), by way of the multiplication rules (3a.1), requires the following notational change. To single out a quaternionic subalgebra, let i, j, k = 1, 2, 3 so that the octonion units are $\{e_0, e_i, e_{i+3}, e_7\}$. Having set $e_0 = 1$, the product rules now read

$$e_i\, e_j = -\,\delta_{ij} + \varepsilon_{ijk}\, e_k = -\,\delta_{ij} - e_{ij} \ , \tag{3a.35}$$

$$e_i\, e_{j+3} = e_7(\,\delta_{ij} + e_{ij}\,) = -\,e_{j+3}\, e_i \ , \tag{3a.36}$$

$$e_{i+3}\, e_{j+3} = -\,\delta_{ij} - \varepsilon_{ijk}\, e_k = -\,\delta_{ij} + e_{ij} \ , \tag{3a.37}$$

$$e_7\, e_j = e_{j+3} = -\,e_j\, e_7 \ , \tag{3a.38}$$

$$[\ e_7\, , e_i\, , e_j\,] = -\,e_7[\,e_i\, , e_j\,] \ , \tag{3a.39}$$

as well as the relation $e_7\, s = \bar{s}\, e_7$.

In the above decomposition, octonion conjugation translates into

$$\bar{\omega} = \overline{(p,\, q\,)} = \overline{(p + e_7 q)} = (\bar{p} - \bar{q}e_7) = (\bar{p} - e_7 q) = (\bar{p}\ , -\,q)\, , \tag{3a.40}$$

$$(p,\, q)\,\overline{(r,\, s)} = (\,p,\, q)\,(\bar{r}, -\,s) = (p\bar{r} + s\bar{q},\, \bar{r}q - \bar{p}s) \tag{3a.41}$$

and

$$\overline{(\,p,\, q)}\,(\,r,\, s) = (\,\bar{p}, -\,q)\,(\,r,\, s) = (\,\bar{p}r + s\bar{q},\, ps - rq\,)\ . \tag{3a.42}$$

Then the quadratic norm of ω, $N(\omega) = \bar{\omega}\omega = \overline{(\,p,\, q)}\,(p,\, q) = \bar{p}p + \bar{q}q$.

Just as $\mathbf{\Omega}$ emerges from a complex extension of quaternions, it can also be seen as a quaternionic extension of complex numbers. To this end, we use the decomposition

$$\omega = \omega_\mu e_\mu = z_0 + z_i e_i \tag{3a.43}$$

where $z_0 = \omega_0 + e_7 w_7$, $z_k = \omega_k + e_7\, \omega_{k+3}$ (k = 1, 2, 3) and the summation convention is implied. Namely, z_0, z_k belong to the complex subalgebra **C** generated by the imaginary unit e_7. Then the octonion conjugate element is $\overline{\omega} = z_0^* - z_i e_i$ and we have the relation $z_i e_i = e_i z_i^*$.

The octonionic product of ω and $\chi \in \mathbf{\Omega}$ becomes

$$\omega\, \chi = (\, z_0 + z_i e_i)\, (\, w_0 + w_j e_j) \tag{3a.44}$$

$$= \left(z_0 w_0 - z_i w_j^*\right) + \left(w_0 z_k - z_0 w_k - \varepsilon_{ijk} z_i^* w_j^*\right) e_k \ . \tag{3a.45}$$

The * operation denotes complex conjugation ($e_7 \to - e_7$) within the complex subalgebra generated by e_7 and it is *not* an automorphism of $\mathbf{\Omega}$. The norm $N(\omega) = \overline{\omega}\omega = z_0^* z_0 - z_i^* z_i$.

Now the octonion algebra $\mathbf{\Omega}$, along with **R**, **C**, **H** is distinguished from all other number systems by Hurwitz's theorem [26, 213, 244]. That famous theorem states that if

1) **A** is a finite dimensional vector space over the field of real numbers,

2) a distributive multiplication is defined in **A** such that

a) $x(\alpha y + \beta z) = \alpha\,(xy) + \beta\,(xz)$
b) $(\alpha x + \beta y)\, z = \alpha\,(xz) + \beta(yz)$
c) a unit elements **I** exists such that x **1** = **1** x = x

where $\alpha, \beta \in$ **R** and $x, y, z \in$ **A**,

3) A has a positive definite symmetric scalar product (x,y) such that $|x| = \sqrt{(x, x)}$, the length of a vector, is defined,

then **A** can only be one of the four composition algebras: the real numbers **R** of dimension 1, the complex numbers **C** of dimension 2, the quaternion **H** of dimension 4 and the octonions $\mathbf{\Omega}$ of dimension 8. A composition algebra is said to be a *division* algebra if the norm is isotropic, i.e. if $N(x) = 0 \rightarrow x = 0$. Thus only **R**, **C**, **H** and $\mathbf{\Omega}$ over **R** are division algebras.

3.a.2. O(8) covariant tensors

From the O(7) numerical tensors $\psi_{\alpha\beta\gamma}$ (3a.1) and $\phi_{\alpha\beta\mu\nu}$, Eqs. (3a.25)-(3a.26), we can construct [239] two completely antisymmetric O(8) tensors f_{abcd} and f'_{abcd} (a, b, c, d = 0, 1, ..., 7) in R^8:

$$f_{\alpha\beta\gamma 0} = -f'_{\alpha\beta\gamma 0} = \psi_{\alpha\beta\gamma} \ ,$$

$$f_{\alpha\beta\mu\nu} = -f'_{\alpha\beta\mu\nu} = \phi_{\alpha\beta\mu\nu} \ . \tag{3a.46}$$

The duality relations between $\psi_{\alpha\beta\gamma}$ and $\phi_{\alpha\beta\mu\nu}$ in R^7 then translate into the self- and antiself-duality of f_{abcd} and f'_{abcd} in R^8:

$$f_{abcd} = \frac{1}{4!}\varepsilon_{abcdnmrs} f_{nmrs} \ , \tag{3a.47}$$

$$f'_{abcd} = -\frac{1}{4!}\varepsilon_{abcdnmrs} f'_{nmrs} \ , \tag{3a.48}$$

$\varepsilon_{abcdnmrs}$ being the totally antisymmetric tensor with $\varepsilon_{012...7} = 1$. Self-duality thus tells us that f_{abcd} is an irreducible O(8) tensor with 35 components. These tensors will play a central role in various vector products available in R^8.

Using the contraction between the ψ's and ϕ's, we obtain

$$\begin{aligned} f^{abcd}f_{nmrd} = {} & \delta^a_n\left(\delta^b_m\delta^c_r - \delta^c_m\delta^b_r\right) + \delta^c_n\left(\delta^a_m\delta^b_r - \delta^b_m\delta^a_r\right) \\ & + \delta^b_n\left(\delta^c_m\delta^a_r - \delta^a_m\delta^c_r\right) + f^{ab}{}_{nm}\delta^c_r \\ & + f^{ca}{}_{nm}\delta^b_r + f^{bc}{}_{nm}\delta^a_r + f^{ab}{}_{rn}\delta^c_m \\ & + f^{bc}{}_{rn}\delta^a_m + f^{ca}{}_{nmrn}\delta^b_m + f^{ab}{}_{mr}\delta^c_n \\ & + f^{bc}{}_{mr}\delta^a_n + f^{ca}{}_{mr}\delta^b_n \ , \end{aligned} \tag{3a.49}$$

or compactly,

$$f^{abcd}f_{nmrd} = \delta^{[a}_n\,\delta^b_m\delta^{c]}_r + \frac{1}{4}f^{[ab}{}_{[nm}\delta^{c]}_{r]} \ . \tag{3a.50}$$

This identity brings down the number of independent components of f_{abcd} to 14 nonzero G_2 invariant elements. They are

$$f_{0123} = f_{0246} = f_{0435} = f_{0367} = f_{0651} = f_{0572} = f_{0714} = 1 \ ,$$

$$f_{4567} = f_{3571} = f_{6172} = f_{5214} = f_{7423} = f_{1346} = f_{2635} = -1 \ . \tag{3a.51}$$

Similarly, we obtain

$$f'^{\,abcd} f'_{nmrd} = \delta_n^{[a}\, \delta_m^{\,b}\, \delta_r^{c]} + \frac{1}{4} f'^{\,[ab}{}_{[nm}\, \delta^{c]}{}_{r]} \ . \tag{3a.52}$$

Further contractions give

$$f^{abcd} f_{nmcd} = 6\, \delta_n^{[a}\, \delta_m^{b]} + 4\, f^{ab}{}_{nm} \ , \tag{3a.53}$$

$$f'^{\,abcd} f'_{nmcd} = 6\, \delta_n^{[a}\, \delta_m^{b]} + 4\, f'^{\,ab}{}_{nm} \ , \tag{3a.54}$$

$$f^{abcd} f_{nbcd} = f'^{\,abcd} f'_{nbcd} = 42\, \delta_n^a \ , \tag{3a.55}$$

$$f^{abcd} f_{abcd} = f'^{\,abcd} f'_{abcd} = 336 \ . \tag{3a.56}$$

We have used the Euclidean metric tensor δ_{ab} to raise, lower indices and to make contractions. There are no O(8) covariant expressions for the contraction $f^{abcd} f'_{nmrd}$ and its descendants.

3.a.3. Exceptional Grassmann algebra

In physical applications it is often useful to define a *split* octonion algebra with its split base units

$$u_0 = \frac{1}{2}(e_0 + i\, e_7) \ , \quad u_0^* = \frac{1}{2}(e_0 - i\, e_7) \ ,$$

$$u_m = \frac{1}{2}(e_m + i\, e_{m+3}) \ , \quad u_m^* = \frac{1}{2}(e_m - i\, e_{m+3}) \quad (m = 1, 2, 3) \tag{3a.57}$$

where $i = \sqrt{-1}$ commutes with any e_α ($\alpha = 1, 2, \ldots, 7$). These basis elements close on the following algebra

$$
\begin{aligned}
&u_i u_j = -u_j u_i = \varepsilon_{ijk} u_k^*, \quad u_i^* u_j^* = -u_j^* u_i^* = \varepsilon_{ijk} u_k \ , \\
&u_i u_j^* = -\delta_{ij} u_0 \ , \qquad u_i^* u_j = -\delta_{ij} u_0^* \ , \\
&u_0 u_i = u_i u_0^* = u_i \ , \qquad u_0^* u_i^* = u_i^* u_0 = u_i^* \ , \\
&u_i u_0 = u_0 u_i^* = 0 \ , \qquad u_i^* u_0^* = u_0^* u_i = 0 \ , \\
&u_0 u_0^* = u_0^* u_0 = 0 \ , \qquad u_0^2 = u_0, \ \ u_0^{*2} = u_0^* \ .
\end{aligned}
\tag{3a.58}
$$

The system (3a.58) is invariant under G_2. Under the action of the SU(3) subgroup of G_2, the split units u_i and u_i^* transform like a triplet and an antitriplet, respectively, while u_0 and u_0^* transform like singlets. From the $\boldsymbol{\Omega}$ algebra we can extract a subalgebra generated by u_i and u_i^* only:

$$
\{u_i \, , u_j\} = 0 \ , \quad \left\{u_i^\dagger \, , u_j^\dagger\right\} = 0, \quad \left\{u_i \, , u_j^\dagger\right\} = \delta_{ij} \ , \tag{3a.59}
$$

here $u_i^\dagger = -u_i^* = \overline{u_i^*}$, the overbar stands for octonionic conjugation. This fermionic Heisenberg algebra shows the three split units u_i to be Grassmann numbers. Being non-associative, these split units give rise to an *exceptional Grassmann algebra*. Their associator is

$$
[u_i \, , u_j \, , u_k] \equiv (u_i \, u_j) \, u_k - u_i \, (u_j \, u_k) = \varepsilon_{ijk} \left(u_0 - u_0^*\right) = i \, \varepsilon_{ijk} \, e_7 \ . \tag{3a.60}
$$

Also

$$
\left[u_i \, , u_j \, , u_k^*\right] = \delta_{ki} \, u_j - \delta_{kj} \, u_i \ . \tag{3a.61}
$$

Like the imaginary octonionic units e_α, the split units cannot be represented by matrices. Unlike $\boldsymbol{\Omega}$, the split octonion algebra contains zero divisors and is therefore *not* a division algebra. In Sect. 3f.4., we will see one of its recent applications in an effective, supersymmetric model of hadrons.

3.b. Octonionic Hilbert Spaces, Exceptional Groups and Algebras

While the first octonionic Hilbert space was discovered in 1933 by Jordan, von

Neumann and Wigner (JNW) [65], neither the geometry nor a potential physical interpretation of such a quantum mechanical space was evident. Nearly twenty years passed before the projective geometry of such a space, the Moufang Plane, and its invariance group were understood by Jordan [245], Chevalley and Schafer [246] and Borel [247]. The associated geometry turned out to be non-Desarguesian. Its invariance group is the exceptional group F_4, replacing the U(n) (Sp(n)) group acting on the complex (quaternonic) quantum mechanical spaces (see Sect.2c). Earlier, Cartan had identified G_2 as the automorphism group of the octonion algebra, which goes in the construction of exceptional projective geometries. Subsequent works of Freudenthal [248], Tits [249], Rozenfeld [250] and Springer [251] showed how the JNW space could be generalized and associated with new, non-Desarguesian geometries. The latter geometries admit as invariance groups the remaining exceptional groups of the E-series, E_6, E_7 and E_8.

Meanwhile new formulations of quantum mechanics were developed [252]. Birkhoff and von Neumann [154] introduced the propositional calculus based on lattice theory and realized by means of projective geometry. The density matrix formalism led naturally to the correspondence with positivity domains. In all these approaches, the exceptional case involving octonions was found to occur. In view of the geometrical interpretation of quantum mechanical spaces, we may say that, modern mathematics has, through the recent generalization of the JNW geometry, provided physicists with new finite quantum mechanical spaces admitting E_6, E_7 and E_8 as their invariance groups. The spaces could be taken as models for charge spaces. Functions of the observables $\hat{0}$ can then be defined provided that the latter obey power associativity . In that case $f(\hat{0})$ is a power series in $\hat{0}$. F_4 and E_6 are automorphism groups of the algebras obeyed by $\hat{0}$. So power associativity of $\hat{0}$, Eqs. (3a.21)-(3a.22), is absolutely an essential feature for such an undertaking. Furthermore, it is proved that the only power associative algebras are alternative ones (including quaternions and octonions) and Jordan algebras (including the exceptional Jordan algebras). Moreover, in a construction of Koecher [253], E_7 and E_8 were realized nonlinearly on functions $f(\hat{0})$. These remarks prompt us to be more specific about octonionic Hilbert spaces and their associated groups.

3.b.1. Octonionic spaces and automorphism groups

In the algebraic formulation of quantum mechanics (Section 2.b.1) only the Jordan multiplication enters. Whether infinitesimal or finite, the unitary transformations U acting on states involve solely Jordan products and multiple Jordan associators. Indeed states, transition probabilities, unitary transformations only involve hermitian matrices and their Jordan products. So, if a way exists to generalize hermitian matrices, projection operators and Jordan products where octonions take the place of complex numbers or quaternions, a generalization of quantum mechanics becomes possible. This is precisely what JNW [65] did when they introduced the unique, exceptional Jordan algebra J_3^{Ω}, of (3 x 3) hermitian matrices over octonions. Here hermiticity is one w.r.t. octonionic conjugation.

We consider such a matrix $J \in J_3^{\Omega}$

$$J = \begin{pmatrix} \alpha & c & \bar{b} \\ \bar{c} & \beta & a \\ b & \bar{a} & \gamma \end{pmatrix} . \tag{3b.1}$$

The elements α, β, γ are scalars and a, b, c are octonions, the overbar denoting $\mathbf{\Omega}$-conjugation. J is hermitian and depends on 27 parameters. So

$$\mathrm{Tr}\, J = \alpha + \beta + \gamma \ , \tag{3b.2}$$

$$\mathrm{Det}\, J = \alpha\, \beta\, \gamma - \alpha\, (\, a\, \bar{a}\,) - \beta\, (\, b\, \bar{b}\,) - \gamma\, (\, c\, \bar{c}\,) + (\, a\, b\,)\, c + \bar{c}\, (\, \bar{b}\, \bar{a}\,) \ . \tag{3b.3}$$

These "hermitian" matrices J are closed under the Jordan product as defined previously in Eq. (2b.12). The transformations (2b.32) therefore remain meaningful. When h_1 and h_2 are traceless Jordan matrices, (2b.12) stands for the 52 parameter exceptional group F_4 when applied to J. We can take as a projection operator a Jordan matrix with unit trace and zero determinant. The quantities Tr J, Tr J^2 and Det J are invariant under F_4. Transition probabilities can be defined by means of Eq. (2b.36) and are left invariant by F_4, they take the place of the unitary transformations for the usual finite Hilbert space.

If P_α is a Jordan matrix of the form (3b.1) obeying Eqs. (2b.10)-(2b.11), it is a projection operator corresponding to a pure state. It is left invariant by the SO(9) subgroup of F_4 ; SO(9) plays the same role as the generalized phase group U(n – 1) x U(1) in an unitary geometry. The quantum mechanical space of all the states corresponds to the elements of the coset space F_4 / SO(9), the Moufang projective plane. SO(9) is the stability group of a point in this projective space and correspondingly is a generalized phase transformation leaving a quantum mechanical state invariant. If such a state corresponds to a particle with a given momentum; a translation should not change the state. The translation operator should therefore result in a particular SO(9) transformation on that state. In fact such a translation singles out a special direction e_7 in octonionic space. So, when we consider jointly F_4 and Poincaré transformations, the stability group leaving e_7 invariant will not be relevant. The resulting subgroup is the group of the Standard Model SU(3) x SU(2) x U(1) $\subset$ SO(9).

For (3 x 3) matrices, it follows from the Freudenthal product (2b.21) that

$$A \times A = A^{-1} \operatorname{Det} A \tag{3b.4}$$

and

$$(A \times A) \cdot A = I \operatorname{Det} A, \qquad \operatorname{Tr}\left[(A \times A) \cdot A\right] = 3 \operatorname{Det} A \ . \tag{3b.5}$$

Therefore, for a projection operator $P_\alpha \in J_3^\Omega$,

$$P_\alpha \times P_\alpha = P_\alpha^2 - P_\alpha - \frac{1}{2} I \operatorname{Tr}\left(P_\alpha^2 - P_\alpha\right) = 0 \tag{3b.6}$$

and

$$\operatorname{Tr}\left(P_\alpha \cdot [P_\alpha \times P_\alpha] \right) = 0 \tag{3b.7}$$

hold.

Now Eq. (2b.21) gives

$$\operatorname{Tr}\left[(A \times B) \cdot C\right] = \operatorname{Tr}\left[(B \times C) \cdot A\right] = \operatorname{Tr}\left[(C \times A) \cdot B\right] \ . \tag{3b.8}$$

Let

$$\Delta(A) = \operatorname{Tr}\left[(A \times A) \cdot A \right] \ , \tag{3b.9}$$

we then obtain

$$6 \operatorname{Tr}\left[(A \times B) \cdot C\right] = \Delta(A + B + C) - \Delta(A + B) - \Delta(B + C) - \Delta(C + A)$$
$$+ \Delta(a) + \Delta(B) + \Delta(C) \ . \tag{3b.10}$$

Next, we elaborate on various quantum mechanics based on the exceptional Jordan algebras and on the groups F_4, E_6 and E_7, respectively.

Let P_α be a (3x3) hermitian octonionic matrix obeying the relations

$$\mathrm{Tr}\, P_\alpha = 1 \ , \ P_\alpha \times P_\alpha = 0 \ . \tag{3b.11}$$

Here

$$P_\alpha = \frac{(aa^\dagger)}{\sqrt{a^\dagger a}} \tag{3b.12}$$

where a stands for an octonionic (3x1) vector and $a^\dagger \equiv (\,\overline{\alpha}_1, \overline{\alpha}_2, \overline{\alpha}_3\,)$, the overbar means octonionic conjugation. We choose α_3 to be self-conjugate, more specifically let

$$\alpha_3 = 1 \, . \tag{3b.13}$$

Then the matrix P_α, generalizing the projection operator (2b.8) in the associative case, depends on 16 independent real parameters. It represents a quantum mechanical state. Hence matrix elements of an operator J can be defined analogously to Eq. (2b.16) ; we then have a new quantum mechanics in the 3-dimensional octonionic Hilbert space of Jordan et al. As the octonions are real, the probabilities $\Pi_{\alpha\beta}$ (J) are positive definite. Let us take the special case of J = 1 and set, as in Eq. (2b.54), $\Pi_{\alpha\beta} = \cos^2 d_{\alpha\beta} = \mathrm{Tr}(P_\alpha \cdot P_\beta)$. The geometrical interpretation of the state P_α is clear, it may be associated with a point with the homogeneous coordinates α_1, α_2, α_3 with the constraint (3b.13). When α_3 is self-conjugate, the inhomogeneous projective coordinates $x_1 = \alpha_1\, \alpha_3^{-1}$ and $x_2 = \alpha_2\, \alpha_3^{-1}$ label a point in the two-dimensional octonionic projective plane, as shown by Moufang in 1933 [254] . $d_{\alpha\beta}$ is the non-Euclidean distance between the points P_α and P_β in the Moufang plane. It is invariant under Eqs. (2b.30) and (2b.32) representing a 52-real parameter F_4 transformation. As in Eq. (2b.17), a complete set of states can be inserted and linear dependence is still defined by Eq. (2b.24). The only difference comes from the lack of associativity of $\mathbf{\Omega}$. We then have

$$\text{i)}\ \ (\,h_2, P_\alpha, h_1) \neq \frac{1}{4}[\,[h_1,h_2]\,, P_\alpha] \ , \tag{3b.14}$$

$$\text{ii)}\ \ \Pi_{\alpha\beta} \neq \left|\langle\beta|\alpha\rangle\right|^2 \ , \tag{3b.15}$$

$$\text{iii)}\ \ P_{\vec{a}} \times P_{\vec{b}} \neq \lambda\, P_{\vec{a}\times\vec{b}} \ . \tag{3b.16}$$

Notably, the identity (2a.82) no longer holds. The last inequality (3b.16) implies the breakdown of Desargues's theorem [218]; the associated projective geometry, the Moufang plane, is therefore non-Desarguesian.

Taking the place of Eq. (2b.40) is a formula of Mostow [255] in the case of non-positive definite metric associated with the real form of F_4. For a Hilbert space with positive definite metric we obtain

$$\cos^2 d_{\alpha\beta} = \Pi_{\alpha\beta} = \frac{(a^\dagger b)(b^\dagger a) + 2\,R_{ab}}{(a^\dagger a)(b^\dagger b)} = \frac{(1+\alpha^\dagger\beta)(1+\beta^\dagger\alpha) + 2\,R_{\alpha\beta}}{(1+\alpha^\dagger\alpha)(1+\beta^\dagger\beta)} \tag{3b.17}$$

with $\alpha \equiv \begin{pmatrix}\alpha_1\\ \alpha_2\end{pmatrix}$, $\beta^\dagger \equiv (\overline{\beta}_1\,\overline{\beta}_2)$ and

$$R_{\alpha\beta} = \mathrm{Sc}\,[\,(\alpha_1\overline{\alpha_2})\;(\beta_2\overline{\beta_1}) - (\overline{\alpha_2}\beta_2)\,(\overline{\beta_1}\alpha_1)\,] \tag{3b.18}$$

by way of Eq. (3b.13). We check that if α and β are elements of an associative algebra, the additional term $R_{\alpha\beta}$ clearly vanishes and the more familiar distance formula (2b.52) is recovered.

For neighboring α and β, namely by setting $\beta = \alpha + d\alpha$, the line element (2b.53) is modified by an additional term

$$d\rho^2 = 2(\,1 + \alpha^\dagger\alpha)^{-2}\,R\,(\alpha\,,\alpha + d\alpha) \tag{3b.19}$$

with

$$R\,(\alpha\,,\alpha + d\alpha) = \mathrm{Sc}\left\{\,(\alpha_1\overline{\alpha_2})(d\alpha_2\overline{\alpha_1}) - [(\overline{d\alpha_1})]\,\alpha_1\,(\overline{\alpha_2}d\alpha_2)\,\right\}\,. \tag{3b.20}$$

Compatible observables are still defined by the cubic associator condition

$$\left(\,\hat{0}_1\;\mathbf{J}\,\hat{0}_1\,\right) = 0\ , \tag{3b.21}$$

with J beings for any element of the Jordan algebra. However, as stated in Section 2.b, we can no longer conclude that they commute since $\left[\,\hat{0}_1\,,\;\hat{0}_1\,\right] = 0$ no longer follows from Eq. (3b.21).

What possible physical implication(s) could such a quantum mechanics have ? To find out, we may examine a scattering formalism in a charge space governed by such an exceptional quantum mechanical space. If the initial state is given by the projection operator P_{in}, after the scattering it turns into the state P_{out}. In the usual quantum mechanics, the two are connected by

$$P_{out} = S\,P_{in}\,S^{-1}\ , \tag{3b.22}$$

S being the unitary S-matrix, S = exp(i A). The hermitian operator A is related to the part of some action functional. Here S defines an automorphism of the algebra of the operators P_{in} or P_{out}. In the exceptional case, P_{out} is similarly related to P_{in} by an automorphism of the exceptional Jordan algebra, a F_4 transformation as shown by Chevalley and Schafer [246]. The latter is defined by the octonionic hermitian matrices A_1 and A_2 such that, by Eq. (2b.32)

$$P_{out} = P_i + (A_1\, P_i\, A_2) + \frac{1}{2}(A_i\,(A_1 P_i A_2)\, A_2) + \ldots \tag{3b.23}$$

A_1 involves only one octonion and A_2 is a purely scalar matrix. Equation (3b.23) can be integrated in the form $P' = U\,P\,U^{\dagger}$ with $U = \exp\Theta$ where Θ is an antihermitian octonionic matrix involving one octonion only. It is seen that the full group is determined by the two traceless, hermitian matrices A_1 and A_2, therefore by 52 parameters altogether. For any element J of the Jordan algebra, the F_4 invariants are I_1 = Tr J (3b.1), I_2 = Tr J^2 and I_3 = Det J = $\frac{1}{3}$ Tr (J · (J x J)) (3b.3). An irreducible representation of F_4 is obtained by taking $I_1 = 0$ i.e. traceless Jordan matrices.

The S-matrix element squared between P_{out} and any incoming state P_f is

$$|\,S_{if}\,|^2 = \mathrm{Tr}\left(P_f \cdot P_{out}\right) = \mathrm{Tr}\left(P_f \cdot P_i\right) + |\,T_{if}\,|^2\ , \tag{3b.24}$$

with the transition probability as

$$|\,T_{if}\,|^2 = \mathrm{Tr}\left\{P_f \cdot (A_1\, P_i\, A_2)\right\} + \frac{1}{2}\mathrm{Tr}\left\{\, P_f \cdot (A_1\,(A_1\, P_i\, A_2)\, A_2)\right\} + \ldots\ . \tag{3b.25}$$

In the non-exceptional cases, we can set

$$\frac{1}{2}[\, A_1, A_2\,] = i\, A\ ,\quad P_i = |\, i\,\rangle\langle\, i\,|\ ,\quad P_f = |\, f\,\rangle\langle\, f\,|\ ; \tag{3b.26}$$

to lowest order in A

$$|\,T_{if}\,|^2 = |\langle\, i\,|\, A\,|\, f\,\rangle|^2\ ; \tag{3b.27}$$

we thus recover the usual expression in perturbation theory. Consequently, Eq. (3b.25) extends this result to the exceptional case:

$$|\,T_{if}\,|^2 = |\langle\, i\,|\, A\,|\, f\,\rangle|^2 + \rho_{if}\,. \tag{3b.28}$$

ρ_{if} is the additional contribution to the transition probability and depends on associators of octonions. This extra term may be thought to arise from some extra force reflecting the non-associative nature of the underlying division ring in exceptional quantum mechanics. Thus, in the context of current physical theories, it was speculated by one of us (F.G.) that the confinement of quarks might show itself algebraically from the existence of such an additional contribution to the scattering matrix in an effective octonionic quantum mechanics.

In octonionic quantum mechanical spaces, the operators act on the hermitian Jordan matrices P representing states. Operators like

$$\Omega\,(A, B)\,P = (A\,P\,B) \tag{3b.29}$$

correspond to F_4 generators and transform like its adjoint 52-dimensional representation. Matrix elements of Ω between the states P and P' can be defined in analogy with $|T_{if}|^2$ as

$$\Omega_{PP'} = \mathrm{Tr}\Big[\ P' \cdot (A\,P\,B)\Big]\ . \tag{3b.30}$$

Higher dimensional F_4 representations can also be introduced by defining operators acting on Ω, and so on. In this manner, we can build a larger Hilbert space, analogous to the Fock space of unitary groups U(n), by including all representations of the underlying invariance group. Thus, if P corresponds to a quark state, then Ω represents a subset of two quark states, etc.

We observe that the infinitesimal F_4 transformation (e.g. from Eq. (3b.23)) of a traceless matrix J, i.e. the associator

$$\delta\,J = (\,h_2\,J\ \ h_1) \equiv [h_2,\,J,\ \ h_1] \tag{3b.31}$$

respects the tracelessness condition for J. It follows that F_4 is also the automorphism group of the wedge algebra of Gell-Mann [256], Michel and Radicati [257] under which traceless hermitian matrices close. In the present context the wedge product is defined by

$$J_1 \wedge J_2 = -\,J_2 \wedge J_1 = J_1 \cdot J_2 - \tfrac{1}{3}\mathrm{Tr}\ (J_1 \cdot J_2) \tag{3b.32}$$

where I is the (3 x 3) unit matrix.

Going beyond F_4, we next give examples of quantum mechanical spaces associated with the exceptional groups of the E series. They generalize the JNW space, namely the Moufang plane associated with F_4.

We consider (3 x 3) *complex* octonionic matrices F, hermitian w.r.t. octonionic conjugation only. Such matrices depend on 27 complex (54 real) parameters. The relevant algebra is not the Jordan algebra, but rather the Freudenthal algebra defined by (2b.21). The product $F_1 \times F_2 = K$ retains its form when F_1, F_2 and K are transformed by the 78-parameter group E_6 such that F_1 and F_2 transform like its 27-dimensional representation and K like the conjugate representation $\overline{27}$. E_6 is thus the automorphism group of the Freudenthal algebra. K is also unchanged if F_1 and F_2 transform like $(\overline{27})$ and K like (27). On the other hand, the Jordan product $F_1 \cdot F_2$ has no definite transformation property. If F is an element of the Freudenthal algebra which closes under the union of the 27 and $\overline{27}$ representations, then it represents a state if

$$F \times F = 0 \ . \tag{3b.33}$$

The infinitesimal automorphism of the algebra is given by

$$\delta F = (H_1 \ , F \ , H_2) + i\, H_3 \cdot F \tag{3b.34}$$

where H_1, H_2 and H_3 are traceless hermitian 3x3 matrices over real octonions. Then if

$$F'_1 = J_1 + \delta J_1 \ , \quad F'_2 = J_2 + \delta J_2 \ , \quad F' = K + \delta K \tag{3b.35}$$

$F_1 \times F_2 = K$ is transformed into

$$F'_1 \times F'_2 = K' \tag{3b.36}$$

showing that Eq. (3b.34) is an automorphism. The invariant quantities under the 3 x 26 = 78-parameter transformation (3b.34) are

$$I_2 = \frac{1}{2} \mathrm{Tr}\ (F^* \cdot F) \quad , \quad I_3 + i\, I'_3 = \mathrm{Det}\, F \ , \tag{3b.37}$$

$$I_4 = \frac{1}{2} \mathrm{Tr} \left\{ (F \times F)^* \cdot (F \times F) \right\} \ , \tag{3b.38}$$

where the superscript "*" denotes complex conjugation, I_2, I_4 are real and non-negative,

I_3, I'_3 are real. Det F is given by Eq. (3b.5).

The norm N(F) of the state F obeying Eq. (3b.33) can be defined as

$$N(F)^2 = 2\,I_2 = \mathrm{Tr}\,(J^* \cdot J) \ . \tag{3b.39}$$

The normalized state α is represented by the projection operator

$$P_\alpha = \frac{J_\alpha}{N(F_\alpha)} \ . \tag{3b.40}$$

The transition probability $\Pi_{\alpha\beta}$ and the distance $d_{\alpha\beta}$ between states α and β are

$$\Pi_{\alpha\beta} = \cos^2 d_{\alpha\beta} = \frac{1}{2} N(\,P_\alpha + P_\beta\,) - \frac{1}{2} N(\,P_\alpha\,) - \frac{1}{2} N(\,P_\beta\,) \ , \tag{3b.41}$$

$$= \frac{1}{2} N(\,P_\alpha + P_\beta\,) - 1 \ . \tag{3b.42}$$

Alternatively, in terms of F_α and F_β

$$\Pi_{\alpha\beta} = \left[2\ N(F_\alpha)\ N(F_\beta)\right]^{-1} \mathrm{Tr}\ (\,F_\alpha^* F_\beta + F_\beta^* F_\alpha) \ . \tag{3b.43}$$

Equation (3b.33), the condition for F to be a state, and $\Pi_{\alpha\beta}$ are invariant under the infinitesimal transformations (3b.34) of E_6.

We also note that, by means of the Freudenthal product and the E_6 invariant scalar product Tr $(\phi_1^* \cdot \phi_2)$, Eq. (3b.34) assumes another form. Using the definition (2b.14), it becomes

$$\delta\ F = \frac{1}{2}\left\{\phi_1\ \phi_2^*\ F\right\} - \frac{1}{2}\left\{\phi_2\,\phi_1^*\ F\right\} \tag{3b.44}$$

with

$$\phi_1 = H_1 + i\,H_2\ ,\ \ \phi_2 = H_2 + i\,B\ ,\ \ H_3 = (H_1 - H_2)\cdot B \ . \tag{3b.45}$$

Also

$$\delta\ F = \phi_1^* \times (\,\phi_2 \times F\,)\ -\ \phi_{12}^* \times (\,\phi_1 \times F\,)\ +\ \frac{1}{4}\phi_1\ \mathrm{Tr}\left(\,\phi_2^* \times F\,\right)$$

$$-\frac{1}{4}\phi_2\,\mathrm{Tr}\left(\phi_1^*\times F\right)+\frac{1}{4}F\,\mathrm{Tr}\left(\phi_1\cdot\phi_2^*-\phi_2\cdot\phi_1^*\right). \tag{3b.46}$$

As yet another example of an exceptional quantum mechanical space, we observe without further elaboration that the algebra of observables can be extended to a *ternary algebra*. A comprehensive treatment of ternary algebras for physicists can be found in the works of Bars and Gunaydin [258].

Let us consider for example the space of boson and fermion states and take as observables the generators of transformations between these states. Generators transforming bosons into bosons and fermions into fermions, respectively, have the properties of ordinary quantum mechanical observables, they are elements of a binary algebra of the Jordan type. Those mapping bosons into fermions and vice versa are of fermionic type, they correspond to odd elements of a superalgebra. Clearly, while such observables are not closed under a binary algebra, they will close under a ternary algebra. In fact, there exists an exceptional ternary algebra whose elements X have 56 complex components. We denote by X^T a row of (3x3) matrices.

$$X^T=\left(f\,I\,,F\,,G^*,\,g^*I\right)\,, \tag{3b.47}$$

where I is the 3x3 unit matrix, f and g are complex while F and G are elements of the Freudenthal algebra. In the fermionic case, f, g and the components of F and G are anticommuting Grassmann numbers. Under E_6, F and G transform like a 27 and f and g are singlets.

The norm of X defined by

$$N(X)^2=\mathrm{Tr}\,(\,f^*f\,I+F^*F+G^*\,G+g^*\,g\,I\,)\,, \tag{3b.48}$$

is manifestly an E_6 invariant. Moreover it is invariant under the 1-parameter transformation

$$X\rightarrow T_\gamma\,X=\exp(\,i\,\Sigma_3\gamma\,)\,X \tag{3b.49}$$

where Σ_3 denotes the spin $\frac{3}{2}$ diagonal matrix with elements $(\frac{3}{2},\frac{1}{2},-\frac{1}{2},-\frac{3}{2})$. Finally, we consider the infinitesimal transformation defined by an element Z of the Freudenthal algebra

$$\delta X = T_Z X \tag{3b.50}$$

where

$$\delta f = i\sqrt{3}\,\mathrm{Tr}(Z^* \cdot F\ ), \tag{3b.51a}$$

$$\delta f = i\sqrt{3}\,Z f + 2i\,Z^* \times G^*\ , \tag{3b.51b}$$

$$\delta G^* = i\,2\,Z \times F + i\sqrt{3}\,Z^* g^*\ , \tag{3b.51c}$$

$$\delta g^* = i\sqrt{3}\,\mathrm{Tr}(Z \cdot G^*)\ . \tag{3b.51d}$$

This is equivalent to acting on the column X the matrix operator

$$\Delta \equiv i(\Sigma_1 + i\Sigma_2)Z^* + i(\Sigma_1 - i\Sigma_2)Z$$

$$= i\begin{pmatrix} 0 & \sqrt{3}\mathrm{Tr}(Z^*) & 0 & 0 \\ \sqrt{3}Z & 0 & 2Z^*\,\mathrm{x} & 0 \\ 0 & 2Z\,\mathrm{x} & 0 & \sqrt{3}Z^* \\ 0 & 0 & \sqrt{3}\mathrm{Tr}(Z) & 0 \end{pmatrix}, \tag{3b.52}$$

Σ_1 and Σ_2 are the remaining spin $\frac{3}{2}$ matrices such that

$$\Sigma_+ = (\Sigma_1 + i\Sigma_2) = \begin{pmatrix} 0 & \sqrt{3} & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & \sqrt{3} \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{3b.53}$$

$$\Sigma_3 = \begin{pmatrix} \frac{3}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & -\frac{3}{2} \end{pmatrix}, \tag{3b.54}$$

$\Sigma_- = (\Sigma_1 - i\Sigma_2)$ is the hermitian conjugate of Σ_+. They obey the SU(2) algebra

$$\left[\Sigma_i\,,\Sigma_j\right] = i\,\varepsilon_{ijk}\Sigma_k \qquad (i, j, k = 1, 2, 3)\ . \tag{3b.55}$$

The transformations defined by γ and Z correspond to the coset $\frac{E_7}{E_6}$. Together with the E_6 transformations they make up the group E_7 having 133 parameters. X is the pseudoreal 56-dimensional representation of E_7; X* is equivalent to X. To see this, let $\widetilde{X} = C X^*$ with

$$C = \exp(i \pi \Sigma_2) = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} . \tag{3b.56}$$

So

$$\widetilde{X}^T = (-gI\, G\, , -F^*\, , f^*) \ . \tag{3b.57}$$

It can be checked that $\widetilde{X}^T$ transforms exactly as X under E_7. Therefore, combining two 56-dimensional representations X_α and X_β, we have two quadratic invariants:

$$\Pi_{\alpha\beta} = \frac{1}{2} N (X_\alpha + X_\beta) - \frac{1}{2} N(X_\alpha) - \frac{1}{2} N(X_\beta) \tag{3b.58}$$

and

$$S_{\alpha\beta} = N (X_\alpha + \widetilde{X}_\beta) - \frac{1}{2} N(X_\alpha) - \frac{1}{2} N(X_\beta) , \tag{3b.59}$$

the symplectic invariant.

We can introduce a ternary operation on X by

$$S = (X\, X\, X) = \Sigma_i\, X\, \widetilde{X^*}^T \cdot\ \Sigma_i X \tag{3b.60}$$

where Σ_i are the spin matrices given by Eq. (3b.53-54) and the summation over the index i is implied. In the multiplication of the matrices, the elements of the Freudenthal algebra are combined with the E_6 invariant scalar product or with the Freudenthal product.

We can now define a state corresponding to a geometrical point in a symplectic geometry. The module X represents a state provided that

$$(X_\alpha\, X_\alpha\, X_\alpha) = 0 \ , \tag{3b.61}$$

then X depends only on an element F of the Freudenthal algebra, its components are proportional to $\sqrt{3}$, F, F x F and Det F. We also get

$$\tilde{X}_\alpha^{*T}\, \Sigma_i\, X_\alpha = 0 \ , \tag{3b.62}$$

which is a quadratic relation taking the place of Eq. (3b.33) for E_6. The state can be normalized by taking N(X) = 1, where the norm is given by Eq. (3b.48). Finally, E_7 invariant transition probabilities can be defined as

$$\Pi_{\alpha\beta} = \frac{1}{2} N (X_\alpha + X_\beta) - \frac{1}{2} N(X_\alpha) - \frac{1}{2} N(X_\beta) = \frac{1}{2} N (X_\alpha + X_\beta) - 1 \ . \tag{3b.63}$$

A ternary algebra for the states X can be defined by polarizing the ternary product (3b.60). This algebra has in fact been axiomatized by Faulker [259, 260, 261] who also showed E_7 to be its automorphism group. We have thus shown the existence of a ternary algebra of observables in a new exceptional quantum mechanical space associated with E_7.

3.b.2. Exceptional algebras, groups and cosets

The preceding discussions make clear the intimate connections amomg octonions, exceptional algebras, groups and their coset spaces. Many excellent treatments of the manifold aspects of these algebras and groups are available in the mathematics and physics literatures. Here we focus primarily on the relations to octonions (see also Ref.[262]). To enliven our presentation with the colors and flavors of the "real world," it will be cast in the context of gauge theories. Exceptional structures do arise naturally in grand unified, superstring and supermembrane theories.

In the case of unified theories, the octonionic connection is realized if the generalized internal symmetry charge space is identified with the exceptional finite dimensional quantum mechanical spaces. With one such space attached to each spacetime point, we can enlarge our Hilbert space to incorporate internal symmetries associated with an exceptional group. We then have a natural extension of Minkowski space for the formulation of a gauge theory based on local invariance w.r.t. one of the exceptional groups. In this subsection, we sketch some properties of exceptional groups and their representations entering into the construction of such gauge theory.

One general feature is the basic role played by the product group SU(3) x SU(3). One of the two SU(3)'s, identifiable as the color group, is a maximal subgroup of G_2; it arises as the automorphism group involving six octonionic units. The other SU(3) can be identified with the Gell-Mann-Neeman unitary flavor symmetry; it is related to the (3 x 3) structure of the JNW exceptional observables. Its existence derives from the triality property of a triplet of octonions. In this manner, the charge space provides us with a natural explanation of color and basic SU(3) flavor. This natural SU(3) x SU(3) structure allows quarks and leptons to sit in one multiplet. The leptons are the SU(3) singlet part of the small representation of the exceptional group while the quarks are given by its color triplet part.

For groups higher than F_4, the maximal subgroup is G x SU(3) with SU(3) $\subseteq$ G. For example G $\approx$ SU(3) x SU(3) for E_6 and G $\approx$ SU(6) for E_7. The small representations of F_4, E_6 and E_7 provide a lepton quark symmetry in that leptons and quarks are represented by local fields belonging to a single representation of the internal symmetry group. This symmetry does not imply equality in number between leptons and quarks. They are different since the fractionally charged quarks are colored and interact with colored gluons while integrally charged leptons are color singlets and have no strong interactions.

Since from the phenomenological standpoint, gauge theories seem to be realized in their spontaneously broken phase in the electroweak sector, the first task is the determination of the possible modes of symmetry breaking for exceptional groups. The standard procedure can be summed up as follows. A given Higgs field can always be brought into some canonical form by a local gauge transformation. The minimization of the Higgs potential then yields invariant relations among the canonical Higgs field components. Such relations can be solved by giving vacuum expectation values (VEV) to some of the field components. A spontaneous breakdown of the group results. The remaining explicit symmetry of the theory is the subgroup leaving invariant the direction chosen by the VEV. The rest of the group now acts nonlinearly. In the presence of several Higgs fields, only the intersection of the corresponding subgroups will survive as the exact symmetry; the corresponding generators will be associated with massless gauge bosons. In a theory incorporating both QCD and QED, with its eight color gluons and a photon, the survival exact gauge groups should be $SU_c(3)$ x $U_{e\&m}(1)$.

To see how the gauge theory approach may be used for exceptional groups, we do the following: we must work out the canonical forms of the relevant group representations, find their algebraic properties, and construct explicitly the action of a finite transformation of the group acting on these representations. This is why we emphasize here the construction of the finite transformations rules for the 26, 27 dimensional representations of F_4, E_6 respectively. They will be needed to determine the canonical forms for higher representations of the same groups, to be used in the would be minimized Higgs potential. We will illustrate below these constructions and do so only for some of the smallest representations. Various applications will also be discussed.

Our procedure is to first derive finite G_2 transformations in octonionic form and then introduce the flavor theory based on the exceptional spaces. The Tits construction of the Magic Square is reviewed. A similar construction from color and flavor is shown to generate the exceptional group. Its advantage consists in showing how the exceptional groups naturally extend such familiar symmetry groups as SU(3), SU(3) x SU(3) and SU(6). We then proceed to F_4 and its 26 dimensional JNW representation. The latter representation unites 3 colored quarks and 4 leptons into a single fermionic multiplet. Canonical forms and finite transformations formulae are given. The Michel-Radicati [257] algebraic treatment of invariant directions is extended to F_4 and the corresponding stability groups are obtained. We go next to E_6, the Freudenthal algebra associated with the 27 dimensional representation is used in writing the finite transformation laws and invariants. However, the study of E_7 and its 56 dimensional representation will be omitted. While E_7 is certainly interesting in its own right from the mathematical viewpoint, as a GUT theory it is incompatible with experiments. As to the group E_8, we will discuss it in the larger context of superstring theory later, in Section 4.

3.b.2.1. Octonionic representation of SO(8), SO(7) and G_2

We recall that, in terms of split units, any octonion ω reads

$$\omega = \omega_0 + \mathbf{e}\cdot\boldsymbol{\omega} = 2\,\mathrm{Re}(\psi) = \psi + \psi^* \tag{3b.64}$$

where

$$\psi = u_0{}^*\psi_0 + u_k{}^*\,\psi_k \tag{3b.65}$$

and

$$\psi_0 = \omega_0 + i\,\omega_7,\quad \psi_k = \omega_k + i\,\omega_{k+3},\quad \psi^2 = \psi_0\,\psi\ . \tag{3b.66}$$

Its octonion conjugate and norm are defined respectively as

$$\overline{\omega} = \omega_0 - \mathbf{e}\cdot\boldsymbol{\omega}\quad\text{and}\quad N(\omega) = \omega\,\overline{\omega}\ . \tag{3b.67}$$

We have the following associated hierarchy of groups. The norm preserving linear transformations form the 28-parameter group SO(8) of $\boldsymbol{\Omega}$. Its 21-parameter SO(7) subgroup leaves the norm $N(\mathbf{e}\cdot\boldsymbol{\omega})$ invariant. The 14-parameter G_2 subgroup of SO(7) leaves the multiplication table (3a.1) invariant. It is therefore the automorphism group of the octonion algebra. Finally, SU(3) is the 8-parameter subgroup of G_2 leaving one octonionic unit, say e_7 (or u_0) invariant.

The seven transformations of the coset SO(8)/SO(7) are represented by

$$T[SO(8)/SO(7)]\,\omega = c\,\omega\,c \quad , \quad N(c) = 1 \ . \tag{3b.68}$$

Expanding c about the unit element, $c = \exp(\frac{1}{2}\mathbf{e}\cdot\boldsymbol{\gamma}) \approx 1 + \frac{1}{2}\mathbf{e}\cdot\boldsymbol{\gamma}$, we get

$$\delta\,\omega = D_\gamma\,\omega = \frac{1}{2}(\,\mathbf{e}\cdot\boldsymbol{\gamma}\,\omega + \omega\,\mathbf{e}\cdot\boldsymbol{\gamma}) \ . \tag{3b.69}$$

We now write the seven transformations of the coset $SO(7)/G_2$ as

$$T[SO(8)/G_2]\,\omega = d\,\omega\,\bar{d} \ , \qquad N(d) = 1. \tag{3b.70}$$

Letting $d \approx 1 + \frac{\mathbf{e}\cdot\boldsymbol{\delta}}{2}$,

$$\delta\omega = D_\delta\omega = \frac{1}{2}\left[\,\mathbf{e}\cdot\boldsymbol{\delta}\,,\,\omega\right] \tag{3b.71}$$

results. The G_2 transformation may be expressed in terms of two unit octonions a and b, $|a| = |b| = 1$, in three different ways, reflecting the triality property of the octonion triples.

If $a \approx 1 + \frac{1}{2}\alpha$, $b \approx 1 + \frac{1}{2}\beta$, we obtain

$$T_1\,[G_2]\,\omega = (\,\bar{b}\,\bar{a}\,)\,[\,b\,(\,a\,\omega\,\bar{a})\,\bar{b}\,]\,(a\,b) \ , \tag{3b.72}$$

$$T_2\,[G_2]\,\omega = (\,\bar{b}\,\bar{a}\,)\,[\,b(\,a\,\omega\,a^2\,)\,b^2\,]\,(\,\bar{b}\,\bar{a}\,)^2 \ , \tag{3b.73}$$

$$T_3[\,G_2]\,\omega = (ab)^2\,[\,\bar{b}^2(\bar{a}^2\omega\,\bar{a})\bar{b}\,]\,(a\,b) \ . \tag{3b.74}$$

The infinitesimal transformation for all these three forms is the same:

$$\delta\omega = D_{\alpha,\beta}\,\omega = \frac{3}{4}\left[\,\mathbf{e}\cdot\boldsymbol{\alpha}\,,\,\mathbf{e}\cdot\boldsymbol{\beta}\,,\,\omega\right] - \frac{1}{4}\left[\left[\mathbf{e}\cdot\boldsymbol{\alpha}\,,\,\mathbf{e}\cdot\boldsymbol{\beta}\right],\,\omega\,\right] , \tag{3b.75}$$

which is the standard form of G_2 as the derivation algebra of $\boldsymbol{\Omega}$. Here we recommend

Schafer's book[241] for a more detailed discussion of Eq. (3b.75) and the derivation algebra.

From either T_1, T_2 or T_3 we clearly have a subgroup of SO(7). The infinitesimal form shows that the transformation does not belong to SO(7)/G_2, which is of the form (3b.71), since (3b.75) involves an extra associator. Thus we have accounted for all the 21 parameters of SO(7). Since the derivations (3b.75) make up a Lie algebra, the three transformations T_i (i = 1, 2, 3), which are exponentiated forms of (3b.75), must be equivalent. They will occur in the transformation formula for F_4.

As a subgroup of G_2, SU(3), taken as $SU(3)_{color}$ in applications, is best viewed in the split octonion basis defined previously. Next, we examine the transformation not of ω but rather of the associated complex octonion ψ. We define

$$\Theta^i_j\,\psi = [u_i{}^*\ u_j\psi] - \frac{1}{3}\,[\,[u_i{}^*\ , u_j]\ , \psi\ ] \tag{3b.76}$$

where

$$\psi = \ u^\dagger\ \Psi + u_0{}^*\ \psi_0 \quad \text{and} \quad u^\dagger = (\ u_1{}^*\ u_2{}^*\ u_3{}^*\ )\ \ ,\ \Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix}. \tag{3b.77}$$

After some algebra, we find [237]

$$\Theta^i_j\,\psi = u_i{}^*\psi_j - \frac{1}{3}\,\delta^i_j\ u_k{}^*\psi_k = -\,u_i{}^*(u_j u^\dagger\Psi) - \frac{1}{3}\,\delta_{ij}\ u^\dagger\Psi\ . \tag{3b.78}$$

Namely,

$$\Theta^1_2\psi = u^\dagger\,\frac{1}{2}\,(\lambda_1 + i\lambda_2)\Psi = -\,u_1^*(u_2 u^\dagger\Psi)\ , \tag{3b.79}$$

$$\Theta^1_3\psi = u^\dagger\,\frac{1}{2}\,(\lambda_4 + i\lambda_5)\Psi = -\,u_1^*(\ u_3 u^\dagger\Psi\ )\ , \tag{3b.80}$$

$$\Theta^2_3\psi = u^\dagger\,\frac{1}{2}\,(\lambda_6 + i\lambda_7)\Psi = -\,u_2^*(\ u_3 u^\dagger\Psi\ )\ , \tag{3b.81}$$

$$\left(\,\Theta^1_1 - \Theta^2_2\,\right)\psi = u^\dagger\,\lambda_3\Psi = -\,u_1^*\ (\ u_1 u^\dagger\Psi\ ) + u_2^*\ (\ u_2\, u^\dagger\Psi\ )\ , \tag{3b.82}$$

$$\Theta^3_3\ \psi = -\,\frac{1}{\sqrt{3}}\,u^\dagger\,\lambda_3\Psi = -\,u_1^*\ \lambda_8\Psi = -\,u^\dagger Y^c\Psi = -\,u_3^\dagger(\ u_3 u^\dagger\Psi\ ) - \frac{1}{3}\,u^\dagger\Psi\ , \tag{3b.83}$$

$$\Theta^1_1 \ \psi = u^\dagger \frac{1}{2} (\lambda_3 + \frac{1}{\sqrt{3}} \lambda_8) \Psi = u^\dagger Q^c \Psi \ . \tag{3b.84}$$

The λ_i are Gell-Mann's (3 x 3) matrices and Q^c, Y^c are the charge and hypercharge operators, respectively. The integrated forms of Eqs. (3b.82) and (3b.83) are

$$\psi' = u^\dagger \exp(i\, \lambda_3 \alpha_3)\, \Psi + u_0^* \, \psi_0 \tag{3b.85}$$

and

$$\psi'' = u^\dagger \exp(\frac{i}{\sqrt{3}} \lambda_8 \alpha_8)\, \Psi + u_0^* \, \psi_0 \, . \tag{3b.86}$$

They are equivalent to the following changes of the split units:

$$\begin{aligned} &u'_0 = u_0 \ \ , \ u'_1 = \exp(-\, i\, \alpha_3)\, u_1 \ , \\ &u'_2 = \exp(i\, \alpha_3)\, u_2 \ , \ u'_3 = u_3 \ , \end{aligned} \tag{3b.87}$$

$$\begin{aligned} &u''_0 = u_0 \ \ , \ \ u''_1 = \exp\left(-\, i \frac{\alpha_8}{3}\right) u_1 \ , \\ &u''_2 = \exp\left(-\, i \frac{\alpha_8}{3}\right) u_2 \ \ , \ \ u''_3 = \exp\left(i \frac{2\alpha_8}{3}\right) u_3 \ \ . \end{aligned} \tag{3b.88}$$

The multiplication table (3a.58) is manifestly invariant under this color transformation, which is therefore an automorphism of $\mathbf{\Omega}$.

Finally, we list the six transformations of $G_2 / SU_c(3)$ in terms of split units:

$$\tau^i \, \psi = \frac{3}{2} [\ u_i \, , (u_0^* - u_0^*) \, , \psi \ ] - \frac{1}{2} [[\ u_i^* \, , (u_0 - u_0^*) \ ] \, , \psi \ ] \tag{3b.89}$$

$$= 2\, u_i^* \, \psi + \psi \, u_i^* = \varepsilon_{ijk} \psi_j u_k + \ \psi_0 u_i^* \ , \tag{3b.90}$$

$$\tau_i \, \psi = \frac{3}{2} [\ u_i \, , (u_0 - u_0^*) \, , \psi \ ] - \frac{1}{2} [[u_i \, , (u_0 - u_0^*)] \, , \psi] \tag{3b.91}$$

$$= [\ u_i , \psi \ ] = u_i \ \psi_0 - \delta_{ij} \, (u_0 - u_0^*) \, \psi_j \ \ . \tag{3b.92}$$

We note that the 14 parameters of G_2 decompose as an 8 ($\Theta^i{}_j$), a 3 (τ^i) and a $\overline{3}$ (τ_i) under SU(3). When ψ_0 is purely imaginary, ψ depends on 7 real parameters, forming a representation of G_2. It decomposes into a SU(3) singlet (ψ_0), a triplet (ψ_i) and an anti-triplet (ψ_i^*).

3.b.2.2. Tits' construction of the Magic Square

In this subsection we will show how color can readily combine with flavors within semi-simple Lie groups if the color SU(3) group is identified with an octonion automorphism. Indeed, the groups F_4, E_6, E_7 and E_8 and some of their subgroups fitting in Freudenthal's Magic Square (see Table I below) can be obtained in a unified way.

Such a unified approach is achieved below by combining the subalgebras of the octonion algebra, namely the four Hurwitz algebras with quadratic norm, with the subalgebras of the exceptional Jordan algebra by way of Tits' construction [249].

We consider the four (3x3) Jordan algebras of Hermitian matrices over $\mathbf{R}$, $\mathbf{C}$, $\mathbf{H}$ and $\mathbf{\Omega}$. The Jordan product reads

$$J_1 \cdot J_2 \equiv \frac{1}{2}(J_1 J_2 + J_2 J_1) \ , \quad (J_1 = J_1^\dagger \ , J_2 = J_2^\dagger) \ . \qquad (3b.93)$$

The dagger denotes transposition combined with the involution of the Hurwitz algebras, namely with complex, quaternionic and octonionic conjugation for $\mathbf{C}$, $\mathbf{H}$ and $\mathbf{\Omega}$, respectively. The automorphism groups of these algebras are SO(3), SU(3) and Sp(6) for J^{R}, J^{C} and J^{Ω}, respectively. The corresponding algebras are the derivative algebras of the Jordan algebras leaving the following quantities invariant

$$C_1 = \mathrm{Tr}\, J \ , \ C_2 = \frac{1}{2}\mathrm{Tr}\, J^2 \ , \ C_3 = \mathrm{Det}\, J = \frac{1}{3}\mathrm{Tr}\, J^3 - C_1 C_2 + \frac{1}{6} C_1^{\,3} \ . \qquad (3b.94)$$

We used the notation of $J^3 \equiv J^2 \cdot J$.

In the case of J^{Ω} we define the matrix

$$J = \begin{pmatrix} \alpha & c & \bar{b} \\ \bar{c} & \beta & a \\ b & \bar{a} & \gamma \end{pmatrix} \qquad (3b.95)$$

with $\alpha, \beta, \gamma \in \mathbf{R}$ and $a, b, c \in \mathbf{\Omega}$. Then

$$C_1 = \alpha + \beta + \gamma \ , \qquad (3b.96)$$

$$C_2 = \frac{1}{2}(\alpha^2 + \beta^2 + \gamma^2) + a\bar{a} + b\bar{b} + c\bar{c} \ , \tag{3b.97}$$

$$C_3 = \alpha\beta\gamma - \alpha a\bar{a} - \beta b\bar{b} - \gamma c\bar{c} + t\,(abc) \ . \tag{3b.98}$$

Here the trace t (abc)≡2 Re (abc) is unambiguous since, by the alternativity property of octonions, we have

$$t\{(ab)c\} = (ab)c + \bar{c}(\bar{b}\bar{a}) = t\{a(bc)\} = a(bc) + (\bar{c}\bar{b})\bar{a} \tag{3b.99}$$

or

$$[a\ b\ c] = [\bar{c}\ \bar{b}\ \bar{a}] \ . \tag{3b.100}$$

To define a derivation algebra in the octonionic case, we are only allowed to use the Jordan product. Non-symmetrized products cannot be defined unambiguously. The Jordan associator

$$(H_1, J, H_2) = (H_1 \cdot J) \cdot H_2 - H_1 \cdot (J \cdot H_2) \tag{3b.101}$$

is antisymmetric in H_1 and H_2. Then

$$(H, J, H) = 0 \quad \text{and} \quad \left(H, J, H^2\right) = 0 \ . \tag{3b.102}$$

The Jordan algebra is **not** alternative, but it is power associative, $(H^p\ J\ H^q) = 0$.

If H is non-singular (det H ≠ 0), we can define the inverse H^{-1} so that we have $H \cdot H^{-1} = I$ and $H^2 \cdot H^{-1} = H$ is satisfied by

$$H^{-1} = (\mathrm{Det}\, H)^{-1}\, H \times H \tag{3b.103}$$

where

$$H \times H \equiv H^2 - H\,\mathrm{Tr}\,H - \frac{1}{2} I\,\mathrm{Tr}\,(H^2 - H\,\mathrm{Tr}\,H) \tag{3b.104}$$

denotes the Freudenthal product and I, the unit matrix.

Then from Eq. (3b.102), we obtain

$$(H, J, H^{-1}) = 0 \ . \tag{3b.105}$$

Observe that (J x J) is the matrix of the minors as we find

$$J \times J = \begin{pmatrix} \beta\gamma - a\bar{a} & \overline{b\bar{a}} - \gamma c & ca - \beta\bar{b} \\ ab - \gamma\bar{c} & \gamma\alpha - b\bar{b} & \overline{c\bar{b}} - \alpha a \\ \overline{ac} - \beta b & bc - \alpha\bar{a} & \alpha\beta - c\bar{c} \end{pmatrix} \tag{3b.106}$$

and

$$(J \times J) \cdot J = I \, \mathrm{Det}\, J \,. \tag{3b.107}$$

We have the identity

$$(H_1 , J , H_2) = -\frac{1}{4}[\,[H_1, H_2] , J\,] + R \tag{3b.108}$$

with R vanishing for special Jordan algebras of hermitian matrices over **R**, **C** or **H**. Explicitly,

$$R = \frac{1}{4}[\, J, H_1, H_2 \,] - \frac{1}{4}[\, J , H_2 , H_1 \,] + \frac{1}{4}[\, H_1 , H_2 , J \,] - \frac{1}{4}[\, H_2 , H_1 , J \,]$$

$$+ \frac{1}{4}[\, H_1 , J , H_2 \,] - \frac{1}{4}[\, H_2 , J , H_1 \,] \tag{3b.109}$$

where we have defined the bracket

$$[\, A, B, C \,] = (A B) C - A (B C) \,, \tag{3b.110}$$

the product here refers to ordinary matrix multiplication.

For special Jordan algebras, the derivation algebra is given by the commutator

$$D_k J = i \, [\, K , J \,] \,. \tag{3b.111}$$

K is hermitian, traceless and obeys the Lie algebra of SO(3), SU(3) and Sp(6) respectively for J^R, J^C and J^H. This derivation can be written as an associator by selecting two hermitian matrices H_1 and H_2 such that $K = \frac{i}{4}\,[\, H_1 , H_2 \,]$.

The derivation then reads

$$D_{H_1H_2} J = (H_1 , J , H_2), \qquad \mathrm{Tr}\, H_1 = \mathrm{Tr}\, H_2 = 0 \ . \tag{3b.112}$$

This formula is also valid in the octonionic case. Due to the extra R term, the derivation depends not only on the combination $[H_1, H_2]$ but also on H_1 and H_2 separately. The commutator $[H_1, H_2]$ depends on 38 parameters since it is a traceless anti-hermitian matrix, having two traceless octonionic matrices in the diagonal and three off-diagonal octonions. The R term where only the associators enter, depends on the 14 parameters of G_2. So the derivations (3b.112) involve altogether 52 parameters, those of the pair (H_1 , H_2). These derivations can be shown to form the Lie algebras of F_4.

The integrated form of Eq. (3b.112) is

$$E_{H_1H_2}J = J + (H_1 , J , H_2) + \frac{1}{2\,!} (H_1, (H_1 , J , H_2) , H_2) + \dots \ . \tag{3b.113}$$

For the special Jordan algebras, we get

$$E_{H_1H_2}J = \exp(iK)\, J \exp(-iK) \tag{3b.114}$$

with $K = \frac{i}{4} [H_1 , H_2]$.

The traceless Jordan matrix (3b.95) with $\alpha + \beta + \gamma = 0$ is now a 26-dimensional irreducible representation of F_4.

Next, we introduce another symmetric algebra which gives closure to traceless hermitian matrices. Take the wedge product

$$J_1 \vee J_2 = J_2 \vee J_1 = J_1 \cdot J_2 - \frac{1}{3} I \,\mathrm{Re}\, (J_1 \cdot J_2) \ . \tag{3b.115}$$

Apart from a factor, it is the product of Michel and Radicati [257] . The wedge associator does not satisfy the second relation in (3b.102); so the product is not power associative. Nevertheless, the four algebras SO(3), SU(3), Sp(6) and F_4 are derivations (associated with automorphism groups) of the wedge algebra of traceless hermitian (3 x 3) matrices corresponding to the four Hurwitz algebras.

Similarly, we can abstract from the power associative, alternative algebras an antisymmetrical algebra of traceless Hurwitz numbers. In the octonionic case, we may define the wedge product

$$e_\alpha \Lambda e_\beta = -e_\beta \Lambda e_\alpha = e_\alpha e_\beta + \delta_{\alpha\beta} = \psi_{\alpha\beta\gamma} e_\gamma , \tag{3b.116}$$

with the $\psi_{\alpha\beta\gamma}$'s being the standard octonionic structure constants. This algebra, lacking the unit element, is *not* power associative. However, it still admits G_2 as an automorphism group; the units e_α form a 7-dimensional representation of G_2. Following the nomenclature of Tits, such a wedge algebra (3b.116) and the traceless part of the alternative algebra are denoted by J^0 and A^0 respectively .

For quaternions, A^0 is defined by

$$e_i \Lambda e_j = - e_j \Lambda e_i = e_i e_j + \delta_{ij} = \varepsilon_{ijk} e_k \qquad (i, j, k = 1, 2, 3) \ . \tag{3b.117}$$

It is simply the familiar vector multiplication for 3-vectors; so A^0 is trivial for the real and complex numbers **R** and **C**. We take next the direct product of J^0 and A^0 (with units ε_α) . Again following Tits, we define the elements of the algebra by

$$L (A^0 , J^0) = \mathrm{Der}\, A^0 + A^0 \otimes J^0 + \mathrm{Der}\, J^0 \ , \tag{3b.118}$$

where Der is the derivation algebra associated with the automorphism group. For an octonionic A^0 the 14 elements of Der A^0 are given by $D_{\alpha\beta}$ in

$$D_{\alpha\beta}\, \omega = \frac{3}{4} \left[\ \varepsilon_\alpha \, \varepsilon_\beta \, \omega \right] - \frac{1}{4} \left[\left[\varepsilon_\alpha \, \varepsilon_\beta \right] \ \omega \right] , \tag{3b.119}$$

since out of the 21 $D_{\alpha\beta}$ only 14 are independent.

For an octonionic J^0, Der J^0 corresponds to

$$G(H_1 , H_2)\, J^0 = (H_1 , J^0 , H_2) \ ; \tag{3b.120}$$

it is the 52-parameter F_4 algebra. In the non-octonionic J^0 cases, Der J^0 only depends on the traceless matrix $K = \frac{i}{4}\ [H_1 , H_2]$. The elements of $A^0 \otimes J^0$ have the form of $\varepsilon_\alpha J^0_\alpha$ where J^0_α are seven traceless Jordan matrices associated with one of the four Hurwitz algebras. Der A^0 and Der J^0 are already Lie algebras for which

$$\mathrm{Der}\, A^0_1 \ \Lambda \ \mathrm{Der}\, A^0_2 = \left[\mathrm{Der}\, A^0_1 , \mathrm{Der}\, A^0_2 \right] \tag{3b.121}$$

and similarly for Der J^0. We now define a skew symmetric algebra L for the elements of $A^0 \otimes J^0$ by way of the following product

$$\varepsilon_\alpha J^0_\alpha \wedge \varepsilon_\beta J^0_\beta = D_{\alpha\beta} + G(J^0_\alpha, J^0_\beta) + (\varepsilon_\alpha \wedge \varepsilon_\beta)(J^0_\alpha \vee J^0_\beta) \, . \tag{3b.122}$$

Thus the elements of L close under this antisymmetric algebra:

$$L_1 \wedge L_2 = \text{Der}\, A^0_1 \wedge \text{Der}\, A^0_2 + D_{\alpha\beta} + (\varepsilon_\alpha \wedge \varepsilon_\beta)(J^0{}_\alpha \vee J^0{}_\beta)$$

$$+ \text{ Der}\, J^0{}_1 \wedge \text{Der}\, J^0{}_2 + G(J^0{}_\alpha, J^0{}_\beta) \, . \tag{3b.123}$$

That we do have a Lie algebra can be checked, by proving the Jacobi identity for its elements.

By taking the traceless products for **R**, **C**, **H** and **Ω** for A^0 and the wedge products for J^R_0, J^C_0, J^H_0 and J^Ω_0, we obtain the 10 Lie algebras of the following symmetrical (along the diagonal) Magic Square of Freudenthal and Tits:

A	Aut A_0 \ J^A_0	J^R_0	J^C_0	J^H_0	J^Ω_0
R	I	SO(3)	SU(3)	Sp(6)	F_4
C	I	SU(3)	SU(3)xSU(3)	SU(6)	E_6
H	SU(2)	Sp(6)	SU(6)	SO(12)	E_7
Ω	G_2	F_4	E_6	E_7	E_8

Table I : Freudenthal-Tits Magic Square

A	$J_N(\mathbf{R})$	$J_N(\mathbf{C})$	$J_N(\mathbf{H})$
R	SO(N)	SU(N)	Sp(2N)
C	SU(N)	SU(N)XSU(N)	SU(2N)
H	Sp(2N)	SU(2N)	SO(4N)

Table II : Magic Square of classical groups

The elements $A^0 \otimes J^{\,0}$ of L belong to the cosets L / (Der A^0xDer $J^{\,0}$). So in the case of exceptional groups, the Tits' construction corresponds to the following decompositions

$$F_4 \supset G_2 \times SO(3), E_6 \supset G_2 \times SU(3) \ ,$$

$$E_7 \supset G_2 \times Sp(6), E_8 \supset G_2 \times F_4 \ . \qquad (3b.124)$$

Note here that the corresponding subgroups are not maximal. We lose one unit of rank for F_4 and two units for E_6, E_7 and E_8. So the last row of the Magic Square is expressed in terms of the Aut A^0 column while its first row represents Aut $J^{\,0}$.

3.b.2.3. The color-flavor construction of the exceptional groups

With applications to particle physics in mind, we move on to a "color-flavor construction" of the exceptional groups. By this procedure, a reduction with respect to maximal subgroups results if the last row of Freudenthal's Magic Square is expressed in terms of the second row and suitable subgroups of Aut A^0.

We begin by considering the subgroup of G_2 which leaves invariant one octonionic unit, e.g. e_7. Clearly it is SU(3), a maximal subgroup. What then is the algebra with SU(3) as its automorphism group? The answer was given by Günaydin [263]. With the six units e_1, ..., e_6 (or the corresponding split units u_i and u_i*) we can define a non-alternative, non-power associative algebra by the following Malcev algebra [264]

$$e_a * e_b = -e_b * e_a = \frac{1}{2}[e_a \cdot e_b, e_7] = e_c\, \phi_{cab7} \ , \tag{3b.125}$$

with ϕ_{abcd} given by Eq. (3a.25) and the indices a, b, c range from 1 to 6.

For the split units we have the skew symmetric multiplication

$$u_i * u_j = \frac{1}{2i}\left[\, u_i u_j \,, u_0 - u_0^* \right] = \varepsilon_{ijk}\, u_k^* \ , \tag{3b.126}$$

$$u_i * u_j^* = \frac{1}{2i}\left[\, u_i u_j^* \,, u_0 - u_0^* \right] = 0 \ . \tag{3b.127}$$

Since u_i and u_j^* are the 3 and $\overline{3}$ representations of SU(3), the SU(3) invariance of the algebra is manifest; the derivation (Malcev) algebra of this algebra of colors is the Lie algebra of SU(3). Also the Malcev algebra for the two quaternion units e_1 and e_3 can be defined by $[\,[\,e_1, e_2\,]\, e_3\,] = e_1 * e_2 = 0$ with automorphism group U(1).

We now turn to the Jordan matrices. By polarizing the Freudenthal product (3b.104), we get the *symmetrical* Freudenthal product

$$F_1 \times F_2 = F_1 \cdot F_2 - \frac{1}{2} F_1 \,\mathrm{Tr}\, F_2 - \frac{1}{2} F_2 \,\mathrm{Tr}\, F_1 - \frac{1}{2} I \left\{ \mathrm{Tr}\left(F_1 \cdot F_2\right) - (\mathrm{Tr}\, F_1)\, (\mathrm{Tr}\, F_2) \right\} \ . \tag{3b.128}$$

Consider the 6-dimensional SU(3) representation which is a (3 x 3) complex symmetric matrix. If the 3-dimensional representation q transforms as

$$q' = U q \quad , \qquad q'^T = q^T U^T \ , \tag{3b.129}$$

then the 6-representation F transforms like qq^T, so that

$$F' = U F U^T \qquad (F = F^T) \ . \tag{3b.130}$$

$F^\dagger = F^*$ transforms like a $\overline{6}$:

$$F'^\dagger = U^* F^\dagger U^\dagger \ . \tag{3b.131}$$

Since U is unitary we also have $F'^{-1} = U^* F^\dagger U^\dagger$. Hence F^{-1} transforms like a $\overline{6}$. With Det F invariant, we find from Eq. (3b.103) that

$$F \times F = F^{-1} \,\mathrm{Det}\, F \qquad (3b.132)$$

also transforms like a $\bar{6}$. Thus

$$F' \times F' = U^{*} \,(F \times F)\, U^{\dagger} \; . \qquad (3b.133)$$

With

$$F_1 \times F_2 = \frac{1}{2}\{ (F_1 + F_2) \times (F_1 + F_2) - F_1 \times F_1 - F_2 \times F_2 \} \; , \qquad (3b.134)$$

it follows that $F_1 \times F_2$ also transforms like a $\bar{6}$. We conclude that, just as the algebra of the octet representation is the wedge product (i.e. Gell-Mann's d-product), the algebra relevant to the 6-representation of SU(3) is the Freudenthal product corresponding to the mapping $\left(6 \times 6 \;\rightarrow \bar{6} \right)$.

At this juncture, we note that this Freudenthal product also holds for 1) the 9-representation (3, 3) of SU(3) x SU(3), a general complex (3x3) matrix, 2) the 15-dimensional representation of SU(6), a 3 x 3 symmetric complex quaternionic matrix, hermitian w.r.t. H-conjugation and 3) the 27-dimensional representation of E_6, a (3x3) complex octonionic matrix, hermitian w.r.t. $\boldsymbol{\Omega}$-conjugation. The Freudenthal product is therefore valid for the (3 x 3) complex representations of the groups in the second column of the Magic Square. It describes the mappings ($6 \times 6 \rightarrow \bar{6}$, $9 \times 9 \rightarrow \bar{9}$, $15 \times 15 \rightarrow \overline{15}$, $27 \times 27 \rightarrow \overline{27}$) for SU(3), SU(3) x SU(3), SU(6) and E_6 respectively. So these groups are the automorphism groups of the corresponding Freudenthal algebras. To multiply invariantly F_1 and $F_2{}^{\dagger}$, from Eqs. (3b.130) and (3b.131) we find

$$\frac{1}{2}\,\mathrm{Tr}\, F_1' \cdot \; F_2'{}^{\dagger} = \frac{1}{2}\,\mathrm{Tr}\, U\, F_1 \cdot \; F_2^{\dagger}\, U^{\dagger} = \frac{1}{2}\,\mathrm{Tr}\, F_1 \cdot \; F_2^{\dagger} \; . \qquad (3b.135)$$

In the case of SU(3) x SU(3)

$$F_1' = U\, F_1\, V^{\dagger} \;\; , \;\; F_2' = U\, F_2\, V^{\dagger} \;\; , \qquad (3b.136)$$

$$F_1'{}^{\dagger} = V\, F_1^{\dagger}\, U^{\dagger} \;\; , \;\; F_2'{}^{\dagger} = V\, F_2^{\dagger}\, U^{\dagger} \;\; , \qquad (3b.137)$$

so that again

$$\left(F_1'^{\dagger}, F_2'\right) = \frac{1}{2}\,\mathrm{Tr}\,(F_1' F_2') = (F_2^{\dagger}, F_2)\ . \tag{3b.138}$$

Indeed, these identities extend to the quaternionic and octonionic cases [244].

We therefore see that Eq. (3b.138) defines a scalar product invariant under the group of the second column of the Magic Square.

Next we consider the 2-dimensional quaternionic Malcev algebra M^H and its 6-dimensional octonionic counterpart M^{Ω} with their corresponding automorphism groups U(1) and SU(3). Simultaneously we also consider the Freudenthal algebras of complex matrices of the second column. We denote these matrices by F^{CR}, F^{CC}, F^{CH} and $F^{C\Omega}$, of dimensions 6, 9, 15 and 27, respectively.

We introduce the elements of a Lie algebra defined by a Malcev algebra and a Freudenthal algebra

$$L\,(M\,, F) = \mathrm{Der}\,M + M\ \otimes F + \mathrm{Der}\,F\ . \tag{3b.139}$$

The Lie product of the derivations algebras was previously defined. An element of M⊗F is

$$M^H \otimes F = u_1\,F^i + u_1^*\,F^{\dagger} \tag{3b.140}$$

in the quaternionic case and

$$M^{\Omega} \otimes F = u_i\,F^i + u_i^*\,F^{i\dagger} \tag{3b.141}$$

in the octonionic case.

The antisymmetric product for the elements M ⊗ F is defined as

$$(M\otimes F)_1 \wedge (M\otimes F)_2 = D\,(M_1\,, M_2) + u_i * u_j\,F_1^i \times F_2^j$$

$$+ u_i^* * u_j^*\,F_1^{i\,\dagger} \times F_2^{j\dagger} + D\,(F_1, F_2)\ . \tag{3b.142}$$

Here $D(M_1, M_2)$ is associated with the mappings $3 \times \bar{3} \rightarrow 8$ for SU(3) and $D(F_1, F_2)$ corresponds to the mappings of the second column of the Magic Square, namely

$$6 \times \bar{6} \to 8 \quad \text{for } SU(3)\,, \tag{3b.143}$$

$$(3,3) \times (\bar{3}, \bar{3}) \to (8, 1) + (1, 8) \quad \text{for } SU(3) \times SU(3)\,, \tag{3b.144}$$

$$15 \times \overline{15} \to 35 \quad \text{for } SU(6)\,, \tag{3b.145}$$

$$27 \times \overline{27} \to 78 \quad \text{for } E_6\,. \tag{3b.146}$$

For SU(3) we have

$$D(F_1, F_2)\, F = \frac{1}{2}(F_1 F_2^\dagger - F_2 F_1^\dagger)\, F + \frac{1}{2} F\,(F_2^\dagger F_1 - F_1^\dagger F_2) \tag{3b.147}$$

$$= \frac{1}{2}(F_1 F_2^\dagger F + F F_2^\dagger F_1) - \frac{1}{2}(F_2 F_1^\dagger F + F F_1^\dagger F_2)\,. \tag{3b.148}$$

This formula also holds for SU(3)xSU(3) with the transformation law (3b.136). In fact Eq.(3b.148) can be rewritten by way of the triple Jordan product:

$$\{J\,F\,K\} = (J \cdot F) \cdot K + J \cdot (F \cdot K) - (J \cdot K) \cdot F\,. \tag{3b.149}$$

In the associative case the simpler form is $\{J\,F\,K\} = \frac{1}{2}(J F K + K F J)$. This product is symmetric in J and K. It defines an isotope Jordan algebra w.r.t. the element F held fixed [244]. We can now reexpress Eq. (3b.148) by means of the triple Jordan product. It is

$$D(F_1, F_2)\, F = \{F_1\, F_2^\dagger\, F\} - \{F_2\, F_1^\dagger\, F\}\,, \tag{3b.150}$$

which is clearly antisymmetric in F_1 and F_2. Since Eq. (3b.149) also holds for the octonionic case, Eq. (3b.150) is well defined for all four Freudenthal algebras in all cases. Taking $M = M^{\Omega}$, we get the Lie algebras of the 4th column of the Magic Square in Table I. They are F_4, E_6, E_7 and E_8 by this construction.

The coset spaces $M \otimes F$ correspond to the decomposition of the exceptional groups w.r.t. the following maximal subgroups

$$F_4 \supset SU(3) \times SU(3)\,,$$

$$E_6 \supset (SU(3) \times SU(3)) \times SU(3) \ ,$$

$$E_7 \supset SU(6) \times SU(3) \ ,$$

$$E_8 \supset E_6 \times SU(3) \ .$$

Here the second group is the automorphism group of the octonionic Malcev algebra and the first group is the automorphism group of the Freudenthal algebra, including the triple Jordan product. This first group could be taken as the flavor group, it occurs in the second column of the Magic Square.

Taking for the Malcev algebra the quaternionic M^H with automorphism group U(1), the same construction then gives the third column of the Magic Square, corresponding to the decompositions

$$Sp(6) \supset SU(3) \times U(1) \ ,$$

$$SU(6) \supset (SU(3) \times SU(3)) \times U(1) \ ,$$

$$SO(12) \supset SU(6) \times U(1) \ ,$$

$$E_7 \supset E_6 \times U(1) \ .$$

In the above fashion we have generated the groups of the Magic Square by combining Malcev algebras with all possible compatible flavor groups. Next, we look at the exceptional groups F_4 and E_6 in more details.

3.b.2.4. The group F_4

Consider a special 26-dimensional representation j_0 which is purely diagonal and traceless:

$$j_0 = \lambda_8 \alpha_0 + \lambda_3 \beta_0 \qquad (3b.151)$$

where α_0 and β_0 can be expressed in terms of the F_4 invariants C_2 and C_3:

$$C_0 = 0 \,, C_2 = \alpha_0{}^2 + \beta_0{}^2 \ , C_3 = \frac{1}{3}\,\mathrm{Tr}\left(j_0{}^3\right) = -\frac{2}{3\sqrt{3}}\,\alpha_0{}^3 + \frac{2}{\sqrt{3}}\,\alpha_0\,\beta_0{}^2 \ . \tag{3b.152}$$

The action of F_4 on j_0 reads

$$J = T\,(\,F_4)\,j_0 \ = T\,(\,F_4\,/\,SO(8)\,)\,T\,(\,SO(8)\,)\,j_0 \ . \tag{3b.153}$$

SO(8), the norm group of $\mathbf{\Omega}$, leaves invariant the diagonal elements of J so that

$$J \ = T\,(\,F_4\,/\,SO(8)\,)\,j_0 \ . \tag{3b.154}$$

Consequently, J depends on the 24 parameters of F_4 / SO(8) and the two invariants of j_0. Conversely, any J depending on 26 parameters can be diagonalized by a F_4 / SO(8) transformation. It follows that j_0, as given by Eq. (3b.151) with Eq. (3b.152), is the canonical form of a traceless Jordan matrix J.

A special Jordan matrix will be given by the invariant relation

$$J \,v\, J = k\,J \ . \tag{3b.155}$$

Using the canonical form and the formulae

$$\lambda_8 \,v\, \lambda_8 = -\frac{1}{\sqrt{3}}\,\lambda_8 \ \ , \ \ \lambda_3 \,v\, \lambda_3 = \frac{1}{\sqrt{3}}\,\lambda_8 \ \ , \ \ \lambda_3 \,v\, \lambda_8 = \frac{1}{\sqrt{3}}\,\lambda_3 \ , \tag{3b.156}$$

we find

$$j_0 \,v\, j_0 = \frac{1}{\sqrt{3}}\left[(\,\beta_0{}^2 - \alpha_0{}^2)\,\lambda_8 + 2\alpha_0\,\beta_0\,\lambda_3\right] = \ k\,(\alpha_0\,\lambda_8 + \beta_0\,\lambda_3) \tag{3b.157}$$

or

$$\beta_0 = 0 \ , \ \beta_0 = \sqrt{3}\,\alpha_0 \ \text{ or } \ \beta_0 = -\sqrt{3}\,\alpha_0 \ . \tag{3b.158}$$

Apart from a constant factor, they correspond to the I, U or V hypercharges.

SO(8) is the subgroup of F_4 leaving j_0 invariant. The subgroup leaving λ_8 (or the corresponding U and V hypercharges) invariant is SO(9). Thus it follows that, in a F_4 gauge theory, a 26-dimensional Higgs field will lead to a spontaneous breaking of F_4 which leaves the SO(9) group invariant with 36 associates massless bosons. The 16 bosons corresponding to the coset space F_4 / SO(9) will acquire mass. This space with stability group SO(9) is the Moufang plane, parametrized by two octonions. Indeed the 26-dimensional representation of F_4 decomposes under SO(9) as 1 + 9 + 16. There the SO(9) singlet is the coefficient of λ_8, 16 is the pair of octonions (rays a and $\bar{b}$) forming a spinor representation of SO(9) and the coefficient of λ_8 combines to form its 9-dimensional vector representation. This example is analogous to the spontaneous breaking of the SU(3) octet to SU(2) x U(1). The latter is the stability group of an element (say along λ_8) satisfying the Michel-Radicati equation (3b.155).

In the present case, the Higgs potential is

$$V(J) = -\frac{\mu^2}{2} C_2 + \frac{\lambda}{4} C_2{}^2 + \frac{k}{3} C_3 \ , \tag{3b.159}$$

C_2 and C_3 are F_4 invariants. The extremum of V(J) w.r.t. J has a solution J = 0. The other solution then satisfies an invariant equation of the second degree, which can only be of the form (3b.155). We find

$$\frac{\partial V}{\partial \alpha_0} = -\mu^2 \alpha_0 + \lambda\, \alpha_0 (\alpha_0^2 + \beta_0{}^2) - \frac{2\,k}{3\sqrt{3}} (\alpha_0^2 - \beta_0{}^2) = 0 \ , \tag{3b.160}$$

$$\frac{\partial V}{\partial \beta_0} = -\mu^2 \beta_0 + \lambda\, \beta_0 (\alpha_0^2 + \beta_0{}^2) + \frac{4\,k}{3\sqrt{3}} \alpha_0 \beta_0 = 0 \ . \tag{3b.161}$$

They are solved by

$$\beta_0 = 0 \ , \alpha_0 = \frac{1}{3\sqrt{3}} \frac{k}{\lambda} \left\{ 1 \pm \left(1 + 27 \frac{\mu^2 \lambda}{k^2} \right)^{\frac{1}{2}} \right\} \tag{3b.162}$$

α_0 takes a value making the matrix of the second derivatives positive definite. If k = 0 we have $\alpha_0 = \frac{\mu}{\sqrt{\lambda}}$. In either case, j_0 and hence J, satisfies the Michel-Radicati equation (3b.55).

We now turn to the infinitesimal and finite F_4 transformations for the 26-dimensional representation. We have already derived the infinitesimal form (3b.112), so that

$$\delta J = (H_1 , J , H_2) \ . \tag{3b.163}$$

The finite transformation is given by the infinite series of multiple associators (3b.113). To see the underlying SU(3) structure, we consider special cases of Eq. (3b.163). We shall express the transformation in terms of the 7 octonionic units e_α and the real Gell-Mann matrices $\lambda_1, \lambda_3, \lambda_4, \lambda_6, \lambda_8, i\lambda_2, i\lambda_5, i\lambda_7$. We next write the 3x8 = 24 transformations of the coset $\frac{F_4}{SO(8)}$ by taking $H_1 = \lambda_3$ or λ_8:

$$\delta_1 J = (\lambda_3 , J , \lambda_1 \alpha_2 + i\lambda_2 \alpha_{1a} e_a) = -\frac{1}{2} [i \alpha_2 \lambda_2 + \alpha_{1a} e_a \lambda_1 , J] , \tag{3b.164}$$

$$\delta_2 J = (\lambda_8 , J , \lambda_4 \alpha_5 + i\lambda_5 \alpha_{4a} e_a) = -\frac{\sqrt{3}}{4} [i \alpha_5 \lambda_5 + \alpha_{4a} e_a \lambda_4 , J] , \tag{3b.165}$$

$$\delta_3 J = (\lambda_8 , J , \lambda_6 \alpha_7 + i\lambda_7 \alpha_{6a} e_a) = -\frac{\sqrt{3}}{4} [i \alpha_7 \lambda_7 + \alpha_6 e_a \lambda_6 , J] . \tag{3b.166}$$

The subgroup SO(8) of F_4 leaves the diagonal elements of J invariant. It consists of 28 transformations decomposable into 14 G_2's and 14 coset space $\frac{SO(8)}{G_2}$'s . The latter's infinitesimal forms can be gotten by taking $H_1 = \lambda_1$ and $H_1 = \frac{\lambda_1 + \sqrt{2}\,\lambda_4}{\sqrt{3}}$ respectively, with H_2 containing in each case a traceless octonion, so that $\frac{SO(8)}{G_2}$ is decomposed into two sets of seven parameter transformations. We find

$$\delta_1' J = (\lambda_1, J , i\lambda_2 \alpha_{3a} e_a) = \frac{1}{2} [\alpha_{3a} e_a \lambda_3 , J] , \tag{3b.167}$$

$$\delta_2' J = (\frac{\lambda_6 + \sqrt{2}\,\lambda_4}{\sqrt{3}}, J , \frac{-i\lambda_7 + i\sqrt{2}\lambda_5}{\sqrt{3}} \alpha_{8a} e_a) = \frac{1}{4\sqrt{3}} [\alpha_{8a} e_a \lambda_8 , J] . \tag{3b.168}$$

For the derivations of (3b.168) we made use of the identity

$$\left[\frac{\lambda_6 + \sqrt{2}\,\lambda_4}{\sqrt{3}} , \frac{-i\lambda_7 + i\sqrt{2}\lambda_5}{\sqrt{3}} \right] = \frac{2i\,\lambda_8}{\sqrt{3}} . \tag{3b.169}$$

The 38 transformations (3b.167)-(3b.168) of F_4 are so chosen that H_1 contains no octonions. Then the associators in Eq. (3b.110) w.r.t. the ordinary matrix product vanish. The Jordan associator is expressed through a double commutator with R = 0 in Eq. (3b.108).

However, the transformations (3b.167)-(3b.168) do not form a Lie algebra. We must add to them the remaining 14 transformations making up a subgroup of SO(7) not contained in Eq. (3b.168). In this case, both H_1 and H_2 in the associator (3b.168) have to be octonionic, and the term R in Eq. (3b.110) contributes. Thus for $H_1 = i\lambda_7 e_1$, $H_2 = i \lambda_7 e_2$, the associator (3b.163) consists of two parts: a transformation belonging to $\frac{SO(7)}{G_2}$ of the form (3b.168) with α_{83} non-zero and a transformation of the form (3b.75) with $\alpha = e_1$, $\beta = e_2$, $x = J$. All the remaining 14 transformations of G_2 result in the infinitesimal variation

$$\delta_3' J = \frac{1}{4}\eta_a\xi_b (3 [e_a , e_b , J] - [[e_a , e_b] , J]) \ . \tag{3b.170}$$

F_4 consists of Eqs. (3b.167)-(3b.168) together with Eq. (3b.170). The SU(3) group corresponds to the parameters α_2, α_5, α_7, α_{17}, α_{47}, α_{67}, α_{37} and α_{87}.

Next, we give an Euler angle decomposition of the finite F_4 transformations. Let

$$A' = \theta (i\lambda_2 \cos\varphi + \lambda_1 \, \mathbf{e} \cdot \boldsymbol{\alpha} \sin\varphi) = \theta N \tag{3b.171}$$

where $a' = \theta (\cos\varphi + \mathbf{e}\cdot \boldsymbol{\alpha} \sin\varphi) = \theta n$, $|a'| = \theta$, a' is an octonion with $\left| \mathbf{e} \cdot \boldsymbol{\alpha} \right|^2 = -1$ and $n \bar{n} = 1$. We get

$$A' = \begin{pmatrix} 0 & a' & 0 \\ -\bar{a}' & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} , \quad N = \begin{pmatrix} 0 & n & 0 \\ -\bar{n} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} . \tag{3b.172}$$

Then $N^2 = -\lambda_1{}^2 = -\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $N^3 = -N$,

$$\exp A' = \exp (\theta N) = I + N^2(\cos \theta - 1) + N \sin\theta \ . \tag{3b.173}$$

If we set $a = n \sin \theta = a' \frac{\sin|a'|}{|a'|}$, $A = N \sin\theta$, then

$$A^2 = N^2 \sin^2 \theta = \lambda_1{}^2(\cos^2 \theta - 1) \ , \tag{3b.174}$$

$$\exp A' = I + A - (1 + \cos\theta)^{-1} A^2 \tag{3b.175}$$

where $A^4 = A^2(\cos 2\theta - 1)$. Thus $\cos^2\theta = 1 - a\bar{a}$, and (3b.175) reads

$$\exp A' = \begin{bmatrix} \sqrt{1-a\bar{a}} & a & 0 \\ -\bar{a} & \sqrt{1-a\bar{a}} & 0 \\ 0 & 0 & 1 \end{bmatrix} = U_3(a) \ , \tag{3b.176}$$

$U_3(A)$ is hermitian, $U_3(a)\, \overline{U}_3^T(a) = I$.

Since A^2 in Eq. (3b.175) is diagonal and scalar, we have

$$J' = \{U_3(a)\, J\}\, U_3(a)^{-1} = U(a) \left\{ J\, U_3(a)^{-1} \right\} . \tag{3b.177}$$

So we can write without ambiguity

$$J' = U(a)\, J\, U_3(a)^{-1} \ ; \tag{3b.178}$$

alternatively, it takes the form of Eq. (3b.113).

Similarly, we define

$$U_1(b') = \exp(i\, \lambda_4\, b_0' + \lambda_6\, \mathbf{e} \cdot \mathbf{b}') \ , \tag{3b.179}$$

$$b = b' \frac{\sin|b'|}{|b'|}$$

and

$$U_2(c') = \exp(i\, \lambda_5\, c_0' + \lambda_4\, \mathbf{e} \cdot \mathbf{c}') \ , \tag{3b.180}$$

$$c = c' \frac{\sin|c'|}{|c'|} \ .$$

So the finite $\frac{F_4}{SO(8)}$ transformation reads

$$T_{F_4/SO(8)}\, J = \left\{U_3(a_3) \left\{U_2(a_2) J\, U_1(a_1)^{-1}\right\} U_2(a_2)^{-1} \right\} U_3(a_3)^{-1} \ . \tag{3b.181}$$

Let

$$V(k) = \exp(\ \lambda_3\, \mathbf{e} \cdot \mathbf{k}) = \begin{bmatrix} k & 0 & 0 \\ 0 & \bar{k} & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad , \ (k\bar{k} = 1)\ , \tag{3b.182}$$

$$W(l\,) = \exp\,(\lambda_8 \mathbf{e} \cdot \boldsymbol{\lambda}) = \begin{bmatrix} l & 0 & 0 \\ 0 & l & 0 \\ 0 & 0 & \bar{l}^2 \end{bmatrix} \quad , \ (l\ \bar{l}\ = 1)\ , \tag{3b.183}$$

then from the infinitesimal transformations we verify that

$$T_{SO(8)\,/\,SO(7)}\, J = V\, J\, V^{-1} = (\ V\, J\ )\, V^{-1} = V\,(\ J\, V^{-1}) \tag{3b.184}$$

and

$$T_{SO(7)\,/\,G_2}\, J = W\, J\, W^{-1} = (\ W\, J\ )\, W^{-1} = W\,(\ J\, W^{-1})\ . \tag{3b.185}$$

For each transformation, we can check by way of the Moufang identities (3a.11)-(3a.13) that Tr (J), Tr $(J^{\,2})$ and Det (J) remain invariant, as they should be, under F_4 transformations. So (3b.181), (3b.184) and (3b.185) constitute the 38 transformations $\frac{F_4}{G_2}$. If $|a| = |b| = 1$, then the G_2 action reads

$$T_{G_2}\, J = W(ab)^{-1} \left\{W(b) \left\{W(a)\, J\, W(a)^{-1}\right\} W(b)^{-1}\right\} W(ab)\ . \tag{3b.186}$$

The reader may check that, under such a transformation, the three octonions in J transform respectively as in formulae (3b.72), (3b.73) and (3b.74), which all have the infinitesimal form (3b.75), characteristic of G_2.

We also note that the 36 transformations of SO(9) are given by the 28 SO(8) transformations (3b.186), (3b.185), (3b.184) and the 8 transformations $\frac{SO(9)}{SO(8)}$ of (3b.178) specified by $U_3(a)$. Then $U_1(a_1)$ and $U_2(a_2)$ together represent the 16-parameter transformations of $\frac{F_4}{SO(9)}$.

Instead of characterizing the 26-dimensional representation by the traceless hermitian matrix J, we can consider a hermitian matrix P with Tr P =1 since C_1 is an invariant. We take the special case of Det P = 0 and of the subdeterminant of P also vanishing. Then P has the form (3b.95) with

$$\alpha + \beta + \gamma = 1 \ , \tag{3b.187}$$

$$\alpha\beta\gamma - \alpha\, a\bar{a} - \beta\, b\bar{b} - \gamma\, c\bar{c} + t(a\, b\, c) = 0 \ . \tag{3b.188}$$

From Eq. (3b.107) it also follows that

$$\det P = \frac{1}{3}\,\mathrm{Tr}\,\{P\cdot(P \times P)\} = 0 \ . \tag{3b.189}$$

The vanishing of the subdeterminant gives

$$P \times P = 0 \ . \tag{3b.190}$$

From Eq. (3b.106) we find

$$\alpha\,\beta - c\,\bar{c} = 0 \quad , \quad \bar{b}\,\bar{a} - \gamma\, c = 0 \tag{3b.191}$$

plus cyclic permutations. Thus $c = \gamma^{-1}\,\bar{b}\,\bar{a}$. We also get

$$\alpha = \gamma^{-1}\, b\,\bar{b} \quad , \quad \beta = \gamma^{-1}\, a\,\bar{a} \ . \tag{3b.192}$$

Then Eq. (3b.187) is satisfied if

$$a\bar{a} + b\bar{b} + \gamma^2 - \gamma = 0 \tag{3b.193}$$

and P reads

$$P = \lambda\,\bar{\lambda}^T = \begin{pmatrix} \gamma^{-1}\bar{b}b & \gamma^{-1}\bar{b}\bar{a} & \bar{b} \\ \gamma^{-1}ab & \gamma^{-1}a\bar{a} & a \\ b & \bar{a} & \gamma \end{pmatrix} \tag{3b.194}$$

with

$$\lambda = \begin{pmatrix} \gamma^{-\frac{1}{2}}\bar{b} \\ \gamma^{-\frac{1}{2}}a \\ \gamma^{\frac{1}{2}} \end{pmatrix} . \tag{3b.195}$$

This theorem is similar to the factorization of a (3 x 3) matrix. In the SU(3) case, with a, b complex, λ would have represented, up to a phase, a 3-dimensional representation of SU(3). And since P transforms in this case as

$$P' = U P U^{\dagger} = U (\lambda \lambda^{\dagger}) U^{\dagger} \ , \tag{3b.196}$$

we would have obtained for the transformation law for λ, $\lambda' = U \lambda$, after invoking associativity, so that

$$U (\lambda \lambda^{\dagger}) U^{\dagger} = (U \lambda) (\lambda^{\dagger} U^{\dagger}) \ . \tag{3b.197}$$

However in the octonionic case, since (3b.197) no longer holds, we do not get a smaller representation of F_4 transforming linearly, even in the case where the (3 x 3) octonionic matrix factorizes as in (3b.194). We note that in (3b.195) we have singled out γ and c. By making cyclic transformations, we can also factorize P by singling out the pairs (α, a) and (β, b).

The matrices P with Tr P = 1, P x P = 0 are important in the geometric interpretation of F_4. The column λ represents the homogeneous coordinates of a point normalized to one, it is represented by a pair of octonions in the spherical version of the Moufang plane.

3.b.2.5. The group E_6

We start by writing the infinitesimal transformation on a complex octonionic (3x3) matrix F, hermitian w.r.t. octonionic conjugation

$$\overline{F}^{T} = F \quad , \ F^{\dagger} = \overline{F^{*}}^{T} = F^{*} \ . \tag{3b.198}$$

Thus F has 27 complex elements, it corresponds to the 27 representation of E_6. If the * operation denotes complex conjugation, F* corresponds to the 27* representation.

We have already found the infinitesimal transformation (3b.150), valid for the representation 6, 9, 15 and 27 of respectively SU(3), SU(3)xSU(3), SU(6) and E_6 in the second column of Freudenthal's Magic Square. It reads

$$\delta F = \frac{1}{2} \{ F_1 F_2^{\dagger} F \} - \frac{1}{2} \{ F_2 F_1^{\dagger} F \} \ . \tag{3b.199}$$

Here the curly bracket denotes the triple Jordan product defined in Eq. (3b.149). Let

$$F_1 = A_1 + i\,B_1 \;\;,\; F_2 = A_2 + i\,B_2 \tag{3b.200}$$

where A_i and B_i are real octonionic hermitian matrices. Then Eq. (3b.199) becomes

$$\delta F = \; i\,C \cdot F + \frac{1}{2}(\,F_2^{+}\,,F\,,F_1\,) - \frac{1}{2}\,(\,F_1^{+}\,,\;F\,,F_2\,) \tag{3b.201}$$

with

$$C = \frac{i}{2}\left(F_1{}^{\dagger} \cdot \; F_2\right) - \frac{i}{2}\left(F_2{}^{\dagger} \cdot \; F_1\right) = \; A_2 \cdot \; B_1 - A_1 \cdot \; B_2 \;. \tag{3b.202}$$

Tr C corresponds to a U(1) transformation commuting with E_6. F_1 and $F_2{}^{\dagger}$ are chosen so that Tr C = 0. Using Eq. (3b.200) we obtain

$$\delta F \; - \; i\,C \cdot F = \left(A_2\;,F\;,A_1\right) \; + \; \left(B_2\;,F\;,B_1\right)\;, \tag{3b.203}$$

which, according to Eq. (3b.120), belongs to F_4 .

Note that if F_1 and F_2 are hermitian and equal, ($F_1 = F_2 = K = K^{\dagger}$), then, in view of Eq. (3b.199), $\delta\,F = 0$. So out of the four J-matrices, A_1, A_2, B_1 and B_2, one is redundant. If we choose $A_1 = A_2 = K$, then one of the associators in Eq. (3b.203) drops out and Eq. (3b.199) reads

$$\delta\,F \; = \; (\;A\;,F\;,B\;) \; + i\,C \cdot F \;, \tag{3b.204}$$

where A, B, C being traceless real octonionic hermitian 3x3 matrices. The associator corresponds to F_4 with 52 parameters and C depends on 26 parameters. So we recover the 78 real parameters of E_6.

The 26 transformations associated with C correspond to the coset $\frac{E_6}{F_4}$, a D = 26 homogeneous space. It is seen as the octonionic generalization of the coset space $\frac{SU(3)\;x\;SU(3)}{SU(3)}$. In the latter case, the coset transformations in the finite form read

$$F' = \exp\!\left(\frac{i}{2}\,C\right) F \exp\!\left(\frac{i}{2}\,C\right)\;. \tag{3b.205}$$

The complex (3x3) matrix $\Gamma = \exp\,(\,\frac{i}{2}C\,)$ has unit determinant and Eq. (3b.205) can be written in terms of the Jordan product only. We find

$$F' = \Gamma F \Gamma = 2\left(\Gamma \cdot F\right) \Gamma - \Gamma^2 \cdot F = \left\{\Gamma F \Gamma\right\} \tag{3b.206}$$

by using the triple Jordan product. Now it can be proved that the expression

$$\mathrm{Det}\left\{\Gamma F \Gamma\right\} = \left(\mathrm{Det}\,\Gamma\right)^2 \mathrm{Det}\,F \tag{3b.207}$$

is also valid in the octonionic case. Hence, if Det $\Gamma = 1$ Eq. (3b.206) leaves Det F invariant, so do the F_4 transformations; the finite E_6 transformations can therefore be written as

$$T_{E_6} = \left\{\Gamma\,(T_{F_4} F)\,\Gamma\right\} \tag{3b.208}$$

and they do not change Det F. Here Γ is a complex hermitian octonionic matrix with unit determinant. (T_{F_4} F) is the finite transformation given by Eq. (3b.113) or by Eqs. (3b.182)-(3b.187).

If F transforms like the 27-dimensional representation of E_6, then $F^\dagger$ and F^{-1} transform like the $\overline{27}$ representation. Therefore F^{-1} DeF also transforms like a $\overline{27}$. But according to Eq. (3b.132), F x F = $G^\dagger$ where G transforms like a 27.

We can form the following invariants

$$I_1 = \frac{1}{2}\mathrm{Tr}\,(F^\dagger \cdot F)\ , \tag{3b.209}$$

$$I_1 + i\, I_3 = \mathrm{Det}\,F\ , \tag{3b.210}$$

$$I_4 = \frac{1}{2}\mathrm{Tr}\,(G^\dagger \cdot G) = \frac{1}{2}\mathrm{Tr}\left\{\left(F \times F\right)^\dagger \cdot F \times F\right\}. \tag{3b.211}$$

We next turn to the canonical form F_0 of the 27-representation. E_6 can be decomposed w.r.t. its SO(8) subgroup as

$$\left(\frac{E_6}{SO(8)}\right) SO(8) = E_6\ . \tag{3b.212}$$

The action of E_6 of F_0 is then given by

$$F = T_{E_6} F_0 = T_{E_6/SO(8)} \{T_{SO(8)}\} \quad . \tag{3b.213}$$

We choose F_0 to be invariant under SO(8); $T_{SO(8)}\, F_0 = F_0$. Hence F is diagonal since SO(8) acts only on the octonionic elements of F_0 and leaves its diagonal elements invariant. In general, F has 27 complex (54 real) elements. The coset space E_6 /SO(8) contains (78-28) = 50 parameters. If we then diagonalize F by the transformation $T^{-1}_{E_6/SO(8)}$, it still depends on 4 independent real elements. The latter are invariants of the group and can be expressed in terms of the above invariants I_i, i = 1, 2, 3, 4 . We can choose the diagonal elements of F to have a common phase, so that they depend on 4 real quantities. We then write

$$F_0 = e^{i\delta} \begin{pmatrix} \rho_0 & 0 & 0 \\ 0 & \sigma_0 & 0 \\ 0 & 0 & \tau_0 \end{pmatrix} \tag{3b.214}$$

where δ_0, ρ_0, σ_0, τ_0 are real and related to the invariants by

$$I_1 = \frac{1}{2} \mathrm{Tr}\,(\, F_0 \cdot F_0^\dagger\,) = \frac{1}{2}\,(\,\rho_0^2 + \sigma_0^2 + \tau_0^2\,)\ , \tag{3b.215}$$

$$I_2 + i\, I_3 = \mathrm{Det}\, F_0 = e^{3i\delta}\, \rho_0 \sigma_0 \tau_0 \ . \tag{3b.216}$$

To find I_4 we evaluate

$$F_0 \times F_0 = e^{2i\delta} \begin{pmatrix} \sigma_0\tau_0 & 0 & 0 \\ 0 & \rho_0\tau_0 & 0 \\ 0 & 0 & \rho_0\sigma_0 \end{pmatrix} \tag{3b.217}$$

and find

$$I_4 = \frac{1}{2} \mathrm{Tr}\, \left\{ (F_0 \cdot F_0)^\dagger \cdot F_0 \times F_0 \right\} = \frac{1}{2}\,(\,\rho_0^2\,\tau_0^2 + \sigma_0^2\tau_0^2 + \rho_0^2\,\sigma_0^2) \ . \tag{3b.218}$$

Conversely, δ, ρ_0, σ_0, τ_0 can be expressed as functions of the invariants I_i, i = 1, 2, 3, 4. Using the canonical form, we get

$$(\,F_0 \times F_0\,) \times (\,F_0 \times F_0\,) = e^{4i\delta} \begin{pmatrix} \rho_0{}^2\sigma_0\tau_0 & 0 & 0 \\ 0 & \rho_0\sigma_0{}^2\tau_0 & 0 \\ 0 & 0 & \rho_0\sigma_0\tau_0{}^2 \end{pmatrix}, \tag{3b.219}$$

$$= (\mathrm{Det}F_0)\, F_0 \ . \tag{3b.220}$$

Hence, by applying an arbitrary E_6 transformation, we obtain Springer's identity

$$(F \times F) \times (F \times F) = F \,\mathrm{Det}\, F \ . \tag{3b.221}$$

All the invariant equations for F can be derived as special cases of the latter identity. Let Det F = 0, we have the relation

$$(F \times F) \times (F \times F) = 0 \ . \tag{3b.222}$$

In this case I_2 and I_3 vanish. We have three possibilities

a) $F \times F = 0$; or $F_0 \times F_0 = 0$. (3b.223)

From Eq. (3b.217) we find $\sigma_0 \tau_0 = \rho_0 \tau_0 = \rho_0 \sigma_0 = 0$, so that two of the three invariants must vanish. Let $\sigma_0 = \tau_0 = 0$. Then

$$F_0 = \rho_0 e^{i\delta} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{3b.224}$$

so that

$$I_2 = I_3 = I_4 = 0, \text{ but } I_1 \neq 0 \ . \tag{3b.225}$$

b) $F \times F \neq 0$, or $F_0 \times F_0 \neq 0$. (3b.226)

Then

$$F_0 = e^{i\delta} \begin{pmatrix} 0 & 0 & 0 \\ 0 & \sigma_0 & 0 \\ 0 & 0 & \tau_0 \end{pmatrix} \tag{3b.227}$$

and

$$I_2 = I_3 = 0\ , \ I_1 \neq 0\ , \ I_4 \neq 0 \, . \tag{3b.228}$$

A special case arises if $\sigma_0 = \tau_0$, giving

$$I_1{}^2 - 2\, I_4 = 0 \ . \tag{3b.229}$$

Next we turn to the case of Det F $\neq 0$, so that $I_2 \neq 0, I_3 \neq 0$. The only possible invariant quadratic relation is

$$F \times F = k\, F^{\dagger}, \quad \text{or} \quad F_0 \times F_0 = k\, F_0{}^{\dagger} \tag{3b.230}$$

which gives the relations $\rho_0{}^2 = \sigma_0{}^2 = \tau_0{}^2$. The latter lead to

$$2\, I_1{}^2 - 3\, I_4 = 0 \tag{3b.231a}$$

and

$$k = \frac{3}{2}\,\frac{I_2 + i\, I_3}{I_1} \quad . \tag{3b.231b}$$

In this case, it follows that either

$$F_0 = a \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{3b.232}$$

or

$$F_0 = b \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{3b.233}$$

We are now set to find the subgroups of E_6 which leave the various canonical forms invariant. If

$$(\, F \times F\,) \times (\, F \times F\,) = C\, F \ , \tag{3b.234}$$

the canonical form is (3b.214) and is left invariant by SO(8). If F x F = 0, the canonical form is (3b.224) and is left invariant by SO(10) x SO(2).

If C = 0 in Eq. (3b.234) but F x F $\neq$ 0, then J_0 has the form (3b.227) and is left invariant by SO(8) x SO(2). When $\sigma_0 = \tau_0$ it is invariant under SO(9)x SO(2).

Finally, if F x F = k $F^{\dagger}$ and F_0 is proportional to the unit matrix as in Eq. (3b.242), F is invariant under F_4 while Eq. (3b.233) is invariant under SO(9).

We now consider a Higgs field F belonging to the 27-representation of E_6. The Higgs potential reads

$$V(F) = - \quad \alpha_1 I_1 + \alpha_2 I_2 + \alpha_3 I_3 + \alpha_4 I_4 + \beta I_1^2 \ . \tag{3b.235}$$

The value F_0 of F which extremizes V(F) will satisfy either $F_0 = 0$ or an equation quadratic in F_0. Hence it can only be of the form (3b.230) with either $k = 0$ or $k \neq 0$. So spontaneous symmetry breaking by way of a 27-dimensional Higgs field can break E_6 down to SO(10) x SO(2), SO(9) or F_4. And if there is no cubic invariant ($\alpha_2 = \alpha_3 = 0$), the only possibility is SO(10) x SO(2).

It is also possible to write trilinear invariant relations by means of the Jordan triple product. Using the canonical form F_0 (3b.214) of F we readily verify that

$$\left(F_0 \times F_0 \right) \times F_0^{\dagger} = \frac{1}{2} \mathrm{Tr} \left(F_0 \cdot F_0^{\dagger} \right) F - \frac{1}{2} \left\{ F_0 F_0^{\dagger} F_0 \right\} . \tag{3b.236}$$

It follows that if F is a 27-representation of E_6, it satisfies the identity

$$\frac{1}{2} \left\{ F F^{\dagger} F \right\} = - (F \times F) \times F^{\dagger} + \frac{1}{2} F \,\mathrm{Tr} \left(F \cdot F^{\dagger} \right) . \tag{3b.237}$$

By replacing F by M + F and separating the terms of the identity which transforms under the phase transformation

$$M \rightarrow e^{i\alpha} M \tag{3b.238}$$

with the phase 1, $e^{i\alpha}$, $e^{i2\alpha}$ and $e^{-i\alpha}$ we derive the new identities

$$\frac{1}{2} \left\{ M F^{\dagger} M \right\} = - \left(M \times M \right) \times F^{\dagger} + \frac{1}{2} M \,\mathrm{Tr} \left(M \cdot F^{\dagger} \right) \tag{3b.239}$$

and

$$\frac{1}{2} \left\{ M M^{\dagger} F \right\} = - M^{\dagger} \times \left(M \times F \right) + \frac{1}{4} M \left(\mathrm{Tr}\, M^{\dagger} F \right) + \frac{1}{4} F \left(\mathrm{Tr}\, M M^{\dagger} \right) . \tag{3b.240}$$

Replacing M by (M+N) in the last identity we find

$$\begin{aligned} \frac{1}{2} \left\{ M N^{\dagger} F \right\} - \frac{1}{2} \left\{ N M^{\dagger} F \right\} = {} & M^{\dagger} \times \left(N \times F \right) - N^{\dagger} \times \left(M \times F \right) + \frac{1}{4} M \left(\mathrm{Tr}\, N^{\dagger} F \right) \\ & - \frac{1}{4} N \left(\mathrm{Tr}\, M^{\dagger} F \right) + \frac{1}{4} F \,\mathrm{Tr} \left(M \cdot N^{\dagger} - N \cdot M^{\dagger} \right) . \end{aligned} \tag{3b.241}$$

This last equation can be used to give another form of the infinitesimal E_6 transformation δF in Eq. (3b.199) in terms of the Freudenthal product, while Eq. (3b.239) leads to a new form for the finite coset transformation E_6 / F_4.

Let us consider the case of Det $F \neq 0$ such that Eq. (3b.230) is satisfied. Then Eq. (3b.237) reads

$$\frac{1}{2}\left\{ F F^{\dagger} F \right\} = -k F^{\dagger} \times F^{\dagger} - I_1 F \ . \tag{3b.242}$$

By way of $F^{\dagger} \times F^{\dagger} = k^* F$, we find

$$\frac{1}{2}\left\{F F^{\dagger} F\right\} = -(I_1 + |k|^2) F \tag{3b.243}$$

so that we have the invariant trilinear relation

$$\left\{ F F^{\dagger} F \right\} = c^2 F \ . \tag{3b.244}$$

Then, such special directions F are left invariant by F_4 or SO(9). The canonical form of Eq. (3b.244) is

$$\left\{ F_0 F_0^{\dagger} F_0 \right\} = e^{i\delta} \begin{pmatrix} \rho_0^3 & 0 & 0 \\ 0 & \sigma_0^3 & 0 \\ 0 & 0 & \tau_0^3 \end{pmatrix} \tag{3b.245}$$

$$= c^2 e^{i\delta} \begin{pmatrix} \rho_0 & 0 & 0 \\ 0 & \sigma_0 & 0 \\ 0 & 0 & \tau_0 \end{pmatrix} \ . \tag{3b.246}$$

If Det $F \neq 0$, we find $\rho_0{}^2 = \sigma_0{}^2 = \tau_0{}^2 = c^2$.

If Det $F = 0$, one or two of ρ_0, σ_0, τ_0 can have zero eigenvalue while the remaining ones have the same absolute value given by c. Then we obtain the special cases (3b.225) and (2b.229) for F_0. The latter are associated with the subgroups SO(10) x SO(2) and SO(9) x SO(2) respectively. The case of $\rho_0 = 0$, $\sigma_0 = -\tau = c$ corresponds to SO(8) x SO(2). The above results summarize various possibilities for the spontaneous symmetry breaking due to a 27-dimensional Higgs field.

To see how the different solutions arise we begin from the potential (3b.235). We find (for Det F $\neq$ 0)

$$\frac{\partial I_1}{\partial \rho_i} = \rho_i \ , \quad \frac{\partial I_2}{\partial \rho_i} = \rho_i^{-1}\, I_2 \ , \quad \frac{\partial I_3}{\partial \rho_i} = \rho_i^{-1}\, I_3 \quad , \quad \frac{\partial I_4}{\partial \rho_i} = 2\, \rho_i\, I_1 - \rho_i^3 \tag{3b.247}$$

after setting $\rho_1 = \rho_0$, $\rho_2 = \sigma_0$, $\rho_3 = \tau_0$. We also have

$$\frac{\partial I_1}{\partial \delta_i} = \frac{\partial I_4}{\partial \delta_i} = 0 \quad , \quad \frac{\partial I_2}{\partial \delta_i} = -\, 3\, I_3 \quad , \quad \frac{\partial I_3}{\partial \delta_i} = -\, 3\, I_2 \ . \tag{3b.248}$$

Thus

$$\frac{\partial V}{\partial \rho_i} = -\, \alpha_i\, \rho_i + \rho_i^{-1}(\, \alpha_2\, I_3 + \alpha_3\, I_2\,) = 0 \quad , \tag{3b.249}$$

$$\frac{\partial V}{\partial \delta} = 3\, (\, -\, \alpha_2\, I_3 + \alpha_3\, I_2\,) = 0 \ . \tag{3b.250}$$

The last equation gives δ through

$$\frac{I_2}{I_3} = \tan\delta = \frac{\alpha_2}{\alpha_3} \, . \tag{3b.251}$$

Equation (3b.249) reads

$$\beta\, \rho_i^4 - (2\, \beta\, I_1 - \alpha_1)\, \rho_i^2 + \alpha_2\, I_2 + \alpha_3\, I_3 = 0 \quad . \tag{3b.252}$$

They can be solved for ρ_i^2 in terms of I_1, I_2 and I_3, giving

$$\rho_1^2 = \rho_2^2 = \rho_3^2 = g\, (\, \alpha_1\, , \alpha_2\, , \alpha_3\, , \beta\, , \, I_1\, , \, I_2\, , \, I_3\,) \tag{3b.253}$$

where the positive roots are chosen

$$I_1 = \frac{1}{2} \sum \rho_i^2 = \frac{3}{2}\, g \ , \tag{3b.254}$$

$$I_2^2 + I_3^2 = \rho_1^2\, \rho_2^2\, \rho_3^2 = g^3 \ , \tag{3b.255}$$

$$I_4 = \frac{1}{2} \left(\, \rho_2^2\, \rho_3^2 + \rho_3^2\, \rho_1^2 + \rho_1^2\, \rho_2^2 \right) = \frac{3}{2}\, g^2 \ . \tag{3b.256}$$

Thus, if Det F $\neq 0$ we must have

$$\frac{2}{3} I_1^2 - I_4 = 0 \quad , \quad |\,\mathrm{Det}\, F|^2 = I_2^2 + I_3^2 = \frac{8}{27} I_1^3 \ . \tag{3b.257}$$

It follows that, due to the connection between Eqs. (3b.230) and (3b.231a), we have

$$F \times F = k\, F^{\dagger} \tag{3b.258}$$

where $k = e^{i\delta} g$ using Eq. (3b.231b).

Moving on to the solutions with Det F = 0 ; then at least one of the ρ_i, say $\rho_3 = \tau_0 = 0$. Then $I_2 = I_3 = 0$ and

$$\frac{\partial I_1}{\partial \rho_0} = \rho_0 \quad , \quad \frac{\partial I_4}{\partial \rho_0} = 2\rho_0\, I_1 - \rho_0^3 \, , \quad I_1 = \frac{1}{2}\left(\rho_0^2 + \sigma_0^2 \right) \ . \tag{3b.259}$$

Thus

$$-\alpha_1\, \rho_0 + \beta\, \rho_0 \left(\rho_0^2 + \sigma_0^2\right) - \beta\, \rho_0^3 = 0 \ , \tag{3b.260}$$

$$-\alpha_1\, \sigma_0 + \beta\, \sigma_0 \left(\rho_0^2 + \sigma_0^2 \right) - \beta\, \sigma_0^3 = 0 \ . \tag{3b.261}$$

If $\sigma_0 \neq 0$, then $\rho_0^2 = \sigma_0^2$. If $\sigma_0 = 0$, then $\rho_0 = \sqrt{\frac{\alpha}{\beta}}$.

The four possibilities are then

$$\text{a) } \rho_0^2 = \sigma_0^2 = \tau_0^2 \neq 0 \ , \tag{3b.262a}$$

$$\text{b) } \rho_0^2 = \sigma_0^2 \neq 0 \, , \ \tau_0 = 0 \ , \tag{3b.262b}$$

$$\text{c) } \rho_0^2 \neq 0 \ , \ \sigma_0 = \tau_0 = 0 \ , \tag{3b.262c}$$

$$\text{d) } \rho_0 = \sigma_0 = \tau_0 = 0 \ , \tag{3b.262d}$$

all of which are covered by

$$\{ F F^{\dagger} F \} = \lambda^2 F \ . \tag{3b.263}$$

Thus spontaneous symmetry breaking with F results in Eq. (3b.263).

In the case (c), Eq. (3b.223) holds and we have

$$\frac{\partial^2 V}{\partial \rho_0^2} = -\alpha_1 \ . \tag{3b.264}$$

So we have a true minimum for $\alpha_1 = -\lambda^2$. The corresponding vacuum is left invariant by SO(10) x SO(2).

In consequence, if the local gauge group is E_6, the basic fermions, namely the leptons and quarks, belong to the 27 and $\overline{27}$ with the following decomposition under the product group SU(3) x SU(3) x $SU_c(3)$:

$$27 = (3, \bar{3}, 1) + (\bar{3}, 1, \bar{3}) + (1, 3, 3),$$

$$\overline{27} = (\bar{3}, 3, 1) + (3, 1, 3) + (1, \bar{3}, \bar{3}) \ .$$

We have given above, in the context of gauge theories, a rather detailed illustration of some of inner workings of the exceptional groups G_2, F_4 and E_6 and their underlying algebraic structures. In order not to make this section any lengthier, we omit a similar analysis for the group E_7. In any case, from the perspective of particle physics, a unified gauge group E_7 is well-known to be phenomenologically untenable.

3.c. Vector Products, Parallelisms on S^7 and Octonionic Manifolds

3.c.1. Vectors products in $\mathbf{R}^8$

We can define [239] in the 8-dimensional Euclidean space R^8 an antisymmetric vector product of order r as a multilinear map $P_r : R^{8r} \to R^8$, one which vanishes if the r vectors are linearly dependent [68, 69, 265]. For the time being, we do **not** require this vector product to be orthogonal to the individual vectors. As stated in Section 2.c.1, it will turn out that such a condition only holds if r = 3 and 7.

We first take up the case of r = 2 . In Sect. 3.a.1, we defined two linearly independent vector products e_{ab} and e'_{ab}

$$e_{ab} = \frac{1}{2} (\overline{e_a} e_b - \overline{e_b} e_a) = - \overline{e_{ab}} , \qquad e'_{ab} = \frac{1}{2} (e_a \overline{e_b} - e_b \overline{e_a}) = - \overline{e'_{ab}} . \tag{3c.1}$$

They are related to the basic commutator by $[e_a , e_b] = (e_{ab} + e'_{ab})$. It is readily checked that e_{ab} and e'_{ab} are self-dual w.r.t. f_{abcd} and f'_{abcd} respectively with eigenvalue 3:

$$\frac{1}{2} f_{abcd} e_{cd} = 3 e_{ab} \qquad \text{and} \qquad \frac{1}{2} f'_{abcd} e'_{cd} = 3 e'_{ab} . \tag{3c.2}$$

These alternatives shall be called f-duality and f'-duality respectively.

It is helpful to further define two purely vectorial entities [266]

$$v_{ab} = e_{ab} - 2 e'_{ab} \qquad \text{and} \qquad v'_{ab} = e'_{ab} - 2 e'_{ab} . \tag{3c.3}$$

They are self-dual w.r.t. the f and f' respectively but with eigenvalue −1:

$$\frac{1}{2} f_{abcd} v_{cd} = - v_{ab} \qquad \text{and} \qquad \frac{1}{2} f'_{abcd} v'_{cd} = - v'_{ab} . \tag{3c.4}$$

By defining the new tensors

$$g_{abcd} = \frac{1}{2} (f_{abcd} - \delta_{c[a} \delta_{b]d}) \quad , \qquad g'_{abcd} = \frac{1}{2} (f'_{abcd} - \delta_{c[a} \delta_{b]d}) ,$$

$$h_{abcd} = \frac{1}{4} (f_{abcd} - 3 \delta_{c[a} \delta_{b]d}) , \qquad h'_{abcd} = \frac{1}{4} (f'_{abcd} - 3 \delta_{c[a} \delta_{b]d}) , \tag{3c.5}$$

the f - and f ' - self-duality become

$$g_{abcd} v_{cd} = g'_{abcd} v'_{cd} = 0 ,$$

$$h_{abcd} e_{cd} = h'_{abcd} e'_{cd} = 0 . \tag{3c.6}$$

The g, g′, h and h′ also satisfy the relations

$$g_{abnm} h_{cdnm} = 0 \qquad , \qquad g'_{abnm} h'_{cdnm} = 0 ,$$

$$h_{abnm}\, h_{cdnm} = -2\, h_{abcd} \ , \qquad h'_{abnm}\, h'_{cdnm} = -2\, h'_{abcd} \ ,$$

$$g_{abnm}\, g_{cdnm} = 4\, g_{abcd} \ , \qquad g'_{abnm}\, g'_{cdnm} = 4\, g'_{abcd} \ . \tag{3c.7}$$

The actions of f, f′, g, g′, h and h' on e, e′, v and v′ are gathered in the following multiplication table (where we read $h_{abcd}\, e_{cd} = 0$ etc.) :

	e_{cd}	e'_{cd}	v'_{cd}	v'_{cd}
$1/2\, f_{abcd}$	$3\, e_{ab}$	$-\, v'_{ab}$	$-\, v_{ab}$	$2v_{ab} + 3\, v'_{ab}$
$1/2\, f'_{abcd}$	$-\, v_{ab}$	$3\, e'_{ab}$	$2v'_{ab} + 3\, v_{ab}$	$-\, v'_{ab}$
h_{abcd}	0	v_{ab}	$-2\, v_{ab}$	v_{ab}
h'_{abcd}	v'_{ab}	0	v'_{ab}	$-2\, v_{ab}$
g_{abcd}	$4\, e_{ab}$	$2\, e_{ab}$	0	$-6\, e_{ab}$
g'_{abcd}	$2\, e'_{ab}$	$4\, e_{ab}$	$-6\, e'_{ab}$	0

Due to the relations

$$-\,\mathrm{Sc}\,(\, e_{ab}\, e_{cd}\,) = -\frac{1}{2}\{\, e_{ab}\, ,\, e_{cd}\,\} = 2\, g_{abcd} \ , \tag{3c.8}$$

$$-\,\mathrm{Sc}\,(\, e'_{ab}\, e'_{cd}\,) = -\frac{1}{2}\left\{\, e'_{ab}\, ,\, e'_{cd}\,\right\} = 2\, g'_{abcd} \ , \tag{3c.9}$$

the g_{abcd} and g'_{abcd} are akin to metric tensors of the space generated by the e_{ab} and e'_{ab} . They also enter in the commutators:

$$[\, e_{ab}\ ,\, e_{cd}\,] = -\, g_{abn\,[c}\ v_{d\,]\,n} + g_{cdn\,[a}\ v_{n\,]\,n} = -2\, g_{abn\,[\,c}\ v_{d]\,n} \ , \tag{3c.10}$$

$$\left[\, e'_{ab}\ ,\, e'_{cd}\,\right] = -2\, g'_{abn\,[\,c}\, v'_{d]\,n} \ . \tag{3c.11}$$

Proceeding to the case of r = 3, we define the antisymmetric triple vector products

$$e_{abc} = \frac{1}{3!}\, e_{[\,a}\ \ e_{bc\,]} = \frac{1}{3}\, e_a\, e_{bc} + e_b\, e_{ca} + e_c\, e_{ab} \ , \tag{3c.12}$$

$$e'_{abc} = \frac{1}{3!}\, e'_{[\,a}\, e_{bc\,]} = \frac{1}{3}\, e'_a\, e_{bc} + e'_b\, e_{ca} + e'_c\, e_{ab} \ . \tag{3c.13}$$

We can also compose the asymmetric products E_{abc} and E'_{abc}

$$E_{abc} = e_a\, e_{bc} \ , \qquad E'_{abc} = e'_{ab}\, e_c \tag{3c.14}$$

such that

$$e_{abc} = \frac{1}{3}\,(E_{abc} + E_{bca} + E_{cab}) \tag{3c.15}$$

and

$$e'_{abc} = \frac{1}{3}\,(E'_{abc} + E'_{bca} + E'_{cab}) \ . \tag{3c.16}$$

The following compact formulae hold:

$$e_{abc} = f_{abcd}\, e_d \ , \tag{3c.17}$$

$$E_{abc} = e_{abc} + \delta_{ab}\, e_c - \delta_{ac}\, e_b = 2\, g_{bcad}\, e_d = e_a\, e_{bc} \ , \tag{3c.18}$$

parallel expressions hold for e'_{abc} and E'_{abc} . The e_{abc} and e'_{abc} are linked by

$$e_{abc} - e'_{abc} = [\, e_a\, , e_b, e_c\,] \ . \tag{3c.19}$$

With the e_{abc} , we make contact with the known antisymmetric vector products of any three octonions x, y and z [68, 69, 265, 267]. They are

$$P_3(x,y,z) \equiv x_a\, y_b\, z_c\, e_{abc} = \frac{1}{2}[\, x\,(\bar{y}\, z) - z\,(\bar{y}\, x)\,] \ , \tag{3c.20}$$

$$P'_3(x,y,z) \equiv x_a\, y_b\, z_c\, e'_{abc} = \frac{1}{2}\Big[\, (x\, \bar{y})\, z - (z\, \bar{y})\, x\,\Big] \ , \tag{3c.21}$$

$$P_3\,(x,y,z) \ - P'_3(x\, , y,\, z) = [\, x,\, y,\, z\,] \ . \tag{3c.22}$$

For r = 4, we similarly define 2 skew symmetric vector products of 4th order

$$e_{abcd} = -\frac{1}{4}\, f_{n\,[bcd}\, \overline{e_{a]}}\, e_n = -\, f_{abcd} + \frac{1}{4!}\, f_{n\,[abc}\, e_{d]n} \ , \tag{3c.23}$$

$$e'_{abcd} = -\frac{1}{4}\, e_n\, \overline{e_{[\,d}}\, f'_{abc\,]\,n} = -\, f_{abcd} - \frac{1}{4!}\, f'_{n\,[abc}\, e'_{d]n} \ . \tag{3c.24}$$

They obey the self and antiself duality relations:

$$e_{abcd} = \frac{1}{4!} \varepsilon_{abcdnmrs} e_{nmrs} \ , \tag{3c.25a}$$

$$e'_{abcd} = -\frac{1}{4!} \varepsilon_{abcdnmrs} e'_{nmrs} \ . \tag{3c.25b}$$

Thus the proof of Eq. (3c.25a) is as follows:

$$\frac{1}{4!} \varepsilon_{abcdnmrs} e_{nmrs} = \frac{1}{4!} \varepsilon_{abcdnmrs} \left(f_{nmrs} + f_{knmr} e_{sk} \right)$$

$$= - f_{abcd} + \frac{1}{(4!)^2} \varepsilon_{abcdnmrs} \varepsilon_{knmrijpq} f_{ijpq} e_{sk}$$

$$= - f_{abcd} - \frac{3!}{(4!)^2} \delta_k^{[a} \delta_i^b \delta_j^c \delta_p^d \delta_q^{s]} f_{ijpq} e_{sk}$$

$$= - f_{abcd} + \frac{1}{4!} f_{[bcds} e_{a]s} = - f_{abcd} + \frac{1}{4!} f_{k[abc} e_{d]k} = e_{abcd} \ . \tag{3c.26}$$

Focusing on the vectorial parts of the e_{abcd} and e'_{abcd}, we define

$$H_{abcd} \equiv -4 \, \mathrm{Vec}(e_{abcd}) = -\frac{1}{3!} f_{n[abc} e_{d]n} \ , \tag{3c.27a}$$

$$H'_{abcd} \equiv -4 \, \mathrm{Vec}(e'_{abcd}) = \frac{1}{3!} f'_{n[abc} e'_{d]n} \ . \tag{3c.27b}$$

Thus,

$$H_{\alpha\beta\mu\nu} = - H'_{\alpha\beta\mu\nu} = -\frac{4}{3} \psi_{[\alpha\beta\mu} e_{\nu]} \ , \tag{3c.28a}$$

$$H_{\alpha\beta\gamma 0} = H'_{\alpha\beta\gamma 0} = 4 \phi_{\alpha\beta\gamma\rho} e_\rho = 2 [e_\alpha, e_\beta, e_\gamma] \ . \tag{3c.28b}$$

Now Kleinfeld [267] had previously introduced another totally skew symmetric, purely vectorial product

$$K_{abcd} \equiv [e_a, e_b, e_c, e_d]$$

$$\equiv [e_a e_b , e_c , e_d] - e_b [e_a , e_c , e_d] - [e_b, e_c, e_d] e_a \tag{3c.29a}$$

where

$$K_{\alpha\beta\gamma 0} = 0 \qquad \text{and} \qquad K_{\alpha\beta\mu\nu} = -\frac{4}{3!}\,\psi_{[\,\alpha\beta\mu}\, e_{\nu]} = H_{\alpha\beta\mu\nu} \ . \tag{3c.29b}$$

We readily verify that the Kleinfeld product is related to the H_{abcd} and H'_{abcd} by

$$K_{abcd} = \frac{1}{2}\Big(H_{abcd} - H'_{abcd}\Big) = 2\,\mathrm{Vec}\Big(e_{abcd} - e'_{abcd}\Big) \ . \tag{3c.30}$$

In components, e_{abcd} reads

$$e_{\alpha\beta\mu\nu} = -\,\phi_{\alpha\beta\mu\nu} - \frac{1}{4}K_{\alpha\beta\mu\nu} = -\,\phi_{\alpha\beta\mu\nu} - \frac{1}{4}[\, e_\alpha\,,\, e_\beta\,,\, e_\mu\,,\, e_\nu\,] \ , \tag{3c.31a}$$

$$e_{\alpha\beta\mu 0} = -\,\psi_{\alpha\beta\mu} - \phi_{\alpha\beta\mu\rho}e_\rho = -\,\psi_{\alpha\beta\mu} - \frac{1}{2}[\, e_\alpha\,,\, e_\beta\,,\, e_\mu\,] \ . \tag{3c.31b}$$

So the duality relation (3c.25a) translates into

$$e_{\alpha\beta\mu\nu} = \frac{1}{3!}\,\varepsilon_{\alpha\beta\mu\nu\rho\sigma\tau\eta}\, e_{\rho\sigma\tau\eta} = -\,\phi_{\alpha\beta\mu\nu} - \frac{1}{4}[\, e_\alpha\,,\, e_\beta\,,\, e_\mu\,,\, e_\nu\,] \ . \tag{3c.32}$$

Separating out the scalar and vector parts, we get

$$\phi_{\alpha\beta\mu\nu} = -\frac{1}{3!}\,\varepsilon_{\alpha\beta\mu\nu\rho\sigma\tau}\,\psi_{\rho\sigma\tau} \tag{3c.33}$$

and

$$[\, e_\alpha\,,\, e_\beta\,,\, e_\mu,\, e_\nu\,] = -\frac{2}{3}\,\varepsilon_{\alpha\beta\mu\nu\rho\sigma\tau}\,[\, e_\rho\,,\, e_\sigma\,,\, e_\tau\,] \ . \tag{3c.34}$$

So the Kleinfeld function is (up to a constant) simply the dual of the associator in R^7.

For $r = 4$, if x,y, z and w are four octonions, their 0(8)-covariant vector products are given by

$$P_4(x,y,z,w) = x_a y_b z_c w_d\, e_{abcd} \ , \tag{3c.35}$$

$$P'_4(x,y,z,w) = x_a y_b z_c w_d\, e'_{abcd} \ . \tag{3c.36}$$

Proceeding to the case of $r = 5$, we define

$$e_{abcdn} \equiv \frac{1}{5}\, e_{[a}\, e_{bcdn]} = -\frac{1}{5!}\, e_{[a}\, f_{bcdn]} + \frac{1}{5!}\, f_{k[bcd}\, e_a\, e_{n]k} \ ,$$

$$= -\frac{1}{4!}\, f_{[abcd}\, e_{n]} \ . \tag{3c.37}$$

By way of the duality properties of the f_{abcd} and f'_{abcd}, one has

$$\varepsilon^{abcdnmrs}\, e_{mrs} = \varepsilon^{abcdnmrs}\, f_{mrst}\, e_t = -\frac{3}{4!}\, f_{[abcd}\, e_{n]} \quad . \tag{3c.38}$$

Then,

$$e_{abcdn} \equiv \frac{1}{5!}\, e_{[a}\, e_{bcdn]}$$

$$= -\frac{1}{4!}\, f_{[abcd}\, e_{n]} = \frac{1}{3!}\, \varepsilon_{abcdnmrs}\, e_{mrs} \quad . \tag{3c.39}$$

Similarly,

$$e'_{abcdn} \equiv \frac{1}{5}\, e'_{[abcd}\, e_{n]} \quad ,$$

$$= -\frac{1}{4!}\, f'_{[abcd}\, e_{n]} = \frac{1}{3!}\, \varepsilon_{abcdnmrs}\, e'_{mrs} \quad . \tag{3c.40}$$

If x, y, z, u, v $\in \mathbf{\Omega}$, 5-fold vector products can then be defined as

$$P_5(x, y, z, u, v) = x_a\, y_b\, z_c\, u_d\, v_n\, e_{abcdn} \quad , \tag{3c.41a}$$

$$P'_5(x, y, z, u, v) = x_a\, y_b\, z_c\, u_d\, v_n\, e'_{abcdn} \quad . \tag{3c.41b}$$

As for r = 6, we define

$$e_{abcdnm} = \frac{1}{6!}\, \bar{e}_{[a}\, e_{bcdnm]} = -\frac{1}{6 \cdot 4!}\, f_{[abcd}\, e_{nm]} \quad , \tag{3c.42a}$$

$$e'_{abcdnm} = \frac{1}{6!}\, e'_{[abcdn}\, \bar{e}_{m]} = -\frac{1}{6 \cdot 4!}\, f'_{[abcd}\, e_{nm]} \quad . \tag{3c.42b}$$

From the duality properties of the f and f' tensors, it follows that

$$\varepsilon^{abcdnmrs}\, e_{rs} = \frac{1}{6}\, \varepsilon^{abcdnmrs}\, f_{rsuv}\, e_{uv} = \frac{2}{6 \cdot 4!}\, f_{[abcd}\, e_{nm]} \tag{3c.43}$$

and

$$\varepsilon^{abcdnmrs}\, e'_{rs} = \frac{2}{6 \cdot 4!}\, f'_{[abcd}\, e'_{nm]} \quad . \tag{3c.44}$$

So

$$e_{abcdnm} = -\frac{1}{2}\, \varepsilon_{abcdnmrs}\, e_{rs} \quad \text{and} \quad e'_{abcdnm} = \frac{1}{2}\, \varepsilon_{abcdnmrs}\, e'_{rs} \quad . \tag{3c.45}$$

The corresponding vector products read

$$P_6(x, y, z, u, v, w) = x_a\, y_b\, z_c\, u_d\, v_n\, w_m\, e_{abcdnm} \quad , \tag{3c.46a}$$

$$P'_6(x, y, z, u, v,w) = x_a\, y_b\, z_c\, u_d\, v_n\, w_m\, e'_{abcdnm} \quad . \tag{3c.46b}$$

Similarly, for r = 7, we define

$$e_{abcdnmr} \equiv \frac{1}{7!}\, e_{[a}\, e_{bcdnmr]} = -\, \varepsilon_{abcdnmrs}\, e_s \quad , \tag{3c.47a}$$

$$e'_{abcdnmr} \equiv \frac{1}{7!}\, e'_{[abcdnm}\, e_{r]} = -\, \varepsilon_{abcdnmrs}\, e_s \quad . \tag{3c.47b}$$

We observe that the two chains of vector products converge for r = 7, an expected result since $\varepsilon_{abcdnmrs}$ is the only fully anti-symmetric 8-index tensor.

If x, y, z, u, v, w, t $\in$ $\mathbf{\Omega}$, the r = 7 vector product then reads

$$P_7(x, y, z, u, v, w, t) = P'_7(x, y, z, u, v, w, t) = x_a\, y_b\, z_c\, u_d\, v_n\, w_m\, t_r\;\, e_{abcdnmr}$$

$$= -\, x_a\, y_b\, z_c\, u_d\, v_n\, w_m\, t_r\;\, \varepsilon_{abcdnmrs}\, e_s \quad . \tag{3c.48}$$

Finally, for r = 8, we write

$$e_{abcdnmrs} \equiv \frac{1}{8!}\, e_{[a}\, e_{bcdnmrs]} = \frac{1}{8!}\, (\, \varepsilon_{kbcdnmrs}\, \delta_{ak} - \varepsilon_{kbcdnmrs}\, e_{ak}) \quad . \tag{3c.49}$$

Note that the second term vanishes as k $\in$ {b, c, d, n, m, r, s} and a $\in$ {b, c, d, n, m, r, s} so that a = k . Consequently,

$$e_{abcdmnrs} = \varepsilon_{abcdmnrs} \tag{3c.50}$$

and similarly,

$$e'_{abcdnmrs} \equiv \frac{1}{8!}\, e'_{[abcdnmr}\, \bar{e}_{s]} = -\, \varepsilon_{abcdnmrs} \quad . \tag{3c.51}$$

The corresponding vector products are pure octonionic scalars and given by

$$\begin{aligned} P_8(x, y, z, u, v, w, p, q) &= -\, P'_8(x, y, z, u, v, w, p, q) \\ &= x_a\, y_b\, z_c\, u_d\, v_n\, w_m\, p_r\, q_s\, e_{abcdnmrs} \quad . \end{aligned} \tag{3c.52}$$

So beyond n = 4, we explicitly find the duals of products defined for n ≤ 4.

We close this section by seeking the norms of the above products

$$r=2: \qquad P_2(x, y) = x_a\, y_b\, e_{ab} \ , \tag{3c.53}$$

$$|P_2(x, y)|^2 = x_a\, y_b\, x_c\, y_d < e_{ab} \mid e_{cd} >$$

$$= <x, x><y, y> - <x, y>^2 = \det\begin{pmatrix} <x, x> & <x, y> \\ <y, x> & <y, y> \end{pmatrix} . \tag{3c.54}$$

One finds that $|P_2(x, y)|^2 = \left|P'_2(x, y)\right|^2$.

$$r = 3: \qquad P_3(x, y, z) = x_a\, y_b\, z_c\ e_{abc} = x_a\, y_b\, z_c\ f_{abcd}\, e_d \ , \tag{3c.55}$$

then

$$|P_3(x,y,z)|^2 = x_a\, y_b\, z_c\ f_{abcd}\, x_n\, y_m\, z_r\, f_{nmrs} < e_d \mid e_s >$$

$$= x_a\, y_b\, z_c\ x_n\, y_m\, z_r\, f_{abcd}\, f_{nmrd}$$

$$= x_a\, y_b\, z_c\, x_{[a}\, y_b\, z_{c]} \ , \tag{3c.56}$$

$$|P_3(x, y, z)|^2 = \left|P'_3(x,y,z)\right|^2 = \det\begin{pmatrix} <x, x> & <x, y> & <x, z> \\ <y, x> & <y, y> & <y, z> \\ <z, x> & <z, y> & <z, z> \end{pmatrix} . \tag{3c.57}$$

r = 4:

$$\begin{aligned} P_4(x, y, z, w) &= x\, y\, z\, w\ e_{abcd} \\ &= -\, f_{abcd}\, x_a\, y_b\, z_c\, w_d + \frac{1}{4!}\, x_a\, y_b\, z_c\, w_d\, f_{kabc}\, e_{dk} \ , \end{aligned} \tag{3c.58}$$

$$|P_4(x, y, z, w)|^2 = \left|P'_4(x, y, z, w)\right|^2 = x_a\, y_b\, z_c\, w_d\, x_{[a}\, y_b\, z_c\, w_{d]} \ , \tag{3c.59}$$

$$= \det\begin{pmatrix} <x, x> & <x, y> & <x, z> & <x, w> \\ <y, x> & <y, y> & <y, z> & <y, w \\ <z, x> & <z, y> & <z, z> & <z, w> \\ <w, x> & <w, y> & <w, z> & <w, w> \end{pmatrix} .$$

r = 5 : if $\omega_1, \omega_2, \ldots, \omega_5 \in \mathbf{\Omega}$, then

$$P_5(\omega_1, \omega_2, \omega_3, \omega_4, \omega_5) = \omega_1^a\, \omega_2^b\, \omega_3^c\, \omega_4^d\, \omega_5^m\; e_{abcdm} \tag{3c.60}$$
$$= \frac{1}{3}\, \omega_1^a\, \omega_2^b\, \omega_3^c\, \omega_4^d\, \omega_5^m\; \varepsilon_{abcdmnrs}\, e_{nrs}\ .$$

One finds that

$$|\,P_5\,|^2 = \det(<\omega_i, \omega_j>) - \frac{1}{4}\, \omega_1^a\, \omega_2^b\, \omega_3^c\, \omega_4^d\, \omega_5^m\, \omega_1^{[\,a}\, \omega_2^b\, \omega_3^c\, \omega_4^d\, \omega_5^m\, f^{nr]nr} \tag{3c.61}$$

where we note the presence of an additional, nonzero second term.

$$r = 6:\ \ P_6(\omega_1, \omega_2, \ldots, \omega_6) = \omega_1^a\, \omega_2^b\, \omega_3^c\, \omega_4^d\, \omega_5^m\, \omega_6^n\; e_{abcdmn}\ \ . \tag{3c.62}$$

Here, like $|P_5|^2$, $|\,P_6\,|^2$ also has, besides det $(<\omega_i\ ,\ \omega_j>)$, a nonzero second piece. Indeed, as stated in the beginning of this section, these anomalous norms for the $r = 5$ and 6 cases are consistent with the results of Eckmann et al. on r-fold vector products (as defined in Eqs. (2c.1)-(2c.2)) on R^n.

Finally, consider $r = 7$:

$$P_7(\omega_1, \omega_2, \ldots, \omega_7) = \omega_1^a\, \omega_2^b \omega_3^c\, \omega_4^d\, \omega_5^m\, \omega_6^n\, \omega_7^r\; e_{abcdmnr}\ , \tag{3c.63}$$
$$= \omega_2^b\, \omega_3^c\, \omega_4^d\, \omega_5^m\, \omega_6^n\, \omega_7^r\; \varepsilon_{abcdmnrs}\, e_s\ \ .$$

Then one finds

$$|\,P_7\,|^2 = \det(<\omega_i, \omega_j>) \tag{3c.64}$$

in agreement with Ref.[68]. This concludes our discussion of octonionic vectors.

3.c.2. Absolute parallelisms on S^7

The *parallelizability* [268] of a Riemann space M is equivalent to the existence on M of a torsion tensor $S^\alpha{}_{\beta\gamma}$ which, when added to the symmetric Levi-Civita connection $\Gamma^\alpha{}_{\beta\gamma}$ "flattens" the space M. Namely, we have a vanishing curvature

$$R^\alpha_{\beta\gamma\rho}\left(\left\{\Gamma^\sigma_{\tau\zeta} + S^\sigma_{\tau\zeta}\right\}\right) = 0\ . \tag{3c.65}$$

If we require the affine connection ($\Gamma + S$) to give the same geodesic as the Riemannian connection Γ, then the torsion S must be completely antisymmetric ($S_{\alpha\beta\gamma} = S_{[\alpha\beta\gamma]}$). (3c.65) then leads to

$$\overset{0}{R}_{\alpha\beta\gamma\rho} = S_{\tau\alpha\beta}\, S^{\tau}_{\gamma\rho} - S_{\tau[\beta\gamma}\, S^{\tau}_{\alpha]\rho} \;, \tag{3c.66}$$

$$S_{\alpha\beta\gamma;\rho} = \partial_{[\rho} S_{\alpha\beta\gamma]} = S_{\tau[\alpha\beta}\, S^{\tau}_{\gamma]\rho} \tag{3c.67}$$

obtained by way of the cyclic identities for $R^{\alpha}{}_{\beta\gamma\rho}(\{\Gamma^{\sigma}{}_{\eta\lambda}\})$.

In 1926 Cartan and Schouten made a remarkable discovery [72]. They found that, besides the trivial case of R^n, the only Riemannian manifolds admitting such " absolute parallelism" (AP) are group manifolds and S^7 . The 7-sphere is therefore not just a parallelizable non-group manifold, but it is the *only* one with the AP property. As we now show, this singular feature of S^7 is uniquely tied to the existence of the octonion algebra in R^8. Our treatment follows closely the elegant and detailed exposition of M. Rooman [75].

Paralleling our quaternionic treatment of the Clifford parallelism on S^3, we parametrize the unit 7-sphere embedded in R^8 by a unit octonion $X = x_\mu e_\mu$, $|X| = 1$. The 7 unit octonions $e_j X$ ($X e_j$) obtained by left (+) (right (–)) multiplication of X by the octonionic imaginary units e_j induce a mapping from S^7 to S^7. By the definition of the scalar product $\langle A \cdot B\rangle = \frac{1}{2}(\bar{A}B + B\bar{A}) = \frac{1}{2}(A\bar{B} + B\bar{A})$ and the $\mathbf{\Omega}$-algebra (3a.1), it then follows that

$$\langle e_iX \cdot e_jX \rangle = \delta_{ij} \;, \qquad \langle e_iX \cdot X \rangle = 0 \quad , \; i, j = 1, 2, ..., 7 \tag{3c.68}$$

and

$$\langle Xe_i \cdot Xe_j \rangle = \delta_{ij} \;, \qquad \langle Xe_i \cdot X \rangle = 0 \quad , \; i, j = 1, 2, ..., 7 \,. \tag{3c.69}$$

So $\{ e_j X \}$ ($\{ X e_j \}$) define tangent $\mathbf{\Omega}$-valued vectors $E_i^{(\pm)}(P) = E_i^{(\pm)\,\mu}(P)\, e_\mu$ at a point P on S^7. In components, we have

$$E_i^{(\pm)\,m}(P) = \pm\, \psi_{inm}\, x_n + \delta_{im}\, x_0 \;\; , \;\; E_i^{(\pm)\,0}(P) = -\, x_0 \;\; , \; (i, n, m) = (1, 2, ..., 7) \tag{3c.70}$$

which explicitly shows the (+) and (–) siebenbeins or septads to be *different*, except at the north or south poles of S^7 ($X = \pm x_0$).

Since the S^7-metric is $g^{-1} = E_i^{(\pm)} \otimes E_i^{(\pm)}$, the set { $e_j X$ } forms a nowhere vanishing orthonormal basis, namely a global Killing frame of S^7 . The above construction then defines the following law of global parallelism: two vectors V and V' tangent to S^7 at X and X' respectively are parallel (V // V') if and only if V has the same coordinates w.r.t. the basis $e_i X$ ($X e_i$) as V' w.r.t. its own basis $e_i X'$ ($X' e_i$), i.e.

$$\langle V \cdot \; e_i X \rangle = \langle V' \cdot \; e_i X' \rangle \tag{3c.71}$$

and

$$\langle V \cdot \; X e_i \rangle = \langle V' \cdot \; X' e_i \rangle \quad , \; i = 1, 2, 3 \, . \tag{3c.72}$$

These relations equivalently read as

$$V \overline{X} = V' \, \overline{X'} \quad , \tag{3c.73a}$$

$$\overline{X} \, V = \overline{X'} \, V' \tag{3c.73b}$$

respectively. This equivalence can be shown by using the definition of the scalar product $\langle A \cdot \; B \rangle$ of two octonions A and B and the fact that V and V' are tangent to S^7, namely $\langle V \cdot \; X \rangle = 0$ and $\langle V' \cdot \; X' \rangle = 0$.

An infinitesimal V at X may be parametrized as $V = [\, \exp(\eta^i e_i) - 1 \,] X$, η^i being the infinitesimal coordinates of V w.r.t. $e_i X$. So, if we let $Y = \exp(\eta^i e_i) X$, $Y' = \exp(\eta'^i e_i) X'$; Eqs. (3c.73a-73b) becomes

$$Y \overline{X} = Y' \, \overline{X'} \quad , \tag{3c.74a}$$

$$\overline{X} \, Y = \overline{X'} \, Y' \quad . \tag{3c.74b}$$

For finite $\eta^{\,i}$ and η'^{i}, Y and Y' correspond to two points on S^7, and Eq. (3c.74) defines the (+) and (–) Cartan-Schouten parallelisms [72] between two segments $\overrightarrow{XY}$ and $\overrightarrow{X'Y'}$ of two geodesics on S^7.

To illustrate the role played by the non-associativity of octonions, it is instructive to calculate the torsion $S_{\alpha\beta\gamma}(X)$ at some point X of S^7. Consider at X a "parallelogram" :

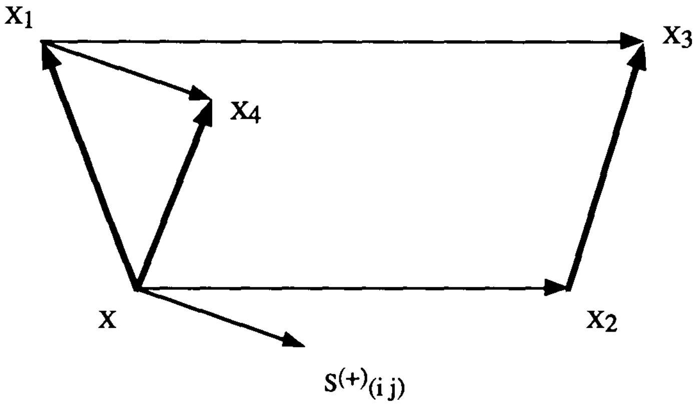

Fig. 3

In the above schematic representation, the four unit octonions

$$X_1 = \exp(\lambda e_i)\,X \quad , \quad X_2 = \exp(\lambda e_j)\,X \ ,$$

$$X_3 = \exp(\lambda\, e_i)\,\{\exp(\lambda\, e_j)\,X\} \ , \ X_4 = \exp(\lambda\, e_i)\exp(\lambda\, e_j)\,\{\exp(-\lambda\, e_i)\,X\} \quad (3c.75)$$

are four points on S^7, λ being a real parameter. The segments $\overrightarrow{X\,X_2}$ and $\overrightarrow{X_1\,X_3}$ are clearly (+) parallel while $\overrightarrow{X\,X_1}$ and $\overrightarrow{X_2\,X_3}$ are generally not parallel. The difference between the second segment $\overrightarrow{X_2\,X_3}$ parallel transported to X and the first, both expanded to second order in $\lambda << 1$ in the tangent hyperplane, is the *translation* $S^{(+)}{}_{ij}$. The latter represents the torsion vector on the "parallelogram" $(e_i\,X\,,\,e_j\,X)$:

$$S^{(+)}{}_{(ij)}(X) = \frac{1}{2} \lim_{\eta \to 0} \frac{1}{\eta^2} \Big\{ \big[\exp(\eta e_i) \exp(\eta e_j) (\exp(\eta \bar{e}_i) X) - X \big] - \big[\exp(\eta e_j) X - X \big] \Big\} \tag{3c.76}$$

giving the manifestly antisymmetric torsion

$$\begin{aligned} S^{(+)}{}_{(i\,j)}(X) &= \frac{1}{2} [\, e_i (e_j X) - e_j (e_i X) \,] \\ &= \psi_{ijk}\, e_k X - [\, e_i, e_j, X \,] \quad . \end{aligned} \tag{3c.77}$$

So the non-zero torsion has two contributions: in addition to a constant piece, connected to the non-commutativity of $\boldsymbol{\Omega}$, there is a *non-constant* piece, connected to the non-associativity of $\boldsymbol{\Omega}$. This associator piece $[\, e_i , e_j , X \,]$ vanishes only at the North and South poles of S^7 where $X = \pm x_0$.

Using the facts that 1) $e_a (e_b X) = - e_b (e_a X)$ for $a \neq b$, 2) an associator is completely antisymmetric and 3) $\langle [X,Y,Z] , X \rangle = 0$ for any X, Y, $Z \in \boldsymbol{\Omega}$, along with Eqs. (3c.68) and (3c.77), it can be shown that

$$\left\langle S^{(+)}{}_{(i\,j)}(X) \cdot X \right\rangle = \left\langle S^{(+)}{}_{(i\,j)}(X) \cdot e_i X \right\rangle = \left\langle S^{(+)}{}_{(i\,j)}(X) \cdot e_j X \right\rangle = 0 \ , \tag{3c.78}$$

$$\left\langle S^{(+)}{}_{(i\,j)}(X) \cdot S^{(+)}{}_{(i\,k)}(X) \right\rangle = \delta_{ik} \ . \tag{3c.79}$$

These relations imply that a) the torsion vectors $S^{(+)}{}_{ij}$, which are unit tangent vectors to S^7, are orthogonal to the frame vectors $e_i X$ and $e_j X$, b) for $j \neq k$, $S^{(+)}{}_{ij}$ and $S^{(+)}{}_{ik}$ are orthogonal. We can therefore define the parallelizing torsion tensor $S^{(+)}{}_{ijk}$ through

$$S^{(+)}{}_{(ij)}(X) = - S^{(+)k}_{ij}(X)\, e_k X \tag{3c.80}$$

or

$$S^{(+)}{}_{ijk}(X) = -\left\langle S^{(+)}_{(ij)}(X) \cdot e_k X \right\rangle . \tag{3c.81}$$

Using Eq. (3c.77) and Eqs. (3c.80)-(3c-81), it can be recast into the compact form

$$S^{(+)}_{ijk}(X) = -\psi_{ijk} - \frac{1}{2}\left\{ e_i, \left[e_j, X, \overline{e_k X} \right]\right\} . \tag{3c.82}$$

While the first term in Eq. (3c.82) is clearly completely antisymmetric, the antisymmetry of the second term is not obvious. The latter antisymmetry in j and k derives from the following properties of associators 1) their total antisymmetry under interchange 2) $[X, Y, \overline{Z}] = -[X, Y, Z]$ and 3) $[XY, \overline{Y}, Z] = -[A, Y, \overline{Y}Z]$ for any X, Y and Z $\in$ **Ω**. By combining this (j, k) antisymmetry with the (i, j) antisymmetry of the torsion vector $S^{(+)}{}_{ij}$ (3c.77), it follows that $S^{(+)}_{ijk}(X)$ is fully antisymmetric i.e. $S^{(+)}_{ijk}(X) \equiv S^{(+)}_{[ijk]}(X)$.

For given i and j

$$S^{(+)}{}_{jk}(NP) = \psi_{jki}\, e_i \ , \ S^{(+)}{}_{ki}(NP) = \psi_{kij}\, e_j \ , \ S^{(+)}{}_{ij}(NP) = \psi_{ijk}\, e_k \tag{3c.83}$$

at the North pole of S^7. Equation (3c.83) clearly shows that there are seven independent triplets of torsion vectors, one for each unbroken line or cycle of the **Ω**-triangle Fig. 2. At any point X of S^7, one has

$$S^{(+)}_{(ij)}(X) = -S^{(+)\,k}_{ij}(X)\, e_k X = e_{k'} X \ ,$$

$$S^{(+)}_{(k'i)}(X) = -S^{(+)k}_{ij}(X)\, S^{(+)}_{(ki)}(X) = e_j X \ ,$$

$$S^{(+)}_{(jk')}(X) = -S^{(+)k}_{ij}(X)\, S^{(+)}_{(jk)}(X) = e_i X \ . \tag{3c.84}$$

So, while they are no longer equal to the basis vectors e_i X, the torsion vectors still form triplets of orthogonal vectors tangent to S^7.

Similarly, the (–) parallelism is defined by the torsion vector

$$S^{(-)}_{(ij)}(X) = \frac{1}{2}\,[\, e_i\,(X\,e_j) - e_j\,(Xe_i)\,]$$

$$= -\,S^{(-)k}_{ij}(X)\,X\,e_k = -\,\psi_{ijk}\,X\,e_k - [\,e_i\,,\,e_j,\,X\,]\ . \qquad (3c.85)$$

The counterparts of Eqs. (3c.78) - (3c.79) are similarly derived.

We should note here an importance feature. In contrast to Lie groups G where $S^{(+)}{}_{ijk}(X) = -\,S^{(-)}{}_{ijk}(X)$ holds everywhere on G, this equality holds true *only* at the Poles of S^7 i.e. at X=±1 or on S^6, i.e. for $X = x_j\, e_j$.

To obtain the covariant derivative of the torsion vector, say $S^{(+)}_{(ij)}$, one begins with the differential $dS^{(+)}_{(ij)}(X) = S^{(+)}_{(ij)\,//} - S^{(+)}_{(ij)}(X)$ where, as illustrated below in Fig. 4, $S^{(+)}_{(ij)\,//}$ is $S^{(+)}_{(ij)}(X')$ at X' parallel transported to the point X .

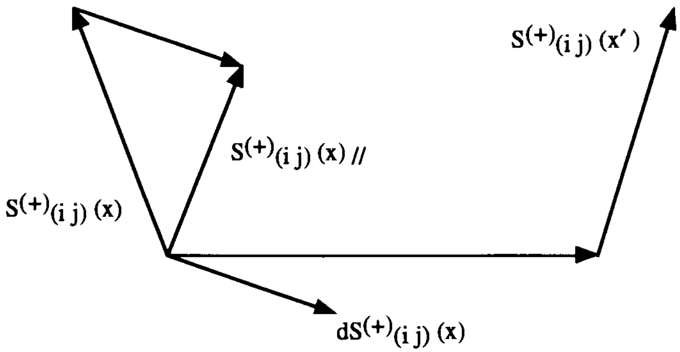

Fig. 4

From the defining relation of (+) parallelism (3c.74a), one finds

$$dS^{(+)}_{(ij)}(X) = S^{(+)}_{(ij)//} - S^{(+)}_{(ij)}(X) = [\,e_i,\,e_j,\,X\,] - \big[\,[e_i,\,e_j,\,X'\,]\,\overline{X'}\,\big]\,X\,, \qquad (3c.86)$$

$S^{(+)}_{(ij)\,|\,i}(X)$, the covariant derivatives of $S^{(+)}_{(ij)}$ are defined through

$$dS^{(+)}_{(ij)} = S^{(+)}_{(ij)\,|\,\alpha}\, dx_\alpha = S^{(+)}_{(ij)\,\|k}\, h^k_\alpha\, dx^\alpha \tag{3c.87}$$

where the h_i^α relate the canonical basis $\{e_j\}$ with the septads $e_j\,X$ of R^8: $e_i\,X = h_i^\alpha e_\alpha$. One finds that

$$S^{(+)}_{(ij)\,|\,m} = -\,S^{(+)k}_{(ij)\,|\,m}\, e_k\, X = -\,\big[\,[e_i, e_jX\,, e_mX\,] + [\,e_i, e_j\,(\,e_m\,X\,)\,, X\,]\,\big]\,X \tag{3c.88}$$

which shows that a) unlike group manifolds, $S^{(+)}_{(ij)\,|\,m}$ does **not** vanish, due to the lack of associativity of $\mathbf{\Omega}$, b) $S^{(+)}_{(ij)\,|\,m} = -\,S^{(+)}_{(im)\,|\,j}$, which, when combined with the complete antisymmetry of $S^{(+)}_{ijk}$, implies that $S^{(+)}_{(ij)\,|\,m}$ is also totally antisymmetric : $S^{(+)}_{ijk\,|\,m} \equiv S^{(+)}_{[ijk\,|\,m]}$.

Analogously, we get the torsion vector of the (–) parallelism in

$$S^{(-)}_{(ij)\,|\,m}(\,X\,) = X\,[\,[X\,e_m, X\,e_j\,, e_i\,] + [\,X, (\,X\,e_m\,)e_j\,, e_i\,]\,]\ . \tag{3c.89}$$

A simplification occurs at the poles of S^7, say at the North Pole ($X = N\,P$):

$$S^{(+)}_{(ij)\,|m}(\,NP\,) = S^{(-)}_{(ij)\,|\,m}(\,NP\,) = -[\,e_i\,, e_j\,, e_m\,] = -\,2\varphi_{ijmn}\, e_n\ . \tag{3c.90}$$

There, the (+) and (–) torsions are equal and are given by the associator. Equation (3c.90) implies that, at the pole, the torsion vectors form a triplet, each vector of which is the torsion vector of the other two directions. From the identity $\langle [X,Y,Z] \cdot X \rangle = 0$, it is clear that $S^{(+)}_{(ij)}(\,NP\,)$ only has nonzero covariant derivatives in directions orthogonal to the triplet.

From the algebraic properties of $\mathbf{\Omega}$, Eqs. (3a.23), (3a.24) alone, it follows that

$$S_i^{(\pm)\,jk}\, S^{(\pm)}_{jkm} = 6\, g_{im}\ , \tag{3c.91}$$

$$S^{(\pm)\,i}_{km}\; S^{(\pm)n}_{ij}\; S^{(\pm)k}_{np} = 3\; S^{(\pm)}_{jmp} \tag{3c.92}$$

are satisfied at the poles of S^7. In fact, a more involved but straightforward computation shows that these relations hold for *any* point X on S^7, as they should. They are known as the Cartan-Schouten equations with the 7-sphere radius $R_0 = 1$. Coupled with the basic Cartan's structural equations, they reproduce the Riemannian geometry of the 7-sphere as a locally symmetric Einstein space with their Riemann, Ricci and scalar curvatures given by $R_{ijkm} = R_0^{-2}(\, g_{ik}\, g_{jm} - g_{im}\, g_{jk}\,)$, $R_{ij} = 6\, R_0^{-2} g_{ij}$ and $R = 42\, R_0^{-2}$ respectively. Since $S^{(\pm)}_{(ij)\,lm}(\,X\,) = -\,2\,\nabla_m\, S^{(\pm)}_{(ij)}(\,X\,)$ where ∇_m denotes the covariant derivative w.r.t. the Riemannian connection $\Gamma^i{}_{mk}$, we obtain

$$\nabla_m\, S^{\pm}_{ijk}(\,X\,) = S^{\pm}_{[ijk;m]}(\,X\,) = S^{\pm}_{r\,[\,ij}(\,X\,)\; S^{\pm\,r}_{k]m}(\,X\,)\ . \tag{3c.93}$$

It is notably non-zero since, unlike the Lie structure constants f_{ijk}, the octonionic constants ψ_{ijk} do *not* obey the Jacobi identity, but rather the more general relation

$$\psi_{ijk}\, \varphi_{ijkr}\, e_r = \psi_{ijk}\, \psi^m_{[ij}\, \psi^m_{k]r} = 0\ . \tag{3c.94}$$

Next, we close this section with a discussion of the remarkable two infinite families of parallelisms on S^7 and their connection to the non-associativity of $\mathbf{\Omega}$.

Unlike the case of Lie groups, there actually exist other – in fact an infinite number of – parallelizing torsions on S^7 beyond the ($\pm$) torsions discussed thus far. To see this, let some torsion tensor equal $\pm\psi_{ijk}$ at diametrically opposite points which are *not* the north and south poles of S^7. Since the S^7-torsions are *not* constant, such a torsion tensor defines an *infinite* number of new ($\pm$) torsions different from $S^{\pm}_{ijk}(\,X\,)$, a two 7-parameter families of absolute parallelisms on S^7. They are generalizations of the parallel transportation of tangent vectors. The ($\pm$) parallelisms on S^7 are given by

$$Y\,(\,\overline{X}\,A\,) = Y'\,(\,\overline{X'}\,A\,)\ , \tag{3c.95}$$

$$(B\overline{X})\,Y = (B\overline{X'})\,Y \tag{3c.96}$$

where A and B, $|A| = |B| = 1$ are *arbitrary* unit octonions. Due to the lack of associativity $(\overline{X}A)\overline{A} \neq \overline{X}$, $X, A \in \boldsymbol{\Omega}$, Eqs. (3c.95) and (3c.96) are clearly *not* equivalent to Eqs. (3c.73) and (3c.74). Generally, different pairs of A (B) and A' (B') ($A \neq -A'$, $B \neq -B'$) give different (+) ((–)) parallelisms, therefore to an infinity of (+) ((–)) parallelisms.

The associated global septads, generalizing those in Eqs. (3c.68)-(3c.69), are given by $(e_i A)(\overline{A}X)$ and $(X\overline{B})(B e_i)$ respectively. They similarly satisfy

$$\langle (e_i A)(\overline{A}X) \cdot (e_j A)(\overline{A}X) \rangle = \langle (X\overline{B})(B e_i) \cdot (X\overline{B})(B e_j) \rangle = \delta_{ij} \ , \tag{3c.97}$$

$$\langle (e_i A)(\overline{A}X) \cdot X \rangle = \langle (X\overline{B})(B e_i) \cdot X \rangle = 0 \ . \tag{3c.98}$$

What is the difference between the above parallelism and that on Lie groups ? For groups, if two segments (X; Y) and (X'; Y') are (+) parallel, the segments (X ; X') and (Y ; Y') are (–) parallel and vice versa. On S^7, due to Eqs. (3c.95) and (3c.96), to two (+) parallel segments (X ; Y) and (X' ; Y') associated with some A, there corresponds a B, in general different from A, such that (X ; X') and (Y ; Y') are (–) parallel, and vice versa.

3.c.3. The almost complex structure on S^6

Of all the spheres S^n, only two, $S^2 \approx Sp(1)/U(1)$ and $S^6 \approx G_2/SU(3)$, have a complex and an almost complex structure respectively. The close connection between octonions, the almost complex structure and the associated non-vanishing torsion tensor of S^6 can be seen by embedding 6-manifolds in R^7. The latter is then identified with the pure imaginary octonions, thus induces an octonionic structure onto 6-manifolds. We shall closely follow the presentation of Günaydin and Warner [269].

In the theory of complex manifolds [2, 70], the defining property of an almost complex structure on a 2n-dimensional real manifold M is a mixed tensor $F^i{}_k$ such that $F^i{}_k F^k{}_j = -\delta^i{}_j$. The torsion tensor of an almost complex structure $F^i{}_k$ is defined [270] by

$$\tau^i{}_{jk} \equiv \frac{1}{2}\left(A^i{}_{mj} F^m{}_k - A^i{}_{mk} F^m{}_j \right) \tag{3c.99}$$

where

$$A^i{}_{jk} = \frac{1}{2}\left(\nabla_j F^i{}_k - \nabla_k F^i{}_j \right) \tag{3c.100}$$

$$= \frac{1}{2}\left(\partial_j F^i{}_k - \partial_k F^i{}_j \right) , \tag{3c.101}$$

∇_j denotes the covariant derivative on M w.r.t. some symmetric affine connection, the symmetry of which implies Eq. (3c.101). If $\tau^i{}_{jk}$ vanishes identically then the almost complex structure is integrable to a complex structure.

To exhibit the almost complex structure in terms of octonions, we consider S^6 as an hypersurface in R^7 viewed as a Cayley space I^7, the space of imaginary octonions. A vector in I^7 is then given by $X = X_A e^A$, $A = 1, 2, \ldots, 7$ where e^A are the seven imaginary octonionic units of $\mathbf{\Omega}$. Due to the existence of the Cayley multiplication in I^7 we can define a scalar product and a vector cross product of two vectors X and Y in I^7. They are

$$\langle X, Y \rangle = \mathrm{Sc}\,(X\, Y) = X^A Y_A \tag{3c.102}$$

and

$$X \wedge Y = \frac{1}{2}\left[X\, Y - Y\, X \right] = \psi^A{}_{BC} X^B Y^C e_A \tag{3c.103}$$

obeying the identities

$$\langle X \wedge Y , Z \rangle = \langle X , Y \wedge Z \rangle \ , \tag{3c.104}$$

$$(X \wedge Y) \wedge Z - \langle X, Z \rangle Y + \langle Y, Z \rangle X = -X \wedge (Y \wedge Z) + \langle X, Z \rangle Y - \langle X, Y \rangle Z \quad . \qquad (3c.105)$$

So the set of all unit normed vectors $X, \langle X, X \rangle = 1$ spans the unit sphere S^6. Let $\hat{n}$ be the unit normal to this sphere and choose a set of basis vectors $\{\hat{e}_i\}$ tangent to S^6 so that the S^6-metric is $g_{ij} = \langle \hat{e}_i, \hat{e}_j \rangle$. Then the almost complex structure on S^6 can be defined [271, 272] as

$$\hat{e}_i \wedge \hat{n} = F^j{}_i \, \hat{e}_j \quad . \qquad (3c.106)$$

Indeed through Eq. (3c.105) it follows that $(\hat{e}_i \wedge \hat{n}) \wedge \hat{n} = F^k{}_j F^j{}_i \hat{e}_k = -\hat{e}_i$, i.e. $F^i{}_k F^k{}_j = -\delta^i{}_j$. From (3c.104) we see that F_{ij} is skew symmetric

$$F_{ij} = \langle \hat{e}_j, \hat{e}_i \wedge \hat{n} \rangle = F^k{}_i \langle \hat{e}_j, \hat{e}_k \rangle = g_{jk} F^k{}_i \qquad (3c.107)$$

$$= \langle \hat{e}_j \wedge \hat{e}_i , \hat{n} \rangle = -\langle \hat{e}_i, \hat{e}_j \wedge \hat{n} \rangle = -F_{ji} \qquad (3c.108)$$

and so is the rank 3 torsion tensor $T_{ijk} = T^m{}_{ij} \, g_{mk}$ defined via the cross product of basis vectors

$$\hat{e}_i \wedge \hat{e}_j = -F_{ij} \hat{n} + T^k{}_{ij} \hat{e}_k \quad ; \qquad (3c.109)$$

$$T_{ijk} = \langle \hat{e}_j \wedge \hat{e}_i , \hat{e}_k \rangle = -T_{jik} = T_{kij} \qquad (3c.110)$$

with $T_{ijk} T^{rjk} = 4 \, \delta^r_i$. The latter follows from the identities

$$T^m_{ij} T^r_{mk} + T^m_{kj} T^r_{mi} = -F_{ij} F^r{}_k - F_{kj} F^r{}_i - \delta^r{}_i \, g_{jk} - \delta^r{}_k \, g_{ij} + 2 \, \delta^r{}_j \, g_{ik} \quad , \qquad (3c.111)$$

$$T^m_{ij} F_{mk} + T^m_{kj} F_{mi} = 0 \qquad (3c.112)$$

which together are clearly equivalent to Eq. (3c.105) .

If the operator ∇_i is the covariant derivative on S^6 with respect to the Levi-Civita connection, one then obtains the following Gauss and Weingarten equations

$$\nabla_i \hat{e}_j = \partial_i \hat{e}_j - \Gamma^k{}_{ij}\, \hat{e}_k = H_{ij}\, \hat{n} \ , \tag{3c.113}$$

$$\nabla_i \hat{n} = \partial_i \hat{n} = - H^j{}_i\, \hat{e}_j \ , \tag{3c.114}$$

H_{ij} is the 2nd fundamental form of the hypersurface. They yield the curvature of the hypersurface in I^7

$$R_{ijkr} = H_{ir}\, H_{jk} - H_{jr}\, H_{ik} \ , \tag{3c.115}$$

$$\nabla_k\, H_{ji} - \nabla_j\, H_{ki} = 0 \tag{3c.116}$$

which for S^6 gives $H_{ij} = \lambda\ g_{ij}$.

By covariant differentiation of Eq. (3c.106), followed by contraction of indices making use of the symmetry of the H and T tensors, we readily obtain

$$\nabla_j\, F^j{}_i = 0 \ . \tag{3c.117}$$

Similarly, differentiation of Eq. (3c.109) gives for S^6

$$\nabla_k\, T^k{}_{ij} = -\, 4\, \lambda\, F_{ij} \tag{3c.118}$$

and

$$\nabla_k\, F_{ij} = \lambda\, T_{ijk} \, . \tag{3c.119}$$

From the foregoing construction, the G_2 invariance of the almost complex structure F_{ij} is apparent. Conversely, it can be shown that the group of isometries leaving the almost complex structure invariant is isomorphic to G_2, the automorphism group of $\mathbf{\Omega}$. Moreover, a G_2 invariant affine connection $\Pi^k{}_{ij}$ can be defined on S^6, one which "parallelizes" the almost complex structure. Specifically,

$$\Pi^k{}_{ij} = \Gamma^k{}_{ij} + \frac{\lambda}{2} T^m{}_{ij} F^k{}_m \tag{3c.120}$$

where $\Gamma^k{}_{ij}$ is the metric connection. Then

$$D_i F_{jk} = \nabla_i F_{jk} + \frac{\lambda}{2} T^m{}_{ij} F^r{}_m F_{rk} + \frac{\lambda}{2} T^m{}_{ik} F^r{}_m F_{jr} = 0 \tag{3c.121}$$

using $F^r{}_m F_{jr} = - g_{mr}$ and Eq. (3c.119) .

Since SU(3) is the subgroup of G_2 which leaves invariant the unit imaginary octonion in Cayley space I^7, it follows that S^6 with the affine connection is isomorphic to the coset space $\frac{G_2}{SU(3)}$.

It is also natural to define a totally antisymmetric tensor

$$S_{ijk} = \frac{1}{3!} \varepsilon_{ijlmnp} T^{mnp} \ , \tag{3c.122}$$

the dual of T_{ijk}. It satisfies $\nabla_i S^{ijk} = 0 \nabla_i S^{ijk} = 0$. The tensors S^{imn} and T_{ijk} obey the following handy identities:

$$T_{imn} T^{jmn} = 4\, \delta^j{}_i = S_{imn} S^{jmn} \ , \tag{3c.123}$$

$$S_{ijk} = F^r{}_i T_{rjk} \ , \tag{3c.124}$$

$$S_{imn} T^{jmn} = 4\, F^j{}_i \ , \tag{3c.125}$$

$$T_{imn} S^{jmn} + S_{imn} T^{jmn} = 0 \ . \tag{3c.126}$$

Finally, from the properties of F_{ij}, T_{ijk} and S_{ijk}, it is easy to show that no nonzero G_2 invariant vector can be made out of them.

For more details on the above formalism and its application to a G_2 invariant compactification of D = 11 supergravity, the reader should consult the paper of Günaydin and Warner [269]. For an alternative, elegant treatment of the almost complex structure on S^6, we recommend the works of Bryant [273].

3.c.4. The Moufang plane

Projective geometry is viewed as more fundamental than Euclidean geometry. The reason rests in its use solely of the concept of *incidence.* It makes no mention whatsoever on metrical properties such as distances or angles. In the following only the relevant details about projective planes will be given, some texts on this topic are Refs.[218] and [274].

The representation theorem for projective spaces asserts that D > 2 projective spaces can be generally represented by vector spaces over a skew (non-commutative) field. If the underlying field is *commutative*, then the Pappus theorem holds. It states :

If D, A, E are distinct points on a line m, and C, F, B are distinct points on another line n, coplanar with m, then three points V, U and W are collinear (see Fig. 5).

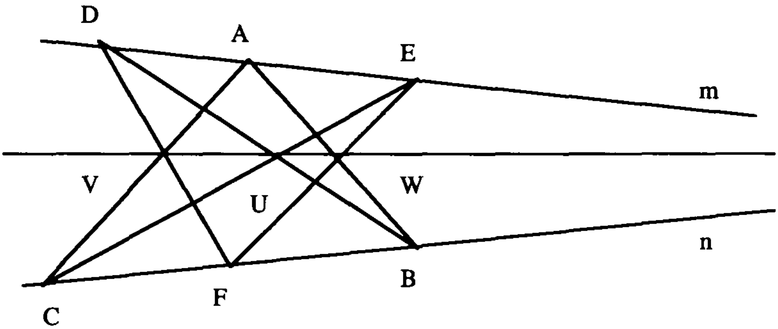

Fig. 5

Another fundamental theorem, Desargues' theorem follows from the projective axioms for $n > 2$. If we denote below the corresponding points in Fig. 6 by their suffixes i.e. P_A by A etc., then Desargues' theorem states that, if two triangles ABC, A'B'C' in any $D > 2$ projective space π are such that the lines AA', BB' and CC' pass through a point E, then the three points of intersections S_1, S_2 and S_3 all lie on a straight line.

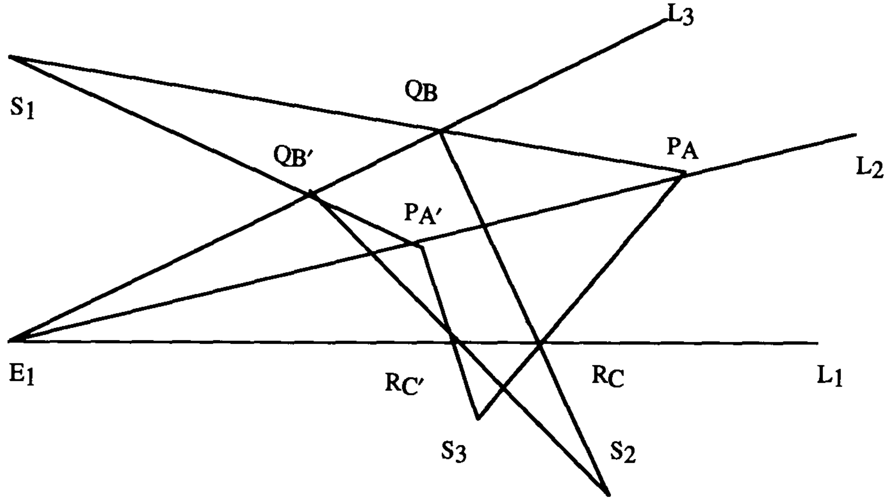

Fig. 6

In 1933 an example of a non-Desarguian projective geometry was given by Ruth Moufang [254], who gave an octonionic affine coordinatization of this Cayley-Moufang plane CaP(2). She showed that, due to the alternativity of $\mathbf{\Omega}$, a weaker theorem than Desargues', the harmonic lock theorem, holds for such an $n = 2$ projective space.

We now illustrate P.Jordan's method of construction of the Moufang plane following the work of Gunaydin et al. [275] and the presentation of Ruegg [276]. The former paper is in turn patterned after the works of Freudenthal [248], Springer [277] and after Jacobson's review article [278].

We define a projective plane as a set of elements called *points*, with a collection of subsets of points called *lines*, obeying the following axioms

a) Any two distinct points are contained in one and only one line.
b) Any two distinct lines intersect at one point
c) There are four points such that no three of which lie on the same line.

From these axioms one can prove that

d) Given three lines intersecting at three points, the line defined by two other points of two of these lines intersects the third line.
e) Any line contains at least three points.

According to Jordan [245], the most general 1-dimensional projection operator (or irreducible idempotent) belonging to the exceptional Jordan algebra J_3^{Ω} can be brought to the form of

$$P = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \begin{pmatrix} \bar{a} & \bar{b} & \bar{c} \end{pmatrix} = \begin{pmatrix} a\bar{a} & a\bar{b} & a\bar{c} \\ b\bar{a} & b\bar{b} & b\bar{c} \\ c\bar{a} & c\bar{b} & c\bar{c} \end{pmatrix} \tag{3c.127}$$

where a,b and $c \in \Omega$ with one of them being purely real, and such that $\mathrm{tr}\, P = a\bar{a} + b\bar{b} + c\bar{c} = 1$.

By way of the alternativity relations $(a\,a)\,b = a\,(a\,b)$, $(a\,\bar{a})\,b = a\,(\bar{a}\,b)$ giving the Moufang identity $a\,(b\,c)\,a = (a\,b)\,(c\,a)$, one finds that

$$P^2 = P \ . \tag{3c.128}$$

Furthermore

$$P \times P = 0 \tag{3c.129}$$

namely, the Freudenthal product is zero.

Now a 2-dimensional projection operator l can be formed with the Freudenthal product of two 1-dimensional projectors P_1 and P_2

$$l_{12} = I - \frac{P_1 \times P_2}{tr[\,P_1 \times P_2\,]} \ , \tag{3c.130}$$

$$l_{12}^2 = l_{12} \ , \ tr(\,l_{12}) = 0 \ , \tag{3c.131}$$

such that

$$P_j \cdot l_{12} = P_j \ , \ \ j = 1, 2 \tag{3c.132}$$

I being the (3 x 3) unit matrix .

In fact the relations

$$tr(\,P_1 \times P_2\,) \cdot P_1 = tr(P_1 \times P_1) \cdot P_2 \ , tr\ (I - l) \cdot P_1 = 0 \tag{3c.133}$$

result from the symmetry of the cubic invariant Tr J^3 of J_3^{Ω} (see Eq. (3b.94)).

To the above relations, we must add the lemma for 1-dimensional projectors P and Q

$$tr(P \cdot Q) = 0 \leftrightarrow P \cdot Q = 0 \tag{3c.134}$$

which shows that Eq. (3c.133) implies Eq. (3c.132).

The Jordan construction of **CaP**(2) is as follows: The points, lines and the plane are represented by 1-dimensional projectors P, 2-dimensional projectors l and the unit matrix I respectively. So a point P is on a line $\leftrightarrow$ $P.l = P$. So (3c.130) is the line passing through P_1 and P_2. The intersection of the two lines $(I - P_1)$ and $(I - P_2)$ is the point $P = \dfrac{P_1 \times P_2}{tr(\,P_1 \times P_2\,)}$.

To proceed further, we recall a key feature of the group F_4 (Sect.3b). Namely, it is the automorphism group of both J_3^{Ω} with its quadratic and cubic invariants and the Freudenthal product (2b.21). Consequently F_4 is also the automorphism group of Jordan's construction of **Ca**P(2). Thus, under F_4 points are mapped into points, lines are mapped into lines, and $P \cdot l = P$ into $P' \cdot l' = P'$. Under an appropriate F_4 transformation, a simplified representation of the projector P is then possible.

If we now define the Jordan matrices

$$E_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \; E_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, E_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad (3c.135)$$

the following lemmas can be proved:

1) There always exists a F_4 transformation bringing P_1 into the form of E_1 with P_2 having the general form (3c.127).

2) Given P_1, P_2 and P_3 with $P_1 \cdot P_2 = P_2 \cdot P_3 = P_3 \cdot P_1 = 0$, then there always is a F_4 transformation bringing them into the form E_1, E_2 and E_3 respectively.

3) Given any two P_1 and P_2, then there always exists a F_4 transformation taking them both into a real form.

4) $\mathrm{tr}\left(P_1 \cdot P_2\right) = 0$ implies $P_1 \; P_2 = 0$.

5) Given two different P_1 and P_3, then P_3 obeys $P_3 \cdot P_1 = P_3 \cdot P_2 = 0$ iff P_3 is a multiple of $P_1 \times P_2$.

6) Indeed any element of J_3^{Ω} can be brought to a diagonal form by a F_4 transformation.

Armed with the above lemmas it can be shown that Jordan's construction fulfills the cited axioms a, b and c of projective geometry. While deferring to Ref.[275] for all the proofs, it is interesting to explicitly show the non-Desarguian nature of the Moufang plane. For that purpose, let us take the three lines

$$L_3 = I - E_3 \quad ; \quad L_2 = I - E_2 \quad ; \quad L_1 = I - \Theta_{23} \tag{3c.136}$$

where E_2 and E_3 are defined as in Eq. (3c.135) and

$$\Theta_{23} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \sin^2\theta & -\sin\theta\cos\theta \\ 0 & -\sin\theta\cos\theta & \cos^2\theta \end{pmatrix} . \tag{3c.137}$$

Their intersection point is represented by the idempotent E_1 since

$$E_2 \times E_3 = \frac{1}{2} E_1 \quad , \quad E_2 \times \Theta_{23} = \frac{\cos^2\theta}{2} E_1 \quad \text{and} \quad E_3 \times \Theta_{23} = \frac{\sin^2\theta}{2} E_1 \ . \tag{3c.138}$$

Next consider two points on each line (see Fig. 6) P_A, $P_{A'}$ on L_3, Q_B and $Q_{B'}$ on L_2 and Q_C and $Q_{C'}$ on L_1 . The matrices $(I - L_{AB})$, $L_{PQ} = \dfrac{P_A \times Q_B}{\operatorname{tr}(P_A \times Q_B)}$ represent the line through P_A (A) and Q_B (B). The intersection point of L_{PQ} (L_{AB}) and $L_{P'Q'}$ $(L_{A'B'})$ is the point S_1. Up to a normalization trace factor

$$S_1 = (P_A \times Q_B) \times (P_{A'} \times Q_{B'}) \ . \tag{3c.139}$$

Similarly, the points S_2 and S_3 are given by $S_2 = (Q_B \times R_C) \times (Q_{B'} \times R_{C'})$ and $S_3 = (P_A \times R_C) \times (P_{A'} \times R_{C'})$.

Were Desargues' theorem to hold, S_1, S_2 and S_3 would lay on one straight line, which translates into the associativity condition

$$[S_1 \times S_2 , S_3] = [S_1, S_2, S_3] = 0 \ . \tag{3c.140}$$

To compute the quantity (3c.140) for the Moufang Plane, Günaydin et al. [275] choose the following representations of the various points :

$$P_A = \frac{1}{2}\begin{pmatrix} 1 & -e_2 & 0 \\ e_2 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \;, \quad P_{A'} = \frac{1}{2}\begin{pmatrix} 1 & -e_1 & 0 \\ e_1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad (3c.141)$$

$$Q_B = \frac{1}{2}\begin{pmatrix} 1 & 0 & e_7 \\ 0 & 0 & 0 \\ -e_7 & 0 & 1 \end{pmatrix} \;, \quad Q_{B'} = \frac{1}{2}\begin{pmatrix} 1 & 0 & e_3 \\ 0 & 0 & 0 \\ -e_3 & 0 & 1 \end{pmatrix}, \qquad (3c.142)$$

$$R_C = \frac{1}{2}\begin{pmatrix} 1 & -\cos\theta\, e_6 & -\sin\theta\, e_6 \\ \cos\theta\, e_6 & \cos^2\theta & \cos\theta\sin\theta \\ \sin\theta\, e_6 & \cos\theta\sin\theta & \sin^2\theta \end{pmatrix}, \qquad (3c.143)$$

$$R_{C'} = \frac{1}{2}\begin{pmatrix} 1 & -\cos\theta & -\sin\theta\, e_5 \\ \cos\theta\, e_5 & \cos^2\theta & \cos\theta\sin\theta \\ \sin\theta\, e_5 & \cos\theta\sin\theta & \sin^2\theta \end{pmatrix}. \qquad (3c.144)$$

A long and tedious calculation gives a nonzero quantity

$$[\, S_1 , S_2 , S_3 \,] = \frac{1}{2^{19}} \cos^2\theta \sin^2\theta \,(\, 3 + \sin\theta - \cos\theta + \sin\theta\cos\theta \,) \;, \qquad (3c.145)$$

which thereby explicitly links the non-associativity of the octonion algebra $\mathbf{\Omega}$ and the non-Desarguian nature of the Moufang Plane.

3.c.5. Spaces with G_2 and Spin (7) holonomy, exceptional calibrated geometries

Riemannian metrics with special holonomy occur frequently as solutions to unified theories. In supersymmetric theories, examples are the complex Kahler, hyperkähler and quaternionic Kähler manifolds with holonomy U(n), Sp(n) and Sp(n)xSp(1) respectively. As to the potentially physically interesting metrics with exceptional holonomy G_2 and Spin(7), their existence had long been in doubt after Berger's classification of holonomy groups of irreducible Riemannian manifolds [78]. Significant progress came in 1987 when Bryant [279] established the existence of Riemannian metrics on open sets of R^7 and R^8 with holonomy group G_2 and Spin(7) respectively. Recently, a few explicit examples of both incomplete as well as complete metrics with exceptional holonomy have been constructed by Bryant and Salamon [76].

Our original intention to briefly survey the above mentioned results was preempted by the timely arrival of two books, one on Riemannian geometry and holonomy groups [76], the other on spinors and calibrations [32]. We therefore refer the interested reader to these fine volumes for a detailed treatment of spaces with exceptional holonomy and of exceptional calibrated geometries.

3.d. Octonionic Function Theory

Since octonions are non-associative, we might expect a function of theory on octonions to differ markedly from quaternion analysis. Luckily, this is not quite the case. From a general viewpoint, one observes that 1) along with the real number field **R**, the division algebras **C**, **H** and **Ω** each constitutes an irreducible Clifford module [280], and 2) theories of functions over a Clifford algebra has been successfully developed over the past two decades [118]. Specifically, in 1973, Dentori and Sce [281] showed that Füeter's theory can be carried over to a significant extend to the theory of function of an octonion variable. Our presentation will be based partly on Refs. [281], [282] and [266].

As in the quaternionic case, we begin by defining the octonionic differential operators D and its conjugate $\overline{D}$:

$$D \equiv e_\mu \frac{\partial}{\partial x_\mu} \quad , \qquad \overline{D} \equiv \bar{e}_\mu \frac{\partial}{\partial x_\mu} \quad . \tag{3d.1}$$

Then $\overline{D}\,D = D\,D \equiv \Box$ is the 8-dimensional Laplacian. Similarly, since **Ω** is non-commutative,we must distinguish three kinds of octonionic analyticity :

$$D\, l\,(x) = 0 \quad , \qquad r\,(x)\, \overleftarrow{D} = 0 \quad , \tag{3d.2a}$$

$$D\,b = b\,\underset{\rightarrow}{D} = 0 \tag{3d.2b}$$

where $l\,(x) = l_\mu(x)\,e_\mu$, $r\,(x) = r_\mu(x)\,e_\mu$ and $b(x) = b_\mu(x)\,e_\mu$ are called left-, right- and left-right analytic respectively. More explicitly, they read

$$Dl\,(x) = (\,\partial_0 + e_\alpha\partial_\alpha\,)\,(\,l_0 + l_\alpha e_\alpha\,)$$

$$= (\,\partial_0\,l_0 - \partial_\alpha l_\alpha\,) + e_\alpha(\,\partial_0 l_0 + \partial_\alpha l_0 + \psi_{\alpha\rho\sigma}\partial_\rho l_\sigma\,) = 0 \quad , \tag{3d.3}$$

$$r\,(x)\,\underset{\leftarrow}{D} = (\,r_0 + r_\alpha e_\alpha\,)\,(\,\underset{\leftarrow}{\partial}_0 + e_\alpha\underset{\leftarrow}{\partial}_\alpha\,)$$

$$= (\,\partial_0\,r_0 - \partial_\alpha r_\alpha\,) + e_\alpha\,(\,\partial_0 r_\alpha + \partial_\alpha r_0 - \psi_{\alpha\rho\sigma}\partial_\rho\,r_\sigma\,) = 0 \quad , \tag{3d.4}$$

$$Dg(x) = (\,\partial_0 g_0 - e_\alpha\partial_\alpha g\,) + e_\alpha(\,\partial_0 g_\alpha + \partial_\alpha g_0 + \psi_{\alpha\rho\sigma}\,\partial_\rho g_\sigma\,) = 0 \quad , \tag{3d.5}$$

$$g(x)\,\underset{\leftarrow}{D} = (\,\partial_0 g_0 - e_\alpha\partial_\alpha g\,) + e_\alpha(\,\partial_0\,g_\alpha + \partial_\alpha g_0 - \psi_{\alpha\rho\sigma}\partial_\rho g_\sigma\,) = 0 \quad . \tag{3d.6}$$

Equivalently,

$$\begin{cases} (\,\partial_0\,l_0 - \partial_\alpha\,l_\alpha\,) = 0 \;, \\ (\,\partial_0\,l_0 + \partial_\alpha\,l_0 + \psi_{\alpha\rho\sigma}\partial_\rho\,l_\sigma) = 0 \,, \end{cases} \tag{3d.7}$$

$$\begin{cases} (\,\partial_0\,r_0 - \partial_\alpha\,r_0\,) = 0 \,, \\ (\,\partial_0\,r_\alpha + \partial_\alpha\;\,r_0 - \psi_{\alpha\rho\sigma}\,\partial_\rho\,r_\sigma\,) = 0 \;, \end{cases} \tag{3d.8}$$

$$\begin{cases} \partial_0\,g_0 - \partial_\alpha\,g_\alpha\; = 0 \;, \\ \partial_0\,g_\alpha + \partial_\alpha\,g_0 = 0 \;, \\ \psi_{\alpha\rho\sigma}\,\partial_\rho\,g_\sigma \;\; = 0 \;. \end{cases} \tag{3d.9}$$

Admittedly, general functions of physical relevance which solve the above equations are hard to find. Yet, as in the quaternionic function theory, there exists a special class of auxiliary $\mathbf{\Omega}$-valued holomorphic functions F(x) from which octonionic analytic functions can be generated. Similarly, these F(x)'s are in turn uniquely tied to their respective complex analytic stem functions. In connection with the Füeter theory (Sect.2d), we remarked as before that the complex structure of the auxiliary functions F(x) with $x \in \mathbf{H}$, Eqs. (2d.14)-(2d.17), is rooted in the complex structure of the 2-sphere $S^2 \approx SU(2) / U(1)$ $((i\mu)^2 = 1)$. Here the complex structure associated with the corresponding octonion analytic functions arises from the almost complex structure of the 6-sphere $S^6 \approx G_2 / SU(3)$ $((i\eta)^2 = 1)$ [273]. Therefore we similarly define the imaginary, purely vectorial unit $\eta \equiv \frac{\mathbf{e}\,\mathbf{x}}{r}$, $\eta^2 = -1$ with $r = \left(\sum_{\alpha=1}^{7} x_\alpha^2\right)^{1/2} \geq 0$ and any octonion $x \in R^8$ has the natural unique orthogonal splitting: $x = x_0 + \eta\, r$.

Just as in the quaternionic case, we introduce the useful projection operators

$$E_\pm = \frac{1 \pm i\eta}{2} \quad . \tag{3d.10}$$

They satisfy the relations

$$E_+ + E_- = 1,\ E_\pm^2 = E_\pm \ ,\ E_+ E_- = E_- E_+ = 0 \ . \tag{3d.11}$$

By considering the complex variable $z = x_0 + i\,r$ ($r \geq 0$), a point in a Poincaré half-plane, we have the decomposition

$$x^n = z^n E_- + z^{*n} E_+ \ , \qquad (n = \text{integer})\ . \tag{3d.12}$$

So if an octonionic function F(x) is expandable in a power series in x with real coefficients, i.e. $F(x) = \sum_n x^n c_n$, $c_n \in \mathbf{R}$, then

$$F(x) = \left(\sum_n z^n c_n\right) E_- + \left(\sum_n z^{*n} c_n\right) E_+ \tag{3d.13}$$

$$= f(z)\, E_- + f(z)^* E_+ \ , \tag{3d.14}$$

with $f(z) = \sum_n z^n c_n = u(x_0, r) + i\, v(x_0, r)$ being complex analytic in the upper-half z plane. Consequently, the octonionic function $F(x) = u(x_0, r) + \eta\, v(x_0, r)$ can be generated from the associated stem function f(z) through the mere replacement of i by the element η of the unit 6-sphere.

While the above functions F(x) are not Ω-analytic, we next show that the derivative function $G(x) = \Box^3 F(x)$ solves Eq. (3d.2b). G(x) is therefore left-right analytic. The close parallel with Fueter's quaternionic analysis suggests that further interesting, special auxiliary functions F(x)'s can be found through the ensuing octonionic formulation of conformal transformations in R^8 (or S^8). This turns out to be indeed the case, as we next examine the transformation group SO(9,1), the Euclidean D=8 conformal group.

We recall that, in R^4 the conformal group is O(5, 1) or spin(5, 1) ≈ SL(2, **H**); a general conformal transformation has the quaternionic form $y = (ax+b)\,(cx+d)^{-1}$. Alternatively $y = M\, x'\, \overline{N} = M\left(\frac{\lambda}{x-A} + \overline{C}\right)^{-1} \overline{N}$ with Vec $\lambda = 0$, $M\overline{M} = N\overline{N} = 1$. Clearly, the unit quaternions M and $\overline{N}$ implement a 4-spacetime rotation, reflecting the semi-simple nature of the Euclidean Lorentz group Spin(4) ≈ Sp(1) x Sp(1) with 6 parameters. λ, the quaternionic scalar parameter represents a dilatation, A, a translation and C, special conformal transformations. They stand for the (6 + 1 + 4 + 4) = 15 parameters of the Euclidean D = 4 conformal group. Similarly, we consider in R^8 the transformation

$$x' = \left(\frac{\lambda}{x-A} + \overline{C}\right)^{-1} \tag{3d.15}$$

with Vec $\lambda = 0$ while x, A and $\overline{C}$ are full octonions. It accounts for only (1 + 8 + 8) = 17 parameters; the remaining part of the conformal transformation is given by a 28-parameter O(8) transformation $T_{O(8)}$:

$$y = T_{O(8)}\, x' \ . \tag{3d.16}$$

In this way the conformal group Spin(9, 1) has (28 + 17) = 45 parameters.

To show the octonionic nature of O(8), we decompose the action in (3d.16) into G_2 and Spin(8) / G_2 ≈ $S^7{}_L \times S^7{}_R$ which, algebraically, are combinations of Spin(8) / Spin(7) ≈ S^7 and Σ^7 ≈ Spin(7) / G_2:

Thus

$$\text{Spin(8)/Spin(7)}: \; x''' = K\, x''\, \overline{K} \qquad (\,|\,K\,| = 1\,)\ , \tag{3d.17}$$

$$\text{Spin(7)/}G_2 \quad : \; x'' = L\, x'\, \overline{L} \qquad (\,|\,L\,| = 1)\ , \tag{3d.18}$$

and

$$G_2: \qquad y = (U\,V)^{-1}\{\,V\,(U\,x''\,U^{-1})\,V^{-1}\}\,(UV)\ , \tag{3d.19}$$

alternatively ,

$$G_2: \qquad y = (U\,V)^{-1}\,\{V\,(\,U\,x''\,U^2\,)\,V^2\,\}\,(\,U\,V\,)^{-2}. \tag{3d.20}$$

Altogether, they give the following Möbius representation

$$y = (\,UV\,)^{-1}\left\{V\left(U\left[K\left(L\left(\frac{\lambda}{x-A}+\overline{C}\right)^{-1}\overline{L}\right)K\right]U^{-1}\right)V^{-1}\right\}(\,UV\,) \tag{3d.21}$$

of a conformal transformation in R^8. Here the key presence of all the nested brackets testifies to the non-associativity of octonions.

Another conformal transformation, not connected to the identity, is obtained by applying an inversion to y :

$$Y(x) = y^{-1} = T_{O(8)}\, x'^{-1} = T_{O(8)}\left(\frac{\lambda}{x-A}+\overline{C}\right)\ . \tag{3d.22}$$

Hence the transformation

$$X(x) = \frac{\lambda}{x-A}+\overline{C} \tag{3d.23}$$

represents the coset space Spin(9, 1) / Spin(8), having (45 – 28) = 17 parameters.

As a pedagogic exercise in non-commutative <u>and</u> non-associative octonionic calculus, we detail below the derivation of some relevant equations satisfied by Y(x) and X(x). We begin with the derivative

$$\partial_\mu X = -\frac{\lambda}{x-A}\,e_\mu\,\frac{1}{x-A}\,. \tag{3d.24}$$

It gives

$$\partial_\mu \partial_\mu X = \left\{ \left(\frac{\lambda}{x-A} e_\mu \frac{1}{x-A} \right) e_\mu \right\} \frac{1}{x-A} + \frac{\lambda}{x-A} \left\{ e_\mu \left(\frac{1}{x-A} e_\mu \frac{1}{x-A} \right) \right\} . \qquad (3d.25)$$

Next, making use of the two Moufang identities a [b (a c)] = (a b a) c and [(c a) b] a = c (a b a) , we obtain the following relations : a [b (a b)] = a [b a b] = (a b a) b and [(b a)b] a = (b a b) a = b(a b a). Applying the latter to the brackets on the left hand side of Eq. (3d.25), we get

$$e_\mu \left(\frac{1}{x-A} e_\mu \frac{1}{x-A} \right) = \left(e_\mu \frac{1}{x-A} e_\mu \right) \frac{1}{x-A} \qquad (3d.26)$$

and

$$\left(\frac{1}{x-A} e_\mu \frac{1}{x-A} \right) e_\mu = \frac{1}{x-A} \left(e_\mu \frac{1}{x-A} e_\mu \right) . \qquad (3d.27)$$

They lead to a simplified expression for the D = 8 Laplacian of X :

$$\Box X = \frac{2\lambda}{x-A} \left(e_\mu \frac{1}{x-A} e_\mu \right) \frac{1}{x-A} \quad . \qquad (3d.28)$$

Moreover, since for any octonion ω, $e_\mu \, \omega \;\; e_\mu = -6 \, \overline{\omega}$, it follows that

$$D\overline{D} X = \Box X = -12 \lambda \frac{1}{|x-A|^2} \frac{1}{x-A} \quad . \qquad (3d.29)$$

We also get

$$D X = e_\mu \partial_\mu X = -\lambda \, e_\mu \left(\frac{1}{x-A} e_\mu \frac{1}{x-A} \right) = -\lambda \left(e_\mu \frac{1}{x-A} e_\mu \right) \frac{1}{x-A} \qquad (3d.30)$$

or

$$D X = 6 \lambda \frac{1}{|x-A|^2} \equiv \rho_0 \quad , \quad \text{Vec}\, \rho_0 = 0 \ , \qquad (3d.31)$$

which defines the quantity ρ_0.

Similarly, the following sequence of useful equations can be derived:

$$\overline{D} \rho_0 = \Box X = -2 \rho_0 \frac{1}{x-A} \quad , \quad D \rho_0 = \Box \overline{X} \ , \qquad (3d.32a)$$

$$-\frac{1}{2}\overline{D}\ln\rho_0 = \frac{1}{x-A} \quad , \tag{3d.32b}$$

$$\square\rho_0 = \frac{-4}{3\lambda}\rho_0^2 \ , \tag{3d.32c}$$

$$\square^2 X = \frac{16}{3\lambda}\rho_0^2\frac{1}{x-A} \quad , \tag{3d.32d}$$

$$\square^3 X = \frac{-32}{3\lambda^2}\rho_0^3\frac{1}{x-A} = -\frac{12^2\,4^2}{\lambda}\frac{1}{|x-A|^6}\frac{1}{x-A} \quad , \tag{3d.32e}$$

$$\square^3 D\frac{1}{x-A} = -\frac{1}{2}\square^4\ln\rho_0 = 0 \ , \tag{3d.32f}$$

$$\square^3 D\,X = \square^3 X\underset{\leftarrow}{D} = 0 \ , \tag{3d.32g}$$

$$\square^4 X = 0 \ . \tag{3d.32h}$$

Consequently, $\square^3 X$ is both left-right- analytic, X is quadri-harmonic. With $\square^4$ being an O(8) invariant,

$$\square^4\,T_{O(8)}\,X = \square^4\,Y = 0 \ . \tag{3d.33}$$

Next we consider the more general meromorphic function made up of a linear superposition of Eq. (3d.23):

$$F(x) = \sum_{i=0}^{n}\left(\frac{\lambda_i}{x-A_i} + \mu_i\right) . \tag{3d.34}$$

It satisfies the following equations

$$D\,F = \rho = \sum_{i=0}^{n}\rho_i \ , \tag{3d.35a}$$

$$\square F = -2\sum_{i=0}^{n}\rho_i\frac{1}{x-A_i} \quad , \tag{3d.35b}$$

$$\square^2 F = \frac{16}{3}\sum_{i=0}^{n}\frac{\rho_i^2}{\lambda_i}\frac{1}{x-A_i} \quad , \tag{3d.35c}$$

$$\square^3 F = -\frac{32}{3}\sum_{i=0}^{n} \frac{\rho_i^3}{\lambda_i^2} \frac{1}{x - A_i} \quad , \tag{3d.35d}$$

$$D\square^3 F = -\frac{32}{3}\square^2 F$$

$$= \frac{32}{3}\sum_{i=0}^{n} \frac{1}{\lambda_i^2} \left(3\,\rho_i^2\,(D\ln\rho_i)\frac{1}{x - A_i} + 6\,\rho_i^3\,\frac{1}{|x - A_i|^2} \right) = 0 \ . \tag{3d.35e}$$

We thus have

$$\square^3 D\,F = \square^3 F \underset{\leftarrow}{D} = 0 \ . \tag{3d.36}$$

In complete analogy to the quaternionic case, let us consider the ansatz for a "gauge potential" of the form

$$\bar{a} = \left(\square^3 F\right)\left(\square^2 D\,F\right)^{-1} = \left(\square^2 D\,F\right)^{-1}\left(\square^3 F\right) \ . \tag{3d.37}$$

Since $D\,F = \rho$, it also reads

$$\bar{a} = \frac{\square^3 F}{\square^2 \rho} = -6\,\frac{\sum_{i=0}^{n} \frac{\rho_i^3}{\lambda_i^2}\frac{1}{x - A_i}}{\sum_{i=0}^{n} \frac{\rho_i^3}{\lambda_i^2}} \tag{3d.38}$$

and

$$\bar{a} = (\bar{D}\square^2\rho)(\square^2\rho)^{-1} = \bar{D}\ln(\square^2\rho) \ . \tag{3d.39}$$

We verify that

$$D\,\bar{a} = (\square^3 D\,F)(\square^2\rho)^{-1} - (\square^3 F)(\square^2\rho)^{-2}(D\square^2\rho) \tag{3d.40}$$

$$= (\square^3 D\,\rho)(\square^2\rho)^{-1}\,\bar{a} \quad , \tag{3d.41}$$

whence the equation

$$D\,\bar{a} + a\,\bar{a} = 0 \ . \tag{3d.42}$$

The latter is the $\mathbf{\Omega}$-counterpart of quaternionic self-dual SU(2) Yang-Mills equation subjected to 't Hooft ansatz.

Similarly, $\bar{a}$ (x) is a $S^7 \to S^7$ mapping as its behavior at spatial infinity is given by $\bar{a} \approx x^{-1} \approx \overline{U}\,\overline{D}\,U$ as $|x| \to \infty$, U being a unit octonion, parametrizing the 7-sphere. To obtain a revealing expression for the winding number, we need the $\mathbf{\Omega}$- extension of the Füeter-Cauchy integral theorem.

If Ω is an 8-dimensional domain in R^8 with boundary $\partial\Omega$, then by Gauss' theorem

$$\int_{\Omega} (\partial_\mu l_\mu)\, d^8x = \int_{\partial\Omega} l_\mu\, d\Sigma_\mu \quad , \tag{3d.43}$$

where $d\Sigma_\mu$ is a component of the surface element $d\Sigma = d\Sigma_\mu e_\mu$.

On the other hand, we have the following decomposition of the product of two octonions $d\Sigma$ and L

$$\begin{aligned}(d\Sigma)\, l &= (d\Sigma_0 + d\Sigma_\alpha e_\alpha)(l_0 + l_\alpha e_\alpha) \\ &= (d\Sigma_0 l_0 - d\Sigma_\alpha l_\alpha) + e_\alpha \left[d\Sigma_0 l_\alpha - d\Sigma_\alpha l_0 + \psi_{\alpha\beta\sigma} d\Sigma_\beta l_\sigma \right] \quad .\end{aligned} \tag{3d.44}$$

By comparing it with Dl and by applying Gauss' theorem to each component we obtain the octonionic form of Eq. (3d.41):

$$\int_{\Omega} d^8x\, D\, l = \int_{\partial\Omega} d\Sigma\, l \ . \tag{3d.45}$$

Similarly,

$$\int_{\Omega} d^8x\, r\, \overleftarrow{D} = \int_{\partial\Omega} r\, d\Sigma \quad , \tag{3d.46}$$

etc.

The corresponding Cauchy-Morera-Füeter theorems are

$$\int_{\partial\Omega\approx S^7} d\Sigma\, l \;=0\;,\quad \int_{\partial\Omega\approx S^7} r\; d\Sigma\;=0\;, \tag{3d.47}$$

$$\int_{\partial\Omega\approx S^7} b\; d\Sigma \;= \int_{\partial\Omega\approx S^7} d\Sigma\; b = 0\;. \tag{3d.48}$$

We mention in passing that octonionic analyticity, in the sense of (3d.48), implies the existence of an infinite number of continuity equations, the Euclidean analogs of conservation laws. Here due to the ring property of the left-right holomorphic function F(x) and the power associativity of octonions, not just F(x) but any F(x) raised to any power n, $[F(x)]^n$, n = 2, 3, 4, ... also solves for the generalized Cauchy-Riemann equations

$$D\, J^{(n)}(x) = D\left(\Box^3[\, B(x)\,]^n\right) = 0\;. \tag{3d.49}$$

They form an infinite set of octonionic continuity equations. The above integral theorem implies that the Euclidean charges

$$q^{(n)}(\tau) = q_\mu^{(n)} e_\mu = \oint_{S^7} f_1^n\, d\Sigma\, f_2^n \tag{3d.50}$$

with $f_1^n = \Box^3 F_1^n$ and $f_2^n = \Box^3 F_2^n$ are independent of τ, parametrizing the family of D = 7 hypersurfaces $S_\tau{}^7$ in S^8. So the $q^{(n)}$ are conserved

$$\frac{dq^{(n)}}{d\tau} = 0\;. \tag{3d.51}$$

To compute the winding number of a topological mapping, the potential $\bar{a}$ (3d.37), we consider a function $r(x)$ on S^8 such that $Dr(x)= \sum R_i\, \delta^8(\,x - C_i)$. Then Stokes' theorem gives

$$\int_{\Omega\approx S^8} d^8x\; r\; \overleftarrow{D} = \int_{S^7} r\; d\Sigma \;= \sum_{i=0}^{n} R_i\;, \tag{3d.52}$$

namely, the sum of the residues at the poles $x_i = C_i$, provided that the latter are located within the domain Ω of R^8. Taking $r = \Box^3 F$ and $\bar{a}(x)$ given by (3d.38), in the infinitesimal neighborhood of the poles, $\bar{a}\,(\,C_i + \varepsilon\,) = -\frac{6}{\varepsilon}$ or

$$\bar{a} = -\frac{6}{x - C_i} + \text{a regular function in } (x - A_i) \tag{3d.53}$$

Since the action of $\Box^3\,D$ on a regular function gives zero,

$$D\Box^3\,\bar{a} = -6\,V\,\delta^8(\,x - C_i\,)\ . \tag{3d.54}$$

$V = \frac{\pi^4 r^7}{3}$ is the volume of a 7-sphere of radius r. By surrounding each pole $x = C_i$ by a small 7-sphere $S_i^{\,7}$, it is clear that the integral $\sum_{i=1}\oint_{S_i^7} d\Sigma\,\Box^3\,\bar{a}$ counts the number of poles in $\bar{a}$. Due to the conformal invariance, one term in the expression for $\bar{a}$, e.g. $(\,x - C_i\,)^{-1}$ can be transformed into a regular function x by a conformal transformation, say by a coordinate inversion. Hence out of (n+1) poles in $\bar{a}$ only n are significant; we obtain for the winding number

$$C_4 = \frac{1}{2\pi^4}\oint_{S^7} d\Sigma\,\Box^3\bar{a} = n \tag{3d.55}$$

or

$$C_4 = \frac{1}{2\pi^4}\oint_{S^7} d\Sigma\,\Box^3\bar{D}\ln(\,\Box^2\rho\,) = n\ . \tag{3d.56}$$

By Stokes' theorem, it takes the 8-dimensional form of

$$C_4 = \frac{1}{2\pi^4}\oint_{S^8} d^8x\,\Box^4\ln\left(\,\Box^2\rho\,\right)\ . \tag{3d.57}$$

Note that these expressions (3d-52)-(3d.54) have their exact counterparts in the quaternionic context of the SU(2) instanton solutions of t' Hooft -Jackiw - Rebbi - Nohl [176].

The above formulae only illustrate a generalized residue theorem due to Dentori and Sce [281]. For a left-analytic function $l(x)$ ($Dl = 0$), it reads

$$l(x) = \frac{1}{48\,(2\pi)^4} \int_{S^7} (\,l(y)\,d\Sigma\,)\,\Box^3 \frac{1}{y-x}\,, \tag{3d.58}$$

where S^7 is a closed hypersurface enclosing the point $x = y$.

While a unified integral formula (2d.43) exists for both left and right **H**-analytic functions, an analogous formula is *not* available in the octonionic case, due to the lack of associativity. Instead one has

$$\oint_{S^7} (\,l(x)\,d\Sigma\,)\,b(x) = 0 \tag{3d.59}$$

where $l(x)$ is left-analytic while $b(x)$ (e.g. $\Box^3 \frac{1}{y-x}$ in Eq. (3d.55)) is <u>both</u> left and right analytic.

To further see this difference between leftness and rightness in the four and eight dimensions, we recall the algebraic decomposition of the respective norm groups of **H** and **Ω**. Recall that $O(4) \approx SU(2)_L \times SU(2)_R \approx S^3{}_L \times S^3{}_R$ while $O(8) \approx Sp(8)/Sp(7) \times Sp(7) / G_2 \times G_2 \approx S^7{}_L \times S^7{}_R \times G_2$. So, in one case, leftness and rightness commute. In the other , they do not and are correlated by G_2, the automorphism group of **Ω**. This manifestation of non-associativity suggests new levels of difficulty as well as further richness and subtleties in structure among octonionic left or right regular functions. We have not elaborated on them since, to the best of our knowledge, nothing is explicitly known. Just as in the topic of exceptional geometries, much work needs to be done in the theory of octonionic functions. Future developments in non-associative geometries and function theory would be accelerated if, as in the spirit of Kaluza-Klein compactification, non-perturbative string theories or in generalized electric/magnetic duality for extended objects [283], some physical relevance could be found for a deeper understanding of eight dimensional space and octonionic analyticity.

3.e. Arithmetics of Octonions

We follow here the treatment of Ref.[284]. Like quaternions, integer octonions ω obey the characteristic rank equation $\omega^2 - 2\,Sc(\omega)\,\omega + N(\omega) = 0$. Therefore integer octonions also have integer or half integer components ω_μ, $\mu = 0, 1, \ldots, 7$. Consider the unit norm integer octonions. Geometrically, they are special points on S^7 ($N(\omega) = 1$)

embedded in R^8. First, it is convenient to define a set of 8 auxiliary octonionic units of quadratic norm $\frac{1}{2}$. They are made out of the seven canonical imaginary units e_i (i = 1, 2, 3, ..., 7):

$$l_1 = \frac{1}{2}(e_1 + e_4)\,,\; l_2 = \frac{1}{2}(e_2 + e_5)\,,\; l_3 = \frac{1}{2}(e_3 + e_6)\,,\; l_8 = \frac{1}{2}(e_0 + e_7)$$
$$l_4 = \frac{1}{2}(e_1 - e_4)\,,\; l_5 = \frac{1}{2}(e_2 - e_5)\,,\; l_6 = \frac{1}{2}(e_3 - e_6)\,,\; l_7 = \frac{1}{2}(e_0 - e_7)\,. \tag{3e.1}$$

They are thus related to the 8 split octonionic units already defined in Eq. (3a.57).

Next, we consider the octonions with one or four non-zero components $\rho_{rs} = \pm\, l_r \pm l_s$ ($r \neq s$) we have $|\rho_{rs}|^2 = 1$. There are a total of 112 such roots. 16 of the latter, namely ($\pm\, l_8 \pm l_7$), ($\pm\, l_1 \pm l_4$), ($\pm\, l_2 \pm l_5$) and ($\pm\, l_3 \pm l_6$) have only one nonzero component equal to ± 1. The remaining 96 have four nonzero components, each with values $\pm\frac{1}{2}$. One readily checks that they form the 112 roots of O(16). Moreover there are unit normed integer octonions of the form

$$\sigma = \frac{1}{2}\,(\pm\, l_1 \pm l_2 \pm l_3 \pm l_4 \pm l_5 \pm l_6 \pm l_7 \pm l_8) \tag{3e.2}$$

with any odd number of minus signs. There are 128 such roots. ρ_{rs} and σ together form the 240 roots of E_8 with those given by σ being in the coset $\frac{E_6}{O(16)}$.

Another useful decomposition of the 240 roots of E_8 is available by way of a ninth octonionic unit :

$$l_0 = \frac{1}{2}\,(e_0 + e_1 + e_2 + e_3)\;. \tag{3e.3}$$

The 72 roots of the form

$$\pm\,(l_\alpha - l_\beta) \quad , \quad (\alpha, \beta = 0, 1, \ldots, 8) \tag{3e.4}$$

are those of the E_6 subgroup of E_8. Since unlike the standard normalization ($\vec{r} \cdot \vec{r} = 2$) these roots are of unit length, the angle θ_{ab} between the roots r_a and r_b is $\cos\theta = \mathrm{Sc}\,(r_a - \bar{r}_b)$.

By choosing the number of principal positive roots of an algebra to equal the rank of the group, we can draw the corresponding Dynkin diagram. This is done by connecting the points of the diagram if $\cos\theta = -\frac{1}{2}$ and leaving them disconnected if $\cos\theta = 0$. Thus, in the O(8) case, the principal roots are

$$r_1 = e_1 \ , \ r_2 = e_2 \ , \ r_0 = -\frac{1}{2}(e_0 + e_1 + e_3) \ , \tag{3e.5}$$

representing the points P_i ($i = 1, 2, 3$) and P_0 of the Dynkin diagram. We then have an O(8) diagram with its three-fold symmetry connected to the three quaternionic units e_i.

The remaining units are obtained from the principal units by Weyl reflections. Thus, if r_a and r_b are two roots, a third root is generated by reflecting r_a w.r.t. the hyperplane with normal r_b. The corresponding division algebra formula is simply

$$r_{ab} = - r_b \bar{r}_a r_b \quad , \quad (|r_a| = |r_b| = 1) \tag{3e.6}$$

which is equivalent to the Weyl reflection formula

$$r^{\mu}_{ab} = |r_b|^2 \bar{r}^{\mu}_a - 2\,\mathrm{Sc}\,(\bar{r}_a r_b)\, r^{\mu}_b = r_a - 2\frac{(r^{\gamma}_a r_{b\nu})\, r^{\mu}_b}{|r_b|^2} \ . \tag{3e.7}$$

From Eq. (3e.5) r_{ab} is then an integer with unit norm and therefore is a root. When $a = b$, r_a turns into $-r_a$, also a root. The whole root system is thus closed under Weyl reflections. Our discussion naturally generalizes to lattice structures generated by discrete Jordan algebras, a topic discussed Ref.[284]. Some further works on number theory of octonions are to be found in the articles of Linnik [285], Coxeter [286], and a beautiful series of papers of Koca [287] and the references therein.

3.f. Some Physical Applications

3.f.1. Exceptional quantum mechanical spaces as charge spaces and unified theories

When Jordan, von Neumann and Wigner [65] discovered the first exceptional quantum mechanical space, they failed to find for it a physical interpretation. Since then such spaces have been generalized in mathematics. They have been understood as non-

Desarguesian geometries associated with the exceptional groups and all admit SU(3) x $SU^C(3)$ as a common subgroup. One of these SU(3)'s arises as a subgroup of the automorphism group G_2 of the octonion algebra while the other SU(3) comes from the (3x3) structure of the Jordan matrices. Since SU(2) x U(1) is a maximal subgroup of SU(3), we can say that these exceptional quantum mechanical spaces have a natural invariance under $SU^C(3)$ x SU(2) x U(1). Obviously the latter is the group of the standard model, the smallest group needed to classify all currently known particles.

Due to space constraint, we refrain from a detailed presentation of applications of exceptional group to unified theories. Due to the nonobservation of proton decay at the predicted rates, grand unified theories [288] may have suffered a possibly premature fall from grace. Even the literature on GUTs based on exceptional groups is vast and varied. Here we shall only illustrate a few of their features closely connected to octonions.

Previously, we have seen that exceptional groups are related to some unique algebras and geometric structures. Specifically, we recall that the (27) representation of E_6 is the only realization of an exceptional Jordan algebra. The coset manifolds E_6 / SO(10) x SO(2) and E_7 / E_6 x SO(2) are the only exceptional Hermitian symmetric spaces or positivity domains; they then provide definite models for new finite Hilbert spaces for internal symmetries. It is this latter aspect of exceptional groups which motivated their original application to particle multiplets.

In fact, exceptional groups are natural generalizations of SU(3) and SU(3) x SU(3) which have so mysteriously emerged in particle physics. The color $SU^C(3)$ group has a natural embedding in the octonionic structure of the exceptional groups. As to the flavor SU(3) group, it is connected with the triality property of the octonion algebra. It is the SU(3) group the Cartan subalgebra of which naturally gives the electric charge and (B-L) (baryon -lepton) quantum number.

While the E-series of exceptional groups consist strictly of E_6 , E_7 and E_8, by truncating their Dynkin diagrams one can define E groups of lower rank, isomorphic to classical groups. There are two different extrapolations to rank 4, namely E_4 and E'_4,

$$E'_4 \approx SO(8)\ ,\ E_4 \approx SU(5)\ \ ,E_5 \approx SO(10)\ , \tag{3f.1}$$

which occur among the first proposed unified gauge groups: E'_4 unifying color and electric charge in supergravity, E_4 and E_5 realizing the oldest successful GUT models.

A remarkable feature of all exceptional groups taken as gauge groups is their freedom from Adler-Bell-Jackiw (ABJ) anomalies. They lead to renormalizable theories with fermions in one representation. Furthermore, in the same context of anomaly freedom, there was the emergence in the last ten years of the D = 10 heterotic superstrings with their unique gauge groups $E_8 \times E_8$ or $SO(32) / Z_2$ [9].

Finally, just as the orthogonal, unitary and symplectic groups are associated respectively with real numbers, complex numbers and quaternions, exceptional groups are uniquely connected to octonions. They realize the most symmetrical Lie groups in that they maximize the ratio $\frac{d}{r}$, d and r being respectively the dimension and the rank of the group.

As an illustration, let us take the standard version of the E_6 grand unified theory. It is the direct extension of the well studied $SO(10) \approx E_5$ model. In the latter model, the basic fermions fit in the 16-dimensional spinor representation of SO(10), which contains the u, d, e, ν_L and a possibly superheavy ν_R, with two other similar families associated with the muon and the tau lepton. The needed basic scalar Higgs fields are in the 10, 16 and 45 dimensional representations of SO(10). The gauge bosons belong to the adjoint (45) representation. Through two-loop diagrams involving gauge bosons and fundamental Higgs particle, an effective Higgs field belonging to the 126-dimensional representation is generated and is responsible for the small masses of the left-handed neutrinos.

A similar construction applies to the standard E_6 model. There are 3 families (27-plets) of quarks and leptons. Since with respect to SO(10) one has the decomposition 27 = 16 + 10 + 1, the model makes use of both the vector and the singlet SO(10) representations, the latter being associated with much heavier fermions.

The gauge bosons belong to the adjoint (78) representation, which includes, besides the 45 bosons of the SO(10) model, a singlet and two 16 and 16' spinor representations.

Two basic Higgs fields are needed. The first (78) has no Yukawa coupling to fermions and can give superheavy masses to leptoquarks mediating proton decay. The second, which is in a (27) representation, can have Yukawa coupling to basic fermions and thus split their masses. If the (27) also acquires some superheavy vacuum expectation values, some of the quarks and leptons could have superheavy masses.

Here one has the possibility of a two-loop Witten diagram. Assuming that the gauge hierarchy works, this diagram will produce an effective Higgs field in the 351' representation of E_6. It was noted that the latter, which embodies the 126 representation of SO(10), gives a) superheavy masses to right handed neutrinos while, upon diagonalization of the mass matrix, b) small masses to left handed neutrinos .

Next we go into the standard E_6 assignment with 3 families where the third family contains the top quark. Let $J^e{}_{27}$ represents the electron family containing the u and d quarks. Under SU(3) x SU(3)′ x SU(3)^c, its decomposition reads

$$L = \begin{pmatrix} \hat{N}_R & \hat{E}_R & \hat{e}_R \\ E^-_R & N_L & \hat{\nu}^e_R \\ e^-_L & \nu^e_L & N'_L \end{pmatrix} , \quad Q^i_L = \begin{pmatrix} u^i_L \\ d^i_L \\ \hat{B}^i_L \end{pmatrix} , \hat{Q}^i_R = \begin{pmatrix} u^i_R \\ d^i_R \\ \hat{B}^i_R \end{pmatrix} , \tag{3f.2}$$

i = 1, 2, 3 is the color index. We have used the notations

$$\psi_L = \frac{1}{2} (I + \gamma_5) \psi , \quad \hat{\psi}_R = i \, \sigma_2 \psi^*_L = \frac{1}{2} (I + \gamma_5) \psi^c \tag{3f.3}$$

for 2-component parts of the Dirac spinor ψ and its charge conjugate ψ^c. If A and B stand for (3 x 3) unitary matrices associated with the first and the second flavor SU(3) groups, respectively, then we have the transformation law

$$L' = B \, L \, A^\dagger \ , \quad Q'^i_L = A \, Q^i_L \ , \quad \hat{Q}'^i_R = B^* \hat{Q}^i_R \ . \tag{3f.4}$$

SU(2) is contained in A, while the neutral SU'(2) is contained in B. The weak SU(2) doublets are

$$\begin{pmatrix} \nu^e_L \\ e^-_L \end{pmatrix} , \begin{pmatrix} N_L \\ E^-_L \end{pmatrix} , \begin{pmatrix} \hat{E}_L \\ \hat{N}_R \end{pmatrix} , \begin{pmatrix} u^i_L \\ d^i_L \end{pmatrix} , \tag{3f.5}$$

while the SU′(2) doublets are

$$\begin{pmatrix} E^-_L \\ e^-_L \end{pmatrix} , \begin{pmatrix} N_L \\ \nu^e_L \end{pmatrix} , \begin{pmatrix} \hat{\nu}^e_R \\ N'_L \end{pmatrix} , \begin{pmatrix} \hat{d}^i_R \\ \hat{B}^i_R \end{pmatrix} . \tag{3f.6}$$

Here E^- is an additional charged lepton, N_L, N'_L and N_R are additional leptons and B^i an additional quark of charge – 1/3. Depending on the vacuum expectations of the Higgs field, these additional fermions could in principle be heavy or superheavy. Standard phenomenology is recovered if the neutral SU'(2) vector bosons are an order of magnitude more massive than the W's and Z's.

Next, we consider the SU(3) subgroup of the flavor SU(3) x SU'(3) obtained by setting A = B. Its Cartan subalgebra consists of the electric charge q and the hypercharge y associated with the (3 x 3) matrices

$$q = \frac{1}{2}\left(\lambda_8 + \sqrt{3}\lambda_3 \right) \ , \ y = \frac{1}{\sqrt{3}} \lambda_8 \ . \tag{3f.7}$$

One obtain the transformation laws induced by these generators by setting

$$q : A = B = \exp(iq\alpha) \ , \tag{3f.8a}$$

$$y : A = B = \exp(iy\phi) \ . \tag{3f.8b}$$

Equation (3f.8a) is just the electric charge U(1) subgroup of the electroweak $SU_W(2)$ x $U_W(1)$. Therefore, we identify U(1) with the generator ($I_8 + I'_8 + \sqrt{3}\, I'_3$) of SU′(3) with the charge operator

$$q = \left(I_3 + \frac{1}{\sqrt{3}} I_8 \right) + \left(I'_3 + \frac{1}{\sqrt{3}} I'_8 \right) . \tag{3f.9}$$

Equation (3f.8b) corresponds to the generator

$$y = \frac{2}{\sqrt{3}}\left(I_8 + I'_8 \right) \ . \tag{3f.10}$$

It induces the transformation

$$\left(e^-_L , \nu^e_L\right) \rightarrow \left(e^-_L , \nu^e_L\right) \exp(- i\phi) \ , \ \begin{pmatrix} \hat{e}_R \\ \nu^e_R \end{pmatrix} \rightarrow \exp(i\phi) \begin{pmatrix} \hat{e}_R \\ \nu^e_R \end{pmatrix} , \tag{3f.11}$$

$$\begin{pmatrix} u^i_L \\ d^i_L \end{pmatrix} \rightarrow \exp(i\frac{\phi}{3}) \begin{pmatrix} u^i_L \\ d^i_L \end{pmatrix} \ , \ \begin{pmatrix} \hat{u}^i_R \\ \hat{d}^i_R \end{pmatrix} \rightarrow \exp(- i\frac{\phi}{3}) \begin{pmatrix} \hat{u}^i_R \\ \hat{d}^i_R \end{pmatrix} . \tag{3f.12}$$

Thus we identify y with the baryon number minus the lepton number y = (B − L).

The additional leptons E, N, N′ have zero hypercharge (B − L = 0) while the extra quark B has hypercharge − 2/3. Consequently, in the standard E_6 scheme, the usual leptons and quark are doublets under the isospin group $\vec{I} + \vec{I}'$. Singlets and triplets are either heavy or superheavy.

Due to the Clebsch-Gordan series

$$(27 \times 27)_{sym.} = \overline{27} + 351' , \tag{3f.13}$$

we can define two invariant tensors d_{abc} and $h_{ab}{}^{mn}$, which project out respectively the $\overline{27}$ and 351′ parts of the symmetric direct product of two 27-plets H^a and H^b. Let H^a denote a $\overline{27}$-Higgs and G_{mn} an effective 351′-Higgs field, then there are the Yukawa couplings of the fundamental 27-plet fermion fields with H^a and G_{mn} :

$$L_{\psi\psi H} = g_H(\psi_L^a)^T i\, \sigma_2 \psi^b\, d_{abc} H^c , \tag{3f.14}$$

$$L_{\psi\psi G} = g_G(\psi_L^a)^T i\, \sigma_2 \psi^b\, h_{ab}^{mn} G_{mn} . \tag{3f.15}$$

Moreover, there can be the vector coupling of ψ to the gauge bosons $V^a_{q\mu}$, namely

$$L_{\bar{\psi}\psi V} = g(\psi_L^a)^\dagger\, \sigma_\mu \psi_L^b\, k_{bm}^{an} V_n^{m\mu} , \tag{3f.16}$$

g is the universal gauge coupling constant, k_{bm}^{an} denotes the invariant tensor projecting out the (78) part of the Clebsch-Gordan series 27 x $\overline{27}$ = 1 + 78 + 650.

Similarly, the gauge coupling of H^a reads

$$L_{H\bar{H}VV} = g^2 \left(k_{bm}^{an}\, V_n^{m\mu}\, H^b \right) \left[k_{am'}^{b'n}\, V_{n'\mu}^{m'} \left(H^b \right)'^\dagger \right] . \tag{3f.17}$$

Finally, we also need the cubic self-coupling of the Higgs H^a:

$$L_{HHH} + L_{\bar{H}\bar{H}\bar{H}} = \mu \left[H^a H^b H^c d_{abc} + (H^a)^\dagger (H^b)^\dagger (H^c)^\dagger d^{abc} \right] \tag{3f.18}$$

with the constant μ having the dimension of a mass.

The above mentioned 2-loop Witten diagram [289] arises via Wick contraction in the integrand

$$L_{\psi\psi H} L_{\bar{\psi}\psi V} L_{\bar{\psi}\psi V} L_{H\bar{H}VV} L_{\bar{H}\bar{H}\bar{H}} , \tag{3f.19}$$

leading to an effective interaction $L_{\psi\psi\bar{H}\bar{H}}$ of the form (3f.15) with

$$G^{\mathrm{eff}}_{mn} = (H^m)^\dagger (H^n)^\dagger \tag{3f.20}$$

and

$$g^{\mathrm{eef}}_{G} \approx g_H\, g^4\, \mu\, K , \tag{3f.21}$$

the factor K coming from the propagators.

Let the VEV's of H be v, the resulting fermion masses are then of order g_H v from $L_{\psi\psi H}$ and induced mass contributions δm are of order $g^{\mathrm{eff}}_G v^2$ from $L_{\psi\psi\bar{H}\bar{H}}$ for fermions left massless in $L_{\psi\psi H}$. Furthermore we need an effective 351' representation to give neutrinos mass. As shown by Witten [289], the neutrinos will acquire masses of order $m_q m_W \alpha_s^{-2} M^{-1}$ from the two-loop diagram, where M, m_q and m_W are the GUT, quark and W masses, respectively.

While the present volume is in no position to review the renewed interest and developments in E_6 grand unification ushered in by superstring phenomenology [290], we may mention that string theories also predict an extra U(1) group surviving below the unification scale. The phenomenological implications of the latter and the neutral current data constraints had been analysed in several quarters. Related works on the extra neutral gauge bosons, the exotic fermions, along with their phenomenology and related symmetry breaking patterns have also been studied, particularly in a unifying (3 x 3) matrix formalism.

3.f.2. S^7 and compactification of D = 11 supergravity

As a model of unification, the D = 4, N = 8 supergravity theory of de Wit and Nicolai [291, 292] is phenomenologically not viable; it does not manifestly contain the spectrum of the Standard Model. In a broader context, it may be viewed as only one of many dimensional reductions of the unique, D = 11, N = 1 supergravity of Cremmer and Julia [293]. Recently, the latter theory has been of renewed interest since it in turn emerges as the zero slope limit of the D = 11 octonionic super 2-membrane theory [294]. Such a membrane theory may be the only quantum mechanically consistent super p-brane theory which encompasses the D = 10 superstrings. To illustrate the workings of octonions, we limit ourselves to some salient aspects of D = 11 supergravity and its spontaneous compactifications. Our brief review follows closely the works of Englert et al. [295, 296, 297]. For more details, the curious reader should consult refs.[298] and [299].

It is enough to recall that, supersymmetry sets stringent restrictions on the dimension of spacetime and on matter couplings. Thus for a consistent supergravity, if the maximum allowed spin is two then the maximum permissible dimension is eleven. In D=11 supergravity, supersymmetry allows only three superfield multiplets, all of them gauge fields. They are 1) the graviton, a 11-bein $E_M{}^A$, then the gravitino with 44 degrees of freedom, 2) one (hence N = 1 supersymmetry) 32-component Rarita-Schwinger-Majorana spinor Ψ_M with 128 degrees of freedom, 3) an antisymmetric rank three gauge field A_{MNP} with its corresponding field strength $F_{MNPQ} \equiv 4\,!\,\partial_{[\,M}A_{NPQ\,]}$. It has 84 to ensure that hallmark of supersymmetry, the equality between bosonic and fermionic degrees of freedom. The symmetries of this theory are 1) d = 11 general covariance, 2) local SO(1,10) Lorentz invariance 3) N = 1 supersymmetry, i.e. invariance under the following local supersymmetry transformations parametrized by ε

$$\delta E_M{}^A = -\frac{i}{2}\bar{\varepsilon}\,\tilde{\Gamma}^A\,\Psi_M \ , \tag{3f.22}$$

$$\delta A_{MNP} = \frac{\sqrt{2}}{8}\bar{\varepsilon}\,\tilde{\Gamma}_{[\,MN}\Psi_{P\,]} \ , \tag{3f.23}$$

$$\delta\Psi_M \equiv \bar{D}_M\varepsilon = D_M\varepsilon + \frac{\sqrt{2}}{288}\,i\left(\tilde{\Gamma}_M^{NPQP} - 8\,\delta_M^N\Gamma^{NPQP}\right)\varepsilon F_{NPQR} \ . \tag{3f.24}$$

$\bar{D}_M$ denotes the supercovariant derivative and ε, a SUSY transformation parameter. The D = 11 $\tilde{\Gamma}$ matrices are (32x32) matrices representable as direct products $\Gamma_\mu = \gamma_\mu \otimes I$, $\tilde{\Gamma}_m = \gamma_5 \otimes \Gamma_m$ of the usual 4x4 γ-matrices and the (8x8) real symmetric matrices Γ_m generating the D=7 Clifford algebra. In the above and in what follows, capital letters denote eleven-dimensional indices (M, N, P,...= 0, 1, ...,10) while greek (μ, ν,... = 0, 1, 2, 3) and small latin letters (m, n, p,...= 4, 5, ..., 10) denote four and seven dimensional indices, respectively.

4) Maxwell-like abelian gauge invariance

$$\delta A_{MNP} = \partial_{[M}\Lambda_{NP]} \tag{3f.25}$$

and finally, 5) invariance under odd number of space or time reflections together with $A_{MNP} \to -A_{MNP}$.

Notably, there is a *unique* action invariant under the above transformations. No other matter couplings, nor the inclusion of a cosmological constant are allowed. In the following discussion of spontaneous compactification, i.e. of possible classical candidates for ground states, we actually only need the bosonic part of that action, A_B; having set the VEV of the fermion field to zero, $\langle \Psi_M \rangle = 0$, the first requirement of maximal symmetry. The relevant action reads:

$$A_B = \int d^{11}x \sqrt{g^{11}} \{ -\frac{1}{2} R - \frac{1}{48} F_{MNPQ}F^{MNPQ}$$

$$+ \frac{\sqrt{2}}{6\,(4!)^2} \frac{1}{\sqrt{g^{11}}} \varepsilon^{M_1M_2\ldots M_{11}} F_{M_1M_2M_3M_4} F_{M_5M_6M_7M_8} A_{M_9M_{10}M_{11}} \} \quad . \tag{3f.26}$$

Capital letters denote D = 11 spacetime indices, the signature of the metric being chosen as (+, − − − ...). It describes a D = 11 theory of Einstein gravity coupled with a 4-index ATGF $F_{M_\alpha M_\beta M_\gamma M_\rho}$ augmented by a Chern-Simons term. The corresponding equations of motions read

$$R_{MN} - \frac{1}{2} g_{MN} R = -\frac{1}{48}\left[8\, F_{MPQR}F_N^{\;PQR} - g_{MN}\, F_{SPQR}F^{SPQR} \right] , \tag{3f.27a}$$

$$F^{MNPQ}{}_{;M} = -\frac{\sqrt{2}}{2\,(4!)^2} \frac{1}{\sqrt{g^{(11)}}} \varepsilon^{M_1\ldots M_8NPQ} F_{M_1M_2M_3M_4} F_{M_5M_6M_7M_8} \; . \tag{3f.27b}$$

A spontaneous compactification (of potential physical relevance) occurs if these classical equations admit solutions which spontaneously split M_{11}, the D = 11 spacetime, into a product $M_4\{x\} x M_7\{y\}$, of a D = 4 spacetime and some D = 7 internal manifold. While Eq. (3f.27) restricts M_4 to be either flat Minkowski space or an Einstein space, there are more options for M_7.

Clearly, a solution in a suitable M_4 with a Lorentz invariant ground state $|\,\Omega >$ can preserve a local supersymmetry only if

$$\langle \Omega \,|\, \delta \Psi_M \,|\, \Omega \rangle \equiv \langle \Omega \,|\, \bar{D}_M \varepsilon(x,y) \,|\, \Omega \rangle = 0 \tag{3f.28}$$

for at least one 32-dimensional Majorana spinor of the form $\varepsilon(x,y) = \varepsilon(x) \otimes \eta(y)$. $\eta(y)$

and $\varepsilon(x)$ denote a (commuting) 8-spinor on M_7 and an arbitrary (anticommuting) 4-component Majorana spinor, respectively.

Lorentz invariance dictates that the only nonzero components of F_{NMPQ} are F^{nmpq} and $F^{\mu\nu\rho\sigma}$ with

$$F^{\mu\nu\rho\sigma} = \frac{f}{\sqrt{g^{(4)}}}\,\varepsilon^{\mu\nu\rho\sigma} \ . \tag{3f.29}$$

Then (3f.28), the condition for unbroken supersymmetry, implies

$$\bar{D}_\mu\, \varepsilon^I \equiv (\, D_\mu + m_7\, \gamma_\mu \gamma_5\,)\, \varepsilon^I = 0 \ , \tag{3f.30}$$

$$\left(\Gamma_m^{npqr} - 8\, \delta_m^n\, \Gamma^{pqr} \right) F_{npqr}\, \eta^I(y) = 0 \ , \tag{3f.31}$$

and

$$\bar{D}_m\, \eta^I \equiv (\, D_m - \frac{m_7}{2}\, \Gamma_m\,)\, \eta^I = 0 \tag{3f.32}$$

with

$$m_7 = \frac{f}{3\sqrt{2}} \ . \tag{3f.33}$$

The covariant derivative D_m is taken w.r.t. the Riemannian metric of M_7. Equation (3f.33), being the Killing equation, means the spinors η^I are covariantly constant, with the index I runs over its solutions, i.e. over the number of independent unbroken supersymmetry.

We see that m_7 or f acts as an order parameter for the spontaneous compactification of M_{11} into $M_4 \otimes M_7$, with the noteworthy feature that four dimensionality of space-time resulting from the rank four nature of the field F_{MNPQ} . The case $f = 0$ gives the well-known compactification on the 7-torus $M_7 = T^7$ with a Minkowski background. Its zero mass sector corresponds to the D = 4, N = 8 supergravity of Cremmer and Julia [293].

The solutions of Eqs. (3f.32) ((3f.30)) are called *Killing spinors* of M_7 (M_4) . From Eq. (3f.32) and with

$$R_{mn} = -\,6\, m_7^2\, g_{mn} \tag{3f.34}$$

and

$$R_{mnpq} = -\, m_7^2(\, g_{mp}g_{nq} - g_{mq}g_{np}) \tag{3f.35}$$

on M_7, the necessary (but not sufficient) condition for the existence of Killing spinors on M_7 is the integrability condition

$$[\, \overline{D}_m\, ,\, \overline{D}_n\,]\, \eta^I(y) = \frac{-1}{4} W_{mn}{}^{ab}\Gamma_{ab}\, \eta^I(y) = 0 \tag{3f.36}$$

where $W_{mn}{}^{ab}$ is the Weyl tensor. And the subgroup of SO(7) generated by the above linear combinations of the SO(7) generators Γ_{ab} is the *holonomy group* $\mathcal{H}$ of the generalized connection in Eq. (3f.32). So the maximum allowed number of unbroken supersymmetries N_{max} equals the number of spinors left invariant by the group $\mathcal{H}$. Clearly the (8x8) matrix commutator $\left[\overline{D}_m\, ,\, \overline{D}_n\right] = 0$ has all the 8 eigenvalues zero; hence the index I runs over 1 to 8 and we have $N_{max} = 8$ supersymmetry. So the maximal symmetric solution (3f.34) is the 7-sphere, $S^7 \approx \frac{SO(8)}{SO(7)}$, of radius $r = m_7^{-1}$, with $W_{mn}{}^{ab} = 0$, $\mathcal{H} = I$ and $N = N_{max} = 8$.

In fact, being parallelizable, S^7 has precisely 8 covariant constant Killing spinors. Its standard metric is SO(8) invariant; so the index I = 1, 2,..., 8 is thereby a SO(8) index, belonging, as it should, to a spinorial representation of SO(8). Also the parameters ε^I in Eq. (3f.30) are precisely the 8 local supersymmetry transformation parameters of D = 4, N = 8 supergravity.

While Eq. (3f.29) always solves Eqs. (3f.27b), (3f.33) and Eq. (3f.34) solves Eq. (3f.27a) only if $F_{mnpq}\, F^{mnpq} = 0$. Consequently, the condition (3f.31) amounts to

$$F_{mnpq} = 0 \quad . \tag{3f.37}$$

Consequently, local supersymmetry can survive spontaneous compactification provided Eqs. (3f.29) and (3f.37) are satisfied. This is the Freund-Rubin solution [300]. The latter yield the geometries given by Eq. (3f.34) in a D = 4 background

$$R_{\mu\nu} = 3\, m_4^2 g_{\mu\nu} \quad , \tag{3f.38a}$$

$$R_{\mu\nu\rho\sigma}(E) = m_4^2(\, g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}) \tag{3f.38b}$$

with $m_4^2 = 4\, m_7^2$. Equation (3f.38b) describes a D = 4 anti-de Sitter space (AdS). Therefore, integrability forces the unique solution of D = 4, AdS x S^7 as the compactification preserving all eight supersymmetries. The zero mass sector of the resulting theory is recognized as the D = 4 gauged N = 8 supergravity of de Wit and Nicolai [291].

Now there exist other solutions to Eq. (3f.27) which break supersymmetry spontaneously (N < 8). Indeed, it was found that the only allowed solutions are the ones with N = 1, 2 and 4, respectively. We briefly mention three such solutions with S^7 topology: the parallelizing "round" 7-sphere, the squashed 7-sphere and their combination, the parallelizing squashed 7-sphere.

The first, Englert 's solution [295], is characterized by Eq. (3f.29) but has $F_{nmpq} \neq 0$. The latter are interpreted as self-generated expectation values of the pseudoscalar fields A_{mpn} in four dimensions. From Eq. (3f.27b) such F_{nmpq} must fulfill the duality equation

$$D_m F^{mnpq} = \frac{\sqrt{2}}{(4!)} \frac{f}{\sqrt{g^7}} \varepsilon^{npqrstu} F_{rstu} \tag{3f.39}$$

or, in terms of forms, $d{*}F = \sqrt{2} f F$. Next, with hindsight from our knowledge of the Cartan-Schouten parallelism on S^7 (Sect.3.c.2), one chooses

$$m_7{}^2 = \frac{3}{10} m_4{}^2 = \frac{f^2}{8} \ , \tag{3f.40}$$

and the ansatz

$$F_{mnpq} = \pm \left(\frac{\sqrt{2}}{m_7} \right) D_m S_{npq} = \pm \frac{\sqrt{2}}{6} \varepsilon_{mnpqrst} S^{rst} \tag{3f.41}$$

with

$$S_{mnp}(y) \equiv m_7 \bar{\psi} \Gamma_{mnp} \psi \ , \quad \bar{\psi} \psi = 1 \tag{3f.42}$$

which then solves Eq. (3f.39); ψ being an arbitrary linear combination of the $\eta^I_{\cdot}$, Eq. (3f.32). In fact the S_{mnp}'s satisfy the Cartan-Schouten equations (3c.91)-(3c.92), the necessary and sufficient conditions for them to be a parallelizing torsion on S^7. That such a torsion has the convenient parametrization (3f.32) as bilinears of the ψ's can be seen clearly from the following representation of the Γ-matrices:

$$\left(\Gamma^m\right)_{n8} = i\delta_n{}^m \ , \quad \left(\Gamma^m\right)_{np} = i\psi_{mnp} \ , \quad \psi_a = \delta_{a8} \ . \tag{3f.43}$$

ψ_{mnp} are the structure constants of $\mathbf{\Omega}$, the octonion algebra.

As first suggested by Duff [301], the Englert solution has the nice interpretation of a version of D = 4, N = 8 supergravity with all supersymmetries spontaneously broken and with as surviving gauge symmetry the exceptional group G_2. This round sphere solution with torsion is also called the "parallelized solution" since the torsion (3f.42) is the one which "parallelizes" S^7.

Remarkably, it turns out that there exists a second Einstein metric on S^7 which led to another solution with N = 1 supersymmetry [302]. This metric is inequivalent to that of the "round" $S^7 \approx SO(8) / SO(7)$ and corresponds to a coset space of a "squashed" $S^7 \approx \dfrac{Sp(2) \times Sp(1)}{Sp(1) \times Sp(1)}$. With its Sp(1) x Sp(1) holonomy group, this squashed sphere is therefore endowed with a quaternionic structure. Specifically, it is an Einstein space homeomorphic to S^7 but one with the distorted metric:

$$ds^2 = -\delta_{ab}\theta^a \otimes \theta^b \ , \quad (a, b = 4, 5, \ldots, 10) \tag{3f.44}$$

with the following convenient identification for the orthonormal frames $\{\theta^a\}$:

$$\begin{array}{l} \theta^4 = \theta^{(1)}, \ \ \theta^5 = \theta^{(2)}, \theta^6 = \theta^{(6)}, \ \theta^7 = \theta^{(4)} \\ \theta^8 = \theta^{(5)}, \ \ \theta^9 = -\theta^{(3)}, \ \theta^{10} = \theta^{(0)} \end{array} \tag{3f.45}$$

Then

$$\theta^{(0)} = \frac{3\,\Lambda}{2\,m_7}\,d\mu \ , \tag{3f.46a}$$

$$\theta^{(i)} = \frac{3\,\Lambda}{4\,m_7}\sin(\mu)\,\left(\sigma^i - \Sigma^i\right) , \tag{3f.46b}$$

$$\theta^{(i+3)} = \frac{3\,\Lambda}{2\,m_7}\left[\cos^2(\tfrac{\mu}{2})\ \sigma^i + \sin^2(\tfrac{\mu}{2})\Sigma^i\right] \tag{3f.46c}$$

with $\Lambda^{-2} = 5$. The six independent 1-forms σ^i, Σ^i (i =1, 2, 3) satisfy their respective $Sp(1) \approx SU(2)$ Cartan structural equations

$$d\sigma^i = -\frac{1}{2}\,\varepsilon^i{}_{jk}\,\sigma^j \wedge \sigma^k \quad , \qquad d\Sigma^i = -\frac{1}{2}\,\varepsilon^i{}_{jk}\,\Sigma^j \wedge \Sigma^k d\Sigma^i = -\frac{1}{2}\,\varepsilon^i{}_{jk}\,\Sigma^j \wedge \Sigma^k \ . \tag{3f.47}$$

It can be readily verified that $R_{ab} = 6\,m_7{}^2\delta_{ab}$.

Though its curvature tensor differs from Eq. (3f.35), the metric (3f.44) still solves for the equations of motions (3f.27) which involve only the Ricci tensor, not the full curvature tensor and $F_{mnpq} = 0$. Its single most interesting feature is the survival of one supersymmetry. Namely, there is one covariantly constant unit norm spinor η:

$$(D^{(s)}_m - \frac{m_7}{2} \Gamma_m) \eta = 0 \tag{3f.48}$$

where $D^{(s)}_m$ is computed with the squashed metric on S^7. Furthermore, it is not only a covariantly constant but simply a constant spinor,

$$\eta^{\dagger}(y) = (1, 0, 0, 0, 0, 0, 0, 0) \quad . \tag{3f.49}$$

Since the constancy feature has interesting implications, it is natural to consider a third solution by parallelizing ($F_{nmpq} \neq 0$) the squashed S^7 and, by so doing, breaks completely its residual supersymmetry. As done previously, the ansatz for the 3-potential is

$$A_{mnp} = \frac{\lambda}{4!} S_{mnp}\,(y) \equiv \frac{\lambda}{4!} m_7 \overline{\psi}\, \Gamma_{mnp} \psi\,, \quad \overline{\psi}\, \psi = 1 \tag{3f.50}$$

where $\psi = \eta^I_-$ as in Eq. (3f.32) and λ, a yet to be determined constant. ψ then solves Eq. (3f.39), provided $m_7 = -\frac{f}{\sqrt{8}}$. If $m_4{}^2 = \frac{10}{3} m_7{}^2$, $\lambda^2 = \frac{2}{m_7{}^2}$, Eqs. (3f.27a) and (3f.27b) are satisfied. With this new solution, the spontaneously induced torsion also "Ricci-flattens" the compact 7-space, i.e.

$$R_{nq}\left(\left\{ {r \atop t\;\; u} \right\} + S^r{}_{tu} \right) = 0 \tag{3f.51}$$

with $\left\{ {r \atop t\; u} \right\}$ denoting the Levi-Civita connection. From (3f.50), the resulting torsion is simply

$$S_{abc} = -\, m_7 \psi_{abc} \quad . \tag{3f.52}$$

In consequence, due to the uniqueness of the Killing spinor, the torsion components are constant over all the squashed 7-sphere, a property shared by all semi-simple compact groups (see Eq. (2c.14)) [303]. In this sense, the squashed S^7 with torsion is even closer to being a group space than the round S^7.

As we mentioned, various compactifications of D = 11 supergravity give rise to different patterns of symmetry breaking and hence to different D = 4 supergravity theories. While these theories provide concrete examples of a geometric Higgs mechanism beautifully tied to the unique octonionic properties of S^7, none readily accommodates the Standard Model. This phenomenological deficiency is also present in the D = 11 supermembrane theory, which is still not well-understood quantum mechanically. D = 11 supergravity also embodies a severe defect, shared by all quantum field theories to date. It predicts a huge cosmological constant, one hundred and twenty or so orders of magnitude larger than the observed one. A satisfactory resolution of this problem [304] will be a major breakthrough in the search for the unification of all known interactions with gravity.

3.f.3. D = 8 self-dualities and octonionic instantons

There are many reasons for seeking explicit, self-dual, solutions to higher dimensional gauge theories [305]. We motivate this subsection by only citing a few of them. From the perspective of dimensional reduction, pure D > 4 Yang-Mills theories give rise to d = 4 Yang-Mills-Higgs systems with spontaneous symmetry breaking. One then has a possible geometric understanding of the Higgs mechanism in the context of D > 4 Riemannian geometry. The D = 4 self-dual Donaldson theory reduces to the D = 3 Chern-Simons, then to D = 2 conformal invariant field theories as well as to lower dimensional completely integrable systems [306]. Similarly, self-dual sectors of D > 4 topological, string and membranes field theories could lead to higher dimensional integrable systems. Thus D > 4 instantons are relevant to the study of topological excitations of Kaluza-Klein theories of particles, strings and membranes. Examples are the exact multi-string solutions of the low energy D = 10 supergravity [228, 307, 308, 309]. Like the BPS monopoles [189, 190] , these solutions saturate the Bogomol'nyi bound for the energy per unit length [228]. They all require an essential ingredient, the knowledge of explicit $D \geq 4$ (super)Yang-Mills instanton solutions.

To explore the possible forms of self-duality in D > 4 spacetimes, it is natural to first model after the 4-dimensional gauge field case. Several groups have studied this problem [305, 310, 311]. The resulting generalized self-duality equations, linear in the field strengths $F_{\mu\nu}$, (μ, ν = 1, 2, ... , D), amounts to a system of first order semi-linear equations for the gauge connection A_μ. By way of the Bianchi identities, they in turn imply the full second order Yang-Mills equations $D^\mu F_{\mu\nu} = 0$. However, there are some key differences with the D = 4 case. Thus in Ward's approach [306] , not all the linear relations leading to the full Y-M equations arise as integrability conditions for his generalization of the first order Belavin-Zhakarov spinor equations [312] , familiar from the method of inverse scattering:

$$V^{\mu}_{a}(\pi)D_{\mu}\psi = 0 \tag{3f.53}$$

where $\mu = 1, 2,..., D$, $a = 1, 2,..., r$. $\psi(x^{\mu}, \pi^{P})$ is a column n-vector. $D_{\mu} = \partial_{\mu} + A_{\mu}$ is the gauge-covariant derivative. The $V^{\mu}_{a}(\pi)$ are homogeneous of degree q in the components of an (m + 1) component vector π, an element of CP_{n}, i.e.

$$V^{\mu}_{a}(\pi) = V^{\mu}_{ab_1b_2\ldots b_q}\pi^{b_1}\ldots\pi^{b_q} \ . \tag{3f.54}$$

In this subsection, following Ward as well as Corrigan et al. [313] , we choose to briefly review a special class A_k (even $r = 2k$, $D = 4k$) of Ward's 1st order equations. This class of solutions are noteworthy since they imply the full Yang-Mills equations and admit a generalized ADHM construction. Moreover, for $q = m = 1$ so that Eq. (3f.54) reads $V^{\mu}_{a}(\pi) = V^{\mu}_{aP}\pi^{P}$, $P = 1, 2$; $a = 1, ..., 2k$. We further assume the invertibility of the matrix V^{μ}_{aP}, seen as a mapping between two 2r-dimensional spaces.

It turns out that, with the only exception of the familiar case of $D = 4$, $k = 1$, the set of equations (3f.53) inevitably breaks SO(D) invariance. The specific nature of the residual symmetry group $G \subset SO(D)$ then depends on one's choice of $\left\{V^{\mu}_{aP}\right\}$.

As to the integrability conditions implied by Eq.(3f.53), they are

$$\left(V^{\mu}_{aP}V^{\nu}_{aQ} + V^{\mu}_{aQ}\,V^{\nu}_{aP}\right)F_{\mu\nu} = 0 \ . \tag{3f.55}$$

They give the eigenvalue or generalized duality equations

$$\frac{1}{2}\,T_{\mu\nu\rho\sigma}\,F_{\rho\sigma} = \lambda\,F_{\mu\nu} \tag{3f.56}$$

with $T_{\mu\nu\rho\sigma}$ being a totally antisymmetric G-invariant numerical tensor and λ a specific eigenvalue. Algebraically λ may take the eigenvalues $1, -\frac{1}{3}, -2k+\frac{1}{3}$, however only the $\lambda = 1$ case is compatible with integrability. As with the case of four dimensions, the full Yang-Mills equations then follow from Eq. (3f.56) and the Bianchi identities.

Leaving further variants and greater details to several works to be cited, we only discuss an explicit D = 8 octonionic realization of Eq. (3f.56) in the $SO(7)^{\pm}$ instanton of Fubini and Nicolai [314]. We shall abide by the definitions and most of the notation of the latter work. Here the key novel invariant residual subgroups of SO(8) are $SO(7)^{\pm}$. The exceptional existence of these non-canonically embedded SO(7) subgroups of SO(8) is a manifestation of triality connected to octonions. In fact, SO(8) has 3 inequivalent 8-dimensional representations, the vector, the left-handed and right-handed spinor representations. Correspondingly, there are three ways of embedding SO(7) into SO(8). They are such that, while the 8-vector reduces to $8 \to 1 + 7$, the other two 8-spinor representations are irreducible: $8 \to 8$. Algebraically, these possibilities are:

$$\begin{aligned} SO(8) &\approx SO(8)/SO(7)^{v} \times SO(7)^{v} \\ &\approx S^7_v \times SO(7)^{v} \\ &\approx S^7_L \times SO(7)^{-} \\ &\approx S^7_R \times SO(7)^{+} \end{aligned} \tag{3f.57}$$

where we distinguish the three types of 7-spheres by the subscripts V, R and S. So the groups $SO(7)^{\pm}$ are seen as the stability groups of the right- and left-handed spinor respectively.

From our previous discussion, there exists a $SO(7)^{\pm}$ invariant realization of the totally antisymmetric tensor $T_{\mu\nu\rho\sigma}$ in the numerical tensor C^{MNPQ} defined by

$$C^{mnp8} \equiv \psi^{mnp} \ , \tag{3f.58}$$

$$C^{mnpq} \equiv \frac{1}{2}\, \eta\, \varepsilon^{mnpqrst}\, \psi_{rst} \ , \quad m, n, p, q, r, s, t = 1, 2, ..., 7 \quad ; \tag{3f.59}$$

ψ^{mnp} being the octonionic structure constants, the phases $\eta = \pm 1$. Such a C_{MNPQ} is self-dual w.r.t. the 8-dimensional Levi-Civita symbol and obeys the relation

$$C^{MNPT}\, C_{QRST} = 6\, \delta^{MNP}_{QRS} - 9\, \eta\, \delta^{[M}_{[Q} C^{NP]}{}_{RS]} \ , \tag{3f.60}$$

i.e. our Eq. (3a.50) in subsection 3.a.3. To split the 28-dimensional vector space of SO(8) into its 21 and 7 orthogonal subspaces, one constructs the projection operators:

$$P_1^{MN}{}_{RS} \equiv \frac{3}{4}\left\{ \delta^{MN}_{RS} + \frac{1}{6}\eta C^{MNRS} \right\} \; , \tag{3f.61}$$

$$P_2^{MN}{}_{RS} \equiv \frac{1}{4}\left\{ \delta^{MN}_{RS} - \frac{1}{2}\eta C^{MNRS} \right\} \; , \tag{3f.62}$$

respectively. With the Γ^{RS} denoting the SO(8) generators, for $\eta = \pm$, then the 21 projected entities

$$G^{MN} \equiv P_1^{MN}{}_{RS}\Gamma^{RS} \tag{3f.63}$$

are generators of $SO^{\pm}(7)$ and are notably "self dual" w.r.t. the numerical tensor C_{MNPQ}, i.e.

$$G^{MN} - \frac{1}{2}C^{MNRS}G_{RS} = P_2^{MN}{}_{RS}G^{RS} = 0 \; . \tag{3f.64}$$

Thus the $SO(7)^+$ subalgebra is explicitly

$$\left[G^{MN} \; , \; G_{PQ} \right] = 6\, G^{[M}_{[P}\delta^{N]}_{Q]} + \frac{1}{2}\left(C_{PQR}{}^{[M}G^{N]R} - C^{MNR}{}_{[P}G_{Q]R]} \right) \; . \tag{3f.65}$$

The second, bracketed term on the RHS of Eq. (3f.65) is unusual; it has no analog in the usual (vector) SO(7) (or for that matter in the SO(n) algebra). We see here the connection between the unique existence in eight dimensions of these exceptional $SO(7)^{\pm}$ subalgebras of SO(8) and octonions.

Next, by the introduction of the field combination

$$\zeta_m \equiv F_{m8} - (\pm)\,\frac{1}{2}\,\psi_{mnp}F_{np} \; , \tag{3f.66}$$

the D = 8 $SO(7)^{\pm}$ Yang-Mills Lagrangian $L = -\frac{1}{4}\,\mathrm{Tr}\,\left(F_{MN}F^{MN} \right)$ can be recast in the alternative forms of

$$L = -\frac{1}{2}\,\mathrm{Tr}\,\xi_m\xi_m - (\pm)\,\frac{1}{2}\,C_{MNPQ}\mathrm{Tr}\left(F_{MN}F^{PQ} \right) \tag{3f.67}$$

or

$$L = -\frac{1}{2}\,\mathrm{Tr}\,\xi_m\xi_m - (\pm)\,\frac{1}{2}\,C_{MNPQ}\partial_{[M}\Phi_{NPQ]} \tag{3f.68}$$

where $\Phi_{NPQ} \equiv 4\,\mathrm{Tr}\left\{A_{[N}\partial_P A_{Q]} + \frac{2}{3}A_{[N}A_P A_{Q]}\right\}$ is a composite Kalb-Ramond field [315]. Though the second term in Eq. (3f.68) is a total derivative and Φ_{NPQ} has precisely the form of a Chern-Simons density, it is nevertheless not a topological charge density. However, it is the exact *algebraic* counterpart of the familiar D = 4 SO(3) Lagrangian density:

$$L = -\frac{1}{2}\,\mathrm{Tr}\,\xi_m\xi_m - (\pm)\,\frac{1}{2}\,\varepsilon_{MNPQ}\partial_{[M}\Phi_{NPQ]}\ , \tag{3f.69}$$

with $\xi_m \equiv F_{m4} \pm \frac{1}{2}\varepsilon_{mnp} F_{np}$; m= 1, 2, 3 and M, N, P, Q = 1, 2, 3, 4. So, just as the vanishing of ξ_m in Eq. (3f.69) implies (anti-) self-duality for the D = 4, SU(2) Yang-Mills, setting $\xi_m = 0$ in Eq. (3f.68) is equivalent to requiring self-duality for the field strength:

$$F_{MN} = \frac{1}{2}\, C_{MNPQ}F^{PQ}\ . \tag{3f.70}$$

A solution to the latter also solves for the full Yang-Mills equations in consequence of the Bianchi identity.

The Fubini-Nicolai octonionic instanton is obtained through a 't Hooft type ansatz for the $SO(7)^+$ gauge potential:

$$A_M(x) = G_{MP}\,\partial_P f(x)\ , \quad f(x) = -\frac{1}{3}\log\phi(x)\ . \tag{3f.71}$$

Substitution into the Y.M. curvature $F_{MN} = \partial_{[M}A_{N]} - [\,A_M, A_N\,]$ results in

$$\begin{aligned} F_{MN} = \Big(2\,\partial_P\partial_{[M}f - 3\,\partial_P\partial_{[M}f - \frac{3}{2}\,\delta_{P[M}\,(\,\partial_Q f\,)^2 \Big) G_{N]P} \\ - \frac{1}{2}\,\partial_P f\partial_Q f\,\psi_{MNPR}G_{QR}\ . \end{aligned} \tag{3f.72}$$

In terms of the projectors (3f.61) and (3f.62), F_{MN} may be expressed as

$$F_{MN} = \sum_{i=1}^{2} P_i{}^{SW}_{MN}\,\phi^i_{SR}G_{WR} \tag{3f.73}$$

with $\phi^{(1)}_{PQ} = 2\,(\,\partial_{PQ}f - \partial_P f \partial_Q f\,) - \frac{3}{2}\,\delta_{PQ}(\,\partial_R f\,)^2$, $\phi^{(2)}_{PQ} = 2\,(\,\partial_{PQ}f - 3\,\partial_P f \partial_Q f\,) - \frac{3}{2}\,\delta_{PQ}(\,\partial_R f\,)^2$. Due to the self-duality of G_{MN}, Eq. (3f.70), the field strength F_{MN} is notably invariant under the shift $\phi^{(2)}_{PQ} \to (\,\phi^{(2)}_{PQ} - h\,\delta_{PQ}\,)$, h being an arbitrary function. Self-duality of F_{MN} implies $P_2^{MN}{}_{RS}F^{RS} = 0$ or $\phi^{(2)}_{PQ} = 0$, i.e.

$$\partial_{PQ}f - 3\,\partial_P f\,\partial_Q f = h\,\delta_{PQ}\ . \tag{3f.74}$$

It translates into an equation for ϕ :

$$\partial_M\partial_N\phi = \frac{\delta_{MN}}{8}\,\partial_R^2\phi\ . \tag{3f.75}$$

Same as its D = 4 SU(2) counterpart, Eq. (3f.75) is uniquely solved by $\phi(x) = 1 + x^2$. Therefore, the $SO(7)^+$ gauge connection reads

$$A_M(x) = -\frac{2}{3}\,\frac{G_{MN}x_N}{1+x^2}\ . \tag{3f.76}$$

On the other hand, unlike the D = 4, 1-instanton solution, setting $P_1^{MN}{}_{RS}F^{RS} = 0$, i.e. $\phi^{(1)}_{PQ} = 0$ only gives the trivial solution $F_{MN} = 0$. Here we are witnessing a general feature of higher (D > 4) dimensional duality equations : the existence of nontrivial solutions is a highly representation dependent phenomenon. Thus there exists an asymmetry between self-duality and anti-self-duality in the sense of Eq. (3f.57). To get the corresponding anti-self dual solution, the $SO(7)^-$ generators should be used. Besides the lack of conformal invariance and the nontopological nature of the "1-instanton" solution, this asymmetry, rooted in the triality of SO(8) and hence in the nonassociativity of octonions, has been an added obstacle to finding D = 8 multi-octonionic instantons, and thereby also to finding octonionic multi-string solutions. We briefly elaborate on this point.

Recently, as a part of concerted efforts to uncover nonperturbative aspects of string field theories in various dimensions, exact multi-string solutions have been discovered for the D = 10 low energy supergravity super-Yang-Mills equations of motions [228, 316]. Thus an extended soliton solution to the equations of motion of the low-energy heterotic field theory has been constructed from the above Fubini-Nicolai instanton. This soliton corresponds to an octonionic superstring in D = 10 Minkowski space. It preserves only one of the sixteen spacetime supersymmetries. Presumably its multi-string counterparts also exist and require the knowledge of the corresponding multi-instanton solutions. In this connection, use could be made of new classes of explicit solutions to Eq. (3f.56) found recently by Ivanova and Popov [317]. Their two step method consists in first reducing the self-duality equations into Ward's nonlinear differential equations in one variables. Then the latter are further reduced to the classical Yang-Baxter equations for which many explicit solutions are known.

As to the general multi-instantons problem, we know from the work of Corrigan et al. [313] on an extended ADHM construction for solutions to Eq. (3f.53), that even a 't Hooft type ansatz for multi-instantons will most likely fail for S^8 as a base space M, as it does for the case $M = HP^2$, the quaternionic projective plane. Indeed, in higher dimensions (for D= 4 k, k = integer), the difficulties in finding multi-instantons to the generic duality equations (3f.53) are manifold. Firstly, it has been proved that there can be no finite action solutions to the D > 4 Yang-Mills equations. Nevertheless, one can impose boundary condition analogous to the one point compactification of R^4 to $S^4 \approx HP^1$, namely that the solutions be extended from $R^D = H^k$ to the quaternionic projective space HP^k. Secondly, even if a discovered solution(s) turns out to be topologically nontrivial, e.g. by a suitable compactification of some dimensions, several nonzero Pontryagin-Chern classes can be defined. Moreover these indices do <u>not</u> always determine the field configuration up to continuous deformations. Therefore, along with higher dimensions come greater arbitrariness, which we don't know how yet to restrict, either on mathematical or on pseudo-physical grounds.

Due to the difficulty in finding multi-D > 4 Yang-Mills instantons, a new direction was tried by formulating a linear self-duality for rank k antisymmetric gauge fields. For reasons of supersymmetry and global topology, (abelian) antisymmetric tensor gauge fields (ATGF) play a prominent role in supergravity, superstring and supermembrane theories, and more recently in higher dimensional topological field theories of links and generalized braids.

Just as the rank 2 Yang-Mills fields have as their sources point particles, rank k > 2 ATGF are coupled to extended objects in spacetime. In superstrings, their presence is the key to the cancellation of gauge and gravitational anomalies, to the various compactification schemes and to the existence of solitonic string solutions [9, 290]. Particularly, the striking duality properties of the ATGF in supergravity suggests the search for non-Abelian extension of these fields at the level of first order (anti-) self-duality equations in the field strength. Moreover the equivalence between D = 2 complex, D = 4 quaternionic analyticity and self-duality (w.r.t. the Levi-Civita ε-tensor) of the rank 1, rank 2 ATGF respectively hints at a possible parallel connection between octonionic analyticity and duality of some D = 8 non-Abelian 4-index ATGF. Henneaux and Teitelboim's proof [318] of the non-existence of non-Abelian Lie algebra valued ATGF further suggests that the sought for antisymmetric gauge field, if it exists at all, must be rather "exceptional" in its algebraic and geometric structure. A natural candidate is a D = 8, 4-index S^7-valued (thus not Lie algebra valued) ATGF. It is the octonionic analog of the quaternionic D = 4 $S^3 \approx SU(2)$ valued rank 2 ATGF or Yang-Mills field.

We begin by postulating a S^7-valued 3-index totally antisymmetric connection $A_{abc} = A^{\mu}_{abc}e_{\mu}$. Next we naively seek its associated 4-index curvature $F_{abcd}(A)$ such that it satisfies

$$F_{abcd} = \pm \widetilde{F}_{abcd} = \pm \frac{1}{4!}\varepsilon_{abcdnmrs}F_{nmrs} \,, \tag{3f.77}$$

a D = 8 linear (anti-) self-duality equations. We recall that though not a group manifold, S^7, the space of unit octonions, is unique in sharing with Lie groups the property of absolute parallelism. As noted above, while the tangent space of $O(4) \approx S_L{}^3 \times S_R{}^3$ is a Lie algebra, that of SO(8) is the vector space $S_L{}^7 \times SO(7)_L$ or $S_R{}^7 \times SO(7)_R$ with *non-commuting* left and right translations e'_{ab} and e_{ab}. This manifestation of non-associativity tells us that the formulation of an associated gauge principle, if it exists at all, must be rather subtle and closely tied to the triality of SO(8). As a first step in that direction, we may restrict ourselves to the (anti-) self-dual sector. Riding on the analogy with the D = 4 SO(4) self-dual Yang-Mills problem, we thus ventured an educated guess [282] by demanding linearity in the spacetime derivative and bilinearity in the potential A_{abc}. The S^7-valued field strength may read

$$F_{abcd} \equiv \partial_{[a}A_{bcd\,]} - C_{abcdnmrstu}[\,A_{nmr},\, A_{stu}\,] \tag{3f.78}$$

where the explicit forms of the constants $C_{abcdnmrstu}$ are trilinear in the f_{abcd} and can be argued on O(8) group theoretical grounds. In fact, A_{nmr} and F_{abcd} belong to the 56 and 35'+35" dimensional representations of O(8), respectively. The structure coefficients $C_{abcdnmrstu}$ are given by the Clebsch-Gordan coefficients of (56 x 56) → (35' + 35"). The latter are known to be nonzero. While this new self-duality has yet to be studied in its general form (3f.78), some partial but telling results have been obtained for a more specific 4-index S^7-valued ATGF:

$$F_{abcd} \equiv \partial_{[a}A_{bcd\,]} + \frac{1}{3}\left(f_{k\,[\,abc}[B'_{d\,]}\,,\, B'_k\,]\right) - \frac{1}{28}[\,H'_{abcd}e_r\,,\; B'_r,\, B'_n e_n\,] \tag{3f.79}$$

with $B'_r \equiv -\frac{1}{6} f'_{rabc} A_{abc}$ and H'_{abcd} is defined in Eq. (3c.27).

We observe that, interestingly, this octonionic ATGF has, besides the commutator nonlinearity of the usual non-Abelian gauge field, an added associator nonlinearity. Its dual is

$$\widetilde{F}_{abcd} \equiv \widetilde{\partial_{[a} A_{bcd\,]}} - \frac{1}{3}\left(f'_{k\,[\,abc}[B'_{d\,]}\,, B'_k]\right) + \frac{1}{28}\left[\, H'_{abcd} e_r,\, B'_r,\, B'_n e_n \,\right] \;. \tag{3f.80}$$

So equating the difference

$$\begin{aligned} F_{abcd} - \widetilde{F}_{abcd} \equiv\; & \partial_{[a} A_{bcd\,]} - \widetilde{\partial_{[a} A_{bcd\,]}} - \frac{2}{3}\left(f'_{k\,[\,abc}[\, B'_{d\,]}\,, B'_k]\right) \\ & - \frac{1}{14}\left[\, H'_{abcd}\, e_r,\, B'_r\,,\, B'_n e_n \,\right] \;, \end{aligned} \tag{3f.81}$$

namely the right hand side of this equation to zero implements the self-duality of F_{abcd}. By a rather laborious series of trials and errors, topologically non-trivial instanton type solutions were obtained through the following ansatz for A_{abc}

$$A_{bcd} = e'_{bc} a_d + e'_{cd} a_b + e'_{db} a_c + \frac{1}{4} H'_{bcdk}\, a_k \;. \tag{3f.82}$$

Then $B'_r = e'_{rk} a_k$ and the self-duality equation simplifies to the octonionic 't Hooft type equation

$$D\,\bar{a} + a\,\bar{a} = 0 \tag{3f.83}$$

whose D=8 holomorphic $S^7 \to S^7$ instanton solutions à la Fueter have already been discussed in details in Sect. 3d. As we recall, the difficulty of realizing a 't Hooft ansatz in a generalized ADHM construction for the generalized linear D > 4 Yang-Mills duality, the success in doing so here for the system (3f.77) could be indicative of the degree of complexity one may expect in order to find a D > 4 counterpart of Cauchy-Riemann-Fueter analyticity.

So far, we have only considered *linear* self-duality. With the exception of the S^7-valued ATFG sketched above, these systems are algebraic, typically characterized by a lack of both D (D = even)-dimensional conformal and SO(D) invariance, as well as global topological characteristics [313]. A way to preserve these invariances and to have topologically nontrivial solutions is to proceed instead to sets of <u>nonlinear</u> relations between the gauge field strength components. Of note among many such undertakings is the systematic program of Tchrakian and collaborators [319, 320, 321] of such nonlinear self-duality equations, which in turn solve for higher order derivative generalizations of Yang-Mills equations (GYM). In fact this extension of self-duality is quite natural from the inspection of the higher Chern characters and from the viewpoint of Kähler geometry. The corresponding instantons are finite action solutions and are labelled by some topological invariant, a higher order Chern characteristics. There have also been several other independent, complementary works along similar lines in the past decade (see for instance Ref.[322] and references therein).

One considers on a $D = 2(p + q)$ (hence even dimensional) manifold M the following duality conditions:

$$F(2p) = {}^*F(2p) \tag{3f.84}$$

where

$$ {}^*F(2p)_{\mu_1 \ldots \mu_{2p}} \equiv \frac{e}{2q!} (i\,\kappa^2)^{q-p} \varepsilon_{\mu_1 \ldots \mu_{2q} \nu_1 \nu_2 \ldots \nu_{2q}} F(2q)^{\nu_1 \ldots \nu_{2q}} \tag{3f.85}$$

e is the determinant of the D-beins on M, F(2p) is the skew p-fold product of the curvature 2-form F(2) and κ a dimensional constant. Here one distinguishes two very different types of duality relations, the *inhomogeneous* ($p \neq q$) from the *homogeneous* ($p = q$) ones. The first kind involves a nonzero power of κ, while the second is independent of κ. The first only holds in Dim $M = 4p$ and derives expectedly from *conformally invariant* GYM systems, the second holds for every even dimension and derives from GYM systems bearing a dimensional scale κ. Also for $p = q$, the 1-instantons of the GYM systems were found in $M \approx$ compactified R^{4p}. While for $p \neq q$, no instanton exist on a flat manifold, for $p = 1$, $q > 1$, explicit examples of one instanton solutions on the base spaces S^{2n}, CP^n and HP^n, have been given in terms of well-known canonical connections over these manifolds [322].

Having in mind the SO(7), SO(8), SO(9) algebras and the connection of their Clifford algebras to octonions, we choose to discuss the 1-instanton solution in one simple, particular case of D=8 ($p = q = 2$) self-duality,

$$F \wedge F = \pm (F \wedge F)^* , \tag{3f.86}$$

first considered by Grossman et al. [323, 324]. The associated generalized Yang-Mills functional is the conformal invariant action

$$A_4 = \int_{S^8} \| F \wedge F \|^2 d^8x \tag{3f.87}$$

on R^8, to be compactified to S^8 for action finiteness.

It is helpful to recall that, in their analysis of the conformal properties of the BPST instanton, Jackiw and Rebbi [325] extended the gauge group SU(2) to SO(4) by putting the 1-instanton and 1-anti-instanton together. Such an extended solution, besides having zero winding number, is SO(5) invariant and soon arose in the spontaneous compactification of Cremmer and Scherk. As was shown by Horvath and Palla, such a topologically trivial solution exists not just for D = 4 but for *any* D dimension for G = SO(D) where the solution has SO(D+1) invariance. So by specializing to G = SO(8) and D = 8, we are reversing the Jackiw-Rebbi procedure by "splitting" the corresponding Horvath and Palla SO(9) solution into two to find the hidden SO(8) instanton and anti-instanton. This approach was that of Ref.[323] which we closely follow below.

We seek a gauge connection $\widehat{A}_\mu$, $\mu = 1,\ldots,9$ on the tangent bundle to $S^8 = \{r \mid r \cdot r = \} \subset R^9$ with the transversality condition $\widehat{A}_\mu \cdot r_\mu = 0$, written in the Ansatz form

$$\widehat{A}_a = i\,\alpha\,\Sigma_{ab} r_b \qquad , \qquad \sum_{a=1}^{9} (r^a)^2 = 1 \tag{3f.88}$$

where the only freedom resides in α, a constant to be determined subsequently, the Σ_{ab} are the matrices of the irreducible 16-dimensional representation of SO(9). They can naturally be expressed in terms of the elements of a Clifford algebra :

$$\Sigma_{\mu\nu} \equiv \frac{1}{4!}\left[\Gamma_\mu , \Gamma_\nu\right] \quad , \quad \Sigma_{\mu 9} \equiv \frac{1}{2}\Gamma_\mu \ , \qquad \mu, \nu = 1, 2,\ldots, 8. \tag{3f.89}$$

where the Γ_μ's satisfy

$$\left\{\Gamma_\mu , \Gamma_\nu\right\} = 2\delta_{\mu\nu} \tag{3f.90}$$

and can be realized as $2^4 \times 2^4$ matrices.

The corresponding 3rd rank Yang-Mills field strength reads

$$\widehat{F}_{abc} = i\, L_{ab}\widehat{A}_c + r_a\left[\, \widehat{A}_b\, ,\, \widehat{A}_c \right] \tag{3f.91}$$

where L_{ba} are the D=9 angular momentum operators

$$L_{ab} \equiv -\, i\, r_a \frac{\partial}{\partial r_b} + i\, r_b \frac{\partial}{\partial r_a} \quad . \tag{3f.92}$$

Then requiring that it solves the usual Yang-Mills equations of motions leads to the constraint $\alpha\ (\alpha{+}1)\ (\alpha{+}2) = 0$, which is the same condition as in D = 4, SO(5) Yang-Mills case. Similarly, the solutions $\alpha = 0, -2$ are trivial, corresponding to pure gauge fields, leaving the $\alpha = -1$ solution.

Next, we proceed to eliminate $\Sigma_{\mu 9}$ from the gauge potential (3f.88) (and also field strength) and in so doing rotate the fields into the SO(8) subalgebra of SO(9). This is achieved via the specific gauge transformation

$$\widehat{A}'_a = U^{-1}\, \widehat{A}_a U + i\, U^{-1}\, r_b L_{ba} U \ . \tag{3f.93}$$

The two possible choices for U are :

$$U_1 = \exp\{\, i\, f(r_9)\, \Sigma_{\mu 9} r_\mu\} \ , \tag{3f.94a}$$

and

$$U_2 = \exp\left\{ i\left[f(r_9) - \frac{\pi}{\sqrt{1 - r_9^2}} \right] \Sigma_{\mu 9} r_\mu \right\} \tag{3f.94b}$$

with $f(r_9) \equiv \dfrac{\cos^{-1} r_9}{\sqrt{1 - r_9^2}}$.

The resulting potentials (with $\alpha = -1$) are such that $\widehat{A}'_9 \equiv 0$ and

$$\widehat{A}'_\mu = -\, i\, \frac{\Sigma_{\mu\nu} r_\nu}{1 + r_9} \quad \widehat{A}'_\mu = -\, i\, \frac{\Sigma_{\mu\nu} r_\nu}{1 + r_9} \ , \tag{3f.95a}$$

$$\widehat{A}'_\mu = -\, i\, \frac{\Sigma_{\mu\nu} r_\nu}{1 - r_9} \tag{3f.95b}$$

for $U = U_1, U_2$, respectively.

Now R^8 can be identified as the complement of the point $r_9 = -1$ given by the stereographic projection on S^8:

$$r_\mu = \frac{2\,x_\mu}{1+x^2} \quad , \quad r_9 = \frac{1-x^2}{1+x^2} \quad , \quad x^2 \equiv x_\mu x_\mu \quad . \tag{3f.96}$$

Then, in the $\widehat{A}'_9 \equiv 0$ gauge, the "conventional" SO(8) connection A'_μ differs from the corresponding $\widehat{A}'_\mu$ over the surface of S^8 by a mere weight factor i.e. $\frac{1+x^2}{2}\,A'_\mu = \widehat{A}'_\mu$. The results are

$$A'_\mu = -\,2\,i\,\frac{\Sigma_{\mu\nu}x_\nu}{1+x^2} \qquad \text{and} \qquad A'_\mu = -\,2\,i\,\frac{\Sigma_{\mu\nu}x_\nu}{\left(1+x^2\right)x^2} \quad . \tag{3f.97}$$

The associated covariant field strengths are

$$F'_{\mu\nu} = \frac{4\,\Sigma_{\mu\nu}}{\left(1+x^2\right)^2} \qquad \text{and} \qquad F'_{\mu\nu} = \frac{4\,\Sigma_{\mu\nu}}{\left(1+x^2\right)^2 x^4} \quad , \tag{3f.98}$$

respectively. As can be checked, this unique spherically symmetric solution not only solves for the usual Yang-Mills action, but also realizes the simplest self-dual (in the generalized sense of (3f.84)) and finite action solution extremizing the higher order action (3f.87) or

$$A_4 = \int_{S^8} \| *\left(F \wedge F\right) \pm \left(F \wedge F\right) \|^2 d^8x \quad . \tag{3f.99}$$

For the solution (3f.98), which is topologically nontrivial, (3f.99) saturates the 4th Chern index

$$C_4 = \frac{1}{96\,\pi^4}\int_{S^8} \| F \wedge F \wedge F \wedge F \|^2 d^8x \quad , \tag{3f.100}$$

labelling the homotopy classes of $\pi_7(SO(8)) \approx Z \oplus Z$, in complete analogy to the SO(4) case. Here $C_4 = 1$. Indeed as shown by Bais and Batenburg [322], the solution (3f.98) is directly given by the canonical Levi-Civita connection on the orthogonal frame bundle of the sphere S^D

$$
\begin{array}{c}
SO(D) \rightarrow SO(D+1) \\
\downarrow \\
S^D
\end{array}
$$

with D = 8. Moreover, it also corresponds precisely to a D-dimensional generalization of Yang's reinterpretation [326] of the D = 4 BPST SO(4) instanton as a D = 5 Dirac monopole.

To exhibit the connection between octonions, (8 x 8) matrices and the generalized self-duality relation (3f.86), we begin by splitting the (16x16) Spin(9) irreducible representation matrices $\{\Sigma_{ab}\}$ by way of the chiral projection $\frac{1}{2}\left(1 \pm \Gamma_9\right)$ into two irreducible 8 x 8 spinor representations of Spin 8:

$$\Sigma^{\pm}_{\mu\nu} \equiv \frac{1}{2}\left(1 \pm \Gamma_9\right)\Sigma_{\mu\nu} \ . \qquad (3f.101)$$

Now a useful representation of the $\left\{\Sigma^{\pm}_{\mu\nu}\right\}$ matrices was given long ago by Gürsey and Günaydin [237] in their (8 x 8) matrix construction of the Cayley algebra Ω. Consider the 8-spinor column matrix

$$[s] \equiv \begin{bmatrix} u \\ u^* \end{bmatrix} \qquad (3f.102)$$

with

$$u \equiv \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_0 \end{pmatrix} \qquad (3f.103)$$

of split octonions (see Sect. 3a.4). We define its conjugate matrix $\lceil \bar{s} \rceil^{\dagger}$ since

$$\frac{1}{2}\lceil \bar{s} \rceil^{\dagger} \equiv \lceil \bar{s} \rceil^{*T} = \left(-\ u_1^*, -\ u_2^*, -u_3^*,\ u_0, -u_1, -u_2, -u_3, -u_1^*\right) \qquad (3f.104)$$

Overbar and * denote octonion and complex conjugation, respectively; T being the usual transposition. Then the product $[s]\lceil \bar{s} \rceil^{\dagger}$, also reads as

$$[s]\lceil \bar{s} \rceil^{\dagger} = \frac{1}{2}\left(1 - i\,\Gamma_A e_A\right) , \qquad A = 1, \ldots, 7 \qquad (3f.105)$$

where the Γ_A's are seven Hermitian (8x8) matrices ($\Gamma_A = \Gamma_A^\dagger$) satisfying Clifford's anticommutation relations

$$\left\{\Gamma_A , \Gamma_B\right\} = 2\, \delta_{AB} \,. \tag{3f.106}$$

They are defined by

$$\Gamma_1 = -\sigma_{112} \,;\; \Gamma_2 = -\sigma_{120} \,;\; \Gamma_3 = \sigma_{132};\; \Gamma_4 = \sigma_{221} \,;$$

$$\Gamma_5 = -\sigma_{220} \,;\; \Gamma_6 = \sigma_{202} \,;\; \Gamma_7 = -\sigma_{300} \tag{3f.107}$$

with

$$\sigma_{ijk} \equiv \sigma_i \otimes \sigma_i \otimes \sigma_k, \tag{3f.108}$$

the symbol $\otimes$ denoting the direct product of the (2 x 2) Pauli matrices σ_1, σ_2, σ_3 and σ_0 being the (2 x 2) unit matrix.

If we now define the 21 skew symmetric Hermitian matrices Γ_{AB} by

$$\Gamma_{AB} = \frac{1}{2i}\left[\Gamma_A, \Gamma_B\right] \tag{3f.109}$$

which then close on the Lie algebra of SO(7). As is well-known, when we include the 7 Γ_A's, the set $\{\Gamma_{AB}\}$ can be enlarged to the SO(8) algebra $\Sigma_{\mu\nu} = \Gamma_{AB} \oplus \Gamma_A$ in two different and inequivalent ways. The resulting 28 generators labelled as

$$\Gamma_{AB} \;,\quad A, B = 1,\ldots, 7,$$

$$\Gamma_{A8} = \pm\, i\, \Gamma_A \;, \tag{3f.110}$$

then form a complete basis for the (8 x 8) skew symmetric matrices. The alternatives in signs reflect the two inequivalent spinor SO(8) representations $\left\{ \Sigma^{\pm}_{\mu\nu} \right\}$.

That the Γ_A's correspond to the left multiplication by the octonionic units eA acting on [s] does not mean that the Cayley algebra Ω is representable by matrices under the usual matrix multiplication rule. However we can define a new product for these matrices such that

$$\Gamma_A \cdot \Gamma_B \equiv \delta_{AB} + i\left\{\Gamma_{AB}, M\right\} + \Gamma_A M \Gamma_B - \Gamma_B M \Gamma_A \qquad (3f.111)$$

with

$$M \equiv -\frac{1}{4\,i}\left(\frac{1}{3!}\right)\psi_{ABC}\Gamma_A\Gamma_B\Gamma_C\ , \qquad (3f.112)$$

then

$$\Gamma_A \cdot \Gamma_B = \delta_{AB} + i\,\psi_{ABC}\Gamma_C\ , \qquad (3f.113)$$

thus giving a Cayley algebra for the imaginary octonion units with basis $e_A \equiv -i\,\Gamma_A$.

If $\Gamma_{ABC\ldots P} \equiv \Gamma_{[A}\Gamma_B\Gamma_C\ldots\Gamma_{P]}$ denotes the antisymmetrized product of Γs, then one verifies the identity

$$\Gamma_{ABCD} = \frac{1}{4!}\,\varepsilon_{ABCDEFG}\,\Gamma_E\,\Gamma_F\,\Gamma_G \qquad (3f.114)$$

whence

$$\Sigma_{8[A}\,\Sigma_{BC]} = \frac{1}{4!}\,\varepsilon_{8ABCDEFG}\,\Sigma_{DE}\,\Sigma_{FG} \qquad (3f.115)$$

and so

$$\Sigma_{[\mu\nu}\Sigma_{\rho\sigma]} = \frac{1}{4!}\,\varepsilon_{\mu\nu\rho\sigma\alpha\beta\gamma\delta}\Sigma_{\alpha\beta}\Sigma_{\gamma\delta}\ . \qquad (3f.116)$$

From Eq. (3f.116) a trivial check shows that Eq. (3f.98) satisfies Eq. (3f.86) where the (+) and (–) correspond to the self-dual (Σ^+) and anti-self-dual (Σ^-) SO(8) representations respectively.

With conformal symmetry, topological nontriviality and symmetry between instanton and anti-instanton hold for the above SO(8) instanton, it was hoped that the search for multi-instanton solutions would be easier than in the case of linear self-duality. Thus far there has been little progress. This is the case even in the restricted class of the axisymmetric solutions of Witten and Peng [325]. Here the self-duality equations reduce to a new generalized Liouville equation. An ADHM typed construction, if possible at all, seems far off in the future.

To close this subsection we should mention a more radical, yet natural extension of self-duality. While Eq. (3f.84) has dealt so far with point particles, it readily admits generalizations to strings and membranes. Here it has been hoped that the topological solutions to the corresponding first order equations in the coordinates of these extended objects will be relevant to the nonperturbative understanding of the string and membrane theory vacua. The richness and interconnectedness of structures are already evident at the classical level.

For example, it has been shown that 1) the D = 3 theory of self-dual membranes gives a D = 3 topological field theory, 2) in the light cone gauge, the self-dual membranes in D= 5 spacetime dimensions are equivalent to D = 4 dimensional self-dual Yang-Mills in the large N limit and for homogeneous (space independent) gauge potentials, 3) the D = 5 spherically symmetric self-dual membranes are integrable and isomorphic to the infinite Toda lattice. More generally, self-dual classical bosonic p-branes exist whenever a p-fold vector cross-product can be defined in R^n. As illustrations [327], the existence of self-dual 2-branes in S^6 and 3-branes in S^7 and S^8 is tied to that of the SO(7) and SO(8) invariant tensors, to the associated calibrated exceptional geometries in D = 7, 8 dimensional spaces, and to the Cauchy-Riemann structures of their nonlinear Dirac-Ampere-Monge equations.

From a broader perspective, we note with Hitchin that D = 2 integrable systems such as the Toda and KdV equations are just lower dimensional reductions of D = 4 self-dual Yang-Mills theory. Similarly, new D=3 integrable systems have been connected to membrane theories. These connections can be appreciated as we recall that vector cross products are natural generalizations of the notion of almost complex structure [328, 329]. From a broader perspective of the generalization of Penrose's twistor theory [3, 5], they are but the multi-faceted manifestation of the rich holomorphic and topological structures of multi-dimensional twistor spaces [77, 231, 273].

3.f.4. Octonionic supersymmetry in hadron physics

A phenomenological manifestation of octonionic structure has been found in connection with quark dynamics inside hadrons and an effective supersymmetry [22, 23, 330]. Previously, it was shown that a broken supersymmetry was also realized in heavy nuclei [331].

We recall that supersymmetry (SUSY) is a symmetry between fermions and bosons [19]. For example, the action of supergravity is invariant under transformations of fields of spin 2, 3/2, 1, 1/2 and 0 among themselves. In a more familiar setting, the low lying hadrons also have the same range of spin values with s = 0, 1 and 2 mesons interlaced with s = 1/2 and 3/2 baryons. Now symmetry groups transforming the s = 0, 1 mesons into one another and at the same time mix the s = 1/2 and 3/2 baryons were introduced long ago. Clearly, such a symmetry must be broken since no degeneracy is observed between particles with different spins. Nevertheless, a casual inspection of the Chew-Frauschi plot (mass of hadrons squared versus spin) of hadronic Regge trajectories reveals the following features: 1) The near linearity of trajectories. 2) Leading mesonic trajectories associated with lowest spin 0, 1 and 2 are mutually parallel. 3) Leading baryonic trajectories with s = 1/2, 3/2 are similarly mutually parallel. 4) Furthermore, mesonic and baryonic trajectories are nearly parallel to one another with a universal slope $\alpha' = 0.9\ (\mathrm{GEV})^{-2}$. 5) The separation between the mesonic trajectories is nearly the same as the one between baryonic trajectories.

What do these parallel characteristics imply? Features (2) and (3) hint at a phenomenological symmetry at work not only between mesons of different spin but also between baryons with different spin. For hadrons made up with the three light flavor quarks u, d and s, this symmetry is described by the group SU(6) x O(3). SU(6) classifies the lowest elements of the trajectories into multiplets while O(3) accounts for the rotational excitations on the leading trajectories. Property (4) is the hallmark of a new symmetry, indeed a supersymmetry (susy), between bosonic mesons and fermionic baryons. Here a manifest supersymmetric observable is the Regge slope, a universal constant of hadronic physics, apparently independent of flavor, spin and statistics.

Property (5) suggests that the same mechanism which breaks the SU(6) symmetry must also be responsible for the breaking of its susy extension. Finally, property (1) implies that the quark binding potential is almost linear and that, in contrast to quarkonia where a nonrelativistic Schrödinger theory suffices, relativistic quantum mechanics should be applied to light quarks.

In the quark model of Gell-Mann and Zweig, mesons and baryons are $(q\bar{q})$ and (qqq) bound states. So it is natural to conceive of any symmetry between baryons and mesons at the quark level as arising from an effective supersymmetry between an antiquark $\bar{q}$ and bound (qq) states or diquarks.

We recall that the quark q, with s = 1/2 and unitary spin of a flavor SU(3) triplet (3), belongs to the SU(6) sextet (6) representation. The low lying baryons are then in its symmetric (56) representation. Since the (56) is contained in 6 x 21 = 56 + 70, the diquark with s = 0 or 1 must be in the (21) of SU(6). Consequently, the sought for hadronic supersymmetry must transform the SU(6) multiplets (6) and (21), both color antitriplets, into each other. It must therefore be 27-dimensional with 6 fermionic and 21 bosonic states. Historically, precisely such a supergroup, now dubbed U(6/21), a generalization of the hadronic SU(6) symmetry, was introduced in as early as 1967 by Miyazawa [20, 21]. The latter also realized that the $(\bar{q})$-(qq) symmetry, and hence also a (q)-$(\bar{q}\,\bar{q})$ symmetry, will generally transform the meson $(q\bar{q})$ not just to baryons (qqq) and antibaryons $(\overline{qqq})$ but also to exotic mesons (qq) $(\overline{qq})$, belonging to the SU(6) representations 1, 35 and 405. The (1) and (35) are 0^+ and 1^+ mesons while the (405) also includes mesons with spin 2^+ and isospin 2 . All the low energy hadrons now sit in the adjoint representation of U(6/21) with both spin and isospin having values 0, 1/2, 1, 3/2 and 2.

The next introduction of supersymmetry in physics was in connection with dual string models of hadrons which naturally give raise to parallel linear baryon and meson trajectories. Unfortunately, these specific Ramond and Neveu-Schwarz models [332, 333] , besides being unrealistic, are only relativistic (i.e. Lorentz invariant) in 10 spacetime dimensions. They did not make a stunning comeback till recent years, albeit as Theories of Everything [290], and their deep connections to division and Jordan algebras will be discussed in the last part of this volume. Through the years the difficult search for a consistent 4-dimensional hadronic (super) string theory has continued, so far without significant success, along the lines of Polyakov [334] and/or in connection with the large N-limit of QCD.

In the meantime, taking dynamical root in the pairing mechanism of the Intermediate Boson Model [335] , a phenomenological nuclear supersymmetry between odd and even nuclei was discovered and formulated by Balantekin, Bars and Iachello [331] in terms of the supergroups U(m/n). It was thus only natural to find a similar scheme for hadronic supersymmetry.

One systematic attempt was initiated by S.Catto and the senior author (F.G.) [22]. Making partial use of the QCD inspired dynamical approach of DeRujula et al. [336], it seeks a dynamical basis for an approximate hadronic Miyazawa supersymmetry [20, 21]. The dynamical details and predictions of this model have been amply discussed in the literature. Faithful to the theme of this volume, we will only focus on the connections of this effective supersymmetric scheme to octonions.

The following algebraic implementation of a needed interquark forces within baryons is grafted into the model. In QCD, both the $(q\bar{q}\,)$ and (qq) color forces between quarks are attractive, the strength of the second being half of the first. The color (anti-)triplet (anti-) quarks are represented by local fields ($\overline{q}^{i}_{\alpha}(y)$) $q^{i}_{\alpha}(x)$, i = 1, 2, 3 labels color while α is the combined spin-flavor index. Consider the two-body systems

$$C^{\beta i}_{\alpha j} = q^{i}_{\alpha}(x_1)\,\overline{q}^{j}_{\beta}(x_2) \quad , \tag{3f.117}$$

$$G^{\beta i}_{\alpha j} = q^{i}_{\alpha}(x_1)\,q^{j}_{\beta}(x_2) \quad . \tag{3f.118}$$

As far as color is concerned, we have the following decomposition

$$3 \times \overline{3} = 1 + 8 \tag{3f.119}$$

so that C is either a color singlet or octet while G is a color antitriplet or sextet. C is realized by colorless mesons and hence observable, the octet state is confined, a non-asymptotic state.

Two (anti-) quarks can bind into a (anti-) diquark. Since

$$3 \times 3 = \overline{3} + 6 \tag{3f.120}$$

the diquark is in the antisymmetric (3) or the symmetric (6) representation of $SU_C(3)$. Similarly the antiquark is in the (3) or (6). Consequently, as far the color group goes, the antidiquark $\overline{D}$ in the (3) and the diquark D in the (3) behave respectively like a quark q and an antiquark $\overline{q}$. So the ($q\overline{q}$), ($q\overline{D}$), (qD) and ($D\overline{D}$) color forces are essentially equal. For two quarks in a S-wave bound state, the wave function, being spatially symmetric, must be antisymmetric in the color-flavor-spin variables. In the (3) of color, the diquark must then belong to the symmetric (21) of the flavor-spin group SU(6). If it is in the (6) of color, it must be in the antisymmetric (15) of SU(6). Now a diquark in the (6) color state cannot combine with a quark to form a color singlet baryon. Therefore, the quarks within a baryon can only bind into color antitriplet diquarks. The color octet part of G can only combine with a third quark to give unobservable color octet and decuplet three quark states. So quark dynamics <u>within</u> hadrons must be such as to suppress the octet (8) part of C and the (6) part of G . Their modified composition laws then read

$$3 \times \overline{3} = 1 \qquad \text{and} \qquad 3 \times 3 = \overline{3} \ . \tag{3f.121}$$

Consequently, if the $SU(3)_C \times SU(6)$ diquark in the representation (6, 15) is formed, it is not a baryonic diquark and is dynamically suppressed. Remarkably, this color dynamical suppression (3f.121) can be algebraically realized by way of octonions.

To best see the behavior of the various states under the color group $SU(3)_C$, we recall very briefly the algebra of split octonion already presented in Sect. 3.a.4. The units in the split basis are:

$$u_0 = \frac{1}{2}(1 + i e_7), \quad u_0^* = \frac{1}{2}(1 - i e_7),$$
$$u_j = \frac{1}{2}(e_j + i e_{j+3}), \quad u_j^* = \frac{1}{2}(e_j - i e_{j+3}), \quad (j = 1, 2, 3) \quad . \tag{3f.122}$$

The automorphism group of the octonion algebra is the 14-parameter group G_2. The imaginary units e_α ($\alpha = 1, 2, ..., 7$) fall into its 7-dimensional representation. Under the $SU(3)_C$ subgroup of G_2 leaving e_7 invariant, u_0 and u^*_0 are singlets, while u_j and u^*_j correspond, respectively to the 3 and $\overline{3}$ representations. The multiplication table takes the manifestly $SU(3)_C$ invariant form of

$$u_0^2 = u_0, \quad u_0 u_0^* = 0,$$
$$u_0 u_j = u_j u_0^* = u_j, \quad u_0^* u_j = u_j u_0 = 0,$$
$$u_i u_j = -u_j u_i = \varepsilon_{ijk} u_{kk}^*, \quad u_i u_j^* = -u_0 \delta_{ij}, \quad (i, j, k = 1, 2, 3) \tag{3f.123}$$

where ε_{ijk} is completely antisymmetric with $\varepsilon_{ijk} = 1$ for ijk = 123, 246, 435, 651, 572, 714, 367. From the octonion algebra, one can extract a subalgebra obeyed by the u_i and u^*_i ($i = 1, 2, 3$) only

$$\{u_i, u_j\} = 0 \quad , \quad \{u_i^\dagger, u_j^\dagger\} = 0 \quad , \quad \{u_i, u_j^\dagger\} = \delta_{ij} \tag{3f.124a}$$

where $u_i^\dagger = -u_i^* = \overline{u}_i^*$, the overbar denoting $\boldsymbol{\Omega}$-conjugation. So the three split units u_i are Grassmann numbers with the $u^\dagger{}_j$'s forming a conjugate set. Together the u's and $u^\dagger$'s form a fermionic Heisenberg algebra, in fact a non-associative exceptional Grassmann algebra; their nonzero associator reads

$$[u_i, u_j, u_k] = (u_i u_j) u_k - u_i (u_j u_i)$$
$$= \varepsilon_{ijk}(u_0 - u_0^*) = i\, \varepsilon_{ijk} e_7 \tag{3f.124b}$$

also

$$[u_i, u_j, u_k^*] = \delta_{ki} u_j - \delta_{kj} u_i \quad . \tag{3f.124c}$$

In consequence, the split units, like the canonical octonionic unit e_α, do not admit any matrix representation.

By combining Eqs. (3f.123) and (3f.124c) we then find

$$(u_i u_j)\, u_k = -\varepsilon_{ijk}\, u_0^* \;; \tag{3f.125}$$

namely that the octonion product leaves only the color singlet part of $3 \times \bar{3}$ and $3 \times 3 \times 3$, so that it is a natural algebra for colored quarks. Indeed the above algebra is such that the product of two triplets yields an antitriplet, while the product of a triplet and an antitriplet yields a singlet. It thus provides precisely the desired dynamical suppression of the octet and sextet states in the combination rules (3f.121).

The split units can be contracted with color indices of triplet or antitriplet fields. For quarks and antiquarks the following transverse octonions can be defined, u_0 and u_0^* being the longitudinal units:

$$q_\alpha = u_i\, q^i_\alpha = \vec{u} \cdot \vec{q}_\alpha \,, \qquad \bar{q}_\alpha = u^\dagger_j\, \bar{q}^j_\beta = -\vec{u}^{\,*} \cdot \vec{q}_\beta \;. \tag{3f.126}$$

We find

$$q_\alpha(1)\, \bar{q}_\beta(2) = u_0\; \vec{q}_\alpha(1) \cdot \vec{\bar{q}}_\beta(2) \;, \tag{3f.127}$$

$$\bar{q}_\alpha(1)\, q_\beta(2) = u_0^*\; \vec{\bar{q}}_\alpha(1) \cdot \vec{q}_\beta(2) \;, \tag{3f.128}$$

$$G_{\alpha\beta}(12) = q_\alpha(1)\, q_\beta(2) = \vec{u}^{\,*} \cdot \vec{q}_\alpha(1) \times \vec{\bar{q}}_\beta(2) \;, \tag{3f.129}$$

$$G_{\beta\alpha}(21) = q_\beta(2)\, q_\alpha(1) = \vec{u}^{\,*} \cdot \vec{q}_\beta(2) \times \vec{\bar{q}}_\beta(1) \;. \tag{3d.130}$$

Due to the anticommutativity of the quark fields,

$$G_{\beta\alpha}(12) = G_{\alpha\beta}(21) = \frac{1}{2}\{q_\alpha(1)\,,\, q_\beta(2)\} \;. \tag{3f.131}$$

Hence if the diquark forms a bound state represented by a field $D_{\alpha\beta}(x)$ at the center of mass location $x = \frac{1}{2}(x_1 + x_2)$, so that when $x_2 \to x_1$, we can replace the argument by x and $D_{\alpha\beta}(x) = D_{\beta\alpha}(x)$. So the local diquark field must be in a symmetric representation of the spin-flavor group. If the latter group is SU(6), the $D_{\alpha\beta}$ is in the 21-dimensional representation, given by $(6 \times 6)_S = 21$. Thus the octonionic fields single out the (21) diquark representation at the expense of the antisymmetric (15) representation, which is also antisymmetric in x_1 and x_2. For more details we refer the reader to the quoted references.

3.g. Historical Notes

3.g.1. Early life of octonions and division algebras (1843-1933)

The fascinating history of octonions [337, 338] is interwoven with some of the most fundamental themes of modern mathematics. Not surprisingly, it also reflects the latter's stormy history.

A natural starting point for a historical sketch of the octonion algebra $\mathbf{\Omega}$ is Diaphantus' formula

$$(a^2+b^2)(p^2+q^2) = (ap - bq)^2 + (aq + bp)^2 \quad , \tag{3g.1}$$

or its generalization

$$(a^2 - D\,b^2)(p^2 - D\,q^2) = (ap + Dbq)^2 - D(aq + bp)^2 \ . \tag{3g.2}$$

For special values of D, (3g.2) was apparently known to early Indian mathematicians. It was not till 1770 when came the striking formula

$$(a^2 + b^2 + c^2 + d^2)(p^2 + q^2 + r^2 + s^2) = x^2 + y^2 + z^2 + t^2 \tag{3g.3}$$

with

$$\begin{aligned} x &= ap + bq + cr + d\,, \\ y &= aq - bp \pm cs \mp dr\,, \\ z &= ar \mp bs - cp \pm dq\,, \\ t &= as \pm br \mp cq - dp\ . \end{aligned} \tag{3g.4}$$

This formula was discovered by Euler [339] in the course of his study of a theorem of Lagrange : every positive integer is a sum of four integral squares. By the dawn of the

19th century, it was realized that (3g.3) is nothing but the factorization of the norm of the product of two complex numbers. It led to an important problem in 19th century mathematics, the quadratic norm problem : Can the product of two sums of n squares be expressed as a sum of n squares?

Hamilton's discovery of quaternions (October 1843) means that this quadratic norm problem has a solution for n = 4. Two months later, Graves [340] discovered the octaves (his name for octonions) as yet another (n = 8) solution. He communicated his new algebra to Hamilton who immediately realized its non-associative nature. However, while in creating quaternions, Hamilton had given up non commutativity, he was still unwilling to forsake the principle of associativity. In any event, he did not help Graves publish his discovery till 1848. In 1845 octonions were rediscovered by Cayley and were published as an appendix to a work on elliptic functions [341]. Thus they came to be commonly known as Cayley numbers.

In contrast to quaternions, the discovery of octonions raised no hopes of physical applications. Subsequently, the octonion algebra was developed solely by mathematicians between 1843 and 1933. In fact the advent of quaternions and octonions marked a rebirth of algebra in Europe [342]. It triggered, especially in Ireland, the discovery of many other algebras. For example, as far as 8-dimensional algebras, there were Hamilton's own biquaternions or complex quaternions. Then, there were Clifford's biquaternions obtained by replacing in Grassmann's algebra the rule $i^2 = 0$ by $i^2 = 1$. Unlike octonions, neither of these algebras are division algebras as they have zero-divisors.

After many failed attempts by mathematicians to find new division algebras (e.g. in a 16 dimensional space over **R**), Frobenius [343] provided in 1878 a partial elucidation of the futility of such an enterprise. He proved that the only associative division algebras over the real field with a quadratic norm were **R**, **C** and **H** . The decisive solution only came in 1898 with Hurwitz's theorem [213]. It states that the *complete* list of solutions of the problem of quadratic norm includes the octonions and no other such algebras. Subsequently, new proofs of this basic theorem were given by Radon, Jordan, von Neumann and Wigner [65] using the theory of group representation, and later on by Freudenthal and by Chevalley. It was not until 1949 that the more general problem, the actual determination of all possible normed algebras was completed by Albert [344, 345]. He showed that the complex numbers, quaternions and octonions were indeed the only possible normed algebras over an arbitrary field of characteristic $\neq 2$.

In related developments, Dickson [346] showed in 1917 how to construct an octonion out of a pair of quaternions by way of a doubling procedure, the Cayley-Dickson process. In 1925, Artin [347] noticed the alternativity property (complete antisymmetry of the associator). Cartan [348, 349] then showed the G_2 group to be the automorphism group of octonions. He and Study also discovered the important Triality property.

From Hurwitz's fundamental theorem, the algebras of quaternions and octonions are therefore put on the same footing, as generalizations of the real and complex numbers. Yet it is only recently that the relevance of division algebras has only been better appreciated as they have emerged ubiquitously and quite unexpectedly in disparate frontiers of modern mathematics and physics.

Thus by the end of the last century, Killing and Cartan had succeeded in the complete classification of semi-simple Lie groups. Notably these groups come in four families. They are the orthogonal, unitary, symplectic groups O(n), SU(n) and Sp(n), which are all infinite in number, and the five exceptional groups G_2, F_4, E_6, E_7 and E_8 . In our century it became gradually clear that these four families are uniquely associated with the four division algebras respectively. Since Lie groups have become the supreme instrument for the analysis of various symmetries in nature, the Hurwitz algebras should be expected to play a central role in the fundamental physical laws.

In 1927 Cartan and Schouten [72] in their pioneering work on the geometries of compact Lie groups showed that, among all spheres, only the zero sphere S^0 or discrete group Z_2, S^1, S^3 and S^7 are parallelizable, a fact connected to the existence of the division algebras **R**, **C**, H and **Ω** in one, two, four and eight dimensional spaces respectively. This discovery was soon followed by Hopf's most striking application of division algebras to topology and vice versa. In 1935, making crucial use of the four division algebras, he constructed the real $S^1 \to S^1$, complex $S^3 \to S^2$, quaternionic $S^7 \to S^4$ and octonionic $S^{15} \to S^8$ fibrations [31]. Investigations of such remarkable maps culminated in Adams' celebrated theorem in K-theory [350, 351]. It states that of all $S^{2n-1} \to S^n$ Hopf maps, only the above admit mappings of Hopf invariant one. The latter theorem is yet another consequence of the existence of four and only four division algebras over **R**. As we will have seen these essential Hopf maps have found applications in recent years in a wide range of areas spanning nematic crystals, chiral soliton models, gauge theories, supergravity and lately anyonic theories of high temperature superconductivity.

Besides the remarkable non-integrable parallelism on S^7, the crowning application of octonions to projective geometry came in 1933 with the discovery by Ruth Moufang

[254] of a non-Desarguesian plane. In the same year, Max Zorn [352] discovered a uniqueness theorem for octonions, the analog of Frobenius uniqueness theorem for quaternions.

3.g.2. Octonionic quantum mechanics, birth of Jordan algebras (1933-1934)

In the same year, 1933, encouraged by the success of quantum mechanics in the atomic domain, some physicists thought that an extension of quantum theory was necessary to explain some newly observed nuclear phenomena. They were the discoveries of the neutron, of nuclear forces and of the apparent violation of energy momentum conservation in beta decay. It was in this spirit that P. Jordan [64] introduced a reformulation of quantum mechanics based on a commutative but non-associative algebra of observables represented by Hermitian complex matrices. In their generalization of quantum mechanics, Jordan, von Neumann and Wigner (JNW) [65] extended these matrices to be valued in the four division algebras. They found that the only **new** kinds of observables are the ones connected to octonions. They further conjectured that the only exceptional solution of the Jordan postulate is given by M_3^8, the algebra of (3x3) Hermitian octonionic matrices. Namely, the exceptional observables can only exist in a finite 3-dimensional Hilbert space. Its real, complex and quaternionic subalgebras are denoted by M_3^1, M_3^2 and M_3^4 with their respective automorphism group O(3), SU(3) and Sp(3,H). The identification of the automorphism group of M_3^8 must wait till 1950s. As to the applicability of the new algebras, JNW 's hope that M_3^8 may help to explain nuclear phenomena and beta decay was soon dashed by the successes of Fermi's theory of β-decay in 1934 . Thus octonions disappeared from the physics scene as soon as they were introduced.

3.g.3. Exceptional life in mathematics (1950-1967)

Shunt from physics, octonions continued to prosper in mathematics. Their study triggered a flowering of the theory of non-associative algebras [241]. From it came the problem of classification of algebras by way of a hierarchy of identities. Thus algebras obeying the alternative identities are called alternative. In 1950, Bruck and Kleinfeld [353] showed that the only alternative, non-associative division algebras were the octonions. Later there were also the work of Shirskov and Slater. In the same year the groups F_4 and E_6 were shown by Chevalley and Schafer [246] to be the automorphism groups of M_3^8 and its complexified form respectively. In the following decade, Freudenthal, Rozenfeld and Tits constructed new geometries connected with the remaining exceptional groups and their subgroups. Each of such geometry is associated with a pair of Hurwitz algebras. These beautiful geometries, their related groups and subgroups are classified and fitted in Magic Squares awaiting physical applications yet to come.

3.g.4. New attempts at applications and exceptional unified theories (1960-1978)

During the sixties, an era marked by the population explosion among hadrons, there was a second revival of octonions in particle theory. Thus Souriau [354, 355], Pais [356] and Tiomno [357] tried to fit the new particles into octonionic multiplets. Others like Goldstine, Horwitz [358] and Biedenharn [359] attempted the construction of octonionic Hilbert spaces. Pais [360] and Penney [361] put forth an octonionic generalization of the Dirac equation. Gamba [362] revived the exceptional M_3^8 algebra in the hope of identifying its SU(3) structure to the 8-fold way of Gell-Mann and Ne'eman. Exceptional observables were further studied by Segal [363], Sherman[364] and Gamba [365], who found them to be physically acceptable in principle. Yet despite all these brave efforts, octonions must wait for more genuine physical incarnations to reemerge in the following decade.

The seventies witness the dawn of a new era in both theoretical and experimental particle physics. There was the electroweak unification based on the group SU(2) x U(1), embodying spontaneous symmetry breaking and renormalizability. There was the establishment of the quark model with the introduction of color symmetry. An exact local color SU(3) gauge theory of strong interactions was born. New quarks and their bound states were predicted and subsequently observed.

Soon the standard SU(3) x SU(2) x U(1) model was embedded into a single simple GUT group. Possibilities were the groups SU(5), SU(6), SO(10), all of which occur in the Magic Square reviewed in Sect.3f. There, SU(3) is connected to transverse octonions $(e_1, e_2, ..., e_6)$, while U(1) is connected to transverse quaternions (j_1, j_2). SO(10) x U(1) appears as the holonomy group of the Kähler manifold E_6 / SO(10) x U(1) etc. These connections suggest linking color symmetry to octonions and taking up the groups E_6, E_7 and E_8 as grand unified groups. Thus the exceptional Jordan algebra found a new interpretation as the basic fermion (lepton-quark) representation of the GUT group E_6. For the very first time, octonionic structures stood a good chance of being relevant to fundamental physics.

During this period supersymmetry was discovered and shown to be compatible with renormalizable quantum field theories by Wess and Zumino [4]. Local supersymmetry automatically allows for a unification of the other basic interactions with gravitation. The first step in this direction was the supersymmetric extension of Einstein's theory by Freedman, van Nieuwenhuizen and Ferrara as well as by Deser and Zumino [366]. The second step consists in enlarging such a supergravity theory to incorporate local internal symmetries. The latter are restricted to having only one graviton, to be the orthogonal group O(n) with n ranging from 2 to 8 . At this juncture, no connection whatsoever seemed to exist between local supersymmetric theories and either quaternionic or octonionic structures. Big surprises were yet to unravel during the next decade.

3.g.5. Extended supergravities, strings and membranes (1978-Present)

Indeed deeper connections to octonions only came with the advent of extended supergravity theories. These theories rekindled the hope of realizing Einstein's dream for a geometrically unified theory of all interactions, of matter and fields including gravity. In 1978 the largest, N = 8 extended supergravity with O(8) internal local symmetry finally got realized in the work of Cremmer, Julia and Scherk [367]. The N = 2, 4 cases had previously been constructed. Moreover different broken or unbroken N = 8 theories can arise from various spontaneous compactifications of a unique pure D = 11, N = 1 supergravity theory [368]. It was thought that the latter theory may be finite i.e. that no counterterms compatible with the symmetry of the action can be added. This sourceless theory completely determines via dimensional reduction all matter couplings of the N = 8 theory in four dimensions. In such a supersymmetric incarnation of the Kaluza-Klein idea, this realization gave substance to Einstein's dream of determining the right hand (matter) side of his equations from a geometric principle, albeit from the geometry of D = 11 superspace.

Though too small a group to accommodate the standard model, the local O(8) symmetry with its SU(3) x U(1) subgroup can nevertheless account for symmetries such as chromodynamics and electrodynamics. Here one sees the first manifestation of octonionic structure among **exact** fundamental symmetries. Indeed O(8) is the norm group of octonions while SU(3) x U(1) is the group associated with octonionic Grassmann numbers.

A first big surprise is that the N = 8 supergravity Lagrangian admits a non-compact form $E_{7(-7)}$ of E_7 with compact subgroup SU(8) as global symmetry group. Now E_7 has both quaternionic and octonionic structures. The other exceptional groups emerge through dimensional reduction of the D = 11 supergravity theory. For instance, reduction to 3 dimensions gives as global symmetry a non-compact form of E_8. Similarly, reduction to 5 and 6 dimensions begets non-compact forms of E_6 and SO(10) respectively. This striking embedding in extended supergravity of the Magic Square was demonstrated by Julia [369]. Further evidence in this direction was later found by Günaydin, Sierra and Townsend [370] in a series of works. They discovered that in D = 3, 4 and 5, N = 2 Maxwell-Einstein supergravity the manifolds of scalar multiplets are coset space of groups of a non-compact form of the Magic Square. The complex and quaternionic counterparts of these structures had been studied by Bagger and Witten [371]. The octonionic structures are new ; they do not arise from dimensional reductions of the D = 11 Cremmer-Julia theory [293]. Instead of $E_{7(-7)}$ with compact subgroup SU(8), one obtains global $E_{7(-25)}$ with maximal compact subgroup E_6 x U(1).

Another manifestation of octonionic structure in D = 11 supergravity came about through the Englert's spontaneous compactification. Here the 11-dimensional space-time splits into a product space of an anti-de Sitter spacetime related to Hermitian quaternions and a round 7- sphere connected to octonions. In fact, by way of the Freund-Rubin Ansatz for the antisymmetric gauge field, the equations of the theory reduce precisely to those studied by Cartan and Schouten in their work on the absolute parallelisms of S^7.

During the last few years the most striking appearance to date of division algebras [372], of octonions in particular [373] and of Jordan algebras [374, 375], first came at the wake of anomaly free superstring theory. Of course, there is the celebrated D = 10 heterotic superstrings with as their gauge groups SO(32) / Z_2 and E_8 x E_8. Be it at the level of the particle spectra or of the associated vertex operators [376], the octonionic connection is abundantly clear. Subsequently, a one-to-one correspondence was discovered between classical superstrings, supermembranes in critical dimensions and the four division algebras. This could alternatively be classified in terms of the Magic Square. It appears that, amongst all critical objects, the only quantum mechanically consistent ones are the octonionic superstring in D = 10 and supermembrane in D = 11. On the other hand, Witten's open bosonic string field theory was shown to embody non-associative structures. Thus the basic relevance of non-associativity, specifically of octonions, seems intimately tied to the quantization of supersymmetric extended objects.

4. Division, Jordan Algebras and Extended Objects

4.a. Dyson's 3-fold Way: Time Reversal and Berry Phases

The energy levels of a complex quantum system such as an atomic nucleus are given by the eigenvalues of the Hamiltonian, a Hermitian operator in an infinite dimensional Hilbert space. Typically, this eigenspectrum is made up of a continuum plus a (possibly large) number of discrete levels. By focusing on the discrete levels, one may truncate, as a first approximation, the associated Hilbert spaces down to ones with finite but possibly large dimensions. The resulting Hamiltonians are then represented in some basis by finite dimensional matrices. In principle, any physical information about these systems can then be deduced from the knowledge of the eigenvalues and eigenfunctions of these truncated hamiltonians.

However, in practice, the hamiltonian of a complex system is not known. Moreover, its density of states becomes so large and, above a certain excitation energy, their intermixing become so strong, that an account of the characteristics of any single state is an impossible task. Ignorant even about the very "nature" of the system at hand, one may try to at least describe its average properties such as the general appearance and the degree of irregularity of the level spectrum.

On pondering the above issues, Wigner [377, 378, 379, 380] (and also von Neumann [381]) formulated a statistical theory of energy levels, a theory in which the hamiltonians are given by random matrices. The distributions of the random matrix elements are only constrained by the symmetry requirements standardly imposed on a Gaussian ensemble of Hamiltonians. To uncover the connection with division algebras in general and with quaternions in particular, we shall confine ourselves to the implications of symmetry constraints on the random Hamiltonians, such as time reversal invariance. Greater details on random matrices are to be found in comprehensive books such as the ones by Mehta [382] and Carmeli [383]. Our short discussion is patterned after these texts and the original articles of Wigner [377, 378, 379, 380, 384, 385] and of Dyson [386, 387].

As random matrices, Hamiltonians are restricted in their general structure by spacetime symmetry requirements. Thus, if a Hamiltonian H is invariant under time and spatial translations, it is time independent and commutes with the total linear momentum. Similarly, parity and rotational invariances imply the commutativity of H with the parity and total angular momentum operators, respectively. On the other hand, if H has none of the above symmetries, it must be a complex Hermitian matrix with complex eigenvalues. Then the unitary group U(n) is the corresponding group of canonical transformations preserving the hermiticity of H under similarity transformations. The resulting *Gaussian unitary ensemble* could be realized by systems without time reversal invariance, such as in the case of an atom or a molecule in a sufficiently strong external magnetic field.

Above all, the constraint of time reversal invariance is specially relevant to the main theme of this volume. We recall the essentials of the associated quantum mechanical canonical transformation.

It is well-known that time-reversal is implemented by an antiunitary operator T. This property follows from the demand that, under the action of T, the position and momentum operators $\mathbf{x}$ and p be even and odd, respectively, i.e. $T\mathbf{x}T^{-1} = \mathbf{x}$, $T\mathbf{p}T^{-1} = -\mathbf{p}$. Then the preservation of the canonical commutation relation $[\mathbf{p}\cdot\mathbf{a}, \mathbf{x}\cdot\mathbf{b}] = -i\,\mathbf{a}\cdot\mathbf{b}$, with $\mathbf{a}$ and $\mathbf{b}\in\mathbf{R}^n$, implies that T anticommutes with i and is therefore an antilinear operator. So T is antiunitary.

If we define a complex conjugation operator K such that $K\psi = \overline{\psi}$, $K^2 = 1$ and TK unitary; then $T = U K$ where U is unitary. A state ψ under time reversal reads

$$\psi^R = T\psi = U\psi^* \; . \tag{4a.1}$$

From this rule and from T-invariance, we have $<\phi, A\psi> = <\phi^R, A^R\psi^R>$, for all vectors $<\phi|$ and $|\psi>$. It follows that

$$A^R = U A^T U^{-1} \; , \tag{4a.2}$$

the superscript T denoting transposition.

Now an operator A is called *self-dual* if $A^R = A$. A physical system is invariant under time reversal if its Hamiltonian is self-dual, namely $H^R = H$.

Under a unitary transformation Ω a state $|\psi> \rightarrow |\psi>' = \Omega|\psi>$, correspondingly

$$T' = \Omega T \Omega^{-1} = \Omega T \Omega^\dagger \tag{4a.3a}$$

and

$$U' = \Omega U \Omega^T \; . \tag{4a.3b}$$

Operating twice with T should leave the system invariant, so $T^2 = \alpha I$ with $|\alpha| = 1$, $T^2 = UKUK = UU^* = \alpha I$. As $U^*U^T = I$, $U = \alpha U^T = \alpha(\alpha U^T)^T = \alpha^2 U$; $\alpha = \pm 1$. So

$$U U^* = I \qquad \text{or} \qquad U U^* = -I \ , \tag{4a.4}$$

the unitary matrix U is either *symmetric* or *antisymmetric*. These alternatives correspond to the cases of integral and half-integral spin systems, respectively.

Indeed, if the particles of our system are spinless, then, in the coordinate representation where the commutators $[U, \mathbf{x}_i] = [U, \mathbf{p}_i] = 0$, U can be chosen to be the unit operator. So $T = K$ if the Schrödinger equation is without spin.

On the other hand, if the particles carry spins, then the choice of U will be determined by the total angular momentum $\mathbf{J}$ with $\mathbf{J}^R = T\,\mathbf{J}\,T^{-1} = -\mathbf{J}$. In the standard representation with J_x and J_z real and J_y pure imaginary, $\{T, J_{x,z}\} = 0$ while $[U, J_y] = 0$. So in the case of one particle with spin $\boldsymbol{\sigma}$, one can represent T by $T = \sigma_y K$ or by Wigner's choice of

$$T = \exp\left[\frac{i}{2}\pi\,\sigma_y\right] K \ . \tag{4a.5}$$

For an N particle system

$$T = \exp\left[\frac{i}{2}\pi\,(\sigma_{1y} + \sigma_{2y} + \sigma_{Ny})\right] K \tag{4a.6}$$

with $T^2 = +1$ for even N, i.e. for an integral total angular momentum in units of $\hbar$ and $T^2 = -1$ for odd N, i.e. a half-integral total angular momentum.

For the $T^2 = 1$ case, there exists a unitary operator Ω such that $U = \Omega\,\Omega^T$, and a transformation $\psi' = \Omega^{-1}\psi$ takes U to I. So, for the integral spin case, a representation can always be chosen where $U = I$, which is preserved by further canonical transformations $\psi' = R\,\psi$, provided that R is a real orthogonal matrix, $R\,R^T = I$. Thus, by an appropriate choice of basis, the (n x n) hamiltonian matrix of a time-reversal invariant integral spin system can always be made real symmetric. The relevant canonical transformation is then the orthogonal group O(n) acting on a real Hilbert space and the relevant ensemble is then the Gaussian orthogonal ensemble.

Now consider the $T^2 = -1$ case where the time-reversal invariant system with a Hamiltonian matrix H is also rotationally invariant, i.e. $[H, \mathbf{J}] = 0$. Then, in the representation of a purely imaginary J_y, $[H, U] = 0$ and H^R reduces to H^T. So a rotationally invariant system with integral or half-integral total spin is represented by a real symmetric matrix H acting on a real Hilbert space with, as appropriate ensemble, the orthogonal Gaussian ensemble.

Next, we consider a T-invariant system with half integral spin (i.e. $T^2 = -1$) but without rotational symmetry. If two state vectors ϕ and ψ satisfy the Schrödinger equation with a time-reversal invariant H, from the antilinear and norm preserving properties of T, we obtain

$$(T\psi, T\phi) = (\psi, \phi)^* = (\phi, \psi). \tag{4a.7}$$

If ϕ is the time-reversed state of ψ: $\phi = T\psi$. Then $T\phi = T^2\psi = -\psi$ and Eq. (4a.7) reads

$$(\phi, T\phi) = -(\phi, \psi) = (\phi, \psi)\ . \tag{4a.8}$$

Hence

$$(\phi, \psi) = (T\psi, \psi) = 0\ , \tag{4a.9}$$

namely, if $T^2 = -1$, ψ and $T\psi$ are always orthogonal and degenerate energy eigenstates. This doublet degeneracy was originally discovered by Kramers in a specific physical context. If an ion with an odd number of electrons is put in an external, possibly strongly inhomogeneous, crystalline electric field, there exists an antiunitary operator T such that $T^2 = -1$ and $[T, H] = 0$. More generally, such a *Kramers degeneracy* can be most readily displayed in the following quaternionic formalism:

As an antisymmetric unitary n x n matrix, U in Eq. (4a.1) can be reduced to the canonical Hua matrix Z_{ij} by a transformation $U' = \Omega\, U\, \Omega^T$. A Hua matrix is a banded diagonal matrix, i.e. one with +1 in the subdiagonal and -1 in the superdiagonal, i.e. $Z_{ij} = \delta_{i,\, j+1} - \delta_{i,\, j-1}$. Further canonical unitary transformations Ω' are possible if $[\Omega', T] = [\Omega', Z K] = 0$, the group of Ω' transformations must be such that

$$Z K \Omega' - \Omega' Z K = 0 \qquad (4a.10)$$

and

$$\Omega' Z \Omega'^T = Z \ . \qquad (4a.11)$$

We encountered the latter condition before, in Sect. 2b. It is but the defining relation for *symplectic* transformations. The canonical group is then the symplectic group Sp(n) acting on an n-dimensional quaternionic Hilbert space; the ensemble is then the Gaussian symplectic ensemble.

The algebra of the symplectic group is naturally expressed in terms of quaternions. Since any (2 x 2) matrix $A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ with complex entries can be cast as a complex quaternion $A = A_\mu e_\mu$ with

$$A_0 = \frac{1}{2}(\alpha + \delta) \ , \ A_1 = \frac{i}{2}(\alpha - \delta) \ , A_2 = -\frac{1}{2}(\beta - \gamma) \ , \ A_3 = \frac{i}{2}(\beta + \gamma) \ , \qquad (4a.12)$$

a general (2Nx2N) matrices Q over **C** can be partitioned into N^2 (2 x 2) blocks, each of which can be expressed in terms of quaternions. So Q can be seen as an (N x N) matrix with complex quaternion elements Q_{ij}, i, j = 1, 2,..., N. In particular, we may write the matrix Z in the simple form:

$$Z = e_2 I \ ; \qquad (4a.13)$$

I being an (N x N) unit matrix .

We now recall our notations in Sect. 2.a for conjugate quaternion $\overline{q} = q_\mu \overline{e}_\mu$, the complex conjugate quaternion $q^* = q_\mu^* e_\mu$ and the Hermitian conjugate quaternion $q^\dagger = \overline{q}^*$. In the quaternionic context, the usual matrix operations of transposition, Hermitian conjugation and time reversal then translate very simply into

$$(Q^T)_{kj} = - e_2 \overline{Q}_{jk} e_2 \ , \quad (Q^\dagger)_{kj} = Q^\dagger_{jk} \ , \ (Q^R)_{kj} = e_2 \ (Q^T)_{kj} \, e_2^{-1} = \overline{Q}_{jk} \qquad (4a.14)$$

respectively. The matrix Q^R is called the *dual* of Q, if $Q = Q^R$, Q is *self-dual*. From Eq. (4a.14), it follows that if $Q^R = Q^\dagger$, then Q_{ij} are real quaternions; Q is then called *quaternion real* or **H**-real .

A unitary matrix Ω, obeying $Z = \Omega Z \Omega^T$ and thus defining the symplectic group, is clearly **H**-real since $\Omega^R = \Omega^\dagger = \Omega^{-1}$. Now for physical systems with half-integral spin the Hamiltonian matrices of time-reversal invariant are both Hermitian and self-dual $H = H^\dagger = H^R$; they are therefore **H**-real.

In reference to Kramers degeneracy, the quaternionic formulation naturally leads to the following theorem: If H is a Hermitian **H**-real (N x N) matrix, then there exists a symplectic matrix Ω such that $H= \Omega^{-1}\, D\, \Omega$ where D is diagonal, real and a scalar, i.e. D consists of N blocks of the form $\begin{pmatrix} D_j & 0 \\ 0 & D_j \end{pmatrix}$. Consequently, the eigenvalues of H come in N equal pairs. That is therefore the generalization of Kramers' result.

In the above we witness an elegant and possibly physical realization of the quaternionic Hilbert space, of quaternionic quantum mechanics.

In summary, the connection between division algebras over the real **R** and the theory of statistical matrices is best given by the following table [383] :

Time Reversal symmetry	Rotational symmetry	H	Canonical group
GOOD	GOOD	**R**	O(n)
GOOD	NOT GOOD integral spin	**R**	O(n)
GOOD	NOT GOOD half-integral spin	**H**	sp(n)
NOT GOOD	GOOD, or NOT GOOD	**C**	U(n)

Such a table displays Dyson's *three-fold way* or the realization of the orthogonal, unitary and symplectic groups, i.e. the groups of (n x n) matrices over the three division algebras **R**, **C** and **H**. They are the canonical transformation groups compatible with the (non-)invariance of the Hamiltonian resulting from the homogeneity, the isotropy of space and the homogeneity of time. As underscored by Dyson [386] , this correspondence can be understood by bringing together two known facts, one from algebra, the other from foundational quantum theory.

On the one hand, we have the classical theorem of Fröbenius: there are only three associative division algebras over the reals, **R** ,**C** and **H**. On the other hand, according to Jordan, Wigner and von Neumann [65], the ground field for quantum mechanics needs not be **C**, as standardly assumed, it can be **R** and **H**. Indeed the moment that anti-unitary operators such as time inversion are included in the quantum mechanical formalism, it is more natural to work with a quantum mechanics over **R**, which then brings out the above 3-foldway in the theory of random matrices, afforded by Fröbenius' theorem. The latter theorem also lies at the root of two other 3-fold ways [386] known before the 1960s. They are Weyl's classification of matrix algebras into orthogonal, unitary and symplectic ones and Wigner's classification of irreducible representations of groups by unitary matrices into potentially real, complex and pseudo-real. So, in our discussion, we obtain a real quaternionic (symplectic) description of half-odd-integer spin systems described by the pseudo-real representations of SU(2). In fact, the quaternionic description naturally enters whenever we deal with a group admitting pseudo-real representations.

In recent years, there has been a renewed interest in global aspects of Kramers' degeneracy and its non-Abelian extension in time-reversal invariant Fermi systems [42]. Specifically, one investigates various topological and geometric invariants in eigenvalue perturbation theory on Berry phases [388]. A canonical example is a spin $\frac{3}{2}$ system in an electric quadrupole field. It would take us too far afield to survey these results here. Besides, a comprehensive review is available in Ref.[389]. Instead, we shall discuss yet another manifestation of the threefold way, more exactly of the fourfoldway (via Hurwitz's theorem) in Chern-Simons-type field theories with *broken* time-reversal invariance. We do so with a discussion of Hopf essential fibrations and their one-to-one connection to division algebras.

4.b. Essential Hopf Fibrations and D≥3 Anyonic Phenomena

4.b.1. Hopf's construction and division algebras

In 1935, Hopf [31] established a surprising and unique link between division algebras **K** = **R**, **C**, **H** and **Ω** and topology. He connected these algebras with the fibrations of S^{2n-1} by a great S^{n-1}-sphere for n = 1, 2, 4 and 8, respectively.

While there are alternative derivations of the Hopf maps – one such will be displayed later – Hopf's original construction [30, 31] is in our view most instructive. It can be directly inferred through the following equivalent statement of Hurwitz's theorem:

The only dimensions n of $\mathbf{R}^n$ with a multiplication $\mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}^n$, denoted by the product $F(X, Y) = X\overline{Y}$ with $X\overline{Y} = 0 \leftrightarrow X = 0$ or $Y = 0$ are $n = 1, 2, 4, 8$, namely these multiplications can be realized by $\mathbf{K}$ with $X, Y \in \mathbf{K}$ i.e. $\mathbf{R}^n \approx \mathbf{K}$.

Then, by a linear identification of the product space $\mathbf{K} \times \mathbf{K}$ with $\mathbf{R}^{2n}$, $F(X,Y)$ where $X, Y \in \mathbf{K}$, defines a bilinear map called the Hopf map

$$H: \mathbf{R}^{2n} \to S^{n+1} \tag{4b.1}$$

with

$$H(X,Y) = (|X|^2 - |Y|^2, 2F(X, Y)) = (|X|^2 - |Y|^2, 2X\overline{Y}) \ . \tag{4b.2}$$

It follows that for $|X|^2 + |Y|^2 = 1$, $|H(X, Y)|^2 = (|X|^2 - |Y|^2)^2 + 4|XY|^2 = 1$. We next consider two spheres. The first S^{2n-1} is the space of pairs (X, Y) of $\mathbf{K}$ with $|X|^2 + |Y|^2 = 1$. The second S^n is the space of all pairs $(s, \mathbf{k})$ consisting of a real number $s = |X|^2 - |Y|^2$ and $\mathbf{k} = 2X\overline{Y} \in \mathbf{K}$. Thus H, Eq. (4b.1), restricts to the map $H: S^{2n-1} \to S^n$ with the spheres S^{2n-1}, S^{2n} and S^{n-1} as the fibre space, the base space and the fiber respectively.

We now parametrize S^n by the standard unit (n+1)-vector $\vec{N}$, $\vec{N}^2=1$. Next, take $K^T = (K_1, K_2)$ with $K_1, K_2 \in \mathbf{K}$ and $K^TK = 1$, to be a unit normed $\mathbf{K}$-valued 2-spinor, parametrizing S^{2n-1}. Then the Hopf map (4b.2) reads

$$\vec{N} = \mathrm{Sc}\left(K^{\dagger}\vec{\gamma}\, K\right) \tag{4b.3}$$

with $K^{\dagger} = (\overline{K}_1, \overline{K}_2)$ and

$$\gamma_\mu = \begin{pmatrix} 0 & e_\mu \\ \overline{e_\mu} & 0 \end{pmatrix}, \ \mu = 0, 1, \ldots, m\text{-}1 \quad \text{and} \quad \gamma_m = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{4b.4}$$

$m = 1, 2, 4$ and 8. This relation between the γ s and the **K**-units e_m mirrors the fact [280] that every orthogonal multiplication $R^k \times R^n \rightarrow R^n$ is a Clifford module C_{k-1}. Specifically, the irreducible Clifford modules C_0, C_1, C_3 and C_7 correspond to **R**, **C**, **H** and **Ω** respectively. Indeed, the γs are just the Dirac matrices of SL(2, **K**) with γ_m being the analog of γ_5 of the familiar 4-dimensional formalism.

Alternatively, with the spheres S^{2n-1} in **K** x **K** and $S^n = \mathbf{K} \cup \{\infty\}$, the Hopf projection map, $\pi : S^{2n-1} \rightarrow S^n$ also takes the form of

$$\pi(X, Y) = \begin{cases} X/Y & \text{provided } Y \neq 0 \\ \infty & \text{if } Y = 0 \end{cases} \qquad (4b.5)$$

where $|X|^2 + |Y|^2 = 1$, $X, Y \in \mathbf{K}$.

Geometrically, $\pi^{-1}(X, Y)$, the pre-image (or inverse) of this Hopf map, is the intersection of S^{2n-1} with an n-dimensional subspace of **K** x **K**, i.e. a great (n – 1) sphere S^{n-1} or a (n – 1) *cycle*. The image of a point on S^n is a S^{n-1}-sphere on S^{2n-1} since $\vec{N}$ (or $\frac{X}{Y}$) is invariant under the phase transformation $K \rightarrow K U$ ($X \rightarrow X U$, $Y \rightarrow Y U$). $U = \overline{U}$, $|U|^2 = 1$, is a unit normed, pure imaginary **K**-number, i.e. $U \in S^0 \approx Z_2$, $S^1 \approx U(1)$, $S^3 \approx SU(2)$ and S^7, a (n – 1)-cycle for n = 1, 2, 4 and 8, respectively. Let us illustrate this explicitly in the case of n = 2 case of the complex Hopf fibration by consider any map $f(x)$: $R^3 \rightarrow S^2$ such that $f(x) \underset{|x| \to \infty}{\longrightarrow} (0, 0, 1)$.

Such a map implies a one-point compactification of R^3 into S^3. Equivalently, we get the map $\tilde{f} : S^3 \rightarrow S^2$, $\pi_3(S^2) \approx Z$. Given $\tilde{f}$, Hopf's construction of H(f), a generator of $\pi_3(S^2)$, goes as follows. Pick any point $\phi_1 = Z_1 / Z_2$ of S^2, $Z_1, Z_2 \in \mathbf{C}$. Its image, $\tilde{f}^{-1}(\phi_1)$ in the covering space S^3, parametrized by the pair $[Z_1, Z_2]$ with $|Z_1|^2 + |Z_2|^2 = 1$, is a loop C_1 or fibre $U = \exp(i\phi)$, because ϕ_1 is invariant under the transformation $Z_i \rightarrow Z_i \exp(i\phi)$. If we now choose any other point ϕ_2 of S^2, then the map $\tilde{f}^{-1}(\phi_2)$ is also a loop C_2 in S^3. If $\phi_1 \neq \phi_2$, then C_1 and C_2, represented by the vectors $\mathbf{x}_1$ and $\mathbf{x}_2$, are disjoint curves.

The <u>Hopf invariant</u> $\gamma(\tilde{f}) = \gamma(f)$ is then simply given by the Gauss linking number of C_1 and C_2 in S^3 (or R^3):

$$\gamma(\tilde{f}) = \gamma(f) = \frac{1}{4\pi} \oint_{C_1} \oint_{C_2} \frac{d\mathbf{x}_1 \times d\mathbf{x}_2 \, \mathbf{x}_{12}}{|\mathbf{x}_{12}|^3} \; . \tag{4b.6}$$

To see how the Hopf map (4b.1)-(4b.2) maps circles in S^3 into single points in S^2 and to check that it has Hopf invariant 1, we do the following. Take the 4-position vector in R^4, $\vec{y} = (y_1, y_2, y_3, y_4)$ with $y_i \in \mathbf{R}$, $i = 1, 2, 3, 4$. S^3 is then given by $\vec{y}^2 = 1$ or $|X|^2 + |Y|^2 = 1$, $X = (y_1 + i\,y_2)$, $Y = (y_3 + i\,y_4)$. The Hopf map $f : S^3 \rightarrow S^2$ is then

$$H(X, Y) = (|X|^2 - |Y|^2, 2X\bar{Y}) = (N_3, N_1 - i\,N_2) \tag{4b.7}$$

where

$$\begin{aligned} N_1 &= 2(y_1 y_3 + y_2 y_4) \; , \\ N_2 &= 2(y_2 y_3 - y_1 y_4) \; , \\ N_3 &= y_1^2 + y_2^2 - y_3^2 - y_4^2 = -1 + 2(y_1^2 + y_2^2) \; . \end{aligned} \tag{4b.8}$$

Then $\mathbf{N}^2 = (y_1^2 + y_2^2 + y_3^2 - y_4^2)^2 = 1$, $\mathbf{N} = (N_1, N_2, N_3) \in S^2$. Next, select any two points of S^2 such as its North and South Poles, $\mathbf{N} = (0, 0, \pm 1)$. Their images, the loops C_1 and C_2, are then described by the curves $y_1^2 + y_2^2 = 1$ and $y_1^2 + y_2^2 = 0$. By deforming S^3 into R^3 through a sterographic projection onto the $y_4 = 0$ plane, with, as the center of projection, the North Pole ($y_4 = 0$), we get the following Fig. 7.

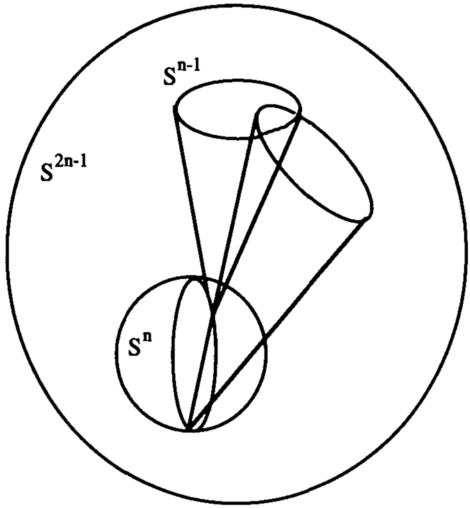

Fig. 7

The curves C_1 and C_2 are clearly linked once; so the Hopf map (4b.7) has invariant one. Similar constructions can be obtained for $\mathbf{K} = \mathbf{H}$ and $\mathbf{\Omega}$, they are left as exercises for the reader. Next, we consider in some details the various expressions for the Hopf invariant.

4.b.2. The many faces of the Hopf invariant

A topological invariant, the Hopf invariant $\gamma(\Phi)$ classifies the maps Φ: $S^{2n-1} \to S^n$. Physically, its presence as an added topological action in the Wilczek-Zee model ($n = 2$) [390] is key to a dynamical realization of exotic spin and statistics through a Berry phase [391]. In essence Ref.[392] is about the many faces of $\gamma(\Phi)$, its various mathematically different and physically revealing forms, its unique connection to division algebras. We begin with a compact derivation of the Whitehead form of the Hopf invariant $\gamma(\Phi)$ [393].

Let $V^{(p)}(M)$ be the space of p-forms on a manifold M, $p \le \dim M$. Next two differential forms are defined. First, on S^n, we select a normalized n-form, the area element ω_n, $\int_{S^n} \omega_n = 1$. Then on S^{2n-1}, by pulling back the Hopf map F, we define a second induced n-form $\widetilde{F}_n = \Phi^*\omega_n \in V^{(n)}(S^{2n-1})$. The latter is closed ($d\widetilde{F}_n = 0$) since $d(F^*\omega_n) = F^*(d\,\omega_n) = 0$ and $d\,\omega_n = 0$; the pull-backs of closed forms are closed. By de Rham's 2nd theorem $H^n(S^{2n-1}) \approx 0$, all closed n-forms on S^{2n-1} are exact; so there is a non-unique (n-1)-form $\widetilde{A}_{n-1} \in V^{(n-1)}(S^{2n-1})$ such that $d\,\widetilde{A}_{n-1} = \widetilde{F}_n$. The integral

$$\gamma(\Phi) = \oint_{S^{2n-1}} \widetilde{A}_{n-1} \wedge \widetilde{F}_n \tag{4b.9}$$

of the exterior product $\widetilde{A}_{n-1} \wedge \widetilde{F}_n$ over S^{2n-1} is therefore defined. It can be readily verified that

a) $\gamma(\Phi)$ is independent of the choice of either $\widetilde{A}_{n-1}$ such that $d\,\widetilde{A}_{n-1} = \widetilde{F}_n$ or of ω_n.

b) $\gamma(\Phi) = 0$ for all maps $\Phi : S^{2n-1} \to S^n$ with n odd.

c) $\gamma(\Phi)$ is invariant for any two smooth and homotopic maps $S^{2n-1} \to S^n$.

Equation (4b.9) is called the Whitehead form of the Hopf invariant $\gamma(\Phi)$. In physics applications, this form of the Hopf index is recognized as the Chern-Simons term for the Kalb-Ramond field A_{n-1}. Then property (a) translates into the gauge invariance of this antisymmetric Abelian gauge field F.

Moreover, Eq. (4b.9) is only one of several equivalent definitions of the Hopf invariant. It is instructive to derive these interesting alternatives forms of $H(\Phi)$. To that end, we parametrize the map $F: S^{2n-1} \to S^n$ by an (n + 1) component unit vector $\vec{N} \in S^n$, ($\vec{N}^2=1$). If $\vec{N}_0$ is an arbitrary fixed point on S^n, then as illustrated above for the complex Hopf fibration, the equation $\vec{N}(x) = \vec{N}_0$ is that of a closed hypercurve $C_0 \approx S^{n-1}$ on S^{2n-1}. Equivalently, the pre-image of $C_0 = F^{-1}(\vec{N}_0)$ of $\vec{N}_0$ is an (n − 1)-cycle in S^{2n-1}. If S_0 is some n-dimensional closed connected submanifold on S^{2n-1} with, as its boundary δS_0, C_0, then $\vec{N}(x)$ maps S_0, known as a Seifert surface, onto the whole n-sphere. The Hopf invariant $\gamma(\vec{N})$ can be defined as the number of times $\vec{N}$ maps S_0 onto S^n. It is the mapping degree of $\vec{N}(x)$ restricted to S_0, from S_0 to S^n, $\vec{N}(x) : \Sigma_0 \to S^n$ and is independent of the point $\vec{N}_0$ of S^n. With $\pi_n(S^n) \approx Z$, the Hopf invariant is an integer. By the theorem of Eilenberg and Niven [43] quoted in Sect.2a representative maps $S^n \to S^n$ for n = 2, 4 and 8 with winding number m are given by X^m with $X \in \mathbf{C}$, $\mathbf{H}$ and $\mathbf{\Omega}$ respectively. Consequently, in addition to (4b.9), we also have a generalized flux and loop integral representation of $\gamma(\vec{N})$ [394, 395]:

$$\gamma(\vec{N}) = \oint_{\Sigma^0 \sim S^n} \widetilde{F}_n = \oint_{C^0 \sim \partial\Sigma^0} \widetilde{A}_{n-1} \tag{4b.10}$$

where $\widetilde{F}_n = d\widetilde{A}_{n-1}$ is the area element n-form of S^n mapped by $\vec{N}$ into S^{2n-1}. As it should be, these $\widetilde{F}_n$ and $\widetilde{A}_{n-1}$ are the same ones occurring in the construction of the Whitehead form of $\gamma(\vec{N})$. In the form (4b.10), the Hopf invariant gives, upon exponentiation, a generalized Aharonov-Bohm-Berry phase factor associated with its antisymmetric U(1) gauge field.

The connection, established by Hopf himself, between the Hopf invariant and Gauss' linking number cannot be simpler: $\gamma(\Phi)$ was originally defined as a linking number ! The map $\mathbf{N}$ represents an element a in the homotopy group $\pi_{2n-1}(S^n)$. Pick two distinct points $\mathbf{N}_1$ and $\mathbf{N}_2$ on S^n, then their pre-images $F(\mathbf{N}_a) = C_a$ (a = 1, 2) are (n – 1)-manifolds in S^{2n-1}. After assigning a natural orientation to these hypercurves, we obtain two (n – 1)-spheres in S^{2n-1} or (n – 1)-cycles C_1 and C_2. They can either be linked or unlinked; $\gamma(\alpha)$ is just the linking number $Lk(\alpha_1, \alpha_2)$ of C_1 and C_2 and depends only on α. $\gamma(\alpha)$ is thus a homomorphism :

$$H: \pi_{2n-1}(S^n) \to Z \tag{4b.11}$$

with the generalized Gauss linking coefficient to be given later (see (4b.19)).

We close this part with a listing of some useful properties of the Hopf invariant [394, 396]:

a) For n odd, H is zero due to the anticommutativity of the linking numbers (see (4b.19)).

b) For n even, Hopf proved that maps of an *even* H always exist.

c) If the map Γ: $S^{2n-1} \rightarrow S^{2n-1}$ has degree p, then $\gamma(\Phi \circ \Gamma) = p\gamma(\Phi)$.

d) If the map Ψ: $S^n \rightarrow S^n$ has degree q, then $\gamma(\Psi \circ \Phi) = q^2\gamma(\Phi)$, the degree of the map $S^n \rightarrow S$, is an element of $\pi_n(S^n)$.

Next,we apply the above properties of H(Φ) to exotic spin and statistics properties of solitons in specific dimensions.

4.b.3. Twists, writhes of solitons and Adams' theorem

In recent years, two overlapping topics have attracted much attention. On the one hand, we have topological quantum field theories (TQFT) in D≥3 spacetime dimensions, pioneered by Witten [397, 398] and ushered in by the works of the mathematicians Donaldson [230, 399], Atiyah[400] and Jones [401]. On the other hand, there are quantum field theories of objects bearing *any* spin and statistics. From the standpoint of physics, these *anyons* may lay at the foundations of the definitive theories of fractional quantum Hall effect and high temperature superconductivity.

In his seminal work on the D = 3 Chern-Simons field theories, Witten [398] showed the beautiful correspondence between the expectation values of the Wilson loops traced by 'colored' *point* sources in spacetime and Jone's polynomials for knots. To obtain the fundamental Skein relation, in addition to doing the standard regularization, he had to regularize or *frame* the Wilson loops. In fact, such a regularization had also been given by Polyakov [402] in his proof of the D = 3 Fermi-Bose transmutation of baby Skyrmions in their *point-like limit.* It is at that very juncture that an interesting overlap take place with the theory of anyons. Our discussion takes off at this meeting point of topology, division algebras, geometry and physics.

First we recall some relevant features of the D = 3 CP_1 σ-model with a Chern-Simons term [390]

$$A = \int d^3x \left[| D_\mu Z |^2 + \frac{\theta}{8\pi^2} \varepsilon^{\mu\nu\lambda} A_\mu F_{\nu\lambda} + A_\mu J^\mu \right] . \qquad (4b.12)$$

$0 \le \theta \le \pi$. The field $Z^T = (Z_1, Z_2)$ with $|Z|^2 = 1$ is a two-component complex spinor and it lives on S^3. The unit normed field $\mathbf{n}$ is given by the complex Hopf projection map taking $Z \in S^3$ to $\mathbf{n} = Z^+ \boldsymbol{\sigma} Z \in S^2$, which is a particular case of the map (4b.3). D_μ is the covariant derivative with $A_\mu = i Z^\dagger \partial_\mu Z$ as the hidden U(1) gauge field and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, the associated the field strength. Since the well-known work of Belavin and Polyakov, this model has been known to admit exact, general classical S^2-solitons. While the third term in Eq. (4b.12) is the Aharonov-Bohm term, the second term alternatively reads as $S_H = \frac{-1}{2}\int d^3x\, A_\mu J^\mu$ (for $\theta = \pi$). Namely, it describes an interaction between A_μ and the conserved topological current $J_\mu = \frac{1}{8\pi} \varepsilon_{\mu\nu\lambda} \varepsilon_{abc} n^a \partial^\nu n^b \partial^\lambda n^c$. The latter is normalized so that the soliton (electric) charge $Q = \int d^2x\, J_0$ is an integer, labelling the elements of $\pi_2(S^2) = Z$. The field boundary condition is such that spacetime is $R^3 \cup (\infty) \approx S^3$; the Chern-Simons action assumes the identity of the Hopf invariant for the maps $\mathbf{n}$: $S^3 \to CP_1 \approx S^2$, classified by the generators of the homotopy group $\pi_3(S^2) \approx Z$.

Wilczek and Zee demonstrated how the interchange of two $Q = 1$-solitons or the rotation of one such soliton around another by 2π gives a statistical (alias spin) phase factor $e^{i\theta}$ to the wave function. So the soliton is an anyon with exotic spin spin $s = \frac{\theta}{2\pi}$ and intermediate statistics. This phase corresponds to a mapping with Hopf invariant 1, a key ingredient in a subsequent higher dimensional analog of the θ-spin and statistics connection.

Polyakov [402] undertook a tractable Wilson loop approach to the large distance behavior of soliton Green functions of system (4b.12). To study the effects induced by the long range Chern-Simons interactions, he approximated [403] the partition function **Z** by

$$\mathbf{Z} = \sum_{(P)}^{\text{all closed paths}} e^{-m\,L(P)} \left\langle \exp\left(i \int dx^\mu A_\mu\right)\right\rangle . \tag{4b.13}$$

P denotes a Feynman path of a soliton, seen as a curve in spacetime R^3. L(P) is the total path length.

The first exponential factor in Eq. (4b.13) is the action of a free relativistic *point-like* soliton of mass m , L(P) is its path-length. $\Phi(P) = \left\langle \exp(i \oint_P A^\mu \, dx_\mu) \right\rangle$, the functional averaging <...> is w.r.t. the Hopf action embodying the Aharonov-Bohm effect. This is typical of topologically massive gauge theories; the Chern-Simons-Hopf action induces magnetic flux on electric charges and vice versa. Being Gaussian, this phase is exactly calculable, thereby the analytic appeal of the point limit approximation. By direct integration of the equation of motion, $\Phi(P)$ is given by the exponentiation of the effective action :

$$\Phi(P) = \frac{1}{N} \exp\left[iS_0 + i \int d^3x \left(\frac{\theta}{4\pi^2} \varepsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho + A_\mu J^\mu \right) \right] . \qquad (4b.14)$$

S_0 is the free point particle action, N is a suitable normalization. The conserved current of a Q = 1 point source is $J_\mu(x) = \int d\tau \, \delta^3(x-y(\tau)) \frac{dy_\mu(\tau)}{d\tau}$. From Eq. (4b.12), the equation for J_μ reads

$$J_\mu(x) = -\frac{\theta}{2\pi^2} \varepsilon_{\mu\nu\rho} \partial^\nu A^\rho . \qquad (4b.15)$$

Upon substitution in Eq. (4b.14) with $\theta = \pi$, we get $\Phi(P) = \exp\left(\frac{i}{2} \int d^3x \, A_\mu J^\mu \right)$. So for the point current and in the Lorentz gauge $\partial^\alpha A_\alpha = 0$, $A\mu$ can be solved and one get:

$$\Phi(P) = \exp\{ i\pi I_G(P) \} , \qquad (4b.16)$$

where

$$I_G (C_\alpha \to C_\beta) = \frac{1}{4\pi} \oint_{C_\alpha} dx^\mu \oint_{C_\beta} dy^\nu \frac{\varepsilon_{\mu\nu\lambda}(x^\lambda - y^\lambda)}{|x - y|^3} \qquad (4b.17)$$

in the limit where the two smooth closed 3-space curves C_α and C_β <u>coincide</u>, namely $C_\alpha = C_\beta = P$, the soliton worldline.

Were C_1 and C_2 in R^3 (or S^3) *disjoint* curves, (4b.17) would just be their Gauss linking coefficient. If $\Omega(M_2)$ denotes the solid angle subtended by C_1 at the point M_2 of C_2, then $I_G = \frac{1}{4\pi}\int_{C_2} d\,\Omega(M_2)$ by Stoke's Theorem. So I_G measures the variation of this solid angle divided by 4π as M_2 runs along C_2; it is the algebraic number of loops of one curve around the other.

However, there is a problem: an indeterminacy is in fact present in the analytic result (4b.17). While its integrand is that of Gauss' invariant, the integration is over one and the same curve. So $I_G(P)$ is *undetermined.* Clearly, this artifact of the point-limit approximation must be corrected by a proper definition or regularization of $I_G(P)$.

Polyakov's regularization consists in trading the delta function in the alternative expression of $\int_P dx^\mu \iint d^2y_\mu \delta(x-y)$ of $4\pi I_G$ for its Gaussian representation, namely $\delta_\varepsilon(x-y)=(2\pi\varepsilon)^{\frac{-3}{2}}\exp\left(-\frac{|x-y|^2}{\varepsilon}\right)$. He found $I_G(P)_{Reg}$ to be -T(P), the total torsion or twist of the curve P in spacetime with $T(P) = \frac{1}{2\pi}\oint_P d\mathbf{x}\cdot(\mathbf{n}\times\frac{\partial\mathbf{n}}{\partial s}) \equiv \frac{1}{2\pi}\int_P \tau(s)\,ds$. The parameters and $\mathbf{n}$ denote respectively the arclength and the principal normal vector to P at the point $\mathbf{x}(s)$. What, we may ask, are the geometric underpinnings of this regularization ?

By substituting the Gaussian, the dominant contribution of the surface integral then comes from an infinitesimal strip Σ_P. Effectively, this procedure turns a spacetime curve into a *ribbon.* Precisely such a technique was performed in the theory of knots, back in 1959, by Calugareanu [404, 405]. The entity $I_G(C_\alpha \rightarrow C_\beta)$ was found then to be perfectly well defined. It yields a new topological invariant SL, the self-linking number for a simple closed *ribbon.*

SL is in fact the linking number of C_β with a twin curve C_α moved an infinitesimally small distance ε along the principal normal vector field to C_β. Being disjoint, these two curves can be linked and unlinked, like the strands of a circular supercoiled DNA molecule. In modern knot theory, this construction is called the *framing* of a curve C_β [406] . Most noteworthy is the existence of the "conservation law": SL = T + W whereby SL, is the algebraic sum of two *differential geometric* characteristics of a closed ribbon, its total torsion or twisting number T and its *writhing number* or *writhe* W. While their sum SL, a topological invariant, must be an integer, T and W are metrical properties of the ribbon and its "axis" respectively; they can take a continuum of values. W, the writhing number or the Gauss integral for the map $\phi : S^1 \times S^1 \to S^2$, is the element solid angle, the pullback volume 2-form $d\Omega_2$ of S^2 under ϕ. Calugareanu's formula reads

$$SL(P) = \frac{1}{4\pi}\int_{P\times P} d\Omega_2 + \frac{1}{2\pi}\int_P \tau\, ds \quad . \tag{4b.18}$$

By the dilatation invariance of W and the map $\mathbf{e}(s,u)$ ($\mathbf{e}^2 = 1$), namely a local Frenet-Serret frame vector attached to the curve, we can express the writhe W as $W = \frac{1}{4\pi}\int_0^L ds \int_0^1 du\, \varepsilon_{abc}\, e^a \partial_s e^b \partial_u e^c$ where a, b, c = (1, 2, 3) and $\partial_s = \delta/\delta s$, $\partial_u = \delta/\delta u$. A conformally invariant action for the frame field $\mathbf{e}$, W is manifestly a Wess-Zumino-Novikov-Witten term [4] and is readily identified as a Berry phase upon exponentiation. It is precisely Polyakov's double integral representation (modulo an integer) for the torsion T(P). This equivalence comes from the equality $W = -T \pmod{Z}$. One of the everyday manifestations of the relation W + T = SL for a ribbon is any coiled phone cord. When the cord is unstressed with its axis curling like a helix in space, its writhe is large while its twist is small. When stretched with its axis almost straight, its twist is large while its writhe is small.

The alternate form $\Phi(P) = \exp(-i\pi\, T(P))\exp(+i\pi\, n)$ by way of the relation $W = -T \pmod{Z}$ is the "spin " phase factor. It is essential to Polyakov 's proof that the 1-solitons of model (4b.12) with $\theta = \pi$ are fermions by obeying a Dirac equation in their point-like limit. For arbitrary θ, one has the more general theory of pointlike anyons carrying fractional spin and intermediate statistics. Thus the relation W + T = SL expresses mathematically the connection between spin and statistics in the geometric point soliton limit.

In the geometry of 2-surfaces, a form of the Gauss-Bonnet theorem is $K = 2\pi\,\chi$. Its key feature, unique in the whole of differential geometry, is the following. Like the formula SL = T + W, it relates entities defined solely in terms of topology, such as the Euler characteristics χ of a closed surface M, and metrical entities defined purely in terms of distances and angles, such as total Gaussian curvature K for M. Not surprisingly, Fuller showed that the Calugareanu formula $SL = T + W$ is but a consequence of the Gauss-Bonnet formula. It is therefore under disguise that the latter theorem plays a pivotal role in a fundamental physics principle, the relation between spin and statistics!

Since the mathematician White obtained the higher dimensional version of Calugareanu's relation from his formulation of Gauss-Bonnet-Chern theorem for Riemannian manifolds, it is natural to extend the analysis of Ref. [402] to the $D > 3$ counterparts of Wilczek-Zee model (4b.12). To do so, we need the generalized Gauss linking number for manifolds.

Extending to higher dimensional manifolds the procedure for linking 3-space curves, we consider two continuous maps f(M) and g(N) from two smooth, oriented, non intersecting manifolds M and N, dim(M) = m and dim(N) = n, into R^{m+n+1}. Let S^{m+n} be a unit (m+n)-sphere centered at the origin of R^{m+n+1} and dW_m be the pull-back volume form of S^{m+n} under the map e: $M \times N \to S^{n+m}$ where to each pair of points $(\mathbf{m},\mathbf{n}) \in M \times N$ we associate the unit vector e in R^{m+n+1}: $e(\mathbf{m},\mathbf{n}) = \dfrac{g(\mathbf{n}) - f(m)}{|g(\mathbf{n}) - f(m)|}$. The degree of this map L is the Gauss linking number of M and N:

$$L(f(M),g(N)) \equiv L(M,N) = \frac{1}{\Omega_{n+m}} \int_{M \times N} d\Omega_{n+m} \,. \tag{4b.19}$$

with $\Omega_n = \dfrac{2\pi^{\frac{n+1}{2}}}{\Gamma\left(\frac{n+1}{2}\right)}$ being the volume of S^n. Due to the non-commutativity of linking numbers, $L(M, N) = (-1)^{(m-1)(n-1)} L(N, M)$, we immediately deduce that the linking number vanishes for even dimensional submanifolds M and N .

Next, we go directly to White's main theorem: Let f: $M^n \to R^{D=2n+1}$ be a smooth embedding of a closed oriented differentiable manifold into Euclidean (2n+1) space. Denote by v a unit vector along the mean curvature vector of M^n. If n is odd (i.e. D = 3, 7, 11, 15, etc.) then

$$SL(f,f_\varepsilon) = \frac{1}{\Omega_{2n}}\int_{M\times M} d\Omega_{2n} + \frac{1}{\Omega_n}\int_M \mathcal{T}^* \, dV \qquad (4b.20)$$

is the self-linking number of a hyper-ribbon made up of M^n and the same manifold deformed a small distance ε along v. The terms on the RHS of Eq. (4b.20) are respectively the writhing and twisting numbers, W and T, of the hyper-ribbon. As for even n (D = 1, 5, 9, etc.), both W and T are zero and therefore S L = 0, such cases are thus uninteresting for application to the physics of nonintegrable phases.

The universality of the formula S L = W + T mirrors that of Gauss-Bonnet-Chern theorem. We naturally expect that, for D>3 solitons in suitable models, White's formula T = – W (mod Z) will similarly link their spin and statistical phases. As applied to physics, it could define and relate the twisting and writhing of odd dimensional closed S^3-, S^5-, S^7-,... hyper-ribbons, the world volumes of topological S^2-, S^4-, S^6-membranes solitons in D = 7, 11, 15, ... dimensional spacetime, respectively. How to cut down this infinity of choices? What are the natural D > 3 σ-model counterparts of system (4b.12), which may admit solitons with exotic spin and statistics?

In seeking for analogs of θ-spin and statistics among higher dimensional extended objects, at least three key features of the CP(1) model (4b.12) should be required of their D >3 counterparts. They are

1) the models admit topological solitons,
2) the presence in the action of an Abelian Chern-Simons-Hopf invariant,
3) the associated Hopf mappings $S^{2n-1} \to S^n$ include ones with Hopf invariant 1.

The first two requirements are embodied in the time component of the key equation (4b.15). After integrating over all of space of both sides of this equation, one obtains the topological charge-magnetic flux coupling, which is at the very basis of the fractional statistics phenomenon in (2+1) dimensions. As to the third requirement, essential to the proof of fractional spin and statistics for one soliton [390], the following striking feature holds true for the Hopf mappings in higher dimensions. While for any even n there always exists a map f : $S^{2n-1} \to S^{2n}$ with only even integer Hopf invariant g(f), possibilities for Hopf maps of invariant 1 are severely limited. This existence question of Hopf maps of invariant 1 has a long and fascinating history, the final answer is provided by a celebrated theorem of Adams [350, 351]:

If there exists a Hopf map Φ: $S^D \to S^{(D+1)/2}$ of invariant γ $(\Phi) = 1$, in fact of invariant γ (Φ) = any integer, then D must equal 1, 3, 7 and 15 ($m = (D+1)/2 = 1$, 2, 4 and 8).

There are therefore altogether four and only <u>four</u> classes of Hopf maps with invariant γ $(\Phi) = 1$. Thus it has been proved that there exists no maps: $S^{31} \to S^{16}$ of Hopf invariant 1 and so there can be no real division algebra of dimension 16 [407].

These unique four families of Hopf maps and their associated hidden (or holonomic) gauge field structures are best seen through a diagram of sphere bundles over spheres [408]:

$$U(1) \approx SO(2)$$
$$\|$$
$$Z_2 = O(1) \approx S^0 \to \mathbf{S^1} \to \mathbf{S^1/Z_2} \approx \mathbf{RP(1)} \approx SO(2)/Z_2$$
$$\|$$
$$SO(2) \approx U(1) \approx S^1 \to \mathbf{S^3} \to \mathbf{S^2} == \mathbf{CP(1)} \approx SU(2)/U(1)$$
$$\|$$
$$SU(2) \approx Sp(1) \approx S^3 \to \mathbf{S^7} \approx SO(8)/SO(7) \to \mathbf{S^4} \approx \mathbf{HP(1)}$$
$$\approx Sp(2)/Sp(1)\psi Sp(1)$$
$$\|$$
$$Spin(8)/Spin(7) \approx S^7 \to \mathbf{S^{15}} \approx Spin(9)/Spin(7) \to \mathbf{S^8} \approx \mathbf{\Omega P(1)}$$
$$\approx Spin(9)/Spin(8).$$

The four rows mirror the one-to-one correspondence between the four (and only four) division algebras over **R** and the real (**R**), complex (**C**), quaternionic (**H**) and octonionic (**Ω**) Hopf bundles (tabulated in bold letters). The first three principal bundles are just the simplest examples of the three infinite sequences of the **K** = **R**, **C**, H universal Stiefel bundles over Grassmannian manifolds. The latter are instrumental in obtaining the general instanton solutions of Atiyah, Drinfeld, Hitchin and Manin [165]. The last bundle stands alone, reflecting the non-associativity of octonions.

The spheres S^p, p = 0, 1, 3 and 7 are the fibers, with the first three being Lie groups while S^7 is a very special coset space, the space of the unit octonions. S^7 has been the focus of much fascination and discoveries in mathematics and in the Kaluza-

Klein compactification of D = 11 supergravity and supermembrane theories. An n-sphere S^n is called *parallelizable* if there is a continuous family of n orthonormal vectors at each its points. The fact that of all spheres, only S^1, S^3 and S^7 are parallelizable is yet another corollary to Adams' theorem. S^r, r = 1, 3, 7, 15 constitute the corresponding fibre spaces. Finally, the sequence of base spaces S^n, n = 1, 2, 4, 8 are equally interesting as **K**-projective lines, as is clear from their coset forms. With their holonomy groups Z_2, SO(2), SO(4) and SO(8) being the norm groups of **R**, **C**, **H** and **Ω**, these spheres can be said to have a real, complex, quaternionic and octonionic Kähler structures [409].

In the past decade Hopf maps f: $S^{2n-1} \rightarrow S^n$, n = 1, 2, 4, 8 with *Hopf invariant one* have found important physical applications in condensed matter physics and in quantum field theory [410, 411]. Even the connection between Hopf maps and nonstandard spin and statistics had long been lurking in the background. Thus, in the D = 2 ϕ^4 field theory, it was shown that the n = 1 real Hopf map realizes the 1-kink soliton, which carries intermediate spin and admits exotic statistics. Besides being the Dirac 1-monopole bundle, the n = 2 complex Hopf map underlies the θ spin and statistics of D = 3 CP(1) model. The n = 4 quaternionic Hopf map is the embedding map for the SO(4) invariant, D = 4 SU(2) Yang-Mills BPST 1-instanton or the SO(5) invariant, D = 5 SU(2) Yang monopole with the Dirac quantization relation $e\,g = \frac{1}{2}$. The n = 8 octonionic Hopf map appears as a SO(9) invariant, D = 8 SO(8) 1-instanton discussed in Sect.3f.5. The latter two maps admit further realizations in terms of U(1) tensor gauge fields associated with extended Dirac monopoles in p-form Maxwellian electrodynamics. Their role in determining the spin and statistics of membranes was the theme of Ref.[392].

Having noted that the field theory realizations of the real and complex Hopf fiberings both admit exotic spin and statistics, a natural question comes to mind. Does this pattern persist in suitable theories built on the remaining two Hopf fibrations, $S^{2n-1} \rightarrow S^n$ for n = 4, 8 ? Clearly, the answer should be sought within a quaternionic D=7 HP(1) ($\approx S^4$) and an octonionic D=15 ΩP(1) ($\approx S^8$) σ-models augmented with their respective Hopf invariant term. Using our outlay of geometric and algebraic tools, these models [392, 412] are studied next.

4.b.4. Division algebra σ-models with a Hopf term

In mathematics, the nonlinear σ-models are known as harmonic maps. One associates with the map Ψ: M $\rightarrow$ N between two Riemannian manifolds M and N an action, called "energy functional" by mathematicians

$$S_0(\Psi) = \int_M a(\Psi)\, d^m x = \frac{1}{2}\int_M |\, d\Psi(x)\,|^2\, d^m x \ . \qquad (4b.21)$$

$d\Psi(x)$ is the differential of Ψ at the point $x \in M$ and $d^m x$, the volume element of M. In a coordinate patch, $|\, d\Psi\,|^2 = g^{ij} \dfrac{\partial \Psi^\alpha}{\partial x^i} \dfrac{\partial \Psi^\beta}{\partial x^j} h_{\alpha\beta}$ is the pullback on M of the metric $ds^2 = h_{\alpha\beta} d\Psi^\alpha d\Psi^\beta$ on N. (4b.21) is just another expression for the standard σ-model action. Ψ is called <u>harmonic</u> if it leads to a vanishing Euler-Lagrange operator (or tension field) div($d\Psi$) $\equiv 0$. It is well-known that the quadratic Hopf map Ψ(X, Y) : $S^{2n-1} \rightarrow S^n$ (n = 2, 4, 8) is a harmonic polynomial map, with <u>constant</u> Lagrangian density $a(\Psi) = 2n$. As such they are the simplest harmonic representatives of maps with Hopf invariant one.

While the D = 3 CP(1) σ-model admits exact finite energy static solitons, the corresponding D = 7 HP(1)($\approx S^4$) and the D = 15 ΩP(1) ($\approx S^8$) σ-models [392, 412], do not because of the well-known Hobart-Derrick scaling argument [413, 414]. In practice, dynamical stability by way of some repulsive interactions can be achieved either by coupling the model to a gauge field or by adding to the standard KP(1) action [415] suitable Skyrme terms, i.e. chiral invariant terms of higher order in the field derivatives. Taking the second alternative, the generic σ-model action with the added Hopf term then reads

$$S_{(n)} = \int_M \partial_\mu \vec{N} \cdot \partial^\mu \vec{N}\, d^{2n-1}x + \frac{\theta}{a}\int_M A_{n-1} \wedge dA_{n-1} + \text{suitable Skyrme terms}; \qquad (4b.22)$$

$$n = 4, 8, \quad M = S^7, S^{15}$$

where the unit vector $\vec{N}$ with **K** = **H** and **Ω** is given by (4b.3). While the composite U(1) ATGF A_{n-1}, is nonlocal in $\vec{N}$, it is local in the 2-spinor K (3.7). Its expression in terms of K will be given later.

The θ-term can be rewritten as

$$S_1 = \frac{1}{(n-1)!} \int d^{2n-1}x \, J^{\mu_1 \ldots \mu_{n-1}} A_{\mu_1 \ldots \mu_{n-1}} \tag{4b.23}$$

i.e. an interaction of the potential A_{n-1} with the topological current $J_{n-1} = -\frac{(n-1)!\theta}{4\pi^2} {}^*F_n$ (n = 4, 8). The conservation of J_{n-1} and its expression in terms of **N** will be obtained solely from the field topology. Since the sources of J_{n-1} are charged solitons, we will first determine what types of solitons are allowed in our KP(1) models.

To a condensed matter physicist, our σ-models are the familiar n-vector models. As field theories of a 5- and 9-unit-vector order parameter $\vec{N}$, they are the quaternionic and octonionic counterparts of the isotropic Heisenberg ferromagnet, albeit in rather exotic higher dimensions. So the nature and dimensionality of their allowed topological defects should only depend on the dimensionalities of the order parameters and of the compactified spacetime. The allowed defects should obey the defect formula of Toulouse and Kleman [410].

Consider a topological defect of spatial dimension d' in D-space or D-Euclidean spacetime. To measure its homotopic charge, we completely "surround" this defect by a submanifold of dimension r such that d' + r + 1 = D. The physical meaning of the contribution 1 on the LHS of this last relation is evident for a vortex line ; it corresponds to the distance in 3-space (D = 3) between the line defect (d' = 1) and its surrounding loop (r = 1).

The topological charge labels the equivalence classes of the group $\pi_r(S^n)$ of mappings $S^r \to S^n$, of the spatial submanifold S^r into the space of the (n+1) unit vector order parameter $\vec{N}$. With r < n and $\pi_r(S^m) \approx 0$ for r < m, $\pi_m(S^m) \approx Z$, topologically stable ($\pi_r(S^n) \neq 0$) defects must have spatial dimension d' = D − 1 − r = D − (n+1). Therefore no stable defect exists for (n+1) > D and (n+1) < 0, but for 0 < (n+1) < D, the so-called *triangle of defects* in the ((n+1), d) plane, defects of various kinds, points, vortices, membranes etc. are allowed. Although the situation is not considered below, for completeness we should mentioned that, if D > 4 as in Kaluza-Klein theories <u>and</u> r > m so that $\pi_r(S^m)$ is generally nontrivial, even a richer variety of defects are possible.

The Toulouse-Kleman formula given above, when applied to our cases of (D, r = n) = (3, 2), (7, 4) and (15, 8), allows 0-, 2- and 6-membrane solitons in the CP(1), HP(1) and ΩP(1) σ-models (3.16), respectively. The soliton's topological charges are the generators of $\pi_n(S^n) \approx Z$, n = 2, 4, 8.

When the solitons are charged 2- and 6-membranes, we expect the associated HP(1) and ΩP(1) σ-models to possess a rank 3 and rank 7 topological conserved current $J^{\mu\rho\sigma}$ and $J^{\mu\rho\sigma\alpha\beta\gamma\lambda}$. Their conservation follows solely from the constraint $\mathbf{N}^2 = 1$ (i.e. $\mathbf{N} \cdot \partial_\mu \mathbf{N} = 0$) and the fact that the dimension n of $\mathbf{N}$ is less than or equal D, the dimension of spacetime. Since here (D, n) = (7, 5), (15, 9), the latter condition is satisfied. Indeed if n $\leq$ D, then the (D x n) matrix [$\partial\mathbf{N}$] must have rank less than n, so we have

$$\varepsilon^{\mu_1 \ldots \mu_D} \varepsilon_{\alpha_1 \ldots \alpha_n} \partial_{\mu_1} N^{\alpha_1} \ldots \partial_{\mu_n} N^{\alpha_n} = 0 \ . \tag{4b.24}$$

Consequently,

$$\partial_{\mu_1} J^{\mu_n \mu_{n+1} \ldots \mu_D} = 0 \tag{4b.25}$$

where

$$J^{\mu_n \mu_{n+1} \ldots \mu_D} = \varepsilon^{\mu_1 \ldots \mu_D} \varepsilon_{\alpha_1 \ldots \alpha_n} \left(\partial_{\mu_1} N^{\alpha_1} \ldots \partial_{\mu_{n-1}} N^{\alpha_{n-1}} \right) N^{\alpha_n} \ . \tag{4b.26}$$

As with the CP(1) model, these 3 and 7-forms, which are conserved currents, when suitably normalized, are just the D = 7 and 15 Hodge duals of the respective 4- and 8-forms antisymmetric gauge field $F_n = dA_{n-1}$ appearing in the Hopf invariant action (3.17): $J_n = -\frac{n!\theta}{4\pi^2} *F_{n+1}$, with the star operation denoting the Hodge dual $*F_{\mu_1 \ldots \mu_{n-p}} = \frac{1}{n!} \varepsilon_{\mu_1 \ldots \mu_n} F^{\mu_{n-p+1} \ldots \mu_n}$.

Following the analysis of Teitelboim, the conserved current (4b.26) can be readily converted into a conservation law. Two equalities are used: a) Stokes' theorem $\int_M d\omega = \int_{\partial M} \omega$, ω is a p-form and M, an oriented compact manifold with boundary ∂M, D = dim M = (p+1) and dim(∂M) = p; b) the relation between the divergence and the exterior derivative: $\partial_\alpha \omega^\alpha_{\mu_1 \ldots \mu_p} = (-1)^p [\, {*d*}\, \omega\,]_{\mu_1 \ldots \mu_p}$. With the latter identity (4b.25) reads

$$d^*J = 0 \ . \tag{4b.27}$$

Its integration over a (D − p + 1)-dimensional manifold M with boundary ∂M gives

$$\oint_{(\partial M)} {}^*J = 0 \ . \tag{4b.28}$$

If ∂M consists of two spacelike hypersurfaces Σ (dim(Σ) = D − p), connected by a remote timelike tube ∂T and since the topological current J in our σ-models is localized in space, the integral over ∂T vanishes and (4b.28) yields the Lorentz invariant and conserved charge

$$Q = \int_\Sigma {}^*J, \tag{4b.29}$$

its value being independent of Σ. Applied to our KP(1) σ-models where the equations of motion forces a θ-dependent linear relation between topological charge and flux, Eq. (4b.29) reduces for (D, p) = (3, 1) to the Skyrmion winding number, the generator of $\pi_2(CP_1) \approx Z$

$$Q = \frac{-1}{4\pi}\int_{S^2} d\Sigma^{ij}\, F_{ij} = \frac{-1}{2\pi}\int_{S^1} dx^i\, A_i = C_1 \quad (\text{ for } q = p\ ) \ . \tag{4b.30}$$

(Note that we are using Roman indices for the spacelike components.)

It is also the first Chern index C_1 of the U(1) bundle, i.e. the complex $S^3 \rightarrow S^2$ Hopf fibration. For (D, p) = (7, 3) and (15,7), the topological charges of the S^4- and

S^8-solitons and the generators of $\pi_4(HP(1)) \approx Z$ and $\pi_8(\Omega P(1)) \approx Z$ are similarly given for $\theta = \pi$ by

$$Q = \int_{S^3} J^{0i_1i_2}\, d\Sigma_{i_3\cdots i_6} = \frac{-1}{4\pi}\int_{S^4} F_{i_3\cdots i_6}\, d\Sigma^{i_3\cdots i_6} = \frac{-1}{2\pi}\int_{S^3} A_{i_3i_4i_5}\, d\Sigma^{i_3i_4i_5} \qquad (4b.31)$$

and

$$Q = \int_{S^7} J^{0i_1\cdots i_6}\, d\Sigma_{i_7\cdots i_{15}} = \frac{-1}{4\pi}\int_{S^8} F_{i_7\cdots i_{15}}\, d\Sigma^{i_7\cdots i_{15}} = \frac{-1}{2\pi}\int_{S^7} A_{i_7\cdots i_{14}}\, d\Sigma^{i_7\cdots i_{14}}, \qquad (4b.32)$$

respectively.

One way to see that Q is equal to n, an integer, is through the above mentioned gauge field connection between our problem and the D = 2 complex, D = 4 quaternionic and D = 8 octonionic instanton. We take for the KP(1) field coordinate, the mapping $K(x) = x^n$, where x is the space position K-number in Σ, K = C, H and Ω. While these maps are not 0-, 2- or 6-membrane solutions to the systems (4b.22), they are the simplest harmonic representatives maps $:S^m \to S^m$ (m = 2, 4 and 8) with topological number Q= n .

As will be clear the charges (4b.31) and (4b.32) can be identified with the 2nd and 4th Chern indices. The latter reflect the relation between the U(1) ATGF and the hidden non-Abelian gauge structure of the σ-models, namely the Sp(1) quaternionic and associated Spin(8) octonionic Hopf fibrations, respectively. Although our analysis of the thin soliton limit deals primarily with the θ-term in Eq. (2b.22), the Hopf term, we consider the $HP(1) \approx S^4$ model in greater details to clarify the hidden gauge connection. A parallel discussion for the $\Omega P(1)$ model is left as an exercise for the reader.

As the coset space Sp(2)/ (Sp(1) x Sp(1)), the quaternionic projective line HP(1) can be parametrized in two ways (see Sect. 2f.3 for details).

1) We use two real quaternions q_1 and q_2 with $|q_1|^2 + |q_2|^2 = 1$, i.e. by a 2-component H-spinor $Q^T = (q_1, q_2)$, $Q^\dagger Q = 1$, coordinatizing the sphere S^7 or by one quaternionic inhomogeneous coordinate $h = q_2 q_1^{-1}$.

2) An alternative parametrization is by the unit 5-vector N defined by the Hopf projection map (4b.3) from S^7 to S^4, $\vec{N} = \mathrm{Sc}\left(Q^\dagger \vec{\gamma} Q\right) = \left(N = \frac{2h}{1+\bar{h}h},\ N_5 = \frac{1-\bar{h}h}{1+\bar{h}h} \right)$. To make manifest the local $Sp(1) \approx SU(2)$ gauge invariance

$$q_\alpha{}' = U(x)\, q_\alpha \qquad \alpha = 1, 2\,; \qquad U(x) \in Sp(1) \tag{4b.33}$$

of the HP(1) model, we introduce the covariant derivative $D_\mu Q = (\partial_\mu + a_\mu)\, Q$. The holonomic Sp(1) gauge field is $a_\mu = \mathbf{a}_\mu \cdot \mathbf{e} = Q^\dagger \partial_\mu Q = \frac{1}{2}\bar{q}_\alpha \overleftrightarrow{\partial_\mu} q_\alpha = \frac{1}{2}\frac{\bar{h}\,\overleftrightarrow{\partial_\mu} h}{1+\bar{h}h}$ is purely vectorial and takes the ADHM form for the 1-SU(2) instanton solution. So the first term in Eq. (4b.22) reads

$$S_{(4)} = \mathrm{Sc}\left[(D_\mu Q)^\dagger (D^\mu Q)\right] \tag{4b.34}$$

and similarly for the Skyrme terms.

As for the Hopf term, we check, after some algebra, that the 3-form $A_{(3)}$ to be the D = 4 Sp(1) Chern-Simons form :

$$\begin{aligned} A_{(3)} &= \frac{1}{3!} A_{[\mu\nu\lambda]}\, dx^\mu\, dx^\nu\, dx^\lambda \\ &= \mathrm{Tr}\,(A \wedge dA + \tfrac{2}{3} A^3)\ , \end{aligned} \tag{4b.35}$$

$$F_{(4)} = dA_{(3)} \tag{4b.36}$$

where $dx^\mu dx^\nu = dx^\mu \Lambda\, dx^\nu$ etc. In components, we get

$$A_{\mu\nu\rho} = \mathrm{Sc}\left(a_{[\mu}\, f_{\nu\rho]} - \frac{2}{3}\, a_{[\mu}\, a_\nu\, a_{\rho]} \right), \tag{4b.37}$$

$$F_{\mu\nu\rho\sigma} = 2\, \mathrm{Sc}\left(f_{\mu\nu} f_{\rho\sigma} + f_{\mu\rho} f_{\sigma\nu} + f_{\mu\sigma} f_{\nu\rho}\right) \tag{4b.38}$$

where a_μ is the **H**-valued Sp(1) gauge potential and $f_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu + [\, a_\mu, a_\nu \,]$, the field strength. In terms of the 2-spinor Q,

$$A_{(3)} = \mathrm{Sc}\left[\, Q^\dagger dQ\, d\left(Q^\dagger\, dQ\right) + \frac{1}{3}\left(Q^\dagger dQ \right)^3 \right] , \tag{4b.39}$$

$$F_{(4)} = \mathrm{Sc}\left[\, dQ^\dagger\, dQ \;+\; \left(Q^\dagger dQ \right)^2 \right] . \tag{4b.40}$$

These expressions clearly show the *local* nature of Hopf term when cast in the bundle space S^7-valued field Q. It is thus locally a <u>total divergence</u> as is already clear from Eq. (4b.10). While a parallel derivation of $A_{(7)}$ and hence of $F_{(8)} = dA_{(7)}$ can be similarly performed for the D = 15 ΩP(1) σ-model, the connection to the D = 8 octonionic instanton problem identifies the 7-form $A_{(7)}$ to be that given by D = 8 Chern-Simons term of a Spin(8) gauge field, i.e.

$$A_{(7)} = \mathrm{Tr}\left[\, A(dA)^3 + \frac{4}{3} A^3(dA)^2 + \frac{6}{5} A^5 dA + \frac{4}{7} A^7 \right] . \tag{4b.41}$$

The above specifics of the σ-models are sufficient for our present purpose. Due to their geometric nonlinearity, our σ-models are analytically intractable in their details. Besides, there is much arbitrariness in the choice of Skyrme terms. The latter, being higher order in the field derivatives, control the shorter distance structure of the solitons. As in CP_1 case, to decode the phase entanglements of the solitons, it is enough to analyse the effective theories obtained in the geometrical Nambu-Goto limit of widely separated membranes. There, the particulars of soliton structures are irrelevant, only the existence but not the details of the Skyrme terms matter.

With greater details to be found in Refs.[412] and [392], it is enough to say that, using the method of Umezawa et al. [416], our membrane solitons can be shown to have a thin London limit [417]. Indeed Polyakov's approximation for the Wilczek-Zee model (4b.12) readily translates into the Chern-Simons-Kalb-Ramond electrodynamics of Nambu-Goto membranes. To obtain the statistical phase, we may consider the propagation of two pairs of membrane-antimembrane. The object is to compute the resulting phase after adiabatically exchanging the two membranes. We get

$$\left\langle \exp\left(\frac{i}{3!} \oint_{P_1} A^{[\mu\nu\lambda]} dx^\mu \wedge dx^\nu \wedge dx^\lambda \right) \exp\left(\frac{i}{3!} \oint_{P_2} A^{[\mu\nu\lambda]} dx^\mu \wedge dx^\nu \wedge dx^\lambda \right) \right\rangle . \tag{4b.42}$$

P_1 and P_2 are S^3 hyper-curves. The functional average $\langle\cdots\rangle$ is taken over the Hopf action (4b.22). Like in the D=3 case, the resulting phase here is the sum of three phases. The first contribution gives the phase $\exp\{2i\,(\frac{\pi^2}{\theta})L\}$ with L being the generalized Gauss' linking coefficient (4b.19) for two S^3-loops. We get $\frac{\pi^2}{\theta}$ for the statistical phase. The other two phase factors $\Phi\,(P_i)$ are given by the expectation value of one hyperloop:

$$\Phi(P) = \left\langle \exp\left(\frac{i}{3!}\oint_P A^{[\mu\nu\lambda]} dx^\mu \wedge dx^\nu \wedge dx^\lambda\right)\right\rangle . \tag{4b.43}$$

In the London-Nielsen-Olesen limit, the effective action is

$$S = S_0 + \frac{\theta}{(3!)^2 a}\int_{S^7} d^7x\; \varepsilon_{\mu\nu\lambda\alpha\beta\gamma\delta}\, A^{\mu\nu\lambda}\partial^\alpha A^{\beta\gamma\delta} + \frac{1}{3!}\int_{S^7} d^7x\; J_{\mu\nu\lambda}A^{\mu\nu\lambda} \tag{4b.44}$$

where S_0 is the free Nambu-Goto action for a relativistic 2-membrane, $0 \le \theta \le \pi$ and a is a constant yet to be fixed. Direct integration of the equation of motion

$$J_{\mu\nu\lambda} + 2\,\frac{\theta}{3!a}\varepsilon_{\mu\nu\lambda\alpha\beta\gamma\delta}\partial^\alpha A^{\beta\gamma\delta} = 0 \tag{4b.45}$$

with $J^{\mu\nu\lambda}(y) = \int d^3x\;\delta^7(x-y)\,\frac{\partial(x^\mu, x^\nu, x^\lambda)}{\partial(\tau, \sigma_1, \sigma_2)}$, gives in the Lorentz gauge $\partial^\alpha A_{\alpha\beta\gamma} = 0$:

$$\frac{1}{2\cdot 3!}\int d^7x\; J_{\beta\gamma\delta}A^{\beta\gamma\delta}(x) \;=\; \frac{a}{4\theta}\frac{1}{\Omega_6}\int_{S^3} d\Sigma_{\beta\gamma\delta}\int_{S^3} d\Sigma_{\mu\nu\lambda}\,\frac{\varepsilon^{\mu\nu\lambda\alpha\beta\gamma\delta}(\,x-y\,)_\alpha}{|\,x-y\,|^7}\;. \tag{4b.46}$$

Here, as in the D = 3 case, the double 3-surface integration is over one and the same hypercurve S^3 ; so the phase (4b.43) is *undetermined*. Naturally, we apply White's tailor made result to get the regularized phase

$$\Phi\,(P \approx S^3) = \exp\left(i\,\frac{a}{4\theta} W(P) \right) \tag{4b.47}$$

where $W(P) = \dfrac{1}{\Omega_6}\displaystyle\int_{S^3\times S^3} d\Omega_6$ is the writhe of the Nambu-Goto S^2-membrane, of the Feynman path P, a S^3 hyper-ribbon in 7-dimensional spacetime. A completely analogous calculation gives the same corresponding result for the octonionic case of P $\approx S^7$ in S^{15} -spacetime .

By appropriately choosing $a = 4\pi^2$, $\Phi(P) = \exp(i\,\dfrac{\pi^2}{\theta}\, W)$. For $\theta = \pi$, one then obtain the exact S^3- (S^7-) counterpart of Polyakov's statistical phase factor $\Phi(P) = \exp(-\pi i T(P)) \exp(\pi i\, n)$, T being the generalized torsion for an S^3-(S^7) ribbon P. For topological reasons given below and in Ref.[412], this phase factor is expected to embody the thin membrane's spin in a functional integral formalism. Consequently, provided that this reasonable conjecture is checked by an explicit construction à la Polyakov of the spin factor for membranes, we will have exhibited a 7- (15-) dimensional analog of the D = 3 Fermi-Bose transmutation. Moreover, as the value of θ is not fixed by the gauge invariance of the Kalb-Ramond field, we have in general the possibility of a fractional statistics and spin connection via the relation $W = -T \pmod{Z}$ for our solitonic membranes.

Without knowing the complexity of higher model dependent soliton structure at shorter distances and / or without performing a detailed canonical quantization of the above $\mathbf{KP}_1$ σ-models, the case for θ-spin and statistics among the membranes can in fact be made on topological grounds. To do so, let us focus on the topology of the configuration space of fields Γ of the above KP(1) σ-models. In the Schrödinger picture, the space Γ of finite energy static solutions is the *mapping space* of all based preserving smooth soliton maps $\vec{N}(x)$: $\mathbf{x} \in S^n \rightarrow \vec{N}(x) \in S^n$, n = 2, 4, 8. Γ is an infinite Lie group [418] with the nontrivial connectivity property:

$$\pi_0(\Gamma = \{\vec{N} : S^n \rightarrow S^n\}) \approx \pi_n(S^n) \approx Z. \tag{4b.48}$$

So Γ is split into an infinite set of pathwise-connected components Γ_α, $\alpha \in Z$, corresponding to the various soliton sectors labelled by the charge Q. Moreover, for our membranes, as with Skyrmions and Yang-Mills instantons, each sector Γ_α has further topological obstructions. G.W. Whitehead [419] in effect showed that all the

Γ_α's in Γ have the *same* homotopy type i.e. $\pi_i(\Gamma_\alpha) \approx \pi_i(\Gamma_\beta)$. Of special relevance to the question of exotic spin and statistics for the 1-soliton sector are the relations [420]

$$\pi_i(\Gamma_1) \approx \pi_i(\Gamma_0) \approx \pi_{i+n}(S^n) \approx Z \quad \text{for } (i, n) = (1, 2), (3, 4), (7, 8). \qquad (4b.49)$$

They follow from the Whitehead's [421] and Hurewicz's [422, 423] isomorphisms, the latter stating that $\pi_i(\Gamma_\alpha) \approx \pi_{i+n}(S^n)$, and reflect the *multi-valuedness* of Γ_i. They suggest the possibility of adding to the KP(1) σ-model action a Hopf invariant $\gamma(\vec{N})$, the generator of the torsion free part of $\pi_{i+n}(S^n)$ ($\pi_3(S^2) \approx Z$, $\pi_7(S^4) \approx Z \oplus Z_{12}$ and $\pi_{15}(S^7) \approx Z \oplus Z_{120}$. Generalizing the CP(1) model { (i, n) = (1, 2) }, the nontriviality of these $\pi_i(\Gamma_1)$ implies the possibilities of Aharonov-Bohm effects of a multiply connected configuration space Γ. It therefore signals for the membrane solitons the existence of a higher dimensional analog of a θ spin and statistics connection.

In the CP(1) case, upon a 2π rotation P of the baby-Skyrmion or an interchange of two baby-Skyrmions, the Hopf term induces a projective spin phase factor $\Phi(P) = \exp\{ i\theta \} = \exp(i\, 2\pi s)$, s being the soliton spin. The equality $\theta = 2\pi s$ for this process of rotation is a physical realization of the homomorphism:

$$\pi_1(SO(2)) \approx \pi_3(S^2) \approx \pi_1(\Gamma_1) \approx Z. \qquad (4b.50)$$

It establishes the equality of the kinematically allowed exotic spin to the dynamically induced θ-spin by way of the Hopf term. Yet (4b.50) is but a special case of the Hopf-Whitehead J-homomorphism [424] $\pi_k(SO(n)) \approx \pi_{k+n}(S^n)$. Generally, we have the following chain of homomorphisms :

$$\pi_i(\Gamma_1) \approx \pi_i(\Gamma_0) \approx \pi_{i+n}(S^n) \approx \pi_i(SO(n)) \approx Z \qquad (4b.51)$$

with (i, n) = (1, 2), (3, 4), (7, 8). $\pi_3(SO(4)) \approx \pi_7(S^4) \approx Z$, $\pi_7(SO(8)) \approx \pi_{15}(S^8) \approx Z$. Clearly, the natural physical interpretation of these topological relations is a dynamically induced exotic spin and statistics connection for the 2- and 6-membranes.

In brief the above analysis only marks a first topological strike at the question of non-standard spin and statistics connection for higher dimensional extended objects.

In view of our current lack of understanding of quantum membranes, it is a small step both in the bosonic functional integral formulation for spinning extended objects and in the study of the θ-vacua phenomenon in Kaluza-Klein compactification.

4.c. The Super-Poincaré Group and Super-Extended Objects

4.c.1. Spinors and super-vectors revisited

The deep relevance of division algebras to global and local supersymmetric theories in critical dimensions has been noticed before. It has been extensively studied by many authors [166, 167, 425, 426, 427, 428]. Our own unified approach complements these studies. It focuses predominantly on the connection investigated by one of us (F.G.) between division algebras and extended objects, particularly superstrings. To set our notations, we begin with a review of some key spacetime properties of spinors and super-vectors.

Let x^μ ($\mu = 0, 1, ..., D-1$) labels a point of D-dimensional Minkowski space M. Let the latter's diagonal Lorentzian metric be $\eta_{\mu\nu}$ ($-\eta_{00} = \eta_{11} = ... = \eta_{D-1,\,D-1}$). Next, M can be extended to a superspace by the adjoining F fermionic coordinates θ_α ($\alpha = 1, 2, ..., F$), represented by Grassmann numbers. The θ_α's transform like the components of a spinor θ under the Lorentz group SO(D−1, 1). They satisfy

$$\{ \theta_\alpha, \theta_\beta \} = 0 , \tag{4c.1}$$

Such a θ can be a Dirac spinor (θ_α complex), a Majorana spinor (θ_α real) or, for even D, a left-handed (θ_L) or a right-handed (θ_R) spinor with the constraint:

$$\frac{1}{2}(1 - \gamma_{D+1}) \theta_L = 0 \quad \text{or} \quad \frac{1}{2}(1 + \gamma_{D+1}) \theta_R = 0 \ ; \tag{4c.2}$$

$$\theta_L = \frac{1}{2}(1 + \gamma_{D+1}) \theta \quad , \ \theta_R = \frac{1}{2}(1 - \gamma_{D+1}) \theta , \tag{4c.3}$$

where γ_ρ ($\rho = 1, ..., D$), the D hermitian generators of a Clifford algebra of (2^n x 2^n) matrices for D = 2n or D = 2n + 1 obey the anti-commutation relations

$$\{ \gamma_\rho, \gamma_\sigma \} = 2\delta_{\rho\sigma} \ . \tag{4c.4}$$

For even D, we set

$$\gamma_{D+1} = \eta_D \, \gamma_1 \gamma_2 \cdots \gamma_D \;, \qquad (\gamma_{D+1} = \gamma_{D+1}^{\dagger}) \,. \tag{4c.5}$$

We also use an anti-hermitian γ_0

$$\gamma_0 = - \gamma_0^{\dagger} = i \gamma_D \,. \tag{4c.6}$$

Hence

$$\{ \gamma_\mu, \gamma_\nu \} = 2 \eta_{\mu\nu} \,, \;\; (\mu, \nu = 0, \ldots, D-1) \,. \tag{4c.7}$$

η_D in Eq. (4c.5) is a phase factor chosen to render γ_{D+1} hermitian; i.e.

$$\eta_D = - i^n = - i^{D/2} \,. \tag{4c.8}$$

We note the important property

$$\eta_{D+8} = \eta_D \;, \tag{4c.9}$$

which is the basis for the dimensional periodicity of eight or Bott periodicity [280] for spinors.

The infinitesimal action of the Lorentz group on θ is

$$\delta \theta = \frac{i}{4} \sigma_{\mu\nu} \, \omega^{\mu\nu} \theta \;, \tag{4c.10}$$

$\omega^{\mu\nu}$ being the infinitesimal Lorentz parameters and $\sigma_{\mu\nu} = \frac{1}{2i} [\gamma_\mu, \gamma_\nu]$.

The Lorentz group will have a *real* representation if γ_μ or the Dirac operator $\partial\!\!\!/ = \gamma^0 \partial_0 + \sum_{n=1}^{D-1} \gamma^n \partial_n$ is purely real or purely imaginary.

So for D = 2 the choice of

$$\gamma^0 = i \gamma^2 = i \sigma_2 \;, \;\; \gamma^1 = \sigma_1 \;, \tag{4c.11}$$

makes the Dirac operator real. It then acts on a real (Majorana) spinor.

$$\gamma^3 = -i\gamma^1\gamma^2 = i\sigma_1\sigma_2 = \sigma_3 , \tag{4c.12}$$

is diagonal. Therefore, we can have a 1-dimensional spinor which is both Majorana and left-handed.

For D = 4, the Majorana representation can be realized by using two commuting sets of Pauli spin matrices σ_i and ρ_i, with

$$\gamma^0 = -i\gamma^4 = i\sigma_2\rho_3 , \quad \gamma^1 = \sigma_1 \quad , \gamma^2 = \rho_2\sigma_2 \quad , \gamma^3 = \sigma_3 . \tag{4c.13}$$

Other Majorana representations can be constructed by interchanging σ_i and ρ_i, or by replacing ρ_3 by ρ_1 in Eq. (4c.13). We also find that $\gamma_5 = \sigma_2\rho_1$, a nondiagonal matrix. Indeed, other alternative Majorana sets all give a non diagonal γ_5. Consequently, for D = 4 we have either a Majorana spinor or a 2-component Weyl spinor (for a diagonal γ_5). However, unlike the D = 2 case, the Majorana and Weyl conditions cannot be simultaneously maintained.

The general theory [429] follows from the properties of the matrices B and C defined by

$$\gamma_\mu^* = B\,\gamma_\mu B^{-1} \;\; , \;\; \gamma_\mu^T = -C\,\gamma_\mu C^{-1} , \tag{4d.14}$$

with the superscript T denoting transposition. For the Minkowskian metric, they are connected by $C = B\gamma^0$, then $B^T = \varepsilon\, B$, $C^T = -\varepsilon\, C$.

From the symmetry and antisymmetry properties of the matrices $\gamma_{\mu\nu\rho}$, antisymmetrical in all their indices

$$\gamma_{\mu\nu} = \frac{1}{2}[\,\gamma_\mu, \gamma_\nu\,] \quad , \quad \gamma_{\mu\nu\rho} = \frac{1}{3}[\,\gamma_\mu, \gamma_{\nu\rho}\,] + \text{cycl. perm.} \tag{4c.15}$$

we get

$$\varepsilon(D) = i^{(n-1)(n-2)} \quad \text{when } D = 2n \text{ or } 2n+1 \tag{4c.16}$$

with

$$\varepsilon = +1 \;\; \text{for } D = 2, 4 \;\; (\bmod 8) \tag{4c.17}$$

$$\varepsilon = -1 \quad \text{for } D = 6, 8 \ (\bmod 8) \ . \tag{4c.18}$$

Real Majorana spinors exist for $\varepsilon = 1$, namely in dimensions 2, 4, 10, 12, etc. In this case, B and γ_{D+1} are simultaneously diagonalizable only for D = 2 (mod 8).

We next construct the D = 10 representation where the Dirac operator is real, B = 1 and γ_{11} diagonal. We pick a (4x4) representation of O(4) $\approx SU_L(2) \times SU_R(2)$ generators in terms of purely imaginary hermitian matrices. As seen in Sect. 2, the latter are associated with left and right multiplication of unit quaternions, respectively. We choose

$$SU(2)_L : \omega_1^L = \rho_2\,\sigma_1 \ , \ \omega_2^L = \sigma_2 \ , \ \omega_3^L = \rho_2\,\sigma_3 \ , \tag{4c.19}$$

$$SU(2)_R : \omega_1^R = \sigma_2\,\rho_1 \ , \ \omega_2^R = \rho_2 \ , \omega_3^R = \sigma_2\,\rho_3 \ . \tag{4c.20}$$

By introducing a third set of Pauli matrices $\{\tau_i\}$, we end up with the following set of 7 anticommuting, purely imaginary (8x8) matrices:

$$\Gamma_i = \tau_3\omega_i^L \ , \ \Gamma_{i+3} = \tau_1\omega_i^R \ , \ \Gamma_7 = \tau_2 \ . \tag{4c.21}$$

On the other hand, we consider a Majorana representation γ_α in D = 4, built out of two sets of Pauli matrices ρ_i's and σ_i's.

$$\gamma_0' = i\gamma_4' = i\rho_2'\sigma_3' \ , \gamma_1' = \rho_1' \ , \gamma_2' = \sigma_2'\rho_2' \ , \gamma_5' = \rho_2'\sigma_1' \ . \tag{4c.22}$$

The stage is now set to write down the desired representation for D = 10. The eleven matrices

$$\gamma_0 = \gamma_\alpha' \quad , \ \gamma_5'\Gamma_a \ , \ \gamma_5'\Gamma_7 \qquad \alpha = 0,\ldots,3 \ ; \ a = 1,\ldots,6 \tag{4c.23}$$

are all real and anticommuting with γ_0', antihermitian and the 10 others hermitian. One choice is

$$\gamma_0 = i\gamma_{10}' = \gamma_0' \ , \gamma_1 = \gamma_2' \ , \gamma_3 = \gamma_5'\Gamma_7 \ , \gamma_{i+a} = \gamma_5'\,\Gamma_a \ , \qquad \gamma_{11} = \gamma_3' = \rho_3' \ . \tag{4c.24}$$

In this representation, the Dirac operator is real and the left-handed Weyl projection operator $\frac{1}{2}(1 + \gamma_{11})$ is real and diagonal. Consequently, we can have a 16-dimensional real left-handed eigenspinor of the Dirac operator.

In summary, we can construct superspaces of minimum dimensions by taking θ to be at the same time Weyl and Majorana spinors in D = 2, 10, Majorana or Weyl spinors in D = 4 and Weyl spinors in D = 6. In D = 3 and D = 11, the spinor θ can be pure Majorana with the respective dimensions 2 and 32. If d_θ and d_s are the dimensions of minimal spinors and supervectors (x, θ) respectively, and $d_x = D$ is the dimension of the vector x^μ with $d_\perp$ or $D - 2$, its transverse dimension, we obtain the following table:

D =	2	3	4	6	10
d_x =	2	3	4	6	10
$d_\perp$ =	0	1	2	4	8
d_θ =	1	2	4	8	16
d_s =	3	5	8	14	26

By mere inspection, mathematically trained eyes will readily recognize in this table that for D = 3, 4, 6 and 10, $d = d_\perp$ is simply the dimension of the division algebra **K** (**R**, **C**, **H** and **Ω**) while d_x is the dimension of (2 x 2) hermitian matrices over **K**. d_θ is the dimension of 2d of a pair of numbers belonging to **K** and d_s is the dimension of a (3 x 3) hermitian matrix over **K** with unit determinant or zero trace. This observation will be instrumental in a reformulation of vectors, spinors and supervectors in the critical dimension $D = d + 2$.

Furthermore, we need to get the form of the spinors in the standard reference frame, one which generalizes the rest frame for a massive particle.

The Dirac equation for an s = 1/2 particle of mass m is

$$\partial\!\!\!/\psi = m\,\psi \; . \tag{4c.25}$$

In momentum space, it reads

$$p\!\!\!/\xi = i\,m\,\xi \; , \tag{4c.26}$$

ξ denotes the Fourier transform of ψ. In the rest frame ($\vec{p} = 0$, $p^0 = m$), $\xi = \xi^{(0)}$, we have for positive energy solutions

$$i\,\gamma^0\xi^{(0)} = \gamma_D\xi^{(0)} = \xi^{(0)} \tag{4c.27}$$

so that

$$\xi^{(0)} = \frac{1}{2}\,(\,1 + \gamma_D)\,\xi^{(0)} = P_+\xi^{(0)} \tag{4c.28}$$

where P_+ is a projecton operator for the "large" components of the Dirac wave function.

Next, consider the vector v_μ and the pseudovector a_μ

$$v_\mu = i\,\overline{\psi}\,\gamma_\mu\,\psi \qquad (\,\overline{\psi} = \psi^+\gamma_D) \tag{4c.29}$$

$$a_\mu = \;i\,\overline{\psi}\,\gamma_{D+1}\,\gamma_\mu\psi\ . \tag{4c.30}$$

If $\tilde{v}^\mu$ is the Fourier transform of v_μ, we must have

$$\partial_\mu\, v^\mu = 0 \quad , \qquad p_\mu\,\tilde{v}^\mu = 0 \tag{4c.31}$$

so that, in the rest frame, $p_0\,\tilde{v}^0 = 0$ and $\tilde{v}^\mu$ only has spatial components.

If $m = 0$, then also

$$\partial_\mu a^\mu = 0 \quad , \quad p_\mu\,\tilde{a}^\mu = 0\ . \tag{4c.32}$$

In the massless case (where no rest system exists), p_μ can be rotated to the standard form of

$$p^0 = p^{D-1} = \kappa \quad , \qquad p^1 = p^2 = \ldots = p^{D-2} = 0 \tag{4c.33}$$

In the latter frame, $\xi = \xi^{(0)}$ satisfies

$$\kappa\,(\,\gamma^0 + \gamma^{D-1}\,)\,\xi^{(0)} = 0\ ,\ \ (\,1 - \gamma^0\gamma^{D-1})\,\xi^{(0)} = 0\ , \tag{4c.34}$$

and

$$\tilde{v}^{(0)} - \tilde{v}^{(D-1)} = 0 \qquad , \qquad \tilde{a}^{(0)} - \tilde{a}^{(D-1)} = 0\ . \tag{4c.35}$$

Furthermore, if ξ is a Majorana spinor, v^μ vanishes. If it is also a Weyl spinor;

$$\frac{1}{2}\,\bar{\xi}\gamma_\mu(\,1+\gamma_{D+1})\,\xi=\frac{1}{2}\,(\,v_\mu-a_\mu)=-\frac{1}{2}\,a_\mu\ . \tag{4c.36}$$

In the massless case, we can also consider another conserved, mixed parity vector

$$w_\mu=\frac{i}{2}\,\overline{\psi}(1+\gamma_{D+1})\overleftrightarrow{\partial}_\mu\psi\ , \tag{4c.37}$$

$$\partial_\mu w^\mu=0\quad,\quad p_\mu\widetilde{w}^\mu=0\ . \tag{4c.38}$$

Under a gauge transformation,

$$\psi\to e^{i\lambda}\psi\quad,\ w_\mu\to w_\mu+\partial_\mu\,\lambda \tag{4c.39}$$

if λ obeys $\Box\,\lambda=0$, then the $(\widetilde{w}^{(0)}-\ \widetilde{w}^{(D-1)})$ component of w in the standard system can be removed by way of the condition (4c.38) and its $(\widetilde{w}^{(0)}-\widetilde{w}^{(D-1)})$ component, using the gauge freedom (4c.39). In the end, for m = 0, only the transverse components of w survive in the standard frame. As for the Weyl spinor, we can always choose the representation where both γ_{D-1} and $\gamma_0\gamma_{D-1}$ are diagonal. There, Eq. (4c.34) leads to another halving of the dimensions of the spinor in the standard frame. Consequently, the physically independent components of $\widetilde{w}$ and ξ become

	D=	3	4	6	10
$d'(\widetilde{w})=$	$d=$	1	2	4	8
$d'(\xi)=$	$d=$	1	2	4	8,

thus establishing a perfect balance ($d'(\widetilde{w})=d'(\xi)$) between the physical degrees of freedom of a vector $\widetilde{w}$ and a spinor ψ in the massless case of dimensions (d+2).

Notably, the above dimensions are precisely those in which space-time supersymmetry can exist in supersymmetric Yang-Mills theories [430] and in classical superstring theories [431]. In that case, supersymmetry results from a Fierz identity for spinors:

$$T(\psi)=\overline{\psi}\,\gamma_\mu\,\psi\,\gamma^\mu\,\psi=0\ . \tag{4c.40}$$

It is valid in the critical dimensions (d + 2), with d = 1, 2, 4 and 8. The actual identity used involves three spinors ψ_1, ψ_2 and ψ_3. Indeed, if we form the quantities

$$T_{ij}=T\,(\,\psi_i+\psi_j)-T\,(\,\psi_i)-T\,(\,\psi_j) \tag{4c.41}$$

$$T_{ijk} = T(\psi_i + \psi_j + \psi_k) - T(\psi_i) - T(\psi_j) - T(\psi_k) - T_{ij} - T_{jk} - T_{ki} \ ; \qquad (4c.42)$$

the vanishing of $T(\psi)$ implies that T_{ijk} also vanishes. We then have the identity

$$\overline{\psi_1}\,\gamma_\mu\,\psi_2\,\gamma^\mu\,\psi_3 + \overline{\psi_2}\,\gamma_\mu\,\psi_3\,\gamma^\mu\,\psi_1 + \overline{\psi_3}\,\gamma_\mu\,\psi_1\,\gamma^\mu\,\psi_2 = 0 \ . \qquad (4c.43)$$

It proves useful in the proof of super-Poincaré invariance of susy Yang-Mills and superstring theories in critical dimensions. Moreover, due to its cyclic structure, Eq. (4c.43) continues to hold whether the spinors are quantized as fermions (with anticommuting Grassmann components) or as bosons (with commuting components).

Notably, in two dimensions, there is a further possibility for supersymmetry. There, a vector v_μ is reduced to its longitudinal components $v_\pm = v_0 \pm v_1$, which are not transformed into one another by Lorentz transformations. Similarly, a D = 2 Weyl-Majorana spinor has two relativistically independent real components ψ_L and ψ_R. So we can have supersymmetry between v_+ and ψ_L or v_- and ψ_R, provided that the longitudinal vector components are dynamical variables.

4.c.2. Vectors as Jordan matrices, Lorentz and Poincaré groups in critical dimensions

Earlier, we discussed at some length (2 x 2) and (3 x 3) hermitian matrices over the division algebras **K** as elements of the Jordan algebras J_2^d and J_3^d. In what follows, it is enough and helpful to repeat some definitions.

If A, B and C are elements of a Jordan algebra, the commutative product A· B is not associative, the associator and the triple product are

$$[\,A\,B\,C\,] = (A \cdot B) \cdot C - A \cdot (B \cdot C) \qquad (4c.44)$$

and

$$\{\,A\,B\,C\,\} = (A \cdot B) \cdot C + A \cdot (B \cdot C) - (C \cdot A) \cdot B \ . \qquad (4c.45)$$

If the algebra is special, i.e. not exceptional, then

$$\{A\,B\,C\} = A\,B\,C \ , \qquad (4c.46)$$

$$\{ A\,B\,C \} - \{ B\,A\,C \} = \frac{1}{4}[[\,A,B\,]\,,C\,] \;. \tag{4c.47}$$

If the symbol He stands for the hermitian part of a matrix product, then

$$A \cdot B = \mathrm{He}(\,AB\,) \tag{4c.48}$$

$$\{ A\,B\,C \} = \mathrm{He}\,(\,A\,B\,C\,) \tag{4c.49}$$

for the special Jordan algebras.

Let

$$V = \begin{pmatrix} v_+ & v_\perp \\ v_\perp^+ & v_- \end{pmatrix} = V^+ \tag{4c.50}$$

where $v_\perp \in \mathbf{K}$ and $\overline{v_\perp}$ is its conjugate, obtained by changing the sign of the $(d-1)$ imaginary units of **K**. The quadratic norm of $\overline{v_\perp}$ is

$$N(v_\perp) = v_\perp \overline{v_\perp} = |\,v_\perp\,|^2 = v_0^2 + v_1^2 + \ldots + v_{D-1}^2 \tag{4c.51}$$

and

$$\mathrm{Det}\,V = v_+ v_- - |\,v_\perp\,|^2 = v_0^2 - v_1^2 - \ldots - v_{D-1}^2 \tag{4c.52}$$

or

$$\mathrm{Det}\,V = -\;v_\mu\, v^\mu \;. \tag{4c.53}$$

We define the hermitian matrix $\overline{V}$ conjugate to V through the requirement

$$V\,\overline{V} = \mathrm{Det}\,V \;. \tag{4c.54}$$

So

$$\overline{V} = V^{-1}\mathrm{Det}\,V = \begin{pmatrix} v_- & -v_\perp \\ -\overline{v}_\perp & v_+ \end{pmatrix} \;. \tag{4c.55}$$

As Det V is invariant under the Lorentz group $O(D-1,\,1)$, V is a vector in a $D = d+2$ Minkowski space and its transverse components are combined to form an element of one of the d-dimensional division algebras.

In the associative cases of $d = 1$, 2 and 4, the Lorentz group acts on the vector V linearly by preserving its hermiticity and leaving its determinant invariant, so that [56]

$$V' = L V L^{\dagger} \quad , \quad \mathrm{Det}\, L = 1 \tag{4c.56}$$

with L being a (2 x 2) matrix over the associative Hurwitz algebras **R**, **C** and **H**.

In the quaternionic case, we must define the determinant of a (2 x 2) quaternionic matrix:

$$L = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \tag{4c.57}$$

with quaternionic entries $a_{\alpha\beta}$ (α, β = 1, 2). One obvious way is to define Det L as that of the (4 x 4) complex matrix $\widetilde{L}$ obtained when the three quaternion units are replaced by Pauli matrices via the correspondence $e_i \to -i\,\sigma_i$. So Det L = Det $\widetilde{L}$, and we get the Study determinant [432]

$$\mathrm{Det}\, L = |a_{11}|^2 |a_{22}|^2 + |a_{12}|^2 |a_{21}|^2 - 2\,\mathrm{Sc}\,(a_{11}\, \bar{a}_{21}\, a_{22}\, \bar{a}_{12}) \,. \tag{4c.58}$$

An equivalent definition is

$$\mathrm{Det}\, L = \det (L\, L^{\dagger}) \quad ; \tag{4c.59}$$

the determinant of a hermitian (2 x 2) matrix being clearly well defined. This gives back Study's expression, which takes the alternative form of

$$\mathrm{Det}\, L = N\, (a_{11}\, a_{22} - a_{11}\, a_{12}\, a_{11}^{-1}\, a_{21}) \,. \tag{4c.60}$$

The group defined by unimodular quaternionic matrices L has as its dimension 15 = {(4 x 4) − 1 }, the number of parameters of the Lorentz group SO(5, 1) ≈ SL(2, **H**). The Lie algebra isomorphisms SO(2, 1) ≈ SL(2, **R**) and SO(3, 1) ≈ SL(2, **C**) are well-known. Such a correspondence naturally suggests that the 10-dimensional Lorentz group SO(9, 1) should be related to SL(2, $\mathbf{\Omega}$), the unimodular linear group over octonions [166, 167, 425, 426, 427, 428]. For a (2 x 2) octonionic matrix L, we can define its determinant through Eq. (4c.59), since no matrix representation exists for octonion units. A unimodular matrix L has 31 parameters. Since linear homogeneous transformations on a vector V, represented by a hermitian matrix, are no longer exhausted by Eq. (4c.56) in the non-associative case, we must also add the transformations of the automorphism group G_2 of dimension 14, expressed by associators [239, 282, 433]. Then the total number of parameters of such an

" SL (2, Ω) " is 45, which is indeed the dimension of SO(9, 1).

The Poincaré group can be represented by the following transformation of the position vector X

$$X' = L\,XL^{\dagger} + A \qquad (X = X^{\dagger}\,,\ A = A^{\dagger}) \tag{4c.61}$$

$$d\,X' = L\,dXL^{\dagger} \qquad (\det L = 1). \tag{4c.62}$$

Det (dx) is then invariant under the group. In the octonionic case, we have

$$X' = L\,(\,T_{G_2}X\,)\,L^{\dagger} + A \tag{4c.63}$$

with, as infinitesimal forms of T_{G_2} X,

$$\delta\,x_{\pm} = 0\ , \tag{4c.64}$$

$$\delta\,x_{\perp} = \frac{1}{2}\,[[\,\mathbf{a}\,,\mathbf{b}\,]\,,x_{\perp}\,] - \frac{3}{2}[\,\mathbf{a},\mathbf{b}\,,x_{\perp}\,]\ . \tag{4c.65}$$

The G_2 transformation is then defined by two purely vectorial octonions, or two unit octonions f and g in the integrated form

$$T_{G_2}x_{\perp} = (\,f\,g\,)^{-1}\left\{g\,(\,f\,x_{\perp}\,f^{-1})\,g^{-1}\right\}(fg) \quad ,\ (\,|\,f\,| = |\,g\,| = 1\,)\ . \tag{4c.66}$$

In all cases, the Lorentz transformations may be written as an SO(D – 1) rotation followed by a (d – 1) parameter boost. They correspond to the decomposition $L = B\,R$, namely into the product of a boost $B = B^{\dagger}$ with DetB = 1, a hermitian matrix, and a rotation R, $R^{\dagger} = R^{-1}$, a unitary matrix.

For D = 6, R is the symplectic matrix Sp(2, H) (≈SO(5) locally), while for D = 4, R is an element of SU(2) (≈ SO(3) locally). There

$$R = \begin{pmatrix} \alpha & -\beta^{*} \\ \beta & \alpha^{*} \end{pmatrix} \quad ,\ |\,\alpha\,|^2 + |\,\beta\,|^2 = 1 \tag{4c.67}$$

while for D = 6 an element of Spin(5) is

$$R = \begin{pmatrix} \alpha & -\alpha\,\overline{\beta}\,\alpha^{-1}u \\ \beta & \overline{\alpha}\,u \end{pmatrix} \quad , \quad |\alpha|^2 + |\beta|^2 = 1 \quad , \quad |u| = 1 \, . \tag{4c.68}$$

In both instances, we can introduce the (2x1) column matrices ξ and $\hat{\xi}$

$$\xi = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \quad , \quad \hat{\xi} = \begin{pmatrix} -\alpha\overline{\beta}\alpha^{-1} \\ \overline{\alpha} \end{pmatrix} \quad , \quad \xi^{+}\xi = 1 \tag{4c.69}$$

and the (2x2) matrix

$$S(\xi) = \begin{pmatrix} \xi & \hat{\xi} \end{pmatrix} \quad , \quad S^{+}S = 1 \, . \tag{4c.70}$$

In the complex case, S is equal to R, an SU(2) group element, hence a point on the unit 3-sphere S^3. In the quaternionic case, from Eq. (4c.69), $S(\xi)$ is seen as a coset element of SO(5) / SO(3) $\approx$ Sp(2) / Sp(1) $\approx S^7$. Explicitly, the Sp(2) element reads

$$R(\xi, u) = S(\xi)\, U(u) \quad , \tag{4c.71}$$

$$U(u) = \begin{pmatrix} 1 & 0 \\ 0 & u \end{pmatrix} \quad , \quad |u| = 1 \, . \tag{4c.72}$$

As U $\in$ SU(2), $S(\xi)$ is the coset element of Sp(2) / Sp(1). Alternatively, we may write

$$R = W(w)\, S(\xi)\, W^{-1}(w) \tag{4c.73}$$

where

$$W(w) = \begin{pmatrix} \overline{w} & 0 \\ 0 & w \end{pmatrix} \quad , \quad |w| = 1 \, . \tag{4c.74}$$

We find

$$\xi_1 = \alpha = \overline{w}\,\xi_1 w \quad , \xi_2 = \beta = w\,\xi_2 w \quad ,$$

$$u = \overline{w}\,\xi_1\, w^2\, \xi_1^{-1}\, \overline{w} \quad ; \tag{4c.75}$$

so again W $\in$ SU(2) $\approx$ O(3).

In the octonionic case, we have seen that the (2 x 2) octonionic matrix L with unit determinant stands for the coset SO(9, 1) / G_2 with 22 elements. W or w has seven parameters and represents the coset space S^7. S(ξ) depends on the spinor ξ constrained by $\xi^+\xi = 1$. Therefore S(ξ) labels a point on S^{15}. The interpretation of Eq. (4c.73) is then

$$(\mathrm{Spin}\ (9) / G_2) / S^7 \approx S^{15} \ . \tag{4c.76}$$

By adjoining G_2 to S^7 we get the group Spin(7) as Spin(7) / $G_2 \approx S^7$. So we can write

$$\mathrm{Spin}(9) / \mathrm{Spin}(7) \approx S^{15} \ . \tag{4c.77}$$

In summary, if we consider the rotation subgroup of the Lorentz group in D = 3, 4, 6 and 10, we are led once again to the essentials Hopf fibrations [30, 31] of the spheres S^1, S^3, S^7 and S^{15} by the fibers $S^0 \approx Z_2$, S^1, S^3 and S^7, respectively. As detailed in Section 4b, these mappings are uniquely linked to the existence of the division algebras **K** .

The above formalism also allows for yet another novel parametrization of the Hopf maps; their more standard forms being given for example in Ref.[392]. To construct this new parametrization, we introduce an element z of the division algebra of dimension d = (D – 2) as the ratio of two components of ξ. ξ transforms as a spinor under a Lorentz transformation or a rotation through left multiplication of the corresponding group elements. Thus, in this generalized stereographic projection, we have

$$z = \beta\, \alpha^{-1} \ , \ \xi = \begin{pmatrix} 1 \\ z \end{pmatrix} \alpha \ . \tag{4c.78}$$

Under the Lorentz group, the projective variable z transforms fractional linearly. Moreover, the normalization condition (4c.68) yields

$$|\alpha| = \gamma (|z|) = (1 + |z|^2)^{-1/2} \ , \ m = \frac{\alpha}{|\alpha|} \ , \ |m| = 1 \tag{4c.79}$$

so that

$$S(\xi) = \gamma(|z|)\begin{pmatrix} 1 & -\bar{z} \\ z & 1 \end{pmatrix}\begin{pmatrix} m & 0 \\ 0 & 1 \end{pmatrix} \tag{4c.80}$$

and

$$R(\xi, u) = Z(z)\,\Delta(m, u) \quad , \quad (Z^{+}Z = 1) \tag{4c.81}$$

with

$$Z(z) = \gamma(|z|)\begin{pmatrix} 1 & -\bar{z} \\ z & 1 \end{pmatrix} \quad , \quad \Delta(m, u) = \begin{pmatrix} m & 0 \\ 0 & u \end{pmatrix} . \tag{4c.82}$$

Specifically, we have:

1. For $D = 4$, $R \in SO(2) \approx S^1$, $SU(2) \approx S^3$, $\Delta \in U(1) \approx S^1$ and $Z \in \mathbf{C}$, or $Z \in S^2$ $\approx CP(1)$. Therefore, we have the complex Hopf map $S^3 \rightarrow S^2$ with fiber S^1.

2. For $D = 6$, $R \in SO(5)$, $\Delta \in SO(4) \approx Sp(1) \times Sp(1) \approx S^3 \times S^3$, consequently $Z \in Sp(2)/Sp(1) \times Sp(1) \approx HP(1) \approx S^4$. Since $S(\xi)$ is the coset $SO(5)/SO(3)$; we may write

$$S(\xi) = Z(z)\,M(m) \quad , \quad M = \begin{pmatrix} m & 0 \\ 0 & 1 \end{pmatrix} . \tag{4c.83}$$

Here $S(\xi)$ labels a point on S^7, $z \in \mathbf{H}$, so that Z represents a point of S^4 and, with $|m| = 1$, $M \in SU(2) \approx S^3$. We thus have an algebraic form of the quaternionic Hopf map $S^7 \rightarrow S^4$ with fiber S^3.

3. Finally, for $D = 10$, $z \in \mathbf{\Omega}$ so that $Z(z) \in Spin(9)/Spin(8) \approx S^8 \approx \Omega P(1)$. So $Z(z)$ labels a point of S^8 while $S(\xi)$ labels one on S^{15}, m being a unit octonion; so it labels a point on S^7. The corresponding map is precisely the last Hopf map $S^{15} \rightarrow S^8$ with fiber S^7.

The final decomposition of the Lorentz matrix reads

$$L = H(v)\,Z(z)\,M(m)\,U(u) \tag{4c.84}$$

with $H = H^{\dagger}$, $\det H = 1$, representing the boost with $(D-1)$ parameters in $D = (d + 2)$ dimensions. $Z(z)$ represents S^d, M corresponding to S^{d-1}, while $S = Z\,M$

parametrizes S^{2d-1}. For D = 10, the matrix L labels the coset SO(9,1) / G_2. Next, we supersymmetrize the Poincaré group.

4.c.3. Super-Poincaré groups and their representations by matrices over K

The generators of the super-Poincaré group [19] consist of $J_{\mu\nu}$ and P_μ, which form its even subgroup, and the odd generators Q_α, which transform like a spinor. Besides the usual commutators of the Poincaré group

$$[J_{\mu\nu}, J_{\rho\sigma}] = i(\eta_{\mu\rho}J_{\nu\sigma} + \eta_{\nu\sigma}J_{\mu\rho} - \eta_{\nu\rho}J_{\mu\sigma} - \eta_{\mu\sigma}J_{\nu\rho}) , \tag{4c.85}$$

$$[J_{\mu\nu}, P_\lambda] = i(\eta_{\mu\lambda}P_\nu - \eta_{\nu\lambda}P_\mu) , \tag{4c.86}$$

$$[P_\lambda, P_\mu] = 0 \tag{4c.87}$$

where $\eta_{\mu\nu}$ is the Minkowskian metric. There are the odd generators Q^i_α and $\overline{Q}^{i\beta}$, (i = 1, 2,..., N) obeying the following relations

$$[J_{\mu\nu}, Q^\alpha_\lambda] = \frac{i}{2}\left(\frac{1+\gamma^{D+1}}{2}\sigma_{\mu\nu}\right)^\beta_\alpha Q^i_\beta , \tag{4c.88}$$

$$[P_\lambda, Q^i_\alpha] = 0 , \tag{4c.89}$$

$$\{Q^i_\alpha, Q^j_\beta\} = 0 , \tag{4c.90}$$

$$[J_{\mu\nu}, \overline{Q}^{i\beta}] = \frac{i}{2}\left(\frac{1-\gamma^{D+1}}{2}\sigma_{\mu\nu}\right)^\beta_\gamma \overline{Q}^{i\gamma} , \tag{4c.91}$$

$$[P_\lambda, \overline{Q}^{i\beta}] = 0 , \tag{4c.92}$$

$$\{\overline{Q}^{i\beta}, \overline{Q}^{j\gamma}\} = 0 \tag{4c.93}$$

$$\{Q^i_\alpha, \overline{Q}^{j\beta}\} = 2i(\gamma^{D+1}\gamma^\lambda)^\beta_\alpha P_\lambda \delta^{ij} . \tag{4c.94}$$

In the above, we take D to be even, Q_α left-handed and $\overline{Q}^\beta$ right-handed. In D = 3,

γ^{D+1} is absent and both γ_λ and $\sigma_{\mu\nu}$ are represented by the three Pauli matrices σ_k. We shall be mainly concerned with the cases of N = 1 and N = 2. For D = 4 and 10, Q_α and $\overline{Q}^\beta$ are combined into a Majorana spinor.

It may be noted that (P_λ, Q_α, $\overline{Q}^\beta$) form the coset of the super-Poincaré algebra w.r.t. the Poincaré algebra π. Γ is also a super Lie algebra, integrable to a supergroup G.

We can find a (5x5) matrix representation of the super-Poincaré group in (d + 2) dimensions over $\mathbf{K}(d)$. We begin by considering the (2 x 2) matrix V associated with the lightlike vector v_μ. From (4c.50)

$$\mathrm{Det}\, V = v_+ v_- - |v_\perp|^2 = 0 \ . \tag{4c.95}$$

Then a Lorentz transformation Λ (V) exists such that, in a specific frame, V assumes the standard form $V^{(0)}$ with

$$v_+^{(0)} = \mu \ , \ v_-^{(0)} = 0 \, , \ v_\perp^{(0)} = 0 \tag{4c.96}$$

i.e.

$$V^{(0)} = \mu \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} . \tag{4c.97}$$

Since V arises from $V^{(0)}$ by a Lorentz transformation,

$$V = L\, V^{(0)}\, L^\dagger = \xi\xi^\dagger \tag{4c.98}$$

where

$$\xi = \sqrt{\mu} \begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix} = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} \tag{4c.99}$$

ξ_1, ξ_2 being elements of $\mathbf{K}(d)$. For D = 10, we adjoint to L a G_2 transformation of the octonionic imaginary units. Conversely, we can show that any (2 x 2) hermitian matrix of the form (4c.98) has vanishing determinant. Thus

$$V = \begin{pmatrix} \xi_1\overline{\xi}_1 & \xi_1\overline{\xi}_2 \\ \xi_2\overline{\xi}_1 & \xi_2\overline{\xi}_2 \end{pmatrix} \tag{4c.100}$$

so that

$$\det V = N(\xi_1)\, N(\xi_2) - N(\xi_1\bar{\xi}_2) = 0 \tag{4c.101}$$

by the multiplicative norm property of division algebras.

Under Lorentz transformations, the column ξ transforms as $\xi' = L\,\zeta$, i.e. like a spinor, provided a G_2 transformation is adjoined to the left multiplication by L in D = 10. Since ξ depends on 2D or (2D – 2) parameters, it is associated with a Majorana spinor in D = 6 and a Weyl-Majorana spinor in D = 10.

We may represent the position vector X in D dimensions by a (2 x 2) hermitian matrix. The Poincaré group then corresponds to the linear transformation

$$X' = L\,(X + A)\,L^{\dagger} \qquad (A = A^{\dagger}), \tag{4c.102}$$

the matrix A being associated with the vector a_μ of translations. This transformation is now expressible in terms of (4 x 4) matrices which are triangular in (2 x 2) submatrices. Indeed, with I being the (2 x 2) unit matrix, we find

$$\begin{pmatrix} I & X' \\ 0 & I \end{pmatrix} = M \begin{pmatrix} I & X \\ 0 & I \end{pmatrix} \widetilde{M} \tag{4c.103}$$

where

$$M = \begin{pmatrix} L & 0 \\ 0 & L^{\dagger -1} \end{pmatrix} \begin{pmatrix} I & \frac{A}{2} \\ 0 & I \end{pmatrix}, \tag{4c.104}$$

$$\widetilde{M} = \begin{pmatrix} I & \frac{A}{2} \\ 0 & I \end{pmatrix} \begin{pmatrix} L^{-1} & 0 \\ 0 & L^{\dagger} \end{pmatrix}. \tag{4c.105}$$

In terms of the Dirac matrices,

$$\rho_1 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \rho_2 = \begin{pmatrix} 0 & -iI \\ iI & 0 \end{pmatrix}, \rho_3 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \tag{4c.106}$$

M satisfies

$$M\overline{M} = I \quad , \quad \overline{M} = i\,\rho_2 M^{\dagger}\,(i\rho_2)^{-1} \tag{4c.107}$$

and

$$\widetilde{M} = \rho_3 M^{-1} \rho_3 \ . \tag{4c.108}$$

A general matrix M with (4c.107) is an element of the conformal group SO(4, 2) ≈ SU(2, 2). The Poincaré group and its extension through the dilatation, the Weyl group, further obey the triangular condition

$$(1 - \rho_3)\,(M\rho_1)\,(1 - \rho_3) = (1 - \rho_3)\, M\, (\rho_1 + i\,\rho_2) = 0 \ . \tag{4c.109}$$

We observe that the coset of the Poincaré group w.r.t. the Lorentz group is an element of the translation group given by

$$\widetilde{M}M = \begin{pmatrix} I & A \\ 0 & I \end{pmatrix} . \tag{4c.110}$$

In this (4 x 4) matrix representation, the generators of the infinitesimal elements of the D = 4 Poincaré group are simply

$$J_{\mu\nu} = \frac{1}{2}\,\sigma_{\mu\nu} = \frac{1}{4i}\left[\,\gamma_\mu, \gamma_\nu\,\right] \ , \tag{4c.111}$$

$$P_\lambda = \frac{1}{4}\,(\,1 + \gamma_5\,)\,\gamma_\lambda \ , \tag{4c.112}$$

in the Weyl representation with γ_5 diagonal. For D = 6 and 10, we have a generalization of $J_{\mu\nu}$ obtained from Eq. (4c.84) when L is close to unity, while P_λ is given by

$$P_\lambda = \frac{1}{2}\begin{pmatrix} 0 & \pi_\lambda \\ 0 & 0 \end{pmatrix} \tag{4c.113}$$

with

$$\pi_0 = I\,,\ \pi_{D-1} = \pi_{d+1} = \sigma_3\,,\ \pi_d = \sigma_1\,, \tag{4c.114}$$

$$\pi_\alpha = i\,\sigma_2\,e_\alpha\ ,\ (\,\alpha = 1, \ldots, d-1\,)\ ; \tag{4c.115}$$

e_α being the imaginary units of **K**.

To represent the super-Poincaré group, we will need (5 x 5) matrices. For D = 4, such a representation has been used in the literature [434] and was subsequently extended to the D = 6, 10 cases. We begin with the construction of the subgroup of translations and supertranslations. Already Γ was seen as a supergroup by itself. We introduce the supertranslation parameters ξ_α representable by a (2 x 1) column matrix over **K**. Similarly, we represent the superspace point (x^μ, θ_α) by the hermitian (2 x 2) matrix X and the (2 x 1) matrix θ. Under supertranslations, the latter transforms as

$$\theta' = \theta + \xi \quad , \quad \text{or} \quad d\theta = 0 \ . \tag{4c.116}$$

Next, we extend the translation group element

$$T(A) = \exp(P_\lambda a_\lambda) = \exp\begin{pmatrix} 0 & \frac{1}{2}A \\ 0 & 0 \end{pmatrix} \tag{4c.117}$$

to the following (5x5) matrix

$$\Gamma(A\,,\xi) = \exp\frac{1}{2}\begin{pmatrix} 0 & \gamma\xi & A \\ 0 & 0 & \gamma\xi^\dagger \\ 0 & 0 & 0 \end{pmatrix} \tag{4c.118}$$

where γ anticommutes with all the Grassmann parameters. This property ensures that the corresponding odd generators obtained by replacing ξ by 1 and e_α, anticommute with the odd-parameters making up the 2d components of ξ. The operator γ is constructed [435, 436] by embedding the Grassmann parameters θ_a (a = 1, 2, ..., N) in a Clifford algebra γ_A (A = 1,..., 2N+1), such that

$$\frac{1}{2}(\gamma_\alpha + i\,\gamma_{\alpha+N}) = \theta_\alpha\,, \qquad \frac{1}{2}(\gamma_\alpha - i\,\gamma_{\alpha+N}) = \frac{\partial}{\partial\theta_\alpha} \ ,$$

$$\gamma = \gamma_{2N+1} = i^N \prod_{a=1}^{N}\left[\theta_a\,, \frac{\partial}{\partial\theta_a}\right] \ . \tag{4c.119}$$

We can now evaluate the supergroup element Γ(A, ξ); noting that { γ, ξ } = 0

$$\Gamma(A\,,\xi) = \begin{pmatrix} I & \frac{1}{2}\gamma\xi & \frac{1}{2}\left(A - \frac{1}{4}\xi\xi^\dagger\right) \\ 0 & 1 & \frac{1}{2}\gamma\xi^\dagger \\ 0 & 0 & I \end{pmatrix} \ . \tag{4c.120}$$

Thus we end up with the combination

$$\psi\,(\,A\,,\,\xi) = A - \frac{1}{4}\,\xi\xi^\dagger \tag{4c.121}$$

occurring in scalar chiral superfields. The latter are functions of y and ξ, but not of $\xi^\dagger$.

The super-Poincaré group element S(A, ξ, ω), with ω denoting the parameters of the Lorentz matrix L(ω), has the form

$$S\,(A, \xi, \omega) = \Gamma(A, \xi)\;\Lambda\;L\,((\omega)) \tag{4c.122}$$

with

$$\Lambda(L) = \begin{pmatrix} L & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \left(L^\dagger\right)^{-1} \end{pmatrix} . \tag{4c.123}$$

If a point in the coset Γ is labelled by X and θ, a superspace point may be represented by the following (5 x 5) matrix

$$\Gamma(\,X, \theta) = \begin{pmatrix} I & \frac{1}{2}\gamma\theta & \frac{1}{2}\,y(x,\theta) \\ 0 & 1 & \frac{1}{2}\gamma\theta^\dagger \\ 0 & 0 & I \end{pmatrix} \tag{4c.124}$$

with the property

$$\Gamma(\,X\,,\theta)^{-1} = \Gamma(\,-X\,,-\theta)\;\;. \tag{4c.125}$$

Generalizing ρ_3, we now introduce the diagonal matrix

$$\mu(\varepsilon) = \mu(\varepsilon)^{-1} = \begin{pmatrix} -I & 0 & 0 \\ 0 & \varepsilon & 0 \\ 0 & 0 & I \end{pmatrix} \;,\;\; \varepsilon^2 = 1 \;\;,\;\; \mu(\,\pm 1) = \mu_\pm\,. \tag{4c.126}$$

Since

$$\Lambda(\omega)\,\mu(\varepsilon)\,\Lambda(\omega)^{-1} = \mu(\varepsilon)\;\;, \tag{4c.127}$$

the coset of the super-Poincaré group w.r.t. the Lorentz group reads

$$Z_\pm = S\,\mu_\pm\,S^{-1} = \Gamma(X,\theta)\,\mu_\pm\Gamma^{-1}(X,\theta) \quad . \tag{4c.128}$$

Explicitly,

$$Z_+ = \begin{pmatrix} -I & \gamma\theta & X - \frac{1}{4}\theta\theta^\dagger \\ 0 & 1 & 0 \\ 0 & 0 & I \end{pmatrix} \tag{4c.129}$$

and

$$Z_- = \begin{pmatrix} -I & 0 & X - \frac{1}{4}\theta\theta^\dagger \\ 0 & -1 & \gamma\theta^\dagger \\ 0 & 0 & I \end{pmatrix} \tag{4c.130}$$

Z_+ and Z_- have elements occurring in both kinds of chiral superfields [371, 437, 438]. In either case, under the action of the super-Poincaré group their transformation law reads

$$Z_\pm' = S(A,\xi,\omega)\,Z_\pm\,S^{-1}(A,\xi,\omega)\,. \tag{4c.131}$$

So, under the Lorentz subgroup we have

$$X' = L\,X\,L^\dagger\,,\ \theta' = L\,\theta\ ,\ \theta'^\dagger = \theta^\dagger L^\dagger\,. \tag{4c.132}$$

Under translations,

$$X'' = X + A\ \ ,\ \theta'' = \theta\ ,\ \theta''^\dagger = \theta^\dagger\,, \tag{4c.133}$$

while under supertranslations,

$$X''' = X + \frac{1}{4}\left(\theta\xi^\dagger - \xi\theta^\dagger\right)\,, \tag{4c.134}$$

$$\theta''' = \theta + \xi\quad ,\ \theta'''^\dagger = \theta^\dagger + \xi^\dagger\,. \tag{4c.135}$$

To compare with the standard D = 4 Dirac formalism, we denote the 2-component complex spinor by θ and the corresponding Majorana spinor by ζ, and

$$\zeta = \zeta^c = \begin{pmatrix} \theta_L \\ \hat{\theta}_L \end{pmatrix} \quad , \quad \hat{\theta}_L = i\,\sigma_2 \theta_L^* \, , \;\; \zeta^c = \gamma_2 \zeta^* = \rho_2 \sigma_2 \zeta^* \, . \qquad (4c.136)$$

It is convenient to work in the Weyl representation :

$$\gamma_4 = i\,\gamma_0 = -\,i\,\gamma^0 = \rho_1 \, , \gamma_n = \gamma^n = \rho_2 \sigma_n \, , \gamma_5 = \rho_3 \; . \qquad (4c.137)$$

Then we have the identities

$$\theta_L\, \theta_L^\dagger = \frac{1}{2}\, \sigma_\mu\, \theta_L^\dagger\, \overline{\sigma}^\mu\, \theta_L \quad , \qquad (4c.138)$$

$$i \overline{\zeta}\, \gamma_5\, \gamma^\mu\, \zeta = -\,2\, \theta_L^\dagger\, \overline{\sigma}^\mu\, \theta_L \qquad (4c.139)$$

with

$$\overline{\zeta} = \zeta^+ \gamma_4 \quad , \quad \overline{\sigma}^n = -\,\sigma^n = -\,\sigma_n \quad , \quad \overline{\sigma}_0 = \sigma_0 = -\,\overline{\sigma}^0 = 1 \; . \qquad (4c.140)$$

The following alternative expressions then arise

$$X + \lambda \theta_L\, \theta_L^\dagger = \sigma_\mu \left(x^\mu - \frac{\lambda}{2}\, \theta_L^\dagger\, \overline{\sigma}^\mu\, \theta_L \right) \qquad (4c.141)$$

or, in terms of the Majorana spinor,

$$X + \lambda \theta_L\, \theta_L^\dagger = \sigma_\mu \left(x^\mu - \frac{i\lambda}{4}\, \overline{\zeta}\, \gamma_5\, \gamma^\mu\, \zeta \right) . \qquad (4c.142)$$

Observe that, owing to the Majorana and Grassmann nature of ξ, the vector current vanishes:

$$\overline{\zeta} \gamma^\mu \zeta = 0 \, . \qquad (4c.143)$$

Hence

$$-\frac{i}{4}\, \overline{\zeta}\, \gamma_5\, \gamma^\mu\, \zeta = \frac{i}{4}\, \overline{\zeta}\, \gamma^\mu\, (\, 1 + \gamma_5\,)\, \zeta \; , \qquad (4c.144)$$

which displays the left-handed structure of the Majorana pseudocurrent.

Another remark relates to the normalization used. To arrive at the commutation relations (4c.94b) in the standard form we need multiply θ and ξ by $\sqrt{2}$. To keep the matrix representation simple we have picked a representation where the coefficient 2 in Eq. (4c.94b) is replaced by one.

We can also generalize the relation (4c.107), obeyed by an element of the Poincaré group, to the super-Poincaré group represented by S (4c.122). To that end the superhermitian conjugate of such a matrix must be so defined that

$$(RS)^\dagger = S^\dagger R^\dagger \ , \tag{4c.145}$$

which requires altering the relative sign of the odd parameters of a matrix like Γ of Eq. (4c.120). Since is odd, $\xi\xi^\dagger$ is antihermitian, we have

$$\Gamma^\dagger(A, \xi) = \begin{pmatrix} I & 0 & 0 \\ -\frac{1}{2}\gamma\xi^\dagger & 1 & 0 \\ \frac{1}{2}(A + \frac{1}{4}\xi\xi^\dagger) & \frac{1}{2}\gamma\xi & I \end{pmatrix} . \tag{4c.146}$$

We readily check that

$$\left[\Gamma(A, \xi)\,\Gamma(B, \eta)\right]^\dagger = \Gamma(B, \eta)^\dagger\,\Gamma(A, \xi)^\dagger \ . \tag{4c.147}$$

Generalizing $i\rho_2$ in Eq. (4c.106), we introduce the (5 x 5) matrix ν:

$$\nu = \begin{pmatrix} 0 & 0 & I \\ 0 & -1 & 0 \\ -I & 0 & 0 \end{pmatrix} \quad , \quad \nu^{-1} = \begin{pmatrix} 0 & 0 & -I \\ 0 & -1 & 0 \\ I & 0 & 0 \end{pmatrix} . \tag{4c.148}$$

Defining $\overline{S} = \nu\ S^+\ \nu^{-1}$, we find

$$\overline{\Lambda} = \Lambda^{-1} \quad , \quad \overline{\Gamma} = \Gamma^{-1} \ , \quad \overline{\Gamma\Lambda} = \overline{\Lambda}\,\overline{\Gamma} \tag{4c.149}$$

so that

$$\overline{S}\,S = S\,\overline{S} = I_{(5)} \ , \tag{4c.150}$$

$I_{(5)}$ stands for the (5 x 5) unit matrix. Eq. (4c.150) must now be complemented by triangularity relations, the generalizations of Eq. (4c.109). Without them, it characterizes the (5 x 5) representation of the superconformal group.

4.c.4. Some Fierz identities and division algebras

By now many papers [373, 439, 440] have underscored the connections between division algebras and the identities (4c.40) and (4c.43), holding for D = 3, 4, 6, 10. We give below an alternative derivation.

We begin by noting that Eq. (4c.43) holds both for the bosonic and fermionic spinors. We first prove it in the bosonic case where the simpler identity holds in critical dimensions. Consider the bosonic spinor S and its hermitian conjugate $S^\dagger$,

$$S = \begin{pmatrix} s_1 \\ s_2 \end{pmatrix} \quad , \qquad S^\dagger = (\, \overline{s_1}\ \overline{s_2}\,) \tag{4c.151}$$

where s_α ($\alpha = 1, 2$) belong to the division algebra $\mathbf{K}(d)$ with d = 1, 2, 4, 8. We form the hermitian matrix h

$$h = S\, S^\dagger = \begin{pmatrix} s_1\overline{s_1} & s_1\overline{s_2} \\ s_2\overline{s_1} & s_1\overline{s_2} \end{pmatrix} . \tag{4c.152}$$

By way of Eq. (4c.55),

$$\bar{h} = \begin{pmatrix} s_2\overline{s_2} & -\, s_1\overline{s_2} \\ -\, s_2\overline{s_1} & s_1\overline{s_1} \end{pmatrix} . \tag{4c.153}$$

Under a Lorentz transformation

$$S \rightarrow L\, S \ , \ h \rightarrow L\, h\, L^\dagger \, , \bar{h} \rightarrow \left(L^\dagger\right)^{-1} \bar{h}\, L^{-1} \ . \tag{4c.154}$$

Therefore, the spinor $T = \bar{h}\, S$ will transform as $t \rightarrow \left(L^\dagger\right)^{-1} t$. If S is left-handed spinor, T is then a right-handed one in even dimensions. Componentwise, we find

$$t_1 = (\, s_2\, \overline{s_2}\,)\, s_1 - (s_1\, \overline{s_2}\,)\, s_2 = N\,(\, s_2\,)\, s_1 - (\, s_1\, \overline{s_2}\,)\, s_2 \ , \tag{4c.155}$$

$$t_2 = (\, s_1\, \overline{s_1}\,)\, s_2 - (s_2\, \overline{s_1}\,)\, s_1 = N\,(\, s_1\,)\, s_2 - (\, s_2\, \overline{s_1}\,)\, s_1 \ . \tag{4c.156}$$

Since the new norm of s is a real number commuting with s_1, we obtain

$$t_1 = s_1 N(s_2) - (s_1 \overline{s_2}) s_2 = -(s_1 \overline{s_2}) s_2 + s_1 (\overline{s_2} s_2) \tag{4c.157}$$

using the norm property $N(s_2) = N(\overline{s_2})$.

By the very definition of the associator, we find

$$t_1 = -[s_1, \overline{s_2}, s_2] \tag{4c.158}$$

which, with $\overline{s_2} = s_2 - 2\,\mathrm{Sc}(s_2)$, gives $t_1 = -[s_1, s_2, s_2]$. Since the associator is skew symmetric for $\mathbf{K}(d)$ and zero for $\mathbf{K}(1)$, $\mathbf{K}(2)$ and $\mathbf{K}(4)$ we have $t_1 = 0$. Similarly, $t_2 = 0$, hence

$$T = \overline{S\,S^\dagger}\,S = 0\,. \tag{4c.159}$$

Next, let us transcribe this identity in the conventional notation. For D = 3, 4 we have the spinor $\psi = \begin{pmatrix} s_1 \\ s_2 \end{pmatrix}$ with s_α real for D = 3, complex and left-handed for D = 4. It can also be associated with the Majorana spinor $\eta = \begin{pmatrix} \psi \\ \hat{\psi} \end{pmatrix}$, $\hat{\psi} = i\,\sigma_2 \psi^*$ with bosonic coordinates.

Then, unlike the fermionic case, Eqs. (4c.43), (4c.44), we have

$$-i\,\bar{\eta}\gamma_5\gamma_\mu\eta = 0 \quad , \quad i\,\bar{\eta}\gamma_\mu\eta = 2\,\psi^+\sigma_\mu\psi = V_\mu \quad , \tag{4c.160}$$

$$-i\,V_\mu\,\gamma^\mu = 2\begin{pmatrix} 0 & \sigma^\mu V_\mu \\ \overline{\sigma}^\mu V_\mu & 0 \end{pmatrix} = 2\begin{pmatrix} 0 & h \\ \bar{h} & 0 \end{pmatrix} \tag{4c.161}$$

So

$$-i\,\bar{\eta}\,\gamma_\mu\,\eta\,\gamma^\mu\,\eta = 2\begin{pmatrix} 0 & h \\ \bar{h} & 0 \end{pmatrix}\begin{pmatrix} \psi \\ \hat{\psi} \end{pmatrix} = 2\begin{pmatrix} h\,\hat{\psi} \\ \bar{h}\,\psi \end{pmatrix} = 0 \tag{4c.162}$$

in consequence of Eq. (4c.159) when the s_α are complex.

In the quaternionic case, we replace the quaternions s_α by (2 x 2) matrices

$$s_\alpha = \begin{pmatrix} a_\alpha & -b^*_\alpha \\ b_\alpha & a^*_a \end{pmatrix} \tag{4c.163}$$

then ψ becomes a (4 x 2) complex matrix :

$$\Psi_L = \begin{pmatrix} a_1 & -b_1^* \\ b_1 & a_1^* \\ a_2 & -b_2^* \\ b_2 & a_2^* \end{pmatrix} . \tag{4c.164}$$

It depends on 8 real parameters. The first column of ψ can be written as a left-handed Weyl spinor $\psi_L = \frac{1}{2}(1 + \gamma_7)\psi$ in a representation where γ is diagonal and ψ a (8 x 1) column. The elements of h now correspond to

$$h_\mu = \frac{1}{2}\overline{\psi_L}\,\gamma_\mu\,\psi_L \quad (\mu = 0, 1, \dots, 5) . \tag{4c.165}$$

Once again the identity (4c.159) takes the form (4c.162).

In the octonionic case, the 16 real elements of the Weyl-Majorana spinor in D = 10 can be expressed in terms of 8 complex numbers and their conjugates. They are obtained by writing the octonions as *split units* u_0 and u_n (n = 1, 2, 3) defined previously. Then

$$S_\alpha = \mathrm{Re}\,(\psi_4^{(\alpha)}\,u_0 + \psi_n^{(\alpha)} u_n) , \tag{4c.166}$$

which sets the correspondence between the 8 complex components $\psi_a^{(\alpha)}$(with $\alpha = 1, 2$; a=1, 2, 3, 4) of the Weyl-Majorana spinor in D = 10. Once again, the octonionic identity (4c.159) assumes the form of (4c.162) for this spinor.

From the identity (4c.40) in the bosonic case, the other Fierz identity (4c.43) follows. However, due to its cyclic nature, Eq. (4c.43) still holds in the fermionic case, though Eq. (4c.40) and therefore Eq. (4c.159) is no longer valid. The proof of the cyclic identity in the fermionic case is straightforward but more involved.

For fermionic spinors with Grassmann-valued components, $s_\alpha \bar{s}_\alpha$ and $\bar{s}_\alpha s_\alpha$ are purely vectorial in the quaternionic and octonionic cases. They also differ from one another. Furthermore, the (2 x 2) matrix $ss^\dagger$ is antihermitian. Nevertheless, the cyclic identity (4c.3) still follows and is at the basis of the proof of the super-Poincaré invariance of the super Yang-Mills and covariant superstring actions. We used above the connection between such an identity with division algebras by a direct construction of a representation of the super-Poincaré group by means of matrices over **K**. Such a construction is only possible in the critical dimensions D = 3, 4, 6 and 10.

4.c.5. N = 2 super-Poincaré groups in critical dimensions

The representation given by Eqs. (4c.120) and (4c.122) was for N = 1 supersymmetry. In the case of N = 2, the Grassmann valued spinor ξ must be replaced by two spinors ξ and η combined into a Grassmann valued (2 x 2) matrix Ξ over the four division algebras

$$\Xi = \begin{pmatrix} \xi & \eta \end{pmatrix} = \begin{pmatrix} \xi_1 & \eta_1 \\ \xi_2 & \eta_2 \end{pmatrix} . \tag{4c.167}$$

Similarly, the odd components of the supervector (X, θ) can be replaced by (X, Θ) where Θ is a (2 x 2) matrix.

The N=2 super-Poincaré group is now represented by a (6 x 6) matrix with elements that can be grouped into (2 x 2) matrices. Instead of Eq. (4c.122), we have

$$S\,(A\,,\Xi\,,\omega) = \Gamma(A\,,\Xi)\,\Lambda\,(L\,(\omega)\,) \tag{4c.168}$$

where

$$\Lambda(L) = \begin{pmatrix} L & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & (L^{\dagger})^{-1} \end{pmatrix} , \tag{4c.169}$$

$$\Gamma(A\,,\Xi) = \begin{pmatrix} I & \frac{1}{2}\gamma\,\Xi & \frac{1}{2}\left(A - \frac{1}{4}\,\Xi\Xi^{\dagger}\right) \\ 0 & I & \frac{1}{2}\gamma\,\Xi^{\dagger} \\ 0 & 0 & I \end{pmatrix} \tag{4c.170}$$

and the superspace point is represented by the (6 x 6) matrix

$$\Gamma(X,\Theta)=\begin{pmatrix} I & \frac{1}{2}\gamma\Theta & \frac{1}{2}Y(x,\Theta) \\ 0 & I & \frac{1}{2}\gamma\Theta^{\dagger} \\ 0 & 0 & I \end{pmatrix} \tag{4c.171}$$

with $Y = X - \frac{1}{4}\Theta\Theta^{+}$.

We can now introduce a new internal symmetry group SU(2) interchanging the two spinors:

$$T(u) = \begin{pmatrix} I & & \\ & U & \\ & & I \end{pmatrix}. \tag{4c.172}$$

Indeed, if we generalize $Z_{\pm}$ of Eqs. (4c.128), (4c.129) by replacing θ by Θ and 1 by I, the supersymmetry transformations become

$$Z'_{\pm} = S(A,\Xi,\omega)\, Z_{\pm} S^{-1}(A,\Xi,\omega)\ , \tag{4c.173}$$

while the internal symmetry transformation reads

$$Z'_{\pm} = T(u)\, Z_{\pm}\, T^{\dagger}(u)\ , \tag{4c.174}$$

amounting to the transformation

$$\Theta \to \Theta U^{\dagger}\ ,\ \Theta^{\dagger} \to U\Theta^{\dagger}\ , \tag{4c.175}$$

which interchanges the two columns of the matrix Θ.

The (6 x 6) matrix associated with the N = 2 super-Poincaré group is also decomposable into four (3 x 3) matrices. Each (3 x 3) matrix can be associated with a Jordan algebra. This new structure accounts for many of the special properties of N = 2 supersymmetry, to be seen subsequently.

4.c.6. Classical superparticles and superstrings

We next review various covariant formulations of classical superparticle and superstring theories in critical dimensions. Specifically, they will be recast so as to exhibit their connection with division algebras and related Jordan algebras. At the advent of the covariant string theories of Siegel and of Green and Schwarz such an algebraic structure relation had been independently noted and investigated by a number of authors. A satisfactory covariant quantization procedure for these models is still missing. The only method which can be confidently applied is the non-covariant quantization in the light cone gauge in $D = 10$. We shall only confine ourselves to the classical models in $D = 3, 4, 6$ and 10.

Consider a superworld-line $Z(\tau)$ with even part $x(\tau)$ and odd part $\theta = \theta(\tau)$ for $N = 1$ and $\Theta = \Theta(\tau)$ for $N = 2$. We shall also deal with a superworld-sheet $Z(\sigma, \tau)$ or $Z(\xi_\alpha)$, ($\alpha = 0, 1$), where $\xi_1 = \tau$ and $\xi_2 = \sigma$, the even part being $x(\xi)$ and the odd part $\theta(\xi)$ for $N = 1$ and $\Theta(\xi)$ for $N = 2$. These will be represented by (5 x 5) or (6 x 6) matrices $Z_\pm(\tau)$ or $Z_\pm(\zeta_\alpha)$ introduced previously. For a superworld-line, we have the supertangent vector $\dot{Z}(\tau)$ corresponding to the super-velocity vector. For a superworld-sheet, we have the vectors $\partial_\alpha Z(\zeta_\beta)$, ($\alpha = 1, 2$). Our next step is to write down super-Poincaré invariants made up of Z and its tangent vectors. Such invariants will be candidates for an action principle.

It is also possible to write the corresponding action for a closed bosonic string with left-movers $X_L(\zeta_\alpha)$ and right-movers $X_R(\zeta_\alpha)$ in $D = 26$. Due to the correspondence between the dimensions of supervectors and those of the bosonic strings in $D = 5, 8, 14$ and 26, namely the dimensions of (3 x 3) traceless (unimodular) Jordan algebras over $\mathbf{K} = \mathbf{R}$, $\mathbf{C}$, $\mathbf{H}$ and $\mathbf{\Omega}$, the bosonic string actions will be naturally expressed in the Jordan algebra formalism. In the case of the heterotic string both supervectors and the associated bosonic vectors come into play, requiring elements of the super-Jordan algebras in addition to those of Jordan algebras. Moreover, the Jordan algebras in critical dimensions have automorphism groups belonging to the Magic Square. Consequently this new formalism establishes a link between superstrings and the exceptional groups and their subgroups. It has been our hope that this algebraic reformulation of superstring theories may open a way to their successful covariant quantization. We begin with the construction of superparticle actions from the super-Poincaré group.

From the super-Poincaré group element we can form two Maurer Cartan forms

$$\lambda = \Gamma^{-1} d\,\Gamma \quad \text{and} \quad \rho = \left(d\,\Gamma\right)\Gamma^{-1} \tag{4c.176}$$

where Γ is defined as in Eq. (4c.120). So

$$\Gamma(A, \theta) = \begin{pmatrix} I & \frac{1}{2}\gamma\theta & \frac{1}{2}\left(A - \frac{1}{4}\theta\theta^{\dagger}\right) \\ 0 & I & \frac{1}{2}\gamma\theta^{\dagger} \\ 0 & 0 & I \end{pmatrix} \tag{4c.177}$$

where the operator γ anticommutes with all the Grassmann numbers and its square is unity. The 1-form λ is invariant under the left action of the supergroup on Γ, so that

$$\Gamma(X, \theta) \to \Gamma(X', \theta') = \Gamma(A, \xi)\,\Gamma(X, \theta)\,, \; \lambda \to \lambda \tag{4c.178}$$

while the 1-form ρ is invariant under the right action on Γ, namely

$$\Gamma(X, \theta) \to \Gamma(X'', \theta'') = \Gamma(X, \theta)\Gamma(A, \xi)\,, \; \rho \to \rho \;. \tag{4c.179}$$

For a superparticle, X and θ are functions of a Lorentz invariant time parameter τ. Then Γ (X, θ) is also a function of the proper time. We denote its time derivative by $\dot{\Gamma} = \frac{d}{d\tau}\Gamma[\,X(\tau)\,, \theta(\tau)\,]$.

To the two Maurer-Cartan 1-forms are associated respectively a left invariant derivative *L* and a right invariant derivative *R*, defined by

$$L(\Gamma) = \Gamma^{-1}\dot{\Gamma} \qquad , \qquad R(\Gamma) = \dot{\Gamma}\,\Gamma^{-1}\,. \tag{4c.180}$$

Next, consider the position super-vectors $Z_{\pm}$ defined by Eq. (4c.128). Z_+ is chiral since it is only a function of y (x, θ) defined as in (4c.121). Z_- is antichiral since it is a function of y (– x, θ) and $\theta^{\dagger}$. The supervelocity vectors are defined as

$$\dot{Z}_{\pm} = \dot{\Gamma}\,\mu_{\pm}\,\Gamma^{-1} - \Gamma\,\mu_{\pm}\Gamma^{-1}\dot{\Gamma}\,\Gamma^{-1} \tag{4c.181}$$

with $\mu_{\pm}$ given by Eq. (4c.126).

Under left supersymmetry transformations, we have

$$\dot{Z}_{\pm} \to \dot{Z}'_{\pm} = \Gamma(A, \xi)\, \dot{Z}_{\pm}\, \Gamma(A, \xi)^{-1} \ . \tag{4c.182}$$

From the above expressions, we can form chiral and anti-chiral functions, invariant under susy transformations corresponding to left action on $\Gamma(X, \theta)$. Let

$$W_{\pm} = \Gamma^{-1} \dot{Z}_{\pm} \Gamma = \left[\Gamma^{-1} \Gamma, \mu_{\pm} \right] = \left[L(\Gamma), \mu_{\pm} \right] \tag{4c.183}$$

where we have used Eq. (4c.180). The corresponding matrices read

$$W_{+} = \begin{pmatrix} 0 & \gamma\,\dot{\theta} & \dot{X} + \frac{1}{4}\left(\theta\dot{\theta}^{\dagger} - \dot{\theta}\theta^{\dagger}\right) \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} , \tag{4c.184}$$

$$W_{-} = \begin{pmatrix} 0 & 0 & \dot{X} + \frac{1}{4}\left(\theta\dot{\theta}^{\dagger} - \dot{\theta}\theta^{\dagger}\right) \\ 0 & 0 & \gamma\dot{\theta}^{\dagger} \\ 0 & 0 & 0 \end{pmatrix} . \tag{4c.185}$$

They are both invariant under translations and supertransformations associated with a left action on $\Gamma(X,\theta)$ and given by Eqs. (4c.132), (4c.134)-(4c.135)

$$X \to X + A + \frac{1}{4}\left(\theta\xi^{\dagger} - \xi\theta^{\dagger}\right) \ , \tag{4c.186}$$

$$\theta \to \theta + \xi \ .$$

In particular, under infinitesimal supertransformations

$$\delta\theta = \varepsilon \quad , \quad (\varepsilon = \delta\xi) \ ,$$

$$\delta X = \frac{1}{4}\left(\theta\varepsilon^{\dagger} - \varepsilon\theta^{\dagger}\right) \ . \tag{4c.187}$$

Under Lorentz transformations given by the matrix Λ in Eq. (4c.123), we have

$$Z_\pm \to \Lambda Z_\pm \Lambda^{-1} \ , \ \dot{Z}_\pm \to \Lambda \dot{Z}_\pm \Lambda^{-1} \ ,$$

$$W_\pm \to \Lambda W_\pm \Lambda^{-1} \ . \tag{4c.188}$$

To obtain super-Poincaré invariants, we need to introduce conjugate triangular matrices with the same transformation property as $W_\pm$ under the Lorentz transformations. Λ is constructed out of the (2 x 2) unimodular matrice L as in Eq. (4c.123). The Lorentz transforms of θ and X and their derivatives are

$$\theta' = L\,\theta\,,\ \ \theta'^\dagger = \theta^\dagger L^\dagger\ ,\ X' = L\,X\,L^\dagger\ , \tag{4c.189}$$

$$\dot{\theta}' = L\,\dot{\theta}\,,\ \ \dot{\theta}'^\dagger = \dot{\theta}^\dagger L^\dagger\ ,\ \dot{X}' = L\,\dot{X}\,L^\dagger\ . \tag{4c.190}$$

$\hat{\theta}$ is right-handed when θ is left-handed. In D = 4, it is defined by $\hat{\theta} \equiv - i\sigma_2\theta^*$ and by Eq. (4c.69) up to a quaternionic SU(2) phase acting from the right in D = 6, and also up to a Spin(7) phase in D = 10, as explained previously. Under the Lorentz group, it transforms as

$$\hat{\theta}' = \left(L^\dagger\right)^{-1}\hat{\theta}\ . \tag{4c.191a}$$

The conjugate matrix $\overline{X}$ defined as in Eq. (4c.55) transforms as

$$\overline{X'} = \left(L^\dagger\right)^{-1}\overline{X}\,L^{-1}\ . \tag{4c.191b}$$

Equation (4c.191) leads to

$$\overline{u}' = \left(L^\dagger\right)^{-1}\overline{u}\,L^{-1} \tag{4c.192}$$

with

$$u = \dot{X} + \frac{1}{4}\left(\theta\,\dot{\theta}^\dagger - \dot{\theta}\,\theta^\dagger\right)\ , \tag{4c.193}$$

$$\overline{u} = \dot{\overline{X}} + \frac{1}{4}\left(\hat{\theta}\,\dot{\hat{\theta}}^\dagger - \dot{\hat{\theta}}\,\hat{\theta}^\dagger\right)\ . \tag{4c.194}$$

$W_\pm$ in Eqs. (4c.184)-(4c.185) are respectively functions of $(\dot{\theta},u)$ and $(\dot{\theta}^\dagger,u)$. We define

$$\widehat{W}_+ = W_+(\hat{\theta}, \bar{u}) \quad , \quad \widehat{W}_- = W_-(\hat{\theta}^\dagger, \bar{u}) \tag{4c.195}$$

with

$$\widehat{W}'_\pm = \left(\Lambda^\dagger\right)^{-1} \widehat{W}_\pm \Lambda^{-1} \quad . \tag{4c.196}$$

By taking the superhermitian conjugate on both sides, it follows that

$$\left(\widehat{W}'_\pm\right)^\dagger = \Lambda \widehat{W}^\dagger_\pm \Lambda^{-1} \quad . \tag{4c.197}$$

Therefore

$$\widehat{W}^\dagger_+ = \begin{pmatrix} 0 & 0 & 0 \\ -\gamma\hat{\theta}^\dagger & 0 & 0 \\ \bar{u} & 0 & 0 \end{pmatrix} \quad , \quad \widehat{W}^\dagger_- = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \bar{u} & \gamma\hat{\theta} & 0 \end{pmatrix} \tag{4c.198}$$

transform in the same way as the matrices $W_\pm$ obeying Eq. (4c.188).

We are now ready to construct super-Poincaré invariants. They are obtained by taking supertraces of products like

$$\widehat{W}^\dagger_+ W_+ = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \hat{\dot{\theta}}\theta & -\gamma\hat{\theta}^{\dagger}\dot{\theta} \\ 0 & \gamma\bar{u}\dot{\theta} & \bar{u}u \end{pmatrix} \quad , \tag{4c.199}$$

$$\widehat{W}^\dagger_+ W_- = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\gamma\hat{\theta}^\dagger u \\ 0 & 0 & \bar{u}u \end{pmatrix} \tag{4c.200}$$

and

$$W_{-}\widehat{W}^{\dagger} = \begin{pmatrix} \bar{u}u & \gamma u \hat{\dot{\theta}} & 0 \\ \gamma \dot{\theta}^{\dagger} \bar{u} & -\dot{\theta}^{\dagger} \hat{\dot{\theta}} & 0 \\ 0 & 0 & 0 \end{pmatrix} . \tag{4c.201}$$

The usual choice is the simplest :

$$\mathrm{Str}\left(\widehat{W}_{+}^{\dagger} W_{-}\right) = \bar{u}\, u \ . \tag{4c.202}$$

The other combinations give additional terms proportional to the invariants

$$L_1^{\theta} = \frac{1}{2}(\hat{\dot{\theta}}^{\dagger} \dot{\theta} + \dot{\theta}^{\dagger} \hat{\dot{\theta}}) \quad , \quad L_2^{\theta} = -\frac{i}{2}(\hat{\dot{\theta}}^{\dagger} \dot{\theta} - \dot{\theta}^{\dagger} \hat{\dot{\theta}}) \ . \tag{4c.203}$$

To have an action invariant under the proper time reparametrization

$$\tau' = f(\tau) \ , \ d\tau' = \dot{f}(\tau) d\tau \tag{4c.204}$$

the einbein determinant e is introduced such that $e' = \dot{f}(\tau)\, e$.

The super-Poincaré and reparametrization invariant superparticle action then takes the standard form

$$I^{s.p.} = \frac{1}{2} \int \frac{d\tau}{e} \bar{u}\, u \ . \tag{4c.205}$$

Its associated Lagrangian reads

$$L^{s.p.} = \frac{1}{2} e^{-1} \left[\dot{X} + \frac{1}{4}(\hat{\theta} \hat{\dot{\theta}}^{\dagger} - \hat{\dot{\theta}}\, \theta^{\dagger}) \right] \left[\dot{X} + \frac{1}{4}(\theta\, \dot{\theta}^{\dagger} - \dot{\theta}\, \theta^{\dagger}) \right] . \tag{4c.206}$$

The latter reproduces the standard superparticle Lagrangian if we introduce the Majorana spinor:

$$\psi = \psi^{c} = \gamma_2 \psi^{*} = 2 \begin{pmatrix} \theta \\ -\hat{\theta} \end{pmatrix} \ , \ \bar{\psi} = \psi^{\dagger} \gamma_4 \ . \tag{4c.207}$$

Then, setting

$$X = x + \vec{\sigma} \cdot \vec{x} \ , \quad u = u_0 + \vec{\sigma} \cdot \vec{u} \ , \tag{4c.208}$$

$$\frac{1}{4}\left(\theta \, \dot{\theta}^\dagger - \dot{\theta} \, \theta^\dagger \right) = k_0 + \vec{\sigma} \cdot \vec{k} \ , \tag{4c.209}$$

we get

$$k_\mu = \frac{i}{2}\left(\overline{\psi}\gamma_\mu\dot{\psi} - \dot{\overline{\psi}}\gamma_\mu\psi \right) \ , \tag{4c.210}$$

$$u_\mu = \dot{x}_\mu + k_\mu \ , \tag{4c.211}$$

$$L^{s.p.} = \frac{1}{2} e^{-1} \bar{u} \, u = \frac{1}{2} e^{-1} (\dot{x}_\mu + k_\mu) (\dot{x}^\mu + k^\mu) \ . \tag{4c.212}$$

Siegel's first order form arises upon the introduction of the momentum

$$p = p_0 + \vec{\sigma} \cdot \vec{p} = p_\mu \sigma^\mu \ , \ \bar{p} = p_0 - \vec{\sigma} \cdot \vec{p} \ . \tag{4c.213}$$

Then

$$L = p_\mu \ (\dot{x}^\mu + k^\mu) - \frac{1}{2} e \, p_\mu \, p^\mu \ . \tag{4c.214}$$

Alternatively,

$$L = \frac{1}{2} \mathrm{Tr}\left[\bar{p} \, u - \frac{1}{2} e \, \bar{p} \, p \right] \ . \tag{4c.215}$$

Variation with respect to p_μ gives

$$p = e^{-1} u \ \ e^{-1}\left[\dot{X} + \frac{1}{4} (\theta \, \dot{\theta}^\dagger - \dot{\theta} \, \theta^\dagger) \right] \ . \tag{4c.216}$$

Therefore L coincides with Eq. (4c.212).

After variation with respect to e, we get

$$\bar{p} \, p = p_\mu \, p^\mu = 0 \ . \tag{4c.217}$$

So the momentum p_μ and the generalized velocity u_μ are light-like. Furthermore, variations w.r.t. $\dot{x}^\mu$ and θ give the equations of motion

$$\dot{p} = \frac{d}{d\tau}(e^{-1} u) = 0 \ , \tag{4c.218}$$

$$\bar{p}\,\dot{\theta} = 0 \quad . \tag{4c.219}$$

As we remarked previously, the constraint is solved by taking

$$p = \alpha\,\alpha^{\dagger} \quad ; \tag{4c.220}$$

α being a left-handed spinor. The novelty here is that $\bar{p} = \hat{\alpha}\,\hat{\alpha}^{\dagger}$ with α defined in Eq. (4c.69). Then p is lightlike in D = 3, 4, 6 and 10. The general solution for α is

$$\alpha = \alpha_0\,U\,(\tau) \quad , \quad (U\,U^{\dagger} = I) \ , \tag{4c.221}$$

where U belongs respectively to U(1), U(2), Sp(7) for D = 3, 4, 10 and α_0 is a constant 2-component spinor.

Consequently, we find

$$p = \alpha_0\,\alpha_0^{\dagger} \quad , \tag{4c.222}$$

$$u = e\,p = e(\tau)\,\alpha_0\,\alpha_0^{\dagger} \quad , \tag{4c.223}$$

$$\dot{\theta} = p\,\hat{\dot{\beta}}(\tau) = \alpha_0\,\alpha_0^{\dagger}\hat{\dot{\beta}}(\tau) \tag{4c.224}$$

where $\beta(\tau)$ is arbitrary. Initial conditions can be picked so that $\beta(o)$ vanishes. Then we get

$$\theta(\tau) = \alpha_0\,\alpha_0^{\dagger}\,\hat{\beta}(\tau) \quad , \tag{4c.225}$$

leading to

$$\dot{X} = b(\tau)\,\alpha_0\,\alpha_0^{\dagger} = b(\tau)\,p \tag{4c.226}$$

with

$$b(\tau) = e(\tau) - \frac{1}{4}\,\alpha_0^{\dagger}\left(\hat{\beta}\hat{\dot{\beta}}^{\dagger} - \hat{\dot{\beta}}\hat{\beta}^{\dagger}\right)\alpha_0 \quad . \tag{4c.227}$$

The equation for $\dot{X}$ can be readily integrated. Equations (4c.225) and (4c.226) then give the world-line of the superparticle.

Another first order Lagrangian can be obtained by using $\alpha(\tau)$ instead of the momentum p. We take

$$\tilde{L} = \hat{\alpha}^\dagger u \hat{\alpha} = \hat{\alpha}^\dagger \left[\dot{X} + \frac{1}{4}\left(\theta \dot{\theta}^\dagger - \dot{\theta} \theta^\dagger\right)\right] \hat{\alpha} \ . \tag{4c.228}$$

Reparametrization invariance now follows from

$$d\tau = \dot{f}(\tau)\, d\tau \quad , \quad \alpha \rightarrow \left[\dot{f}(\tau)\right]^{-1/2} \alpha \ . \tag{4c.229}$$

Variation w.r.t. α gives the constraint u $\hat{\alpha} = 0$, which is solved by

$$u = e^{-1}(\tau)\, \alpha\, \alpha^\dagger \ . \tag{4c.230}$$

Varying X and θ give the equations of motion

$$\frac{d}{d\tau}(\alpha\alpha^\dagger) = 0 \qquad , \qquad \hat{\alpha}\, \hat{\alpha}^\dagger\, \dot{\theta} = 0 \ . \tag{4c.231}$$

They are the same as Eqs. (4c.216) - (4c.219). A Lagrangian of the form (4c.228) has the advantage of resembling the interaction term of a left-handed spinor field with a left-handed current. A term corresponding to a kinetic Dirac-type Lagrangian and interactions w.r.t. the fermionic current may also be added, leading to the more general Lagrangian

$$L = \hat{\alpha}^\dagger\, \dot{\alpha} + \hat{\alpha}^\dagger\, u\, \hat{\alpha} + a\, \hat{\alpha}^\dagger\, \dot{\theta} + \text{h.c.} \ . \tag{4c.232}$$

The latter is precisely of Dirac type. It is still an open question whether such a generalization can lead to a consistent covariant quantization.

We note that L can be compactly written in terms of a 5-dimensional representation ψ of the super-Poincaré group

$$\psi = \begin{pmatrix} \alpha \\ \gamma a \\ -\tilde{\alpha} \end{pmatrix} \quad , \quad \hat{\psi}^\dagger = \begin{pmatrix} \hat{\alpha}^\dagger & \gamma a & \alpha^\dagger \end{pmatrix} \ . \tag{4c.233}$$

It transforms under the supergroup element $\Gamma(A, \xi)$ as

$$\psi' = \Gamma(A, \xi)\,\psi \tag{4c.234}$$

and

$$\psi'' = \Lambda\,\psi \tag{4c.235}$$

under Lorentz transformations.

Then, the Lagrangian

$$L = \widehat{\psi}^{\dagger}\widehat{W}_{+}^{\dagger}\,\psi + \text{h.c.} \tag{4c.236}$$

has exactly the same structure as Eq. (4c.232), apart from the term $\dot{\alpha}a$, a total derivative. If ψ is not self-conjugate and a is complex, then a dynamical term in a is also added. The generalization to N = 2 supersymmetry can be accomplished by replacing θ and α by 2 x 2 matrices.

4.c.7. Actions for superstrings

For the superstring the position supervector (X, θ) is a function of τ and σ, the real coordinates on the worldsheet. One can also use the light cone variables

$$\xi_{\pm} = \frac{1}{\sqrt{2}}(\tau \pm \sigma) = \frac{1}{\sqrt{2}}(\xi_0 \pm \xi_1) \tag{4c.237}$$

or, in the Euclidean formulation, the complex variables $z = \frac{1}{\sqrt{2}}(\tau + i\sigma)$ and $\bar{z} = \frac{1}{\sqrt{2}}(\tau - i\sigma)$.

Under worldsheet parity, we have

$$\xi_{+} \leftrightarrow \xi_{-} \quad , \quad z \leftrightarrow \bar{z} \ . \tag{4c.238}$$

The super-Poincaré group elements Γ(X, θ) and the chiral elements $Z_{\pm}$ are now functions of $\xi_{\pm}$ or z and $\bar{z}$. Instead of the super-velocity W, we have the matrices

$$W_{\alpha}^{\pm} = \left[\Gamma^{-1}\partial_{\alpha}\Gamma\ , \mu_{\pm}\right] \tag{4c.239}$$

and their conjugates $\widehat{W}^{\dagger}_{\alpha}$ defined as in Eq. (4c.198). The reparametrization invariant element $e^{-1}d\tau$ is now replaced by

$$d^2\xi\sqrt{g}\, g^{\alpha\beta} = d^2\xi\, \sqrt{g}\, (G^{-1})_{\alpha\beta} \tag{4c.240}$$

with the metric tensor

$$G = G^T = \begin{pmatrix} g_{00} & g_{01} \\ g_{01} & g_{11} \end{pmatrix} = g_{00}\frac{1+\tau_3}{2} + g_{11}\frac{1-\tau_3}{2} + \tau_1 g_{01} \ . \tag{4c.241}$$

Its negative determinant ($g > 0$) is

$$-g = \mathrm{Det}\, G = \overline{G}\, G = g_{00}\, g_{11} - g_{01}^2 \ . \tag{4c.242}$$

In the above we have introduced a set of Pauli matrices τ_i which commute with the σ_i used for the D-dimensional spacetime vectors.

It follows that the matrix $i\sqrt{g}\, g^{\alpha\beta}$ has determinant -1 and is unimodular. For a general (2x2) matrix A, we have defined an $\overline{A}$ such that

$$A\,\overline{A} = \mathrm{Det}\, A \ , \ (\, \overline{A} = \tau_2\, A^T\, \tau_2) \ . \tag{4c.243}$$

The Lorentz metric on the worldsheet is represented by the diagonal matrix τ_3. We can introduce a zweibein matrix E by demanding

$$G = E\, \tau_3 E^T \tag{4c.244}$$

so that $\sqrt{g} = \mathrm{Det}\, E$.

E can always be chosen symmetric, then

$$\overline{E} = \tau_2\, E\, \tau_2 \ \ , \ E = E^T \tag{4c.245}$$

and

$$\frac{1}{\sqrt{g}}\overline{E}\ \ \tau_2 E^{-1} = E^{-1}\tau_2 E^{-1} = \frac{1}{\sqrt{g}}\tau_2 \ . \tag{4c.246}$$

Thus, we have the projection operators

$$P_{\pm} = \frac{1}{2}\left(G^{-1} \pm \frac{i\tau_2}{\sqrt{g}} \right) = \frac{1}{2} E^{-1} (\tau_3 \pm i\,\tau_2) E^{-1} \tag{4c.247}$$

with

$$Det\, P_{\pm} = P_{\pm} \overline{P}_{\pm} = 0 \quad , \quad P_{+} G\, P_{-} = P_{-} G\; P_{+} = 0 \tag{4c.248}$$

and

$$P_{+} G\, P_{+} = E^{-1} \frac{\tau_3 + i\,\tau_2}{2} \frac{\tau_3 - i\,\tau_2}{2} \tau_3 E^{-1} = P_{+} \quad , \tag{4c.249}$$

$$P_{-} G\, P_{-} = E^{-1} \frac{\tau_3 - i\,\tau_2}{2} \frac{\tau_3 + i\,\tau_2}{2} \tau_3 E^{-1} = P_{-} \quad . \tag{4c.250}$$

The metric G can be replaced by τ_3 if we define

$$\Pi_{\pm} = E\, P_{\pm} E = \frac{\tau_3 \pm i\,\tau_2}{2} \quad . \tag{4c.251}$$

We verify that

$$\Pi_{\pm} \overline{\Pi}_{\pm} = -\left(\Pi_{\pm}\right)^2 = 0 \quad , \quad \Pi_{\pm} \tau_3 \Pi_{\pm} = \Pi_{\pm} \quad , \tag{4c.252}$$

$$\Pi_{+} \tau_3 \Pi_{-} = \left(\Pi_{+}\right)^2 \tau_3 = 0 \quad , \quad \Pi_{-} \tau_3 \Pi_{+} = \left(\Pi_{-}\right)^2 \tau_3 = 0 \quad . \tag{4c.253}$$

Due to the reparametrization invariance of the action, there always exists a gauge where G and E reduce to τ_3 and the unit matrix, respectively.

We note that the matrix W_α^+ has $\partial_\alpha \theta$ and

$$u_\alpha = \partial_\alpha X + \frac{1}{4}\left[\theta \partial_\alpha \theta^\dagger - \partial_\alpha \theta \theta^\dagger \right] \tag{4c.254}$$

as elements, while those of W_α^- are u_α and $\partial_\alpha \theta^\dagger$. These matrices are invariant under the left action of the supergroup element $\Gamma(A, \xi)$ on $\Gamma(X, \theta)$. On the other hand, the chiral matrices

$$\widetilde{W}_\alpha^{\pm} = \left[\left(\partial_\alpha \Gamma\right) \Gamma^{-1} , \mu_{\pm} \right] \tag{4c.255}$$

are invariant under the right action

$$\Gamma(X,\theta)\rightarrow\Gamma(X,\theta)\Gamma(B,\eta) \tag{4c.256}$$

but transform under the left action supersymmetry.

The elements of $\widetilde{W}^+_\alpha$ are $\partial_\alpha\theta$ and

$$v_\alpha = \partial_\alpha X - \frac{1}{4}\left[\theta\partial_\alpha\theta^\dagger - \partial_\alpha\theta\theta^\dagger\right] \tag{4c.257}$$

while those of $\widetilde{W}^-_\alpha$ are v_α and $\partial_\alpha\theta^\dagger$. Explicitly,

$$\widetilde{W}^+_\alpha = \begin{pmatrix} 0 & \gamma\partial_\alpha\theta & v_\alpha \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad , \quad \widetilde{W}^-_\alpha = \begin{pmatrix} 0 & 0 & v_\alpha \\ 0 & 0 & \gamma\partial_\alpha\theta^\dagger \\ 0 & 0 & 0 \end{pmatrix} . \tag{4c.258}$$

The conjugate matrices transforming under Lorentz transformations like the w_α and $\widetilde{w}_\alpha$ are $\widetilde{w}^\dagger_\alpha$ and $\widehat{\widetilde{w}}^\dagger_\alpha$, as defined in Eq. (4c.198). Then we have the invariants

$$\mathrm{Str}\left(\widetilde{W}^\dagger_{+\alpha}\widetilde{W}^-_\alpha\right) = \bar{u}_\alpha u^\alpha \tag{4c.259}$$

under left acting supersymmetry, the corresponding one $\bar{v}_\alpha v^\alpha$ constructed out of the $\widetilde{w}_\pm$ for the right acting supersymmetry and the invariants

$$\partial_\alpha\hat{\theta}^\dagger\partial^\alpha\theta \quad , \quad \partial_\alpha\theta^\dagger\partial^\alpha\hat{\theta} \tag{4c.260}$$

for both kinds of supersymmetries.

As shown by Green and Schwarz, there is a second invariant action for superstrings, namely

$$I_2 = -\frac{1}{2}\mathrm{Tr}\int d^2\xi\, \varepsilon^{\alpha\beta}\bar{u}_\alpha v_\beta \ . \tag{4c.261}$$

The first invariant generalizing the superstring action is

$$I_1 = -\frac{1}{4}\,\mathrm{Tr}\int d^2\xi\sqrt{g}\;g^{\alpha\beta}\bar{u}_\alpha u_\beta \quad . \tag{4c.262}$$

Both actions are reparametrization invariant. I_2 is also invariant under left acting supersymmetry though the vector v is not. To see this, we write

$$u_\alpha = \partial_\alpha X + N_\alpha \qquad , \qquad v_\alpha = \partial_\alpha X - N_\alpha \tag{4c.263}$$

where

$$N_\alpha = \frac{1}{4}\left[\,\theta\partial_\alpha\theta^\dagger - \partial_\alpha\theta\theta^\dagger\,\right] \;. \tag{4c.264}$$

Then

$$L_2 = -\frac{1}{4}\,\varepsilon^{\alpha\beta}\,\mathrm{Tr}\,\left(\,\bar{u}_\alpha\, v_\beta\,\right) = -\frac{1}{4}\,\mathrm{Tr}\,(\,\overline{u_0}\,v_1 - \overline{u_1}\,v_0) \tag{4c.265}$$

$$= -\frac{1}{4}\,\mathrm{Tr}\,(\,\overline{u_0}\,v_1 - \overline{v_0}\,u_1\,)\;\; . \tag{4c.266}$$

Only cross terms are left, so

$$L_2 = \frac{1}{2}\,\mathrm{Tr}\left[(\,\partial_0\,\overline{X}\,)\,N_1 - \,\overline{N_0}\,(\partial_1\,X)\,\right] = \frac{1}{2}\,\varepsilon^{\alpha\beta}\mathrm{Tr}\left[\,\left(\partial_\alpha\,\overline{X}\right)N_\beta\,\right] \quad , \tag{4c.267}$$

which is the usual form of L_2.

Under left acting supersymmetry with infinitesimal parameter ε, we have

$$\delta\theta = \varepsilon \;\;,\; \delta\partial_\alpha\theta = 0 \quad , \tag{4c.268}$$

$$\delta(\,\partial_\alpha X) = \frac{1}{4}\left(\,\partial_\alpha\theta\,\varepsilon^\dagger - \varepsilon\partial_\alpha\theta^\dagger\,\right) \;\;,\;\; \delta N_\alpha = -\,\delta\;(\,\partial_\alpha X\,)\;\; . \tag{4c.269}$$

So

$$\delta\,L_2 = \frac{1}{2}\,\varepsilon^{\alpha\beta}\mathrm{Tr}\;\left[\,\left(\delta\overline{N}_\alpha\right)N_\beta + (\partial_\alpha\overline{X})\;\delta N_\beta\,\right]\;. \tag{4c.270}$$

The second term is a total derivative since

$$\varepsilon^{\alpha\beta}\mathrm{Tr}\left[\,\left(\partial_\alpha\,\overline{X}\right)\delta N_\beta\,\right] = -\frac{1}{4}\,\varepsilon^{\alpha\beta}\mathrm{Tr}\left[\left(\partial_\alpha\,\overline{X}\right)\left[\,\varepsilon\partial_\beta\theta^\dagger - \partial_\beta\theta\,\varepsilon^\dagger\,\right]\right] \tag{4c.271}$$

$$= -\frac{1}{4}\,\partial_\alpha\left\{\,\varepsilon^{\alpha\beta}\mathrm{Tr}\left[\,\overline{X}\left\{\,\varepsilon\partial_\beta\theta^\dagger - \partial_\beta\theta\,\varepsilon^\dagger\,\right\}\right]\right\}\;. \tag{4c.272}$$

It makes no contribution to the action.

The first term is

$$\delta L_2 = \frac{1}{32}\varepsilon^{\alpha\beta}\,\mathrm{Tr}\left\{\left[(\partial_\beta\theta)\,\varepsilon^\dagger - \varepsilon\left(\partial_\beta\theta^\dagger\right)\right]\left[\theta\left(\partial_\beta\theta^\dagger\right) - (\partial_\beta\theta)\,\theta^\dagger\right]\right\} \ . \tag{4c.273}$$

Except for a total divergence term, it only involves antisymmetric combinations of θ, $\partial_0\theta$ and $\partial_1\theta$ which, due to the identities (4c.41), (4c.43), vanish in the critical dimensions 3, 4, 6 and 10.

Any combination of I_1 and I_2 is invariant under left acting supersymmetry, usually identified with global supersymmetry. However, only combinations with relative coefficients ± 1 have a fermionic gauge invariance, also present in the superparticle case. The underlying reason is that only such combination brings in the projection operators $P_\pm$ (4c.247). These projectors play an essential role in the proof of the gauge invariance connected to a local form of right acting supersymmetry.

If the worldsheet is taken as the boundary of a three-manifold, then, as shown by Curtright et al., the action I_2 can be interpreted as a topological (Wess-Zumino-Novikov-Witten type) term associated with the super-Poincaré group. Note that I_2 is symmetric between u and v, hence it is invariant not only w.r.t. left acting supersymmetry but also w.r.t. right-handed supersymmetry with parameters η, as in Eq. (4c.256). Combining both transformations, we find the matrix

$$W_\alpha = \left[\,(\hat{\Gamma}^\dagger)^{-1}\partial_\alpha\hat{\Gamma}^\dagger + \Gamma^{-1}\partial_\alpha\Gamma\ ,\mu_\pm\right] \tag{4c.274}$$

or

$$W_\alpha = \begin{pmatrix} 0 & \gamma\,\partial_\alpha\theta & u_\alpha \\ 0 & 0 & 0 \\ \bar{v}_\alpha & \gamma\,\partial_\alpha\hat{\theta} & 0 \end{pmatrix} \ . \tag{4c.275}$$

Then L_2 is given by a supertrace

$$L_2 = \frac{1}{4}\varepsilon^{\alpha\beta}\,\mathrm{Str}\,(\,W_\alpha\,\mu_+ W_\beta\,) \ . \tag{4c.276}$$

4.c.8. Local symmetries of superstring actions

On the one hand, we have seen how the supersymmetric particle and string theories involve a light-like momentum with (D – 2) transverse directions and a spinor with 2(D – 2) components in the critical dimensions D. On the other hand, worldsheet supersymmetry demands equal numbers of physical bosonic and fermionic degrees of freedom. Since the transverse vector components represent the physical bosonic degrees of freedom, (D – 2) components of the spinor are not independent and must be eliminated by a gauge transformation. As shown by Siegel, such a gauge symmetry exists for the superparticle and can be generalized to the superstring provided the two invariant actions I_1 and I_2 have fixed relative coefficients equal to ± 1.

First, let us consider this local symmetry at the level of the superparticle. The Lagrangian is

$$L = \frac{1}{2} e^{-1} \bar{u} u \tag{4c.277}$$

where

$$u = \dot{X} + \frac{1}{4} (\theta\dot{\theta}^{\dagger} - \dot{\theta}\theta^{\dagger}) \ . \tag{4c.278}$$

The constant light like momentum is

$$p = e^{-1}u \ , \ pp = 0 \, , \ \dot{p} = 0 \tag{4c.279}$$

and the equation of motion for θ reads

$$\bar{p}\,\dot{\theta} = 0 \ . \tag{4c.280}$$

Recall that, under a right acting transformation ,

$$\delta\theta = \eta \quad , \quad \delta X = \frac{1}{4}\left(\eta\theta^{\dagger} - \theta\eta^{\dagger} \right) , \tag{4c.281}$$

instead of the left acting transformation

$$\delta\theta = \varepsilon \quad , \quad \delta X = \frac{1}{4}\left(\varepsilon\theta^{\dagger} - \theta\varepsilon^{\dagger} \right) \ , \tag{4c.282}$$

L is invariant under the ε transformation if e is invariant. This is not so for the ε transformation if e is unchanged.

Next we take up a local η transformation with the specific form

$$\eta(\tau) = P\,\widehat{K}(\tau) \quad , \quad \widehat{\eta} = -\overline{P}\,K(\tau) \ , \tag{4c.283}$$

with κ being a left-handed spinor. We obtain

$$\delta\theta = \eta \quad , \quad \delta\dot{\theta} = \dot{\eta} \ . \tag{4c.284}$$

So, if $N = \frac{1}{4}\left(\theta\dot{\theta}^{\dagger} - \dot{\theta}\,\theta^{\dagger} \right)$, its variation is

$$\delta N = \frac{1}{4}\left(\eta\dot{\theta}^{\dagger} - \dot{\theta}\eta^{\dagger} + \theta\dot{\eta}^{\dagger} - \dot{\eta}\theta^{\dagger} \right) \ . \tag{4c.285}$$

On the other hand, we have

$$\delta\dot{X} = \frac{1}{4}\left(\eta\dot{\theta}^{\dagger} - \dot{\theta}\eta^{\dagger} + \dot{\eta}\theta^{\dagger} - \theta\dot{\eta}^{\dagger} \right) \tag{4c.286}$$

leading to

$$\delta u = \delta\dot{X} + \delta N = \frac{1}{2}\left(\eta\,\dot{\theta}^{\dagger} - \dot{\theta}\eta^{\dagger} \right) \tag{4c.287}$$

with no dependence on $\dot{\eta}$. The action is invariant if

$$\delta(e^{-1}\,\overline{u}\,u) = e^{-1}\left[-\frac{\delta e}{e}\,\overline{u}\,u + \left(\delta\overline{u}\right)u + \overline{u}\,\delta u \right] = 0 \ , \tag{4c.288}$$

which may be recast as

$$\overline{u}\,u\,\delta\,e = e^{2}\,\mathrm{Tr}(\,\overline{p}\delta u\,) = \frac{e^{2}}{2}\,\mathrm{Tr}\left(\overline{p}\,\eta\,\dot{\theta}^{\dagger} - \overline{p}\dot{\theta}\eta^{\dagger} \right) \ . \tag{4c.289}$$

By way of Eqs. (4c.280) and (4c.283), we get

$$\overline{u}\,u\,\delta\,e = \frac{1}{2}\,e^{2}\,\mathrm{Tr}(\,\overline{p}p\,\widehat{\kappa}\dot{\theta}\,) = -\frac{1}{2}\,\overline{u}u\,\dot{\theta}^{\dagger}\widehat{\kappa} \qquad , \tag{4c.290}$$

ending up with

$$\delta e = \frac{1}{2} \dot{\theta}^{\dagger} \hat{\kappa} \tag{4c.291}$$

together with

$$\delta \theta = p\, \hat{\kappa} \quad , \quad \delta u = \frac{1}{2} p\, \hat{\kappa}\, \dot{\theta}^{\dagger} \quad . \tag{4c.292}$$

In the light cone gauge, the lightlike vector p takes the form

$$p = (\, p_1 + p_3 \,) \frac{1 + \sigma_3}{2} = \begin{pmatrix} p_+ & 0 \\ 0 & 0 \end{pmatrix} . \tag{4c.293}$$

So

$$\begin{pmatrix} \delta\theta_1 \\ \delta\theta_2 \end{pmatrix} = p\, \hat{\kappa} = \begin{pmatrix} p_+ & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} -\kappa_2^* \\ \kappa_1^* \end{pmatrix} = \begin{pmatrix} -p_+\kappa_2^* \\ 0 \end{pmatrix} . \tag{4c.294}$$

Therefore θ_1 can be made to vanish by a κ local transformation. Only half of the components of θ i.e. θ_2, survives. θ_2 is an element of one of the Hurwitz division algebras and has (D − 2) components in critical dimensions, the same number of components as the transverse vectors in the same dimensions. Consequently, the worldsheet supersymmetry is restored in the light cone gauge.

Moreover, by exploiting the reparametrization invariance, we can make e = 1, giving p = u and

$$\frac{d}{d\tau} \left[\dot{X} + \frac{1}{4} (\theta\dot{\theta}^{\dagger} - \dot{\theta}\theta^{\dagger}) \right] = 0 \tag{4c.295}$$

as equation of motion, together with $\bar{u}\, \dot{\theta} = 0$ as a constraint.

We now turn to the Green-Schwarz superstring. The symmetric matrix associated with the worldsheet metric factor $\sqrt{g}\, g^{\alpha\beta}$ can be represented by

$$H = \sqrt{g}\, G^{-1} = F\, \tau_3\, F \quad , \quad (\, F\,\overline{F} = 1 \,) \tag{4c.296}$$

F is symmetric and unimodular. The determinant of H is minus one, as it should be.

F is connected to the metric E (4c.244) by

$$F = F^T = (g)^{-\frac{1}{4}}\overline{E} = (g)^{-\frac{1}{4}}\tau_2\, E\, \tau_2 \ . \tag{4c.297}$$

Then the superstring action reads

$$I = \frac{1}{4}\,\mathrm{Tr}\int d^2\xi\,(\,u^\dagger F\,\tau_3\,F\,u - u^\dagger F\,i\tau_2\,F\,v\,) \tag{4c.298}$$

since

$$F\,i\,\tau_2\,F \;=\; i\,F\,\overline{F}\,\tau_2 = i\,\tau_2 \ . \tag{4c.299}$$

Here

$$u = \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} \ , \quad v = \begin{pmatrix} v_0 \\ v_1 \end{pmatrix} \ , \quad u^\dagger = \bar{u}^T = (\bar{u}_0\ \bar{u}_1\,) \ . \tag{4c.300}$$

We shall be consistent about our notations. The symbol tr denotes the trace w.r.t. the worldsheet Pauli matrices τ_i, while Tr is the trace over the space-time Pauli matrices σ_i. Str denotes the supertrace of (5 x 5) matrices over **K**, representing the super-Poincaré group transformations.

The worldsheet vector N_α defined in Eq. (4c.264) and the gradient $\partial_\alpha X$ are representable by column matrices

$$N = \begin{pmatrix} N_0 \\ N_1 \end{pmatrix} \qquad , \qquad \nabla\chi = \begin{pmatrix} \partial_0\chi \\ \partial_1\chi \end{pmatrix} \tag{4c.301}$$

so that $v = u - 2N$, $u = \nabla\chi + N$.

The action has the alternative form

$$I = \frac{1}{2}\,\mathrm{Tr}\int d^2\xi\left[\,\frac{1}{2}\tilde{u}^\dagger\, F\tau_3 F\,\tilde{u} + (\nabla\chi)^\dagger i\tau_2 N\,\right] \tag{4c.302}$$

giving rise to equations involving the projection operators $\Pi_\pm$ (4c.251). Due to the reparametrization invariance of the action, there is a worldsheet coordinate system where F reduces to one. Generally, we can define $\tilde{u} = F\,u$ to cast the action into the simple form

$$I = \frac{1}{2}\,\mathrm{Tr}\int d^2\xi\left[\frac{1}{2}\,\tilde{u}^\dagger\,\tau_3\,\tilde{u} + (\nabla\chi)^\dagger i\tau_2 N\right] . \tag{4c.303}$$

We can also go to the light cone coordinates via the transformation

$$\begin{pmatrix}\xi_+\\ \xi_-\end{pmatrix} = \frac{1 + i\,\tau_2}{\sqrt{2}}\begin{pmatrix}\xi_0\\ \xi_1\end{pmatrix} = \exp\left\{i\,\frac{\pi}{4}\,\tau_2\right\}\begin{pmatrix}\xi_0\\ \xi_1\end{pmatrix} . \tag{4c.304}$$

So, introducing the new variables

$$\hat{u}' = \exp\left\{i\,\frac{\pi}{4}\,\tau_2\right\}u \;\;,\;\; \nabla' = \exp\left\{i\,\frac{\pi}{4}\,\tau_2\right\}\nabla \;,\;\; N' = \exp\left\{i\,\frac{\pi}{4}\,\tau_2\right\}N \;\;, \tag{4c.305}$$

results in

$$I = \mathrm{Tr}\int d\xi_+\,d\xi_-\left[\frac{1}{2}\,\tilde{u}'^\dagger\;\tau_1\,\tilde{u}' + (\nabla'\chi)^\dagger\; i\;\tau_2 N'\right] \tag{4c.306}$$

$$= \frac{1}{2}\,\mathrm{Tr}\int d\xi_+\,d\xi_-\left[\,\partial_-\overline{X}\partial_+X + \overline{N}_-\,N_+ + 2\left(\partial_-\overline{X}\right)N_+\,\right] \tag{4c.307}$$

as the simplest expression for the covariant superstring action when the special gauge $\tilde{u}_\pm$ has been replaced by $u_\pm$. Here

$$N_\pm = \frac{1}{4}\left[\,\theta\left(\partial_\pm\theta^\dagger\right) - \left(\partial_\pm\theta\right)\theta^\dagger\,\right] . \tag{4c.308}$$

For N = 2 supersymmetry, we have an additional term involving another spinor θ' and a term $(\partial_+\overline{X})\,N_-$.

The light cone expression for the projectors $\Pi_\pm$ are

$$\Pi'_\pm = \exp\left(-\,i\,\frac{\pi}{4}\,\tau_2\right)\Pi_\pm\,\exp\left(i\,\frac{\pi}{4}\,\tau_2\right) = \frac{\tau_1 \pm i\tau_2}{2} . \tag{4c.309}$$

Therefore, Eq. (4c.307) displays the projection operators when the Lagrangian reads

$$L = \frac{1}{2} \mathrm{Tr} \left(\nabla'^{T} \overline{X}\, \Pi'_{-} \nabla' X + N'^{\dagger} \Pi'_{-} N' + 2\, \nabla'^{T} \overline{X}\, \Pi'_{-} N' \right) \ . \quad (4c.310)$$

The resulting equations of motion are

$$\left(\partial_{-} \overline{X} + \overline{N}_{-} \right) \partial_{+} \theta = \overline{u}_{-} \partial_{+} \theta = u'^{\dagger} \Pi'_{-} \nabla' \theta = 0 \ , \quad (4c.311)$$

$$\partial_{+} \partial_{-} X - \partial_{-} N_{+} = 0 \quad (4c.312)$$

along with another equation expressing the vanishing of the energy-momentum tensor.

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